Equations de reaction diffusion non-locale

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Équations de réaction-diffusion non-locale

THÈSE

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Je dédie cette thèse
à mon Armelle et à Maude.

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Résumé

Cette thèse est consacrée à l'étude des équations de réaction diffusion non-locale du type $\frac{\partial u}{\partial t} - (J \star u - u) = f(u)$. Ces équations non-linéaires apparaissent naturellement en physique et en biologie voir [28, 37, 48]. On s'intéresse plus particulièrement aux propriétés (existence, unicité, monotonie) des solutions du type front progressif. Trois classes de non-linéarités f (bistable, ignition, monostable) sont étudiées. L'existence dans les cas bistable et ignition est obtenue via une technique d'homotopie. Le cas monostable nécessite une autre approche. L'existence est obtenue via approximation des équations sur des semi-intervales infinis $(-r, +\infty)$. L'unicité et la monotonie des solutions sont quand à elles obtenues par méthode de glissement. Le comportement asymptotique ainsi que des formules pour les vitesses sont aussi établis.

Mots-clés: fronts progressifs, sur et sous-solution, principe du maximum, équations de réaction-diffusion non-locale, méthode de glissement.

Abstract

This PHD Thesis is devoted to the study of the exitence, uniqueness and qualitative behavior of travelling wave solutions of non-local reaction diffusion equations $\frac{\partial u}{\partial t} - (J \star u - u) = f(u)$. Such nonlinear equations arise in population dynamics or in neural network when considering non-local diffusion, see [28, 37, 48]. We treat three different classes of nonlinearities f (bistable, ignition, monostable), which are commonly used in the litterature. Existence for bistable and ignition nonlinearity are obtained using a homotopy argument. For monostable nonlinearity, existence is obtained through approximation problem set on semi infinite interval $(-r, +\infty)$. Uniqueness and monotonicity of the travelling wave are obtained using sliding techniques. Asympotic behavior and speed formula are also investigate.

Keywords: travelling front, super and sub-solution, maximum principle, non-local reaction-diffusion equations, sliding techniques

Introduction générale

1 Introduction et motivation de la thèse

La modélisation et l'analyse mathématique des systèmes biologiques est d'un grand intérêt pour mieux comprendre notre environnement ainsi que son évolution. De nombreuses analogies entre les réacteurs chimiques et certains systèmes biologiques ont conduit les chercheurs à introduire des modèles du type "réaction-diffusion" dans la description de ceux-ci. Notamment, au niveau d'une population, les individus interagissent et se déplacent librement, ainsi il n'est guère étonnant d'obtenir des modèles pour la dynamique d'une population similaire à ceux décrivant une réaction chimique. Ces modèles de réaction-diffusion sont essentiellement fondés sur le système d'équations suivant :

$$u_t - \gamma \Delta u = f(u) \quad \text{sur } \mathbb{R}^n \times \mathbb{R}^+, \tag{1}$$

où u est un vecteur à m-composantes (chaque composante représentant la mesure d'une espèce qui se diffuse), γ est une matrice de diffusion et Δ est l'opérateur de Laplace. La fonction vectorielle f est un terme généralement non-linéaire décrivant toutes les réactions et interactions considérées.

C'est depuis les premiers travaux de Fisher (1930)[35] sur la propagation d'un gène mutant au sein d'une population donnée que ces systèmes et leur généralisation ont donné lieu à d'intenses recherches et se sont montrés très robustes dans la description de phénomènes variés. On les retrouve, entre autres, dans la description de phénomènes liés à la dynamique des populations, l'écologie, les réseaux neuronaux, la combustion, la chimiotaxie ... et bien d'autres encore.

Ces systèmes d'équations sont caractérisés par l'existence de fronts progressifs décrivant l'évolution en temps long du phénomène considéré : ainsi la proportion u des gènes mutants dans l'équation de Fisher (m=1)

$$u_t - \gamma \Delta u = u(1-u) \quad \text{sur } \mathbb{R}^n \times \mathbb{R}^+ \tag{2}$$

se propage (après un certain temps) à vitesse constante et le long d'un profil déterminé (aussi appelé onde progressive). Ces fronts progressifs sont des solutions $u(x,t)$ de (1) qui peuvent s'écrire sous la forme $u(x,t) = \phi(x.e + ct)$ où e et c sont respectivement un vecteur de S^{n-1} et un réel en général inconnu. La fonction ϕ est une fonction scalaire qui vérifie l'équation différentielle suivante :

$$\begin{cases} \phi'' - c\phi' + f(\phi) = 0 & \text{sur } \mathbb{R} \\ \phi(x) \to 0 & \text{quand } x \to -\infty \\ \phi(x) \to 1 & \text{quand } x \to +\infty \end{cases}$$

Un front progressif est alors un couple (ϕ, c) où c représente la vitesse de l'onde et ϕ son profil.

L'équation (1) est, en fait, une approximation locale dans laquelle on suppose que les quantités étudiées à l'instant t et au point x ne se diffusent que vers ses voisins immédiats.

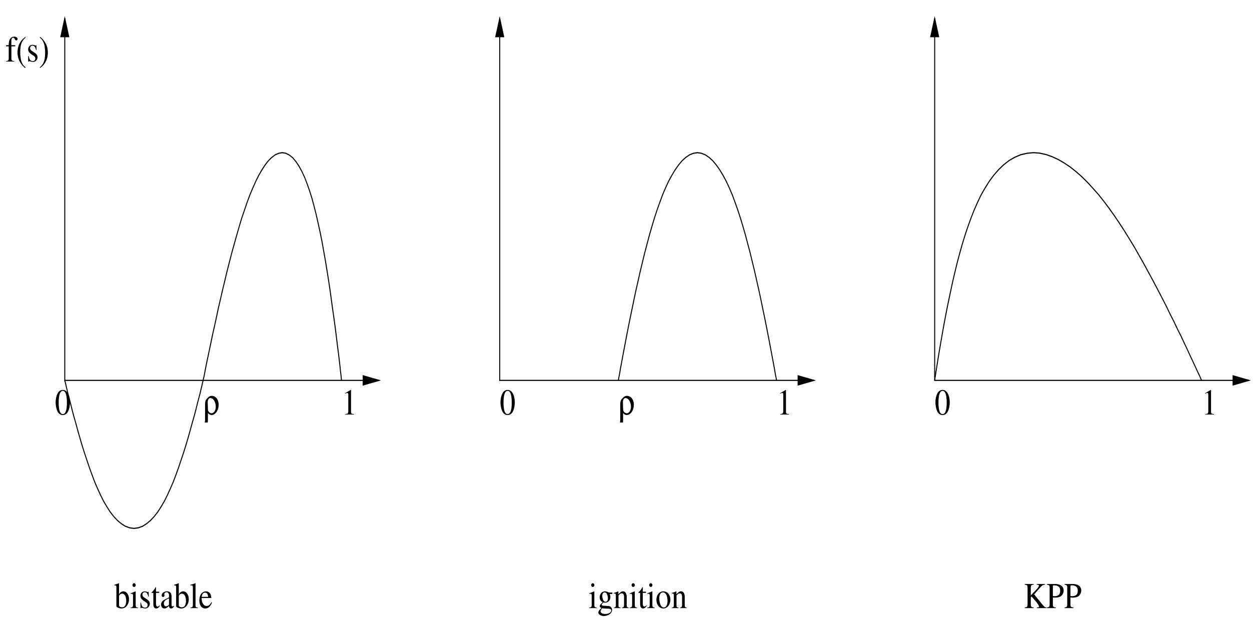
Je me suis intéressé dans cette thèse aux propriétés (existence, unicité, monotonie) des fronts progressifs dans des modèles prenant en compte les effets non locaux de la diffusion. Dans ces modèles (introduits très tôt par Kolmogorov-Petrovskii-Piskounov (1937) [46]), le terme de diffusion $\gamma\Delta u$ est remplacé par un opérateur intégral de la forme $J\star u-u$, où J peut être assimilé à une densité de probabilité et $J\star u(x) = \int_\Omega J(x-y)u(y)dy$. Les modèles étudiés rendent aussi compte de termes de diffusion mixte, c'est-à-dire composés d'une partie non-locale $J\star u-u$ et d'une partie locale modélisée par un opérateur de diffusion classique. Pour fixer les idées, j'étudie l'équation intégrodifférentielle suivante :

$$u_t - \gamma\Delta u - \beta(J\star u - u) = f(u) \quad \text{sur } \mathbb{R}^n \times \mathbb{R}^+, \tag{3}$$

où γ et β sont deux réels positifs ou nuls, J est une fonction continue, positive, paire et d'intégrale 1. Lors de cette analyse, j'ai considéré les trois classes de nonlinéarités f suivantes, communément utilisées dans la littérature :

- A1 f est bistable i.e. il existe $\rho > 0$ tel que f vérifie
  - $f|_{(0,\rho)} < 0$ et $f|_{(\rho,1)} > 0$
  - $f(0) = f(1) = 0$ et $f'(1) < 0$;
- A2 f est du type ignition i.e. il existe $\rho > 0$ tel que
  - $f|_{(0,\rho)} \equiv 0$ et $f|_{(\rho,1)} > 0$
  - $f(0) = f(1) = 0$ et $f'(1) < 0$;
- B f est monostable i.e. $f(0) = f(1) = 0$,$f|_{(0,1)} > 0$ et $f'(1) < 0$.

Le graphique ci-dessous résume les propriétés de ces trois classes de nonlinéarités :



Les nonlinéarités du type A1 interviennent plutôt dans la description de réactions chimiques, notamment pour expliquer les transitions de phases ainsi que la propagation d'interface. En effet, les états $s = 0$ et $s = 1$ représentent les états stables du système et les fronts progressifs décrivent la transition à vitesse constante d'un état stable vers un autre. La vitesse de la transition c n'est alors rien d'autre que la vitesse de l'onde progressive. Le prototype de fonction bistable est donné par $f(u) = u(a - u)(u - 1)$. Pour plus de détails, on peut se référer aux articles de Fife[30, 32] ; Fife, Mc Leod [34] ; Chen [16] et le livre de Murray [48] ainsi qu'aux références qu'ils contiennent. Les types A2 et B interviennent plutôt en combustion où généralement le terme de réaction est de la forme $g(s) = (1 - s)e^{-\frac{E}{s}}$. Ainsi, pour E assez grand, une nonlinéarité du type ignition (A2) apparaît comme une bonne approximation du terme de réaction g. Pour plus d'information voir Berestycki, Larrouturou [7] ; Kanel' [41] et Zeldovich, Frank-Kamenetskii [62]. Dans la classe des fonctions monostables, il existe une sous-classe qui joue un grand rôle dans la modélisation en dynamique de populations. Cette sous-classe, introduite par Kolmogorov-Petrovski-Piskounov [46], est caractérisée par les hypothèses supplémentaires $f'(0) > 0$ et $f'(0)s \geq f(s)$. On fera donc référence à des nonlinéarités du type KPP pour cette sous-classe. On remarquera que la nonlinéarité utilisée par Fisher est de ce type.

Lorsque $\beta = 0$, l'existence et l'unicité ou la multiplicité de fronts progressifs pour ces trois classes sont bien connues, voir [4, 7, 35, 34, 41, 46, 61, 62]. Notamment, les résultats d'existence et d'unicité diffèrent suivant les nonlinéarités considérées. En effet, dans le cas d'une nonlinéarité monostable il existe une infinité de solutions, contrastant avec l'existence d'un unique front (ϕ, c) dans les autres situations. On résume les résultats d'existence et d'unicité par le théorème suivant :

Théorème 0.1.1. *(Cas $\beta = 0$) Soit f une fonction $C^1(\mathbb{R})$,*

- *Si f est du type A1 ou A2, alors il existe un front progressif (ϕ, c) solution de* (1).*De plus, ce front est unique à translation près c'est-à-dire, si $(\tilde{\phi}, \tilde{c})$ est un autre front progressif solution de* (1) *alors $c = \tilde{c}$ et il existe un réel τ tel que $\tilde{\phi}(.) = \phi(. + \tau)$.*
- *Si f est du type B, alors il existe un réel $c^* > 0$ tel que pour toute vitesse $c \geq c^*$, il existe un front progressif (ψ, c) solution de* (1)*, et pour toute vitesse $c < c^*$, il n'existe pas de front progressif solution de* (1).

2 Intérêt de cette modélisation

L'un des intérêts de cette modélisation non-locale de la diffusion est qu'elle permet de tenir compte de bon nombre d'interactions à longue distance jusqu'alors ignorées. Notamment, lors de la dispersion d'une population soumise à une pression sélective, le terme $J(x - y)dy$ est considéré comme la probabilité d'un individu à la position y de migrer vers la position x. En posant $p(x, t)$ la densité de population au temps t et à la position x, la proportion d'individus qui migrent vers la position x par unité de temps est donnée par $(\int_{\mathbb{R}} J(x - y)p(y, t)dy)\delta t$. En supposant que la densité globale de population reste constante, les effets dus à la migration sur la répartition des individus sont donnés par :

$$\delta p(x, t) = \left(\int_{\mathbb{R}} J(x - y)p(y, t)dy - p(x, t) \right) \delta t \tag{4}$$

En combinant les effets migratoires et ceux dus à la sélection, on obtient l'équation suivante :

$$\frac{\partial p(x,t)}{\partial t} = \int_{\mathbb{R}} J(x-y)p(y,t)dy - p(x,t) + f(p). \tag{5}$$

Cette dernière est en fait un cas particulier de l'équation (3) (i.e $\gamma = 0$ et $\beta = 1$).

2.1 Quelques exemples de modèles

Cet opérateur de diffusion, par son caractère très général, se retrouve dans de nombreux modèles. Voici quelques exemples que l'on peut trouver dans la littérature.

A Transition de phase non-locale

Dans [5], Bates, Fife, Ren and Wang introduisent et étudient un modèle général de transition de phase non-locale qu'ils modélisent par l'équation suivante

$$\frac{\partial U}{\partial t} = J \star U - U + f(U) \text{ pour } (\xi, t) \quad \in \mathbb{R} \times \mathbb{R}^+,$$

où f est une nonlinéarité du type bistable (i.e. du type A1). Cette équation est une généralisation de l'équation d'Allen-Cahn classique. Notamment elle permet d'étudier des phénomènes de propagation d'interfaces dans des cristaux. Une manière d'obtenir cette équation est de considérer le "flot gradient L^2" de l'énergie libre de Helmholtz défini par $E(u) = \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)(u(x)-u(y))^2 dxdy + \int_{\mathbb{R}} F(u)dx$ avec J un noyau positif symétrique, d'intégrale 1 et $F' = f$. Cette énergie libre est une généralisation naturelle de l'énergie de Ginzburg-Landau classique. On peut donc la voir comme une équation analogue à l'équation de réaction-diffusion classique (1) mais dans un cadre non-local.

Dans les modèles d'Ising, cette équation sort naturellement comme équation limite quand la densité des particules est considérée très grande. On parle alors de propagation d'instanton. Pour plus d'information sur ce sujet voir [24, 26, 27].

B "Morphogénèse"

L'étude de la formation de cellules fait intervenir des modèles "activateur-inhibiteur". Certains de ces modèles comme

$$\begin{cases} \frac{\partial A}{\partial t} - A_{xx} = f(A) - I \\ -I_{xx} + I = A \end{cases}$$

peuvent se reformuler en une équation du type (3). En effet, on peut remarquer que la seconde équation peut s'inverser et ainsi on peut expliciter l'inhibiteur I en fonction de l'activateur A. On obtient alors $I = J \star A$ avec $J \star A = \int_{\mathbb{R}} e^{-|\xi - y|}A(y)dy$ et le système précédent se réécrit

$$\frac{\partial A}{\partial t} = A_{xx} + J \star A - A + g(A) \quad \text{pour } (\xi, t) \in \mathbb{R} \times \mathbb{R}^+ \tag{6}$$

où $g(A) = f(A) + A$. Ces modèles ont été introduits pour expliquer la formation de certaines structures complexes. Pour plus d'information on peut se référer au livre de James Murray "Mathematical Biology"[48].

C Réseaux neuronaux

On retrouve une forme non-linéaire de l'équation (3) dans l'étude des réseaux neuronaux. Dans une étude sur la propagation d'une excitation à travers une membrane, Ermentrout et McLeod [28] propose le modèle suivant : on considère un réseau neuronal unidimensionnel, uniformément réparti en espace et qui varie continûment en temps. Ce type de réseau peut être obtenu en collant bout à bout une série de cellules neuronales. Si on définit u comme le potentiel membranaire à la position x et au temps t et si on suppose que la réponse à une excitation d'une cellule neuronal est modélisée par une fonction non-linéaire S du potentiel u, la propagation du potentiel membranaire à travers le réseau est alors régie par l'équation suivante :

$$\frac{\partial U}{\partial t} = J \star S(U) - U \quad \text{pour } (\xi, t) \in \mathbb{R} \times \mathbb{R}^+ \tag{7}$$

où $S \in C^1(\mathbb{R}), S' > 0$ dans $[0,1]$, $S(0) = 0, S(1) = 1, S'(0) < 1, S'(1) < 1$ et J est un noyau positif, symétrique, régulier et d'intégrale 1. Dans le cas d'une onde stationnaire, on remarquera que $Lv := J \star v - v$ avec $v := S(u)$ et $f(v) := v - S^{-1}(v)$.

2.2 Équations limites et justification de modèles discrets.

Un autre point d'intérêt de ce cadre non-local est de pouvoir retrouver l'équation classique de réaction-diffusion comme équation limite pour une suite de noyaux J_ϵ appropriée. En effet, en dimension 1, en prenant Ψ une fonction paire à support compact de masse unité et $J_\epsilon(x) := \frac{1}{\epsilon}\Psi(\frac{x}{\epsilon})$ avec $\epsilon > 0$ petit. Un rapide calcul montre que

$$\begin{aligned} J_\epsilon \star u - u &= \frac{1}{\epsilon}\int \Psi(\frac{1}{\epsilon}y)(u(x-y) - u(x))\, dy = \int \Psi(z)(u(x-\epsilon z) - u(x))\, dz \\ &= -\epsilon \int \Psi(z)u'(x)z\, dz + \epsilon^2 \int z^2\Psi(z)u''(x)\, dz + o(\epsilon^2) = d\epsilon^2 u''(x) + o(\epsilon^2), \end{aligned}$$

où $d = \int_{\mathbb{R}} \Psi(z)z^2\, dz$ représente le deuxième moment de Ψ. Ainsi on peut voir l'équation de réaction-diffusion classique comme la première approximation de l'équation (3).

De même, d'un point de vue numérique, l'espace n'est plus considéré continu et cette approche permet de justifier certains résultats sur l'existence de solutions dans le cadre de problèmes mélangeant les descriptions continues et discrètes. En effet, on retrouve par exemple, un Laplacien discret en construisant explicitement une suite de noyaux J_ϵ adéquate. Toujours, en dimension 1, soit Ψ une fonction paire à support compact de masse unité et $\Psi_\epsilon(x) := \frac{1}{\epsilon}\Psi(\frac{x}{\epsilon})$ avec $\epsilon > 0$. On définit J_ϵ de la manière suivante $J_\epsilon(x) := \frac{1}{2}\left(\Psi_\epsilon(x+h) + \Psi_\epsilon(x-h)\right)$ où h est le pas de discrétisation. Un simple calcul montre que :

$$\begin{aligned} J_\epsilon \star u - u &= \frac{1}{2\epsilon}\int (\Psi(\frac{x-h-y}{\epsilon}) + \Psi(\frac{x+h-y}{\epsilon}))(u(y) - u(x))\, dy \\ &= \frac{1}{2\epsilon}\int \Psi(\frac{z}{\epsilon})\left((u(x-h-z) + u(x+h-z) - 2u(x)\right)\, dz \\ &= \frac{1}{2}\int \Psi(t)((u(x-h-\epsilon t) + u(x+h-\epsilon t) - 2u(x))\, dt \\ &\to \frac{1}{2}(u(x-h) + u(x+h) - 2u(x)) = h^2\Delta_h u \quad \text{quand } \epsilon \to 0. \end{aligned}$$

3 Résultats obtenus

Les résultats obtenus se divisent en deux parties : on établit tout d'abord l'existence et l'unicité de la solution dans le cas d'une nonlinéarité ignition (i.e. f du type A2) et on étend les résultats d'existence et d'unicité pour le cas d'une fonction bistable précédemment obtenus par Bates, Fife, Ren, Wang [5] et Xinfu Chen [17]. Puis en utilisant d'autres méthodes et en collaboration avec Louis Dupaigne, nous avons pu montrer l'existence d'une demi-droite de solutions pour des nonlinéarités monostables. Les techniques développées lors de ce travail conjoint, nous ont permis d'autre part de caractériser la vitesse des fronts progressifs par une formule variationnelle et d'obtenir des estimations du comportement à l'infini des solutions. Par ailleurs, via des techniques de glissement, j'ai pu établir le comportement monotone des solutions dans la majeure partie des cas et souvent montrer l'unicité de ces solutions. Cependant le comportement exact des solutions en $-\infty$ dans le cas de nonlinéarités monostables n'est toujours pas connu. Ce comportement nous permettrait de complètement caractériser les solutions du problème.

3.1 Construction de fronts progressifs

Formulons maintenant de manière précise l'équation vérifiée par un front progressif (en dimension un) : si f est une nonlinéarité telle que $f(0) = f(1) = 0$, nous cherchons des couples (u, c) tels que $u \in C^1(\mathbb{R})$ et $c \in \mathbb{R}$ soient solution de

$$\begin{cases} J \star u - u - cu' + f(u) & = & 0 & \text{dans } \mathbb{R} \\ u(x) & \to & 0 & \text{quand } x \to -\infty \\ u(x) & \to & 1 & \text{quand } x \to +\infty. \end{cases} \tag{8}$$

$J \star u$ est le produit de convolution usuel et nous supposerons que $J \in W^{1,1}(\mathbb{R})$ est une fonction positive, paire et d'intégrale 1.
Dans le modèle de Fisher, ces hypothèses reviennent à postuler que la dispersion des gènes est homogène dans tout l'habitat et ne dépend que de la distance entre deux niches de population données.

Nous imposons également une condition plus technique de décroissance à l'infini sur J : nous supposons qu'il existe $\lambda \in \mathbb{R}$ tel que

$$\int_{\mathbb{R}} J(z)e^{\lambda z}\, dz < \infty. \tag{H1}$$

Pour certains résultats cette condition de décroissance peut être affaiblie : on imposera seulement que le premier moment de J soit fini, en d'autres termes :

$$\int_{\mathbb{R}} J(z)|z|\, dz < \infty. \tag{H2}$$

Les premiers résultats sur l'équation (8) sont dûs à Bates, Fife, Ren et Wang : pour une nonlinéarité de type bistable (par exemple $f(u) = u(1-u)(u-1/2)$), ils ont montré l'existence d'un front (u, c) où u est une fonction croissante. De plus, ce front est unique à translation près : si (v, c') est une autre solution croissante, alors $c = c'$ et il existe $\tau \in \mathbb{R}$ tel que $u(x) = v(x + \tau)$.

En utilisant une méthode de continuité développée par Bates, Fife, Ren et Wang, (cf Chap1) j'ai pu étendre les résultats d'existence et d'unicité à des nonlinéarités de type ignition (i.e. f s'annule sur un intervalle $[0, \theta]$ puis reste strictement positive sur $[\theta, 1]$). Les estimations obtenues, permettent aussi d'affaiblir certaines hypothèses techniques postulées par Bates et al. et d'obtenir l'existence d'au moins une solution dans le cas d'une nonlinéarité monostable (i.e. du type B).

Comme dans le cas (connu) d'une diffusion locale, l'unicité du front n'est plus valable pour une nonlinéarité de type monostable. Ainsi la méthode de continuité ne s'applique plus. Néanmoins, en introduisant de nouveaux outils, avec Louis Dupaigne, nous avons prouvé l'existence d'une vitesse minimale $c^* > 0$ telle que l'on puisse construire des fronts pour toute vitesse $c \geq c^*$, alors qu'il n'existe aucun front (croissant) de vitesse $c < c^*$.

Pour résumer, on obtient le théorème d'existence et d'unicité suivant :

Théorème 0.3.1.
Soit f une fonction $C^1(\mathbb{R})$,

- *Si f est du type A1 ou A2, alors il existe un front progressif (ϕ, c) solution de* (8).*De plus, ce front est unique à translation près, c'est-à-dire, si $(\tilde{\phi}, \tilde{c})$ est un autre front progressif solution de* (8) *alors $c = \tilde{c}$ et il existe un réel τ tel que $\tilde{\phi}(.) = \phi(. + \tau)$.*
- *Si f est du type B alors il existe un réel $c^* > 0$, tel que pour toute vitesse $c \geq c^*$, il existe un front progressif (ψ, c) solution de* (8) *et pour toute vitesse $c < c^*$ il n'existe pas de front progressif croissant solution de* (8).

Remarque 0.3.1. *Les résultats établis pour le cas $\gamma = 0$, restent valables lorsque le coefficient de diffusion locale γ est non nul.*

3.2 Comportement à l'infini, caractérisation de la vitesse

Dans les trois cas évoqués et sous l'hypothèse d'intégrabilité exponentielle du noyau, quand un front existe, nous obtenons également son comportement asymptotique au voisinage de $\pm\infty$. On montre qu'il existe des constantes $C, \lambda, \mu, \nu > 0$ telles que

$$C^{-1}e^{-\lambda x} \leq 1 - u \leq Ce^{-\mu x} \qquad \text{quand } x \to +\infty$$
$$\text{et} \quad Ce^{\nu x} \leq u \qquad \text{quand } x \to -\infty.$$

Deplus, si f est du type bistable, ignition ou monostable telle que $f'(0) > 0$, alors il existe des constantes C, ν' telles que

$$u \leq Ce^{\nu' x} \qquad \text{quand } x \to -\infty.$$

Enfin, nous obtenons des formules min-max caractérisant la vitesse du front (dans les cas bistable et ignition), respectivement la vitesse minimale du front (dans le cas monostable) :

$$c^* = \min_{w \in X} \sup_{x \in \mathbb{R}} \left\{ \frac{J * w - w + f(w)}{w'} \right\},$$

où $X = \{w | w' > 0, w(-\infty) = 0, w(+\infty) = 1\}$. Par ailleurs, on établit aussi la dépendance monotone de la vitesse c^* en fonction de la nonlinéarité f quand f est une nonlinéarité bistable ou ignition. On obtient de cette propriété de monotonie une autre caractérisation de la vitesse minimale c^* dans le cas monostable. Plus précisément, pour f une nonlinéarité monostable, il existe une suite de fonction $(f_n)_{n \in \mathbb{N}}$ de type ignition approximant f tel que

$$\lim_{n \to +\infty} c_n = c^*,$$

où c_n est la vitesse du front associée à f_n.

3.3 Monotonie et unicité par méthode de glissement

La méthode de glissement développée par H. Berestycki et L. Nirenberg [11] permet entre autre chose d'obtenir une propriété de monotonie des solutions positives dans le cas de problèmes classiques. Cette méthode est essentiellement fondée sur une propriété de principe du maximum vérifiée par l'opérateur. Je me suis intéressé au développement de cette méthode dans un cadre abstrait recouvrant celui d'équations intégrodifférentielles. J'ai obtenu par cette méthode des résultats de monotonie pour trois grandes classes de nonlinéarités (bistable, ignition, monostable). Pour clarifier les choses, on étudie le comportement des solutions positives des deux classes de problèmes suivants :

$$Lu = -f(u) \ \text{ sur } \ \mathbb{R} \tag{9}$$
$$u(x) \to 0 \ \text{ quand } \ x \to -\infty \tag{10}$$
$$u(x) \to 1 \ \text{ quand } \ x \to +\infty \tag{11}$$

et

$$Lu = -f(u) \ \text{ sur } \ (a, +\infty) \tag{12}$$
$$u(a) < 1 \tag{13}$$
$$u(\xi) \to 1 \ \text{ quand } \ \xi \to +\infty \tag{14}$$

où L est un opérateur abstrait qui vérifie :

$$\exists k \in \mathbb{N} \qquad L : C^k(\mathbb{R}) \to C^0(\mathbb{R})$$

On suppose aussi que L vérifie les propriétés suivantes :
- Soit $U_h(.) := U(. + h)$, alors pour tout $h > 0$ on a $L[U_h](x) \leq L[U](x + h)$ pour tout $x \in \mathbb{R}$.
- Si v est une constante positive, alors $L[v](x) \leq 0$ pour tout $x \in \mathbb{R}$.

On suppose par ailleurs que L vérifie la propriété suivante qui est une version renforcée du principe du maximum fort :

Hypothèse 1. *Principe du maximum*
Soit u une fonction régulière, si u atteint un minimum (resp. maximum) en un point x de $\mathbb{R}$ alors on a l'alternative suivante :
- *Soit $L[u](x) > 0 \ \ (resp. \ \ < 0)$*
- *Soit $L[u](x) \leq 0 \ \ (resp. \ \ \geq 0)$ et u est identiquement égale à une constante.*

Par ailleurs, on suppose que les nonlinéarités f sont des fonctions régulières et choisies de telle sorte que les solutions u vérifient $0 < u < 1$. On établit trois théorèmes correspondant aux trois types de nonlinéarités f :

Théorème 0.3.2. *Soit $f \in C^1((0, 1))$ telle que pour un $\rho > 0$ on ait $f|_{(0,\rho)} \equiv 0$, $f|_{(\rho,1)} > 0$, $f(0) = f(1) = 0$ et $f'(1) < 0$. Alors toute solution positive et régulière de* (9)-(11) *est strictement croissante.*

Théorème 0.3.3. *Soit $f \in C^1((0,1))$ telle que $f|_{(0,1)} > 0$, $f(0) = f(1) = 0$ et $f'(1) < 0$. Si u est une solution positive, régulière de* (9)-(11) *et strictement croissante sur $(-\infty, -M)$ pour un M positif alors u est strictement croissante sur $\mathbb{R}$.*

Théorème 0.3.4. *Soit $f \in C^1((0,1))$ telle que $f|_{(0,1)} \geq 0$, $f(1) = 0$ et $f'(1) < 0$. Alors toute solution positive et régulière de* (12)-(14) *est strictement croissante.*

Les techniques développées s'adaptent assez aisément en dimension supérieure et permettent d'obtenir des théorèmes analogues aux théorèmes 0.3.2 et 0.3.3 pour des problèmes posés sur des cylindres infinis $\Sigma = \omega \times \mathbb{R}$ du type :

$$Lu = -f(u) \ \text{ sur } \ \Sigma \tag{15}$$
$$u(x', x_N) \to 0 \quad \text{quand} \quad x_N \to -\infty \tag{16}$$
$$u(x', x_N) \to 1 \quad \text{quand} \quad x_N \to +\infty \tag{17}$$

où ω est un ouvert borné de $\mathbb{R}^{N-1}$, et les convergences sont supposées uniformes par rapport à x'. On obtient de même un résultat similaire au théorème 0.3.4, en considérant des problèmes sur des demi-cylindres infinis du type $\Sigma = \omega \times (a, +\infty)$. Les démonstrations de ces généralisations ne sont pas présentées dans cette thèse.

3.4 Commentaires et perspectives

De cette analyse ressort en premier lieu le caractère général des solutions du type front d'onde. En effet, ces solutions persistent dans le cadre de modèles plus généraux prenant en compte des interactions plus complexes et jusqu'alors ignorées. Une conséquence surprenante est que les théorèmes obtenus sont à peu de chose près les mêmes que ceux existant pour des équations de réaction-diffusion classiques. Néanmoins, ces résultats étaient prévisibles puisque dans certains cas limites, on retrouve les modèles classiques. Ces résultats soulignent la pertinence des modèles classiques dans l'étude des phénomènes de propagation.

La principale difficulté rencontrée lors de cette étude a été le développement d'approches utilisant les outils classiques d'analyse s'adaptant aussi bien à des équations différentielles que celles considérées dans cette thèse. En effet, le caractère non-local de nos opérateurs ainsi que leur faible régularité rendent très difficile, voir impossible, une démarche classique d'approximation sur des sous-domaines bornés. Notamment, dans le cas où le terme de diffusion locale est absent (i.e. $\gamma = 0$) la plupart des approches simples et directes envisagées ont échoué.

Les perspectives de recherches sont multiples : En effet, je me suis aperçu tout au long de cette thèse de la forte connexion des opérateurs intégraux considérés avec diverses branches de l'analyse moderne et de la modélisation. Notamment, du point de vue de l'analyse fonctionnelle, les espaces $X = \{u \in L^p(\mathbb{R}^n) | \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J(x-y)(u(x) - u(y))^p dxdy < \infty\}$ considérés font l'objet d'études importantes et la compréhension des propriétés de ceux-ci reste un des objectifs actuels.

Par ailleurs, une voie naturelle de recherches est d'obtenir des résultats concernant d'autres types de front, par exemple les fronts pulsatoires. Ces fronts pulsatoires apparaissent naturellement lors de l'incorporation dans les modèles d'une inhomogénéité périodique. Cette inhomogénéité périodique peut se manifester au niveau de l'opérateur de diffusion, sur les fonctions de réaction ou de manière structurelle (i.e. le domaine sur lequel se pose l'équation est périodique). De nombreux travaux ont déjà été réalisés dans cette voie pour des opérateurs différentiels classiques, voir [6]. Une généralisation à un cadre non-local de ces résultats semble donc envisageable.

Une autre perspective intéressante est l'étude de modèles incorporant des termes de réactions non-locaux. Dans le modèle de Fisher par exemple, le terme de sélection $f(u) = \lambda u(1-u)$ est alors remplacé par $f(u) = \lambda u(1 - K \star u)$ où K est une autre densité de probabilité. On obtient alors une équation de la forme

$$u_t - \gamma \Delta u - \beta(J \star u - u) = \lambda u(1 - K \star u) \quad \text{sur } \mathbb{R}^n \times \mathbb{R}^+. \tag{18}$$

Ces équations ont été récemment étudiées du point de vue probabiliste par Méleard [37] et également par Perthame et Souganidis [50] dans le cas particulier où $J = K$.

Enfin, l'étude effective du problème parabolique (3) avec donnée initiale à support compact ainsi que la simulation numérique de la propagation de fronts sont en cours.

4 Plan de la thèse

Cette thèse comprend deux parties composées elles même de deux chapitres. La première partie (Chap1 et Chap2) concerne principalement l'existence et suivant les cas l'unicité ou la multiplicité de fronts progressifs. Le cas d'une nonlinéarité ignition est étudié dans le chapitre 1. Le chapitre 2 pour sa part est dédié au cas monostable. La seconde partie (Chap3 et Chap4) concerne les propriétés qualitatives des fronts progressifs. L'étude sur le comportement des fronts en $\pm\infty$ ainsi que la dérivation de formules variationnelles pour la vitesse minimale c^* sont obtenues dans le chapitre 3. Le comportement monotone des solutions est quand à lui traité dans le dernier Chapitre (Chap 4). Suite à des développements récents dans la manière de voir ces équations, bon nombre de preuves peuvent être simplifiées. Les différents chapitres sont mathématiquements indépendants et donneront lieu à des publications. Les chapitres 1 et 4 sont précédés de notes aux Comptes Rendus de l'Académie de Sciences résumant respectivement pour la première les résultats d'existences et d'unicités des fronts d'ondes et pour la seconde l'aspect monotone de ces fronts.

Première partie

Nonlocal reaction-diffusion equations

Note au CRAS

Chapitre 1

Travelling waves in a nonlocal reaction diffusion equation with ignition nonlinearity

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Sommaire

1.1 Introduction

We present here a follow up of previous work by Bates, Fife, Ren and Wang [5] on a bistable integrodifferential equation modeling biological and chemical phenomenas. Instead of the bistable nonlinearity, we explore the ignition and monostable cases. Typically we study the evolution equation below :

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}} J(x-y)u(y)dy - Ku(x) + f(u) \tag{1.1}$$

Where f and J are sufficiently smooth functions and J satisfies :
- $J(x) = J(-x) \geq 0$,

- $K = \int_{\mathbb{R}} J(y)dy > 0$.

The following analysis uses some properties that the linear operator A, defined by $Au = \int_{\mathbb{R}} J(x-y)u(y)dy - Ku(x)$, shares with the Laplace operator, such as a form of maximum principle. Therefore, we see that our equation (1.1) can be seen as a non-local analog of the usual reaction-diffusion equation.

$$\frac{\partial u}{\partial t} - \Delta u = f(u) \tag{1.2}$$

As in (1.2), depending on the nonlinearity f involved, the evolution equation (1.1) may model some combustion, chemical or biological phenomenas involving media with properties varying in space. The possible interest of such an equation lies in the fact that much more general types of interactions in the medium can be accounted for. Another interesting point of (1.1) lies in the fact that our equation is a gradient flow for a natural generalization of the usual Ginzburg-landau functional, i.e.(1.1) is given by $\frac{\partial u}{\partial t} = -\nabla E(u)$. Where $E(u)$ is the following functional :

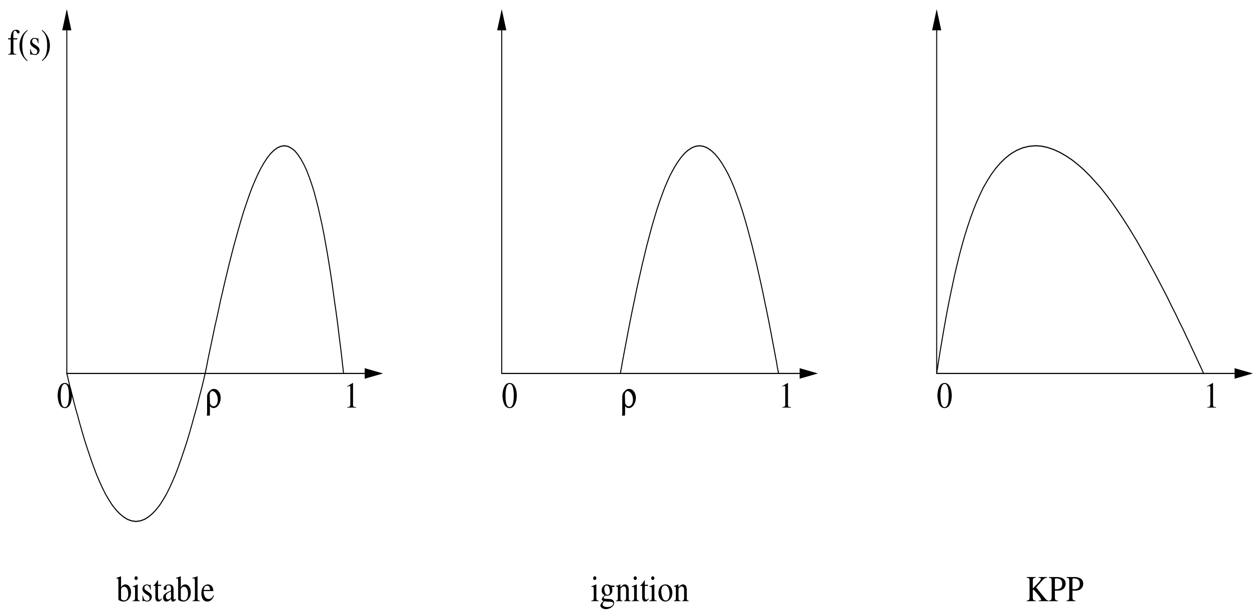
$$E(u) = \frac{1}{4}\int\int_{\mathbb{R}^2} J(x-y)(u(x)-u(y))^2 dxdy - \int_{\mathbb{R}} F(u(x))dx,$$

where $F(t) = \int_0^t f(s)ds$ for all $t \in \mathbb{R}$.

Integrodifferential equations with many of the properties of (1.1) have been also derived and studied from the point of view of certain continuum limits in dynamic Ising models [43, 44, 45, 24, 25, 26, 27]. For an excellent review, see [54].

It is well known (see [11, 12, 14, 31, 34, 46] and references therein) that equation (1.2) possesses travelling-wave solutions (i.e. solutions of the form $u(x,t) = \hat{u}(x+ct)$ for some velocity c, with u having limits 0 and 1 as x goes to $-\infty$ and $+\infty$), when

1. f is of bistable type, i.e. for some $\rho > 0$,f satisfies
   - $f|_{(0,\rho)} < 0$ and $f|_{(\rho,1)} > 0$
   - $f(0) = f(1) = 0$ and $f'(1) < 0$
2. f is of ignition type, i.e. for some $\rho > 0$, f satisfies
   - $f|_{(0,\rho)} \equiv 0$ and $f|_{(\rho,1)} > 0$
   - $f(0) = f(1) = 0$ and $f'(1) < 0$
3. f is of KPP type, i.e. $f(0) = f(1) = 0$,$f|_{(0,1)} > 0$ and $f'(1) < 0$.



Since we know the existence and uniqueness of travelling wave solutions of (1.2) when f is of bistable, KPP, or ignition type, our principal motivation is to establish similar results for our equation (1.1).

So we are lead to consider the following problem : we seek a function $\hat{u}(\xi)$ and a constant c satisfying :

$$\begin{cases} \int_{\mathbb{R}} J(x-y)u(y)dy - Ku - cu' = -f(u) \\ u(-\infty) = 0 \\ u(+\infty) = 1 \end{cases} \tag{1.3}$$

Results of existence and uniqueness for (1.3) have been obtained in [5] by Bates, Fife, Ren and Wang and later by Xinfu Chen [17]. In particular they show :

Theorem 1.1.1. *Bates, Fife, Ren, Wang*
If we also assume that :
– $J \in C^1(\mathbb{R})$, $\int_{\mathbb{R}} J = 1$, $\int_{\mathbb{R}} J(s)|s|ds < +\infty$ *and* $J' \in L^1(\mathbb{R})$
– f *bistable*, $f \in C^2(\mathbb{R})$ *and* $f'(0) < 0$.
then there exists a unique solution, up to translation, of (1.3).

In their proof, they use a homotopy argument. They construct a homotopy between equation (1.2) and their equation and use an implicit function theorem and some a-priori estimates to get the existence. The uniqueness of the speed and profile of the solution is obtained via a maximum principle and suitable sub- and super-solutions.

By using similar ideas, I was able to prove the following result :

Theorem 1.1.2.
If we also assume that :
– $J \in C^1(\mathbb{R})$, $\int_{\mathbb{R}} J = 1$, $\int_{\mathbb{R}} J(s)|s|ds < +\infty$ *and* $J' \in L^1(\mathbb{R})$
– *f is of ignition type,* $f \in C^{1,\alpha}(\mathbb{R})$.
then there exists an increasing function u and a constant c solution of (1.3)

Furthermore, we obtain an uniqueness result for c and u. More precisely

Theorem 1.1.3.
Let f be an ignition non-linearity, then, up to translation, (1.3) *has a unique increasing solution (u,c) i.e. if (u,c) and $(v, \bar{c})$ are increasing solutions of* (1.3) *then $c = \bar{c}$ and $u(x) = v(x + \tau)$ for some fixed $\tau \in \mathbb{R}$.*

I was also able to establish existence of a speed c^* and a function u solution of (1.3) in the KPP case only by replacing the assumption

$$\int_{\mathbb{R}} J(s)|s|ds < +\infty \tag{H}$$

by the following stronger one :

$$\forall \lambda > 0 \int_{\mathbb{R}} J(s)e^{\lambda s}ds < +\infty \tag{H^*}$$

This assumption imposes very restricted behavior for J at infinity. This means that J must decay faster than any exponential function. (H^*) is satisfied, for example by any kernel J with compact support or by the standard Gaussian distribution.

The corresponding existence result can be state as :

Theorem 1.1.4.
If we assume that

- $J \in C^1(\mathbb{R})$, $\int_{\mathbb{R}} J = 1$, $J' \in L^1(\mathbb{R})$ *and J satisfies* (H^*),
- *f is a KPP function* ,$f \in C^{1,\alpha}(\mathbb{R})$,

then there exists an increasing function u and a constant c^* solution of (1.3). *Moreover if $c < c^*$,* (1.3) *has no increasing solution.*

The proof of this theorem uses a standard limiting procedure and is strongly related to the asymptotic behavior at infinity of the solution of (1.3). In fact we will see in section 4 that under assumption (H^*), the solution of Theorem1.1.2 has exponential behavior at infinity i.e.

$$\begin{aligned} Ae^{\delta_1 x} \leq u(x) \leq Be^{\lambda_1 x} \quad for \;\; x \to -\infty \\ and \\ Ce^{-\delta_0 x} \leq 1 - u(x) \leq De^{-\lambda_0 x} \quad for \;\; x \to +\infty \end{aligned}$$

for suitable $\lambda_0, \lambda_1, \delta_0, \delta_1 > 0$.
Existence results for $c > c^*$ as expected in the KPP theory and some the characterization of c^* are still open and give rise to intensive research.
Remark that we can rewrite problem (1.3) as the convolution equation below :

$$J \star u - u - cu' = -f(u) \tag{1.4}$$

$$u(-\infty) = 0 \tag{1.5}$$

$$u(+\infty) = 1 \tag{1.6}$$

In this article we will use the equations (1.4), (1.5) (1.6) instead of (1.3). The paper is organized as follows : Section 2 is devoted to the proof of Theorem 1.1.2, we then prove Theorem 1.1.3 in Section 3. Some continuity properties of the speed are derived in Section 4. Section 5 is devoted to the asymptotic behavior of solution given by Theorem 1.1.2. Finally, we prove Theorem 1.1.4 in the last section.

1.2 Existence of solutions in the ignition case

As I previously mentioned, Theorem 1.1.2 will be proved using a continuation method. Let me describe the main ideas of this method. We can break it down into three steps :

– First, we embed our equation (1.4) in a family of equations continuously parametrized by $\theta \in [0,1]$, in such way that when $\theta = 0$, the equation possesses a unique travelling wave up to translation, and when $\theta = 1$ the equation is (1.4).

– Then, using a continuation argument given by the implicit function theorem, we pass in increments from 0 to 1, obtaining a sequence of functions for all values in the process.

– Finally, we extract a converging sequence when θ goes to 1.

The family, we will use, is the following :

$$\begin{cases} \theta(J \star u - u) + (1-\theta)u'' - c_\theta u' = -f(u) \\ u(-\infty) = 0 \\ u(+\infty) = 1 \end{cases} \qquad (1_\theta)$$

When $\theta = 0$, it is well known that (1_θ) has a unique solution (u, c_0), up to translation. Furthermore $u' > 0$ on $\mathbb{R}$, see for example [11, 12].

Remark 1.2.1. *: Theorem 1.1.2 remains valid for our family of equations.*

So the first step of our method is now complete. Let us proceed to the second step.
The existence for all θ will be proven through a series of lemmas. First of all, we extend f to the whole line $\mathbb{R}$ by letting

$$\tilde{f}(x) = \begin{cases} 0 & if \;\; x \in (-\infty, 0) \\ f(x) & if \;\; x \in [0,1] \\ f'(1)(x-1) & if \;\; x \in (1, +\infty) \end{cases}$$

and study the new family of equations :

$$\begin{cases} \theta(J \star u - u) + (1-\theta)u'' - c_\theta u' = -\tilde{f}(u) \\ u(-\infty) = 0 \\ u(+\infty) = 1 \end{cases} \qquad (\tilde{1}_\theta)$$

In the first lemma we show that a solution $(u_{\tilde{\theta}}, c_{\tilde{\theta}})$ of $(\tilde{1}_\theta)$ is also a solution of (1_θ) and vice versa. We show that $\tilde{u}_\theta$ takes its value between 0 and 1 for all $\theta \in [0,1]$.

Lemma 1.2.1.
Let $\theta \in [0,1]$ and let (u,c) be a non constant solution of $\tilde{1}_\theta$ then $0 < u(\xi) < 1 \;\; \forall \xi \in \mathbb{R}$.

proof :

If $\theta = 0$, then the equation becomes a second order elliptic equation. Therefore, we may apply the maximum principle. A simple computation then shows that $u \leq 1$. Indeed, if not, u

reaches its maximum at some point ξ_0. At this point $\tilde{f}(u(\xi_0)) < 0$, so u satisfies the following inequality :

$$u'' - cu' \geq 0 \quad in \quad V,$$

where V is some closed neighborhood of ξ_0. The maximum principle implies that u is identically constant in V, so by iteration of the process we get that $u \equiv cste = u(\xi_0)$ on $\mathbb{R}$, yielding a contradiction.
We very easily get $u \geq 0$, by observing that from our assumption on f we have $f(u) \geq 0$, thus u satisfies :

$$u'' - cu' \leq 0 \quad on \quad \mathbb{R}$$

Then, the maximum principle implies that $u \geq 0$.
Now, if $\theta > 0$,
suppose that u reaches its maximum at ξ_0, and that $u(\xi_0) \geq 1$, so at this point we have :

$$u''(\xi_0) \leq 0 \tag{1.7}$$

$$u(y) - u(\xi_0) \leq 0 \Rightarrow \int_{\mathbb{R}} J(y-\xi_0)(u(y)-u(\xi_0))dy \leq 0 \tag{1.8}$$

$$\tilde{f}(u(\xi_0)) \leq 0 \tag{1.9}$$

and u satisfies :

$$\theta(J \star u - u)(\xi_0) + (1-\theta)u''(\xi_0) = -\tilde{f}(u(\xi_0)) \geq 0 \tag{1.10}$$

We thus have $(J \star u - u)(\xi_0) = 0$ and

$$\begin{aligned} 0 = (J \star u - u)(\xi_0) &= \int_{\mathbb{R}} J(y-\xi_0)(u(y)-u(\xi_0))dy \\ &\Rightarrow u(y) = u(\xi_0) \quad \forall \ y \in \quad \xi_0 + suppJ \end{aligned}$$

By iteration of the process, we obtain $u(y) = u(\xi_0) \quad \forall \ y \in \mathbb{R}$. This contradicts $u \not\equiv cst$.
We prove that $u > 0$ in the same way.

$\square$

Now assume that (u_0, c_0) is a solution of 1_{θ_0} for some $\theta_0 \in [0,1)$ and that $u_0' > 0$ on $\mathbb{R}$. We shall use the Implicit Function Theorem to obtain a solution for $\theta > \theta_0$. We take perturbations in the space :

$$X_0 = \{uniformly \ continuous \ functions \ on \ \mathbb{R} \ which \ vanish \ at \ \pm\infty\}$$

We define $L = L(u_0, c_0, \theta_0)$ as the following linear operator :

$$Lv = \theta_0(J \star v - v) + (1-\theta_0)v'' - c_0 v' + f'(u_0)v \tag{1.11}$$

where

$$dom \ L = X_2 = \{v \in X_0 / v'' \in X_0\}$$

Lemma 1.2.2.
L has 0 as simple eigenvalue.

Proof :

If $\theta = 0$, the result is already known, see [13] and references therein
If $\theta > 0$, then we know that 0 is an eigenvalue of L since u_0' solves $Lu_0' = 0$. We only have to prove the simplicity of the eigenvalue 0. We show this by contradiction . Suppose that ϕ is another eigenfunction with eigenvalue 0, $\phi \not\equiv Au'$, and assume also that ϕ is positive at some point. We set

$$\phi_\beta = u + \beta\phi \quad with \quad \beta \in \mathbb{R} \quad and$$
$$\bar{\beta} = sup\{\beta < 0 \mid \exists \xi \in \mathbb{R}, \phi_\beta(\xi) < 0\}$$

$\bar{\beta}$ is well defined since there exists $\xi \in \mathbb{R}$ such that $\phi(\xi) > 0$. For $\beta < \bar{\beta}$ let ξ_β be a point of negative minimum of ϕ_β. At this point we have the following relations :

$$J \star \phi_\beta(\xi_\beta) - \phi_\beta(\xi_\beta) \geq 0, \;\; \phi_\beta''(\xi_\beta) \geq 0 \;\; and \;\; \phi_\beta'(\xi_\beta) = 0. \tag{1.12}$$

In fact $J \star \phi_\beta(\xi_\beta) - \phi_\beta(\xi_\beta) > 0$, since otherwise $\phi_\beta \equiv cte$. Thus

$$f'(u_0(\xi_\beta))\phi_\beta(\xi_\beta) = \phi_\beta(\xi_\beta) - J \star \phi_\beta(\xi_\beta) - (1 - \theta_0)\phi_\beta''(\xi_\beta) < 0.$$

This implies $f'(u_0(\xi_\beta)) > 0$, so ξ_β lies in a compact subset $[a, b]$ of $\mathbb{R}$.

Choose now $(\beta_n)_{n\in\mathbb{N}}$, a sequence which converges to $\bar{\beta}$. Let $(\xi_{\beta_n})_{n\in\mathbb{N}}$ be the corresponding sequence of negative minimum. Since $\forall n \xi_{\beta_n} \in [a, b]$, $(\xi_{\beta_n})_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. We can therefore extract a converging sub-sequence $(\xi_{\beta_{n_k}})_{k\in\mathbb{N}}$ such that $\xi_{\beta_{n_k}} \to \bar{\xi}$.
We obtain at $\bar{\xi}$:

$$f'(u(\bar{\xi})\phi_{\bar{\beta}}(\bar{\xi}) = \theta_0(\phi_{\bar{\beta}}(\bar{\xi}) - J \star \phi_{\bar{\beta}}(\bar{\xi})) - (1 - \theta_0)\phi_{\bar{\beta}}''(\bar{\xi}) \leq 0 \tag{1.13}$$
$$0 = \phi_{\bar{\beta}}(\bar{\xi}) \leq \phi_{\bar{\beta}}(\xi) \;\; \forall \xi \in \mathbb{R} \tag{1.14}$$

So

$$0 = \theta_0(\phi_{\bar{\beta}}(\bar{\xi}) - J \star \phi_{\bar{\beta}}(\bar{\xi})) - (1 - \theta_0)\phi_{\bar{\beta}}''(\bar{\xi}) \leq 0,$$

which implies that $0 = \phi_{\bar{\beta}}(\bar{\xi}) - J \star \phi_{\bar{\beta}}(\bar{\xi})$ and as in the previous proof, we obtain $\phi_{\bar{\beta}} \equiv cste = 0$ which provides the desired contradiction .

□

The formal adjoint of L is given by :

$$L^* v = \theta_0(J \star v - v) + (1 - \theta_0)v'' - c_0 v' + f'(u_0)v$$

This operator satisfies the same property of simplicity of its eigenvalue 0. Moreover, 0 is isolated since the same holds true for the operator A :

$$Av = v'' - cv' + f'(u)v$$

(see [2, 8]) and we can show that the added term $\theta_0(J\star v - v)$ leaves the essential spectrum unchanged. We now have conditions for the existence of a solution of (1.11) via the Fredholm Alternative : There exists a solution of $Lu = f$ iff $\int f\phi^* = 0$ where ϕ^* is the eigenfunction associated to the eigenvalue 0 of L^*.

We can now state the continuation result :

Lemma 1.2.3.
Let (u_0, c_0) be a solution of (1_{θ_0}) such that $u_0' > 0$. Then there exists $\eta > 0$ such that for all $\theta \in [\theta_0, \theta_0 + \eta)$, the problem (1_θ) has a solution (u,c).

Proof :

We will use the Implicit Function Theorem. Without loss of generality we may assume that $u_0(0) = \rho$. For $w = (v, c) \in X_2 \times \mathbb{R}$ and $\theta \in \mathbb{R}$, we define G $\in C^1(X_2 \times \mathbb{R} \times \mathbb{R}, X_2 \times \mathbb{R})$ by :

$$G(w, \theta) = [\theta(J\star(u_0+v)-(u_0+v))+(1-\theta)(u_0+v)''-(c_0+c)(u_0+v)'+f((u_0+v)), (u_0+v)(0)].$$

At $(0, \theta_0)$ we have $G(0, \theta_0) = (0, u_0)$ and

$$DG := \frac{\partial G}{\partial w}(0, \theta_0) = \begin{pmatrix} L & u_0' \\ \delta & 0 \end{pmatrix},$$

where $\delta v = v(0)$. We will show that DG is invertible. To this end, let $(h, b) \in X_0 \times \mathbb{R}$: we want to show the existence of a unique $w = (v, c) \in X_2 \times \mathbb{R}$ solving $DG(w) = (h, b)$.

$$DG(w) = (h, b) \Longleftrightarrow \begin{cases} Lv - cu_0' = h \\ v(0) = b \end{cases} \tag{$\sharp$}$$

If we let $c = \frac{\int \phi^\star h}{\int \phi^* u_0'}$, we get, via the Fredholm alternative, a solution v of $Lv = h + cu_0'$. In fact, *any* solution of $L\tilde{v} = h + cu_0'$ can be written as $\tilde{v} = v + \alpha u_0', \alpha \in \mathbb{R}$. But since we must also have $v(0) = b$, $w = (v, c)$ is the only solution of ($\sharp$). This shows that DG is invertible. We then apply the Implicit Function Theorem to prove the lemma.

□

Remark 1.2.2. *: The solution u_θ, obtained by the Implicit Function Theorem also satisfies the boundary conditions at infinity* (1.5) *and* (1.6).

In order to prove the previous continuation lemma, we needed the condition $u_0' > 0$. Thus, if we want to apply this lemma we have to show that for all $\theta \in [\theta_0, \theta_0 + \eta]$, any smooth solution u_θ of (1_θ) previously construted satisfies $u_\theta' > 0$.

Lemma 1.2.4.
Let $\theta \in [\theta_0, \theta_0 + \eta)$ and let (u, c) be a smooth solution of (1_θ) given by Lemma 1.2.3. Then $u'(\xi) > 0 \ \forall \xi \in \mathbb{R}$.

Proof :

We first prove that $u'(\xi) \geq 0 \ \forall \xi \in \mathbb{R}$.
We use a contradiction argument. Assume there exist $\theta \in [\theta_0, \theta_0 + \eta)$ such that there exists $\xi \in \mathbb{R}$ with $u_\theta'(\xi) < 0$. Let us define

$$\tilde{\theta} = \inf\{\theta > \theta_0 \mid \exists \xi \in \mathbb{R}, u'_\theta(\xi) < 0\}.$$

$\theta_0 + \eta > \tilde{\theta} \geq \theta_0$ is well defined. Since $\tilde{\theta} < \theta_0 + \eta$, $u_{\tilde{\theta}}$ exists. From the definition of $\tilde{\theta}$, there exists a decreasing sequence $\theta_n \searrow \tilde{\theta}$ and a non positive minimum ξ_{θ_n} of u'_{θ_n}.
At this minimum u'_{θ_n} satisfies :

$$f'(u_{\theta_n}(\xi_{\theta_n}))u'_{\theta_n}(\xi_{\theta_n}) = \theta_n(u'_{\theta_n}(\xi_{\theta_n}) - J \star u'_{\theta_n}(\xi_{\theta_n})) - (1-\theta_n)u'''_{\theta_n}(\xi_{\theta_n}) < 0. \tag{1.15}$$

So $f'(u_{\theta_n}(\xi_{\theta_n})) > 0$ and therefore we have $\rho \leq u_{\theta_n}(\xi_{\theta_n}) \leq 1-\gamma$. By observing that from Lemma 1.2.3 $u_{\theta_n} \to u_{\tilde{\theta}}$ uniformaly, the sequence $(\xi_{\theta_n})_{n\in\mathbb{N}}$ must stay in a compact subset $[a,b]$ of $\mathbb{R}$. Extract now a subsequence which converges to $\tilde{\xi}$. By letting n goes to ∞ in (1.15) along this subsequence we get

$$0 = \tilde{\theta}(u'_{\tilde{\theta}}(\tilde{\xi}) - J \star u'_{\tilde{\theta}}(\tilde{\xi})) - (1-\tilde{\theta})u'''_{\tilde{\theta}}(\tilde{\xi}) \leq 0.$$

As in the previous lemma, it follows that $(u'_{\tilde{\theta}}(\tilde{\xi}) - J \star u'_{\tilde{\theta}}(\tilde{\xi})) = 0$. This implies $u'_{\tilde{\theta}} \equiv cte = 0$ which is a contradiction.
Thus $u'_\theta \geq 0$. The previous inequality also shows $u'_\theta > 0$.

□

Next we prove some apriori estimates on the solution u_θ of (1_θ) for all $\theta \in [0,1)$ which will be useful later. We have

Lemma 1.2.5.
If for some $0 < \bar{\theta} < 1$ and all $\theta \in [0, \bar{\theta})$, there exists a solution (u_θ, c_θ) of (1_θ) then $\{u_\theta \mid \theta \in [0,\bar{\theta})\}$ is bounded in $C^3(\mathbb{R})$.

Proof :

By Lemma 1.2.1, we already know that u is bounded. Fix $\theta \in [0,\bar{\theta})$ and let (u_θ, c_θ) be a solution of (1_θ) obtained using Lemma 1.2.3.
Lemma 1.2.4 shows that u'_θ achieves its maximum at some point ξ of $\mathbb{R}$.
At ξ we get :

$$u''_\theta(\xi) = 0, \tag{1.16}$$

$$-c_\theta u'_\theta(\xi) = -f(u_\theta(\xi)) - \theta(J \star u_\theta(\xi) - u_\theta(\xi)), \tag{1.17}$$

$$\Rightarrow \| c_\theta u'_\theta \|_\infty = | -f(u_\theta(\xi)) - \theta(J \star u_\theta(\xi) - u_\theta(\xi)) | \leq K = (2 + \sup_{x\in[0,1]}(f(x))), \tag{1.18}$$

$$\| c_\theta u'_\theta \|_\infty \leq K = (2 + \sup_{x\in[0,1]}(f(x))). \tag{1.19}$$

Remark 1.2.3. *: The bound K does not depend on θ.*

Using (1_θ) we get a bound for u''_θ. Indeed,

$$(1-\theta)u''_\theta = c_\theta u'_\theta - \theta(J \star u_\theta - u_\theta) - f(u_\theta), \tag{1.20}$$

$$| (1-\theta)u''_\theta | \leq | c_\theta u'_\theta | + | \theta(J \star u_\theta - u_\theta) | + | f(u_\theta) |, \tag{1.21}$$

$$\Rightarrow \| u''_\theta \|_\infty \leq \frac{2K}{1-\bar{\theta}}. \tag{1.22}$$

The C^3 bound follows by differentiating (1_θ) and some a priori bounds on the speed c_θ that we provide in the next lemma.

□

Remark 1.2.4. *: We can also get a bound for $c_\theta u''_\theta$ in terms of u'_θ.*
In fact, we have

$$c_\theta \parallel u''_\theta \parallel_\infty \leq (2+\tilde{K}) \parallel u'_\theta \parallel_\infty .$$

Remarks 1.2.1. *: Since f is $C^{1,\alpha}$, in order to get $C^{2,\alpha}$ bounds, it is sufficient to have a positive bound from below for the speed c_θ.*

We need now some estimates on the speed c_θ, which will ensure that we can get a solution as $\theta \to \bar{\theta}$.

Lemma 1.2.6.
With the same assumptions as those of the previous lemma, the set $A_{\bar{\theta}} = \{c_\theta \mid \theta \in [0,\bar{\theta})\}$ is bounded and there exists a constant $e > 0$ so that $c_\theta \geq e \ \forall \ \theta \in [0,\bar{\theta})$.

Proof :

We will first show that the set $A_{\bar{\theta}}$ is bounded.We begin with a computation :

Claim 1.2.1. $(J \star u_\theta - u_\theta) \in L^1(\mathbb{R})$
$\|(J \star u_\theta - u_\theta)\|_{L^1} \leq \int_{\mathbb{R}^2}(J(x-y)|u_\theta(y) - u_\theta(x)|dydx \leq \int_{\mathbb{R}} J(z)|z|dz$
and $\int_{\mathbb{R}}(J \star u_\theta - u_\theta) = 0$

Proof :

Let us compute $\int_{\mathbb{R}} |(J \star u_\theta - u_\theta)|$.

$$\int_{\mathbb{R}} |(J \star u_\theta - u_\theta)| \leq \int_{\mathbb{R}^2} J(x-y)|u_\theta(y) - u_\theta(x)|dydx. \tag{1.23}$$

Since u_θ is smooth and increasing we have :

$$|u_\theta(y) - u_\theta(x)| = |x-y| \int_0^1 u'_\theta(y + s(x-y))ds.$$

Plug this equality in (1.23) to obtain :

$$\int_{\mathbb{R}^2} J(x-y)|u_\theta(y) - u_\theta(x)|dydx = \int_{\mathbb{R}^2} J(x-y)|x-y| \int_0^1 u'_\theta(x + s(y-x))dsdydx. \tag{1.24}$$

Make the change of variables $z = x - y$ so that equation (1.24) becomes :

$$\int_{\mathbb{R}^2} J(z)|z| \int_0^1 u'_\theta(x - sz)dsdzdx. \tag{1.25}$$

Because all the terms are positive, we may apply Tonnelli's theorem and permute the order of integration. We obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} J(z)|z| \int_0^1 u_\theta'(x-sz)dsdzdx &= \int_0^1 \int_{\mathbb{R}^2} J(z)|z|u_\theta'(x-sz)dxdzds &(1.26)\\
&= \int_0^1 \int_{\mathbb{R}} J(z)|z|[u_\theta(+\infty)-u_\theta(-\infty)]dzds &(1.27)\\
&= \int_{\mathbb{R}} J(z)|z|dz \leq \infty &(1.28)
\end{aligned}$$

Hence $J\star u_\theta - u_\theta$ is integrable and we have a bound on its L^1 norm.
Let us now compute

$$\int_{\mathbb{R}} J\star u_\theta - u_\theta dx.$$

We have :

$$\int_{\mathbb{R}^2} J(x-y)(u_\theta(y)-u_\theta(x))dydx.$$

Let $z=x-y$ so that :

$$\int_{\mathbb{R}^2} J(z)(u_\theta(x-z)-u_\theta(x))dzdx = \int_{\mathbb{R}^2} J(z)(u_\theta(y)-u_\theta(y+z))dydz.$$

Make the change of variable $z\to -z$ in the left integral to obtain :

$$\int_{\mathbb{R}^2} J(z)(u_\theta(x+z)-u_\theta(x))dzdx = \int_{\mathbb{R}^2} J(z)(u_\theta(y)-u_\theta(y+z))dydz.$$

Then, apply Fubini's theorem to the last integral and get

$$\int_{\mathbb{R}^2} J(z)(u_\theta(x+z)-u_\theta(x))dzdx = \int_{\mathbb{R}^2} J(z)(u_\theta(y)-u_\theta(y+z))dzdy,$$

which shows that

$$2\int_{\mathbb{R}^2} J(z)(u_\theta(x+z)-u_\theta(x))dzdx = 0.$$

□

Integrate now equation (1_θ) over $\mathbb{R}$, to get a bound from below for the speed c_θ. Indeed we have :

$$\begin{aligned}
\theta\int_{\mathbb{R}}(J\star u_\theta - u_\theta) + (1-\theta)\int_{\mathbb{R}} u_\theta'' - c_\theta\int_{\mathbb{R}} u_\theta' &= -\int_{\mathbb{R}} f(u_\theta)\\
\theta\int_{\mathbb{R}}(J\star u_\theta - u_\theta) + (1-\theta)(u_\theta'(+\infty)-u_\theta'(-\infty)) - c_\theta(u_\theta(+\infty)-u_\theta(-\infty)) &= -\int_{\mathbb{R}} f(u_\theta)\\
\Rightarrow \theta\int_{\mathbb{R}}(J\star u_\theta - u_\theta) - c_\theta &= -\int_{\mathbb{R}} f(u_\theta)
\end{aligned}$$

From Claim 1.2.1 we have $\theta\int_{\mathbb{R}}(J\star u_\theta - u_\theta)=0$, therefore

$$c_\theta = \int_{\mathbb{R}} f(u_\theta) \geq 0. \tag{1.29}$$

This gives a bound from below for the speed c.

Remark 1.2.5. *: Note that this compution holds for any smooth solution of Problem* (1_θ).

We now establish an upper bound for c_θ. First we claim that :

Claim 1.2.2. *For every* $\theta \in [0, \bar{\theta})$ *we have* $c_\theta \leq \frac{(1-\theta)u_\theta'(0)+\theta\kappa}{\rho}$ *where* κ *is a constant independent of* θ.

Proof :

Without loss of generality we may assume that $u_\theta(0) = \rho$ for every $\theta \in [0, \bar{\theta})$.
Now integrate our equation (1_θ) over $\mathbb{R}^-$, to get :

$$\underbrace{\theta \int_{\mathbb{R}^-} (J \star u - u)dy}_{I_1} \quad + \quad \underbrace{(1-\theta)\int_{\mathbb{R}^-} u''dy}_{I_2} \quad - \quad \underbrace{c\int_{\mathbb{R}^-} u'dy}_{I_3} \quad = \quad \underbrace{-\int_{\mathbb{R}^-} f(u)dy}_{I_4}$$

A quick computation shows that :
- $I_2 = (1-\theta)u'(0)$
- $I_3 = c\rho$
- $I_4 = 0$

So we have,

$$c_\theta \rho = (1-\theta)u_\theta'(0) + I_1.$$

Since $J \star u_\theta - u_\theta \in L^1(\mathbb{R})$, we have $|I_1| \leq \theta\kappa$ where κ is the $L^1(\mathbb{R}^-)$ norm of $J \star u_\theta - u_\theta$.
We then obtain the required bound from Claim 1.2.1 .
More precisely we have :

$$c_\theta \rho \leq (1-\theta)u_\theta'(0) + \theta\kappa, \tag{1.30}$$

where $\kappa = \int_{\mathbb{R}} J(z)|z|dz$. This ends the proof of the claim.

□

Now multiply (1.30) by c_θ to obtain :

$$c_\theta^2 \rho - c_\theta \theta \kappa \leq (1-\theta)c_\theta u_\theta'(0). \tag{1.31}$$

From the apriori estimate (1.19) we have :

$$c_\theta^2 \rho - c_\theta \theta \kappa \leq (1-\theta)K. \tag{1.32}$$

This last equation shows that c_θ has an upper bound, which ends the proof of the boundness of A_θ.

□

Remark 1.2.6. *: Theses two claims remain true for a bistable non-linearity, and give an alternate proof of the corresponding lemma in [5].*

Now, we prove the following claim, and this will complete the proof of the lemma.

Claim 1.2.3. *There exists a constant $e > 0$ such that $\forall \theta \in [0, \bar{\theta})$, $c_\theta \geq e$.*

To prove this claim, we will use contradiction argument.

Proof :

Fix $\epsilon > 0$ and assume there exists a sequence $(\theta_n)_{n\in\mathbb{N}} \to \tilde{\theta}$ such that :
$c_{\theta_n} > 0 \to 0$ and $u_{\theta_n}(0) = \rho + \epsilon$.

$(u_{\theta_n})_{n\in\mathbb{N}}$ is a uniformly bounded sequence of monotone increasing functions, so according to Helly's theorem there exists a sub-sequence, still denoted $(u_{\theta_n})_{n\in\mathbb{N}}$, which converges pointwise to a monotone increasing function denoted by $\bar{u}$.

Since f is positive, $(f(u_{\theta_n}))_{n\in\mathbb{N}}$ is a sequence of positive functions. Apply Fatou's lemma to this sequence to get :

$$0 \leq \int_{-\infty}^{+\infty} f(\bar{u})ds \leq \liminf_{n\to\infty} \int_{-\infty}^{+\infty} f(u_{\theta_n})ds = \liminf_{n\to\infty} c_{\theta_n} = 0. \tag{1.33}$$

Thus $f(\bar{u}) = 0$ almost everywhere, so :

$$\bar{u}(\xi) \in [0, \rho] \cup \{1\} \;\; a.e. \tag{1.34}$$

Next, we multiply equation (1_{θ_n}) by u'_{θ_n} and integrate it. We obtain :

$$\theta_n \int_{\mathbb{R}} (J \star u_{\theta_n} - u_{\theta_n})u'_{\theta_n} + (1-\theta_n) \int_{\mathbb{R}} u''_{\theta_n} u'_{\theta_n} - c_{\theta_n} \int_{\mathbb{R}} (u'_{\theta_n})^2 = -\int_{\mathbb{R}} f(u_{\theta_n})u'_{\theta_n} dx,$$

$$\theta_n \int_{\mathbb{R}} (J \star u_{\theta_n} - u_{\theta_n})u'_{\theta_n} + \frac{(1-\theta_n)}{2}((u'_{\theta_n})^2(+\infty) - (u'_{\theta_n})^2(-\infty)) - c_{\theta_n} \int_{\mathbb{R}} (u'_{\theta_n})^2 = -\int_0^1 f(s)ds.$$

Therefore u_{θ_n} and c_{θ_n} satisfy the following equality

$$\theta_n \int_{\mathbb{R}} (J \star u_{\theta_n} - u_{\theta_n})u'_{\theta_n} + c_{\theta_n} \int_{\mathbb{R}} (u'_{\theta_n})^2 = \int_0^1 f(s)ds. \tag{1.35}$$

We can easily show by integration by parts that the first integral in (1.35) is zero, therefore :

$$c_{\theta_n} \int_{\mathbb{R}} (u'_{\theta_n})^2 = \int_0^1 f(s)ds. \tag{1.36}$$

In the same way if we multiply equation (1_{θ_n}) by $1 - u_{\theta_n}$ and integrate it, we obtain :

$$-\int_{\mathbb{R}} f(u_{\theta_n})(1-u_{\theta_n}) = \theta_n \int_{\mathbb{R}} (J\star u_{\theta_n} - u_{\theta_n})(1-u_{\theta_n}) + (1-\theta_n) \int_{\mathbb{R}} (1-u_{\theta_n})u''_{\theta_n} - c_{\theta_n} \int_{\mathbb{R}} (1-u_{\theta_n})u'_{\theta_n},$$

$$\Leftrightarrow -\int_{\mathbb{R}} f(u_{\theta_n})(1-u_{\theta_n}) = \frac{\theta_n}{2} \int_{\mathbb{R}^2} J(x-y)(u_{\theta_n}(x) - u_{\theta_n}(y))^2 + (1-\theta_n) \int (u'_{\theta_n})^2 - \frac{c_{\theta_n}}{2},$$

$$\Leftrightarrow -2 \int_{\mathbb{R}} f(u_{\theta_n})(1-u_{\theta_n}) = \theta_n \int_{\mathbb{R}^2} J(x-y)(u_{\theta_n}(x) - u_{\theta_n}(y))^2 + 2(1-\theta_n) \frac{\int_0^1 f(s)ds}{c_{\theta_n}} - c_{\theta_n}.$$

Therefore, u_{θ_n} satisfies

$$-c_{\theta_n}^2+c_{\theta_n}[\theta_n\int_{\mathbb{R}^2}J(x-y)(u_{\theta_n}(x)-u_{\theta_n}(y))^2+2\int_{\mathbb{R}}f(u_{\theta_n})(1-u_{\theta_n})]+2(1-\theta_n)\int_0^1 f(s)ds=0. \tag{1.37}$$

According to Claim 1.2.1, (1.37) implies that c_{θ_n} is the positive root of a second order polynomial, i.e.

$$c_{\theta_n}=\frac{D_{\theta_n}+\sqrt{(D_{\theta_n})^2+K_{\theta_n}}}{2}, \tag{1.38}$$

where

$$\begin{aligned} D_{\theta_n} &= \theta_n\int_{\mathbb{R}^2}J(x-y)(u_{\theta_n}(x)-u_{\theta_n}(y))^2+2\int_{\mathbb{R}}f(u_{\theta_n})(1-u_{\theta_n}),\\ K_{\theta_n} &= 8(1-\theta_n)\int_0^1 f(s)ds. \end{aligned}$$

From our assumption on the sequence $(c_{\theta_n})_{n\in\mathbb{N}}$, we have $D_{\theta_n}\to 0$. This implies

$$\theta_n\int_{\mathbb{R}^2}J(x-y)(u_{\theta_n}(x)-u_{\theta_n}(y))^2\to 0. \tag{1.39}$$

Now we apply Fatou's lemma in (1.39) and obtain :

$$0\le\int_{\mathbb{R}^2}J(x-y)(\bar{u}(x)-\bar{u}(y))^2\le\liminf_{n\to\infty}\int_{\mathbb{R}^2}J(x-y)(u_n(x)-u_n(y))^2=0.$$

Then, $\bar{u}(x)=\bar{u}(y)$ $a.e.$ for all $x-y\in supp(J)$. Therefore $\bar{u}\equiv cste$ almost everywhere. Since $\rho+\epsilon=u_{\theta_n}(0)\to\bar{u}(0)$ and $\bar{u}$ is a monotone increasing function, we have $\bar{u}\equiv\rho+\epsilon$, which contradicts (1.34).

□

Remark 1.2.7. *: The formula obtained for the speed c remains true for any solution of* (1_θ) *with non zero speed. Moreover the proof is independent of* $\bar{\theta}$*, thus the lemma remains true with* $\bar{\theta}=1$*.*

Remark 1.2.8. *: This lemma combined with the apriori estimates guarantees that we get a smooth solution as* $\theta\to\bar{\theta}$*.*

We can now obtain a solution to our problem (1.4),(1.5),(1.6). Let me describe how our machinery works. Since for $\theta=0$ there exists a positive increasing solution u_0, we may apply Lemmas 1.2.2 and 1.2.3 to get the existence of a solution to the problem 1_θ for $\theta\in[0,\eta)$ for some η positif. Now, let us defined

$$\bar{\theta}=\sup\{\theta>0|\text{ There exists a positive increasing solution } u_\theta \text{ of } (1_\theta)\}.$$

We have $\bar{\theta}\ge\eta$. We will show that $\bar{\theta}\ge 1$. We argue by contradiction, assume that $\bar{\theta}<1$. Let $(\theta_n)_{n\in\mathbb{N}}$ such that $\theta_n\nearrow\bar{\theta}$ and for each n (1_{θ_n}) has a positive increasing solution denoted by (u_n,c_n). Recall that (u_n,c_n) satisfies

$$\theta_n(J\star u_n-u_n)+(1-\theta_n)u_n''-c_nu_n'=-f(u_n) \tag{1.40}$$

$$u_n(-\infty)=0 \tag{1.41}$$

$$u_n(+\infty)=1 \tag{1.42}$$

Without loss of generality we may also normalize u_n by $u_n(0) = \rho$. From Lemmas 1.2.1, 1.2.5 and 1.2.6 there exists a positive constant C independant of n such that for each n we have $\|u_n\|_{C^3(\mathbb{R})} \leq C$ and $e \leq c_n \leq C$. Since $(u_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence of positive increasing function, according to Helly's theorem and C^3 estimates, there exists a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, which converges pointwise and C^2_{loc} to a positive non-decreasing smooth function $\bar{u}$. Since $(c_n)_{n\in\mathbb{N}}$ is bounded, up to extraction, $(u_n, c_n) \to (\bar{u}, \bar{c})$. Therefore by letting $n \to \infty$ in (1.40) we end up with

$$\bar{\theta}(J \star \bar{u} - \bar{u}) + (1 - \bar{\theta})\bar{u}'' - \bar{c}\bar{u}' = -f(\bar{u}). \tag{1.43}$$

Next we show that $\bar{u}$ satisfies the desired boundary conditions. Namely we have

Lemma 1.2.7.
Let $\bar{\theta} \leq 1$, the function $\bar{u}$ previously constructed satisfies the boundary conditions (1.41) *and* (1.42).

Assume for the moment that Lemma 1.2.7 is proved. From Lemma 1.2.7 $\bar{u}$ is a non-trivial solution and by Lemma 1.2.4, $\bar{u}$ is therefore a positive increasing solution of $(1_{\bar{\theta}})$. Now since we have assumed that $\bar{\theta} < 1$, then Lemma 1.2.2 and 1.2.3 holds with $\bar{u}$ instead of u_0. Thus there exists a positive increasing solution of (1_θ) for $\theta \in [0, \bar{\theta} + \eta)$ for some positive η, contradicting the defintion of $\bar{\theta}$. Thus $\bar{\theta} \geq 1$ and (1_θ) has a solution for every $\theta \in [0, 1)$.

We obtained a solution for $\theta = 1$ in the same way, let $(\theta_n)_{n\in\mathbb{N}}$ such that $\theta_n \nearrow 1$ and $(u_n, c_n)_{n\in\mathbb{N}}$ be the corresponding normalized sequence of solution. From Lemma 1.2.6, we have $c_n > e$ therefore according to Remark 1.2.1 and Lemma 1.2.5, we have $\|u_n\|_{C^2(\mathbb{R})} \leq C$ and $e \geq c_n \geq C$ for some positive constant C. Again, from Helly's theorem and apriori estimates there exists a non decreasing function $\tilde{u}$ and a constant $\tilde{c}$ such that $u_{\theta_n} \to \tilde{u}$ pointwise and $c_n \to \tilde{c}$. From the C^2 estimates, up to extraction, we have $u_{\theta_n} \to \tilde{u}$ in C^1_{loc}. Therefore $\tilde{u}$ satisfies the following equation

$$J \star \tilde{u} - \tilde{u} - \tilde{c}\tilde{u}' = -f(\tilde{u}).$$

It remains to prove that $\tilde{u}$ satisfies the boundary condition (1.41) and (1.42). This will be done with the proof of Lemma 1.2.7.

Proof of Lemma 1.2.7 :

Since $u_{\bar{\theta}}$ is a positive bounded nondecreasing function, it admits limits $l^{\pm}$ at $\pm\infty$. Standard theory and the Lebesgue dominated convergence theorem imply that theses limits are zeros of the function f and that $u'_{\bar{\theta}} \to 0$ as $\xi \to \pm\infty$. Observe that by construction we have $u_{\bar{\theta}}(0) = \rho$. Thus $l^+ = \{1, \rho\}$ and $l^- \in [0, \rho]$. First we claim that $(\bar{u}, \bar{c})$ must satisfies the following

Claim 1.2.4. $\bar{\theta} \int_0^1 \int_{\mathbb{R}} J(z) z \bar{u}(tz) dz dt = \bar{c}\rho - (1 - \bar{\theta})\bar{u}'(0)$

Proof :

Recall that the solution $\bar{u}$ is obtained as a limit of a sequence of $(u_n)_{n\in\mathbb{N}}$ and by Lemma 1.2.6 $\bar{c} > 0$. For this sequence, we have

$$\theta_n(J \star u_n - u_n) + (1 - \theta_n)u_n'' - c_n u_n' = 0 \ \ on \ \ \mathbb{R}^-. \tag{1.44}$$

Let us integrate equation (1.44) over $\mathbb{R}^-$ to get

$$\int_{-\infty}^{0} \theta_n (J \star u_n - u_n) = c_n \rho - (1-\theta_n) u_n'(0). \tag{1.45}$$

Observe that

$$\begin{aligned}
\int_{-\infty}^{0} (J \star u_n - u_n) &= \int_{-\infty}^{0}\int_{\mathbb{R}} J(x-y)(u_n(y)-u_n(x))dydx, &(1.46)\\
&= \int_0^1\int_{-\infty}^{0}\int_{\mathbb{R}} J(z) z u_n'(x+tz) dzdxdt, &(1.47)\\
&= \int_0^1\int_{\mathbb{R}} J(z) z \int_{-\infty}^{0} u_n'(x+tz) dxdzdt. &(1.48)
\end{aligned}$$

Let $v_n = \int_{-\infty}^{0} u_n'(x+tz)dx$ so that

$$\theta_n \int_0^1\int_{\mathbb{R}} J(z) z v_n(t,z) dzdt = c_n\rho - (1-\theta_n)u_n'(0). \tag{1.49}$$

Since $|J(z)zv_n(t,z)| = J(z)|z|u_n(tz) \leq J(z)|z|$, we can apply Lebesgue's dominated convergence theorem . Therefore, in the limit we get

$$\bar{\theta}\int_0^1\int_{\mathbb{R}} J(z) z \bar{u}(tz) dzdt = \bar{c}\rho - (1-\bar{\theta})\bar{u}'(0), \tag{1.50}$$

since $u_n' \to \bar{u}'$ on every compact set. This ends the proof of the claim.

□

Now, assume that $l^+ = \rho$, then $\bar{u}$ satisfies

$$\bar{\theta}(J\star\bar{u} - \bar{u}) + (1-\bar{\theta})\bar{u}'' - \bar{c}\bar{u}' = 0 \text{ in } \mathbb{R} \tag{1.51}$$

Since $\max u = u(0) = \rho$, and u satisfies (1.51) we may apply the maximum principle and thus $u \equiv \rho$. Therefore, we have $u_n'(0) \to 0$ and since J is even

$$\int_0^1\int_{\mathbb{R}} J(z) z \bar{u}(tz) dzdt = \rho\int_{\mathbb{R}} J(z) z dz = 0. \tag{1.52}$$

From Claim (1.2.4), this implies $\bar{c}\rho = 0$ which is a contradiction. Thus $l^+ = 1$. It now remains to prove that $l^- = 0$. We argue again by contradiction. Suppose $l^- = \sigma > 0$, where σ is any zero of f less than ρ. Since $\bar{u}$ satisfies equation (1.43), in particular

$$\bar{\theta}(J\star\bar{u} - \bar{u}) + (1-\bar{\theta})\bar{u}'' - \bar{c}\bar{u}' = 0 \text{ in } \mathbb{R}^-. \tag{1.53}$$

Integrate equation (1.53) over $\mathbb{R}^-$

$$\int_{-\infty}^{0} \bar{\theta}(J\star\bar{u} - \bar{u}) = \bar{c}(\rho-\sigma) - (1-\bar{\theta})u'(0). \tag{1.54}$$

Since $\bar{u}$ is smooth and non trivial, a quick computation shows that

$$\int_{-\infty}^{0} (J\star\bar{u} - \bar{u}) = \int_0^1\int_{\mathbb{R}} J(z) z \bar{u}(tz) dzdt. \tag{1.55}$$

The proof is now straightforward. Substitute (1.55) in equality (1.54) to get

$$\bar{\theta}\int_0^1 \int_{\mathbb{R}} J(z)z\bar{u}(tz)dzdt = \bar{c}(\rho-\sigma) - (1-\bar{\theta})\bar{u}'(0). \tag{1.56}$$

This last inequality contradicts Claim 1.2.4. This ends the proof of the lemma and completes the proof of the existence of a solution since the all proofs holds if $\bar{\theta} = 1$.

□

1.3 Uniqueness

In this section we present a result about the uniqueness of monotone solutions of (1_θ). We have the following theorem :

Theorem 1.3.1.
Let $\theta \in [0,1]$. Problem (1_θ) admits a unique increasing solution (u, c) i.e. if $(v, \bar{c})$ is another increasing solution of (1_θ) then $c = \bar{c}$ and $u(.) = v(. + \tau)$ for some $\tau \in \mathbb{R}$.

The main tools in this proof is a construction of appropriate sub and super solution which trap our solution.

Proof :

Fix $\theta \in [0,1]$, and let $(v, \bar{c})$ and (u, c) be two increasing solutions of (1_θ). We break down our proof into two steps :

– first step : we prove that $c = \bar{c}$.

– second step : we show that there exists $\tau \in \mathbb{R}$ such that $v(\xi) \equiv u(\xi - \tau)$.

1.3.1 Proof of the first step

In order to prove the uniqueness of the speed c, we argue by contradiction. Assume for example that $\bar{c} > c$. Without loss of generality, we may also assume

$$v(0) = u(0) = \frac{\rho}{2}. \tag{1.57}$$

Now we define some quantities that we will use to construct our sub- and super-solutions. Let $\delta <\mid \bar{c} - c \mid$ positive, such that

$$f'(p) < -2\delta \quad \forall p \quad such \;\; that \mid p - 1 \mid < \delta. \tag{1.58}$$

Let $\mu \in (0, \frac{\delta}{2})$ and $a(s) = \mu e^{-\delta s}$.
Choose $M > 0$ and $K > 0$ such that :

$$\mid u(\xi) - 1 \mid < \frac{\delta}{2} \;\; \forall \xi > M, \tag{1.59}$$

$$u'(\xi) > K \;\; in \;\; [-1, M+1]. \tag{1.60}$$

Finally define the following function :

$$b(s) = \frac{\mu\bar{\delta}}{K}(1 - e^{-\delta s}), \tag{1.61}$$

where $\bar{\delta} = 1 + \frac{\max\{f'(p) \quad -1 \leq p \leq 2\}}{\delta}$.

We will further impose that $\mu \leq \min\{\frac{\varrho}{2}, \frac{K}{\bar{\delta}}\}$.
Now we define our sub- and super-solutions as follow :

$$\tilde{u}(\xi, s) = u(\xi + b(s) + (c - \bar{c})s) + a(s), \tag{1.62}$$

$$\tilde{v}(\xi, s) = v(\xi - z), \tag{1.63}$$

where $z > z_0 = \inf\{z \in \mathbb{R} \mid u(\xi) + a(0) > v(\xi - z) \ \ \forall \xi \in \mathbb{R}\}$ is fixed. Let $w(\xi, s) = (\tilde{u} - \tilde{v})(\xi, s)$. w satisfy the next equations :

$$-\frac{\partial w}{\partial s} + \theta(J \star w - w) + (1-\theta)w_{\xi\xi} - \bar{c}w_{\xi} = -a'(s) - u'(\xi, s)b'(s) + f(v(\xi, s)) - f(u(\xi, s)) \tag{1.64}$$

$$w(\xi, 0) > 0 \ \ \forall \xi \in \mathbb{R} \tag{1.65}$$

$$w(\pm\infty, s) = a(s) \ \ \forall s \in \mathbb{R} \tag{1.66}$$

By (1.65), (1.66) and continuity there exists $s_0 = \sup\{s > 0 \mid w(\xi, s) > 0 \ \ \forall \xi \in \mathbb{R}\}$.

Claim 1.3.1. $s_0 = +\infty$.

Proof :

We argue by contradiction. If not, there exist $s_0 < +\infty$ and $\xi_0 \in \mathbb{R}$ such that

$$0 = w(\xi_0, s_0) = \min_{\mathbb{R}} w(\xi, s_0). \tag{1.67}$$

Next, we prove a kind of localization of minimum lemma. More precisely

Lemma 1.3.1.
Let $\xi_0 \in \mathbb{R}$ and s_0 previously defined by (1.67)*, then we have $\xi_0 + (c - \bar{c})s_0 > -1$.*

Proof :

w satisfies the equation :

$$\theta(J \star w - w) + (1 - \theta)w_{\xi\xi} - cw_{\xi} = f(v) - f(u) + (c - \bar{c})v'. \tag{1.68}$$

So at (ξ_0, s_0) we have,

$$\theta(J \star w - w)(\xi_0, s_0) + (1 - \theta)w_{\xi\xi}(\xi_0, s_0) \geq 0,$$
$$w_{\xi}(\xi_0, s_0) = 0.$$

Then $f(v) - f(u) + (c - \bar{c})v' \geq 0$, which implies that $f(v(\xi_0, s_0)) > 0$. Thus,

$$\begin{aligned} v(\xi_0, s_0) &= u(\xi_0 + b(s_0) + (c - \bar{c})s_0) + a(s_0) > \rho \\ \Rightarrow u(\xi_0 + b(s_0) + (c - \bar{c})s_0) &> \rho - a(s_0) > \frac{\rho}{2} \\ \Rightarrow \xi_0 + (c - \bar{c})s_0) &> u^{-1}(\frac{\rho}{2}) - b(s_0) \\ \Rightarrow \xi_0 + (c - \bar{c})s_0) &> -1. \end{aligned}$$

□

This lemma provides a bound from below for the minimum of w. Moreover w satisfies (1.64), thus at (ξ_0, s_0) we have :

$$-\frac{\partial w(\xi_0, s_0)}{\partial s} + \theta(J \star w - w)(\xi_0, s_0) + (1 - \theta)w_{\xi\xi}(\xi_0, s_0) - \bar{c}w_{\xi}(\xi_0, s_0) \geq 0. \tag{1.69}$$

Then

$$Q := -a'(s_0) - u'(\xi_0, s_0)b'(s_0) + f(u(\xi_0, s_0) + a(s_0)) - f(u(\xi_0, s_0)) \geq 0, \tag{1.70}$$

$$= \mu e^{-\delta s_0}[\delta - \frac{\delta\bar{\delta}}{K}u'(\xi_0 + b(s_0) + (c - \bar{c})s_0) + f'(d)] \geq 0. \tag{1.71}$$

Lemma 1.3.1 leads us to consider two cases :

- 1^{st} case : $\xi_0 + (c - \bar{c})s_0 \in [-1, M]$.

  Then $\xi_0 + (c - \bar{c})s_0 + b(s_0) \in [-1, M + 1]$ and Q would satisfy :

  $$0 > \mu e^{-\delta s_0}[\delta(1 - \frac{u'(\xi_0, s_0)}{K}) - \frac{u'(\xi_0, s_0)}{K}\max\{f'(p) \quad -1 \leq p \leq 2\} + f'(d)],$$

  which contradicts (1.71).

- 2^{nd} case : $\xi_0 + (c - \bar{c})s_0 > M$.

  Then $\xi_0 + (c - \bar{c})s_0 + b(s_0) > M$ and Q would then verify :

  $$\mu e^{-\delta s_0}[\delta - \frac{\delta\bar{\delta}u'(\xi_0, s_0)}{K} + f'(d)] < \mu e^{-\delta s_0}[-\delta - \frac{\delta\bar{\delta}u'(\xi_0, s_0)}{K}] < 0,$$

  which also contradicts (1.71) and prove the claim.

□

Therefore w is positive for any couple $(\xi, s) \in \mathbb{R} \times \mathbb{R}^+$. Take now $\xi = (\bar{c} - c)s - b(s) \geq 0$ and let s go to $+\infty$. We get

$$\lim_{s \to +\infty} w((\bar{c} - c)s - b(s), s) = u(0) - 1 < 0,$$

which is a contradiction. Thus $c \geq \bar{c}$. By switching the role of u and v in the proof we achieve $\bar{c} \geq c$ and therefore $\bar{c} = c$. This ends the proof of the first step.

□

Now we turn our attention to the second part of the uniqueness lemma.

1.3.2 Proof of the second step

Remember that the second step is the following lemma :

Lemma 1.3.2.
Let u and v be two solution of same speed c of (1_θ). Then there exists $\tau \in \mathbb{R}$ such that $u(\xi) = v(\xi+\tau)$ for all $\xi \in \mathbb{R}$.

Proof :

We will use once again a contradiction argument. The analysis performed in the first step remains valid for u and v. So we have :

$$w(\xi, s) = u(\xi + b(s)) + a(s) - v(\xi - z) > 0 \ \ \forall \ \ (\xi, s) \in \mathbb{R} \times \mathbb{R}^+.$$

Let s go to $+\infty$ to obtain :

$$u(\xi + \frac{\mu\bar{\delta}}{K}) \geq v(\xi - z) \ \ \forall \ \ \xi \in \mathbb{R}.$$

Next let $z \searrow z_0$:

$$u(\xi) \geq v(\xi - (z_0 + \frac{\mu\bar{\delta}}{K})) \ \forall \ \ \xi \in \mathbb{R}.$$

We can therefore find a minimal $\bar{z}$ such that

$$u(\xi) \geq v(\xi - \bar{z}) \ \forall \ \ \xi \in \mathbb{R}. \tag{1.72}$$

Claim 1.3.2. *We claim that if $u \not\equiv v$ then we have a strict inequality in* (1.72) *and $\bar{z} > 0$ since $u(0) = v(0)$.*

Proof :

If not, there exists a point ξ_0 such that $w(\xi) = u(\xi) - v(\xi - \bar{z}) \geq w(\xi_0) = 0 \ \ \forall \ \ \xi \in \mathbb{R}$.
At this point, w verifies :

$$0 \leq \theta(J \star w - w)(\xi_0) + (1-\theta)w''(\xi_0) = f(v(\xi_0 - \bar{z})) - f(u(\xi_0)) = f(u(\xi_0)) - f(u(\xi_0)) = 0.$$

If $\theta \neq 0$, $(J \star w - w)(\xi_0) = 0$ then, as we have seen before this implies $w \equiv 0$ contradicting our assumption. If $\theta = 0$ the maximum principle provides the same result.

□

Fix $\eta > 0$ and define :

$$z(\eta) = inf\{z | u(\xi) \geq v(\xi - z) - \eta \ \ \forall \ \ \xi \in \mathbb{R}\} \tag{1.73}$$

This implies that $z(\eta) < \bar{z}$ and $\lim_{\eta \to 0} z(\eta) = \bar{z}$. For a fixed $N > 0$ we claim :

Claim 1.3.3. *There exist $\eta(N) > 0$ such that for all $\eta \in [0, \eta(N))$ we have :*

$$u(\xi) > v(\xi - z(\eta)) - \eta \ \ for \ \ \ | \xi | \leq N. \tag{1.74}$$

Proof :

If $\eta(N) > 0$ doesn't exist, then there exists sequences $\xi_n \to \bar{\xi} \in [-N, N]$ and $\eta_n \to 0$ such that :

$$u(\xi_n) = v(\xi_n - z(\eta_n)) - \eta_n. \tag{1.75}$$

At the limit we get :

$$u(\bar{\xi}) = v(\bar{\xi} - \bar{z}), \tag{1.76}$$

which contradicts Claim 1.3.2.

□

Set $a(s) = \mu e^{-\delta s}$ with $\mu < \eta(M)$ where M and δ are defined by (1.58),(1.59) and (1.60). Since $z(\mu) < \bar{z}$, we can choose $\epsilon > 0$ such that $2\epsilon < \bar{z} - z(\mu)$. Define a new function as follows :

$$p(\xi, s) \equiv u(\xi) + a(s) - v(\xi - (\bar{z} - \epsilon)). \tag{1.77}$$

By construction of p, we get $p(\xi, 0) > 0$. We claim that :

Claim 1.3.4. $p(\xi, s) > 0 \ \forall \ (\xi, s) \in \mathbb{R} \times \mathbb{R}^+$

Proof :

First we handle the case $\theta \neq 0$. As in the proof of uniqueness of the speed we argue by contradiction. Assume there exists (ξ_0, s_0) such that

$$0 = p(\xi_0, s_0) \leq p(\xi, s) \ \forall \ (\xi, s) \in \mathbb{R} \times [0, s_0). \tag{1.78}$$

At this point p verifies

$$-\frac{\partial p}{\partial s}(\xi_0, s_0) + \theta(J \star p - p)(\xi_0, s_0) + (1-\theta)p_{\xi\xi}(\xi_0, s_0) - cp_\xi(\xi_0, s_0) \geq 0, \tag{1.79}$$

which implies

$$-a'(s_0) + f(v(\xi_0, s_0) - f(u(\xi_0, s_0) \geq 0, \tag{1.80}$$
$$\mu e^{-\delta s_0}(\delta + f'(d)) \geq 0, \tag{1.81}$$

where $d \in [u(\xi_0), u(\xi_0) + a(s_0)]$.

As in the proof of uniqueness of c we need a localization lemma to continue the proof.

Lemma 1.3.3.
With (ξ_0, s_0) *defined by* (1.78) *we have :*

1. $\mid \xi_0 \mid > M$,

2. $\xi_0 > 0$.

Proof of lemma 1.3.3 :

p also satisfy at (ξ_0, s_0) the following equation :

$$\theta(J \star p - p)(\xi_0, s_0) + (1-\theta)p_{\xi\xi}(\xi_0, s_0) - cp_\xi(\xi_0, s_0) = f(v(\xi_0, s_0)) - f(u(\xi_0, s_0)) \geq 0.$$

Then $f(u(\xi_0) + a(s_0)) > 0$ since otherwise we would have :

$$0 \leq \theta(J \star p - p)(\xi_0, s_0) + (1-\theta)p_{\xi\xi}(\xi_0, s_0) = 0, \tag{1.82}$$
$$\Rightarrow (J \star p - p)(\xi_0, s_0) = 0, \tag{1.83}$$
$$\Rightarrow p(\xi, s_0) \equiv 0, \tag{1.84}$$
$$\Rightarrow u(\xi) = v(\xi - (\bar{z} - \epsilon)) - a(s_0) \;\; \forall \xi \in \mathbb{R}. \tag{1.85}$$

Passing to the limit as $\xi \to +\infty$ in (1.85), we would then get a contradiction.
Since $f(u(\xi_0) + a(s_0)) > 0$ we have :

$$u(\xi_0) + a(s_0) > \rho, \tag{1.86}$$
$$\Rightarrow u(\xi_0) > \rho - a(s_0) > \frac{\rho}{2}, \tag{1.87}$$
$$\Rightarrow \xi_0 > u^{-1}(\frac{\rho}{2}) = 0, \tag{1.88}$$

which proves the second part of the lemma.

Now we show the first part of the lemma. Since

$$u(\xi) \geq v(\xi - (\bar{z} - \epsilon)) - \mu e^{-\delta s_0}, \;\; \text{for all} \;\; \xi \in \mathbb{R}, \tag{1.89}$$

we have $z(\mu e^{-\delta s_0}) \leq \bar{z} - \epsilon$. Moreover, $\mu e^{-\delta s_0} < \eta(M)$. Thus by Claim 1.3.3 we have

$$u(\xi) > v(\xi - z(\mu e^{-\delta s_0})) - \mu e^{-\delta s_0} \;\; for \;\; \mid \xi \mid \leq M. \tag{1.90}$$

Since v is increasing and $z(\mu e^{-\delta s_0}) \leq \bar{z} - \epsilon$, we have

$$u(\xi) > v(\xi - (\bar{z} - \epsilon)) - \mu e^{-\delta s_0} \;\; for \;\; \mid \xi \mid \leq M. \tag{1.91}$$

Since (ξ_0, s_0) is a zero of p, (1.91) implies $\mid \xi_0 \mid > M$. This concludes the proof of the lemma. $\square$

Now we return to the proof of Claim 1.3.4. From the localization lemma 1.3.3 we deduce that $\mid d - 1 \mid \leq \delta$. Thus, we have $f'(d) + \delta < 0$ which contradicts (1.81). This ends the proof of Claim 1.3.4 in the case $\theta \neq 0$.

In the case $\theta = 0$, for every fixed s, take k positive large enough such that p will verify the following in-equation :

$$\begin{cases} p_{\xi\xi}(\xi, s) - cp_\xi(\xi, s) - kp \leq 0 \\ p(\pm\infty) > 0 \end{cases}$$

Apply the maximum principle, to get $p(\xi, s) > 0 \ \forall \xi \in \mathbb{R}$. This proves Claim 1.3.4.

□

Now, we return to the proof of Lemma 1.3.2. By passing to the limit as $s \to +\infty$ in $p(\xi, s)$ we obtain :

$$u(\xi) \geq v(\xi - (\bar{z} - \epsilon)) \ \ \forall \ \xi \in \mathbb{R}.$$

This contradicts the definition of $\bar{z}$ and shows, at the same time, that $u \equiv v_{\bar{z}}$.

□

1.4 Continuity of the speed c_θ

This section is devoted to some continuity property of the speed $c_\theta(\rho)$ as a function of θ and ρ. This is a direct consequence of the uniqueness theorem.

Theorem 1.4.1.
$c_\theta(\rho)$ is a continuous function of θ and f_ρ in the following sense for any $\theta_0 \in [0,1]$ and $f_{\rho_0} > 0$ then for any sequences $f_{\rho_n} \to f_{\rho_0}$ uniformaly and $\theta_n \to \theta_0$), we have $c_{\theta_n}^{\rho_n} \to c_{\theta_0}^{\rho_0}$.

The main tool in this proof is the previous uniqueness theorem 1.3.1.

proof :

From Theorems 1.1.2 and 1.1.3, we know that for every f_ρ with $\rho > 0$ and $\theta \in [0,1]$ there exists a unique increasing solution $(u_\theta^\rho, c_\theta^\rho)$ to the following problem,

$$\begin{cases} (1-\theta)(u_\theta^\rho)'' + \theta(J \star u_\theta^\rho - u_\theta^\rho) - c_\theta^\rho (u_\theta^\rho)' + f_\rho(u_\theta^\rho) = 0 & in \ \mathbb{R} \\ u_\theta^\rho \to 0 & x \to -\infty \\ u_\theta^\rho \to 1 & x \to +\infty \end{cases} \tag{1.92}$$

Fixed $f_{\rho_0} > 0$ and $\theta_0 \in [0,1]$, we will shows that for any sequences $f_{\rho_n} \to f_{\rho_0}$ uniformaly and $\theta_n \to \theta_0$, we have $c_{\theta_n}^{\rho_n} \to c_{\theta_0}^{\rho_0}$ this will show the continuity of the speed. Let $u_{\theta_n}^{\rho_n}$ be the normalized associated solution, i.e $u_{\theta_n}^{\rho_n}(0) = \frac{1}{2}$. From the apriori estimates obtained in Lemma 1.2.6, we have $c_{\theta_n}^{\rho_n}$ bounded as $(\rho_n, \theta_n) \to (\rho_0, \theta_0)$. We can extract a sequence, which converges to some value γ. From the apriori estimates 1.2.5 on $u_{\theta_n}^{\rho_n}$, there also exists a subsequence which converges to a smooth nondecreasing function u solution of the following problem with speed γ.

$$\begin{cases} \epsilon_0 u'' + J \star u - u - \gamma u' + f_{\theta_0}(u) = 0 & in \ \mathbb{R} \\ u \to 0 & x \to -\infty \\ u \to 1 & x \to +\infty \end{cases} \tag{1.93}$$

According to Theorem 1.1.3, the speed and the profile are unique. Therefore, $\gamma = c_{\theta_0}^{\rho_0}$ and $u(x) = u_{\theta_0}^{\rho_0}(x + \tau)$. Since $c_{\theta_n}^{\rho_n}$ is precompact and has only γ as accumulation point, the sequence $c_{\theta_n}^{\rho_n}$ converge to γ as n goes to infinity. This ends the proof of Theorem 1.4.1.

□

1.5 Asymptotic behavior of solutions

In this section we establish the asymptotic behavior of the solution u and its derivative, with the extra assumption (H^*). We prove that u (resp. $|1-u|$) has exponential behavior near $-\infty$ (resp $+\infty$). We prove also that u' has exponential behavior near $\pm\infty$. This asymptotic behavior is one of the key-points in the proof of existence of travelling-front solutions in the monostable case. To summarize :

Proposition 1.5.1.
Let u be a solution of (1.4) (1.5) *and* (1.6) *where J satisfies* (H^*). *Then u satisfies :*

(i) *There exists positive constants A, B,* λ_0 *and* δ_0 *such that*

$$Be^{-\delta_0 y} \leq 1 - u \leq Ae^{-\lambda_0 y} \ \ when \ \ y \ \ \rightarrow +\infty.$$

(ii) *There exists positive constants K, K',* λ_1 *and* δ_1 *such that*

$$K'e^{\delta_1 y} \leq u \leq Ke^{\lambda_1 y} \ \ when \ \ y \ \ \rightarrow -\infty.$$

(iii) *There exist two positive constants* $\tilde{K}$ *and* λ_2 *such that*

$$u' \leq \tilde{K}e^{-\lambda_2|y|} \ \ when \ \ |y| \ \ \rightarrow +\infty.$$

Remarks 1.5.1. *: Since u goes to 0 and 1 when x goes to* $\pm\infty$*, we only need to prove (iii) to get the inequalities* $u \leq Ke^{\lambda_1 y}$ *and* $1 - u \leq Ae^{-\lambda_0 y}$.

The proof is based on the constant use of a comparison principle and the construction of appropriate barrier functions. The comparison principle that we use is the following :

Theorem 1.5.1. *Comparison Principle*
Assume b(x) in $L^\infty(\mathbb{R})CAP$:$C(\mathbb{R})$
Let u and v be two smooth functions ($C^{1,\alpha}(\mathbb{R})$*) and* ω *a connected subset of* $\mathbb{R}$. *Assume that u and v satisfy the following conditions :*

- $Lv = J \star v - v - b(x)v' \geq 0$ *in* $\omega \subset \mathbb{R}$
- $Lu = J \star u - u - b(x)u' \leq 0$ *in* $\omega \subset \mathbb{R}$
- $u \geq_{\not\equiv} v$ *on* $\mathbb{R} - \omega$
- *if* ω *is an unbounded domain, also assume that* $\lim_{\pm\infty} u - v \geq 0$.

Then $u \geq v$ *on all* $\mathbb{R}$.

Proof :

Let $w = u - v$, so w will satisfy :
- $w \geq 0$, $w \not\equiv 0$ in $\mathbb{R} - \omega$,
- $Lw \leq 0$ in ω.

Now, we argue by contradiction. Assume that w achieves a negative minimum at x_0. By assumption this point x_0 is in ω and is a global minimum of w. So, at this point, w satisfies this following two inequalities :

$$0 \geq Lw(x_0) = (J \star w - w)(x_0) = \int_{\mathbb{R}} J(x_0 - z)(w(z) - w(x_0))dz, \tag{1.94}$$

$$Lw(x_0) = (J \star w - w)(x_0) = \int_{\mathbb{R}} J(x_0 - z)(w(z) - w(x_0))dz \geq 0. \tag{1.95}$$

This implies that $w \equiv w(x_0)$ on $\mathbb{R}$ which contradicts our assumption $w \geq_{\not\equiv} 0$ on $\mathbb{R} - \omega$ $\square$

Remarks 1.5.2. *: This comparison principle also holds if we consider an integrodifferential operator with negative zero terms such as* $Lv = J \star v - v - b(x)v' + c(x)v$, $c(x) \leq 0$.

We now present the proof of the third assertion in Proposition 1.5.1. We will first focus on the behavior of u' at $-\infty$.

Proof of (iii) in Proposition 1.5.1

For $\lambda > 0$, let $g(x, \lambda) = De^{\lambda x}$. Let L be the following integrodifferential operator :

$$Lv = J \star v - v - cv'. \tag{1.96}$$

A quick computation of Lg shows that g satisfies :

$$Lg(x) = g(x)(\int_{\mathbb{R}} J(z)e^{\lambda z}dz - c\lambda - 1). \tag{1.97}$$

Observe that the condition (H^*) gives sense to $\int_{\mathbb{R}} J(z)e^{\lambda z}dz$, so

$$Lg = h(\lambda)g(x), \tag{1.98}$$

where

$$h(\lambda) = \int_{\mathbb{R}} J(z)e^{\lambda z}dz - c\lambda - 1. \tag{1.99}$$

Remarks 1.5.3. *: The assumption* (H^*) *ensures that* $h(\lambda)$ *is a smooth function of* λ.

We will now choose our λ such that g will be a super-solution of L. We claim :

Claim 1.5.1. *There exists* $\lambda > 0$ *such that* $Lg \leq 0$.

We only have to prove that there exists $\lambda > 0$ such that $h(\lambda) \leq 0$.

Proof :

We first compute $h(0)$.
Since J satisfies (H^*) we have $J(z)e^{\lambda z} \leq J(z)e^{z}$ for $\lambda \leq 1$, which is an integrable function. Observe that for every z,

$$J(z)e^{\lambda z} \to J(z) \;\; when \;\; \lambda \to 0.$$

Therefore we can apply Lebesgue's theorem to obtain

$$h(0) = \int_{\mathbb{R}} J(z)dz - 1 = 0. \tag{1.100}$$

If we show that $h'(0) < 0$ the claim will be proved. Again since J satisfies (H^*), we can use Lebesgue's derivation theorem, to get :

$$h'(\lambda) = \int_{\mathbb{R}} J(z)ze^{\lambda z}dz - c. \tag{1.101}$$

We claim that
$$\int_{\mathbb{R}} J(z)ze^{\lambda z} \to 0 \;\; as \;\; \lambda \to 0. \tag{1.102}$$

Indeed for every point z,
$$J(z)ze^{\lambda z} \to J(z)z \;\; as \;\; \lambda \to 0.$$

As above we have $J(z)ze^{\lambda z} \leq J(z)|z|e^{z}$, which is integrable according to (H^*). Apply Lebesgue's theorem to obtain
$$h'(0) = \int_{\mathbb{R}} J(z)zdz - c = 0.$$

Since $J(z)z$ is an odd function, we get :
$$h'(0) = -c < 0.$$

This ends the proof of Claim 1.5.1.
We define λ_1 as the first non-trivial zero of the function h. λ_1 is well defined since $h(\lambda) \to +\infty$ as $\lambda \to +\infty$.

□

Remark 1.5.1. *: It follows from its definition that $\lambda_1(c, J)$ is an increasing function of c. This will be very useful later.*

Now, by taking $D = \| u' \|_\infty$, we achieve $g(x) > u'(x)$ on $\mathbb{R}^+$. From the translation invariance, without loss of generality, we may assume that $u(0) = \rho$. Therefore u satisfies $Lu = 0$ in $\mathbb{R}^-$.
We now use the comparison principle 1.5.1 with u' and g to obtain the desired inequality :
$$\| u' \|_\infty \exp(\lambda_1 x) > u'(x) \;\; on \;\; \mathbb{R}.$$

□

We use the same idea to obtain the behavior of u' near $+\infty$. We present briefly it proof.

Proof :

As above we define for $\lambda > 0$, $g(x, \lambda) = \| u' \|_\infty \; e^{-\lambda x}$. By our definition we have $g(x, \lambda) \geq_{\not\equiv} u'$ for every $x \in \mathbb{R}^-$. Then we just have to show the inequality on $\mathbb{R}^+$. As in our previous proof, we construct our barrier function and a linear problem to obtain the inequality :
$$g(x, \lambda) \geq_{\not\equiv} u' \;\; for \;\; every \;\; x \in \mathbb{R}. \tag{1.103}$$

Let L be the following integrodifferential operator :
$$Lv = J \star v - v - cv' + \frac{f'(1)}{2}v. \tag{1.104}$$

A quick computation shows that g satisfies
$$Lg(x) = g(x, \lambda)(\int_{\mathbb{R}} J(z)e^{\lambda z}dz + c\lambda - 1 + \frac{f'(1)}{2}) \text{ for } x \in \mathbb{R}. \tag{1.105}$$

As before, we study the function

$$l(\lambda) = (\int_{\mathbb{R}} J(z)e^{\lambda z}dz + c\lambda - 1 + \frac{f'(1)}{2}) \tag{1.106}$$

in a neighborhood of 0. A quick computation gives :

$$l(0) = \frac{f'(1)}{2} < 0$$

and

$$l'(\lambda) > 0.$$

So l has a first zero. Let λ_0 be this first zero.

Remark 1.5.2. *: The function $\lambda_0(c, J)$ is a decreasing function of c.*

As $u \to 1$ when $x \to +\infty$, by translation invariance and regularity of f, we may assume, without loss of generality, that $f'(u(x)) \leq \frac{f'(1)}{2}$ for all positive x. Thus u' satisfies in $\mathbb{R}^+$

$$Lu(x) = -(f'(u(x)) - \frac{f'(1)}{2}) \geq 0. \tag{1.107}$$

The comparison principle 1.5.1 with L, u' and g gives the desired conclusion.

□

By our two computations, u' satisfies near $\pm\infty$:
- $u' \leq \| u' \|_\infty e^{\lambda_1 x}$,
- $u' \leq A \| u' \|_\infty e^{-\lambda_0 x}$.

Thus by taking $\lambda_2 = \min\{\lambda_0, \lambda_1\}$, and $\tilde{K} = \sup\{A \| u' \|_\infty, \| u' \|_\infty\}$ we get :

$$u' \leq \tilde{K}e^{-\lambda_2 |x|}.$$

This ends the proof of (iii) in Proposition 1.5.1.

□

As we have previously mentioned, we can get some asymptotic behavior of u by integrating the above inequality. So we have :
- $u(x) \leq \frac{\|u'\|_\infty}{\lambda_1} e^{\lambda_1 x} \;\; for \;\; x \;\; near \;\; -\infty$,
- $1 - u(x) \leq \frac{\|u'\|_\infty}{\lambda_0} e^{-\lambda_0 x} \;\; for \;\; x \;\; near \;\; +\infty$.

To complete the proof of Proposition 1.5.1, we establish the two inequalities below :

$$Be^{-\delta_0 x} \leq 1 - u(x) \;\; for \;\; x \;\; near \;\; +\infty, \tag{1.108}$$

$$K'e^{\delta_1 x} \leq u(x) \;\; for \;\; x \;\; near \;\; -\infty \tag{1.109}$$

We use the same technique as in the study of the behavior of u'. We first deal with (1.108). Define $g(x, \delta) = Be^{-\delta |x|}$, for $\delta > 0$. Let L be the following operator :

$$Lv = J \star v - v - cv' + \frac{f(u)}{u - 1}v. \tag{1.110}$$

Choose $k > 0$ such that $-k < \frac{f(u)}{u-1} < k$. Such a k exists because f is Lipschitz continuous on [0,1].

Now, we compute Lg for $x > 0$, this gives :

$$Lg = B\int_{\mathbb{R}} J(y)e^{-\delta|x-y|}dy - g(x) + c\delta g(x) + \frac{f(u)}{u-1}g(x), \tag{1.111}$$

$$\geq g(x)(\int_{\mathbb{R}} J(y)e^{-\delta|y|}dy - 1 + c\delta + \frac{f(u)}{u-1}), \tag{1.112}$$

$$\geq g(x)(\int_{\mathbb{R}} J(y)e^{-\delta|y|}dy - 1 + c\delta - k). \tag{1.113}$$

We are looking for a value of δ such that $Lg \geq 0$.
Observe that

$$m(\delta) = \int_{\mathbb{R}} J(y)e^{-\delta|y|}dy - 1 + c\delta - k \geq 0$$

for $\delta \geq \frac{k+1}{c}$. Therefore, for $\delta_0 = \frac{k+1}{c}$ we achieve $Lg(x, \delta_0) > 0$.
Let the constant B be defined by $B = 1 - u(0)$. Then for all positive δ, we have :

$$1 - u(y) \geq (1 - u(0))e^{-\delta|y|} \ \ for \ \ all \ \ y \in \mathbb{R}^-. \tag{1.114}$$

By our choice of L we have $L(1-u) = 0$ in $\mathbb{R}$. Apply the comparison principle 1.5.1 with $1 - u$ and $g(x, \delta_0)$, to get :

$$1 - u(y) \geq (1 - u(0))e^{-\delta_0|y|} \ \ for \ \ all \ \ y \ \ in \ \ \mathbb{R}.$$

Now we will end the proof of the proposition by showing that u satisfies (1.109).

Define the following function $g(x, \delta)$ for $\delta > 0$

$$g(x, \delta) := \begin{cases} \frac{\rho}{2}e^{\delta x} & for \ \ x < 0 \\ \frac{\rho}{2} & for \ \ x > 0. \end{cases}$$

Let L be the following operator :

$$Lv = J \star v - v - cv' - kv. \tag{1.115}$$

Choose $k > 0$ such that $-k < \frac{f(u)}{u} < k$. Such a k exists because f is Lipschitz continuous on [0,1].

Now, we compute Lg for $x < 0$, this gives :

$$Lg = g(x, \delta)\int_{-\infty}^{-x} J(z)e^{\delta z}dz + \frac{\rho}{2}\int_{-x}^{+\infty} J(z)dz - g(x, \delta) - c\delta g(x, \delta) - kg(x, \delta), \tag{1.116}$$

$$= g(x, \delta)(\int_{-\infty}^{-x} J(z)e^{\delta z}dz + e^{-\delta x}\int_{-x}^{+\infty} J(z)dz - 1 - c\delta - k), \tag{1.117}$$

$$\geq g(x, \delta)(\int_{0}^{-x} J(z)e^{\delta z}dz + e^{-\delta x}\int_{-x}^{+\infty} J(z)dz - 1 - c\delta - k). \tag{1.118}$$

Choose R and δ_1 such that $supp(J)CAP$:$[0, R] \neq \emptyset$ and

$$\int_0^R J(z)e^{\delta_1 z}dz - 1 - c\delta_1 - k \geq 0 \tag{1.119}$$

From (1.118) and (1.119) for $x \leq -R$, we have

$$Lg \geq g(x,\delta_1)(\int_0^R J(z)e^{\delta_1 z}dz + e^{-\delta_1 x}\int_{-x}^{+\infty} J(z)dz - 1 - c\delta_1 - k) \geq 0$$

Observe that u and any translation of u satisfy on $\mathbb{R}$

$$Lu = -(\frac{f(u)}{u} + k)u \leq 0.$$

Therefore we can assume that $u(-R) = \frac{\rho}{2}$. Since u is increasing we have $g(x,\delta_1) \leq \frac{\rho}{2} < u(x)$ for $x > -R$. Using the comparison principle 1.5.1 with u and g on $(-\infty,-R)$, then yields to $g(x,\delta_1) \leq u$ on $\mathbb{R}$, which proves (1.109) and ends the proof of the proposition.

□

Remark 1.5.3. *: By its definition, $\delta_0(c)$ is a decreasing function of the speed c. This will be useful in the proof of existence of travelling waves in the case where f is monostable.*

Remark 1.5.4. *: One can observe in the derivation of the exponential behavior at $+\infty$ of a solution u that the only assumption required on f is $f'(1) < 0$. Therefore this computation still holds for nonlinearities f which belong to the bistable and monostable case. To derive* (1.109)*, the only required assumption on f is that f is lipschitz, therefore the same argument will hold for bistable and monostable nonlinearity.*

1.6 Existence of c^*

In this section we establish the existence of solutions in the KPP case. We prove the existence of one value of c, c^*, for which there is a solution u, $0 < u < 1$ of

$$\begin{cases} J \star u(x) - u(x) - cu'(x) = -f(u(x)) \\ u(-\infty) = 0 \\ u(+\infty) = 1, \end{cases} \tag{1.120}$$

where f is a monostable non-linearity. To achieve this, we use a standard procedure of approximation of the non-linearity f by functions f_{ϵ_n} of ignition type (i.e type B). So, for each $n \in \mathbb{N}$ Theorems 1.1.2 and 1.1.3 yields to existence of a solution (u_n, c_n) and a limiting procedure will give the result.

This approximation can be easily obtained by multiplying f by a "cut-off" function g_{ϵ_n}. Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers which goes to 0 as n goes to infinity. We require that g_{ϵ_n} satisfy the following assumption :

- $g_{\epsilon_n} \in C_0^\infty(\mathbb{R})$,
- $0 \leq g_{\epsilon_n} \leq 1$,
- $g_{\epsilon_n}(s) \equiv 0$ for $s \leq \epsilon_n$ and $g_{\epsilon_n}(s) \equiv 1$ for $s \geq 2\epsilon_n$,
- g_{ϵ_n} is a monotone increasing sequence of function (i.e. $g_{\epsilon_n} \leq g_{\epsilon_p}$ for $p \geq n$).

Our approximation function $f_{\epsilon_n} = fg_{\epsilon_n}$ is now well-defined. We may now apply Theorems 1.1.2 and 1.1.3, according to which there is a unique solution (u_n, c_n) of :

$$\begin{cases} J \star u(x) - u(x) - cu'(x) = -f_{\epsilon_n}(u(x)) \\ u(-\infty) = 0 \\ u(+\infty) = 1. \end{cases} \tag{1.121}$$

u_n is unique modulo translation. After translation, we may normalize u_n so that :

$$u_n(0) = \frac{1}{2}. \tag{1.122}$$

Now, we come to the main point : to obtain upper and lower bounds for c_n. To derive the lower bound, we use the same technique as in Lemma 1.2.6. We claim the following :

Claim 1.6.1. *There exists a positive constant $e > 0$ such that for all $n \in \mathbb{N}$, $c_n \geq e$*

Proof

We use a contradiction argument. Assume there exists a subsequence of $(c_n)_{n\in\mathbb{N}}$ which converges to 0. Let $(u_n)_{n\in\mathbb{N}}$ be the corresponding subsequence of solutions. First integrate equation (1.121) over $\mathbb{R}$. A straightforward computation shows that :

$$c_n = \int_{\mathbb{R}} f_{\epsilon_n}(u_n). \tag{1.123}$$

$(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of positive, increasing functions, so by Helly's Theorem, there exists a sub-sequence which converge pointwise to a non-decreasing non-negative function $\bar{u}$. By Fatou's Lemma, from (1.123) we get :

$$0 = \liminf c_n \geq \int_{\mathbb{R}} f(\bar{u}) \geq 0. \tag{1.124}$$

Hence, $\bar{u}$ takes its values in the set $\{0, 1\}$ for almost every $x \in \mathbb{R}$. $\bar{u}$ is a non-decreasing function, so by the normalization, $\bar{u}$ must satisfy :

$$\bar{u} \equiv 0 \;\; almost \;\; everywhere \;\; in \;\; \mathbb{R}^-, \tag{1.125}$$

$$\bar{u} \equiv 1 \;\; almost \;\; everywhere \;\; in \;\; \mathbb{R}^+. \tag{1.126}$$

Now, multiply (1.121) by $(1 - u_n)$ and integrate over $\mathbb{R}$. An easy computation leads us to :

$$c_n = 2\int_{\mathbb{R}} f_{\epsilon_n}(u_n)(1 - u_n) + \int\int_{\mathbb{R}^2} J(x - y)(u_n(x) - u_n(y))^2 dxdy. \tag{1.127}$$

Again, apply Fatou's Lemma in (1.127) to get :

$$0 = \liminf c_n \geq 2\int_{\mathbb{R}} f(\bar{u})(1 - \bar{u}) + \int\int_{\mathbb{R}^2} J(x - y)(\bar{u}(x) - \bar{u}(y))^2 dxdy \geq 0. \tag{1.128}$$

Thus

$$\int_{\mathbb{R}} f(\bar{u})(1 - \bar{u}) = 0 \tag{1.129}$$

and

$$\int\int_{\mathbb{R}^2} J(x - y)(\bar{u}(x) - \bar{u}(y))^2 dxdy = 0. \tag{1.130}$$

It follows from (1.130) that $\bar{u}$ must satisfy $\bar{u}(x) = \bar{u}(y)$ for almost every couple $(x, y) \in \mathbb{R}^2, \;\; x \neq y$. But this contradicts (1.125) and (1.126) and ends the proof.

□

Now we derive an upper bound for c_n. This will be done with the aid of a non linear comparison principle and comparison function denoted by w.

Lemma 1.6.1. *Non-Linear comparison principle*
Let u and v be two smooth bounded and increasing functions, $(u, v \in C^{1,\alpha})$, and let Ω be a compact subset of $\mathbb{R}$. If u and v satisfy the following :

$$J \star u - u - u' + f(u) \geq 0 \;\; on \;\; \mathbb{R}, \tag{1.131}$$
$$J \star v - v - v' + f(v) \leq 0 \;\; on \;\; \mathbb{R}, \tag{1.132}$$
$$v(x) - u(x) \geq_{\not\equiv} 0 \quad on \quad \mathbb{R} - \Omega, \tag{1.133}$$

then $v(x) - u(x) > 0$ on $\mathbb{R}$.

The derivation of the upper bound is organized as follows. First we prove the non-linear comparison principle. We then establish the following lemma.

Lemma 1.6.2.
There exists a positive constant κ such that for all n, $\kappa \geq c_n > 0$.

Proof of the comparison principle :

First, observe that if $\lim_{x\to-\infty} v(x) \geq \lim_{x\to+\infty} u(x)$ there is nothing to prove. So, assume that

$$\lim_{x\to-\infty} v(x) < \lim_{x\to+\infty} u(x). \tag{1.134}$$

Choose $\tau > 0$ such that $w_\tau = v(x+\tau) - u(x) \geq 0$ for all $x \in \mathbb{R}$. Such τ exists according to (1.133) and the assumption that u and v are increasing functions.
We define $\tau_0 = \inf\{\tau | v(x+\tau) - u(x) \geq 0 \;\; on \;\; \mathbb{R}\}$. We claim :

Claim 1.6.2. $\tau_0 \leq 0$.

Observe that by proving this claim we prove the comparison lemma.

Proof :

If not, τ_0 is positive and by its definition and according to (1.133) and the assumption that u and v are increasing, there exists x_0 such that for all $x \in \mathbb{R}$

$$w_{\tau_0}(x) \geq w_{\tau_0}(x_0) = 0. \tag{1.135}$$

Therefore x_0 is a global minimum of w_{τ_0}. Furthermore at x_0 we have $u(x_0) = v(x_0 + \tau_0)$. Moreover $x_0 \in \Omega$ because $w_\tau > v(x) - u(x) \geq 0$ for every $\tau > 0$ and $x \in \mathbb{R} - \Omega$.
In addition, at x_0, w_{τ_0} satisfies :

$$(J \star w_{\tau_0} - w_{\tau_0} - w'_{\tau_0})(x_0) \geq 0 \tag{1.136}$$
$$(J \star w_{\tau_0} - w_{\tau_0} - w'_{\tau_0})(x_0) \leq f(u(x_0)) - f(v(x_0 + \tau_0)) = 0. \tag{1.137}$$

Thus $(J \star w_{\tau_0} - w_{\tau_0})(x_0) = 0$, which implies that $w_{\tau_0} \equiv 0$ and contradicts $w_{\tau_0} > v(x) - u(x) \geq 0$ for every $x \in \mathbb{R} - \Omega$. This ends the proof of Claim 1.6.2 and of Lemma 1.6.1.

□

Now we present the proof of Lemma 1.6.2

Proof :

Once again we use a contraction argument. Suppose that $(c_n)_{n\in\mathbb{N}}$ is not bounded, so $c_n \to +\infty$.

Fix $\delta > \delta_0$ and $\lambda < \lambda_1$ where δ_0 and λ_1 are defined in the previous section. This is possible according to Remarks 1.5.1,1.5.2 and 1.5.3 on the monotony of δ_0 and λ_1 with respect to the speed c. Now we construct our comparison function w for an adequate g. Let w be a positive increasing function, such that w satisfies the following :

- $w(x) = e^{\lambda x}$ for $x \in (-\infty, -N]$,
- $w(x) \leq e^{\lambda x}$ on $\mathbb{R}$,
- $w(x) = 1 - e^{-\delta x}$ for $x \in [N, +\infty)$,
- $w(0) = \frac{1}{2}$.

for given positive λ and δ such that (H^*) holds. Let $x_0 = e^{-\lambda N}$ and $x_1 = 1 - e^{-\delta N}$. We have $0 < x_0 < x_1 < 1$. We construct a positive function g defined on $(0, 1)$ which satisfies $g(w) \geq f(w)$. Since f is smooth near 0 and 1 we may achieve for κ large enough, say $\kappa \geq \kappa_0$

$$g(s) = \lambda(\kappa - \lambda)s \geq f(s) \;\; for \;\; s \in [0, x_0] \tag{1.138}$$

and

$$g(w) = \delta(\kappa - \delta)(1 - s) \geq f(s) \;\; for \;\; s \in [x_1, 1]. \tag{1.139}$$

Therefore $g(s) \geq f(s)$ for s in [0,1], with g defined by :

$$g = \begin{cases} \lambda(\kappa_0 - \lambda)s & for \;\; 0 \leq s \leq x_0 \\ h(x) & for \;\; x_0 < s < x_1 \\ \delta(\kappa_0 - \delta)(1 - s) & for \;\; x_1 \leq s \leq 1 \end{cases} \tag{1.140}$$

where h is any smooth positive function greater than f on $[x_0, x_1]$ so that g is smooth i.e. $(g \in C^2)$.

Remark 1.6.1. *The definition of g depends only on the nonlinearity f*

Since g is well defined, we see that for large κ, w is a supersolution.
Observe that : according to (1.140) for $x \leq -N$ i.e. for $w \leq e^{-\lambda N}$, we have

$$\begin{aligned} J \star w - w - \kappa w' + g(w) &= J \star w - e^{\lambda x} - \lambda\kappa\, e^{\lambda x} + \lambda(\kappa_0 - \lambda)e^{\lambda x}, \\ &\leq J \star e^{\lambda x} - e^{\lambda x} - \lambda\kappa e^{\lambda x} + \lambda(\kappa_0 - \lambda)e^{\lambda x}, \\ &\leq e^{\lambda x}[\int_{\mathbb{R}} J(z)e^{\lambda z}dz - 1 - \lambda(\kappa - \kappa_0) - \lambda^2], \\ &\leq 0, \end{aligned}$$

for κ large enough. Furthermore for large κ and $w \geq 1 - e^{-\delta N}$,

$$\begin{aligned} J \star w - w - \kappa w' + g(w) &= J \star w - (1 - e^{-\delta x}) - \delta\kappa\, e^{-\delta x} + \delta(\kappa_0 - \delta)e^{-\delta x}, \\ &\leq 1 - 1 + e^{-\delta x} - \delta\kappa e^{-\delta x} + \delta(\kappa_0 - \delta)e^{-\delta x}, \\ &\leq e^{-\delta x}[1 - \delta(\kappa - \kappa_0) - \delta^2], \\ &\leq 0. \end{aligned}$$

Thus by taking κ large enough, we achieve

$$g(w) \geq f(w) \qquad \text{and} \qquad J \star w - w - \kappa w' + g(w) \leq 0$$
$$\text{for} \quad 0 \leq w \leq e^{-\lambda N} \quad \text{and} \quad w \geq 1 - e^{-\delta N}.$$

For the remaining values of w, i.e. for $x \in [-N, N]$, $w' > 0$ and we may therefore increase κ further if necessary to achieve

$$J \star w - w - \kappa w' + g(w) \leq 0 \ \ on \ \ \mathbb{R}. \tag{1.141}$$

Having now chosen κ such that $g \geq f$, we observe that by Proposition 1.5.1 and our choice of w :

$$w > u_n \quad for \ |x| \ large, \ \ say \ for \ |x| > a_n. \tag{1.142}$$

Since $c_n \to +\infty$, there exists n_0 such that for every $n \geq n_0$, $c_n > \kappa$. Fix $p \geq n_0$. Since $c_p > \kappa$ and $g \geq f$ we have :

$$J \star w - w - c_n w' + f_n(w) < 0 \ \ on \ \ \mathbb{R}.$$

We now apply the non-linear comparison principle and conclude that $w > u_p$ on $\mathbb{R}$. But this contradicts the normalization $w(0) = u_p(0) = \frac{1}{2}$. The lemma is then proved.

□

We now return to our solution (c_n, u_n). Since the speed c_n is bounded, there exists c^* such that a subsequence of $(c_n)_{n\in\mathbb{N}}$ converge to c^*. We may then proceed as in Section 2. By the apriori estimates and Helly's theorem, there exists a non decreasing function $\bar{u}$ such that $u_n \to \bar{u}$ pointwise. From the apriori estimates obtained in Section 2, since we have a positive upper bound for the speed c_n, the sequence of functions $(u_n)_{n\in\mathbb{N}}$ is bounded in the $C^{2,\alpha}_{loc}$ topology. Therefore we can extract a sub-sequence of $(u_n)_{n\in\mathbb{N}}$ which converges (in $C^{2,\beta}_{loc}$) to a function u. Since the limit is unique, we have $\bar{u} = u$. Thus u is a solution of :

$$\begin{cases} J \star u - u - c^* u' + f(u) = 0 \quad on \ \ \mathbb{R} \\ u(0) = \frac{1}{2} \end{cases} \tag{1.143}$$

We have to verify that u satisfies the right boundary conditions. Since u is an increasing bounded function, u achieves finite limits at $\pm\infty$. Furthermore, $u'(\xi) \to 0$ and $(J \star u - u)(\xi) \to 0$ when $\xi \to \pm\infty$. Thus these limits must be zeros of the function f. Hence u satisfies

$$\begin{cases} J \star u - u - c^* u' + f(u) = 0 \qquad on \ \ \mathbb{R} \\ u(0) = \frac{1}{2} \\ u(-\infty) = 0 \quad and \quad u(+\infty) = 1 \end{cases} \tag{1.144}$$

and we have established the existence of a solution (u, c^*) of (1.120).

□

Chapitre 2

On a nonlocal reaction diffusion equation arising in population dynamics

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Sommaire

2.1 Introduction

In 1930, Fisher [35] suggested to model the spatial spread of a mutant in a given population by the following reaction-diffusion equation :

$$u_t - \Delta u = u(1-u), \tag{2.1}$$

where u represents the gene fraction of the mutant. Dispersion of the genetic characters is assumed to follow a diffusion law while the logistic term $u(1-u)$ takes into account the saturation of this dispersion process.

Since then, much attention has been drawn to reaction-diffusion equations, as they have proved to give a robust and accurate description of a wide variety of phenomena, ranging from combustion to bacterial growth, nerve propagation or epidemiology. We point the interested reader to [31, 48, 41] and their many references.

In this work, we consider a variant of (2.1) where diffusion is modeled by a convolution operator. Going back to the early work of Kolmogorov - Petrovskii- Piskounov (see [46]), dispersion of the gene fraction at point $y \in \mathbb{R}^n$ should affect the gene fraction at $x \in \mathbb{R}^n$ by a factor $J(x,y)u(y)dy$ where $J(x,\cdot)$ is a probability density. Restricting to a one-dimensional setting and assuming that such a diffusion process depends only on the distance between two niches of the population, we end up with the equation

$$u_t - (J \star u - u) = f(u), \tag{2.2}$$

where $J : \mathbb{R} \to \mathbb{R}$ is a nonnegative even function of mass one. More precisely, we assume in what follows that

$$J \in C^1(\mathbb{R}), \qquad J \geq 0, \qquad J(x) = J(-x) \qquad \text{and} \qquad \int_{\mathbb{R}} J = 1, \tag{H1}$$

$$\int_{\mathbb{R}} J(z)|z|dz < +\infty. \tag{H2}$$

For some of our results, we need a stronger assumption on the decay of J at infinity. Namely we suppose that J decays faster than any exponential in the following sense :

$$\forall \lambda > 0, \ \int_{\mathbb{R}} J(z)e^{\lambda z}dz < +\infty. \tag{H3}$$

The nonlinearity f in (2.2) can be chosen more generally than in equation (2.1). In the literature, three types of nonlinearities appear, according to the underlined application : we always assume that $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$, $f'(1) < 0$ and

- we say that f is of bistable type if there exists $\theta \in (0,1)$ such that
$$f < 0 \text{ in } (0,\theta), \quad f(\theta) = 0 \quad \text{and} \quad f > 0 \text{ in } (\theta, 1)$$
- f is of ignition type if there exists $\theta \in (0,1)$ such that
$$f|_{[0,\theta]} \equiv 0, \qquad f|_{(\theta,1)} > 0 \qquad \text{and } f(1) = 0.$$
- f is of monostable type if
$$f > 0 \text{ in } (0,1)$$

Observe that equation (2.1) falls in the monostable case. In the present article, we will focus on the monostable nonlinearity. (2.1) can also be seen as a first order approximation of (2.2). Indeed if any given niche of the species is assumed to interact mostly with close-by neighbours, the diffusion term is of the form $J_\epsilon(x) := \frac{1}{\epsilon}J(\frac{1}{\epsilon}x)$, where J is compactly supported and $\epsilon > 0$ is small. We then have

$$\begin{aligned} J_\epsilon \star u - u &= \frac{1}{\epsilon}\int J(\frac{1}{\epsilon}y)(u(x-y) - u(x))\, dy = \int J(z)(u(x-\epsilon z) - u(x))\, dz \\ &= -\epsilon \int J(z)u'(x)z\, dz + \epsilon^2 \int z^2 J(z)u''(x)\, dz + o(\epsilon^2) = c\epsilon^2 u''(x) + o(\epsilon^2), \end{aligned}$$

where we used the fact that J is even in the last equality.

We also observe that equation (2.2) can be related to the class of problems studied in [59, 60]. However, our approach differs in at least two different ways : firstly, from the technical point of view, inverting the operator $u \to u_t - (J * u - u)$ in any reasonable space yields no *a priori* regularity property on the solution u and the compactness assumptions made in [60] no longer hold in our case.

Secondly, whereas the author favored discrete models over continuous ones to describe the dynamics of certain populations, we remain interested in the latter. In particular, we have in mind the following application to adaptative dynamics : in [37], the authors study a probabilistic model describing the microscopic behavior of the evolution of genetic traits in a population subject to mutation and selection. Averaging over a large number of individuals in the initial state, they then derive in the limit a deterministic equation, a special case of which can be written as

$$\partial_t u = J * u - u + (1 - K * u)u, \tag{2.3}$$

where $J(x)$ is a kernel taking into account mutation about trait x and $K(x)$ is a competition kernel, measuring the "intensity" of the interaction between x and y. Equation (2.2) can therefore be seen as a first step towards solving (2.3).

The aim of this article is the study of so-called travelling-wave solutions of equation (2.2) i.e. solutions of the form

$$u(x,t) = U(x + ct),$$

where $c \in \mathbb{R}$ is called the wave speed and U the wave profile, which is required to solve the equation

$$\begin{cases} J \star U - U - cU' + f(U) = 0 & \text{in } \mathbb{R} \\ U(x) \to 0 & \text{as } x \to -\infty \\ U(x) \to 1 & \text{as } x \to +\infty. \end{cases} \tag{2.4}$$

Such solutions are expected to give the asymptotic behavior in large time for solutions of (2.2) with say compactly supported initial data : in the Fisher equation, this is equivalent to saying that the mutant propagates (after some time) at constant speed and along the profile U. It is therefore of interest to prove existence of such solutions.

The first results in this direction are due to Schumacher [53], who considered the monostable nonlinearity, under the extra assumption that $f(r) \geq h_0 r - K r^{1+\alpha}$, for some $h_0, K, \alpha > 0$ and all $r \in [0,1]$. In this case, his results imply existence of travelling waves with arbitrary speed $c \geq c^*$, where c^* is the smallest $c \in \mathbb{R}$ such that the $\rho_c : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho_c(\lambda) = -\lambda c + \int J(z) e^{\lambda z}\, dz - 1 + f'(0),$$

vanishes for some $\lambda \in \mathbb{R}$.

Furthermore if $c > c^*$ and under some extra assumptions on f, he shows that the profile u of the associated travelling wave is unique up to translation.

Recently, Carr and Chmaj [15] completed the work of Schumacher. For the "KPP" nonlinearity (i.e. if f is monostable and $f(r) \leq f'(0) r$ for all $r \in [0,1]$) and if J has compact support, they show that the above uniqueness result can be extended to $c = c^*$.

Concerning the bistable nonlinearity, Bates-Fife-Ren-Wang [5] and Chen [17] showed that in this case sthere exists an increasing travelling wave U with speed c solving (2.4). Furthermore if V is another nondecreasing travelling wave with speed c' then $c = c'$ and $V(x) = U(x+\tau)$ for some $\tau \in \mathbb{R}$.

Coville [20] then looked at the case of ignition nonlinearities and proved again the existence and uniqueness (up to translation) of an increasing travelling wave (U, c). He also obtained the existence of at least one travelling-wave solution in the monostable case.

Our first theorem extends some of the afore-mentioned results of Schumacher to the general monostable case :

Theorem 2.1.1.
Assume (H1) and (H3) hold and assume that f is of monostable type. Then there exists a constant $c^* > 0$ (called the minimal speed of the travelling wave) such that for all $c \geq c^*$, there exists an increasing solution $U \in C^1(\mathbb{R})$ of (2.4) *while no nondecreasing travelling wave of speed $c < c^*$ exists.*

Our second result shows that the behavior of the travelling front u near $\pm\infty$ is governed by exponentials. Namely we have the following proposition

Proposition 2.1.1.
Assume (H1) and (H3) hold. Then given any travelling-wave solution (u, c) of (2.4) *with f monostable, the following assertions hold :*

1. *There exists positive constants A,B, λ_0 and δ_0 such that*

$$Be^{-\delta_0 y} \leq 1 - u(y) \leq Ae^{-\lambda_0 y} \quad as \;\; y \;\; \rightarrow +\infty$$

2. *If $f'(0) > 0$ then there exists two positive constants K and λ_1 such that*

$$u(y) \leq Ke^{\lambda_1 y} \quad as \;\; y \;\; \rightarrow -\infty$$

The proof of Theorem 2.1.1 essentially uses elementary analysis and is based on the study of two auxiliary problems and the construction of adequate super and subsolutions. Let us briefly explain the idea of this proof. We break it down in to three steps.

We first start by showing existence and uniqueness of a solution for the following auxiliary problem

$$\begin{cases} \epsilon u'' + \int_r^{+\infty} J(x-y)u(y)dy - u - cu' + f(u) = -h_r(x) & \text{for } x \in (r, +\infty) \\ \quad u(r) = \theta & \\ \quad u \rightarrow 1 \qquad x \rightarrow +\infty, & \end{cases} \tag{2.5}$$

with $r \in \mathbb{R}$, $\epsilon > 0$, $\theta \in (0,1)$, $h_r(x) = \theta \int_{-\infty}^r J(x-y)dy$ and $c \in (\kappa, \infty)$ for some $\kappa > 0$. The existence is obtained via an iterative scheme using a comparison principle and good sub and supersolutions. In the second step, with a standard limiting procedure, we prove Theorem 2.1.1 for the following problem

$$\begin{cases} \epsilon u'' + J \star u - u - cu' + f(u) = 0 & in \;\; \mathbb{R} \\ u \rightarrow 0 & x \rightarrow -\infty \\ u \rightarrow 1 & x \rightarrow +\infty, \end{cases} \tag{2.6}$$

for $\epsilon > 0$. Finally, in the last step we send $\epsilon \to 0$ and extract converging subsequences. Though elementary in nature, the proof of this result requires a number of lemmas which we list and prove in Section 2.2. We construct sub and supersolutions in Section 2.3. Section 2.4 deals with existence and uniqueness of solutions of (2.5). Then we construct in Section 2.5 a solution of (2.6) for every $c \geq c^*(\epsilon)$. Finally existence of solutions for all $c \geq c^*$ is given in Section 2.6. The last section is devoted to the proof of Proposition 2.1.1.

2.2 Linear theory

We start this section with a maximum principle for integro-differential operators defined on the semi infinite interval $\Omega = (r, +\infty)$ of the kind :

$$L_r u := \epsilon u'' + \int_r^{\infty} J(x-y)u(y)dy - u + b(x)u' + c(x)u, \tag{2.7}$$

where $\epsilon \geq 0$, $r < 0$, $\int_{\mathbb{R}} J = 1$, $\overset{\circ}{supp}(J)\ CAP\ :\Omega \not\equiv \emptyset$ and $c(x) \leq 0$, $b(x)$ and $c(x)$ are bounded functions on $\mathbb{R}$. We shall also consider operators of the form $L_r + h_r(x)$, where h_r is a function defined for $x \in \Omega$ by

$$h_r(x) = \theta \int_{-\infty}^{r} J(x-y)dy, \tag{2.8}$$

where $\theta \in \mathbb{R}$ is some given constant.

We prove a strong maximum principle that applies to smooth functions :

Theorem 2.2.1. *Strong Maximum Principle*
Let $u \in C^2(\Omega) CAP\ :C^0(\bar{\Omega})$ satisfy

$$L_r u \geq 0 \; on \quad \Omega \qquad (resp.\; L_r u \leq 0 \; \; on \; \; \Omega), \tag{2.9}$$

then u may not achieve a positive maximum (resp. negative minimum) without being constant.

Similarly we have

Theorem 2.2.2. *Strong Maximum Principle*
Let $u \in C^2(\Omega) CAP\ :C^0(\bar{\Omega})$ satisfy

$$\begin{cases} L_r u + h_r(x) \geq 0 \quad on \;\; \Omega & (resp.\;\; L_r u + h_r(x) \leq 0 \;\; on \;\; \Omega) \\ u(r) = \theta \leq u(x) \quad on \;\; \Omega & (resp.\;\; u(x) \geq \theta \;\; on \;\; \Omega) \end{cases}$$

Then u may not achieve a positive maximum (resp. negative minimum) without being constant.

As a straightforward consequence, we have the following practical corollary :

Corollary 2.2.1.
Let $u \in C^2(\Omega) CAP\ :C^0(\bar{\Omega})$ satisfy

$$\begin{cases} L_r u + h_r(x) \geq 0 & on \;\; \Omega \\ u(r) = \theta \leq 0 & \\ \limsup_{x \to +\infty} u(x) \leq 0. & \end{cases}$$

Then

– *Either* $u < 0$ *in* Ω
– *Or* $u \equiv 0$.

Remarks 2.2.1.
Similarly if $L_r u + h_r(x) \leq 0$, $u(r) = \theta \geq 0$ *and* $\liminf_{x \to +\infty} u(x) \geq 0$ *then* u *is either positive or identically* 0.

Remark 2.2.1. *A result such as Corollary 2.2.1 can also be directly derived for the operator* L_r.

Proof of Theorem 2.2.1 :

We argue by contradiction and assume that u is nonconstant and achieves a positive maximum at some point $x_0 \in \Omega$. Since $\int_{\mathbb{R}} J(z)dz = 1$ we can rewrite (2.9) as

$$L_r u = \epsilon u'' + \int_r^{+\infty} J(x-y)[u(y) - u(x)]dy + b(x)u' + \bar{c}(x)u, \tag{2.10}$$

with $\bar{c}(x) = c(x) - \int_{-\infty}^{r} J(x-y)dy$.

At the positive point of maximum x_0 we then have on the one hand

$$\epsilon u''(x_0) \leq 0, \quad \int_r^{+\infty} J(x_0 - y)[u(y) - u(x_0)]dy \leq 0 \quad \text{and} \quad \bar{c}(x_0)u(x_0) \leq 0. \tag{2.11}$$

On the other hand by our assumption

$$\epsilon u''(x_0) + \int_r^{+\infty} J(x_0 - y)[u(y) - u(x_0)]dy + \bar{c}(x_0)u(x_0) \geq 0 \tag{2.12}$$

(2.11) and (2.12) imply that $\epsilon u''(x_0) = \bar{c}(x_0)u(x_0) = 0$ and

$$\int_r^{\infty} J(x_0 - y)[u(y) - u(x_0)]\, dy = 0. \tag{2.13}$$

By assumption, J is a continuous nonnegative function with $\overset{\circ}{supp}(J)\ CAP\ :\Omega \not\equiv \emptyset$. Let $0 < a < b$ be some constants such that $[-b, -a] \cup [a, b] \subset supp(J)$ and $[a, b] \subset \Omega$. Theses constants exist since J is an even function and $r < 0$. Thus we deduce from (2.13) that $u(y) = u(x_0)$ for all y in the set $(x_0 + supp(J))CAP\ :\Omega$ and therefore in the set $(x_0 + [-b, -a] \cup [a, b])CAP\ :\Omega$. Next we show that $u = u(x_0)$ for $y \in [x_0, +\infty)$. Let $z \in x_0 + [a, b]$, observe that at the point z, u achieves a positive maximum since $u(z) = u(x_0)$. We may thus argue as above and conclude that

$$u(y) = u(x_0) \text{ for all } y \in (x_0 + [-b, -a] \cup [-(b-a), b-a] \cup [a, b] \cup [a+b, 2b])CAP\ :\Omega. \tag{2.14}$$

Since $r < 0$, we have $u(y) = u(x_0)$ for all $y \in x_0 + [0, b-a]$. Now repeat all the computations with $z = x_0 + b - a$ instead of x_0 to obtain that $u(y) = u(x_0)$ for all $y \in x_0 + [0, 2(b-a)]$. Therefore by repeating infinitely many times this process we obtain $u = u(x_0)$ for $y \in [x_0, +\infty)$. Next we show that $u = u(x_0)$ for $y \in [r, x_0)$. From (2.14), we also have $u(y) = u(x_0)$ for all $y \in (x_0 + [-(b-a), 0])CAP\ :\Omega$. If $x_0 - (b-a) \leq r$ then we are done. Otherwise we can again repeat the computations with $z = x_0 - (b-a)$ and obtain $u(y) = u(x_0)$ for all $y \in (x_0 + [-2(b-a), 0])CAP\ :\Omega$. We then obtain the result by doing so infinitely many times. Eventually we end up with $u(y) = u(x_0)$ for all y in Ω, which is a contradiction.

$\square$

Remark 2.2.2. *Observe that if $\Omega \subset supp(J)$, the above proof is simplified and we obtain directly that $\tilde{u}(y) = cte$ without using the construction with the compact set $[a, b]$.*

Proof of Theorem 2.2.2

Define

$$\tilde{u}(x) := \begin{cases} u(x) & on \;\; \Omega \\ u(r) = \theta & on \;\; \mathbb{R} \setminus \Omega \end{cases}$$

and observe that we can rewrite equation (2.7) as

$$\begin{cases} \tilde{L}_r \tilde{u} \geq 0 & in \;\; \Omega \\ \tilde{u}(x) \geq \theta & on \;\; \Omega \end{cases}$$

where $\tilde{L}_r \tilde{u} = \epsilon \tilde{u}'' + J \star \tilde{u} - \tilde{u} + b(x)\tilde{u}' + c(x)\tilde{u}$. We now argue by contradiction and assume that $\tilde{u}$ achieves a positive maximum at some point $x_0 \in \Omega$ and is nonconstant. Since $\int_{\mathbb{R}} J(z)dz = 1$ we have :

$$\tilde{L}_r \tilde{u} = \epsilon \tilde{u}'' + \int_{-\infty}^{+\infty} J(x-y)[\tilde{u}(y) - \tilde{u}(x)]dy + b(x)\tilde{u}' + c(x)\tilde{u}. \tag{2.15}$$

Since $u(x) \geq \theta$ on Ω we have $u(x_0) > \theta$ and u achieves a global positive maximum at x_0. Arguing as in the proof of Theorem 2.2.1 we end up with $u \equiv u(x_0)$ on $\bar{\Omega}$ which is a contradiction since $u(r) = u(x_0) > \theta = u(r)$.

□

Next, we provide an elementary lemma to construct solutions to constant-coefficient Dirichlet problems of the form

$$\begin{aligned} L_r u = f \quad & on \;\; \Omega = (r, +\infty) \\ u(r) = 0 \quad & \\ u(x) \to 0 \quad & \text{as} \;\; x \to +\infty \end{aligned}$$

Lemma 2.2.1. *Let $f \in C_0(\Omega) CAP : L^2(\Omega)$ and L_r defined by*

$$L_r v = \epsilon v'' + \int_r^{+\infty} J(x-y)v(y)dy - v - cv' - dv,$$

where $\epsilon > 0$, $c, d \in \mathbb{R}$, $d > 0$. Then there exists a unique solution $v \in C_0(\mathbb{R}) CAP : L^2(\Omega)$ (additionally $v \in C^2(\Omega)$) of

$$\begin{cases} L_r v = f & in \;\; \Omega \\ v(r) = 0 & \\ v \to 0 & x \to +\infty \end{cases} \tag{2.16}$$

Proof

Uniqueness follows from the maximum principle. Let $X = H_0^1(\Omega)$ and define the following bilinear form $\mathcal{A}(u, v)$ for $u, v \in X$ by

$$\mathcal{A}(u,v) = \epsilon \int_\Omega u'v' + \frac{1}{2} \int_\Omega \int_\Omega J(x-y)(u(y) - u(x))(v(y) - v(x))dydx + c \int_\Omega u'v + d \int_\Omega uv$$

We will show that $\mathcal{A}$ is coercive and continuous in X. Existence will then be given by the Lax-Milgram Lemma. Observe that since $\frac{1}{2}\int_\Omega\int_\Omega J(x-y)(u(y)-u(x))^2dydx \geq 0$ we have

$$\mathcal{A}(u,u) \geq \epsilon\int_\Omega (u')^2 + c\int_\Omega u'u + d\int_\Omega u^2 = \epsilon\int_\Omega (u')^2 + d\int_\Omega u^2$$

Thus $\mathcal{A}$ is coercive in X. It remains to prove the continuity of $\mathcal{A}$. Observe that by a density argument, it is sufficient to prove the continuity for smooth functions. Let ϕ and ψ be two smooth functions with compact support in Ω.

$$\frac{1}{2}\int_\Omega\int_\Omega J(x-y)(\phi(y)-\phi(x))(\psi(y)-\psi(x))dydx \leq \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)|\phi(y)-\phi(x)||\psi(y)-\psi(x)|dydx$$

From the Fundamental Theorem of Calculus and Cauchy-Schwartz inequality we have :

$$\begin{aligned}\int_{\mathbb{R}^2} J(x-y)|\phi(y)-\phi(x)||\psi(y)-\psi(x)|dydx &\leq \int_{\mathbb{R}^2}\int_0^1\int_0^1 J(z)z^2|\phi'(x+tz)||\psi'(x+sz)|dzdxdtds\\ &\leq \int_{\mathbb{R}}\int_{[0,1]^2} J(z)z^2\int_{\mathbb{R}}|\phi'(h)||\psi'(h+(s-t)z)|dhdsdzdt\\ &\leq \int_{\mathbb{R}}\int_{[0,1]^2} J(z)z^2\ dzdtds\|\phi'\|_{L^2(\mathbb{R})}\|\psi'\|_{L^2(\mathbb{R})}\\ &\leq (\int_{\mathbb{R}} J(z)z^2\ dz)\|\phi'\|_{L^2(\mathbb{R})}\|\psi'\|_{L^2(\mathbb{R})}\end{aligned}$$

which shows the continuity of $\mathcal{A}$.

□

2.3 Existence of sub and supersolutions

In this section, we construct various nonnegative sub and supersolutions. We start with the construction of a supersolution of the following integro-differential equation :

$$\begin{cases} \epsilon u'' + J\star u - u - cu' + f(u) = 0 & in \ \ \mathbb{R} \\ u \to 0 & x \to -\infty \\ u \to 1 & x \to +\infty, \end{cases} \tag{2.17}$$

where ϵ is a fixed nonnegative parameter. Using the exponential integrability assumption on J, we are able to construct a supersolution for a certain positive speed $c = \kappa$. We start with the construction of the supersolution.

Let N be a large positive constant and w be a positive increasing function satisfying

- $w(x) = e^{\lambda x}$ for $x \in (-\infty, -N]$,
- $w(x) \leq e^{\lambda x}$ on $\mathbb{R}$,
- $w(x) = 1 - e^{-\delta x}$ for $x \in [N, +\infty)$,
- $w(0) = \frac{1}{2}$.

for given positive λ and δ. Let $x_0 = e^{-\lambda N}$ and $x_1 = 1 - e^{-\delta N}$. We have $0 < x_0 < x_1 < 1$. We now construct a positive function g defined on $(0,1)$ which satisfies $g(w) \geq f(w)$. Since f is smooth near 0 and 1, we have for κ large enough, say $\kappa \geq \kappa_0$,

$$\lambda(\kappa-\lambda)s \geq f(s) \ \ for \ \ s \in [0, x_0], \tag{2.18}$$

and

$$\delta(\kappa-\delta)(1-s) \geq f(s) \;\; for \;\; s \in [x_1, 1]. \tag{2.19}$$

Therefore we can achieve $g(s) \geq f(s)$ for s in [0,1], with g defined by :

$$g(s) = \begin{cases} \lambda(\kappa_0-\lambda)s & for \;\; 0 \leq s \leq x_0 \\ l(s) & for \;\; x_0 < s < x_1 \\ \delta(\kappa_0-\delta)(1-s) & for \;\; x_1 \leq s \leq 1 \end{cases} \tag{2.20}$$

where l is any smooth positive function greater than f on $[x_0, x_1]$ such that g is of class C^1.

Remark 2.3.1. *The definition of g depends only on the nonlinearity f.*

Since we have a well-defined function g, we can see that for large κ, w is a supersolution. According to (2.20), for $x \leq -N$ i.e. for $w \leq e^{-\lambda N}$, we have

$$\begin{aligned} \epsilon w'' + J\star w - w - \kappa w' + g(w) &= \epsilon\lambda^2 e^{\lambda x} + J\star w - e^{\lambda x} - \lambda\kappa\, e^{\lambda x} + \lambda(\kappa_0-\lambda)e^{\lambda x} \\ &\leq \epsilon\lambda^2 e^{\lambda x} + J\star e^{\lambda x} - e^{\lambda x} - \lambda\kappa e^{\lambda x} + \lambda(\kappa_0-\lambda)e^{\lambda x} \\ &\leq e^{\lambda x}[\int_{\mathbb{R}} J(z)e^{\lambda z}dz - 1 - \lambda(\kappa-\kappa_0) - \lambda^2(1-\epsilon)] \\ &\leq 0 \end{aligned}$$

for κ large enough, say $\kappa \geq \kappa_1 = \frac{\int_{\mathbb{R}} J(z)e^{\lambda z}dz - 1 + \lambda\kappa_0 - \lambda^2(1-\epsilon)}{\lambda}$.
Furthermore for $w \geq 1 - e^{-\delta N}$ we have,

$$\begin{aligned} \epsilon w'' + \; J\star w - w - \kappa w' + g(w) &= \epsilon\delta^2 e^{-\delta x} + J\star w - (1-e^{-\delta x}) - \delta\kappa\, e^{-\delta x} + \delta(\kappa_0-\delta)e^{-\delta x} \\ &\leq \epsilon\delta^2 e^{-\delta x} + 1 - 1 + e^{-\delta x} - \delta\kappa e^{-\delta x} + \delta(\kappa_0-\delta)e^{-\delta x} \\ &\leq e^{-\delta x}[1 - \delta(\kappa-\kappa_0) - \delta^2(1-\epsilon)] \\ &\leq 0 \end{aligned}$$

for κ large enough, say $\kappa \geq \kappa_2 = \frac{1+\delta\kappa_0-\delta^2(1-\epsilon)}{\delta}$. Thus by taking $\kappa \geq \sup\{\kappa_1, \kappa_2\}$, we achieve

$$g(w) \geq f(w) \qquad \text{and} \qquad J\star w - w - \kappa w' + g(w) \leq 0$$
$$\text{for} \quad 0 \leq w \leq \; e^{-\lambda N} \quad \text{and} \quad w \geq 1 - e^{-\delta N}.$$

For the remaining values of w, i.e. for $x \in [-N, N]$, $w' > 0$ and we may therefore increase κ further if necessary, to achieve

$$\epsilon w'' + \; J\star w - w - \kappa w' + g(w) \leq 0 \;\; on \;\; \mathbb{R} \tag{2.21}$$

Remark 2.3.2. *An easy consequence of this construction is that for $\kappa(\epsilon) := \sup\{\kappa_1, \kappa_2, \kappa_3\}$ where $\kappa_3 = \sup_{x\in[-N,N]}\{\frac{\epsilon|w''|+|J\star w - w| + g(w)}{w'}\}$, w is a supersolution of* (2.17) *and $\kappa(\epsilon)$ is a nondecreasing function of ϵ.*

Observe that given any $r \in \mathbb{R}$ and $\theta \in (0,1)$, some translation of w is also a supersolution of the following problem :

$$\begin{cases} L_r u + f(u) = 0 & \text{for } x \in (r, +\infty) \\ u(r) = \theta & \\ u(x) \to 1 & \text{as} \quad x \to +\infty, \end{cases} \tag{2.22}$$

where

$$L_r u = \epsilon u'' - cu' + \int_r^\infty J(x-u)u(y)\, dy - u. \tag{2.23}$$

Next we show that some constants are respectively super and subsolution of the following problem

$$\begin{cases} L_r u + h_r(x) + f(u) = 0 & \text{for } x \in (r, +\infty) \\ u(r) = \theta & \\ u \to 1 & x \to +\infty, \end{cases} \tag{2.24}$$

with $h_r(x)$ defined by (2.8) A simple computation of $L_r(\theta) + h_r(x)$ and $L_r(1) + h_r(x)$ yields

$$L_r(\theta) + h_r(x) + f(\theta) = \int_r^{+\infty} J(x-y)\theta dy - \theta + \theta \int_{-\infty}^r J(x-y)dy + f(\theta) \geq 0,$$

$$L_r(1)+h_r(x)+f(1) = \int_r^{+\infty} J(x-y)dy - 1 + \theta \int_{-\infty}^r J(x-y)dy = (\theta-1)\int_{-\infty}^r J(x-y)dy + f(1) \leq 0,$$

so that θ and 1 are respectively a sub and a supersolution of (2.24).

2.4 Construction of a solution of (2.25)

In this section, we will show that for the speed $c = \kappa$, $\epsilon > 0$ and for any $\theta \in (0,1)$ there exists an increasing solution u_r of Problem (2.25) below. Moreover this solution is unique.

$$\begin{cases} L_r u + h_r(x) + f(u) = 0 & \text{for } x \in (r, +\infty) \\ u(r) = \theta & \\ u \to 1 \quad x \to +\infty, \end{cases} \tag{2.25}$$

with $L_r u$ defined by (2.23) and $h_r(x)$ by (2.8). For the uniqueness proof see the appendix of the thesis. The existence of a solution is obtained via an iterative scheme using sub and super solutions.

2.4.1 Preliminaries

Fix $c = \kappa > 0$, $\epsilon > 0$ and $1 > \theta > 0$. Recall that the constants θ and 1 are respectively a sub and a supersolution of equation (2.25) (see Section 2.3). Let now $g \in C_c^\infty(\mathbb{R})$ be a nonnegative function with $\|g\|_{L^1(\mathbb{R})} = 1$ and $G(x) = \int_{-\infty}^x g(t)\, dt$. We can chose g such that $G(r) = \theta$. Now for $\lambda > 0$,$r \in \mathbb{R}$ define

$$\begin{array}{rccc} T_{\lambda,r}: & C_0(\Omega) CAP :L^2(\Omega) & \to & C_0(\Omega) CAP :L^2(\Omega) \\ & v & \mapsto & z, \end{array}$$

where z is the unique solution of

$$\begin{cases} L_r z - \lambda z = F(v,x) \qquad in \quad \Omega \\ z(r) = 0 \\ z(x) \to 0 \qquad x \to +\infty, \end{cases} \tag{2.26}$$

where $F(v,x) = -f(v+G) - \lambda v - L_r G - h_r(x)$. Now, using Lemma 2.2.1, to prove that z is well-defined, it is enough to show that $v \in L^2(\Omega) CAP :C_0(\Omega) \Longrightarrow F(v,x) \in L^2(\Omega) CAP :C_0(\Omega)$.

On the one hand since $G(x) = 1$ for $x >> 1$, it follows that $1 - G \in L^2(\Omega) CAP :C_0(\Omega)$. On the other hand given $v \in L^2(\Omega) CAP :C_0(\Omega)$, since $f(1) = 0$,

$$|f(v+G)| \le \|f'\|_\infty |v + G - 1| \in L^2(\Omega) \qquad \text{and} \qquad \lim_{+\infty} f(v+G) = 0,$$

so that $f(v+G) \in L^2(\Omega) CAP :C_0(\Omega)$. Clearly $G', G'' \in L^2(\Omega)$. Finally, the following lemma applied to $u = G$ shows that $\int_r^{+\infty} J(x-y)G(y)dy \;-\; G \;\in\; L^2(\Omega)$ and $h_r(x) \in\; L^2(\Omega)$, so we can conclude that z solving (2.26) is well-defined.

Lemma 2.4.1. *Let $u \in C^1(\Omega) CAP :L^\infty(\Omega)$. Then*

$$\| \int_r^{+\infty} J(x-y)u(y)dy - u\|_{L^2(\Omega)} \le C(\|u'\|_{L^2(\Omega)} + \|j\|_{L^2(\Omega)}).$$

with $j(x) = \int_{-\infty}^r J(x-y)\ dy \in L^2(\Omega)$

Remark 2.4.1. *The lemma also implies that $h_r(x) \in L^2(\Omega)$ since $\theta j(x) = h_r(x)$.*

Proof of the lemma

Using the fundamental theorem of calculus, we have that

$$\begin{aligned} \int_r^{+\infty} J(x-y)u(y)dy - u(x) &= \int_r^{+\infty} J(x-y)(u(y)-u(x))dy - u(x)\int_{-\infty}^r J(x-y)dy \\ &= \int_{r-x}^{+\infty} J(z)(u(x+z)-u(x))dz - u(x)\int_{-\infty}^{r-x} J(z)dz \\ &= \int_{r-x}^{+\infty} J(z)z\left(\int_0^1 u'(x+tz)\ dt\right)\ dz - u(x)j(x), \end{aligned}$$

with $j(x) \;=\; \int_{-\infty}^{r-x} J(z)dz$. By standard estimation and the Cauchy-Schwartz inequality, it follows that

$$\begin{aligned} |\int_r^{+\infty} J(x-y)u(y)dy - u(x)|^2 &\le 2\left[\left(\int_{r-x}^{+\infty} J(z)z\left(\int_0^1 u'(x+tz)\ dt\right)\ dz\right)^2 + u^2(x)j^2(x)\right] \\ &\le 2\left[\int_{r-x}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)dt\ dz \cdot \int_{r-x}^{+\infty} J(z)|z|\ dz + u^2 j^2\right] \\ &\le C\left[\int_{r-x}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)dt\ dz + u^2 j^2(x)\right] \end{aligned}$$

Define $\Gamma(x) := \int_{r-x}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)dt\ dy$. We then have

$$|\int_r^{+\infty} J(x-y)u(y)dy - u(x)|^2 \leq C\left[\Gamma(x) + j^2(x)\right]$$

We thus need to show that Γ is in $L^1(\Omega)$ and satisfies

$$\|\Gamma\|_{L^1(\Omega)} \leq C\|u'\|^2_{L^2(\Omega)}, \tag{2.27}$$

and that j is in $L^2(\Omega)$. The latter comes easily form the definition of j and the exponential decay of J. Namely, we can bound j from above in the following way :

$$0 \leq j(x) = \int_{-\infty}^{r-x} J(z)dz \leq e^{\delta(r-x)}\int_{\mathbb{R}} J(z)e^{-\delta z}dz \leq Ke^{\delta(r-x)} \in L^2(\Omega)$$

The main difficulty is to show that Γ is in $L^1(\Omega)$ and satisfies (2.27). We show (2.27) through a direct computation.
By definition of Γ and since all the integrands are positive, using Tonelli's Theorem, we have :

$$\begin{aligned}
\int_r^{+\infty}\Gamma(x)dx &= \int_0^1\int_r^{+\infty}\int_{r-x}^{+\infty} J(z)|z|(u')^2(x+tz)dzdxdt\\
&= \int_0^1\int_r^{+\infty}\int_{-\infty}^{+\infty} J(z)|z|(u')^2(x+tz)\chi_{\{z>r-x\}}dzdxdt\\
&= \int_0^1\int_{-\infty}^{+\infty} J(z)|z|\left(\int_r^{+\infty}(u')^2(x+tz)\chi_{\{x>r-z\}}dx\right)dzdt\\
&= \int_0^1\int_{-\infty}^{+\infty} J(z)|z|\left(\int_{r+tz}^{+\infty}(u')^2(s)\chi_{\{s>r-z+tz\}}ds\right)dzdt\\
&= \int_0^1\int_{-\infty}^{+\infty} J(z)|z|\left(\int_{-\infty}^{+\infty}(u')^2(s)\chi_{\{s>r-z+tz\}}\chi_{\{s>r+tz\}}ds\right)dzdt
\end{aligned}$$

Where χ_O is the characteristic function of the subset O.
Observe now that $\chi_{\{s>r-z+tz\}}\chi_{\{s>r+tz\}} \leq \chi_{\{s>r\}}$. Hence we end up with :

$$\begin{aligned}
\int_r^{+\infty}\Gamma(x)dx &\leq \int_0^1\int_{-\infty}^{+\infty} J(z)|z|\left(\int_{-\infty}^{+\infty}(u')^2(s)\chi_{\{s>r\}}ds\right)dzdt\\
&\leq \left(\int_{-\infty}^{+\infty} J(z)|z|dz\right)\|u'\|^2_{L^2(\Omega)},
\end{aligned}$$

which is the desired conclusion.

□

Remark 2.4.2. *Observe that $T_{\lambda,r}$ is still well define for any function G such that $G(r) = \theta$, $1-G \in L^2(\Omega)$ and $L_rG \in L^2(\Omega)$.*

2.4.2 Iteration procedure

We claim that there exists a sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ satisfying

$$u_0 = G \text{ and for } n \in \mathbb{N}\setminus\{0\},$$

$$\begin{cases} L_r u_{n+1} - \lambda u_{n+1} = -f(u_n) - \lambda u_n - h_r(x) & in \quad \Omega \\ u_{n+1}(r) = \theta \\ u_{n+1}(x) \to 1 \qquad x \to +\infty. \end{cases} \tag{2.28}$$

We proceed as follows. Using the substitution $v_n = u_n - G$, (2.28) reduces to

$$\begin{cases} L_r v_{n+1} - \lambda v_{n+1} = F(v_n, x) & in \quad \Omega \\ v_{n+1}(r) = 0 \\ v_{n+1}(x) \to 0 \qquad x \to +\infty, \end{cases} \tag{2.29}$$

where $F(v, x) = -f(v + G) - \lambda v - L_r G - h_r(x)$. Therefore we have $v_{n+1} = T_{\lambda,r} v_n$. Now, using the previous subsection and induction, to prove that v_n is well-defined, it is enough to show that $v_0 \in L^2(\Omega) CAP :C_0(\Omega)$ which is trivial since $v_0 = 0$.

2.4.3 Passing to the limit as $n \to \infty$

Recall that the constants θ and 1 are respectively a subsolution and a supersolution of (2.25).
It follows easily from induction and the Maximum Principle (Theorem 2.2.2) that for all $n \in \mathbb{N} \setminus \{0\}$,

$$\theta \le u_n \le 1. \tag{2.30}$$

Choosing $\lambda > 0$ so large that $-f - \lambda$ is nonincreasing, we prove next by induction that

$$x \to u_n(x) \text{ is a nondecreasing function.} \tag{2.31}$$

First define the following sequence of function :

$$\tilde{u}_n(x) := \begin{cases} \theta & if \;\; x \in \mathbb{R} \setminus \Omega \\ u_n(x) & if \;\; x \in \Omega. \end{cases}$$

We will prove that $(\tilde{u}_n)_n$ are nondecreasing functions, which implies (2.31). Observe that $\tilde{u}_{n+1}$ solves the following problem

$$\begin{cases} L_r \tilde{u}_{n+1} - \lambda \tilde{u}_{n+1} + \int_{-\infty}^{r} J(x-y) \tilde{u}_{n+1}(y) dy = -(f+\lambda)(\tilde{u}_n(x)) & in \quad \Omega \\ \tilde{u}_{n+1}(r) = \theta \\ \quad \tilde{u}_{n+1} \to 1 \qquad x \to +\infty, \end{cases} \tag{2.32}$$

which can be rewriten as

$$\begin{cases} \tilde{u}''_{n+1} - c\tilde{u}'_{n+1} + J \star \tilde{u}_{n+1} - \tilde{u}_{n+1} - \lambda \tilde{u}_{n+1} = -(f+\lambda)(\tilde{u}_n(x)) & in \quad \Omega \\ \tilde{u}_{n+1}(r) = \theta \\ \quad \tilde{u}_{n+1} \to 1 \qquad x \to +\infty. \end{cases} \tag{2.33}$$

For $n = 0$, we already know that $\tilde{u}_0$ is nondecreasing. Fix now $n \ge 1$ and assume that $\tilde{u}_{n-1}$ is nondecreasing. Also given any positive τ, let $w(x) = \tilde{u}_n(x + \tau) - \tilde{u}_n(x)$. It follows from Equation (2.33) and the assumption that $\tilde{u}_{n-1}$ and $f + \lambda$ are nondecreasing that

$$\epsilon w'' + J \star w - cw' - (1 + \lambda) w(x) \le 0 \qquad \text{in } \Omega, \tag{2.34}$$

$$w(x) \ge 0 \;\text{ for } x \in \mathbb{R} \setminus \Omega, \tag{2.35}$$

$$w(\infty) = 0, \tag{2.36}$$

whence by the Maximum Principle $w \geq 0$. In particular, $\tilde{u}_n(x+\tau) - \tilde{u}_n(x) \geq 0$ for any positive τ. This shows that $\tilde{u}_n$ is nondecreasing.

Using (2.30), (2.31) and Helly's lemma, it follows that a subsequence of $\{u_n\}$ converges pointwise to a nondecreasing function u satisfying

$$\theta \leq u \leq 1.$$

By the dominated convergence theorem, we have for all $x \in \Omega$

$$\int_r^{+\infty} J(x-y)u_n(y)dy - u_n(x) \to \int_r^{+\infty} J(x-y)u(y)dy - u(x), \qquad \text{as} \qquad n \to \infty.$$

Rewriting (2.28) as

$$\epsilon u''_{n+1} - cu'_{n+1} = u_{n+1} - \int_r^{+\infty} J(x-y)u_{n+1}(y)dy - \lambda(u_n - u_{n+1}) - f(u_n) - h_r(x), \quad (2.37)$$

observing that the right-hand side in the above equation is uniformly bounded and using elliptic regularity, we conclude that $\{u_n\}$ is bounded e.g. in $C^{1,\alpha}(\omega)$, where $\alpha \in (0,1)$ and ω is an arbitrary bounded open subset of Ω. Repeating the argument implies that $\{u_n\}$ is bounded in $C^{2,\alpha}(\omega)$. Hence $u \in C^2(\Omega)$ and we can pass to the limit in the equation to obtain that u satisfies

$$\epsilon u'' - cu' + \int_r^{+\infty} J(x-y)u(y)dy - u + f(u) + h_r(x) = 0 \qquad \text{in } \Omega. \qquad (2.38)$$

Observing that $u_n(r) = \theta$ and that u_n converges pointwise to u, we easily conclude that $u(r) = \theta$. To complete the construction of the solution, we prove that $u(+\infty) = 1$. Indeed, since u is uniformly bounded and nondecreasing, u achieves its limit at $+\infty$. Using standard estimates we easily get from (2.38) that u satisfies $f(u(+\infty)) = 0$. Hence $u(+\infty) = 1$. We have thus constructed an increasing solution u of (2.25). Observe that the construction holds for any $\theta \in (0,1)$, since from Section 2.3 we can construct appropriate sub and supersolutions of (2.25).

□

We can now turn our attention to the construction of a solution of (2.39), which will be proved in the next section.

2.5 Construction of solutions of (2.39) for all $c \geq c^*(\epsilon)$

In this section, we show that there exists $c^*(\epsilon)$ such that for each $c \geq c^*(\epsilon)$ there exists a positive increasing solution of the following problem

$$\begin{cases} \epsilon u'' + J \star u - u - cu' + f(u) = 0 & in \ \ \mathbb{R} \\ \quad u \to 0 & x \to -\infty \\ \quad u(x) \to 1 & x \to +\infty, \end{cases} \qquad (2.39)$$

Namely we have the following

Theorem 2.5.1.
Let $\epsilon > 0$, then there exists a positif real number $c^*(\epsilon)$ such that for all $c \geq c^*(\epsilon)$ there exists a positive smooth increasing solution u_ϵ of (2.39). *Futhermore if $c < c^*(\epsilon)$, then problem* (2.39) *has no increasing solution.*

The proof of Theorem 2.5.1 will be broken down in two parts. In the first part, Subsection 2.5.1, we construct a solution of Problem (2.39) for a specific value of the speed $c = \kappa$, using solutions of approximate problems constructed in the previous section and a standard limiting procedure. Then in the second part (Subsections 2.5.2 and 2.5.3) we define the minimal speed $c^*(\epsilon)$ and construct solutions of (2.39) for speeds $c \geq c^*(\epsilon)$.

2.5.1 Construction of one solution of (2.39) for $c = \kappa$

For the construction of the solution, we use the approximate problem below

$$\begin{cases} L_r u + h_r(x) + f(u) = 0 & \text{for } x \in (r, +\infty) \\ \quad u(r) = \theta \\ \quad u(x) \to 1 \qquad x \to +\infty, \end{cases} \tag{2.40}$$

From the previous section, for any real number r and any $\theta \in (0, 1)$ there exists a unique solution of (2.40). For fixed $r < 0$, we claim that the solution of (2.40) satisfies the following normalization.

Claim 2.5.1.
There exists $\theta_0 \in (0, 1)$ such that the corresponding solution $u_r^{\theta_0}$ of (2.40) *with $\theta = \theta_0$ satisfies the normalization $u_r^{\theta_0}(0) = \frac{1}{2}$.*

Remark 2.5.1. *This normalization has no meaning when r is no longer negative.*

Proof of Claim 2.5.1

We start with the definition of the following set of acceptable values of θ.

$$\Theta = \{\theta | u_r^{\theta}(0) > \frac{1}{2}\}$$

Choosing any $\theta \geq \frac{1}{2}$ and observing that u_r^{θ} is increasing we have $[\frac{1}{2}, 1) \subset \Theta$. The uniqueness of the solution u_r^{θ} and standard a priori estimates imply that $\theta \to u_r^{\theta}(0)$ is a continuous over $[0, 1]$. By continuity, we can therefore conclude that

– Either there exists a positive θ_0 such that $u_r^{\theta_0}(0) = \frac{1}{2}$

– Or $(0, 1) \subset \Theta$.

We show that the latter case can not occur which will prove the claim. For this, we argue by contradiction. Suppose that $(0, 1) \subset \Theta$. Let $(\theta_n)_{n\in\mathbb{N}}$ a sequence such that $\theta_n \to 0$. Let $(u_n)_{n\in\mathbb{N}}$ be the corresponding sequence of solution of (2.40) with $\theta = \theta_n$. Using Helly's Lemma and standard *a priori* estimates, we can extract a subsequence, still denoted $(u_n)_{n\in\mathbb{N}}$ which converges to a nondecreasing function u. Since $u_n(0) > \frac{1}{2}$, $u(0) \geq \frac{1}{2}$ and u is thus a non-trivial function, satisfying the following equation

$$\begin{cases} L_r u + f(u) = 0 & \text{for } x \in (r, +\infty) \\ \quad u(r) = 0 \\ \quad u(x) \to 1 \qquad x \to +\infty. \end{cases} \tag{2.41}$$

Observe that the function w constructed in Section 2.3 is a subsolution of (2.41). One can show that $w > u$, which provides a contradiction since $\frac{1}{2} \leq u(0) < w(0) = \frac{1}{2}$. See the appendix for details.

□

With the latter normalization, we are ready for the construction of a solution of (2.39). Let $(r_n)_{n\in\mathbb{N}} = (-n)_{n\in\mathbb{N}}$ and $(u_n^{\theta_n})_{n\in\mathbb{N}}$ be the sequence of solutions of the corresponding approximate problem (2.40) with r replaced by r_n and $\theta = \theta_n$, where $(\theta_n)_{n\in\mathbb{N}}$ is such that we have the normalization $u_n^{\theta_n}(0) = \frac{1}{2}$. Define $(h_n)_{n\in\mathbb{N}}$ by

$$h_n(x) = \theta_n \int_{-\infty}^{r_n} J(x-y)dy. \tag{2.42}$$

From Claim 2.5.1 and the previous section such sequences are well defined. Clearly, $h_n \to 0$ pointwise, as $n \to \infty$. Observe now that $(u_n^{\theta_n})_{n\in\mathbb{N}}$ is a uniformly bounded sequence of increasing functions, therefore using Helly's lemma, there exists a subsequence which converges pointwise to a nondecreasing function u. Since $\epsilon > 0$, using local $C^{2,\alpha}$ estimates, up to extraction, the subsequence converge in $C^{2,\alpha}_{loc}$. Therefore $u \in C^{2,\alpha}$ and satisfies

$$\epsilon u'' + J \star u - u - cu' + f(u) = 0 \qquad \text{in} \qquad \mathbb{R}. \tag{2.43}$$

From the normalization and the fact that $f(\frac{1}{2}) \neq 0$, u is not trivial. Since u is increasing and bounded, u achieves its limits $l^{\pm}$ at $\pm\infty$. A standard argument implies that $f(l^-) = 0$ therefore $l^- = 0$ since f is a nonnegative function and $l^- \leq \frac{1}{2}$. Similarly $l^+ = 1$. Therefore we have constructed a non trivial solution of (2.39).

Remark 2.5.2. *Since we know an explicit subsolution on a semi-infinite interval, we can redo this construction with any increasing smooth supersolution of* (2.39) ψ *which satisfies* $\psi'', \psi' \in L^2(\mathbb{R})$ *and* $1 - \psi \in L^2(\mathbb{R}^+)$ *.*

Let us now turn our attention to the second part of the proof.

2.5.2 Definition of $c^*(\epsilon)$

Define

$$c^*(\epsilon) := \inf\{c > 0 : \qquad \text{(2.39) admits an increasing solution}\} \tag{2.44}$$

By the previous section, $c^*(\epsilon)$ is well defined. Obviously, from the definition of $c^*(\epsilon)$, there is no increasing solution to (2.39) for speeds $c < c^*(\epsilon)$. Our goal in this subsection is to provide a solution of (2.39) for all $c \geq c^*(\epsilon)$.

First we observe that (2.39) has a solution for $c = c^*(\epsilon)$. Indeed, by definition of $c^*(\epsilon)$, there exists a sequence of speeds c_n converging to $c^*(\epsilon)$. The corresponding solutions u_n of (2.39) are increasing (and uniformly bounded by 1) so that we may apply Helly's lemma and elliptic regularity as in the previous section to conclude that u_n converges to an increasing solution of (2.39) for $c = c^*(\epsilon)$, which we denote by u_ϵ. Boundary conditions for u_ϵ are obtained as in Subsection 2.5.1.

Fix now $c > c^*(\epsilon)$ and observe that $w := u_\epsilon$ is a smooth increasing supersolution of (2.39) (with speed c). Assume for a moment that u_ϵ satisfies $u_\epsilon'', u_\epsilon' \in L^2(\mathbb{R})$ and $1 - u_\epsilon \in L^2(\mathbb{R}^+)$, then by Remark 2.5.2 the construction of Subsection 2.5.1 applies. Therefore we get a solution of (2.39) for all $c \geq c^*(\epsilon)$ which ends the proofs of Theorem 2.5.1. In the next subsection we prove that u_ϵ satisfies the desired L^2 estimates.

2.5.3 L^2 estimates on u_ϵ

We start out by showing that u'_ϵ and u''_ϵ vanish at infinity and drop ϵ subscripts for convenience. It follows from (2.39) that for some $\rho(x) \in C_0(\mathbb{R})$,

$$\epsilon u'' = cu' + \rho(x). \tag{2.45}$$

Fix $\delta > 0$ and let $R > 0$ be such that $|\rho(x)| < \delta$ for $|x| > R$. Suppose by contradiction that for some $x_0 > R$ we have $cu'(x_0) - \delta > 0$. Then (2.45) implies that $u''(x_0) > 0$ and in fact that $u'' > 0$ on $[x_0, \infty)$, contradicting $0 \leq u \leq 1$. Hence $cu' \leq \delta$ on $[R, \infty)$ i.e.

$$\lim_{+\infty} u' = 0.$$

Similarly, if for some $x_0 < -R$, $cu'(x_0) - \delta > 0$, either u remains convex in $(-\infty, x_0]$, which is possible only if $\lim_{-\infty} u' = 0$, either there exists $x_1 < x_0$ such that $u''(x_1) = 0$, whence u is concave in $(-\infty, x_1]$ in view of (2.45), which is again impossible. Hence

$$\lim_{-\infty} u' = 0$$

and

$$\lim_{-\infty} u'' = 0 \qquad by \qquad (2.45)$$

□

Next we show that $u', u'' \in L^2(\mathbb{R})$. Indeed, multiplying (2.39) by u' and integrating over $\mathbb{R}$ yields

$$c\int (u')^2 = \int f(u)u' = \int_0^1 f(s)\, ds < \infty.$$

Multiplying (2.39) by u'' and integrating over $\mathbb{R}$ we get

$$\epsilon \int (u'')^2 + \int (J \star u - u)\, u'' - c\int u'u'' = \int f(u)u''.$$

Integration by parts and uniform bounds yield

$$\begin{aligned}
\epsilon \int (u'')^2 &= -\int (J \star u - u)\, u'' - \int f(u)u'' && (2.46)\\
&= \int (J \star u' - u')\, u' + \int f'(u)\,(u')^2 && (2.47)\\
&\leq C_0 \int u' + C_1 \|u'\|^2_{L^2(\mathbb{R})} && (2.48)
\end{aligned}$$

where C_0 and C_1 are positive constants.

□

Finally, we show that $1 - u \in L^2(\mathbb{R}^+)$. We need the following lemma :

Lemma 2.5.1. $J \star u - u \in L^1(\mathbb{R})$. *More precisely,*

$$\|J \star u - u\|_{L^1} \leq \int_{\mathbb{R}} J(z)|z|dz \qquad \textit{and} \qquad \int_{\mathbb{R}} (J \star u - u) = 0$$

Moreover, $f(u) \in L^2(\mathbb{R})$.

Proof :

Clearly,

$$\int_{\mathbb{R}} |(J \star u - u)| \leq \int_{\mathbb{R}^2} J(x-y)|u(y) - u(x)|dy\ dx. \tag{2.49}$$

Since $u \in C^1(\mathbb{R})$,

$$|u(y) - u(x)| = |x-y| \int_0^1 u'(y + s(x-y))ds.$$

Plug this equality in (2.49) to obtain :

$$\int_{\mathbb{R}^2} J(x-y)|u(y) - u(x)|dydx = \int_{\mathbb{R}^2} J(x-y)|x-y| \int_0^1 u'(x + s(y-x))ds\ dy\ dx \tag{2.50}$$

Make the change of variables $z = x - y$, so that the right-hand side of (2.50) becomes :

$$\int_{\mathbb{R}^2} J(z)|z| \int_0^1 u'(x - sz)ds\ dz\ dx \tag{2.51}$$

Because all terms are positive, we may apply Tonnelli's Theorem and permute the order of integration. We obtain

$$\begin{aligned}\int_{\mathbb{R}^2} J(z)|z| \int_0^1 u'(x - sz)ds\ dz\ dx \qquad &= \textstyle\int_0^1 \int_{\mathbb{R}^2} J(z)|z|u'(x-sz)dx\ dz\ ds \\ &= \textstyle\int_0^1 \int_{\mathbb{R}} J(z)|z|[u(+\infty) - u(-\infty)]dz\ ds \\ &= \textstyle\int_{\mathbb{R}} J(z)|z|dz < \infty\end{aligned}$$

These last computations show that $J \star u - u$ is an integrable function and give a bound on its L^1 norm. Let us now compute $\int_{\mathbb{R}} (J \star u - u)\ dx$. We have

$$\int_{\mathbb{R}} (J \star u - u)dx = \int_{\mathbb{R}^2} J(x-y)(u(y) - u(x))dy\ dx.$$

Let $z = x - y$ so that

$$\int_{\mathbb{R}^2} J(z)(u(x-z) - u(x))dz\ dx = \int_{\mathbb{R}^2} J(z)(u(y) - u(y+z))dy\ dz.$$

Make the change of variable $z \to -z$ in the left integral and obtain

$$I_1 := \int_{\mathbb{R}^2} J(z)(u(x+z) - u(x))dz\ dx = \int_{\mathbb{R}^2} J(z)(u(y) - u(y+z))dy\ dz =: I_2.$$

Fubini's theorem applied to the last integral shows that $I_1 = -I_2$, hence $I_1 = I_2 = 0$, which proves the first part of the lemma.

Next, we show that $(1-u) \in L^2(\mathbb{R}^+)$. We multiply (2.39) by $1 - u$ and integrate over $\mathbb{R}$

$$\epsilon \int (u')^2 - \int (J \star u - u)u + c/2 + \int f(u)(1-u) = 0$$

Using Lemma 2.4.1 and choosing R so large that $f(u) \geq \frac{|f'(1)|}{2}(1-u)$ on $[R, \infty)$,

$$\frac{|f'(1)|}{2} \int_R^{\infty} (1-u)^2 \leq \int_{-\infty}^{\infty} f(u)(1-u) \leq C(\|u'\|^2_{L^2(\mathbb{R})} + 1) < \infty, \tag{2.52}$$

which ends the proof of the L^2 estimates of u.

□

2.6 Existence of a solution for $\epsilon = 0$

In the previous section, we were able to prove that for every positive ϵ, the following problem :

$$\begin{cases} \epsilon u'' + J \star u - u - cu' + f(u) = 0 & in\ \mathbb{R} \\ u \to 0 & x \to -\infty \\ u \to 1 & x \to +\infty, \end{cases} \tag{2.53}$$

admits a semi infinite interval of solution, i.e for $c \geq c^*(\epsilon)$ there exists a positive increasing solution of (2.53). We will see that the same holds true of the following problem.

$$\begin{cases} J \star u - u - cu' + f(u) = 0 & in\ \mathbb{R} \\ u \to 0 & x \to -\infty \\ u \to 1 & x \to +\infty, \end{cases} \tag{2.54}$$

The idea is to let $\epsilon \to 0$ in the equation and to extract a converging sequence of solutions. The main problem is to control $c^*(\epsilon)$ when $\epsilon \to 0$. We prove the following :

Lemma 2.6.1.
For every positive ϵ_0, there exists $\nu_0 > 0$ such that $c^*(\epsilon) \leq \nu_0$ for all $\epsilon \in [0, \epsilon_0)$

Proof :

According to Remark 2.3.2, $\kappa(\epsilon)$ is an nondecreasing function of ϵ, therefore $\kappa(\epsilon) \leq \kappa(\epsilon_0)$. The conclusion easily follows from the definition of $c^*(\epsilon)$, i.e. $c^*(\epsilon) \leq \kappa(\epsilon)$.

□

We can now derive existence of solution for (2.54) for every speed c greater than ν_0. More precisely we have the following :

Theorem 2.6.1.
There exists ν_0 such that for every speed c greater than ν_0, there exists a solution u with speed c of the equation (2.54)*.*

Proof :

According to the previous lemma, for ϵ small, say $\epsilon \leq \epsilon_0$, equation (2.53) has a solution u_ϵ for every c greater than ν_0 and $\epsilon \leq \epsilon_0$. Without loss of generality we assume that for all ϵ, $u_\epsilon(0) = \frac{1}{2}$. From standard a-priori estimates, u_ϵ is a bounded smooth increasing function. Let $\epsilon \to 0$ along a sequence. As in the previous section, uniform *a priori* estimates and Helly's theorem applied to u_ϵ, provide the existence of a monotone increasing solution u of

$$J \star u - u - cu' + f(u) = 0 \ \ in \ \ \mathbb{R}. \tag{2.55}$$

The solution cannot be trivial, according to the normalisation $\frac{1}{2} = u_\epsilon(0) \to u(0)$. Boundary conditions are obtained as in Section 2.5 .

□

We can define another minimal speed

$$c^{**} = \inf\{c|\forall c' \geq c \ \ (2.54) \ \ \text{has a positive increasing solution of speed} \ \ c'\}. \tag{2.56}$$

This minimal speed is well defined according to the previous theorem.

Remark 2.6.1. *A quick computation, shows that*

$$c^{**} \leq \liminf_{\epsilon \to 0} c^*(\epsilon).$$

Nevertheless to complete the characterization of the set of solutions of (2.54), we have to prove that there exists no travelling-wave solutions of speed c less than c^{**}. In other words, if we defined :

$$c^* = \inf\{c| \ \ (2.54) \ \ \text{has a positive increasing solution of speed}\ c\}, \tag{2.57}$$

we have to show that $c^* = c^{**}$. Clearly we have $c^{**} \geq c^*$, the main problem is to prove $c^{**} \leq c^*$. This will be done with the aid of the monotony of the speed of truncated problems $c_\theta(\epsilon)$ and its continuous behavior at zero. More precisely, consider equation (2.58) below

$$\begin{cases} \epsilon u'' + J \star u - u - cu' + (f\chi_\theta)(u) = 0 & in \ \ \mathbb{R} \\ u \to 0 & x \to -\infty \\ u \to 1 & x \to +\infty, \end{cases} \tag{2.58}$$

where $\epsilon \geq 0$, $\theta > 0$ and let χ_θ be such that
- $\chi_\theta \in C_0^\infty(\mathbb{R})$,
- $0 \leq \chi_\theta \leq 1$,
- $\chi_\theta(s) \equiv 0$ for $s \leq \theta$ and $\chi_\theta(s) \equiv 1$ for $s \geq 2\theta$.

We have the following existence and uniqueness theorem

Theorem 2.6.2.
There exists a unique smooth increasing solution u_θ with speed $c_\theta(\epsilon)$ to (2.58). *Moreover the speed $c_\theta(\epsilon)$ is positive and satisfies*

$$c_\theta(\epsilon) < c^*(\epsilon) \tag{2.59}$$

$$\lim_{\theta \to 0} c_\theta(\epsilon) = c^*(\epsilon). \tag{2.60}$$

A proof of Theorem 2.6.2 can be found in [20, 22], so we do not include it. A natural corollary of this theorem is the continuity of the speed $c_\theta(\epsilon)$ with respect to ϵ and θ. Namely, we have

Corollary 2.6.1.
Under the above assumptions, the following application

$$\begin{array}{ccc} (0,1) \times [0,1] & \to & \mathbb{R}^+ \\ (\theta, \epsilon) & \mapsto & c_\theta(\epsilon) \end{array}$$

is continuous.

Suppose, for a moment that the continuity in θ and ϵ holds, then we can easily conclude the proof of $c^* = c^{**}$. Namely, suppose that $c^* < c^{**}$. Then choose c such that $c^* < c < c^{**}$. Since $c_\theta < c^*$ for every positive θ, we have $c_\theta < c^* < c$. Fix $\theta > 0 :$ since $c_\theta(\epsilon)$ is a continuous function of ϵ, one has on the one hand $c_\theta(\epsilon) < c$ for ϵ small, say $\epsilon \in [0, \epsilon_0]$. On the other hand, according to Remark 2.6.1, we may achieve,

$$c_\theta(\epsilon) < c < c^*(\epsilon) \forall \epsilon \in [0, \epsilon_0]. \tag{2.61}$$

From this last inequality, and according to (2.60), for each $\epsilon \in (0, \epsilon_0]$ there exists a positive $\theta(\epsilon) \leq \theta$ such that $c = c_{\theta(\epsilon)}(\epsilon)$. Let $u_{\theta(\epsilon)}$ be the normalized associated solution.
Now we take a sequence θ_n which goes to 0. From the above construction for each n there exists $\epsilon_n \leq \theta_n$, and $\theta(\epsilon_n) \leq \theta_n$ such that $c = c_{\theta(\epsilon_n)}(\epsilon_n)$ and $u_{\theta(\epsilon_n)}$ is the corresponding normalized solution. From our construction we have,

$$\theta(\epsilon_n) \to 0.$$

Use now, as usual, uniform *a priori* estimates and Helly's theorem to get a solution $\bar{u}$ of the following problem

$$\begin{cases} J \star \bar{u} - \bar{u} - c\bar{u}' + f(\bar{u}) = 0 & in \ \mathbb{R} \\ \bar{u}(x) \to 0 & x \to -\infty \\ \bar{u}(x) \to 1 & x \to +\infty, \end{cases} \tag{2.62}$$

with $c > c^*$. So we get a non trivial solution of (2.54) for the speed c. Since c is arbitrary, there exists a non trivial solution of (2.54) for any speed $c > c^*$, which contradicts the definition of c^{**}.

□

Now, let us turn our attention to the continuity of $c_\theta(\epsilon)$, which will complete the proof.

Proof of Corollary 2.6.1

We know from Theorem 2.6.2 that for every $\epsilon \geq 0$ and $\theta > 0$ there exists a unique solution $(u_\theta^\epsilon, c_\theta^\epsilon)$ to the following problem,

$$\begin{cases} \epsilon(u_\theta^\epsilon)'' + J \star u_\theta^\epsilon - u_\theta^\epsilon - c(u_\theta^\epsilon)' + f_\theta(u_\theta^\epsilon) = 0 & in \ \mathbb{R} \\ u_\theta^\epsilon \to 0 & x \to -\infty \\ u_\theta^\epsilon \to 1 & x \to +\infty, \end{cases} \tag{2.63}$$

Fix $\epsilon_0 \geq 0$ and $\theta_0 > 0$, we will show that for any sequence $(\epsilon_n, \theta_n) \to (\epsilon_0, \theta_0)$, we have $c_{\theta_n}^{\epsilon_n} \to c_{\theta_0}^{\epsilon_0}$. This will show the continuity of the speed. Let $u_{\theta_n}^{\epsilon_n}$ be the normalized associated solution, i.e $u_{\theta_n}^{\epsilon_n}(0) = \frac{1}{2}$. Since $c_\theta(\epsilon) > 0$ and using (2.59), we have $c_{\theta_n}^{\epsilon_n}$ bounded as $(\epsilon_n, \theta_n) \to (\epsilon_0, \theta_0)$. We can extract a sequence of speeds, which converges to some value γ. From the *a priori* estimates on $u_{\theta_n}^{\epsilon_n}$, there also exists a subsequence which converges to a smooth function u solution of the following problem with speed γ.

$$\begin{cases} \epsilon_0 u'' + J \star u - u - \gamma u' + f_{\theta_0}(u) = 0 & in \ \mathbb{R} \\ u \to 0 & x \to -\infty \\ u \to 1 & x \to +\infty, \end{cases} \tag{2.64}$$

According to Theorem 2.6.2, the speed and the profile are unique. Therefore, $\gamma = c_{\theta_0}^{\epsilon_0}$ and $u(x) = u_{\theta_0}^{\epsilon_0}(x + \tau)$. Since $c_{\theta_n}^{\epsilon_n}$ is precompact and has a unique accumulation point, the sequence $c_{\theta_n}^{\epsilon_n}$ must converge to $c_{\theta_0}^{\epsilon_0}$. This ends the proof of the continuity and by means the characterization of the minimal speed c^*.

□

2.7 Asymptotic behavior of solutions

In this section we establish the asymptotic behavior of the solution u near $\pm\infty$ provided J satisfies $(H3)$. The behavior of the function near $+\infty$ has been already obtained in a previous work by one of the authors [20], therefore we only deal with the behavior of u near $-\infty$.

Remark 2.7.1. *The behavior of u near $\pm\infty$ for bistable and ignition type nonlinearities was also obtained in [20].*

We use the same strategy as in [11] and start by proving the following lemma

Lemma 2.7.1. *Assume that (H1) and (H3) hold. Also assume that f is of KPP-type i.e. $f'(0) > 0$. Let u be an increasing solution of problem (P). Then there exists $\epsilon > 0$ such that*

$$\int_{-\infty}^{\infty} u(x)e^{-\epsilon x}\, dx < \infty.$$

Proof

Let $\zeta \in C^\infty(\mathbb{R})$ be a nonnegative nondecreasing function such that $\zeta \equiv 0$ in $(-\infty, -2]$ and $\zeta \equiv 1$ in $[-1, \infty)$. For $N \in \mathbb{N}$, let $\zeta_N = \zeta(x/N)$. Multiplying (P) by $e^{-\epsilon x}\zeta_N$ and integrating over $\mathbb{R}$, we get

$$\int (J * u - u)(e^{-\epsilon x}\zeta_N) - \int cu'(e^{-\epsilon x}\zeta_N) + \int f(u)(e^{-\epsilon x}\zeta_N) = 0 \tag{2.65}$$

Since J is even,

$$\begin{aligned}
\int (J * u - u)(e^{-\epsilon x}\zeta_N) &= \int (J * (e^{-\epsilon x}\zeta_N) - e^{-\epsilon x}\zeta_N)u \\
&= \int u(x)e^{-\epsilon x}\left(\int J(y)e^{\epsilon y}\zeta_N(x - y)\, dy - \zeta_N(x)\right)\, dx \\
&= \int u(x)e^{-\epsilon x}\left(\int J(y)e^{-\epsilon y}\zeta_N(x + y)\, dy - \zeta_N(x)\right)\, dx \\
&\geq \int u(x)e^{-\epsilon x}\left(\int_{-R}^{\infty} J(y)e^{-\epsilon y}\, dy\, \zeta_N(x - R) - \zeta_N(x)\right)\, dx,
\end{aligned} \tag{2.66}$$

where we used the monotone behaviour of ζ_N in the last inequality and where $R > 0$ is chosen as follows : first pick $0 < \alpha < f'(0)$ and $R > 0$ so large that

$$f(u)(x) \geq \alpha u(x) \qquad \text{for } x \leq -R. \tag{2.67}$$

Next, one can increase R further if necessary so that $\int_{-R}^{\infty} J(y)\, dy > (1 - \alpha/2)$. By continuity we obtain for some $\epsilon_0 > 0$ and all $0 < \epsilon < \epsilon_0$,

$$\int_{-R}^{\infty} J(y)e^{-\epsilon y}\, dy \geq (1-\alpha/2)e^{\epsilon R}. \tag{2.68}$$

Collecting (2.66) and (2.68), we then obtain

$$\begin{aligned}\int (J*u-u)(e^{-\epsilon x}\zeta_N) &\geq \int u(x)e^{-\epsilon x}\left((1-\alpha/2)e^{\epsilon R}\zeta_N(x-R)-\zeta_N(x)\right)\, dx\\ &\geq (1-\alpha/2)\int u(x+R)e^{-\epsilon x}\zeta_N(x)\, dx - \int u(x)e^{-\epsilon x}\zeta_N(x)\, dx\\ &\geq -\alpha/2\int u(x)e^{-\epsilon x}\zeta_N(x)\, dx, \end{aligned} \tag{2.69}$$

where we used the monotone behaviour of u in the last inequality.

We now estimate the second term in (2.65) :

$$\begin{aligned}\int u'\zeta_N e^{-\epsilon x}\, dx &= \epsilon\int u\zeta_N e^{-\epsilon x} \; - \int u\zeta_n' e^{-\epsilon x}\, dx\\ &\leq \epsilon\int u\zeta_N e^{-\epsilon x}\; . \end{aligned} \tag{2.70}$$

Finally using (2.67), the last term in (2.65) satisfies

$$\int f(u)\zeta_N e^{-\epsilon x}\, dx \geq \alpha\int_{-\infty}^{-R} u\zeta_N e^{-\epsilon x}\, dx - C. \tag{2.71}$$

By (2.65), (2.69), (2.70) and (2.71) we then obtain

$$(\alpha/2 - c\epsilon)\int_{-\infty}^{-R} u\zeta_N e^{-\epsilon x}\, dx \leq C.$$

Choosing $\epsilon < \alpha/(2c)$ and letting $N \to \infty$ proves the lemma.

□

Using Lemma 2.7.1 it is now easy to see that $u(x) \leq Ce^{\epsilon x}$ for all $x \in \mathbb{R}$. Suppose indeed this is not the case and let $x_n \in \mathbb{R}$ be such that $u(x_n) > ne^{\epsilon x_n}$. Since $0 \leq u \leq 1$, we may pick a subsequence x_{n_k} such that $x_{n_{k+1}} < x_{n_k} - 1$. But since u is nondecreasing,

$$\begin{aligned}\int u(x)e^{-\epsilon x}\, dx &\geq \sum_{k\geq 1}\int_{x_{n_k}}^{x_{n_{k-1}}} u(x)e^{-\epsilon x}\, dx\\ &\geq \sum_{k\geq 1} n_k \int_{x_{n_k}}^{x_{n_k-1}} e^{\epsilon(x_{n_k}-x)}\, dx\\ &\geq \sum_{k\geq 1} n_k/\epsilon\left(1-e^{-\epsilon}\right) = \infty.\end{aligned}$$

□

Annexe A

Uniqueness and monotony in integrodifferential equations on a semi-infinite interval

A.1 Uniqueness and Monotony of solutions of integrodifferential equations on semi infinite domain

In this section we show that solution of the following problem are monotone increasing and if there exists this solution is unique.

$$L_r u + f(u) + h_r(\xi) = 0 \text{ on } \Omega \tag{A.1}$$
$$0 \leq u(r) = \theta < 1 \tag{A.2}$$
$$u(\xi) \to 1 \text{ as } \xi \to +\infty \tag{A.3}$$

Where $L_r u = \epsilon u'' + \int_r^{+\infty} J(\xi - y)u(y)dy - u - cu'$, $h_r(\xi) = \theta \int_{-\infty}^r J(\xi - y)dy$, Ω is any semi-infinite interval of the form $(r, +\infty)$ and $f \in C^1((0,1))$ satisfies $f(1) = 0$ and $f'(1) < 0$. We start with some transformation of the problem (A.1)-(A.3) to a equivalent problem posed on function defined on all $\mathbb{R}$. Let us extented continuously the solution u the following way,

$$\tilde{u} := \begin{cases} \theta & if \;\; \xi \in (-\infty, r] \\ u(\xi) & if \;\; \xi \in (r, +\infty). \end{cases}$$

A quick computation shows that the function $\tilde{u}$ satisfies the following equations :

$$L\tilde{u} + f(\tilde{u}) = 0 \text{ on } \Omega \tag{A.4}$$
$$\tilde{u}(\xi) = \theta \text{ as } \xi \leq r \tag{A.5}$$
$$\tilde{u}(\xi) \to 1 \text{ as } \xi \to +\infty, \tag{A.6}$$

with $L\tilde{u} := \epsilon\tilde{u}'' - cu' + J \star \tilde{u} - \tilde{u}$. Observe now that any solution of (A.4)-(A.6) is a solution of (A.1)-(A.3). Therefore by showing the uniquess of solution of (A.4)-(A.6), we obtain the uniqueness of solution of (A.1)-(A.3). In what follows, we only deal with the latter equations and for convinience we drop the tilde subscript on the function u. Observe that our operator L satisfies a strong maximum principle on Ω and is translation invariant i.e. L satisfies

$\forall h > 0,\;\; L[(u)_h](x) = L[u](x+h) \;\; \forall x \in \Omega.$

For positive solution of (A.4)-(A.6), we prove the following,

Theorem A.1.1.
Let u be a nonnegative, smooth solution of (A.4)-(A.6)*, then the solution is unique. Moreover the solution is monotone increasing on Ω*

The proof of Theorem A.1.1 rely on the following two results. First we show that solution of (A.4)-(A.6) have a monotone behavior. Namely,

Theorem A.1.2.
Let u be a nonnegative, smooth solution of (A.4)-(A.6)*, then the solution is monotone increasing on Ω.*

Sencondly, let consider the following problem for function $z \in C^0(\mathbb{R}) CAP : C^2(\Omega)$,

$$Lz + f(z) = 0 \;\text{ on }\; \Omega \tag{A.7}$$
$$z(\xi) \to 1 \;\text{ as }\; \xi \to +\infty, \tag{A.8}$$

we have a comparison result. Namely,

Lemma A.1.1. *Nonlinear comparison principle*
Let z and v be respectively a smooth positive, nondecreasing function satisfying,

$$Lz + f(z) \geq 0 \;\; on \;\; \Omega \tag{A.9}$$
$$Lv + f(v) \leq 0 \;\; on \;\; \Omega \tag{A.10}$$
$$lim_{\xi \to +\infty} z(\xi) \leq 1, \tag{A.11}$$
$$lim_{\xi \to +\infty} v(\xi) = 1, \tag{A.12}$$
$$v(\xi) > z(r) \;\; on \;\; \Omega \tag{A.13}$$
$$v - z \geq 0 \;\; on \;\; \mathbb{R} \setminus \Omega \tag{A.14}$$

then $z \leq v$ on $\mathbb{R}$.

The two functions z and v in the lemma are respectively called subsolution and supersolution.

Proof of the Uniqueness :

The uniqueness simply rely on this two results. Indeed, let u_1 and u_2 be two solution of (A.4)-(A.6) then u_1 and u_2 are increasing function from Theorem A.1.2. Since θ is a subsolution of (A.4)-(A.6) we have $u_1(\xi) > \theta$ and $u_2(\xi) > \theta$ for $\xi \in \Omega$. From the definition of u_1 and u_2 we have $u_1(\xi) - u_2(\xi) = 0$ for $\xi \in \mathbb{R} \setminus \Omega$. Therefore, since u_1 and u_2 can be alternatively consider as positive, nondecreasing, sub and supersolution, Lemma A.1.1 implies that $u_1 \leq u_2$ and $u_2 \leq u_1$. Thus we must have $u_1 \equiv u_2$ which proves the uniqueness.

□

The main problem is to prove Theorem A.1.2 and Lemma A.1.1. The proof of theses results is based on the idea developed in [21].

The first section is devoted to the proof of Theorem A.1.2 and the second concerned with the proof of Lemma A.1.1.

A.2 Monotonicity, proof of Theorem A.1.2

As in [21], we break down our proof into three steps :

– first step : we prove that for any solution u of (A.4)-(A.6) there exists a positive τ such that
$$u(\xi+\tau) \geq u(\xi) \;\; \forall \xi \in \bar{\Omega}.$$

– second step : we show that for any $\tilde{\tau} \geq \tau$, u satisfies
$$u(\xi+\tilde{\tau}) \geq u(\xi) \;\; \forall \xi \in \bar{\Omega}.$$

– third step : we prove that
$$\inf\{\tau > 0 | \forall \tilde{\tau} > \tau, \;\; u(\xi+\tilde{\tau}) \geq u(\xi) \;\; \forall \xi \in \bar{\Omega}\} \leq 0.$$

We easily see that the last step provided the conclusion of the theorem. The next three subsection are devoted to each step of the proof. We will use also some notation all along this paper. We set $\bar{\Omega} := [r, +\infty)$.

A.2.1 Proof of the first step

The first step will be achieve with this following lemma.

Lemma A.2.1.
Let u a positive solution of (A.4)-(A.6) *then there exists a positive τ such that* $u(\xi+\tau) \geq u(\xi) \;\; \forall \xi \in \bar{\Omega}$.

Proof of lemma A.2.1

Let u a positive solution of (A.4)-(A.6). Since the constant θ and 1 are respectively sub and supersolution of (A.4)-(A.6), using the comparison principle, without loss of generality, we may also assume that $u(r) < u < 1$.

We start with some definitions of quantities that we will use all along the proof.
Let δ positive, such that

$$f'(p) < -2\delta \quad \forall p \quad such \;\; that 1-p < \delta. \tag{A.15}$$

Choose $M > 0$ such that :

$$\mid u(\xi) - 1 \mid < \frac{\delta}{2} \;\; \forall \xi > M. \tag{A.16}$$

The proof of Lemma A.2.1 is mainly based on the following technical lemma which will be proved later on.

Lemma A.2.2.
Let u be a positive solution of (A.4)-(A.6) *satisfying* (A.16). *If there exists positive constant $a \leq \frac{\delta}{2}$ and b such that u satisfies :*

$$u(\xi+b) \geq u(\xi) \;\; \forall \xi \in [r, M+1], \tag{A.17}$$
$$u(\xi+b) + a > u(\xi) \;\; \forall \xi \in \Omega. \tag{A.18}$$

Then we have $u(\xi+b) \geq u(\xi)$ $\forall \xi \in \Omega$.

Proof of Lemma A.2.1

Assume for the moment that Lemma A.2.2 holds. Then to prove Lemma A.2.1 we just have to find appropriate constants a and b which satisfy (A.17) and (A.18). We claim that we can find such constants.

Since u satisfies (A.6) and is constant on $(-\infty, r]$, there exists a constant D such that on the set $(-\infty, M+1]$ we have for every $b \geq D$

$$u(\xi + b) \geq u(\xi) \;\; \forall \xi \in (-\infty, M+1].$$

Now take $a = \frac{\delta}{2}$, it's easy to see that we can find a $b > D$ such that $u(\xi+b)+a > u(\xi) \;\; \forall \xi \in \Omega$. This ends the construction of the constants a and b.

□

Now we turn our attention to the technical Lemma A.2.2.

Proof of Lemma A.2.2

From our assumption on $a > 0$ and b we have

$$u(\xi + b) + a > u(\xi) \; \forall \xi \in \Omega. \tag{A.19}$$

Let's define

$$a^* = \inf\{a > 0 \mid u(\xi + b) + a > u(\xi) \; \forall \xi \in \Omega\}. \tag{A.20}$$

We claim that

Claim A.2.1. $a^* = 0$.

Observe that from the Claim we end up with $u(\xi + b) \geq u(\xi) \; \forall \xi \in \Omega$ which is the desired conclusion.

Proof of claim A.2.1

We argue by contradiction.
If not, since $\lim_{\xi \to +\infty} u(\xi + b) + a^* - u(\xi) = a^* > 0$ and $u(\xi + b) + a^* - u(\xi) \geq a^* > 0$ for $\xi \in (-\infty, r]$, there exists $\xi_0 \in \Omega$ such that $u(\xi_0 + b) + a^* = u(\xi_0)$.
Let $w(\xi) := u(\xi + b) + a^* - u(\xi)$, then we have

$$0 = w(\xi_0) = \min_{\mathbb{R}} w(\xi) \tag{A.21}$$

Observe that w satisfies also the following equation :

$$Lw \leq -f(u(\xi + b)) + f(u(\xi)), \tag{A.22}$$

$$w(\xi) \geq a^* \;\; \text{for } \xi \in (-\infty, r] \tag{A.23}$$

$$w(+\infty) = a^*. \tag{A.24}$$

Next, we will prove a kind of localization of minimum lemma. More precisely

Claim A.2.2. *Let $\xi_0 \in \mathbb{R}$ be the minimum of w, then we have $\xi_0 > M + 1$.*

Proof of Claim A.2.2

From the assumption on a and b, we know that $u(\xi+b) \geq u(\xi) \quad \forall \xi \in (-\infty, M+1]$. Therefore, we have

$$w(\xi) = u(\xi+b) + a^* - u(\xi) > 0 \;\; \forall \xi \in (-\infty, M+1].$$

Thus, $\xi_0 > M+1$.

□

By the maximum principle property and since $\xi_0 \in \Omega$, at its minimum ξ_0, w satisfies :

$$f(u(\xi_0)) - f(u(\xi_0) - a^*) \geq Lw(\xi_0) > 0, \tag{A.25}$$

then

$$Q = f(u(\xi_0)) - f(u(\xi_0) - a^*) > 0, \tag{A.26}$$
$$= f'(d)a^* > 0, \tag{A.27}$$

for some $d \in]u(\xi_0) - a^*, u(\xi_0)[$.
From Claim A.2.2, we have $\xi_0 > M+1$, therefore (A.16) and $a^* \leq \frac{\delta}{2}$ implies $|d-1| < \delta$.
Thus, Q would verify :

$$Q = f'(d)a^* < -2a^*\delta < 0,$$

which contradicts (A.27). This implies that $a^* = 0$, which ends the proof of Claim A.2.2.

□

Now, we turn our attention to the second step in the proof of theorem A.1.2.

A.2.2 Proof of the second step

We achieve the second step with the following proposition.

Proposition A.2.1.
Let u be a positive solution of (A.4)-(A.6) *satisfying* (A.16). *If there exists τ such that*

$$u(\xi+\tau) \geq u(\xi) \;\; \forall \xi \in \Omega. \tag{A.28}$$

Then, for all $\tilde{\tau}$ we have, $u(\xi+\tilde{\tau}) \geq u(\xi) \;\; \forall \xi \in \Omega$.

From Lemma A.2.1, there exists a such τ. The proof of the proposition is based on the two following technical lemmas.

Lemma A.2.3.
Let u a positive solution of (A.4)-(A.6) *and $\tau > 0$ such that $u(\xi+\tau) \geq u(\xi) \;\; \forall \xi \in \bar{\Omega}$.*
Then, we have $u(\xi+\tau) > u(\xi) \;\; \forall \xi \in \bar{\Omega}$.

Lemma A.2.4.
Let u be a positive solution of (A.4)-(A.6) *satisfying* (A.16) *and $\tau > 0$ such that $u(\xi+\tau) > u(\xi) \;\; \forall \xi \in \bar{\Omega}$.*
Then, there exists $\epsilon_0(\tau) > 0$ such that for all $\tilde{\tau} \in [\tau, \tau+\epsilon_0]$, we have

$$u(\xi+\tilde{\tau}) > u(\xi) \;\; \forall \xi \in \bar{\Omega}. \tag{A.29}$$

Proof of Proposition A.2.1 :

Assume that the two technicals lemmas holds. We know from the first step that we can find a positive τ, such that,

$$u(\xi+\tau) \geq u(\xi) \;\; \forall \xi \in \bar{\Omega}.$$

Therefore from Lemmas A.2.3 and A.2.4, we can construct a interval $[\tau, \tau+\epsilon]$, such that for all $\tilde{\tau} \in [\tau, \tau+\epsilon]$ we have

$$u(\xi+\tilde{\tau}) \geq u(\xi) \;\; \forall \xi \in \bar{\Omega}.$$

Let's defined the following quantity,

$$\bar{\gamma} = \sup\{\gamma | \forall \hat{\tau} \in [\tau, \gamma], \;\; u(\xi+\hat{\tau}) \geq u(\xi) \;\; \forall \;\; \xi \in \bar{\Omega}\}. \tag{A.30}$$

We claim that $\bar{\gamma} = +\infty$, if not, $\bar{\gamma} < +\infty$ and by continuity we have

$$u(\xi+\bar{\gamma}) \geq u(\xi) \;\; \forall \;\; \xi \in \bar{\Omega}. \tag{A.31}$$

Recall that from the definition of $\bar{\gamma}$, we have

$$\forall \hat{\tau} \in [\tau, \bar{\gamma}] \;\; u(\xi+\hat{\tau}) \geq u(\xi) \;\; \forall \;\; \xi \in \bar{\Omega}. \tag{A.32}$$

Therefore to get a contradiction, it is sufficient to construct ϵ_0 such that for all $\epsilon \in [0, \epsilon_0]$, we have

$$u(\xi+\bar{\gamma}+\epsilon) \geq u(\xi) \;\; \forall \;\; \xi \in \bar{\Omega}. \tag{A.33}$$

Since $\bar{\gamma} > 0$, then we can apply Lemma A.2.3 to have,

$$u(\xi+\bar{\gamma}) > u(\xi) \;\; \forall \;\; \xi \in \bar{\Omega}, \tag{A.34}$$

we can now apply Lemma A.2.4, to find the desired $\epsilon > 0$. Therefore, from the definition of $\bar{\gamma}$ we get

$$\forall \hat{\tau} \in [\tau, +\infty], \;\; u(\xi+\hat{\tau}) \geq u(\xi) \;\; \forall \;\; \xi \in \bar{\Omega}.$$

Which proves Proposition A.2.1.

□

Proof of lemma A.2.3 :

To prove

$$u(\xi+\tau) > u(\xi) \;\; \forall \xi \in \bar{\Omega}, \tag{A.35}$$

we argue by contradiction. Assume there exists a point $\xi_0 \in \bar{\Omega}$ such that

$$w(\xi) = u(\xi+\tau) - u(\xi) \geq w(\xi_0) = 0 \;\; \forall \;\; \xi \in \bar{\Omega}.$$

We have to consider two cases :

- Either $\xi_0 \in (r, +\infty)$, and at this point, w verifies :

  $$0 \leq Lw(\xi_0) \leq f(u(\xi_0)) - f(u(\xi_0+\tau)) = f(u(\xi_0)) - f(u(\xi_0)) = 0.$$

  Then, from the maximum principle property, we end up with $w \equiv 0$. Therefore, u is τ periodic, which is impossible since $lim_{\xi \to +\infty} u(\xi) = 1$.
- Or $\xi_0 = r$, and then we have $u(r) = u(r+\tau)$ which is also impossible since $u(\xi) > u(r)$ for all $\xi \in \Omega$.

Since both cases leads to a contradiction, a such ξ_0 doesn't exists. Therefore (A.35) holds.

$\square$

We can turn our attention to the proof of Lemma A.2.4.

Proof of lemma A.2.4 :

Let u a positive solution of (A.4)-(A.6), which satisfies (A.16) and

$$u(\xi + \tau) > u(\xi) \ \ \forall \xi \in \bar{\Omega} \tag{A.36}$$

for a given τ. Since u is continuous and satisfies (A.36), we can find ϵ_0, such that for all $\epsilon \in [0, \epsilon_0]$, we have :

$$u(\xi + \tau + \epsilon) > u(\xi) \ \text{ for } \ \xi \in [r, M+1]. \tag{A.37}$$

Let $a = \frac{\delta}{2}$, then for all $\epsilon \in [0, \epsilon_1]$, we have

$$u(\xi + \tau + \epsilon) + a > u(\xi) \ \ \forall \xi \in \bar{\Omega}, \tag{A.38}$$

for some ϵ_1. Let $\epsilon_3 = \min\{\epsilon_0, \epsilon_1\}$. Observe that for all $\epsilon \in [0, \epsilon_3]$, $b := \tau + \epsilon$ and a satisfies assumptions (A.17) and (A.18) of Lemma A.2.2. Therefore we can apply Lemma A.2.2 for each $\epsilon \in [0, \epsilon_3]$ and get

$$u(\xi + \tau + \epsilon) \geq u(\xi) \ \ \forall \xi \in \bar{\Omega}. \tag{A.39}$$

Thus, we end up with

$$u(\xi + \tilde{\tau}) \geq u(\xi) \ \ \forall \xi \in \bar{\Omega} \tag{A.40}$$

for all $\tilde{\tau} \in [\tau, \tau + \epsilon_3]$. This ends the proof of Lemma A.2.4.

$\square$

A.2.3 Proof of the third step

From Lemma A.2.1 and Proposition A.2.1, we can define the following quantity :

$$\tau^* = \inf\{\tau > 0 | \ \forall \tilde{\tau} > \tau, \ \ u(\xi + \tilde{\tau}) \geq u(\xi) \ \ \forall \xi \in \bar{\Omega}\}. \tag{A.41}$$

We end the proof of theorem A.1.2, with the following lemma

Lemma A.2.5.
Let u a positive solution of (A.4)-(A.6)*, satisfying* (A.16)*. Then, we have $\tau^* \leq 0$*

Observe that this lemma implies the monotony of u, which concludes the proof of Theorem A.1.2.

Proof of lemma A.2.5

We argue by contradiction, suppose that $\tau^* > 0$. We will show that for ϵ small enough, we still have,

$$u(\xi + \tau^* - \epsilon) \geq u(\xi) \ \text{ for all } \ \ \xi \in \bar{\Omega} \tag{A.42}$$

and then from the previous step, we will have for all $\tilde{\tau} \geq \tau^* - \epsilon$

$$u(\xi) \leq u(\xi + \tilde{\tau}) \ \text{ for all } \ \ \xi \in \bar{\Omega}, \tag{A.43}$$

which contradicts the definition of τ^*.
Now, we start the construction. From the definition of τ^* and by continuity, we have

$$u(\xi + \tau^*) \geq u(\xi) \ \text{ for all } \ \ \xi \in \bar{\Omega}. \tag{A.44}$$

Therefore, from Lemma A.2.3, we have

$$u(\xi+\tau^*) > u(\xi) \text{ for all } \xi \in \bar{\Omega}. \tag{A.45}$$

Thus, on the compact [r,M+1], we can find $\epsilon_1 > 0$ such that,

$$\forall \epsilon \in [0, \epsilon_1) \;\; u(\xi+\tau^*-\epsilon) > u(\xi) \quad \text{on the compact [r,M+1].} \tag{A.46}$$

Now fix $\epsilon \in (0, \epsilon_1)$, then we can easily find a positive constant $a \leq 1$ such that

$$u(\xi+\tau^*-\epsilon) + a > u(\xi) \text{ for all } \xi \in \bar{\Omega}. \tag{A.47}$$

We can then apply Lemma A.2.2 to obtain the desired result.

□

A.3 Nonlinear comparison principles, proof of Lemma A.1.1

In this section we prove the nonlinear comparison principle i.e Lemma A.1.1. The proof is based on the three following lemma which will be proved later on.

Lemma A.3.1. *Let z and v be respectively two positive, increasing sub and supersolution, satisfying* (A.9)-(A.14)*. Then there exists a positive τ such that $v(\xi+\tau) \geq z(\xi)$ for all $\xi \in \bar{\Omega}$.*

Lemma A.3.2.
Let z and v be respectively smooth positive nondecreasing sub and supersolution satisfying (A.9)-(A.14)*. If there exists positive constant $a \leq \frac{\delta}{2}$ and b such that z and v satisfy :*

$$v(\xi+b) > z(\xi) \;\; \forall \xi \in [r, M+1], \tag{A.48}$$
$$v(\xi+b) + a > z(\xi) \;\; \forall \xi \in \Omega. \tag{A.49}$$

Then we have $v(\xi+b) \geq z(\xi)$ $\forall \xi \in \Omega$.

Lemma A.3.3. *Let z and v be respectively two positive, increasing sub and supersolution satisfying* (A.9)-(A.14) *such that for a positive τ, $v(\xi+\tau) \geq z(\xi)$. Then $v(\xi+\tau) > z(\xi)$ for all $\xi \in \bar{\Omega}$.*

Assume for the moment that these three lemmas holds.

Proof of Lemma A.1.1

We start with some definitions of quantities that we will use all along the proof.
Let δ positive, such that

$$f'(p) < -2\delta \quad \forall p \quad such \;\; that 1-p < \delta. \tag{A.50}$$

If $\lim_{\xi\to+\infty} z(\xi) = 1$, choose $M > 0$ such that :

$$1 - v(\xi) < \frac{\delta}{2} \;\; \forall \xi > M, \tag{A.51}$$
$$1 - z(\xi) < \frac{\delta}{2} \;\; \forall \xi > M. \tag{A.52}$$
$$\tag{A.53}$$

Otherwise we choose M such that

$$v(\xi) > z(\xi) \;\; \forall \xi > M. \tag{A.54}$$

From Lemma A.3.1, there exist a positive τ such that $v(\xi + \tau) \geq z(\xi)$. Therefore the following quantity is well defined.

$$\tau^* = \inf\{\tau > 0 | \;\; v(\xi + \tau) \geq z(\xi) \; \forall \xi \in \bar{\Omega}\}. \tag{A.55}$$

We claim

Claim A.3.1. $\tau^* = 0$.

Proof :

We argue by contradiction. If not then $\tau^* > 0$. We will show that for ϵ small enough, we still have,

$$v(\xi + \tau^* - \epsilon) \geq z(\xi) \;\; \text{for all} \quad \xi \in \bar{\Omega}, \tag{A.56}$$

which contradicts the definition of τ^*.

Let's start the construction of the desired ϵ.

By continuity we have

$$v(\xi + \tau^*) \geq z(\xi) \;\; \text{for all} \quad \xi \in \bar{\Omega}. \tag{A.57}$$

Using A.3.3 on (A.56), we indeed have

$$v(\xi + \tau^*) > z(\xi) \;\; \text{for all} \quad \xi \in \bar{\Omega}. \tag{A.58}$$

Since z and v are monotone increasing we can find ϵ_0 such that for all $\epsilon \in [0, \epsilon_0]$,

$$v(\xi + \tau^* - \epsilon) > z(\xi) \;\; \text{for all} \quad \xi \in [r, M+1]. \tag{A.59}$$

Let consider the two following cases :

- $\lim_{\xi \to +\infty} z(\xi) < 1$:
  From (A.59) and (A.54) we have our desire contradiction. Indeed, from (A.59) $v(\xi + \tau^* - \epsilon) \geq z(\xi)$ for all $\xi \in [r, +\infty]$ and in that case from (A.54) M was chosen such that $v(\xi + \tau^* - \epsilon) \geq v(\xi) \geq z(\xi)$ for all $\xi > M$.
- $\lim_{\xi \to +\infty} z(\xi) = 1$:
  In this case for $a = \frac{\delta}{2}$, using (A.52) and (A.53) we achieve,

  $$v(\xi + \tau^* - \epsilon) + a > z(\xi) \;\; \text{for all} \quad \xi \in \bar{\Omega}.$$

  Then use Lemma A.3.2 to get the desired result.

□

To conclude the section we have to prove Lemma A.3.1,Lemma A.3.2 and Lemma A.3.3. Since the proof of these three lemmas are, with minor changes, the same as the proof of Lemma A.2.1, Lemma A.2.2 and Lemma A.2.3, we omit it.

□

Deuxième partie

Qualitative properties of fronts solving nonlocal reaction-diffusion equations

Chapitre 3

Min-max formulas for front speeds in a nonlocal reaction-diffusion equation

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Sommaire

3.1 Introduction

In this article, we are concerned with variational formulas characterizing the speed c of travelling fronts u arising in the study of a nonlocal reaction-diffusion model. More precisely, we study the solutions (u, c) of the following one dimensional integro-differential equation

$$\begin{cases} \epsilon u'' + J \star u - u - cu' + f(u) = 0 & \text{on } \mathbb{R} \\ u(x) \to 0 & \text{as } x \to -\infty \\ u(x) \to 1 & \text{as } x \to +\infty \end{cases} \tag{P}$$

where $J \star u(x) = \int_{\mathbb{R}} J(x-y)u(y)dy$, J is an even positive kernel of mass one, ϵ a nonnegative real number (which we emphasize *can* be taken equal to zero) and f is a given nonlinearity. We will always assume in what follows that J satisfies the following

$$J \in W^{1,1}(\mathbb{R}), \qquad J \geq 0, \qquad J(x) = J(-x) \qquad \text{and} \qquad \int_{\mathbb{R}} J = 1. \tag{H1}$$

The unknowns of this problem are the real number c, which represents the speed of the front, and u the profile of the front. The speed c can also be viewed as a nonlinear eigenvalue of the problem. Travelling-front solutions are expected to give the asymptotic behavior in large time for solutions of the following evolution problem (3.1), with say compactly supported initial data.

$$\frac{\partial u}{\partial t} = \epsilon u_{xx} + J \star u - u + f(u) \tag{3.1}$$

It is therefore of interest to characterize the speed of these solutions. Such types of equation were derived in the early work of Kolmogorov - Petrovskii- Piscounov (see [46]) on the spread of a gene . The dispersion of the gene fraction at point $y \in \mathbb{R}^n$ should affect the gene fraction at $x \in \mathbb{R}^n$ by a factor $J(x,y)u(y)dy$ where $J(x,\cdot)$ is a probability density. Restricting to a one-dimensional setting and assuming that such a diffusion process depends only on the distance between two niches of the population, we end up with the equation

$$u_t = J \star u - u + f(u), \tag{3.2}$$

where $J : \mathbb{R} \to \mathbb{R}$ is a nonnegative even function of mass one.

Equation (3.1) also appears in the context of pattern formation in Activator - Inhibitor systems such as

$$\begin{cases} \frac{\partial u}{\partial t} - u_{xx} = f(u) - v \\ -v_{xx} + v = u \end{cases}$$

Observe that we can inverse the second equation. We can thus rewrite v in terms of u. Namely we have $v = J \star u$ with $J(x) = \frac{1}{2}e^{-|x|}$, so that the system can be reformulated as

$$\frac{\partial u}{\partial t} = u_{xx} + J \star u - u + g(u) \text{ for } (\xi,t) \in \mathbb{R} \times \mathbb{R}^+ \tag{3.3}$$

where $g(u) = f(u) + u$. For more information, see the excellent book of J. Murray [48] and [49]. In these two models the operator $\mathcal{A}(u) := J \star u - u$ represents the nonlocal diffusion of the species through its environment.

In this work, we study three types of nonlinearity f that we present below : $f \in C^1([0,1])$ and

- Case A1 $\quad$ f is of bistable type if for some $\rho > 0$, f satisfies
  - $f|_{(0,\rho)} < 0$ and $f|_{(\rho,1)} > 0$
  - $f(0) = f(1) = 0$ and $f'(1) < 0$
- Case A2 $\quad$ f is of ignition type if for some $\rho > 0$,
  - $f|_{(0,\rho)} \equiv 0$ and $f|_{(\rho,1)} > 0$
  - $f(0) = f(1) = 0$ and $f'(1) < 0$
- Case B $\quad$ f is of monostable type if $f(0) = f(1) = 0$,$f|_{(0,1)} > 0$ and $f'(1) < 0$

These three types of nonlinearities are commonly used in the literature to describe models of phase transition, nerve propagation, combustion, population dynamics and ecology : see [4, 7, 24, 31, 34, 35, 41, 46, 54, 61, 59, 62]. Under some additional assumption on the kernel J, existence and in some cases uniqueness of travelling-wave solutions have been investigated by Bates, Fife, Ren, Wang [5] and Chen [17] in the bistable case and completed by the work of one of the present authors [20], for the ignition case. The monostable case is the object of a forthcoming publication [23]. We summarize these results in the following theorem.

Theorem 3.1.1. *[5, 20, 23]*

– *Assume that J satisfies (H1) and the following*

$$\int_{\mathbb{R}} J(z)|z|dz < +\infty. \tag{H2}$$

and let f be a nonlinearity of type A1 or A2. Then the problem (P) admits a monotone solution (u, c). Furthermore this solution is unique in the following sense, if (v, c') is another solution then $c = c'$ and $u(x) = v(x + \tau)$ for a fixed τ. Moreover u satisfies $u' > 0$.

– *Let f be a monostable function and assume further that J satisfies the following integrability condition*

$$\forall \lambda > 0, \quad \int_{\mathbb{R}} J(z)e^{\lambda z}dz < +\infty. \tag{H3}$$

then there exists a minimal speed $c^* > 0$ such that for all $c \geq c^*$, there exits a monotone function u such that (u, c) is a solution of (P), while there exists no monotone solution (u, c) with speed $c < c^*$.

Remark 3.1.1. *The condition (H3) can be weakened : it is enough to assume that (H3) holds for one given value of λ.*

These results of existence and uniqueness are similar to those of the standard reaction-diffusion problem below,

$$\frac{\partial u}{\partial t} = \Delta u + g(u) \qquad \text{in} \qquad \mathbb{R}^n \times \mathbb{R}^+ \tag{3.4}$$

This is due to the fact that the nonlocal operator shares many properties of the Laplacian and in some limiting cases reduces to it. Namely letting $J_\epsilon(x) := \frac{1}{\epsilon}J(\frac{1}{\epsilon}x)$, for J compactly supported and $\epsilon > 0$ small, we have

$$\begin{aligned} J_\epsilon \star u - u &= \frac{1}{\epsilon}\int J(\frac{1}{\epsilon}y)(u(x-y) - u(x))\, dy = \int J(z)(u(x - \epsilon z) - u(x))\, dz \\ &= -\epsilon \int J(z)u'(x)z\, dz + \epsilon^2 \int z^2 J(z)u''(x)\, dz + o(\epsilon^2) = c\epsilon^2 u''(x) + o(\epsilon^2), \end{aligned}$$

where we used the fact that J is even in the last equality.

The characterization by min-max formulas of the wave speed c in the context of (3.4) is well known. In one space dimension, equation (3.4) reduces to an ordinary differential equation and the speed of planar fronts satisfies the following min-max formulas

– For f of type A1 or A2, the unique speed c^* satisfies

$$c^* = \min_{w \in X} \sup_{x \in \mathbb{R}} \{\frac{w'' + f(w)}{w'}\} \tag{3.5}$$

$$c^* = \max_{w \in X} \inf_{x \in \mathbb{R}} \{\frac{w'' + f(w)}{w'}\} \tag{3.6}$$

– For f of type B, the minimal speed c^* satisfies

$$c^* = \min_{w \in X} \sup_{x \in \mathbb{R}} \{\frac{w'' + f(w)}{w'}\}, \tag{3.7}$$

where $X = \{\chi \in C^0(\mathbb{R}) | \chi$ is increasing, $\chi(+\infty) = 1 \;\; and \;\; \chi(-\infty) = 0\}$.

With some extra assumption on the nonlinearity f, an explicit formula for the minimal speed can be given. Kolmogorov,Petrovskii and Piscounov [46] proved in 1937 that the minimal speed c^* is given by $c^*_{KPP} = 2\sqrt{f'(0)}$ when f is monostable and satisfies $\frac{f(s)}{s} \leq f'(0)$ for $s \in (0,1)$. This formula was recovered by Berestycki and Nirenberg [11] using a different approach. Weinberger in [61] generalized it to time-discrete models. In the opposite situation, when f approaches a Dirac mass centered at one, Zeldovich and Frank-Kamenetskii [62] were able to give an asymptotic formula for the flame's front speed. In this case the speed is given by $c^*_{ZFK} \simeq \sqrt{\int_0^1 f(s)ds}$. More recently, Berestycki, Nikolaenko and Sheurer [14] have shown that the asymptotic speed c^*_{ZFK} holds for planar-front solutions of a system of ordinary differential equations. Other asymptotic formulas were also derived in turbulent combustion : see Clavin and Williams [19]. Clavin [18] also explains the transition from c^*_{KPP} to c^*_{ZFK}. Min-max formulas for travelling fronts in systems of ODE's also exist, see Kan-On [42], Mischaikow- Hudson [47], and Volpert, Volpert, Volpert [57, 58]. The proofs of these formulas were in most cases deeply related to shooting methods, phase plane analysis and good asymptotics. Since our equation is nonlocal, we can not carry out most of theses techniques. Nevertheless we can provide min-max formulas for the speed of travelling fronts of (3.1), analog to those above. Namely we have the following variational characterization of the wave-speed :

Theorem 3.1.2.

– *Assume (H1)and (H2) hold. Then if c^* denotes the unique front speed in the "bistable" case, we have*

$$c^* = \min_{w \in X} \sup_{x \in \mathbb{R}} \{ \frac{\epsilon w'' + J \star w - w + f(w)}{w'} \} \tag{3.8}$$

$$c^* = \max_{w \in X} \inf_{x \in \mathbb{R}} \{ \frac{\epsilon w'' + J \star w - w + f(w)}{w'} \} \tag{3.9}$$

– *Assume (H1) and (H3) holds. Then if c^* denotes the minimal speed in the monostable case, we have*

$$c^* = \min_{w \in X} \sup_{x \in \mathbb{R}} \{ \frac{\epsilon w'' + J \star w - w + f(w)}{w'} \} \tag{3.10}$$

where $X = \{\chi \in C^0(\mathbb{R}) | \chi$ is increasing, $\chi(+\infty) = 1 \;\; and \;\; \chi(-\infty) = 0\}$.

We should also mention a result of Hamel [39] which generalize min-max formulas to the setting of multidimensional travelling fronts in a cylinder and those of Heinze, Papanicolaou and Stevens [40] giving a simple proof (in the usual bistable case) of those variational formulas for quite general operators.

Remark

The technique developed in this paper also applies to the traditional reaction-diffusion problem, thus providing an alternate proof of these formulas. In the monostable case, the existence of solutions (u, c) with speed $c > c^*$, as in the reaction-diffusion case, has not yet appeared. In a forthcoming paper the present authors will carry out this analysis. In the

KPP-like situation we were not able to give an exact explicit formula and only provide an upper bound in terms of a spectral formula. Namely we show

$$c^*_{KPP} \leq \inf_{\lambda>0}\{\frac{1}{\lambda}[\epsilon\lambda^2 + \int_{\mathbb{R}} J(z)e^{\lambda z}dz - 1 + f'(0)]\}. \tag{3.11}$$

Method and plan

The proof relies on two simple ideas :
– The construction of solutions via the method of sub and supersolutions.
– Let u and v be a sub and a supersolution of a bistable problem then translations of u and v are ordered.

Though elementary in nature, the proof of these results requires a number of lemmas which we list and prove in Section 3.2. In Sections 3.3 and 3.4, we present the construction of a solution via the method of sub and supersolutions. Theorem 3.1.2 is the object of the Sections 3.5 and 3.6. Section 3.5 deals with the min-max formula in the bistable case while Section 3.6 is concerned with the monostable case and the proof of inequality (3.11).

3.2 Linear theory

We start this section with two maximum principles for integro-differential operators defined on the real line of the kind :

$$Lu = \epsilon u'' + J \star u - u + b(x)u' + d(x)u, \tag{3.12}$$

where $\epsilon \geq 0$, $\int_{\mathbb{R}} J = 1$, $d(x) \leq 0$ and $b(x)$ and $d(x)$ are bounded functions on $\mathbb{R}$. We first prove a strong maximum principle that applies to smooth functions :

Theorem 3.2.1. *Strong Maximum Principle*
Let $u \in C^2(\mathbb{R})$ satisfy

$$Lu \geq 0 \;\; in \;\; \mathbb{R} \quad (respectively, \quad Lu \leq 0 \quad in\, \mathbb{R}).$$

Then u may not achieve a positive maximum (resp. negative minimum) without being constant.

This theorem immediately implies the following practical corollary :

Corollary 3.2.1.
Let $u \in C^2(\mathbb{R})$ satisfy

$$\begin{cases} Lu \geq 0 & on \;\; \mathbb{R} \\ u \to 0 & |x| \to +\infty. \end{cases}$$

Then
– *either $u < 0$*
– *either $u \equiv 0$.*

Remarks 3.2.1.
Similarly, if $Lu \leq 0$ then u is either positive or identically 0.

The proof of the corollary is a straightforward application of the theorem. Let us prove the Strong Maximum Principle.

Proof :

We argue by contradiction. Suppose that u is a nonconstant function and achieves a positive maximum somewhere, say at x_0.
Since u is a C^2 function, we have $d(x_0)u(x_0) \leq 0$, $u'(x_0) = 0$ and $u''(x_0) \leq 0$. Furthermore, since $\int_{\mathbb{R}} J(z)dz = 1$ and $u(y) - u(x_0) \leq 0$ for every y in $\mathbb{R}$, we have $J \star u(x_0) - u(x_0) = \int_{\mathbb{R}} J(x_0 - y)(u(y) - u(x_0))dy \leq 0$. Therefore, we have at the point x_0 :

$$\epsilon u''(x_0) + J \star u(x_0) - u(x_0) + b(x_0)u'(x_0) + d(x_0)u(x_0) \leq 0 \tag{3.13}$$

and by our assumption

$$\epsilon u''(x_0) + J \star u(x_0) - u(x_0) + b(x_0)u'(x_0) + d(x_0)u(x_0) \geq 0 \tag{3.14}$$

These two equations imply that $d(x_0)u(x_0) = 0$, $u''(x_0) = 0$ and

$$J \star u(x_0) - u(x_0) = \int J(x_0 - y)(u(y) - u(x_0))\, dy = 0. \tag{3.15}$$

By assumption, J is a smooth nonnegative function with $\overset{\circ}{supp(J)} \not\equiv \emptyset$. Thus we deduce from (3.15) that $u(y) = u(x_0)$ for all y in the set $x_0 + supp(J)$. If J is supported by $\mathbb{R}$ we obtain a contradiction immediately. If not, we can repeat the previous calculation for every y in $x_0 + supp(J)$, thus u is constant on the set $y + supp(J)$ where y belongs to $x_0 + supp(J)$. By doing so infinitely many times, we cover all of $\mathbb{R}$ and thus end up with $u(y) = u(x_0)$ for all y in $\mathbb{R}$, which is a contradiction.

□

Provided ϵ is nonzero, we also have the following weak maximum principle :

Theorem 3.2.2. *Weak Maximum Principle*
Suppose $\epsilon > 0$ and let $u \in H^1(\mathbb{R})$ sastisfy the following inequality in the weak sense :

$$Lu \geq 0 \;\; on \;\; \mathbb{R}.$$

Then for any compact subset ω of $\mathbb{R}$,

$$\sup_{\omega} u \leq \sup_{\partial\omega} u^+$$

Remark 3.2.1. *A function $u \in H^1(\mathbb{R})$ satisfies $Lu \geq 0$ in the weak sense if for all nonnegative $\phi \in C_c^\infty(\mathbb{R})$,*

$$\int_{\mathbb{R}} -\epsilon u'\phi' + b(x)u'\phi + d(x)u\phi + (J \star u - u)\phi \geq 0.$$

We shall use the following easy corollary :

Corollary 3.2.2. *Let u satisfy the assumptions of the above theorem, then u is nonpositive.*

Proof of the corollary

It is sufficient to show that for every positive δ, $u \leq \delta$. Now fix δ positive.
In one space dimension, $H^1(\mathbb{R}) \hookrightarrow C_0(\mathbb{R})$, so u must go continuously to zero at infinity. Whence there exists r_0 such that $|u| \leq \delta$ for every $|x| \geq r_0$. In particular $u^+(\pm r_0) \leq \delta$ and we may apply Theorem 3.2.2 with the compact set $\omega = [-r_0, r_0]$.
We end up with $u|_\omega \leq \sup_{\partial\omega} u^+ \leq \delta$. Thus $u \leq \delta$ on $\mathbb{R}$.

$\square$

We now proceed with the proof of Theorem 3.2.2.

Proof of Theorem 3.2.2

The proof follows that of Theorem 8.1 in Gilbarg and Trudinger's book [38]. For convenience of the reader, we provide its details. Let ω be a compact subset of $\mathbb{R}$. Assume by contradiction that

$$\sup_\omega u > \sup_{\partial\omega} u^+ = l.$$

Define a bilinear operator $\mathcal{L}$ on $H_0^1(\omega) \times H_0^1(\omega)$ by

$$\mathcal{L}(u, z) = \int_\omega \epsilon u' z' - b(x) u' z - (J * u - u) z - d(x) u z \, dx \tag{3.16}$$

By assumption, u satisfies $\mathcal{L}(u, z) \leq 0$ for all nonnegative $z \in H_0^1(\omega)$ i.e.

$$\int_\omega \epsilon u' z' - (J * u - u) z \leq \int_\omega b(x) u' z + \int_\omega d(x) u z \tag{3.17}$$

Now let k be such that $\sup_\omega u > k \geq l$. The function $v := (u - k)^+$ is nontrivial and satisfies :

$$v = \begin{cases} u - k & when \;\; u > k \\ 0 & otherwise, \end{cases} \tag{3.18}$$

$$v' = Dv = \begin{cases} Du & when \;\; u > k \\ 0 & otherwise, \end{cases} \tag{3.19}$$

so that $\Gamma := supp \;\; Dv \subset \{u > k\} CAP :supp \;\; Du$ and $v \in H_0^1(\omega)$. Also, since $d(x) \leq 0$,

$$\int_\omega d(x) u v = \int_\omega d(x) v^2 + k \int_\omega d(x) v \leq 0,$$

so that applying (3.17) with $z = v$ we obtain

$$\int_\omega \epsilon v' v' - (J * u - u) v \leq C \int_\Gamma |v'| v \tag{3.20}$$

Claim 3.2.1.
$\int_\omega (J * u - u) v \leq 0$

Proof :

Extend v by

$$\tilde{v} = \begin{cases} v & in \;\; \omega \\ 0 & otherwise. \end{cases}$$

Clearly $\int_\omega (J*u-u)v = \int_{\mathbb{R}}(J*u-u)\tilde{v}$ and we only need to prove that $\int_{\mathbb{R}}(J*u-u)\tilde{v} \leq 0$. Observe that for any constant α we have $J*\alpha-\alpha=0$ hence

$$\begin{aligned}
\int_{\mathbb{R}}(J*u-u)\tilde{v}(x)dx &= \int_{\mathbb{R}}(J*(u-k)-(u-k))\tilde{v}(x)dx \\
&= \int_{\mathbb{R}} J*(u-k)\tilde{v}(x) - \int_{\mathbb{R}}(\tilde{v})^2(x)dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)(u-k)(y)\tilde{v}(x)dydx - \int_{\mathbb{R}}(\tilde{v})^2(x)dx
\end{aligned}$$

Since $(u-k)(y)\tilde{v}(x) \leq (u-k)^+(y)\tilde{v}(x)$ we have

$$\begin{aligned}
\int_{\mathbb{R}}(J*u-u)\tilde{v}(x)dx &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)(u-k)^+(y)\tilde{v}(x)dydx - \int_{\mathbb{R}}(\tilde{v})^2(x)dx \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)\tilde{v}(y)\tilde{v}(x)dydx - \int_{\mathbb{R}}(\tilde{v})^2(x)dx \\
&\leq \frac{1}{2}\left(2\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)\tilde{v}(y)\tilde{v}(x)dydx - \int_{\mathbb{R}}(\tilde{v})^2(x)dx - \int_{\mathbb{R}}(\tilde{v})^2(y)dy\right) \\
&\leq -\frac{1}{2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)[\tilde{v}(y)^2 - 2\tilde{v}(y)\tilde{v}(x)+\tilde{v}(x)^2]dydx\right) \\
&\leq -\frac{1}{2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)[\tilde{v}(y)-\tilde{v}(x)]^2dydx\right) \leq 0
\end{aligned}$$

and the claim is proved.

□

From our claim we deduce the following inequality :

$$\begin{aligned}
\int_\omega \epsilon(v')^2 &\leq C\int_\Gamma |v'|v \\
&\leq C\|v'\|_{L^2(\omega)}\|v\|_{L^2(\Gamma)}
\end{aligned}$$

and end up with :

$$\|v'\|_{L^2(\omega)} \leq C\|v\|_{L^2(\Gamma)} \tag{3.21}$$

By the one-dimensional Sobolev embedding on compact subsets,

$$\|v\|_{L^\infty(\omega)} \leq C\|v'\|_{L^2(\omega)}. \tag{3.22}$$

Thus v is in $L^\infty(\omega)$ and since

$$\|v\|_{L^2(\Gamma)} \leq |\Gamma|^{\frac{1}{2}}\|v\|_{L^\infty(\omega)}, \tag{3.23}$$

we can combine the last three equations to obtain

$$\|v\|_{L^2(\Gamma)} \leq C|\Gamma|^{\frac{1}{2}}\|v\|_{L^2(\Gamma)}. \tag{3.24}$$

Therefore we have

$$C^{-\frac{1}{2}} \leq |\Gamma| = |supp\ \ Dv| \leq |supp\ \ Du CAP :\{u > k\}|, \tag{3.25}$$

where C is a constant which only depends on $|\omega|$, ϵ and $\|b(x)\|_{L^\infty(\omega)}$.
Since C is independent of k, one can let k go to $\sup_\omega u$. By the dominated convergence theorem, the right-hand side of (3.25) converges to $|supp\ \ Du CAP :\{u = \sup_\omega u\}|$
This implies that there exists a set of positive measure ω^+ where u takes its maximum value and Du is not identically zero. Since u is in H^1, $Du = 0$ a.e. on its level sets and we obtain a contradiction.
This ends the proof.

□

Next, we provide an elementary lemma to construct solutions to constant-coefficient linear equations of the form $Lu = f$.

Lemma 3.2.1.
Let $f \in C_0(\mathbb{R}) CAP :L^2(\mathbb{R})$ and L defined by

$$Lv = \epsilon v'' + J \star v - v + bv' + dv,$$

where $\epsilon \geq 0$, $b, d \in \mathbb{R}$, $d < 0$. Then there exists a unique solution $v \in C_0(\mathbb{R}) CAP :L^2(\mathbb{R})$ (additionally $v \in C^1(\mathbb{R})$ if $b \neq 0$, $v \in C^2(\mathbb{R})$ if $\epsilon > 0$) of

$$\begin{cases} Lv = f & in\ \ \mathbb{R} \\ v \to 0 & x \to -\infty \\ v \to 0 & x \to +\infty \end{cases} \tag{3.26}$$

Proof :

We assume first that $(\epsilon, b) \neq (0, 0)$. Uniqueness follows from the maximum principle. Next, applying Fourier transform to (3.26), we obtain

$$(-\epsilon|\xi|^2 + \hat{J}(\xi) - 1 + ib\xi + d)\hat{v} = \hat{F}$$

Since $\|J\|_{L^1} = 1$, $|\hat{J}| \leq 1$ and since J is even, $\hat{J}$ is real-valued, so that

$$\begin{aligned} |-\epsilon|\xi|^2 + \hat{J}(\xi) - 1 + ib\xi + d| &\geq |-\epsilon|\xi|^2 + \hat{J}(\xi) - 1 + d| \\ &= \epsilon|\xi|^2 + 1 - \hat{J}(\xi) + |d| \geq |d| > 0. \end{aligned}$$

If w is defined by

$$w := (-\epsilon|\xi|^2 + \hat{J}(\xi) - 1 + ib\xi + d)^{-1}\hat{F}, \tag{3.27}$$

it follows that $w \in L^2(\mathbb{R})$ and that $v := \mathcal{F}^{-1}(w) \in L^2(\mathbb{R})$ solves (3.26) in the sense of distributions. By the dominated convergence theorem,$J * v \in C(\mathbb{R})$ and by elliptic regularity applied to the operator $\tilde{L}v = Lv - J \star v$, v has the appropriate regularity for (3.26) to hold in the classical sense.
Also, since either ϵ or b is nonzero, (3.27) implies that $\xi w \in L^2(\mathbb{R})$ so that $v \in H^1_0(\mathbb{R}) \subset C_0(\mathbb{R})$.

When $\epsilon = b = 0$, (3.26) can be rewritten as

$$v = \frac{1}{1 + |d|}\,(J \star v + f) \tag{3.28}$$

It follows easily from the dominated convergence theorem that $J \star v \in C_0(\mathbb{R})$ whenever $v \in C_0(\mathbb{R})$, so that the right-hand side of (3.28) is a (strict) contraction in $C_0(\mathbb{R})$ and admits a unique fixed point. The fact that $v \in L^2(\mathbb{R})$ can be obtained as above.

□

3.3 Construction of a solution of (P)

In this section, we construct an increasing solution of the following problem

$$\begin{cases} \epsilon u'' + J \star u - u - cu' + f(u) = 0 \quad \text{on } \mathbb{R} \\ u \to 0 \quad as \;\; x \to -\infty \\ u \to 1 \quad as \;\; x \to +\infty \end{cases} \tag{3.29}$$

using ordered sub and supersolutions.

Theorem 3.3.1.
Assume there exist two nonnegative smooth functions w and ψ such that w and ψ are respectively a super and a subsolution of (3.29)*, satisfying $\psi \leq w$. If w is increasing, $w \in L^2(\mathbb{R}^-)$ and $1 - w \in L^2(\mathbb{R}^+)$, then there exists a positive increasing solution u of* (3.29).

Remark 3.3.1. *For $\epsilon > 0$, since the weak maximum principle holds, the previous theorem remains true if w and ψ are only assumed to be weak super and subsolutions of* (3.29).

Remark 3.3.2. *Alternatively, the assumption of monotonicity and L^2 integrability on w can be dropped and replaced by the same assumption on ψ.*

We break down the proof into two steps. In the first step we construct a sequence of functions starting from the supersolution. In the second we prove that this sequence has a subsequence which converges to a solution of (3.29).

Proof :

3.3.1 Iteration procedure

Let w and ψ be the two nonnegative functions such that w and ψ are respectively a super and a subsolution of (3.29). Also let $\lambda > 0$ be a parameter to be fixed later on. We claim that there exists a sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ satisfying

$$u_0 = w \text{ and for } n \in \mathbb{N} \setminus \{0\},$$

$$\begin{cases} \epsilon u''_{n+1} + J \star u_{n+1} - (1+\lambda)u_{n+1} - cu'_{n+1} = -f(u_n) - \lambda u_n & in \;\; \mathbb{R} \\ u_{n+1} \to 0 & x \to -\infty \\ u_{n+1} \to 1 & x \to +\infty. \end{cases} \tag{3.30}$$

We proceed as follows : let $g \in C_c^\infty(\mathbb{R})$ be a nonnegative function with $\|g\|_{L^1(\mathbb{R})} = 1$ and $G(x) = \int_{-\infty}^x g(t)\, dt$. Using the substitution $v_n = u_n - G$, (3.30) reduces to

$$\begin{cases} Lv_{n+1} - \lambda v_{n+1} = F(v_n, x) & in \;\; \mathbb{R} \\ v_{n+1} \to 0 & x \to -\infty \\ v_{n+1} \to 0 & x \to +\infty, \end{cases} \tag{3.31}$$

where $Lv = \epsilon v'' + J \star v - v - cv'$ and $F(v, x) = -f(v + G) - \lambda v - LG$.

Now, using Lemma 3.2.1 and induction, to prove that v_n is well-defined, it is enough to show that $v_0 \in L^2(\mathbb{R}) CAP :C_0(\mathbb{R})$ and that $v \in L^2(\mathbb{R}) CAP :C_0(\mathbb{R}) \Longrightarrow F(v, x) \in L^2(\mathbb{R}) CAP :C_0(\mathbb{R})$.

On the one hand since $G(x) = 0$ whenever $-x >> 1$ (and similarly $G(x) = 1$ for $x >> 1$), it follows from the definition of w that $v_0 = w - G \in L^2(\mathbb{R}) CAP :C_0(\mathbb{R})$.

On the other hand given $v \in L^2(\mathbb{R}) CAP :C_0(\mathbb{R})$, since $f(0)=0$,

$$|f(v+G)| \leq \|f'\|_\infty |v+G| \in L^2(\mathbb{R}^-) \qquad \text{and} \qquad \lim_{-\infty} f(v+G) = 0$$

and since $f(1)=0$,

$$|f(v+G)| \leq \|f'\|_\infty |v+G-1| \in L^2(\mathbb{R}^+) \qquad \text{and} \qquad \lim_{+\infty} f(v+G) = 0,$$

so that $f(v+G) \in L^2(\mathbb{R}) CAP :C_0(\mathbb{R})$. Clearly $G', G'' \in L^2(\mathbb{R})$. Finally, the following lemma applied to $u = G$ shows that $J \star G - G \in L^2(\mathbb{R})$ and we can conclude that u_n solving (3.30) is well-defined.

Lemma 3.3.1.
Let $u \in C^1(\mathbb{R}) CAP :L^\infty(\mathbb{R})$. Then

$$\|J \star u - u\|_{L^2(\mathbb{R})} \leq C \|u'\|_{L^2(\mathbb{R})}.$$

Proof of the lemma

Using the fundamental theorem of calculus, we have that

$$J \star u(x) - u(x) = \int J(y)(u(x-y) - u(x))dy = \int J(y) y \left(\int_0^1 u'(x-ty)\, dt \right) dy.$$

By the Cauchy-Schwartz inequality, it follows that

$$\begin{aligned} |J \star u(x) - u(x)|^2 &\leq \int_{\mathbb{R}} \int_0^1 J(y)|y|(u')^2(x-ty) dt\, dy \cdot \int_{\mathbb{R}} \int_0^1 J(y)|y| dt\, dy \\ &\leq C \int_{\mathbb{R}} \int_0^1 J(y)|y|(u')^2(x-ty) dt\, dy, \end{aligned}$$

hence

$$\|J \star u - u\|^2_{L^2(\mathbb{R})} \leq C \int_{\mathbb{R}} J(y)|y| \left(\int_0^1 \int_{\mathbb{R}} (u')^2(x-ty) dx\, dt \right) dy \leq C \|u'\|^2_{L^2(\mathbb{R})}.$$

□

3.3.2 Passing to the limit as $n \to \infty$

Since ψ and w are ordered function i.e. $\psi \leq w$, it follows easily from induction and the maximum principle that for all $n \in \mathbb{N} \setminus \{0\}$,

$$\psi \leq u_n \leq w \tag{3.32}$$

Also, if $\tau > 0$ and $z_n(x) = u_n(x+\tau) - u_n(x)$,we have

$$\begin{cases} \epsilon z''_{n+1} + J \star z_{n+1} - (1+\lambda) z_{n+1} - c z'_{n+1} = -(f+\lambda)(u_n(x+\tau)) + (f+\lambda)(u_n(x)) & in \quad \mathbb{R} \\ z_{n+1} \to 0 \qquad |x| \to \infty. \end{cases} \tag{3.33}$$

Choosing $\lambda > 0$ so large that $-f - \lambda$ is nonincreasing, it follows from induction, the maximum principle and the fact that w is nondecreasing that for each $n \in \mathbb{N}$, $z_n \leq 0$ i.e.

$$x \to u_n(x) \text{ is a nondecreasing function.} \tag{3.34}$$

Using (3.32), (3.34) and Helly's lemma, it follows that a subsequence of $\{u_n\}$ converges pointwise to a nondecreasing function u satisfying

$$\psi \leq u \leq w$$

By the dominated convergence theorem, $J \star u_n - u_n \to J \star u - u$. Rewriting (3.30) as

$$\epsilon u''_{n+1} - cu'_{n+1} = u_{n+1} - J \star u_{n+1} - \lambda(u_n - u_{n+1}) - f(u_n), \tag{3.35}$$

observing that the right-hand side in the above equation is uniformly bounded and using elliptic regularity, we conclude that $\{u_n\}$ is bounded e.g. in $C^{1,\alpha}(\omega)$, where $\alpha \in (0,1)$ and ω is an arbitrary bounded open subset of $\mathbb{R}$. Hence $u \in C^2(\mathbb{R})$ and by Helly's lemma,

$$u_n \to u \qquad \text{uniformly in } \mathbb{R}. \tag{3.36}$$

Differentiating (3.35), we obtain similarly local $C^{2,\alpha}$ bounds on u_n so that

$$u_n \to u \qquad \text{in } C^{2,\alpha}_{loc}. \tag{3.37}$$

Using (3.37), it is now a trivial matter to pass to the limit in the equation. Furthermore, since $\psi \leq u \leq w$, u also has the desired limits at infinity of (3.29) and we have thus constructed an increasing solution u of (3.29).

□

3.4 L^2 estimates of solutions of (3.38)

Our goal in this section is to provide L^2 estimates of monotone solutions of (3.38).

$$\begin{cases} \epsilon u'' + J \star u - u - cu' + f(u) = 0 \quad \text{on } \mathbb{R} \\ u \to 0 \quad as \;\; x \to -\infty \\ u \to 1 \quad as \;\; x \to +\infty \end{cases} \tag{3.38}$$

Let (u, c) be a smooth solution of (3.38). We start out by showing that u' vanishes at infinity. It follows from (3.38) that for some $h(x) \in C_0(\mathbb{R})$,

$$\epsilon u'' = cu' + h(x). \tag{3.39}$$

Fix $\delta > 0$ and let $R > 0$ be such that $|h(x)| < \delta$ for $|x| > R$. Suppose by contradiction that for some $x_0 > R$ we have $cu'(x_0) - \delta > 0$. Then (3.39) implies that $u''(x_0) > 0$ and in fact that $u'' > 0$ on $[x_0, \infty)$, contradicting $0 \leq u \leq 1$. Hence $cu' \leq \delta$ on $[R, \infty)$ i.e.

$$\lim_{+\infty} u' = 0.$$

Similarly, if for some $x_0 < -R$, $cu'(x_0) - \delta > 0$, either u remains convex in $(-\infty, x_0]$, which is possible only if $\lim_{-\infty} u' = 0$, either there exists $x_1 < x_0$ such that $u''(x_1) = 0$, whence u is concave in $(-\infty, x_1]$ in view of (3.39), which is again impossible. Hence

$$\lim_{-\infty} u' = 0.$$

Remark 3.4.1. *As a direct consequence of* (3.39), *we also have* $\lim_{\pm\infty} u'' = 0$.

Now we show that $u' \in L^2(\mathbb{R})$. Indeed, multiplying (3.38) by u' and integrating over $\mathbb{R}$ yields

$$c \int (u')^2 = \int f(u)u' = \int_0^1 f(s)\, ds < \infty. \tag{3.40}$$

Next, we show that $f(u) \in L^2(\mathbb{R})$. We need the following lemma :

Lemma 3.4.1.
$J \star u - u \in L^1(\mathbb{R})$. *More precisely,*

$$\|J \star u - u\|_{L^1} \leq \int_{\mathbb{R}} J(z)|z|dz \qquad \textit{and} \qquad \int_{\mathbb{R}} (J \star u - u) = 0$$

Proof :

Clearly,

$$\int_{\mathbb{R}} |(J \star u - u)| \leq \int_{\mathbb{R}^2} J(x-y)|u(y) - u(x)|dy\, dx. \tag{3.41}$$

Since $u \in C^1(\mathbb{R})$,

$$|u(y) - u(x)| = |x-y| \int_0^1 u'(y + s(x-y))ds$$

Plug this equality in (3.41) to obtain :

$$\int_{\mathbb{R}^2} J(x-y)|u(y) - u(x)|dydx = \int_{\mathbb{R}^2} J(x-y)|x-y| \int_0^1 u'(x + s(y-x))ds\, dy\, dx \tag{3.42}$$

Make the change of variables $z = x - y$, so that the right-hand side of (3.42) becomes :

$$\int_{\mathbb{R}^2} J(z)|z| \int_0^1 u'(x - sz)ds\, dz\, dx \tag{3.43}$$

Because all terms are positive, we may apply Tonnelli's Theorem and permute the order of integration. We obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} J(z)|z| \int_0^1 u'(x - sz)ds\, dz\, dx &= \textstyle\int_0^1 \int_{\mathbb{R}^2} J(z)|z|u'(x - sz)dx\, dz\, ds \\
&= \textstyle\int_0^1 \int_{\mathbb{R}} J(z)|z|[u(+\infty) - u(-\infty)]dz\, ds \\
&= \textstyle\int_{\mathbb{R}} J(z)|z|dz \leq \infty
\end{aligned}$$

These last computations show that $J \star u - u$ is an integrable function and gives a bound on its L^1 norm. Let us now compute $\int_{\mathbb{R}} (J \star u - u)\, dx$. We have

$$\int_{\mathbb{R}} J \star u - udx = \int_{\mathbb{R}^2} J(x-y)(u(y) - u(x))dy\, dx.$$

Let $z = x - y$ so that

$$\int_{\mathbb{R}^2} J(z)(u(x-z) - u(x))dz\, dx = \int_{\mathbb{R}^2} J(z)(u(y) - u(y+z))dy\, dz.$$

Make the change of variable $z \to -z$ in the left integral and obtain

$$I_1 := \int_{\mathbb{R}^2} J(z)(u(x+z) - u(x))dz\, dx = \int_{\mathbb{R}^2} J(z)(u(y) - u(y+z))dy\, dz =: I_2.$$

Fubini's theorem applied to the last integral shows that $I_1 = -I_2$, hence $I_1 = I_2 = 0$.

□

Next, we integrate (3.38) over $[R, \infty)$, where $R > 0$ is chosen so large that $f(u(x)) > 0$ for $x > R$. We get

$$-\epsilon u'(R) + \int_R^\infty (J \star u - u) + cu(R) + \int_R^\infty f(u) = 0.$$

Using Lemma 3.4.1, we conclude that $f(u) \in L^1(R, \infty)$. Working similarly on $(-\infty, -R)$, it follows that $f(u) \in L^1(\mathbb{R})$.

Multiplying (3.38) by u'' and integrating over $\mathbb{R}$ yields

$$\epsilon \int (u'')^2 + \int (J * u - u)u'' + \int f(u)u'' = 0.$$

Since $J*u-u$ and $f(u)$ are integrable and since u'' is bounded (by Remark 3.4.1), we conclude that $u'' \in L^2(\mathbb{R})$, which, using (3.38),(3.40) and Lemma 3.3.1, implies that $f(u) \in L^2(\mathbb{R})$.

We finally prove that $u \in L^2(\mathbb{R}^-)$ and $1 - u \in L^2(\mathbb{R}^+)$. Using Lemma 3.2.1, we know that there exists $w \in L^2(\mathbb{R})$ such that $v := w + G$ (with $G \in C^\infty(\mathbb{R})$, $G \equiv 0$ in a neighbourhood of $-\infty$ and $G \equiv 1$ in a neighbourhood of $+\infty$) solves

$$\begin{cases} \epsilon v'' + J \star v - v - cv' + f(u) = 0 \quad \text{on } \mathbb{R} \\ v \to 0 \quad as \ x \to -\infty \\ v \to 1 \quad as \ x \to +\infty \end{cases} \tag{3.44}$$

Since both u and v solve (3.44), it follows from the maximum principle that $u \equiv v$ i.e. u has the desired integrability.

□

3.5 Min-max formula : cases A1 and A2

In this section we prove the min-max formula for the asymptotic speed in the case where the non linearity is of bistable or ignition type. The proof relies on the construction of appropriate sub and supersolutions for the problem (P), and a uniqueness theorem which holds for solutions of (P) only when f is of bistable or ignition type.

We will prove the following :

Theorem 3.5.1.
Let $X = \{w \in C^0(\mathbb{R}) |\ w \text{ increasing}, \quad w(+\infty) = 1 \text{ and } w(-\infty) = 0\}$, then the (unique) front speed is given by

$$c^* = \min_{w'>0,\ w \in X} \max_{x \in \mathbb{R}} \{ \frac{\epsilon w'' + J \star w - w + f(w)}{w'} \}. \tag{3.45}$$

Proof of Theorem 3.5.1 :

Define c^1 by :

$$c^1 = \min_{w'>0,\ w\in X} \max_{x\in\mathbb{R}} \{ \frac{\epsilon w'' + J\star w - w + f(w)}{w'} \}, \tag{3.46}$$

with X as above.

Then we just have to show that

$$c^* = c^1. \tag{3.47}$$

Since we know from the previous section that there exists an increasing solution (u^*, c^*) to (P), taking $w = u^*$ in the definition of c^1 yields

$$c^1 \leq c^*.$$

The main difficulty lies in the proof of the reverse inequality $c^1 \geq c^*$. We argue by contradiction and assume that $c^1 < c^*$. Let c be such that $c^1 \leq c < c^*$. From the definition of c^1, there exists a positive increasing function w which satisfies

$$\begin{cases} \epsilon w'' + J\star w - w - cw' + f(w) \leq 0 & in\ \mathbb{R} \\ w \to 0 & x \to -\infty \\ w \to 1 & x \to +\infty, \end{cases} \tag{3.48}$$

Since $c < c^*$, and $(u^*)' > 0$, u^* satisfies :

$$\begin{cases} \epsilon(u^*)'' + J\star u^* - u^* - c(u^*)' + f(u^*) = (c^* - c)(u^*)' \geq 0 & in\ \mathbb{R} \\ u^* \to 0 & x \to -\infty \\ u^* \to 1 & x \to +\infty, \end{cases} \tag{3.49}$$

Observe that any translation of u^* and w are also respectively a sub and a supersolution of the same problem. Therefore, if we can order two translations of u^* and w, we will be done. Indeed, from the a priori estimates of Section 3.4 and Theorem 3.3.1, there would exist a positive solution of the following problem :

$$\begin{cases} \epsilon u'' + J\star u - u - cu' + f(u) = 0 & in\ \mathbb{R} \\ u \to 0 & x \to -\infty \\ u \to 1 & x \to +\infty, \end{cases} \tag{3.50}$$

which contradicts the uniqueness theorem 3.1.1.

□

The proof of Theorem 3.5.1 thus reduces to finding ordered translations of w and u^*. We claim the following

Lemma 3.5.1.
There exists constants a and b such that $w(s+a) \geq u^*(s+b)$

Proof of Lemma 3.5.1 :

Without loss of generality, we may always assume $w(0) = u^*(0) = \frac{\theta}{2}$.

Now we define some quantities that we will use to construct sub and supersolutions. Let α positive, such that

$$f'(p) < -2\alpha \quad \text{whenever} \quad | p-1 |< \alpha \tag{3.51}$$

Let $\mu \in (0, \frac{\alpha}{2})$ and define $a(s) = \mu e^{-\alpha s}$.
Choose $M > 0$ and $K > 0$ such that :

$$w(\xi) - 1 < \frac{\alpha}{2} \quad \text{in} \quad (M-1, +\infty) \tag{3.52}$$

$$w'(\xi) > K \quad \text{in} \quad [-1, M+1] \tag{3.53}$$

Define the following function :

$$b(s) = \frac{\mu\bar{\alpha}}{K}(1 - e^{-\alpha s}),$$

where $\bar{\alpha} = 1 + \frac{\max\{f'(p) \quad -1 \leq p \leq 2\}}{\alpha}$.

We will assume further that $\mu \leq \min\{\frac{\theta}{2}, \frac{K}{\bar{\alpha}}\}$.
We now define a sub and a supersolution as follows :

$$\tilde{w}(\xi, s) = w(\xi + b(s)) + a(s) \tag{3.54}$$

$$\tilde{u}(\xi, s) = u^*(\xi - \tau) \tag{3.55}$$

where $\tau > 0$ is taken so large that

$$w(\xi) + a(0) > u^*(\xi - \tau).$$

Let $z(\xi, s) = (\tilde{w} - \tilde{u})(\xi, s)$. z satisfies the next equations :

$$-\frac{\partial z}{\partial s} + \epsilon z'' + (J \star z - z) - cz_\xi \leq -a'(s) - w'(\xi, s)b'(s) + f(\tilde{u}(\xi, s)) - f(\tilde{w}(\xi, s) - a(s)) \tag{3.56}$$

$$z(\xi, 0) > 0 \ \ \forall \xi \in \mathbb{R} \tag{3.57}$$

$$z(\pm\infty, s) = a(s) \ \ \forall s \in \mathbb{R} \tag{3.58}$$

From (3.57,3.58), by continuity, there exists $s_0 = \sup\{s > 0 \mid z(\xi, s) > 0 \ \ \forall \xi \in \mathbb{R}\}$.

Claim 3.5.1. $s_0 = +\infty$.

proof of Claim 3.5.1 :

We argue by contradiction. If not, $s_0 < +\infty$ and there exits $\xi_0 \in \mathbb{R}$ such that

$$0 = z(\xi_0, s_0) = \min_{\mathbb{R}} z(\xi, s_0) \tag{3.59}$$

Next, we use a kind of localization of minimum lemma. More precisely we claim

Claim 3.5.2. *Under the previous assumptions, we have* $\xi_0 > -1$

proof of Claim 3.5.2 :

Let $Z(\xi) = z(\xi, s_0)$, then Z satisfies :

$$\epsilon Z'' + J \star Z - Z - cZ_\xi = f(\tilde{u}(\xi, s_0)) - f(\tilde{w}(\xi, s_0) - a(s_0))$$

So at ξ_0 we have,

$$Z''(\xi_0) \geq 0$$
$$(J \star Z - Z)(\xi_0) > 0$$
$$Z_\xi(\xi_0) = \tilde{w}(\xi_0, s_0) - \tilde{u}(\xi_0, s_0) = 0$$

Thus $f(\tilde{u}(\xi_0, s_0)) - f(\tilde{w}(\xi_0, s_0) - a(s_0)) > 0$, which implies $f(\tilde{u}(\xi_0, s_0)) > 0$.
Recall, that

$$\begin{aligned} \tilde{u}(\xi_0, s_0) &= \tilde{w}(\xi_0, s_0) \\ \Rightarrow u^*(\xi_0 - \tau) &= w(\xi_0 + b(s_0)) + a(s_0) \end{aligned}$$

Thus,

$$\begin{aligned} u(\xi_0, s_0) = w(\xi_0 + b(s_0)) + a(s_0) &> \theta \\ \Rightarrow w(\xi_0 + b(s_0)) > \theta - a(s_0) &> \frac{\theta}{2} \\ \Rightarrow \xi_0 > w^{-1}(\frac{\theta}{2}) &- b(s_0) \\ \Rightarrow \xi_0 &> -1 \end{aligned}$$

□

Remark 3.5.1. *Claim 3.5.2 bounds from below the minimum of* z.

Now, observe that, at (ξ_0, s_0), z satisfies :

$$-\frac{\partial z(\xi_0, s_0)}{\partial s} + \epsilon z'' + (J \star z - z)(\xi_0, s_0) - cz_\xi(\xi_0, s_0) \geq 0$$

and

$$-\frac{\partial z(\xi_0, s_0)}{\partial s} + [\epsilon z'' + (J \star z - z) - cz_\xi](\xi_0, s_0) \leq -a'(s_0) - w'(\xi_0, s_0)b'(s_0) + f(\tilde{u}(\xi_0, s_0)) - f(\tilde{w}(\xi_0, s_0) - a(s_0))$$

So we end up with

$$Q = -a'(s_0) - w'(\xi_0, s_0)b'(s_0) + f(\tilde{u}(\xi_0, s_0)) - f(\tilde{w}(\xi_0, s_0) - a(s_0)) \geq 0$$

Since at (ξ_0, s_0) we have,

$$\tilde{u}(\xi_0, s_0) = \tilde{w}(\xi_0, s_0)$$

and f is a smooth function, we can rewrite Q as

$$Q = \mu e^{-\alpha s_0}[\alpha - \frac{\alpha\bar{\alpha}}{K} w'(\xi_0 + b(s_0)) + f'(d)] \geq 0 \tag{3.60}$$

for some $d \in [\tilde{w}(\xi_0, s_0) - a(s_0), \tilde{w}(\xi_0, s_0)]$.
Now, from Claim 3.5.2, we are lead to considering two cases :

1. case : $\xi_0 \in [-1, M]$

   Then, Q would satisfy :

$$0 > \mu e^{-\alpha s_0}[\alpha(1 - \frac{w'(\xi_0, s_0)}{K}) - \frac{w'(\xi_0, s_0)}{K} \max\{f'(p) \quad -1 \leq p \leq 2\} + f'(d)]$$

   which contradicts (3.60).

2. case : $\xi_0 > M$

   Then, Q would then verify :

$$\mu e^{-\alpha s_0}[\alpha - \frac{\alpha\bar{\alpha}w'(\xi_0, s_0)}{K} + f'(d)] < \mu e^{-\alpha s_0}[-\alpha - \frac{\alpha\bar{\alpha}w'(\xi_0, s_0)}{K}] < 0$$

   which also contradicts (3.60), thus proving Claim 3.5.1.

□

From Claim 3.5.1, we have $z(\xi, s) \geq 0$ for all $(\xi, s) \in \mathbb{R} \times \mathbb{R}^+$. Let s go to infinity : we end up with $w(\xi - a) \geq u^*(\xi - b)$, where $a = \frac{\mu\bar{\alpha}}{K}$ and $b = \tau$. This ends the proof of Lemma 3.5.1.

□

3.6 Min-max formula : the monostable case

In this section we prove the min-max formula for the minimal speed in the case where the non linearity f is monostable. We are concerned with the following problem.

$$\begin{cases} \epsilon u'' + J \star u - u - cu' + f(u) = 0 & in \ \mathbb{R} \\ u \to 0 & x \to -\infty \\ u \to 1 & x \to +\infty, \end{cases} \tag{3.61}$$

where f is monostable and J has a fast decay near infinity. Uniqueness of solutions no longer holds in this situation. Nevertheless, the min-max formula still holds :

Theorem 3.6.1.
Let $X = \{w \in C^0(\mathbb{R}) | w(+\infty) = 1 \ \ and \ \ w(-\infty) = 0\}$, then we have

$$c^* = \min_{w'>0, \ w \in X} \max_{x \in \mathbb{R}} \{\frac{\epsilon w'' + J \star w - w + f(w)}{w'}\}. \tag{3.62}$$

Proof :

We define c^1 as in the previous section :

$$c^1 = \min_{w'>0, \ w \in X} \max_{x \in \mathbb{R}} \{\frac{\epsilon w'' + J \star w - w + f(w)}{w'}\} \tag{3.63}$$

where $X = \{w \in C^0(\mathbb{R}) | w(+\infty) = 1 \ \ and \ \ w(-\infty) = 0\}$.

Then again we just have tho show,

$$c^* = c^1, \tag{3.64}$$

As in the previous section, since we know from [20] that there exists an increasing solution of (3.61), for the speed c^*, we obviously have $c^1 \leq c^*$. The main difficulty again lies in the proof of $c^1 \geq c^*$. Before, showing $c^1 \geq c^*$, we will characterize the behavior of the speed of solutions of (3.61) when f is of ignition type.

Lemma 3.6.1.
Let $\epsilon \geq 0$, let f and g be two functions of ignition type, such that $f \geq g$, $f \not\equiv g$, then the corresponding speed c_f, c_g satisfy $c_f > c_g$.

From this monotone charaterization of the speed, we easily obtain the following corollary :

Corollary 3.6.1.
There exists a sequence of approximations $(f_n)_{n\in\mathbb{N}}$ of f such that for each n f_n is of ignition type and the corresponding speed satisfies

$$\lim_{n\to+\infty} c_n = c^*$$

Proof of Corollary 3.6.1 :

Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0 as n goes to infinity. And let χ_{ϵ_n} satisfy the following assumptions :

– $\chi_{\delta_n} \in C_0^\infty(\mathbb{R})$
– $0 \leq \chi_{\delta_n} \leq 1$
– $\chi_{\delta_n}(s) \equiv 0$ for $s \leq \delta_n$ and $\chi_{\delta_n}(s) \equiv 1$ for $s \geq 2\delta_n$
– χ_{δ_n} is a monotone increasing sequence of function (i.e.$\chi_{\delta_n} \leq \chi_{\delta_p}$ for $p \geq n$)

Now define a new function $f_{\delta_n} = f\chi_{\delta_n}$. Since f_{δ_n} is of ignition type, there exists a unique travelling wave solution (u_n, c_n) of (3.65), cf [17].

$$\begin{cases} \epsilon u_n'' + J \star u_n - u_n - \tilde{c}_n u_n' + f_{\delta_n}(u_n) = 0 & in \ \mathbb{R} \\ u_n \to 0 & x \to -\infty \\ u_n \to 1 & x \to +\infty, \end{cases} \tag{3.65}$$

Observe that $f(s) \geq f_{\delta_n}$, therefore (u^*, c^*) satisfies

$$\begin{cases} \epsilon(u^*)'' + J \star u^* - u^* - c^*(u^*)' + f_{\delta_n}(u^*) \leq 0 & in \ \mathbb{R} \\ u^* \to 0 & x \to -\infty \\ u^* \to 1 & x \to +\infty, \end{cases} \tag{3.66}$$

By Lemma 3.6.1, $\{c_n\}$ is an increasing sequence.Now we will show that this sequence is bounded by c^*. We claim the following

Claim 3.6.1. $\forall n \in \mathbb{N} \ \ c_n \leq c^*$

Proof :

We argue by contradiction, suppose not. Then, there exists $c_n > c^*$. Since u_n is monotone increasing, u_n satisfies

$$\begin{cases} \epsilon u_n'' + J \star u_n - u_n - c^* u_n' + f_{\delta_n}(u_n) \geq 0 & in \ \mathbb{R} \\ u_n \to 0 & x \to -\infty \\ u_n \to 1 & x \to +\infty, \end{cases} \tag{3.67}$$

Therefore (u_n, c^*) is a subsolution and (u^*, c^*) a supersolution of (3.65). Since f_{δ_n} is of ignition type we can apply Lemma 3.5.1 to get constants a and b such that $u^*(s+a) \geq u_n(s+b)$. Then, as in the previous section, we can apply Theorem 3.3.1, which implies the existence of a non trivial solution (u, c^*) to (3.65) which contradicts the uniqueness of the solution (u_n, c_n). This proves Claim 3.6.1.

□

Since c_n is a bounded increasing sequence, it converges to a constant γ. From standard a-priori estimates, there exists a subsequence still denoted u_n which converges to an increasing function $\bar{u}$ solution of (3.61).

Since $c^* = \inf\{c > 0 | \ \text{(3.61)} \quad \text{has a positive increasing solution}\}$, we must have $\gamma = c^*$, which proves Corollary 3.6.1.

□

Now, let us prove Lemma 3.6.1.

Proof of Lemma 3.6.1

Again, we argue by contradiction. Assume that $c_f < c_g$. Then, since they are increasing, u_f and u_g will be respectively a super and subsolution of

$$\begin{cases} \epsilon w'' + J \star w - w - c_g w' + f(w) \leq 0 & in \ \mathbb{R} \\ w \to 0 & x \to -\infty \\ w \to 1 & x \to +\infty, \end{cases} \tag{3.68}$$

Since f is of ignition type, we can use Lemma 3.5.1 and Theorem 3.3.1 to get a non trivial solution (u, c_g) of (3.68), which violates the uniquess theorem 3.1.1. The strict inequality follows by the same argument.

□

We are now ready to prove the last inequality

$$c^1 \geq c^* \tag{3.69}$$

Proof of inequality (3.69) **:**

We argue by contradiction, assuming that (3.69) is not true : there exists $c > 0$ such that $c_1 \leq c < c^*$. Therefore by the definition of c^1, there exists a positive increasing function w such that

$$\begin{cases} \epsilon w'' + J \star w - w - c w' + f(w) \leq 0 & in \ \mathbb{R} \\ w \to 0 & x \to -\infty \\ w \to 1 & x \to +\infty, \end{cases} \tag{3.70}$$

Now, by Corollary 3.6.1, there exists $\delta_n > 0$ and u_n increasing and $c_n > 0$ such that

$$\begin{cases} \epsilon u_n'' + J \star u_n - u_n - c_n u_n' + f_{\delta_n}(u_n) = 0 & in \ \mathbb{R} \\ u_n \to 0 & x \to -\infty \\ u_n \to 1 & x \to +\infty. \end{cases} \tag{3.71}$$

Therefore, if we replace c_n by c in (3.71), w and u_n become a super and a subsolution of the problem. We can then apply Lemma 3.5.1 and Theorem 3.3.1 to get a solution of (3.71) with speed c. But this contradicts the uniquess of the speed for problems with ignition non-linearities.

□

This ends the proof of the min-max formula in the monostable case.

We can give a more precise bound for the minimal speed, if in addition to the common assumption that f is monostable, we assume further that $f'(0)s \geq f(s)$. This new assumption is known as the KPP assumption. When there is no integral terms, then it is known that $c^* = 2\sqrt{f'(0)}$, which can be also formulated as

$$c^* = \min_{\lambda>0}\{\frac{1}{\lambda}(\lambda^2 + f'(0))\}$$

. We derive a similar formula when there is an integral term. Namely, we have

$$c^* \leq \min_{\lambda>0}\{\frac{1}{\lambda}(\epsilon\lambda^2 + \int_{\mathbb{R}} J(z)e^{\lambda z}dz - 1 + f'(0))\} = \gamma.$$

There are hints that in fact there is equality in the above equation, but we were not able to prove it. The proof relies on the same ideas : one assumes that the inequality is false then picks a constant $c \in (\gamma, c^*)$, finds good super and subsolution for an ignition-type problem and concludes with the existence and uniqueness theorem. We omit the details of the proof and just present the construction of the super solution. A straight forward computation shows that exponential functions are eigenfunctions of the operator $Lw + f'(0)w := \epsilon w'' + J \star w - w - cw' + f'(0)w$, i.e.($[L + f'(0)]e^{\lambda x} = h(\lambda)e^{\lambda x}$).
Therefore, since f is of KPP type, exponential solution satisfy

$$L(e^{\lambda x}) + f(e^{\lambda x}) \leq h(\lambda)e^{\lambda x} \tag{3.72}$$

where $h(\lambda) = \epsilon\lambda^2 + \int_{\mathbb{R}} J(z)e^{\lambda z}dz - 1 - c\lambda + f'(0)$
Now use the definitions of γ and c to find some λ such that $h(\lambda) \leq 0$. Then argue as above : since there exists a supersolution of the monostable problem (3.61), and $c_n \to c^*$, we get a contradiction.

□

Note au CRAS

Chapitre 4

On uniqueness and monotonicity of solutions of non-local reaction diffusion equations

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Sommaire

4.1 Introduction and main result

In this note, I investigate monotonicity and uniqueness of positive solutions of the following integrodifferential problem

$$J \star u - u - cu' + f(u) = 0 \quad \text{in} \quad \mathbb{R}, \tag{4.1}$$
$$u(x) \to 0 \quad \text{as} \quad x \to -\infty, \tag{4.2}$$
$$u(x) \to 1 \quad \text{as} \quad x \to +\infty, \tag{4.3}$$

where J is an even nonnegative continuous function on $\mathbb{R}$ with $\int_{\mathbb{R}} J(z)dz = 1$, c is a real constant, $J \star u(x) = \int_{\mathbb{R}} J(x-y)u(y)dy$ is the standard convolution and $f : \mathbb{R} \to \mathbb{R}$ is appropriately smooth, with $f(0) = f(1) = 0$.

Remark 4.1.1. *The previous problem is invariant under translation, which means that for any real τ, $u_\tau := u(. + \tau)$ is still a solution of* (4.1)-(4.3).

Such a problem arises in the study of so-called *Travelling Fronts* (solutions of the form $u(x,t) = \phi(x + ct)$) of the following nonlocal phase-transition problem

$$\frac{\partial u}{\partial t} - (J \star u - u) = f(u) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+. \tag{4.4}$$

The constant c is called the speed of the front and is usually unknown. The operator $Lu = \int_{\mathbb{R}} J \star u - u$ can be viewed as a diffusion operator. This kind of equation was originally introduced in 1937 by Kolmogorov, Petrovskii and Piskunov [46] as a way to derive the Fisher equation (i.e (4.5) below with $f(s) = s(1-s)$)

$$\frac{\partial U}{\partial t} = U_{xx} + f(U) \;\; \text{for (x,t)} \;\; \in \mathbb{R} \times \mathbb{R}^+. \tag{4.5}$$

In the literature, much attention has been drawn to reaction-diffusion equations like (4.5), as they have proved to give a robust and accurate description of a wide variety of phenomena, ranging from combustion to bacterial growth, nerve propagation or epidemiology. We point the interested reader to the following articles for more informations : [6, 7, 31, 35, 46, 48, 62].

For nonlinearities f satisfying $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$ and for some $\epsilon > 0$, $f'(s) \le 0$ in $s < \epsilon$ and in $1 - \epsilon < s$, monotonicity and uniqueness of travelling-front solutions of the reaction-diffusion equation (4.5) is well known, see [6, 9, 10, 11, 34, 55]. By uniqueness of travelling wave solution, we mean that if (u, c) and (v, c') are travelling-wave solution of (4.5) then $c = c'$ and $u(x) = v(x + \tau)$ for some real τ. Observe that the Fisher nonlinearity ($f(s) = s(1-s)$) does not satisfy theses assumptions. For this kind of nonlinearity, it is known that several travelling-wave solutions exist, see [11, 46]. However, in that case, using a precise exponential asymptotic expansion of the solutions in a neighborhood $(-\infty, M)$ of $-\infty$, Berestycki and Nirenberg obtained in [11] the monotonicity and uniqueness up to translation of the travelling-wave solutions of (4.5) (i.e. If (u, c) and (v, c) are travelling-wave solution of (4.5) then $u(x) = v(x + \tau)$ for some real τ.

For the nonlocal equation (4.1)-(4.3), existence, uniqueness and monotonicity were first obtained by Bates, Fife, Ren and Wang [5] and later by Chen [17] for a bistable nonlinearity f, i.e. a nonlinearity $f \in C^1(\mathbb{R})$ satisfying for some $\rho > 0$, $f|_{(0,\rho)} < 0$, $f|_{(\rho,1)} > 0$, $f(0) = f(1) = 0$, $f'(0) < 0$ and $f'(1) < 0$. In that case, they showed the following

Theorem 4.1.1. *[5]*
Assume that $J \in C^1(\mathbb{R})$ is a positive, even, integrable function with unit mass. Let (u, c) and (v, c') be solutions of (4.1)-(4.3) *with bistable nonlinearity and assume that (u, c) is monotone increasing, then $c = c'$. Moreover, if either u or v is continuous or if v is monotone then $v(x) = u(x + \tau)$ for some $\tau \in \mathbb{R}$.*

Notice that Theorem 4.1.1 contains two distinct uniqueness results. Indeed, it states that the speed is unique and the profile u is unique. It also shows how rigid Problem (4.1)-(4.3) is, since it has a positive solution only for one real value of c.

Our first result is a generalization of the uniqueness result for the continuous profile contained in Theorem 4.1.1 to more general nonlinearities,

Theorem 4.1.2.
Assume that $J \in C^0(\mathbb{R})$ is a positive, even, integrable function with unit mass. Let $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$ be such that f satisfies for some $\epsilon > 0$, $f'(s) \leq 0$ when $s < \epsilon$ and when $1 - \epsilon < s$. Let (u, c) and (v, c) be two continuous solutions of (4.1)-(4.3) *then $u(.) = v(. + \tau)$ for some real τ. Moreover, the solution u is monotone increasing.*

Note that the assumption on f in the previous theorem covers the case of bistable nonlinearities and that in the case of continuous solutions, the existence of a monotone solution u is not needed anymore.

Also observe that our theorem does not cover the case of discontinuous solutions of (4.1)-(4.3) which appears when the speed $c = 0$. However when $c = 0$, there is an example of existence of several discontinuous positive solutions of (4.1)-(4.3). A generalization of Theorem 4.1.2 for monotone discontinuous solutions is currently under investigation.

Our second result concerns the uniqueness of the speed c for continuous solutions when they exist. Namely, we have

Theorem 4.1.3.
Assume that $J \in C^0(\mathbb{R})$ is a positive even integrable function with unit integral. Let $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$ be such that for some $\epsilon > 0$, $f'(s) \leq 0$ when $s < \epsilon$ and when $1 - \epsilon < s$. Let (u, c) and (v, c') be two continuous positive solutions of (4.1)-(4.3)*, then $c = c'$.*

As we previously mentioned, the assumptions made on f in Theorem 4.1.2, do not cover the case of the Fisher nonlinearity. Our next result deals with the monotonicity of solutions in that case. With some extra assumption on the behavior of the solution in some neighborhood $(-\infty, M)$ of $-\infty$, we show that the solutions are monotone increasing. More precisely we have

Theorem 4.1.4.
Assume that $J \in C^0(\mathbb{R})$ is a positive even integrable function with unit mass. Let $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$ be such that for some $\epsilon > 0$, $f'(s) \leq 0$ when $1 - \epsilon < s$. Let (u, c) be a positive continuous solution of (4.1)-(4.3)*, such that u is monotone increasing in some neighborhood $(-\infty, M)$ of $-\infty$ then u is monotone increasing in all of $\mathbb{R}$.*

Assuming that the solutions u are monotone increasing in some neighborhood $(-\infty, M)$ of $-\infty$ may seem strange, however such behavior holds true for travelling-front solutions of the reaction-diffusion equation (4.5). Indeed, when f is monostable, ($f \in C^1(\mathbb{R})$, such that $f(0) = f(1) = 0, \;\; f|_{(0,1)} > 0$) with $f'(0) > 0$, the positive solutions (u, c) of (4.5) satisfies the following expansion near $-\infty$,

$$u(x) = Ce^{\lambda_0 x} + o(e^{\lambda_0 x})$$
$$u'(x) = \lambda_0 Ce^{\lambda_0 x} + o(e^{\lambda_0 x}),$$

where C is a positive constant and λ_0 is one of the positive roots of $\lambda^2 - c\lambda + f'(0)$. u' is therefore strictly positive in a neighborhood $(-\infty, M)$ of $-\infty$, which is our needed assumption. For details on the proof of this expansion, see [1, 11]. For Equations (4.1)-(4.3), it seems that such exponential expansion of the solution no longer stands in general. The assumption on the monotone behavior of the solution in Theorem 4.1.4 fills the lack of such exponential expansion.

4.1.1 General remarks and comments

For the uniqueness of the speed c (Theorem 4.1.3), we originally required that the solutions are continuous. It appears that the proof of this result can easily be adapted to solutions u with a finite number of discontinuities. This is briefly discussed in the end of Section 4.3.

In the case of monostable nonlinearities, the uniqueness of the speed c no longer holds, see [11, 23]. However, the uniqueness up to translation of the travelling-fronts of (4.5) still holds. We expect to have similar results for positive solution of (4.1)-(4.3), but we were not able to prove it.

Theorems 4.1.2-4.1.4 stand for more general operators than $Lu := J \star u - u - cu'$. Namely, our proofs hold for operators of the form

$$Lu := \alpha u'' + \beta \int_{\mathbb{R}} J(x-y)\big(u(y) - u(x)\big)dy - cu' - du, \tag{4.6}$$

where α, β and d are non-negative real numbers such that $\alpha + \beta > 0$ and J a positive continuous integrable kernel such that $[-b, -a] \cup [a, b] \subset supp(J)$ for some $0 \leq a < b$. Observe that when $\alpha \neq 0$, even in the case of stationary travelling fronts (i.e. $c = 0$), there is no need to consider discontinuous travelling fronts since the local elliptic regularity implies that solutions are smooth. Note also that the kernel J does not need to be an even function.

In our analysis for operators satisfying Lemma 4.1.1 below, we have also observed that some assumptions on L can be weakened, in particular the translation invariance. We summarize below the required condition on L

(H1) For all positive functions U, let $U_h(.) := U(.+h)$. Then for all $h > 0$ we have $L[U_h](x) \leq L[U](x+h) \;\; \forall x \in \mathbb{R}$.

(H2) Let v a positive constant then we have $L[v] \leq 0$.

Operators satisfying these two conditions are easily constructed. For example, let J be a positive, even, continuous, integrable kernel of mass one, then the operator $Lu := \int_{-r}^{+\infty} J(x-y)u(y)dy - u$ where $r > 0$, is *not* translation-invariant but satisfies (H1) and (H2).

Most of the results that we obtain can be generalized to multidimensional situations. For example, Theorems 4.1.2 -4.1.4 can be generalized to the following problem

$$\epsilon \Delta u + \beta \int_{\Sigma} J(x-t, y-s)(u(t,s) - u(x,y))dtds + \gamma(y)u_x + f(u) = 0 \quad \text{on} \quad \Sigma \tag{4.7}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{at} \quad \partial\Sigma \tag{4.8}$$

$$u(x,y) \to 0 \quad \text{uniformly in } y \text{ as} \quad x \to -\infty \tag{4.9}$$

$$u(x,y) \to 1 \quad \text{uniformly in } y \text{ as} \quad x \to +\infty, \tag{4.10}$$

where $\Sigma := \mathbb{R} \times \Omega$ is a cylinder.
Here ϵ and β are nonnegative constants, $\Omega \subset \mathbb{R}^{n-1}, (n \geq 2)$ is a bounded domain with a $C^{2,\alpha}$ boundary for some $\alpha > 0$ if $n > 2$, ν is the outward normal to the boundary of $\mathbb{R} \times \Omega$,

$\gamma(y) : \Omega \to \mathbb{R}$ is a smooth function and the spatial coordinates are denoted by (x, y) where $x = x_1$ and $y = (x_2, ..., x_n)$. $J(x, y)$ is a positive, continuous, integrable kernel on Σ such that the support of J contains a set of the form $([-b, -a] \cup [a, b]) \times \omega$ for some $0 \leq a < b$, where ω is an open subset of Ω containing 0.
When $\beta = 0$, such kind of equations arise in combustion theory to describe the propagation of flames in a tube. The term $\gamma(y)$ is usualy compose of two terms $\gamma(y) = c + \gamma_1(x)$, where c is the unknown speed of the flame and $\gamma_1(x)$ is a given driving flow. The operator $Lu := \beta \int_\Sigma J(x - t, y - s)(u(t, s) - u(x, y))dtds$ is a natural multidimensional generalization of the one dimensional diffusion operator $J \star u - u$.

4.1.2 Method and plan

To prove Theorems 4.1.2-4.1.4, we use a sliding technique introduced by Berestycki-Nirenberg in [11], combined with some ideas of Alikakos-Bates-Chen [3] (see also Chen [17] and [34]) and Vega [55]. We also use extensively a strong maximum principle that holds for the operator $Lu := J \star u - u - cu'$:

Theorem 4.1.5. *Maximum Principle*
Let u be a smooth (C^1) function on $\mathbb{R}$, such that

$$L[u](x) \geq 0 \ \ (resp. \ \ L[u](x) \leq 0) \quad in \quad \mathbb{R}.$$

Then u cannot achieve a global maximum (resp. a global minimum) without being constant.

and some property attached to our operator L :

Lemma 4.1.1.
Let u be a smooth (C^1) function. If u achieves a global minimum (resp.a global maximum) at some point ξ then the following holds :
- *Either $L[u](\xi) > 0 \ \ (resp. \ \ L[u](\xi) < 0)$*
- *Or $L[u](\xi) = 0$ and u is identically constant.*

Remark 4.1.2. *The assumption on the regularity of u in Theorem 4.1.5 and Lemma 4.1.1 have to be adjusted to the regularity requirements of L. More precisely, if $Lu := J \star u - u$, C^1 regularity is not really needed. Indeed in that case Lemma 4.1.1 and the Maximum Principle stand for u continuous by parts and with a finite number of discontinuities. Observe that operators of the form $Lu := \alpha u'' + \beta \int_{\mathbb{R}} J(x - y)\big(u(y) - u(x)\big)dy - cu' - du$ also satisfy Theorem 4.1.5 and Lemma 4.1.1 when $\beta \neq 0$ under the additional assumption $u \in C^2$.*

Remark 4.1.3. *The maximum principle for $Lu = J \star u - u$ needs that u achieves a global extremum on $\mathbb{R}$. It will be untrue if we only assume that u achieves a local extremum. For the Laplacian operator the maximum principle holds even if we assume that u achieves a local extremum. This difference is easily explained by the global/local nature of our operator and the Laplacian operator. This also implies that local analysis will fails for our operator.*

Remark 4.1.4. *For general multidimensional operators $Lu := \int_\Sigma J(x-t, y, s)(u(t, s)-u(x, y))dtds$ the strong maximum principle stated as in Theorem 4.1.5 no longer holds. However, our proofs will still hold only by assuming that at a global minimum (x_0, y_0) (resp.a global maximum), we have*
- *Either $L[u](x_0, y_0) > 0 \ \ (resp. \ \ L[u](x_0, y_0) < 0)$*
- *Or $L[u](x_0, y_0) = 0$ and $u(x, y) = u(x_0, y_0)$ on $\mathbb{R} \times \{y\}$ for some $y \in \bar{\Omega}$.*

This conditions enables to consider a much greater varitiety of kernels.

Details of the proof of the maximum principle and the previous lemma can be found in [23]. Let me describe in a few words the idea of the method. We compare translations of two solutions u and v on $\mathbb{R}$. We show that for some real τ, we have

$$u(.+\tau) \geq v(.) \text{ on all } \mathbb{R}. \tag{4.11}$$

Then, using standard procedures we obtain the desired conclusion. To obtain (4.11), a global approach is needed, since we deal with nonlocal operators. The method used by Berestycki-Nirenberg [11, 9] and Vega [55] fails in our case because it rely on comparison results either on compact set or semi infinite cylinders, which can't be obtained in our case.

This note is organized as follows : Section 4.2 is devoted to some prelinimary results which will be used extensively in the other sections. Uniqueness and monotonicity of travelling front solution (i.e. Theorem 4.1.2,4.1.3) is proved in Section 4.3. Theorem 4.1.4 is then proved in Section 4.4. In the last section we examine some aspect of the multidimensional problem.

Remark 4.1.5. *Even though the Laplacian (i.e. $L := \Delta$) does not satisfy Lemma 4.1.1, one can show that our proof of Theorems 4.1.2-4.1.3 holds for this operator.*

4.2 Preliminary results, Nonlinear Comparison Principle

In this section we present some useful results concerning sub and supersolutions of the problem

$$Lu + f(u) = 0 \text{ on } \mathbb{R} \tag{4.12}$$

$$u(x) \to 0 \text{ as } x \to -\infty \tag{4.13}$$

$$u(x) \to 1 \text{ as } x \to +\infty, \tag{4.14}$$

where $f \in C^1$ satisfies the set of conditions in Theorem 4.1.2. For the sake of simplicity, we will only consider translation-invariant operators L satisfying Lemma 4.1.1 and (H2) . For convenience, we introduce the notation $u_\tau := u(.+\tau)$. As briefly mentioned in the introduction, all our proofs rely on a comparison result on translations of two solutions. So we start by showing a Nonlinear Comparison Principle which will enable us to order translations of sub and supersolutions of (4.12)-(4.14). More precisely we have the following result.

Theorem 4.2.1. *Nonlinear Comparison Principle*
Let f satisfy the assumptions of Theorem 4.1.2. Let u and v be two smooth (C^1) functions on $\mathbb{R}$, such that

$$Lu + f(u) \leq 0 \quad on \quad \mathbb{R} \tag{4.15}$$

$$Lv + f(v) \geq 0 \quad on \quad \mathbb{R} \tag{4.16}$$

$$\lim_{x\to-\infty} u(x) \geq 0, \quad \lim_{x\to-\infty} v(x) \leq 0 \tag{4.17}$$

$$\lim_{x\to+\infty} u(x) \geq 1, \quad \lim_{x\to+\infty} v(x) \leq 1. \tag{4.18}$$

Then there exists a positive real τ such that $u_\tau \geq v$. Moreover, either $u_\tau > v$ or $u_\tau \equiv v$.

Remark 4.2.1. *Observe that by the Maximum Principle (i.e. Theorem 4.1.5) and since $f(s) \geq 0 \;\; \forall s \leq 0$, the supersolution u is necessarily positive. Similarly, since $f(s) \leq 0 \;\; \forall s \leq 1$, the Maximum Principle implies that $v < 1$.*

Before proving Theorem 4.2.1, we start with some definitions of quantities that we will use all along this section.
Let $0 < \delta \leq \frac{\epsilon}{2}$ such that

$$f'(p) \leq 0 \quad \text{for } p < \delta \;\text{ and }\; 1 - p < \delta. \tag{4.19}$$

Choose $M > 0$ so that

$$1 - u(x) < \frac{\delta}{2} \;\; \forall x > M \tag{4.20}$$

$$\text{and} \;\; v(x) < \frac{\delta}{2} \;\; \forall x < -M. \tag{4.21}$$

The proof of Theorem 4.2.1 is mainly based on the following technical lemma, which will be proved later on.

Lemma 4.2.1.
Let u and v be as in Theorem 4.2.1 and satisfy Conditions (4.20) *and* (4.21) *above. If there exists a positive constant b such that u and v satisfy :*

$$u(x+b) > v(x) \;\; \forall x \in [-M-1, M+1] \tag{4.22}$$

$$\textit{and} \;\; u(x+b) + \frac{\delta}{2} > v(x) \;\; \forall x \in \mathbb{R}, \tag{4.23}$$

then we have $u(x+b) \geq v(x) \; \forall x \in \mathbb{R}$.

Proof of Theorem 4.2.1 :

Note that if $\inf_{\mathbb{R}} u > \max_{\mathbb{R}} v$, the theorem trivially holds. In the sequel, we assume that $\inf_{\mathbb{R}} u \leq \max_{\mathbb{R}} v$. Assume for a moment that Lemma 4.2.1 holds. To prove Theorem 4.2.1, by construction of M and δ, we just have to find an appropriate constant b which satisfies (4.22) and (4.23). Since u and v satisfy (4.17)-(4.18) using Remark 4.2.1, there exists a constant D such that on the compact set $[-M-1, M+1]$, we have for every $b \geq D$

$$u(x+b) > v(x) \;\; \forall x \in [-M-1, M+1].$$

Now, we claim that there exists $b \geq D$ such that $u(x+b) + \frac{\delta}{2} > v(x) \;\; \forall x \in \mathbb{R}$.
If not then we have,

$$\forall b \geq D \;\;\text{ there exists }\; x(b) \;\text{ such that }\; u(x(b)+b) + \frac{\delta}{2} \leq v(x(b)). \tag{4.24}$$

Since u is nonnegative and v satisfies (4.17) there exists a positive constant A such that

$$u(x+b) + \frac{\delta}{2} > v(x) \quad \text{for all } b > 0 \text{ and } \;\; x \leq -A. \tag{4.25}$$

Take now a sequence $(b_n)_{n \in \mathbb{N}}$ which tends to $+\infty$. Let $x(b_n)$ be the point defined by (4.24). Thus we have for that sequence

$$u(x(b_n) + b_n) + \frac{\delta}{2} \leq v(x(b_n)). \tag{4.26}$$

According to (4.25) we have $x(b_n) \geq -A$. Therefore the sequence $x(b_n) + b_n$ converges to $+\infty$. Pass to the limit in (4.26) to get

$$1 + \frac{\delta}{2} \leq \lim_{n \to +\infty} u(x(b_n) + b_n) + \frac{\delta}{2} \leq \limsup_{n \to +\infty} v(x(b_n)) \leq 1,$$

which is a contradiction. Therefore there exists a $b > D$ such that

$$u(x + b) + \frac{\delta}{2} > v(x) \ \ \forall x \in \mathbb{R}.$$

Since we have found our appropriate constant b, we can apply Lemma 4.2.1 to obtain

$$u(x + \tau) \geq v(x) \ \ \forall x \in \mathbb{R},$$

with $\tau = b$. It remains to prove that either $u_\tau > v$ or $u_\tau \equiv v$. We argue as follows. Let $w := u_\tau - v$, then either $w > 0$ or w achieves a non-negative minimum at some point $x_0 \in \mathbb{R}$. If such x_0 exists then at this point we have $w(x) \geq w(x_0) = 0$ and

$$0 \leq Lw(x_0) \leq f(v(x_0)) - f(u(x_0 + \tau)) = f(v(x_0)) - f(v(x_0)) = 0. \tag{4.27}$$

Then using Lemma 4.1.1, we obtain $w \equiv 0$, which means $u_\tau \equiv v$. This ends the proof of Theorem 4.2.1.

□

Remark 4.2.2. *Note that the construction of b still stands if we only assume that u is continuous and $v < 1$ has a finite number of discontinuities.*

We now turn our attention to the proof of Lemma 4.2.1.

Proof of Lemma 4.2.1

Let u and v be respectively a super and a subsolution of (4.12)-(4.14) satisfying (4.20) and (4.21). Let $a > 0$ be such that

$$u(x + b) + a > v(x) \, \forall x \in \mathbb{R}. \tag{4.28}$$

Note that for b defined by (4.22) and (4.23), any $a \geq \frac{\delta}{2}$ satisfies (4.28). Define

$$a^* = \inf\{a > 0 \mid u(x + b) + a > v(x) \, \forall x \in \mathbb{R}\}. \tag{4.29}$$

We claim that

Claim 4.2.1. *$a^* = 0$.*

Observe that Claim 4.2.1 implies that $u(x + b) \geq v(x) \ \forall x \in \mathbb{R}$, which is the desired conclusion.

Proof of claim 4.2.1

We argue by contradiction. If $a^* > 0$, since $\lim_{x \to \pm\infty} u(x + b) + a^* - v(x) \geq a^* > 0$, there exists $x_0 \in \mathbb{R}$ such that $u(x_0 + b) + a^* = v(x_0)$.
Let $w(x) := u(x + b) + a^* - v(x)$, then

$$0 = w(x_0) = \min_{\mathbb{R}} w(x). \tag{4.30}$$

Observe that w also satisfies the following equations :

$$Lw \leq f(v(x)) - f(u(x+b)) \tag{4.31}$$
$$w(+\infty) \geq a^* \tag{4.32}$$
$$w(-\infty) \geq a^*. \tag{4.33}$$

Since $w \geq_{\not\equiv} 0$, by Lemma 4.1.1

$$Lw(x_0) > 0. \tag{4.34}$$

By our assumption, $u(x+b) > v(x)$ on $[-M-1, M+1]$. Hence $|x_0| > M+1$.
Let us define

$$Q(x) := f(v(x)) - f(u(x+b)). \tag{4.35}$$

We now have to consider the two following cases :

- $x_0 < -M-1$:
  At x_0 we have
  $$Q(x_0) = f(v(x_0)) - f(v(x_0) - a^*) \leq 0, \tag{4.36}$$
  since f is non-increasing for $s \leq \epsilon$, $a^* > 0$ and $v \leq \frac{\delta}{2} < \epsilon$ for $x < -M$. Now, combining (4.31), (4.34) and (4.36) yields the following contradiction
  $$0 < Lw(x_0) \leq Q(x_0) \leq 0.$$
- $x_0 > M+1$:
  We argue similarly for that case. At x_0 we have
  $$Q(x_0) = f(u(x_0+b) + a^*) - f(u(x_0+b)) \leq 0, \tag{4.37}$$
  since f is non-increasing for $s \geq 1-\epsilon$, $a^* > 0$ and $1-\epsilon < 1-\frac{\delta}{2} \leq u$ for $x > M$. Again, combining (4.31),(4.34) and (4.37) yields the contradiction
  $$0 < Lw(x_0) \leq Q(x_0) \leq 0.$$

Hence $a^* = 0$, which ends the proof of Claim 4.2.1.

□

Remark 4.2.3. *One can observe that the proof of Lemma 4.2.1 still holds for any $\delta < \frac{\epsilon}{2}$ and M such that* (4.19)-(4.21) *hold. In particular, since u and v satisfies* (4.17)-(4.18),*Lemma 4.2.1 holds if we increase M.*

General remarks and comments :

One can observe that most of the arguments used in the above two proofs hold for the Laplacian operator ($L = \Delta$). Only the final argument in the alternative fails. We can still obtain a contradiction, in this case, by arguing as follows :
Set $z := u(x+b) + a^*$ and $\Omega^- = \{x < -M-1 | w(x) = 0\}$.
If $x_0 < -M-1$, we have $\Omega^- \neq \emptyset$ and

$$Q(x) := f(v(x)) - f(z(x) - a^*) = f'(\theta(x))(v - z) + a^* f'(\theta(x)),$$

for some $\theta(x) \in [\min\{v(x), z(x) - a^*\}, \max\{v(x), z(x) - a^*\}]$.
Since $x_0 < -M - 1$, we have

$$z(x_0) - a^* < v(x_0) < \frac{\delta}{2} \leq \frac{\epsilon}{4}.$$

Therefore on a small neighborhood $V(x_0)$ of x_0, $z(x) - a^* < \frac{\epsilon}{2}$ and $v(x) < \frac{\epsilon}{2}$.
Hence, on $V(x_0)$ we have $f(\theta(x)) \leq 0$. By observing that $w = z - v$, from (4.31)-(4.33), we then have on $V(x_0)$

$$Lw + f'(\theta(x))w \leq a^* f'(\theta(x)) \leq 0 \text{ on } V(x_0) \tag{4.38}$$

$$w \geq 0 \text{ on } V(x_0). \tag{4.39}$$

Apply now the usual Strong Maximum Principle (i.e. Theorem 4.1.5) to obtain $w \equiv 0$ on $V(x_0)$. Observe that the previous computation holds for any $x \in \Omega^-$, therefore Ω^- is an open subset of $(-\infty, -M-1)$. Since w is continuous, Ω^- is obviously a closed subset of $(-\infty, -M-1)$. By connectedness, we then have $\Omega^- = (-\infty, -M-1)$, which is a contradiction since $lim_{x\to-\infty} w(x) \geq a^* > 0$. A similar argument can be used for the case $x_0 > M + 1$.

In the case of a continuous supersolution u and a subsolution v with a finite number of discontinuities,the first part of Theorem 4.2.1 holds. Similarly, this result also holds if the subsolution v is continuous and the supersolution u has a finite number of discontinuities.
Recall that u and v satisfy (4.20)-(4.21) for a positive M. Let us assume that v is discontinuous, the proof in the other case is similar. Since v has finite discontinuities, we can increase M further if necessary so that all the point of discontinuities of v ly in $(-M, M)$. By doing so, $w := u(. + b) + a - v$ is then continuous on $(-\infty, -M] \cup [M, +\infty)$ for all positive a, b. Using Remark 4.2.2, there exists b such that u and v satisfy (4.22)-(4.23). As in the proof of Lemma 4.2.1 we can define

$$a^* = \inf\{a > 0 \mid u(x + b) + a > v(x) \ \forall x \in \mathbb{R}\}.$$

If $a^* > 0$, since $w > a^*$ on $(-M - 1, M + 1)$ and w is continuous on $(-\infty, -M] \cup [M, +\infty)$, w achieves a global minimum at some point $x_0 \in \mathbb{R} \setminus [-M - 1, M + 1]$. Since Lemma 4.1.1 holds for discontinuous functions with a finite number of discontinuities, arguing as in the continuous case we end up with $u(. + b) \geq v$. If u is discontinuous and v continuous, we choose M such that all the points of discontinuity of u ly in $(-M, M)$. We argue as above, with the function $\tilde{w} := u - v(. - b) - a$ instead of w.

Theorem 4.2.1 and Lemma 4.2.1 will be used extensively in other proofs.

4.3 Uniqueness and monotonicity of solutions of the integrodifferential equation on $\mathbb{R}$

In this section we present the proof of Theorems 4.1.2 and 4.1.3. We show that positive solutions of the following problem are unique up to translation and are always monotone.

$$Lu + f(u) = 0 \ \text{ on } \ \mathbb{R} \tag{4.40}$$

$$u(x) \to 0 \ \text{ as } \ x \to -\infty \tag{4.41}$$

$$u(x) \to 1 \ \text{ as } \ x \to +\infty, \tag{4.42}$$

where $f \in C^1$ satisfies the assumptions of Theorem 4.1.2. For a sake of simplicity, in the sequel we will only consider continuous solutions and translation-invariant operators L satisfying Lemma 4.1.1 and (H2). Using the comparison principle and the translation invariance, without loss of generality, we may also assume that the solutions satisfy

$$0 < u < 1. \tag{4.43}$$

We break down this section in three subsections. In the first two subsections, we show that the solution is unique up to translation and monotone, which proves Theorem 4.1.2. The last subsection deals with nonexistence of the solution of (4.40)-(4.43), and as a corollary, we obtain the uniqueness of the speed c of a travelling wave, which proves Theorem 4.1.3.

4.3.1 Uniqueness up to translation :

Let u and v be two solutions of (4.40)-(4.43).

First we define the following real number :

$$\tau^* = \inf\{\tau \geq 0 | \ u_\tau(x) \geq v(x) \ \ \forall x \in \mathbb{R}\}. \tag{4.44}$$

Since u and v are solutions of (4.40)-(4.43), they satisfy the assumptions of Theorem 4.2.1 and therefore τ^* is well defined and has an upper bound. Since v is positive and u satisfies (4.41), there exists $\tau_0 > 0$ such that $u(-\tau) < v(0) \ \ \forall \tau \geq \tau_0$. Therefore τ^* is bounded from above. By continuity, we have at τ^*, $u_{\tau^*} \geq v$. We claim the following

Claim 4.3.1. $u_{\tau^*}(x) = v(x)$*, for all* $x \in \mathbb{R}$*.*

Proof :

We argue by contradiction and assume that $w := u_{\tau^*} - v \geq_{\not\equiv} 0$. We will show that for ϵ small enough, we have

$$u_{\tau^*-\epsilon}(x) \geq v(x) \ \text{ for all } \ \ x \in \mathbb{R}, \tag{4.45}$$

which will contradict the definition of τ^*.

Let us start the construction of our desired ϵ. We first show that $w > 0$. Assume that there exists x_0 in $\mathbb{R}$ such that w achieves a nonnegative minimum at this point. Then we have $w(x) \geq w(x_0) = 0$ and

$$0 \leq Lw(x_0) = f(v(x_0)) - f(u(x_0 + \tau^*)) = f(v(x_0)) - f(v(x_0)) = 0. \tag{4.46}$$

Using Lemma 4.1.1, we obtain $w \equiv 0$, which contradicts $u_{\tau^*} \geq_{\not\equiv} v$. Therefore, we must have $w > 0$.

Choose $M > 0$ and $\delta < \frac{\epsilon}{2}$ as in Section 4.2 such that u and v satisfy (4.20) and (4.21). By continuity and since $u_{\tau^*} > v$, we can then find $\epsilon_1 > 0$ such that

$$\forall \epsilon \in [0, \epsilon_1) \ \ u(x + (\tau^* - \epsilon)) > v(x) \ \ \ \text{for all } x \in [-M-1, M+1]. \tag{4.47}$$

We claim the following

Claim 4.3.2. *There exists* $\epsilon \in (0, \epsilon_1]$ *such that* u *and* v *satisfy*

$$u(x + (\tau^* - \epsilon)) + \frac{\delta}{2} > v(x) \ \ \textit{for all} \ \ \ x \in \mathbb{R}. \tag{4.48}$$

Proof :

We argue by contradiction. If (4.48) fails, then for all $\epsilon \in (0, \epsilon_1)$ there exists $x(\epsilon) \in \mathbb{R}$ such that

$$u(x(\epsilon) + (\tau^* - \epsilon)) + \frac{\delta}{2} \leq v(x(\epsilon)). \tag{4.49}$$

Now take a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ which tends to zero. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by (4.49). Thus u satisfies for each positive integer n

$$u(x_n + (\tau^* - \epsilon_n)) + \frac{\delta}{2} \leq v(x_n). \tag{4.50}$$

Since u and v satisfy (4.20) and (4.21), $(x_n)_{n\in\mathbb{N}}$ stays in the compact [-M,M]. Therefore we can extract a subsequence of $(x_n)_{n\in\mathbb{N}}$ which converges to some $\bar{x} \in \mathbb{R}$. Letting now n go to $+\infty$ in (4.50) we end up with

$$u(\bar{x} + \tau^*) + \frac{\delta}{2} \leq v(\bar{x}), \tag{4.51}$$

which contradicts $u_{\tau^*} \geq v$.

□

Fix now $\epsilon \in (0, \epsilon_1)$, such that (4.48) holds. Observe that $b := \tau^* - \epsilon$ satisfies assumptions (4.22) and (4.23) of Lemma 4.2.1. Therefore by Lemma 4.2.1 we end up with the desired contradiction

$$u(x + (\tau^* - \epsilon)) \geq v(x) \text{ for all } \ x \in \mathbb{R}.$$

This ends the proof of Claim 4.3.1 and at the same time proves the uniqueness up to translation.

□

4.3.2 Monotonicity of the solution

Now, we show the second part of Theorem 4.1.2 on the monotone behavior of the solution of (4.40)-(4.42). More precisely we show

Theorem 4.3.1. *Let f be as in Theorem 4.1.2, then the solution u of* (4.40)-(4.42) *is monotone increasing.*

We break down our proof into three steps :

– first step : we prove that for any solution u of (4.40)-(4.42) there exists a positive τ such that

$$u(x + \tau) \geq u(x) \ \ \forall x \in \mathbb{R}.$$

– second step : we show that for any $\tilde{\tau} \geq \tau$, u satisfies

$$u(x + \tilde{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R}.$$

– third step : we prove that

$$\inf\{\tau > 0 | \forall \tilde{\tau} > \tau, \ \ u(x + \tilde{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R}.\} \leq 0.$$

We easily see that the last step provides the conclusion of Theorem 4.3.1.

Proof of Theorem 4.3.1 :

First step :

The first step is easily obtained from Theorem 4.2.1 by observing that u is a sub and a supersolution of (4.40)-(4.42). Therefore we have $u_\tau \geq u$ for one positive τ.

Second step :

Choose $0 < \delta \leq \frac{\epsilon}{2}$ and M such that

$$f'(p) \leq 0 \quad \text{for } p < \delta \quad \text{and} \quad 1 - p < \delta \tag{4.52}$$

and so that u satisfies

$$1 - u(x) < \frac{\delta}{2} \ \ \forall x > M, \tag{4.53}$$

$$\text{and} \quad u(x) < \frac{\delta}{2} \ \ \forall x < -M. \tag{4.54}$$

We achieve the second step with the following proposition.

Proposition 4.3.1.
Let u be a positive solution of (4.40)-(4.43) *satisfying* (4.53) *and* (4.54). *If there exists $\tau > 0$ such that*

$$u(x+\tau) \geq u(x) \ \ \forall x \in \mathbb{R}, \tag{4.55}$$

then for all $\tilde{\tau} \geq \tau$ we have, $u(x+\tilde{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R}$.

From the previous step we know that such a τ exists.
The proof of Proposition 4.3.1 is based on the following two technical lemmas which will be proved later on.

Lemma 4.3.1.
Let u be a positive solution of (4.40)-(4.43) *and $\tau > 0$ such that $u(x+\tau) \geq u(x) \ \ \forall x \in \mathbb{R}$. Then, we have $u(x+\tau) > u(x) \ \ \forall x \in \mathbb{R}$.*

Lemma 4.3.2.
Let u be a positive solution of (4.40)-(4.43) *satisfying* (4.53) *and* (4.54) *and $\tau > 0$ such that $u(x+\tau) > u(x) \ \ \forall x \in \mathbb{R}$.*
Then, there exists $\epsilon_0(\tau) > 0$ such that for all $\tilde{\tau} \in [\tau, \tau+\epsilon_0]$, we have

$$u(x+\tilde{\tau}) > u(x) \ \ \forall x \in \mathbb{R}. \tag{4.56}$$

Proof of Proposition 4.3.1

We know from the first step that we can find a positive τ such that,

$$u(x+\tau) \geq u(x) \ \ \forall x \in \mathbb{R}.$$

Therefore by Lemmas 4.3.1 and 4.3.2 we can construct an interval $[\tau, \tau+\epsilon]$, such that for all $\tilde{\tau} \in [\tau, \tau+\epsilon]$ we have

$$u(x+\tilde{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R}.$$

Let us define the quantity

$$\bar{\gamma} = \sup\{\gamma | \forall \hat{\tau} \in [\tau, \gamma], \ \ u(x+\hat{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R}\}. \tag{4.57}$$

We claim that $\bar{\gamma} = +\infty$. If not, $\bar{\gamma} < +\infty$ and by continuity we have

$$u(x+\bar{\gamma}) \geq u(x) \ \ \forall x \in \mathbb{R}. \tag{4.58}$$

Recall that from the definition of $\bar{\gamma}$ we have

$$\forall \hat{\tau} \in [\tau, \bar{\gamma}], \ \ u(x+\hat{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R}. \tag{4.59}$$

Therefore to get a contradiction it is sufficient to construct ϵ_0 such that for all $\epsilon \in [0, \epsilon_0]$ we have

$$u(x+(\bar{\gamma}+\epsilon)) \geq u(x) \ \ \forall x \in \mathbb{R}. \tag{4.60}$$

Since $\bar{\gamma} > 0$, we can apply Lemma 4.3.1 to get

$$u(x+\bar{\gamma}) > u(x) \ \ \forall x \in \mathbb{R}. \tag{4.61}$$

We can now apply Lemma 4.3.2 to find the desired $\epsilon > 0$. Therefore, from the definition of $\bar{\gamma}$ we get

$$\forall \hat{\tau} \in [\tau, +\infty], \ \ u(x+\hat{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R},$$

which proves Proposition 4.3.1.

□

We now turn our attention to the proofs of the two technical lemmas. We start with the proof of Lemma 4.3.1.

Proof of Lemma 4.3.1

To prove that

$$u(x+\tau) > u(x) \ \ \forall x \in \mathbb{R}, \tag{4.62}$$

we argue by contradiction. Assume there exists a point x_0 such that

$$w(x) = u(x+\tau) - u(x) \geq w(x_0) = 0 \ \ \forall \ \ x \in \mathbb{R}.$$

At this point, w satisfies :

$$Lw(x_0) = f(u(x_0)) - f(u(x_0+\tau)) = f(u(x_0)) - f(u(x_0)) = 0.$$

By Lemma 4.1.1 we get $w \equiv Cte$. Since $w(x_0) = 0$, we have $w \equiv 0$. Therefore we have $u(x+\tau) = u(x)$ for all x in $\mathbb{R}$, which says that u is τ periodic.

Now, since $\tau > 0$, we have for any positive integer N,

$$u(0) = u(N\tau). \tag{4.63}$$

Letting N go to infinity in (4.63), we end up with

$$1 = u(0) < 1,$$

which is a contradiction. Therefore (4.62) holds for every x in $\mathbb{R}$.

□

We now turn our attention to the proof of Lemma 4.3.2.

Proof of Lemma 4.3.2

Let u be a positive solution of (4.40)-(4.42), which satisfies

$$u(x+\tau) > u(x) \;\; \forall x \in \mathbb{R}, \tag{4.64}$$

for a given $\tau > 0$. Observe that since $0 < u < 1$ satisfies (4.20) and (4.21) we have for all $\epsilon > 0$,

$$u(x+\tau+\epsilon) + \frac{\delta}{2} > u(x) \;\; \forall x \in \mathbb{R} \setminus [-M, M]. \tag{4.65}$$

Since u is continuous and satisfies (4.64), we can find ϵ_0, such that for all $\epsilon \in [0, \epsilon_0]$, we have

$$u(x+\tau+\epsilon) > u(x) \;\; \text{for} \;\; x \in [-M-1, M+1]. \tag{4.66}$$

Therefore for all $\epsilon \in [0, \epsilon_0]$, we have

$$u(x+\tau+\epsilon) + \frac{\delta}{2} > u(x) \;\; \forall x \in \mathbb{R}. \tag{4.67}$$

Observe that for all $\epsilon \in [0, \epsilon_0]$, $b := \tau + \epsilon$ satisfies assumptions (4.22) and (4.23) of Lemma 4.2.1. Therefore we can apply Lemma 4.2.1 for each $\epsilon \in [0, \epsilon_0]$ and get

$$u(x+\tau+\epsilon) \geq u(x) \;\; \forall x \in \mathbb{R}. \tag{4.68}$$

Thus, we end up with

$$u(x+\tilde{\tau}) \geq u(x) \;\; \forall x \in \mathbb{R}, \tag{4.69}$$

for all $\tilde{\tau} \in [\tau, \tau+\epsilon_0]$. This ends the proof of Lemma 4.3.2.

□

Third step :

By the first step and Proposition 4.3.1, we can define the quantity

$$\tau^* = \inf\{\tau > 0 | \;\; \forall \tilde{\tau} > \tau, \;\; u(x+\tilde{\tau}) \geq u(x) \;\; \forall x \in \mathbb{R}\}. \tag{4.70}$$

We end the proof of Theorem 4.3.1, by proving the following claim

Claim 4.3.3. $\tau^* \leq 0$.

Proof :

We follow the arguments used in the previous subsection on the uniqueness up to translation. We argue by contradiction and assume that $\tau^* > 0$. We will show that for ϵ small enough, we still have,

$$u(x + (\tau^* - \epsilon)) \geq u(x) \;\; \text{for all} \;\;\; x \in \mathbb{R}. \tag{4.71}$$

Using the previous step, we will have for all $\tilde{\tau} \geq \tau^* - \epsilon$

$$u(x+\tilde{\tau})) \geq u(x) \;\; \text{for all} \;\;\; x \in \mathbb{R}, \tag{4.72}$$

which will contradict the definition of τ^*.

The construction of ϵ is obtained as follows. By the definition of τ^* and by continuity, we have

$$u(x+\tau^*) \geq u(x) \;\; \text{for all} \;\;\; x \in \mathbb{R}. \tag{4.73}$$

Since $\tau^* > 0$, by Lemma 4.3.1, we have

$$u(x+\tau^*) > u(x) \text{ for all } x \in \mathbb{R}. \tag{4.74}$$

Therefore, on the compact [-M-1,M+1], we can find $\epsilon_1 > 0$ such that

$$\forall \epsilon \in [0, \epsilon_1) \;\; u(x + (\tau^* - \epsilon)) > u(x) \quad \text{for all } x \in [-M-1, M+1]. \tag{4.75}$$

The arguments used in the proof of Claim 4.3.2 apply to u, therefore there exists $\epsilon \in (0, \epsilon_1]$ such that u satisfies

$$u(x + (\tau^* - \epsilon)) + \frac{\delta}{2} > u(x) \text{ for all } x \in \mathbb{R}. \tag{4.76}$$

Fix now $\epsilon \in (0, \epsilon_1)$, such that (4.76) holds. Again, observing that $b := \tau^* - \epsilon$ satisfies assumptions (4.22) and (4.23) of Lemma 4.2.1 with u as sub and supersolution, we conclude that

$$u(x + (\tau^* - \epsilon)) \geq u(x) \text{ for all } x \in \mathbb{R}. \tag{4.77}$$

This ends the proof of Claim 4.3.3 and at the same time proves Theorem 4.3.1.

□

4.3.3 Nonexistence and applications

In this subsection, we obtain nonexistence results. More precisely, we have the following nonexistence result.

Theorem 4.3.2. *Let f be as in Theorem 4.1.2. If there exists a continuous sub or supersolution u of Problem* (4.40)-(4.42)*, such that u is not a solution of* (4.40)-(4.42) *then there exists no solution of Problem* (4.40)-(4.42).

Theorem 4.3.2 come as a consequence of the uniqueness up to translation of the solution. The uniqueness of the speed of a travelling front (Theorem 4.1.3) is then obtained as a corollary of Theorem 4.3.2 and the monotonicity of the solution. Indeed, let us assume that Theorem 4.3.2 holds and assume by contradiction that there exist (u, c) and (v, c') two continuous solutions of (4.40)-(4.42) with different speeds ($c \neq c'$).
Recall that

$$Lu = \alpha u'' + \beta \int_{\mathbb{R}} J(x-y)\big(u(y) - u(x)\big)dy - cu' - du.$$

We note L_c and $L_{c'}$ the operator L with parameter respectively c and c'. By the previous subsection we have $u' > 0$ and $v' > 0$. Note that u satisfies the set of equations

$$L_{c'}u + f(u) = (c - c')u' \text{ on } \mathbb{R} \tag{4.78}$$

$$u(x) \to 0 \text{ as } x \to -\infty \tag{4.79}$$

$$u(x) \to 1 \text{ as } x \to +\infty. \tag{4.80}$$

Since $u' > 0$, u is not a solution of (4.40)-(4.42) with speed c' and is a either a sub or a supersolution of this problem. Theorem 4.3.2 then provides a contradiction. Thus we must have $c = c'$.

Let us turn our attention to the proof of Theorem 4.3.2.

Proof of Theorem 4.3.2

Without loss of generality we can assume that u is a supersolution of (4.40)-(4.42). We argue with a contradiction argument. Let us assume that there exists a continuous solution v to (4.40)-(4.42). Since u and v are respectively a super and a subsolution of (4.40)-(4.42), the argument developed in the proof of the uniqueness up to translation (i.e. Subsection 4.3.1) holds. We then have $u_\tau = v$ for some real τ, which is a contradiction.

□

Remark :

When $c' = 0$ and v has finitely many discontinuities, the proof of the uniqueness of the speed still holds. Indeed, assume by contradiction that (u, c) is another solution with $c \neq 0$. Following the previous argumentat, u is continuous and is either a supersolution or a subsolution of (4.40)-(4.42) with speed $c' = 0$. We can assume that u is a supersolution. The proof in the other case is similar. Using the observation in Section 4.2 on the first part of Theorem 4.2.1, there exists $\tau > 0$ such that $u_\tau > v$. Define as in Subsection 4.3.1

$$\tau^* = \inf\{\tau \geq 0| \ \ u_\tau(x) \geq v(x) \ \ \forall x \in \mathbb{R}\}.$$

Since the Maximum Principle holds for u_τ and v, working as in Subsection 4.3.1 yields a contradiction. Thus we must have $c = 0$.

4.4 Monotonicity of solutions of the integrodifferential equation : the monostable case

In this section, we present a proof of Theorem 4.1.4. Recall that we are interested in the monotonicity of solutions of the following problem.

$$Lu = -f(u) \ \text{ on } \ \mathbb{R} \tag{4.81}$$

$$u(x) \to 0 \ \text{ as } \ x \to -\infty \tag{4.82}$$

$$u(x) \to 1 \ \text{ as } \ x \to +\infty, \tag{4.83}$$

where $f \in C^1(\mathbb{R})$ satisfies $f(0) = f(1) = 0$ and $f'(s) \leq 0$ in $1 - \epsilon < s$ for some $\epsilon > 0$. We start as in Subsection 4.3.2 by breaking down our proof in three steps.

- first step : we prove that for any solution u of (4.81)-(4.83) there exists a positive τ such that
$$u(x) \geq u(x - \tau) \ \ \forall x \in \mathbb{R}.$$
- second step : we show that for any $\tilde{\tau} \geq \tau$, u satisfies
$$u(x) \geq u(x - \tilde{\tau}) \ \ \forall x \in \mathbb{R}.$$
- third step : we prove that
$$\inf\{\tau > 0 | \forall \tilde{\tau} > \tau, \ \ u(x) \geq u(x - \tilde{\tau}) \ \ \forall x \in \mathbb{R}\} \leq 0.$$

We easily see that the last step provides the conclusion of the theorem. The next three subsections are devoted to each step of the proof.

Proof of the first step

We show that most of the technical lemmas developed in the previous section can be adapted to this situation. First we show the following

Lemma 4.4.1.
Let u be a positive solution of (4.81)-(4.83)*, such that u is increasing in a neighborhood $(-\infty, -M)$ of $-\infty$. Then there exists a positive τ such that*

$$u(x) \geq u(x-\tau) \;\; \forall x \in \mathbb{R}.$$

Remark 4.4.1. *Since f does not satisfy $f'(s) \leq 0$ when $s < \epsilon$ for some $\epsilon > 0$, Theorem 4.2.1 does not readily apply. However, in the case of monotonicity, an analogue of Lemma 4.2.1 can be obtained with minor change to the proof.*

Proof of Lemma 4.4.1

Let u be a positive solution of (4.81)-(4.83).

We start with the definition of quantities that we will use all along the proof.
Let δ positive be such that

$$f'(p) \leq 0 \quad \forall p \quad such \;\; that \;\; 1-p < \delta. \tag{4.84}$$

Choose $M > 0$ such that :

$$\begin{aligned} &\mid u(x) - 1 \mid < \frac{\delta}{2} && \forall x > M, && (4.85)\\ &u(x) < \frac{\delta}{2} && \forall x < -M, && (4.86)\\ &u(x) > u(\tilde{x}) && \forall \tilde{x} < x \leq -M. && (4.87)\end{aligned}$$

Again the proof of Lemma 4.4.1 is mainly based on the following technical lemma which will be proved later on.

Lemma 4.4.2.
Let u be a positive solution of (4.81)-(4.83) *satisfying* (4.85)-(4.87) *. Assume there exist positive constants a and b such that u satisfies :*

$$u(x) > u(x-b) \;\; \forall x \in (-\infty, M+1] \tag{4.88}$$

$$u(x) + a > u(x-b) \;\; \forall x \in \mathbb{R}. \tag{4.89}$$

Then we have $u(x) \geq u(x-b)$ $\forall x \in \mathbb{R}$.

Proof of Lemma 4.4.1

Assume for the moment that Lemma 4.4.2 holds. Then to prove Lemma 4.4.1 we just have to find appropriate constants a and b which satisfy (4.88) and (4.89).
Since we chose M such that u is increasing on $(-\infty, -M]$, then for every positive b, u satisfies

$$u(x) > u(x-b) \;\; \forall x \in (-\infty, -M-1].$$

Now, since u satisfies (4.82) and (4.83), there exists a constant D such that on the compact set $[-M-1, M+1]$ we have for every $b \geq D$,

$$u(x) > u(x-b) \;\; \forall x \in [-M-1, M+1].$$

Therefore, for b greater than D, u satisfies

$$u(x) > u(x-b) \;\; \forall x \in (-\infty, M+1].$$

Now take $a = 1$ and observe that $u(x) + a > u(x-b) \;\; \forall x \in \mathbb{R}$. This ends the construction of the constants a and b.

□

Now we turn our attention to Lemma 4.4.2.

Proof of Lemma 4.4.2

From our assumption on $a > 0$ and b we have

$$u(x) + a > u(x-b) \; \forall x \in \mathbb{R}. \tag{4.90}$$

Let us define

$$a^* = \inf\{a > 0 \mid u(x) + a > u(x-b) \; \forall x \in \mathbb{R}\}. \tag{4.91}$$

We claim that

Claim 4.4.1. $a^* = 0$.

Observe that by Claim 4.4.1 we end up with $u(x) \geq u(x-b) \; \forall x \in \mathbb{R}$ which is the desired conclusion.

Proof of Claim 4.4.1

As in Section 4.2 we argue by contradiction.
If not, since $\lim_{x\to\pm\infty} u(x) + a^* - u(x-b) = a^* > 0$, there exists $x_0 \in \mathbb{R}$ such that $u(x_0) + a^* = u(x_0 - b)$.
Let $w(x) := u(x) + a^* - u(x-b)$, then we have

$$0 = w(x_0) = \min_{\mathbb{R}} w(x). \tag{4.92}$$

Observe that w also satisfies the following equation :

$$Lw = f(u(x-b)) - f(u(x)) \tag{4.93}$$
$$w(+\infty) = a^* \tag{4.94}$$
$$w(-\infty) = a^*. \tag{4.95}$$

By the construction of a and b we have

$$w(x) = u(x) + a^* - u(x-b) > 0 \;\; \forall x \in (-\infty, M+1].$$

Thus, $x_0 > M + 1$.

By the maximum principle property, at its minimum x_0, w satisfies :

$$f(u(x_0) + a^*) - f(u(x_0)) = Lw(x_0) > 0. \tag{4.96}$$

Thus

$$Q = f(u(x_0) + a^*) - f(u(x_0)) > 0 \tag{4.97}$$
$$Q = f'(d)a^* > 0, \tag{4.98}$$

for some $d \in]u(x_0), u(x_0) + a^*[$.
Since $x_0 > M + 1$, (4.85) implies that $1 - d < \delta$.
Thus, Q would verify :

$$Q = f'(d)a^* \leq 0,$$

which contradicts (4.98). Hence $a^* = 0$, which ends the proof of Claim 4.4.1.

□

Now, we turn our attention to the second step in the proof of Theorem 4.1.4.

Proof of the second step

As in Subsection 4.3.2 we achieve the second step with the following proposition.

Proposition 4.4.1.
Let u be a positive solution of (4.81)-(4.83) *satisfying* (4.85)-(4.87). *If there exists $\tau > 0$ such that*

$$u(x) \geq u(x - \tau) \ \ \forall x \in \mathbb{R}. \tag{4.99}$$

Then, for all $\tilde{\tau}$ we have, $u(x) \geq u(x - \tilde{\tau}) \ \ \forall x \in \mathbb{R}$.

As in Subsection 4.3.2, the proof of the proposition is based on the two following technical lemmas.

Lemma 4.4.3.
Let u be a positive solution of (4.81)-(4.83) *and $\tau > 0$ be such that $u(x) \geq u(x - \tau) \ \ \forall x \in \mathbb{R}$.*
Then, we have $u(x) > u(x - \tau) \ \ \forall x \in \mathbb{R}$.

Lemma 4.4.4.
Let u be a positive solution of (4.81)-(4.83) *satisfying* (4.85)-(4.87) *and $\tau > 0$ be such that*
$u(x) > u(x - \tau) \ \ \forall x \in \mathbb{R}$.
Then, there exists $\epsilon_0(\tau) > 0$ such that for all $\tilde{\tau} \in [\tau, \tau + \epsilon_0]$, we have

$$u(x) > u(x - \tilde{\tau}) \ \ \forall x \in \mathbb{R}. \tag{4.100}$$

We omit the details of the proofs since essentially all the arguments developed in the previous section work. We proceed to the last step.

Proof of the third step

By Lemma 4.4.1 and Proposition 4.4.1, we can define the quantity

$$\tau^* = \inf\{\tau > 0| \ \forall \tilde{\tau} > \tau, \ u(x) \geq u(x - \tilde{\tau}) \ \forall x \in \mathbb{R}\}. \tag{4.101}$$

We end the proof of Theorem 4.1.4 with the following lemma

Lemma 4.4.5.
Let u be a positive solution of (4.81)-(4.83) *satisfying* (4.85)-(4.87). *Then, we have $\tau^* \leq 0$.*

Proof of Lemma 4.4.5

Again, we argue by contradiction, suppose that $\tau^* > 0$. We will show that for ϵ small enough, we still have

$$u(x) \geq u(x - (\tau^* - \epsilon)) \ \text{ for all } \ x \in \mathbb{R}. \tag{4.102}$$

Then by the previous step, we will have for all $\tilde{\tau} \geq \tau^* - \epsilon$,

$$u(x) \geq u(x - \tilde{\tau}) \ \text{ for all } \ x \in \mathbb{R}, \tag{4.103}$$

which contradicts the definition of τ^*.

Now, we start the construction. By definition of τ^* and by continuity, we have

$$u(x) \geq u(x - \tau^*) \ \text{ for all } \ x \in \mathbb{R}. \tag{4.104}$$

Therefore, by Lemma 4.4.3, we have

$$u(x) > u(x - \tau^*) \ \text{ for all } \ x \in \mathbb{R}. \tag{4.105}$$

Thus, on the compact [-M,M], we can find $\epsilon_1 > 0$ such that,

$$\forall \epsilon \in [0, \epsilon_1) \ \ u(x) > u(x - (\tau^* - \epsilon)) \ \ \forall x \in [-M - 1, M + 1]. \tag{4.106}$$

Since u is increasing on $(-\infty, -M]$, we indeed have

$$\forall \epsilon \in [0, \epsilon_1) \ \ u(x) > u(x - (\tau^* - \epsilon)) \quad \text{on } (-\infty, M + 1]. \tag{4.107}$$

Now fix $\epsilon \in (0, \epsilon_1)$. We can easily find a positive constant a such that

$$u(x) + a > u(x - (\tau^* - \epsilon)) \ \text{ for all } \ x \in \mathbb{R}. \tag{4.108}$$

We can then apply Lemma 4.4.2 to obtain the desired result.

□

4.5 The multidimensional case

In this section, we study the extension of the uniqueness results to multidimensional problems. Let us consider the following integrodifferential problem :

$$\epsilon\Delta u + \theta \int_{\Sigma} J(x-t,y,s)(u(t,s)-u(x,y))dtds + \beta(y)u_x + f(u) = 0 \quad \text{on} \quad \Sigma \tag{4.109}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Sigma \tag{4.110}$$

$$u(x,y) \to 0 \quad \text{uniformly in } y \text{ as} \quad x \to -\infty \tag{4.111}$$

$$u(x,y) \to 1 \quad \text{uniformly in } y \text{ as} \quad x \to +\infty. \tag{4.112}$$

As we briefly mentioned in the introduction, for general operator $Lu := \theta \int_{\Sigma} J(x-t,y,s)(u(t,s)-u(x,y))dsdt + \beta(y)u_x$ the Strong Maximum Principle as state in Theorem 4.1.5 no longer holds. However, for operator of the form $Lu = \theta \int_{\Sigma} J(x-t,y-s)(u(t,s)-u(x,y))dsdt+\beta(y)u_x$ with a kernel $J(x,y)$ such that the support of J contains a set of the form $([-b,-a]\cup[a,b])\times\omega$ for some $0 \le a < b$, where ω is an open subset of Ω containing 0, we can show the following,

Theorem 4.5.1. *Multidimensional Maximum Principle*
Let u be a continuous function such that $L[u](x,y) \ge 0$ $(resp.\ L[u](x,y) \le 0)$ on Σ. Assume that u achieves at a global maximum (resp. a global minimum) at some point $(x_0,y_0) \in \Sigma$, then there exists $y \in \bar{\Omega}$ such that $u(x,y) = u(x_0,y_0)$ on $\mathbb{R}\times\{y\}$.

We obtain as a consequence of this maximum principle the following characterization of such operators,

Lemma 4.5.1.
Let u be a smooth function on $\bar{\Sigma}$. If u achieves a global minimum (resp.a global maximum) at some point $(x_0,y_0) \in \bar{\Sigma}$ then the following holds :
- *Either $L[u](x_0,y_0) > 0$ $(resp.\ L[u](x_0,y_0) < 0)$*
- *Or $L[u](x_0,y_0) = 0$ and $u(x,y) = u(x_0,y_0)$ on $\mathbb{R}\times\{y\}$ for some $y \in \bar{\Omega}$.*

Remark 4.5.1. *The multidimensional Maximum Principle also holds for operators of the form*

$$L := \epsilon\Delta u + \theta \int_{\mathbb{R}\times\Omega} J(x-t,y-s)(u(t,s)-u(x,y))dsdt + \beta(y)u_x \tag{4.113}$$

provided that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Sigma$.

Lemma 4.5.1 can be proved just as Theorem 4.5.1.

Proof of Theorem 4.5.1 :

Recall that

$$Lu = \epsilon\Delta u + \theta \int_{\mathbb{R}\times\Omega} J(x-t,y-s)(u(t,s)-u(x,y))dsdt + \beta(y)u_x.$$

First assume that $\epsilon = 0$. Observe that since we only consider the derivatives of u in the direction x, and that u is continuous, $L[u](x,y)$ is well defined on $\bar{\Sigma}$. Assume that $L[u](x,y) \ge 0$ and achieves a global maximum at $(x_0,y_0) \in \bar{\Sigma}$. Then at this point we have

$$\int_{\mathbb{R}\times\Omega} J(x_0-t,y_0-s)(u(t,s)-u(x_0,y_0))dtds \le 0$$

and

$$u_x(x_0,y_0) = 0.$$

Which implies that

$$\int_{\mathbb{R}\times\Omega} J(x_0 - t, y_0 - s)(u(t,s) - u(x_0,y_0))dtds = 0.$$

Therefore $u(t,s) = u(x_0,y_0) = M$ for all $(t,s) \in \Sigma$ such that $(x_0 - t, y_0 - s) \in supp(J)$. In particular since $([-b,-a]\cup[a,b])\times\{0\} \subset J$ we have $u(t,y_0) = M$ for all $t \in x_0+[-b,-a]\cup[a,b]$. Next we show that $u(x,y_0) = M$ for $x \in [x_0,+\infty)$. Let $z \in x_0+[a,b]$, observe that at the point $(z, y_0$, u achieves a positive maximum since $u(z,y_0) = u(x_0,y_0)$. We may thus argue as above and conclude that

$$u(x,y_0) = M \text{ for all } x \in x_0 + [-b,-a] \cup [-(b-a), b-a] \cup [a,b] \cup [a+b, 2b]. \tag{4.114}$$

Thus we have $u(x,y_0) = u(x_0,y_0)$ for all $x \in x_0 + [0, b-a]$. Now repeat all the computations with $z = x_0 + b - a$ instead of x_0 to obtain that $u(x,y_0) = u(x_0,y_0)$ for all $x \in x_0 + [0, 2(b-a)]$. Therefore by repeating infinitely many times this process we obtain $u(x,y_0) = M$ for $x \in [x_0,+\infty)$. By using $z = x_0 - (b-a)$ in the previous computation, we obtain $u(x,y_0) = M$ for $x \in (-\infty, x_0]$. Therefore $u(x,y_0) = M$ on $\mathbb{R}\times\{y_0\}$.

If $\epsilon > 0$ we argue as follows. As in the above proof assume that $L[u](x,y) \geq 0$ and achieves a global maximum at $(x_0,y_0) \in \bar{\Sigma}$. If $(x_0,y_0) \in \mathbb{R}\times\Omega$, then the previous argument holds and u is constant on $\mathbb{R}\times\{y_0\}$.
If $(x_0,y_0) \in \mathbb{R}\times\partial\Omega$, then we have the following alternative

- Either $\int_{\mathbb{R}\times\Omega} J(x_0-t, y_0-s)(u(t,s)-u(x_0,y_0))dtds = 0$ and then the previous argument holds.
- Or $\int_{\mathbb{R}\times\Omega} J(x_0-t, y_0-s)(u(t,s)-u(x_0,y_0))dtds < 0$.
  In that case since $\int_{\mathbb{R}\times\Omega} J(x-t, y_0-s)(u(t,s)-u(x,y))dtds$ is a continuous function on $\bar{\Sigma}$, in a small neighborhood $V(x_0,y_0) = B_r(x_0,y_0)CAP :\bar{\Sigma}$, we have

  $$\epsilon\Delta u + \beta(y)u_x \geq -\int_{\mathbb{R}\times\Omega} J(x-t, y-s)(u(t,s) - u(x,y))dtds \geq 0.$$

  Applying the Hopf Lemma to $Mu = \epsilon\Delta u + \beta(y)u_x$, we obtain a contradiction since $\frac{\partial u}{\partial\nu} = 0$. Therefore $u = u(x_0,y_0)$ on $\mathbb{R}\times\{y_0\}$.

$\square$

Remark 4.5.2. *In the case $\epsilon = 0$, the assumption on the normal derivative is not required. However, in that case there is no Hopf Lemma available.*

Remark 4.5.3. *The multidimentional Maximum Principle holds for Kernel of the form $J(x,y,s) = k(x)\tilde{k}(y,s)$ with*

- *$k \in L^1(\mathbb{R})$ is a positive continuous kernel such that $[-b,-a] \cup [a,b] \subset supp(k)$ for some $0 \leq a < b$.*
- *$\tilde{k}(y,s)$ is a positive continuous kernel, which satisfy the following properties :*

$$\forall\, y \in \bar{\Omega}\ \exists\, s_y \in \bar{\Omega} \quad \textit{such that} \quad \tilde{k}(y,s_y) \neq 0$$

Remark 4.5.4. *Whether generalizations of our Maximum Principle to operators such as $L := \int_{\mathbb{R}\times\Omega} J(x-t, y-s)(u(t,s)-u(x,y))dtds + d(y)u_y$ hold, is still open. An equivalent of the Hopf Lemma for that case must be established in order to treat the cases of extrema achieved on the boundary of the cylinder.*

Using the multidimensional Maximum Principle, Lemma 4.5.1 and the ideas developed in Section 4.2 we have

Theorem 4.5.2. *Multidimensional Nonlinear Comparison Principle*
Let f satisfy the assumptions of Theorem 4.1.2. Let u and v be two smooth (C^1) functions on Σ, such that

$$Lu + f(u) \leq 0 \quad on \quad \Sigma \tag{4.115}$$
$$Lv + f(v) \geq 0 \quad on \quad \Sigma \tag{4.116}$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad on \quad \mathbb{R} \times \partial\Omega \tag{4.117}$$
$$\lim_{x \to -\infty} u(x,y) \geq 0, \quad \lim_{x \to -\infty} v(x,y) \leq 0 \text{ uniformly in } y \tag{4.118}$$
$$\lim_{x \to +\infty} u(x,y) \geq 1, \quad \lim_{x \to +\infty} v(x,y) \leq 1 \text{ uniformly in } y\ . \tag{4.119}$$

Then there exists a positive real τ such that $u_\tau \geq v$. Moreover, either $u_\tau > v$ on $\bar{\Sigma}$ or $u_\tau \equiv v$ on $\mathbb{R} \times \{y\}$ for some $y \in \bar{\Omega}$.

As in Section 4.2, Theorem 4.5.2 is proved using the following construction.
Let $0 < \delta \leq \frac{\epsilon}{2}$ such that

$$f'(p) \leq 0 \quad \text{for } p < \delta \quad \text{and} \quad 1 - p < \delta. \tag{4.120}$$

Choose $M > 0$ so that

$$1 - u(x,y) < \frac{\delta}{2} \quad \forall (x,y) \in (M, +\infty) \times \bar{\Omega} \tag{4.121}$$
$$\text{and} \quad v(x,y) < \frac{\delta}{2} \quad \forall (x,y) \in (-\infty, -M) \times \bar{\Omega}. \tag{4.122}$$

Lemma 4.5.2.
Let u and v be as in Theorem 4.5.2 and satisfy Conditions (4.121) *and* (4.122). *If there exists a positive constant b such that u and v satisfy :*

$$u(x+b,y) > v(x,y) \quad \forall (x,y) \in [-M-1, M+1] \times \bar{\Omega} \tag{4.123}$$
$$\text{and } u(x+b,y) + \frac{\delta}{2} > v(x,y) \quad \forall (x,y) \in \bar{\Sigma}, \tag{4.124}$$

then we have $u(x+b,y) \geq v(x,y)$ $\forall (x,y) \in \bar{\Sigma}$.

As we have already observed in the previous analysis, the proofs of Theorems 4.1.2-4.1.3 only rely on a nonlinear comparison principle, a technical lemma such as Lemma 4.2.1 and a good characterization of $L[u](x)$ at a global extremum of u. The generalization of these two theorems will therefore be straightforward using their multidimensional analog.

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