L-infini déformations et cohomologie de Hochschild

Frank Schuhmacher

Résumé

L_∞-déformations

Rappelons quelques faits et définitions concernant les L_∞-algèbres avant de donner une introduction à la théorie des déformations qu'on va développer dans le Chapitre 1.

L_∞-algèbres Une L_∞-algèbre sur un anneau k de caractéristique 0 est par définition un module $\mathbb{Z}$-gradué L muni d'une suite $\mu_* = (\mu_n)_{n\geq 0}$ de morphismes $\mu_n : L^{\otimes n} \longrightarrow L$ gradués antisymétriques de degré $2-n$ telle que, pour chaque $n \geq 0$ et $a_1, \ldots, a_n \in L$, on a la condition suivante:

$$\sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} (-1)^{k(l-1)} \chi(\sigma) \mu_l(\mu_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), a_{\sigma(k+1)}, \ldots, a_{\sigma(n)}) = 0 \qquad (*_n)$$

Ici, $\mathrm{Sh}(k,n)$ dénote l'ensemble des (k,n)-shuffles, c'est-à-dire l'ensemble des toutes les permutations $\sigma \in \Sigma_n$ telles que $\sigma(1) < \ldots < \sigma(k)$ et $\sigma(k+1) < \ldots \sigma(n)$. Le terme $\chi(\sigma) = \chi(\sigma, a_1, \ldots, a_n)$ est le signe qui est défini par la condition

$$a_{\sigma(1)} \wedge \ldots \wedge a_{\sigma(n)} = \chi(\sigma) a_1 \wedge \ldots \wedge a_n.$$

Souvent on ne considère que les L_∞-algèbres (L, μ_*) telles que $\mu_0 = 0$. Dans ce cas, la condition $(*_1)$ signifie juste que (L, μ_1) est un DG module, la condition $(*_2)$ signifie que μ_1 est une différentielle vis-à-vis du produit μ_2. La condition $(*_3)$ signifie que (L, μ_2) satisfait la condition de Jacobi pour les algèbres de Lie graduées à homotopie (donnée par μ_3) près. Donc les L_∞-algèbres généralisent les algèbres de Lie différentielles graduées (qu'on va appeler DGL algèbres).

Grâce au fait général suivant, les DGL algèbres et les L_∞-algèbres jouent un rôle important dans la théorie des déformations: à un objet donné, on peut associer une DGL algèbre qui gouverne sa théorie des déformations. Deux objets ont une théorie des déformations équivalente dès que leurs DGL algèbres associées respectives sont équivalentes comme L_∞-algèbres.

Construction bar et morphismes La construction bar associe à une L_∞-algèbre (L, μ_*) une DG coalgèbre libre dont le module sous-jacent est l'algèbre graduée symétrique $S(L[1])$ sur le shifting $L[1]$ de L. Posons $M := L[1]$. Les morphismes μ_n

définissent des morphismes gradués symétriques

$$Q_n^M :\downarrow \mu_n \uparrow^n : M^{\otimes n} \longrightarrow M$$

de degré 1. Une telle suite $(Q_n^M)_{n\geq 0}$ définit de manière unique une codérivation Q^M sur $S(M)$ et la suite de conditions $(*_n)_{n\geq 1}$ est équivalente à la seule condition $(Q^M)^2 = 0$. La construction bar nous donne ainsi une correspondance bijective entre les L_∞-algèbres et les DG coalgèbres libres. On définit les morphismes de L_∞-algèbres de telle manière que cette correspondance devient fonctorielle. C'est-à-dire qu'**un morphisme** $f : L \longrightarrow L'$ de L_∞-algèbres est une suite $f_n : L^{\otimes n} \longrightarrow L'$ de morphismes $(n \geq 1)$ gradués symétriques de degré $(1-n)$ telle que la suite $(\downarrow f_n \uparrow^n)_{n\geq 1}$ définit un morphisme $S(L[1]) \longrightarrow S(L'[1])$ de DG coalgèbres (voir Section 1.1).

DGL algèbres à homotopie près Une classe importante de L_∞-algèbres est donnée par les DG modules, homotopiquement équivalents à une DGL algèbre. On peut montrer que, si (M, d^M) est un DG module, si $(L, d, [\cdot,\cdot])$ est une DGL algèbre et si $f : M \longrightarrow L$ est une équivalence d'homotopie, alors il existe une structure μ_* de L_∞-algèbre sur M telle que $\mu_1 = d^M$ et il existe un morphisme $f_* : M \longrightarrow L$ de L_∞-algèbres tel que $f_1 = f$. Comme cas particulier, on obtient l'existence de la structure de L_∞-algèbre sur la cohomologie H d'une DGL algèbre $(L, d, [\cdot,\cdot])$ dont le complexe (L, d) est scindé. Ce résultat a été obtenu par des méthodes non-constructives. En vue des applications à la théorie des déformations, on voudrait avoir une description explicite d'une telle structure μ_* sur H. Dans cette thèse on démontre le résultat suivant:

Théorème 0.0.1. *Soit η un scindage du complexe (L, d) satisfaisant les conditions $\eta^2 = 0$, $\eta d\eta = \eta$ et $d\eta d = d$. Les morphismes gradués antisymétriques $\mu_n : H^{\otimes n} \longrightarrow H$ de degré $(2-n)$ qui suivent définissent une structure de L_∞-algèbre sur H:*

$$\begin{aligned}
\mu_1 &:= 0\\
\mu_2 &:= (1 - d\eta)[\cdot,\cdot]\\
&\vdots\\
\mu_n &:= (\frac{-1}{2})^{n-1} \sum_{\phi\in \mathrm{Ot}_n} e(\phi)\phi((1-[d,\eta])[\cdot,\cdot], \eta[\cdot,\cdot], \dots, \eta[\cdot,\cdot]) \circ \alpha_n
\end{aligned}$$

Ici la somme porte sur tous les arbres binaires ϕ de n feuilles et $\phi((1-[d,\eta])[\cdot,\cdot], \eta[\cdot,\cdot], \dots, \eta[\cdot,\cdot])$ désigne la forme n-linéaire qui est donnée en associant la forme bilinéaire $(1-[d,\eta])[\cdot,\cdot]$ à la racine de l'arbre ϕ et la forme $\eta[\cdot,\cdot]$ à toute autre ramification de ϕ. Le symbole α_n désigne l'antisymétrisation et $e(\phi)$ est un signe qui dépend de la géométrie de ϕ.

On obtient aussi une description du morphisme $f : H \longrightarrow L$:

Théorème 0.0.2. *Les morphismes gradués antisymétriques $f_n : H^{\otimes n} \longrightarrow L$ de degré $(1-n)$ qui suivent définissent une L_∞-équivalence $H \longrightarrow L$:*

$$
\begin{aligned}
f_1 &:= \textit{inclusion}\\
f_2 &:= -\eta[\cdot,\cdot]\\
&\vdots\\
f_n &:= -(\frac{-1}{2})^{n-1} \sum_{\phi\in \mathrm{Ot}(n)} e(\phi)\phi(\eta[\cdot,\cdot],\dots,\eta[\cdot,\cdot])\circ\alpha_n.
\end{aligned}
$$

Un Théorème de décomposition Un fait qui est extrêmement utile pour cette théorie est que la catégorie de L_∞-algèbres satisfait l'axiome M1 de Quillen pour les catégories des modèles. Le fait analogue pour les A_∞-algèbres a été prouvé par K.Lefèvre [28] et dans cette thèse on le prouve pour les L_∞-algèbres:

Proposition 0.0.3. *Soit*

$$
\begin{tikzcd}
A \arrow[r, "c"] \arrow[d, "f"] & C \arrow[d, "e"] \\
B \arrow[r, "d"] & D
\end{tikzcd}
$$

un diagramme commutatif de L_∞-algèbres. Supposons que (a) le morphisme f est injectif et scindé, (b) le morphisme e est surjectif et scindé, (c) soit f soit e est un L_∞-quasi-isomorphisme. Alors il existe un morphisme $g : B \longrightarrow C$ de L_∞-algèbres tel que tout le diagramme

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[d] \\
B \arrow[r] \arrow[ur, "g"] & D
\end{tikzcd}
$$

commute.

En posant, dans ce diagramme $A := C := H$ et $B := L$, où L et H sont choisis comme en Théorème 0.0.2 on obtient l'existence d'un morphisme $g : L \longrightarrow H$ tel que $g \circ f = \mathrm{Id}_H$. Donc, H est un facteur direct de L. En utilisant de nouveau la Proposition 0.0.3, on montre le résultat plus précis:

Théorème 0.0.4. *Soit $F := \mathrm{Kern}(1-[d,\eta])$ le complément de $H = \mathrm{Kern}([d,\eta])$ dans (L,d). Il y a alors un isomorphisme*

$$(L,d,[\cdot,\cdot]) \cong (H,\mu_*) \oplus (L,d,0)$$

dans la catégorie des L_∞-algèbres.

Une conséquence directe est que si on travaille sur un corps, alors l'existence d'un L_∞-quasi-isomorphisme entre deux DGL-algèbres définit une relation d'équivalence sur la classe des DGL algèbres qu'on appelle L_∞-équivalence.

DG variétés formelles et singularités On motive l'interprétation géométrique des L_∞-algèbres en utilisant une construction du Chapitre 2:

Dans ce paragraphe on suppose que $\mathbb{K}$ est un corps de caractéristique zéro. On va montrer que, pour toute singularité X qui est plongée dans le germe lisse $(\mathbb{K}^n, 0)$, il existe un $\mathbb{K}$-espace vectoriel gradué $M = \oplus_{i\geq 0} M^i$ avec $M^0 = \mathbb{K}^n$ et une codifférentielle Q^M sur la coalgèbre graduée libre $S(M)$ telle que X est l'espace des zéros de l'application analytique $M^0 \longrightarrow M^1$. C'est-à-dire: pour $k \geq 0$, soit $f_k : M^0 \longrightarrow M^1$ l'application $x \mapsto Q_k^M(x, \ldots, x)$. Alors X est l'espace des zéros de

$$f := \sum_{k\geq 1} \frac{1}{k!} f_k.$$

On obtient (M, Q^M) en "prédualisant" une résolution de Tyurina de l'algèbre analytique associée à X. En plus, le couple (M, Q^M) est uniquement défini à L_∞-équivalence près. Autrement dit:

Proposition 0.0.5. *Il existe un foncteur de la catégorie des singularités dans la localisation de la catégorie des L_∞-algèbres par les L_∞-quasi-isomorphismes.*

Suivant une suggestion de Kontsevich, pour une DG coalgèbre libre $(S(M), Q^M)$, on appelle le couple (M, Q^M) une **DG variété formelle**. On appelle les codérivations sur $S(M)$ des **(super) champs vectoriels**.

Déformations Soient (M, Q^M) et (B, Q^B) deux DG variétés formelles. Une déformation de (M, Q^M) à base (B, Q^B) est un super champs vectoriel Q sur $B \times M$ tel que

(i) $Q|_{\{0\}\times M} = 0$.

(ii) Le super champs vectoriel $\tilde{Q} := Q^M + Q^B + Q$ est une DG structure sur $B \times M$, c'est-à-dire $\tilde{Q}^2 = 0$.

(iii) La projection $B \times M \longrightarrow B$ est un morphisme de DG variétés formelles.

On dénote une déformation de (M, Q^M) comme triplet (B, Q^B, Q). Une déformation est dite **triviale**, si la projection $B \times M \longrightarrow M$ respecte les structures $\tilde{Q}$ and Q^M.

Pour un élément b de $S(B)$ fixé, la "fibre" $Q^M + Q(b, \_)$ peut être interprétée comme perturbation du super champs vectoriel Q^M sur M, mais c'est seulement une DG structure, si $Q^B(b) = 0$.

Un **morphisme** entre les déformations (B, Q^B, Q) et $(B', Q^{B'}, Q')$ est une paire (F, f) où F est un morphisme entre les DG variétés formelles $(B \times M, \tilde{Q} := Q^B +$

$Q^M + Q)$ et $(B' \times M, \tilde{Q}' := Q^B + Q^M + Q')$ et f est un morphisme entre les DG variétés formelles (B, Q^B) et $(B', Q^{B'})$ tel que le diagramme

$$\begin{tikzcd} B \times M \arrow[r] \arrow[d] & B' \times M \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

est cartésien et le diagramme

$$\begin{tikzcd} M \arrow[d] \arrow[dr] & \\ B \times M \arrow[r] & B' \times M \end{tikzcd}$$

commute.

Deux déformations sont dites **équivalentes**, s'il existe des morphismes dans les deux sens.

La déformation universelle Une déformation (U, Q^U, Q) de (M, Q^M) est dite **universelle** si, pour chaque autre déformation (B, Q^B, Q'), il y a un morphisme $(F, f) : (B, Q^B, Q') \longrightarrow (U, Q^U, Q)$ où f est défini de manière unique.

La construction de la déformation universelle est très naturelle dans ce contexte. Sa base est donnée par le shift (U, Q^U) de la DGL algèbre $L = \mathrm{Coder}(S(M))$ qui s'appelle **complexe tangent**. Définissons le super champs vectoriel Q sur $U \times M$ (qui est linéaire dans U) de la manière suivante:

$$Q_{n+1}(u, m_1, \ldots, m_n) := (\uparrow u)(m_1, \ldots, m_n)$$

pour $u \in U$ et $m_i \in M$. On va montrer que $(Q^U + Q^M + Q)^2 = 0$, donc le triplet (U, Q^U, Q) est bien une déformation de (M, Q^M). Pour une déformation arbitraire (B, Q^B, Q') de (M, Q^M), on définit un morphisme $f : B \longrightarrow U$ en posant

$$(\uparrow f_r(b))_s(m) := Q'_{r+s}(b, m)$$

pour $b \in B^{\odot r}$ et $m \in M^{\odot s}$. Ensuite, on prouve que la paire $(f \times \mathrm{Id}, f)$ définit un morphisme de déformations. On obtient le théorème:

Théorème 0.0.6. *Le triplet (U, Q^U, Q) est une déformation universelle de (M, Q^M).*

Une déformation semi-universelle Une déformation (V, Q^V, Q) de (M, Q^M) est dite **semi-universelle** si, pour toute déformation (B, Q^B, Q'), il existe un morphisme

$$(B, Q^B, Q') \longrightarrow (V, Q^V, Q)$$

de déformations et si (V, Q^V) est minimale, c'est-à-dire que $Q_1^V = 0$.

On va construire une déformation semi-universelle de la manière suivante : supposons que le complexe tangent $L = \mathrm{Coder}(S(M))$ est scindé. Alors il existe une L_∞-structure μ_* sur la cohomologie H de $L = \mathrm{Coder}(S(M))$. Le shift (V, Q^V) de (H, μ_*) donne la base d'une déformation semi-universelle. Un "changement de base" par le morphisme $V \longrightarrow U$ définit une déformation (V, Q^V, Q') à partir de la déformation universelle (U, Q^U, Q). Comme (V, Q^V) est minimale, pour que (V, Q^V, Q') soit semi-universelle, il suffit qu'il y ait un morphisme $(U, Q^U, Q) \longrightarrow (V, Q^V, Q')$ de déformations. Ceci peut être réalisé en utilisant la décomposition du Théorème 0.0.4 et de nouveau la Proposition 0.0.3. Donc on obtient:

Théorème 0.0.7. *La déformation (V, Q^V, Q') de (M, Q^M) est semi-universelle.*

Application aux déformations de singularités Dans le Chapitre 2, on appliquera le Théorème 0.0.7 à la théorie formelle des déformations de singularités. Pour une singularité isolée (formelle ou convergente) donnée X on va obtenir d'une manière explicite une singularité formelle S qui est la base d'une déformation semi-universelle de X.

Comme expliqué ci-dessus, pour une singularité X, il existe une DG variété formelle (M, Q^M) qui contient X comme sous-espace. Inversement, il existe un foncteur $\mathcal{V}$ qui associe un germe d'espace formel à chaque DG-variété qui est de dimension finis en chaque degré. On va prouver qu'une déformation semi-universelle formelle de X est donnée par le foncteur $\mathcal{V}$ appliqué à une déformation semi-universelle de (M, Q^M).

Cohomologie de Hochschild

Le théorème HKR Rappelons le théorème classique de Hochschild, Kostant et Rosenberg et sa preuve dans le cas particulier où l'on ne considère que des algèbres libres, avant d'expliquer les généralisations de ce théorème qu'on va démontrer au Chapitre 3.

Le **complexe naïf de Hochschild** $C_\bullet^{\mathrm{naive}}(A)$ d'une k-algèbre A est le complexe simpliciale de A-algèbres, défini par $C_n^{\mathrm{naive}}(A) := A^{\otimes n+1}$, $n \geq 0$, dont la composante $b_n : A^{\otimes n+1} \longrightarrow A^{\otimes n}$ de la différentielle b est la somme alternée $\sum_{i=0}^{n}(-1)^i d_i$ des morphismes

$$d_i : a_0 \otimes \ldots \otimes a_n \mapsto a_0 \otimes \ldots \otimes a_i \cdot a_{i+1} \otimes \ldots \otimes a_n,$$

pour $0 \leq i < n$ et

$$d_n : a_0 \otimes \ldots \otimes a_n \mapsto a_n \cdot a_o \otimes \ldots \otimes a_{n-1}.$$

L'homologie naïve de Hochschild $\mathrm{HH}_\bullet^{\mathrm{naive}}(A/k)$ de A sur k est l'homologie du complexe naïf de Hochschild.

Posons $R := A \otimes_k A$. Le **bar complexe** $C_\bullet^{\mathrm{bar}}(A)$ de A sur k est le complexe de R-algèbres, défini par $C_n^{\mathrm{bar}}(A) := A^{\otimes n+2}$, dont la n-ième composante de la différentielle b' est définie par $b'_n := \sum_{i=0}^{n-1}(-1)^i d_i$. Le bar complexe est une résolution simpliciale de A sur R, mais on verra que le complexe $C^{\mathrm{bar}}(A)^\bullet := C_{-\bullet}^{\mathrm{bar}}(A)$ porte aussi une structure d'une DG algèbre[1] sur R. La corrélation bien connue entre le complexe de Hochschild et le bar complexe est l'isomorphisme

$$C^{\mathrm{naive}}(A) \cong C^{\mathrm{bar}}(A) \otimes_R A,$$

où A est considéré comme R-module par la multiplication $\mu : R \longrightarrow A$. Si A est plat comme k-module, alors le bar complexe est une résolution plate de A sur R, d'où l'isomorphisme

$$\mathrm{HH}_n^{\mathrm{naive}}(A) \cong \mathrm{Tor}_n^R(A, A).$$

Si A est la k-algèbre libre $k[x_1, \ldots, x_n]$, à part la "résolution standard" $C^{\mathrm{bar}}(A)$, il y a une deuxième DG résolution naturelle de A sur R, notamment, le complexe de Koszul $K(X)$ sur R, donné par la séquence régulière $X = \{\frac{1}{2}(x_i \otimes 1 - 1 \otimes x_i) | \; i = 1, \ldots, n\}$. On construit un morphisme de complexes

$$K(X) \longrightarrow C^{\mathrm{bar}}(A)$$

de R-module plats. On applique le foncteur $\_ \otimes_R A$ et obtient un quasi-isomorphisme $K(X) \otimes_R A \longrightarrow C^{\mathrm{naive}}(A)$. L'observation que $K(X) \otimes_R A$ est égale au complexe $\wedge^\bullet \Omega_A$ donne directement le théorème HKR pour les algèbres libres:

Théorème 0.0.8. *Pour $A = k[x_1, \ldots, x_n]$, on a un quasi-isomorphisme*

$$\wedge^\bullet \Omega_A \longrightarrow C_\bullet^{\mathrm{naive}}(A),$$

où $\wedge^\bullet \Omega_A$ est muni de la différentielle triviale.

En vue des applications géométriques, on généralise ce théorème dans trois directions:

(1) Il faut admettre que A est une DG algèbre libre. Dans ce cas, $\wedge \Omega_A$ et $C^{\mathrm{naive}}(A)$ sont bigradués et seulement la différentielle verticale de $\wedge \Omega_A$ est triviale.

(2) Il faut généraliser la DG version pour des "complexes simpliciaux" d'algèbres, c.à.d. pour des foncteurs $\mathcal{N} \longrightarrow \mathfrak{Alg}$, où $\mathcal{N}$ est une petite catégorie (un "complexe simplicial"). Dans les applications géométriques, $\mathcal{N}$ apparaîtra comme le "nerf" d'un recouvrement d'un espace (voir ci-dessous).

(3) Il faut admettre que A appartient à une classe d'algèbres topologiques, par exemple la classe des algèbres analytiques. Dans cet exemple, les produits tensoriels doivent être remplacés par des produits tensoriels analytiques, et les modules de différentielles de Kähler doivent être remplacés par leur contreparties analytiques.

[1] c.à.d. d'une algèbre graduée $B = \coprod_{i \leq 0} B^i$, muni d'une différentielle d^B de degré 1, telle que $ab = (-1)^{g(a)g(b)} ba$ et $d^B(ab) = d^B(a)b + (-1)^{g(a)} a d^B(b)$, pour des éléments homogènes $a, b \in B$.

Pour réaliser (1), on généralise la construction de Koszul pour les algèbres graduées commutatives: si R est une algèbre graduée commutative et si $X \subseteq R$ est un sous-ensemble g-fini[2], on définit le **complexe de Koszul** de X sur R comme l'algèbre double graduée commutative[3] libre $R[E]$, où E est un ensemble de générateurs libres qui contient, pour chaque $x \in X$, un élément $e(x)$ de bidegré $(g(x), -1)$, et dont la différentielle verticale[4] est définie par $e(x) \mapsto x$.

On donne la définition et caractérisation suivante des séquences régulières dans une algèbre graduée commutative:

Définition et Théorème 0.0.9. *Supposons que l'anneau de base $\mathbb{K}$ contient $\mathbb{Q}$. Soit $X \subseteq R$ un sous-ensemble et soit I l'idéal dans R engendré par X. Supposons que, pour chaque sous-ensemble $Y \subseteq X$, on a $\cap_{n\geq 1} I^n R/(Y) = 0$. Alors X est appelé une **séquence régulière**, si au moins une des conditions équivalentes équivalentes suivantes est satisfaite:*

(i) Soit T un ensemble de générateurs d'algèbre libres qui contient pour chaque $x \in X$ un élément $t(x)$ du même degré que x. Le morphisme $R/I[T] \longrightarrow \mathrm{gr}_I(R) = R/I \oplus I/I^2 \oplus \dots$ de R/I-algèbres, qui applique $t(x)$ sur la classe de x dans I/I^2 est un isomorphisme d'algèbres différentielles graduées sur R/I.

(ii) Pour chaque élément $x \in X$ et chaque idéal $J \subseteq R$ généré par un sous-ensemble $Y \subseteq X$ tel que $x \notin Y$, on a la condition suivante: si $g(x)$ est pair, alors x n'est pas un diviseur de zéro dans R/J. Si $g(x)$ est impair, alors l'annulateur de x dans R/J est l'idéal généré par la classe de x.

(iii) Le complexe de Koszul $K(X)$ est une DG résolution de $R/(X)$ sur R.

(iv) $H^{-1}(K(X)) = 0$.

En utilisant l'implication (ii)⇒(iii) de ce théorème, on démontrera une DG version du Théorème 0.0.8 comme suit : Si $A = k[x_i]_{i\in I}$ est une DG algèbre libre, alors $R := A \otimes_k A$ est une DG algèbre libre dont les éléments de la forme $\frac{1}{2}(x_i \otimes 1 - 1 \otimes x_i)$ forment une séquence régulière $X \subseteq R$ dans le sens de Définition et Théorème 0.0.9. Donc le complexe de Koszul $K(X)$ de X sur R est une DG résolution libre de A sur R.

On a un morphisme $K(X) \longrightarrow C^{\mathrm{bar}}(A)$ de complexes de R-modules gradués, qui par contre ne respecte pas les différentielles horizontales. Mais on observe l'effet remarquable suivant : après tensorisation par A sur R, on obtient un morphisme

$$\wedge\Omega_A \cong K(X) \otimes_R A \longrightarrow C^{\mathrm{bar}}(A) \otimes_R A = C^{\mathrm{naive}}(A)$$

[2] La lettre g dénote toujours la graduation.

[3] c.à.d. une algèbre avec une double graduation négative qui est une algèbre graduée commutative par la graduation totale

[4] c.à.d. de bidegré $(0, +1)$

qui respecte aussi les différentielles horizontales. Ceci prouve la DG généralisation du Théorème 0.0.8.

En ce qui concerne les généralisations (2) et (3) : On définit un **complexe simplicial** $\mathcal{N}$ comme une classe de sous-ensembles d'un ensemble I telle que $\{i\} \in \mathcal{N}$ pour chaque $i \in I$, telle que $\emptyset \notin \mathcal{N}$ et telle que $\alpha \in \mathcal{N}$ et $\beta \subseteq \alpha$ implique $\beta \in \mathcal{N}$. Les éléments de $\mathcal{N}$ engendrent une catégorie dont les seuls morphismes sont les inclusions. Pour une catégorie $\mathcal{C}$ d'algèbres (où de modules), on dénote la catégorie des foncteurs covariants $\mathcal{N} \longrightarrow \mathcal{C}$ par $\mathcal{C}^{\mathcal{N}}$. On dénote la catégorie des algèbres graduées commutatives dont la composante de degré zéro appartient à $\mathcal{C}$ par $\mathrm{gr}(\mathcal{C})$ (pour les énoncés plus précis, voir Section 3.1.2).

En Section 3.1, nous allons introduire des “bonnes paires de catégories” $(\mathcal{C}, \mathcal{M})$ qui consistent en une catégorie $\mathcal{C}$ d'algèbres et une catégorie de modules $\mathcal{M}$ sur les algèbres dans $\mathcal{C}$ qui satisfont une liste d'axiomes, disant par exemple que $\mathcal{M}$ est une catégorie additive dans laquelle il existe des produits tensoriels. Les exemples les plus importants sont:

(1) La paire $(\mathcal{C}, \mathcal{M})$ où $\mathcal{C}$ est la catégorie des $\mathbb{K}$-algèbres (Noethériennes) et $\mathcal{M}$ la catégorie des modules sur les algèbres dans $\mathcal{C}$.

(2) La paire $(\mathcal{C}, \mathcal{M})$, où $\mathcal{C}$ est la catégorie des algèbres analytiques, c.à.d les algèbres des sections sur un compact de Stein. Dans ce cas les algèbres dans $\mathcal{C}$ sont muni d'une topologie DNF (duale nucléaire de Fréchet). Et $\mathcal{M}$ est par définition la catégorie des DNF modules sur les algèbres dans $\mathcal{C}$.

Une algèbre libre de n générateurs dans ce contexte est une algèbre qui représente un sous-foncteur du foncteur $\mathrm{Id}_{\mathcal{C}}^{n} : \mathcal{C} \longrightarrow$ (ensembles) qui est défini par un “marquage” (voir Section 3.1.1). Pour obtenir les généralisations (2) et (3), on va réduire la preuve esquissée ci-dessus sur les axiomes qui définissent les bonnes paires de catégories. En plus, on va faire toutes les constructions dans la preuve d'une manière $\mathcal{N}$-compatible. On obtient ainsi le Théorème 0.0.8 pour le cas où A est une DG algèbre dans $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$.

L'invariance d'homotopie du complexe de Hochschild Si $(\mathcal{C}, \mathcal{M})$ est une bonne paire de catégories dont les produits tensoriels sont des produits tensoriels topologiques, pour les algèbres A dans $\mathcal{C}$, le complexe naïf de Hochschild $C^{\mathrm{naive}}(A)$ n'a pas toutes les propriétés qu'on voudrait. Par exemple, si $\mathcal{C}$ est la catégorie des algèbres analytiques sur $\mathbb{C}$, le complexe $C^{\mathrm{naive}}(A)$ n'est pas un complexe de A-modules projectifs. On définit donc un complexe de Hochschild modifié:

Définition et Théorème 0.0.10. *Soit a une k-algèbre. Choisissons une DG résolution A de a sur k et posons $R := A \otimes_k A$. Choisissons en plus une DG résolution S de A sur R. On définit le **complexe de Hochschild** $\mathbb{H}(a/k)$ de a sur k comme l'objet $S \otimes_R a$ dans la catégorie d'homotopie $K(\mathcal{M}(a))$. Le complexe $\mathbb{H}(a/k)$ est bien défini à un isomorphisme (non canonique) dans $K(\mathcal{M}(a))$ près.*

La preuve de l'invariance d'homotopie est basée sur le fait que deux DG résolutions libres g-fini d'un morphisme de DG algèbres sont homotopiquement équivalentes. Ce fait est bien connu en algèbre homologique et on le généralisera pour les bonnes paires de catégories.

Comparons cette définition avec la définition du complexe cotangent $\mathbb{L}_{a/k}$ comme l'objet $\Omega_A \otimes_A a$ dans $K(\mathcal{M}(a))$. Par un théorème de Bingener et Kosarew [2], le complexe cotangent est également bien défini à un isomorphisme dans $K(\mathcal{M}(a))$ près. Nous allons prouver le théorème suivant :

Théorème 0.0.11. *Il y a un isomorphisme $\wedge\mathbb{L}_{a/k} \longrightarrow \mathbb{H}(a/k)$ dans la catégorie $K(\mathcal{M}(a))$.*

Pour prouver ce théorème, il faut construire, pour chaque choix des résolutions A et S, un morphisme

$$S \otimes_R a \longrightarrow \wedge\Omega_A \otimes_A a$$

dans $\mathrm{gr}(\mathcal{C})$ qui est une équivalence d'homotopie. Par le théorème de comparaison de DG résolutions (voir Théorème I.8.4 de [2]), il existe un quasi-isomorphisme $S \longrightarrow \operatorname{tot} C^{\mathrm{bar}}(A)$ sur R. On obtient des quasi-isomorphismes

$$S \otimes_R A \approx \operatorname{tot} C^{\mathrm{naive}}(A) \approx \wedge\Omega_A$$

de DG algèbres libres. Pour le deuxième quasi-isomorphisme, on utilise la DG version du Théorème 0.0.8. L'application du foncteur $\_ \otimes_A a$ terminera la preuve.

On généralise les Définition et Théorème 0.0.10 et le Théorème 0.0.11 pour les morphismes $k \longrightarrow a$ dans $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$, où $\mathcal{C}$ appartient à une bonne paire de catégories et où $\mathcal{N}$ est un complexe simplicial.

Schémas et espaces complexes La construction du complexe de Hochschild pour les (morphismes de) schémas Noethériens et les espaces complexes ressemble à la construction du complexe tangent. Pour un espace X et un recouvrement $X = \cup_{i\in I} X_i$, **le nerf** du recouvrement est défini comme

$$\mathcal{N} := \{\alpha \subseteq I |\ X_\alpha := \cap_{i\in\alpha} X_i \neq \emptyset\}.$$

Le nerf est une catégorie dont les seules morphismes sont les inclusions. La famille $X_* := (X_\alpha)_{\alpha\in\mathcal{N}}$ avec les inclusions $p_{\beta\alpha} : X_\alpha \longrightarrow X_\beta$, pour $\beta \subseteq \alpha$, peut être vue comme foncteur contravariant $\mathcal{N} \longrightarrow$ (espaces). Si X est un schéma et $\cup X_i$ un recouvrement ouvert par des schémas affines, on va étudier la catégorie des $\mathcal{O}_{X_*}$-modules, c'est-à-dire la catégorie des familles $\mathcal{F}_* = (\mathcal{F}_\alpha)_{\alpha\in\mathcal{N}}$ où chaque $\mathcal{F}_\alpha$ est un $\mathcal{O}_{X_\alpha}$-module avec de morphismes $p^*_{\alpha\beta}\mathcal{F}_\alpha \longrightarrow \mathcal{F}_\beta$, pour $\alpha \subseteq \beta$, qui sont compatibles. La catégorie des $\mathcal{O}_{X_*}$-modules quasi-coherentes est isomorphe à la catégorie de A_*-modules dans $(\mathbb{K} - \mathfrak{Mod})^{\mathcal{N}}$, où A_* est défini par $A_\alpha := \Gamma(X_\alpha, \mathcal{O}_{X_\alpha})$.

Dans toutes les constructions de ce paragraphe, on peut remplacer un schéma Noethérien par un espace complexe et un ouvert affine par un compact de Stein pour obtenir la théorie analytique.

On peut plonger la catégorie des $\mathcal{O}_X$-modules par le foncteur $j^* : \mathcal{F} \mapsto (\mathcal{F}|_{X_\alpha})_{\alpha\in\mathcal{N}}$ comme sous-catégorie dans la catégorie des $\mathcal{O}_{X_*}$-modules. Soit j_α l'inclusion $X_\alpha \longrightarrow X$, pour $\alpha \in \mathcal{N}$. Pour construire l'adjoint j_*, on a besoin d'une construction de Čech qui à un $\mathcal{O}_{X_*}$-module $\mathcal{G}_*$ associe un complexe $\check{C}^\bullet(\mathcal{G}_*)$ de $\mathcal{O}_X$-modules où

$$\check{C}^n(\mathcal{G}_*) = \prod_{|\alpha|=n} j_{\alpha *}\mathcal{F}_\alpha$$

dont la différentielle est définie de la manière usuelle. Le foncteur j_* donne les sections globales

$$j_*\mathcal{G}_* = H^0(C^\bullet(\mathcal{G}_*)).$$

Avant de donner la définition du complexe de Hochschild pour un morphisme $f : X \longrightarrow Y$ de type fini de schémas Noethériens (ou un morphisme d'espaces complexes), il faut la définition suivante:

Definition 0.0.12. Si $f : X \longrightarrow Y$ est un morphisme de type fini de schémas Noethérien, on définit une **résolvante** de X sur Y comme quadruple $(X_*, Y_*, P_*, \mathcal{A}_*)$ qui consiste en (1) le complexe simplicial Y_* associé à un recouvrement affine, localement fini $(Y_j)_{j\in J}$ de Y; (2) le complexe simplicial $X_* = (X_\alpha)_{\alpha\in\mathcal{N}}$ associé à un recouvrement affine, localement fini $(X_{ji})_{j\in J, i\in I_j}$ de X. Ce recouvrement est choisi de telle façon que, pour $j \in J$ fixé, la famille $(X_{ji})_{i\in I_j}$ est un recouvrement de $f^{-1}(Y_j)$; (3) un complexe simplicial $P_* = (P_\alpha)_{\alpha\in\mathcal{N}}$ sur la même catégorie d'indice; (4) un diagramme commutatif de la forme:

$$\begin{tikzcd} X_* \arrow[r, "\iota"] \arrow[d, "\bar{f}"] & P_* \arrow[dl, "g"] \\ Y_* & \end{tikzcd}$$

Ici, $\bar{f} = (\bar{f}, \tau)$ est le morphisme induit de complexes simpliciales; ι est une immersion fermé et g est un morphisme lisse[5]; (5) une résolution libre, g-fini $\mathcal{A}_*$ de $\mathcal{O}_{X_*}$ telle que $\mathcal{A}^0_* = \mathcal{O}_{P_*}$.

Il n'est pas difficile de démontrer l'existence d'une résolvante pour chaque morphisme $f : X \longrightarrow Y$ de type fini. Il est plus difficile de démontrer l'invariance d'homotopie dans la définition suivante:

[5] Ceci signifie que, pour chaque $\alpha \in \mathcal{N}$ et chaque $p \in P_\alpha$ le germe $\mathcal{O}_{P_\alpha,p}$ est une algèbre libre sur $\mathcal{O}_{Y_{\tau(\alpha)},y}$.

Définition et Théorème 0.0.13. *Pour un morphisme $f : X \longrightarrow Y$ de type fini, on définit le* ***complexe de Hochschild*** *de X sur Y comme suit: Soit $(X_*, Y_*, P_*, \mathcal{A}_*)$ une résolvante de X sur Y et $\mathcal{S}_*$ une DG résolution de $\mathcal{A}_*$ sur $\mathcal{R}_* := \mathcal{A}_* \otimes_{\mathcal{O}_{Y_*}} \mathcal{A}_*$. Le complexe simplicial de Hochschild de X sur Y est l'objet*

$$\mathbb{H}_*(X/Y) := \mathcal{S}_* \otimes_{\mathcal{A}_*} \mathcal{O}_{X_*}$$

dans la catégorie d'homotopie $K(X_*)$. À isomorphisme dans $K(X_*)$ près, le complexe simpliciale de Hochschild ne dépend pas du choix de $\mathcal{A}$ et $\mathcal{S}$.

Le complexe de Hochschild de X sur Y est l'objet

$$\mathbb{H}(X/Y) := \check{C}^\bullet(\mathbb{H}_*(X/Y))$$

dans la catégorie dérivée $D(X)$. À isomorphisme dans $D(X)$ près, le complexe de Hochschild ne dépend pas du choix de la résolvante $(X_*, Y_*, P_*, \mathcal{A}_*)$.

On définit **la n-ième homologie de Hochschild** $\mathrm{HH}_n(X/Y)$ de X sur Y comme la (-n)-ième hyper-cohomologie du complexe $\mathbb{H}(X/Y)$ et la **n-ième cohomologie de Hochschild** de X sur Y avec des valeurs dans un $\mathcal{O}_X$-module $\mathcal{F}$ comme

$$\mathrm{HH}^n(X/Y, \mathcal{F}) := \mathrm{Ext}^n_X(\mathbb{H}(X/Y), \mathcal{F}).$$

La version simpliciale du Théorème 0.0.11 donne directement un quasi-isomorphime $\wedge\mathbb{L}(X/Y)_* \longrightarrow \mathbb{H}(X/Y)_*$. L'application du foncteur de $\check{C}$ech qui est exact, donne le théorème suivant:

Théorème 0.0.14. *Pour un morphisme $X \longrightarrow Y$ d'espace complexes ou un morphisme de type fini de schémas Noethériens, il y a un isomorphisme*

$$\wedge\mathbb{L}(X/Y) \longrightarrow \mathbb{H}(X/Y)$$

dans la catégorie $D(X)$.

Pour les morphismes d'espaces complexes, ce théorème a été prouvé d'une manière complètement différente par Buchweitz et Flenner [7]. Comme conséquences directes, on obtient la décomposition suivante pour l'homologie de Hochschild

$$\mathrm{HH}_n(X/Y) \cong \prod_{p-q=n} H^q(X, \wedge^p\mathbb{L}(X/Y)),$$

et la décomposition suivante pour la cohomologie de Hochschild

$$\mathrm{HH}^n(X/Y, \mathcal{F}) \cong \prod_{p+q=n} Ext^p_X(\wedge^q\mathbb{L}(X/Y), \mathcal{F}).$$

On observe que la $(n-1)$-ième cohomologie tangente

$$T^{n-1}(X/Y) = \mathrm{Ext}^{n-1}_X(\mathbb{L}(X/Y), \mathcal{F})$$

est un facteur directe de la n-ième cohomologie de Hochschild.

Pour les variétés (algébriques ou analytiques) lisses, le complexe $\mathbb{L}(X)$ est une résolution du faisceau cotangent Ω_X. Donc on obtient un isomorphisme $\mathbb{H}(X) \cong \wedge\Omega_X$ dans la catégorie $D(X)$. On peut on déduire les "décompositions de Hodge" :

Corollaire 0.0.15. *Si X est un schéma Noethérien lisse ou une variété analytique, on a*

$$\mathrm{HH}^n(X) \cong \prod_{i+j=n} H^i(X, \wedge^j \mathcal{T}_X), \tag{0.1}$$

$$\mathrm{HH}_n(X) \cong \prod_{i-j=n} H^j(X, \wedge^i \Omega_X). \tag{0.2}$$

Ici, $\mathcal{T}_X$ dénote le faisceau tangent.

Pour les schémas lisses, l'isomorphisme (0.1) a été prouvé d'une manière différente par Yekutieli [49]. Pour les schémas lisses, l'isomorphisme (0.2) a été prouvé avant (en utilisant la λ-décomposition du complexe naïf de Hochschild) par Weibel [46].

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Introduction

The context of this thesis

The classic questions of the analytic deformation theory (for example the question of the existence of semi-universal deformations for isolated singularities, compact complex spaces or complex structures on compact complex manifolds) are completely understood and resolved. However, there still remain reasons for considering extensions of the classic deformation theories. The two principal motivations are discussed below:

I. Within the classic deformation theory, the following undesired phenomena appear: (a) moduli spaces (i.e. bases of semi-universal deformations) are almost always singular. This results in obstructing the extension of small deformations to larger ones. (b) Deformation functors are almost never representable. This means that infinitely close to the base point of a local moduli space, there are different points with isomorphic fibers. Several attempts have been made to avoid this phenomena. The two main attempts are: (1) To change from classical spaces to graded spaces (like supermanifolds) or to DG spaces[6]. This can be done either by extending just the category of the to-be-deformed objects alone or extending the category of the to-be-deformed objects *and* the base category. (2) To give up some of the properties of the to-be-deformed object, for example, the commutativity of the structure sheaf of the space. This attempt is related to motivation II.

There is hope to get moduli spaces with better behavior in such a generalized context. The classic moduli space should be an analytic subspace of the generalized moduli space, defined by some natural conditions. This ideas is sometimes referred to as *derived moduli program.*

The derived moduli program has not yet produced a general method for extending deformation problems to achieve smooth moduli spaces or representable deformation functors. There are only a few examples[7] where smooth moduli spaces are found in this way.

In this thesis, we develop two approaches which are inspired by the derived moduli program.

[6]DG spaces are spaces whose structure sheaf carry a differential graded structure.

[7]One example is the deformation of "generalized" complex structures on a complex manifold with trivial first Chern class [1].

II. The second motivation for the extension of the classic (analytic) deformation theory is to consider deformations of commutative spaces in the framework of noncommutative geometry. As the example of deformation quantization shows, it is interesting to consider noncommutative structures that are close to commutative structures. Therefore, a noncommutative deformation theory of commutative initial spaces should be developed. This can serve as tool for the construction of noncommutative schemes, which are not yet well understood.

Classic deformation theory of schemes and complex spaces requires elaborate homologic tools. The first step to achieving a noncommutative extension is to develop adequate homological tools for the noncommutative theory. A geometric version of the Hochschild Complex would prove useful for this purpose.

The content of this thesis

Here, only a brief insight into the contents of this thesis is given, since each chapter has a detailed introduction.

The first part (Chapter 1 and 2), focuses on the creation of a deformation theory for L_∞-algebras or their geometric counterpart, the formal DG manifolds. The base of a deformation of a formal DG manifolds is also, in this theory, a DG manifold. On one hand, this is interesting because of the fundamental role played by L_∞-algebras in deformation theory. On the other hand, formal DG manifolds are DG generalizations of (formal) singularities. This means that the deformation theory of formal DG manifolds is an extended deformation theory of singularities. It is demonstrated that the differential graded Lie algebra (DGL) L of super vectorfields on a formal DG manifold M gives rise to a universal deformation and that a "minimal model" for L gives rise to a "generalized moduli space" of M, containing the classic (formal) moduli space as (formal) analytic subspace. This generalized moduli space is still obstructed. However, it holds all of the information contained in the DGL structure on L. Thus, the generalized moduli space contains more invariants of the initial object M than the classic moduli space.

The **methods** applied in the first part belong to abstract deformation theory, deformations of singularities, combinatorics of trees and to the theory of model categories[8].

The second part (Chapter 3) of this thesis will focus on developing simultaneously the theory of the Hochschild (cochain) complex for complex spaces and algebraic varieties. Especially, the relation between the Hochschild cochain complex and the tangent complex is considered closely. It is shown that Hochschild cohomology contains tangent cohomology plus additional information. This result leads to interesting

[8] in the sense of Quillen [40]

interpretations and speculations about the deformation of complex spaces and varieties in terms of noncommutative geometry. For manifolds and smooth varieties, this leads to a "Hodge-decomposition" of Hochschild (co-) homology.

The **methods** applied in the second part of this thesis mainly belong to homologic and homotopic algebra: DG resolvents, simplicial complexes (which are associated to coverings of spaces), algebraic Hochschild theory, a generalized Koszul construction, a generalization of the Hochschild-Kostant-Rosenberg theorem and Čech constructions. Arguments are reduced, in this part, to a system of axioms, defining "good pairs of categories". This axiomatic approach makes the theory accessible for the algebraic **and** the analytic situation. The fact that the axioms for good pairs of categories hold in the analytic situation is based on the theory of Stein spaces, for example, the fact that the sections over a Stein compact form a Noetherian algebra is applied implicitly in this part.

Which results are new?

The following list contains the *new* results of this thesis. More precise explications concerning this statements are given in the introductions to the chapters.

In homological algebra: Regular sequences in graded commutative algebras are defined and characterized (Theorem 3.1.28). **In algebraic geometry:** For a morphism $X \longrightarrow Y$ of schemes of finite type, the homotopy invariance of the Hochschild cochain complex is proved (Proposition 3.2.6). The existence of a quasi-isomorphism between the relative Hochschild complex and the exterior algebra of the relative tangent complex is proved (Theorem 3.3.13). A decomposition theorem, saying that relative tangent cohomology is a direct factor of relative Hochschild cohomology is deduced (Corollary 3.3.14). **In analytic geometry:** For analytic manifolds, a "Hodge-decomposition" of Hochschild homology and a "Hodge-decomposition" of Hochschild cohomology is demonstrated (Corollary 3.3.17). **For L_∞-algebras:** An L_∞-structure on the cohomology $H(L)$ of a differential graded Lie algebra L is constructed in such a way that $H(L)$ and L are L_∞-equivalent[9] (Theorem 1.4.4). Deformations of L_∞-algebras with L_∞-bases are defined and a (semi-) universal deformation is constructed explicitly (Theorem 1.2.9 and Theorem 1.5.13).

For which statements are new proofs provided?

The following list contains familiar statements for which a new proof is provided. References to the original proofs are given in the introductions to the chapters and in the text.

[9] At this point, we must precise that Huebschmann/ Stasheff [19] stated (without proof) a recursion formula for the structure on $H(L)$ and for the morphism $H(L) \longrightarrow L$ and that the A_∞-analogue was shown by various authors.

In analytic geometry: For a morphism $X \longrightarrow Y$ of complex spaces, the invariance of the relative Hochschild (co-) homology is shown (Proposition 3.3.8); the existence of a quasi-isomorphism between the relative Hochschild complex and the exterior algebra of the relative tangent complex is demonstrated (Theorem 3.3.13); a decomposition theorem, saying that relative tangent cohomology is a direct factor of relative Hochschild cohomology is shown (Corollary 3.3.14). **In algebraic geometry:** For smooth varieties M, a "Hodge-decomposition" of Hochschild homology and a "Hodge-decomposition" of Hochschild cohomology is demonstrated (Corollary 3.3.17). **In singularity theory:** A formal semi-universal deformation for isolated singularities is constructed with L_∞-methods (Chapter 2).

Chapter 1

Deformation of L_∞-algebras

Introduction

L_∞-algebras (see Section 1.1) play a crucial role in deformation theory. They are natural generalizations of differential graded Lie algebras (DGL's). Deformation problems can always be described by DGL's (see [22], for instance). The importance of L_∞-algebras in deformation theory is due to the fact that two different deformation problems are equivalent, if the corresponding DGL's are equivalent as L_∞-algebras. This was one ingredient of Kontsevich's [22] proof that any Poisson manifold has a deformation quantization. L_∞-algebras also build a bridge between algebra and geometry. A simple shift in degrees makes a formal DG-manifold out of an L_∞-algebra (see Section 1.1). This observation is also due to Kontsevich. If a deformation problem is governed by a DGL L, then the (formal) local moduli space, if it exists, is an "analytic subspace" of the formal DG-manifold corresponding to L.

In the other direction, to each DGL L, an abstract deformation functor Def_L can be defined. In the classical theory Def_L is a set-valued functor on the category of (Artinian, local) algebras. Recent studies in mirror symmetry ([23], [34]) have led to an extension of this functor first to graded and then to differential graded Artinian algebras. The aim of this extension is to produce smooth (in a sense) formal moduli spaces with tangent space isomorphic to the whole cohomology of L. But sometimes, it is not evident (or even possible) to give an algebraic or geometric meaning to the objects obtained as deformations of an initial object by the extended deformation functor. (However, at other times, this is possible. For the classic deformations of associative algebras, the extended deformation functor produces A_∞-algebras.)

The deformation theory of L_∞-algebras (or, in geometric terms, of formal DG-manifolds) presented in this thesis is in fact an extended deformation theory of (formal) singularities. Instead of working with deformation functors, we present a completely geometric (extended) deformation theory of formal DG manifolds. The bases of deformations are also formal DG manifolds. We will observe in Section 1.2 that this extended deformation theory is still obstructed.

The deformations of a given formal DG-manifold $M = (M, Q^M)$ are governed by the DGL L of formal vectorfields on M (see Section 1.2). The degree 1 shift of L is again a formal DG manifold denoted by U. Two nice observations are made here: The first is that the transition from M to U doesn't change the category. (This one is trivial.) The second (Theorem 1.2.9) is that U is the base of a universal deformation of M. For the construction of a semiuniversal deformation of M, we have to construct an L_∞-structure on the cohomology H of (L, d) such that H and L are equivalent as L_∞-algebras. H with such an L_∞-structure is called a minimal model for L.

Hence, the essence of this chapter is the following general method for the construction of (formal) analytic moduli spaces: Take a minimal representative in the class of L_∞-algebras modulo L_∞-equivalence of the DGL controlling the deformation problem. In Chapter 2, we will apply this method to construct a formal moduli space for isolated singularities.

The content of this chapter: In Section 1.1, we remind the definitions of L_∞-algebras and of their correspondence with differential graded coalgebras. We state the conditions for a sequence of maps, to define an L_∞-morphism. We will prove those conditions in the Appendix since they are hard to find in the literature. Then, we remind Kontsevich's geometric point of view (=formal DG manifolds) of L_∞-algebras. In Section 1.2, we define deformations of formal DG manifolds with formal DG bases and morphisms of those. Our definition generalizes the one of Fialowski and Penkava [38]. We show that for an arbitrary formal DG manifold M, the differential graded Lie algebra $\mathrm{Coder}(S(M), S(M))$ (which we call tangent complex of M) is a base of a universal deformation of M. In Section 1.3, we give an ad-hoc combinatorial introduction to binary trees. In a sense, binary trees contain the algebraic structure of L_∞-algebras (see [45]). In Section 1.4, they are used to define an L_∞-structure μ_* on the cohomology H of a differential graded Lie algebra $L = (L, d, [\cdot, \cdot])$ (admitting a splitting). Furthermore, again with the help of binary trees, in Section 1.4, we construct explicitly an L_∞-quasi-isomorphism between (H, μ_*) and $(L, d, [\cdot, \cdot])$. In Section 1.5, we prove that $(L, d, [\cdot, \cdot])$ is as L_∞-algebra isomorphic to the direct sum of the L_∞-algebras (H, μ_*) and (F, d), where F is the complement of H in L. As a consequence, we can show that for each formal DG manifold M such that $L = \mathrm{Coder}(S(M), S(M))$ splits, the shift V of (H, μ_*) is the base of a semi-universal deformation of M.

1.1 L_∞-algebras and coalgebras

In this chapter, we shall always work over a commutative ground ring $\mathbb{K}$ with unit of characteristic zero.

1.1.1 Graded symmetric and exterior algebras

For a graded module W, the graded symmetric algebra $S(W)$ is defined as the tensor algebra $T(W) = \oplus_{n\geq 0} W^{\otimes n}$ modulo the relations $w_1 \otimes w_2 - (-1)^{w_1 w_2} w_2 \otimes w_1 = 0$. We denote the graded symmetrical product by $\odot$. The algebra $T(W)$ (resp. $S(W)$) is bigraded. The graduation on $T(W)$ (resp. $S(W)$) defined by $g(w_1 \otimes \ldots \otimes w_n) = g(w_1) + \ldots + g(w_n)$ (resp. $g(w_1 \odot \ldots \odot w_n) = g(w_1) + \ldots g(w_n)$), where g is the graduation of W, will be called **linear graduation**. The one defined by $g(w_1 \otimes \ldots \otimes w_n) = n$ (resp. $g(w_1 \odot \ldots \odot w_n) = n$) will be called **polynomial graduation**. Set $S_+(W) := \oplus_{n\geq 1} W^{\odot n}$. On $S_+(W)$, there is a natural $\mathbb{K}$-linear comultiplication $\Delta^+ : S_+(W) \longrightarrow S_+(W) \otimes S_+(W)$, given by

$$w_1 \odot \ldots \odot w_n \mapsto \sum_{j=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}(j,n)} \epsilon(\sigma, w_1, \ldots, w_n) w_{\sigma(1)} \odot \ldots \odot w_{\sigma(j)} \otimes w_{\sigma(j+1)} \odot \ldots \odot w_{\sigma(n)}.$$

Here, $\epsilon(\sigma) := \epsilon(\sigma, w_1, \ldots, w_n)$ is defined such that $w_{\sigma(1)} \odot \ldots \odot w_{\sigma(n)} = \epsilon(\sigma) w_1 \odot \ldots \odot w_n$. Note that we have $\mathrm{Kern}\, \Delta^+ = W$. On $S(W)$, there is a $\mathbb{K}$-linear comultiplication Δ, defined by $\Delta(1) := 1 \otimes 1$ and $\Delta(w) := w \otimes 1 + \Delta'(w) + 1 \otimes w$, for $w \in \mathrm{S}_+(W)$. Note that Δ is injective.

For a graded module L, the graded exterior algebra $\bigwedge^+ L$ without unit is defined as the tensor algebra $T_+(L) = \oplus_{n\geq 1} L^{\otimes n}$ modulo the relations $a_1 \otimes a_2 + (-1)^{a_1 a_2} a_2 \otimes a_1 = 0$. We denote the graded exterior product by $\wedge$. By $L[1]$, we denote the graded module with $L[1]^i = L^{i+1}$ and by $\downarrow$ the canonical map $L \longrightarrow L[1]$ of degree -1. Set $\uparrow := \downarrow^{-1}$. Remark that for each $n \geq 1$, there is an isomorphism

$$\begin{aligned} \downarrow^n \colon \bigwedge\nolimits^n L &\longrightarrow L[1]^{\odot n} \\ a_1 \wedge \ldots \wedge a_n &\mapsto (-1)^{(n-1)\cdot a_1 + \ldots + 1 \cdot a_{n-1}} \downarrow a_1 \odot \ldots \odot \downarrow a_n. \end{aligned}$$

Its inverse map is given by $(-1)^{\frac{n(n-1)}{2}} \uparrow^n$. In this formula, we deduce the sign from the Koszul convention. More generally, for homogeneous graded morphisms f, g of graded modules, we set $(f \otimes g)(a \otimes b) := (-1)^{ga} f(a) \otimes g(b)$. In the exponent, a always means the degree of an homogeneous element (or morphism) a and ab means the product of degrees and not the degree of the product.

For $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in L$, we define the sign $\chi(\sigma) := \chi(\sigma, a_n, \ldots, a_n)$ in such a way that

$$a_{\sigma(1)} \wedge \ldots \wedge a_{\sigma(n)} = \chi(\sigma) a_1 \wedge \ldots \wedge a_n.$$

The following statement about the correlation between χ and ϵ is an easy exercise:

Lemma 1.1.1. *For $a_1, \dots, a_n \in L$, we have*

$$\chi(\sigma, a_1, \dots, a_n) = (-1)^{(n-1)(a_1+a_{\sigma(1)})+\dots+1\cdot(a_{n-1}+a_{\sigma(n-1)})}\epsilon(\sigma, \downarrow a_1, \dots, \downarrow a_n).$$

For a graded module V, we define two different actions of the symmetric group Σ_n on $V^{\otimes n}$: The first one is given by

$$\begin{aligned}\Sigma_n \times V^{\otimes n} &\longrightarrow V^{\otimes n}\\ (\sigma, v_1 \otimes \dots \otimes v_n) &\mapsto \epsilon(\sigma, v_1, \dots, v_n) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}.\end{aligned}$$

Here, the application of a σ commutes with the canonical projection $V^{\otimes n} \longrightarrow V^{\odot n}$. The second one is given by

$$\begin{aligned}\Sigma_n \times V^{\otimes n} &\longrightarrow V^{\otimes n}\\ (\sigma, v_1 \otimes \dots \otimes v_n) &\mapsto \chi(\sigma, v_1, \dots, v_n) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}.\end{aligned}$$

Here, the application of a σ commutes with the canonical projection $V^{\otimes n} \longrightarrow \wedge^n V$. When we work with symmetric powers, we use the first action; when we work with exterior powers, we use the second one. Since the context shall always be clear, we don't distinguish both actions by different notation. We will use the anti-symmetrisation maps:

$$\alpha_n := \sum_{\sigma \in \Sigma_n} \sigma. : V^{\otimes n} \longrightarrow V^{\otimes n}.$$

When $\sigma.$ denotes the first action, α_n can be seen as map $V^{\odot n} \longrightarrow V^{\otimes n}$; when $\sigma.$ denotes the second action, α_n can be seen as map $\wedge^n V \longrightarrow V^{\otimes n}$. Furthermore, for both cases, we will use the maps

$$\alpha_{k,n} := \sum_{\sigma \in \mathrm{Sh}(k,n)} \sigma. : V^{\otimes n} \longrightarrow V^{\otimes n}.$$

For the natural projection $\pi : W^{\otimes n} \longrightarrow W^{\odot n}$ (resp. $V^{\otimes n} \longrightarrow \bigwedge^n V$), we have

$$\pi \circ \alpha = n! \,\mathrm{Id}\,.$$

1.1.2 Free differential graded coalgebras

Let (C_1, Δ_1) and (C_2, Δ_2) be coalgebras. Remember that a module homomorphism $F : C_1 \longrightarrow C_2$ is a coalgebra morphism, if and only if the diagram

$$\begin{tikzcd} C_1 \arrow[r, "F"] \arrow[d, "\Delta_1"] & C_2 \arrow[d, "\Delta_2"] \\ C_1 \otimes C_1 \arrow[r, "F\otimes F"] & C_2 \otimes C_2 \end{tikzcd} \tag{1.1}$$

commutes. Each coalgebra morphism $F : (S(W), \Delta) \longrightarrow (S(W'), \Delta)$ satisfies $F(1) = 1$. The restriction $F \mapsto F|_{S_+(W)}$ is a one-to-one correspondence between coalgebra morphisms $(S(W),\Delta) \longrightarrow (S(W'),\Delta)$ and coalgebra morphisms $F\colon (S_+(W),\Delta^+) \longrightarrow (S_+(W'),\Delta^+)$.

The next proposition gives a one-to-one correspondence between coalgebra maps $F : S(W) \longrightarrow S(W')$ and sequences of linear maps $F_n : S_n(W) \longrightarrow W'$, $n \geq 1$. We fix the following notations:

$$\begin{aligned} \hat{F}_n :=& F|_{W^{\odot n}} : W^{\odot n} \longrightarrow S(W') \\ F_{k,l} :=& \operatorname{pr}_{W'^{\odot l}} \circ \hat{F}_k : W^{\odot k} \longrightarrow W'^{\odot l} \\ F_n :=& F_{n,1} : W^{\odot n} \longrightarrow W' \end{aligned}$$

Sometimes, we shall consider the maps F_n as antisymmetric maps $W^{\otimes n} \longrightarrow W'$ instead of maps $W^{\odot n} \longrightarrow W'$. For each multi-index $I = (i_1,\ldots,i_k) \in \mathbb{N}^k$, we set $I! := i_1! \cdot \ldots \cdot i_k!$ and $|I| := i_1 + \ldots + i_k$ and

$$F_I := \frac{1}{I!k!}(F_{i_1} \odot \ldots \odot F_{i_k}) \circ \alpha_n.$$

Here, by $F_{i_1} \odot \ldots \odot F_{i_k}$, we mean the composition of $F_{i_1} \otimes \ldots \otimes F_{i_k}$ and the natural projection $W'^{\otimes k} \longrightarrow W'^{\odot k}$.

Proposition 1.1.2. *For $n \geq 1$, we have that*

$$\hat{F}_n = \sum_{k=1}^{n} \sum_{I\in\mathbb{N}^k}^{|I|=n} F_I. \tag{1.2}$$

The proof can be found in the appendix.

A coalgebra homomorphism $F : S(W) \longrightarrow S(W')$ is called **strict**, if $F_n = 0$ for each $n \geq 2$.

For a coalgebra (C,Δ), remember that a module homomorphism $Q : C \longrightarrow C$ is a coderivation, if and only if the diagram

$$\begin{tikzcd} C \arrow[r, "Q"] \arrow[d, "\Delta"] & C \arrow[d, "\Delta"] \\ C \otimes C \arrow[r, "Q\otimes 1+1\otimes Q"] & C \otimes C \end{tikzcd} \tag{1.3}$$

commutes. By the next proposition, there is a one-to-one correspondence between coderivations $Q : S(W) \longrightarrow S(W)$ of degree $+1$ and sequences of linear maps $Q_n : S_n(W) \longrightarrow W$ of degree $+1$. We fix the following notations:

$$\begin{aligned} \hat{Q}_n :=& Q|_{W^{\odot n}} : W^{\odot n} \longrightarrow S(W) \\ Q_{k,l} :=& \operatorname{pr}_{W^{\odot l}} \circ \hat{Q}_k : W^{\odot k} \longrightarrow W^{\odot l} \\ Q_n :=& Q_{n,1} : W^{\odot n} \longrightarrow W \end{aligned}$$

Proposition 1.1.3. *Let Q be a coderivation of degree $+1$ on the graded coalgebra $(S(W), \Delta)$. Then, $Q(1) = Q_0(1) \in W$ and for $n \geq 1$ and $w_1, \ldots, w_n \in W$, we have*

$$\hat{Q}_n(w_1, \ldots, w_n) = \sum_{l=0}^{n} \sum_{\sigma \in \mathrm{Sh}(l,n)} \epsilon(\sigma) Q_l(w_{\sigma(1)}, \ldots, w_{\sigma(l)}) \odot w_{\sigma(l+1)} \odot \ldots \odot w_{\sigma(n)}, \quad (1.4)$$

where the $l = 0$ term must be interpreted as $Q_0(1) \odot w_1 \odot \ldots \odot w_n$.

The proof can be found in the appendix. Remark that there is a 1:1 - correspondence between coderivations of degree $+1$ on $(S_+(W), \Delta^+)$ and coderivations of Q degree $+1$ on $(S(W), \Delta)$ with $Q(1) = 0$.

Corollary 1.1.4. *Let Q be a coderivation of degree $+1$ on the coalgebra $S(W)$, Q' a coderivation of degree $+1$ on the coalgebra $S(W)$ and $F := S(W) \longrightarrow S(W')$ a morphism of coalgebras. Then, for $n \geq 1$ and $1 \leq l \leq n+1$, $1 \leq k \leq n$, we have*

$$Q_{n,l} = (Q_{n-l+1} \otimes 1 \otimes \ldots \otimes 1) \circ \alpha_{n-l+1,n}$$

and

$$F_{n,k} = \sum_{i_1 + \ldots + i_k = n} F_I.$$

The map F respects the coderivations Q and Q' if and only if $F(Q(1)) = Q'(1)$ and for each $n \geq 1$, we have

$$\sum_{k=1}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I| = n}} Q'_k \circ F_I = \sum_{k+l=n+1} F_l \circ (Q_k \otimes 1 \otimes \ldots \otimes 1) \circ \alpha_{k,n}. \quad (1.5)$$

On the right hand - side, the sum is over all $l \geq 1$ and $k \geq 0$. The term $(Q_0 \otimes 1 \otimes \ldots \otimes 1)(w_1 \otimes \ldots \otimes w_n)$ must be interpreted as $Q_0(1) \otimes w_1 \otimes \ldots \otimes w_n$.

1.1.3 L_∞-algebras

Remember that a module L with a sequence of maps $\mu_n : \bigwedge^n L \longrightarrow L$ of degree $2-n$, for $n \geq 0$, is called an L_∞**-algebra** if the coderivation Q (of degree $+1$) on $S(W)$, defined by the maps

$$Q_n := (-1)^{n(n-1)/2} \downarrow \circ \mu_n \circ \uparrow^n : W^{\odot n} \longrightarrow W$$

is a codifferential, i.e. $Q^2 = 0$.

Proposition 1.1.5. *The condition $Q^2 = 0$ just means that for each $n > 0$ and elements $w_1, \ldots, w_n \in W$, the term*

$$(Q^2)_n(w_1, \ldots, w_n) = \sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} \epsilon(\sigma) Q_l(Q_k(w_{\sigma(1)}, \ldots, w_{\sigma(k)}), w_{\sigma(k+1)}, \ldots, w_{\sigma(n)})$$

disappears. This condition is equivanlent to the equations

$$\sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} (-1)^{k(l-1)} \chi(\sigma) \mu_l(\mu_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), a_{\sigma(k+1)}, \ldots, a_{\sigma(n)}) = 0 \quad (1.6)$$

for each $n \geq 0$ *and* $a_1, \ldots, a_n \in L$.

In the literature, μ_0 is mostly assumed to be trivial. If this is the case, (L, μ_1) is a DG module.

Definition 1.1.6. An L_∞-algebra $(L, \mu_n)_{n \geq 1}$ is called **minimal**, if $\mu_1 = 0$. It is called **linear**, if $\mu_i = 0$ for $i \geq 2$.

Now let (L, μ_*) and (L', μ'_*) be L_∞-algebras. Set $W := L[1]$, $W' := L'[1]$ and denote the induced codifferentials on $S(W)$ and $S(W')$ by Q and Q'. A sequence of maps $f_n : \bigwedge^n L \longrightarrow L'$; $n \geq 1$ of degree $1 - n$ is called L_∞**-morphism**, if the maps $F_n := W^{\odot n} \longrightarrow W'$ induced by f_n (explicitly: $F_n = (-1)^{n(n-1)/2} \downarrow \circ f_n \circ \uparrow^n$) define a morphism $F : S(W) \longrightarrow S(W')$ of differential graded coalgebras. Rewrite condition (1.5) in terms of f_n and μ_n:

$$\begin{aligned} & \mu'_1 \circ f_n - \textstyle\sum_{i+j=n} \frac{(-1)^i}{2} \mu'_2(f_i, f_j) \circ \alpha_{i,j} \\ + \quad & \textstyle\sum_{k=3}^n \sum_{\substack{I \in \mathbb{N}^k \\ |I| = n}} (-1)^{k(k-1)/2 + i_1(k-1) + \ldots + i_{k-1} \cdot 1} \mu'_k \circ f_I \\ = \quad & \textstyle\sum_{k+l=n+1} (-1)^{k(l-1)} f_l \circ (\mu_k \otimes 1 \otimes \ldots \otimes 1) \circ \alpha_{k,n} \end{aligned} \quad (1.7)$$

For the case where L' is a differential graded Lie-algebra, i.e. $\mu'_k = 0$ for $k = 0$ and $k \geq 3$, set $d := \mu'_1$ and $[\ ,\ ] := \mu'_2$. We get the following conditions for the maps f_n to define an L_∞-morphism (see Definition 5.2 of [27]):

$$\begin{aligned} & df_n(a_1, \ldots, a_n) - \\ & \textstyle\sum_{i+j=n} \sum_\sigma \chi(\sigma)(-1)^{i+(j-1)(a_{\sigma(1)} + \ldots + a_{\sigma(i)})} [f_i(a_{\sigma(1)}, \ldots, a_{\sigma(i)}), f_j(a_{\sigma(i+1)}, \ldots, a_{\sigma(n)})] \\ & = \textstyle\sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} (-1)^{k(l-1)} \chi(\sigma) f_l(\mu_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), a_{\sigma(k+1)}, \ldots, a_{\sigma(n)}), \end{aligned}$$

where $a_1, \ldots, a_n \in L$ and the second sum goes over all σ in $\mathrm{Sh}(i, n)$ such that $\sigma(1) < \sigma(i+1)$.

Definition 1.1.7. A morphism $f : (L, \mu_n)_{n \geq 1} \longrightarrow (L', \mu'_n)_{n \geq 1}$ of L_∞-algebras is called an L_∞**-quasi-isomorphism**, if f_1 is a quasi-isomorphism of differential graded modules.

1.1.4 L_∞-algebras and formal DG manifolds

In this section, we explain briefly the geometric point of view of L_∞-algebras, as proposed by Kontsevich [22]. First, recall the definition of **pointed modules** (see Section II.6 of [2]): A pointed module is a pair $(M, *)$ of a module M and an element $* \in M$. We restrict ourselves to the case where $*$ is just the zero element. For modules M and N, a **homogeneous polynomial of degree** p on M with values in N is a mapping $\tilde{f} : M \longrightarrow N$ of the form $f \circ \Delta^{(p)}$, where f is a p-multilinear form $M \times \ldots \times M \longrightarrow N$ and $\Delta^{(p)}$ is the diagonal $m \mapsto (m, \ldots, m)$. The polarization formula (Lemma II.6.2 of [2]) says that $f \mapsto \tilde{f}$ is a 1:1-correspondence between symmetrical p-multilinear forms $M \times \ldots \times M \longrightarrow N$ and homogeneous polynomials of degree p on M with values in N. For pointed modules $M = (M, 0)$ and $N = (N, 0)$, a formal map $f : M \longrightarrow N$ is a formal sum $f = \sum_{p \geq 1} \tilde{f}_p$, where $\tilde{f}_p$ is a homogeneous polynomial of degree p. Pointed modules together with formals maps form a category. By the polarization formula and Proposition 1.1.2, there is a 1:1-correspondence between formal maps $f : M \longrightarrow N$ and morphisms $S(M) \longrightarrow S(N)$ of (non-graded free) coalgebras.

For the definition of formal supermanifolds, modules are replaces by $\mathbb{Z}$-graded modules and symmetric multilinear forms by graded symmetric multilinear forms.

Definition 1.1.8. A **formal supermanifold** is a pair $M = (M, 0)$ of a $\mathbb{Z}$-graded module M and its zero element. A **formal map** $f : M \longrightarrow N$ of degree j of formal supermanifolds is a sequence $(f_p)_{p \geq 1}$, where f_p is a graded symmetric multilinear form $M \times \ldots \times M \longrightarrow N$ of linear degree j. The **composition** $f \circ g$ of formal maps $g : L \longrightarrow M$ and $f : M \longrightarrow N$ is defined as the sequence $(g_p)_{p \geq 1}$ with

$$g_p = \sum_{k=1}^{p} \sum_{\substack{I \in \mathbb{N}^k \\ |I| = p}} f_k \circ g_I.$$

A **morphism** of formal supermanifolds is a formal map of degree zero.

It is clear by this definition that the category of formal supermanifolds is equivalent to the category of free, graded coalgebras with coalgebra maps of degree zero.

Definition 1.1.9. A **vectorfield** of degree j on a formal supermanifold M is a coderivation of degree j on $S(M)$.

By Proposition 1.1.3, a vectorfield on M can be interpreted as formal map $M \longrightarrow M$. The M on the right hand-side of the arrow should be considered as tangent space of M. The graded commutator defines the structure of a graded Lie algebra on $\mathrm{Coder}(S(M), S(M))$. Therefore, there is a bracket $[\cdot, \cdot]$ of vectorfields.

Let (M, Q^M) and (N, Q^N) be formal supermanifolds with vectorfields. A formal map $f : M \longrightarrow N$ is called **Q-equivariant**, if the induced map $S(M) \longrightarrow S(N)$ of coalgebras commutes with Q^M and Q^N. Remark that in the case where M and N

are non-graded free $\mathbb{K}$-modules of finite dimension, this definition coincides with the classic definition and the Q-equivariance just means that

$$Q^N \circ f = Df \circ Q^M.$$

Definition 1.1.10. A **formal DG manifold** is a pair (M, Q^M) of a formal supermanifold M and a vectorfield Q^M of degree 1 such that $[Q^M, Q^M] = 0$. Morphisms of formal DG manifolds are Q-equivariant maps of formal supermanifolds (sometimes we call them L_∞-morphisms). Denote the category of formal DG manifolds by `DG-Manf`.

By the previous subsection, the lifting $L \mapsto L[1]$ gives an isomorphism between the category of L_∞-algebras and the category of formal DG manifolds, and the functor $M \mapsto S(M)$ gives an isomorphism between the category of formal DG manifolds and the category of free differential graded coalgebras.

We use the following superscripts to denote full subcategories of `DG-Manf`:
L ("local"): the subcategory of all (M, Q^M) in `DG-Manf` such that $Q_0^M = 0$;
M ("minimal"): the subcategory of all (M, Q^M) in $\texttt{DG-Manf}^L$ such that $Q_1^M = 0$;
G ("g-finite"): the subcategory of all (M, Q^M) in $\texttt{DG-Manf}^L$ such that $H(M, Q_1^M)$ is g-finite;
C ("convergent"): the subcategory of all (M, Q^M) in $\texttt{DG-Manf}^G M$ such that the mapping $M_0 \longrightarrow M_1$ induced by Q^M converges.

1.2 Deformation of L_∞-Algebras

Fialowski and Penkava [38] have defined a deformation theory of L_∞-algebras such that the base of a deformation is an algebra with augmentation. The new approach here is to take L_∞-algebras also as bases of deformations. Since the geometric language is more elegant, we will talk about formal DG manifolds instead of L_∞-algebras. Thus, the objects that we deform are DG structures, i.e. degree 1 vectorfields Q with $Q^2 = 0$ on formal supermanifolds.

In our setting, the "fiber" of a deformation of a DG structure on M does not give a DG structure on M in general, but only a degree 1 vectorfield. But it is easy to find those points of the basis B of a deformation of M for which the associated deformation of Q^M is again a DG structure. They just correspond to the zero locus of the vectorfield Q^B.

A very nice fact for this deformation theory is that we get a universal deformation for free: The deformations of a DG manifold M are governed by the differential graded Lie algebra of vectorfields on M, i.e. the DGL L of coderivations on $S(M)$ with graded commutator as bracket $[\cdot, \cdot]$ and differential $d = [\cdot, Q^M]$. In contrast to Fialowski/ Penkava, we use the linear grading on L (see Section 1.1). Set $U := L[1]$ and denote the vectorfield corresponding to the DGL structure of L by Q^U. We will see that (U, Q^U) is the base of a universal deformation of M.

1.2.1 Definitions

Definition 1.2.1. Let (M, Q^M) in `DG-Manf` and (B, Q^B) in $\texttt{DG-Manf}^L$ be formal DG manifolds. A **deformation** of M with base B, or more exactly a deformation of the DG structure Q^M, is a degree 1 vectorfield Q on $B \times M$ with $Q_0 = 0$ such that

(i) $Q|_{\{0\}\times M} = 0$.

(ii) The vectorfield $\tilde{Q} := Q^M + Q^B + Q$ is a DG structure on $B \times M$.

(iii) The projection $B \times M \longrightarrow B$ is a homomorphism of formal DG manifolds.

We denote deformations of (M, Q^M) as triples (B, Q^B, Q). Remark that condition (i) is equivalent to the condition that the inclusion $M \longrightarrow B \times M$ is a morphism of formal DG manifolds. Condition (iii) is equivalent to the condition

$$\mathrm{im}(Q) \subseteq \{0\} \times M.$$

A deformation is trivial, if the projection $B \times M \longrightarrow M$ respects the DG structures $\tilde{Q}$ and Q^M.

Definition 1.2.2. A **morphism** of deformations (B, Q^B, Q) and $(B', Q^{B'}, Q')$ of (M, Q^M) is a pair (F, f), where F is a morphism of formal DG manifolds $(B \times M, \tilde{Q} := Q^B + Q^M + Q)$ and $(B' \times M, \tilde{Q}' := Q^B + Q^M + Q')$ and f is a morphism of formal DG manifolds (B, Q^B) and $(B', Q^{B'})$ such that the diagram

$$\begin{tikzcd} B \times M \arrow[r] \arrow[d] & B' \times M \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

is cartesian and the diagram

$$\begin{tikzcd} M \arrow[d] \arrow[dr] & \\ B \times M \arrow[r] & B' \times M \end{tikzcd}$$

commutes.

Definition 1.2.3. Two deformations are called **equivalent**, if there exist homomorphisms in both senses.

Convention: For elements $m_1, \dots, m_n$ in a module M and for shuffles $\sigma \in \mathrm{Sh}(k, n)$, we will sometimes write m'_σ for $m_{\sigma(1)} \odot \dots \odot m_{\sigma(k)}$ and m''_σ for $m_{\sigma(k+1)} \odot \dots \odot m_{\sigma(n)}$ and sometimes just m instead of $m_1 \odot \dots \odot m_n$.

Proposition 1.2.4. ***(Base change)*** *Suppose that $(B', Q^{B'}, Q')$ is a deformation of (M, Q^M) and $f : B \longrightarrow B'$ a homomorphism of formal DG manifolds with $B = (B, Q^B)$ in* **DG-Manf**^{L}*. Via*

$$Q_n(b_1, \ldots, b_r, m_1, \ldots, m_s) := \sum_{t=1}^{r} \sum_{\substack{I \in \mathbb{N}^t \\ |I|=r}} Q'_{s+t}(f_I(b_1, \ldots, b_r), m_1, \ldots, m_s),$$

for $r \geq 1$ with $r+s = n$ and $b_i \in B'$, $m_j \in M$, we can define a deformation (B, Q^B, Q) of (M, Q^M) and $(f \times \mathrm{Id}, f)$ is a morphism of deformations.

Proof. We have to show that $(Q^2)_n(b_1, \ldots, b_r, m_1, \ldots, m_s) = 0$, for $b_1, \ldots, b_r \in B$ and $m_1, \ldots, m_s \in M$. First, let $s \geq 1$. Then

$$\begin{aligned}(Q^2)_n(b, m) = &\sum_{k=r}^{n} \sum_{\sigma \in \mathrm{Sh}(k-r,s)} \epsilon(\sigma) Q^M_{n-k+1}(Q_k(b, m'_\sigma), m''_\sigma) \\ &+ \sum_{k=1}^{r} \sum_{\sigma \in \mathrm{Sh}(k,r)} \epsilon(\sigma) Q_{n+1-k}(Q^B_k(b'_\sigma), b''_\sigma, m) \\ &+ \sum_{k=0}^{s} \sum_{\sigma \in \mathrm{Sh}(k,s)} \epsilon(\sigma) Q_{n+1-k}(b, Q^M_k(m'_\sigma), m''_\sigma) \\ &+ \sum_{k+l=n+1} \sum_{p=1}^{r-1} \sum_{\sigma \in \mathrm{Sh}(p,r)} \sum_{\tau \in \mathrm{Sh}(k+p-r,s)} \epsilon(\sigma)\epsilon(\tau) Q_l(b'_\sigma, Q_k(b''_\sigma, m'_\tau), m''_\tau).\end{aligned}$$

Using the definition of Q and the assumption that f is a DG morphism, after changing the order of summation, this sum takes the following form:

$$\begin{aligned}&\sum_{t=1}^{r} \sum_{\substack{I \in \mathbb{N}^t \\ |I|=r}} \sum_{p=0}^{s} \sum_{\sigma \in \mathrm{Sh}(p,s)} \epsilon(\sigma) Q^M_{s-p+1}(Q'_{p+t}(f_I(b), m'_\sigma), m''_\sigma) \\ &\quad + \sum_{p=1}^{r} \sum_{t=1}^{p} \sum_{I', I'', u} \sum_{\sigma \in \mathrm{Sh}(p,r)} \epsilon(\sigma) Q'_{s+u+1}(Q^B_t(f_{I'}(b'_\sigma)), f_{I''}(b''_\sigma), m) \\ &\quad + \sum_{t=1}^{r} \sum_{\substack{I \in \mathbb{N}^t \\ |I|=r}} \sum_{p=0}^{s} \sum_{\sigma \in \mathrm{Sh}(p,s)} \epsilon(\sigma) Q'_{s-p+t+1}(f_I(b), Q^M_p(m'_\sigma), m''_\sigma) \\ &+ \sum_{p=1}^{r-1} \sum_{q=0}^{s} \sum_{t=1}^{p} \sum_{I', I'', u} \sum_{\sigma \in \mathrm{Sh}(p,r)} \sum_{\tau \in \mathrm{Sh}(q,s)} \epsilon(\sigma)\epsilon(\tau) Q'_{t+s-q+1}(f_{I'}(b'_\sigma), Q'_u(f_{I''}(b''_\sigma), m'_\tau), m''_\tau),\end{aligned}$$

where in the second and forth term, the sum is taken over all $I' \in \mathbb{N}^t$ such that $|I'| = p$, over all $u = 1, \ldots, r-p$ and all $I'' \in \mathbb{N}^u$ such that $|I''| = r-p$. But this sum equals

$$\sum_{t=1}^{r} \sum_{\substack{I \in \mathbb{N}^t \\ |I|=r}} (\tilde{Q})^2_{s+t}(f_I(b_1, \ldots, b_r), m_1, \ldots, m_s),$$

which is zero. The case $s = 0$ goes in the same manner.

Now, $f \times \mathrm{Id}$ is a map of formal DG manifolds and for a diagram

$$\begin{tikzcd}
A \arrow[dr, dotted] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\
& B \times M \arrow[r, "f\times \mathrm{Id}"] \arrow[d] & B' \times M \arrow[d] \\
& B \arrow[r, "f"] & B'
\end{tikzcd}$$

one can see that $j := g \times (\mathrm{pr}_M \circ h)$ is a DG morphism completing the diagram commutatively. Hence, the quadratic diagram is cartesian and the pair $(f \times \mathrm{Id}, f)$ is a morphism of deformations. $\square$

Corollary 1.2.5. *If (F, f) is a morphism $(B, Q^B, Q) \longrightarrow (B', Q^{B'}, Q')$ of deformations and f an isomorphism, then (F, f) is also an isomorphism.*

Proof. The deformation (B, Q^B, Q) is natural isomorphic to the deformation obtained by base change. For the latter one, the statement is clear. $\square$

1.2.2 A Universal deformation

Definition 1.2.6. A deformation (U, Q^U, Q) of (M, Q^M) is called **universal**, if for each deformation (B, Q^B, Q'), there exists a morphism $(F, f) : (B, Q^B, Q') \longrightarrow (U, Q^U, Q)$, where f is uniquely defined. A deformation (V, Q^V, Q) is called **semi-universal**, if for each deformation (B, Q^B, Q'), there exists a homomorphism

$$(B, Q^B, Q') \longrightarrow (V, Q^V, Q)$$

of deformations and if (V, Q^V) is minimal (in the sense of Definition 1.1.6).

Let L be the differential graded Lie algebra $\mathrm{Coder}(S(M), S(M))$ with bracket

$$[s, t] = s \circ t - (-1)^{st} t \circ s,$$

for homogeneous s, t and differential $d(s) := (-1)^s [s, Q^M]$.

Definition 1.2.7. The complex (L, d) is called the **tangent complex** of M.

Set $U := L[1]$ and denote the vectorfield corresponding to the DGL structure on L by Q^U. There is a canonical construction of a deformation Q of M with base U:

Define multilinear maps $q_n : U \otimes M^{\otimes n-1} \longrightarrow M$ of degree $+1$ by

$$u \otimes m_1 \otimes \ldots \otimes m_{n-1} \mapsto (\uparrow u)(m_1 \odot \ldots \odot m_{n-1})$$

and denote the symmetrisation of the map $(U \times M)^{\otimes n} \longrightarrow U \times M$ induced by $\frac{1}{n!} q_n$ by Q_n. Hence, we get a vectorfield Q of degree $+1$ on $U \times M$ such that $Q|_{\{0\} \times M} = 0$.

Proposition 1.2.8. *The vectorfield $\tilde{Q} := Q^M + Q^U + Q$ is a DG vectorfield on $U \times M$ and the projection $U \times M \longrightarrow U$ respects the DG structures $\tilde{Q}$ and Q^U.*

Proof. Remember that we have

$$(\tilde{Q}^2)_n = \sum_{k+l=n+1} \tilde{Q}_l \circ (\tilde{Q}_k \otimes 1 \otimes \ldots \otimes 1) \circ \alpha_{k,n}.$$

Since Q^M and Q^U are L_∞-structures, we have $(\tilde{Q}^2)_n(a_1, \ldots, a_n) = 0$ if all a_i belong to M or if all a_i belong to U. Hence, it is enough to show that $\tilde{Q}^2$ is zero on products of the form

$$w_1 \odot \ldots \odot w_i \odot m_1 \odot \ldots \odot m_n,$$

for $i = 1, 2$ and $n \geq 1$. In the case $i = 1$, write w instead of w_1. We have

$$\begin{aligned}
&(\tilde{Q}^2)_{n+1}(w, m_2, \ldots, m_n) = \\
&\quad \sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} \epsilon(\sigma, m) Q_l^M(Q_{k+1}(w, m'_\sigma), m''_\sigma) \\
&\quad + \sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,l)} \epsilon(\sigma, m) \cdot (-1)^w Q_l(w, Q_k^M(m'_\sigma), m''_\sigma) \\
&\quad + Q_n(Q_1^U(w), m) = \\
&\quad \sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} \epsilon(\sigma, m) Q_l^M((\uparrow w)(m'_\sigma), m''_\sigma) \\
&\quad + \sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,l)} \epsilon(\sigma, m) \cdot (-1)^w (\uparrow w)(Q_k^M(m'_\sigma), m''_\sigma) \\
&\quad + (-1)^{w+1}((\uparrow w) \circ Q^M - Q^M \circ (\uparrow w))(m),
\end{aligned}$$

which is zero since the first two factors are minus the last factor. For $i = 2$, we have

$$\begin{aligned}
Q_{n+2}(w_1, w_2, m_1, \ldots, m_n) &= Q_{n+1}(Q_2^U(w_1, w_2), m) \\
&+ \sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} \epsilon(\sigma, m)(-1)^{w_1} Q_{l+1}(w_1, Q_{k+1}(w_2, m'_\sigma), m''_\sigma) \\
&+ \sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} \epsilon(\sigma, m)(-1)^{1+w_1 w_2 + w_2} Q_{l+1}(w_2, Q_{k+1}(w_1, m'_\sigma), m''_\sigma),
\end{aligned}$$

which is equal to the sum

$$
\begin{aligned}
&(-1)^{w_1+1}[\uparrow w_1, \uparrow w_2](m)+ \\
&\qquad \sum_{k+l=n+1}\sum_{\sigma\in \mathrm{Sh}(k,n)} \epsilon(\sigma,m)(-1)^{w_1}(\uparrow w_1)((\uparrow w_2)(m'_\sigma), m''_\sigma) \\
&\qquad + \sum_{k+l=n+1}\sum_{\sigma\in \mathrm{Sh}(k,n)} \epsilon(\sigma,m)(-1)^{1+w_1w_2+w_2}(\uparrow w_2)((\uparrow w_1)(m'_\sigma), m''_\sigma),
\end{aligned}
$$

which is zero, since the first factor is the negative of the last two factors. □

Hence, (U, Q^U, Q) is a deformation of (M, Q^M).

Theorem 1.2.9. *The triple (U, Q^U, Q) is a universal deformation of Q^M. More precisely, the mapping $Q' \mapsto f$, where*

$$(\uparrow f_n(b_1 \odot \ldots \odot b_n))_k(m_1,\ldots,m_k) := Q'_{n+k}(b_1,\ldots,b_n,m_1,\ldots,m_k)$$

defines a one-to-one correspondence between deformations of M with base (B, Q^B) and morphisms $B \longrightarrow U$ of formal DG manifolds. In other words, U represents the functor

$$B \mapsto \{\textit{deformations of } M \textit{ with base } B\}.$$

Proof. We have to show that $(Q^M + Q^B + Q')^2 = 0$, if and only if the family $(f_n)_n$ defines a map $f : S(B) \longrightarrow S(U)$ of differential graded coalgebras, i.e. if and only if for each n and $b_1,\ldots,b_n \in B$, the equation

$$
\begin{aligned}
Q_1^U(f_n(b)) + \frac{1}{2}\sum_{i+j=n}\sum_{\sigma\in \mathrm{Sh}(i,n)} \epsilon(\sigma,b) Q_2^U(f_i(b'_\sigma), f_j(b''_\sigma)) = \\
\sum_{k+l=n+1}\sum_{\sigma\in \mathrm{Sh}(k,n)} \epsilon(\sigma,b) f_l(Q_k^B(b'_\sigma), b''_\sigma) \quad (2.8)
\end{aligned}
$$

holds. In equation (2.8), we apply both sides on terms $m_1 \odot \ldots \odot m_r \in M^{\odot r}$ and use the definition of f. Then, the condition on f is equivalent to the condition that the following term is zero:

$$
\begin{aligned}
&\sum_{k+l=r+1}\sum_{\tau\in \mathrm{Sh}(k,r)} \epsilon(\tau,m) Q_l^M(Q'_{n+k}(b,m'_\tau), m''_\tau) \\
&\qquad +(-1)^{b_1+\ldots+b_n}\sum_{k+l=r+1}\sum_{\tau\in \mathrm{Sh}(k,r)} \epsilon(\tau,m) Q'_{n+l}(b, Q_k^M(m'_\tau), m''_\tau) \\
&+ \sum_{\substack{i+j=n\\ i,j\geq 1}}\sum_{\sigma\in \mathrm{Sh}(i,n)} \epsilon(\sigma,b)(-1)^{b_{\sigma(1)}+\ldots+b_{\sigma(i)}} \sum_{k=0}^{n}\sum_{\tau\in \mathrm{Sh}(k,r)} \epsilon(\tau,m) Q'_{i+l}(b'_\sigma, Q'_{j+k}(b''_\sigma, m'_\tau), m''_\tau) \\
&\qquad + \sum_{k+l=r+1}\sum_{\sigma\in \mathrm{Sh}(k,n)} \epsilon(\sigma,b) Q'_{n+r}(Q_k^B(b'_\sigma), b''_\sigma, m)
\end{aligned}
$$

But this term just equals

$$(Q^M + Q^B + Q')^2(b_1, \ldots, b_n, m_1, \ldots, m_r).$$

This proves the second part of the theorem.

To prove the first part, we show that the map $F := (f \times \mathrm{Id}) : B \times M \longrightarrow U \times M$ respects the DG structures, i.e. that for $n \geq 0$, the following equality holds:

$$\begin{aligned} &\tilde{Q}_1 F_n + \frac{1}{2} \sum_{i+j=n} \tilde{Q}_2 \circ F_i \otimes F_j \circ \alpha_{i,n} \\ &\quad + \sum_{k=3}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} \frac{1}{I!k!} \tilde{Q}_k \circ (F_{i_1} \otimes \ldots \otimes F_{i_k}) \circ \alpha_n \\ &\quad = \sum_{k+l=n+1} F_l \circ (\tilde{Q}'_k \otimes 1 \otimes \ldots \otimes 1) \circ \alpha_{k,n} \end{aligned} \tag{2.9}$$

Remark that F_n takes the following values on products $b_1 \odot \ldots \odot b_r \odot m_1 \odot \ldots \odot m_{n-r}$, with $r < n$, $b_i \in B$ and $m_j \in M$:

$$\begin{aligned} F_n(b_1, \ldots, b_r, m_1, \ldots, m_{n-r}) &= 0 \qquad \text{for} \qquad 0 < r < n \\ F_n(b_1, \ldots, b_n) &= f_n(b_1, \ldots, b_n) \\ F_n(m_1, \ldots, m_n) &= 0 \qquad \text{for} \qquad m > 1 \\ F_1(m_1) &= m_1 \end{aligned}$$

Applying the left hand-side of equation (2.9) on $b_1 \odot \ldots \odot b_r \odot m_1 \odot \ldots \odot m_{n-r}$, we only get the term

$$Q_{1+n-r}(f_r(b_1, \ldots, b_r), m_1, \ldots, m_r).$$

Applying the right hand-side of equation (2.9) on $b_1 \odot \ldots \odot b_r \odot m_1 \odot \ldots \odot m_{n-r}$, we only get the term

$$Q'_n(b_1, \ldots, b_r, m_1, \ldots, m_r).$$

By our construction, both terms coincide. □

At the end of Section 1.5, we will be able to construct a semiuniversal deformation of an L_∞-algebra with split tangent complex (see Theorem 1.5.13).

1.2.3 Infinitesimal deformations and obstructions

In deformation theory, an infinitesimal deformation is always a deformation over the double point. What do we mean by double point in our context ?

Definition 1.2.10. The **n-fold point** is the formal DG manifold (B, Q^B) with $B = \mathbb{C}e_0 \oplus \mathbb{C}e_1$, $deg(e_i) = i$ and

$$Q_j^B(e_0, \dots, e_0) := \begin{cases} e_1 & \text{for} \quad j = n, \\ 0 & \text{for} \quad j \neq n. \end{cases}$$

Fix an arbitrary formal DG manifold (M, Q^M) with tangent complex $(L, d, [\cdot, \cdot])$ and corresponding (U, Q^U), which is the base of a universal deformation of M. Recall that for given $B = (B, Q^B)$, deformations of M with base B correspond to morphisms $B \longrightarrow U$ of formal DG manifolds.

Proposition 1.2.11. *Let $D = (D, Q^D)$ be the double point. There is a 1:1- correspondence between morphisms $D \longrightarrow U$ and formal maps $q : D^0 \longrightarrow U^0$ such that $q_1(e_0) \in \mathrm{Kern}(Q_1^U)$. The morphism $D \longrightarrow U$ is linear if and only if q is linear.*

Proof. We can extend a given $q : D^0 \longrightarrow U^0$ in the following way to a morphism $D \longrightarrow U$:

$$\begin{aligned} q_1(e_1) :=& Q_1^U(q_2(e_0, e_0)) + Q_2^U(q_1(e_0), q_1(e_0)) \\ q_2(e_1, e_0) :=& Q_1^U(q_3(e_0, e_0, e_0)) + Q_2^U(q_2(e_0, e_0), q_1(e_0)) \\ q_3(e_1, e_0, e_0) :=& Q_1^U(q_4(e_0, e_0, e_0, e_0)) + Q_2^U(q_2(e_0, e_0), q_2(e_0, e_0)) \\ &+ \quad Q_2^U(q_3(e_0, e_0, e_0), q_1(e_0)) \\ &\vdots \end{aligned}$$

□

Thus, a deformation over the double point is just a formal map $\mathbb{C}e_0 \longrightarrow U$. We want to call only the minimal "infinitesimal deformations".

Definition 1.2.12. Let e_i be a vector of degree $i \in \mathbb{Z}$. A deformation of (M, Q^M) with base $\mathbb{C}e_i$ is called **infinitesimal deformation in degree** i, if the corresponding morphism $q : \mathbb{C}e_i \longrightarrow U$ is linear.

We see that we can interpret $\mathrm{Kern}(Q_1^U) \cap U^i$ as space of infinitesimal deformations in degree i. It will be clear after Section 1.5.3 that elements of $\mathrm{Im}(Q_1^U) \cap U^i$ define trivial infinitesimal deformations (they are in the kernel of the map $g : U \longrightarrow V$, see Section 1.5.3). Hence, the homology $H^{i+1}(L)$ can be interpreted as set of equivalence classes of infinitesimal deformations in degree i of (M, Q^M).

Proposition 1.2.13. *Let $T = (T, Q^T)$ be the triple point and $q : T^0 \longrightarrow U^0$ a formal map such that $q(e_0) \in \mathrm{Kern}(Q_1^U)$. We can extend q to a morphism $T \longrightarrow U$ if and only if*

$$Q_1^U(q_2(e_0, e_0)) + Q_2^U(q_1(e_0), q_1(e_0)) = 0.$$

Proof. In the same manner as Proposition 1.2.11. □

In particular, an "infinitesimal deformation" $q : \mathbb{C}e_0 \longrightarrow U$ can be extended to a deformation over the triple point, if and only if

$$Q_2^U(q_1(e_0), q_1(e_0)) \in \mathrm{Im}(Q_1^U).$$

Since $Q_2^U(q_1(e_0), q_1(e_0)) \in \mathrm{Kern}(Q_1^U)$, obstructions belong to $H^1(U) = H^2(L)$. More generally, for each even i, the obstructions for extending an infinitesimal deformation in degree i belong to $H^{i+1}(U)$.

1.3 Trees

Trees were first used by Kontsevich/ Soibelman [23] to describe the A_∞-structure that a DG module, homotopy equivalent to a differential graded algebra inherits. We have a similar objective, but for $L-\infty$-algebras instead of A_∞-algebras. Trees in our definition are always binary trees. We give a definition of binary trees and assign several invariants to them, which are important in order to get good signs later on.

1.3.1 Definitions

Definition 1.3.1. A **(binary) tree** with n leaves consists of a pair $\phi = (\phi, V)$ where $V = \{K_0, \dots, K_{n-2}\}$ denotes a set of **ramifications** and ϕ denotes a map $\phi : \{K_1, \dots, K_{n-1}\} \longrightarrow \{K_0, \dots, K_{n-1}\}$ such that for each $i = 0, \dots, n-2$, we have:

(1) The inverse image $\phi^{-1}(K_i)$ contains at most 2 elements.

(2) There is an $n \geq 0$ such that $\phi^n(K_i) = K_0$.

K_0 is called **root** of ϕ.

There is a tree with one leaf and no ramification which will always be denoted by τ.

Definition 1.3.2. An **orientation** of a tree (ϕ, V) is a family $\pi = (\pi_K)_{K \in V}$ of inclusions $\pi_K : \phi^{-1}(K) \longrightarrow \{1, 2\}$. The triple $\phi = (\phi, V, \pi)$ is called an **oriented tree.**

Definition and Proposition 1.3.3. *For each oriented tree (ϕ, V, π), there is a natural ordering on the set V: For $K \in V \setminus K_0$, suppose that $\phi^m(K) = K_0$. We set*

$$v(K) := \frac{\pi_{\phi(K)}(K)}{3^m} + \frac{\pi_{\phi^2(K)}(\phi(K))}{3^{m-1}} + \dots + \frac{\pi_{\phi^m(K)}(\phi^{m-1}(K))}{3}.$$

Set $v(K_0) := 0$. Then $v : V \longrightarrow \mathbb{R}$ is injective, hence it induces an ordering on V.

When we write down the value $v(K)$ of a ramification K in its 3-ary decomposition, we just get an algorithm, how to get from the root K_0 to K. For example 0.1121 means "go (in the driving direction) right-right-left-right". When (ϕ, V, π) is an oriented tree with n leaves, we can extend the map ϕ to a map $\tilde{\phi} : V \setminus K_0 \cup \{1, \ldots, n\} \longrightarrow V$ such that

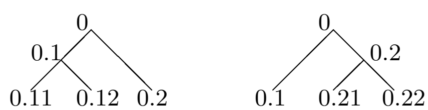
- For $1 \leq i < j \leq n$, we have $\tilde{\phi}(i) \leq \tilde{\phi}(j)$.
- For each $K \in V$, $\tilde{\phi}^{-1}(K)$ has exactly 2 elements.

The numbers $1, \ldots, n$ stand for the leaves of ϕ. Furthermore, we can extend the map $v : K \longrightarrow [0, 1)$ on $\tilde{V} := V \cup \{1, \ldots, n\}$ in such a way that the 3-ary decomposition of $v(i)$ describes the way from the root to the i-th leaf of ϕ, for $i = 1, \ldots, n$. Then we have $v(i) < v(j)$, for $1 \leq i < j \leq n$. In consequence, we have an ordering on $\tilde{V}$.

Definition 1.3.4. Two trees (ϕ, V) and (ϕ', V') are called **equivalent**, if there is a bijection $f : V \longrightarrow V'$ of the ramification sets such that $f \circ \phi = \phi' \circ f$. Two oriented trees (ϕ, V, π) and (ϕ', V', π') are called **oriented equivalent**, if there is a bijection $f : V \longrightarrow V'$ of the ramification sets such that $f \circ \phi = \phi' \circ f$ and $\pi' \circ f = \pi$.

When we draw oriented trees, we shall put elements K' of $\phi^{-1}(K)$ down left of K if $\pi_K(K') = 1$ and down right of K if $\pi_K(K') = 2$.

Example 1.3.5. The following trees with three leaves are equivalent but not oriented equivalent:

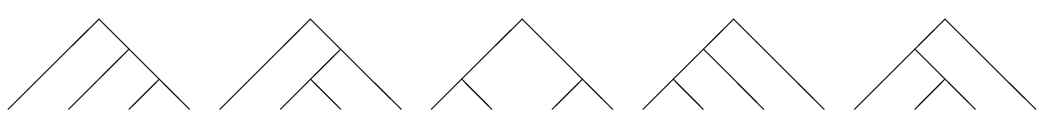


For each ramification and each leaf, we have indicated its value.

Set $\mathrm{Ot}(n)$ to be the set of equivalence classes of oriented trees with n leaves.

Example 1.3.6. (1) The set $\mathrm{Ot}(2)$ contains just one element. We denote it by β.

(2) The set $\mathrm{Ot}(4)$ contains just the following elements:



Definition 1.3.7. For a tree (ϕ, V) and $K \in V$, there is a tree $\phi|_K$ with root K and ramifications $\{K' \in V : \ \phi^n(K') = K \text{ for an } n \geq 0\}$.

We have to introduce several invariants:

For a tree ϕ with $n > 1$ leaves and $1 \leq i \leq n$, set $w_\phi(i)$ to be the difference of the number $s_\phi(i)$ of ramifications of ϕ which are smaller than i and $i - 1$. ($i - 1$ is the number of leaves of ϕ, smaller than i.) For $K \in V$, set $w_\phi(K) := w_{\phi-\phi|_K}(K)$, where on the right hand-side, K is considered as leaf of $\phi - \phi|_K$.

Remark 1.3.8. For $K \in V$, the integer $w_\phi(K)$ is just the number of 1's arising in the 3-ary decomposition of $v(K)$.

Now, for each tree ϕ with at least 2 leaves, set $e(\phi) := (-1)^{w_\phi(1)+\ldots+w_\phi(n)}$. Set $e(\tau) := 1$

Example 1.3.9. (1) $e(\beta) = -1$

(2) For the first tree in Example 1.3.5, we have $e(\phi) = -1$; For the second tree in Example 1.3.5, we have $e(\phi) = +1$;

Now, let L be a graded module, ϕ an oriented tree with n leaves and $B = (b_K)_{K\in V}$ a family of bilinear maps $L \otimes L \longrightarrow L$. Recursively, we want to define a multilinear map

$$\phi(B) : L^{\otimes n} \longrightarrow L.$$

- If ϕ has one leaf, i.e. B is empty, we set $\phi(B) := \mathrm{Id}$.
- If ϕ has only two leaves, i.e. $V = \{K_0\}$, for a bilinear map $b_0 : L \otimes L \longrightarrow L$, we set $\phi(b_0) := b_0$.
- If $\phi^{-1}(K_0)$ contains exactly one element, say K_1, and $\pi_{K_0}(K_1) = 1$, we set
$$\phi(B) := b_0 \circ (\phi|_{K_1}((b_K)_{K\in V\setminus K_0}) \otimes 1).$$
- If $\phi^{-1}(K_0)$ contains exactly one element, say K_1, and $\pi_{K_0}(K_1) = 2$, we set
$$\phi(B) := b_0 \circ (1 \otimes \phi|_{K_1}((b_K)_{K\in V\setminus K_0})).$$
- If $\phi^{-1}(K_0) = \{K_1, K_2\}$ with $\phi_{K_0}(K_1) = 1$ and $\phi_{K_0}(K_2) = 2$, we set
$$\phi(B) := b_0 \circ (\phi|_{K_1}((b_K)_{K\in V_1}) \otimes \phi|_{K_2}((b_K)_{K\in V_2})).$$

  Here, V_1 denotes the ramification set of $\phi|_{K_1}$ and V_2 the ramification set of $\phi|_{K_2}$.

1.3.2 Operations on trees

Addition Let (ϕ, V, π) and (ϕ', V', π') be oriented trees with disjoint ramification sets. Let R be a point in neither one of them. Set $V'' := V \cup V' \cup \{R\}$. We define a map $\psi : V'' \setminus R \longrightarrow V''$ by $\psi|_{V\setminus K_0} := \phi$, $\psi|_{V'\setminus K_0'} := \phi'$ and $\psi(K_0) := \psi(K_0') := R$.

There is a family $(\pi''_K)_{K\in W}$ of inclusions $\pi''_K : \psi^{-1}(K) \longrightarrow \{0,1\}$ with $\pi''_K = \pi_K$, for $K \in V$, $\pi''_K = \pi'_K$ for $K \in V'$ and $\pi''_R(K_0) = 0$ and $\pi''_R(K_0') = 1$. Now, we set

$$(\phi, V, \pi) + (\phi', V', \pi') := (\psi, V'', \pi'').$$

It is obvious how to define the addition of non-oriented trees. The addition of oriented trees is not commutative. The addition of non-oriented trees is commutative.

Example 1.3.10. We have $\tau + \tau = \beta$. Furthermore, each tree can be reconstructed by addition out of copies of τ.

Subtraction Let (ϕ, V) be a tree with n leaves and $K \in V$. Let l be the number of leaves of $\phi|_K$. The definition of a tree $\phi - \phi|_K$ with $n - l + 1$ leaves is quite obvious.

Composition Let (ϕ, V, π) be an oriented tree with n leaves and let $(\psi^{(1)}, V^{(1)}, \pi^{(1)}), \ldots, (\psi^{(n)}, V^{(n)}, \pi^{(n)})$ be oriented trees. Let W be the disjoint union of V and all $V^{(i)}$. For $K \in V$ set $n(K) := 2 - |\phi^{-1}(K)|$. (This is the number of leaves belonging to K.) Let $K_1 < \ldots < K_l$ all elements K of V with $n(K) > 0$. We define a map $\Phi : W \setminus K_0 \longrightarrow W$ as follows: For $K \in V \setminus K_0$, set $\Phi(K) := \phi(K)$. For $K \in V^{(i)} \setminus K_0^{(i)}$, set $\Phi(K) := \psi^{(i)}(K)$. And define the values of Φ on the $K^{(i)}$, setting

$$(\Phi(K_0^{(1)}), \ldots, \Phi(K_0^{(n)})) := (\underbrace{K_1, \ldots, K_1}_{n_1 \text{ times}}, \ldots, \underbrace{K_l, \ldots, K_l}_{n_l \text{ times}}).$$

Then, (Φ, W) is a tree with a canonical orientation π', given as follows: For each i, $K \in V^{(i)}$ and $K' \in \Phi^{-1}(K)$, we set $\pi'(K') := \pi^{(i)}(K')$. For $K \in V$ and $K' \in \Phi^{-1}(K) \cap V$, we set $\pi'(K') := \pi(K')$. It remains to define π'_{K_i} on elements of $\Phi^{-1}(K_i) \setminus V$, for $i = 1, \ldots, l$. So, if $n(K_i)$ equals 2, then $\Phi^{-1}(K_i) \setminus V$ has two elements, say $K_0^{(j)}$ and $K_0^{(k)}$ with $j < k$. Set $\Phi_{K_i}(K_0^{(j)}) := 1$ and $\Phi_{K_i}(K_0^{(k)}) := 2$. If $n(K_i)$ equals 1, then $\Phi^{-1}(K_i)$ has one element in V, say K and one element which is not in V, say K'. Set $\Phi_{K_i}(K') := 1$ if $\phi_{K_i}(K) = 2$ and $\Phi_{K_i}(K') := 2$ if $\phi_{K_i}(K) = 1$.

We will denote this decomposition by $\Phi = \phi \circ (\psi^{(1)}, \ldots, \psi^{(n)})$.

The next lemma follows directly from the definitions:

Lemma 1.3.11. *In this situation, suppose that there is a family $B = (b_K)_{K \in W}$ of of bilinear maps $L \otimes L \longrightarrow L$. Set $B^{(0)} := (b_K)_{K \in V}$ and $B^{(i)} := (b_K)_{K \in V^{(i)}}$, for $i = 1, \ldots, n$. We have*

$$\phi \circ (\psi^{(1)}, \ldots, \psi^{(n)})(B) = (-1)^{exponent} \phi(B^{(0)}) \circ (\psi^{(1)}(B^{(1)}) \otimes \ldots \otimes \psi^{(n)}(B^{(i)})),$$

where the exponent is the sum

$$(\sum_{K \in V^{(1)}} b_K)(\sum_{K \in V}^{V > 1} b_K) + \ldots + (\sum_{K \in V^{(n-1)}} b_K)(\sum_{K \in V}^{V > n-1} b_K).$$

We remind that $V > i$ means that the value $v(V)$ is greater than the value $v(i)$ of the i-th leaf of ϕ.

1.4 L_∞-equivalence of L and $H(L)$

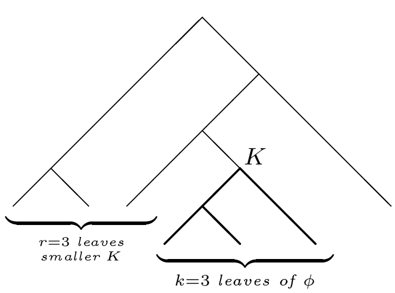
Let $L = (L, d, [\cdot,\cdot])$ be a differential graded Lie algebra, where the differential d is of degree $+1$. Suppose that there is a splitting η, i.e. a map of degree -1 such that $d\eta d = d$. Furthermore, suppose that $\eta^2 = 0$ and $\eta d\eta = \eta$. When we use a Lie bracket on $\mathrm{Hom}(L, L)$, we mean the graded commutator.

In this section, we want to construct an L_∞-algebra structure μ_* on $H := H(L, d)$ with $\mu_1 = 0$ such that $(L, d, [\cdot,\cdot])$ and (H, μ_*) are L_∞-equivalent. The multilinear forms μ_n will be constructed using trees as in the last section. In the A_∞-context, the existence of an A_∞-structure on the cohomology of a DG algebra A had already be shown (for A connected) by Kadeishvili [21], Gugenheim/ Stasheff [15] and (in the general case) by Merkulov [35]. Merkulov gives a recursion formula for construction of the higher products. A similar recursion formula for the L_∞-case can be found in the article [19] of Huebschmann and Stasheff. Kontsevich and Soibelman [23] rewrote the higher terms obtained by Merkulov's construction (A_∞-case) in terms of decorated trees. Their formulas are still recursion formulas.

In contrast to Kontsevich/ Soibelman, we only work with binary trees and give explicit (non-recursion) formulas for the terms μ_n. Using the invariants $e(\phi)$ defined in the last section, we are able to control signs terms.

We have to make some preparations. First of all, there is the following simple but important lemma:

Lemma 1.4.1. *Let $n \geq 3$ be a natural number. There is a 1:1-correspondence between triples (Φ, K, σ), where $\Phi = (\Phi, V, \pi)$ is an oriented tree with n leaves, K is a ramification in V, σ a permutation in Σ_n and 6-tuples $(k, \phi, \psi, \rho, \gamma, \delta)$, where k is a natural number with $2 \leq k \leq n-1$, ϕ is a tree in $\mathrm{Ot}(k)$, ψ is a tree in $\mathrm{Ot}(l)$ where $l := n + 1 - k$, ρ is a shuffle in $\mathrm{Sh}(k, n)$ and $\gamma \in \Sigma_l$, $\delta \in \Sigma_k$ are permutations.*



Example:
The fine lines represent the tree ψ and the fat lines the tree ϕ.
In the sequel, the first r leaves of Φ will be associated to the indexes $\sigma(1) = \rho(\gamma(1) + k - 1), ..., \sigma(r) = \rho(\gamma(r) + k - 1)$, the following k leaves to the indexes $\sigma(r+1) = \rho(\delta(1)), ..., \sigma(r+k) = \rho(\delta(k))$ and the remaining leaves to the indexes $\sigma(r+k+1) = \rho(\gamma(r+2) + k - 1), ..., \sigma(n) = \rho(\gamma(l) + k - 1)$.

To the triple (Φ, K, σ), we associate the following data: Set k to be the number of leaves of $\Phi|_K$, $\phi := \Phi|_K$, $\psi := \Phi - \phi$. Let r be the number of leaves F of Φ with $F < K$. The shuffle ρ is chosen in such a way that $\{\rho(1), ..., \rho(k)\} = \{\sigma(r+1), ..., \sigma(r+k)\}$. The permutation δ is defined by $\delta(i) := \rho^{-1}(\sigma(r+i))$, for $i = 1, ..., k$ and γ is defined in the following way:

$$\gamma(i) := \begin{cases} \rho^{-1}(\sigma(i)) - k + 1 & \text{for} \quad i = 1, ..., r \\ 1 & \text{for} \quad i = r+1 \\ \rho^{-1}(\sigma(i+k-1)) - k + 1 & \text{for} \quad i = r+2, ..., l \end{cases}$$

In the other direction, to the 6-tuple $(k, \phi, \psi, \rho, \gamma, \delta)$, we associate the following data: Set $r := \gamma^{-1}(1) - 1$. Then Φ is the composition

$$\Phi = \psi \circ (\underbrace{\tau, ..., \tau}_{r \text{ times}}, \phi, \tau, ..., \tau),$$

where τ again stands for the tree with one leaf. The ramification K is the root of ϕ, considered as ramification of Φ and σ is given by

$$\sigma(i) := \begin{cases} \rho(\gamma(i) + k - 1) & \text{for} \quad i = 1, ..., r \\ \rho(\delta(i - r)) & \text{for} \quad i = r+1, ..., r+k \\ \rho(\gamma(i - (k-1)) + k - 1) & \text{for} \quad i = r+k+1, ..., n. \end{cases}$$

Now suppose that such corresponding tuples $(\Phi, \hat{K}, \sigma)$ and $(k, \phi, \psi, \rho, \gamma, \delta)$ are given. Let V' be the ramification set of ψ and V'' the ramification set of ϕ. Then $V := V' \cup V''$ is the ramification set of Φ. Again, set $r := \gamma^{-1}(1) - 1$. Remark that the ordering on V depends on γ. We define a permutation $\tilde{\gamma} \in \Sigma_{l-1}$ by

$$\tilde{\gamma}(i) := \begin{cases} \gamma(i) - 1 & \text{for} \quad i = 1, ..., r \\ \gamma(i+1) - 1 & \text{for} \quad i = r+1, ..., l-1. \end{cases}$$

Lemma 1.4.2. *We keep all notation from above. Let $B = (b_K)_{K \in V}$ be a family of homogeneous bilinear forms $L \otimes L \longrightarrow L$. Denote the subfamilies $(b_K)_{K \in V'}$ and $(b_K)_{K \in V''}$ by B' and B''. Set W to be the set of all ramifications $K \in V$ such that $K > \hat{K}$. Then we have*

$$\psi(B') \circ \gamma \circ (\phi(B'') \circ \delta \otimes 1 \otimes ... \otimes 1) \circ \rho$$

$$= (-1)^{r+rk} \psi(B') \circ (\underbrace{1 \otimes ... \otimes 1}_{r \text{ times}} \otimes \phi(B'') \otimes 1 \otimes ... \otimes 1) \circ \sigma$$

$$= (-1)^{r+rk+\sum_{K \in W} b_k \cdot B''} \Phi(B) \circ \sigma.$$

Proof. Let $a_1, ..., a_n$ be homogeneous elements of L. We get

$$(\psi(B') \circ \gamma \circ (\phi(B'') \circ \delta \otimes 1 \otimes ... \otimes 1) \circ \rho)(a_1 \otimes ... \otimes a_n) =$$

$$= \chi(\rho, a_1, ..., a_n) \chi(\delta, a_{\rho(1)}, ..., a_{\rho(k)})$$

$$\cdot (\psi(B') \circ \gamma)(\underbrace{\phi(B'')(a_{\rho(\delta(1))}, ..., a_{\rho(\delta(k))})}_{=:u_1} \otimes \underbrace{a_{\rho(k+1)}}_{u_2} \otimes ... \otimes \underbrace{a_{\rho(n)}}_{u_l})$$

$$= \chi(\rho, a_1, ..., a_n) \chi(\delta, a_{\rho(1)}, ..., a_{\rho(k)}) \chi(\gamma, u_1, ..., u_l) \psi(B')(u_{\gamma(1)}, ..., u_{\gamma(l)}).$$

Using the following three formulas

$$\chi(\sigma, a_1, ..., a_n) = (-1)^{kr+(a_{\sigma(1)}+...+a_{\sigma(r)})(a_{\rho(1)}+...+a_{\rho(k)})}\chi(\rho, a_1, ..., a_n)\cdot$$
$$\chi(\delta, a_{\rho(1)}, ..., a_{\rho(k)})\chi(\tilde{\gamma}, a_{\rho(k+1)}, ..., a_{\rho(n)}),$$
$$u_{\gamma(1)} \otimes ... \otimes u_{\gamma(l)} = (-1)^{B''(a_{\sigma(1)}+...+a_{\sigma(r)})}\cdot$$
$$(1 \otimes ... \otimes 1 \otimes \phi(B'') \otimes 1... \otimes 1)(a_{\sigma(1)} \otimes ... \otimes a_{\sigma(n)}),$$
$$\chi(\gamma, u_1, ..., u_l) = (-1)^{r+u_1(u_{\gamma(1)}+...+u_{\gamma(r)})}\chi(\tilde{\gamma}, u_2, ..., u_l),$$

this expression is just

$$(-1)^{kr+r}\chi(\sigma, a_1, ..., a_n)\psi(B')((1 \otimes ... \otimes 1 \otimes \phi(B'') \otimes 1 \otimes ... \otimes 1)(a_{\sigma(1)} \otimes ... \otimes a_{\sigma(n)})).$$

The second equality of this Lemma is just a special case of Lemma 1.3.11. $\square$

We turn to the construction of an L_∞-structure on $H(L)$.

Proposition 1.4.3. *The map $[d, \eta] = d\eta + \eta d$ is a projection, i.e. $[d, \eta]^2 = [d, \eta]$. And $H := \mathrm{Kern}[d, \eta]$ is as module, isomorphic to $H(L)$. Remark that under the assumption of the beginning of this section, we have*

$$H = \mathrm{Kern}\, d \cap \mathrm{Kern}\, \eta.$$

The bracket on L induces a Lie-bracket on $H(L)$ and the induced bracket on H (via the isomorphism $H \longrightarrow H(L)$) is just given by $(1 - d\eta)[\cdot, \cdot] = (1 - [d, \eta])[\cdot, \cdot]$.

For simplicity, we set $g := \eta[\cdot, \cdot]$.

Theorem 1.4.4. *The following graded anti-symmetric maps $\mu_n : H^{\otimes n} \longrightarrow H$ of degree $2 - n$ define the structure of an L_∞-algebra on H:*

$$\begin{aligned} \mu_1 &:= 0 \\ \mu_2 &:= (1 - d\eta)[\cdot, \cdot] \\ &\vdots \\ \mu_n &:= (\frac{-1}{2})^{n-1} \sum_{\phi \in \mathrm{Ot}_n} e(\phi)\phi((1 - [d, \eta])[\cdot, \cdot], g, \ldots, g) \circ \alpha_n \end{aligned}$$

Here, the sum is taken over all trees ϕ with n leaves and $\phi((1 - [d, \eta])[\cdot, \cdot], g, \ldots, g)$ is the n-linear form obtained by assigning the bilinear form $(1 - [d, \eta])[\cdot, \cdot]$ to the root of the tree ϕ and the bilinear form g to each other ramification. The sign $e(\phi)$ is defined in Section 1.3. and α_n is the anti-symmetrisation map.

Proof. We must show that

$$\sum_{k+l=n+1}(-1)^{k(l-1)}\mu_l\circ(\mu_k\otimes 1\otimes\ldots\otimes 1)\circ\alpha_{k,n}=0. \tag{4.10}$$

Up to the factor $(-1)^{n-1}$ this sum has the form

$$\sum_k\sum_{\phi,\psi}\sum_{\rho,\gamma,\delta}(-1)^{k(l-1)}e(\phi)e(\psi)\psi((1-[d,\eta])[\cdot,\cdot],g,\ldots,g)\circ\gamma\circ$$
$$\circ\,(\phi((1-[d,\eta])[\cdot,\cdot],g,\ldots,g)\circ\delta\otimes 1\otimes\ldots\otimes 1)\circ\rho, \tag{4.11}$$

where k ranges from from 2 to $n-1$, $l=n+1-k$, ϕ and ψ vary in $\mathrm{Ot}(k)$ and $\mathrm{Ot}(l)$, ρ in $\mathrm{Sh}(k,n)$, γ and δ in Σ_l and Σ_k. For corresponding tuples $(k,\phi,\psi,\rho,\gamma,\delta)$ and $(\Phi,\hat{K},\sigma)$ as in Lemma 1.4.1, we denote as usual $r:=\gamma^{-1}(1)-1$ and by t the number of ramifications of ψ, greater than $r+1$. Using

$$e(\Phi)=(-1)^{w_\Phi(\hat{K})(k-1)}e(\phi)e(\psi)$$
$$w_\Phi(\hat{K})=l-1-r-t$$

and Lemma 1.4.2, the expression (4.11) can be expressed as

$$\sum_{\Phi\in\mathrm{Ot}(n)}\sum_{\hat{K}\in V\setminus K_0}e(\Phi)(-1)^{r+w_\Phi(\hat{K})}\Phi(B)\circ\alpha_n,$$

where $B=(B_K)_{K\in V}$ is the family with $b_{K_0}=b_{\hat{K}}=(1-[d,\eta])[\cdot,\cdot]$ and $b_k=\eta[\cdot,\cdot]$ for $K\neq K_0,\hat{K}$. To show that the last term is zero, it is enough to show the following two conditions:

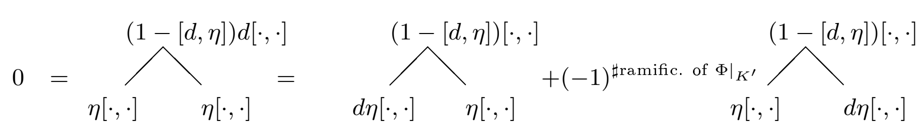
$$\sum_{\Phi\in\mathrm{Ot}(n)}\sum_{K\in V\setminus K_0}\sum_{\sigma\in\Sigma_n}(-1)^{r+w_\Phi(K)}e(\Phi)\Phi((1-[d,\eta])[\cdot,\cdot],g,\ldots,g,\underbrace{[\cdot,\cdot]}_{\text{pos. }K},g,\ldots,g)\circ\sigma=0. \tag{4.12}$$

For each tree Φ, we have

$$\sum_{K\in V\setminus K_0}(-1)^{r+w_\Phi(K)}e(\Phi)\Phi((1-[d,\eta])[\cdot,\cdot],g,\ldots,g,\underbrace{[d,\eta][\cdot,\cdot]}_{\text{position }K},g,\ldots,g)\circ\sigma=0. \tag{4.13}$$

The first condition follows by the Jacobi-identity and an easy combinatorial argument. In equation (4.13) the term annihilate each other since the differential d trickles down the branches of Φ:

Initiation of the trickling: Suppose that $\Phi^{-1}(K_0)$ contains an element K' with $\pi_{K_0}(K')=1$. We have the following picture:



Here, we only have drawn the top of the tree Φ for the case where $\Phi^{-1}(K_0)$ consists of two elements K', K'' and the corresponding bilinear forms. It is quite obvious how this goes when $\Phi^{-1}(K_0)$ has only one element, since $d|_H = 0$.

Going-on of the trickling at a ramification $K \in V$: We illustrate the case, where $\Phi^{-1}(K)$ has two elements K', K'' with $\pi_K(K') = 1$.

$$\begin{array}{c} \eta d[\cdot,\cdot] \\ \diagup \quad \diagdown \\ \eta[\cdot,\cdot] \qquad \eta[\cdot,\cdot] \end{array} \; - \; \begin{array}{c} \eta[\cdot,\cdot] \\ \diagup \quad \diagdown \\ d\eta[\cdot,\cdot] \qquad \eta[\cdot,\cdot] \end{array} \; -(-1)^{\sharp\text{ramific. of }\Phi|_{K'}} \begin{array}{c} \eta[\cdot,\cdot] \\ \diagup \quad \diagdown \\ \eta[\cdot,\cdot] \qquad d\eta[\cdot,\cdot] \end{array} \; = \; 0$$

Iterating the trickling down to the leaves and using $d|_H = 0$, we see that all terms in the sum are annihilated. $\square$

Remark 1.4.5. The restriction of μ defines a formal map $H^1 \longrightarrow H^2$. One can show that this is just the Kuranishi map as defined in [26].

Theorem 1.4.6. *The following anti-symmetric maps $f_n : H^{\otimes n} \longrightarrow L$ of degree $1-n$ define an L_∞-equivalence $H \longrightarrow L$ (i.e. an L_∞-quasi-isomorphism).*

$$\begin{aligned} f_1 &:= \textit{inclusion} \\ f_2 &:= -g \\ &\ \vdots \\ f_n &:= -(\frac{-1}{2})^{n-1} \sum_{\phi \in \mathrm{Ot}(n)} e(\phi)\phi(g, \ldots, g) \circ \alpha_n. \end{aligned}$$

Proof. For $n \geq 0$, we have to prove the equation

$$df_n - \sum_{i+j=n} \frac{(-1)^i}{2}[f_i, f_j]\alpha_{i,n} = \sum_{k+l=n+1} (-1)^{k(l-1)} f_l \circ (\mu_k \otimes 1 \otimes \ldots \otimes 1) \circ \alpha_{k,n}$$

For $l = 1$, the right hand-side is just μ_n. Since

$$df_n = (\frac{-1}{2})^{n-1} \sum_{\phi \in \mathrm{Ot}(n)} e(\phi)\phi(-d\eta[\cdot,\cdot], g, \ldots, g) \circ \alpha_n,$$

it is sufficient to show the following three identities:

$$- \sum_{i+j=n} \frac{(-1)^i}{2}[f_i, f_j]\alpha_{i,n} = (\frac{-1}{2})^{n-1} \sum_{\phi \in \mathrm{Ot}(n)} e(\phi)\phi([\cdot,\cdot], g, \ldots, g) \circ \alpha_n \tag{4.14}$$

$$f_l \circ (\phi([\cdot,\cdot], g, \ldots, g) \circ \alpha_k \otimes 1 \otimes \ldots \otimes 1) \circ \alpha_{k,n} = 0 \text{ for } l > 1, k + l = n + 1. \tag{4.15}$$

$$
\begin{aligned}
&(\frac{-1}{2})^{n-1} \sum_{\phi \in \mathrm{Ot}(n)} e(\phi)\phi(\eta d[\cdot,\cdot], g, \ldots, g) \circ \alpha_n = \\
&- \sum_{\substack{k+l=n+1 \\ l\geq 2}} (-1)^{k(l-1)} \sum_{\phi\in\mathrm{Ot}(k)} (\frac{-1}{2})^{k-1} e(\phi) \quad \cdot \\
&f_l \circ (\phi(d\eta + \eta d)[\cdot,\cdot], g, \ldots, g) \circ \alpha_k \otimes 1 \otimes ... \otimes 1) \circ \alpha_{k,n}.
\end{aligned}
\tag{4.16}
$$

Proof of equation (4.16): The right hand-side of equation (4.16) is

$$
(\frac{-1}{2})^{n-1} \sum_{k+l=n+1}^{k,l\geq 2} \sum_{\phi,\psi} \sum_{\gamma,\delta,\rho} (-1)^{k(l-1)} e(\phi)e(\psi)\psi(g,...,g) \circ \gamma \circ
$$
$$
(\phi([d,\eta][\cdot,\cdot], g, ..., g) \circ \delta \otimes 1 \otimes ... \otimes 1) \circ \rho.
$$

As in the proof of Theorem 1.4.4, this expression takes the form

$$
(\frac{-1}{2})^{n-1} \sum_{\Phi\in\mathrm{Ot}(n)} \sum_{\hat{K}\in V} \sum_{\sigma\in\Sigma_n} (-1)^{r+w_\Phi(\hat{K})} e(\Phi)\Phi(B)\sigma,
$$

where $B = (b_K)_{K\in V}$ is the family with $b_{\hat{K}} = [d,\eta][\cdot,\cdot]$ and $b_K = \eta[\cdot,\cdot]$ for $K \neq \hat{K}$.

Hence to show equation (4.16), it is enough to show that for each tree ϕ, we have

$$
\Phi(\eta d[\cdot,\cdot], g, ..., g) = \sum_{K\in V\setminus K_0} (-1)^{r+w_\Phi(K)} \Phi(B).
$$

This is true by the same trickling argument as in Theorem 1.4.4.

Proof of equation (4.15): This is again the Jacobi-identity and some combinatorics.

Proof of equation (4.14):

$$
\begin{aligned}
&\sum_{i+j=n} \frac{(-1)^i}{2} [\cdot,\cdot] \circ (f_i \otimes f_j) \circ \alpha_{i,n} = \\
&= \sum_{i+j=n} (-\frac{1}{2})^{i-1+j-1} \sum_{\phi\in\mathrm{Ot}(i),\psi\in\mathrm{Ot}(j)} \frac{(-1)^i}{2} e(\phi)e(\psi)(\phi+\psi)([\cdot,\cdot], g, ..., g) \circ \alpha_n \\
&\qquad = -(-\frac{1}{2})^{n-1} \sum_{\Phi\in\mathrm{Ot}(n)} e(\Phi)\Phi([\cdot,\cdot], g, ..., g) \circ \alpha_n.
\end{aligned}
$$

□

Almost in the same manner, one can realize the following: If $L = (L, d, [\cdot,\cdot])$ is a DGL and $f_1 : (M, d^M) \longrightarrow (L, d)$ a homotopy equivalence between DG modules, then there is an L_∞-algebra structure μ_* on M with $\mu_1 = d^M$ and an L_∞-quasi-isomorphism $f : (M, \mu_*) \longrightarrow (L, d, [\cdot,\cdot])$, extending f_1. In other words: An up to homotopy differential graded Lie algebra is an L_∞-algebra. In the A_∞-context, this was already shown by Markl [31].

1.5 Decomposition theorem for differential graded Lie algebras

In this section, we want to achieve two things: (a) the construction of an inverse map of the quasi-isomorphism $f : (H, \mu_*) \longrightarrow (L, d, [\cdot,\cdot])$ constructed in Section 1.4; (b) the construction of a semi-universal deformation (V, Q^V, Q'), for a given formal DG manifold. As consequence of (a), we get the following decomposition theorem $(L, d, [\cdot,\cdot]) \cong (H, \mu_*) \oplus (F, d, 0)$ for DGLs, where F is the complement of H in L and the sum is taken in the category of L_∞-algebras. The existence of such a decomposition was already stated by Kontsevich (see [22]) and an A_∞-analogue was proved by Kadeishvili (see [21]). In fact, each L_∞- (resp. A_∞-algebra) over a field is isomorphic to the direct sum of a minimal and a linear contractible one. Our Proposition 1.5.6 is analogue to the corresponding statement for A_∞-algebras, which was proved by Lefevre (see [28]). The proof here is almost a transcription of Lefevre's proof.

1.5.1 Obstructions

Consider the formal DG manifolds (W, Q) and (W', Q'). For any $n \geq 0$, there is a differential δ of degree $+1$ on the graded module $\mathrm{Hom}(W^{\otimes n}, W')$, given by $\delta(g) = Q'_1 \circ g - (-1)^g g \circ Q_{n,n}$. Now, let $f : W \longrightarrow W'$ be a morphism of formal supermanifolds. Set

$$r(f_1, \ldots, f_{n-1}) := \sum_{l=1}^{n-1} f_l \circ Q_{n,l} - \sum_{k=2}^{n} \sum_{i_1+\ldots+i_k=n} Q'_k \circ f_I.$$

Recall that f is an L_∞-homomorphism, if for each $n \geq 1$, we have $\delta(f_n) = r(f_1, \ldots, f_{n-1})$. If this condition is satisfied only for $n \leq m$, we call f (or the family $(f_1, \ldots, f_m)$) an L_m-**homomorphism**.

Lemma 1.5.1. *Suppose that f is an L_{n-1}-homomorphism. Then $\delta(r(f_1, \ldots, f_{n-1})) = 0$.*

The proof of Lemma 1.5.1 is done in the appendix. The proof of the next lemma is an easy exercise:

Lemma 1.5.2. *Let $e : W \longrightarrow W'$ and $f : V' \longrightarrow V$ be strict L_∞-morphisms and let $g : V \longrightarrow W$ be any L_∞-morphism. Then*

(1) $r((gf)_1, \ldots, (gf)_{n-1}) = r(g_1, \ldots, g_{n-1}) \circ f_1^{\otimes n}$,

(2) $r((eg)_1, \ldots, (eg)_{n-1}) = e_1 \circ r(g_1, \ldots, g_{n-1})$.

1.5.2 Constructions

Proposition 1.5.3. *Let $f : M \longrightarrow M'$ be a morphism of formal supermanifolds. Suppose, there is a module homomorphism $g' : M' \longrightarrow M$ such that $g' \circ f_1 = \mathrm{Id}_M$. Then, there is a morphism $g : M' \longrightarrow M$ of formal supermanifolds such that $g_1 = g'$ and $gf = \mathrm{Id}_M$. If f_1 is an isomorphism with inverse g and if f is Q-equivariant and if g' respects $Q_1^{M'}$ and Q_1^M, then g can be chosen Q-equivariant as well.*

Proof. One can check directly that the sequence of maps defined by

$$g_n := -\sum_{k=2}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} g_1 \circ f_k \circ (g_I),$$

for $n \geq 2$, define a morphism of formal supermanifolds with the desired property. □

Lemma 1.5.4. *Let $f : V \longrightarrow W$ be a morphism of formal DG manifolds.*

(i) If f_1 is split injective, then there is a formal DG manifold W' and an L_∞-isomorphism $\kappa : W \longrightarrow W'$ such that $\kappa \circ f$ is strict.

(ii) If f_1 is split surjective, then there is a formal DG manifold V' and an L_∞-isomorphism $\kappa : V' \longrightarrow V$ such that $f \circ \kappa$ is strict.

Proof. (i) As module, set $W' := W$. We have to construct an isomorphism $\kappa : S(W) \longrightarrow S(W')$ of graded coalgebras and then, we can define the DG structure on W' via $Q^{W'} := \kappa \circ Q^W \circ \kappa^{-1}$. Set $\kappa_1 := \mathrm{Id}$. Inductively, we define maps $\kappa_n : W^{\odot n} \longrightarrow W'$ such that for $2 \leq m \leq n$, we have

$$(\kappa \circ f)_m = \sum_{k=1}^{m} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} \kappa_k \circ f_I = 0.$$

Let $g : W \longrightarrow V$ be a module homomorphism with $g \circ f_1 = \mathrm{Id}_V$. When $\kappa_1, \ldots, \kappa_n$ is already constructed, set

$$\kappa_{n+1} := -\sum_{k=1}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n+1}} \kappa_k \circ f_I \circ g^{\odot n+1}.$$

Obviously, $(\kappa \circ f)_m = 0$, for $2 \leq m \leq n+1$. (ii) goes in a similar way. □

For our situation, we have the following more explicit statement:

Lemma 1.5.5. *Let $f : H \longrightarrow L$ be the L_∞-quasi-isomorphism constructed in Section 1.4. Consider the morphisms $\kappa_n : L^{\otimes n} \longrightarrow L$, defined by $\kappa_1 := \mathrm{Id}$ and $\kappa_n := -f_n \circ \mathrm{pr}_H^{\otimes n}$, for $n \geq 2$. Then, κ is an L_∞-morphism and $\kappa \circ f$ is strict. Furthermore, $(\kappa^{-1})_1 = \mathrm{Id}$ and $(\kappa^{-1})_n = f_n \circ \mathrm{pr}_H^{\otimes n}$, for $n \geq 2$.*

Proof. For $n \geq 2$, we have

$$(\kappa \circ f)_n = \sum_{k=1}^{n} \sum_{\substack{I \in \mathbb{N}^K \\ |I|=n}} \kappa_k \circ f_I = f_n - f_n = 0.$$

The second statement is as easy to prove. □

The following important proposition says that the quadruple (category of L_∞-algebras; class of L_∞-quasi-isomorphisms; class of those L_∞-morphisms f such that f_1 is split injective; class of those L_∞-morphism f such that f_1 is split surjective) satisfies Quillen's Axiom M1 (see [40]) for model categories.

Proposition 1.5.6. *Let*

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "f"] & C \arrow[d, "e"] \\ B \arrow[r, "d"] & D \end{tikzcd}$$

a commutative diagram of L_∞-algebras. Suppose that f is split injective and that e is split surjective and that either f or e is an L_∞-quasi-isomorphism. Then, there is an L_∞-morphism $g : B \longrightarrow C$ such that the complete diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] \arrow[ur, "g"] & D \end{tikzcd}$$

commutes.

Proof. By Lemma 1.5.4, we may suppose that e and f are strict. Inductively, we will construct morphisms $g_n : B^{\odot n} \longrightarrow C$ such that

(i) $\delta(g_m) + r(g_1, \ldots, g_{m-1}) = 0,$

(ii) $g_m \circ f_1^{\otimes m} = c_m,$

(iii) $e_1 \circ g_m = d_m,$

for each $m \leq n$. Choose maps $u : (D, Q_1^D) \longrightarrow (C, Q_1^C)$ and $v : (B, Q_1^B) \longrightarrow (A, Q_1^A)$ of DG-modules such that $v \circ f_1 = \mathrm{Id}_A$ and $e_1 \circ u = \mathrm{Id}_D$. A candidate for g_1 can easily be found. Suppose that $g_1, .., g_{n-1}$ are already constructed. Then

$$\beta := c_n v^{\odot n} + u d_n - u e_1 c_n v^{\odot n}$$

satisfies conditions (i) and (ii). By Lemma 1.5.2, we get

$$\begin{aligned} (\delta(\beta) + r(g_1, \ldots, g_{n-1})) \circ f_1^{\odot n} &= \\ \delta(\beta \circ f_1^{\odot n}) + r((gf)_1, \ldots, (gf)_{n-1}) &= \\ \delta(c_n) + r(c_1, \ldots, c_{n-1}) &= 0. \end{aligned}$$

On the other side, again by Lemma 1.5.2, we have

$$\begin{aligned} e_1 \circ (\delta(\beta) + r(g_1, \dots, g_{n-1})) = \\ \delta(e_1\beta) + r((eg)_1, \dots, (eg)_{n-1}) = \\ \delta(d_n) + r(d_1, \dots, d_{n-1}) = 0. \end{aligned}$$

Hence, $\delta(\beta) + r(g_1, \dots, g_{n-1})$ has a factorization

$$B^{\otimes n} \xrightarrow{p} \mathrm{Cokern}(f_1^{\odot n}) \xrightarrow{q} \mathrm{Kern}(e_1) \xrightarrow{i} C,$$

where i is the natural inclusion and p the natural epimorphism. By Lemma 1.5.1, $\delta(\beta) + r(g_1, \dots, g_{n-1})$ is a cycle, so $\delta(q) = 0$, i.e. q is a map of complexes. Now, either $\mathrm{Cokern}(f_1^{\odot n})$ or $\mathrm{Kern}(e_1)$ is contractible. Hence $q = \delta(h)$, for a morphism $h : \mathrm{Cokern}(f_1^{\odot n}) \longrightarrow \mathrm{Kern}(e_1)$ of graded modules. Then $g_n := \beta - i \circ h \circ p$ satisfies the conditions (i)-(iii). □

Corollary 1.5.7. *There is a map $g : (L, d, [\cdot, \cdot]) \longrightarrow (H, \mu_*)$ of L_∞-algebras such that $g \circ f = \mathrm{Id}_H$.*

Corollary 1.5.8. *Let M be an L_∞-manifold and (B, Q^B, Q) a deformation of M such that (B, Q_1^B) is contractible and $Q_1 = 0$. Then (B, Q^B, Q) is a trivial deformation.*

Proof. There is a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & (B \times M, Q^B + Q^M + Q) \\ \downarrow & & \downarrow \\ (B \times M, Q^B + Q^M) & \longrightarrow & B \end{array}$$

where the vertical left arrow induces an injective quasi-isomorphism of DG-modules and the vertical right arrow induces a surjective map of DG modules. By Proposition 1.5.6, there is a map $q : (B \times M, Q^B + Q^M) \longrightarrow (B \times M, Q^B + Q^M + Q)$ with $q_1 = \mathrm{Id}$, completing the diagram commutatively. In particular, q establishes an isomorphism of the given deformation and of the trivial deformation of M with base B. □

Proposition 1.5.9. *There exists*

(a) a homomorphism $\iota : (F, d, 0) \longrightarrow (L, d, [\cdot, \cdot])$ of L_∞-algebras such that ι_1 is the natural inclusion,

(b) a homomorphism $p : (L, d, [\cdot, \cdot]) \longrightarrow (F, d, 0)$ of L_∞-algebras such that $p \circ \iota = \mathrm{Id}_F$.

Proof. (a) Suppose that there are already homomorphisms $\iota_m : F^{\otimes m} \longrightarrow L$, for $m \leq n-1$, which form an L_{n-1}-homomorphism. We have to find an ι_n such that $\delta(\iota_n) = r(\iota_1, \ldots, \iota_{n-1})$. Since (F, d) is contractible, $\mathrm{Hom}(F^{\otimes n}, L)$ is acyclic, so the existence of ι_n follows by Lemma 1.5.1.
(b) By Lemma 1.5.4, we can assume that ι is strict. Set $p_1 := \mathrm{pr}_F = [d, \eta]$. Now assume that $p_1, \ldots, p_{n-1}$ are already constructed such that they define an L_{n-1}-homomorphism $p' : L \longrightarrow F$ such that $(p' \circ \iota)_m = 0$, for $m \leq n-1$. We have to find $p_n : L^{\otimes n} \longrightarrow F$ such that

$$\delta(p_n) + r(p_1, \ldots, p_{n-1}) = 0,$$
$$p_n \circ \iota^{\otimes n} = 0.$$

We may chose $p_n := \eta r(p_1, \ldots, p_{n-1})$. Then, since $r(p_1, \ldots, p_{n-1}) \in \mathrm{Kern}\,\delta$, we have $\delta(p_n) = [\eta, d] \circ r(p_1, \ldots, p_{n-1}) = r$, and again by Lemma 1.5.2, we get

$$p_n \circ \iota^{\otimes n} = \eta \circ r(p_1, \ldots, p_{n-1}) \circ \iota^{\otimes n} = 0.$$

So inductively, the map p can be constructed. □

As a consequence, we get the expected decomposition theorem for differential graded Lie algebras admitting a splitting:

Theorem 1.5.10. *We have an isomorphism of L_∞-algebras*

$$f \times \iota : H \times F \longrightarrow L.$$

Corollary 1.5.11. *If $(L, d, [\cdot,\cdot])$ and $(L', d', [\cdot,\cdot])$ are differential graded Lie algebras such that (L, d) and (L', d') are split, then, for each L_∞-quasi-isomorphism $f : L \longrightarrow L'$, there exists an L_∞-morphism $g : L' \longrightarrow L$ such that f_1 and g_1 are inverse maps on the homology. In particular, if $\mathbb{K}$ is a field, then "L_∞-quasi-isomorphic" is an equivalence relation.*

1.5.3 A semiuniversal deformation

Proposition 1.5.12. *(i) Let $M = (M, Q^M)$ be a formal DG manifold and N a Q^M-closed submodule of M, i.e. $Q_j^M(n_1, \ldots, n_j) \in N$, for all $j \geq 1$ and $n_1, \ldots, n_j \in N$. Then, $(N, Q^M|_N)$ is a formal DG manifold and the inclusion $N \longrightarrow M$ is a morphism in* `DG-Manf`*.*

(ii) Let (B, Q^B, Q) be a deformation of (M, Q^M). Suppose that (B, Q^B) is a direct sum of formal DG manifolds $(B', Q^{B'})$ and $(B'', Q^{B''})$. Then, the triple $(B'', Q^{B''}, Q|_{B'' \times M})$ is also a deformation of M and the canonical map

$$(B'' \times M, Q^{B''} + Q^M + Q|_{B'' \times M}) \longrightarrow (B \times M, Q^B + Q^M + Q) \qquad (5.17)$$

defines a morphism of deformations.

(iii) If in the situation of (ii), $(B', Q_1^{B'})$ is contractible, then the map (5.17) is an equivalence of deformations.

Proof. The statements (i) and (ii) are easy to see. To show (iii), we apply Proposition 1.5.6 to the commutative diagram

$$\begin{tikzcd} (B'' \times M, Q^{B''} + Q^M + Q|_{B'' \times M}) \arrow[r] \arrow[d] & (B \times M, Q^B + Q^M + Q) \arrow[d] \\ (B' \times (B'' \times M), Q^{B'} + Q^{B''} + Q^M + Q|_{B'' \times M}) \arrow[r] & (B, Q^B) \end{tikzcd}$$

of formal DG manifolds. We get an isomorphism

$$(B, Q^B, Q|_{B'' \times M}) \longrightarrow (B, Q^B, Q)$$

of deformations with base B. Obviously, the left one is equivalent to the deformation $(B'', Q^{B''}, Q|_{B'' \times M})$. □

For the rest of this subsection, we work in the setting of Section 1.2. Thus L is the DGL $\mathrm{Coder}(S(M), S(M))$, for some formal DG manifold (M, Q^M). Again, we must assume that the complex (L, d) has a splitting η. Equip the cohomology H of (L, d) with the L_∞-structure μ_* constructed in Section 1.4. Set $U := L[1]$, $V := H[1]$ and denote the morphism $V \longrightarrow U$ induced by the quasi-isomorphism $H \longrightarrow L$ constructed in Section 1.4 again by f.

Let (U, Q^U, Q) be the universal deformation of M (see Section 1.2). Again, set $\tilde{Q} := Q^U + Q^M + Q$. By base change $f : V \longrightarrow U$, we get a deformation (V, Q^V, Q') of (M, Q^M). Explicitly, on products $v_1 \odot \ldots \odot v_r \odot m_1 \odot \ldots \odot m_s$ with $r, s \geq 1$ and $n = r + s$, the perturbation Q' is given by

$$Q'_n(v_1, \ldots, v_r, m_1, \ldots, m_s) = (\uparrow f_r(v_1, \ldots, v_r))_s(m_1, \ldots, m_s).$$

Set $\tilde{Q}' := Q^V + Q^M + Q'$.

Theorem 1.5.13. *The deformation (V, Q^V, Q') is semi-universal.*

Proof. Since (H, μ_*) is minimal and (U, Q^U, Q) is universal, we only have to show that there exists a morphism of deformations from (U, Q^U, Q) to (V, Q^V, Q'). This is a consequence of Theorem 1.5.10 and Proposition 1.5.12. □

Corollary 1.5.14. *If the tangent complexes of two formal DG manifolds M, M' are split and L_∞-quasi-isomorphic, then M and M' have a common base of a semi-universal deformation.*

1.6 Appendix

1.6.1 Some calculations

Proof of Proposition 1.1.2: Induction on n
The case $n = 1$ follows by the commutativity of diagram 1.1 and $\mathrm{Kern}(\Delta^+) = W'$. Now suppose that the formula is proved for all $m \leq n-1$. Then we have

$$(F \otimes F \circ \Delta^+)(w_1, \ldots, w_n) =$$

$$\sum_{j=1}^{n-1} \sum_{\tau \in \mathrm{Sh}(j,n)} \epsilon(\tau, w_1, \ldots, w_n) \hat{F}_j(w_{\tau(1)}, \ldots, w_{\tau(j)}) \otimes \hat{F}_{n-j}(w_{\tau(j+1)}, \ldots, w_{\tau(n)}) =$$

$$\sum_{j=1}^{n-1} \sum_{k,k'} \sum_{I,I'} \frac{1}{k!k'!I!I'!}[(F_{i_1} \odot \ldots \odot F_{i_k}) \circ \alpha_j \otimes (F_{i'_1} \odot \ldots \odot F_{i'_{k'}}) \circ \alpha_{n-j}] \circ \alpha_{j,n}(w_1 \odot \ldots \odot w_n),$$

where k ranges over $1, ..., j$; k' over $1, ..., n-j$, I takes all values in $\mathbb{N}^k$ such that $|I| = j$ and I' takes all values in $\mathbb{N}^{k'}$ such that $|I'| = n-j$. The last expression equals

$$\sum_{k=2}^{n} \sum_{l=1}^{k-1} \sum_{I \in \mathbb{N}^k}^{|I|=n} \frac{1}{I!l!(k-l)!}[(F_{i_1} \odot \ldots \odot F_{i_l}) \otimes (F_{i_{l+1}} \odot \ldots \odot F_{i_k})] \circ \alpha_n(w_1, ..., w_n). \quad (6.18)$$

On the other hand we have

$$\Delta^+(\sum_{k=2}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} \sum_{\sigma \in \Sigma_n} \frac{1}{I!k!} \epsilon(\sigma) F_{i_1}(w_{\sigma(1)}, \ldots, w_{\sigma(i_1)}) \odot \ldots \odot F_{i_k}(w_{\sigma(n-i_k+1)}, \ldots, w_{\sigma(n)})) =$$

$$\sum_{k=2}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} \sum_{\sigma \in \Sigma_n} \frac{1}{I!k!} \epsilon(\sigma) \sum_{l=1}^{k-1} \sum_{\tau \in \mathrm{Sh}(l,k)} \epsilon(\sigma, F_{i_1}(\Diamond), \ldots, F_{i_k}(\Diamond)) F_{i_{\tau(1)}}(\Diamond) \odot \ldots \odot F_{i_{\tau(l)}}(\Diamond) \otimes$$

$$F_{i_{\tau(l+1)}}(\Diamond) \odot \ldots \odot F_{i_{\tau(k)}}(\Diamond) =$$

$$\sum_{k=2}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} \sum_{l=1}^{k-1} \binom{k}{l} \sum_{\sigma \in \Sigma_n} \frac{1}{I!k!} \epsilon(\sigma, w_1, \ldots, w_n) F_{i_1}(\Diamond) \odot \ldots \odot F_{i_l}(\Diamond) \otimes$$

$$F_{i_{l+1}}(\Diamond) \odot \ldots \odot F_{i_k}(\Diamond).$$

Here, we have set $F_{i_m}(\Diamond) := F_{i_m}(w_{\sigma(i_1+\ldots+i_{m-1}+1)}, \ldots, w_{\sigma(i_1+\ldots+i_m)})$. Since $\binom{k}{l} \cdot \frac{1}{k!} = \frac{1}{l!(k-l)!}$, we see that both sides coincide. Hence, by the commutativity of diagram (1.1), the difference

$$F_n(w_1, ..., w_n) \quad -$$

$$\sum_{k=2}^{n} \sum_{i_1+\ldots+i_k=n} \sum_{\sigma \in \Sigma_n} \frac{1}{I!k!} \epsilon(\sigma) F_{i_1}(w_{\sigma(1)}, \ldots, w_{\sigma(i_1)}) \odot \ldots \odot F_{i_k}(w_{\sigma(n-i_k+1)}, \ldots, w_{\sigma(n)})$$

belongs to $\mathrm{Kern}(\Delta^+) = W'$. Thus it is just the term $F_n(w_1, ..., w_n)$, and the induction step is done. $\square$

Proof of Proposition 1.1.3: Induction on n. Set $q_0 := Q_{0,1}(1)$. (1) By the commutativity of diagram (1.3), comparing terms of polynomial degree zero, we have that $Q_{i,0} = 0$, for each $i \geq 0$. (2) By the commutativity of diagram (1.3), comparing terms of polynomial degree i and linear degree $+1$, we have that $Q_{0,i} = 0$, for $i \neq 1$. (3) By similar arguments, we see that for $w \in W$, we have $Q_{1,2}(w) = q_0 \odot w$ and $Q_{1,i} = 0$, for $i \geq 3$. Thus the cases $n = 0, 1$ are done.
Now suppose that the statement is proved for all $m \leq n-1$. Then $(Q \otimes 1 + 1 \otimes Q)(\Delta(w_1, \ldots, w_n))$ can be written in the form

$$\sum_{l=0}^{n-1} \sum_{k=1}^{n-l} \sum_{\sigma \in \mathrm{Sh}(k+l-1,n)} \sum_{\tau \in \mathrm{Sh}(l,k+l-1)} \epsilon(\sigma)\epsilon(\tau, w_{\sigma(1)}, \ldots, w_{\sigma(k+l-1)}) \quad \cdot$$

$$Q_l(w_{\sigma(\tau(1))}, \ldots, w_{\sigma(\tau(l))}) \odot \odot w_{\sigma(\tau(l+1))} \odot \ldots \odot w_{\sigma(\tau(k+l-1))} \otimes w_{\sigma(k+l)} \odot \ldots \odot w_{\sigma(n)}$$

$$+ \quad \sum_{l=0}^{n-1} \sum_{k=1}^{n-l} \sum_{\sigma \in \mathrm{Sh}(k,n)} \sum_{\tau \in \mathrm{Sh}(l,n-k)} \epsilon(\sigma)\epsilon(\tau, w_{\sigma(k+1)}, \ldots, w_{\sigma(n)}) w_{\sigma(1)} \odot \ldots \odot w_{\sigma(k)} \otimes$$

$$\otimes Q_l(w_{\sigma(k+\tau(1))}, \ldots, w_{\sigma(k+\tau(l))}) \odot w_{\sigma(k+l+1)} \odot \ldots \odot w_{\sigma(n)}.$$

This is just the sum over k and l of the following expression:

$$\sum_{\sigma \in \mathrm{Sh}(l,k+l-1,n)} \epsilon(\sigma) Q_l(w_{\sigma(1)}, \ldots, w_{\sigma(l)}) \odot w_{\sigma(l+1)} \odot \ldots \odot w_{\sigma(k+l-1)} \otimes w_{\sigma(k+l)} \odot \ldots \odot w_{\sigma(n)}$$

$$+ \sum_{\sigma \in \mathrm{Sh}(k,k+l,n)} \epsilon(\sigma)(-1)^{w_{\sigma(1)}+\ldots+w_{\sigma(k)}} \epsilon(\tau, w_{\sigma(1)}, \ldots, w_{\sigma(k+l-1)}) w_{\sigma(1)} \odot \ldots \odot w_{\sigma(k)} \otimes$$

$$\otimes Q_l(w_{\sigma(k+1)}, \ldots, w_{\sigma(k+l)}) \odot w_{\sigma(k+l+1)} \odot \ldots \odot w_{\sigma(n)}.$$

Here, by $\mathrm{Sh}(l,m,n)$ we mean the set of all permutations $\sigma \in \Sigma_n$, such that $\sigma(1) < ... < \sigma(l)$ and $\sigma(l+1) < ... < \sigma(m)$ and $\sigma(m+1) < ... < \sigma(n)$. On the other side,

$$\Delta(\sum_{l=0}^{n-1} \sum_{\sigma \in \mathrm{Sh}(l,n)} \epsilon(\sigma) Q_l(w_{\sigma(1)}, \ldots, w_{\sigma(l)}) \odot w_{\sigma(l+1)} \odot \ldots \odot w_{\sigma(n)})$$

can be written as sum over k and l of expressions of the form

$$\sum_{\sigma \in \mathrm{Sh}(l,n)} \sum_{\tau \in \mathrm{Sh}(k,l-n+1)} \epsilon(\sigma)\epsilon(\tau, u) u_{\tau(1)} \odot \ldots \odot u_{\tau(k)} \otimes u_{\tau(k+1)} \odot \ldots \odot u_{\tau(n-l+1)},$$

where we have set

$$(u_1, \ldots, u_{n-l+1}) := (Q_l(w_{\sigma(1)}, \ldots, w_{\sigma(l)}), w_{\sigma(l+1)}, \ldots, w_{\sigma(n)}).$$

We see easily that on both sides we have the same sums. This completes the induction step. $\square$

Proof of Lemma 1.5.1: From our hypothesis, for each $m \leq n-1$, we have

$$Q_1 \circ f_m - f_m \circ Q_{m,m} = \sum_{\substack{k+l=m+1 \\ k\geq 2}} f_l \circ Q_{m,l} - \sum_{k=2}^{m} \sum_{\substack{I\in\mathbb{N}^k \\ |I|=m}} Q_k \circ f_I. \tag{6.19}$$

Furthermore, we can generalize the fact that Q is an L_∞-structure to the the following equations: Let m, k, l be natural numbers such that $m \geq 1$ and $k+l = m+1$. Then

$$Q_{l,l} \circ Q_{m,l} + Q_{m,l} \circ Q_{m,m} + \sum_{\substack{r+s=k+1 \\ r,s\geq 2}} Q_{m+1-s,l} \circ Q_{m,m+1-s} = 0. \tag{6.20}$$

Of course, they are also correct for Q'. Now, we apply δ on the first summand of $r(f_1, ..., f_{n-1})$. Using equations (6.20) and (6.19), it takes the following form:

$$\sum_{k=2}^{n} \sum_{\substack{I\in\mathbb{N}^k \\ |I|=n}} (Q'_k \circ f_I \circ Q_{n,n} - Q'_1 \circ Q'_k \circ f_I) =$$

$$\sum_k \sum_I Q'_k \circ Q'_{k,k} \circ f_I + \sum_k \sum_I \sum_{\substack{r+s=k+1 \\ r,s\geq 2}} Q'_r \circ Q'_{k,r} \circ f_I + \sum_K \sum_I Q'_k \circ f_I \circ Q_{n,n} =$$

$$-\sum_{k=2}^{n-1} \sum_I \sum_{\nu=1}^{k} \sum_{r=2}^{i_\nu} \sum_{j_1+...+j_r=i_\nu} Q'_k \circ (1^{\otimes \nu-1} \otimes Q'_r \otimes 1^{\otimes k-\nu}) \circ f_{(i_1,...,i_{\nu-1},j_1,...,j_r,i_{\nu+1},...,i_k)}$$

$$+ \quad \sum_{\bar{k}=2}^{n} \sum_{\substack{|\bar{I}|\in\mathbb{N}^{\bar{k}} \\ |\bar{I}|=n}} \sum_{\substack{\bar{r}+\bar{s}=\bar{k}+1 \\ \bar{r},\bar{s}\geq 2}} \sum_{\bar{u}+\bar{v}=\bar{r}-1} Q'_{\bar{r}} \circ (1^{\otimes\bar{u}} \otimes Q'_{\bar{s}} \otimes 1^{\bar{v}}) \circ f_{\bar{I}}$$

$$+\sum_{k=2}^{n} \sum_I \sum_{\nu=1}^{k} \sum_{\substack{s+t=i_\nu+1 \\ t\geq 2}} Q'_k \circ f_{(i_1,...,i_{\nu-1},s,i_{\nu+1},...,i_k)} \circ (1^{\otimes i_1+...+i_{\nu-1}} \otimes Q_{i_\nu,s} \otimes 1^{\otimes i_{\nu+1}+...+i_k}).$$

The first and second summand annihilate each other, since we have the following 1:1 - correspondence of index sets:

$$(k, I, \nu, r, J) \mapsto (\bar{k} = k+r-1, \bar{I} = (i_1, ..., i_{\nu-1}, j_1, ..., j_r, i_{\nu+1}, ..., i_k), \bar{r} = k, \bar{u} = \nu - 1),$$

$$(k = \bar{r}, I = (\bar{i}_1, ..., \bar{i}_{\bar{u}}, \bar{i}_{\bar{u}+\bar{s}+1}, ..., \bar{i}_{\bar{k}}), \nu + \bar{u} + 1, r = \bar{s}, J = (\bar{i}_{\bar{u}+1}, ..., \bar{i}_{\bar{u}+\bar{s}})) \leftarrow (\bar{k}, \bar{I}, \bar{r}, \bar{u}).$$

We apply δ on the second summand of $r(f_1, ..., f_{n-1})$:

$$\delta(\sum_{k=2}^{n} \sum_{\substack{I\in\mathbb{N}^k \\ |I|=n}} Q'_k \circ f_I) = \sum_k \sum_I (Q'_k \circ f_I \circ Q_{n,n} - Q'_1 \circ Q'_k \circ f_I) =$$

$$\sum_k \sum_I Q'_k \circ f_I \circ Q_{n,n} - \sum_k \sum_I Q'_k \circ Q'{}_{k,k} \circ f_I - \sum_k \sum_I \sum_{\substack{u+v=k+1 \\ u,v\geq 2}} Q'_u \circ Q'{}_{k,u} \circ f_I =$$

$$\sum_{k=2}^{n-1} \sum_I \sum_{\nu=1}^{k} \sum_{r=2}^{i_\nu} \sum_{j_1+...+j_r=i_\nu} Q'_k \circ (1^{\otimes \nu-1} \otimes Q'_r \otimes 1^{\otimes k-\nu}) \circ f_{(i_1,...,i_{\nu-1},j_1,...,j_r,i_{\nu+1},...,i_k)}$$

$$-\sum_k \sum_I \sum_{\substack{s+t=i_\nu+1 \\ t\geq 2}} Q'_k \circ f_{(i_1,...,i_{\nu-1},s,i_{\nu+1},...,i_k)} \circ (1^{\otimes i_1+...+i_{\nu-1}} \otimes Q_{i_\nu,s} \otimes 1^{\otimes i_{\nu+1}+...+i_k})$$

$$-\sum_{\bar{k}} \sum_{\bar{I}} \sum_{\substack{u+v=\bar{k}+1 \\ u,v\geq 2}} \sum_{c+d=u-1} Q'_u \circ (1^{\otimes c} \otimes Q'_v \otimes 1^{\otimes d}) \circ f_I.$$

The first and third summand annihilate each other, since we have the following 1:1 - correspondence of index sets:

$$(k, I, \nu, r, J) \mapsto (\bar{k} = k - 1 + r, \bar{I} = (i_1, ..., i_{\nu-1}, j_1, ..., j_r, i_{\nu+1}, ..., i_k), u = k, c = \nu - 1)$$

and

$$(k = u, I = (\bar{i}_1, ..., \bar{i}_c, \bar{i}_{c+v}, ..., \bar{i}_{\bar{k}}), \nu = c + 1, r = v, J = (\bar{i}_{c+1}, ..., \bar{i}_{c+v-1})).$$

The second term is just the remaining term above. So the statement is proved. $\square$

Chapter 2

Deformation of singularities via L_∞-algebras

Introduction

In this chapter, we apply the following general idea for the construction of moduli spaces to isolated singularities: Take the differential graded Lie algebra L describing a deformation problem (for isolated singularities, this is the tangent complex) and find a minimal representative M of L in the class of L_∞-algebras (see Chapter 1). In geometric terms, M is a formal DG-manifold, containing the moduli space as analytic substructure.

We show the existence of a functor F from the category of analytic space germs to the localization of the category of L_∞-algebras by L_∞-equivalence. For a singularity X, we take the semi-universal L_∞-deformation (V, Q^V) of $F(X)$ constructed in Chapter 1. If X is an isolated singularities, then the components V^i are of finite dimension. The restriction of the vectorfield Q^V defines a formal map (Kuranishi-map) $V^0 \longrightarrow V^1$ whose zero locus gives the formal moduli space.

2.1 Definitions and reminders

In the whole chapter, we work over a ground field $\mathbb{K}$ of characteristic zero.

Denote the category of formal (resp. convergent) complex analytic space germs by $\mathfrak{Anf}$ (resp. $\mathfrak{An}$). Denote the category of isomorphism classes of formal DG manifolds by `DG-Manf`. We use the same superscripts to denote full subcategories of `DG-Manf` as in Section 1.1.4.

We call a morphism $f = (f_n)_{n\geq 1}$ in $\texttt{DG-Manf}^L$ **weak equivalence**, if the morphism f_1 of DG vectorspaces is a quasi-isomorphism, i.e. if the corresponding morphism of L_∞-algebras is an L_∞-equivalence. Recall that by Theorem 4.4 and Lemma 4.5 of

[22], weak equivalences define an equivalence relation in $\texttt{DG-Manf}^L$ and that in each equivalence class, there is a **minimal model**, i.e. an object belonging to $\texttt{DG-Manf}^M$.

2.2 The functors F and $\mathcal{V}$

In this section we explain how to represent (formal) singularities by formal DG manifolds.

Let $\mathcal{C}$ be the category of formal analytic algebras, $A \in \mathrm{Ob}(\mathcal{C})$ and $R = (R, d^R)$ a **resolvent** of A over $\mathbb{K}$, i.e. a g-finite free DG-algebra in $\mathrm{gr}(\mathcal{C})$ such that $H^0(R, d^R) \cong A$ and $H^j(R, d^R) = 0$, for $j < 0$. For $l \geq 0$, let I_l be an index set containing one index for each free algebra generator of R of degree $-l$. Consider the disjoint union I of all I_l as graded set such that $g(i) = l$, for $i \in I_l$. Fix an ordering on I, subject to the condition $i < j$, if $g(i) < g(j)$.

Thus, as graded algebra, $R = k[[X^0]][X^-]$, where $X^0 = \{x_i |\, i \in I,\, g(i) = 0\}$ and $X^- = \{x_i |\, i \in I, g(i) \geq 1\}$ are sets of free algebra generators with $g(x_i) = -g(i)$.

Set $M := \coprod_{i \in I} k e_i$ to be the free, graded $\mathbb{K}$-vectorspace with base $\{e_i : i \in I\}$, where $g(e_i) = g(i)$. Consider $S(M) = \coprod_{n \geq 0} M^{\odot n}$ as graded coalgebra as in Section 1.1. Set

$$S(M)^* := \mathrm{Hom}_{\mathbb{K}-\mathfrak{Mod}}(S(M), k) = \prod_{j \geq 0} \mathrm{Hom}_{\mathbb{K}-Mod}(M^{\odot j}, k).$$

We identify products $x_{i_1} \cdot \ldots \cdot x_{i_l}$ in R with the maps $M^{\odot l} \longrightarrow k$, defined by $e_{i_1} \cdot \ldots \cdot e_{i_l} \mapsto 1$ and $e_{j_1} \cdot \ldots \cdot e_{j_l} \mapsto 0$, for $\{j_1, \ldots, j_l\} \neq \{i_1, \ldots, i_l\}$. Especially, we identify each constant $\lambda \in k$ with the map $k \longrightarrow k$, sending 1 to λ. We have

$$R^j = \prod_{n \geq 0} \mathrm{Hom}^j(M^{\odot n}, k)$$

and $R = \coprod_{j \leq 0} R^j$. The differential d^R of R extends naturally to $\bar{R} := \prod_{j \leq 0} R^j$. As complexes, R and $\bar{R}$ are identical, but not as graded modules. We identify $\bar{R} = S(M)^*$. Set

$$\mathrm{Der}(R) := \coprod_{i \in \mathbb{Z}} \mathrm{Der}^i(R, R) \quad \text{and} \quad \mathrm{Coder}(S(M)) := \coprod_{i \in \mathbb{Z}} \mathrm{Coder}^i(S(M), S(M)).$$

Denote $\mathrm{Diff}(R)$ (resp. $\mathrm{Codiff}(S(M))$) the submodule of differentials (resp. codifferentials). The following proposition explains why, for a formal DG manifold W, the complex $\mathrm{Coder}(S(W), S(W))$ is called tangent complex of W.

Proposition 2.2.1. *Take R and M as above. The natural map*

$$\mathrm{Coder}(S(M)) \longrightarrow \mathrm{Der}(\bar{R}),$$
$$Q \mapsto s^Q$$

where $s^Q(g) = g \circ Q$, is bijective and the restriction gives rise to an isomorphism

$$\mathrm{Codiff}(S(M)) \longrightarrow \mathrm{Diff}(\bar{R}).$$

Proof. The injectivity is clear. Surjectivity: We have to find a coderivation Q of degree j on $S(M)$ such that, for $u \in S(M)^*$, we have $s(u) = u \circ Q$.

For each $i \in I$, set $f_i := s(x_i)$. Then, f_i is a product $((f_i)_n)_{n\geq 1}$ with $(f_i)_n \in \mathrm{Hom}^{-g(i)+1}(M^{\odot n}, k)$. We define the coderivation Q by

$$Q_n(m_1, \ldots, m_n) := \sum_{i\in I} (f_i)_n(m_1, \ldots, m_n) \cdot e_i,$$

for homogeneous $m_1, \ldots, m_n \in M$. In fact, the non-vanishing terms in the sum satisfy the condition $g(m_1) + \ldots + g(m_n) = g(i)$, hence the sum is finite. To show that for $u \in S(M)^*$, we have $s(u) = u \circ Q$, it is enough to show that for all $i \in I$, $s(x_i) = x_i \circ Q$. But by definition, for $m_1, \ldots, m_n \in M$, we have

$$(x_i \circ Q)_n(m_1, \ldots, m_n) = (f_i)_n(m_1, \ldots, m_n) = (s(x_i))(m_1, \ldots, m_n).$$

The second statement is a direct consequence of the first. $\square$

As consequence, the differential d^R on R induces a codifferential Q^M on $S(M)$. We consider the pair (M, Q^M) as formal DG manifold in $\texttt{DG-Manf}^{LG}$. It has the following property: The restriction of Q^M to M^0 defines a formal map $M^0 \longrightarrow M^1$. Its zero locus is isomorphic to X.

Summarizing the above construction, to each formal space germ X with associated formal analytic algebra A, we can construct a formal DG manifold (M, Q^M), containing X as "subspace". Of course, (M, Q^M) depends on the choice of the resolvent (R, d^R). But we will show that (M, Q^M) is well defined up to weak equivalence, i.e. there exists a functor

$$F : \mathfrak{Anf} \longrightarrow \texttt{DG-Manf}^{LG}/\approx$$

into the localisation of the category of local, g-finite formal DG manifolds by weak equivalences. Remark that such a functor F can't be defined explicitely. We make use of the existence of a functional class, mapping each formal analytic algebra to a DG resolvent, which follows by set-theory.

Lemma 2.2.2. *If $W = (W, d)$ is a DG $\mathbb{K}$-vectorspace and if the dual complex $\mathrm{Hom}(W, k)$ is acyclic, then W is acyclic. Consequently, if $f : V \longrightarrow W$ is a morphism of DG $\mathbb{K}$-vectorspaces such that the dual complex $f^* : W^* \longrightarrow V^*$ is a quasi-isomorphism, then f is a quasi-isomorphism.*

Proof. Assume that M is cyclic, i.e. there is an n and an element $a \in M^n$ such that $d^n(a) = 0$ and $a \notin \operatorname{Im} d^{n-1}$. Let B' be a base of $\operatorname{im} d^{n-1}$. We extend $B' \cup \{a\}$ to a base B of M^n. Let $p : M^n \longrightarrow k$ be the projection on the coordinate a of B. Then, $d^*(p) = p \circ d^{n-1} = 0$ and $p(a) = 1$, hence $p \notin \operatorname{Im} d^*$. Contradiction ! $\square$

Lemma 2.2.3. *Let $f : M \longrightarrow M'$ be a morphism in* `DG-Manf`^{L} *such that the corresponding map $S(M) \longrightarrow S(M')$ is a quasi-isomorphism of complexes. Then, f is a weak equivalence.*

Proof. By the Decomposition Theorem for L_∞-algebras (see Lemma 4.5 of [22]) and Lemma 1.5.4, we may assume that M is minimal and that f is strict. In this case, the homomorphism $f : S(M) \longrightarrow S(M')$ of DG coalgebras is a direct sum of maps of complexes $f_1 : M \longrightarrow M'$ and

$$\sum_{j\geq 2} f_1^{\odot j} : \coprod_{j\geq 2} M^{\odot j} \longrightarrow \coprod_{j\geq 2} M'^{\odot j}.$$

Since the sum is a quasi-isomorphism, both factors are quasi-isomorphisms. $\square$

Corollary 2.2.4. *Let $g : (M, Q^M) \longrightarrow (M', Q^{M'})$ be a morphism of formal DG manifolds in* `DG-Manf`^{LG} *and suppose that the dual map If $S(M')^* \longrightarrow S(M)^*$ is a quasi-isomorphism of free DG coalgebras, then g is a weak equivalence.*

Proof. This follows by Lemma 2.2.2 and 2.2.3. $\square$

Thus, we have proved the functoriality of F. Next, we define a functor

$$\mathcal{V} : \texttt{DG-Manf}^{LG} \longrightarrow \mathfrak{Anf}$$

as already mentioned above: For a DG manifold (M, Q^M) in `DG-Manf`^{LG}, set $\mathcal{V}(M, Q^M)$ to be the zero locus of the formal map $M^0 \longrightarrow M^1$, induced by Q^M.

More precisely, if we define "homogeneous polynomials" $f_n : M^0 \longrightarrow M^1$ by $f_n(x) := Q_n^M(x, \ldots, x)$, then X is the zero locus of the analytic map $f := \sum_{n\geq 1} \frac{1}{n!} f_n$.

The category $\mathfrak{Anf}/\cong$ of isomorphism classes of formal analytic space germs, is the category of formal space germs without fixed coordinates. In fact, $\mathcal{V}$ defines a functor

$$\mathcal{V} : \texttt{DG-Manf}^{LG}/\approx \longrightarrow \mathfrak{Anf}/\cong .$$

Since the functor F factors through $\mathfrak{Anf}/\cong$ and $\mathcal{V} \circ F$ is naturally isomorphic to the identity functor on $\mathfrak{Anf}/\cong$, we see that $\mathfrak{Anf}/\cong$ is equivalent to a full subcategory of `DG-Manf`$^{LG}/\approx$.

2.3 Deformations and embedded deformations

In this section we recall the classical result that each deformation of a singularity is equivalent to an embedded deformation.

Consider a complex space germ X with corresponding analytic algebra $\mathcal{O}_X$. Suppose that X is embedded in the smooth space germ P with corresponding analytic algebra R^0. Let $R = (R, d^R)$ be a g-finite, free algebra resolution of $\mathcal{O}_X$ such that $R^0 = \mathcal{O}_P$.

For any space germ $(S, \mathcal{O}_S)$, set $R_S := R\hat{\otimes}_{\mathbb{C}}\mathcal{O}_S$ and

$$\mathcal{C}(S) := \{\delta \in \mathrm{Der}^1(R_S, R_S)|\ \delta(0) = 0 \text{ and } (d^R + \delta)^2 = 0\}$$

Furthermore, let $\mathcal{D}(S)$ be the equivalence class of deformations of X with base S, i.e. the equivalence class of all flat morphisms $\mathcal{X} \longrightarrow S$ such that there is a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ * \arrow[r] & S \end{tikzcd} \tag{3.1}$$

where $*$ denotes the single point. Then, $\mathcal{C}$ and $\mathcal{D}$ are fibered grouppoids over $\mathfrak{An}$ and we define a morphism $G : \mathcal{C} \longrightarrow \mathcal{D}$ as follows: For $\delta \in \mathcal{C}(S)$, let $\mathcal{X}$ be the space germ with $\mathcal{O}_{\mathcal{X}} = H^0(R_S, d^R + \delta)$ and $\mathcal{X} \longrightarrow S$ the composition of the closed embedding $\mathcal{X} \longrightarrow S \times P$ and the canonical projection $S \times P \longrightarrow S$. Obviously, there is a cartesian diagram (3.1). I. e. $G(\delta) := \mathcal{X} \longrightarrow S$ is a deformation of X. We want to remind the proof of the well-known fact that G is surjective.

Let $(A, \mathfrak{m})$ be a local analytic algebra, B a graded, g-finite free A-algebra and C a flat DG-algebra over A. For A-modules M, we set $M' := M\hat{\otimes}_A A/\mathfrak{m}$. The following statement is a special case of Proposition 8.20 in Chapter I of [2]:

Proposition 2.3.1. *Let $v' \in \mathrm{Der}^1_{B'_0}(B', B')$ be a differential and $\phi' : B' \longrightarrow C'$ a surjective quasi-isomorphism of DG-algebras over A'. Then, there is a differential $v \in \mathrm{Der}^1_{B_0}(B, B)$, lifting v' and a surjective quasi-isomorphism $\phi : B \longrightarrow C$ of DG-algebras over A, lifting ϕ'.*

Corollary 2.3.2. *For all S in $\mathfrak{An}$, $G(S) : \mathcal{C}(S) \longrightarrow \mathcal{D}(S)$ is surjective.*

Proof. For $\mathcal{X} \longrightarrow S$ in $\mathcal{D}(S)$, we have to find a $\mathcal{O}_S$-derivation $\delta : R_S \longrightarrow R_S$ of degree 1 with $\delta(0) = 0$ such that $d^R + \delta$ is a differential and a surjective quasi-isomorphism $(R_S, d^R + \delta) \longrightarrow \mathcal{O}_{\mathcal{X}}$. Since $R_S\hat{\otimes}_{\mathcal{O}_S}\mathbb{C} = R$ and $\mathcal{O}_{\mathcal{X}}\hat{\otimes}_{\mathcal{O}_S}\mathbb{C} = \mathcal{O}_X$, the existence follows by Proposition 2.3.1, if we set $A := \mathcal{O}_S, B := R_S$ and $C := \mathcal{O}_{\mathcal{X}}$. $\square$

In the literature (see [2], for instance), the deformation functor is defined such that a space germ S maps to the quotient of $\mathcal{C}(S)$ by the Lie group, associated to the Lie algebra $\mathrm{Der}^0(R_S, R_S)$. In fact, G factors through this quotient and the first factor is even “minimal smooth”. For the construction here, we don't need to consider this group action to get semi-universal deformations. One can say that the group action is replaced by the going - over to a minimal model.

2.4 A semi-universal formal deformation

In this section, we apply the new method for the construction of a semi-universal formal deformation to isolated singularities X. We need a technical lemma. The symbol $\otimes$ always denotes the (formal) analytic tensor product over the ground field $\mathbb{K}$.

Lemma 2.4.1. *Let (R, d^R) and (S, d^S) be g-finite resolvents of the (formal) analytic algebras $\mathcal{O}_X$ and $\mathcal{O}_Y$, respectively. Set $R_Y := R \otimes \mathcal{O}_Y$. Suppose that $\delta \in \mathrm{Der}^1_{\mathcal{O}_Y}(R_Y, R_Y)$ is a derivation such that $d^R + \delta$ is a differential on R_Y. Then, on $R \otimes S$, there exists a derivation $\gamma \in \mathrm{Der}^1_S(R \otimes S, R \otimes S)$ such that the following two conditions hold:*

(a) The projection $1 \otimes \pi : R \otimes S \longrightarrow R \otimes \mathcal{O}_Y$ commutes with γ and δ.

(b) The derivation $d^R + d^S + \gamma$ on $R \otimes S$ is a differential.

Proof. Since $\mathrm{Der}_S(R \otimes S, R \otimes S) \cong \mathrm{Der}_{\mathbb{K}}(R, R \otimes S)$, for the definition of γ, we only need to choose its values on the free generators of the graded algebra R. We proceed inductively. For free algebra generators $x \in R$ of degree zero, set $\gamma_0(x) := 0$. For free algebra generators $x \in R$ of degree -1, we define $\gamma_{-1}(x) \in R^0 \otimes S^0$ in such a way that $(1 \otimes \pi)(\gamma_{-1}(x)) = \delta(x)$. For free algebra generators $x \in R$ of degree -2, we define $\gamma'(x) \in R^{-1} \otimes S^0$ in such a way that $(1 \otimes \pi)(\gamma'(x)) = \delta(x)$. Then we have

$$(d^R + \gamma)(d^R + \gamma')(x) \in \mathrm{Kern}(1 \otimes \pi) = \mathrm{Im}(1 \otimes d^S).$$

Thus we can choose $\gamma''(x) \in R^0 \otimes S^{-1}$ such that

$$d^S(\gamma''(x)) + (d^R + \gamma_{-1})(d^R + \gamma'_{-2})(x) = 0.$$

We set $\gamma_{-2}(x) := \gamma'(x) + \gamma''(x)$. We can varify directly that $(d^R + d^S + \gamma)^2(x) = 0$.

We have the following scheme:

$$
\begin{tikzcd}
{} \arrow[r] \arrow[dr] & R^0 \otimes S^{-3} \arrow[dr] & & & \\
{} \arrow[r] \arrow[dr] \arrow[u, dotted] & R^{-1} \otimes S^{-2} \arrow[r] \arrow[dr] \arrow[u, dotted] & R^0 \otimes S^{-2} \arrow[dr] & & \\
{} \arrow[r] \arrow[dr] \arrow[u, dotted] \arrow[uu, dotted] & R^{-2} \otimes S^{-1} \arrow[r] \arrow[dr] \arrow[u, dotted] \arrow[uu, dotted] & R^{-1} \otimes S^{-1} \arrow[r] \arrow[dr] \arrow[u, dotted] & R^0 \otimes S^{-1} \arrow[dr] & \\
{} \arrow[r] \arrow[u, dotted] \arrow[uu, dotted] \arrow[uuu, dotted] & R^{-3} \otimes S^0 \arrow[r] \arrow[u, dotted] \arrow[uu, dotted] & R^{-2} \otimes S^0 \arrow[r] \arrow[u, dotted] \arrow[uu, dotted] & R^{-1} \otimes S^0 \arrow[r] \arrow[u, dotted] & R^0 \otimes S^0
\end{tikzcd}
$$

The horizontal arrows denote the derivation $d^R + \gamma'$. The dotted arrows denote (in function of their slope) the derivations $\gamma'', \gamma''', \ldots$ and the down-right arrows denote the differential d^S.

If $\gamma_1, \ldots, \gamma_k$ are already constructed, then on free algebra generators $x \in R$ of degree $k-1$, choose $\gamma'(x) \in R^k \otimes S^0$ in such a way that $(1 \otimes \pi)(\gamma'(x)) = \delta(x)$. As above, we can choose $\gamma''(x) \in R^{k+1} \otimes S^{-1}$ such that

$$d^S(\gamma''(x)) + (d^R + \gamma')(\gamma'(x)) = 0.$$

We can choose $\gamma'''(x) \in R^{k+2} \otimes S^{-2}$ such that

$$d^S(\gamma'''(x)) + (d^R + \gamma')(\gamma''(x)) + \gamma''(\gamma'(x)) = 0.$$

Going on like this, we can finally choose $\gamma^{(-k+1)}(x) \in R^0 \otimes S^k$ such that

$$d^S(\gamma^{(-k+1)}(x)) + (d^R + \gamma')(\gamma^{(-k)}(x)) + \gamma''(\gamma^{(-k-1)}(x)) + \ldots + \gamma^{(-k+1)}(\gamma'(x)) = 0.$$

Then, we set $\gamma_{k-1}(x) := \gamma'(x) + \ldots + \gamma^{(-k+1)}(x)$. We get immediately by construction that $(d^R + d^S + \gamma)^2(x) = 0$. $\square$

Now, suppose that the space germ X is an isolated singularity. (If X is only a formal space germ, then suppose that the tangent cohomology of X is g-finite, instead.) Let (M, Q^M) be a formal DG-manifold in $\texttt{DG-Manf}^{LG}$ that represents $F(X)$. As in Section 2.2, denote the resolvent of the analytic algebra $\mathcal{O}_X$, having $S(M)^*$ as completion, by (R, d^R).

Theorem 1.5.13 describes a semiuniversal deformation (V, Q^V, Q) of (M, Q^M). Recall that as graded module, we have $V = H[1]$, where H denotes the cohomology of the tangent complex $\mathrm{Coder}(S(M), S(M)) \cong \mathrm{Der}(R, R)$, i.e. the tangent cohomology of X. It is well-known that for isolated singularities, the tangent cohomology H is g-finite.

We apply the functor $\mathcal{V}$ to the morphism $(V \times M, Q^V + Q^M + Q) \longrightarrow (V, Q^V)$ and get a morphism $\mathcal{Z} \longrightarrow Z$ in $\mathfrak{Anf}$.

Theorem 2.4.2. *The morphism $\mathcal{Z} \longrightarrow Z$ is a semi-universal formal deformation of the space germ X.*

Proof. Let

$$\begin{tikzcd} \mathcal{Y} \arrow[d] & X \arrow[l] \arrow[d] \\ Y & * \arrow[l] \end{tikzcd}$$

be any formal deformation of X. By Corollary 2.3.2, we may assume that this deformation is embedded, i.e. that $\mathcal{Y}$ is a subgerm of $M^0 \times Y$ and $\mathcal{O}_{\mathcal{Y}} = H^0(R_Y, d^R + \delta)$, for a certain $\delta \in \mathcal{C}(Y)$ (see Section 2.3) and that the morphism $\mathcal{Y} \longrightarrow Y$ is the composition of the inclusion $\mathcal{Y} \longrightarrow M^0 \times Y$ and the projection $M^0 \times Y \longrightarrow Y$.
Let (N, Q^N) be the formal DG manifold $F(Y)$. Let (S, d^S) be a DG resolvent of $\mathcal{O}_Y$, dual to (N, Q^N), i.e. the completion of (S, d^S) is $S(N)^*$. By Lemma 2.4.1, there exists a derivation $\gamma \in \mathrm{Der}^1(R \otimes S, R \otimes S)$ such that $d^R + d^S + \gamma$ is a differential on $R \otimes S$ and such that $H^0(R \otimes S, d^R + d^S + \gamma) = \mathcal{O}_{\mathcal{Y}}$.

Using Proposition 2.2.1, we can translate the situation from the language of DG algebras to the geometric language of formal DG manifolds. Like d^R is dual to Q^M and d^S is dual to Q^N, the derivation γ is dual to a certain super-vectorfield Q_γ on $N \times M$. Furthermore, the triple (N, Q^N, Q_γ) is a deformation of the formal DG manifold (M, Q^M). Since (V, Q^V) is semi-universal, we get a morphism

$$\begin{tikzcd} (N \times M, Q^N + Q^M + Q_\gamma) \arrow[r] \arrow[d] & (V \times M, Q^V + Q^M + Q) \arrow[d] \\ (N, Q^N) \arrow[r] & (V, Q^V) \end{tikzcd}$$

of deformations. Application of the functor $\mathcal{V}$ gives a cartesian diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

which obviously respects the distinguished fiber $X \longrightarrow *$. Hence, the diagram is a morphism of deformations. This shows that $\mathcal{Z} \longrightarrow Z$ is versal. Since Z is a formal analytic subgerm of $V^0 = H^1$, we have $\dim(TZ) \leq \dim H^1$. Thus, necessarily $\mathcal{Z} \longrightarrow Z$ is semi-universal (see Chapter 2.6 of [37]). $\square$

Chapter 3

Hochschild cohomology for complex spaces and Noetherian schemes

Introduction

One motivation for considering Hochschild cohomology in geometry is its possible application in noncommutative deformation theory. Even if the geometrical objects (complex spaces, schemes) considered in this chapter belong to classical commutative geometry, it is interesting to find out how they can be deformed into noncommutative spaces, and in which way classical deformation theory is contained in noncommutative deformation theory[1]. Classical deformations of schemes and complex spaces are governed by tangent cohomology (see for example [37]), and since deformations of (noncommutative) associative algebras are governed by Hochschild cohomology (see [11]), we may assume that Hochschild cohomology of schemes and complex spaces governs their noncommutative deformation theories. This leads to the question of how their Hochschild and tangent cohomologies are related, which is the subject of this chapter.

Hochschild complexes and Hochschild homology for schemes were first defined by Weibel/ Geller [47]. They defined the Hochschild complex as sheafification of the cyclic bar complex. An alternative definition is due to Yekutieli [49]. Hochschild cohomology for schemes was defined by Gerstenhaber/ Schack [11]. Kontsevich [22] suggested to define Hochschild cohomology of a complex manifold X as $\mathrm{Ext}_{X^2}(\mathcal{O}_X, \mathcal{O}_X)$. Swan [44] showed that for schemes, the Gerstenhaber/ Schack- definition is equivalent to Kontsevich's definition. The definition of Hochschild complexes and Hochschild (co)homology for complex spaces is due to Buchweitz/ Flenner [7]. Weibel and Geller's definition of Hochschild complexes is not recommendable for complex spaces. This is

[1]So far, deformation theory for noncommutative schemes has not been developed and noncommutative complex spaces with singularities are not jet defined.

due to the fact that the (cyclic) bar complex of an *analytic* algebra A (in which tensor products are replaced by analytic tensor products) is not a complex of projective A-modules. Thus, if we defined the Hochschild cohomology as cohomology of the dual of the cyclic bar complex of A, we would not arrive at the desired Ext-interpretation. Instead, we must define the Hochschild complex piecewise (on Stein compacts) via free algebra resolutions, and globally via a Čech construction (see [7] or Section 3.3).

In order to avoid having to prove each statement for each situation, we unify the algebraic and analytic theories. To do so, we follow an approach due to Bingener/ Kosarew [2] who extracted the common features of both situations and listed them as axioms for **"admissible pairs of categories"**. These are pairs $(\mathcal{C}, \mathcal{M})$ where $\mathcal{C}$ is a suitable category of algebras and $\mathcal{M}$ a category of modules over $\mathcal{C}$. Admissible pairs of categories enable the description of "affine" spaces. For example $\mathcal{C}$ may be the category of sections on Stein compacts or the category of sections on affine schemes (i.e. the category of algebras). In the this chapter we will mainly talk about admissible pairs of categories. These are interesting by themselves, as they are useful in many more situations (see Examples 3.1.1) than affine schemes and Stein compacts. We will show how to apply results for admissible pairs to schemes and complex spaces only in Section 3.3.

The **main results** for admissible pairs $(\mathcal{C}, \mathcal{M})$ of categories are the following: (1) In Section 3.1.8 we characterize regular sequences in graded commutative algebras in $\mathcal{C}$. (2) We define the Hochschild complex for simplicial algebras in $\mathcal{C}$ (i.e. functors from a small category $\mathcal{N}$ to $\mathcal{C}$) and show that the Hochschild complex is homotopy invariant (Proposition 3.2.2). (3) We prove a HKR theorem for free commutative graded DG algebras in $\mathcal{C}$, if the ground ring k contains $\mathbb{Q}$. Loday's textbook [29] contains a sketch of proof of the special case of this theorem where $\mathcal{C}$ is just the category of k-algebras. A corollary of this HKR theorem is the following Quillen-type theorem for a k-algebra in $\mathcal{C}$: There is a quasi-isomorphism

$$\mathbb{H}(a/k) \approx \wedge\mathbb{L}(a/k)$$

from the Hochschild complex to the exterior algebra of the tangent complex. (Quillen [41] stated this result in the case where $\mathcal{C}$ is the category of k-algebras.) We generalize this theorem for objects in $\mathcal{C}^{\mathcal{N}}$, and, in Section 3.3, we deduce the main result of this chapter by means of a Čech construction. (4) If $X \longrightarrow Y$ is a morphism of complex spaces (paracompact and separated) or a separated morphism of finite type of Noetherian schemes in characteristic zero, then there exists a quasi-isomorphism

$$\mathbb{H}(X/Y) \approx \wedge\mathbb{L}(X/Y) \tag{0.1}$$

over $\mathcal{O}_X$, where $\mathbb{H}(X/Y)$ is the relative Hochschild complex of X over Y (see Section 3.3) and $\mathbb{L}(X/Y)$ is the relative cotangent complex. From the main result we deduce several decomposition theorems for Hochschild (co)homology. (5) Hochschild cohomology contains tangent cohomology:

$$\mathrm{HH}^n(X/Y, \mathcal{M}) \cong \coprod_{i+j=n} \mathrm{Ext}^i(\wedge^j\mathbb{L}(X/Y), \mathcal{M}). \tag{0.2}$$

The left side is the n-th Hochschild cohomology of X over Y with values in $\mathcal{M}$. The right side contains the $(i-1)$-th relative tangent cohomology $\mathrm{Ext}^{i-1}(\mathbb{L}(X/Y), \mathcal{M})$ as a direct factor. For complex spaces, this decomposition as well as equation (0.1) has already been proved in a completely different way by Buchweitz/Flenner [7]. (6) The second corollary is a decomposition theorem for the Hochschild cohomology of complex analytic manifolds and smooth schemes in characteristic zero:

$$\mathrm{HH}^n(X) \cong \coprod_{i+j=n} H^i(X, \wedge^j \mathcal{T}_X). \tag{0.3}$$

On the right, we have the sheaf cohomology of the exterior powers of the tangent complex. A proof of this result for complex analytic manifolds has been announced (but not yet published) by Kontsevich. For smooth schemes, decomposition (0.3) was proved in a different way by Yekutieli [49]. A similar statement for quasi-projective smooth schemes is due to Gerstenhaber/Schack [11] and Swan [44]. (7) If X is a smooth scheme in characteristic zero, or a manifold, then we can deduce the "Hodge decomposition" of the Hochschild homology:

$$\mathrm{HH}_n(X) \cong \coprod_{i-j=n} H^j(X, \wedge^i \Omega_X) \tag{0.4}$$

Remark that the difference with the Hodge composition of the De Rham cohomology is that we sum over the **columns** of the "Hodge diamond" instead of over the **lines**. For schemes, this result was shown in a different way (using the λ-decomposition of the Hochschild complex) by Weibel [48].

Conventions: For a ring k, we denote the category of k-modules by k-$\mathfrak{Mod}$. A ring A together with a ring homomorphism $k \longrightarrow A$ is called an algebra over k or a k-algebra. For a morphism $f : A \longrightarrow B$ in any category, we denote the kernel of f in the categorical sense (see [46]) by $\mathrm{kern}\, f$, i.e. $\mathrm{kern}\, f$ is a morphism $K \longrightarrow A$, where K is an object, determined up to a canonical isomorphism. By $\mathrm{Kern}\, f$, we mean the object K. We use the notions cokern, Cokern, im and Im in the same way. For example, we have $\mathrm{Im}\, f \cong \mathrm{Kern}(\mathrm{cokern}\, f)$. We write $\approx$ for quasi-isomorphisms and $\simeq$ for homotopy equivalences. We use the letter D to denote derived categories and K to denote homotopy categories, i.e. the localization of categories by homotopy equivalences.

The differential of a DG object is always of degree $+1$. If the degree $g(a)$ of a homogeneous element a of a graded ring or module arises in an exponent, we just write a instead of $g(a)$. By convention, $(-1)^{ab}$ means $(-1)^{g(a)\cdot g(b)}$ and not $(-1)^{g(ab)}$.

3.1 Admissible pairs of categories

In order to describe geometric objects locally by means of algebraic objects, one has to handle pairs of categories $(\mathcal{C}, \mathcal{M})$, where $\mathcal{C}$ is a category of algebras and $\mathcal{M}$ a category of modules over algebras in $\mathcal{C}$. If the algebraic calculus should include the local description of commutative schemes, complex spaces and even infinite dimensional spaces, like Banach analytic spaces, then the frame of admissible pairs of categories is a good choice. Before listing the axioms defining an admissible pair of categories (figured out by Bingener and Kosarew in [2]), we give several examples:

Examples 3.1.1. The following pairs $(\mathcal{C}, \mathcal{M})$ are admissible pairs of categories:

(1) Let $\mathcal{C}^{(0)}$ be the category of all commutative $\mathbb{K}$-algebras and $\mathcal{M}^{(0)}$ the category of modules over algebras in $\mathcal{C}^{(0)}$.

(2) Recall that a Stein compact (see [14]) is a compact subset X of a complex space, admitting a base of open neighborhoods $\mathfrak{U}$ such that each $U \in \mathfrak{U}$ is a Stein space. Let $\mathcal{C}^{(1)}$ be the category of all analytic $\mathbb{C}$-algebras, i.e. the category of all sections of the structure sheaf of a Stein compact. Then each algebra in $\mathcal{C}^{(1)}$ is a DFN-algebra, i.e. a topological algebra with respect to the dual Frechet nuclear topology (see [39], for instance) and each homomorphism of such algebras is continuous. Let $\mathcal{M}^{(1)}$ be the category of all DFN-modules over algebras in $\mathcal{C}^{(1)}$.

(3) In the first example, we can replace $\mathcal{C}^{(0)}$ by the category of all Noetherian, commutative $\mathbb{K}$-algebras.

(4) In the second example, we can replace $\mathcal{C}^{(1)}$ by the category of local analytic algebras.

(5) For $\epsilon \in (0,1]$, let $\mathcal{C}^{(\epsilon)}$ be the category of commutative complete PO-algebras in the sense of [37] and $\mathcal{M}^{(\epsilon)}$ the category of all complete PO-modules and PO_ϵ-homomorphisms.

The reader, only interested in schemes or algebraic varieties, doesn't have to care about the following definition and may always take $(\mathcal{C}, \mathcal{M})$ as in Example (1) instead.

We fix a commutative ground ring $\mathbb{K}$ with unit (in our main reference [2], $\mathbb{K}$ is the field $\mathbb{Q}$, so here we start with a more general setting). Denote by $\mathcal{C}$ a category of commutative $\mathbb{K}$-algebras and by $\mathcal{C}$-$\mathfrak{Mod}$ the category of all modules over algebras in $\mathcal{C}$. For objects A, B in $\mathcal{C}$ and M in A-$\mathfrak{Mod}$, N in B-$\mathfrak{Mod}$, a homomorphism $M \longrightarrow N$ in $\mathcal{C}$-$\mathfrak{Mod}$ is a pair (ϕ, f), where $\phi : A \longrightarrow B$ is a homomorphism in $\mathcal{C}$ and $f : M \longrightarrow N_{[\phi]}$ is a homomorphism in A-$\mathfrak{Mod}$. Let $\mathcal{M}$ be a subcategory of $\mathcal{C}$-$\mathfrak{Mod}$. For algebras A in $\mathcal{C}$, we denote the intersection of $\mathcal{M}$ and A-$\mathfrak{Mod}$ by $\mathcal{M}(A)$ and the subcategory of $\mathcal{C}$ of all algebras with a morphism $A \longrightarrow B$ by $\mathcal{C}_A$. The pair $(\mathcal{C}, \mathcal{M})$ is called an **admissible pair of categories** if the following conditions hold:

(1) In $\mathcal{C}$ there exist finite fibered sums that we denote as usual by $A \otimes_K^{\mathcal{C}} B$.

(2) If $\phi : A \longrightarrow B$ is a homomorphism in $\mathcal{C}$ and N a module in $\mathcal{M}(B)$, then N is via ϕ an object of $\mathcal{M}(A)$, and for each module M in $\mathcal{M}(A)$, $\mathrm{Hom}_{\mathcal{M}(A)}(M, N_{[\phi]})$

is the set of all homomorphisms $f : M \longrightarrow N$ in $\mathcal{M}$ such that (ϕ, f) is a homomorphism in $\mathcal{C}$-$\mathfrak{Mod}$.

(3) Let A be an algebra in $\mathcal{C}$. Then $\mathcal{M}(A)$ is an additive category, in which kernels and cokernels exist. Further, $\mathcal{C}_A$ is a subcategory of $\mathcal{M}(A)$ and the functor of $\mathcal{M}(A)$ in A-$\mathfrak{Mod}$ commutes with kernels and finite direct sums.

(4) Let $\phi : A \longrightarrow B$ a homomorphism in $\mathcal{C}$ and $u : M \longrightarrow N$ a homomorphism in $\mathcal{M}(B)$. Let L (resp. L') be the kernel of u (resp. $u_{[\phi]}$) in $\mathcal{M}(B)$ resp. $\mathcal{M}(A)$. Then the canonical map $L' \longrightarrow L_{[\phi]}$ is an isomorphism in $\mathcal{M}(A)$.

(5) Let A be an algebra in $\mathcal{C}$ and N a module in $\mathcal{M}(A)$. For each finite family M_i; $i \in I$ of modules in $\mathcal{M}(A)$, there is a given $\mathbb{K}$-submodule

$$\mathrm{Mult}_{\mathcal{M}(A)}(M_i, i \in I; N)$$

of the module $\mathrm{Mult}_A(M_i, i \in I; N)$ of A-multilinear forms $\prod_{i \in I} M_i \longrightarrow N$, which is functorial in each M_i and N and has the following properties:

(5.1) Let i_0 be an element of I and $u : M'_{i_0} \longrightarrow M_{i_0}$ a homomorphism in $\mathcal{M}(A)$. Set $M''_{i_0} := \mathrm{Cokern}(u)$ and $M'_i := M''_i := M_i$, for $i \in I \setminus \{i_0\}$. The sequence

$$\begin{aligned} 0 \to \mathrm{Mult}_{\mathcal{M}(A)}(M''_i, i \in I; N) \to \\ \to \mathrm{Mult}_{\mathcal{M}(A)}(M_i, i \in I; N) \to \mathrm{Mult}_{\mathcal{M}(A)}(M'_i, i \in I; N) \end{aligned}$$

induced by u is exact.

(5.2) For modules $M, N \in \mathcal{M}(A)$, there is a canonical isomorphism

$$\mathrm{Mult}_{\mathcal{M}(A)}(M; N) \longrightarrow \mathrm{Hom}_{\mathcal{M}(A)}(M; N).$$

(5.3) For M in $\mathcal{M}(A)$, the multiplication map $\mu_M : A \times M \longrightarrow M$ is in $\mathrm{Mult}_{\mathcal{M}(A)}(A \times M; M)$.

(5.4) If $\sigma : I \longrightarrow J$ is a bijective map, then the restriction of the isomorphism

$$\mathrm{Mult}_A(M_i, i \in I; N) \longrightarrow \mathrm{Mult}_A(M_{\sigma^{-1}(j)}, j \in J; N)$$

defined by σ, defines an isomorphism

$$\mathrm{Mult}_{\mathcal{M}(A)}(M_i, i \in I; N) \longrightarrow \mathrm{Mult}_{\mathcal{M}(A)}(M_{\sigma^{-1}(j)}, j \in J; N).$$

(5.5) Each homomorphism $\phi : A \longrightarrow B$ in $\mathcal{C}$ induces a cartesian diagram

$$\begin{tikzcd} \mathrm{Mult}_{\mathcal{M}(B)}(M_i, i \in I; N) \arrow[r] \arrow[d] & \mathrm{Mult}_{\mathcal{M}(A)}((M_i)_{[\phi]}, i \in I; N_{[\phi]}) \arrow[d] \\ \mathrm{Mult}_B(M_i, i \in I; N) \arrow[r] & \mathrm{Mult}_A((M_i)_{[\phi]}, i \in I; N_{[\phi]}) \end{tikzcd}$$

(5.6) For each $i \in I$, let $L_j,\ j \in J_i$ a nonempty finite family of modules in $\mathcal{M}(A)$. Set $J := \amalg_{i\in I} J_i$. The canonical map

$$(\prod_{i\in I} \mathrm{Mult}_{\mathcal{M}(A)}(L_j, {}_{j\in J_i}; M_i)) \times \mathrm{Mult}_{\mathcal{M}(A)}(M_i, {}_{i\in I}; N) \longrightarrow \mathrm{Mult}_A(L_j, {}_{j\in J}; N)$$

factorises through $\mathrm{Mult}_{\mathcal{M}(A)}(L_j, j \in J; N)$.

(5.7) The functor $N \mapsto \mathrm{Mult}_{\mathcal{M}(A)}(M_i, i \in I; N)$ on $\mathcal{M}(A)$ is represented by a module $\bigotimes_{i\in I}{}_A^{\mathcal{M}} M_i$ in $\mathcal{M}(A)$.

(5.8) If I is a disjoint union $\cup_{j\in J} I_j$ with $I_j \neq \emptyset$ for all j, then the canonical homomorphism

$$\bigotimes_{i\in I}{}_A^{\mathcal{M}} M_i \longrightarrow \bigotimes_{j\in J}{}_A^{\mathcal{M}} (\bigotimes_{i\in I_j}{}_A^{\mathcal{M}} M_i)$$

is an isomorphism in $\mathcal{M}(A)$.

(5.9) The canonical map $A \otimes_A^{\mathcal{M}} M \longrightarrow M$ is an isomorphism in $\mathcal{M}(A)$.

(6) Let $\phi : A \longrightarrow B$ be a homomorphism in $\mathcal{C}$ and M a module in $\mathcal{M}(A)$ and N a module in $\mathcal{M}(B)$. The module $N_{[\phi]} \otimes_A^{\mathcal{M}} M$ is via the canonical A-bilinear map[2]

$$B \times N_{[\phi]} \otimes_A^{\mathcal{M}} M \longrightarrow N_{[\phi]} \otimes_A^{\mathcal{M}} M$$

a module in $\mathcal{M}(B)$. The analogue statement holds for $M \otimes_A^{\mathcal{M}} N_{[\phi]}$.

(7) Let $k \longrightarrow A$ and $k \longrightarrow B$ be two homomorphisms in $\mathcal{C}$ and ϕ (resp. ψ) the canonical maps of A (resp. B) in $C := A \otimes_k^{\mathcal{M}} B$. Let M be a module in $\mathcal{M}(k)$ and $\rho : C \times M \longrightarrow M$ an element of $\mathrm{Mult}_{\mathcal{M}(k)}(C \times M; M)$ such that

(a) The map ρ extends the multiplication of k on M.

(b) The module M is via ρ a C-module.

(c) The module $M_{[\phi]}$ belongs to $\mathcal{M}(A)$ and $M_{[\psi]}$ belongs to $\mathcal{M}(B)$.

Then M is in $\mathcal{M}(C)$.

(8) For algebras A and B in $\mathcal{C}_k$, the canonical map $A \otimes_k^{\mathcal{M}} B \longrightarrow A \otimes_k^{\mathcal{C}} B$ is an isomorphism in $\mathcal{M}(k)$.

Axioms (1) - (8) hold in the algebraic, as well as in the analytic context. The difference between these contexts is manifest in the difference between Axiom (S1) and (S1'):

Axioms. Let A be an algebra in $\mathcal{C}$.

(S1) If $u : M \longrightarrow N$ is a homomorphism of finite A-modules in $\mathcal{M}(A)$, then the cokernel of u in $\mathcal{M}(A)$ coincides with the cokernel of u in A-$\mathfrak{Mod}$ and for $N = A$ the cokernel of u is an algebra in $\mathcal{C}_A$ with respect to the canonical projection $A \longrightarrow \mathrm{Cokern}(u)$.

[2] The existence of this map is a consequence of (2), (5.7) and (5.8).

(S1') For any homomorphism $u : M \longrightarrow N$ of A-modules, the cokernel of u in $\mathcal{M}(A)$ coincides with the cokernel of u in A-$\mathfrak{Mod}$ and for $N = A$ the cokernel of u is an algebra in $\mathcal{C}_A$ with respect to the canonical projection $A \longrightarrow \mathrm{Cokern}(u)$.

(S2) Bijective homomorphisms in $\mathcal{M}(A)$ are isomorphisms.

Examples 3.1.2. Again, consider the Examples 3.1.1.

(1) The pair $(\mathcal{C}^{(0)}, \mathcal{M}^{(0)})$ is an admissible pair of categories that satisfies Axioms (S1') and (S2).

(2) Let $\mathrm{Mult}_{\mathcal{M}^{(1)}}(\ )$ be the group of all continuous multilinear forms. Then $(\mathcal{C}^{(1)}, \mathcal{M}^{(1)})$ is an admissible pair of categories that satisfies Axioms (S1) and (S2).

The following lemma is an easy exercise. The proof can be found in the Appendix.

Lemma 3.1.3. *Let $(\mathcal{C}, \mathcal{M})$ be an admissible pair of categories. Let k be an algebra in $\mathcal{C}$ and A, B, M and N modules in $\mathcal{M}(k)$.*

(1) For elements $a, a' \in A$ and $b, b' \in B$, we have $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

(2) Each homomorphism $h \in \mathrm{Hom}_{\mathcal{M}(k)}(A \otimes_k^{\mathcal{M}} B, N)$ is uniquely determined by its values on the elements of the form $a \otimes b$.

Remark that in the graded commutative context (see below) we will have $(a \otimes b)(a' \otimes b') = (-1)^{ba'} aa' \otimes bb'$, for homogeneous a, b, a', b'.

3.1.1 Free modules and algebras

To motivate the definition of markings on categories of algebras and modules: Let X be a compact n-dimensional polydisk. This is a Stein compact. We want to consider the algebra $A = \mathrm{colim}_{U \supset X} \Gamma(U, \mathcal{O}_{\mathbb{C}^n})$ of global sections on X as a free algebra in $\mathcal{C}^{(1)}$, generated by n algebra generators $x_1, \ldots, x_n$. If B is the algebra of sections on a second polydisk Q, we can't just choose n elements of B as images of $x_1, \ldots, x_n$ to define a homomorphism in $\mathcal{C}^{(1)}$, since the corresponding map $Q \longrightarrow \mathbb{C}^n$ would not, in general, land in X. There are restrictions on the values that each x_i may take. A marking on the category $\mathcal{C}$ is such a restriction. A non-restrictive marking is called canonical. Usually, when the modules in $\mathcal{M}$ are non-graded, the marking on $\mathcal{M}$ will be the canonical one.

Let $(\mathcal{C}, \mathcal{M})$ be an admissible pair of categories.

Free algebras: A **marking** on $\mathcal{C}$ is a family $(F_\tau)_{\tau \in \mathsf{T}}$ of subfunctors $F_\tau : \mathcal{C} \longrightarrow (\text{sets})$ of the identity functor such that $F_\tau(B)$ contains 0, for all τ, and all objects B in $\mathcal{C}$. For a given object k of $\mathcal{C}$ and a family $(\tau_i)_{i \in I}$, we consider the functor $F_{I,k} : A \mapsto \prod_{i \in I} F_{\tau_i}(A)$ on the category $\mathcal{C}_k$. If $F_{I,k}$ is representable, i.e. there is a k-algebra A and a canonical bijection

$$b : \mathrm{Hom}_k^{\mathcal{C}}(A, B) \longrightarrow \prod_{i \in I} F_{\tau_i}(B)$$

for each algebra B in $\mathcal{C}_k$, then A together with the family $(e_i)_{i\in I} = b(\mathrm{Id}_A)$ is called the **free algebra** over k with free algebra generators $e_i,\ i \in I$. We will write $A = k\langle e_i\rangle_{i\in I}$. The marking F is called **representable**, if $F_{I,k}$ is representable for each k in $\mathcal{C}$ and each finite family $(\tau_i)_{i\in I}$.

Free modules: A **marking** on $\mathcal{M}$ is a family $(G_u)_{u\in U}$ of subfunctors $G_u : \mathcal{M} \longrightarrow$ (sets) of the identity functor such that, for each $u \in U$, the following condition holds: For each homomorphism $\phi : A \longrightarrow B$ in $\mathcal{C}$ and each module W in $\mathcal{M}(B)$, we have $G_u(W_{[\phi]}) = G_u(W)$. For a given algebra A in $\mathcal{C}$ and a family $(u_i)_{i\in I}$, we consider the functor $G_{I,A} : N \mapsto \prod_{i\in I} G_{u_i}(N)$ on the category $\mathcal{M}(A)$. If $G_{I,A}$ is representable, i.e. there is an A-module M and a canonical bijection

$$b : \mathrm{Hom}_{\mathcal{M}(A)}(M, N) \longrightarrow \prod_{i\in I} G_{u_i}(N)$$

for each A-module N, then M together with the family $(e_i)_{i\in I} = b(\mathrm{Id}_M)$ is called the **free module** over A with free module generators $e_i,\ i \in I$. We will write $M = \coprod_{i\in I} Ae_i$. The marking G is called **representable**, if $G_{I,A}$ is representable, for each A in $\mathcal{C}$ and each finite family $(u_i)_{i\in I}$.

A marking on $(\mathcal{C}, \mathcal{M})$ is a pair (F, G) of a marking $F = (F_\tau)_{\tau\in\mathsf{T}}$ on $\mathcal{C}$ and a marking $G = (G_u)_{u\in U}$ on $\mathcal{M}$ together with a map $\eta : \mathsf{T} \longrightarrow U$ such that $F_\tau(A) \subseteq G_{\eta(\tau)}(A)$, for each A in $\mathcal{C}$ and each τ in T.

Axioms. Let (F, G) be a marking on $(\mathcal{C}, \mathcal{M})$.

(F1) The functor F is representable.

(F2) Let k be an algebra in $\mathcal{C}$ and $A = k\langle e_i\rangle_{i\in I}$ be a free k-algebra in $\mathcal{C}$. The canonical homomorphism $k[e_i]_{i\in I} \longrightarrow k\langle e_i\rangle_{i\in I}$ in k-$\mathfrak{Mod}$ is flat and the functor $M \mapsto A \otimes_k^{\mathcal{M}} M$ is exact on the category of finite modules in $\mathcal{M}(k)$.

(F3) Let A be like in (F2) and $A' = k\langle e'_i\rangle_{i\in I'}$ be another free k-algebra in $\mathcal{C}$ with $I \subseteq I'$. Then A' is flat over A via the homomorphism $A \longrightarrow A'$ with $e_i \mapsto e'_i$.

(F4) The functor G is representable.

(F5) For each $u \in U$ and each A in $\mathcal{C}$, G_u is a right exact functor on $\mathcal{M}(A)$.

(F6) Let A be an algebra in $\mathcal{C}$ and $E = \coprod_{i\in I} Ae_i$ be a free A-module with respect to G with finite basis $(e_i)_{i\in I}$ and let M be a module in $\mathcal{M}(A)$. The canonical homomorphism $M^I \longrightarrow M \otimes_A E$ in A-$\mathfrak{Mod}$ is bijective.

(F7) Let k be an algebra in $\mathcal{C}$ and $A = k\langle e_i\rangle_{i\in I}$ be a free k-algebra in $\mathcal{C}$ with finite I. Then $\Omega_{A/k}$ is a free A-module with base $de_i \in G_{\eta(\tau_i)}(\Omega_{A/k})$; $i \in I$.

Remark that Axiom (F2) implies that free algebra generators (of degree 0) are not zero divisors.

Definition 3.1.4. The marking (F, G) is called **good**, if Axioms (F1), (F4), (F5), (F6) and (F7) hold. An admissible pair of categories $(\mathcal{C}, \mathcal{M})$ equipped with a good marking (F,G) is called a **good pair of categories** if it satisfies Axioms (S1) and (S2).

Examples 3.1.5. (1) On the admissible pair $(\mathcal{C}^{(0)},\mathcal{M}^{(0)})$ of Example 3.1.1, we work with the **canonical marking**, i.e. $F(A)=A$, for each algebra A in $\mathcal{C}$, and $G(M)=M$, for each module M in $\mathcal{M}(A)$. With this marking, $(\mathcal{C}^{(0)},\mathcal{M}^{(0)})$ is a good pair of categories, that satisfies additionally Axioms (F2) and (F3).

(2) Consider the admissible pair $(\mathcal{C}^{(1)},\mathcal{M}^{(1)})$. For A in $\mathcal{C}^{(1)}$ and $t \in T := (0,\infty)$, let $F_t(A)$ be the set of all elements of A such that the Gelfand-transformation (see [5] for the definition) $\chi(A) \longrightarrow \mathbb{C}$ factorises through $\{z \in \mathbb{C} : \ |z| \le t\}$. Further, let G be the canonical marking on $\mathcal{M}^{(1)}$. Then the pair $(\mathcal{C}^{(1)},\mathcal{M}^{(1)})$, together with the marking (F,G) is a good pair of categories, that satisfies Axioms (F2) and (F3).

(3) If $\mathcal{C}$ is the category of local analytic algebras and $\mathcal{M}$ the category of DFN-modules over $\mathcal{C}$, then G is set to be the canonical marking and for objects A, we set $F(A)$ to be the maximal ideal $\mathfrak{m}_A$ of A. Then $(\mathcal{C},\mathcal{M})$ is a good pair of categories that satisfies Axioms (F2) and (F3).

3.1.2 Graded objects

Let $(\mathcal{C},\mathcal{M})$ be an admissible pair of categories. As in [2], we can construct a new admissible pair $(\mathrm{gr}(\mathcal{C}),\mathrm{gr}(\mathcal{M}))$ as follows: Let $\mathrm{gr}(\mathcal{C})$ be the category of graded commutative[3] rings $A=\coprod_{i\le 0} A^i$ with A^0 in $\mathcal{C}$, all A^i in $\mathcal{M}(A^0)$ such that the multiplication maps $A^i \times A^j \longrightarrow A^{i+j}$ belong to $\mathrm{Mult}_{\mathcal{M}(A^0)}(A^i \times A^j, A^{i+j})$. A homomorphism $\phi : A \longrightarrow B$ in $\mathrm{gr}(\mathcal{C})$ is a homomorphism of graded commutative rings such that ϕ^0 is a homomorphism in $\mathcal{C}$ and all $\phi^i : A^i \longrightarrow B^i$ are ϕ^0- linear homomorphisms in $\mathcal{M}$.

Let $\mathrm{gr}(\mathcal{M})$ be the category over $\mathrm{gr}(\mathcal{C})$ whose objects over an algebra A in $\mathrm{gr}(\mathcal{C})$ are the graded A-modules $M=\coprod_{i\in\mathbb{Z}} M^i$, with $M^i=0$, for almost all $i>0$, such that each M^i is in $\mathcal{M}(A^0)$ and the maps $A^i \times M^j \longrightarrow M^{i+j}$ belong to $\mathrm{Mult}_{\mathcal{M}(A^0)}(A^i \times M^j, M^{i+j})$. If B is another algebra in $\mathrm{gr}(\mathcal{C})$ and N is a module in $\mathrm{gr}(\mathcal{M})(B)$, then $\mathrm{Hom}_{\mathrm{gr}(\mathcal{M})}(M,N)$ is the set of all pairs (ϕ,f), where $\phi : A \longrightarrow B$ is a homomorphism in $\mathrm{gr}(\mathcal{M})$ and $f : M \longrightarrow N$ is a ϕ-linear homomorphism of degree zero, such that all $f^i : M^i \longrightarrow N^i$ are homomorphisms in $\mathcal{M}$ over ϕ^0.

For modules $M_1,\ldots,M_n$ and N in $\mathrm{gr}(\mathcal{M})(A)$, let $\mathrm{Mult}_{\mathrm{gr}(\mathcal{M})(A)}(M_1\times\ldots\times M_n, N)$ be the $\mathbb{K}$-module of all maps $f : M_1\times\ldots\times M_n \longrightarrow N$ with the following properties:

(1) For $k_1,\ldots,k_n$ in $\mathbb{Z}$, the restriction $f|_{M_1^{k_1}\times\ldots\times M_n^{k_n}}$ factorises through a map in $\mathrm{Mult}_{\mathcal{M}(A^0)}(M_1^{k_1}\times\ldots\times M_n^{k_n}, N^{k_1+\ldots+k_n})$.

[3] i.e. $ab=(-1)^{g(a)g(b)}ba$, for homogeneous $a,b\in A$

(2) For elements $a \in A$ and $m_i \in M_i$, we have
$f(m_1, \dots, m_r a, m_{r+1} \dots m_n) = f(m_1, \dots, m_r, a m_{r+1}, \dots, m_n)$, for $1 \le r < n$, and $f(m_1, \dots, m_n a) = f(m_1, \dots, m_n) a$

We just have made use of the fact that we can make each M in $\mathrm{gr}(\mathcal{M})(A)$ a graded symmetrical A-bimodule by setting $m \cdot a := (-1)^{g(m)g(a)} a \cdot m$, for homogeneous elements $a \in A$ and $m \in M$.

To define **free algebras in** $\mathrm{gr}(\mathcal{C})$, we modify the definition in Section 3.1.1 as follows: There is a map $g : \mathsf{T} \longrightarrow \mathbb{Z}_{\le 0}$ and each functor F_τ is a subfunctor of the functor $A \mapsto A^{g(\tau)}$. In this context, if F is representable, then each functor $F_{I,A}$ is representable, if for $n \le 0$, the set of all τ_i with $g(\tau_i) = n$ is finite. In this case the free algebra $A\langle e_i \rangle_{i \in I}$ is called **g-finite** free algebra. Of course, the degree $g(e_i)$ of a free generator e_i is just $g(\tau_i)$.

To define free modules in $\mathrm{gr}(\mathcal{M})$, we modify the definition in Section 3.1.1 as follows: There is a map $g : U \longrightarrow \mathbb{Z}$ and each functor G_u is a subfunctor of the functor $M \mapsto M^{g(u)}$. In this graded context, when G is representable, then each functor $G_{I,A}$ is representable, if for each n, the set of all u_i with $g(u_i) = n$ is finite. In this case the free module $\coprod_{i \in I} A e_i$ is called **g-finite** free A-module. We have $g(e_i) = g(u_i)$ for $i \in I$.

To define a marking on $(\mathrm{gr}(\mathcal{C}), \mathrm{gr}(\mathcal{M}))$, we have to add in Section 3.1.1 the condition that the map $\eta : \mathsf{T} \longrightarrow U$ is compatible with g.

Example 3.1.6. If G is a marking on $\mathcal{M}$, then we get a marking $\mathrm{gr}(G) = (\mathrm{gr}(G))_{u' \in U'}$ on $\mathrm{gr}(\mathcal{M})$ in the following way: Set $U' := U \times \mathbb{Z}$. For A in $\mathrm{gr}(\mathcal{C})$, M in $\mathrm{gr}(\mathcal{M})$ and $u' = (u, n) \in U'$ set $\mathrm{gr}_{u'}(G)(M) := G_u(M^n)$. Here we have $g(u') = n$.

If (F, G) is a marking on $(\mathcal{C}, \mathcal{M})$, then we get a marking $\mathrm{gr}_G(F) = (\mathrm{gr}_G(F)_{\tau'})_{\tau' \in \mathsf{T}'}$ in the following way: Let T' be the disjoint union of $\mathsf{T} \times \{0\}$ and $U \times \mathbb{Z}_{<0}$. For A in $\mathrm{gr}(\mathcal{C})$ and $\tau' = (\tau, n)$, we set $\mathrm{gr}_G(F)_{\tau'}(A) = F_\tau(A^0)$, if $n = 0$, and $\mathrm{gr}_G(F)_{\tau'}(A) = G_\tau(A^n)$, if $n < 0$.

If (F, G) is a marking on $(\mathcal{C}, \mathcal{M})$, then $(\mathrm{gr}_G(F), \mathrm{gr}(G))$ is a marking on $(\mathrm{gr}(\mathcal{C}), \mathrm{gr}(\mathcal{M}))$ with the map $\eta' : \mathsf{T}' \longrightarrow U'$ given by $(\tau, 0) \mapsto (\eta(\tau), 0)$ and $(u, n) \mapsto (u, n)$, for $n < 0$.

Remark that by Lemma I.7.6 of [2], free algebra generators of negative degree behave much like polynomial variables[4] and if $(\mathcal{C}, \mathcal{M})$ is a good pair of categories, then $(\mathrm{gr}(\mathcal{C}), \mathrm{gr}(\mathcal{M}))$ is a good pair of categories as well.

If $(\mathcal{C}, \mathcal{M})$ is an admissible pair of categories that satisfies Axiom (S2), then by Proposition I.6.9 of [2], the admissible pair $(\mathrm{gr}(\mathcal{C}), \mathrm{gr}(\mathcal{M}))$ also satisfies Axiom (S2). In general, this is not true for Axiom (S1). But we have:

[4]For a more precise statement, see Proposition 3.1.19.

Proposition 3.1.7. *Let $(\mathcal{C}, \mathcal{M})$ satisfy Axiom (S1) and let A be an object of $\mathrm{gr}(\mathcal{C})$ such that all A^i are finite A^0-modules. Then, for g-finite modules M, N in $\mathrm{gr}(\mathcal{M})(A)$ each homomorphism $f : M \longrightarrow N$ in $\mathrm{gr}(\mathcal{M})(A)$ is **strict**, i.e. the cokernel of f in $\mathrm{gr}(\mathcal{M})$ coincides with the set-theoretical cokernel.*

Proposition 3.1.8. *Let $(\mathcal{C}, \mathcal{M})$ be an admissible pair of categories with a marking (F, G), where G is canonical. Suppose that Axiom $(S1)$ holds. Let k be an algebra in $\mathcal{C}$ and let M_1, M_2 and N be modules in $\mathcal{M}(k)$ such that M_1 and M_2 are finite k-modules with $M_i \subseteq N$ and $M_1 \cap M_2 = \{0\}$. Then we have*

(1) The inclusions $M_i \hookrightarrow N$ are homomorphisms in $\mathcal{M}(k)$.

(2) The sum $M_1 + M_2$ is in $\mathcal{M}(k)$.

(3) The inclusions $M_i \longrightarrow M_1 + M_2$ are homomorphisms in $\mathcal{M}(k)$.

(4) The projections $p_i : M_1 + M_2 \longrightarrow M_i$ are homomorphisms in $\mathcal{M}(k)$.

(5) $M_1 + M_2 = M_1 \oplus M_2$.

In $(\mathrm{gr}(\mathcal{C}), \mathrm{gr}(\mathcal{M}))$ the same statement is true if we suppose that all k^i are finite k^0-modules and M_1, M_2 and N are g-finite.

The proof is very simple. It can be found in the Appendix. In the sequel, we will denote the full subcategory of $\mathrm{gr}(\mathcal{C})$ generated by all algebras A such that each A^i is a finite A^0-module by $\mathrm{gr}'(\mathcal{C})$.

Lemma 3.1.9. *(1) Suppose that $(\mathcal{C}, \mathcal{M})$ is a good pair of categories and that k is an algebra in $\mathrm{gr}'(\mathcal{C})$. Let $R = k\langle T\rangle$ be a free g-finite algebra over k in $\mathrm{gr}(\mathcal{C})$. We have the following decomposition*

$$R = k \oplus \sum_{t \in T} Rt$$

in the category $\mathrm{gr}(\mathcal{M})(k)$.

(2) Suppose that additionally the marking G on $\mathcal{M}$ is trivial and that Axiom (F2) holds. Then, for each $n \geq 0$, R decomposes as

$$R = k \oplus \sum_{t_1 \in T} t_1 k \oplus \ldots \oplus \sum_{t_1, \ldots, t_n \in T} t_1 \cdot \ldots \cdot t_n k \oplus \sum_{t_1, \ldots, t_{n+1} \in T} t_1 \cdot \ldots \cdot t_{n+1} R.$$

Proof. We only prove (1) and leave the proof of (2) as exercise. We can form the free g-finite R-module $M = \coprod_{t \in T} Re(t)$, where to each free algebra generator $t \in \mathrm{gr}_G(F)_{\tau'}$ we have associated a free module generator $e(t) \in \mathrm{gr}(G)_{\eta(\tau')}(M)$. Consider the homomorphism $M \longrightarrow R$ in $\mathrm{gr}(\mathcal{M})(R)$ with $e(t) \mapsto t$. By Proposition 3.1.7, the cokernel map of this homomorphism coincides with the cokernel map in R-$\mathfrak{Mod}$, which is just the projection $p : R \longrightarrow R/(T)$ and $R/(T)$ is an algebra in $\mathrm{gr}(\mathcal{C})$. Now there is a diagram

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[dr, "p"'] & k \arrow[d] \\ & R/(T) \end{tikzcd}$$

in $\mathrm{gr}(\mathcal{C})$, where $\pi : R \longrightarrow k$ is the homomorphism given by $t \mapsto 0$ for $t \in T$ and the homomorphism $k \longrightarrow R/(T)$ is the canonical inclusion. The diagram commutes, since in both directions an element $t \in T$ goes to 0. So we get $\mathrm{Kern}(\pi) = (T)$. But obviously, we have $R = k \oplus \mathrm{Kern}(\pi)$. □

3.1.3 Balanced and convex markings

Let k be an algebra in $\mathrm{gr}(\mathcal{C})$ and $A := k\langle T\rangle$ a free algebra over k in $\mathrm{gr}(\mathcal{C})$ with a g-finite set T of free generators $t \in F_{\tau(t)}(R^{g(t)})$. Then $A \otimes_k A$ is a free algebra over k with two free algebra generators $t_1 = t \otimes 1$ and $t_2 = 1 \otimes t$, for each $t \in T$. For $t \in T$, set $t^+ := \frac{1}{2}(t_1 + t_2)$ and $t^- := \frac{1}{2}(t_1 - t_2)$. Let T^+ be the set of all t^+ and T^- be the set of all t^-.

We say that the marking F on $\mathcal{C}$ is **balanced**, if for each $\tau \in \mathsf{T}$, A in $\mathcal{C}$ and $t \in F_\tau(A)$, we have $-t \in F_\tau(A)$. We say that the marking F is **convex** if for each $\tau \in \mathsf{T}$, A in $\mathcal{C}$, $t_1, t_2 \in F_\tau(A)$ and $a, b \in \mathbb{K}$ with $a + b = 1$, we have $at_1 + bt_2 \in F_\tau(A)$.

Proposition 3.1.10. *If the marking $\mathrm{gr}_G(F)$ on $\mathrm{gr}(\mathcal{C})$ is balanced and convex, we have $A \otimes_k A \cong k\langle T^+ \cup T^-\rangle$.*

Example 3.1.11. (1) The trivial marking on $\mathcal{C}$ is balanced and convex.

(2) If $\mathcal{C}$ is the category $\mathcal{C}^{(1)}$ of (local) analytic algebras and $\mathcal{M}$ the category $\mathcal{M}^{(1)}$ of DFN-modules over $\mathcal{C}^{(1)}$, then the marking F on $\mathcal{C}$ (see Example 3.1.5) is balanced and convex.

Proof. The first example is trivial. For the second example, we show that if a free generator t is in $F_\tau(R)$, then t^+ and t^- are in $F_\tau(R \otimes_B R)$: Here, τ stands for a positive real number and $F_\tau(R)$ is the set of all $r \in R$ such that, for each character $\xi \in \mathcal{X}(R)$, we have $|\xi(r)| \le \tau$. Now, $t_1 = t \otimes 1$ and $t_2 = 1 \otimes t$ belong to $F_\tau(R \otimes_B R)$, so for each character $\xi \in \mathcal{X}(R \otimes_B R)$, we have $|\xi(t_1)| \le \tau$ and $|\xi(t_2)| \le \tau$. Hence $|\xi(t^+)| = |\frac{1}{2}(\xi(t_1) + \xi(t_2))| \le \tau$ and $|\xi(t^-)| = |\frac{1}{2}(\xi(t_1) - \xi(t_2))| \le \tau$. The case of local analytic algebras is clear, since maximal ideals are additively closed. □

3.1.4 Simplicial complexes

Let I be an index set. Then a set $\mathcal{N}$ of subsets of I is called **simplicial complex** over I, if $\emptyset \notin \mathcal{N}$; if for all $i \in I$, we have $\{i\} \in \mathcal{N}$ and if every nonempty subset of an element in $\mathcal{N}$ is again in $\mathcal{N}$.

For an element α of a simplicial complex $\mathcal{N}$ over I, containing n elements, set $|\alpha| := n - 1$. Then for $n \ge 0$, the set $\mathcal{N}^{(n)}$ of all $\alpha \in \mathcal{N}$ with $|\alpha| \le n$ is again a simplicial complex over I.

A simplicial complex $\mathcal{N}$ can be seen as category, where $\mathrm{Hom}(\alpha, \beta)$ contains only the inclusion $\alpha \subseteq \beta$ if $\alpha \subseteq \beta$ and is empty in all other cases.

Let $\mathcal{A}$ be a category. An **$\mathcal{N}$-object** in $\mathcal{A}$ is a covariant functor $\mathcal{N} \longrightarrow \mathcal{A}$. The $\mathcal{N}$-objects in $\mathcal{A}$ again form a category, denoted by $\mathcal{A}^{\mathcal{N}}$. If $(\mathcal{C}, \mathcal{N})$ is an admissible pair of categories and $A = (A_\alpha)_{\alpha \in \mathcal{N}}$ an object of $\mathcal{C}^{\mathcal{N}}$, then we denote the category of $\mathcal{N}$-objects $M = (M_\alpha)_{\alpha \in \mathcal{N}}$ in $\mathcal{M}^{\mathcal{N}}$ with $M_\alpha \in \mathrm{ob}(\mathcal{M}(A_\alpha))$ by $\mathcal{M}^{\mathcal{N}}(A)$.

Let $(\mathcal{C}, \mathcal{M})$ be an admissible pair of categories and $\mathcal{N}$ a simplicial complex. Let $((F_\tau)_{\tau \in \mathsf{T}}, (G_u)_{u \in U})$ be a marking on $(\mathcal{C}, \mathcal{M})$. Then, for each pair (α, τ) in $\mathcal{N} \times \mathsf{T}$, there is a functor $F_{\alpha,\tau} : A \mapsto F_\tau(A_\alpha)$. For a family $(\alpha_i, \tau_i), i \in I$ of elements of $\mathcal{N} \times \mathsf{T}$ and A in $\mathcal{C}^{\mathcal{N}}$, there is a set-valued functor $B \mapsto \prod_{i \in I} F_{\alpha_i, \tau_i}(B)$. We denote it by $F_{I,A}$.

Definition and Proposition 3.1.12. *Suppose that for $\alpha \in \mathcal{N}$, the free A_α-algebra $A'_\alpha = A_\alpha \langle e_i^{(\alpha)} \rangle_{\alpha_i \subseteq \alpha}$ in the free generators $e_i^{(\alpha)} \in F_{\tau_i}(A'_\alpha)$ exists. For $\alpha \subseteq \beta$, let $\rho_{\alpha\beta} : A'_\alpha \longrightarrow A'_\beta$ be the homomorphism in $\mathcal{C}$ over A_α, given by $e_i^{(\alpha)} \mapsto e_i^{(\beta)}$. Then $A' = (A'_\alpha)_{\alpha \in \mathcal{N}}$ is an algebra in $\mathcal{C}^{\mathcal{N}}$, and together with the family $(e^{(\alpha_i)})_{i \in I}$, it represents the functor $F_{I,A}$. We call it the **free A-algebra** in the free generators $e_i := e_i^{(\alpha_i)} \in F_{\alpha_i, \tau_i}(A')$ and denote it by $A\langle e_i \rangle_{i \in I}$.*

For each pair $(\alpha, u) \in \mathcal{N} \times U$, there is a functor $G_{\alpha,u} : M \mapsto G_u(M_\alpha)$. For a family $(\alpha_i, u_i), i \in I$ of elements of $\mathcal{N} \times U$ and A in $\mathcal{C}^{\mathcal{N}}$, there is a set-valued functor $N \mapsto \prod_{i \in I} G_{\alpha_i, u_i}(M)$. We denote it by $G_{I,A}$.

Definition and Proposition 3.1.13. *Fix a family $(\alpha_i, u_i), i \in I$ of elements of $\mathcal{N} \times U$ and an algebra A in $\mathcal{C}^{\mathcal{N}}$. Suppose that for each $\alpha \in \mathcal{N}$, the free A_α-module $M_\alpha = \coprod_{\alpha_i \subseteq \alpha} A_\alpha e_i^{(\alpha)}$ in the free generators $e_i^{(\alpha)} \in G_{u_i}(M_\alpha)$ exists. For $\alpha \subseteq \beta$, let $\rho_{\alpha\beta} : M_\alpha \longrightarrow M_\beta$ be the homomorphism in $\mathcal{M}$ over A_α, given by $e_i^{(\alpha)} \mapsto e_i^{(\beta)}$. Then $M = (M_\alpha)_{\alpha \in \mathcal{N}}$ is a module in $\mathcal{M}^{\mathcal{N}}$ and together with the family $(e^{(\alpha_i)})_{i \in I}$ it represents the functor $G_{I,A}$. We call it the **free A-module** with free generators $e_i := e_i^{(\alpha_i)} \in G_{\alpha_i, u_i}(A')$ and denote it by $\coprod_{i \in I} A e_i$.*

To distinguish the non-simplicial from the simplicial context, we call the first one **affine**.

3.1.5 Resolvents

Fix an admissible pair $(\mathcal{C}, \mathcal{M})$ of categories with marking (F, G). For a DG module K in $\mathrm{gr}(\mathcal{M})(R)$ with differential d, we define the **i-th homology $H^i_{\mathcal{M}}(K)$ of K in $\mathcal{M}(R)$** as cokernel of the natural map $\mathrm{Im}(d^{i-1}) \longrightarrow \mathrm{Kern}(d^i)$ (image and kernel formed in the category $\mathcal{M}(R)$). If K is **separated**, i.e. the cokernels of the map $d : K^i \longrightarrow K^{i+1}$ and of the induced maps $K^i \longrightarrow \mathrm{Kern}(d^{i+1})$ and $\mathrm{Kern}(d^i) \longrightarrow K^i$ coincide with their cokernels formed in the category R-$\mathfrak{Mod}$, then $H^i_{\mathcal{M}}(K)$ is as R-module isomorphic to the $i-th$ cohomology of K, considered as complex in $R - \mathfrak{Mod}$. We call K **acyclic**, if $H^i_{\mathcal{M}}(K) = 0$ for all i.

Lemma 3.1.14. *Suppose that $(\mathcal{C},\mathcal{M})$ satisfies Axiom (S1) and all K^i are finite R-modules. The complex K is acyclic if and only if K is acyclic as complex in $R-\mathfrak{Mod}$.*

A **DG resolution** of an object B in $\mathcal{M}$ is a DG module M in $\mathrm{gr}(\mathcal{M})$ such that $H^i_{\mathcal{M}}(M) = 0$, for $i < 0$, and $H^0_{\mathcal{M}}(M) = B$.

We recall Definition I.8.1 of [2].

Definition 3.1.15. Let $A \longrightarrow B$ be a homomorphism of DG objects in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$. A **resolvent of B over** A is a free DG algebra R over A in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ (with respect to the marking $\mathrm{gr}_G(F)$) together with a morphism $R \longrightarrow B$ of DG objects in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ which is a surjective quasi-isomorphism on each $\alpha \in \mathcal{N}$.

In this chapter, we will mostly work in a Noetherian context, i.e. we will mostly assume that the following axiom is satisfied:

Axiom (N) Each algebra A in $\mathcal{C}$ is Noetherian and each finite module M in $\mathcal{M}(A)$ is a quotient of a finite free A-module.

If the good pair $(\mathcal{C},\mathcal{M})$ satisfies Axioms (N) and (F2) and if A and B belong to $\mathrm{gr}'(\mathcal{C})$ and if A^0 is a quotient of a g-finite B^0-module C in $\mathcal{C}^{\mathcal{N}}$ such that each C_α is a finite free B^0_α-algebra, then such resolvents exist by Proposition I.8.7 and I.8.8 of [2] or alternatively by Lemma 3.1.23 below.

The next proposition is of great importance for this work. In the algebraic context, the statement is well-known and was used by Quillen and others. The difference in the analytic context is that a free DG algebra over a ring k is not, in general, a complex of free k-modules as long as there are *analytic* algebra generators, i.e. free algebra generators of degree zero.

Suppose that the marking G on $\mathcal{M}$ is trivial and that Axiom (N) is satisfied.

Proposition 3.1.16. *Let $A \longrightarrow B$ be a homomorphism of DG algebras in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$. For two g-finite resolvents R_1 and R_2 of B over A, there exists a homomorphisms $R_1 \longrightarrow R_2$ in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$, which is a homotopy equivalence over A.*

Proof. We first prove the affine case and then sketch the generalization to $\mathcal{N}$-objects in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$, then. **First affine case:** Suppose that $R_1^0 = R_2^0$.
Set $A' := A \otimes_{A^0} R_1^0$. Then R_1 and R_2 are resolvents of B over A'. By Proposition I.8.1 of [2], there is a quasi-isomorphism $R_1 \longrightarrow R_2$ in $\mathrm{gr}(\mathcal{C})$ over A'. Since $R_1^0 = R_2^0 = A'^0$, R_1 and R_2 are free A'-modules in $\mathrm{gr}(\mathcal{M})$. Hence the quasi-isomorphism is already a homotopy-equivalence.
Second affine case: Suppose that R_2^0 is a finite free algebra over R_1^0 in $\mathcal{C}$.
By induction, we can restrict ourselves to the case where $R_2^0 = R_1^0\langle e\rangle$ is just a free algebra in one generator. Consider the free R_1^0-algebra $R := R_1^0\langle e, f\rangle$ in $\mathrm{gr}(\mathcal{C})$, generated by a free generator e of degree 0 and a free generator f of degree -1. We define a

differential on R by setting $f \mapsto e$. By Lemma 3.1.9, we have $R_1^0\langle e\rangle = R_0 \oplus eR_1^0\langle e\rangle$. So by Axiom (S2), the differential gives an isomorphism $fR_1^0\langle e\rangle \longrightarrow eR_1^0\langle e\rangle$. With this in mind, we can easily construct a contracting homotopy on R. Now $R_1' := R_1 \otimes_{R_1^0} R$ is homotopic over R_1 to R_1. More precisely, the inclusion $R_1 \longrightarrow R_1'$ and the projection $R_1' \longrightarrow R'$ are homotopy equivalences. By the first case, there is a homotopy-equivalence $R_1' \longrightarrow R_2$ in $\mathrm{gr}(\mathcal{C})$.
General affine case: Let R_3 be a free g-finite resolution of B over $R_1 \otimes_A R_2$. Now R_3 is free over R_1 and R_2 and by the second case, we get a homotopy-equivalence $R_1 \longrightarrow R_3 \longrightarrow R_2$.

In the simplicial case, a free algebra in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ over an algebra A in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ is not a free module in $\mathrm{gr}(\mathcal{M})^{\mathcal{N}}(A)$, even if all free algebra generators are of strictly negative degree. The point is that even A itself is not free as A-module. But a free algebra over A with free algebra generators of negative degree is as A-module in $\mathrm{gr}(\mathcal{M})^{\mathcal{N}}$ a direct sum $A \oplus M$ with a free A-module M. To prove the simplicial case, we must first generalize the Comparison Theorem (see Theorem 2.2.6 of [46]) to (free) DG resolutions in $\mathrm{gr}(\mathcal{M})^{\mathcal{N}}$ which is strait forward. Secondly, observe that for a DG algebra A in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ and free A-modules M, N, each quasi-isomorphism $\mathrm{Id} \times f : A \oplus M \longrightarrow A \oplus N$ is even a homotopy equivalence. With those tools we can generalize the first case above. To avoid overloading, we have put a detailed proof of the simplicial case in the Appendix. The second and third case go just as in the affine situation. $\square$

3.1.6 Double graded objects

Let $(\mathcal{C}, \mathcal{M})$ be an admissible pair of categories. We define the pair $(\mathrm{gr}^2(\mathcal{C}), \mathrm{gr}^2(\mathcal{M}))$ as follows: The objects of $\mathrm{gr}^2(\mathcal{C})$ are the double graded rings $A = \coprod_{i,j \le 0} A^{i,j}$ with $A^{0,0}$ in $\mathcal{C}$ and all $A^{i,j}$ in $\mathcal{M}(A^{0,0})$ such that

(1) for $a \in A^{i,j}$ and $b \in A^{k,l}$ we have $ab = (-1)^{(i+j)(l+k)} ba$,

(2) the multiplication maps $A^{i,j} \times A^{k,l} \longrightarrow A^{i+j,k+k}$ belong to $\mathrm{Mult}_{\mathcal{M}(A^{0,0})}(A^{i,j} \times A^{k,l}, A^{i+j,k+l})$.

Following the ideas of Section 3.1.2, we can define $\mathrm{Hom}_{\mathrm{gr}^2(\mathcal{C})}(A, B)$, for objects A, B in $\mathrm{gr}^2(\mathcal{C})$, the category $\mathrm{gr}^2(\mathcal{M})$, $\mathrm{Hom}_{\mathrm{gr}^2(\mathcal{M})}(M, N)$ for objects M, N of $\mathrm{gr}^2(\mathcal{M})$ and $\mathrm{Mult}_{\mathrm{gr}^2(\mathcal{M})(A)}(M_1, \dots, M_n, N)$ for modules $M_1, \dots, M_n, N$ in $\mathrm{gr}^2(\mathcal{M})(A)$. We don't make this definitions explicit here.

Proposition 3.1.17. *Let A be an object of $\mathrm{gr}^2(\mathcal{C})$ and M, N objects of $\mathrm{gr}^2(\mathcal{M})$. For (p,q) in $\mathbb{Z} \times \mathbb{Z}$, set $T^{p,q} := \coprod_{i+j=p,k+l=q} M^{i,k} \otimes^{\mathcal{M}}_{A^{0,0}} N^{j,l}$. Then $T := \coprod_{p,q} T^{p,q}$ is a tensor product of A and B in $\mathrm{gr}^2(\mathcal{M})(A^{0,0})$. T can be seen in two different ways as an object of $\mathrm{gr}^2(\mathcal{M})(A)$. Consider the homomorphism $u : A \otimes_{A^{0,0}} T \longrightarrow T$ in $\mathrm{gr}^2(\mathcal{M})(A^{0,0})$, sending $a \otimes m \otimes n$ to $ma \otimes n - m \otimes an$. u can be seen in two manners as homomorphism in $\mathrm{gr}^2(\mathcal{M})(A)$. Both of them induce the same A-module structure on $\bar{T} := \mathrm{Cokern}(u)$. We see that $\bar{T}$ is a tensor product of M and N in $\mathrm{gr}^2(\mathcal{M})(A)$.*

Proposition 3.1.18. *The pair* $(\mathrm{gr}^2(\mathcal{C}), \mathrm{gr}(\mathcal{M})^2)$ *is an admissible pair of categories.*

Proof. Analogue to the proof of Proposition 6.9 of [2]. □

Convention: When we consider an object K of $\mathrm{gr}(\mathcal{M})$ as object of $\mathrm{gr}^2(\mathcal{M})$, we set $K^{i,0} = K^i$ and $K^{i,j} = 0$ for $j \neq 0$.

In the same manner as above, we can define a marking $(\mathrm{gr}^2_G(F), \mathrm{gr}^2(G))$ on the pair $(\mathrm{gr}^2(\mathcal{C}), \mathrm{gr}^2(\mathcal{M}))$:
Define the index set T'' as $\mathsf{T} \times \{0,0\} \cup U \times (\mathbb{Z}^{\leq 0} \times \mathbb{Z}^{\leq 0}) \setminus (0,0)$. For $\tau'' = (\tau, 0, 0) \in \mathsf{T}''$ and $A \in \mathrm{gr}^2(\mathcal{C})$ set $\mathrm{gr}^2_G(F)_{\tau''}(A) := F_\tau(A^{0,0})$ and for $\tau'' = (u,p,q)$ in T'' with $(p,q) \neq (0,0)$ set $\mathrm{gr}^2_G(F)_{\tau''}(A) := G_u(A^{p,q})$. Define the index set U'' as $U \times \mathbb{Z} \times \mathbb{Z}$. For $u'' = (u,p,q) \in U''$ and $M \in \mathrm{gr}^2(\mathcal{M})$, set $\mathrm{gr}(G)_{u''}(M) := G_u(M^{p,q})$.

Proposition 3.1.19. *(1) Let A be an algebra in $\mathrm{gr}^2(\mathcal{C})$ and $A' = A\langle e_i\rangle_{i\in I}$ a free algebra over A, with respect to the marking $\mathrm{gr}^2_G(F)$. Suppose that the bidegree of each e_i is different to 0. Then the canonical homomorphism $A[e_i]_{i\in I} \longrightarrow A'$ in A-$\mathfrak{Alg}$ is bijective.*

(2) If (F,G) is good, then $(\mathrm{gr}^2_G(F), \mathrm{gr}^2(G))$ is good as well.

Proof. Analogue to the proof of Lemma I.7.6 of [2]. □

Definition 3.1.20. A **DG algebra** in $\mathrm{gr}^2(\mathcal{C})$ is an algebra A in $\mathrm{gr}^2(\mathcal{C})$ equipped with a (vertical) $A^{0,0}$-derivation[5] $v : A \longrightarrow A$ of bidegree $(0,1)$ with $s^2 = 0$. A **DDG algebra** in $\mathrm{gr}^2(\mathcal{C})$ is a DG algebra A in $\mathrm{gr}^2(\mathcal{C})$ equipped with a (horizontal) derivation h of bidegree $(1,0)$ that anti-commutes with v such that $h^2 = 0$. A **homomorphism** of (D)DG algebras is a morphism in $\mathrm{gr}^2(\mathcal{C})$ that commutes with the vertical (and horizontal) differentials.

The definition of **(D)DG modules** over D(DG) algebras is straightforward (pay attention to Koszul signs).

Lemma 3.1.21. *Let $K = (K,h,v)$ be a DG algebra in $\mathrm{gr}^2(\mathcal{C})$. Consider a free algebra $K\langle E\rangle$ over K in $\mathrm{gr}^2(\mathcal{C})$ with a set $E = \{e_i : i \in I\}$ of free algebra generators with $e_i \in \mathrm{gr}^2_G(F)_{\tau''_i}(K\langle E\rangle)$, for a certain $\tau''_i \in \mathsf{T}''$. For each i, if $g(x_i) \neq (0,0)$ choose an element $h_i \in G_{u_i}(K\langle E\rangle^{g(x_i)+(1,0)})$ and an element $v_i \in G_{u_i}(K\langle E\rangle^{g(x_i)+(0,1)})$, where u_i is the first component of $\tau_i = (u_i, g(x_i))$. Then, setting $h(e_i) := h_i$ and $v(e_i) := v_i$, we get an extension of the horizontal and the vertical derivation h and v of K. This extensions make $K\langle E\rangle$ a DDG algebra, if and only if, for each i, we have*

(1) $h(v_i) + v(h_i) = 0$ and

(2) $h(h_i) = v(v_i) = 0$.

[5] This means that for homogeneous $a, b \in A$ we have $v(ab) = v(a) + (-1)^a a v(b)$. In the exponent, by a, we mean the total degree of a.

Proof. Inductively, we can reduce the proof to the case where E consists of a single element e of bidegree (p,q). In this case, it is an easy calculation. □

Definition 3.1.22. A **DG resolvent** of an algebra B in $\mathrm{gr}(\mathcal{C})$ is a free DG algebra A in $\mathrm{gr}^2(\mathcal{C})$ such that, for all i, the i-th row is a surjective DG module resolution of B^i. A **DDG resolvent** of a DG algebra B in $\mathrm{gr}(\mathcal{M})$ is a DDG algebra A in $\mathrm{gr}^2(\mathcal{M})$ which is a DG resolvent of B such that the map $A^{*,0} \longrightarrow B$ is a homomorphism of DG algebras in $\mathrm{gr}(\mathcal{C})$.

For a homomorphism $A \longrightarrow B$ of DG algebras in $\mathrm{gr}(\mathcal{C})$, to get a resolvent R of B over A it is enough to construct a DDG resolvent K of B which is free over A as object of $\mathrm{gr}^2(\mathcal{C})$. Then we can choose R as total complex $\mathrm{tot}(K)$. This leads to the question of the existence of DDG resolvents. The following remark provides a positive answer:

Lemma 3.1.23. *Suppose that for the pair $(\mathcal{C},\mathcal{M})$ the Axioms (N) and (F2) hold. Let $K=(K,h,v)$ be a DDG algebra in $\mathrm{gr}^2(\mathcal{C})$ and $u : K^{*,0} \longrightarrow A$ a homomorphism of DG algebras in $\mathrm{gr}(\mathcal{C})$. Suppose that each $A \in \mathrm{gr}'(\mathcal{C})$ and that each $K^{i,j}$ a finite $K^{0,0}$-module.*

(1) If A^0 is a quotient of a free $K^{0,0}$-algebra, then there exists a free DDG algebra $L = K\langle F\rangle$ over K, where F is a g-finite set of generators of bidegree $(k,0)$; $k \leq 0$, and a surjective homomorphism $L^{*,0} \longrightarrow A$ over $K^{*,0}$.

(2) Suppose that u is surjective and that for a fixed $p < 0$, we have $u^{p+1} = \mathrm{cokern}(v^{p+1,-1})$. There exists a free DDG algebra $L = K\langle F \cup G\rangle$ over K with finite sets F and G of generators of bidegree $(p,-1)$ and $(p+1,-1)$, respectively such that we still have $u^{p+1} = \mathrm{cokern}(v^{p+1,-1})$, and additionally $u^p = \mathrm{cokern}(v^{p,-1})$ holds.

(3) Fix $p \leq 0$ and $q \leq -1$. Suppose that we have $H^{q+1}(K^{p+1,*}) = 0$. There exists a free DDG algebra $L = K\langle F\cup G\rangle$ over K spanned by finite sets F and G of generators of bidegree (p,q) and $(p+1,q)$, respectively such that we still have $H^{q+1}(K^{p+1,*}) = 0$, and additionally $H^{q+1}(K^{p,*}) = 0$ holds.

Proof. The proofs of (1) - (3) are very similar, so we are only carrying the proof of (3). We choose G such that there is an epimorphism $\pi : \amalg_{g\in G} K^{0,0} g \longrightarrow \mathrm{Kern}(v^{p+1,q+1}) \cap \mathrm{Kern}(h^{p+1,q+1})$. Set $v(g) := \pi(g)$ and $h(g) := 0$. We choose F such that there is an epimorphism $\pi' : \amalg_{f\in F} K^{0,0} f \longrightarrow \mathrm{Kern}(v^{p,q+1})$. Set $v(f) = \pi'(f)$ and choose $h(f)$ in $\amalg K^{0,0} g$ in such a way that $v(h(f)) = -h(v(f))$. □

Definition 3.1.24. For a g-finite free DG module $M = \coprod_{e\in E} Ae$ in $\mathrm{gr}(\mathcal{M})$ with differential d (this construction can be done more generally in $\mathrm{gr}(\mathcal{M})^{\mathcal{N}}$), we define the **exterior algebra** $\bigwedge_A M$ to be the free DDG algebra $A\langle \hat{E}\rangle$ in $\mathrm{gr}^2(\mathcal{C})$, where $\hat{E}$ contains for each $e \in E$ a free algebra generator $\hat{e}$ of bidegree $(g(e),-1)$. The vertical differential of $\bigwedge_A M$ is set to be trivial, and the horizontal differential h is defined in such a way that the assignment $e \mapsto \hat{e}$ identifies M as DG module with the line $A\langle\hat{E}\rangle^{*,-1}$. The total complex $\wedge_A M := \mathrm{tot}(\bigwedge_A M)$ has the structure of a DG algebra in $\mathrm{gr}(\mathcal{C})$ and corresponds to the ordinary definition of the exterior algebra.

In this situation, let $\wedge_A^j M$ be the DG module in $\mathrm{gr}(\mathcal{M})$ with $(\wedge_A^j M)^n = A\langle\hat{E}\rangle^{(n,-j)}$, for all $j \geq 0$. As usual, for $e_1, \dots, e_j \in E$, write $e_1 \wedge \dots \wedge e_j$ for the element in $\wedge_A^j M$, corresponding to $\hat{e}_1 \cdots \hat{e}_n \in A\langle\hat{E}\rangle^{*,-j}$.

In particular, we have $\wedge_A^0 M \cong A$ and $\wedge_A^1 M \cong M$ and

$$\wedge_A M = \mathrm{tot}(\bigwedge\nolimits_A M) = \coprod_{j \geq 0} \wedge^j M[j]. \tag{1.5}$$

3.1.7 The bar complex and the naive Hochschild complex

For the reader's convenience and to motivate our definitions in Section 3.2, in this section, we state several statements about the bar complex and the classical Hochschild complex. The classical Hochschild complex (or cyclic bar complex) is called "naive Hochschild complex" in this thesis, since we will define a more elaborate "Hochschild complex" which is better for the treatment of algebras in good pairs of categories in Section 3.2. In the algebraic context, the statements of this subsection are well-known and they apply directly to admissible pairs of categories.

Let $(\mathcal{C}, \mathcal{M})$ be an admissible pair of categories. Consider a homomorphism $k \longrightarrow A$ of DG algebras in $\mathrm{gr}(\mathcal{C})$. The tensor product $R := A \otimes_k^{\mathrm{gr}(\mathcal{C})} A$ is a DG algebra with differential $d^R = d^A \otimes 1 + 1 \otimes d^A$, and the natural "multiplication" map $\mu : A \otimes_k^{\mathcal{C}} A \longrightarrow A$ respects the differentials.

Let M be a DG A-bimodule in $\mathrm{gr}(\mathcal{M})$, which is a symmetrical k-bimodule. We can consider M as DG object of $\mathrm{gr}(\mathcal{M})(R)$, where the scalar multiplication $R \times M \longrightarrow M$ satisfies $(a \otimes a', m) \mapsto (-1)^{a'm} ama'$, for homogeneous elements[6] $a, a' \in A$ and $m \in M$. To see this, we have to apply Axioms (5.3), (5.5) and (5.6). The same axioms must be used to define the mappings in the sequel.

For $n = 0, 1, \dots$ set $C_n^{\mathrm{naive}}(A, M) := M \otimes A^{\otimes n}$ and $C_n^{\mathrm{bar}}(A, M) := M \otimes A^{\otimes n} \otimes A$. (All tensor products are formed in the category $\mathrm{gr}(\mathcal{M})(k)$.) Consider the homomorphisms

$$\begin{aligned} d_i : M \otimes A^{\otimes n} &\longrightarrow M \otimes A^{\otimes n-1} \\ a_0 \otimes \dots \otimes a_n &\mapsto a_0 \otimes \dots \otimes a_i \cdot a_{i+1} \otimes \dots \otimes a_n \end{aligned}$$

for $i = 0, \dots, n-1$, and the homomorphism d_n, defined by

$$a_0 \otimes \dots \otimes a_n \mapsto (-1)^{a_n(a_1 + \dots + a_{n-1})} a_0 \cdot a_n \otimes a_1 \otimes \dots \otimes a_{n-1}.$$

[6]In the exponents we write sometimes just a instead of $g(a)$ for homogeneous elements. ab then means $g(a) \cdot g(b)$ and not $g(ab)$, which is just $g(a) + g(b)$.

Each d_i is a homomorphisms in $\mathrm{gr}(\mathcal{M})(A)$, if we regard the tensor products $M \otimes A \otimes \ldots \otimes A$ as A-modules by left-multiplication on the first factor. Remark that when M is only a A-right module, we consider it as an antisymmetrical A-bimodule by setting $m \cdot a := (-1)^{ma} a \cdot m$.

Set $b'_{n-1} := d_0 - \ldots + (-1)^{n-1} d_{n-1}$ and $b_n := b + (-1)^n d_n$. Exactly as in the algebraic case (see paragraph III.2.1 of [2]), b defines a differential on $C_\bullet^{\mathrm{naive}}(A, M)$, i.e. $b^2 = 0$. The pair $(C_\bullet^{\mathrm{naive}}(A, M), b)$ is called **naive Hochschild complex**. b' defines a differential on $C_\bullet^{\mathrm{bar}}(A, M)$. The pair $(C_\bullet^{\mathrm{bar}}(A, M), b')$ is called **bar complex**. For later use, we set $C^{\mathrm{bar}}(A, M)^{-n} := C_n^{\mathrm{bar}}(A, M)$ and $C^{\mathrm{naive}}(A, M)^{-n} := C_n^{\mathrm{naive}}(A, M)$, for $n \geq 0$.

Observe that $C_\bullet^{\mathrm{bar}}(A, M)$ is even a complex in $\mathrm{gr}(\mathcal{M})(R)$, when we define the R-module structure on $M \otimes A^{\otimes n} \otimes A$ by

$$(a \otimes a') \cdot (m \otimes \alpha \otimes a_{n+1}) = (-1)^{a(a'+m+\alpha)} a' m \otimes \alpha \otimes a \cdot a_{n+1}$$

for homogeneous elements a, a', α and m. In the sequel, we write $C_\bullet^{\mathrm{naive}}(A)$ for $C_\bullet^{\mathrm{naive}}(A, A)$ and $C_\bullet^{\mathrm{bar}}(A)$ for $C_\bullet^{\mathrm{bar}}(A, A)$.

In $\mathrm{gr}(\mathcal{M})(k)$ there exist homomorphisms $C_n^{\mathrm{bar}}(A) \longrightarrow C_{n+1}^{\mathrm{bar}}(A)$, sending elements $a_1 \otimes \ldots \otimes a_n$ to $1 \otimes a_1 \otimes \ldots \otimes a_n$. They define a contracting homotopy for the bar complex. Hence the bar complex $C^{\mathrm{bar}}(A)$ is acyclic. By Theorem III.2.2 of [2], we can even define a DDG algebra structure on $C^{\mathrm{bar}}(A)$. We explain how to form the product $*$ on the total complex $\mathrm{tot}(C^{\mathrm{bar}}(A))$: For elements $c_1, \ldots, c_k$ in A and a permutation $\sigma \in \Sigma_k$, we introduce the sign $\rho(\sigma) := \rho(\sigma, c_1, \ldots, c_k)$ in such a way that we have the following relation

$$\uparrow c_{\sigma(1)} \odot \ldots \odot \uparrow c_{\sigma(k)} = \rho(\sigma) \uparrow c_1 \odot \ldots \odot \uparrow c_k$$

in the the symmetric algebra $S(A[1])$. Here, $\uparrow$ is just the shift $A \mapsto A[1]$ (see Section 1.1). On bihomogeneous elements $a_0 \otimes \ldots \otimes a_{n+1} \in C_n^{\mathrm{bar}}(A)$ and $b_0 \otimes \ldots \otimes b_{m+1} \in C_m^{\mathrm{bar}}(A)$, the product $*$ has the form

$$(a_0 \otimes \ldots \otimes a_{n+1}) * (b_0 \otimes \ldots \otimes b_{m+1}) = \pm \sum_{\sigma^{-1} \in \mathrm{Sh}(n, m+n)} \rho(\sigma) a_0 b_0 \otimes c_{\sigma(1)} \otimes \ldots \otimes c_{\sigma(n+m)} \otimes a_{n+1} b_{m+1},$$

where $\pm$ is the sign

$$(-1)^{b_0(a_1 + \ldots + a_{n+1} + n) + a_{n+1}(b_1 + \ldots + b_n + m)}$$

and $(c_1, \ldots, c_{m+n}) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$. Hence $\mathrm{tot}(C^{\mathrm{bar}}(A))$ is a DG algebra resolution of A over R. But it can only serve as resolvent in the algebraic case since:

Attention: In the analytic case, $\mathrm{tot}(C^{\mathrm{bar}}(A))$ is **not** a free object in $\mathrm{gr}(\mathcal{C})$.

Recall two well-known relations between the bar complex and the naive Hochschild complex. We consider R as A-bimodule via $a(a_1 \otimes a_2) = aa_1 \otimes a_2$ and $(a_1 \otimes a_2)a = a_1 \otimes a_2 a$.

Proposition 3.1.25. *We have an isomorphism $C_\bullet^{\text{naive}}(A, R) \longrightarrow C_\bullet^{\text{bar}}(A)$ of complexes in $\text{gr}(\mathcal{M})(A)$, which is in the n-th component given by*

$$\begin{aligned} C_n^{\text{naive}}(A, R) &\longrightarrow C_n^{\text{bar}}(A) \\ (a \otimes a') \otimes \alpha &\mapsto (-1)^{a(a'+\alpha)} a' \otimes \alpha \otimes a \end{aligned}$$

with $\alpha \in A^{\otimes n}$. Furthermore, we have an isomorphism $C_\bullet^{\text{naive}}(A, M) \longrightarrow M \otimes_R C_\bullet^{\text{bar}}(A)$ of complexes in $\text{gr}(\mathcal{M})(A)$, where the differential of the second complex is given by $1 \otimes b'$. The n-th component has the following form:

$$\begin{aligned} C_\bullet^{\text{naive}}(A, M) &\longrightarrow M \otimes_R C_\bullet^{\text{bar}}(A) \\ m \otimes \alpha &\mapsto m \otimes 1 \otimes \alpha \otimes 1. \end{aligned}$$

The classical Hochschild cochain complex is called **naive Hochschild cochain complex** in this thesis. It is defined as the complex $C^\bullet(A, M) = (C^\bullet(A, M), \beta)$ where $C^0(A, M) = M$ and $C^n(A, M) = \text{Hom}_k(A^{\otimes n}, M)$, for $n = 1, 2, \dots$ The differential β is given by:

$$\begin{aligned} \beta(f)(a_1, \dots, a_{n+1}) = a_1 f(a_2, \dots, a_{n+1}) - f(a_1 \cdot a_2, \dots, a_{n+1}) + \dots \\ +(-1)^n f(a_1, \dots, a_n a_{n+1}) + (-1)^{n+1} f(a_1, \dots, a_n) a_{n+1}. \end{aligned}$$

We will define a more elaborate Hochschild cochain complex in Section 3.2.

Proposition 3.1.26. *If M is a graded symmetric A-bimodule, then there exists an isomorphism of complexes*

$$\text{Hom}_k(A^{\otimes n}, M) \longrightarrow \text{Hom}_A(C_n^{\text{naive}}(A), M),$$

where the differential on the left complex is β and the differential on the right complex is the one induced by the differential b on $C_\bullet^{\text{naive}}(A)$. Furthermore, we have an isomorphism of complexes

$$\text{Hom}_R(C_\bullet^{\text{bar}}(A), M) \longrightarrow \text{Hom}_k(A^{\otimes n}, M),$$

sending an $f : C_n^{\text{bar}} \longrightarrow M$ to the mapping $a_1 \otimes \dots \otimes a_n \mapsto f(1 \otimes a_1 \otimes \dots \otimes a_n \otimes 1)$.

3.1.8 Regular sequences

In this section we want to define a regular sequence for the graded commutative context. In our definition the question as to weather a sequence is regular won't depend on the order of its elements. We suppose that the ground ring $\mathbb{K}$ contains the rational numbers.

We work with a good pair of categories $(\mathcal{C}, \mathcal{M})$, equipped with a marking $((F_t)_{t\in T}, (G_u)_{u\in U})$, which induces the marking $((\mathrm{gr}_G(F))_{\tau'\in T'}, (\mathrm{gr}(G))_{u'\in U'})$ on $(\mathrm{gr}(\mathcal{C}), \mathrm{gr}(\mathcal{M}))$ and the marking $((\mathrm{gr}^2_G(F))_{\tau''\in \mathsf{T}''}, (\mathrm{gr}^2)_{u''\in U''})$ on $(\mathrm{gr}^2(\mathcal{C}), \mathrm{gr}^2(\mathcal{M}))$. Assume that Axiom (F3) holds.

Definition 3.1.27. Let R be an algebra in $\mathrm{gr}(\mathcal{C})$. We call a g-finite subset X of R a **handy sequence** if for each x, there is an $u(x) \in U$ such that

$$x \in \mathrm{gr}(G)_{(u(x),g(x))}(R) = G_{u(x)}(R^{g(x)}).$$

When $R = (R, s)$ is a DG algebra, then a handy sequence $X \subseteq$ is called **handy s-sequence** if we have $s(X) \subseteq (X)$. For a handy sequence $X \subseteq R$, let E be a set of free algebra generators, containing for each $x \in X$ a generator $e(x) \in \mathrm{gr}^2_G(F)_{(u(x),g(x),-1)}(R\langle E\rangle)$ of bidegree $(g(x), -1)$. Then we call the free DG algebra $K(X) := R\langle E\rangle$ in $\mathrm{gr}^2(\mathcal{C})$ over R, whose differential (of bidegree $(0, -1)$) is given by $e(x) \mapsto x$, the **Koszul complex** of X over R.

For practical reasons, when we work with a handy sequence $X = \{x_i :\ i \in J\}$, we define an ordering on the index set J, subject to the condition $g(x_i) \leq g(x_j)$, for $i \leq j$. Remark that for a handy sequence $X \subseteq R$ and each subset $Y \subseteq X$, the quotient[7] $R/(Y)$ exists in $\mathrm{gr}(\mathcal{C})$. And if R is a DG algebra (R, s) and X is s-handy, then the quotient $R/(X)$ is also a DG algebra.

Definition and Theorem 3.1.28. *Suppose that $\mathbb{Q} \subseteq \mathbb{K}$.*
Let $X \subseteq R$ be a handy sequence and let I be the ideal $(X) \subseteq R$. Suppose that for each subset $Y \subseteq X$, we have $\cap_{n\geq 1} I^n R/(Y) = 0$. Then X is called a ***regular sequence****, if one of the following equivalent conditions holds:*

(i) Let T be a set of free algebra generators that contains for each $x \in X$, an element $t(x)$ with $g(t(x)) = g(x)$. The map $R/I[T] \longrightarrow \mathrm{gr}_I(R) = R/I \oplus I/I^2 \oplus \ldots$ in $\mathrm{gr}(\mathbb{Q} - \mathfrak{Alg})_{R/I}$, sending $t(x)$ to the class of x in I/I^2 is an isomorphism of (differential) graded R/I-algebras.

(ii) For each $x \in X$ and for each ideal $J \subseteq R$, which is generated by a subset $Y \subseteq X$ with $x \notin Y$, we have: If $g(x)$ is even, then x is not a zero divisor in R/J. If $g(x)$ is odd, then the annulator of x in R/J is just the ideal, generated by the class of x.

(iii) The Koszul complex $K(X)$ is a DG resolvent of $R/(X)$ over R.

(iv) $H^{-1}(K(X)) = 0$.

Proof. The implication (iii)⇒(iv) is trivial.
Proof of (i)⇒(ii) For an element $r \in R$, let $n(r)$ be the greatest n such that r is

[7] By "quotient", we mean the cokernel in $\mathrm{gr}(\mathcal{M})$ of the embedding $(X) \hookrightarrow R$.

contained in I^n and let $\mathrm{in}(r)$ be the class of r in $I^{n(r)}/I^{n(r)+1} \subseteq \mathrm{gr}_I(R)$. For elements $r, r' \in R$, we have that:

$$\mathrm{in}(r) \cdot \mathrm{in}(r') = rr' + I^{n(r)+n(r')+1}. \tag{1.6}$$

Claim: A subset $X \subseteq R$ satisfies condition (ii), if the subset $\{\mathrm{in}(x) : \ x \in X\} \subseteq \mathrm{gr}_I(R)$ satisfies condition (ii).

Proof of the claim: First step: For $x \in X$, if $g(x)$ is even and $\mathrm{in}(x)$ is not a zero divisor, then x is not a zero divisor. If $g(x)$ is odd and the annulator of $\mathrm{in}(x)$ in $\mathrm{gr}_I(R)$ is the ideal, generated by $\mathrm{in}(x)$, then the annulator of x in R is the ideal generated by x.

The even case follows immediately by (1.6). In the odd case, let r be in the annulator of x, i.e $rx = 0$. By (1.6), we get $\mathrm{in}(x) \cdot \mathrm{in}(r) = 0$. By the assumption, there is an $a_1 \in R$ such that $\mathrm{in}(r) = \mathrm{in}(x) \cdot \mathrm{in}(a_1)$. This implies that $r_1 := r - xa_1$ is in $I^{n(r)+1}$ and $n(r_1) \geq n(r) + 1$. Since $x^2 = 0$, we have $r_1 x = rx = 0$, and in the same way we find a $a_2 \in R$ with $r_2 := r_1 - xa_2 \in I^{n(r-1)+1}$. Inductively, for each $m > n(r)$, we find $a_1, \dots, a_k$ such that $r_k := r - x(a_1 + \ldots + a_k) \in I^m$. Thus r belongs to $\cap_{k \geq 0}((x) + I^k)$, which, by the condition $\cap_{n \geq 1} I^n R/(x) = 0$, equals (x).

Second step: For $x \in X$, if weather $g(x)$ is even and $\mathrm{in}(x)$ is not a zero divisor, or $g(x)$ is odd and the annulator of $\mathrm{in}(x)$ in $\mathrm{gr}_I(R)$ is $(\mathrm{in}(x))$, then $(x) \cap I^{n(x)+n} = xI^n$, for each $n \geq 0$.

One inclusion and the even case are easy to see. Suppose that $g(x)$ is odd and that rx is in $I^{n(x)+n}$. We have to find $r' \in I^n$ such that $xr = xr'$. If $r \in I^n$, we are done. Otherwise, we have $n(r) < n$ and $\mathrm{in}(r) \cdot \mathrm{in}(x) = rx + I^{n(r)+n(x)+1} = 0$. So there exists a $y \in R$ such that $\mathrm{in}(r) = \mathrm{in}(x) \cdot \mathrm{in}(y)$. This means that $r_1 := r - xy$ is in $I^{n(r)+1}$ and we have $r_1 x = rx$. Inductively, we find an $r' := r_{n-n(r)}$ such that $r' \in I^n$ and $r'x = rx$.

As consequence, taking $\bar{R} := R/(x)$ and $\bar{I} := I/(x)$, we get an isomorphism

$$\mathrm{gr}_I(R)/(\mathrm{in}(x)) \cong \mathrm{gr}_{\bar{I}}(\bar{R}).$$

We deduce inductively that for $\bar{R} := R/(x_1, \dots, x_s)$ and $\bar{I} := I/(x_1, \dots, x_s)$, we get an isomorphism

$$\mathrm{gr}_I(R)/(\mathrm{in}(x_1), \dots, \mathrm{in}(x_s)) \cong \mathrm{gr}_{\bar{I}}(\bar{R}).$$

Last step: When $g(x)$ is even, we have to show that x is not a zero divisor in $R/(x_1, \dots, x_s)$. We know that $\mathrm{in}(x)$ is not a zero divisor in $\mathrm{gr}_I(R)/(\mathrm{in}(x_1), \dots, \mathrm{in}(x_s))$ $\cong \mathrm{gr}_{\bar{I}}(\bar{R})$. By the first step, the assumption follows. For the odd case, we use the analogue argument. This proves the claim.

If (i) is true, it is clear that $\{\mathrm{in}(x) : \ x \in X\}$, which corresponds to the set T, satisfies condition (ii) and by the claim, X satisfies condition (ii).

Proof of (ii)⇒(iii) We have to show that for $p \leq 0$, the p-th row of the double complex $K(X)$ is a DG resolution in $\mathcal{M}$ over R^0 of the p-th component of $R/(X)$. For this we can suppose that X is finite with $g(x) \geq p$, for all $x \in X$. Say $X = \{x_1, \ldots, x_n\}$. We have that $K(X) = K(x_1) \otimes \ldots \otimes K(x_n)$.

Each $K(X)^{(p,q)}$ is obviously a finite R-module, so by Lemma 3.1.14, we only have to show that $K(X)^{(p,*)}$ is a resolution of $(R/(X))^p$ in the category of R-modules. We show this by induction on n.
For n=1, we write x instead of x_1 and e instead of e_1. Set $m := g(x)$. Remark that if m is even, then the total degree $m-1$ of e is odd, so in this case we have $e^2 = 0$. If m is odd, then the total degree of e is even, so $e^2 \neq 0$. In the first case, $K(x)$ is just the complex

$$\begin{array}{ccccccccc}
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots \\
\downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & \\
0 & \ldots & 0 & R^0e & \ldots & R^{m-1}e & R^me & \ldots \\
\downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & \\
R^0 & \ldots & R^{m+1} & R^m & \ldots & R^{2m-1} & R^{2m} & \ldots \\
\downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & \\
R^0 & \ldots & R^{m+1} & R^m/R^0x & \ldots & R^{2m-1}/R^{m-1}x & R^{2m}/R^mx & \cdots
\end{array}$$

s is injective since x_1 is not a zero divisor in R, hence the rows are exact. In the second case, $K(x)$ is the complex

$$\begin{array}{ccccccccc}
0 & \ldots & 0 & 0 & \ldots & 0 & R^0e^2 & \ldots \\
\downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & \\
0 & \ldots & 0 & R^0e & \ldots & R^{m-1}e & R^me & \ldots \\
\downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & \\
R^0 & \ldots & R^{m+1} & R^m & \ldots & R^{2m-1} & R^{2m} & \ldots \\
\downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & \\
R^0 & \ldots & R^{m+1} & R^m/R^0x & \ldots & R^{2m-1}/R^{m-1}x & R^{2m}/R^mx & \cdots
\end{array}$$

In R^i, for $i < m$, there is no element that annulates x, so up to the row $m-1$, the situation is as above. In the m-th row, the kernel of $R^me \longrightarrow R^{2m}$ is just R^0xe, so it coincides with the image of the map $R^0e^2 \longrightarrow R^me$. Remark that here, we use that 2 is invertible in R. Inductively we see that all rows are exact. Here, we use that all naturals are invertible.

Now suppose that the statement is proved for n. Set $L := K(x_1, \ldots, x_n) = R\langle e_1, \ldots, e_n\rangle$ and $K(X) = K(x_1, \ldots, x_{n+1})$. We write x and e instead of x_{n+1} and e_{n+1}. $K(x)$ is (as object of $\mathrm{gr}^2(\mathcal{M})(R)$) a direct sum $K_0 \oplus K_{-1} \oplus K_{-2} \oplus \ldots$, where in the case where x is even, we have $K_0 \cong R$, $K_{-1} \cong R[-m, 1]$ and $K_s = 0$ for $s < -1$ and in the odd case, we have $K_s = R[sm, -s]$ for all $s \leq 0$. Hence, we have

$K_q^{p,q} \cong R^{p+qm}$, for all p, and in the even case for $-1 \leq q \leq 0$ and in the odd case for $q \leq 0$. The differential on $K(x)$ is given by the maps $d_q : K_q^{*,q} \longrightarrow K_{q+1}^{*,q+1}$, where $d_q^p : K_q^{p,q} \longrightarrow K_{q+1}^{p,q+1}$ is just the multiplication by x.

We have

$$\begin{aligned} K(X)^{p,q} &= (K(x) \otimes_R L)^{p,q} \\ &= (K_0 \otimes_R L)^{p,q} \oplus (K_{-1} \otimes_R L)^{p,q} \oplus \dots \\ &\cong L^{p,q} \oplus L^{p-m,q+1} \oplus \dots, \end{aligned}$$

where in the even case the sum has only two factors. Hence in the even case, for $p \leq 0$, the complex $K(X)^{p,*}$ is the total complex of the double complex

$$\begin{tikzcd}
{} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & \\
L^{p,-2} \arrow[d, "\delta"] & L^{p-m,-2} \arrow[l, "x"] \arrow[d, "\delta"] & 0 \arrow[l] \arrow[d] & {} \arrow[l, dotted] \\
L^{p,-1} \arrow[d, "\delta"] & L^{p-m,-1} \arrow[l, "x"] \arrow[d, "\delta"] & 0 \arrow[l] \arrow[d] & {} \arrow[l, dotted] \\
L^{p,0} & L^{p-m,0} \arrow[l, "x"] & 0 \arrow[l] & {} \arrow[l, dotted]
\end{tikzcd}$$

In the odd case, $K(X)^{p,*}$ is the total complex of the double complex

$$\begin{tikzcd}
{} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & \\
L^{p,-2} \arrow[d, "\delta"] & L^{p-m,-2} \arrow[l, "x"] \arrow[d, "\delta"] & L^{p-2m,-2} \arrow[l, "x"] \arrow[d, "\delta"] & {} \arrow[l, dotted] \\
L^{p,-1} \arrow[d, "\delta"] & L^{p-m,-1} \arrow[l, "x"] \arrow[d, "\delta"] & L^{p-2m,-1} \arrow[l, "x"] \arrow[d, "\delta"] & {} \arrow[l, dotted] \\
L^{p,0} & L^{p-m,0} \arrow[l, "x"] & L^{p-2m,0} \arrow[l, "x"] & {} \arrow[l, dotted]
\end{tikzcd}$$

The first double complex is a DDG resolution in $\mathrm{gr}^2(\mathcal{M})(R^0)$ of the DG module

$$(R/(x_1, \dots, x_n))^p \leftarrow (R/(x_1, \dots, x_n))^p \leftarrow 0 \leftarrow \dots$$

where the left arrow stands for multiplication by x. But this DG module is a resolution of $(R/(x_1, \dots, x_n, x))^p$ over R^0, since $g(x)$ is even. So $K(X)^{p,*}$ is a resolution of $(R/(x_1, \dots, x_n, x))^p$. The second double complex is a DDG resolution in $\mathrm{gr}^2(\mathcal{M})(R^0)$ of the DG module

$$(R/(x_1, \dots, x_n))^p \leftarrow (R/(x_1, \dots, x_n))^p \leftarrow (R/(x_1, \dots, x_n))^p \leftarrow \dots$$

where the arrows stand for multiplication by x. But this DG module is a resolution of $(R/(x_1, \ldots, x_n, x))^p$ over R^0, since $g(x)$ is odd. So $K(X)^{p,*}$ is a resolution of $(R/(x_1, \ldots, x_n, x))^p$. For both cases, the induction step is done.

Proof of (iv)⇒(i) Without restriction, we can suppose that $\mathcal{C}$ is the category of commutative $\mathbb{Q}$-algebras. For each $j \geq 0$, we have to show that the j-th homogeneous component $(R/I[T])_j$ in the T-grading of $R/I[T]$ maps isomorphically to I^j/I^{j+1}.

We will already make use of the implication (ii)⇒(iii). Set $S := \mathbb{Q}[T]$. We consider R as S-algebra via the map $t(x) \mapsto x$. Obviously $T \subseteq S$ satisfies condition (ii), so by (iii), the Koszul complex $K_S(T)$ is a DG resolution of $\mathbb{Q}$ over S.

We consider the exact sequence

$$0 \longrightarrow (T)^j/(T)^{j+1} \longrightarrow S/(T)^{j+1} \longrightarrow S/(T)^j \longrightarrow 0$$

of graded S-modules. The quotient $(T)^j/(T)^{j+1}$ is a free graded g-finite $\mathbb{Q}$-vectorspace, which is a S-module via the canonical map $S \longrightarrow \mathbb{Q}$. We write $\coprod_{i \in J} \mathbb{Q}e_i$ for it. Now $\coprod_{i \in J} K_S(T)e_i$ is a free resolution of $\coprod_{i \in J} \mathbb{Q}e_i$ over S. So we get

$$\mathrm{Tor}_1^S((T)^j/(T)^{j+1}, R) = H^{-1}(\coprod_{i \in J}(K_S(T)e_i \otimes_S R)) = \coprod_{i \in J} H^{-1}(K(X)e_i) = 0.$$

By the property of left derived functors, there is an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Tor}_1^S(S/(T)^{j+1}, R) \longrightarrow \mathrm{Tor}_1^S(S/(T)^j, R) \longrightarrow \\ (T)^j/(T)^{j+1} \otimes_S R \longrightarrow S/(T)^{j+1} \otimes_S R \longrightarrow S/(T)^j \otimes_S R \longrightarrow 0. \end{aligned}$$

By induction on j and the exactness of the first line, we see that $\mathrm{Tor}_1^S(S/(T)^j, R) = 0$ for any $j \geq 0$. The second line gives rise to a short exact sequence

$$0 \longrightarrow (R/I[T])_j \longrightarrow R/I^j \longrightarrow R/I^{j+1} \longrightarrow 0,$$

which implies the desired isomorphism. □

Remark 3.1.29. The assumption $\mathbb{Q} \subseteq \mathbb{K}$ is used only to prove the implications (ii)⇒(iii) and (iv)⇒(i). The assumption that for each subset $Y \subseteq X$ we have $\cap_{n \geq 1} I^n R/(Y) = 0$ is used only to prove (i)⇒(ii). So if you want to get rid of it, use condition (ii) for the definition of regular sequences. It can be stated in a slightly modified manner which depends on the order of the elements of X.

Definition 3.1.30. Let R be a DG algebra in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$. Let $(\alpha_i, u_i, g_i)_{i \in J}$ be a family in $\mathcal{N} \times U'$ and $X = \{x_i : \ i \in J\}$ a family of elements with $x_i \in G_{u_i}(R_{\alpha_i}^{g_i})$ such that, for $\beta, \beta' \subseteq \alpha$, the sets $\{\rho_{\beta\alpha}(x_i) : \ \alpha_i = \beta\}$ and $\{\rho_{\beta'\alpha}(x_i) : \ \alpha_i = \beta'\}$ are disjoint. Suppose that

$$X_\alpha := \cup_{\beta \subseteq \alpha} \{\rho_{\beta\alpha}(x_i) : \ \alpha_i = \beta\}$$

is a regular (resp. handy) (s_α-)sequence in R_α, for each α. Then X is called a **regular (s-)sequence** (resp. handy (s-)sequence) in R.

Corollary 3.1.31. *If $R = (R, s)$ is a DG algebra in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ and X a handy s-sequence in R, then $K(X)$ is a DG algebra in $\mathrm{gr}^2(\mathcal{C})^{\mathcal{N}}$ and if X is regular, then $K(X)$ is a DG resolution of $R/(X)$ over R.*

Remark 3.1.32. When R carries the structure of a DG algebra (R, s), one would like the Koszul complex to carry the structure of a DDG module. In general, this is not the case.
If X is an s-handy sequence then, since $I = (X)$ is s-stable, then the algebra $\mathrm{gr}_I(R)$ has the structure of a DG algebra in $\mathrm{gr}(\mathfrak{Alg})$ such that each I^n/I^{n+1} is a DG submodule of $\mathrm{gr}_I(R)$. If, for example, R is already a free DG algebra in $\mathrm{gr}(\mathbb{Q}-\mathfrak{Alg})$ with a set X of free algebra generators, i.e. $R = R/I[X]$, then the differential of $\mathrm{gr}_I(R) = R$ differs in general from the differential s. In this way we get a modified differential $\tilde{s}$ on R. In a similar way we get a modified differential, when R is a free DG algebra in $\mathrm{gr}(\mathcal{C})$, for any good pair of categories $(\mathcal{C}, \mathcal{M})$. In geometric language, going over from s to $\tilde{s}$ is a deformation to the normal cone.
The interest of this deformation of s comes from the fact that the Koszul construction of R over X is compatible with the differential $\tilde{s}$, i.e. $K(X)$ is a DDG resolvent of R/X over $(R, \tilde{s})$. This can be of interest for the construction of the Hochschild complex.

3.1.9 The universal module of differentials

Fix an admissible pair of categories $(\mathcal{C}, \mathcal{M})$. Let $k \longrightarrow A$ be a morphism of DG algebras in $\mathrm{gr}(\mathcal{C})$.

As in paragraph I.6.12 of [2], we define the **universal module $\Omega_{A/k}$ of k-differentials** as the first homology of the complex $C_\bullet^{\mathrm{naive}}(A)$, i.e. the cokernel in $\mathrm{gr}(\mathcal{M})(A)$ of the map $b_2 : A \otimes_k A \otimes_k A \longrightarrow A \otimes_k A$, sending $a \otimes b \otimes c$ to $ab \otimes c - a \otimes bc + (-1)^{bc} ac \otimes b$, for homogeneous elements $a, b, c \in A$. $\Omega_{A/k}$ is a DG module over A and there is an A-derivation $d : A \longrightarrow \Omega_{A/k}$ (i.e. a homomorphism of DG k-modules, which is a derivation), sending elements $a \in A$ to the class of $a \otimes 1$. $\Omega_{A/k}$ is universal in the sense that, for each A-module M in $\mathrm{gr}(\mathcal{M})$, the natural map $\mathrm{Hom}_{\mathrm{gr}(\mathcal{M})(A)}(\Omega_{A/k}, M) \longrightarrow \mathrm{Der}_k(A, M)$ is bijective.

Set $R := A \otimes_k^{\mathcal{C}} A$ and denote the kernel of the multiplication map $\mu : R \longrightarrow A$ in the category $\mathrm{gr}(\mathcal{M})(R)$ by I. (Attention: In general A is the cokernel of the inclusion $I \hookrightarrow R$ only in the category R-$\mathfrak{Mod}$.) A natural question is if $\Omega_{A/k}$ is isomorphic to the "quotient" I/I^2. But we already need several assumptions for the existence of I/I^2 in the category $\mathrm{gr}(\mathcal{M})(A)$. An answer which is sufficient for our purpose is given by the following proposition. For our examples, the statement is well-known. The proof for the general case can be found in the appendix.

Proposition 3.1.33. *Suppose that $(\mathcal{C}, \mathcal{M})$ is a good pair of categories satisfying Axiom (S1) and that the marking G on $\mathcal{M}$ is canonical. Suppose that $A \in \mathrm{gr}'(\mathcal{C})$ (i.e. all A^i are finite A^0-modules) and that I is a g-finite R-module. Then we have $A = R/I := \mathrm{Cokern}(I \hookrightarrow R)$ and I/I^2 is in a natural way a module in $\mathrm{gr}(\mathcal{M})(A)$.*

Furthermore, the map $A \longrightarrow I/I^2$, sending $a \in A$ to the class of $a \otimes 1 - 1 \otimes a$ is a k-derivation and the quotient I/I^2 is a universal module of derivations. In particular $I/I^2 \cong \Omega_{A/k}$.

Denote the differential of $R = A \otimes_k A$ by s. The next Proposition is a consequence of Proposition 3.1.33 and Definition and Theorem 3.1.28:

Proposition 3.1.34. *Take the assumptions of Proposition 3.1.33. Suppose that the ideal $I \subseteq R$ is generated by an s-regular sequence $X \subseteq R$. Then $\Omega_{A/k}$ is a free DG A-module in $\mathrm{gr}(\mathcal{M})$, generated by a set $E = \{e(x) | \; x \in X\}$, containing one free module generator, for each element $x \in X$.*

The definitions and statements of this subsection carry over directly to $\mathcal{N}$-objects in $\mathrm{gr}(\mathcal{C})$ and $\mathrm{gr}(\mathcal{M})$.

3.1.10 A HKR-type theorem for DG algebras

Use a good pair of categories $(\mathcal{C}, \mathcal{M})$, satisfying Axioms (S1) and (F3) such that the marking F on $\mathcal{C}$ is balanced and convex. Again, suppose that the ground ring $\mathbb{K}$ contains $\mathbb{Q}$. Let k be an algebra in $\mathcal{C}^{\mathcal{N}}$. Suppose that the marking G on $\mathcal{M}$ is trivial. Recall that this implies that each free algebra in $\mathrm{gr}'(\mathcal{C})^{\mathcal{N}}$ with free algebra generators only of strictly negative degree is a direct sum of A and a free A-module.

Consider a free g-finite DG algebra $A = k\langle x_i, i \in I\rangle$ in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$. By Proposition 3.1.34, as graded module, we have $\Omega_A := \Omega_{A/k} = \coprod_{i \in I} Ae_i$, where e_i is a free module generator of degree $g(x_i)$. Write d for the canonical derivation $A \to \Omega_A$ and denote as usual the composition of d with the projection $\Omega_A \longrightarrow Ae_i$ by $\partial/\partial x_i$. For $n \in \mathbb{N}$, we can define A-linear maps

$$\begin{aligned} \epsilon_n : \wedge^n \Omega_A &\longrightarrow C_n^{\mathrm{naive}}(A) \\ e_{i_1} \wedge \ldots \wedge e_{i_n} &\mapsto \sum_{\sigma \in \Sigma_n} \chi(\sigma) \cdot 1 \otimes x_{i_{\sigma(1)}} \otimes \ldots \otimes x_{i_{\sigma(n)}}. \end{aligned}$$

(for the exterior powers, see Definition 3.1.24). Here, $\chi(\sigma) = \chi(\sigma, e_{i_1}, \ldots, e_{i_n})$ is a sign satisfying

$$e_{i_1} \wedge \ldots \wedge e_{i_n} = \chi(\sigma) \cdot e_{i_{\sigma(1)}} \wedge \ldots \wedge e_{i_{\sigma(n)}}.$$

Verify (or see [29]) that the image of ϵ_n is in the kernel of the differential of the naive Hochschild complex, i.e. we have a map of complexes $\epsilon : \wedge^\bullet \Omega_A \longrightarrow C_\bullet^{\mathrm{naive}}(A)$ of differential graded A-modules, where the vertical differential on $\wedge^\bullet \Omega_A$ is set to be trivial. On the other hand, there is a morphism

$$\alpha : C_\bullet^{\mathrm{naive}}(A) \longrightarrow \wedge^\bullet \Omega_A$$

which is, on each simplex in $\mathcal{N}$, in the n-th component given by

$$\alpha_n(a_0 \otimes \ldots \otimes a_n) = \frac{1}{n!} a_0 \cdot da_1 \wedge \ldots \wedge da_n = \frac{1}{n!} a_0 \cdot \sum_{i_1, \ldots, i_n} \pm \frac{\partial a_1}{\partial x_{i_1}} \cdots \frac{\partial a_n}{\partial x_{i_n}} e_{i_1} \cdots e_{i_n},$$

where the sign is obtained using the Koszul-convention. Remark that $\alpha \circ \epsilon = \mathrm{Id}$. The following theorem is a DG-generalization of the classical Hochschild - Kostant - Rosenberg Theorem as stated for example in Loday's textbook [29]. It is the central result of this chapter.

Theorem 3.1.35. *The map ϵ (and consequently α) is a quasi-isomorphism of complexes of DG-modules in $\mathrm{gr}(\mathcal{M})^{\mathcal{N}}$ over A. If A is concentrated in degree zero, there is no need to assume that $\mathbb{Q} \subseteq k$.*

Proof. The proof is Loday's sketch of proof for the DG situation together with Definition and Theorem 3.1.28. It works as follows: The maps ϵ and α are morphisms of double complexes. Since we are only interested in the vertical differential, we can forget about the differential on each $\wedge^j \Omega_A$ and $A^{\otimes j+1}$. This is necessary, since we make use of the Koszul construction (see Definition and Theorem 3.1.28), which is not compatible with horizontal differentials.

Set M to be the free, graded k-module $\coprod_{i \in I} ke_i$. Set R to be the graded algebra $A \otimes_k A$. We can identify the Koszul complex $K(X)$ over R of the regular sequence $\{\frac{1}{2}(x_i \otimes 1 - 1 \otimes x_i) |\ i \in I\} \subseteq R$ (since the marking is balanced and convex, this is a set of free generators of a free subalgebra of R, hence it is regular) with the DG algebra $A \otimes_k A \otimes \wedge^\bullet M$ in $\mathrm{gr}^2(\mathcal{C})$. The (vertical) differential of the latter is defined by $e_i \mapsto \frac{1}{2}(1 \otimes x_i - x_i \otimes 1)$. We have a map of complexes

$$\begin{aligned} A \otimes_k A \otimes \wedge^\bullet M &\longrightarrow C_\bullet^{\mathrm{bar}}(A) \\ a \otimes a' \otimes e_{i_1} \wedge \ldots \wedge e_{i_n} &\mapsto (-1)^{a'(e_{i_1}+\ldots+e_{i_n})} \sum_{\sigma \in \Sigma_n} \chi(\sigma) a \otimes x_{i_{\sigma(1)}} \otimes \ldots \otimes x_{i_{\sigma(n)}} \otimes a' \end{aligned}$$

of graded R-modules. By Definition and Theorem 3.1.28 and Axiom (F3), the left complex is a flat resolutions of A over R. By Section 3.1.7, the right complex is a flat resolution of A over R. Applying the functor $\cdot \otimes_R A$, we just get the map ϵ:

$$\wedge^\bullet \Omega_A \cong K(X) \otimes_R A \longrightarrow C_\bullet^{\mathrm{naive}}(A)$$

By flatness, this is still a quasi-isomorphism. $\square$

Example 3.1.36. Let $A = \mathbb{C}\{x_1, \ldots, x_n\}$ be a convergent power series algebra. The m-fold complete tensor power $A^{\hat{\otimes} m+1}$ can be identified with the convergent power series algebra

$$\mathbb{C}\{x_1^{(0)}, \ldots, x_n^{(0)}, \ldots, x_1^{(m)}, \ldots, x_n^{(m)}\}$$

in $(m+1) \cdot n$ variables. The m-th component of the quasi-isomorphism $\alpha : C^{\mathrm{naive}}(A) \longrightarrow \wedge^\bullet \Omega_A$ can be written as

$$\alpha_m(f) = \sum_{i_1 < \ldots < i_m} \frac{\partial^m f}{\partial x_{i_1}^{(1)} \ldots \partial x_{i_m}^{(m)}}(x, \ldots, x) dx_{i_1} \wedge \ldots \wedge dx_{i_m}.$$

3.2 Hochschild complex and Hochschild cohomology

3.2.1 Homotopy invariance of the Hochschild complex

In the (algebraic) literature, for an algebra homomorphism $k \longrightarrow a$, the Hochschild complex of a over k is defined as our naive Hochschild complex $C^{\text{naive}}(a)$. However, as observed by Buchweitz and Flenner, there are reasons to define the Hochschild complex in the case where a and k are analytic algebras in a different manner. In this case, if a is flat over k, then the bar complex $C^{\text{bar}}(a)$ is still a complex of flat a-modules but even if k is just the field $\mathbb{C}$, the bar complex $C^{\text{bar}}(a)$ is not a complex of projective a-modules. Thus, for the definition of Hochschild homology, the naive Hochschild complex would do, but for the definition of Hochschild cohomology as cohomology of the a-dual of the Hochschild complex, it is not a good choice. Thus, for a morphism $k \longrightarrow a$ of algebras in good pairs of categories, we will give a different definition of the Hochschild complex $\mathbb{H}(a/k)$ of a over k. We will see that in the flat case it coincides up to a quasi-isomorphism with the naive Hochschild complex.

The definitions of this section were inspired by the article [7]. For simplicity, we restrict ourselves to the Noetherian context. Fix a simplicial complex $\mathcal{N}$ and a good pair of categories $(\mathcal{C}, \mathcal{M})$ with marking (F, G), where G is the canonical marking of $\mathcal{M}$. Suppose that the Axioms (N) and (F2) are satisfied.

Using a Čech construction (for more details, see paragraph I.10.1 of [2]), we get a functor $\check{C}^\bullet : \text{gr}(\mathcal{M})^{\mathcal{N}} \longrightarrow \text{gr}(\mathbb{K} - \mathfrak{Mod})$, sending a DG module $M \in \text{gr}(\mathcal{M})^{\mathcal{N}}$ to the total complex $\text{tot}^{\prod}(C^\bullet(M^\bullet))$ of the double complex

$$\begin{tikzcd}
\vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] & \\
C^0(M^{-1}) \arrow[r] \arrow[d] & C^1(M^{-1}) \arrow[r, dotted] \arrow[d] & {} \\
C^0(M^0) \arrow[r] & C^1(M^0) \arrow[r, dotted] & {}
\end{tikzcd}$$

where $C^p(M^q) = \coprod_{|\alpha|=p} M^q$ and the differentials as usual. This functor sends quasi-isomorphisms to quasi-isomorphisms. We will write $\check{H}^n(M)$ for $H^n(\check{C}^\bullet(M))$.

Let $k \longrightarrow a$ be a finite morphism of $\mathcal{N}$-objects in $\mathcal{C}$, i.e. a is a quotient of a free k-algebra b in $\mathcal{C}^{\mathcal{N}}$ such that, for each $\alpha \in \mathcal{N}$, the algebra b_α is a a free finite k_α-algebra. (More generally, we may assume that $k \longrightarrow a$ is a morphism in $\text{gr}(\mathcal{C})^{\mathcal{N}}$, as long as there exists a g-finite resolvent of a over k.) By Proposition 8.8 of [2], there exists a g-finite resolvent of a over k. Fix such a resolvent A. Set $R := A \otimes_k^{\text{gr}(\mathcal{C})} A$ and consider A as an algebra over R by the multiplication map $\mu : R \longrightarrow A$. Let S be a free g-finite resolvent[8] of A over R.

[8] Again by [2], such a resolvent exists. We can even construct it in such a way that $S^0 = R^0$.

Definition 3.2.1. The **simplicial Hochschild complex** $\mathbb{H}_*(a/k)$ of a over k is the object represented by the complex $S \otimes_R a$ in the homotopy category $K^-(\mathcal{M}^{\mathcal{N}}(a))$. The **Hochschild complex** $\mathbb{H}(a/k)$ is the object represented by the complex $\check{C}^\bullet(\mathbb{H}_*(a/k))$ in the derived category $D(\mathbb{K} - Mod)$.

Proposition 3.2.2. *The simplicial Hochschild complex $\mathbb{H}_*(a/k)$ is a well defined object in $K^-(\mathcal{M}^{\mathcal{N}}(a))$. More precisely, for two choices A_i, S_i of resolvents $(i = 1, 2)$, there exists a morphism $S_1 \otimes_{R_1} a \longrightarrow S_2 \otimes_{R_2} a$ in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}(a)$, which is a homotopy equivalence. Consequently, $\mathbb{H}(a/k)$ is a well-defined object in $D(\mathbb{K} - Mod)$.*

Proof. For $i = 1, 2$, let A_i be a g-finite resolvent of a over k, $R_i := A_i \otimes_k A_i$ and let S_i be a g-finite resolvent of A_i over R_i. We have to find a homotopy equivalence $S_1 \otimes_{R_1} a \simeq S_2 \otimes_{R_2} a$ over a. By Proposition 3.1.16, there is a homomorphism $A_1 \longrightarrow A_2$ in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ which is a homotopy equivalence over k. Hence,

$$R_1 \approx A_1 \otimes_k a \simeq A_2 \otimes_k a \approx R_2.$$

By Proposition 3.1.16, the quasi-isomorphism $R_1 \approx R_2$ is even a homotopy equivalence over k. Thus we get a quasi-isomorphism

$$S_1 \cong S_1 \otimes_{R_1} R_1 \longrightarrow S_1 \otimes_{R_1} R_2$$

over R_1. Both S_2 and $S_1' := S_1 \otimes_{R_1} R_2$ are resolvents of a over R_2. Hence, there is a homomorphism $S_1' \longrightarrow S_2$ in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$, which is a homotopy equivalence over R_2. We can tensor both sides over R_2 with a and still get a homotopy equivalence $S_1 \otimes_{R_1} a \longrightarrow S_2 \otimes_{R_2} a$. □

Recall that the notion $C^{\mathrm{bar}}(a) = C^{\mathrm{bar}}(a)^\bullet$ stands for the complex $C^{\mathrm{bar}}_{-\bullet}(a)$.

Proposition 3.2.3. *Suppose that a is flat over k. Then there is an isomorphism $\mathbb{H}_*(a/k) \longrightarrow C^{\mathrm{naive}}(a)$ in the derived category $D(a - \mathfrak{Mod})$. More precisely, for each representation $S \otimes_R a$ of $\mathbb{H}_*(a)$, there exists a morphism $S \otimes_R a \longrightarrow C^{\mathrm{naive}}(a)$ in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$, which is a quasi-isomorphism over a.*

Proof. Set $r := a \otimes_k a$. Since a is flat over k, R is a resolvent of r and $C^{\mathrm{bar}}(a)$ a flat resolution of a over r. Another one is $s := S \otimes_R r$. By Theorem I.8.4 of [2], there is a morphism $s \longrightarrow C^{\mathrm{bar}}(a)$ in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ over r. By flatness, we get a quasi-isomorphism

$$C^{\mathrm{naive}}(a) \cong C^{\mathrm{bar}}(a) \otimes_r a \cong s \otimes_r a \cong S \otimes_R a.$$

□

Definition 3.2.4. We define the **n-th Hochschild homology** of a over k as $\check{H}^{-n}(\mathbb{H}_*(a/k))$.

Definition 3.2.5. Let M be an object of $\mathcal{M}^{\mathcal{N}}$ over a. We define the **Hochschild cochain complex** of a over k with values in M to be the complex

$$\mathrm{Hom}_a^{\mathcal{N}}(\mathbb{H}_*(a/k), M),$$

with the differential induced by the differential of $\mathbb{H}_*(a/k)$. We define the **Hochschild cohomology** $\mathrm{HH}(a/k, M)$ of a over k with values in M to be the cohomology of the Hochschild cochain complex.

Proposition 3.2.6. *The Hochschild cochain complex is well defined up to homotopy equivalence.*

Proof. This is a consequence of Lemma I.3.7 of [2] and Proposition 3.2.2. □

3.2.2 A Quillen-type theorem

We work over a good pair of categories with the same properties as in Section 3.1.10.

Recall that for an algebra a over k in $\mathcal{C}^{\mathcal{N}}$ with resolvent A in $\mathrm{gr}'(\mathcal{C})^{\mathcal{N}}$, the **cotangent complex** $\mathbb{L}(a/k)$ of a over k is defined as the class of $\Omega_A \otimes_A a$ in the homotopy category $K(\mathcal{M}^{\mathcal{N}}(a))$. By Theorem III.2.4 of [2], the homotopy class does not depend on the resolvent A.

Theorem 3.2.7. *Consider a homomorphism $k \longrightarrow a$ in $\mathcal{C}^{\mathcal{N}}$. There exists a homotopy equivalence*

$$\wedge\mathbb{L}_{a/k} \longrightarrow \mathbb{H}(a/k)$$

in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ over a. More precisely, for any choice of resolvents A and S as in Section 3.2.1, there is a morphism

$$S \otimes_R a \longrightarrow \wedge\Omega_A \otimes_A a$$

in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ which is a homotopy equivalence over a.

Proof. The total complex of $C^{\mathrm{bar}}(A)^{\bullet} = C^{\mathrm{bar}}_{-\bullet}(A)$ is a DG algebra and a resolution of A over R. By Theorem I.8.4 of [2], we get a morphism $S \longrightarrow \mathrm{tot}\, C^{\mathrm{bar}}(A)$ in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ over R such that the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & \mathrm{tot}\, C^{\mathrm{bar}}(A) \arrow[d] \\ & A \end{tikzcd}$$

commutes. By flatness of S and $C^{\mathrm{bar}}(A)$, after tensoring over R with A, we get a quasi-isomorphism

$$S \otimes_R A \longrightarrow \mathrm{tot}(C^{\mathrm{naive}}(A))$$

in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$. Applying Theorem 3.1.35, we get a quasi-isomorphism

$$S \otimes_R A \longrightarrow \wedge\Omega_A$$

of free DG-algebras in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ over A. By Proposition 3.1.16, this quasi-isomorphism is already a homotopy equivalence. Tensoring over A with a gives the desired result. □

Remark that Quillen's proof does not work in our situation, since the category $\mathcal{M}$ is not, in general, Abelian.

Corollary 3.2.8. *If a is already free over k (in this case there is no need to assume that $\mathbb{Q} \subseteq a$) and $A = a$, then $\Omega_{a/k}$ is an object of $\mathcal{C}_a^{\mathcal{N}}$ and we get isomorphisms*

$$\wedge_a^n \Omega_{a/k} \cong H_n(\mathbb{H}(a/k)).$$

Dually, with $T_{A/k} := \mathrm{Hom}_A(\Omega_{A/k}, A)$ we get

$$H^n(\mathrm{Hom}_a(\mathbb{H}(a/k), a)) \cong \wedge^n T_{a/k}.$$

3.2.3 Decomposition of Hochschild (co-) homology

Let M be a module in $\mathcal{M}^{\mathcal{N}}(a)$.

Theorem 3.2.9. *We have the following decomposition of Hochschild (co)homology:*

$$\mathrm{HH}_n(a/k) \cong \prod_{i-j=n} \check{H}^i(\wedge^j \mathbb{L}(a/k)), \tag{2.7}$$

$$\mathrm{HH}^n(a/k, M) \cong \prod_{i+j=n} H^i(\mathrm{Hom}_a(\wedge_a^j \mathbb{L}(a/k), M)). \tag{2.8}$$

Proof. The first isomorphism is a direct consequence of Theorem 3.2.7. We show the second one:

$$\begin{aligned}
\mathrm{HH}^n(a/k, M) &= H^n(\mathrm{Hom}_a(\mathbb{H}(a/k), M)) \cong H^n(\mathrm{Hom}_a(\wedge\mathbb{L}(a/k), M)) \\
&\cong H^n(\mathrm{Hom}_A(\mathrm{tot}(\wedge^{\cdot} \Omega_{A/k}), M)) \cong H^n(\mathrm{Hom}_A(\coprod_{j\geq 0} \wedge_A^j \Omega_{A/k}[j], M)) \\
&\cong H^n(\prod_j \mathrm{Hom}_A(\wedge_A^j \Omega_{A/k}[j], M)) \cong \prod_{j\geq 0} H^{n-j}(\mathrm{Hom}_A(\wedge^j \Omega_A, M)) \\
&\cong \prod_{i+j=n} H^i(\mathrm{Hom}_A(\wedge_A^j \Omega_{A/k}, M)).
\end{aligned}$$

The first equality holds by definition. The second one follows by Theorem 3.2.7. The other equalities are elementary. □

Remark that tangent cohomology is a direct factor on the right hand-side of isomorphism (2.8).

3.3 Application to complex spaces and varieties

In this section, all schemes and complex spaces are supposed to be paracompact and separated. For more details on many of the constructions, we refer to [6] and [7].

First, we will sketch the correlation between the theory of coherent sheaves on schemes or complex spaces and the theory of $\mathcal{N}$-objects in good pairs of categories. The main tools that we need here are:

(1) Instead of considering a space X, we consider the simplicial scheme, associated to an affine covering of X. By an affine subspace, we mean an open affine subscheme in the case of schemes and a Stein compact[9] in the case of complex spaces. There are functors that make simplicial modules out of sheaves of modules and functors doing the inverse.

(2) Let X be a complex space or a Noetherian scheme. For affine subsets $U \subseteq X$, we use the equivalence of categories of coherent $\mathcal{O}_U$-modules and finite modules over the ring $\Gamma(U, \mathcal{O}_X)$. (Remember that $\Gamma(U, \mathcal{O}_X)$ is Noetherian, when X is a complex space.) This equivalence is given by Cartan's theorem A in the analytic case and by Exercise II.2.4 of [16] in the algebraic case.

More generally, let X be a ringed space and $(X_i)_{i \in I}$ a locally finite covering of X. The **nerve** $\mathcal{N}$ of this covering is the set of all subsets $\alpha \subseteq I$ such that $\cap_{i\in\alpha} X_i \neq \emptyset$. This set $\mathcal{N}$ is a simplicial complex in the sense of Section 3.1.4. Further, there is a contravariant functor from $\mathcal{N}$ in the category of ringed spaces, mapping an object α to the object $\mathcal{X}_\alpha := \cap_{i\in\alpha} X_i$. For $\alpha \subseteq \beta$, denote the inclusion $X_\beta \longrightarrow X_\alpha$ by $p_{\alpha\beta}$. Such a functor is called **simplicial complex of ringed spaces**. Let $X_* = (X_\alpha)_{\alpha\in\mathcal{N}}$ be a simplicial complex of ringed spaces. Following [9], we define the category of $\mathcal{O}_{X_*}$-modules as follows: Its objects are families $\mathcal{F}_* = (\mathcal{F}_\alpha)_{\alpha\in\mathcal{N}}$ with $\mathcal{F}_\alpha$ in $\mathcal{M}od(X_\alpha)$, together with compatible maps $p^*_{\alpha\beta}\mathcal{F}_\alpha \longrightarrow \mathcal{F}_\beta$. For $\mathcal{O}_{X_*}$-modules $\mathcal{F}, \mathcal{G}$, we set $\mathrm{Hom}_{X_*}(\mathcal{F}, \mathcal{G})$ to be the set of compatible families $f_\alpha : \mathcal{F}_\alpha \longrightarrow \mathcal{G}_\alpha$. We denote this category by $\mathcal{M}od(X_*)$. The full subcategory of those $\mathcal{F}_*$, where each $\mathcal{F}_\alpha$ is coherent is denoted by $\mathcal{C}oh(X_*)$.

Definition 3.3.1. Let A and B be simplicial complexes over the index sets A_0 and B_0. Suppose that $X_* = (X_\alpha)_{\alpha\in A}$ and $Y_* = (Y_\beta)_{\beta\in B}$ are simplicial complexs of ringed spaces. A **morphism** $f : X_* \longrightarrow Y_*$ consists of a mapping $\tau : A_0 \longrightarrow B_0$ such that for $\alpha \in A$, we get $\tau(\alpha) \in B$, and a family of compatible maps $f_\alpha : X_\alpha \longrightarrow Y_{\tau(\alpha)}$.

As in [9], we can form the adjoint functors

$$f^* : \mathcal{M}od(Y_*) \longrightarrow \mathcal{M}od(X_*) \quad \text{and}$$
$$f_* : \mathcal{M}od(X_*) \longrightarrow \mathcal{M}od(Y_*).$$

For $\mathcal{F}$ in $\mathcal{M}od(Y_*)$ and $\alpha \in A$, we have $(f^*\mathcal{F})_\alpha := f^*_\alpha \mathcal{F}_{\tau(\alpha)}$. The construction of f_* is more complicated. For the general case, we refer to [9]. We need only a particular case which is explained below.

[9] Remember that a Stein compact is a compact subset of a complex space, having a base of neighborhoods, consisting of Stein spaces. A Stein compact is only a pseudocomplex space.

Examples 3.3.2. (1) If X is a Noetherian scheme or a complex space and $(X_i)_{i\in I}$ is a covering by affine subspaces, then by the separated condition, all X_α are affine. Now let $(\mathcal{C},\mathcal{M})$ be the good pair $(\mathcal{C}^{(0)},\mathcal{M}^{(0)})$ or $(\mathcal{C}^{(1)},\mathcal{M}^{(1)})$ (see Example 3.1.1). Then, $a_* := (\Gamma(X_\alpha,\mathcal{O}_{X_\alpha}))_{\alpha\in\mathcal{N}}$ is an $\mathcal{N}$-object in $\mathcal{C}$ and there is a 1:1-correspondence between the objects of $\mathcal{C}oh(X_*)$ and the $\mathcal{N}$-objects M_* in $\mathcal{M}$ over a_* such that each M_α is finite over a_α.

(2) If X is a complex space and the covering $(X_i)_{i\in I}$ is locally finite and chosen in such a way that each X_i admits a closed embedding into a polydisc P_α, then we get another simplicial complex of Stein compacts: Set $P_\alpha := \prod_{i\in\alpha} P_i$. For $\alpha\subseteq\beta$, we have the projection $P_\beta \longrightarrow P_\alpha$. This makes $P_* = (P_\alpha)_{\alpha\in\mathcal{N}}$ a simplicial complex of Stein compacts and there is a closed embedding $X_* \longrightarrow P_*$.

(3) Let X be a scheme of finite type over a ring $\mathbb{K}$ and $(X_i)_{i\in I}$ an open affine covering of X. We can construct a new simplicial scheme: Set $a_\alpha := \Gamma(X_\alpha,\mathcal{O}_{\mathcal{X}_\alpha})$, for $\alpha\in\mathcal{N}$. For each α, there is a free, finitely generated algebra $\mathbb{K}[T]$ that maps surjectively onto a_α. We get a closed embedding $X_\alpha \longrightarrow \mathrm{Spec}(\mathbb{K}[X]) =: P_\alpha$. As above, we get a simplicial complex P_* and a closed embedding $X_* \longrightarrow P_*$.

The inclusions $j_\alpha : X_\alpha \longrightarrow X$ give rise to a map $j : X_* \longrightarrow X$ of simplicial complexs of ringed spaces. Next, we will study the adjoint functors j_* and j^*:
j^* is just the exact functor, mapping an $\mathcal{O}_X$-module $\mathcal{F}$ to the $\mathcal{O}_{X_*}$-module $(\mathcal{F}|_{X_\alpha})_{\alpha\in\mathcal{N}}$. To describe j_*, we consider the Čech -functor: For an $\mathcal{O}_{X_*}$-module $\mathcal{F}_*$, set

$$\check{C}^p(\mathcal{F}_*) := \prod_{|\alpha|=p} j_{\alpha *}\mathcal{F}_\alpha$$

and define a differential on $\check{C}^\bullet(\mathcal{F}_*)$ in the usual sense. Then, $j_*\mathcal{F}_*$ is just $H^0(\check{C}^\bullet(\mathcal{F}_*))$.
j_*j^* is the identity functor. One can prove the adjointness of j^* and j_* directly by a gluing argument. Since j^* is an exact functor and j_* is right adjoint to j^*, we see that j_* transforms injective objects in $\mathcal{M}od(X_*)$ into injective objects in $\mathcal{M}od(X)$.

For each $\alpha\in\mathcal{N}$, we define a functor $p_{\alpha *} : \mathfrak{Mod}(X_\alpha) \longrightarrow \mathfrak{Mod}(X_*)$ via

$$(p_{\alpha *}\mathcal{F}_\alpha)_\beta := \begin{cases} p_{\beta\alpha_*}\mathcal{F}_\alpha & \text{for} \quad \beta\subseteq\alpha \\ 0 & \text{for} \quad \text{all other cases} \end{cases}$$

By Proposition 2.26 of [6], each $\mathcal{O}_{X_*}$-module admits an injective resolution by modules of the form $\prod_{\alpha\in\mathcal{N}} p_{\alpha *}\mathcal{I}_\alpha$ with injective $\mathcal{O}_{X_\alpha}$-modules $\mathcal{I}_\alpha$. We will use the following properties of the functor $\check{C}^\bullet$:

Lemma 3.3.3. *(1) For $p\geq 0$, the functor $\check{C}^p$ is exact.*

(2) If $\mathcal{F}_\alpha$ is an $\mathcal{O}_{X_\alpha}$-module, then $\check{C}^\bullet(p_{\alpha *}\mathcal{F}_\alpha)$ is a resolution of $j_*(p_{\alpha *}\mathcal{F}_\alpha)$.

(3) If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\check{C}^\bullet(j^*\mathcal{F})$ is a resolution of $\mathcal{F}$.

We generalize a part of Proposition 2.28 of [6] for the case where X is only assumed to be a ringed space and X_* is the simplicial complex of ringed spaces associated to an open or closed covering $(X_i)_{i\in I}$ of X:

Proposition 3.3.4. *The functor $j^* : D(X) \longrightarrow D(X_*)$ embeds $D(X)$ as a full and exact subcategory into $D(X_*)$ and $\check{C}^\bullet = Rj_*$ is an exact right adjoint. In particular, for $\mathcal{F}, \mathcal{G} \in D(X)$ and $\mathcal{M}_* \in D(X_*)$, there are functorial isomorphisms*

$$\mathrm{Ext}^k_X(\mathcal{F},\mathcal{G}) \cong \mathrm{Ext}^k_{X_*}(j^*\mathcal{F}, j^*\mathcal{G}) \quad \textit{and}$$
$$\mathrm{Ext}^k_{X_*}(j^*\mathcal{F},\mathcal{M}_*) \cong \mathrm{Ext}^k_X(\mathcal{F},\check{C}^\bullet(\mathcal{M})).$$

If all the maps $p^*_{\alpha\beta}(\mathcal{M}_\alpha) \longrightarrow \mathcal{M}_\beta$, for $\alpha \subseteq \beta$ in $\mathcal{N}$, are quasi-isomorphisms, then the natural map

$$j^*\check{C}^\bullet(\mathcal{M}_*) \longrightarrow \mathcal{M}_*$$

is a quasi-isomorphism, and in consequence, for all n, there are isomorphisms

$$\mathrm{Ext}^n_{X_*}(\mathcal{M}_*, j^*\mathcal{F}) \cong \mathrm{Ext}^n_X(\check{C}^\bullet(\mathcal{M}_*),\mathcal{F}).$$

Proof. For the proof that $\check{C}^\bullet$ is the right derived functor of j_*, we use an injective resolution $\mathcal{I}_*$ of an $\mathcal{O}_{X_*}$-module $\mathcal{F}_*$ of the same form as above. We have

$$(Rj_*)(\mathcal{F}_*) = (j_*\mathcal{I}_*)^\bullet = \prod j_*(p_{\alpha *}\mathcal{I}_\alpha)^\bullet \approx \prod \check{C}^\bullet(p_{\alpha *}\mathcal{I}_\alpha) = \check{C}^\bullet(\mathcal{I}^\bullet_*) \approx \check{C}^\bullet(\mathcal{F}_*).$$

We only prove the first formula for Ext. Here, $\mathcal{I}^\bullet_*$ denotes an injective resolution of $j^*\mathcal{G}$.

$$\begin{aligned}\mathrm{Ext}^n_{X_*}(j^*\mathcal{F}, j^*\mathcal{G}) =& H^n(\mathrm{Hom}_{X_*}(j^*\mathcal{F},\mathcal{I}^\bullet_*)) = H^n(\mathrm{Hom}_X(\mathcal{F}, j_*\mathcal{I}^\bullet_*))\\ =& \mathrm{Ext}^n_X(\mathcal{F}, j_*\mathcal{I}^\bullet_*) = \mathrm{Ext}^n_X(\mathcal{F}, (Rj_*)(j^*\mathcal{G}))\\ =& Ext^n_X(\mathcal{F}, \check{C}^\bullet(j^*\mathcal{G})) = \mathrm{Ext}^n_X(\mathcal{F},\mathcal{G}).\end{aligned}$$

□

In the sequel, let $\mathcal{X}$ be a complex space or a scheme of finite type over a Noetherian ring.

The structure sheaf $\mathcal{O}_\mathcal{X}$ defines an $\mathcal{N}$-Object $a = a_*$ in $\mathcal{C}$. In the algebraic case each $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ defines an $\mathcal{N}$-object $F = F_*$ in $\mathcal{M}$ over a. In the analytic case each coherent $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ defines an $\mathcal{N}$-object $F = F_*$ in $\mathcal{M}$ over a. Here, $(\mathcal{C},\mathcal{M})$ stands for $(\mathcal{C}^{(0)},\mathcal{M}^{(0)})$ in the algebraic case and for $(\mathcal{C}^{(1)},\mathcal{M}^{(1)})$ in the analytic case (see Example 3.1.1).

We make the following convention to avoid the distinction between analytic and algebraic tensor products:

Convention: Let $f : X_* \longrightarrow Y_*$ be a morphism of simplicial complexs of Stein compacts and let $\mathcal{F},\mathcal{G}$ be graded objects in $\mathcal{M}od(X_*)$, coherent in each degree. By $\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}$, we mean the object in $\mathcal{M}od(X_*)$, which is given by the sheafification of the object T_* in $\mathrm{gr}(\mathcal{C})^\mathcal{N}$, defined as follows:

For $\alpha \in \mathcal{N}$, set $B_\alpha := \Gamma(Y_{\tau(\alpha)}, \mathcal{O}_{Y_{\tau(\alpha)}})$, $F_\alpha := \Gamma(X_\alpha, \mathcal{F}_\alpha)$ and $G_\alpha := \Gamma(X_\alpha, \mathcal{G}_\alpha)$. Then, F_α and G_α are modules over B_α via the comorphism of f_α. Set $T_\alpha := F_\alpha \otimes_{B_\alpha} G_\alpha$. This defines a simplicial DG algebra T_*.

In the same manner, we define the tensor product $\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G}$, when $\mathcal{F}$ and $\mathcal{G}$ are modules over a sheaf of $\mathcal{O}_{\mathcal{X}_*}$-modules $\mathcal{R}$, coherent in each degree.

3.3.1 Hochschild cohomology for complex spaces and varieties

Let $f : X \longrightarrow Y$ be a morphism of complex spaces or a morphism of finite type of Noetherian schemes.

A **resolvent** of X over Y is a collection of the following things:
(1) The simplicial complex Y_* associated to a local finite affine covering $(Y_j)_{j\in J}$ of Y; (2) the simplicial complex $X_* = (X_\alpha)_{\alpha\in\mathcal{N}}$ associated to a local finite affine covering $(X_{ji})_{j\in J, i\in I_j}$ of X. This covering is chosen in such a way that, for a fixed $j \in J$, the family $(X_{ji})_{i\in I_j}$ is a covering of $f^{-1}(Y_j)$; (3) a simplicial complex $P_* = (P_\alpha)_{\alpha\in\mathcal{N}}$ with the same index category; (4) a commutative diagram of the following form:

$$\begin{tikzcd} X_* \arrow[r, "\iota"] \arrow[d, "\bar{f}"'] & P_* \arrow[dl, "g"] \\ Y_* & \end{tikzcd}$$

Here, $\bar{f} = (\bar{f}, \tau)$ is the induced map of simplicial complexs; ι is a closed embedding and g is a smooth map[10]; (5) a free resolution $\mathcal{A}_*$ of $\mathcal{O}_{X_*}$ as sheaf of DG-algebras on P_* with $\mathcal{A}^0_* = \mathcal{O}_{P_*}$ such that in each degree there is only a finite number of free algebra generators.

If $\mathcal{A}_* \longrightarrow \mathcal{B}_*$ is a morphism of sheaves of DG-algebras, coherent in each degree, on a simplicial space X_*, where each X_α is affine, then, going over to global sections, we can construct a free resolution S_* of $B_* := (\Gamma(X_\alpha, \mathcal{B}_\alpha))_{\alpha\in\mathcal{N}}$ over $A_* := (\Gamma(X_\alpha, \mathcal{A}_\alpha))_{\alpha\in\mathcal{N}}$, at least when B^0_* is a quotient of a free algebra over A^0_* in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$. This follows by Proposition I.8.8 of [2]. Sheafifying S_*, we get a free resolution $\mathcal{S}_*$ of $\mathcal{B}$ over $\mathcal{A}$. Using this remark, it is easy to deduce the existence of resolvents in both situations we are going to consider.

Example 3.3.5. Suppose that X is smooth and Y is just the single point Spec($\mathbb{C}$). For $i \in I$, we can choose $P_i = X_i$. Then, X_α is a diagonal in P_α and A can be chosen to be a Koszul resolution of $a = (\Gamma(X_\alpha, \mathcal{O}_{X_\alpha}))_{\alpha\in\mathcal{N}}$ over $A^0 = (\Gamma(P_\alpha, \mathcal{O}_{P_\alpha}))_{\alpha\in\mathcal{N}}$. In this case, one can prove that for each α, Ω_{A_α} is a module resolution of Ω_{a_α}. It follows that for $\alpha \subseteq \beta$, the restriction maps $\mathbb{L}_\alpha(a/\mathbb{C}) \longrightarrow \mathbb{L}_\beta(a/\mathbb{C})$ are quasi-isomorphisms. Consequently, the canonical map $\mathbb{L}(X) \longrightarrow \Omega_X$ is a quasi-isomorphism.

[10] This means that for each $\alpha \in \mathcal{N}$ and each $p \in P_\alpha$ the stalk $\mathcal{O}_{P_\alpha,p}$ is free (in the analytic case as local analytic algebra) over $\mathcal{O}_{Y_{\tau(\alpha)},y}$.

Let $(X_*, Y_*, P_*, \mathcal{A}_*)$ be a resolvent of the morphism $f : X \longrightarrow Y$. Set $\mathcal{R} := \mathcal{A} \otimes_{\mathcal{O}_{Y_*}} \mathcal{A}$ and let $\mathcal{S}$ be a free resolution of $\mathcal{A}$ over $\mathcal{R}$.

The following definition coincides for complex spaces with the corresponding definition in [7]:

Definition 3.3.6. The **simplicial Hochschild complex** of X over Y is the object in the derived category $D(X_*)$ of $\mathcal{O}_{X_*}$-modules, represented by

$$\mathbb{H}_*(X/Y) := \mathcal{S} \otimes_{\mathcal{R}} \mathcal{O}_{X_*}.$$

The **Hochschild complex** of X over Y is defined as the object in $D(X)$, represented by

$$\mathbb{H}(X/Y) := \check{C}^\bullet(\mathbb{H}_*(X/Y)).$$

When Y is just the simple point, we will write $\mathbb{H}(X)$ instead of $\mathbb{H}(X/Y)$.

To show the independence of the Hochschild complex of the choice of the resolvent, we have to use the following version of Lemma I.13.7 of [2]:

Lemma 3.3.7. *Let $f : X \longrightarrow X'$ be a flat homomorphism of complex spaces (resp. schemes) and $(X_i)_{i\in I}$ and $(X'_i)_{i\in I'}$ be compact locally finite coverings of X and X' by Stein compacts (resp. open affine subsets). Let $\tau : I \longrightarrow I'$ be a mapping, such that $f(X_i) \subseteq X_{\tau(i)}$ for all $i \in I$. Denote the associated simplicial complexs by X_* and X'_*. Then, f defines a homomorphism $(\bar{f}, \tau)$ of simplicial complexs of ringed spaces. Let $\mathcal{G}^\bullet$ be a complex in $\mathcal{C}oh(X')$ such that, for $\alpha \subseteq \beta$, the restriction map $p^*_{\alpha\beta}\mathcal{G}^\bullet_\alpha \longrightarrow \mathcal{G}^\bullet_\beta$ is a quasi-isomorphism. Then, the canonical homomorphism*

$$f^*\check{C}(\mathcal{G}^\bullet) \longrightarrow \check{C}(\bar{f}^*\mathcal{G}^\bullet)$$

is a quasi-isomorphism.

Proposition 3.3.8. *The definition of $\mathbb{H}(X/Y)$ depends neither on the resolvent $(Y_*, X_*, P_*, \mathcal{A}_*)$ nor on the choice of the resolvent $\mathcal{S}$.*

Proof. Let $(Y_*, X_*, P_*, \mathcal{A}_*)$ and $(\tilde{Y}_*, \tilde{X}_*, \tilde{P}_*, \tilde{\mathcal{A}}_*)$ be two resolvents, $\mathcal{S}$ a free resolution of $\mathcal{A}$ over $\mathcal{A} \otimes \mathcal{A}$ and $\tilde{\mathcal{S}}$ a resolvent of $\tilde{\mathcal{A}}$ over $\tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}}$. We have to show that there is a quasi-isomorphism

$$\check{C}(\tilde{\mathcal{S}} \otimes_{\tilde{\mathcal{R}}} \mathcal{O}_{\tilde{X}_*}) \longrightarrow \check{C}(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{O}_{X_*}).$$

First case: Suppose that $Y_* = \tilde{Y}_*$, $X_* = \tilde{X}_*$ and $P_* = \tilde{P}_*$. By Proposition 3.2.2, there is a quasi-isomorphism

$$\tilde{\mathcal{S}} \otimes_{\tilde{\mathcal{R}}} \mathcal{O}_{\tilde{X}_*} \longrightarrow \mathcal{S} \otimes_{\mathcal{R}} \mathcal{O}_{X_*}$$

in $\mathcal{M}od(X_*)$. Applying the Čech functor, this case is proved.

General case: Let Y'_* be the simplicial complex associated to the covering $\{Y_j\} \cup \{Y'_j\}$, and let X'_* be the simplicial complex associated to the covering $\{X_{ij}\} \cup \{X'_{ij}\}$.

We construct P'_* in the canonical way and can find a resolvent $\mathcal{A}'$ such that $(Y'_*, X'_*, P'_*, \mathcal{A}'_*)$ forms another resolvent of $f : X \longrightarrow Y$. There is a commutative diagram

$$\begin{tikzcd} X_* \arrow[r, "h"] \arrow[d, "f_*"] & X'_* \arrow[d, "f'_*"] \\ Y_* \arrow[r] & Y'_* \end{tikzcd}$$

By the first case, there is a quasi-isomorphism

$$h^*(\mathcal{S}' \otimes_{\mathcal{R}'} \mathcal{O}_{X'_*}) \approx \mathcal{S} \otimes_{\mathcal{R}} \mathcal{O}_{X_*}.$$

By Lemma 3.3.7, there is a quasi-isomorphism

$$\check{C}(\mathcal{S}' \otimes_{\mathcal{R}'} \mathcal{O}_{X'_*}) \approx \check{C}(h^*(\mathcal{S}' \otimes_{\mathcal{R}'} \mathcal{O}_{X'_*})).$$

Hence, we get $\check{C}(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{O}_{X_*}) \approx \check{C}(\mathcal{S}' \otimes_{\mathcal{R}'} \mathcal{O}_{X'_*})$. In the same way we get $\check{C}(\tilde{\mathcal{S}} \otimes_{\tilde{\mathcal{R}}} \mathcal{O}_{\tilde{X}_*}) \approx \check{C}(\mathcal{S}' \otimes_{\mathcal{R}'} \mathcal{O}_{X'_*})$. □

As in [7], we define the **n-th Hochschild cohomology of $\mathcal{X}$ over $\mathcal{Y}$** with values in the sheaf $\mathcal{F}$ as $\mathrm{Ext}^n_{\mathcal{X}}(\mathbb{H}(\mathcal{X}/\mathcal{Y}), \mathcal{F})$. We define the **n-th Hochschild homology** of X over Y as $H^{-n}(X, \mathbb{H}(Y/Y))$. At least in the case where $\mathcal{F}$ is coherent, we want to show that this definition is equal to the following one, which seems to be more natural from the viewpoint of good pairs of categories:

Definition 3.3.9. [alternative]
Suppose that $\mathcal{F}$ is coherent. Let a be the algebra $(\Gamma(X_\alpha, \mathcal{O}_{X_\alpha}))_{\alpha\in\mathcal{N}}$ in $\mathcal{C}^{\mathcal{N}}$, let k be the algebra $((\Gamma(Y_{\tau(\alpha)}, \mathcal{O}_{Y_{\tau(\alpha)}}))_{\alpha\in\mathcal{N}}$ in $\mathcal{C}^{\mathcal{N}}$. Then to f, there is associated a homomorphism $k \longrightarrow a$ in $\mathcal{C}^{\mathcal{N}}$. Let F be the module $(\Gamma(X_\alpha, \mathcal{F}_\alpha))_{\alpha\in\mathcal{N}}$. We define the n-th **Hochschild cohomology** of X over Y with values in $\mathcal{F}$ as

$$\mathrm{HH}^n(\mathcal{X}/\mathcal{Y}, \mathcal{F}) := H^n(\mathrm{Hom}_a(\mathbb{H}_*(a/k), F)).$$

We define the n-th **Hochschild homology** of X over Y as

$$\mathrm{HH}_n(\mathcal{X}/\mathcal{Y}) := \check{H}^{-n}(\mathbb{H}_*(a/k)).$$

We see directly that the Hochschild cohomology is concentrated in non-negative degrees, whereas Hochschild homology, in general has positive and negative degrees.

Lemma 3.3.10. *For $\mathcal{M}_* := \mathbb{H}_*(\mathcal{X}/\mathcal{Y})$, the assumption of the second part of Proposition 3.3.4 is satisfied, i.e., for $\alpha \subseteq \beta$, the maps $p^*_{\alpha\beta}(\mathcal{M}_\alpha) \longrightarrow \mathcal{M}_\beta$ are quasi-isomorphisms.*

Proof. See Lemma 1.7 of [7]. □

Corollary 3.3.11. *For coherent $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{F}$, the two definitions of Hochschild (co)homology coincide, i.e.*

$$\mathrm{HH}^n(\mathcal{X}/\mathcal{Y},\mathcal{F}) \cong \mathrm{Ext}^n_{\mathcal{X}}(\mathbb{H}(\mathcal{X}/\mathcal{Y}),\mathcal{F}) \qquad \text{and}$$

$$\mathrm{HH}_n(X/Y) \cong H^n(X,\mathbb{H}(X/Y)).$$

Proof. Since $\mathbb{H}_*(\mathcal{X}/\mathcal{Y})$ is a complex of free $\mathcal{O}_{\mathcal{X}_*}$-modules, by Proposition 3.3.4 we get

$$\begin{aligned}&\mathrm{Ext}^n_{\mathcal{X}}(\mathbb{H}(\mathcal{X}/\mathcal{Y}),\mathcal{F}) \cong \mathrm{Ext}^n_{\mathcal{X}_*}(\mathbb{H}_*(\mathcal{X}/\mathcal{Y}),j^*\mathcal{F})\\ &\cong H^n(\mathrm{Hom}_a(\mathbb{H}_*(a/k),F_*)) = \mathrm{HH}^n(\mathcal{X}/\mathcal{Y},\mathcal{F}).\end{aligned}$$

The second isomorphism is obtained as follows:

$$\begin{aligned}H(X,\mathbb{H}(X/Y)) &\cong H(\mathrm{tot}^{\Pi}\,\Gamma(X,\check{C}^\bullet(j^*\mathbb{H}(X/Y))))\\ &\cong H(\mathrm{tot}^{\Pi}\,\Gamma(X,\check{C}^\bullet(j^*\check{C}^\bullet\mathbb{H}_*(X/Y))))\\ &\cong H(\mathrm{tot}^{\Pi}\,\Gamma(X,\check{C}^\bullet\mathbb{H}_*(X/Y))) \cong H(\check{C}^\bullet\mathbb{H}_*(a/k)) = \check{H}(\mathbb{H}_*(a/k)).\end{aligned}$$

In the third step, we have made use of Remark 3.3.10. $\square$

In the absolute case, i.e. in the case where $Y = \operatorname{Spec}\mathbb{C}$, Definition 3.3.6 is up to quasi-isomorphism equivalent to the definition proposed by Weibel/ Geller [47].

Proposition 3.3.12. *Let X be a Noetherian scheme of finite type over a field or a complex space. Let $\mathcal{C}^{\mathrm{naive}}(X)$ be the complex of sheaves in $\mathfrak{Mod}(X)$ associated to the presheaf $U \mapsto C^{\mathrm{naive}}(\Gamma(U,\mathcal{O}_X))$. (In the analytic case, the naive Hochschild complex is formed, using the analytic tensor product, of course.) There exists a quasi-isomorphism of sheaves*

$$\mathbb{H}(X) \longrightarrow \mathcal{C}^{\mathrm{naive}}(X).$$

Proof. Choose a resolvent $(X_*,P_*,\mathcal{A}_*)$ of X. Let $\mathcal{S}$ be a resolvent of $\mathcal{A}$ over $\mathcal{R} = \mathcal{A}\otimes\mathcal{A}$. Let a, A, R and S be the simplicial algebras in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ corresponding to $\mathcal{O}_{X_*}, \mathcal{A}, \mathcal{R}$ and $\mathcal{S}$. By Lemma 3.3.3, there is a quasi-isomorphism

$$\check{C}(j^*\mathcal{C}^{\mathrm{naive}}(X)) \longrightarrow C^{\mathrm{naive}}(X).$$

Now, $j^*\mathcal{C}^{\mathrm{naive}}(X)$ corresponds to $C^{\mathrm{naive}}(a)$. Set $r := a\otimes_k a$. In the absolute case, R is a resolvent of r over k, hence $S\otimes_R r$ is a resolvent of a over r. Thus, there is a quasi-isomorphism

$$S \longrightarrow C^{\mathrm{bar}}(a).$$

Since $C^{\mathrm{bar}}(a)$ is a complex of flat r-modules, after tensoring over r with a, we get a quasi-isomorphism $S\otimes_r a \longrightarrow C^{\mathrm{naive}}(a)$, i.e. a quasi-isomorphism

$$\mathcal{S}\otimes_{\mathcal{R}}\mathcal{O}_{X_*} \longrightarrow j^*C^{\mathrm{naive}}(X)$$

in $\mathfrak{Mod}(X_*)$. Applying the Čech functor, we get the desired result. $\square$

3.3.2 The decomposition theorem

The quasi-isomorphism $\wedge\mathbb{L}_{a/k} \longrightarrow \mathbb{H}(a/k)$ in $\mathrm{gr}(\mathcal{M})^{\mathcal{N}}$ over a in Theorem 3.2.7 defines a quasi-isomorphism

$$\wedge\mathbb{L}_*(X/Y) \longrightarrow \mathbb{H}_*(X/Y)$$

in $\mathcal{M}od(X_*)$. Since the Čech -functor is exact, we get the following Quillen-type theorem:

Theorem 3.3.13. *There is an isomorphism*

$$\wedge\mathbb{L}(X/Y) \longrightarrow \mathbb{H}(X/Y)$$

in the derived category $D(X)$.

Corollary 3.3.14. *There are natural decompositions*

$$\mathrm{HH}^n(X/Y,\mathcal{M}) \cong \coprod_{p+q=n} \mathrm{Ext}^p_X(\wedge^q\mathbb{L}(X/Y),\mathcal{M})$$
$$\mathrm{HH}_n(X/Y) \cong \prod_{p-q=n} H^q(X,\wedge^p\mathbb{L}(X/Y)).$$

For complex spaces, this is just Theorem 4.2 of [7]. There is another nice description of Hochschild cohomology of complex spaces or Noetherian schemes over a field K in any characteristic:

Remark 3.3.15. $\mathrm{HH}^n(X) = \mathrm{Ext}_{X^2}(\mathcal{O}_X,\mathcal{O}_X)$.

Proof. We use the letter $\mathbb{K}$ for the field K or for the complex numbers, depending on the context. With the notation as above, we get:

$$\begin{aligned}\mathrm{HH}^n(\mathcal{X}) =& H^n(\mathrm{Hom}_a(S\otimes_R a,a)) = H^n(\mathrm{Hom}_R(S,a))\\ =& H^n(\mathrm{Hom}_{a\otimes_{\mathbb{K}}a}(S\otimes_R(a\otimes_{\mathbb{K}}a),a)) = H^n(\mathrm{Hom}_{\mathcal{O}_{X^2_*}}(\mathcal{S}\otimes_{\mathcal{R}}\mathcal{O}_{X^2_*},\mathcal{O}_{X_*}))\\ =& \mathrm{Ext}^n_{\mathcal{O}_{\mathcal{X}^2_*}}(\mathcal{O}_{\mathcal{X}_*},\mathcal{O}_{\mathcal{X}_*}) = \mathrm{Ext}_{X^2}(\mathcal{O}_X,\mathcal{O}_X).\end{aligned}$$

Here, we have used that $\mathcal{S}\otimes_{\mathcal{R}}\mathcal{O}_{X^2_*}$ is a free resolution of $\mathcal{O}_{X_*}$ over $\mathcal{O}_{X^2_*}$. □

3.3.3 Hochschild cohomology for manifolds and smooth varieties

Theorem 3.3.16. *Let X be a complex analytic manifold or a smooth scheme of finite type over a field $\mathbb{K}$ of characteristic zero. There exists an isomorphism*

$$\mathbb{H}(X) \cong \wedge\Omega_X$$

in the derived category $D(X)$.

Proof. This is a direct consequence of Theorem 3.3.13 and Example 3.3.5

For the case that $\mathbb{K}$ is algebraically closed, we have the following alternative proof: Define the morphism

$$\alpha : C^{\text{naive}}(X) \longrightarrow \wedge \Omega_X$$

locally in the n-th component as $\alpha_n : a_0 \otimes \ldots \otimes \ a_n \mapsto \frac{1}{n!} a_0 \cdot a_1 \wedge \ldots \wedge a_n$. Since this maps are natural, they define a map of complexes of $\mathcal{O}_X$-modules. For smooth manifolds, α is a quasi-isomorphism (on the stalks), by Theorem 3.1.35. For smooth schemes, α is a quasi-isomorphism (on the stalks), by the classical HKR theorem (the local rings $\mathcal{O}_{X,x}$ are geometrically regular, hence smooth; see the Remark on page 318 of [46]). Thus the theorem is true by Proposition 3.3.12. □

Remark that for smooth schemes in positive characteristic, by [44] Lemma 2.4, there are natural isomorphisms $\wedge^n_{\mathcal{O}_X} \Omega_X \longrightarrow H_n(\mathbb{H}(X))$, but I don't know if they are induced by a quasi-isomorphism of complexes.

Corollary 3.3.17. *Let X be a complex analytic manifold or a smooth scheme of finite type over a field $\mathbb{K}$ of characteristic zero. There is the following decomposition of Hochschild (co)homology:*

$$\mathrm{HH}^n(X) \cong \coprod_{i+j=n} H^i(\mathcal{X}, \wedge^j \mathcal{T}_{\mathcal{X}}) \tag{3.9}$$

$$\mathrm{HH}_n(X) \cong \coprod_{i-j=n} H^j(X, \wedge^i \Omega_X). \tag{3.10}$$

Proof. We consider $\wedge_{\mathcal{O}_X} \Omega_X$ as complex in negative degrees, so $\wedge \Omega_X = \coprod_{j \geq 0} \wedge^j \Omega_X[j]$ and

$$\mathrm{HH}^n(X) = \mathrm{Ext}^n_X(\mathbb{H}(X), \mathcal{O}_X) \cong \coprod_{j \geq 0} \mathrm{Ext}^{n-j}_X(\wedge^j \Omega_X, \mathcal{O}_X).$$

By Theorem 7.3.3 of [13], there exists a (bounded) spectral sequence with $E_2^{p,q} = H^p(X, \mathcal{E}xt^q_X(\wedge^j \Omega_X, \mathcal{O}_X))$, converging to $\mathrm{Ext}_X(\wedge^j \Omega_X, \mathcal{O}_X)$. But $\wedge^j \Omega_X$ is a locally free $\mathcal{O}_X$-module, so $\mathcal{E}xt^q_X(\wedge^j \Omega_X, \mathcal{O}_X)$ is zero for $q > 0$ and $\mathcal{H}om_{\mathcal{X}}(\wedge^j \Omega_X, \mathcal{O}_X)$ for $q = 0$. Hence, the spectral sequence degenerates at once and we get

$$\mathrm{Ext}^q_X(\wedge^j \Omega_X, \mathcal{O}_X) = H^q(X, \mathcal{H}om_X(\wedge^j \Omega_X, \mathcal{O}_X)).$$

There is a natural isomorphism of sheaves

$$\wedge^j \mathcal{T}_X = \wedge^j \mathcal{H}om_X(\Omega_X, \mathcal{O}_X) \longrightarrow \mathcal{H}om_X(\wedge^j \Omega_X, \mathcal{O}_X),$$

which, by Proposition 7, p. 154 of [3], is an isomorphism. As consequence,

$$\mathrm{HH}^n(X) \cong \coprod_{j \geq 0} H^{n-j}(X, \wedge^j \mathcal{T}_X) \cong \coprod_{i+j=n} H^i(X, \wedge^j \mathcal{T}_X).$$

The second equality is a direct consequence of Theorem 3.3.16. □

A proof of this result for complex analytic manifolds has been announced (but not yet published) by Kontsevich. For smooth schemes, decomposition (3.9) was proved in a different way by Yekutieli [49]. A similar statement for quasi-projective smooth schemes is due to Gerstenhaber/Schack [11] and Swan [44]. For smooth schemes, decomposition (3.10) was proved in a different way by Weibel [48].

3.4 L_∞-structure of the Hochschild cochain complex

This section contains some open questions and conjectures. Furthermore, we give a very brief idea for an interpretation of Hochschild cohomology of complex spaces in terms of non-commutative deformation theory. The exposition of this section is not assumed to be exhaustive. We only want to indicate how the subject could be developed further. We make use of the notions and constructions of Chapters 1 and 2.

Let X be complex space with fixed locally finite covering by Stein compacts with nerf $\mathcal{N}$. Denote the associated simplicial complex of Stein compacts by $X_* = (X_\alpha)_{\alpha\in\mathcal{N}}$ (see Section 3.3). Set $a = a_* := (\Gamma(X_\alpha, \mathcal{O}_{X_\alpha}))_{\alpha\in\mathcal{N}}$ and fix a resolvent $A = A_*$ of a over $\mathbb{C}$. Classical deformation theory of X is described by the differential graded Lie algebra $(L^{\tan}, d^{\tan}, [\cdot,\cdot])$, where

$$L^{\tan} = \mathrm{Der}_{\mathbb{C}}(A, A) \cong \mathrm{Hom}_A(\Omega_A, A)$$

is the tangent complex of X (for fixed covering, $L^{\tan}$ is independent up to homotopy of the choice of A; otherwise, $L^{\tan}$ is independent up to quasi-isomorphism of the choice of the covering and the resolvent A; see Chapter IV of [2]), $[\cdot,\cdot]$ is the graded commutator and $d^{\tan} = [\cdot, d^A]$. Roughly speaking, deformations of X are obtained by simultaneously perturbing the differential d^{A_α} on A_α, for each $\alpha \in \mathcal{N}$. This effects simultaneous perturbations of $\mathcal{O}_{X_\alpha} = H^0(A_\alpha, d^{A_\alpha})$. "Simultaneous" means that the perturbed Stein compacts $\tilde{X}_\alpha$ can be glued together to a complex space $\tilde{X}$, which is a deformation X.

To interpret the Hochschild cohomology of X in terms of deformation theory, we need a DGL structure on the **Hochschild cochain complex**

$$\mathrm{Hom}_A(S \otimes_R A, A)$$

(where $R := A \otimes A$ and S is a resolvent of A over R) or at least an L_∞-structure (it was noticed by Merkulov [32] and others that a deformation functor can be defined for each L_∞-algebra). We suggest the following: By Theorem 3.2.7, there exists a homotopy equivalence

$$\mathrm{Hom}_A(S \otimes_R A, A) \longrightarrow \mathrm{Hom}_A(\wedge\Omega_A, A). \tag{4.11}$$

On $L := \mathrm{Hom}_A(\wedge\Omega_A, A) \cong \wedge\mathrm{Hom}_A(\Omega_A, A)$, there is a DGL structure $(L, d, [\cdot,\cdot])$, where the differential d is induced by d^A and the graded Lie bracket $[\cdot,\cdot]$ is the **Schouten-Nijenhuis-bracket** (see [22]). It has the following form:

$$[\xi_1\wedge\ldots\wedge\xi_k, \eta_1\wedge\ldots\wedge\eta_l] = \sum_{i,j} \pm[\xi_i,\eta_j]\wedge\xi_1\wedge\ldots\wedge\hat{\xi}_i\wedge\ldots\wedge\xi_k\wedge\eta_1\wedge\ldots\wedge\hat{\eta}_j\wedge\ldots\wedge\eta_l$$

Remark that the inclusion $L^{\tan} \longrightarrow L$ is a DGL morphism. With the method of Section 1.4, we can define an L_∞-algebra structure on $\mathrm{Hom}_A(S \otimes_R A, A)$ such that the linear map (4.11) is the first term of an L_∞-equivalence.

By Proposition 3.1.35 , the natural maps

$$\alpha_n : C_n^{\text{naive}}(A) \longrightarrow \wedge^n \Omega_a$$
$$a_0 \otimes \ldots \otimes a_n \mapsto a_0 \cdot da_1 \wedge \ldots \wedge da_n$$

define a homotopy equivalence of complexes of simplicial DG A-modules. Dually, we have a quasi-isomorphism

$$\phi : L = \mathrm{Hom}_A(\wedge \Omega_A, A) \longrightarrow \mathrm{Hom}_A(C_{-\bullet}^{\text{naive}}(A), A).$$

The complex $\mathrm{Hom}_A(C_{-\bullet}^{\text{naive}}(A), A)$ is also a DGL. Its Lie bracket is the so-called **Gerstenhaber bracket**. Instead of giving an explicit formula for the Gerstenhaber bracket, we recall that the complex $\mathrm{Hom}_A(C^{cycl}(A), A)$ is isomorphic to the complex $\mathrm{Coder}(T(A[1]), T(A[1]))$, where $T(A[1])$ denotes the (simplicial) graded tensor algebra of the shift of the graded module A. This follows in analogy to the correspondence between formal super-vectorfields on a formal DG manifold M and coderivations on the graded symmetric algebra $S(M)$, see Section 1.1. There is a codifferential $Q^{A[1]}$ on $T(A[1])$ with first order term induced by the differential d^A and second order term induced by the multiplication $A \otimes A \longrightarrow A$. (One should think of it as odd vectorfield on the simplicial non-commutative formal superspace $A[1]$.) The DGL structure on $\mathrm{Coder}(T(A[1]), T(A[1]))$ is given by the graded commutator as bracket and differential $[\,\cdot, Q^{A[1]}]$. The DGL structure on $\mathrm{Hom}_A(C_{-\bullet}^{\text{naive}}(A), A)$ is the induced one.

Attention: The DG map ϕ does not respect the brackets! In other words: Schouten-Nijenhuis is not compatible with Gerstenhaber.

But: The restriction to $L^{\tan}$ does respect the brackets, thus we have two different extensions of the DGL $L^{\tan}$, i.e. two possibilities to extend (on the abstract level of deformation functors) the classical deformation theory.

It was the main point in Kontsevich's (Fields-price-winning) proof for the possibility of quantization of each Poisson structure on manifolds that this lack can be cured in the situation, where A is replaced by the algebra Γ of global sections on a Poisson manifold. I.e. there exists a morphism

$$\Phi : \wedge \mathrm{Hom}_\Gamma(\Omega_\Gamma, \Gamma) \longrightarrow \mathrm{Hom}_\Gamma(C^{\text{naive}}(\Gamma), \Gamma)$$

in the category of L_∞-algebras with $\Phi_1 = \phi$. This is the so-called Formality Theorem.

Question 1. Is there a Formality Theorem for our situation ?

A positive answer would imply that the two mentioned extensions of deformation theory of complex spaces are in fact equivalent. For one of them, we have the following "geometric" interpretation:

In the same way as the underlying module M of the free coalgebra $S(M)$ is considered as formal germ of a supermanifold (see Section 1.1.4), the underlying module of the free coalgebra $T(M)$ should probably be considered as formal germ of a non-commutative superspace and coderivations on $T(M)$ should again be considered as vectorfields.

It is well-known (see e.g. [33], [38]) that the differentials described by the differential graded Lie algebra $\mathrm{Coder}(T(A[1]), T(A[1]))$ are the deformations of the A_∞-structure of A, or in "geometric" terms deformations of the simplicial complex $A[1]$ of formal non-commutative DG manifolds. A deformation theory for non-commutative formal DG manifolds can be developed in analogy to the theory in Section 1.2.

Question 2. Is the base of a universal deformation of a formal non-commutative DG manifold M given by by the universal enveloping algebra of the DGL $\mathrm{Coder}(T(M), T(M))$?

The following question is motivated by Hinich and Schechtman's observation [17] that the Čech complex of a sheaf of DGL has the structure of an L_∞-algebra:

Question 3. Is any simplicial complex of A_∞-algebras (resp. L_∞-algebras) again an A_∞- (resp. L_∞) algebra ?

In the next question, F is the functor $\mathfrak{Anf} \longrightarrow$ `DG-Manf` of Section 2.2.

Question 4. If Question 3 is true, can we reconstruct the complex space X by the L_∞-algebra corresponding to $(F(X_\alpha))_{\alpha\in\mathcal{N}}$?

3.5 Appendix

In this Appendix, we give some details ommited in Section 3.1.

Proof of Lemma 3.1.3

Let k be an algebra in $\mathcal{C}$. and A, B, M and N modules in $\mathcal{M}(k)$. By Axiom (5.2), there is a natural isomorphism $\sim$: $\mathrm{Mult}_{\mathcal{M}(k)}(A \times B, M) \longrightarrow \mathrm{Hom}_{\mathcal{M}(k)}(A \otimes_k^{\mathcal{M}} B, M)$. This means that each morphism $f : M \longrightarrow N$ in $\mathcal{M}(k)$ induces a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Mult}_{\mathcal{M}(k)}(A \times B, M) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{M}(k)}(A \otimes_k^{\mathcal{M}} B, M) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
\mathrm{Mult}_{\mathcal{M}(k)}(A \times B, N) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{M}(k)}(A \otimes_k^{\mathcal{M}} B, N)
\end{array}$$

We denote the inverse map of $\sim$ also by $\sim$.

Lemma 3.5.1. *For $h \in \mathrm{Hom}_{\mathcal{M}(k)}(A \otimes_k^{\mathcal{M}} B, N)$, we have $\tilde{h} = h \circ \tilde{\mathrm{Id}}_{A \otimes B}$.*

Proof. In the diagram above, set $M := A \otimes_k^{\mathcal{M}} B$ and $f := h$. We have $h = h^*(\mathrm{Id}_{A \otimes B})$. So $\tilde{h}$ is the image of $\mathrm{Id}_{A \otimes B}$ by going through the diagram starting up right, going down left. The result choosing the other way is $h \circ \tilde{\mathrm{Id}}_{A \otimes B}$. □

Now suppose that A and B are k-algebras in $\mathcal{C}$. There are two ways to see the elements $a \otimes b$ in $A \otimes_k^{\mathcal{M}} B = A \otimes_k^{\mathcal{C}} B$: By the universal property of fibered products, there is a natural homomorphism of k-algebras $\alpha : A \otimes_k^{\mathfrak{Alg}} B \longrightarrow A \otimes_k^{\mathcal{C}} B$.

Lemma 3.5.2. *For elements $a \otimes b$ of $A \otimes_k^{\mathfrak{Alg}} B$, we have $\alpha(a \otimes^{\mathfrak{Mod}} b) = \tilde{\mathrm{Id}}_{A \otimes_k^{\mathcal{M}} B}((a, b))$.*

Proof. We see that α is just the image of $\mathrm{Id}_{A \otimes_k^{\mathcal{M}} B}$ by the composition of the mappings

$$\mathrm{Hom}_{\mathcal{M}}(A \otimes_k^{\mathcal{M}} B, A \otimes_k^{\mathcal{M}} B) \cong \mathrm{Mult}_{\mathcal{M}}(A \times B, A \otimes_k^{\mathcal{M}} B) \hookrightarrow$$

$$\mathrm{Mult}_{k-\mathfrak{Mod}}(A \times B, A \otimes_k^{\mathcal{M}} B) \cong \mathrm{Hom}_{k-\mathfrak{Mod}}(A \otimes_k^{\mathfrak{Mod}} B, A \otimes_k^{\mathcal{M}} B).$$

□

The first point of Lemma 3.1.3 is a direct consequence of Lemma 3.5.2. The second point of Lemma 3.1.3 is a direct consequence of Lemma 3.5.1.

Proof of Proposition 3.1.8

(1) There are free finite k-modules L_i in $\mathcal{M}(k)$ and homomorphisms $\phi_i : L_i \longrightarrow N$ such that the inclusions $M_i \hookrightarrow N$ is Kern(cokern(ϕ_i)). (2) We have $M_1 + M_2 \hookrightarrow N =$ Kern(cokern($\phi_1 + \phi_2$)). (3) $M_i \hookrightarrow N$ factorises through Kern(cokern($\phi_1 + \phi_2$)). (4) The projection $M_1 + M_2 \longrightarrow M_1$ is the kernel of the inclusion $M_2 \longrightarrow M_1 + M_2$ in k-$\mathfrak{Mod}$, so as well in $\mathcal{M}(k)$. (5) Consider homomorphisms $f_i : M_i \longrightarrow P$ in $\mathcal{M}(k)$. We define a homomorphism $M + N \longrightarrow P$ as $f_1 \circ p_1 + f_2 \circ p_2$. Then the diagram

$$\begin{tikzcd} M_1 + M_2 \arrow[dr] & M_1 \arrow[l] \arrow[d, "f_1"] \\ M_2 \arrow[u] \arrow[r, "f_2"] & P \end{tikzcd}$$

in $\mathcal{M}(k)$ commutes. The graded case follows in the same manner.

Proof of simplicial version of Proposition 3.1.16

We need two lemmas to prove a simplicial version of Proposition 3.1.16. The first one is a simplicial version of the Comparison Theorem (for the affine case, see Theorem 2.2.6 of [46]).

Lemma 3.5.3. *Let A be a DG algebra in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$. Let $P = \coprod_{i \in I} Ae_i$ be a free DG A-module in $\mathrm{gr}(\mathcal{M})^{\mathcal{N}}$ with a homomorphism $P^0 \longrightarrow M$ of A^0-modules in $\mathcal{M}^{\mathcal{N}}(A^0)$. Let N be an A^0-modules in $\mathcal{M}^{\mathcal{N}}$ and Q in $\mathrm{gr}(\mathcal{M})^{\mathcal{N}}(A)$ a DG-resolution of* N. *Let $\phi : M \longrightarrow N$ be a A^0-homomorphism in $\mathcal{M}^{\mathcal{N}}$. Then there exists a homomorphism $f : P \longrightarrow Q$, lifting ϕ and it is unique up to a chain homotopy.*

Proof. The existence of such an f is not hard to prove. But we only make use of the uniqueness. We only prove this part here: Let f and g two DG-homomorphisms lifting ϕ. Inductively, we construct families $\{s_\alpha : \ |\alpha| \leq m\}$ of compatible homotopy maps $s_\alpha : P_\alpha \longrightarrow Q_\alpha[-1]$ satisfying

$$g^n - f^n = d_Q \circ s_\alpha^n + (-1)^n s_\alpha^{n+1} \circ d_P.$$

Suppose that the free generator e_i is associated to the pair (α_i, z_i) with $\alpha_i \in \mathcal{N}$ and $z_i < 0$. For $m = 0$ and each β in $\mathcal{N}$ with $|\beta| = 0$, we see that P_β is free DG-module in $\mathrm{gr}(\mathcal{M})(A_\alpha)$, and we can construct $s_\beta^\bullet$ just as in the affine case. Now, suppose that $\{s_\alpha : \ |\alpha| \leq m\}$ is already constructed. Then, for each $\beta \in \mathcal{N}$ with $|\beta| = m + 1$, we have

$$P_\beta = \coprod_{\alpha_i \subseteq \beta} A_\beta e_i.$$

For $\alpha \subseteq \beta$, denote the restriction map $P_\alpha \longrightarrow \mathrm{P}_\beta$ by $\rho_{\alpha\beta}$. For free generators e_i with $\alpha_i \subseteq \beta$ but $\alpha_i \neq \beta$, set $s_\beta(e_i) := \rho_{\alpha_i\beta}(s_{\alpha_i}(e_i))$. Then, we get

$$(g_\beta - f_\beta)(e_i) = \rho_{\alpha_i\beta}((g_{\alpha_i} - f_{\alpha_i})(e_i)) = \rho_{\alpha_i\beta}([s_{\alpha_i}, d_{\alpha_i}](e_i)) = [s_\beta, d_\beta](e_i).$$

For free algebra generators e_i with $\alpha_i = \beta$ and say $n = z_i = g(e_i)$, exactly as in the affine case, by induction on n, we can find elements m_i in $P_\beta^{z_i - 1}$ such that

$$(g_\beta - f_\beta)(e_i) = s_\beta(d(e_i)) + (-1)^n d(m_i).$$

Then we set $s_\beta(e_i) := m_i$.
In this manner, we get a family $(s_\alpha)_{\alpha \in \mathcal{N}}$ of compatible chain homotopies. $\square$

Lemma 3.5.4. *Let A be a DG algebra in $\mathrm{gr}(\mathcal{C})^{\mathcal{N}}$ such that each A^i is a finite A^0-module. Let $M = \coprod_{i \in I} Ae_i$ and $N = \coprod_{j \in J} Ae_j$ two g-finite free DG A-modules in $\mathrm{gr}(\mathcal{M})^{\mathcal{N}}$ such that all generators e_i and e_j are of negative degree. Suppose that there is a quasi-isomorphism*

$$f = \mathrm{Id}_A \oplus f' : A \oplus M \longrightarrow A \oplus N.$$

Then f is already a homotopy equivalence. More precisely, there is a homomorphism

$$g = \mathrm{Id}_A \oplus g' : A \oplus N \longrightarrow A \oplus N$$

of DG-modules and a map $s_* : M \longrightarrow M[-1]$ of graded modules such that $s_*^0 = 0$ and $g \circ f - \mathrm{Id} = [s, d]$.

Proof. Consider the following diagram, where the first line is just the mapping cone $\mathrm{cone}(f) = N \oplus M[1]$ of f and the vertical maps are the canonical inclusions:

$$\begin{tikzcd}
\cdots \arrow[r] & M^{-1} \oplus N^{-2} \arrow[r] & M^0 \oplus N^{-1} \arrow[r] & N^0 \\
\cdots \arrow[r] & N^{-2} \arrow[r] \arrow[u, "\iota"] & N^{-1} \arrow[r] \arrow[u, "\iota"] & N^0 \arrow[u, "\iota"]
\end{tikzcd}$$

Since f is a quasi-isomorphism, the mapping cone of f is acyclic, so the first line is a resolution of the module $\{0\}$. The map ι of DG-modules is a lifting of the trivial map $0 \longrightarrow 0$. The zero map $N \longrightarrow \mathrm{cone}(f)$ is a second candidate for such a lifting. So we are almost in the situation of the uniqueness statement in the Comparison Theorem. The only difference is, that $N = A \oplus N'$ is not a free module in $\mathrm{gr}(\mathcal{M})^{\mathcal{N}}$. But to construct a chain homotopy $\sigma : N \longrightarrow \mathrm{cone}(f)[-1] = N[-1] \oplus M$ for $0 \simeq \iota$, we can set $\sigma|_A$ to be the composition of the inclusions $A \longrightarrow A \oplus M' = M$ and $M \longrightarrow \mathrm{cone}(f)[-1]$. On the free generators of N', the map σ can be defined exactly as in the proof of the comparison theorem. So we can work with a family of maps

$$\sigma^n = (g^n, t^n) : N^n \longrightarrow M^n \oplus N^{n-1}$$

for $n \leq 0$, satisfying the condition

$$\iota^n = \delta^{n-1} \sigma^n + (-1)^n \sigma^{n+1} d^n.$$

Here, d denotes the differential of N and δ the differential of $\mathrm{cone}(f)$. The evaluation of this conditions shows that g is a chain map $N \longrightarrow M$ and that t is a chain homotopy for $\mathrm{Id}_N \simeq f \circ g$.
In an analogous manner, we get a chain map $h : M \longrightarrow N$ with $\mathrm{Id}_M \simeq g \circ h$. We see easily that then, we have $h \simeq f$, so we get $\mathrm{Id}_M \simeq g \circ f$. □

Of course, we can also show that two free module resolutions of a module in $\mathrm{gr}(\mathcal{M})^{\mathcal{N}}$ are homotopy equivalent over the base ring. Now the proof of the simplicial version of Proposition 3.1.16 is as follows: For the first step we have to use Lemma 3.5.4. The second and third step are easy to generalize.

Proof of Proposition 3.1.33

First of all, we have to ask if we can consider the R/I-module I/I^2 as an object of $\mathrm{gr}(\mathcal{M})(A)$.

Lemma 3.5.5. *Let $R = (R, s)$ be a DG-object in $\mathrm{gr}(\mathcal{C})$ such that all R^i are finite R^0-modules. Consider an ideal $I \subseteq R$ which is generated by a handy s-sequence $X = \{x_j : j \in J\}$ in R. Then, I is a DG-object of $\mathrm{gr}(\mathcal{M})(R)$ and I/I^2 is isomorphic as R/I-module to a DG-object of $\mathrm{gr}(\mathcal{M})(R/I)$.*

Proof. For each $x \in X$, we choose a free module generator $e(x)$ with $g(e(x)) = g(x)$ and we see that I is the image[11] of the map from the free module $\coprod_{x \in X} Re(x)$ to R defined by $e(x) \mapsto x$. So I is already an object of $\mathrm{gr}(\mathcal{M})(R)$. But since I is s-stable by assumption, I is a DG-module.

For each pair i, j in J with $i \leq j$, we choose a free module generator e_{ij} with $g(e_{ij}) = g(x_i) + g(x_j)$. We get a homomorphism $\coprod_{i \leq j} Re_{ij} \longrightarrow R$ of modules in $\mathrm{gr}(\mathcal{M})(R)$ by sending e_{ij} to the product $x_i x_j$. This homomorphism factorises through I, so there is a homomorphism $\pi : \coprod_{i \leq j} Re_{ij} \longrightarrow I$ and there is an isomorphism of R-modules $\mathrm{Cokern}\, \pi \cong I/I^2$. It it easy to see that the differential s induces a differential on $\mathrm{Cokern}\, \pi$ that makes it a DG-module in $\mathrm{gr}(\mathcal{M})(R)$.
Now, I/I^2 is also an R/I-module and in $R/I - \mathfrak{Mod}$ the objects $\mathrm{Cokern}\, \pi$ and $\mathrm{Cokern}\, \pi \otimes_R^{\mathrm{gr}(\mathcal{M})} R/I$ are isomorphic. And the latter is an object of $\mathrm{gr}(\mathcal{M})(R/I)$. □

In the sequel, by I/I^2 in fact we mean $\mathrm{Cokern}\, \pi \otimes_R^{\mathrm{gr}(\mathcal{M})} R/I$.

Proposition 3.5.6. *Let $k \longrightarrow A$ be a homomorphism of DG-objects in $\mathrm{gr}(\mathcal{C})$. Suppose that all A^i are finite A^0-modules and that $I := \mathrm{Kern}(\mu : A \otimes_k^{\mathcal{C}} A \longrightarrow A)$ is generated by an s-handy sequence X in $R := A \otimes_k A \longrightarrow A$. Here, s denotes the differential of R induced by the differential of A. Then, by $\bar{a} \mapsto [a]$, we get an isomorphism $I/I^2 \longrightarrow \Omega_{A/k}$ in $\mathrm{gr}(\mathcal{M})(R)$ whose inverse is given by $[\alpha] \mapsto \overline{\alpha - \iota_1 \mu(\alpha)}$. Here, $\overline{a}$ denotes the class in I/I^2 represented by a and $[\alpha]$ denotes the class in $\Omega_{A/k}$ represented by α.*

[11] We remind that by image we mean the kernel of the cokernel map.

Proof. First we have to show that the map $I/I^2 \longrightarrow \Omega_{A/k}$, $\bar{a} \mapsto [a]$ is well defined. There is a homomorphism $\eta : I \otimes_R I \longrightarrow I$ in $\mathrm{gr}(\mathcal{M})(R)$ with $a \otimes b \mapsto ab$. Consider the homomorphism $\xi : I \longrightarrow \Omega_{A/k}$, $a \mapsto [a]$. For the well-definedness, it is enough to prove that $\xi \circ \eta = 0$. Since the bar complex $C^{\mathrm{bar}}_{\bullet}(A/k)$ is acyclic, we see that b' gives rise to an epimorphism $A^{\otimes 3} \longrightarrow I$. Hence, it is enough to show that the map

$$\begin{aligned} A^{\otimes 6} &\longrightarrow \Omega_{A/k} \\ a \otimes b \otimes c \otimes d \otimes e \otimes f &\mapsto [(ab \otimes c - a \otimes bc)(de \otimes f - d \otimes ef)] \end{aligned}$$

is zero. But the argument in the brace on the right hand-side is just the image of

$$\begin{aligned} &(-1)^{cd+ce+db} adbe(cf \otimes 1 \otimes 1 - 1 \otimes c \otimes f) - \\ &(-1)^{bd+cd} adb(cef \otimes 1 \otimes 1 - 1 \otimes c \otimes ef) - \\ &(-1)^{bd+be+cd+ce} ade(bcf \otimes 1 \otimes 1 - 1 \otimes bc \otimes f) + \\ &(-1)^{bd+cd} ad(bcef \otimes 1 \otimes 1 - 1 \otimes bc \otimes ef) \end{aligned}$$

by the map b_2.

Secondly we have to show that the map $\Omega_{A/k} \longrightarrow I/I^2$, $[\alpha] \mapsto \overline{\alpha - s_1\mu(\alpha)}$ is well defined. But there is a derivation

$$\begin{aligned} \delta : A &\longrightarrow I/I^2 \\ a &\mapsto \overline{1 \otimes a - a \otimes 1}. \end{aligned}$$

So by the universal property of $\Omega_{A/k}$ (see the proof of Lemma I.6.13 of [2]), there is a map $\Omega_{A/k} \longrightarrow I/I^2$ sending a class $[a \otimes b]$ to $a\delta(b) = a \cdot \overline{1 \otimes b - b \otimes 1}$ and we see that this map is just the map we want.

To see that the both given maps are inverse to each other, we remark that elements of the form $a \otimes 1$ in $A \otimes A$ are in the image of b_2, so they represent the zero class. $\square$

I/I^2 has the structure of an A-module in $\mathrm{gr}(\mathcal{M})(A)$. The multiplication $A \times I/I^2 \longrightarrow I/I^2$ is inherited by the multiplication $a \cdot \alpha = \iota_1(a) \cdot \alpha$ on $A \otimes A$. But on $A \otimes A$ there is also a left multiplication $\alpha \cdot a := \alpha \cdot \iota_2(a)$. Remark that the left- and right multiplication induced on I/I^2 make I/I^2 an antisymmetrical A-bimodule.

Now, let $R = (R, s)$ be a DG-object of $\mathrm{gr}(\mathcal{C})$ and suppose that all R^i are finite R^0-modules. Let $I \subseteq R$ be an ideal which is generated by a regular s-sequence $X \subseteq R$. Say $X = \{x_i : i \in J\}$. s defines a differential δ on R/I that we denote again by s. Consider the free module $\coprod_{i \in J} R/Ie_i$, where the e_i are free module generators of degree $g(x_i)$.

Proposition 3.5.7. *We can make $\coprod_{i \in J} R/Ie_i$ a DG-module, by defining a differential t in the following sense: For $i \in J$, there is a finite family of elements $a_{ij} \in R$ such that $s(x_i) = \sum_j a_{ij} x_j$. Now we set $\delta(a) := s(a)$, for elements $a \in R/I$ and $\delta(e_i) := \sum_j \overline{a_{ij}} e_j$.*

Proof. To show that this defines a differential on $\coprod_{i\in J} R/Ie_i$, we only have to show, that $\delta^2(e_i)=0$, for $i\in J$. But, since s is a differential on R, we have

$$0=s^2(x_i)=s(\sum_j a_{ij}x_j)=\sum_{j,k}(-1)^{a_{ij}}a_{ij}a_{jk}x_k+\sum_j s(a_{ij})x_j.$$

We can reorganize the coefficients and get a sum $\sum_{k=1}^m b_k x_k=0$ where the x_k are pairwise different. Remark that $\sum_{k=1}^m \overline{b_k}e_k$ is just $\delta^2(e_i)$. To show that this sum is zero, we have to show that each b_k belongs to I. But assume that one b_k, say b_m does not belong to I, then $\overline{b_m}$ is a nonzero annulator of x_m in $R/(x_1,\ldots,x_{m-1})$ and it doesn't belong to Rx_m. This contradicts the hypothesis that X is regular. $\square$

In the algebraic case, the following proposition is an immediate consequence of condition (i) in Definition and Theorem 3.1.28.

Proposition 3.5.8. *In this situation, the assignment*

$$\coprod_{x\in X} R/Ie(x)\longrightarrow I/I^2,\quad e(x)\mapsto\bar{x}$$

gives rise to an isomorphism of DG-objects in $\mathrm{gr}(\mathcal{M})(R/I)$.

Proof. It is clear that the map commutes with the differentials.
Obviously the map is well defined and surjective. By Axiom (S2), we only have to show that the map is injective. So let $\sum_{i=1}^m e_i\bar{a}_i$ be an element of the kernel of this map. Then we have $\sum\overline{a_ix_i}=0$, i.e. $\sum a_ix_i\in I^2$. We must show that all a_i are elements of I. Let Y be a finite subset of X such that $\sum a_ix_i$ is a sum $\sum_{y,y'\in Y}a(yy')yy'$ with $a(yy')\in R$. Now as in the well-known nongraded case, when we assume that one a_i, say a_m is not in I, we can deduce that a_m is a zero divisor in R/J, where $J\subseteq R$ is the ideal generated by $Y\setminus x_m$. This leads to a contradiction! $\square$

The condition on A in the following corollary is something like a smoothness condition.

Corollary 3.5.9. *Suppose that all A^i are finite A^0-modules. If the kernel of the multiplication map $R:=A\otimes_k A\longrightarrow A$ is generated by a regular s-sequence X in R then there is a natural isomorphism of DG-modules in* $\mathrm{gr}(\mathcal{M})(A)$

$$\Omega_{A/k}\longrightarrow\coprod_{x\in X}Ae(x).$$

Here, X denotes the regular s-sequence in R and to $x\in X$ we have associated a free module generator $e(x)$ with $g(e(x))=g(x)$. The differential on the right is given by the rule $e(x)\mapsto\sum\bar{a}_{xy}e(y)$, where for $x\in X$ the family a_{xy} is chosen in a way such that $s(x)=\sum a_yy$ and $\bar{a}$ denotes the residue class in $R/(X)\cong A$ of an element $a\in R$.

From this statement, we can deduce the corresponding simplicial statement.

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