Marchés financiers avec une infinité d'actifs, couverture quadratique et délits d'initiés

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Introduction

Comme son titre l'indique, cette thèse consiste en une série d'applications assez variées du calcul stochastique aux mathématiques financières. Elle est structurée en quatre chapitres:

1. Chapitre 1: A note on extremality and completeness in financial markets with infinitely many risky assets;
2. Chapitre 2: Arbitrage and completeness in financial markets with given N-dimensional distributions;
3. Chapitre 3: Mean-variance hedging in large financial markets;
4. Chapitre 4: Some results on quadratic hedging with insider trading.

Dans la section suivante, on va donner une brève description de leur contenu.

0.1 Présentation des résultats principaux.

0.1.1 Chapitre 1: A note on extremality and completeness in financial markets with infinitely many risky assets

Dans le Chapitre 1, on étudie les relations existantes entre la complétude des marchés financiers *avec une infinité d'actifs* (MFIA) et l'extremalité des mesures martingale équivalentes.

Tout d'abord, on considère un marché financier *fini*, c.-à-d. présentant un actif sans risque et un nombre fini d'actifs risqués. Plus formellement, on se donne un espace de probabilités $(\Omega, \mathcal{F}, P)$ muni d'une filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfaisant les conditions habituelles (complétude et continuité à droite), où $T > 0$ est un horizon fini fixé, et on suppose $\mathcal{F}_0$ triviale et $\mathcal{F}_T = \mathcal{F}$. On suppose aussi que le processus de l'actif sans risque S^0 satisfait $S^0 \equiv 1$. D'autre part, l'évolution des prix des actifs risqués est décrite par une semimartingale $S = (S_t) = (S_t^1, ..., S_t^d)$ à valeurs dans $\mathbb{R}^d$.

Dans ce marché, non seulement on peut investir dans ces actifs "de base", mais aussi dans des produits dérivés plus complexes (appellés actifs contingents), qui peuvent être formalisés par un certain sous-espace de $L^0(\Omega, \mathcal{F}, P)$, l'ensemble de toutes les fonctions mesurables à valeurs réelles.

Un problème fondamental des mathématiques financières est l'évaluation de ces produits dérivés. Une façon de le résoudre consiste à utiliser la notion d'arbitrage.

Un arbitrage est une stratégie auto-financée qu'un investisseur peut suivre avec zéro investissement initial et telle que son processus de valeur en T est presque sûrement positif et strictement positif avec probabilité strictement positive. On dit que le marché satisfait AOA si il n'admet aucun arbitrage.

On sait très bien que pour un marché financier fini, AOA est équivalent à l'existence d'une mesure martingale (locale) équivalente Q, c.-á-d. une mesure de probabilité equivalente (à P) sous laquelle S est une martingale (locale) (pour une formulation plus précise de ce résultat, voir Delbaen and Schachermayer (1994)). De plus, cette mesure Q est unique si et seulement si le marché est complet; dans ce cas, tout actif contingent F est atteignable (ou simulable), c.-à-d. il existe une stratégie auto-financée ϑ telle que $V_T^\vartheta = F$. Par conséquent, dans un marché financier complet, tout actif contingent F admet un unique prix compatible avec la condition AOA et il est donné par l'espérance $\mathbb{E}_Q[F]$, où Q est l'unique mesure martingale équivalente du marché.

La différence principale entre le cas fini et le cas infini a été soulignée par Artzner et Heath (1995). Dans leur article, ils ont construit un marché financier à deux dates et avec une infinité dénombrable d'actifs risqués, admettant deux mesures martingale équivalentes sous lesquelles le marché est approximativement complet, c.-á-d. l'espace de tous les valeurs finales des stratégies est dense dans l'ensemble $L^1(\Omega, \mathcal{F}, Q)$ de tous les actifs contingents, où Q est une mesure martingale équivalente. Alors, avec cette notion de complétude, dépendant de la mesure martingale Q, l'équivalence entre complétude et unicité de la mesure martingale équivalente n'est plus valable. En effet, cette équivalence est valable si l'une des deux conditions est satisfaite:

1. tout processus de prix est à trajectoires continues (voir Delbaen (1992));

2. la filtration sous-jacente, disons $(\mathcal{F}_t)$, est strictement continue, c.-à-d. pour tout temps d'arrêt τ, on doit avoir $\mathcal{F}_\tau = \mathcal{F}_{\tau-}$ (voir Pratelli (1996)).

Plus tard, pour étendre cette équivalence à un MFIA, Bättig (1999) a proposé une notion de complétude invariante par changement de mesure de probabilité équivalente et donc indépendante de la condition AOA: complétude veut dire que l'image d'un certain opérateur agissant sur les stratégies simples est dense dans l'espace des variables aléatoires bornées $L^\infty(P)$ muni de la topologie faible*. On précisera plus tard cette notion. De ce point de vue, l'exemple de Artzner et Heath (1995) n'est plus si pathologique: si le marché est complet (au sens de Bättig) alors il existe au plus une mesure martingale équivalente, et le marché de Artzner et Heath devient incomplet.

Dans le premier chapitre, on étudie le lien existant entre la complétude des MFIA et l'extrémalité des mesures martingales équivalentes. Notre point de départ est la caractérisation de la complétude des MFIA obtenue par Bättig (1999).

Il considère une famille $\mathcal{V} = \{(Z_t^\alpha)_{t\in[0,1]}\}_{\alpha\in A\cup\{\Delta\}}$ de processus de prix, où A est un ensemble d'actifs risqués (éventuellement infini) et Δ indique l'actif sans risque. Il suppose, comme d'habitude, $Z_t^\Delta \equiv 1$. De plus, chaque agent peut investir dans un nombre fini d'actifs selon des strategies simples du type $(x, (H^\alpha)_{\alpha\in A})$, où $x \in \mathbb{R}$ est la valeur initiale du portefeuille et H^α est donné par

$$H_t^\alpha = \sum_{i=1}^{n_\alpha} h_{i-1}^\alpha \mathbf{1}_{\left(\tau_{i-1}^\alpha, \tau_i^\alpha\right]}(t)$$

où $0 \leq \tau_0^\alpha \leq ... \leq \tau_{n_\alpha}^\alpha \leq 1$ sont des temps d'arrêt, $h_i^\alpha \in L^\infty(\mathcal{F}_{\tau_i^\alpha}, P)$ et $H^\alpha \equiv 0$ sauf pour un nombre fini de $\alpha \in A$. Il désigne par $\mathbf{Y}$ l'espace de toutes ces stratégies et par $\mathbf{C}$=$L^\infty(\mathcal{F}_1, P)$ l'espace de tous les actifs contingents. Alors l'opérateur linéaire $T : \mathbf{Y} \to \mathbf{C}$ donné par

$$T\left(x, (H^\alpha)_{\alpha \in A}\right) = x + \sum_{\alpha \in A} \int_0^1 H_t^\alpha dZ_t^\alpha$$

est bien défini et donne la valeur à l'instant 1 d'une certaine stratégie.

Il désigne par $\mathbf{M}$ le dual topologique de $\mathbf{C}$, c.-à-d. l'espace de toutes les mesures signées définies sur $\mathcal{F}_1$ absolument continues par rapport à P. Il introduit deux autres opérateurs, π_0 et T^* en posant $\pi_0\left(x, (H^\alpha)_{\alpha \in A}\right) = x$ et pour $\mu \in \mathbf{M}$

$$(T^*\mu)\left(x, (H^\alpha)_{\alpha \in A}\right) = \int T\left(x, (H^\alpha)_{\alpha \in A}\right) d\mu$$

et il munit $\mathbf{Y}$ de la topologie plus fine, qui rend continues les fonctionnelles linéaires $\{T^*\mu\}_{\mu \in \mathbf{M}} \cup \{\pi_0\}$. En notant $\mathbf{X}$ le dual topologique de $\mathbf{Y}$, il est facile de voir que T^* est l'opérateur adjoint de T. Enfin, $\mathcal{A}_1$ indique l'ensemble de tous les actifs contingents atteignables, c.-à-d.

$$\mathcal{A}_1 = \left\{ x + \sum_{\alpha \in A} \int_0^1 H_t^\alpha dZ_t^\alpha : \left(x, (H^\alpha)_{\alpha \in A}\right) \in \mathbf{Y} \right\}.$$

Il adopte les deux définitions suivantes de complétude de marché:

1. (*complétude*) le marché est dit *complet* si $\mathcal{A}_1 = \mathrm{Im}T$ est dense dans $\mathbf{C}$ par rapport à la topologie faible*;

2. (*Q-complétude*) en supposant qu'il existe une mesure martingale (locale) équivalente Q, le marché est *Q-complet* si $\mathcal{A}_1 = \mathrm{Im}T$ est dense dans $\mathbf{C}$ par rapport à la topologie $L^1(\mathcal{F}_1, Q)$.

Bättig (1999) montre que si le marché est complet, alors il existe au plus une mesure martingale (locale) équivalente. La preuve de ce résultat est basée sur l'équivalence entre la complétude de marché et l'injectivité de l'opérateur adjoint T^*.

Dans le premier chapitre, on démontre le même résultat mais avec des techniques différentes et plus élémentaires, basées sur la notion d'extrémalité de mesures. En particulier, on obtient une version du théorème de Douglas-Naimark pour un système dual $\langle X, Y\rangle$ d'espaces topologiques réels localement convexes munis de la topologie faible $\sigma(X, Y)$, et on l'applique notamment à l'espace L^∞ avec les topologies faibles $\sigma(L^\infty, L^p)$, $p \geq 1$. Grâce à ces résultats, nous obtenons des conditions équivalentes à la complétude du marché et basées sur la notion d'extrémalité de mesure. Plus rigoureusement, soit $\mathcal{K}$ un ensemble convexe de mesures finies, alors une mesure $Q \in \mathcal{K}$ est extrémale dans $\mathcal{K}$, si et seulement si $Q = \beta Q_1 + (1-\beta)Q_2$ pour $Q_1, Q_2 \in \mathcal{K}$ et $\beta \in (0,1)$ entraîne $Q = Q_1 = Q_2$. Notre résultat principal est la caractérisation suivante de la complétude de marché (voir Theorem 11):

le marché est complet si et seulement si toute mesure de probabilité $Q \ll P$ est extrémale dans l'ensemble de toutes les mesures martingale équivalentes à Q.

Cette caractérisation nous permet de donner une preuve nouvelle et plus simple de la version de Bättig du deuxième théorème fondamental de l'évaluation des actifs contingents.

Enfin, le premier chapitre contient une discussion plus générale des différents types de complétude, dépendant de la topologie définie sur $\mathbf{C}$, et une application à l'exemple pathologique construit par Artzner et Heath (1995).

0.1.2 Chapitre 2: Arbitrage and completeness in financial markets with given N-dimensional distributions

Motivé par la formule de Breeden et Litzenberger (1978), qui établit l'équivalence entre les prix des options calls européens et les marginales du processus des prix sous la mesure de probabilité risque-neutre, on s'intéresse au problème suivant: on se donne un marché financier, modelisé par un processus $S = (S_t)_{t \in \mathcal{T}}$, et une famille

$$\mathbf{M}_N = \{\mu_{t_1,\dots,t_N} : t_1, \dots, t_N \in \mathcal{T}\}$$

de mesures de probabilité sur $\mathcal{B}(\mathbb{R}^N)$, où N est un entier positif et $\mathcal{T}$ l'espace des temps; on cherche ensuite des proprietés équivalentes à l'existence et l'unicité d'une mesure martingale (locale) équivalente (MME) Q telle que le processus de prix S ait sous Q les marginales de dimension N (abbr. N-marginales) $\mathbf{M}_N$ données. On appelle ces deux conditions équivalentes *N-mixed no free lunch* et *N-complétude du marché*, respectivement. Elles se basent sur une classification des actifs contingents par rapport à leur dépendance trajectorielle de S et sur la notion de stratégie N-mélangée (en anglais "N-mixed strategy"). On montre aussi que pour le modèle de Black-Scholes avec sauts, l'ensemble des MMEs avec 1-marginales données se réduit à un seul élément.

Une telle étude est motivée par le fait que l'observation des prix des calls donne un aperçu de la loi des prix des actifs. En effet, si on suppose que sur le marché on peut trouver une famille d'options call pour tout strike k et avec maturité t, et si on désigne par $C(k,t)$ le prix d'un call européen avec strike k et maturité t, la formule de Breeden-Litzenberger établit une relation entre la famille des prix des calls $\{C(k,t) : k > 0\}$ et la loi (sous une certaine mesure martingale équivalente Q) de S_t à t fixé (voir Breeden et Litzenberger (1978) ou Dupire (1997)):

$$\frac{\partial}{\partial k+} C(k,t) = -Q(S_t > k).$$

On peut donc affirmer que des prix des calls on peut déduire la loi de S_t sous Q. On pourrait même généraliser cette remarque, en disant que plus le marché est liquide plus on peut en tirer d'informations sur la loi de S sous Q.

Si le marché S considéré est incomplet, alors il existe une infinité de mesures martingales équivalente et, en utilisant l'information additionnelle sur la loi de S, on peut espérer de réduire l'ensemble des mesures martingale équivalentes à un seul élément. D'habitude, dans la littérature sur les marchés incomplets, choisir la "bonne" mesure martingale équivaut à trouver la mesure qui minimise une certaine "distance" entre la probabilité objective P et l'ensemble $\mathcal{M}$. Par exemple, en utilisant l'entropie relative, on obtient le mesure à entropie minimale (voir Frittelli (2000a)); d'autre part, en utilisant la distance L^2, on obtient la mesure à variance optimale (voir Delbaen et Schachermayer (1996)). Tous ces critères sont étroitement liés à la maximisation d'une certaine fonction d'utilité (voir par exemple Frittelli (2000b)). Le choix de la "bonne" mesure martingale dépend des préférences de l'agent économique.

Au lieu de ces critères "subjectifs", on pourrait utiliser l'information "objective" contenue dans le prix de marché de quelques actifs contingents "liquides", et concentrer l'attention sur les mesures martingale compatibles avec cette information.

Donc, en suivant cette approche, il serait plutôt naturel, une fois fixés un marché financier S et une famille de probabilités $\mathbf{M}_N$ définies sur $\mathcal{B}(\mathbb{R}^N)$, de s'intéresser aux deux questions suivantes:

1. est-qu'il existe une mesure martingale équivalente Q telle que les N-marginales de S soient dans $\mathbf{M}_N$?
2. Si une telle mesure existe, est-ce qu'elle est unique?

Dans le deuxième chapitre, on montrera (voir Theorem 34, quand l'espace de probabilité est fini et l'espace des temps est discret, et Theorem 44, pour le cas à temps continu) qu'une mesure Q avec ces propriétés existe si et seulement si on ne peut construire aucune position d'arbitrage à travers une strategie admissible et en investissant de façon statique dans les actifs contingents dépendants d'au plus N coordonnées temporelles du processus de prix S (par exemple une option lookback calculée sur une grille de N instants de temps).

D'autre part, Q est l'unique mesure martingale sous la quelle les lois N-dimensionnelles du processus de prix sont dans $\mathbf{M}_N$ si et seulement si tout actif contingent peut être atteint par un investissement initial, une stratégie admissible et, encore, en investissant statiquement en actifs contingents dépendant d'au plus N coordonnées temporelles de S (voir Theorem 38, pour le cas à temps discret, et Subsection 2.3.3 pour le cas à temps continu).

Enfin, comme application, on montrera que, si on se donne une famille de 1-marginales $\mathbf{M}$, le modèle de Black-Scholes avec sauts, quand tous ses coefficients sont fonctions déterministes du temps, admet au plus une mesure martingale équivalente Q dans Υ [1], sous laquelle les 1-marginales du processus de prix sont dans $\mathbf{M}$ (Proposition 52).

0.1.3 Chapitre 3: Mean-variance hedging in large financial markets

Dans le troisième chapitre, on considère un problème de couverture moyenne-variance (CMV) dans un "large financial market", c.-à.-d. un marché financier avec une infinité

[1] Υ est le sousensemble des mesures martingale équivalentes induite par des paramètres h_t fonctions détérministes du temps (voir la section 2.3.4).

dénombrable d'actifs risqués modélisés par une suite de semimartingales continues.

Comme dans le cas fini, si le marché est incomplet, il existe au moins un actif contingent qui n'est atteignable par aucune stratégie. On a donc le problème d'évaluer cet actif (évaluation ou "pricing") et de gérer le risque dû à son achat ou sa vente (couverture ou "hedging"). Dernièrement, plusieurs techniques d'évaluation et de couverture des actifs contingeants dans les marchés incomplets ont été developpées. On se concentre ici sur l'approche CMV, qui consiste à chercher une stratégie auto-financée qui minimise le risque résidu entre l'actif contingent et la valeur du portefeuille. Du point de vue mathématique, il s'agit d'une projection sur l'espace des intégrales stochastiques. Plus rigoureusement, on se donne un "large financial market" $X = \{S^0, S\}$, où S^0 est le processus de prix de l'actif sans risque $S_t^0 = \exp \int_0^t r_s ds$ et $S = (S^i)_{i \geq 1}$ est une suite de semimartingales à valeurs réelles, qui modélisent la dynamique des prix des actifs risqués, et un actif contingent $F \in L^2(P)$. On voudrait résoudre le problème de minimisation (introduit par Föllmer et Sondermann (1986)) suivant:

$$\min_{\vartheta \in \Theta} \mathbb{E} \left[F - V_T^{x,\vartheta} \right]^2, \tag{1}$$

où

$$V_T^{x,\vartheta} = S_T^0 \left(x + \int_0^T \vartheta_t d \left(S/S^0 \right)_t \right) \tag{2}$$

est la valeur finale du portefeuille autofinancé dans les actifs de base, avec l'investissement initial x et les quantités ϑ investies dans les actifs risqués. L'existence d'une solution est liée au fait que l'espace des intégrales stochastiques choisi soit fermé dans $L^2(P)$, où P est la "vraie" probabilité du marché. Evidemment, dans un marché fini ce problème a bien un sens et il a été résolu par des méthodes différentes par Rheinländer et Schweizer (1997) et Gourieroux et al. (1998).

Dans leur article, Gourieroux et al. (1998) introduisent la notion d'extension artificielle, qui consiste à ajouter aux actifs de base X un numéraire, défini comme un portefeuille autofinancé dans X tel que son processus de valeur soit strictement positif. Ce numéraire est utilisé soit comme facteur d'actualisation soit comme actif sur lequel on peut investir, élargissant ainsi la famille des actifs risqués. Ils montrent l'invariance de l'ensemble des mesures martingale équivalentes et aussi que cet extension ne change pas l'ensemble des opportunités d'investissement, défini comme l'ensemble des actifs contingents atteignables. Mais leur résultat principal est la preuve qu'on peut transformer le problème de CMV initial en un autre plus simple, grâce à un bon changement de numéraire (*hedging numéraire*) et à la méthode de l'extension artificielle. Plus rigoureusement, en indiquant par $\widetilde{a}$ ce numéraire, par $V(\widetilde{a})$ son processus de valeur, et par $X(\widetilde{a}) = (S^0/V(\widetilde{a}), S/V(\widetilde{a}))$ le processus de prix de la famille d'actifs de base renormalisée par rapport au nouveau numéraire, il montrent que le problème de minimisation initial est équivalent à

$$\min_{\phi(\widetilde{a}) \in \Phi(\widetilde{a})} \mathbb{E}_{\widetilde{P}(\widetilde{a})} \left[\frac{F}{V_T(\widetilde{a})} - x - \int_0^T \phi_t(\widetilde{a}) \, dX_t(\widetilde{a}) \right]^2 \tag{3}$$

où $\phi_t(\widetilde{a})$ désigne les quantités investies dans les actifs de base d'un portefeuille autofinancé par rapport à la famille élargie, et $\widetilde{P}(\widetilde{a})$ est une probabilité équivalente à P, définie de façon unique par $\widetilde{a}$, et telle que $X(\widetilde{a})$ soit une martingale locale sous $\widetilde{P}(\widetilde{a})$. Cette dernière propriété leur permet de résoudre (3) par la decomposition de Galtchouk-Kunita-Watanabe.

On considère maintenant un marché financier avec une infinité dénombrable d'actifs risqués. Motivés par le Capital Asset Pricing Model (CAPM) et le Arbitrage Pricing Theory (APT), Kabanov et Kramkov (1994) introduisent la notion de large financial market comme une suite de marchés finis de dimension croissante. Ils se donnent une suite de bases stochastiques $\mathbf{B}^n := (\Omega^n, \mathcal{F}^n, \mathbf{F}^n = (\mathcal{F}^n_t), P^n)$, $n \geq 1$, avec, pour simplifier, la tribu initiale triviale. Les prix des actifs risqués sont modelisés par une semimartingale $S^n = (S^n_t)$ définie sur $\mathbf{B}^n$ et à valeurs dans $\mathbb{R}^d_+$ pour un certain $d = d(n)$. Ils fixent aussi une suite d'horizons temporels T^n. La suite $\mathbf{M} = \{(\mathbf{B}^n, S^n, T^n)\}$ est appellée large financial market. Ils étudient les propriétés de AOA et en particulier la relation existante entre l'arbitrage asymptotique et la contiguité de la mesure martingale équivalente.

Dans le troisième chapitre, en utilisant la théorie de l'intégration stochastique pour une suite de semimartingales développée par De Donno et Pratelli (2003), on montre qu'il est possible donner un sens au problème CMV même en présence d'une infinité dénombrable d'actifs. En particulier, on montre que l'ensemble des opportunités d'investissement est fermé dans $L^2(P)$ (Proposition 63) et qu'il est invariant par changement de numèraire (Proposition 67), mais on n'a pas d'expression explicite de la bijection correspondante. Cette différence fondamentale avec le cas fini est dûe au fait qu'on a élargi l'espace des stratégies en incluant les intégrands généralisés (à valeurs dans l'espace des opérateurs non nécéssairement bornés) et donc on ne peut plus multiplier les stratégies par le processus de prix.

On considère aussi, pour tout $n \geq 1$, le marché formé des premiers n actifs risqués et on montre que les solutions aux problèmes CMV n-dimensionnels correspondants convergent dans $L^2(P)$, quand n tend vers l'infini, vers l'unique solution du problème infini-dimensionnel initial (Proposition 72).

0.1.4 Chapitre 4: Some results on quadratic hedging with insider trading

Dans le dernier chapitre, on considère le problème de couverture dans un marché financier sous l'hypothèse AOA et avec deux catégories d'investisseurs, qui basent leur décisions sur deux differents flots d'informations concernant l'évolution future des prix, décrites par deux filtrations $\mathbf{F}$ et $\mathbf{G} = \mathbf{F} \vee \sigma(G)$ où G est une variable aléatoire donnée (à valeurs dans un espace polonais $(U, \mathcal{U})$) représentant l'information additionnelle.

On se concentre ici sur deux types d'approche quadratique pour couvrir un actif contingent donné $X \in L^2(P, \mathcal{F}_t)$ avec $t < T$: minimisation du risque local (abbr. MRL) et CMV. En utilisant des techniques de grossissement initial des filtrations, on va pouvoir résoudre le problème de couverture pour les deux investisseurs et comparer leurs stratégies optimales sous les deux approches.

Plus précisement, sous la condition

$$P[G \in \cdot\, | \mathcal{F}_t](\omega) \sim P[G \in \cdot\,]$$

pour tout $t \in [0, T)$ et P-p.p. $\omega \in \Omega$, on sait d'après Jacod (1985) qu'il existe un processus

$(\omega, t, x) \mapsto p_t^x(\omega)$, avec les bonnes propriétés de mesurabilité, et tel que la mesure $p_t^x P[G \in dx]$ sur $(U, \mathcal{U})$ soit une version de la loi conditionnelle $P[G \in dx|\mathcal{F}_t]$. De plus, on a le résultat suivant, qui est dû à Amendinger et al. (1998): il existe une $(P, \mathbf{G}^0)$-martingale locale $\widetilde{N}$ issue de 0, $(P, \mathbf{G}^0)$-orthogonale à $\widetilde{M}$ (c.-à.-d. $\langle \widetilde{M}^i, \widetilde{N} \rangle = 0$ pour $i = 1, ..., d$) et telle que

$$\frac{1}{p_t^G} = \mathcal{E}\left(-\int \left(\mu^G\right)' d\widetilde{M} + \widetilde{N}\right)_t, \quad t \in [0, T). \tag{4}$$

Donc, pour la MRL, on montre que pour toute variable aléatoire additionnelle G telle que la martingale orthogonale $\widetilde{N}$ dans la representation (4) soit identiquement nulle, les deux investisseurs adoptent la même strategie minimisant le risque local et que le processus de coût de l'agent ordinaire peut s'exprimer comme la projection sur $\mathbf{F}$ de celui de l'initié (Proposition 84).

Pour ce qui concerne l'approche CMV, on montre que les "stratégies optimales" des deux agents sous leurs mesures respectives à variance optimale coïncident, c.-à-d.

$$\vartheta_s^{opt,\mathbf{G}} = \vartheta_s^{opt,\mathbf{F}}, \quad s \in [0, t],$$

pour tout $t \in [0, T)$. On étudie aussi un modèle à volatilité stochastique assez général, incluant les modèles de Hull et White, Heston et Stein et Stein. Dans ce contexte plus spécifique et pour une variable aléatoire G mesurable par rapport à la filtration engendrée par la volatilité, on obtient (Proposition 92) la caractérisation feedback suivante du processus $\xi^{MVH} = \vartheta^{MVH,\mathbf{G}} - \vartheta^{MVH,\mathbf{F}}$ défini comme différence entre la strategie optimale de l'initié et celle de l'ordinaire:

$$\xi_s^{MVH} = \rho_s^{opt}\left(V_{s-}^{opt,\mathbf{G}} - V_{s-}^{opt,\mathbf{F}} + \int_0^s \xi_u^{MVH} dS_u\right), \quad s \in [0, t] \tag{5}$$

où $V_s^{opt,\mathbf{H}} := \mathbb{E}^{opt}[X|\mathcal{H}_s]$ pour $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$ et $\rho_s^{opt} := \zeta_s^{opt,\mathbf{F}}/Z_s^{opt,\mathbf{F}} = \zeta_s^{opt,\mathbf{G}}/Z_s^{opt,\mathbf{G}}$, $s \in [0, t]$. $Z^{opt,\mathbf{G}}$ et $\zeta^{opt,\mathbf{H}}$ sont, respectivement, le processus de densité de la mesure à variance optimale et l'intégrand dans sa représentation intégrale par rapport à S, sous la filtration $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$.

Chapter 1

A note on extremality and completeness in financial markets with infinitely many risky assets

This chapter is based on the technical report n. 655 of the Laboratoire de Probabilités et Modèles Aléatoires of the Universities of Paris VI and VII, entitled "A weak version of Douglas theorem with applications to finance", submitted to the review *Journal of Applied Probability*. I am indebted to Marc Yor for his help, support and suggestions, and to Massimo Marinacci for his interest in this work.

1.1 Introduction

Artzner and Heath (1995) constructed a market with an infinite number of equivalent martingale probability measures, which is complete under two of such measures, the extremal ones. This market has the essential property that the set of risky assets is infinite, in other words it is a large financial market. The possibility of such an economy implies that the equivalence between completeness and uniqueness of the equivalent martingale measure is not verified in an infinite assets setting.

Bättig (1999), Jin, Jarrow and Madan (1999) and Jarrow and Madan (1999) adopted a different notion of market completeness in order to extend this equivalence even to a large financial market. They give a definition of completeness which is independent from the notion of no-arbitrage, and show that if the market is complete, then there exists at most one equivalent martingale signed measure and if the market is arbitrage-free, then this signed measure is in fact a probability. In order to demonstrate them, these authors have to verify the surjectivity of a certain operator and then the injectivity of its adjoint.

Here we will examine the interplay existing between the extremality of martingale probability measures and the various notions of market completeness introduced by Artzner and Heath (1995) and Jin, Jarrow and Madan (1999). For this we will need two versions of the Douglas Theorem, which is a functional analysis result connecting the density of the subsets of some space L^p with the extremality on a certain subset of measures of the underlying probability. Now, we quote them without proofs, for which one can consult Douglas (1964) (Theorem 1, p. 243) or Naimark (1947) for the first and Yor (1976) (Proposition 4 of the Appendice, p. 306) for the second.

Theorem 1 *Let $(\Omega, \mathcal{F}, P)$ a probability space and let F be a subspace of $L^1(P)$ such that $1 \in F$. The following two assertions are equivalent:*

1. *F is dense in $(L^1(P), \|\cdot\|_1)$;*
2. *if $g \in L^\infty(P)$ satisfies $\int fgd\mu = 0$ for each $f \in F$, then $g = 0$ P-a.s.;*
3. *P is an extremal point of the set*

$$\widetilde{\Xi}_1(P) = \left\{Q \in \mathcal{P} : \textit{for each } f \in F,\ f \in L^1(Q) \ \textit{ and } \mathbb{E}_Q(f) = \mathbb{E}_P(f)\right\},$$

where $\mathcal{P}$ is the space of all probability measures over $(\Omega, \mathcal{F})$.

We denote $ba(\Omega, \mathcal{F})$, or simply ba, the space of additive bounded measures on the measurable space $(\Omega, \mathcal{F})$. It is well known that one can identify ba with the topological dual of the space $L^\infty(P)$ equipped with the strong topology. Finally, with an obvious notation, one has the decomposition $ba = ba^+ - ba^+$. For further information on ba, one can consult Dunford and Schwartz (1957).

Theorem 2 *Let $ba^+(P) = \{\nu \in ba^+; \nu \ll P\}$, and let F be a subspace of $L^\infty(P)$ such that $1 \in F$. The following two assertions are equivalent:*

1. *F is dense in $(L^\infty(P), \|\cdot\|_\infty)$;*
2. *every additive measure $\nu \in ba^+(P)$ is an extremal point of the set*

$$\Xi_{ba}(\nu) = \left\{\lambda \in ba^+(\nu) : \textit{for each } f \in F,\ \lambda(f) = \nu(f)\right\}.$$

We note that the spaces L^p considered in the previous theorems are equipped with their respective strong topologies.

In Section 1.2 we obtain a version of the Douglas-Naimark Theorem for a dual system $\langle X, Y \rangle$ of ordered locally convex topological real vector spaces, and we apply it to the special case $\langle X, Y \rangle = \langle L^\infty, L^p \rangle$ for $p \geq 1$.. In Subsection 1.2.3 we obtain also a Douglas-Naimark Theorem for L^∞ with L^p-norm topologies for $p \geq 1$, which we will use for the discussion of the completeness of the AH-market.

In Section 1.3, we apply these results to mathematical finance. In particular, in subsections 1.3.2 and 1.3.3 we give new proofs of the versions of the Second Fundamental Theorems of Asset Pricing (abbr. SFTAP) obtained by Jarrow, Jin and Madan (1999) and Bättig (1999), based on the notion of extremality of measures thanks to the results established in Section 1.2. The advantage of this approach is that it permits to work directly on the equivalent martingale measures set of the market, using only some elementary geometrical argument. In Subsection 1.3.4 we discuss the completeness of the Artzner and Heath market with respect to several topologies and we obtain a more general construction of it.

1.2 A weak version of the Douglas-Naimark Theorem

1.2.1 Weak Douglas-Naimark Theorem for a dual system

We recall some basic facts about duality for a locally convex topological real vector space (abbr. LCS). Let X, Y be a pair of real vector spaces, and let f be a bilinear form on $X \times Y$, satisfying the separation axioms:

$$\begin{aligned} f(x_0, y) &= 0 \text{ for each } y \in Y \text{ implies } x_0 = 0, \\ f(x, y_0) &= 0 \text{ for each } x \in X \text{ implies } y_0 = 0. \end{aligned}$$

The triple (X, Y, f) is called a dual system or duality (over $\mathbb{R}$). To distinguish f from other bilinear forms on $X \times Y$, f is called the canonical bilinear form of the duality, and is usually denoted by $(x, y) \mapsto \langle x, y \rangle$. The triple (X, Y, f) is more conveniently denoted by $\langle X, Y \rangle$.

If $\langle X, Y \rangle$ is a duality, the mapping $x \mapsto \langle x, y \rangle$ is, for each $y \in Y$, a linear form f_y on X. Since $y \mapsto f_y$ is linear and, by virtue of the second axiom of separation, biunivocal, it is an isomorphism of Y into the algebraic dual X^* of X; thus Y can be identified with a subspace of X^*. Note that under this identification, the canonical bilinear form of $\langle X, Y \rangle$ is induced by the canonical bilinear form of $\langle X, X^* \rangle$.

We recall that the weak topology $\sigma(X, Y)$ is the coarsest topology on X for which the linear forms f_y, $y \in Y$, are continuous; by the first axiom of separation, X is a LCS under $\sigma(X, Y)$.

Let $\langle X, Y \rangle$ be a duality between LCS's and let $K \subset X$ be a cone, which introduce in X a natural order $\leq$, which we call K-order, i.e. $x \leq x'$ if $x' - x \in K$. Now, we set

$$H_K = \{y \in Y : \langle x, y \rangle \geq 0 \text{ for each } x \in K\}$$

and we observe that it is a cone contained in Y. If there is no ambiguity about the cone K we will consider, we will simply write H instead of H_K.

We assume that Y is a vector lattice. We recall that a vector lattice is an ordered vector space Y over $\mathbb{R}$ such that for each pair $(y_1, y_2) \in Y$, $\sup(y_1, y_2)$ and $\inf(y_1, y_2)$ exist. Thus, we can define the positive and the negative part of each $y \in Y$ by

$$\begin{aligned} y^+ &= \sup(0, y) \\ y^- &= \sup(0, -y) \end{aligned}$$

and its absolute value $|y| = \sup(y, -y)$ which satisfies $|y| = y^+ + y^-$. Finally, we have $y = y^+ - y^-$.

We need some additional notation. If $y \in Y$ and $F \subset X$, we set

$$\Xi_{y,F} = \{z \in H_K : \langle x, z \rangle = \langle x, y \rangle \text{ for every } x \in F\}.$$

If there is no confusion about the subset F, we will simply write Ξ_y. Finally, we set $K_0 = K \backslash \{0\}$ and $H_0 = H \backslash \{0\}$.

For more information on topological vector spaces, see e.g. Schaefer (1966) or Narici and Berenstein (1985).

Theorem 3 *Let F be a subspace of X. The following assertions are equivalent:*

1. *F is dense in $(X, \sigma(X, Y))$;*

2. *every $y \in H_0$ is extremal in Ξ_y.*

Proof. Firstly, we show that 1. implies 2.. It is known (e.g. exercise 9.108(a) in Narici and Berenstein (1985), p.222) that F is dense in $(X, \sigma(X, Y))$ if and only if, for every $y \in Y$, $\langle x, y \rangle = 0$ for each $x \in F$ implies $y = 0$. Now, we proceed by contradiction and we assume that there exists $y \in H_0$ not extremal in Ξ_y, i.e. we can write $y = \alpha y_1 + (1 - \alpha) y_2$ where $\alpha \in (0, 1)$ and $y_i \in \Xi_y$ for $i = 1, 2$. Then, we have

$$\langle x, y_1 \rangle = \langle x, y_2 \rangle = \langle x, y \rangle ,$$

which implies

$$\langle x, y_1 - y_2 \rangle = 0.$$

Then $y_1 = y_2 = y$.

In order to demonstrate the other direction of the equivalence, we note that it is sufficient to show, for every $y \in Y$, that if $\langle x, y \rangle = 0$ for each $x \in F$, then $y = 0$. We assume that there exist $y_0 \in Y$ and $x_0 \in X \backslash F$ such that $\langle x, y_0 \rangle = 0$ for every $x \in F$ and $\langle x_0, y_0 \rangle \neq 0$. Since Y is a vector lattice, we can write $y_0 = y_0^+ - y_0^-$ and $|y_0| = y_0^+ + y_0^- \in H_0$, where $y_0^+, y_0^- \in H_0$. Now, we observe that

$$|y_0| = \frac{1}{2} \left(2y_0^+ + 2y_0^- \right)$$

and, since $2y_0^+, 2y_0^- \in \Xi_{|y_0|}$, we have, by the extremality hypothesis, that $|y_0| = 2y_0^+ = 2y_0^-$, which implies $y_0 = 0$. $\square$

We note that we have used the assumption that Y is a lattice only in the second part of the proof. Then, even if Y is not a lattice, 1. still implies 2..

1.2.2 The space L^∞ equipped with weak topologies

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, where μ is a positive finite measure.

Now, we want to apply Theorem 3 to the special case $X = L^\infty(\mu)$ equipped with a family of weak topologies. In this case, the order we consider is the usual one, i.e. for each $f, g \in L^\infty(\mu)$, $f \geq g$ if $f(\omega) \geq g(\omega)$ for every $x \in \Omega$. In other words, we choose $K = L^\infty_+(\mu)$.

Corollary 4 *Let F be a subspace of $L^\infty(\mu)$, where μ is a non null finite positive measure and let $p \geq 1$. The following assertions are equivalent:*

1. *F is dense in $(L^\infty(\mu), \sigma(L^\infty, L^p))$;*

2. *every* $g \in L^p(\mu)$*, such that* $g \geq 0$ *and* $\mu(\{g > 0\}) > 0$*, is extremal in*

$$\Xi_p(g) = \left\{ h \in L^p(\mu) : h \geq 0 \text{ and } \int fh d\mu = \int fg d\mu \text{ for each } f \in F \right\};$$

3. *every non null finite positive measure* $\nu \ll \mu$*, such that* $\frac{d\nu}{d\mu} \in L^p(\mu)$*, is extremal in*

$$\Xi_p(\nu) = \left\{ \rho \in \mathcal{M}_p(\nu) : \int f d\rho = \int f d\nu \text{ for each } f \in F \right\}$$

where $\mathcal{M}_p(\nu)$ *is the space of finite positive measures* ρ *absolutely continuous with respect to* ν *and such that* $\frac{d\rho}{d\nu} \in L^p(\nu)$*.*

Proof. It is an immediate application of Theorem 3. □

Corollary 5 *Under the same assumptions of Corollary 4, if* μ *is a probability measure and* $1 \in F$*, the following two assertions are equivalent:*

1. F *is dense in* $(L^\infty(\mu), \sigma(L^\infty, L^p))$*;*

2. *every probability measure* $\nu \ll \mu$*, such that* $\frac{d\nu}{d\mu} \in L^p(\mu)$*, is extremal in*

$$\widetilde{\Xi}_p(\nu) = \left\{ \rho \in \mathcal{P}_p(\nu) : \int f d\rho = \int f d\nu \text{ for each } f \in F \right\}$$

where

$$\mathcal{P}_p(\nu) = \{\rho \in \mathcal{M}_p(\nu) : \rho(1) = 1\}.$$

Proof. If F is dense in $(L^\infty(\mu), \sigma(L^\infty, L^p))$, then, by Corollary 4, every probability measure $\nu \ll \mu$ such that $\frac{d\nu}{d\mu} \in L^p(\mu)$ is extremal in $\Xi_p(\nu) \supset \widetilde{\Xi}_p(\nu)$ and it follows that ν is extremal in $\widetilde{\Xi}_p(\nu)$. Then, 1. implies 2..

Now, we assume that there exists a positive finite measure $\nu \ll \mu$, such that $\frac{d\nu}{d\mu} \in L^p(\mu)$, which satisfies $\nu = \alpha\rho_1 + (1-\alpha)\rho_2$ with $\alpha \in (0,1)$ and $\rho_i \in \Xi_p(\nu)$ for $i = 1, 2$. By setting $\overline{\nu} = \frac{\gamma}{\nu(1)}$, we have

$$\overline{\nu} = \alpha \frac{\rho_1}{\nu(1)} + (1-\alpha) \frac{\rho_2}{\nu(1)},$$

and, since $1 \in F$, $\rho_1(1) = \rho_2(1) = \nu(1)$ and so $\overline{\rho}_i = \frac{\rho_i}{\nu(1)} \in \widetilde{\Xi}_p(\overline{\nu})$ for $i = 1, 2$. □

1.2.3 The space L^∞ equipped with L^p-norm topologies

In this subsection, we obtain a Douglas-Naimark Theorem for $L^\infty(\mu)$ equipped with L^p-norm topologies, i.e. the topologies induced by the norms $\|\cdot\|_p$ for $p \geq 1$. In the proof we will essentially use the same argument as in the proof of Theorem 3.

Theorem 6 *Let F be a subspace of $L^\infty(\mu)$ such that $1 \in F$ and let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The following assertions are equivalent*

1. *F is dense in $(L^\infty(\mu), \|\cdot\|_p)$;*

2. *for each $g \in L^q$ such that $g \geq 0$ and $\int g d\mu = 1$, the probability $\nu = g \cdot \mu$ is extremal in $\widetilde{\widetilde{\Xi}}_q(\nu)$.*

Proof. We first show that 1. implies 2.. The Hölder inequality shows that if F is dense in $L^\infty(\mu)$ for the $L^p(\mu)$-topology, then it is dense even for the $\sigma(L^\infty, L^q)$-topology. So Corollary 5 applies and the thesis follows.

In order to show that 2. implies 1. it is sufficient to show that if $h \in L^q(\mu)$ verify $\int fhd\mu$ for each $f \in F$, then $h = 0$ μ-a.s.. We assume, without loss of generality, that $\nu = |h| \cdot \mu$ is a probability. As usually, we denote by h^+ and h^- the positive and negative parts, respectively, of h. Hence, we have

$$\int fh^+ d\mu = \int fh^- d\mu$$

for every $f \in F$. Hence

$$\nu = |h| \cdot \mu = \frac{1}{2}\left[(2h)^+ \cdot \mu + (2h)^- \cdot \mu\right]$$

is a middle-sum of two points of the set $\widetilde{\widetilde{\Xi}}_q(\nu)$. But, by assumption, ν is an extremal point of $\widetilde{\widetilde{\Xi}}_q(\nu)$ and then $|h| = 2h^+ = 2h^-$ μ-a.s., which implies $h = 0$ μ-a.s.. $\square$

An immediate consequence of Corollary 5 and Theorem 6 is the following

Corollary 7 *Let $F \subset L^\infty(\mu)$ be a subspace such that $1 \in F$ and let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. F is dense in $(L^\infty(\mu), \sigma(L^\infty, L^p))$ if and only if it is dense in $(L^\infty(\mu), \|\cdot\|_q)$.*

Remark 8 *For the case $p = 1$, being $L^\infty(\mu)$ dense in $(L^1(\mu), \|\cdot\|_1)$, we have the same equivalence as in Theorem 1.*

1.3 Applications to finance

1.3.1 The model

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We consider a financial market where the set of trading dates is given by $\mathcal{T} \subseteq [0, 1]$ with $\mathcal{T} = \{0, 1\}$ or $\mathcal{T} = [0, 1]$ and we denote $\mathcal{S}$ the set of discounted price processes of this economy, i.e. $\mathcal{S}$ is a family of stochastic processes indexed by $\mathcal{T}$ and adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}$, where $\mathcal{F}_0$ is the trivial σ-field and $\mathcal{F}_1 = \mathcal{F}$. For simplicity, we assume that, for each $S = (S_t)_{t \in \mathcal{T}} \in \mathcal{S}$, $S_0 = 1$. We note that the set

$\mathcal{S}$ may be infinite. In the continuous-time case, we will always suppose that $\mathbb{F}$ satisfies the usual conditions and each price process $S \in \mathcal{S}$ is càdlàg.

Following Jin, Jarrow and Madan (1999) and Bättig and Jarrow (1999), we identify the set of contingent claims with the space of essentially bounded random variables $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$ equipped with some topology τ. We call P the true probability of the market.

Finally, throughout the sequel, $\mathbb{R}$ will be the set of real numbers and, if $\mathcal{A}$ is an arbitrary subset of L^∞, $v.s.(\mathcal{A})$ will denote the vector space generated by $\mathcal{A}$. We recall that $\mathcal{A}$ is total in L^∞ if, by definition, $v.s.(\mathcal{A})$ is dense in L^∞.

Now, we give two notions of market completeness for the discrete and the continuous-time cases.

Definition 9 *(discrete-time case) Let $\mathcal{T} = \{0, 1\}$. The market $\mathcal{S}$ is said to be τ-complete if the set*

$$\mathcal{Y}_d = v.s.\left((\mathcal{S}_1 \cup \mathbb{R}) \cap L^\infty\right)$$

where $\mathcal{S}_1 = \{S_1; S \in \mathcal{S}\}$, is total in L^∞ for the topology τ.

Definition 10 *(continuous-time case) Let $\mathcal{T} = [0, 1]$. The market $\mathcal{S}$ is said to be τ-complete if the set*

$$\mathcal{Y}_c = v.s.\left((\mathcal{I} \cup \mathbb{R}) \cap L^\infty\right)$$

is dense in L^∞ for the topology τ, where

$$\mathcal{I} = \left\{Y\left(S_\tau - S_\sigma\right) : \sigma \le \tau \ \mathbb{F}\text{-stopping times}, Y \in L^\infty\left(\mathcal{F}_\sigma, P\right), S \in \mathcal{S}\right\}.$$

Throughout the sequel we will assume that the sets $\mathcal{Y}_d$ and $\mathcal{Y}_c$ are nonempty.

The space L^∞ will be equipped with the strong topology, i.e. the topology induced by the supremum norm $\|\cdot\|_\infty$, and the weak topologies $\sigma(L^\infty, L^p)$ for $p \ge 1$. If the market is τ-complete with $\tau = \|\cdot\|_\infty$ or $\tau = \sigma(L^\infty, L^1)$, we will say that it is strongly complete or, respectively, weakly* complete.

A detailed discussion of the economic interpretation of the topology $\sigma(L^\infty, L^1)$ can be found in Bättig and Jarrow (1999).

If we apply Corollary 4 to these notions of market completeness, we obtain immediately the following equivalence.

Theorem 11 *Let $p \ge 1$. The following two assertions are equivalent:*

1. *The market is $\sigma(L^\infty, L^p)$-complete;*

2. *every probability measure $P \ll Q$ such that $\frac{dQ}{dP} \in L^p(P)$ is extremal in $\widetilde{\Xi}_p(Q)$.*

Proof. Choose $F = \mathcal{Y}_d$ for the discrete time case and $F = \mathcal{Y}_c$ for the continuous time one. $\square$

1.3.2 The SFTAP: the discrete-time case

In this subsection we will treat the case $\mathcal{T} = \{0, 1\}$. Finally, we denote by $\mathcal{M}$ the set of all martingale probability measures for $\mathcal{S}$ and we set

$$\mathcal{M}^a = \{Q \in \mathcal{M} : Q \ll P\}$$

and

$$\mathcal{M}^e = \{Q \in \mathcal{M} : Q \sim P\}.$$

In this case a process $S \in \mathcal{S}$ is a Q-martingale if $S_1 \in L^1(Q)$ and $\mathbb{E}_Q(S_1) = 1$.

Thanks to Theorem 11, we can re-demonstrate, using only some geometrical argument based on the notion of extremality, two results which have been initially obtained by Jarrow, Jin and Madan (1999).

Theorem 12 *Let $p \geq 1$ and let the market be $\sigma(L^\infty, L^p)$ -complete. Then, there exists at most one $Q \in \mathcal{M}^e$ such that $\frac{dQ}{dP} \in L^p(P)$.*

Proof. We assume that there exists two equivalent martingale probability measures Q_1 and Q_2 for $\mathcal{S}$. Since $\mathcal{M}^e$ is a convex set, for each $\alpha \in [0, 1]$, $Q_\alpha = \alpha Q_1 + (1 - \alpha)Q_2$ is an equivalent martingale probability measure for $\mathcal{S}$. But, since the market is $\sigma(L^\infty, L^p)$-complete, by Theorem 11, every Q_α must be extremal in $\Xi_p(Q_\alpha) = \mathcal{M} \supseteq [Q_1, Q_2]$, which is a contradiction if we choose $\alpha \in (0, 1)$. $\square$

Let ν be a finite signed measure over the measurable space $(\Omega, \mathcal{F})$. We will say that $S = (1, S_1) \in \mathcal{S}$ is a ν-martingale if S_1 is $|\nu|$-integrable and $\nu(S_1) = \int S_1 d\nu = 1$.

We denote by $\mathcal{M}^s$ the space of all finite signed measures ν which are absolutely continuous with respect to P, such that $\nu(\Omega) = 1$ and each $S \in \mathcal{S}$ is a ν-martingale.

Theorem 13 *Let $\mathcal{M}^s$ be nonempty. The following two assertions are equivalent:*

1. *the market is weakly*-complete;*
2. *$\mathcal{M}^s$ is a singleton.*

Proof. Firstly, we show that 2. implies 1.. We fix $\nu \in \mathcal{M}^s$, which exists by assumption, and assume that the market is not weakly*-complete, i.e. by Theorem 11 there exists a probability $Q \ll P$ such that

$$Q = \alpha Q_1 + (1 - \alpha) Q_2$$

where $\alpha \in (0, 1)$ and $Q_i \in \widetilde{\Xi}_1(Q)$ for each $i = 1, 2$. Now, we set

$$\nu_i = Q_i - Q + \nu$$

for $i = 1, 2$. Then, since $\nu_i(S_1) = \mathbb{E}_{Q_i}(S_1) - \mathbb{E}_Q(S_1) + \nu(S_1) = 1$, for each $i = 1, 2$, ν_i is martingale signed measure for $\mathcal{S}$. Furthermore, since $Q_1 \leq \frac{1}{\alpha} Q$ and $Q_2 \leq \frac{1}{1-\alpha} Q$, we have

$Q_i \ll Q \ll P$ for every $i = 1, 2...$ Then, since $|\nu_i| \leq Q_i + Q + |\nu|$, we have $|\nu_i| \ll P$ for each $i = 1, 2$. This shows that ν is not unique in $\mathcal{M}^s$ and so 2. implies 1..

To show that 1. implies 2., proceed by contradiction and suppose that $\mathcal{M}^s \supseteq \{\nu_1, \nu_2\}$, with $\nu_1 \neq \nu_2$. Observe now that, by the definition of $\mathcal{M}^s$, $\nu_1(S_1) = \nu_2(S_1) = 1$ and $\nu_1(\Omega) = \nu_2(\Omega) = 1$, that is

$$\nu_1^+(S_1) - \nu_1^-(S_1) = \nu_2^+(S_1) - \nu_2^-(S_1) = 1$$

and

$$\nu_1^+(\Omega) - \nu_1^-(\Omega) = \nu_2^+(\Omega) - \nu_2^-(\Omega) = 1,$$

where ν_i^+ and ν_i^- $(i = 1, 2)$ are, respectively, the positive and the negative part of ν_i in its Hahn-Jordan decomposition. This implies

$$\nu_1^+(S_1) + \nu_2^-(S_1) = \nu_2^+(S_1) + \nu_1^-(S_1)$$

and

$$\nu_1^+(\Omega) + \nu_2^-(\Omega) = \nu_2^+(\Omega) + \nu_1^-(\Omega) =: k > 0.$$

Thus, define the two probability measures Q_1 and Q_2 as follows:

$$Q_1 = \frac{\nu_1^+ + \nu_2^-}{k}, \quad Q_2 = \frac{\nu_2^+ + \nu_1^-}{k}.$$

Observe that $Q_1 = Q_2$ on $\mathcal{Y}_d$ and define $Q := \alpha Q_1 + (1 - \alpha) Q_2$ for some real $\alpha \in (0, 1)$. It is straightforward to verify that $Q \ll P$ (since $|\nu_i| \ll P$, for $i = 1, 2$) and that Q_1 and Q_2 are absolutely continuous to Q, which implies that $Q_1, Q_2 \in \widetilde{\Xi}_1(Q)$. We have so built a probability measure Q absolutely continuous to P that is not extremal in $\widetilde{\Xi}_1(Q)$. Finally, Theorem 11 applies and gives that 1. $\Rightarrow$ 2. $\square$

We recall that a necessary and sufficient condition for the existence of a martingale equivalent probability (resp. the existence of a finite signed measure) for $\mathcal{S}$ is the absence of free lunch with free disposal (resp. free lunch). For the precise definition of these two conditions, see Jin, Jarrow and Madan (1999). Here, we note only that, under the absence of free lunch, there can exist arbitrage opportunities.

1.3.3 The SFTAP: the continuous-time case

Here we pass to the continuous-time case, i.e. we take $\mathcal{T} = [0, 1]$, for which our main reference is Bättig (1999). We suppose that the filtration $\mathbb{F}$ satisfies the usual conditions and that each price process $S \in \mathcal{S}$ is càdlàg (right continuous with left limit).

We denote $\mathcal{M}_{loc}$ the set of all local martingale probability measures for $\mathcal{S}$ and we set

$$\mathcal{M}_{loc}^a = \{Q \in \mathcal{M}_{loc} : Q \ll P\}$$

and

$$\mathcal{M}_{loc}^e = \{Q \in \mathcal{M}_{loc} : Q \sim P\}.$$

If we use exactly the same argument as in the proof of Theorem 12, we obtain its analogue in the continuous-time case. In order to avoid repetitions, we omit its proof.

Theorem 14 *Let $p \geq 1$ and let the market be $\sigma(L^\infty, L^p)$ -complete. Then, there exists at most one $Q \in \mathcal{M}_{loc}^e$ such that $\frac{dQ}{dP} \in L^p(P)$.*

Now, let ν be a signed finite measure over $(\Omega, \mathcal{F})$ such that $\nu(\Omega) = 1$. We will say that $S \in \mathcal{S}$ is a ν-local martingale if $\nu(f) = 0$ for all $f \in \mathcal{I}$ and ν-integrable. We let $\mathcal{M}_{loc}^s$ denote the space of all finite signed measures ν which are absolutely continuous to P and such that $\nu(\Omega) = 1$ and each $S \in \mathcal{S}$ is a ν-local martingale .

Remark 15 *For a complete treatment of martingales under a finite signed measure but with a definition slightly different from ours, one can consult Beghdadi-Sakrani (2003); for a striking extension to signed measures of Lévy's martingale characterization of Brownian Motion, see Ruiz de Chavez (1984).*

Theorem 16 *Let $\mathcal{M}_{loc}^s$ be nonempty. The following two assertions are equivalent:*

1. *the market is weakly*-complete;*
2. *$\mathcal{M}_{loc}^s$ is a singleton.*

Proof. One may proceed exactly in the same way as in the proof of Theorem 35. □

1.3.4 The Artzner-Heath example

In this subsection, we study the τ-completeness of an Artzner-Heath market (abbr. AH-market), which is a slight generalization of the pathological economy constructed in Artzner and Heath (1995). Now we give its precise definition. We use the same notation as in the previous section.

Definition 17 *We say that a financial market $\mathcal{S}$ is of the AH-type or that it is an AH-market if, P_0 and P_1 being two different equivalent probability measures,*

$$\mathcal{M} = [P_0, P_1] = \{P_\alpha = \alpha P_0 + (1-\alpha) P_1; \alpha \in [0,1]\}.$$

In this market we can choose P_0 as the true probability measure. So, applying the different versions of Douglas Theorem, one has the following result.

Proposition 18 *An AH-market satisfies the following three properties:*

1. *it is $\|\cdot\|_1$-complete under P_α if and only if $\alpha \in \{0,1\}$;*
2. *it is not strongly complete under P_α for each $\alpha \in [0,1]$;*
3. *it is not weakly* complete under P_α for each $\alpha \in [0,1]$.*

Proof. The first and the third property are simple consequences of, respectively, Theorem 1 and Theorem 11. In order to prove the second property, we assume that there exists $\alpha \in [0,1]$ such that the market is complete w.r.t. P_α. By Theorem 2, this is equivalent to the extremality of every $\nu \in ba^+(P_\alpha)$ in $\Xi_{ba}(\nu)$. But, if we choose $\nu = P_\beta$ for $\beta \in (0,1)$, P_β has to be extremal in $\Xi_{ba}(P_\beta) \supset [P_0, P_1]$, which is obviously absurd. □

Remark 19 *The previous proposition is a generalization of Proposition 4.1 of Artzner and Heath (1995) and of the content of Section 6 of Jarrow, Jin and Madan (1999).*

Now, we give a little more general construction of an AH-market than the original one contained in Artzner and Heath (1995).

Firstly, we set $(\Omega, \mathcal{F}) = (\mathbb{Z}^*, \mathcal{P}(\mathbb{Z}^*))$, where $\mathbb{Z}^*$ is the set of integers different from zero, and $\mathcal{S} = \{S^n : n \in \mathbb{Z}\}$. Now, we assume that every random variable S_1^n has a two-points support, i.e.

$$\begin{aligned} \mathrm{supp} S_1^n &= \{n, n+1\} \text{ for } n > 0 \\ S_1^n(k) &= S_1^{-n}(-k) \text{ for } n < 0 \\ S_1^0 &= \frac{1}{K(p_1+q_1)}\left(\mathbf{1}_{\{1\}} + \mathbf{1}_{\{-1\}}\right). \end{aligned} \tag{1.1}$$

Remark 20 *The hypothesis on the support of the price processes is not at all restrictive. In fact, thanks to Lemme A of Dellacherie (1968), we know that the extremality of a probability P in the set of martingale probabilities for a process $S = (1, S_1) \in \mathcal{S}$, implies $S \equiv 1$ or that the support of the law of S_1 is a two-points set. In financial terms the result of Dellacherie means that, in a two-period setting, the only market model which is both arbitrage-free and complete is the binomial one.*

Then, we fix two different equivalent probabilities P_0 and P_1 over $(\Omega, \mathcal{F})$ and we denote, for every $n \in \mathbb{Z}^*$, $p_n^0 = P_0(\{n\})$ and $p_n^1 = P_1(\{n\})$.

We want that every process $S \in \mathcal{S}$ is a martingale under P_0 and P_1. Then, we have

$$Q(\{n\})\, S_1^n(n) + Q(\{n+1\})\, S_1^n(n+1) = 1 \text{ for every } n > 0, \tag{1.2}$$

for $Q \in \{P_0, P_1\}$, i.e. for every $n > 0$

$$\begin{aligned} p_n^0 S_1^n(n) + p_{n+1}^0 S_1^n(n+1) &= 1, \\ p_n^1 S_1^n(n) + p_{n+1}^1 S_1^n(n+1) &= 1. \end{aligned} \tag{1.3}$$

We solve (1.3) with respect to S_1^n and we found for each $n > 0$

$$S_1^n = \frac{\left(p_{n+1}^0 - p_{n+1}^1\right)\mathbf{1}_{\{n\}} - \left(p_n^0 - p_n^1\right)\mathbf{1}_{\{n+1\}}}{p_n^1 p_{n+1}^0 - p_n^0 p_{n+1}^1}. \tag{1.4}$$

Hence, we have constructed a class $\mathcal{S}$ of processes, which are martingales under both P_0 and P_1, i.e.

$$[P_0, P_1] \subset \mathcal{M}.$$

Now, we fix S_1^n and interpret (1.3) as an equation with respect to the vector Q and we found that its solutions are of the form $P_\lambda = \lambda P_0 + (1-\lambda)P_1$ for $\lambda \in \mathbb{R}$. This implies, for this kind of market,

$$\mathcal{M} \subset \{P_\lambda; \lambda \in \mathbb{R}\}.$$

Now, we look for some conditions on P_0 and P_1 such that P_λ is not a probability when $\lambda \notin [0,1]$.

Lemma 21 *Let P_0 and P_1 be two different equivalent probabilities over an arbitrary measurable space $(\Omega, \mathcal{F})$. The following two properties are equivalent:*

1. $P_\lambda = \lambda P_0 + (1-\lambda)P_1$ *is a probability if and only if* $\lambda \in [0,1]$*;*
2. $\frac{dP_0}{dP_1}$ *and* $\frac{dP_1}{dP_0}$ *are not bounded.*

Proof. Firstly, we assume that the Radon-Nikodym derivative $\frac{dP_1}{dP_0}$ is bounded, i.e. there exists a constant $M > 1$ such that $\frac{dP_1}{dP_0} \leq M$ almost surely. Let $f : \Omega \to \mathbb{R}$ be a measurable, positive and bounded function. If $\lambda > 1$, we have

$$\begin{aligned} \int f dP_\lambda &= \int f \left(\lambda + (1-\lambda) \frac{dP_1}{dP_0} \right) dP_0 \\ &\geq \int f \left(\lambda + (1-\lambda) M \right) dP_0. \end{aligned}$$

Then, if one chooses $\lambda > 1$ such that $\lambda + (1-\lambda)M \geq 0$, i.e. $\lambda \leq \frac{M}{M-1}$, P_λ is a probability measure. If we assume that the other Radon-Nikodym derivative is bounded, then we found that there exists $\lambda < 0$ such that P_λ is a probability.

Now, let $\frac{dP_1}{dP_0}$ be unbounded, i.e. for every $M > 0$

$$P_\alpha \left(\frac{dP_1}{dP_0} \geq M \right) > 0 \ \text{ for } \alpha = 0, 1.$$

Let $f = \mathbf{1}_{\{\frac{dP_1}{dP_0} \geq M\}}$ and $\lambda > 1$. Then, we have

$$\begin{aligned} \int f dP_\lambda &= \int_{\left\{\frac{dP_1}{dP_0} \geq M\right\}} \left(\lambda + (1-\lambda) \frac{dP_1}{dP_0} \right) dP_0 \\ &\leq (\lambda + (1-\lambda) M) P_0 \left(\frac{dP_1}{dP_0} \geq M \right) \\ &< 0 \end{aligned}$$

for M sufficiently large. For the case $\lambda < 0$, we proceed exactly in the same way, using the fact that $\frac{dP_1}{dP_0}$ is supposed unbounded. $\square$

Finally, thanks to Lemma 20, we have the following result, which is a generalization of the construction contained in Section 3 of Artzner and Heath (1995).

Proposition 22 *Let P_0 and P_1 two different equivalent probability measures on $(\mathbb{Z}^*, \mathcal{P}(\mathbb{Z}^*))$ which satisfy condition 2 of Lemma 21. Then the class $\mathcal{S}$ defined by (1.1) and (1.4) is an AH-market, i.e.*

$$\mathcal{M} = [P_0, P_1] .$$

Example 23 *(Artzner and Heath (1995)) Let $0 < p < q < 1$ be two real numbers. We set, for every $n > 0$,*

$$\begin{aligned} P_0(\{n\}) &= Kp^n \mathbf{1}_{\{n>0\}} + Kq^{-n}\mathbf{1}_{\{n<0\}} \\ P_1(\{n\}) &= P_0(\{-n\}) \quad \textit{for every } n \in \mathbb{Z}^*, \end{aligned}$$

where K is a renormalizing constant. In this case, it is obvious that

$$\lim_{n\to+\infty} \frac{dP_1}{dP_0}(n) = \lim_{n\to+\infty} \left(\frac{q}{p}\right)^n = +\infty$$

and

$$\lim_{n\to-\infty} \frac{dP_0}{dP_1}(n) = \lim_{n\to-\infty} \left(\frac{p}{q}\right)^n = +\infty.$$

So the previous proposition applies, and we find that for

$$S_1^n = \frac{\left(q^{n+1} - p^{n+1}\right)\mathbf{1}_{\{n\}} + \left(q^n - p^n\right)\mathbf{1}_{\{n+1\}}}{Kp^n q^n (q-p)} \tag{1.5}$$

the set of all equivalent martingale probabilities for S is equal to the segment $[P_0, P_1]$.

1.4 Conclusions

In this chapter, we have established in a very easy way a weak version of the Douglas-Naimark theorem, which relates the density (with respect to the weak topology) of a subspace of a vector topological locally convex space with the extremality of a certain family of linear functionals. Then, in Subsection 1.2.2, we have applied this result to the space $L^\infty(\mu)$ equipped with the topologies $\sigma(L^\infty, L^p)$ for $p \geq 1$, where μ is a probability measure, and in Subsection 1.2.3 we have shown an analogue result for the spaces $(L^\infty(\mu), \|\cdot\|_p)$, $p \geq 1$. Finally, thanks to these results, we have obtained, in Section 1.3, a condition equivalent to the market completeness and based on the notion of extremality of measures, which has permitted us to give new elementary proofs of the SFTAP and to discuss the completeness of a more general construction of the Artzner-Heath example.

Chapter 2

Arbitrage and completeness in financial markets with given N-dimensional distributions

This chapter is based on the technical report n. 766 of the Laboratoire de Probabilités et Modèles Aléatoires of the Universities of Paris VI and VII, entitled "Financial markets with given marginals via fundamental theorems of asset pricing", to appear in the review *Decisions in Economics and Finance*, date of publication: May 2004. I am grateful to Marc Yor for helpful suggestions and remarks.

2.1 Introduction

In this chapter we are interested in the existence and uniqueness of an equivalent martingale measure (EMM) Q with the following additional property: under Q the finite-dimensional distributions of order N (abbr. N-dds) of the price process must belong in a *given* family of probabilities on $\mathcal{B}(\mathbb{R}^N)$, where N is a fixed non negative integer.

More precisely, given an arbitrage-free and incomplete financial market consisting of a single stock whose price evolution is described by a process $S = (S_t)_{t\in\mathcal{T}}$ (defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$) and given a family $\mathbf{M}_N = \{\mu_{t_1,...,t_N} : t_1, ..., t_N \in \mathcal{T}\}$ of probability measures on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$, we search for economically meaningful properties which are equivalent to the existence and uniqueness of an EMM Q under which the N-dds (N being fixed) of S are precisely given by $\mathbf{M}_N$.

Such a problem is motivated by the following remark: suppose that the European call prices have arisen as the expected pay-off under a EMM Q in the financial market S, where the instantaneous interest rate r_t is a deterministic function of time, that is we can represent the price $C(k, t)$ of a call with strike price k and maturity t as

$$C(k, t) = \beta_t \mathbf{E}_Q[(S_t - k)^+],$$

where $\beta_t := \exp\left(-\int_0^t r_s ds\right)$ is a discounting factor, e.g. the current price (time 0) of a zero-coupon bond with maturity t. When we take a right-derivative with respect to k we find that (see Breeden and Litzenberger (1978) or Dupire (1997))

$$\frac{\partial}{\partial k+}C(k,t) = -\beta_t Q(S_t > k).$$

Thus from call prices it is possible to infer the Q-distribution of S_t for all t.

When the instantaneous interest rate r_t is random, the above formula still holds, but to obtain from the previous formula the law of each S_t under Q the agent has to know calls *and* zero-coupon bonds prices.

Obviously, behind this reasoning there is the assumption that on the market there exists a family of zero-coupon bonds and call options traded with all maturities $t \in \mathcal{T}$, and (for call options) all potential strike prices k. This assumption seems to be quite realistic. In fact many authors (see e.g. Dupire (1997) or Hobson (1998)) observe that call options market is now so liquid that one can realistic treat calls no longer as derivatives but as primary assets, whose prices are fixed exogeneously by market sentiments .

This viewpoint drives us to interpret $\mathbf{M}_{J,N}$ as a family of exogeneous measures, obtained by the observation of market prices.

From this discussion we can also argue, at least theoretically, that when the given financial market is incomplete, it would be possible to reduce (even to a singleton) the set of EMMs, by considering only those that match these N-dds.

This approach could facilitate even considerably the choice of the "good" EMM. Furthermore, if the set of EMMs matching the N-dds inferred from the market is still infinite, one could even apply the various criteria which have been developed in, e.g., Delbaen and Schachermayer (1996) (variance-optimal martingale measure), Föllmer and Schweizer (1991) (minimal martingale measure) and Frittelli (2000) (minimal entropy martingale measure, see also Miyahara (1996)). This would provide a kind of mixed approach in order to select the "right" EMM: use first some "objective" additional information on the distribution of the stock inferred from the market prices and then one of the above "subjective" criteria. Some authors already begin to explore this research field, e.g. see Goll and Ruschendorf (2002) for minimal distance martingale measures, Tierbach (2002) for the mean-variance hedging approach and the book by Föllmer and Schied (2002) (pp. 298-308) for the super-hedging approach.

In the financial literature, there exist several articles whose topics are closely related with ours. For instance, Hobson (1998) finds bounds on the prices of exotic derivatives (in particular, lookback options), in terms of the market prices of call options. This is achieved without making explicit assumptions about the dynamics of the price process of the underlying asset, but rather by inferring information about the potential distribution of asset prices from the call prices. Also, quite connected to this approach is the article by Madan and Yor (2002), that contains three explicit constructions of martingales that all have the Markov property and pre-specified marginal densities (see also Carr, Geman, Madan and Yor (2003)). On the other hand, Brigo and Mercurio (2000) construct stock-price processes with the same marginal lognormal law as that of geometric brownian motion and also with the same transition density between two instants in a given discrete-time grid.

The main difference between this part of the literature and our approach is that we fixed also the market model and not only the N-dds and, to avoid trivialities, we assume that the market under consideration is incomplete.

The remainder of this paper is structured as follows. Section 2.2 contains the fundamental theorems of asset pricing (abbr. FTAP) with given N-dds in a market model, where the underlying probability space is finite; we find that the existence of a EMM with given N-dds is equivalent to a property of no-arbitrage, which is stronger than the usual one (see Harrison and Pliska (1981) or Schachermayer (2001)) in the sense that it even allows to trade (statically) in some non-replicable contingent claims. On the other hand, the uniqueness of such a measure is equivalent to the replicability of all contingent claims by a dynamic strategy in S and a static strategy in contingent claims depending on at most N time-coordinates of the underlying price process, i.e. a weaker market completeness condition.

Section 2.3 presents the FTAP in a market with finite horizon and one risky asset, whose price dynamics is modelled by a continuous-time, real-valued and locally bounded semimartingale.

Finally, in Subsection 2.3.4 we apply our approach to the Black-Scholes model with jumps and we find that, given a family of marginals such that there exists a EMM matching them, and under some standard assumptions on the coefficients, the subset of all such EMMs belonging to Υ [1] reduces to a singleton.

2.2 The finite case

2.2.1 The model

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t : t \in \mathcal{T}\}$ where $\mathcal{T} = \{0, ..., T\}$ for T positive integer chosen as a fixed finite horizon, i.e.

$$\Omega = \{\omega_1, \omega_2, ..., \omega_K\}$$

is a finite set, the σ-algebra $\mathcal{F}$ is the power set of Ω and we may assume without loss of generality that P assigns strictly positive value to all $\omega \in \Omega$. We also assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$.

We consider a financial market with $d \geq 1$ risky assets modelled by an $\mathbb{R}^{d+1}$-valued stochastic process $S = (S_t)_{t\in\mathcal{T}} = (S_t^0, S_t^1, ..., S_t^d)_{t\in\mathcal{T}}$ based on and adapted to the filtered stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, P)$. We shall assume that the cash account S^0 satisfies $S_t^0 \equiv 1$ for all $t \in \mathcal{T}$. As usual, this means that the stock prices are expressed in discounted terms.

We denote by $\mathcal{H}$ the set of trading strategies for the financial market S, i.e. the set of all $\mathbb{R}^d$-valued stochastic processes $H = (H_t)_{t\in\mathcal{T}} = (H_t^1, ..., H_t^d)_{t\in\mathcal{T}}$ which are predictable with respect to the given filtration, i.e. each H_t is $\mathcal{F}_{t-1}$-measurable.

[1] Υ is the subset of all equivalent martingale measures Q^h corresponding to some parameter h deterministic function of time (see 2.3.4).

We then define the stochastic integral $(H \cdot S)$ as the real valued process $((H \cdot S)_t)_{t \in \mathcal{T}}$ given by $(H \cdot S)_0 = 0$ and

$$(H \cdot S)_t = \sum_{j=1}^{t} H_j \Delta S_j, \quad t = 1, ..., T, \tag{2.1}$$

where $\Delta S_t = S_t - S_{t-1}$.

We observe that, having defined the zero coordinate S^0 of S to be identically equal to 1 so that $\Delta S_t^0 \equiv 0$, this coordinate do not contribute to the stochastic integral (2.1).

We denote by $\mathcal{M}^a$ (resp. $\mathcal{M}^e$) the set of absolutely continuous (resp. equivalent) martingale measures, i.e. the set of all P-absolutely continuous (resp. P-equivalent) probability measures Q on $\mathcal{F}$ such that S is a martingale under Q.

Throughout the paper we make the following standing assumption: *$\mathcal{M}^e$ is not empty and it does not reduce to a singleton, i.e. the market is arbitrage-free and it is not complete.*

Let N be a positive integer less than K and let

$$\mathbf{M}_N = \{\mu_{t_1,\dots,t_N} : t_i \in \mathcal{T}, 1 \leq i \leq N\}$$

be a family of probability measures on $\mathcal{B}(\mathbb{R}^{N \times d})$.

We let $\mathcal{M}_N^a$ denote the subset of $\mathcal{M}^a$ formed by the probability measures under which the law of every N-dimensional vector $(S_{t_1}, ..., S_{t_N})$ is precisely $\mu_{t_1,\dots,t_N}$, and by

$$\mathcal{M}_N^e = \{Q \in \mathcal{M}_N^a : Q \sim P\}$$

we denote the set of all *EMMs with N-dds in* $\mathbf{M}_N$, that is the subset of $\mathcal{M}_N^a$ containing the probability measures that are equivalent to P.

Throughout the sequel, we will always suppose *the consistency of the set $\mathbf{M}_N$ with respect to the martingale property*, i.e. that there exists a martingale, on some stochastic base, such that its N-dds are in $\mathbf{M}_N$.

When $N = 1$, a necessary and sufficient condition for this property can be found in Strassen (1965) (see also the book by Föllmer and Schied (2002), pp. 103-111). We quote it without proof.

Theorem 24 *(Strassen (1965), Theorem 8, pp. 434-435) Let d a positive integer and $(\mu_n)_{n \geq 0}$ be a sequence of probability measures on the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. A necessary and sufficient condition for the existence of a d-dimensional martingale, say $(M_n)_{n \geq 0} = (M_n^1, \cdots, M_n^d)_{n \geq 0}$, on some filtered stochastic base, such that the distribution of M_n is μ_n for all $n \geq 0$ is that all μ_n have means, i.e. $\int |x| \mu_n(dx) < \infty$, and that for any concave function $\psi : \mathbb{R}^d \to \mathbb{R}$ μ_n-integrable for each $n \geq 1$, the sequence $(\mu_n(\psi))_{n \geq 0}$ is non-increasing (the values of the integrals may be $-\infty$).*

Remark 25 *When N is an arbitrary integer less than K, a necessary condition to the existence of a martingale M such that the law of every N-dimensional vector $(M_{t_1}, \ldots, M_{t_N})$ is $\mu_{t_1,\ldots,t_N}$ is the following: for every concave function $\phi : \mathbb{R}^N \to \mathbb{R}$, every N-uple $\mathbf{t}_N = (t_1, \ldots, t_N)$ and every $s \in \mathcal{T}$, one must have*

$$\int_{\mathbb{R}^N} \phi(\mathbf{x}) \mu_{\mathbf{t}_N}(d\mathbf{x}_N) \leq \int_{\mathbb{R}^N} \phi(\mathbf{x}_N) \mu_{\mathbf{t}_N \wedge s}(d\mathbf{x}_N),$$

where $\mathbf{t}_N \wedge s = (t_1 \wedge s, \ldots, t_N \wedge s)$. Indeed, it suffices to use the conditional Jensen inequality. Generalizing this to the d-dimensional case is not difficult and is left to the reader.

In this paper, we treat only the case $d = 1$ (i.e. only one risky asset), the multidimensional case being a straightforward extension.

The following proposition shows that, in order to reduce the set of EMMs and facilitate the choice of the "good" one, fixing a nonempty subset of marginals (i.e. 1-dds) of the price process is not a useless operation, unless the price process is trivial, i.e. almost surely constant.

We work on the canonical space $\Omega = \{x_1, x_2, \ldots, x_m\}^{\mathcal{T}}$ (x_i real for all $i = 1, \ldots, m$, with m a positive integer such that $mN = K$), $\mathcal{F} = \mathcal{P}(\Omega)$. S will be the coordinate process, that is $S_t(\omega) = \omega_t$ for all $t \in \mathcal{T}$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_N) \in \Omega$, and P a probability measure on $\mathcal{F}$. This is the right setting for the application of Strassen's theorem (Theorem 24), stating in particular that under the convex-order condition there exists a martingale measure Q (not necessarily P-equivalent) for the coordinate process S.

Proposition 26 *Let $\mathcal{M}^e$ be not empty. The following statements hold:*

1. *If the process S is a.s. constant under P, then for every given family of marginals $\mathbf{M}_1 = (\mu_t)_{t \in \mathcal{T}}$, either $\mathcal{M}_1^a = \emptyset$ or $\mathcal{M}_1^a = \mathcal{M}^a$.*

2. *Let $\mathbf{M}_1 = (\mu_t)_{t \in \mathcal{T}}$ be a given family of marginals such that $\mathcal{M}_1^a$ is not empty. If $\mathcal{M}^a = \mathcal{M}_1^a$, then S is P-a.s. constant.*

Proof.

1. We assume that $S_t = c$ P-a.s. for some real $c \in \{x_1, \ldots, x_m\}$ and for all $t \in \mathcal{T}$. Then, either $\mu_t \neq \delta_c$ for some $t \in \mathcal{T}$ or $\mu_t = \delta_c$ for all $t \in \mathcal{T}$. In the first case $\mathcal{M}_1^a = \emptyset$, in the second one $\mathcal{M}_1^a = \mathcal{M}^a$. Indeed, obviously $\mathcal{M}_1^a \subseteq \mathcal{M}^a$, and, on the other hand, if R is in $\mathcal{M}^a$ then $S_t = c$ for all $t \in \mathcal{T}$ R-a.s., which implies $R(S_t \in dx) = \mu_t(dx) = \delta_c(dx)$ for all $t \in \mathcal{T}$.

2. We assume without loss of generality that $P(\{\omega\}) > 0$ for all $\omega \in \Omega$, and proceed by contradiction. Let Q be a given measure in $\mathcal{M}^e \subseteq \mathcal{M}_1^a$. If S is not degenerate under P, there exist a set A and an instant $t_0 \in \mathcal{T}$ such that $Q(S_{t_0} \in A) > 0$. So, we can define, for every such A, on $\mathcal{F}$ a probability measure $Q_{t_0,A}(\bullet) = Q(\bullet | S_{t_0} \in A)$. We observe now that $Q_{t_0,A}$ is a P-absolutely continuous martingale measure for the process $S^{(t_0)} := (S_t)_{t \geqslant t_0}$, i.e. it is an element of $\mathcal{M}^a(S^{(t_0)})$. We assume that

$\mathcal{M}^a(S^{(t_0)}) = \mathcal{M}^a_1(S^{(t_0)})$. Since $Q \in \mathcal{M}^a_1 \subseteq \mathcal{M}^a_1(S^{(t_0)})$, this assumption implies in particular that

$$Q(S_{t_0} \in A, S_{t_0} \in B) = Q(S_{t_0} \in A)Q(S_{t_0} \in B) \tag{2.2}$$

for all A such that $Q(S_{t_0} \in A) > 0$ and for all B. This equality implies that S_{t_0} is independent from itself and so S_{t_0} must be degenerate under Q and so even under P, which is absurd. To complete the proof, it remains to show that $\mathcal{M}^a = \mathcal{M}^a_1$ implies $\mathcal{M}^a(S^{(t_0)}) = \mathcal{M}^a_1(S^{(t_0)})$ for every instant t_0. We proceed again by contradiction, by assuming that there exists a probability $Q' \in \mathcal{M}^a(S^{(t_0)}) \setminus \mathcal{M}^a_1(S^{(t_0)})$ for some $t_0 \in \mathcal{T}$. We denote by $(\nu_t)_{t \geqslant t_0}$ its family of marginals, that is by assumption different from $(\mu_t)_{t \geqslant t_0}$ and satisfies the convex-order condition in Strassen's theorem. We consider now the following family of marginals:

$$\nu'_t := \nu_{t_0} \mathbf{1}_{\{t=0,\dots,t_0-1\}} + \nu_t \mathbf{1}_{\{t \geqslant t_0\}}, \quad t \in \mathcal{T}. \tag{2.3}$$

Also the family $(\nu'_t)_{t \in \mathcal{T}}$ satisfies the convex-order condition of Strassen's theorem, so that there exists, on the canonical space, a probability measure R such that the coordinate process S is an R-martingale and, for all $t \in \mathcal{T}$, $R(S_t \in dx) = \nu'(dx)$. Moreover, since the space Ω is finite, R is P-absolutely continuous, that is $R \in \mathcal{M}^a \setminus \mathcal{M}^a_1$. The proof is now complete. □

2.2.2 Path-dependent contingent claims and N-mixed trading strategies

We identify the set of all contingent claims to the space $L^0(P) = L^0(\Omega, \mathcal{F}, P)$ of all a.s. finite random variables defined on $(\Omega, \mathcal{F}, P)$, and we introduce a classification of its elements based on the notion of *path-dependence*, which has been introduced by Peccati (2001) in a slightly different framework.

In order to formalize this notion, we need to define the following spaces: let Π_0 be the whole real line $\mathbb{R}$ and, for $N \in \{1, ..., T\}$, let

$$\Pi_N := v.s. \left\{ \varphi(S_{t_1}, ..., S_{t_N}) : \varphi \in L^0(\mathbb{R}^N),\, t_1 \leq ... \leq t_N \in \mathcal{T} \right\}$$

denote the vector space spanned by all random variables that depend on at most N time-coordinates of the process S, where $L^0(\mathbb{R}^N)$ is the space of all real-valued Borel-measurable functions defined on $\mathbb{R}^N$. Obviously, if $N' \leq N$ we have $\Pi_{N'} \subseteq \Pi_N$.

We say that a contingent claim $f \in L^0(P)$ has a *path-dependence degree (abbr. pdd)* less than or equal to $N \in \mathcal{T}$ if $f \in \Pi_N$. Furthermore, we say that f has pdd N if it belongs in $\Pi_N \backslash \Pi_{N-1}$.

Example 27 *Two examples of contingent claims with pdd equal to 1 are an European call (or put) option with maturity t and strike price k, whose pay-off function is $(S_t - k)^+$ and, assuming the price process S positive, an Asian option paying the mean value obtained by the spot price over any subset J of $\mathcal{T}$, whose pay-off function is $(1/|J|) \sum_{t \in J} S_t$.*

Example 28 *More generally, two examples of contingent claims with pdd equal to N are a lookback option calculated over any subset J of $\mathcal{T}$ and with strike price k whose pay-off function is* $(\sup_{t\in J} S_t - k)^+$, *and an asian option with maturity N and strike price k with pay-off* $(1/|J|)\sum_{t\in J}(S_t - k)^+$.

Remark 29 *We observe that the spaces Π_N are the analogues, in the finite space case, of the Föllmer-Wu-Yor spaces*

$$\Pi_N(X) = v.s.\{f(X_{t_1}, ..., X_{t_N}) : f \in L^\infty(\mathbb{R}^N), 0 \leq t_1 < ... < t_N \leq T\},$$

where $X = (X_t)_{t\in[0,T]}$ is a standard Brownian Motion, introduced by Föllmer, Wu and Yor (2000) for the study of weak Brownian Motions (see also Peccati (2003), for their financial interpretation in terms of space-time chaos).

We can now introduce the new notion of N-mixed trading strategy.

Definition 30 *A* N-mixed trading strategy *is a triplet (x, H, ψ), where $x \in \mathbb{R}$ is an initial investment, $H \in \mathcal{H}$ is a dynamic trading strategy in S and ψ is a contingent claim with pdd less then or equal to N.*

The denomination "mixed trading strategy" comes from the fact that it is a combination of a dynamic strategy in the underlying and a static strategy in a certain contingent claim (i.e. buy it at $t = 0$ and keep it until the end).

Finally, we define another family of spaces, related to the sets Π_N and with a clear financial interpretation: for $N = 0$ we set

$$\mathcal{G}_0 := \{0\}$$

and for $N \geq 1$

$$\mathcal{G}_N := \{\psi - \mathbf{E}_N[\psi] : \psi \in \Pi_N\} \tag{2.4}$$

where for all contingent claims $\psi = \sum_{i=1}^p \varphi_i(S_{t_1^{(i)}}, ..., S_{t_N^{(i)}}) \in \Pi_N$ we have denoted

$$\begin{aligned} \mathbf{E}_N[\psi] &:= \sum_{i=1}^p \mu_{t_1^{(i)},...,t_N^{(i)}}(\varphi_i) \\ &:= \sum_{i=1}^p \int \varphi_i(x_1, ..., x_N)\, \mu_{t_1^{(i)},...,t_N^{(i)}}(dx_1, ..., dx_N) \end{aligned}$$

the price of the contingent claim ψ based on the N-dds $\mathbf{M}_N$, which can be viewed as the price observed on the market.

The elements of $\mathcal{G}_N$ are the gains which an investor can obtain by pursuing the N-mixed strategies $(0, 0, \psi)$, i.e. by investing statically in the contingent claim ψ at the market price $\mathbf{E}_N[\psi]$.

2.2.3 The first FTAP with given N-dds

In this subsection we study the problem of the existence of an EMM with given N-dds $\mathbf{M}_N$ for the financial model previously described. We will find that this is equivalent to a stronger notion of no-arbitrage, involving also the contingent claims with pdd less than or equal to N.

Following Schachermayer (2001), we denote by

$$\mathcal{K} = \{(H \cdot S)_T : H \in \mathcal{H}\} \tag{2.5}$$

the set of attainable contingent claims at price zero.

On the other hand, the vector subset of $L^0(P)$ defined by

$$\mathcal{K}_N^0 = v.s.\,(\mathcal{K} \cup \mathcal{G}_N) \tag{2.6}$$

is called *the set of contingent claims N-attainable at price zero*, i.e. the set of all random variables $f \in \mathcal{K}_N^0$ of the form

$$f = (H \cdot S)_T + (\psi - \mathbf{E}_N[\psi]) \tag{2.7}$$

for some $H \in \mathcal{H}$ and $\psi \in \Pi_N$. They are precisely those contingent claims that one may replicate by pursuing some N-mixed trading strategy $(0, H, \psi)$.

We call the cone $\mathcal{C}_N$ in $L^0(P)$ defined by

$$\mathcal{C}_N = \left\{g \in L^0(P) : \text{there is } f \in \mathcal{K}_N^0, f \geq g\right\} \tag{2.8}$$

the set of contingent claims N-superreplicable at price zero, i.e. the set of all contingent claims $g \in L^0(P)$ that an investor may replicate with zero initial investment, by pursuing some N-mixed trading strategy $(0, H, \psi)$ and then, eventually, "throwing away money".

Definition 31 *A financial market S satisfies the* N-mixed no-arbitrage condition (N-MNA) *if*

$$\mathcal{K}_N^0 \cap L_+^0(P) = \{0\} \tag{2.9}$$

or, equivalently,

$$\mathcal{C}_N \cap L_+^0(P) = \{0\} \tag{2.10}$$

where 0 denotes the function identically equal to zero.

The previous definition formalizes a more refined notion of arbitrage: an N-mixed arbitrage possibility is a N-mixed trading strategy $(0, H, \psi)$, such that the replicated contingent claim

$$f = (H \cdot S)_T + (\psi - \mathbf{E}_N[\psi])$$

is non-negative and not identically equal to zero. So, if a financial market does not allow for this type of arbitrage, we say that it satisfies N-MNA.

Remark 32 *If a market model satisfies N-MNA, then it also satisfies N'-MNA for each $0 \leq N' \leq N$, where*

$$\mathbf{M}_{N'} = \left\{ \mu_{t_1,...,t_{N'-1},t_{N'},...,t_{N'}}; t_1, ..., t_{N'} \in \mathcal{T} \right\},$$

but the converse does not necessarily hold.

Remark 33 *By the definition of $\mathcal{G}_0$, the condition* 0*-MNA is the usual condition of no-arbitrage (e.g. see Harrison and Pliska (1981) or Schachermayer (2001)).*

The first FTAP establishes the equivalence between the condition of no-arbitrage and the existence of an EMM for the stock price process S (e.g. see Harrison and Pliska (1981) or Schachermayer (2001)). The next theorem generalizes this equivalence to our setting. It claims that the existence condition is equivalent precisely to N-MNA.

Theorem 34 *For a financial market S the following are equivalent:*

1. *S satisfies N-MNA,*
2. *$\mathcal{M}_N^e \neq \emptyset$.*

To prove this result we need the following preliminary lemma:

Lemma 35 *For a probability measure Q on $(\Omega, \mathcal{F})$ the following are equivalent:*

1. *$Q \in \mathcal{M}_N^a$,*
2. *$\mathbf{E}_Q[f] = 0$, for all $f \in \mathcal{K}_N^0$,*
3. *$\mathbf{E}_Q[g] \leq 0$, for all $g \in \mathcal{C}_N$.*

Proof. The implication 1. ⇒ 2. is obvious. On the other hand if 2. holds we have

$$\mathbf{E}_Q \left[(H \cdot S)_T + \varphi \left(S_{t_1}, ..., S_{t_n} \right) - \mu_{t_1,...,t_n} (\varphi) \right] = 0$$

for $H \in \mathcal{H}$, $\varphi \in L^0(\mathbb{R}^N)$ and $t_1, ..., t_N \in \mathcal{T}$. So, if we take alternatively $H \equiv 0$ or $\varphi \equiv 0$, we obtain, respectively, the martingale and the N-dds property of the price process S.

The equivalence of 2. and 3. is straightforward. □

Proof. *(of Theorem 34)* We use the separation Hahn-Banach Theorem as in the proof of Theorem 2.8, Schachermayer (2001):

2. ⇒ 1. If there is some $Q \in \mathcal{M}_N^e$ then by Lemma 35 we have that

$$\mathbf{E}_Q [g] \leq 0, \quad \text{for } g \in \mathcal{C}_N.$$

But, if there were some $g \in \mathcal{C}_N \cap L_+^0 \backslash \{0\}$, then, since $Q \sim P$, we would have

$$\mathbf{E}_Q [g] > 0$$

a contradiction.

1. $\Rightarrow$ 2. We consider the convex hull of the unit vectors $(\mathbf{1}_{\{\omega_i\}})_{i=1}^K$ in $L^0_+(P)$, i.e. the set

$$\mathcal{P} = \left\{ \sum_{i=1}^K \lambda_i \mathbf{1}_{\{\omega_i\}} : \lambda_i \geq 0, \sum_{i=1}^K \lambda_i = 1 \right\}.$$

$\mathcal{P}$ is a convex, compact subset of $L^0_+(P)$ disjoint from $\mathcal{K}^0_N$. Hence there exists a random variable $\phi \in L^0(P)$ and two constants $\alpha < \beta$ such that

$$\mathbf{E}[\phi f] \leq \alpha < \beta \leq \mathbf{E}[\phi h] \quad \text{for } f \in \mathcal{K}^0_N \text{ and } h \in \mathcal{P}.$$

As $\mathcal{K}^0_N$ is a linear space, we may replace, without loss of generality, α by 0. Hence, we may normalize ϕ such that $\mathbf{E}[\phi] = 1$ and, by setting $dQ/dP = \phi$ and thanks to Lemma 35, it is now easy to verify that $Q \in \mathcal{M}^e_N$. $\square$

2.2.4 The second FTAP with given N-dds

In mathematical finance, one says that a market S is complete if each contingent claim $f \in L^0(P)$ can be replicated, with a certain initial investment $x \in \mathbb{R}$, by a predictable (dynamic) trading strategy $H \in \mathcal{H}$, i.e. one can write

$$f = x + (H \cdot S)_T .$$

Here we introduce a weaker notion of market completeness named N-*completeness*, allowing the agents to invest even in some non-replicable contingent claims with a certain path-dependence on the stock S.

Definition 36 *A financial market S is N-*complete *if for each contingent claim $f \in L^0(P)$ there exists a N-mixed trading strategy (x, H, ψ) such that*

$$f = x + (H \cdot S)_T + (\psi - \mathbf{E}_N[\psi]) . \tag{2.11}$$

Economically speaking, a financial market S is N-complete if every contingent claim may be attained by a combination of an initial investment $x \in \mathbb{R}$, a predictable trading strategy $H \in \mathcal{H}$ and a contingent claim ψ with pdd less than or equal to N.

Remark 37 *Two easy consequences of the definition are the following:*

1. *if the market is N-complete, then it is N'-complete for all $0 \leq N \leq N'$;*
2. *a market which is N-complete is even complete if and only if $\mathcal{G}_N \subset \mathcal{K}$.*

The second FTAP is a very well-known result which relates, under the assumption of no-arbitrage (which is equivalent to the existence of an EMM), the market completeness and the uniqueness of the EMM, so that the problem of evaluating a contingent claim reduces to take its expected value with respect to this measure.

Here, we state and prove an analogue of this theorem, but with the new notion of N-completeness.

Theorem 38 *For a financial market S satisfying the condition N-MNA the following are equivalent:*

1. *$\mathcal{M}_N^e$ consists of a single element Q,*

2. *the market S is N-complete.*

Proof. First we remark that $L^0(P) = L^0(Q)$ for each probability $Q \sim P$. For the implication 2. $\Rightarrow$ 1. note that (2.11) implies that, for elements $Q_1, Q_2 \in \mathcal{M}_N^e$, we have $\mathbf{E}_{Q_1}[f] = \mathbf{E}_{Q_2}[f]$ for every $f \in L^0(P)$ and so $Q_1 = Q_2$.

For the implication 1. $\Rightarrow$ 2., we denote $\mathcal{N}_N$ the subspace of all contingent claims $f \in L^0(P)$ which may be represented as in (2.11) and we proceed by contradiction. By assumption $\mathcal{N}_N \subsetneq L^0(P)$. So, there exists a contingent claim $g \in L^0(P) \backslash \{0\}$ which is orthogonal to $\mathcal{N}_N$, and we can define a probability measure $\widetilde{Q} \neq Q$, by setting

$$\frac{d\widetilde{Q}}{dQ} = 1 + \frac{g}{2\left\|g\right\|_\infty}.$$

It is easy to verify that $\widetilde{Q} \in \mathcal{M}_N^e$. $\square$

2.3 The continuous-time case

2.3.1 Terminology and definitions

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0,T]\}$ satisfying the usual conditions of right continuity and completeness, where $T > 0$ is a fixed finite horizon. We also assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. Let $S = (S_t)_{t \in [0,T]}$ be a real-valued, $\mathbb{F}$-adapted, càdlàg, locally bounded semimartingale modelling the discounted price of a risky asset. Furthermore let be given a family of probability measures on $\mathcal{B}(\mathbb{R}^N)$

$$\mathbf{M}_N = \{\mu_{t_1,...,t_N} : t_1 \leq ... \leq t_N \in [0,T]\}$$

where N is a fixed positive integer, such that the usual consistency condition with respect to the martingale property holds, i.e. there exists a filtered probability space and a martingale with precisely these laws as N-dds (see remark below).

In these sections we will investigate the problem of the existence and uniqueness of an equivalent local martingale measure (ELMM) Q for the price process S such that

$$Q\left(S_{t_1} \in dx_1, ..., S_{t_N} \in dx_N\right) = \mu_{t_1,...,t_N}\left(dx_1, ..., dx_N\right)$$

for every $t_1 \leq ... \leq t_N \in [0,T]$.

In the continuous-time case, a necessary and sufficient condition for the consistency of a N-dds family with respect to the martingale property has been obtained by Kellerer (1972) for $N = 1$:

Theorem 39 *(H. G. Kellerer (1972), Theorem 3, p. 120) Let $(\mu_t)_{t\in[0,T]}$ be a family of probability measures on $\mathcal{B}(\mathbb{R}^d)$, with first moment, such that for $s<t$ μ_t dominates μ_s in the convex order, i.e. for each convex function $\phi:\mathbb{R}^d\to\mathbb{R}$ μ_t-integrable for each $t\in[0,T]$, we must have*

$$\int_{\mathbb{R}^d}\phi d\mu_t\geq\int_{\mathbb{R}^d}\phi d\mu_s.$$

Then there exists a d-dimensional Markov process (M_t) with these marginals for which it is a submartingale. Furthermore if the means are independent of t then (M_t) is a martingale.

Proof. See Kellerer (1972), page 120.

Remark 40 *As a matter of fact, Kellerer (1972) proved the above result in the case $d=1$, but its proof is essentially based on his Theorem 1 which holds for any family μ_t of probability measures and each μ_t is defined on a* polish *space E_t.*

Remark 41 *When N is arbitrary, it is immediate to adapt the necessary condition formulated in Remark 25 to the continuous case.*

A probability measure Q on $(\Omega,\mathcal{F})$ is called an *ELMM for S with N-dds* $\mathbf{M}_N$ if $Q\sim P$ and S is a local martingale under Q such that, for each N-uple of instants $t_1\leq ...\leq t_N\in[0,T]$, the law of the vector $(S_{t_1},...,S_{t_N})$ is given by $\mu_{t_1,...,t_N}$.

We denote by $\mathcal{M}^e_{loc}$ the set of ELMMs and by $\mathcal{M}^e_{N,loc}$ the subset of $\mathcal{M}^e_{loc}$ containing the ELMMs for S with N-dds $\mathbf{M}_N$.

Finally, $\mu_{t_1,...,t_N}(\varphi)$ will frequently denote the integral $\int_{\mathbb{R}^N}\varphi(x)\mu_{t_1,...,t_N}(dx)$, for any bounded measurable function $\varphi:\mathbb{R}^N\to\mathbb{R}$.

As in the finite space case, we can introduce a classification on the space of contingent claims $L^\infty(P)=L^\infty(\Omega,\mathcal{F},P)$, based on the notion of path-dependence. We denote by Π_0 the whole real line and for $N\geq 1$, we set

$$\Pi_N\equiv v.s.\left\{\varphi\left(S_{t_1},...,S_{t_N}\right):\varphi\in L^\infty\left(\mathbb{R}^N\right),t_1\leq ...\leq t_N\in[0,T]\right\},$$

where $L^\infty(\mathbb{R}^N)$ is the space of all Borel-measurable essentially bounded functions $\varphi:\mathbb{R}^N\to\mathbb{R}$.

We say that a contingent claim $f\in L^\infty(P)$ has a *path-dependence degree* (abbr. pdd) less than or equal to N if $f\in\Pi_N$. On the other hand, we will say that a contingent claim $f\in L^\infty(P)$ has a pdd equal to N if $f\in\Pi_N\backslash\Pi_{N-1}$. Obviously, if $0\leq N'\leq N$ we have $\Pi_{N'}\subseteq\Pi_N$.

One can easily adapt the finite-case examples (subsection 2.2) to the continuous-time model.

2.3.2 The first FTAP with given N-dds

In this subsection we obtain an extension of the Kreps-Yan first FTAP (see Theorem 4.7 in Schachermayer (2001), p. 43), which states the equivalence between the no-free lunch condition and the existence of an ELMM for S.

We recall that an *admissible trading strategy* is a predictable, S-integrable process $H = (H_t)_{t\in[0,T]}$ such that there exists a constant $a \in \mathbb{R}$ which satisfies $H_t \geq -a$ for each $t \in [0, T]$, and such that $(H \cdot S)_T$ is bounded. We denote by $\mathcal{A}$ the set of all such strategies.

$\mathcal{A}^{simple}$ denotes the subset of $\mathcal{A}$ formed by all *simple* trading strategies H, i.e. of the type

$$H = \sum_{i=1}^{n} h_i \mathbf{1}_{]\tau_{i-1},\tau_i]},$$

where $0 = \tau_0 \leq \tau_1 \leq ... \leq \tau_n \leq T$ are stopping times such that each stopped process S^{τ_k} is uniformly bounded and h_i are $\mathcal{F}_{\tau_{i-1}}$-measurable bounded real-valued random variables, and $\mathcal{K}$ the set

$$\mathcal{K} = \left\{ (H \cdot S)_T : H \in \mathcal{A}^{simple} \right\}$$

of all attainable contingent claims at price zero.

Definition 42 *A* N-mixed admissible (simple) trading strategy *is a triplet* (x, H, ψ)*, where* $x \in \mathbb{R}$ *is an initial investment,* $H \in \mathcal{A}\,(\mathcal{A}^{simple})$ *is a admissible (simple) trading strategy in* S *and* ψ *is a contingent claim with pdd less then or equal to* N*.*

We call the subspace $\mathcal{K}_N^0$ of $L^\infty(P)$ defined by

$$\mathcal{K}_N^0 := v.s. \left(\mathcal{K} \cup \mathcal{G}_N^0 \right), \tag{2.12}$$

the set of contingent claims N*-attainable at price zero*, where

$$\mathcal{G}_N^0 = \left\{ \psi - \mathbf{E}_N \left[\psi \right] : \psi \in \Pi_N \right\}.$$

Then, all random variables $f \in \mathcal{K}_N^0$ are of the form

$$\begin{aligned} f &= (H \cdot S)_T + (\psi - \mathbf{E}_N [\psi]) && (2.13) \\ &= (H \cdot S)_T + \sum_{i=1}^{p} \left[\varphi_i \left(S_{t_1^{(i)}}, ..., S_{t_N^{(i)}} \right) - \mu_{t_1^{(i)},...,t_N^{(i)}} (\varphi_i) \right] && (2.14) \end{aligned}$$

for some $H \in \mathcal{A}^{simple}$, $\varphi_i \in B(\mathbb{R}^N)$, and $t_1^{(i)}, ..., t_N^{(i)} \in [0, T]$ for $1 \leq i \leq p$, i.e. they are precisely those contingent claims that an economic agent may replicate with zero initial investment, by pursuing some simple admissible N-mixed trading strategy $(0, H, \psi)$.

On the other hand, the elements of $\mathcal{G}_N^0$ are the gains which an investor can obtain by buying some contingent claim in Π_N (i.e. whose pdd is less than or equal to N) at the price given by $\mathbf{M}_N$.

We call the cone $\mathcal{C}_N$ in $L^\infty(P)$ defined by

$$\mathcal{C}_N := \left\{ g \in L^\infty (P) : f \geq g \text{ for some } f \in \mathcal{K}_N^0 \right\} \tag{2.15}$$

the set of contingent claims N*-superreplicable at price zero*, i.e. the set of all contingent claims that may be replicated with zero initial investment, by pursuing some N-mixed trading strategy $(0, H, \psi)$ and then, eventually, "throwing away money".

Definition 43 *We say that a financial market S satisfies the* N-mixed no-free lunch condition (N-MNFL) *if*

$$\overline{\mathcal{C}_N} \cap L^{\infty}(P) = \{0\} \tag{2.16}$$

where the closure is taken with respect to the weak topology $\sigma(L^{\infty}, L^1)$ of $L^{\infty}(P)$.*

By following exactly the same steps as in the proof of the Yan-Kreps Theorem (e.g. see Schachermayer (2001), Theorem 4.7, page 43), we can arrive, without any additional difficulty, to the following result, whose proof is so omitted.

Theorem 44 *The following properties are equivalent:*

1. *S satisfies N-MNFL,*
2. *$\mathcal{M}^e_{N,loc} \neq \emptyset$.*

2.3.3 The second FTAP with given N-dds

We identify the set of contingent claims with the space $L^{\infty}(P)$ of all essentially bounded random variables and we denote by τ any topology on this space. We recall that L^{∞} is invariant under equivalent change of probability.

Definition 45 *Let τ be some topology on $L^{\infty}(P)$. A financial market S is* N-complete for *τ if the set*

$$\mathcal{K}_N := \left\{x + \mathcal{K}^0_N : x \in \mathbb{R}\right\}$$

is dense in L^{∞} equipped with the topology τ.

$\mathcal{K}_N$ is the set of all essentially bounded contingent claims that can be perfectly replicated by a N-mixed admissible strategy (x, H, ψ), where x is the initial investment, H is an admissible (dynamic) strategy in S and ψ is a contingent claim with pdd less or equal to N.

Now, we establish an analogue of the second FTAP, with the new notion of N-completeness and for two kind of topologies on the set of contingent claims. Recall that, given a subset C of a vector space E, an element $x \in E$ is extremal in C if the relation $x = \alpha y_1 + (1-\alpha) y_2$ with $y_1, y_2 \in C$ and $\alpha \in (0,1)$ implies $x = y_1 = y_2$.

Theorem 46 *Let $Q \in \mathcal{M}^e_{N,loc}$. The following are equivalent:*

1. *Q is extremal in $\mathcal{M}^e_{N,loc}$,*
2. *the market S is N-complete for $L^1(Q)$-topology.*

Proof. It is an easy application of Theorem 1 in Douglas (1964) (see also Naimark (1947)). □

Corollary 47 *Let $Q \in \mathcal{M}^e_{N,loc}$. If $\mathcal{M}^e_{N,loc} = \{Q\}$, then the market S is N-complete for $L^1(Q)$-topology.*

The previous corollary means that if there exists an ELMM with given N-dds $\mathbf{M}_N$ and if this measure is unique, then each contingent claim can be approximately replicated (with respect to the topology induced by $L^1(Q)$) by a N-mixed trading strategy.

At present, we do not know if the converse of Corollary 47 holds true. Nonetheless, we are able to give a partial answer to this problem if we change the topology on the contingent claims set L^∞.

To prove the next results, we need the following functional analytical result (for more general results of this type with some applications to finance, see Campi (2001)):

Lemma 48 *Let P be a probability measure on $(\Omega, \mathcal{F})$ and let F be a subspace of $L^\infty(P)$ containing the unit function 1. F is dense in $(L^\infty(P), \sigma(L^\infty, L^1))$ if and only if every probability measure $Q \ll P$, is an extremal point of the set of all probability measures $R \ll Q$ on $(\Omega, \mathcal{F})$ such that*

$$\int f dR = \int f dQ \quad \text{for all } f \in F.$$

Proof. See Campi (2001). □

Theorem 49 *Let the market S be N-complete for the weak* topology. Then, there exists at most one $Q \in \mathcal{M}^e_{N,loc}$.*

Proof. Assume that there exists two different probability measures $Q_0, Q_1 \in \mathcal{M}^e_{N,loc}$. Since $\mathcal{M}^e_{N,loc}$ is a convex set, for each $\alpha \in [0,1]$ $Q_\alpha = \alpha Q_1 + (1-\alpha) Q_0$ is in $\mathcal{M}^e_{N,loc}$. But, since the market S is N-complete for the weak* topology, by Lemma 48, every Q_α must be extremal in the set of all probability measures $R \ll Q_\alpha$ on $(\Omega, \mathcal{F})$ such that $\int f dR = \int f dQ$ for all $f \in \mathcal{K}_N$. This set containing $[Q_0, Q_1]$, we have obtained a contradiction. □

To have an equivalence in Theorem 49, we have to consider a measures set larger than $\mathcal{M}^e_{N,loc}$. Thus, if we denote by $\mathcal{M}^s_{N,loc}$ the set of all finite *signed measures* $\nu \ll P$ on $(\Omega, \mathcal{F})$ such that $\nu(\Omega) = 1$ and $\int f d\nu = 0$ for every $f \in \mathcal{K}_N$, we have the following

Theorem 50 *Let $\mathcal{M}^s_{N,loc}$ be nonempty. The following are equivalent:*

1. *the market S is N-complete for the weak* topology;*

2. *$\mathcal{M}^s_{N,loc}$ is a singleton.*

Proof. 2. $\Rightarrow$ 1. Given a measure $\nu \in \mathcal{M}^s_{N,loc}$, which exists by assumption, we proceed by contradiction. Assume that the market is not N-complete for the weak* topology, i.e. by Lemma 48 there exists a probability $Q_\alpha \ll Q$ such that

$$Q_\alpha = \alpha Q_1 + (1-\alpha) Q_2$$

where $\alpha \in (0,1)$ and, for each $i = 1,2$, Q_i is a probability measure on $(\Omega, \mathcal{F})$ absolutely continuous to Q such that

$$\int f dQ_i = \int f dQ \quad \text{for all } f \in \mathcal{K}_N.$$

Now, consider the measures

$$\nu_i = Q_i - Q + \nu$$

for $i = 1,2$. For each $i = 1,2$, $\nu_i \in \mathcal{M}^s_{N,loc}$. Furthermore, since $Q_1 \leq \frac{1}{\alpha} Q_\alpha$ and $Q_2 \leq \frac{1}{1-\alpha} Q_\alpha$, we have $Q_i \ll Q_\alpha \ll Q$ for every $i = 1,2$. Then, since $|\nu_i| \leq Q_i + Q_\alpha + |\nu|$, we have $|\nu_i| \ll Q$ for each $i = 1,2$. This shows that ν is not unique in $\mathcal{M}^s_{N,loc}$ and so 2. implies 1..

1. $\Rightarrow$ 2. Proceed by contradiction and suppose that $\mathcal{M}^s_{N,loc} \supseteq \{\nu_1, \nu_2\}$, with $\nu_1 \neq \nu_2$. Observe now that, by the definition of $\mathcal{M}^s_{N,loc}$, $\nu_1(f) = \nu_2(f) = 1$ for all $f \in \mathcal{K}_N$ and $\nu_1(\Omega) = \nu_2(\Omega) = 1$, that is

$$\nu_1^+(f) - \nu_1^-(f) = \nu_2^+(f) - \nu_2^-(f) = 1, \quad \text{for all } f \in \mathcal{K}_N,$$

and

$$\nu_1^+(\Omega) - \nu_1^-(\Omega) = \nu_2^+(\Omega) - \nu_2^-(\Omega) = 1,$$

where ν_i^+ and ν_i^- $(i = 1,2)$ are, respectively, the positive and the negative part of ν_i in its Hahn-Jordan decomposition. This implies

$$\nu_1^+(f) + \nu_2^-(f) = \nu_2^+(f) + \nu_1^-(f), \quad \text{for all } f \in \mathcal{K}_N,$$

and

$$\nu_1^+(\Omega) + \nu_2^-(\Omega) = \nu_2^+(\Omega) + \nu_1^-(\Omega) =: k > 0.$$

Thus, define the two probability measures Q_1 and Q_2 as follows:

$$Q_1 = \frac{\nu_1^+ + \nu_2^-}{k}, \quad Q_2 = \frac{\nu_2^+ + \nu_1^-}{k}.$$

Observe that $Q_1 = Q_2$ on $\mathcal{Y}_d$ and define

$$Q := \alpha Q_1 + (1-\alpha) Q_2$$

for some real $\alpha \in (0,1)$. It is straightforward to verify that $Q \ll P$ (since $|\nu_i| \ll P$, for $i = 1,2$) and that Q_1 and Q_2 are absolutely continuous to Q. We have so built a probability measure Q absolutely continuous to P that is not extremal in the set of all probability measures $R \ll Q$ on $(\Omega, \mathcal{F})$ such that $\int f dR = \int f dQ$ for all $f \in \mathcal{K}_N$. Finally, Lemma 48 applies and gives that 1. $\Rightarrow$ 2. $\square$

2.3.4 An application: the Black-Scholes model with jumps

Now, as an application of our approach, we will study the Black-Scholes model with jumps (BSJ) and we will show that when its coefficients (drift, volatility, intensity and jump size) are time-deterministic functions, it admits at most one EMM with pre-specified marginals (1-dds). Our presentation of BSJ is based on the survey article by Runggaldier (2002).

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which are defined a Wiener process $(W_t)_{t\in[0,T]}$ and a Poisson process $(N_t)_{t\in[0,T]}$ with deterministic intensity $(\lambda_t)_{t\in[0,T]}$, so the compensated process $M_t = N_t - \int_0^t \lambda_s ds$ is a P-martingale adapted to the natural filtration of N. Furthermore we assume that these two processes are independent. As usual we will work on the augmented filtration

$$\mathcal{F}_t = \sigma\left(W_s, N_s : s \leq t\right) \quad t \in [0,T]$$

jointly generated by the Wiener and Poisson processes.

We now suppose that the discounted price process $(S_t)_{t\in[0,T]}$ satisfies the following SDE

$$dS_t = S_{t-}\left[a_t dt + \sigma_t dW_t + \gamma_t dN_t\right] \quad t \in (0,T] \tag{2.17}$$

with $S_0 > 0$ constant and where a_t, σ_t, φ_t are three time-deterministic functions such that:

- $\int_0^T |a_t|\, dt < \infty$;
- $\int_0^T \sigma_t^2 dt < \infty$, and $\quad \sigma_t > 0$ for all $t \in [0,T]$;
- $\int_0^T |\gamma_t|\, dt < \infty$ and $\gamma_t > -1$ for all $t \in [0,T]$.

By applying Itô's formula to the process $\ln S_t$ we find

$$S_t = S_0 e^{\int_0^t \left(a_s - \frac{1}{2}\sigma_s^2\right)ds + \int_0^t \sigma_s dW_s + \int_0^t \ln(1+\gamma_s)dN_s}. \tag{2.18}$$

It is well-known that this model admits an infinite number of EMMs Q^h, depending on a parameter $h = (h_t)$ and which can be represented as

$$\frac{dQ^h}{dP} = e^{\int_0^T \left(\lambda_t(1-h_t) - \frac{1}{2}\vartheta_t^2\right)dt + \int_0^T \vartheta_t dW_t + \int_0^T \ln h_t dN_t}, \tag{2.19}$$

for any couple of predictable processes (h, ϑ) where $h_t \geq 0$ is arbitrary and

$$\vartheta_t = \sigma_t^{-1}\left(-a_t - h_t\lambda_t\gamma_t\right). \tag{2.20}$$

Then, under the EMM Q^h the dynamic of the price process $(S_t)_{t\in[0,T]}$ is given by

$$dS_t = S_{t-}\left[\sigma_t d\widetilde{W}_t^h + \gamma_t d\widetilde{M}_t^h\right] \tag{2.21}$$

where for every $t \in [0,T]$

$$\widetilde{W}_t^h := W_t - \int_0^t \vartheta_s ds \tag{2.22}$$

and

$$\widetilde{M}_t^h := N_t - \int_0^t h_s \lambda_s ds \tag{2.23}$$

are, respectively, a Wiener process and a compensated Poisson process under Q^h. Moreover, we denote Υ the set of all equivalent martingale measures Q^h corresponding to some parameter $h = (h_t)$ deterministic function of time.

Remark 51 *We point out that Υ is a quite remarkable set, because it contains the pricing measure used by Merton (1976) and also the main equivalent martingale measures investigated in the incomplete markets literature: the Föllmer-Schweizer minimal martingale measure, the Frittelli-Miyahara minimal entropy martingale measure and the Esscher transform martingale measure (see the paper by Henderson and Hobson (2001)).*

We are now able to study the set $\Upsilon(\mathbf{M}_1)$ of EMMs belonging to Υ with given marginals $\mathbf{M}_1 = (\mu_t)$. The main result is the following:

Proposition 52 *Let $\mathbf{M}_1 = \{\mu_t : t \in [0,T]\}$ be a family of marginals for the price process S such that $\Upsilon(\mathbf{M}_1)$ is not empty. If*

$$\sigma_t > 0, \quad \gamma_t > -1, \quad \lambda_t > 0$$

where a_t, σ_t, γ_t are deterministic functions of time, there exists only one EMM $Q \in \Upsilon$ for S such that for all $t \in [0,T]$

$$Q\left(S_t \in dx\right) = \mu_t\left(dx\right).$$

Furthermore we have the following formulae for $h = (h_t)$: if $n \in \mathbf{Z} \setminus \{0\}$ and $(1+\gamma_t)^n - (n-1)\gamma_t - 1 \neq 0$ then

$$h_t = \frac{\frac{d}{dt} m_t^{(n)} - \frac{\sigma_t^2}{2} n(n-1) m_t^{(n)}}{\lambda_t[(1+\gamma_t)^n - (n-1)\gamma_t - 1] m_t^{(n)}}, \tag{2.24}$$

where for $t \in [0,T]$,

$$m_t^{(n)} = \int_{\mathbb{R}} x^n \mu_t\left(dx\right).$$

Proof. Letting $f \in C^2(\mathbb{R})$, we apply Itô's formula for discontinuous semimartingales to $f(S_t)$ and obtain

$$\begin{aligned} f\left(S_t\right) &= f\left(S_0\right) + \int_0^t f'\left(S_s\right) S_s a_s ds \\ &\quad + \int_0^t f'\left(S_s\right) S_{s-} \sigma_s dW_s + \frac{1}{2} \int_0^t f''\left(S_s\right) \sigma_s^2 S_s^2 ds \\ &\quad + \int_0^t \left[f\left(S_{s-}\left(1+\gamma_s\right)\right) - f\left(S_{s-}\right)\right] dN_s. \end{aligned} \tag{2.25}$$

By the relations (2.22) and (2.23) for a given value of the parameter h, we have

$$\begin{aligned} f(S_t) &= f(S_0) + \int_0^t f'(S_s)\, S_s a_s ds \\ &\quad + \int_0^t f'(S_s)\, S_{s-}\sigma_s(d\widetilde{W}_s^h + \vartheta_s ds) + \frac{1}{2}\int_0^t f''(S_s)\,\sigma_s^2 S_s^2 ds \\ &\quad + \int_0^t [f(S_{s-}(1+\gamma_s)) - f(S_{s-})]\,(d\widetilde{M}_s^h + h_s\lambda_s ds). \end{aligned} \tag{2.26}$$

If we choose $f(x) = x^n$, $n \in \mathbb{Z} \setminus \{0\}$, take the expectation with respect to the EMM $Q^h \in \mathcal{M}_1^e$, and use relation (2.20), we obtain

$$\begin{aligned} \mathbf{E}_{Q^h}[S_t^n] &= S_0^n + (n-1)\int_0^t \mathbf{E}^{Q^h}[S_s^n](a_s - \frac{a_s + h_s\lambda_s\gamma_s}{\sigma_s}\sigma_s)ds \\ &\quad + \frac{n(n-1)}{2}\int_0^t \mathbf{E}_{Q^h}[S_s^n]\,\sigma_s^2 ds \\ &\quad + \int_0^t \mathbf{E}_{Q^h}[S_s^n]\, h_s\lambda_s[(1+\gamma_s)^n - 1]ds \\ &= S_0^n + \frac{n(n-1)}{2}\int_0^t \mathbf{E}_{Q^h}[S_s^n]\,\sigma_s^2 ds \\ &\quad + \int_0^t \mathbf{E}_{Q^h}[S_s^n]\, h_s\lambda_s[(1+\gamma_s)^n - (n-1)\gamma_s - 1]ds. \end{aligned}$$

Since under Q^h the price process has marginals (μ_t), we have

$$\mathbf{E}_{Q^h}[S_t^n] = m_t^{(n)} := \int_{\mathbb{R}} x^n \mu_t(dx)$$

and then

$$\begin{aligned} m_t^{(n)} &= m_0^{(n)} + \frac{n(n-1)}{2}\int_0^t \sigma_s^2 m_s^{(n)} ds \\ &\quad + \int_0^t h_s\lambda_s\,[(1+\gamma_s)^n - (n-1)\gamma_s - 1]\, m_s^{(n)} ds. \end{aligned} \tag{2.27}$$

Thus, the application $t \mapsto m_t^{(n)}$ is absolutely continuous, and we can take its Radon-Nikodym derivative with respect to t and, having assumed $(1+\gamma_t)^n - (n-1)\gamma_t - 1 \neq 0$, after an easy calculation, we obtain

$$h_t = \frac{\frac{d}{dt}m_t^{(n)} - \frac{\sigma_t^2}{2}n(n-1)m_t^{(n)}}{\lambda_t[(1+\gamma_t)^n - (n-1)\gamma_t - 1]m_t^{(n)}}, \quad dt\text{-a.e.},$$

which is well-defined since $m_t^{(n)} > 0$. This ends the proof. $\square$

Remark 53 *Actually, formula (2.24) implies that in order to reduce the subset Υ of EMMs of the model to a singleton it suffices to know, for example, the second positive moments of the price process S under one of them. In this case (i.e. $n = 2$) the condition $(1+\gamma_t)^n - (n-1)\gamma_t - 1 \neq 0$ reduces to assume $\gamma_t \neq 0$ for all $t \in [0,T]$.*

Remark 54 *We can interpret equation (2.27) as a family of constraints on the moments of the marginals μ_t considered as unknown. Formally, if we assume that h is a deterministic function of time and that every μ_t satisfies $\mu_t(dx) = g(t,x)dx$ with $g(t,x)$ sufficiently regular, from (2.27) and by several simple integrations by parts, it is easy to see that $g(t,x)$ must be the solution of the following EDP:*

$$\begin{aligned}
-\frac{\sigma_t^2}{2}\frac{d^2}{dx}(x^2 g(t,x)) \quad &+ \quad h_t\lambda_t\gamma_t\frac{d}{dx}(xg(t,x)) + \frac{d}{dt}g(t,x) \\
&= \quad h_t\lambda_t\frac{1}{1+\gamma_t}g(t,\frac{x}{1+\gamma_t}) + (\gamma_t - 1)h_t\lambda_t g(t,x).
\end{aligned}$$

Remark 55 *If the coefficients of the model are constant we have an even stronger conclusion: to reduce the EMMs subset Υ to a singleton, it suffices to know only one marginal, e.g. the terminal one μ_T. Indeed, in this case the price process S is the exponential of a Lévy process and it is well known that any finite-dimensional distributions of a Lévy process is uniquely determined by any of its (one-dimensional) marginals.*

2.4 Conclusions

In this chapter, given a financial market S, and a family of probability measures $\mathbf{M}_N = \{\mu_{t_1,...,t_N} : t_1, ..., t_N \in \mathcal{T}\}$ on $\mathcal{B}(\mathbb{R}^N)$, we have obtained equivalent conditions for the existence and uniqueness of an EMM Q such that the Q-distribution of every vector $(S_{t_1}, ..., S_{t_N})$ is $\mu_{t_1,...,t_N}$.

When the probability space is finite (Section 2.2), the existence of such a measure Q is equivalent to the following no-arbitrage property: one cannot construct any arbitrage position by trading dynamically in the underlying and statically in a contingent claim with pdd less or equal to N, bought at the price given by $\mathbf{M}_N$. We have called this condition *N-mixed no-arbitrage*. On the other hand, Q is unique in the set of all EMMs that match the set $\mathbf{M}_N$, if and only if each contingent claim may be replicated by a predictable trading strategy and a contingent claim with pdd less or equal to N.

When the probability space is arbitrary and the price of the stock is modelled by a locally bounded real-valued semimartingale (Section 2.3), we had to consider a topological notion of N-mixed no-arbitrage, that we have named *N-mixed no-free-lunch*. In this setting, by considering two different topologies on the set of all contingent claims, we have defined two corresponding notions of N-completeness. We have so established two versions of the second FTAP.

In the last subsection, we have shown that for the Black-Scholes model with jumps, when the coefficients are deterministic and given a family of marginals for the price process, the subset of Υ of all EMMs induced by a parameter h deterministic in time and under which the price process has exactly the pre-specified marginals reduces to a singleton.

We finally remark that our main results have an immediate generalization if, more generally, we fix the N-dds of the price process only on a given *subset* (finite or infinite) J of $\mathcal{T}$. This easy extension is left to the reader. Future work will be devoted to investigate, with this approach, other concrete and more general examples.

Chapter 3

Mean-variance hedging in large financial markets

This chapter is based on the homonymous technical report n. 758 of the Laboratoire de Probabilités et Modèles Aléatoires of the Universities of Paris VI and VII, submitted to the review *Mathematical Finance*. I would like to thank Huyên Pham for fruitful suggestions and Marc Yor for his remarks. I am also grateful to Marzia De Donno and Maurizio Pratelli for their interest in this work.

3.1 Introduction

In this chapter we study the hedging problem for a future stochastic cash flow F, delivered at time T, in a large financial market.

Let us consider first a market consisting of $n+1$ primitive assets $X = \{S^0, S\}$: one bond with price process $S^0_t = \exp(\int_0^t r_s ds)$ and n risky assets whose price process is a continuous n-dimensional semimartingale $S = (S^1, \cdots, S^n)$. A criterion for determining a "good" hedging strategy is to solve the *mean-variance hedging* (MVH) problem introduced by Föllmer and Sondermann (1986):

$$\min_{\vartheta \in \Theta} \mathbb{E}\left[F - V_T^{x,\vartheta}\right]^2, \tag{3.1}$$

where

$$V_T^{x,\vartheta} = S_T^0 \left(x + \int_0^T \vartheta_t d\left(S/S^0\right)_t \right) \tag{3.2}$$

is the terminal value of a self-financed portfolio in the primitive assets, with initial investment x and quantities ϑ invested in the risky assets.

This problem has been solved by Föllmer and Sondermann (1986) and Bouleau and Lamberton (1989) in the martingale case: S^0 deterministic and S/S^0 is a martingale under the objective probability P thanks to direct application of the Galtchouk-Kunita-Watanabe (abbr. GKW) projection theorem. Extensions to more general cases were later studied by

Gourieroux, Laurent and Pham (1998) (abbr. GLP), Rheinländer and Schweizer (1997) and Pham et al. (1998), in a continuous-time framework under more or less restrictive conditions.

In particular GLP, by adding a numéraire as an asset to trade in, show how self-financed portfolios may be expressed with respect to this extended assets family, without changing the set of attainable contingent claims. They introduce the *hedging numéraire* $V(\widetilde{a})$, relate it to the variance-optimal martingale measure $\widetilde{P}$ and, using this numéraire both as a deflator and to extend the primitive assets family $\{S^0, S\}$, transform the original MVH problem (3.1) into an equivalent and simpler one; this transformed quadratic optimization problem is solved by the GKW projection theorem under a martingale measure for the hedging numéraire extended assets family $\{V(a), S^0, S\}$. This procedure gives an explicit description of the optimal trading strategy for the original MVH-problem.

Here, motivated by the papers by Kabanov and Kramkov (1994, 1998), Klein and Schachermayer (1996) and, more recently, Bjork and Näslund (1998) and De Donno (2002), we seek to extend this approach to a *large financial market*, i.e. a market with one bond with price process $S^0_t = \exp(\int_0^t r_s ds)$ and countably many risky assets whose price process is a sequence of continuous semimartingales $S = (S^i)_{i\geq 1}$.

The main problem of this extension is that we have to adopt a stochastic integration (SI) theory with respect to a sequence of semimartingales, i.e. with respect to a semimartingale taking values in the space $\mathbb{R}^{\mathbb{N}}$ of all real sequences, which is much more delicate to use than the vectorial one. Mikulevicius and Rozovskii (1998, 1999) developed a SI theory for cylindrical martingales, i.e. martingales taking values in a topological vector space (see also De Donno (2002), De Donno and Pratelli (2002) and De Donno et al. (2003) for financial applications). More recently, De Donno and Pratelli (2003) have proposed a stochastic integral for a sequence of semimartingales, generalizing the SI theory by Mikulevicius and Rozovskii in this particular case. We will use their construction for making our MVH problem meaningful.

This chapter is organized as follows. In Section 3.2, we recall some basic facts on stochastic integration with respect to a sequence of semimartingales $S = (S^i)_{i\geq 1}$. Moreover, we obtain an infinite-dimensional version of the GKW-decomposition theorem.

In Section 3.3, we describe the model and define the set Θ of trading strategies, by using the SI theory of the previous section, and we show that the set of all attainable contingent claims $G(x, \Theta)$ is closed in $L^2(P)$ for every initial investment $x \in \mathbb{R}$.

In Section 3.4, following GLP, we define a numéraire as a self-financed portfolio based on the primitive assets family with unit initial investment and whose value is strictly positive at every date; then we generalize the artificial extension method to our setting, by showing invariance properties of state-price densities and especially of self-financed portfolios: the infinite-dimensional Memin's theorem turns out to be crucial to show this property. We recall here that the original finite version of Memin's theorem (see Memin (1980)) states that, if S is an $\mathbb{R}^d$-valued semimartingale, then the set of all stochastic integrals $\int \xi dS$, for ξ predictable and S-integrable, is closed with respect to the Emery topology.

In Section 4.4, we show how to use the artificial extension method for solving the MVH problem in this framework. The main difference with respect to the finite assets case is that here we have not an explicit expression of the correspondences relating the solutions of the initial and the transformed MVH-problem. Finally in Section 3.6, we define the finite-dimensional MVH problems, corresponding to consider only the market formed by the bond S^0 and the first n risky assets $\overline{S}^n := (S^1, \ldots, S^n)$, $n \geq 1$; we show that the sequence of solutions to these finite-dimensional problems converges to the solution of the original one (3.1).

3.2 Some preliminaries on stochastic integration with respect to a sequence of semimartingales

In this section we will follow very closely the treatment of the stochastic integration for countable many semimartingales, that the reader can find in De Donno and Pratelli (2003). For unexplained notations we refer to Jacod (1979).

Furthermore, we remark that there is a huge literature on stochastic integration for martingales and semimartingales taking values in infinite-dimensional spaces. Here we quote only the pioneering work by Kunita (1970) and the book by Métivier (1982).

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$ satisfying the usual conditions of right-continuity and completeness, where $T > 0$ is a fixed finite horizon. Letting $p \geq 1$, we will denote by $\mathcal{H}^p(P)$ the set of all real-valued martingales M on the given filtered probability space, such that $M^* = \sup_{t \in [0,T]} |M_t|$ is in $L^p(P)$ (see Jacod (1979)). We recall that $\mathcal{H}^p(P)$, equipped with the norm

$$\|M\|_{\mathcal{H}^p(P)} = \|M^*\|_{L^p(P)}$$

is a Banach space. Moreover, we denote by $\mathcal{S}(P)$ the space of real semimartingales, equipped with Emery's topology (see Emery (1979)). $\mathcal{S}(P)$ is a complete metric space.

Let $S = (S^i)_{i \geq 1}$ be a sequence of semimartingales. We denote by E the set of all real-valued sequences (i.e. $\mathbb{R}^{\mathbb{N}}$), endowed with the topology of pointwise convergence and by E' its topological dual, namely the space of all signed measures on $\mathbb{N}$ which have a finite support; each of them can be identified with a sequence with all but finitely many components equal to 0. We will denote by e^i the element, both of E' and E such that $e^i_j = \delta_{i,j}$ (where $\delta_{i,j}$ is the Dirac delta); $\langle \cdot, \cdot \rangle_{E',E}$ will denote the duality between E' and E.

We denote by $\mathcal{U}$ the set of not necessarily bounded operators on E and, for all $h \in \mathcal{U}$, we denote by $\mathcal{D}(h)$ the domain of h ($\mathcal{D}(h) \subset E$). We say that a sequence $(h^n) \subset E$ converges to $h \in \mathcal{U}$ if $lim_n h^n(x) = h(x)$, for every $x \in \mathcal{D}(h)$.

We say that a process ξ taking values in $\mathcal{U}$ is *predictable* if there exists a sequence (ξ^n) of E'-valued predictable processes, such that for all (ω, t), and for all $x \in \mathcal{D}(\xi_t(\omega))$, one has $\xi_t(\omega) = lim_n \xi^n_t(\omega)$.

Finally, a E'-valued predictable process ξ is called *simple integrand* if it has the form $\xi = \sum_{i \leqslant n} \xi^i e^i = (\xi^1, \xi^2, \cdots, \xi^n, 0, \cdots)$, where ξ^i are real-valued predictable bounded processes. We note that one can define the stochastic integral of a simple integrand ξ with respect to

S in the following obvious way:

$$\int \xi dS = \int \sum_{i \leqslant n} \xi^i dS^i.$$

We are now able to give the following definition:

Definition 56 *(De Donno and Pratelli (2003)) Let ξ be a predictable $\mathcal{U}$-valued process. We say that ξ is* integrable *with respect to S if there exists a sequence (ξ^n) of simple integrands such that:*

1. *ξ^n converges to ξ pointwise;*
2. *$\int \xi^n dS$ converges to a semimartingale Y in $\mathcal{S}(P)$.*

We call ξ a generalized integrand *and define $\int \xi dS := Y$. Moreover, we denote by $L(S,\mathcal{U})$ the set of generalized integrands.*

This notion of stochastic integral is well-defined, as shown by De Donno and Pratelli (2003) (Proposition 5.1), and moreover there is a infinite-dimensional extension of Memin's theorem (Théorème III.4 in Memin (1980)), which states that the set of stochastic integrals with respect to a semimartingale is closed in $\mathcal{S}(P)$:

Theorem 57 *(De Donno and Pratelli (2003), Theorem 5.2) Let be given a sequence of semimartingales $S = (S^i)_{i \geqslant 1}$ and a sequence (ξ^n) of generalized integrands: assume that $(\int \xi^n dS)$ is a Cauchy sequence in $\mathcal{S}(P)$. Then there exists a generalized integrand ξ such that $\int \xi^n dS \to \int \xi dS$.*

In the sequel, we will need an infinite-dimensional version of the GKW-decomposition for a sequence of continuous local martingales. For this reason, we briefly recall the Mikulevicius and Rozovskii (1998) theory of SI for a sequence of locally square integrable martingales and show how to extend it to a sequence of continuous local martingales.

We assume that $S^i = M^i \in \mathcal{H}^2(P)$ for all $i \geqslant 1$. It is easy to see that there exist:

1. an increasing predictable real-valued process (A_t) with $\mathbb{E}[A_T] < \infty$,
2. a family $C = (C^{ij})_{i,j \geqslant 1}$ of predictable real-valued process, such that C is symmetric and non-negative definite, in the sense that $C^{ij} = C^{ji}$ and $\sum_{i,j \leqslant l} x_i C^{ij} x_j \geqslant 0$, for all $l \in \mathbb{N}$, for all $x \in \mathbb{R}^l$, $dPdA$ a.s.,

such that

$$\left\langle M^i, M^j \right\rangle_t (\omega) = \int_0^t C^{ij}_{s,\omega} dA_s(\omega). \tag{3.3}$$

For a simple integrand $\xi = \sum_{i \leqslant n} \xi^i e^i$, the Itô isometry holds:

$$\mathbb{E}\left[\left(\int_0^T \xi_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^T \sum_{i,j \leqslant n} \xi^i_s \xi^j_s d\left\langle M^i, M^j \right\rangle_s\right] = \mathbb{E}\left[\int_0^T \sum_{i,j \leqslant n} \xi^i_s \xi^j_s C^{ij}_s dA_s\right]. \tag{3.4}$$

Consider C for fixed (ω, t) and assume for simplicity that C is definite positive. The above Itô isometry makes it natural to define on E' a norm by setting:

$$|x|^2_{E'_{t,\omega}} = \langle x, C_{t,\omega} x \rangle_{E',E} = \sum_{i,j=1}^{\infty} x_i C^{ij}_{t,\omega} x_j, \tag{3.5}$$

where the sum contains a finite number of terms. The norm is induced by an obvious scalar product, which makes E' a pre-Hilbert space. This norm depends on (ω, t): for simplicity, we omit ω, but we keep t to remind us about this dependence and denote by E'_t the space E' with norm induced by C_t. It is not difficult to see that E'_t is not complete, but we can take its completion which we denote by H'_t and which is an Hilbert space. H'_t is generically not included in E, hence the canonical injection from E' to E cannot be extended to an injection from H'_t to E.

Moreover, H'_t can be characterized as the topological dual of the completion H_t of the pre-Hilbert space $C_t E'$ with respect to the norm induced by the scalar product

$$(C_t x, C_t y)_{C_t E'} = \langle x, C_t y \rangle_{E',E} = \langle y, C_t x \rangle_{E',E}.$$

The following theorem is essentially due to Mikulevicius and Rozovskii (1988) (see their Proposition 11, p. 145). But we prefer to follow the formulation given by De Donno and Pratelli (2003) of this result, since it fits better into their more general theory of SI for a sequence of semimartingales, as introduced before.

Theorem 58 *(De Donno and Pratelli (2003), Theorem 3.1) Let ξ be a $\mathcal{U}$-valued process such that:*

1. $\mathcal{D}(\xi_{\omega,t}) \supset H_{\omega,t}$ *for all* (ω, t)*;*
2. $\xi_{\omega,t}|_{H_{\omega,t}} \in H'_{\omega,t}$*;*
3. $\xi_t(C_t e_n)$ *is predictable for all* n*;*
4. $\mathbb{E}[\int_0^T |\xi_t|^2_{H'_t} dA_t] < \infty$.

Then, there exists a sequence ξ^n of simple integrands, such that $\xi^n_{\omega,t}$ converges to $\xi_{\omega,t}$ in $H'_{\omega,t}$ for all (ω, t) and $\int \xi^n dM$ is a Cauchy sequence in $\mathcal{H}^2(P)$. As a consequence, we can define the stochastic integral $\int \xi dM$ as the limit of the sequence $\int \xi^n dM$.

Remark 59 *(Mikulevicius and Rozovskii (1998), De Donno (2002)) When C_t is only non negative-definite, (3.5) defines a seminorm on E'. The construction of H'_t and of the stochastic integral can be carried on replacing E' with the quotient space $E'/\ker C_t$.*

Remark 60 *The set of all stochastic integrals $\int \xi dM$, with ξ fulfilling the four conditions of Theorem 58, is a closed set in $\mathcal{H}^2(P)$ and coincides with the stable subspace generated by M in $\mathcal{H}^2(P)$. It is an immediate extension of the analogous result in the finite-dimensional case.*

When $M = (M^i)_{i\geqslant 1}$ is a sequence of continuous *local* martingales, it is quite easy to extend the previous construction. Indeed, from Dellacherie (1978) we know that there exists a *uniform* localization for the sequence (M^i), i.e. a sequence (τ_n) of stopping times such that $\tau_n \to T$ and $M^i_{\cdot\wedge\tau_n}$ is a bounded martingale *for all* $i \geqslant 1$. To see this, it suffices to apply Théorème 2 and Théorème 3 of Dellacherie (1978), p. 743, to the sequence M.

This property allows us to define by localization, in the usual way, a stochastic integral with respect to M and for all $\mathcal{U}$-valued processes ξ such that:

1. $\mathcal{D}(\xi_{\omega,t}) \supset H_{\omega,t}$ for all (ω, t);
2. $\xi_{\omega,t}|_{H_{\omega,t}} \in H'_{\omega,t}$;
3. $\xi_t(C_t e_n)$ is predictable for all n;
4. $\int_0^T |\xi_t|^2_{H'_t} dA_t < \infty$ P-a.s..

Finally, again by using the uniform localization, it is easy to prove a GKW-decomposition theorem in our infinite-dimensional setting:

Proposition 61 *Let $M = (M^i)_{i\geqslant 1}$ be a sequence of continuous local martingales and N be a real-valued local martingale. Then, there exist an integrand ξ satisfying conditions 1.-4. above, and a real-valued local martingale L vanishing at zero and orthogonal to each M^i, such that*

$$N = N_0 + \int \xi dM + L. \tag{3.6}$$

Proof. Apply the Dellacherie uniform localization and Remark 60 and proceed exactly as, for instance, in Jacod (1979), Théorème (4.27) if N is locally square-integrable. Otherwise, use the following argument by Ansel and Stricker (1993) (D.K.W. cas 3): write N as the sum $N = N^c + N^d$, where N^c and N^d are its continuous and purely discontinuous parts. N^d is orthogonal to all continuous local martingales and N^c is locally bounded and then it can be written as $N^c = \int \xi dM + U$ with U orthogonal to M and ξ satisfying conditions 1.-4.. To conclude, it suffices to set $L = U + N^d$. $\square$

3.3 The model

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$ satisfying the usual conditions of right-continuity and completeness, where $T > 0$ is a fixed finite horizon. We also assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. There exists countably many primitive assets modelled by a sequence of real valued processes $X = (S^i)_{i\geq 0}$: one bond whose price process is given by $S^0_t = \exp \int_0^t r_s ds$, with r a progressively measurable process with respect to $\mathbb{F}$, interpreted as the instantaneous interest rate, and countably many risky assets, whose price processes S^i are continuous semimartingales for every $i \geq 1$. In the rest of the paper, we shall make the standing assumption on the bond process S^0:

$$S^0_T, \frac{1}{S^0_T} \in L^\infty(P), \tag{3.7}$$

which is equivalent to assuming that $|\int_0^T r_s ds| \leq c$, P a.s. for some positive constant c.

An important notion in mathematical finance is the set of *equivalent martingale measures*, also called risk-neutral measures. We let

$$\mathcal{M}_2 = \left\{ Q \ll P : \frac{1}{S_T^0} \frac{dQ}{dP} \in L^2(P), \text{every } S^i/S^0 \text{ is a } Q\text{-local martingale} \right\}$$

denote the set of P-absolutely continuous probability measures Q on $\mathcal{F}$ with square integrable state price density $(1/S_T^0)dQ/dP$ and such that each component of the sequence S/S^0 is a local martingale under Q. By

$$\mathcal{M}_2^e = \{Q \in \mathcal{M}_2 : Q \sim P\}$$

we denote the subset of $\mathcal{M}_2$ formed by the probability measures that are equivalent to P.

Throughout the paper, we make the natural standing assumption:

$$\mathcal{M}_2^e \neq \emptyset. \tag{3.8}$$

This assumption is related to some kind of no-arbitrage condition and we refer to Delbaen (1992) and to the seminal paper by Kreps (1981) for a general version of this fundamental theorem of asset pricing for a potentially infinite family of price processes. For the more specific framework of large financial markets, one could see also Kabanov and Kramkov (1994, 1998) and Klein and Schachermayer (1996).

We now define the space of trading strategies and the related notion of *self-financed portfolio.*

We denote by Θ the space of all generalized integrands $\vartheta \in L(S/S^0, \mathcal{U})$ such that $\int_0^T \vartheta_t d(S/S^0)_t$ is in $L^2(P)$ and for all $Q \in \mathcal{M}_2^e$ $\int \vartheta d(S/S^0)$ is a Q-martingale.

We observe (see the discussion in De Donno (2002), pp. 8-11) that in general one cannot define the value process of a trading strategy ϑ in the usual way: the expression $\vartheta_t \cdot (S/S^0)_t$ is not always well-defined. This is because ϑ_t takes values in the space $\mathcal{U}$ which, in most cases, is strictly bigger than E', and so we cannot use duality to define a product between the strategy and the price process. For this reason we give the following:

Definition 62 *For a trading strategy $\vartheta \in \Theta$ the value process of the corresponding self-financed portfolio with respect to the primitive asset family $\{S^0, S\}$ and with initial value $x \in \mathbb{R}$ is given by*

$$V_t = V_t^{x,\vartheta} = S_t^0 \left(x + \int_0^t \vartheta_s d\left(S/S^0\right)_s \right) \quad t \in [0,T] \tag{3.9}$$

Finally we denote by $G_T(x, \Theta)$ the set of *investment opportunities* (or attainable claims) with initial value $x \in \mathbb{R}$:

$$G_T(x, \Theta) := \left\{ S_T^0 \left(x + \int_0^T \vartheta_s d\left(S/S^0\right)_s \right) : \vartheta \in \Theta \right\} \subseteq L^2(P).$$

Proposition 63 *The set $G_T(x,\Theta)$ is closed in $L^2(P)$.*

Proof. Let ϑ^n be a sequence in Θ such that $S_T^0(x+\int_0^T \vartheta_s^n d(S/S^0)_s)$ converges in $L^2(P)$ to a random variable V. Take some $Q \in \mathcal{M}_2^e$ and set $Y_t^n := \int_0^t \vartheta_s^n d(S/S^0)_s$, $t \in [0,T]$. By (3.7) and since the identity mapping from $L^2(P)$ to $L^1(Q)$ is in fact a continuous operator into $\mathcal{H}^1(Q)$, there exists a real constant $c > 0$ such that

$$\|(Y_t^n)\|_{\mathcal{H}^1(Q)} \leq c \, \|Y_T^n\|_{L^2(P)} \, .$$

This shows that the sequence of Q-martingales $(Y_t^n)_{t\in[0,T]}$ converges in $\mathcal{H}^1(Q)$. Now, since convergence in $\mathcal{H}^1(Q)$ implies that in $\mathcal{S}(Q)$ (see Theorem 14 in Protter (1980), p. 208) and Emery's topology is invariant under a change of an equivalent probability measure (see Théorème II.5 in Memin (1980), p. 20), the sequence $(Y_t^n)_{t\in[0,T]}$ converges in $\mathcal{S}(P)$ too. Now, thanks to Theorem 57 we can exhibit a generalized integrand $\vartheta \in L(S/S^0, \mathcal{U})$ such that

$$V = S_T^0 \left(x + \int_0^T \vartheta_s d\left(S/S^0\right)_s \right).$$

The other two properties, characterizing the set Θ, are obviously satisfied by the process ϑ (use the same arguments as in Delbaen and Schachermayer (1996b), proof of Theorem 1.2). $\square$

This proposition makes our MVH-problem meaningful, ensuring the existence and uniqueness of its solution.

3.4 Numéraire and artificial extension

Definition 64 *A* numéraire *is defined as a self-financed portfolio with respect to the primitive assets family $\{S^0, S\}$, characterized by a trading strategy $a \in \Theta$ and a value process $V(a) = V^{1,a}$ as in (3.9) with unit initial value and intermediate values assumed to be almost surely strictly positive.*

Remark 65 *To avoid misunderstandings of our notation, observe that we have denoted by "a"* the strategy *used as numéraire, instead of the integrand in its exponential representation as in GLP.*

To such a numéraire a we can associate a new countable family of assets consisting of this numéraire and the primitive assets. This assets family is called, as in GLP, an *a-extended assets family* and its price process is given by $\{V(a), X\}$. Its price process renormalized in the new numéraire is

$$\{1, X\,(a)\} := \{1, X/V\,(a)\}\,.$$

Notice that when $a = 0$, $V(a)$ is the initial bond price process S^0 and $X(a) = (1, S/S^0)$.

The notion of equivalent martingale measures may also be applied with respect to the a-extended assets family $\{V(a), X\}$. Given a numéraire $a \in \Theta$, we define

$$\mathcal{M}_2(a) = \left\{ Q(a) \ll P : \frac{1}{V_T(a)} \frac{dQ(a)}{dP} \in L^2(P), \text{every } X(a)^i \text{ is a } Q(a)\text{-local martingale} \right\}$$

and

$$\mathcal{M}_2^e(a) = \{Q(a) \in \mathcal{M}_2(a) : Q(a) \sim P\}.$$

As in GLP Proposition 3.1, and with the same proof, we have the following characterization of the set $\mathcal{M}_2^e(a)$ of *equivalent a-martingale measures* in terms of the set $\mathcal{M}_2^e$ of equivalent martingale measures:

Proposition 66 *Let $a \in \Theta$ be a numéraire and $V(a)$ its value process. There is a one-to-one correspondence between $\mathcal{M}_2(a)$ (resp. $\mathcal{M}_2^e(a)$) and $\mathcal{M}_2$ (resp. $\mathcal{M}_2^e$): $Q(a) \in \mathcal{M}_2(a)$ (resp. $\mathcal{M}_2^e(a)$) if and only if there exists $Q \in \mathcal{M}_2$ (resp. $\mathcal{M}_2^e$) such that*

$$\frac{dQ(a)}{dP} = \frac{V_T(a)}{S_T^0} \frac{dQ}{dP}. \tag{3.10}$$

We now define the notion of trading strategy and the self-financed portfolio with respect to the a-extended assets family.

We denote by $\Phi(a)$ the space of trading strategies with respect to the a-extended assets family $\{V(a), X\}$, i.e. the set of all $\phi(a) \in L(X(a), \mathcal{U})$ such that $V_T(a) \int_0^T \phi_t(a) dX_t(a) \in L^2(P)$ and $V(a) \int \phi(a) dX(a)$ is a local $Q(a)$-martingale for all $Q(a) \in \mathcal{M}_2^e(a)$.

For a trading strategy $\phi(a) \in \Phi(a)$ the value process of the corresponding self-financed portfolio with respect to the a-extended assets family $\{V(a), X\}$ and with initial value $x \in \mathbb{R}$ is given by

$$V_t = V_t^{x,\phi(a)} = V_t(a) \left(x + \int_0^t \phi_s(a) dX_s(a) \right) \quad t \in [0, T] \tag{3.11}$$

Let us denote by $G_T(x, \Phi(a))$ the set of terminal values of self-financed portfolios with respect to the a-extended assets family $\{V(a), X\}$, and with initial value x:

$$G_T(x, \Phi(a)) := \left\{ V_T(a) \left(x + \int_0^T \phi_s(a) dX_s(a) \right) : \phi(a) \in \Phi(a) \right\}.$$

In the finite assets case (GLP, Proposition 3.2, p. 186-188) the artificial extension leaves invariant the investment opportunity set, and there are explicit expressions for the correspondences linking the investment opportunities sets. We briefly recall this result: let a^n be a n-dimensional numéraire and $V(a^n) = V^{1,a^n}$ its value process,

- to a self-financed portfolio (V^n, ϑ^n) with respect to $\{S^0, \overline{S}^n\}$, corresponds the self-financed portfolio $(V^n, \phi^n(a^n)) = (V^n, (\eta^n(a^n), \vartheta^n(a^n)))$ w.r.t. $\{V(a^n), S^0, S\}$ given by

$$\eta_t^n(a^n) = \frac{V_t^n - \vartheta_t^n \overline{S}_t^n}{S_t^0} \quad \text{and} \quad \vartheta_t^n(a^n) = \vartheta_t^n, \quad t \in [0, T]; \tag{3.12}$$

- to a self-financed portfolio $(V^n, \phi^n(a^n)) = (V^n, (\eta^n(a^n), \vartheta^n(a^n)))$ w.r.t. $\{V(a^n), S^0, S\}$ corresponds the self-financed portfolio (V^n, ϑ^n) w.r.t. $\{S^0, \overline{S}^n\}$ given by

$$\vartheta_t^n = \vartheta_t^n(a^n) + a_t^n \frac{V_t^n - \phi_t^n(a^n)\overline{X}_t^n}{V_t(a^n)}, \quad t \in [0,T]. \tag{3.13}$$

Note that the above expressions involve some scalar product between strategies and price processes, which in our infinite-dimensional setting are not well-defined. This makes very difficult to find the infinite-dimensional analogues of the GLP-correspondences above. Nonetheless, the following proposition states their existence for our large financial market and, as a direct consequence, the invariance of the investment opportunities set under a change of numéraire.

Proposition 67 *Let $a \in \Theta$ be a numéraire and $V(a)$ its value process.*

1. *Let $\vartheta \in \Theta$ be a trading strategy with respect to the primitive assets family $\{S^0, S\}$ and let V denote the value process of the corresponding self-financed portfolio. Then there exists a trading strategy $\phi(a) \in \Phi(a)$ with respect to the a-extended assets family $\{V(a), X\}$* with the same value process V.

2. *Let $\phi(a) = (\eta(a), \vartheta(a)) \in \Phi(a)$ be a trading strategy with respect to the a-extended assets family $\{V(a), X\}$ and let V denote the value process of the corresponding self-financed portfolio. Then there exists a trading strategy $\vartheta \in \Theta$ with respect to the primitive assets family $\{S^0, S\}$* with the same value process V.

3. *We have then in particular that*

$$G_T(x, \Theta) = G_T(x, \Phi(a)). \tag{3.14}$$

Proof. 1. Let $\vartheta \in \Theta$ be a trading strategy with respect to the primitive assets family $\{S^0, S\}$ with value process $V = S^0(V_0 + \int \vartheta d(S/S^0))$. By Definition 56 there exists a sequence ϑ^n of simple integrands w.r.t. S/S^0 such that $\int \vartheta^n d(S/S^0)$ converges in $\mathcal{S}(P)$ to $\int \vartheta d(S/S^0)$.

We associate to each approximating strategy ϑ^n a self-financed portfolio with respect to the primitive assets family, whose value process is given by

$$V_t^n = S_t^0 \left(V_0 + \int_0^t \vartheta_s^n d\left(S^n/S^0\right)_s \right), \quad t \in [0,T].$$

By GLP, Proposition 3.2 (i), there exists a trading strategy $\phi^n(a) = (\eta^n(a), \vartheta^n(a))$ given by (3.12) with $V^n(a)$ instead of $V^n(a^n)$ and with the same value process V^n, i.e.

$$S_t^0 \left(V_0 + \int_0^t \vartheta_s^n d\left(S/S^0\right)_s \right) = V_t(a) \left(V_0 + \int_0^t \phi_s^n(a) dX_s(a) \right), \quad t \in [0,T].$$

By the multidimensional version of Proposition 4 in Emery (1979), we have that

$$\frac{S^0}{V(a)}\left(V_0+\int \vartheta^n d\left(S/S^0\right)\right) \to \frac{S^0}{V(a)}\left(V_0+\int \vartheta d\left(S/S^0\right)\right)$$

in $\mathcal{S}(P)$, as $n\to\infty$, and so the sequence $\int \phi^n(a)dX(a)$ is convergent in $\mathcal{S}(P)$. Now, by the infinite-dimensional version of Memin's theorem (Theorem 57) there exists a generalized integrand $\phi(a)\in L(X(a),\mathcal{U})$ such that $V_0+\int\phi^n(a)dX(a)\to V_0+\int\phi(a)dX(a)$ in $\mathcal{S}(P)$, as $n\to\infty$, and obviously for all $t\in[0,T]$

$$S_t^0\left(V_0+\int_0^t \vartheta_s d\left(S/S^0\right)_s\right)=V_t\left(a\right)\left(V_0+\int_0^t \phi_s\left(a\right)dX_s\left(a\right)\right).$$

Finally, by Proposition 66 and since $\vartheta\in\Theta$, the process $\int\phi(a)dX(a)$ is a local $Q(a)$-martingale for all $Q(a)\in\mathcal{M}_2^e(a)$, and also $V_T(a)\int_0^T\phi_s(a)dX_s(a)\in L^2(P)$, i.e. $\phi(a)\in\Phi(a)$.

2. Let $\phi(a)\in\Phi(a)$ be a trading strategy with respect to the a-extended assets family $\{V(a),X\}$ with value process of the corresponding self-financed portfolio given by $V(a)(V_0+\int\phi(a)dX(a))$.

By definition of $\Phi(a)$, there exists a sequence of simple integrands $\phi^n(a)=(\eta^n(a),\vartheta^n(a))$, with $\eta^n(a)$ real-valued, converging pointwise to $\phi(a)$ and such that

$$\int\phi^n(a)dX(a)\to\int\phi(a)dX(a)$$

in $\mathcal{S}(P)$ as $n\to\infty$.

Denote by V^n the value process of the approximating self-financed portfolio corresponding to $\phi^n(a)$, i.e. $V^n=V(a)(V_0+\int\phi_s^n(a)dX(a))$, and consider the following sequence of strategies ϑ^n with respect to $\{S^0,S\}$ defined by the GLP correspondence (3.13):

$$\vartheta_t^n=\vartheta_t^n(a)+a_t\psi_t^n(a),\quad t\in[0,T],$$

where

$$\psi^n=\frac{V^n-\phi^n(a)\overline{X}^n}{V(a)}.$$

We remark that the process ϑ^n takes values in $\mathcal{U}$. Now, if we proceed as in the second part of the proof of Proposition 3.2 in GLP (observe that, by definition of generalized integrand, $\phi^n(a)$ is bounded, which implies $\psi^n(a)$ locally bounded), we obtain

$$d(V^n/S^0)_t=\psi_t^n d(V(a)/S^0)_t+\vartheta_t^n(a)d(S/S^0)_t.$$

Being $d(V(a)/S^0)_t=a_td(S/S^0)_t$ with $a\in L(S/S^0,\mathcal{U})$, if we approximate a by a sequence a^k of simple integrands converging pointwise to a and such that $\int a^kd(S/S^0)\to\int ad(S/S^0)$ in $\mathcal{S}(P)$, also the sequence $\int\psi^n(a)a^kd(S/S^0)$ converges in $\mathcal{S}(P)$ with n fixed and k tending to infinity and then, by Theorem 57, there exists a generalized integrand ζ^n such that

$$\psi_t^n(a)d\left(\frac{V(a)}{S^0}\right)_t=\zeta_t^nd\left(\frac{S}{S^0}\right)_t,$$

and moreover, since $\psi^n(a)a^k$ converges pointwise to $\psi^n(a)a$, $\zeta^n = \psi^n(a)a$. Furthermore

$$S_t^0 \left(V_0 + \int_0^t \vartheta_s^n d(S/S^0)_s \right) = V_t(a) \left(V_0 + \int_0^t \phi_s^n(a) dX_s(a) \right), \quad t \in [0,T].$$

Finally, by letting n tend to infinity and by using the same argument (infinite-dimensional version of Memin's theorem) as in the previous part of the proof (after having inverted the rôles of ϑ^n and $\phi^n(a)$), one can easily show that there exists a strategy $\vartheta \in \Theta$, whose value process equals V. The proof of the point 2 is now complete.

3. It is clear from the first two points of this proposition. □

As we have discussed just before this proposition, since it is not possible in this setting to define a product between strategies and price processes, we are not able to find an explicit expression for the infinite-dimensional GLP correspondences. We only know that the two strategies sets are related by the equality of their value processes, i.e. given a strategy ϑ (resp. $\phi(a)$) its corresponding strategy $\phi(a)$ (resp. ϑ) satisfies the following equation:

$$S_t^0 \left(V_0 + \int_0^t \vartheta_s d(S/S^0)_s \right) = V_t(a) \left(V_0 + \int_0^t \phi_s(a) dX_s(a) \right), \quad t \in [0,T].$$

The previous proposition ensures the existence of a unique solution to this equation when ϑ (resp. $\phi(a)$) is fixed. Nonetheless, we observe that if, given a trading strategy ϑ w.r.t. $\{S^0, S\}$, its approximating sequence ϑ^n is such that $\vartheta^n \overline{S}^n$ converges pointwise to some process U, then the corresponding trading strategy $\phi(a) = (\eta(a), \vartheta(a))$ is given by

$$\eta_t(a) = \frac{V_t - U_t}{S_t^0} \quad \vartheta_t(a) = \vartheta_t, \quad t \in [0,T].$$

Analogously, if, given a trading strategy $\phi(a)$ w.r.t. $\{V(a), S^0, S\}$, its approximating sequence $\phi^n(a)$ is such that $\phi^n(a)\overline{X}^n$ converges pointwise to a process W, then the corresponding trading strategy ϑ is given by

$$\vartheta_t = \vartheta_t(a) + a_t \frac{V_t - W_t}{V_t(a)}, \quad t \in [0,T].$$

Remark 68 *In Section 3.6, we will see that there exists a sequence of predictable trading strategies $\vartheta^{n,*}$, that both solve the MVH-problem arisen by considering only the first n risky assets, and its value processes converge to the value process of ϑ^*, solution to problem (3.1), in $L^2(P)$ as n tends to infinity.*

3.5 The MVH problem

We would like to apply the artificial extension method introduced by GLP to the following "large" mean-variance hedging optimization problem:

$$J(x,F) := \min_{\vartheta \in \Theta} \mathbb{E} \left[F - S_T^0 \left(x + \int_0^T \vartheta_t d\left(S/S^0\right)_t \right) \right]^2. \qquad (\mathcal{H}(x))$$

where $F \in L^2(P)$ and $x \in \mathbb{R}$ are fixed. In financial terms, given an $\mathcal{F}_T$-measurable contingent claim $F \in L^2(P)$, we are looking for a self-financed portfolio with respect to the primitive assets family $\{S^0, S\}$, with initial investment x, that minimizes the expected square of the hedging residual. Mathematically speaking, we would like to project the $\mathcal{F}_T$-measurable square integrable random variable $F - S_T^0 x$ on the closed subspace $G_T(0, \Theta)$ of $L^2(P)$. There exists a unique solution $\vartheta^* = \vartheta^*(x, H)$ to the problem $(\mathcal{H}(x))$ called the *optimal hedging strategy*, with associated value process

$$V_t^* = V_t^*(x, F) = S_T^0 \left(x + \int_0^T \vartheta_t^*(x, F)\, d\left(S/S^0\right)_t \right).$$

The couple (V^*, ϑ^*) is called *optimal hedging portfolio.*

Let us consider first the following optimization problem:

$$\min_{\vartheta \in \Theta} \mathbb{E}\left[S_T^0 \left(1 + \int_0^T \vartheta_t d\left(S/S^0\right)_t \right) \right]^2 \tag{$\mathcal{P}$}$$

which is a particular case of $(\mathcal{H}(x))$ for a zero cash flow $F = 0$ and with initial wealth $x = 1$. Since, by Proposition 8, $G_T(1, \Theta)$ is a non-empty closed convex set in $L^2(P)$ under the standing assumptions (3.7) and (3.8), problem $(\mathcal{P})$ has a solution $\widetilde{a} \in \Theta$,which leads to a unique terminal wealth $V_T(\widetilde{a}) = V_T^{1,\widetilde{a}} = S_T^0(1 + \int_0^T \widetilde{a}_t d(S/S^0)_t)$. Let us consider also the dual quadratic problem of $(\mathcal{P})$:

$$\min_{Q \in \mathcal{M}_2} \mathbb{E}\left[\frac{1}{S_T^0} \frac{dQ}{dP} \right]^2 . \tag{$\mathcal{D}$}$$

Under the standing assumption (3.7) and (3.8), the set

$$\mathbb{D}_2 := \left\{ \frac{1}{S_T^0} \frac{dQ}{dP} : Q \in \mathcal{M}_2 \right\}$$

is a non-empty closed convex set in $L^2(P)$, and therefore problem $(\mathcal{D})$ admits a unique solution $\widetilde{P} \in \mathcal{M}_2$.

Our aim, as in GLP, is to prove that $V_T(\widetilde{a})$ is strictly positive so that it can be chosen as a numéraire. It is possible to extend all results of GLP, Section 4, to our case, i.e. for countably many risky assets. We summarize them in the following:

Theorem 69 *Assume (3.7) and (3.8).*

1. *The variance-optimal martingale measure (abbr. VOMM) $\widetilde{P}$ solution to problem $(\mathcal{D})$ is related to the terminal wealth $V_T(\widetilde{a})$ corresponding to the solution $\widetilde{a}$ of problem $(\mathcal{P})$ by*

$$\frac{1}{S_T^0} \frac{d\widetilde{P}}{dP} = \mathbb{E}\left[\frac{1}{S_T^0} \frac{d\widetilde{P}}{dP} \right]^2 V_T(\widetilde{a}). \tag{3.15}$$

2. *$\widetilde{P}$ is equivalent to P; that is , $\widetilde{P} \in \mathcal{M}_2^e$.*

3. The self-financed portfolio value process $V(\widetilde{a})$ is strictly positive:

$$V_t(\widetilde{a}) > 0, \quad P \ a.s., \, t \in [0,T] \, .$$

Proof. The proof is exactly as in the case of a finite number of risky assets, for whom see GLP, Theorem 4.1, for point 1 and Theorem 4.2 for points 2 and 3. □

Since the self-financed portfolio value process $V(\widetilde{a})$ is strictly positive, we can use it as a numéraire that we call the *hedging numéraire*. From (3.15), the VOMM is then related to the hedging numéraire by

$$\frac{d\widetilde{P}}{dP} = \frac{V_T\left(\widetilde{a}\right) S_T^0}{\mathbb{E}\left[V_T\left(\widetilde{a}\right) S_T^0\right]} \tag{3.16}$$

Following GLP, we will solve problem $(\mathcal{H}\,(x))$ by transforming it into a simpler one corresponding to the martingale case thanks to the artificial extension method. Let us consider the hedging numérare $\widetilde{a}$ and the associated $\widetilde{a}$-extended assets family $\{V(\widetilde{a}), S^0, S\}$. We can define the equivalent $\widetilde{a}$-martingale measure $\widetilde{P}(\widetilde{a})$ given by the relation

$$\frac{d\widetilde{P}\left(\widetilde{a}\right)}{dP} = \frac{V_T\left(\widetilde{a}\right)^2}{\mathbb{E}\left[V_T\left(\widetilde{a}\right)\right]^2} \tag{3.17}$$

and we call it the *variance-optimal $\widetilde{a}$-martingale measure.* Let us consider the quadratic optimization problem

$$J^{\widetilde{a}}(x,F) = \min_{\phi(\widetilde{a})\in\Phi(\widetilde{a})} \mathbb{E}_{\widetilde{P}(\widetilde{a})}\left[\frac{F}{V_T\left(\widetilde{a}\right)} - x - \int_0^T \phi_t\left(\widetilde{a}\right) dX_t\left(\widetilde{a}\right)\right]^2 \tag{$\mathcal{H}^{\widetilde{a}}\,(x)$}$$

It is easy to see (this is a straightforward extension of Proposition 5.1 in GLP) that *problems* $(\mathcal{H}\,(x))$ *and* $(\mathcal{H}^{\widetilde{a}}\,(x))$ *are equivalent* in the following sense: if ϑ^* and $\phi^*(\widetilde{a})$ are the unique solutions of, respectively, problem $(\mathcal{H}\,(x))$ and problem $(\mathcal{H}^{\widetilde{a}}\,(x))$, then they have the same value process, i.e.

$$S_t^0\left(V_0 + \int_0^t \vartheta_s^* d(S/S^0)_s\right) = V_t(\widetilde{a})\left(V_0 + \int_0^t \phi_s^*(\widetilde{a}) dX_s(\widetilde{a})\right), \quad t \in [0,T]. \tag{3.18}$$

Moreover, the relation (5.2) in GLP, between their minimal quadratic risks, is still verified, i.e. we still have

$$J\left(x,F\right) = \mathbb{E}\left[V_T\left(\widetilde{a}\right)\right]^2 J^{\widetilde{a}}\left(x,F\right). \tag{3.19}$$

Now since $\widetilde{P}(\widetilde{a}) \in \mathcal{M}_2^e(\widetilde{a})$, the continuous process $X(\widetilde{a})$ is a locally square integrable martingale under $\widetilde{P}(\widetilde{a})$. Furthermore, being F square integrable under P, the claim $F/V_T(\widetilde{a})$ is square integrable under $\widetilde{P}(\widetilde{a})$. The infinite-dimensional GKW-projection theorem (Proposition 61) implies that there exists a $\mathcal{U}$-valued predictable process $\phi^F(\widetilde{a})$ satisfying

$$\mathbb{E}_{\widetilde{P}(\widetilde{a})}\left[\left\langle \int \phi^F\left(\widetilde{a}\right) dX\left(\widetilde{a}\right)\right\rangle_T\right] < \infty$$

and a real-valued square integrable $\widetilde{P}(\widetilde{a})$-martingale $\widetilde{R}(\widetilde{a})$, orthogonal to $X(\widetilde{a})$ under $\widetilde{P}(\widetilde{a})$, such that

$$\frac{F}{V_T(\widetilde{a})} = \mathbb{E}_{\widetilde{P}(\widetilde{a})}\left[\frac{F}{V_T(\widetilde{a})}\right] + \int_0^T \phi^F(\widetilde{a})\, dX(\widetilde{a}) + \widetilde{R}_T(\widetilde{a}) .. \tag{3.20}$$

It is now easy to see (use the same argument as in Lemma 5.1 of GLP) that the solution $\phi^*(\widetilde{a})$ to problem $(\mathcal{H}^{\widetilde{a}}(x))$ is given by the integrand in the decomposition (4.18), i.e. $\phi^*(\widetilde{a}) = \phi^F(\widetilde{a})$, and that the associated minimal quadratic risk of problem $(\mathcal{H}^{\widetilde{a}}(x))$ is given by

$$J^{\widetilde{a}}(x, F) = \left(\mathbb{E}_{\widetilde{P}(\widetilde{a})}\left[\frac{F}{V_T(\widetilde{a})}\right] - x\right)^2 + \mathbb{E}_{\widetilde{P}(\widetilde{a})}\left[\widetilde{R}_T(\widetilde{a})\right]^2 . \tag{3.21}$$

We now summarize how to "theoretically" solve our initial infinite-dimensional MVH-problem $(\mathcal{H}(x))$: compute the hedging numéraire $\widetilde{a}$ and consider the MVH-problem $(\mathcal{H}^{\widetilde{a}}(x))$ corresponding to the price process $X(\widetilde{a})$, the strategies set $\Phi(\widetilde{a})$ and the probability $\widetilde{P}(\widetilde{a})$, which is a martingale measure for the new integrator; the GKW-projection theorem gives its unique solution $\phi^*(\widetilde{a})$. Now, in order to find the optimal strategy ϑ^*, *solve with respect to ϑ the following stochastic equation*:

$$S_t^0\left(V_0 + \int_0^t \vartheta_s d(S/S^0)_s\right) = V_t(\widetilde{a})\left(V_0 + \int_0^t \phi_s^*(\widetilde{a}) dX_s(\widetilde{a})\right), \quad t \in [0, T]. \tag{3.22}$$

Observe that Proposition 67 ensures the existence of a unique solution for this equation.

We conclude this section by considering the problem

$$\min_{x \in \mathbb{R}} J(x, F), \tag{$\mathcal{H}$}$$

which corresponds to the projection of F on the closed subspace $\{G_T(x, \Theta) : x \in \mathbb{R}\}$ of $L^2(P)$ (to see this use the same argument as in GLP, p. 195). The solution $x^*(F)$ to problem $(\mathcal{H})$ is called the *approximation price* for F (see Schweizer (1996)) and it is obviously a generalization of the usual arbitrage-free price for F. From (3.19) and (3.21) one can easily deduce that the approximation price for F is given by

$$x^*(F) = \mathbb{E}_{\widetilde{P}(\widetilde{a})}\left[\frac{F}{V_T(\widetilde{a})}\right]$$

and so, by Proposition 66,

$$x^*(F) = \mathbb{E}_{\widetilde{P}}\left[\frac{F}{S_T^0}\right]. \tag{3.23}$$

This shows that, even in a market with countably many assets, the VOMM can be interpreted as a viable price system corresponding to a mean-variance criterion, and also extends Theorem 5.2 of GLP.

3.6 Finite-dimensional MVH problems

For all $n \geq 1$, we denote by $\mathbb{F}^n = \{\mathcal{F}^n_t : t \in [0,T]\}$ the (completed) filtration generated by the n-dimensional primitive assets family $\{S^0, \overline{S}^n\}$, $\mathcal{F}^n = \mathcal{F}^n_T$, by P^n the restriction on $\mathcal{F}^n$ of the probability measure P and we set

$$\begin{aligned}\mathcal{M}^n_2 &= \Big\{Q^n \text{ probability measure on } \mathcal{F}^n : Q^n \ll P^n, \\ &\qquad \frac{1}{S^0_T}\frac{dQ^n}{dP} \in L^2(P), \overline{S}^n/S^0 \text{ is a local } Q^n\text{-martingale}\Big\}\end{aligned}$$

and

$$\mathcal{M}^{n,e}_2 = \{Q^n \in \mathcal{M}^n_2 : Q^n \sim P^n\}.$$

Assumption (3.8) ensures that, for all $n \geq 1$, the set $\mathcal{M}^{n,e}_2$ is not empty. Recall that $\overline{S}^n = (S^1, \cdots, S^n)$, $n \geq 1$.

In this section we consider for all $n \geq 1$ the following n-dimensional mean-variance hedging (n-MVH) problem:

$$\min_{\vartheta^n \in \Theta^n} \mathbb{E}\left[F - S^0_T\left(x + \int_0^T \vartheta^n_t d\left(\overline{S}^n/S^0\right)_t\right)\right]^2, \qquad (\mathcal{H}_n(x))$$

where we have denoted by Θ^n the set of all $\mathbb{R}^n$-valued $\overline{S}^n/S^0$-integrable $\mathbb{F}^n$-predictable processes ϑ^n such that $S^0_T \int_0^T \vartheta^n_t d(\overline{S}^n/S^0)_t \in L^2(\mathcal{F}^n, P)$ and for all $Q^n \in \mathcal{M}^n_2$, the process $\int \vartheta^n d(\overline{S}^n/S^0)$ is a Q^n-martingale, $F \in L^2(P)$ and $x \in \mathbb{R}$ being fixed.

All the objects that we have introduced in the previous two sections have their n-dimensional counterparts, their notations and financial interpretations will be self-evident.

The aim of this section is to study the asymptotical behaviour, as $n \to \infty$, of the sequence $(\vartheta^{n,*})_{n\geq 1}$, where $\vartheta^{n,*}$ is the unique solution to problem $(\mathcal{H}_n(x))$.

Let us consider the following finite-dimensional dual problem associated to the assets family $\{S^0, \overline{S}^n\}$, $n \geq 1$:

$$\min_{Q^n \in \mathcal{M}^n_2} \mathbb{E}\left[\frac{1}{S^0_T}\frac{dQ^n}{dP^n}\right]^2. \qquad (\mathcal{D}_n)$$

Under the standing assumptions (3.7) and (3.8), problem $(\mathcal{D}_n)$ admits a unique solution $\widetilde{P}^n \in \mathcal{M}^n_2$, which we call *$n$-dimensional variance-optimal martingale measure* (abbr. n-VOMM).

Now some other notations from Delbaen and Schachermayer (1996a): for all $n \geq 1$, we denote by K^n_0 the subspace of $L^\infty(\mathcal{F}^n, P)$ spanned by the stochastic integrals of the form

$$f_n = h'_n\left(\left(\overline{S}^n/S^0\right)_{\tau_2} - \left(\overline{S}^n/S^0\right)_{\tau_1}\right)$$

where $\tau_1 \leq \tau_2$ are $\mathbb{F}^n$-stopping times such that the stopped process $(\overline{S}^n/S^0)^{\tau_2}$ is bounded and h_n is a bounded $\mathbb{R}^n$-valued $\mathcal{F}^n_{\tau_1}$-measurable function.

We denote by $\widehat{K^n_0}$ the closure of K^n_0 in $L^2(P)$ and by $\widehat{K^n}$ the closure of the span of K^n_0 and the constants in $L^2(P)$, i.e. $\widehat{K^n} = \overline{\text{span}(K^n_0, 1)}$.

On the other hand we denote by K_0 the subspace of $L^\infty(\mathcal{F}^n, P)$ given by the union of all K_0^n, i.e. $K_0 := \cup_{n\geq 1} K_0^n$, by $\widehat{K_0}$ the closure of K_0 in $L^2(P)$ and by $\widehat{K}$ the closure of the span of K_0 and the constants in $L^2(P)$, i.e. $\widehat{K} = \overline{\text{span}(K_0, 1)}$. Obviously a probability measure Q^n on $\mathcal{F}^n$ (resp. a probability measure Q on $\mathcal{F}$) is a local martingale measure for $\overline{S}^n/S^0$ (resp. for S/S^0) if and only if $\mathbb{E}_{Q^n}[f_n] = 0$ for every $f_n \in K_0^n$ (resp. $\mathbb{E}_Q[f] = 0$ for every $f \in K_0$).

We recall the following characterizations of the VOMM $\widetilde{P}$ and the n-VOMM $\widetilde{P}^n$ (here we identify any measure Q with the linear functional $\mathbb{E}_Q[\cdot]$ and linear functionals on $L^2(P)$ with elements of $L^2(P)$):

Lemma 70 (Lemma 2.1(c) in Delbaen and Schachermayer (1996a)) *Assume (3.7) and (3.8).*

1. *$\widetilde{P}$ is the unique element of $\widehat{K}$ vanishing on $\widehat{K_0}$ and equaling* 1 *on the constant function* 1*;*

2. *For all $n \geq 1$, $\widetilde{P}^n$ is the unique element of $\widehat{K^n}$ vanishing on $\widehat{K_0^n}$ and equaling* 1 *on the constant function* 1*.*

We have then the following

Proposition 71 *Assume (3.7) and (3.8). The sequence $\widetilde{P}^n$ converges in $L^2(P)$, as $n \to \infty$, to the VOMM $\widetilde{P}$ solution to problem $(\mathcal{D})$.*

Proof. It is an immediate application of the projection theorem for Hilbert spaces. □

Now, we consider the following n-MVH problem:

$$\min_{\vartheta^n \in \Theta^n} \mathbb{E}\left[S_T^0\left(1 + \int_0^T \vartheta_t^n d\left(\overline{S}^n/S^0\right)_t\right)\right]^2 \tag{$\mathcal{P}_n$}$$

and we set

$$G_T^n(x, \Theta^n) := \left\{S_T^0\left(x + \int_0^T \vartheta_s^n d\left(\overline{S}^n/S^0\right)_s\right) : \vartheta^n \in \Theta^n\right\}.$$

We remark that for all $n \geq 1$

$$G_T^n(x, \Theta^n) \subseteq G_T^{n+1}\left(x, \Theta^{n+1}\right) \subseteq G_T(x, \Theta).$$

It is well known that, under the assumptions (3.7) and (3.8), $G_T^n(0, \Theta^n)$ is closed in $L^2(P)$ and so each problem $(\mathcal{P}_n)$ has a unique solution $\widetilde{a}^n \in \Theta^n$, which leads to a unique terminal wealth $V_T^n(\widetilde{a}^n) = V_T^{1,\widetilde{a}^n} = S_T^0(1 + \int_0^T \widetilde{a}_t^n d(\overline{S}^n/S^0)_t)$. As in GLP and in the previous section, the n-VOMM $\widetilde{P}^n$ satisfies the following properties (see Theorems 4.1 and 4.2 in GLP):

1. the n-VOMM $\widetilde{P}^n$ is related to the terminal wealth $V_T^n(\widetilde{a}^n)$ corresponding to the solution $\widetilde{a}^n$ of problem $(\mathcal{P}_n)$ by

$$\frac{1}{S_T^0}\frac{d\widetilde{P}^n}{dP^n} = \mathbb{E}\left[\frac{1}{S_T^0}\frac{d\widetilde{P}^n}{dP^n}\right]^2 V_T^n(\widetilde{a}^n) \tag{3.24}$$

2. $\widetilde{P}^n \in \mathcal{M}_2^{n,e}$

3. $V_t^n(\widetilde{a}^n) > 0, \quad P^n$ a.s., $t \in [0,T]$.

We can so use the self-financed portfolio value process $V^n(\widetilde{a}^n)$ as a numéraire that we call *n-dimensional hedging numéraire.*. We define the *n-dimensional variance-optimal $\widetilde{a}^n$-martingale measure* by the relation

$$\frac{d\widetilde{P}^n\left(\widetilde{a}^n\right)}{dP^n} = \frac{V_T^n\left(\widetilde{a}^n\right)^2}{\mathbb{E}_n\left[V_T^n\left(\widetilde{a}^n\right)\right]^2}$$

and consider the n-dimensional analogue of problem $(\mathcal{H}^{\widetilde{a}}\,(x))$:

$$\min_{\phi^n(\widetilde{a}^n)\in\Phi^n(\widetilde{a}^n)} \mathbb{E}_{\widetilde{P}^n(\widetilde{a}^n)}\left[\frac{F}{V_T^n\left(\widetilde{a}^n\right)} - x - \int_0^T \phi_t^n\left(\widetilde{a}^n\right) d\overline{X_t\left(\widetilde{a}^n\right)}^n\right]^2 \tag{$\mathcal{H}_n^{\widetilde{a}^n}\,(x)$}$$

where $\overline{X(\widetilde{a}^n)}^n = \overline{X}^n/V^n(\widetilde{a}^n)$ and $\Phi^n(\widetilde{a}^n)$ is the set of all $\mathbb{R}^{n+1}$-valued $\overline{X(\widetilde{a}^n)}^n$-integrable predictable processes such that $V_T^n(\widetilde{a}^n)\int_0^T \phi_t^n(\widetilde{a}^n)d\overline{X_t(\widetilde{a}^n)}^n \in L^2(\mathcal{F}^n, P)$ and for all $Q^n(\widetilde{a}^n) \in \mathcal{M}_2^n(\widetilde{a}^n)$ (which has an obvious meaning), the process $\int \phi^n(\widetilde{a}^n)d\overline{X(\widetilde{a}^n)}^n$ is a local $Q^n(\widetilde{a}^n)$-martingale.

We recall that, for all $n \geq 1$, the solution to problem $(\mathcal{H}_n^{\widetilde{a}^n}\,(x))$ is given by the $\mathbb{R}^{n+1}$-valued predictable integrand $\phi^{n,*}(\widetilde{a}^n)$ satisfying the integrability condition

$$\mathbb{E}_{\widetilde{P}^n(\widetilde{a}^n)}\left[\left\langle\int \phi^{n,*}\left(\widetilde{a}^n\right) d\overline{X\left(\widetilde{a}^n\right)}^n\right\rangle_T\right] < \infty$$

in the following GKW-decomposition

$$\frac{F}{V_T^n\left(\widetilde{a}^n\right)} = \mathbb{E}_{\widetilde{P}^n(\widetilde{a}^n)}\left[\frac{F}{V_T^n\left(\widetilde{a}^n\right)}\right] + \int_0^T \phi^{n,*}\left(\widetilde{a}^n\right) d\overline{X\left(\widetilde{a}^n\right)}^n + \widetilde{R}_T^n\left(\widetilde{a}^n\right), \tag{3.25}$$

where $\widetilde{R}^n(\widetilde{a}^n)$ is a real-valued square integrable $(\mathbb{F}^n, \widetilde{P}^n(\widetilde{a}^n))$-martingale, orthogonal to $\overline{X(\widetilde{a}^n)}^n$ under $\widetilde{P}^n(\widetilde{a}^n)$, and then the associated minimal quadratic risk of problem $(\mathcal{H}_n^{\widetilde{a}^n}\,(x))$ is given by

$$J^{\widetilde{a}^n}\left(x,F\right) = \left(\mathbb{E}_{\widetilde{P}^n(\widetilde{a}^n)}\left[\frac{F}{V_T^n\left(\widetilde{a}^n\right)}\right] - x\right)^2 + \mathbb{E}_{\widetilde{P}^n(\widetilde{a}^n)}\left[\widetilde{R}_T^n\left(\widetilde{a}^n\right)\right]^2. \tag{3.26}$$

Proposition 72 *Let $\vartheta^{n,*}$ and $\phi^{n,*}$ be solutions to problems (respectively) $(\mathcal{H}_n(x))$ and $(\mathcal{H}_n^{\widetilde{a}^n}(x))$ for all $n \geq 1$. Then we have the following assertions:*

1. *$\int_0^T \vartheta_t^{n,*} d(\overline{S}^n/S^0)_t$ converges to $\int_0^T \vartheta_t^* d(S/S^0)_t$ in $L^2(P)$ as $n \to \infty$, where ϑ^* is the solution to problem $(\mathcal{H}(x))$;*

2. *$V_T^n(\widetilde{a}^n) \int_0^T \phi_t^{n,*}(\widetilde{a}^n) d\overline{X_t(\widetilde{a}^n)}^n$ converges to $V_T(\widetilde{a}) \int_0^T \phi_t^*(\widetilde{a}) dX_t(\widetilde{a})$ in $L^2(P)$ as $n \to \infty$, where ϕ^* is the solution to problem $(\mathcal{H}^{\widetilde{a}}(x))$.*

Proof. 1. It suffices to note that for all $n \geq 1$, $S_T^0 \int_0^T \vartheta_t^{n,*} d(\overline{S}^n/S^0)_t$ is the orthogonal projection of $S_T^0 \int_0^T \vartheta_t^* d(S/S^0)_t$ onto the subspace $G_T^n(0, \Theta^n)$ closed in $L^2(P)$.

2. By Proposition 3.2 of GLP and Proposition 67 we have, respectively, that for all $n \geq 1$

$$S_T^0 \left(x + \int_0^T \vartheta_t^{n,*} d\left(\overline{S}^n/S^0\right)_t \right) = V_T^n(\widetilde{a}^n) \left(x + \int_0^T \phi_t^{n,*}(\widetilde{a}^n)\, d\overline{X_t(\widetilde{a}^n)}^n \right)$$

and

$$S_T^0 \left(x + \int_0^T \vartheta_t^* d\left(S/S^0\right)_t \right) = V_T(\widetilde{a}) \left(x + \int_0^T \phi_t^*(\widetilde{a})\, dX_t(\widetilde{a}) \right).$$

Then assertion 2. follows easily. □

Proposition 72 suggests the following natural procedure to compute, at least asymptotically, the optimal strategy ϑ^*:

1. instead of $(\mathcal{H}(x))$, consider the n-dimensional problem $(\mathcal{H}_n(x))$;

2. solve the equivalent problem associated $(\mathcal{H}_n^{\widetilde{a}^n}(x))$, by means of the GKW-projection theorem;

3. come back to the solution $\vartheta^{n,*}$ of the original n-dimensional problem $(\mathcal{H}_n(x))$, by means of the correspondence given in GLP, Theorem 5.1, p. 198;

4. let n tend to infinity.

Remark 73 *Firstly, we point out that, with respect to the Emery topology, used to prove Proposition 67, the $L^2(P)$-convergence works only for the finite-dimensional* optimal *hedging strategies. Then, to establish the correspondence between the sets Θ and $\Phi(a)$ of* all strategies*, one has to deal with convergence in $\mathcal{S}(P)$.*

3.7 Conclusions

In this chapter, we have generalized the artificial extension method, developed in a market with a finite number of risky assets by GLP, to a large financial market. The more delicate point is the use of the SI theory (by De Donno and Pratelli (2003)) with respect to a sequence of semimartingales to define the "good" set of trading strategies and to extend the invariance property (under the change of numéraire) of the set of all attainable contingent claims. Indeed, since in the infinite-dimensional setting the strategies are allowed to take values in the space $\mathcal{U}$ of non necessarily bounded operators on E, it is not possible multiply the strategies and the price process S. This observation has two consequences: firstly, we can not define the value process in the usual way but directly as a stochastic integral (Definition 62); secondly, to pass from the solution of the artificial MVH problem to the original one, we have to solve equation (3.22), whose solution is known explicitly in the finite case, but it is not in this infinite-dimensional setting.

We have also studied (Section 3.6) the asymptotic behaviour of the solutions of the finite-dimensional MVH problems and shown their convergence in $L^2(P)$ sense to the optimal strategy ϑ^*, which suggests a practical method to approximate it.

A further generalization of this approach could be to a market with a continuum of stochastic processes modelling the dynamics of forward rates, i.e. to a process taking values in the space $C([0,T])$ of all continuous real functions defined on the time interval $[0,T]$. Applications to more concrete models, e.g. interest rate or "large" stochastic volatility models, will be the subject of a future work.

Chapter 4

Some results on quadratic hedging with insider trading

This chapter is based on the homonymous technical report n. 841 of the Laboratoire de Probabilités et Modèles Aléatoires of the Universities of Paris VI and VII, submitted to the review *Stochastic Processes and their Applications*. I wish to thank Marc Yor for his support and also Huyên Pham and Francesca Biagini for their interest in this work.

4.1 Introduction

In this chapter we begin the study of an hedging problem for a future stochastic cash flow X (delivered at some instant $t < T$, where T is a given finite horizon) in an arbitrage-free and incomplete financial market characterized by the presence of two kinds of investors, which have different levels of information on the future price evolution.

When the given financial market is complete, every contingent claim can be *perfectly replicated* by a *self-financing* portfolio strategy based on the underlying assets, usually modelled by an $\mathbb{R}^d$-valued semimartingale S. In this case, one can reduce to zero the risk of the claim by a suitable dynamic strategy. In the incomplete case, this is no longer possible for a general claim. Every agent then faces the problem of managing the risk they incur by buying or selling the claim.

In the mathematical finance literature, there are two main quadratic approaches to tackle this difficulty: *local risk minimization (abbr. LRM)* and *mean-variance hedging (abbr. MVH)*. Since one cannot ask simultaneously for the perfect replication of a given general claim by a portfolio strategy and the self-financing property of this strategy, we have to relax one of these two conditions. The LRM keeps then the replicability and relaxes the self-financing condition, by requiring it only on average. On the other hand, the MVH keeps the self-financing condition and relaxes the replicability, by requiring it approximately in L^2-sense.

To be a little more precise, Föllmer and Sondermann (1986) introduced the risk minimization approach, which consists in comparing strategies by means of a risk measure in terms of a conditional mean square error process. When the price process is a (local) martingale under P, it was shown that a unique risk-minimizing strategy exists and it can be computed using the Galtchouck-Kunita-Watanabe (abbr. GKW) decomposition (for a short review on this topic, see Ansel and Stricker (1993)). The case of a semimartingale price process is much more delicate and it induced Schweizer (1988) to introduce the concept of LRM. Existence of a LRM-strategy is now related to the existence of a so-called Föllmer-Schweizer decomposition, which can be viewed as a generalization of the GKW-decomposition and characterized by means of the minimal martingale measure (abbr. MMM) introduced by Föllmer and Schweizer (1991).

On the other hand, in the MVH approach, one looks for self-financing strategies which minimize the residual risk between the contingent claim and the terminal portfolio value. Again, existence and construction of an optimal strategy in the martingale case are stated by means of the GKW-decomposition of the given claim we search to hedge. In the semimartingale case, we have two kinds of characterization of the optimal strategy obtained by Gourieroux et al. (1998) (by means of a suitable change of numéraire) and by Rheinländer and Schweizer (1997), who obtained a representation of it in a feedback form. Anyway, in both papers, the variance-optimal martingale measure (abbr. VOMM), introduced by Schweizer (1996) plays a fundamental rôle.

All these papers deal with financial market models in which all agents have the same information flow, represented by a filtration which in most cases is generated either by the underlying price processes or by the driving brownian motions, as in the classical diffusion models as well as in the stochastic volatility models.

An important and natural development of this study is the introduction, in a general semimartingale model, of an *insider*. While the ordinary agent chooses his trading strategy according to the "public" information flow $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$, the insider possesses from the beginning additional information about the outcome of some random variable G and therefore has the large filtration $\mathbf{G} = (\mathcal{G}_t)_{t\in[0,T]}$ with $\mathcal{G}_t = \cap_{\epsilon>0}(\mathcal{F}_{t+\epsilon} \vee \sigma(G))$ at his disposal. For instance, the insider may know the price of a stock at time T, or the price range of a stock at time T, or the price of a stock at time T distorted by some noise and so on.

In the past few years, there has been an increasing interest in asymmetry of information, and the enlargement of filtrations techniques, developed by the French School of Probability, revealed a crucial mathematical tool to investigate this topic. The reader could look at the paper by Brémaud and Yor (1978), the Lecture Notes by Jeulin (1980) and the series of papers in the Séminaire de Calcul Stochastique (1982/83) of the University Paris VI published in 1985, containing among others the important paper by Jacod (1985).

On the other hand, the mathematical finance literature focuses mainly on the problem of portfolio optimization of an insider. We refer here to Karatzas and Pikovsky (1996), Amendinger et al. (1998), Grorud and Pontier (1998) and Imkeller et al. (2001). All these works consider the differential of utility between the two agents (as previously described) and one important conclusion is that the differential is the relative entropy of the additional r.v. G with respect to the original probability measure P. We quote also a recent paper by Biagini and Øksendal (2002), which adopts a different approach based on forward integrals with respect to the brownian motion, and a preprint by Baudoin and Nguyen-Ngoc (2002), who study a financial market where the price process may jump and there is an insider possessing some weak anticipation on the future evolution of a stock (i.e. he knows the law of some functional of the price process).

The present chapter uses the same probabilistic tools as in these articles, but deals with the hedging problem of a given contingent claim $X \in L^2(P)$ in a general semimartingale financial market admitting the phenomenon of asymmetry of information as formalized above. In particular, we would compare the hedging strategies of the ordinary agent and the insider, when they both adopt the LRM or the MVH approach. We will search to answer the following natural questions: for what kind of additional information will the two agents pursue the same optimal hedging strategies? How are the two optimal strategies and the two intrinsic risks of the claim different? The remainder of the chapter is structured as follows.

In Section 4.2 we collect the main results about initial enlargement of filtrations. In particular, we recall that if the additional r.v. G satisfies $P[G \in \cdot|\mathcal{F}_t](\omega) \sim P[G \in \cdot]$ for all $t \in [0,T)$, then there exists a version of the conditional density $(p_t^x)_{t\in[0,T)}$ of G possessing good measurability properties (Jacod (1985) and Amendinger (2000)). We quote also a result by Jacod (1985) who states that, under the above assumptions, S is also a semimartingale with respect to the enlarged filtration $\mathbf{G}$ and provides its canonical decomposition. Finally, we recall the representation of p^G and its inverse as a stochastic exponential (Amendinger et al. (1998)).

Section 4.3 deals with LRM for a claim $X \in L^2(P,\mathcal{F}_t)$ with $t < T$ given. We first review the definitions of cost process and locally risk minimizing strategy (abbr. LRM-strategy) and then its characterization in terms of the Föllmer-Schweizer decomposition and the minimal martingale measure. We then establish a relation between the MMMs of the ordinary agent and the insider and we use it to compare the LRM-strategies for a large class of r.v.s G. More precisely, we show that for such a G the two agents pursue the same optimal strategy and the cost process of the ordinary agent is just the projection on his filtration $\mathbf{F}$ of that of the insider.

In Section 4.4 we investigate the MVH approach with insider trading. After having recalled the main features of this approach, in particular the Rheinländer-Schweizer feedback representation of the optimal strategy $\vartheta^{MVH,\mathbf{H}}$ for $\mathbf{H} \in \{\mathbf{F},\mathbf{G}\}$, we compare the MVH-strategies in the martingale case, when the price process S is a (local) P-martingale under both $\mathbf{F}$ and $\mathbf{G}$, and we that their optimal strategies are equal. Then, we show that this equality still hold for the "optimal strategies" of the two agents calculated under their respective VOMMs. Unfortunately, we are not able to compare the MVH-strategies in the general case, but nonetheless we can give a feedback representation of the difference process $\xi^{MVH} = \vartheta^{MVH,\mathbf{G}} - \vartheta^{MVH,\mathbf{F}}$ in a quite general stochastic volatility model (including Hull and White, Stein and Stein and Heston models) for all r.v.s G that are measurable with respect to the filtration generated by the volatility process.

4.2 Preliminaries on initial enlargement of filtrations

Let a probability space $(\Omega, \mathcal{F}, P)$ be given and equipped with a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions of completeness and right continuity, where $T \in [0, \infty]$ is a fixed time horizon. We also assume that $\mathcal{F}_0$ is trivial.

Given an $\mathcal{F}$-measurable random variable G taking values in a Polish space $(U,\mathcal{U})$, we denote by $\mathbf{G} = (\mathcal{G}_t)_{t\in[0,T]}$ the filtration $\mathbf{F}$ initially enlarged by G and made right-continuous, i.e.

$$\mathcal{G}_t := \bigcap_{\epsilon>0} (\mathcal{F}_{t+\epsilon} \vee \sigma(G)) \quad t \in [0, T].$$

Furthermore, we set $\mathbf{F}^0 := (\mathcal{F}_t)_{t\in[0,T)}$ and $\mathbf{G}^0 := (\mathcal{G}_t)_{t\in[0,T)}$; note the difference between $[0, T]$ and $[0, T)$. For a given $t \in [0, T)$, we will frequently use also the notations $\mathbf{F}^t := (\mathcal{F}_s)_{s\in[0,t]}$ and $\mathbf{G}^t := (\mathcal{G}_s)_{s\in[0,t]}$.

Now, we make the following *fundamental technical assumption*:

$$P[G \in \cdot\,|\mathcal{F}_t](\omega) \sim P[G \in \cdot\,] \tag{4.1}$$

for all $t \in [0, T)$ and P-a.e. $\omega \in \Omega$. In other words we are assuming that the regular distributions of G given $\mathcal{F}_t$, $t \in [0, T)$, are equivalent to the law of G for P-almost all $\omega \in \Omega$. It is known that, under this assumption, also the enlarged filtration $\mathbf{G}$ satisfies the usual conditions (Proposition 3.3 in Amendinger (2000)).

We now quote a result by Amendinger (2000), which is based on a previous lemma by Jacod (1985), and which states that there exists "nice" version of the conditional density process resulting from the previous assumption. By $\mathcal{O}(\mathbf{H}^0)$ ($\mathbf{H}^0 \in \{\mathbf{F}^0, \mathbf{G}^0\}$) we will denote the optional σ-field corresponding to the filtration $\mathbf{H}^0$.

Lemma 74 *Under assumption (4.1), there exists a strictly positive $\mathcal{O}(\mathbf{F}^0) \otimes \mathcal{U}$-measurable process $(\omega, t, x) \mapsto p^x_t(\omega)$, which is right-continuous with left-limits (RCLL) in t and such that*

1. *for all $x \in U$, p^x is a $(P, \mathbf{F}^0)$-martingale, and*
2. *for all $t \in [0, T)$, the measure $p^x_t P[G \in dx]$ on $(U,\mathcal{U})$ is a version of the conditional distributions $P[G \in dx|\mathcal{F}_t]$.*

We now assume that on the stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$ a continuous, $\mathbf{F}$-adapted, $\mathbb{R}^d$-valued semimartingale $S = (S_t)_{t\in[0,T]}$ is defined, which models the discounted price evolution of d risky assets and with canonical decomposition $S = S_0 + M + A$, where $M \in \mathbb{H}^2_{0,loc}(\mathbf{F})$ and A is an $\mathbf{F}$-predictable process with locally square-integrable variation $|A|$.

For $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$, we will denote by $\mathcal{M}_2(\mathbf{H})$ (resp. $\mathcal{M}^e_2(\mathbf{H})$) the set of all $(P, \mathbf{H})$-absolutely continuous (resp. equivalent) (local) martingale measures with square-integrable Radon-Nikodym densities. More formally

$$\mathcal{M}_2(\mathbf{H}) = \big\{Q \ll P : dQ/dP \in L^2(P), S \text{ is a } (Q, \mathbf{H})\text{-local martingale}\big\}$$

and

$$\mathcal{M}_2^e(\mathbf{H}) = \{Q \in \mathcal{M}_2(\mathbf{H}) : Q \sim P\},$$

where $L^2(P) = L^2(P, \mathcal{F})$. In order to stress the dependence from the underlying probability measure, we will write sometimes $\mathcal{M}_2^e(\mathbf{H}, P)$.

We make the following standing assumption:

$$\mathcal{M}_2^e(\mathbf{H}) \neq \emptyset, \tag{4.2}$$

for $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$. By Girsanov's theorem, the existence of an element $Q \in \mathcal{M}_2^e(\mathbf{F})$ implies that the predictable process A in the canonical decomposition of S must have the form:

$$A_t = \int_0^t \lambda_s' d\langle M\rangle_s, \quad t \in [0,T],$$

for some predictable $\mathbb{R}^d$-valued process λ. We denote

$$\widehat{K}_t = \int_0^t \lambda_s' d\langle M\rangle_s \lambda_s, \quad t \in [0,T],$$

and call this the *mean-variance tradeoff process of* S *under* $\mathbf{F}$ ($\mathbf{F}$-MVT process).

The following fundamental results by Amendinger (2000), Jacod (1985) and Amendinger et al. (1998), respectively, will be very useful in the sequel of the paper.

Theorem 75 *Let Q be in $\mathcal{M}_2^e(\mathbf{F})$ and let Z denote its density process with respect to P. Moreover, let $p^G = (p^x)_{|x=G}$. Then, under assumptions (1) and (2), the following assertions hold for every $t \in [0,T]$:*

1. $\widetilde{Z} := Z/p^G$ *is a* $(P, \mathbf{G}^0)$*-martingale, and*
2. *the* $[0,t]$-martingale preserving probability measure (abbr. t-MPM) *(under initial enlargement)*

$$\widetilde{Q}_t(A) := \int_A \frac{Z_t}{p_t^G} dP \quad \text{for } A \in \mathcal{G}_t \tag{4.3}$$

has the following properties

   (a) the σ-algebra $\mathcal{F}_t$ and $\sigma(G)$ are independent under $\widetilde{Q}_t$,

   (b) $\widetilde{Q}_t = Q$ on $(\Omega, \mathcal{F}_t)$, and $\widetilde{Q}_t = P$ on $(\Omega, \sigma(G))$, i.e. for all $A \in \mathcal{F}_t$ and $B \in \mathcal{U}$,

$$\widetilde{Q}_t[A \cap \{G \in B\}] = Q[A]P[G \in B] = \widetilde{Q}_t[A]\widetilde{Q}_t[B]$$

3. *for every $p \in [1,\infty]$, $\mathbb{H}^p_{(loc)}(Q, \mathbf{F}^t) = \mathbb{H}^p_{(loc)}(\widetilde{Q}_t, \mathbf{F}^t) \subseteq \mathbb{H}^p_{(loc)}(\widetilde{Q}_t, \mathbf{G}^t)$.*

Proof. See Amendinger (2000), Theorem 3.1 and Theorem 3.2, p. 104. □

Remark 76 *Theorem 75 implies that, under assumption (4.2) for* $\mathbf{H} = \mathbf{F}$*, there exists an equivalent local martingale measure for* S *also under the enlarged filtration* $\mathbf{G}$*, whose Radon-Nikodym derivative with respect to* P *is not necessarily in* $L^2(P)$*. Assumption (4.2) is then necessary also for* $\mathbf{H} = \mathbf{G}$*.*

The next theorem (due to J. Jacod) claims that under the fundamental assumption (4.1), the price process S is also a $\mathbf{G}^0$-semimartingale and it gives its canonical decomposition under the enlarged filtration.

Theorem 77 *For* $i = 1, ..., d$*, there exists a* $\mathcal{P}(\mathbf{F}^0)$*-measurable function* $(\omega, x, t) \mapsto (\mu_t^x(\omega))^i$ *such that*

$$\langle p^x, M^i \rangle = \int (\mu^x)^i p_-^x \, d \langle M^i \rangle .$$

For every such function $(\mu^\cdot)^i$*, we consider* $(\mu^G)^i = (\mu^x)^i_{|x=G}$ *and we have*

1. $\int_0^t |(\mu_s^G)^i| d\langle M^i \rangle_s < \infty$ $P-a.s.$ *for all* $t \in [0, T)$*, and*
2. M^i *is a* $(P, \mathbf{G}^0)$*-semimartingale, and the continuous local* $(P, \mathbf{G}^0)$*-martingale in its canonical decomposition is*

$$\widetilde{M}_t^i := M_t^i - \int_0^t \left(\mu_s^G\right)^i d \left\langle M^i \right\rangle_s , \quad t \in [0, T). \tag{4.4}$$

Proof. See Théorème 2.1 of Jacod (1985). □

This theorem with the standing assumption (4.2) for $\mathbf{H} = \mathbf{G}$ implies that the finite variation process $\widetilde{A}$ in the canonical decomposition of S under $\mathbf{G}$ must satisfy

$$\widetilde{A}_t = \int_0^t \left(\lambda_s + \mu_s^G\right)' d \left\langle \widetilde{M} \right\rangle_s = \int_0^t \left(\lambda_s + \mu_s^G\right)' d \left\langle M \right\rangle_s , \quad t \in [0, T],$$

and then the corresponding $\mathbf{G}$-MVT process of S is given by

$$\widehat{K}_t^{\mathbf{G}} = \int_0^t \left(\lambda_s + \mu_s^G\right)' d \left\langle M \right\rangle_s \left(\lambda_s + \mu_s^G\right) , \quad t \in [0, T].$$

Finally, the theorem quoted below gives a stochastic exponential representation of the conditional density p^G and its inverse.

Theorem 78 1. *There exists a local* $(P, \mathbf{G}^0)$*-martingale* $\widetilde{N}$ *null at* 0*, which is* $(P, \mathbf{G}^0)$*-orthogonal to* $\widetilde{M}$ *(i.e.* $\langle \widetilde{M}^i, \widetilde{N} \rangle = 0$ *for* $i = 1, ..., d$*) and such that*

$$\frac{1}{p_t^G} = \mathcal{E} \left(- \int \left(\mu^G\right)' d\widetilde{M} + \widetilde{N} \right)_t , \quad t \in [0, T). \tag{4.5}$$

2. *Given $x \in U$, there exists a local $\mathbf{F}^0$-martingale N^x null at 0 which is orthogonal to S and such that*
$$p_t^x = \mathcal{E}\left(\int \mu^x dS + N^x\right)_t, \quad t \in [0, T]. \tag{4.6}$$

Proof. See Proposition 2.9, p. 270, of Amendinger et al. (1998). □

Remark 79 *In the sequel, without further mention, all equalities between strategies or integrands will hold a.s. $d\langle M\rangle dP$.*

4.3 The LRM approach

4.3.1 Preliminaries and terminology

We collect in this subsection the main definitions and results of the LRM approach and to do this, we will essentially follow the two survey papers by Pham (2000) and Schweizer (2001). All the objects we will introduce in this section refer to the initially non-trivial filtration $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$.

A portfolio strategy is a pair $\varphi = (V, \vartheta)$ where V is a real-valued adapted process such that $V_T \in L^2(P)$ and ϑ belongs to $\Theta = \Theta^{\mathbf{H}}$, which denotes the set of all $\mathbf{H}$-predictable, $\mathbb{R}^d$-valued, S-integrable processes ϑ such that $\int_0^T \vartheta_s dS_s \in L^2(P)$ and $\int \vartheta dS$ is a $(Q, \mathbf{H})$-martingale for all $Q \in \mathcal{M}_2^e(\mathbf{H})$, which is closed in $L^2(P)$.

We now associate to each portfolio strategy $\varphi = (V, \vartheta)$ a process, which will be very useful in the sequel in describing the main features of the LMR approach: the *cost process* $C(\varphi)$.

The cost process of a portfolio strategy $\varphi = (V, \vartheta)$ is defined by
$$C_t(\varphi) = V_t - \int_0^t \vartheta_u dS_u, \quad t \in [0, T].$$

A portfolio strategy φ is called *self-financing* if its cost process $C(\varphi)$ is constant P a.s.. It is called *mean self-financing* if $C(\varphi)$ is a martingale under P.

Fix now a square-integrable, $\mathcal{F}_T$-measurable contingent claim X. We say that a portfolio strategy $\varphi = (V, \vartheta)$ is X-*admissible* if $V_T = X$, P a.s.. Therefore, an X-admissible portfolio strategy φ is called *locally risk minimizing* (abbr. LRM-strategy) if the corresponding cost process $C(\varphi)$ belongs to $\mathbb{H}^2(P, \mathbf{H})$ and is orthogonal to S under $(P, \mathbf{H})$. There exists a LRM-strategy if and only if X admits a decomposition:
$$X = X_0 + \int_0^T \vartheta_t^X dS_t + L_T^X, \quad P \text{ a.s.}, \tag{4.7}$$

where X_0 is $\mathcal{H}_0$-measurable, $\vartheta^X \in \Theta$ and $L^X \in \mathbb{H}^2(P, \mathbf{H})$ is orthogonal to S. Such a decomposition is called *Föllmer-Schweizer decomposition* of X under $(P, \mathbf{H})$, and the portfolio strategy $\varphi^{LRM} = (V^{LRM}, \vartheta^{LRM})$ with $\vartheta^{LRM} = \vartheta^X$ and

$$V_t^{LRM} = X_0 + \int_0^t \vartheta_s^X dS_s + L_t^X, \quad P \text{ a.s.}, \quad t \in [0,T].$$

is a LRM-strategy for X.

There exists also a very useful characterization of the LRM-strategy by means of the Galtchouk-Kunita-Watanabe decomposition (abbr. GKW-decomposition) of X under a suitable equivalent martingale measure, namely the *minimal martingale measure* (abbr. MMM) introduced by Föllmer and Schweizer (1991). We recall now some basic facts about this measure and its very deep relation with the LRM approach.

We denote by $Z^{min,\mathbf{H}}$, for $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$, the *minimal martingale density under* $\mathbf{H}$, i.e. for the ordinary agent

$$Z_t^{min,\mathbf{F}} = \mathcal{E}\left(-\int \lambda dM\right)_t, \quad t \in [0,T),$$

and for the insider

$$Z_t^{min,\mathbf{G}} = \mathcal{E}\left(-\int \left(\lambda + \mu^G\right) d\widetilde{M}\right)_t, \quad t \in [0,T).$$

Since our goal is comparing the LRM-strategies, we have to assume that, given a contingent claim $X \in L^2(\mathcal{F}_t)$ for some $t < T$, there exists a LRM-strategy (to hedge X) for the ordinary agent as well as for the insider. We make then the following

Assumption 80 *$Z^{min,\mathbf{H}}$ is a uniformly integrable $\mathbf{H}^0$-martingale satisfying $R_2(P)$ for $\mathbf{H}^0 \in \{\mathbf{F}^0, \mathbf{G}^0\}$, i.e. for all $t \in [0,T)$ there exists a constant $C > 0$ such that*

$$\mathbb{E}\left[\left(\frac{Z_t^{min,\mathbf{H}}}{Z_s^{min,\mathbf{H}}}\right)^2 \middle| \mathcal{H}_s\right] \leq C, \quad s \in [0,t].$$

Since Delbaen et al (1997) we know that this assumption is equivalent to assuming the existence of a Föllmer-Schweizer decomposition (and so of a unique LRM-strategy) for every $X \in L^2(P, \mathcal{F}_t)$, for any $t \in [0,T)$, under both $\mathbf{F}$ and $\mathbf{G}$.

Moreover, under Assumption 80, we can define on $\mathcal{F}_t$, for all $t \in [0,T)$, a P-equivalent $\mathbf{H}$-martingale measure $P^{min,\mathbf{H}}$ for S, given by

$$\left.\frac{dP^{min,\mathbf{H}}}{dP}\right|_{\mathcal{H}_t} = Z_t^{min,\mathbf{H}},$$

which is called *minimal martingale measure* for S under $\mathbf{H}$ (abbr. $\mathbf{H}$-MMM).

We now quote without proof (for whom we refer to Föllmer and Schweizer (1991), Theorem 3.14, p. 403) the following fundamental result relating the MMM and the LRM-strategy:

Theorem 81 *(We drop here, for simplicity, the dependence on* $\mathbf{H}$*) Let* X *be a contingent claim in* $L^2(P, \mathcal{F}_t)$ *for some* $t \in [0,T)$*. The LRM-strategy* φ^{LRM}*, hence also the corresponding Föllmer-Schweizer decomposition (4.7), is uniquely determined. It can be computed in terms of the MMM* P^{min}*: if* $(V_s^{min,X})_{s\in[0,t]}$ *denotes a right-continuous version of the* P^{min}*-martingale* $(\mathbb{E}[X|\mathcal{H}_s])_{s\in[0,t]}$ *with GKW-decomposition*

$$V_s^{min,X} = V_0^{min,X} + \int_0^s \vartheta_u^{min,X} dS_u + L_s^{min,X}, \quad s \in [0,t],$$

then the portfolio strategy $\varphi^{min,X} = (V^{min,X}, \vartheta^{min,X})$ *is the LRM-strategy for* X *and its cost process is given by* $C(\varphi^{LRM}) = \mathbb{E}^{min}[X|\mathcal{H}_0] + L^{min,X}$*.*

4.3.2 Comparing the LRM-strategies

In this subsection, we want to compare the LRM-strategies of the two differently informed agents. We start with a simple but very useful lemma establishing a relation between the respective MMMs. We recall that if Q is any P-absolutely continuous martingale measure for S and Z its density process under $\mathbf{F}$, then $\widetilde{Q}$ and $\widetilde{Z}$ denote respectively the corresponding MPM and its density process (under $\mathbf{G}$).

Lemma 82 *The minimal martingale densities* $Z^{min,\mathbf{H}}$ *for* $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$ *satisfy the following relation:*

$$\mathcal{E}(\widetilde{N}) Z^{min,\mathbf{G}} = \widetilde{Z^{min,\mathbf{F}}}, \tag{4.8}$$

where $\widetilde{N}$ *is the local* $(P, \mathbf{G}^0)$*-martingale, null at* 0 *and* $(P, \mathbf{G}^0)$*-orthogonal to* S *appearing in Theorem 78.*

Proof. By developing the stochastic exponential, we find immediately that

$$\begin{aligned} Z^{min,\mathbf{G}} &= \mathcal{E}\left(-\int \left(\lambda + \mu^G\right) d\widetilde{M}\right) \\ &= \mathcal{E}\left(-\int \lambda dM\right) \mathcal{E}\left(-\int \mu^G d\widetilde{M}\right) \\ &= Z^{min,\mathbf{F}} \mathcal{E}\left(-\int \mu^G d\widetilde{M}\right). \end{aligned}$$

If we multiply both sides of the above equality by $\mathcal{E}(\widetilde{N})$ and apply Yor's formula on stochastic exponentials, we have

$$\mathcal{E}(\widetilde{N}) Z^{min,\mathbf{G}} = Z^{min,\mathbf{F}} \mathcal{E}\left(-\int \mu^G d\widetilde{M} + \widetilde{N} + \left[\int \mu^G d\widetilde{M}, \widetilde{N}\right]\right).$$

Since $\widetilde{M}$ is continuous and orthogonal to $\widetilde{N}$, we have

$$\left[\int \mu^G d\widetilde{M}, \widetilde{N}\right] = \left\langle \int \mu^G d\widetilde{M}, \widetilde{N} \right\rangle = 0$$

Then the representation of $1/p^G$ provided by Theorem 78 implies

$$\mathcal{E}(\widetilde{N})Z^{min,\mathbf{G}} = Z^{min,\mathbf{F}}\frac{1}{p^G} = \widetilde{Z^{min,\mathbf{F}}}$$

and the proof is now complete. □

Remark 83 *The previous lemma states in particular that if the orthogonal part $\widetilde{N}$ in the stochastic exponential representation (4.5) of the conditional density p^G vanishes, then the MMM of the insider is just the MPM corresponding to the MMM of the ordinary agent.*

We now compare the LRM-strategies of both agents when the additional r.v. G is such that $\widetilde{N} = 0$. The next proposition shows that in this case they will adopt the same behaviour and their cost processes satisfy a simple projection relation.

Proposition 84 *Assume $\widetilde{N} = 0$ and let X be a contingent claim in $L^2(P, \mathcal{F}_t)$ for some $t < T$. Then:*

1. $\vartheta_s^{LRM,\mathbf{F}} = \vartheta_s^{LRM,\mathbf{G}}$ *for all $s \in [0, t]$;*
2. $L_t^{min,\mathbf{F}} + (\mathbb{E}^{min,\mathbf{F}}[X] - \mathbb{E}^{min,\mathbf{G}}[X|\mathcal{G}_0]) = L_t^{min,\mathbf{G}}$.

In particular, $C_s(\varphi^{LRM,\mathbf{F}}) = \mathbb{E}[C_s(\varphi^{LRM,\mathbf{G}})|\mathcal{F}_s]$ for all $s \in [0, t]$.

Proof. Associate firstly to X the $(P^{min,\mathbf{G}}, \mathbf{G})$-martingale $X_s^{min,\mathbf{G}} := \mathbb{E}^{min,\mathbf{G}}[X|\mathcal{G}_s]$, $s \leq t$, and consider its GKW-decomposition under $(P^{min,\mathbf{G}}, \mathbf{G})$:

$$X_s^{min,\mathbf{G}} = \mathbb{E}^{min,\mathbf{G}}[X|\mathcal{G}_0] + \int_0^s \vartheta_u^{min,\mathbf{G}} dS_u + L_s^{min,\mathbf{G}}, \quad s \in [0, t], \tag{4.9}$$

where $\vartheta^{min,\mathbf{G}} \in L^1(S, P^{min,\mathbf{G}})$ and $L^{min,\mathbf{G}}$ is a $(P^{min,\mathbf{G}}, \mathbf{G})$-martingale, orthogonal to S. On the other hand consider the $(P^{min,\mathbf{F}}, \mathbf{F})$-martingale $X_s^{min,\mathbf{F}} := \mathbb{E}^{min,\mathbf{F}}[X|\mathcal{F}_s]$, $s \leq t$. Its GKW-decomposition under $(P^{min,\mathbf{F}}, \mathbf{F})$ is given by

$$X_s^{min,\mathbf{F}} = \mathbb{E}^{min,\mathbf{F}}[X] + \int_0^s \vartheta_u^{min,\mathbf{F}} dS_u + L_s^{min,\mathbf{F}}, \quad s \in [0, t], \tag{4.10}$$

where $\vartheta^{min,\mathbf{F}} \in L^1(S, P^{min,\mathbf{F}})$ and $L^{min,\mathbf{F}}$ is a $(P^{min,\mathbf{F}}, \mathbf{F})$-martingale, orthogonal to S. Observe now that $\vartheta^{min,\mathbf{F}} \in L^1(S, P^{min,\mathbf{G}})$ and moreover, since $P^{min,\mathbf{G}} = \widetilde{P^{min,\mathbf{F}}}$, item 3 of Theorem 75 implies that $L^{min,\mathbf{F}}$ is also a $(P^{min,\mathbf{G}}, \mathbf{G})$-martingale orthogonal to S and so is $L^{min,\mathbf{F}} + (\mathbb{E}^{min,\mathbf{F}}[X] - \mathbb{E}^{min,\mathbf{G}}[X|\mathcal{G}_0])$. Finally, since the two processes we are considering have the same terminal value X, the uniqueness property of the LRM-strategies implies the first two items of the proposition. The claimed relation between the cost processes is now quite clear. Indeed, since $L^{min,\mathbf{H}}$ is a local $(P, \mathbf{H})$-martingale for $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$ (see Ansel and

Stricker (1992) or Schweizer (1995)), the usual localization procedure allows us to assume, without loss of generality, that it is a true $(P, \mathbf{H})$-martingale and then, for all $s \in [0,t]$,

$$\begin{aligned} C_s(\varphi^{LRM,\mathbf{F}}) &= \mathbb{E}^{min,\mathbf{F}}[X] + L_s^{min,\mathbf{F}} = \\ &= \mathbb{E}\left[\mathbb{E}^{min,\mathbf{F}}[X] + L_t^{min,\mathbf{F}}|\mathcal{F}_s\right] = \\ &= \mathbb{E}\left[\mathbb{E}^{min,\mathbf{G}}[X|\mathcal{G}_0] + L_t^{min,\mathbf{G}}|\mathcal{F}_s\right] = \\ &= \mathbb{E}\left[\mathbb{E}^{min,\mathbf{G}}[X|\mathcal{G}_0] + L_s^{min,\mathbf{G}}|\mathcal{F}_s\right] = \\ &= \mathbb{E}\left[C_s(\varphi^{LRM,\mathbf{G}})|\mathcal{F}_s\right]. \end{aligned}$$

The proof is now complete. □

Remark 85 *The conclusion of Proposition 84 is not so surprising. Indeed, under the MPM corresponding to the insider MMM the additional r.v. G is independent to the claim X, which is assumed to be $\mathcal{F}_t$-measurable. Then, in this case the additional knowledge of the insider does not produce any effect on his behaviour.*

Even if it is clearly hard to check the assumption $\widetilde{N} \equiv 0$ on G in a general incomplete market, it is nonetheless not difficult to exhibit several examples of such r.v.s. Indeed, it suffices to consider the stochastic volatility model described in Subsection 4.3 with G equaling the terminal value of the first driving brownian motion W_T^1 or $G = \mathbf{1}_{\{W_T^1 \in (a,b)\}}$ with $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, or $G = \alpha W_T^1 + (1-\alpha)\varepsilon$ where the random variable ε is independent of $\mathcal{F}_T$ and normally distributed with mean 0 and variance $\sigma^2 > 0$, and α is a real number in $(0,1)$. To verify this the reader could easily adapt the computations contained in the paper by Amendinger et al. (1998) to the incomplete market setting provided by our stochastic volatility model.

4.4 The MVH approach

4.4.1 Preliminaries and terminology

Given a contingent claim $X \in L^2(P)$ and an initial investment $h \in L^2(\mathcal{H}_0)$, we are interested in the following two quadratic optimization problems:

$$\min_{\vartheta^{\mathbf{H}} \in \Theta^{\mathbf{H}}} \mathbb{E}\left[X - h - \int_0^T \vartheta_t^{\mathbf{H}} dS_t\right]^2 \tag{4.11}$$

for $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$ and where the $\mathbf{H}$-admissible strategies set $\Theta^{\mathbf{H}}$ is as in the previous section.

The financial interpretation is the usual one: two investors search to replicate (approximately, in the L^2-sense) a given future cash-flow X by trading dynamically in the underlying S.

The *ordinary investor* uses only the information contained in the filtration $\mathbf{F}$, e.g. if $\mathbf{F}$ is the natural filtration of S, he observes only the market prices of the underlying assets. On the other hand, the *informed agent* or *insider*, has an additional information which is described by the random variable G, so that the filtration, on which he bases his decisions, is given by $\mathbf{G}$.

From a mathematical viewpoint, this corresponds to project the random variable X onto the following subset of $L^2(P)$

$$\mathcal{G}(h, \Theta^{\mathbf{H}}) := \left\{ h + \int_0^T \vartheta_t^{\mathbf{H}} dS_t : \theta^{\mathbf{H}} \in \Theta^{\mathbf{H}} \right\},$$

that is named *set of investment* **H**-*opportunities.* Since $\mathcal{G}(h, \Theta^{\mathbf{H}})$ is closed in $L^2(P)$ then problem (4.11) is meaningful and it admits a unique solution that we will denote by $\vartheta^{MVH,\mathbf{H}}$, for $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$.

We are interested also in the following minimization problem:

$$J^{\mathbf{H}}(X) := \min_{h \in L^2(\mathcal{H}_0)} J^{\mathbf{H}}(h, X) \tag{4.12}$$

where

$$J^{\mathbf{H}}(h, X) := \min_{\vartheta^{\mathbf{H}} \in \Theta^{\mathbf{H}}} \mathbb{E}\left[X - h - \int_0^T \vartheta_t^{\mathbf{H}} dS_t \right]^2 \quad h \in L^2(\mathcal{H}_0),$$

is the associated risk function of the investor with information $\mathbf{H}$.

The solution h^{MVH} to this problem is named *approximation price of* X (see Schweizer (1996)).

Assume now that $P \in \mathcal{M}_2^e(\mathbf{H})$. In this case $\Theta^{\mathbf{H}} = L^2(S, P, \mathbf{H})$ (see Remark 5.3 in Pham (2000)). We recall that every contingent claim $X \in L^2(P)$ admits a unique GKW-decomposition

$$X = \mathbb{E}[X|\mathcal{H}_0] + \int_0^T \vartheta_t^{\mathbf{H},X} dS_t + L_T^{\mathbf{H},X}$$

where $\mathcal{H}_0$ is the initial σ-field of $\mathbf{H}$ and $L_T^{\mathbf{H},X}$ is the terminal value of the uniformly integrable $(P, \mathbf{H})$-martingale $(L_t^{\mathbf{H},X})_{t\in[0,T]}$, which orthogonal to S under $(P, \mathbf{H})$ and whose initial value is zero.

Proposition 86 *Assume that* $P \in \mathcal{M}_2^e(\mathbf{H})$.

1. *There exists a unique solution* $\vartheta^{MVH,\mathbf{H}}$ *to problem (4.11), for all* $h \in L^2(\mathcal{H}_0)$, *given by the process* $\vartheta^{\mathbf{H},X}$ *in the decomposition (4.4.1), and*

$$J^{\mathbf{H}}(h, X) = \mathbb{E}\left[\mathbb{E}[X|\mathcal{H}_0] - h\right]^2 + \mathbb{E}\left[L_T^{\mathbf{H},X}\right]^2 \quad h \in L^2(\mathcal{H}_0),$$

2. *the approximation price for the agent is given by* $h^{MVH} = \mathbb{E}[X|\mathcal{H}_0]$, *and*

$$J^{\mathbf{H}}(X) = \mathbb{E}\left[L_T^{\mathbf{H},X}\right]^2.$$

Proof.

1. By using GKW-decomposition of X with respect to the filtration $\mathbf{H}$, and conditioning to $\mathcal{H}_0$, which is not necessarily trivial, one obtains

$$\begin{aligned}\mathbb{E}\left[X - h - \int_0^T \vartheta_t^{\mathbf{H}} dS_t\right]^2 &= \mathbb{E}\left[\mathbb{E}[X|\mathcal{H}_0] - h + \int_0^T \left(\vartheta_t^{\mathbf{H},X} - \vartheta_t^{\mathbf{H}}\right) dS_t + L_T^{\mathbf{H},X}\right]^2 \\ &= \mathbb{E}\left[\mathbb{E}[X|\mathcal{H}_0] - h\right]^2 + \mathbb{E}\left[\int_0^T \left(\vartheta_t^{\mathbf{H},X} - \vartheta_t^{\mathbf{H}}\right) dS_t\right]^2 + \\ &\quad + \mathbb{E}\left[L_T^{\mathbf{H},X}\right]^2. \end{aligned} \tag{4.13}$$

   Then the strategy $\vartheta^{\mathbf{H},X}$ solves problem (4.11) and we also have the desired formula for the associated value function $J^{\mathbf{H}}(h, X)$, for all $h \in L^2(\mathcal{H}_0)$.

2. By relation (4.13),

$$J^{\mathbf{H}}(h, X) = \mathbb{E}\left[\mathbb{E}[X|\mathcal{H}_0] - h\right]^2 + \mathbb{E}\left[L_T^{\mathbf{H},X}\right]^2,$$

   that implies $h^{MVH} = \mathbb{E}[X|\mathcal{H}_0]$ and concludes the proof of the proposition. $\square$

If P is not an $\mathbf{H}$-martingale measure, Rheinländer and Schweizer (1997) and Gourieroux et al. (1998) (see also Pham (2000)) have nonetheless obtained two characterizations of the solution of problem (4.11), under the assumption $\mathcal{H}_0$ trivial. But it is very easy to check that all those results still hold even without this assumption. We now recall some basic facts of the first approach.

We know since Delbaen and Schachermayer (1996) that, being the price process S continuous, the *variance optimal martingale measure (abbr. VOMM)* can be defined as the unique martingale probability measure $P^{\mathbf{H},opt}$ solution to the problem

$$\min_{Q \in \mathcal{M}_2(\mathbf{H})} \mathbb{E}\left[\frac{dQ}{dP}\right]^2, \tag{4.14}$$

and that this measure is in fact equivalent to P. Moreover, the process

$$Z_t^{\mathbf{H},opt} := \mathbb{E}^{\mathbf{H},opt}\left[\left.\frac{dP^{\mathbf{H},opt}}{dP}\right| \mathcal{H}_t\right], \quad t \in [0, T]$$

can be written as

$$Z_t^{\mathbf{H},opt} = Z_0^{opt} + \int_0^t \zeta_s^{\mathbf{H},opt} dS_s, \quad t \in [0, T] \tag{4.15}$$

for some constant Z_0^{opt} (independent from the underlying filtration) and some process $\zeta^{\mathbf{H},opt} \in \Theta^{\mathbf{H}}$. The following theorem contains the characterization of the optimal mean-variance strategy for a given contingent claim $X \in L^2(P)$ in a feedback form.

Theorem 87 *Let $X \in L^2(P)$ be a contingent claim and let $h \in L^2(\mathcal{H}_0)$ be an initial investment. The GKW-decomposition of X under $(P^{\mathbf{H},opt}, \mathbf{H})$ with respect to S is*

$$X = \mathbb{E}^{\mathbf{H},opt}[X|\mathcal{H}_0] + \int_0^T \vartheta_s^{\mathbf{H},opt} dS_s + L_T^{\mathbf{H},opt} = V_T^{\mathbf{H},opt} \tag{4.16}$$

with

$$V_t^{\mathbf{H},opt} = \mathbb{E}^{\mathbf{H},opt}[X|\mathcal{H}_t] = \mathbb{E}^{\mathbf{H},opt}[X|\mathcal{H}_0] + \int_0^t \vartheta_s^{\mathbf{H},opt} dS_s + L_t^{\mathbf{H},opt}, \quad t \in [0,T].$$

Then, the mean-variance optimal strategy for X is given by

$$\vartheta_t^{MVH,\mathbf{H}} = \vartheta_t^{\mathbf{H},opt} - \frac{\zeta_t^{\mathbf{H},opt}}{Z_t^{\mathbf{H},opt}} \left(V_{t-}^{\mathbf{H},opt} - h - \int_0^t \vartheta_s^{MVH,\mathbf{H}} dS_s \right), \quad t \in [0,T]. \tag{4.17}$$

Moreover the approximation price for X is given by $h^{MVH} = \mathbb{E}^{\mathbf{H},opt}[X|\mathcal{H}_0]$.

For the proof of this result and many remarks, the reader may look at the survey article by Schweizer (2001).

4.4.2 Comparing the optimal MVH-strategies

The martingale case under both F and G

Firstly we assume that the price process S is a P-martingale with respect to both $\mathbf{F}$ and $\mathbf{G}$. Given an instant $t \in [0,T)$ and a contingent claim $X \in L^2(P, \mathcal{F}_t)$ we compare the strategies and the risk functions of the informed and the ordinary agent. This means that we are considering a MVH-problem for the ordinary agent and the insider *until time $t < T$*.

For a given $t \in [0,T)$, we will denote by $\vartheta^{MVH,\mathbf{H}}(X)$ the optimal strategy for an $\mathbf{H}$-investor to hedge the claim X. Moreover, we fix two initial investments for the agents, $c \in \mathbb{R}$ for the ordinary one and $g \in L^2(\mathcal{G}_0) = L^2(G)$ for the informed one. It is important to point out that in this case the information drift μ^G vanishes.

The next technical result states a relation between the insider optimal hedging strategies $\vartheta^{MVH,\mathbf{G}}(X)$ under P and the integrand $\widetilde{\vartheta}^{X/\widetilde{Z}_t,\mathbf{G}}$ in the GKW-decomposition of the claim $X/\widetilde{Z}_t$ under the corresponding MPM $\widetilde{P}$.

Lemma 88 *Assume that $P \in \mathcal{M}_2^e(\mathbf{G})$ and let $X \in L^2(P, \mathcal{F}_t)$ for a given $t \in [0,T)$. Then*

$$\vartheta^{MVH,\mathbf{G}}(X) = \widetilde{Z}_- \widetilde{\vartheta}^{X/\widetilde{Z}_t,\mathbf{G}}$$

and

$$J^{\mathbf{G}}(g,X) = \mathbb{E}[\mathbb{E}[X|\mathcal{G}_0] - g]^2 + \mathbb{E}\left[\int_0^t \widetilde{Z}_{s-} dL_s^{\mathbf{G},\widetilde{X}} + \int_0^t V_{s-}^{\mathbf{G}} dN_s\right]^2,$$

where $V_s^{\mathbf{G}} := \mathbb{E}[X|\mathbf{G}_s]$, $\widetilde{\vartheta}^{X/\widetilde{Z}_t,\mathbf{G}}$ is the integrand with respect to S in the GKW-decomposition of $X/\widetilde{Z}_t$ under $(\widetilde{P}, \mathbf{G})$, $L^{\mathbf{G},\widetilde{X}}$ is a $(P,\mathbf{G})$-martingale strongly orthogonal to S, and N as in Theorem 78.

Proof. We start by considering the $(P, \mathbf{G})$-martingale $V_s^{\mathbf{G}} := \mathbb{E}[X|\mathcal{G}_s]$, $s \in [0,t]$. Since $\widetilde{V}^{\mathbf{G}} := V^{\mathbf{G}}/\widetilde{Z}$ is a local $(\widetilde{P}, \mathbf{G}^t)$-martingale, we can write the following GKW-decomposition

$$\widetilde{V}_s^{\mathbf{G}} = V_0^{\mathbf{G}} + \int_0^s \widetilde{\vartheta}_u^{\mathbf{G},\widetilde{X}} dS_u + \widetilde{L}_s^{\mathbf{G},\widetilde{X}}, \quad s \in [0,t], \tag{4.18}$$

where $\widetilde{\vartheta}^{\mathbf{G},\widetilde{X}} \in L_{loc}(S, \widetilde{P}, \mathbf{G}^t)$ and $\widetilde{L}^{\mathbf{G},\widetilde{X}}$ is a $(\widetilde{P}, \mathbf{G}^t)$-martingale orthogonal to S.

Integration by parts formula gives

$$dV_s^{\mathbf{G}} = d\left(\widetilde{V}^{\mathbf{G}}\widetilde{Z}\right)_s = \widetilde{Z}_{s-}d\widetilde{V}_s^{\mathbf{G}} + \widetilde{V}_{s-}^{\mathbf{G}}d\widetilde{Z}_s + \left[\widetilde{Z}, \widetilde{V}^{\mathbf{G}}\right]_s.$$

By using the decomposition (4.18) and since, by Theorem 78, $\widetilde{Z}$ satisfies $d\widetilde{Z}_s = \widetilde{Z}_{s-}dN_s$ (in this easy case the process μ of Theorem 78 is null), where N is a local $(P, \mathbf{G}^t)$-martingale orthogonal to S, we also have

$$dV_s^{\mathbf{G}} = \widetilde{Z}_{s-}\widetilde{\vartheta}_s^{\mathbf{G},\widetilde{X}}dS_s + \widetilde{Z}_{s-}d\widetilde{L}_s^{\mathbf{G},\widetilde{X}} + \widetilde{V}_{s-}^{\mathbf{G}}\widetilde{Z}_{s-}dN_s + \widetilde{Z}_{s-}d\left[N, \widetilde{L}^{\mathbf{G},\widetilde{X}}\right]_s.$$

Now, we use Girsanov's Theorem to write

$$\widetilde{L}^{\mathbf{G},\widetilde{X}} = L^{\mathbf{G},\widetilde{X}} + A^{\mathbf{G},\widetilde{X}}$$

where $L^{\mathbf{G},\widetilde{X}} := \widetilde{L}^{\mathbf{G},\widetilde{X}} - \frac{1}{\widetilde{Z}_-}\langle \widetilde{L}^{\mathbf{G},\widetilde{X}}, \widetilde{Z}\rangle$ is a local $(P, \mathbf{G}^t)$-martingale, orthogonal to S and $A^{\mathbf{G},\widetilde{X}} = \frac{1}{\widetilde{Z}_-}\langle \widetilde{L}^{\mathbf{G},\widetilde{X}}, \widetilde{Z}\rangle$.

But since $V^{\mathbf{G}}$ is a $(P, \mathbf{G}^t)$-martingale, we must have $\widetilde{Z}_s d(A^{\mathbf{G},\widetilde{X}} + [N, \widetilde{L}^{\mathbf{G},\widetilde{X}}])_s = 0$ and so

$$dV_s^{\mathbf{G}} = \widetilde{Z}_{s-}\widetilde{\vartheta}_s^{\mathbf{G},\widetilde{X}}dS_s + \widetilde{Z}_{s-}dL_s^{\mathbf{G},\widetilde{X}} + \widetilde{V}_{s-}^{\mathbf{G}}\widetilde{Z}_{s-}dN_s.$$

This concludes the proof of the lemma. □

Finally, the next proposition gives a complete answer to the comparison problem in the martingale case.

Proposition 89 *Assume that $P \in \mathcal{M}_2^e(\mathbf{G})$.*

1. *If $X \in L^2(P, \mathcal{F}_t)$, then*

$$\vartheta_s^{MVH,\mathbf{G}} = \vartheta_s^{MVH,\mathbf{F}}, \quad s \in [0,t].$$

2. *The risk functions of both agents satisfy*

$$J^{\mathbf{F}}(X) - J^{\mathbf{G}}(X) = \mathbb{E}\left[\mathbb{E}[X] - \mathbb{E}[X|\mathcal{G}_0]\right]^2.$$

Proof.

1. To the random variable $X \in L^2(\mathcal{F}_t)$ we associate the $(P, \mathbf{F}^t)$-martingale $V_s := V_s^{\mathbf{F}} := \mathbb{E}[X|\mathcal{F}_s]$, for which the GKW-decomposition holds:

$$V_s = V_0 + \int_0^s \vartheta_u^{\mathbf{F},X} dS_u + L_s^{\mathbf{F},X} \quad s \in [0,t] \tag{4.19}$$

where $\vartheta^{\mathbf{F},X} \in \Theta^{\mathbf{F}}$ and $L^{\mathbf{F},X}$ is a $(P, \mathbf{F}^t)$-martingale, strongly orthogonal to S for $(P, \mathbf{F}^t)$. Moreover, $Y_s := V_s p_s^G$ is a $(\widetilde{P}, \mathbf{G}^t)$-local martingale and its GKW-decomposition under $(\widetilde{P}, \mathbf{G}^t)$ is given by

$$Y_s = Y_0 + \int_0^s \widetilde{\vartheta}_u^{\mathbf{G},Y} dS_u + \widetilde{L}_s^{\mathbf{G},Y} \quad s \in [0,t]. \tag{4.20}$$

By (4.6) the process p_s^G satisfies

$$p_s^G = 1 + \int_0^s p_{u-}^G dN_u^G$$

and by the integration by parts formula applied to Y_s, we obtain

$$\begin{aligned} Y_s &= V_s p_s^G = Y_0 + \int_0^s p_{u-}^G \vartheta_u^{X,\mathbf{F}} dS_u + \int_0^s p_{u-}^G dL_u^{X,\mathbf{F}} \\ &+ \int_0^s V_{u-} p_{u-}^G dN_u^G + \left[V, p^G\right]_s . \end{aligned}$$

Since Y is a $(\widetilde{P}, \mathbf{G}^t)$-local martingale, the finite variation part in the above decomposition vanishes and then

$$\begin{aligned} Y_s &= V_s p_s^G = Y_0 + \int_0^s p_{u-}^G \vartheta_u^{X,\mathbf{F}} dS_u + \int_0^s p_{u-}^G dL_u^{X,\mathbf{F}} \\ &+ \int_0^s V_{u-} p_{u-}^G dN_u^G. \end{aligned} \tag{4.21}$$

If we compare this orthogonal decomposition with (4.20), we obtain that

$$\widetilde{\vartheta}_s^{Y,\mathbf{G}} = p_{s-}^G \vartheta_s^{X,\mathbf{F}}.$$

We finally apply Lemma (88) and we have

$$\begin{aligned} \vartheta_s^{MVH,\mathbf{G}}(X) &= \widetilde{Z}_{s-} \widetilde{\vartheta}_s^{X/\widetilde{Z}_t,\mathbf{G}} \\ &= \widetilde{Z}_{s-} \widetilde{\vartheta}_s^{Y_t,\mathbf{G}} \\ &= \widetilde{Z}_{s-} p_{s-}^G \vartheta_s^{X,\mathbf{F}} \\ &= \vartheta_s^{X,\mathbf{F}}. \end{aligned}$$

2. From the GKW-decompositions of X under $\mathbf{F}$ and $\mathbf{G}$, one can deduce

$$\begin{aligned} L_t^{\mathbf{F},X} &= X - \mathbb{E}[X] - \int_0^t \vartheta_s^{\mathbf{F},X} dS_s \\ &= (\mathbb{E}[X|\mathcal{G}_0] - \mathbb{E}[X]) + X - \mathbb{E}[X] - \int_0^t \vartheta_s^{\mathbf{G},X} dS_s \\ &\quad + \int_0^t \left(\vartheta_s^{\mathbf{G},X} - \vartheta_s^{\mathbf{F},X}\right) dS_s \\ &= (\mathbb{E}[X|\mathcal{G}_0] - \mathbb{E}[X]) + \int_0^t \left(\vartheta_s^{\mathbf{G},X} - \vartheta_s^{\mathbf{F},X}\right) dS_s + L_t^{\mathbf{G},X}. \end{aligned}$$

By item 1 of this proposition, we have

$$\mathbb{E}\left[L_t^{\mathbf{F},X}\right]^2 = \mathbb{E}[\mathbb{E}[X|\mathcal{G}_0] - \mathbb{E}[X]]^2 + \mathbb{E}\left[L_t^{\mathbf{G},X}\right]^2,$$

that is

$$J^{\mathbf{F}}(X) = J^{\mathbf{G}}(X) + \mathbb{E}[\mathbb{E}[X|\mathcal{G}_0] - \mathbb{E}[X]]^2.$$

The proof is now complete. □

Remark 90 *If* both *investors are allowed to minimize only over all pairs* $(c, \vartheta) \in \mathbb{R} \times \Theta^{\mathbf{H}}$ *(*$\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$*), then the risk functions are equal, i.e.* $J^{\mathbf{F}}(X) = J^{\mathbf{G}}(X)$.

The semimartingale case

For the general case, that is S is a continuous $(P, \mathbf{F})$-semimartingale, the Rheinländer-Schweizer feedback representation (4.17) of the optimal MVH-strategies suggests to compare

- the "optimal strategies" $\vartheta^{opt,\mathbf{F}} := \vartheta^{X,P^{opt,\mathbf{F}}}$ and $\vartheta^{opt,\mathbf{G}} := \vartheta^{X,P^{opt,\mathbf{G}}}$ of the ordinary agent and the insider under their own VOMMs $P^{opt,\mathbf{F}}$ and $P^{opt,\mathbf{G}}$, and
- the ratios $\zeta^{opt,\mathbf{F}}/Z^{opt,\mathbf{F}}$ and $\zeta^{opt,\mathbf{G}}/Z^{opt,\mathbf{G}}$ in the Rheinländer-Schweizer backward representation (4.17).

We assume that both agents start with the same initial investment $c \in \mathbb{R}$. We begin by the first item and, to do this, we will use the results of the previous subsection. Before this, we need some more results on the VOMM $P^{opt,\mathbf{H}}$ ($\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$), for which our main reference remains the paper by Delbaen and Schachermayer (1997).

Let $K_0^{\mathbf{H}}$ denote the subspace of $L^\infty(P)$ spanned by the "simple" stochastic integrals of the form

$$f = \phi'\left(S_{\tau_2} - S_{\tau_1}\right)$$

where $\tau_1 \leqslant \tau_2$ are stopping-times (with respect to the filtration $\mathbf{H}$) such that the stopped process S^{τ_2} is bounded and ϕ is a bounded $\mathbb{R}^d$-valued $\mathcal{H}_{\tau_1}$-measurable function. In this paper, S is assumed to be a continuous semimartingale under both $\mathbf{F}$ and $\mathbf{G}$ and so a probability measure Q on $\mathcal{F}$ is a local $\mathbf{H}$-martingale measure for S iff Q vanishes on K_0.

Moreover, by $\widehat{K}^{\mathbf{H}}$ we denote the closure of the span of $K_0^{\mathbf{H}}$ and the constants in $L^2(P)$:

$$\widehat{K}^{\mathbf{H}} := \overline{\text{span}\left(K_0^{\mathbf{H}}, 1\right)}.$$

By Delbaen-Schachermayer (1997) (Lemma 2.1) and our standing assumption (4.2), we know that $P^{opt,\mathbf{H}}$ is the unique element of $\widehat{K}^{\mathbf{H}}$ vanishing on $\widehat{K}_0^{\mathbf{H}}$ and equaling 1 on the constant function 1. (Here we have identified any measure Q with the linear functional $\mathbb{E}_Q[\cdot]$ and linear functionals on $L^2(P)$ with elements of $L^2(P)$)

Now, since $\widehat{K}^{\mathbf{F}} \subseteq \widehat{K}^{\mathbf{G}}$, it is easy to see, by a standard Hilbert space argument, that $P^{opt,\mathbf{F}}$ is just the projection of $P^{opt,\mathbf{G}}$ into $\widehat{K}^{\mathbf{F}}$.

Indeed, denote by f this projection, i.e. $f := \pi(P^{opt,\mathbf{G}}, \widehat{K}^{\mathbf{F}})$. Then, we have $\mathbb{E}[fg] = \mathbb{E}^{opt,\mathbf{G}}[g] = 0$ for all $g \in \widehat{K}_0^{\mathbf{G}}$ and, since $1 \in \widehat{K}^{\mathbf{G}}$, $\mathbb{E}[f] = \mathbb{E}[f1] = \mathbb{E}^{opt,\mathbf{G}}[1] = 1$. By the previously mentioned Lemma 2.1 in Delbaen-Schachermayer (1997), we conclude that $f = P^{opt,\mathbf{F}}$. Furthermore this property of the VOMM does not depend on the structure of the filtration $\mathbf{G}$.

A first consequence of this remark is that, for the ordinary agent, solving the MVH-problem under either $P^{opt,\mathbf{F}}$ or $P^{opt,\mathbf{G}}$ leads to the same optimal strategy, i.e. $\vartheta^{\mathbf{F},P^{opt,\mathbf{F}}} = \vartheta^{\mathbf{F},P^{opt,\mathbf{G}}}$.

Finally, since $P^{opt,\mathbf{G}}$ is a local martingale measure for S under both $\mathbf{F}$ and $\mathbf{G}$, item 1. of Proposition 89 applies and provides the equality between $\vartheta^{opt,\mathbf{G}}$ and $\vartheta^{opt,\mathbf{F}}$. We have so proved the following:

Proposition 91 *If $X \in L^2(P, \mathcal{F}_t)$ for some $t \in [0, T)$, then for all $s \leq t$*

$$\vartheta_s^{opt,\mathbf{G}} = \vartheta_s^{opt,\mathbf{F}}. \tag{4.22}$$

Comparing now the VOMM ratios in our general framework is a quite difficult problem. We are able to give an answer by considering some particular insider's information in some particular incomplete model. In fact, in the next subsection, we will see that in a given stochastic volatility model (including Hull and White, Heston and Stein and Stein models) if the additional r.v. G is measurable with respect to the filtration generated by the volatility process, then the two VOMM ratios coincide. This result will allow us to obtain a feedback representation for the difference process between the two optimal strategies $\vartheta^{MVH,\mathbf{F}}$ and $\vartheta^{MVH,\mathbf{G}}$.

4.4.3 Stochastic volatility models

We consider the following stochastic volatility model for a discounted price process S:

$$dS_t = \sigma(t, S_t, Y_t) S_t [\lambda(t, S_t, Y_t) dt + dW_t^1] \tag{4.23}$$

where W^1 is a brownian motion and Y is assumed to satisfy the following SDE

$$dY_t = \alpha(t, S_t, Y_t) dt + \gamma(t, S_t, Y_t) dW_t^2 \tag{4.24}$$

with W^2 another brownian motion independent from the first one. The coefficients are assumed to satisfy the usual hypotheses ensuring the existence of a unique strong solution and of an equivalent local martingale measure with square integrable Radon-Nikodym density. Furthermore, we assume that the underlying filtration $\mathbf{F} = (\mathcal{F}_t)$ is that generated by the two driving brownian motions, i.e. $\mathcal{F}_t = \sigma(W_s^1, W_s^2 : s \leqslant t)$ for all $t \in [0,T]$, and that λ does not depend on the process S, that is $\lambda(t, S_t, Y_t) = \lambda(t, Y_t)$. We point out that this assumption is satisfied by the Hull and White, Heston and Stein and Stein models (e.g. see Hobson (1998b)).

We will denote by $\mathbf{F}^1 = (\mathcal{F}_t^1)$ (resp. $\mathbf{F}^2 = (\mathcal{F}_t^2)$) the filtration generated by W^1 (resp. W^2).

We assume that the additional random variable G is $\mathcal{F}_T^2$-measurable, e.g. $G = W_T^2$, $G = \mathbf{1}_{(W_T^2 \in [a,b])}$ with $a < b < \infty$ or $G = Y_T$ when Y and W^2 generate the same filtration (for example, in the Hull and White model).

In this case, the VOMM is the same for the ordinary and the informed agent. Indeed, by Biagini et al. (2000) (Theorem 1.16), we have for $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$,

$$\frac{dP^{\mathbf{H},opt}}{dP} = \frac{\mathcal{E}\left(-\int_0^\cdot \beta_t^{\mathbf{H}} dS_t\right)_T}{\mathbb{E}\left[\mathcal{E}\left(-\int_0^\cdot \beta_t^{\mathbf{H}} dS_t\right)_T\right]} \tag{4.25}$$

with $\beta_t^{\mathbf{H}} = \frac{\lambda(t,Y_t) - h_t^{\mathbf{H}}}{\sigma(t,S_t,Y_t)S_t}$. So, we focus on the process $h^{\mathbf{H}}$. Now, by assumption the process λ does not depend on S and then, again by Biagini et al. (2002) (Section 2), $h^{\mathbf{F}} = 0$.

Moreover, being G $\mathcal{F}_T^2$-measurable and since W^1 and W^2 are independent, the dynamics of S does not change if we pass from $\mathbf{F}$ to $\mathbf{G}$. Indeed, since in this case assumption (4.1) is equivalent to assume $P(G \in \cdot \,|\mathcal{F}_t^2) \sim P(G \in \cdot\,)$ for all $t \in [0,T)$, it is easy to see that the conditional density process $(p_s^G)_{s\in[0,T)}$ can be chosen $\mathbf{F}_2^0$-optional, where $\mathbf{F}_2^0 := (\mathcal{F}_t^2)_{t\in[0,T)}$. The equality

$$d\left\langle p^G, \int \sigma(u, S_u, Y_u) S_u dW_u^1 \right\rangle_t = \sigma(t, S_t, Y_t) S_t d\left\langle p^G, W^1\right\rangle_t = 0, \quad t \in [0,T),$$

implies, thanks to Theorem 77, $\mu^G \equiv 0$.

So, always by Biagini et al. (2002) (Section 2), $h^{\mathbf{G}} = 0$. This implies $\beta^{\mathbf{F}} = \beta^{\mathbf{G}}$ and then $P^{\mathbf{F},opt} = P^{\mathbf{G},opt} =: P^{opt}$.

Proposition 92 *Let G be $\mathcal{F}_T^2$-measurable and $X \in L^2(P, \mathcal{F}_t)$ with $t < T$. Then*

$$\vartheta_s^{MVH,\mathbf{G}} = \vartheta_s^{MVH,\mathbf{F}} + \xi_s^{MVH}, \quad s \in [0,t], \tag{4.26}$$

where the process ξ^{MVH} has the following backward representation:

$$\xi_s^{MVH} = \rho_s^{opt}\left(V_{s-}^{opt,\mathbf{G}} - V_{s-}^{opt,\mathbf{F}} + \int_0^s \xi_u^{MVH} dS_u\right), \quad s \in [0,t] \tag{4.27}$$

where $V_s^{opt,\mathbf{H}} := \mathbb{E}^{opt}[X|\mathcal{H}_s]$ for $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$ and $\rho_s^{opt} := \zeta_s^{opt,\mathbf{F}}/Z_s^{opt,\mathbf{F}} = \zeta_s^{opt,\mathbf{G}}/Z_s^{opt,\mathbf{G}}$, $s \in [0,t]$.

Proof. Since $P^{opt,\mathbf{F}} = P^{opt,\mathbf{G}} = P^{opt}$, it is easy to remark that by isometry $\zeta_s^{opt,\mathbf{F}} = \zeta_s^{opt,\mathbf{G}}$ and so $\zeta_s^{opt,\mathbf{F}}/Z_s^{opt,\mathbf{F}} = \zeta_s^{opt,\mathbf{G}}/Z_s^{opt,\mathbf{G}} =: \rho_s^{opt}$ for $s \leqslant t$. Indeed, since by localization we can assume that S is a true martingale under P^{opt}, it suffices to note that $Z_T^{opt,\mathbf{F}} = Z_T^{opt,\mathbf{G}}$ implies $\int_0^T \zeta_s^{opt,\mathbf{F}} dS_s = \int_0^T \zeta_s^{opt,\mathbf{G}} dS_s$ and so, by isometry, we have

$$\mathbb{E}^{opt}\left[\int_0^T \left(\zeta_s^{opt,\mathbf{F}} - \zeta_s^{opt,\mathbf{G}}\right)^2 d\left\langle S\right\rangle_s\right] = 0.$$

Then, by Proposition 89, the optimal strategies of the two agents under the VOMM are equal, i.e. $\vartheta^{\mathbf{F},opt} = \vartheta^{\mathbf{G},opt}$.

Finally, by comparing the backward representations of the two optimal hedging strategies $\vartheta^{MVH,\mathbf{F}}$ and $\vartheta^{MVH,\mathbf{G}}$, we have the claimed representation of the difference process ξ^{MVH}. □

4.5 Conclusions

This chapter represents a first attempt to analyze the sensitivity of the hedging strategies to a change of the information flow. We have studied this problem for the locally risk minimization and the mean-variance hedging separately. We have shown in particular that if both agents use the first approach and the additional information of the insider satisfies a certain property, namely the orthogonal part in the stochastic exponential representation of its conditional density process vanishes, their hedging strategies coincide and the cost processes of the ordinary investor is just the projection on his filtration $\mathbf{F}$ of the insider cost process.

On the other hand, the asymmetry of information in the MVH approach is much more delicate to investigate. Motivated by the feedback characterization of the optimal strategies yielded by Rheinländer and Schweizer (1997), we have shown that the integrands in the GKW-decomposition of a claim X under the respective VOMMs of the two agents are equal. Finally, we have obtained a feedback representation for the difference between the hedging strategies in a rather general stochastic volatility model where the additional r.v. G is measurable with respect to the filtration generated by the volatility process.

The problem of comparing the hedging strategies of the two investors in the semimartingale case and for all r.v. G satisfying assumption (4.1) remains open in the LRM as well as in the MVH approach.

Moreover, a natural development of this study would be to investigate the hedging problem in a financial market with an insider possessing either a weak anticipation on the future evolution of the stock price (Baudoin (2003) and Baudoin and Nguyen-Ngoc (2002)) or an additional dynamical information (as in Corcuera et al. (2002)).

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