Calcul Stochastique Covariant à Sauts & Calcul Stochastique à Sauts Covariants

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CALCUL STOCHASTIQUE COVARIANT À SAUTS

& CALCUL STOCHASTIQUE À SAUTS COVARIANTS

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A Papé.

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Résumé

Calcul Stochastique Covariant à Sauts & Calcul Stochastique à Sauts Covariants

Nous proposons un calcul stochastique covariant pour des semimartingales dans le fibré tangent TM au dessus d'une variété M. Une connexion sur M permet de définir une dérivée intrinsèque d'une courbe (Y_t), C^1 dans TM, la dérivée covariante. Plus précisément, c'est la dérivée de (Y_t) vue dans un repère mobile, se déplaçant parallèlement le long de sa courbe (x_t) projetée sur M. Avec le principe de transfert, Norris définit l'intégration covariante le long d'une semimartingale dans TM. Nous décrivons le cas où la semimartingale saute dans TM, en utilisant les travaux de Norris et les résultats de Cohen sur le calcul stochastique à sauts sur une variété. Nous comprenons, que, selon l'ordre dans lequel on compose la fonction qui donne les sauts et la connexion, on obtient un *calcul stochastique covariant à sauts* ou *un calcul stochastique à sauts covariants*. Tous deux dépendent du choix de la connexion et des objets (interpolateurs et connecteurs) décrivant les sauts au sens de Stratonovich ou d'Itô. Nous étudions les choix qui rendent équivalents les deux calculs. Sous certaines conditions, on retrouve les résultats de Norris lorsque (Y_t) est continue. Le cas continu est décrit par un calcul covariant continu d'ordre deux, formalisme défini à l'aide de la notion de connexion d'ordre deux.

Abstract

Stochastic Covariant Calculus with jumps & Stochastic Calculus with covariant jumps

We propose a stochastic covariant calculus for càdlàg semimartingales in the tangent bundle TM over a manifold M. A connection on M allows us to define an intrinsic derivative of a C^1 curve (Y_t) in TM, the covariant derivative. More precisely, it is the derivative of (Y_t) seen in a frame moving parallelly along its projection curve (x_t) on M. With the transfer principle, Norris defined the stochastic covariant integration along a continuous semimartingale in TM. We describe the case where the semimartingale jumps in TM, using Norris's work and Cohen's results about stochastic calculus with jumps on manifolds. We see that, depending on the order in which we compose the function giving the jumps and the connection, we obtain a *stochastic covariant calculus with jumps* or a *stochastic calculus with covariant jumps*. Both depend on the choice of the connection and of the tools (interpolation and connection rules) describing the jumps in the meaning of Stratonovich or Itô. We study the choices that make equivalent the two calculus. Under suitable conditions, we recover Norris's results when (Y_t) is continuous. The continuous case is described by a covariant continuous calculus of order two, a formalism defined with the notion of connection of order two.

Contents

Introduction

Cette thèse a pour objectif de définir un calcul stochastique covariant avec sauts, sur une variété différentiable M. Les motivations de ce travail sont donc à la fois probabilistes et géométriques. Les équations différentielles covariantes sont fréquement utilisées, en sciences physiques par exemple, pour décrire de façon intrinsèque l'évolution d'un processus dans un repère mobile. Certains problèmes, notamment dans les théories de Jauge, nécessitent l'étude de ces équations différentielles covariantes dans un cadre stochastique. D'autre part, le calcul à sauts sur une variété permet la discrétisation d'équations différentielles stochastiques (e.d.s.). Ainsi, une application importante du formalisme que nous développons est la discrétisation d' e.d.s. covariantes.

Nos travaux s'appuient d'une part sur les résultats de Cohen dans [1] et [2], portant sur le calcul stochastique à sauts sur une variété, et d'autre part sur le formalisme donné par Norris dans [8] pour un calcul stochastique continu covariant. Nous prolongeons ces travaux en suivant deux démarches : dans la première partie de la thèse, nous étendons le calcul covariant de Norris au cas à sauts, tandis que, dans la deuxième partie, nous étendons le calcul à sauts de Cohen au cas covariant. Une conclusion importante de ce travail est que, de manière générale, ces deux approches ne sont pas équivalentes. Une comparaison est faite dans la partie III, entre les calculs stochastiques *covariant à sauts* (partie I) et *à sauts covariants* (partie II). Nous appliquons enfin ce formalisme à l'étude d'e.d.s. dans la dernière partie. On y retrouve les deux démarches possibles, qui permettent de définir des *e.d.s covariantes à sauts* ou des *e.d.s à sauts covariants.*

Dans cette introduction, nous commencerons par donner les rappels nécessaires pour comprendre les termes "calcul covariant" et "calcul stochastique à sauts ". Ceux-ci doivent permettre de mieux situer notre travail parmi ceux déjà réalisés, et de comprendre en quoi deux démarches sont naturelles pour étendre ces résultats. Après ces rappels, nous présenterons les idées directrices et les résultats principaux de chacune des quatre parties qui composent cette thèse.

	Calcul classique	Calcul covariant
Calcul Stochastique continu	*Meyer, Schwartz, Emery*	*Elworthy, Norris*
Calcul Stochastique à sauts	*Cohen, Picard*	*Partie I* *Partie II*

Noter que, s'il nous est nécéssaire de faire ces rappels et de préciser certaines hypothèses ou notations dès l'introduction, ils seront réécrits de façon plus précise et rigoureuse dans une partie en préambule de la thèse.

Rappels de géométrie et de probabilités

On considère une variéte différentielle M, C^∞, sans bords, possèdant un atlas dénombrable. D'après le théorème de Withney, M est plongeable dans $\mathbb{R}^m$. Les formules que nous donnerons en coordonnées ne dépendront pas du choix de ce plongement (sauf précision). Nous utiliserons dans ces formules la convention de sommation d'Einstein.
Pour tout V dans le fibré tangent TM au dessus de M, la notation $V = (x, v)$ signifie que x est la projection de V sur M et v sa partie dans la fibre T_xM. Tout processus z sera noté (z_t), où l'on suppose toujours $t > 0$. La dérivée par rapport au temps sera notée indifférement $\frac{\partial z_t}{\partial t}$ ou $\dot{z}_t$.

1. Rappels sur le calcul covariant, passage au cas stochastique

Connexion

Pour faire du calcul covariant sur une variété M, il faut se donner une connexion p. Celle-ci consiste en une règle de séparation, pour chaque $V = (x, v)$ dans le fibré tangent TM, de l'espace tangent T_VTM en un espace vertical et un espace horizontal. Plus précisément, pour chaque V de TM, p_V projette T_VTM sur T_vT_xM, espace vertical en V. L'espace horizontal en V est le noyau $Ker\ p$. Le déplacement horizontal est décrit par une application linéaire, appelée transport parallèle, à partir de laquelle est définie la notion de dérivée covariante.

Transport parallèle

Soit (x_t) une courbe C^1 sur M et $u_o \in GL(\mathbb{R}^m, T_{x_0}M)$ donné. Le relèvement horizontal de (x_t), partant de u_o, est l'unique courbe (u_t) telle que (x_t, u_t) soit l'unique solution dans $GL(\mathbb{R}^m, TM)$ de l'équation différentielle

$$p_{(x_t,u_t)}(\dot{x}_t, \dot{u}_t) = 0, \quad u_0 = u_o, \qquad (\textit{c'est à dire} \ : \forall z \in \mathbb{R}^m, \ \ p_{(x_t,u_t(z))}(\dot{x}_t, \dot{u}_t(z)) = 0). \tag{1}$$

Le transport parallèle $\tau^{//}$ le long de (x_t) est alors défini pour tous s, t par l'application linéaire

$$\tau^{//}_{st} = u_t u_s^{-1} \ : \ T_{x_s}M \to T_{x_t}M.$$

Dérivée covariante

Soit $(Y_t) = ((x_t, y_t))$ une courbe C^1 sur TM. Nous voulons dériver sa partie (y_t) dans la fibre, mais il n'est pas satisfaisant de définir cette dérivée en utilisant des coordonnées locales en x_t. Pour dériver (y_t) de façon intrinsèque, on voudrait pouvoir écrire l'accroissement infinitésimal $''\ y_{t+h} - y_t\ ''$, le diviser par h et prendre sa limite quand h tend vers 0. Mais $y_{t+h} \in T_{x_{t+h}}M$ n'est pas dans le même espace tangent que $y_t \in T_{x_t}M$. Il faut donc transporter parallèlement y_{t+h} dans l'espace $T_{x_t}M$. On obtient alors ce que l'on appelle la dérivée covariante de (Y_t)

$$\frac{\partial^\nabla Y_t}{\partial t} = \lim_{h \to 0} \frac{\tau^{//}_{t+h,t}\ y_{t+h} - y_t}{h}, \tag{2}$$

où $(\tau_{s,t}^{//}) : T_{x_s}M \to T_{x_t}M$ est le transport parallèle le long de la projection (x_t) de (Y_t) sur M.

La dérivée covariante au point t est donc la dérivée en la variable s de l'application $f_t(s) = \tau_{s,t}^{//} y_s$, prise en $s = t$. Elle appartient à $T_{y_t}(T_{x_t}M)$, l'espace vertical en Y_t.
N.B. : Notre notation (non usuelle) pour la dérivée covariante, avec un exposant $\mathcal{V}$ pour "vertical", fait référence à cela.

Nous utiliserons une expression équivalente à (2), dans laquelle la dérivée covariante s'écrit comme la projection par la connexion p de la dérivée usuelle $\dot{Y}_t$

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = p_{Y_t}(\dot{Y}_t). \tag{3}$$

Principe de transfert (Elworthy, Malliavin)

Pour passer du cas déterministe au cas stochastique sur une variété, on utilise ce que l'on appelle le principe de transfert. Celui-ci consiste à faire une analogie entre le vecteur tangent $\dot{z}_t$ d'une courbe C^1 et la variation infinitésimale ∂z_t au sens de Stratonovich d'une semimartingale continue.

Si (z_t) est une semimartingale continue sur $\mathbb{R}^m$, on peut définir une intégrale d'Itô et une intégrale de Stratonovich. On exprime l'une en fonction de l'autre à l'aide d'une formule de conversion Itô / Stratonovich. Pour un processus h, prévisible et localement borné, elle s'écrit

$$\int_0^t h(z_s)\partial z_s = \int_0^t h(z_s)dz_s + \frac{1}{2} < h(z), z >_t . \tag{4}$$

Si f est une fonction C^2 dans $\mathbb{R}^m$, on peut écrire la formule d'Itô, avec une intégrale de Stratonovich, comme

$$\int_0^t df(z_s)\partial z_s = f(z_t) - f(z_0). \tag{5}$$

Si f est une fonction C^2 sur une variété U, et (z_t) une courbe déterministe tracée sur U, on a

$$\int_0^t df(z_s)\dot{z}_s ds = f(z_t) - f(z_0). \tag{6}$$

L'analogie entre les formules (5) et (6) conduit au principe de transfert. Il permet de définir le calcul de Stratonovich sur une variété. Le calcul d'Itô résulte ensuite d'une formule de conversion Stratonovich / Itô sur U, généralisant (4). Cette formule nécessite la donnée d'une connexion sans torsion sur U (voir [3]).

Soit $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ un espace de probabilité, muni d'une filtration $(\mathcal{F}_t)_{t\geq 0}$ qui vérifie les conditions habituelles.
Rappelons qu'un processus (z_t) à valeurs dans U est appelé semimartingale continue (resp. càdlàg) si, pour toute fonction f dans $C^2(U, \mathbb{R})$, $f(z_t)$ est une semimartingale continue (resp. càdlàg) réelle.

(*NB : "càdlàg" signifie "continue à droite et ayant une limite à gauche".*)
Nous utiliserons également la notion de processus prévisible localement borné sur U, dont on peut trouver une définition dans [3].

Calcul stochastique continu covariant (Elworthy, Norris)

Soit p une connexion sur M. Dans [8], Norris définit des intégrales covariantes de Stratonovich et d'Itô le long d'une semimartingale continue sur TM, en appliquant le principe de transfert à la formule suivante, équivalente à (2),

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = u_t \frac{\partial}{\partial t}(u_t^{-1} y_t), \tag{7}$$

où u_t est le relèvement horizontal de (x_t) (partant d'un u_0 donné, mais la dérivée covariante ne dépend pas de ce u_0).

A travers cette formule, la dérivée covariante est vue comme un vecteur de $T_{x_t}M$. Ainsi, Norris intègre une forme α d'ordre un sur M, contre une semimartingale continue $(Y_t) = ((x_t, y_t))$ sur TM.

On définit le relèvement horizontal de la semimartingale (x_t), partant de u_o donné, en appliquant le principe de transfert à (1). C'est l'unique semimartingale (x_t, u_t) de $GL(\mathbb{R}^m, TM)$, solution de l'e.d.s.

$$p_{(x_t,u_t)}(\partial x_t, \partial u_t) = 0, \quad u_0 = u_o. \tag{8}$$

L'intégrale de Stratonovich covariante de α le long de (Y_t) est alors définie par Norris comme

$$\int \alpha_{x_t} u_t \partial(u_t^{-1} y_t). \tag{9}$$

L'intégrale d'Itô covariante de α le long de (Y_t) est définie par Norris comme

$$\int \alpha_{x_t} u_t d(u_t^{-1} y_t). \tag{10}$$

2. Rappels sur le calcul stochastique d'ordre deux sur une variété

Les rappels suivant sont donnés sur une variété différentielle U, que l'on prendra par la suite égale à M ou TM.

Calcul stochastique d'ordre deux continu (Meyer, Schwartz, Emery)

Le long d'une semimartingale (z_t), continue sur U, on intègre ordinairement une forme α d'ordre un (par exemple $\alpha = df$ dans (6)). En ce sens, les calculs de Stratonovich et d'Itô sont des calculs d'ordre un. Mais les variations infinitésimales ∂z_t (Stratonovich) et dz_t (Itô) n'ont pas de sens intrinsèque. Puisque la formule d'Itô fait apparaître non seulement la dérivée première mais aussi la dérivée seconde de la fonction f à intégrer, c'est à dire son 2-jet

$$d^2f(z_t) = \frac{\partial f}{\partial z^i}(z_t)dz^i + \frac{\partial^2 f}{\partial z^i \partial z^j}(z_t)dz^i.dz^j,$$

l'objet infinitésimal intrinsèque qu'il est naturel d'intègrer est assimilable à un opérateur différentiel d'ordre deux

$$d\!I z_t = dz_t^i \frac{\partial}{\partial z^i} + \frac{1}{2} d < z^i, z^j >_t \frac{\partial^2}{\partial z^i \partial z^j}.$$

Dans [7] et [10], Meyer et Schwartz proposent ainsi un calcul stochastique d'ordre deux sur une variété, permettant d'écrire la formule d'Itô de manière compacte et intrinsèque

$$f(z_t) - f(z_0) = \int_0^t d^2f(z_t) d\!I z_t. \tag{11}$$

L'ensemble de ces opérateurs d'ordre deux au plus, sans terme constant, au point z, est appelé espace tangent d'ordre deux en z, et noté $\tau_z U$. Le fibré tangent d'ordre deux, τU, est l'union de tous ces espaces tangents d'ordre deux. Les éléments de son dual, $\tau^* U$, sont les formes Θ d'ordre deux (Θ telle que $\Theta_z \in \tau_z^* U$, par exemple $\Theta = d^2 f$ dans (11)).

Ainsi, on définit plus généralement, l'intégrale stochastique $\int \Theta d\!I z$ d'une forme d'ordre deux Θ contre une semimartingale continue (z_t), de manière unique et intrinsèque. Les intégrales de Stratonovich et d'Itô sont alors vues comme des cas particuliers d'une intégrale stochastique d'ordre deux, à l'aide d'une application (d_s pour Stratonovich, G pour Itô, données dans [3]), transformant une forme d'ordre un en une forme d'ordre deux.

L'**intégrale de Stratonovich** est donnée par

$$\int \alpha \ \partial z = \int d_s \alpha \ d\!I z. \tag{12}$$

Dans un plongement, cette expression donne

$$\int \alpha \ \partial z = \int (\alpha_{z_t})_i dz_t^i + \frac{1}{2} \int \frac{\partial \alpha_i}{\partial z^j}(z_t) d < z^i, z^j >_t .$$

L'**intégrale d'Itô** est donnée par

$$\int \alpha \ dz = \int G(\alpha) \ d\!I z. \tag{13}$$

Dans un plongement, cette expression donne

$$\int \alpha \ dz = \int (\alpha_{z_t})_i dz_t^i + \frac{1}{2} \int (\alpha_{z_t})_k \Gamma_{ij}^k (z_t) d < z^i, z^j >_t,$$

où les symboles de Christoffel (Γ_{ij}^k) sont ceux de la connexion requise pour le calcul d'Itô.
N.B. : Par abus de langage, nous appellerons également G cette connexion.

Calcul stochastique d'ordre deux à sauts (Cohen, Picard)

Dans [1], Cohen étend le formalisme de Meyer et Schwartz au cas d'une semimartingale (z_t) càdlàg sur U. Il définit, de manière unique et intrinsèque, une intégrale stochastique d'ordre deux à sauts $\int \theta \overset{\triangle}{dI} z$ le long de (z_t). Le processus intégrant $\theta_{z_{t^-}}$ au dessus de (z_{t^-}) est une fonction réelle sur U, deux fois différentiable au voisinage de z_{t^-} et nulle en ce point
(par exemple $\theta_{z_{t^-}}(z) = f(z) - f(z_{t^-})$ pour $f \in C^2(U, \mathbb{R})$).

L'expression de cette intégrale dans un plongement est

$$\int_0^t \theta_{z_{s^-}} \overset{\triangle}{dI} z_s = \int_0^t \frac{\partial \theta_{z_{s^-}}}{\partial z^i}(z_{s^-}) dz_s^i + \frac{1}{2}\int_0^t \frac{\partial^2 \theta_{z_{s^-}}}{\partial z_s^i \partial z_s^j}(z_{s^-}) d < z^{i^c}, z^{j^c} >_s$$

$$+ \sum_{0 \leq s \leq t} (\theta_{z_{s^-}}(z_s) - \frac{\partial \theta_{z_{s^-}}}{\partial z^i}(z_{s^-}) \triangle z_s^i).$$

Les sauts de l'intégrale sont décrits par θ et les parties "diffusion" et "drift" par le 2-jet de θ. L'intégrale $\int \theta \overset{\triangle}{dI} z$ coïncide donc avec $\int d^2\theta dI z$ lorsque (z_t) est supposée continue.

Nous appellerons θ une fonction de saut. Retenons que l'intégrale $\int \theta \overset{\triangle}{dI} z$ est parfaitement définie lorsque l'on décrit un seul de ses sauts :

$$\theta_{z_{t^-}}(z_t) \text{ est le saut de } \int \theta \overset{\triangle}{dI} z \text{ quand } (z_t) \text{ saute du point } z_{t^-} \text{ au point } z_t\ .$$

Les intégrales de Stratonovich et d'Itô à sauts peuvent alors être définies comme des cas particuliers d'une intégrale d'ordre deux à sauts, à l'aide d'une application ($\overset{\triangle}{I}$ pour Stratonovich, $\overset{\triangle}{\gamma}$ pour Itô), transformant une forme d'ordre un en une fonction de sauts.

Pour l'intégrale de Stratonovich, lorsque la semimartingale (z_t) saute de z_{t^-} à z_t, on prolonge la trajectoire entre ces deux points par exemple à l'aide d'une géodésique $(I(s, z_{t^-}, z_t))_{s \in [0,1]}$ sur U, si celle-ci existe et est unique (plus généralement, on suppose l'existence d'un interpolateur I sur U, notion introduite par Cohen dans [1]). Le saut de l' intégrale est donné par l'intégration de α le long de cette géodésique.

Ainsi, l '**intégrale de Stratonovich à sauts** est définie par

$$\int \alpha \ \overset{\triangle}{\partial} z = \int \overset{\triangle}{I}(\alpha) \ \overset{\triangle}{dI} z, \tag{14}$$

où le saut de l'intégrale quand (z_t) saute de z_{t^-} à z_t est

$$\overset{\triangle}{I}(\alpha)_{z_{t^-}}(z_t) = \int_0^1 \alpha_{I(s, z_{t^-}, z_t)} \dot{I}(s, z_{t^-}, z_t)\ ds. \tag{15}$$

Si la semimartingale (z_t) est continue, l'intégrale ne dépend plus de l'interpolateur I et on retrouve

$$\int \alpha \overset{\triangle}{\partial} z = \int \alpha \partial z.$$

Pour l'intégrale d'Itô, lorsque la semimartingale (z_t) saute de z_{t^-} à z_t, on représente ce saut dans $T_{z_{t^-}} M$, par exemple à l'aide du vecteur vitesse en 0, $\gamma(z_{t^-}, z_t) = \exp^{-1}_{z_{t^-}}(z_t)$, à la géodésique joignant ces deux points sur U, si celle-ci existe et est unique (plus généralement, on suppose l'existence d'un connecteur γ sur U, notion introduite par Picard dans [9]). Le saut de l'intégrale est donné par l'application de α contre le saut de (z_t).

Ainsi, l '**intégrale d'Itô à sauts** est définie par

$$\int \alpha \overset{\triangle}{d} z = \int \overset{\triangle}{\gamma}(\alpha) \overset{\triangle}{dI} z, \tag{16}$$

où le saut de l'intégrale quand (z_t) saute de z_{t^-} à z_t est

$$\overset{\triangle}{\gamma}(\alpha)_{z_{t^-}}(z_t) = \alpha_{z_{t^-}} \gamma(z_{t^-}, z_t). \tag{17}$$

Si la semimartingale (z_t) est continue, l'intégrale ne dépend que du 2-jet du connecteur γ, qui fournit une connexion G, et on retrouve

$$\int \alpha \overset{\triangle}{d} z = \int \alpha dz.$$

Partie I : Calcul Stochastique covariant à sauts

(Part I : Covariant Stochastic calculus with jumps)

Note sur le calcul stochastique covariant continu

Comme on l'a vu dans les rappels, Norris applique le principe de tranfert à la formule (7), équivalente à (3). En réalité, cette équivalence contient une identification. En effet, La bonne façon d'écrire (7) serait

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = du_t(y_t)\frac{\partial}{\partial t}(u_t^{-1}y_t),$$

qui appartient à l'espace tangent $T_{y_t}(T_{x_t}M)$, mais l'application u_t étant linéaire, on identifie la différentielle $du_t(y_t)$ au point y_t, à u_t lui-même, ce qui donne une dérivée covariante dans l'espace tangent $T_{x_t}M$. On peut décrire cette identification comme une identification de l'espace vertical $T_{y_t}(T_{x_t}M)$ avec $T_{x_t}M$. En effet, pour chaque $V = (x, v)$ dans TM, l'espace tangent $T_v(T_xM)$ est l'ensemble des dérivées $\dot\gamma(0)$ en 0, aux courbes $(\gamma(s) = v + su)_{u\in T_xM}$ passant par v, dans T_xM. En écrivant $\dot\gamma(0) = u$, où $\dot\gamma(0) \in T_v(T_xM)$ et $u \in T_xM$, on identifie les espaces tangents $T_v(T_xM)$ et T_xM.

Cette identification, canonique à l'ordre un, est beaucoup plus délicate à traduire dans le cas à sauts, ainsi qu'a l'ordre deux. Nous ne ferons donc pas cette identification dans notre travail, ce qui a pour conséquence que nous intègrerons des formes sur TM et non des formes sur M.

Soit p une connexion sur M. Nous appliquons le principe de transfert à (3), qui ne contient pas l'identification. Cette expression a l'avantage de nous montrer le calcul covariant comme se déduisant du calcul classique par projection sur l'espace vertical. Ainsi, on peut écrire l'intégrale covariante d'une forme α sur TM le long d'une courbe (Y_t) de TM comme une intégrale classique, en faisant agir p sur α par "pull-back"

$$\int \alpha_{Y_t}\frac{\partial^{\mathcal{V}} Y_t}{\partial t}dt = \int p^*(\alpha)_{Y_t}\dot Y_t dt.$$

Ainsi, nous définissons l'intégration covariante de α le long d' une semimartingale continue (Y_t) sur TM, au sens de Stratonovich ou d'Itô, comme suit.

Définition : *L'* **intégrale de Stratonovich covariante continue** *est*

$$\int \alpha\ \partial^{\mathcal{V}} Y = \int p^*(\alpha)\ \partial Y \tag{18}$$

(c'est à dire $\int \alpha\partial^{\mathcal{V}} Y = \int d_s p^*(\alpha)\ dIY$*).*

Nous verrons dans la partie III que notre définition coïncide avec celle de Norris, lorsque l'on fait l'identification de l'espace vertical décrite ci-dessus.

Définition : *Soit $\mathbb{G}$ une connexion sur TM, définissant le calcul d'Itô sur TM.*
L' **intégrale d'Itô covariante continue** *est*

$$\int \alpha \; d^{\mathcal{V}} Y = \int p^*(\alpha) \; dY \tag{19}$$

(c'est à dire $\int \alpha d^{\mathcal{V}} Y = \int \mathbb{G}(p^*(\alpha)) \; d\mathbb{I} Y$).

Noter que l'intégrale d'Itô covariante nécessite la donnée de deux connexions : la connexion p sur M décrit le calcul covariant, et la connexion $\mathbb{G}$ sur TM permet de définir le calcul d'Itô. Dans le formalisme proposé par Norris, une seule connexion est requise. Nous verrons dans la partie III que notre définition coïncide avec celle de Norris, lorque l'on fait l'identification de l'espace vertical décrite ci-dessus, et lorsque la connexion $\mathbb{G}$ est plate.

Calcul Stochastique covariant à sauts

Nous nous intéressons maintenant au cas où la semimartingale (Y_t) est càdlàg dans TM. D'après les rappels précédents, il suffit de décrire un saut de l'intégrale covariante quand (Y_t) saute. On applique pour cela les formules (15) et (17) à la forme $p^*\alpha$.

Définition : *On suppose l'existence d'un interpolateur $\mathbb{I}$ sur TM.*
L' **intégrale de Stratonovich covariante à sauts** *est*

$$\int \alpha \; \overset{\triangle}{\partial^{\mathcal{V}}} \; Y = \int \overset{\triangle}{\mathbb{I}} \; (p^*(\alpha)) \; \overset{\triangle}{d\mathbb{I}} \; Y,$$

où le saut de l'intégrale, lorsque la semimartingale (Y_t) saute de Y_{t^-} à Y_t, est donné par

$$\overset{\triangle}{\mathbb{I}} \; (p^*\alpha)_{Y_{t^-}}(Y_t) = \int_0^1 \alpha_{\mathbb{I}(s, Y_{t^-}, Y_t)} \frac{\partial^{\mathcal{V}} \mathbb{I}}{\partial s}(s, Y_{t^-}, Y_t) \; ds. \tag{20}$$

Proposition : *Si la semimartingale (Y_t) est continue, l'intégrale ne dépend plus de l'interpolateur et on retrouve*

$$\int \alpha \; \overset{\triangle}{\partial^{\mathcal{V}}} \; Y = \int \alpha \partial^{\mathcal{V}} Y.$$

Définition : *On suppose l'existence d'un connecteur $\boldsymbol{\gamma}$ sur TM.*
L' **intégrale d'Itô covariante à sauts** *est*

$$\int \alpha \; \overset{\triangle}{d^{\mathcal{V}}} \; Y = \int \overset{\triangle}{\boldsymbol{\gamma}} \; (p^*(\alpha)) \; \overset{\triangle}{d\mathbb{I}} \; Y,$$

où le saut de l'intégrale, lorsque la semimartingale (Y_t) saute de Y_{t^-} à Y_t, est donné par

$$\overset{\triangle}{\boldsymbol{\gamma}} \; (p^*\alpha)_{Y_{t^-}} (Y_t) = \alpha_{Y_{t^-}} (0, p_{Y_{t^-}}(\boldsymbol{\gamma}(Y_{t^-}, Y_t))), \tag{21}$$

avec, pour tout U dans l'espace vertical $T_v(T_xM)$,

$$\alpha_V(0, U) = (\alpha_V)|_{T_v(T_xM)}(U). \tag{22}$$

Proposition : *Si la semimartingale (Y_t) est continue, l'intégrale ne dépend que du 2-jet du connecteur γ, qui fournit une connexion $\mathbb{G}$, et on retrouve*

$$\int \alpha \overset{\triangle}{d^{\mathcal{V}}} Y = \int \alpha d^{\mathcal{V}} Y.$$

Nous venons d'étudier séparément le cas de l'intégrale de Stratonovich covariante et celui de l'intégrale d'Itô covariante. Dans le cas non covariant, aussi bien le cas continu de Meyer et Schwartz que le cas à sauts de Cohen, les intégrales de Stratonovich et d'Itô sont des cas particuliers d'une intégrale stochastique d'ordre deux. Nous voulons inclure de même les intégrales covariantes à sauts de Stratonovich et d'Itô dans un formalisme plus général. Ce formalisme donnera, dans le cas continu, un calcul stochastique covariant continu d'ordre deux qui n'a encore jamais, a notre connaissance, été défini. Nous en proposons une définition, qui doit inclure les intégrales covariantes continues (18) et (19).

Calcul Stochastique covariant continu d'ordre deux

Nous partons de la notion de connexion d'ordre deux $\bar{p}$. Nous la définissons comme une règle de séparation, pour chaque $V = (x, v)$ dans TM, du fibré tangent d'ordre deux, $\tau_V TM$, en un espace vertical d'ordre deux $\tau_v(T_xM)$ et un espace horizontal d'ordre deux $Ker\ \bar{p}$. La restriction de $\bar{p}$ à l'ordre un est une connexion d'ordre un. Nous pouvons ensuite définir l' intégrale covariante d'ordre deux d'une forme d'ordre deux Θ contre une semimartingale continue (Y_t) de TM. En faisant une analogie avec les formules (18) et (19) de l'ordre un, nous appliquons le pull-back de la connexion $\bar{p}$ à la forme Θ.

Définition : *L'***intégrale stochastique covariante continue d'ordre deux** *est*

$$\int \Theta\ d\!I^{\mathcal{V}} Y = \int \bar{p}^*(\Theta)\ d\!IY.$$

Les intégrales de Stratonovich et d'Itô covariantes continues peuvent alors être vues comme des cas particuliers d'une intégrale covariante continue d'ordre deux.

Théorème : *Il existe une connexion d'ordre deux, notée $d_s p$, dont la restriction à l'ordre un est p, et telle que*

$$\int \alpha \partial^{\mathcal{V}} Y = \int (d_s p)^* (d_s \alpha) d\!I Y \tag{23}$$

(c'est à dire $\int \alpha \partial^{\mathcal{V}} Y = \int d_s \alpha d\!I^{\mathcal{V}} Y$ pour la connexion $d_s p$).

En particulier, on obtient alors une formule analogue à (12) pour le cas covariant

$$\int \alpha \partial^{\nu} Y = \int d_s \alpha dI^{\nu} Y.$$

Dans le cas du calcul d'Itô, nous donnons un théorème analogue lorsqu'une seule connexion est utilisée.

Théorème : *Supposons que la connexion $\mathbb{G}$ décrivant le calcul d'Itô sur TM soit plate. Alors, il existe une connexion d'ordre deux, notée $\tilde{p}$, dont la restriction à l'ordre un est p, et telle que*

$$\int \alpha d^{\nu} Y = \int \tilde{p}^*(\alpha) dIY \tag{24}$$

(c'est à dire $\int \alpha d^{\nu} Y = \int \alpha dI^{\nu} Y$ pour la connexion $\tilde{p}$.)

Partie II : Calcul stochastique à sauts covariants

(Part II: Stochastic calculus with covariant jumps)

Nous proposons un formalisme général, appelé calcul stochastique à sauts covariant, qui permet de retrouver le calcul stochastique covariant d'ordre deux, décrit ci-dessus, lorsque l'on intègre contre une semimartingale continue.

Transport sur M

La démarche de cette partie consiste à partir directement du saut pour rendre celui-ci "covariant". Reprenons un raisonnement analogue a celui que nous avons fait pour comprendre la notion de dérivée covariante. Soit $(Y_t) = ((x_t, y_t))$ une semimartingale càdlàg sur TM. Nous voulons décrire le saut de sa partie (y_t) dans la fibre, mais il n'est pas satisfaisant de définir ce saut en utilisant des coordonnées locales. Pour en donner une définition intrinsèque, on voudrait pouvoir écrire le saut $''\ y_t \ - \ y_{t^-}\ ''$. Mais $y_t \in T_{x_t}M$ n'est pas dans le même espace tangent que $y_{t^-} \in T_{x_{t^-}}M$. Il nous faut donc transporter y_t dans l'espace $T_{x_{t^-}}M$, par exemple parallèlement le long de la géodésique, si elle existe et est unique, qui lie les points x_{t^-} et x_t sur M. Le saut covariant de (Y_t) sera alors représenté par un saut dans l'espace tangent $T_{x_{t^-}}M$

$$\tau^{//}_{x_t, x_{t^-}}(y_t) \ - \ y_{t^-}.$$

Si θ une fonction de saut sur TM, on définira l'intégrale stochastique à sauts covariants de θ le long de (Y_t) par

$$\int \theta \overset{\triangle}{dI}{}^{\nu} Y = \int (\tau^{//})^*(\theta) \overset{\triangle}{dI} Y,$$

où le saut de l'intégrale, lorsque la semimartingale (Y_t) saute de Y_{t^-} à Y_t, sera donné par

$$(\tau^{//})^*(\theta)_{Y_{t^-}}(Y_t) = \theta_{Y_{t^-}}(\tau^{//}_{x_t, x_{t^-}}(y_t) - y_{t^-}).$$

Nous voulons définir une notion plus générale de transport τ sur M. Le cas où (Y_t) est continue sera donné par l'intégrale $\int d^2(\tau^*(\theta))d\!IY$. Celle-ci doit correspondre à une intégrale covariante continue d'ordre deux, ce qui implique que le 2-jet de τ soit une connexion d'ordre deux.

Définition : *Un transport est une application régulière τ qui fournit pour chaque $V = (x, v)$ de TM une application $\tau_V : W \in TM \to \tau_V(W) \in T_xM$, régulière sur un voisinage de la diagonale $\{W \in TM, W = V\}$, et telle que son 2-jet en $W = V$,*

$$\tilde{p}_V = d^2\tau_V(V), \tag{25}$$

soit une connexion d'ordre deux sur M. Ceci implique en particulier que la restriction de $d^2\tau_V(V)$ à l'ordre un, c'est-à-dire la différentielle de τ_V en $W = V$,

$$p_V = d\tau_V(V), \tag{26}$$

est une connexion (d'ordre un) sur M.

Notons d'une part que, pour $V = (x, v)$ et $W = (y, w)$ dans TM, le vecteur transporté $\tau_{(x,v)}(y, w)$ ne dépend pas d'une courbe tracée sur M entre les points x et y, mais seulement des points eux-mêmes. D'autre part, τ_V est autorisé à dépendre du vecteur v basé en x. Cette dépendance pourrait correspondre à une contrainte physique dans certains problèmes, pour lesquels cette notion de transport s'adapterait mieux que le transport parallèle classique. Par exemple, si M est une variété riemanienne munie d'une distance δ, l'application définie pour tout $V = (x, v)$ de TM par

$$\forall W = (y, w) \in TM, \quad \tau_V(W) = \tau^{/\!/}_{yx}w + \delta^4(x, y)v,$$

(où $\tau^{/\!/}_{yx}$ est le transport parallèle le long de la géodésique joignant x à y, dont on suppose l'existence et l'unicité), est un transport sur M.

Une part importante de cette partie II est consacrée à l'étude des propriétés d'un transport ainsi qu'à leurs liens éventuels. La notion de transport linéaire (*pour tous $V \in TM$, $y \in M$, l'application $w \to \tau_V(y, w)$ est linéaire*) et idempotent (*pour tout $V = (x, v) \in TM$, l'application $w \to \tau_V(x, w)$ est l'identité de T_xM*) correspond à celle proposée par Picard dans [9]. Un tel transport est l'équivalent d'une connexion pour les sauts, au sens où il fournit pour chaque x de M une projection τ_x de l'espace TM sur T_xM, espace "vertical" pour les sauts. Les propriétés de linéarité, d'inversibilité, ou encore celle qu'un transport "aller-retour" d'un point à un autre soit l'identité, ont des conséquences importantes pour les connexions p et $\tilde{p}$ (données par (26) et (25)). Nous proposons également une définition de la courbure d'un transport τ, qui permet de retrouver la notion classique de courbure pour la connexion p .

Calcul stochastique à sauts covariants

Soit τ un transport sur M. Nous définissons l'intégrale stochastique d'ordre deux à sauts covariants d' une fonction θ de saut sur TM, le long d'une semimartingale càdlàg $(Y_t) = (((x_t, y_t))$

sur TM. Quand (Y_t) saute de Y_{t^-} à Y_t, le saut covariant de (Y_t) est un saut de y_{t^-} à $\tau_{Y_{t^-}}(Y_t)$.

Définition : *L'***intégrale stochastique à sauts covariants** *est*

$$\int \theta \ \overset{\triangle\nu}{d\!I} \ Y = \int \tau^*(\theta) \ \overset{\triangle}{d\!I}\, Y,$$

où le saut de l'intégrale, quand (Y_t) saute de Y_{t^-} à Y_t, est donné par

$$\tau^*(\theta)_{Y_{t^-}}(Y_t) = \theta_{Y_{t^-}} \circ \tau_{Y_{t^-}}(Y_t). \tag{27}$$

Remarque : *Noter que cette définition n'utilise que la partie de $\theta_{Y_{t^-}}$ dans l'espace tangent au dessus de (x_{t^-})*

$$\tau^*(\theta)_{Y_{t^-}}(Y_t) = \theta_{Y_t}(x_{t^-}, \tau_{Y_{t^-}}(Y_t)).$$

Appliquons maintenant ce formalisme aux calculs de Stratonovich et d'Itô.

Pour le calcul de Stratonovich, lorsque la semimartingale (Y_t) fait un saut covariant de y_{t^-} à $\tau_{Y_{t^-}}(Y_t)$, on intègre α le long de l'interpolation linéaire $(\ s(\tau_{Y_{t^-}}(Y_t) - y_{t^-}) + y_{t^-}\ )_{s\in[0,1]}$ de la trajectoire dans $T_{x_{t^-}}M$ entre ces deux points.

Définition : *L'* **intégrale de Stratonovich à sauts covariants** *est*

$$\int \alpha \ \overset{\triangle\nu}{\partial} \ Y = \int \tau^*(\overset{\triangle}{\mathbb{I}}\,(\alpha)) \ \overset{\triangle}{d\!I}\, Y,$$

où le saut de l'intégrale, lorsque la semimartingale (Y_t) saute de Y_{t^-} à Y_t, est donné par

$$\tau^*(\overset{\triangle}{\mathbb{I}}\,(\alpha))_{Y_{t^-}}(Y_t) = \int_0^1 \alpha_{\mathbb{I}(s,Y_{t^-},(x_{t^-},\tau_{Y_{t^-}}(Y_t)))} ds\ (0, \tau_{Y_{t^-}}(Y_t)) - y_{t^-}). \tag{28}$$

Remarque : *Ici, $\tau_{Y_{t^-}}(Y_t)) - y_{t^-}$ représente la dérivée en s de $\mathbb{I}(s) = s(\tau_{Y_{t^-}}(Y_t) - y_{t^-}) + y_{t^-}$, c'est donc un élément de $T_{y_{t^-}}(T_{x_{t^-}}M)$, et on lui applique α en suivant* (22).

Proposition : *Lorsque la semimartingale (Y_t) est continue, on obtient*

$$\int \alpha \ \overset{\triangle\nu}{\partial} \ Y = \int \tilde{p}^*(d_s\alpha) d\!IY. \tag{29}$$

Pour le calcul d'Itô, lorsque la semimartingale (Y_t) fait un saut covariant de y_{t^-} à $\tau_{Y_{t^-}}(Y_t)$, on applique α, en suivant (22), au saut représenté dans $T_{y_{t^-}}(T_{x_{t^-}})M$ par $\tau_{Y_{t^-}}(Y_t) - y_{t^-}$.

Définition : *L'* **intégrale d'Itô à sauts covariants** *est*

$$\int \alpha \ \overset{\triangle\nu}{d} \ Y = \int \tau^*(\overset{\triangle}{\gamma}\,(\alpha)) \ \overset{\triangle}{d\!I}\, Y,$$

où le saut de l'intégrale, lorsque la semimartingale (Y_t) saute de Y_{t-} à Y_t, est donné par

$$\tau^*(\overset{\triangle}{\gamma}(\alpha))_{Y_{t-}}(Y_t) = \alpha_{Y_{t-}}\ (0, \tau_{Y_{t-}}(Y_t)) - y_{t-}). \tag{30}$$

Proposition : *Lorsque la semimartingale (Y_t) est continue, on obtient*

$$\int \alpha \overset{\triangle\nu}{d}\ Y = \int \tilde{p}^*(\alpha) d\!I Y. \tag{31}$$

Remarque : *Plus généralement, on pourrait supposer l'existence d'un interpolateur $\mathbb{I}$ et d'un connecteur $\boldsymbol{\gamma}$ sur TM comme dans la partie I, mais seule leur partie dans $T_{x_{t-}}M$ serait utilisée. Autrement dit, on se donnerait une famille d'interpolateurs ou de connecteurs sur les espaces tangents.*

Partie III : Comparaison entre les deux calculs stochastiques : covariant à sauts ou à sauts covariants.

(Part III : Comparison of the two stochastic calculus : covariant with jumps or with covariant jumps)

Dans cette partie, nous comparons les définitions des intégrales de Stratonovich et d'Itô "covariantes à sauts" de la partie I et celle "à sauts covariants" de la partie II.

Un premier niveau de comparaison peut être fait en ce qui concerne la différence entre les objets utilisés pour décrire le caractère covariant dans chacune des 2 parties. Selon le problème donné, nous préfèrerons partir d'une connexion p sur M, comme dans la partie I, ou d'un transport τ sur M, comme dans la partie II.

Pour comparer les deux définitions, nous supposerons dans cette partie que

- *Le transport τ de la partie II est le transport parallèle le long des géodésiques :*

$$\forall V = (x, v) \in TM, \forall W = (y, w) \in TM : \quad \tau_V(W) = \tau^{/\!/}_{yx}(w)$$

- *La connexion p de la partie I est donnée par la différentielle de ce transport :*

$$p_V = d\tau_V(V)$$

Afin d'être cohérent avec la partie II, l' interpolateur (resp. connecteur) dans TM, utilisé dans la partie I, doit correspondre à l'interpolateur lineaire (resp. au connecteur linéaire) lorsque le point de base n'est pas interpolé. Pour cette comparaison, nous considèrerons donc l' interpolateur et le connecteur géodésiques sur TM, qui sont donnés par

$$\mathbb{I}(s,V,W) = (\ \exp_x(s \exp_x^{-1} y)\ ,\ \tau_{0s}^{//}[s(\tau_{10}^{//}(w) - v) + v]\ ),$$

$$\boldsymbol{\gamma}(V,W) = \dot{\mathbb{I}}(0,V,W),$$

où $(\tau_{0s}^{//})$ est le transport parallèle le long de la géodésique $(\exp_x(s \exp_x^{-1} y))$ qui lie x à y.

Comparaison dans le cas d'une semimartingale générale

Il voulons étudier les relations entre les objets sur M ou TM (connexion, transport, interpolateur, connecteur) qui rendent les définitions des parties 1 et 2 équivalentes pour toute semimartingale sur TM.

La différence entre les définitions des parties 1 et 2 résulte de l'ordre dans lequel on fait agir sur la forme α, l' application $\overset{\triangle}{\mathbb{J}}$ qui décrit le saut de l'intégrale ($\overset{\triangle}{\mathbb{J}}=\overset{\triangle}{\mathbb{I}}$ pour Stratonovich, $\overset{\triangle}{\mathbb{J}}=\overset{\triangle}{\boldsymbol{\gamma}}$ pour Itô) avec les applications qui décrivent le calcul covariant (p ou $\tau^{//}$). L'équivalence entre ces définitions se traduit donc par une commutativité entre ces applications :

$$\begin{array}{ccccc}
\alpha & \xrightarrow{\overset{\triangle}{\mathbb{J}}=\overset{\triangle}{\mathbb{I}}\ ou\ \overset{\triangle}{\boldsymbol{\gamma}}} & & & \overset{\triangle}{\mathbb{J}}\ (\alpha) \\
\Big\downarrow \text{connexion } p^* & & & & \Big\downarrow \text{transport } (\tau^{//})^* \\
p^*(\alpha) & \xrightarrow{\overset{\triangle}{\mathbb{J}}} & \overset{\triangle}{\mathbb{J}}\ (p^*(\alpha)) & \overset{?}{=} & (t^{//})^*(\overset{\triangle}{\mathbb{J}}\ (\alpha))
\end{array} \tag{32}$$

Nous allons voir que cette commutativité donne bien l'égalité des intégrales covariantes d'Itô des parties I et II. En revanche, les intégrales covariantes de Stratonovich ne coïncident pas.

Cas du calcul d'Itô

La commutativité entre la connexion et $\boldsymbol{\gamma}$ s'écrit

$$\overset{\triangle}{\boldsymbol{\gamma}}\ (p^*(\alpha))\ \ =\ \ (\tau^{//})^*(\overset{\triangle}{\boldsymbol{\gamma}}\ (\alpha)).$$

En utilisant (21) et (30), l'égalité des deux sauts se traduit par

$$p_{Y_{t^-}} \boldsymbol{\gamma}(Y_{t^-}, Y_t) = \tau_{x_t, x_{t^-}}^{//} y_t - y_{t^-}. \tag{33}$$

Cette condition est vérifiée lorsque $\boldsymbol{\gamma}$ est le connecteur géodesique. L'intégrale d'Itô covariante à sauts coïncide donc avec l'integrale d'Itô à sauts covariants.

Lorsque (Y_t) est continue, les intégrales coïncident également, et on montre que l'on retrouve la définition (10) de Norris, lorsque l'on fait l'identification de l'espace vertical.

Noter qu'une seule connexion est alors requise, la connexion p qui décrit le calcul covariant, la connexion nécessaire au calcul d'Itô étant en réalité la connexion plate sur un espace vectoriel. Dans notre cas, c'est la connexion $\mathbb{G}$, donnée par le 2-jet du connecteur géodésique, qui est plate. Dans le cas de Norris, le relèvement horizontal (u_t) est défini à l'aide de p, et le calcul d'Itô s'applique à la semimartingale $u_t^{-1}y_t$ dans $\mathbb{R}^m$. La deuxième connexion utilisée est donc la connexion plate sur $\mathbb{R}^m$.

Cas du calcul de Stratonovich

La commutativité entre la connexion et $\mathbb{I}$ s'écrit

$$\overset{\triangle}{\mathbb{I}}\ (p^*(\alpha)) \ = \ (\tau^{//})^*(\overset{\triangle}{\mathbb{I}}\ (\alpha)).$$

Avec l'exemple du transport parallèle et de l'interpolateur géodésique, le saut (20) s'écrit

$$\int_0^1 \alpha_{\mathbb{I}(s,Y_{t-},Y_t)}\ (0, \tau^{//}_{x_{t-},\pi\mathbb{I}(s)}(\tau^{//}_{x_t,x_{t-}} y_t - y_{t^-}))\ ds.$$

Le saut (28) s'écrit :

$$\int_0^1 \alpha_{\mathbb{I}(s,Y_{t-},(x_{t-},\tau^{//}_{x_t,x_{t-}} y_t))} ds\ (0, \tau^{//}_{x_t,x_{t-}} y_t - y_{t^-}).$$

Dans la première définition, la forme α est intégrée le long de tout l'interpolateur $\mathbb{I}$ dans TM. Dans la deuxième, le transport ramène tout dans l'espace de départ du saut $T_{x_{t-}}M$. On intègre donc α dans $T_{x_{t-}}M$, et seule la partie de $\mathbb{I}$ dans cette fibre est utilisée. Autrement dit, changer la partie de $\mathbb{I}$ sur M changerait la première définition mais pas la deuxième.

En conséquence, l'intégrale de Stratonovich covariante à sauts et l'intégrale de Stratonovich à sauts covariants sont en général différentes.

On montre dans cette thèse que les définitions restent différentes dans le cas continu. En conséquence du principe de transfert, c'est l'intégrale de Stratonovich covariante à sauts de la partie I qui coïncide avec la définition (9) de Norris, lorsque l'on fait l'identification de l'espace vertical.

Égalité pour l'intégration de semimartingales particulières

Nous avons introduit deux intégrales stochastiques dites "covariantes". Nous attendons de ces intégrales , d'une part qu'elles correspondent à l'intégrale classique le long de (y_t) lorsque (Y_t) reste dans un espace vectoriel, d'autre part qu'elles soient nulles lorsque (Y_t) est une courbe C^1 horizontale dans TM. En fait, on peut généraliser la notion de courbe horizontale aux semimartingales. On dira qu'une semimartingale $(Y_t) = ((x_t, y_t))$ dans TM est horizontale si elle vérifie

$$y_t = \tau^{//}_{x_t,x_{t-}} y_{t^-}, \text{ si } x_t \neq x_{t^-}, \quad \partial^{\mathcal{V}} Y_t = 0 \text{ si } x_t = x_{t^-}.$$

Proposition : *Si $(Y_t) = ((x_t, y_t))$ est à valeurs dans un espace vectoriel (i.e. si (x_t) est une courbe constante), alors les intégrales stochastiques covariantes à sauts et à sauts covariants coïncident avec l'intégrale le long de la semimartingale (y_t)*

$$\int \alpha \overset{\triangle\nu}{\partial} Y = \int \alpha \overset{\triangle}{\partial^{\nu}} Y = \int \alpha \overset{\triangle}{\partial} y$$

$$\int \alpha \overset{\triangle\nu}{d} Y = \int \alpha \overset{\triangle}{d^{\nu}} Y = \int \alpha \overset{\triangle}{d} y.$$

Si (Y_t) est une semimartingale horizontale dans TM, alors les intégrales stochastiques, covariantes à sauts, et à sauts covariants, sont nulles.

Conclusion de la comparaison :

Les deux intégrations stochastiques covariantes définies dans les parties I et II généralisent toutes deux l'intégration covariante d'une courbe C^1. Elles coïncident lorsqu'on intègre deux classes de semimartingales particulières, les semimartingales verticales (à valeurs dans un espace vectoriel) et les semimartingales horizontales. Dans le cas de semimartingales générales, elles diffèrent.

Le calcul stochastique covariant à sauts de la partie I présente l'avantage d'être compatible avec le calcul de Stratonovich comme avec celui d'Itô, puisqu'il redonne le formalisme de Norris lorsqu'on intègre contre une semimartingale continue. Cependant, il n'inclut pas ces calculs de Stratonovich et d'Itô à sauts dans un formalisme plus général.

De cette motivation naît le calcul stochastique à sauts covariants de la partie II. Il est plus séduisant au niveau géométrique, puisqu'il donne l'expression d'un saut covariant, à partir d'un transport, que l'on peut voir comme une connexion "d'ordre 0". Il s'applique très bien au calcul d'Itô, mais il est mal adapté au calcul de Stratonovich.

Dans le cas de l' intégration contre une semimartingale continue, on peut, à l'aide de la notion de connexion d'ordre deux, écrire les intégrales stochastiques covariantes des parties I et II comme des intégrales covariantes d'ordre deux. Ceci rejoint le principe de Schwartz, par lequel toute intégration stochastique sur une variété est en réalité une intégration d'ordre deux.

Partie IV : e.d.s. covariantes à sauts, ou à sauts covariants.

(Part IV : Covariant s.d.e. with jumps, or with covariant jumps)

Cette dernière partie est consacrée à l'étude d'e.d.s. entre variétés.
Soient N et M deux variétés différentielles. Rappelons que nous nous intéressons, dans le cas

covariant, à des semimartingales à valeurs dans les fibrés tangents. Ainsi, on se donne une semimartingale directrice (W_t) càdlàg sur TN, un coefficient régulier $e : TN \times TM \rightarrow \mathcal{L}(TN, TM)$, et un point initial Y_o sur TM . La solution (Y_t) sera une semimartingale càdlàg sur TM.

En nous inspirant de ce qui précède, nous pouvons définir :

- des "e.d.s. covariantes à sauts"
  de Stratonovich
$$\overset{\triangle}{\partial^{\mathcal{V}}} Y_t \; = \; e(\; W_{t^-}, Y_{t^-} \;) \; \overset{\triangle}{\partial^{\mathcal{V}}} W_t, \quad Y_0 = Y_o \tag{34}$$
  ou d'Itô
$$\overset{\triangle}{d^{\mathcal{V}}} Y_t \; = \; e(\; W_{t^-}, Y_{t^-} \;) \; \overset{\triangle}{d^{\mathcal{V}}} W_t, \quad Y_0 = Y_o \tag{35}$$
- des "e.d.s. à sauts covariants"
  de Stratonovich
$$\overset{\triangle \mathcal{V}}{\partial} Y_t \; = \; e(\; W_{t^-}, Y_{t^-} \;) \; \overset{\triangle \mathcal{V}}{\partial} W_t, \quad Y_0 = Y_o \tag{36}$$
  ou d'Itô
$$\overset{\triangle \mathcal{V}}{d} Y_t \; = \; e(\; W_{t^-}, Y_{t^-} \;) \; \overset{\triangle \mathcal{V}}{d} W_t, \quad Y_0 = Y_o. \tag{37}$$

Pour cela, nous écrivons chaque e.d.s. comme une e.d.s. à sauts, en suivant le formalisme de Cohen dans [2], comme
$$\overset{\triangle}{d\!I}\; Y = \phi(Y, \overset{\triangle}{d\!I}\; W).$$
ϕ est la fonction qui donne les sauts de la solution. Plus précisement , $W \rightarrow \phi(W_{t^-}, Y_{t^-}, W)$ est une fonction deux fois différentiable au voisinage de W_{t^-} et nulle en ce point.
Lorsque (W_t) saute de W_{t^-} à W_t, la solution (Y_t) saute de Y_{t^-} à $\phi(W_{t^-}, Y_{t^-}, W_t)$. La semimartingale (Y_t) est ensuite entièrement déterminée à partir de ϕ et de son 2-jet en W_{t^-}.
Les théorèmes d'existence et d'unicité sont montrés dans [2].

Selon que l'on suit la démarche de la partie I ou celle de la partie II, on décrira le saut des e.d.s. (34) et (35) en se donnant des connexions, et des interpolateurs ou connecteurs, sur les variétés N et M, et le saut des e.d.s. (36) et (37) en se donnant des transports sur N et sur M.

Rappelons que, dans [8], Norris définit des e.d.s covariantes de Stratonovich
$$\partial^{\mathcal{V}} Y_t \; = \; e(\; W_t, Y_t \;) \; \partial^{\mathcal{V}} W_t, \quad Y_0 = Y_o \tag{38}$$
et des e.d.s. covariantes d'Itô
$$d^{\mathcal{V}} Y_t \; = \; e(\; W_t, Y_t \;) \; d^{\mathcal{V}} W_t, \quad Y_0 = Y_o \tag{39}$$
dirigées par une semimartingale continue $(W_t) = ((x_t, w_t))$ dans TM, dont la solution $(Y_t) = ((x_t, y_t))$ est une semimartingale dans TM au dessus de (x_t). Cette semimartingale (x_t) est connue, puisque c'est aussi la projection de (W_t) sur M. Dans ce cas, (y_t) est solution d'une e.d.s.

linéaire. En particulier, il y a existence et unicité.
Ainsi, la résolution d'une équation différentielle covariante nécessite que l'on se donne la projection de la solution (Y_t) sur la variété d'arrivée. On retrouve cela dans le cas à sauts. Le saut d'une e.d.s. covariante de Stratonovich ou d'Itô dirigée par une semimartingale càdlàg $(W_t) = ((z_t, w_t))$ dans TN est donné de façon unique si le saut de la projection (x_t) de (Y_t) sur M est connu. La solution $(Y_t) = ((x_t, y_t))$, semimartingale càdlàg dans TM, est alors unique.

On suppose donc que la semimartingale (Y_t) a une projection (x_t) sur M qui est connue. Ainsi, la semimartingale directrice d'une équation différentielle covariante peut être vue comme une semimartingale $((x_t, W_t))$ dans $M \times TN$. Afin que la solution (Y_t) et le processus directeur sautent aux mêmes temps, nous supposerons également que (x_t) saute aux mêmes temps que (W_t).

D'après la conclusion de la partie III, l'e.d.s. (36) n'est pas une e.d.s. de Stratonovich. Nous préférerons donc étudier les trois autres e.d.s covariantes. Présentons ici les e.d.s. (34) et (37), qui sont les mieux adaptées, la première au calcul de Stratonovich et la deuxième à celui d' Itô, d'après la conclusion de la partie III.

Definition : *Soient $\mathbb{I}^{(N)}$ un interpolateur sur TN, $I^{(M)}$ un interpolateur sur M, $p^{(N)}$ une connexion sur N et $p^{(M)}$ une connexion sur M. Le sens de l'e.d.s. de Stratonovich covariante à sauts* (34)*, est donné par*

$$\overset{\triangle}{\partial^{\mathcal{V}}} Y = e(W, Y)\ \overset{\triangle}{\partial^{\mathcal{V}}} W, \quad Y_0 = Y_o \quad \Leftrightarrow \quad \overset{\triangle}{d\!I}\ Y = \Phi_{\mathcal{S}}(Y, \overset{\triangle}{d\!I}\ W), \quad Y_0 = Y_o,$$

avec

$$\Phi_{\mathcal{S}}((W_{t^-}, Y_{t^-}), W_t) = \Phi_{\mathcal{S}}(1) = (x_t, \phi_{\mathcal{S}}(1))$$

où $(\Phi_{\mathcal{S}}(s))$ est définie au dessus de $(I^{(M)}(s, x_{t^-}, x_t))$, comme la solution de l'équation différentielle

$$\frac{\partial^{\mathcal{V}} \Phi_{\mathcal{S}}(s)}{\partial s} = e(\mathbb{I}^{(N)}(s, W_{t^-}, W_t), \Phi_{\mathcal{S}}(s))\ \frac{\partial^{\mathcal{V}} \mathbb{I}^{(N)}(s, W_{t^-}, W_t)}{\partial s}, \quad \Phi_{\mathcal{S}}(0) = Y_{t^-}. \tag{40}$$

Noter que les dérivées covariantes sont définies à l'aide des connexions $p^{(N)}$ et $p^{(M)}$.

Proposition : *Quand la semimartingale (Y_t) est continue, l'e.d.s* (34) *est équivalente à l'e.d.s. de Stratonovich covariante continue* (38).

En ce qui concerne l'e.d.s. (37), qui demande la donnée de deux transports, demander que la solution existe et soit unique impose une condition d'inversibilité sur le transport de la variété d'arrivée M (*τ est dit inversible sur M si, pour tous $V = (x, v)$ et $W = (y, w)$ dans TM, l'application $\tau_V(y, .) : T_yM \to T_xM$ est inversible*).

Définition : *Soient $\tau^{(N)}$ un transport sur N et $\tau^{(M)}$ un transport inversible sur M. Le sens de l'e.d.s. d'Itô à sauts covariants* (37)*, est donné par*

$$\overset{\triangle \mathcal{V}}{d}\ Y = e(W, Y)\ \overset{\triangle \mathcal{V}}{d}\ W, \quad Y_0 = Y_o \quad \Leftrightarrow \quad \overset{\triangle}{d\!I}\ Y = \Phi_{\mathcal{I}}(Y, \overset{\triangle}{d\!I}\ W), \quad Y_0 = Y_o,$$

où

$$\Phi_{\mathcal{I}}((W_{t^-}, Y_{t^-}), W_t) = (\ x_t\ ,\ \tau^{(M)}_{Y_{t^-}}(x_t, .)^{-1}[y_{t^-} + e(W_{t^-}, Y_{t^-})(\tau^{(N)}_{W_{t^-}}(W_t) - w_{t^-})]\ ). \quad (41)$$

Proposition : *Prenons pour* $\tau^{(N)}$ *et* $\tau^{(M)}$ *les transports parallèles le long des géodésiques. Alors, quand la semimartingale* (Y_t) *est continue, l'e.d.s* (37) *est équivalente à l'e.d.s. d'Itô covariante continue* (39).

Nous appliquons ensuite ce formalisme à quelques cas particuliers et exemples.
Tout d'abord, puisqu' une intégrale stochastique est la solution d'une e.d.s. particulière, nous vérifions que ce formalisme redonne les intégrales des parties I et II. Pour cela, nous considèrerons $M = \mathbb{R}$, munie du transport "plat" $\tau^{(M)}_V(y, w) = w$ et de la connexion plate $p^{(M)}$, ainsi que le coefficient $e(W, Y) = \alpha_W$, où α est une forme sur TN.

Proposition : *Sous les conditions précédentes, les e.d.s.* (34), (35), *et* (37) *ont pour solution les intégrales stochastiques dont les sauts sont respectivement donnés par* (20), (21), *et* (30).

Nous étudions l'exemple du relèvement horizontal au dessus d'une semimartingale càdlàg, solution d'une e.d.s. covariante de Stratonovich (généralisation de (1)), qui correspond à la définition donnée par Cohen dans [1].
La notion de semimartingale horizontale proposée dans la partie III est rigoureusement définie comme la solution de l'e.d.s. à sauts $\overset{\triangle}{\partial^{\mathcal{V}}} Y_t = 0$.

Discrétisation d'une e.d.s. covariante

Pour chaque ϕ décrivant la solution d'une des e.d.s. covariantes, le théorème de discrétisation proposé par Cohen dans [2] s'applique :

Soit (Y_t) l'unique solution sur $[0, T]$ de l'e.d.s

$$\overset{\triangle}{d\!I}\ Y = \phi(Y, \overset{\triangle}{d\!I}\ W), \quad Y_0 = Y_o$$

Soit la suite (Y^n_t) définie par

- $Y^n_0 = Y_o$
- $Y^n_t = \phi(W_{T^k_n}, Y^n_{T^k_n}, W_t)$ pour $t \in [T^n_k, T^n_{k+1}]$ et $k < k(n)$
- $Y^n_t = \phi(W_{T^{k(n)}_n}, Y^n_{T^{k(n)}_n}, W_t)$ si $t \geq T^n_{k(n)}$

pour une subdivision $\sigma^n = (T^k_n)$ qui tend vers l'identité (voir [2]).

Alors (Y^n_t) converge uniformément sur tout compact en probabilité vers (Y_t) sur $[0, T]$.

Reminders of Geometry and Probability

0.1 Reminders of Geometry

In this section, we specify the geometric tools needed. We introduce the definition of linear connection on M, yielding the notions of parallel transport and covariant derivative. Then we give the definition of geodesic, providing our simplest examples. In the end, we recall the notions of interpolation and connection rules, used to define Stratonovich and Itô calculus with jumps on a manifold.

0.1.1 Geometric assumptions and notations

- In all the paper, M and U are C^∞ manifolds without boundary. They are assumed to have a countable atlas, so the Withney's imbedding theorem allows to imbed these paracompact manifolds in $\mathbb{R}^m$. All the formulas given in coordinates are expressed in this imbbeding, but never depend on it, otherwise it will be specified. Coordinates are always used in the form described in Appendix A. The Einstein summation convention is used.

- We work on the manifold called M. However, everytime we recall general results, intended to be applied later to M or TM, the manifold is called U. Note that, when another manifold is needed, it is called W.

- For all z in U, T_zU is the tangent space to U at z. It is the set of (first order) tangent vectors at z on U, that is differential operators of order one with no constant term. $TU = \cup_{z\in U} T_zU$ denotes the tangent bundle.

  α is a first order form on U if, for every z in U, $\alpha_z \in T_z^*U$. For instance, the differential df of a smooth function f on U is a first order form on U.

  For every smooth $\phi : U \to W$, for every z in U, we denote by $d\phi(z) : T_zU \to T_{\phi(z)}W$ and call also the differential of ϕ at z, the linear tangent map defined, for all A in T_zM and every smooth f, by $d\phi(z)(A)(f) = A(\phi \circ f)$. Its expression is given in coordinates in Appendix B.

- For all z in U, τ_zU is the tangent space of order two to U at z. It is the set of second order tangent vectors at z on U, that is differential operators of order at most two, with no constant term. $\tau U = \cup_{z\in U} \tau_zU$ denotes the tangent bundle of order two.

  Θ is a second order form on U if, for every z in U, $\Theta_z \in \tau_z^*U$. For instance, the 2-jet d^2f of a smooth function f on U is a second order form on U.

  For every smooth $\phi : U \to W$, for every z in U, we denote by $d^2\phi(z) : \tau_zU \to \tau_{\phi(z)}W$ and call also the 2-jet of ϕ at z, the linear tangent map of order two defined, for all $\mathbb{A}$ in τ_zM and every smooth f, by $d^2\phi(z)(\mathbb{A})(f) = \mathbb{A}(\phi \circ f)$ (see [3] for more details). Its expression is given in coordinates in Appendix B.

- For all z in U, $\overset{\triangle}{\tau^*}_z U$ denotes the set of real functions θ_z on U, twice differentiable at z and such that $\theta_z(z) = 0$. $\overset{\triangle}{\tau^*} U$ is defined by $\overset{\triangle}{\tau^*} U = \cup_{z \in U} \overset{\triangle}{\tau^*}_z U$.(see [1] for more details)
- π denotes the projection of TM onto M. For every V in TM, the notation $V = (x, v)$ means that x is the projection πV of V on M and v is the part of V in the fibre T_xM above x.

  For every smooth $\phi : TM \to W$, for all x in M, the map $\phi(x, .) : T_xM \to W$ denotes the restriction of ϕ to the tangent space T_xM.
- For every smooth curve (z_t) on U, the derivative of z_t with respect to the time t will be denoted either by $\dot{z}_t$ or by $\frac{\partial z}{\partial t}$ ($\dot{z}_t = \dot{z}^i_t \frac{\partial}{\partial z^i}$).

0.1.2 Connection of order one on M

There are different equivalent ways to define a connection on a manifold. We specify here our approach. We first recall the general definition of a connection in a principal fibre bundle, following [11] and [5]. Actually, we will work in the particular case of a linear connection in the principal fibre bundle $L(M)$ of linear frames on M, with the linear group $Gl_m(\mathbb{R})$. More precisely, we consider a connection on the associated vector bundle TM with standard fibre $\mathbb{R}^m$. Nevertheless, note that the results of part I could be written more generally with a linear connection in a principal fibre bundle over M.

Our presentation of connection may slightly differ from the usual one. Hence, certain proofs needs to be given. However, since this paragraph consists in reminders, when a proof is too long and technical, it is written in Appendix D.

Connection in a principal fibre bundle E over M

Let E be a principal fibre bundle over M with Lie group G and Lie algebra $\mathcal{G}$.
For all $V = (x, v)$ in E, the fibre through V is G_V, submanifold of E isomorphic to $\mathcal{G}$. Then the tangent space to E at V, T_VE, contains a canonical subset $\mathcal{V}_VE = T_VG_V$ (rigorously $di(V)(T_VG_V)$ with the inclusion $i : G_V \hookrightarrow E$) called the vertical space at V. There is no canonical way of choosing for each V a supplementary space of this vertical space. A connection on E is precisely a way of making this choice. It defines a map p_V projecting T_VE onto $\mathcal{V}_VE$. Its kernel $kerp_V = \mathcal{H}_VE$ is called the horizontal space at V. The notion of parallel linear displacement in these horizontal spaces comes from their right invariance under the action of the group G given by the second point of the definition.

Definition 0.1 *A connection p on the principal fibre bundle E over M with Lie group G is a smooth map $p : V \in E \to p_V$ such that, for all V in E*

1. *p_V projects T_VE onto the vertical space $\mathcal{V}_VE = T_vG_V$.*

2. *The horizontal spaces at V, $\mathcal{H}_V E = kerp_V$, satisfy*

$$\forall V \in E, \forall g \in G, \quad \mathcal{H}_{R_g V} E = (R_g)_* \mathcal{H}_V E$$

where R is the right multiplication on G.

Connection in M

In the following, we consider a connection on the vector bundle TM, with standard fibre $\mathbb{R}^m$, associated to the principal bundle $E = L(M)$ with the linear group $Gl_m(\mathbb{R})$. We define here such a connection, called a connection on M. We follow Spivak ([11]).

For all $V = (x, v)$ in TM, the fibre through V is $T_x M$. The tangent space $T_V TM$ to TM at V contains a canonical subset $T_v(T_x M)$ (rigorously $di(V)(T_V(x, T_x M))$ with the inclusion $i : (x, T_x M) \hookrightarrow TM$). This space is called the vertical space at V, and denoted by $\mathcal{V}_V TM$. A connection on M is a way of choosing for each V a supplementary space $\mathcal{H}_V TM$ of this vertical space, called the horizontal space at V.

Definition 0.2 *A connection on M is a smooth map $p : V \in TM \to p_V$ such that , for every $V = (x, v)$ in TM*

1. *p_V projects $T_V TM$ onto the vertical space $\mathcal{V}_V TM = T_v(T_x M)$.*
2. *$\forall \lambda \in \mathbb{R}$, $p_{\bar{\lambda} V} \circ d\bar{\lambda}(V) = \lambda p_V$ where $\bar{\lambda}(V) = (x, \lambda.v)$.*

Remark 0.1 *By the first point of the definition, p_V is a projection so it is linear. This linearity is expressed by the properties*

$$\forall U_1, U_2 \in T_V TM, \; p_V(U_1 + U_2) = p_V(U_1) + p_V(U_2) \tag{42}$$

and

$$\forall U \in T_V TM, \; p_{\bar{\lambda} V}(d\bar{\lambda}(V)(U)) = \lambda p_V(U),$$

that is the second point of the definition. This second point expresses the right invariance of the horizontal spaces in the vectorial case. Actually, using definition 0.1*, this property should be written on the horizontal spaces $\mathcal{H}_V E = kerp_V$ at V as follows*

$$\mathcal{H}_{\bar{\lambda} V} TM = d\bar{\lambda}(V)(\mathcal{H}_V TM). \tag{43}$$

The equivalence between this formulation and those of the second point is proved in Appendix D.

Expressions in coordinates

We use the coordinates described in Appendix A.

Proposition 0.1 *A smooth map $p : V \in TM \to p_V$ is a connection on M if and only if, for all $V = (x, v)$ in TM,* *1. p_V is a linear map, such that, for every U in $T_v(T_xM)$, written $U = dv^i(U)\frac{\partial}{\partial v^i}$,*

$$p_V(U) = dv^i(U)\frac{\partial}{\partial v^i}. \tag{44}$$

2. We have

$$\forall \lambda \in \mathbb{R}, \forall V = (x, v) \in TM,\ p_{\bar{\lambda}V}(\frac{\partial}{\partial x^i}) = \lambda p_V(\frac{\partial}{\partial x^i}).$$

This implies that $v \to p_V(\frac{\partial}{\partial x^i})$ is a linear map, so there exist smooth functions (Γ^i_{jk}) such that

$$\forall v \in T_xM,\ p_V(\frac{\partial}{\partial x^j}) = \Gamma^i_{jk}(x)v^k\frac{\partial}{\partial v^i}. \tag{45}$$

Proof : See Appendix D.

Every vector $U = dx^i(U)\frac{\partial}{\partial x^i} + dv^i(U)\frac{\partial}{\partial v^i}$ on T_VTM can be uniquely written as $U = U^{\mathcal{V}} + U^{\mathcal{H}}$ with a vertical component $U^{\mathcal{V}}$ and an horizontal one $U^{\mathcal{H}}$. Note that if U is a differentiable vector field, so are $U^{\mathcal{V}}$ and $U^{\mathcal{H}}$. According to (44) and (45), they are expressed as follows

$$U^{\mathcal{V}} = p_V(U) = [dv^i(U) + \Gamma^i_{jk}(x)v^k dx^j(U)]\frac{\partial}{\partial v^i}, \tag{46}$$

$$U^{\mathcal{H}} = U - p_V(U) = dx^i(U)\frac{\partial}{\partial x^i} - dx^k(U)\Gamma^i_{kj}(x)v^k\frac{\partial}{\partial v^i}. \tag{47}$$

Remark 0.2 *Seeing the tangent space T_VTM at $V = (x, v)$ as a product space $T_xM \times T_v(T_xM)$, we will sometimes write*

$$p_V(U) = p_V(dx^i(U)\frac{\partial}{\partial x^i}, dv^i(U)\frac{\partial}{\partial v^i}).$$

It implies, using the linearity of p_V and (44), that

$$p_V(U) = dx^i(U)p_V(\frac{\partial}{\partial x^i}, 0) + dv^i(U)\frac{\partial}{\partial v^i}.$$

As a consequence, if the part $dx^i(U)\frac{\partial}{\partial x^i}$ of U on T_xM and its vertical part $p_V(U)$ are known, we recover the part $dv^i(U)\frac{\partial}{\partial v^i}$ of U in $T_v(T_xM)$.

Torsion and curvature of p

If M is a Riemmanian manifold, the notions of torsion and curvature can be defined for a connection p. We present here briefly the statements used in the paper, following [5].

Torsion of p

The torsion of p is represented by the torsion tensor T given in coordinates by

$$T^i_{jk}(x) = \Gamma^i_{jk}(x) - \Gamma^i_{kj}(x).$$

Definition 0.3 *p is a torsion-free connection on M if the functions (Γ^j_{ik}) are symmetric*

$$\Gamma^j_{ik}(x) = \Gamma^j_{ki}(x),$$

or equivalently if p satisfies

$$p_{\frac{\partial}{\partial x^k}}(\frac{\partial}{\partial x^j}) = p_{\frac{\partial}{\partial x^j}}(\frac{\partial}{\partial x^k}). \tag{48}$$

Remark 0.3 *For a torsion-free connection, the (Γ^j_{ik}) are called Christoffel symbols of the connection p. Note that we will also call Christoffel symbols the coefficients for a non torsion-free connection.*

In the paper, we use the notion of connection for two different purposes :

- Connections are used to do covariant calculus. They are not supposed to be torsion-free. In part III, these connections are deduced from a transport τ. We will see examples of τ providing a torsion-free connection.
- Other connections are required to do Itô calculus on a manifold. Emery explains in [3] that only the symmetric part of the connection is involved for the Itô calculus. Thus, without loss of generality, these connections are assumed to be torsion-free.

We will use the following proposition, which can be found in [11].

Proposition 0.2 *Let p and q be two connections on M. If the geodesics for p and for q are the same, then the torsion-free parts of p and q equal. In particular, if p and q are torsion-free connections, we get $p = q$.*

Curvature of p

The curvature of a connection is a way to measure the difference between a vector and its parallel transport along an infinitesimal loop. It is represented by the curvature tensor R given in coordinates by

$$R^i_{jml}(x) = dx^i R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^l}) = \frac{\partial \Gamma^i_{lj}}{\partial x^m}(x) - \frac{\partial \Gamma^i_{mj}}{\partial x^l}(x) + \Gamma^i_{mk}(x)\Gamma^k_{lj}(x) - \Gamma^i_{lk}(x)\Gamma^k_{mj}(x).$$

Definition 0.4 *p is said to be a flat connection if its curvature vanishes for every x in M.*

0.1.3 Covariant derivative of a C^1 curve

Let p be a connection on M and $(Y_t) = ((x_t, y_t))$ a curve of class C^1 in TM. Its covariant derivative is the usual derivative seen in a frame moving parallelly along its projection curve (x_t) on M. This displacement is described by the notion of parallel transport along (x_t), defined as follows.

Definition 0.5 *The horizontal lift (u_t) of (x_t), starting from a given $u_o \in GL(\mathbb{R}^m, T_{x_0}M)$, is the unique curve $((x_t, u_t))$ in $GL(\mathbb{R}^m, TM)$ $(\forall t, u_t \in GL(\mathbb{R}^m, T_{x_t}M))$ satisfying the differential equation*

$$p_{(x_t,u_t)}(\dot{x}_t, \dot{u}_t) = 0, \quad u_0 = u_o. \tag{49}$$

(This equation means

$$\forall z \in \mathbb{R}^m, \quad p_{(x_t,u_t(z))}(\dot{x}_t, \dot{u}_t(z)) = 0, \quad u_0 = u_o.) \tag{50}$$

The parallel transport $\tau^{//}$ along (x_t) is defined, for all s, t, by the linear map

$$\tau^{//}_{st} = u_t u_s^{-1} : T_{x_s}M \to T_{x_t}M.$$

Remark 0.4 *Note that the parallel transport satisfies in coordinates*

$$\frac{\partial (\tau^{//}_{st})^i_k}{\partial s} = (\tau^{//}_{st})^i_r \Gamma^r_{lk}(x_s)\ \dot{x}^l_s, \tag{51}$$

where $(\tau^{//}_{st})^i_k = dx^i(\tau^{//}_{st}(\frac{\partial}{\partial x^k}))$.

Definition 0.6 *The covariant derivative of a C^1 curve $(Y_t) = ((x_t, y_t))$ on TM is defined by*

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = \lim_{h \to 0} \frac{\tau^{//}_{t+h,t}(y_{t+h}) - y_t}{h}. \tag{52}$$

Remark 0.5 *Our notation for the covariant derivative is not the usual one. The exponent $\mathcal{V}$ stands for "vertical", since the covariant derivative is valued in the vertical space $T_{y_t}(T_{x_t}M)$.*

We won't use the definition of covariant derivative in this form. The following equivalent formulations are more significant. Formula (53) makes us understand the covariant derivative as the projection of the usual derivative $\dot{Y}_t$ by p_{Y_t}. Consequently, we shall see the covariant case as the image by the connection p of the classical case.
The formulation (55) of the covariant derivative will be useful for many proofs. It is the most geometrically significant formula : the process (y_t) is transported along (x_t) up to a given $T_{x_s}M$ ($s = 0$ for instance), where it is derivated, then this derivative is transported back to $T_{x_t}M$.

Proposition 0.3 *The covariant derivative of the curve $(Y_t) = ((x_t, y_t))$ is equivalently given by*

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = p_{Y_t}(\dot{Y}_t) = [\ \dot{y}^i_t\ +\ \Gamma^i_{jk}(x_t)\ y^k_t\ \dot{x}^j_t\ ]\ \frac{\partial}{\partial v^i}, \tag{53}$$

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = (\frac{\partial\ \tau^{//}_{st} y_s}{\partial s})_{|s=t}, \tag{54}$$

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = u_t \frac{\partial}{\partial t}(u_t^{-1} y_t), \qquad or \qquad \forall s,\ \frac{\partial^{\mathcal{V}} Y_t}{\partial t} = \tau^{//}_{st} \frac{\partial}{\partial t}(\tau^{//}_{ts} y_t). \tag{55}$$

Proof : (52) $\Leftrightarrow$ (54)
(52) is written

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = \lim_{h\to 0} \frac{\tau^{//}_{t+h,t}(y_{t+h}) - y_t}{h} = \lim_{h\to 0} \frac{\tau^{//}_{t+h,t}(y_{t+h}) - \tau^{//}_{t,t} y_t}{h}.$$

Setting $f_t(s) = \tau^{//}_{s,t} y_s$, we get

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = \lim_{h\to 0} \frac{f_t(t+h) - f_t(t)}{h} = (\frac{\partial f_t(s)}{\partial s})_{|s=t} = (\frac{\partial\ \tau^{//}_{st} y_s}{\partial s})_{|s=t}.$$

(54) $\Leftrightarrow$ (55)
The definition of parallel transport implies

$$\frac{\partial\ (\tau^{//}_{st} y_s)}{\partial s} = \frac{\partial\ (u_t u_s^{-1} y_s)}{\partial s} = du_t(u_t u_s^{-1} y_s) \frac{\partial\ (u_s^{-1} y_s)}{\partial s}.$$

At time $s = t$ we get

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = du_t(y_t) \frac{\partial}{\partial t}(u_t^{-1} y_t).$$

Since u_t is a linear map, its differential $du_t(y_t)$ at point y_t is identified with u_t itself, and we get

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = u_t \frac{\partial}{\partial t}(u_t^{-1} y_t).$$

Moreover, for all s, we get

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = u_t \frac{\partial}{\partial t}(u_t^{-1} y_t) = u_t u_s^{-1} \frac{\partial}{\partial t}(u_s u_t^{-1} y_t) = \tau^{//}_{st} \frac{\partial}{\partial t}(\tau^{//}_{ts} y_t).$$

(53) $\Leftrightarrow$ (55)
Writing $y_t = u_t u_t^{-1} y_t$, we have

$$\dot{y}_t = \frac{\partial}{\partial t}(u_t u_t^{-1} y_t) = u_t\ \frac{\partial}{\partial t}(u_t^{-1} y_t) + \dot{u}_t\ u_t^{-1} y_t = \frac{\partial^{\mathcal{V}} Y_t}{\partial t} + \dot{u}_t\ u_t^{-1} y_t.$$

Then, using the linearity of p_{Y_t}, we have

$$p_{Y_t}(\dot{Y}_t) = p_{(x_t,y_t)}(\dot{x}_t, \dot{y}_t) = p_{(x_t,y_t)}(0, \frac{\partial^{\mathcal{V}} Y_t}{\partial t}) + p_{(x_t, u_t u_t^{-1} y_t)}(\dot{x}_t, \dot{u}_t\ u_t^{-1} y_t).$$

Now, applying (50) to $z = u_t^{-1} y_t \in \mathbb{R}^m$, the second term in the previous equality is null. As the first term is concerned, recall that p verifies $p_{Y_t}(0, U) = U$. Consequently, we get

$$p_{Y_t}(\dot{Y}_t) = \frac{\partial^{\mathcal{V}} Y_t}{\partial t}.$$

Moreover, by (46) we can write

$$p_{Y_t}(\dot{Y}_t) = \ [\ \dot{y}_t^i\ +\ \Gamma^i_{jk}(x_t)\ y_t^k\ \dot{x}_t^j\ ]\ \frac{\partial}{\partial v^i}.\square$$

Remark 0.6 *Note the meaning of remark* 0.2 *with regard to the covariant derivative. If the curve* (x_t) *on* M *is known, so is the part* $(\dot{x}_t)$ *of* $(\dot{Y}_t)$ *on* $T_{x_t}M$*. Then* $(\dot{y}_t)$ *is deduced from the covariant derivative of* (Y_t) *and conversely. On account of this property, if* (Y_t) *is solution of a linear covariant differential equation and its projection* (x_t) *is known, then* $(\dot{y}_t)$ *is solution of a linear differential equation. So unicity stands for the solution* (y_t) *and hence for the solution* (Y_t)*.*

Remark 0.7 *Note that, in the two following cases*

1. p *is the flat connection in* $\mathbb{R}^m$*,*
2. $(Y_t) = ((x_t, y_t))$ *is vertical, that is the curve* (x_t) *is constant,*

we have

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = \frac{\partial y_t}{\partial t}$$

0.1.4 Geodesics on M and TM

Geodesics and exponential map on M

In this paragraph, M is a manifold endowed with a connection p. The proofs which are not written here, are in [5].

Definition 0.7 *A curve* (x_t) *of class* C^1 *in* M *is a geodesic if its tangent vector* $(\dot{x}_t)$ *is parallelly transported along* (x_t)*, that is if*

$$\forall t, \ \frac{\partial^{\mathcal{V}}(x_t, \dot{x}_t)}{\partial t} = 0.$$

It is equivalently written in coordinates

$$\ddot{x}_t^k \ = \ -\Gamma_{ij}^k(x_t)\ \dot{x}_t^i\ \dot{x}_t^j.$$

Proposition 0.4 *Let* $\tau^{//}$ *be the parallel transport along* (x_t)*. Then* (x_t) *is a geodesic if and only if* $\dot{x}_t = \tau_{0t}^{//}\dot{x}_0$*.*

Proof : Using (55), the relation $\frac{\partial^{\mathcal{V}}(x_t,\dot{x}_t)}{\partial t} = 0$ is equivalent to $u_t \frac{\partial}{\partial t}(u_t^{-1}\dot{x}_t) = 0$, where (u_t) is the horizontal lift of (x_t) starting from a given u_0. This equation means that, for all t, $[u_t^{-1}\dot{x}_t = u_0^{-1}\dot{x}_0$, that is $\dot{x}_t = u_t u_0^{-1} x_0 = \tau_{0t}^{//}\dot{x}_0$. □

The following proposition allows us to define the exponential map on M.

Proposition 0.5 *The geodesic starting from* x *in* M *with initial tangent vector* v *in* T_xM *exists and is unique.*

Definition 0.8 *The exponential map is defined at every point* x *in* M *as follows.*
For all v *in* T_xM*,* $\exp_x(v)$ *is the point at time* $t = 1$ *of the geodesic on* M *starting from* x *with initial tangent vector* v*.*

The exponential map is generally only defined on a subset of T_xM for each x in M. It is thus a local diffeomorphism from N_x to a neighbourhood of x in M. Consequently, the geodesic joining a point x to another point y in M exists and is unique for y in N_x. Then, it is written with the exponential map as

$$x_t = \exp_x(t \exp_x^{-1}),$$

where the initial tangent vector is

$$\dot{x}_0 = \exp_x^{-1}(y).$$

A manifold on which any two points can be joined by a unique geodesic is said to be simply convex.

We conclude with some useful properties about the exponential map.

Proposition 0.6 *The parallel transport along a geodesic* (x_t) *with initial tangent vector* $\dot{x}_0$ *is related to the exponential map as follows*

$$d \exp_x(t\dot{x}_0)(\dot{x}_0) = \tau_{0t}^{//}(\dot{x}_0).$$

Proof : According to proposition 0.4, $\dot{x}_t = \tau_{0t}^{//}\dot{x}_0$. But $\dot{x}_t = \frac{\partial}{\partial t}(\exp_x(t\dot{x}_0)) = d\exp_x(t\dot{x}_0)(\frac{\partial}{\partial t}(t\dot{x}_0)) = d\exp_x(t\dot{x}_0)(\dot{x}_0)$. □

Note that the map $d\exp_x(v) : T_xM \to T_{\exp_x(v)}M$ is not the parallel transport along the geodesic from x with initial tangent vector $v \in T_xM$. Actually, this map is related to the notion of Jacobi fields. It will be used to provide an example of transport in part II.

Proposition 0.7 *The differential and the 2-jet of the exponential map at* 0 *are*

$$d\exp_x(0) = Id_{T_xM}, \ d^2\exp_x(0) = Id_{\tau_xM}.$$

Consequently, we get also

$$d\exp_x^{-1}(x) = Id_{T_xM}.$$

Proof : The two first assertions are proved in Appendix C. Then, writing $d(\exp_x^{-1} \circ \exp_x)(0) = Id_{T_xM}$, we get $d\exp_x^{-1}(\exp_x(0)) \circ d\exp_x(0) = Id_{T_xM}$. It follows $d\exp_x^{-1}(\exp_x(0)) = d\exp_x^{-1}(x) = d\exp_x(0)^{-1} = Id_{T_xM}$. □

Geodesics on TM

When doing covariant calculus, we consider semimartingales on TM. Therefore, the Itô calculus involves torsion-free connections on TM. Let us present here the classical example of the horizontal lift of a torsion-free connection p, for which geodesics are parallel transports. Note that another example is given by the complete lift of p, for which geodesics are Jacobi Fields. (see [12]). We won't give the definition of the horizontal lift of p, but only define the geodesics with respect to this horizontal lift, following Yano and Ishihara in [12], where proofs are given.

Proposition 0.8 *Let p be a torsion-free connection on M. Let $(Y_t) = ((x_t, y_t))$ be a curve of class C^1 in TM. It is a geodesic with respect to the horizontal lift of p if its projection (x_t) on M is a geodesic with respect to p and if the covariant derivative of its tangent part is parallelly transported along (x_t), that is if*

$$(a_1)\ \frac{\partial^{\mathcal{V}}}{\partial t}(\ x_t, \dot{x}_t\ ) = 0 \quad \text{and} \quad (b_1)\ \frac{\partial^{\mathcal{V}}}{\partial t}(\ x_t, \frac{\partial^{\mathcal{V}} Y_t}{\partial t}\ ) = 0.$$

This is equivalent to

$$(a_2)\ \dot{x}_t = \tau_{0t}^{//}\ \dot{x}_0 \quad \text{and} \quad (b_2)\ \frac{\partial^{\mathcal{V}} Y_t}{\partial t} = \tau_{0t}^{//}\ (\ \frac{\partial^{\mathcal{V}} Y_t}{\partial t}\ )|_{t=0}.$$

Proof : The expressions (a_1) and (b_1) comes from [12]. Let us prove the equivalence with (a_2) and (b_2). The equivalence $(a_1) \Leftrightarrow (a_2)$ is given by proposition 0.4.
Using the formulation (55) for the covariant derivative, (b_1) is equivalent to

$$\frac{\partial}{\partial t}(\ u_t^{-1} \frac{\partial^{\mathcal{V}} Y_t}{\partial t}\ ) = 0,$$

where (u_t) is the horizontal lift of (x_t) starting from a given u_0. This equality means that, for all t,

$$u_t^{-1} \frac{\partial^{\mathcal{V}} Y_t}{\partial t} = u_0^{-1} \frac{\partial^{\mathcal{V}} Y_t}{\partial t}|_{t=0},$$

that is (b_2)

$$\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = \tau_{0t}^{//} \frac{\partial^{\mathcal{V}} Y_t}{\partial t}|_{t=0}. \square$$

Example 0.1 *If (x_t) is a geodesic on M, the curve $(Y_t) = ((x_t, \dot{x}_t)) = ((x_t, \tau_{0t}^{//} \dot{x}_0))$ is a geodesic on TM. Indeed, (a_1) is satisfied since (x_t) is a geodesic and it implies (b_1), that is $\frac{\partial^{\mathcal{V}}}{\partial t}(x_t, \frac{\partial^{\mathcal{V}}(x_t, \dot{x}_t)}{\partial t}) = \frac{\partial^{\mathcal{V}}}{\partial t}(x_t, 0) = 0$.*

Example 0.2 *If M is simply convex, a geodesic joining two points on TM is given as follows, with a geodesic on M and a linear interpolation parallelly transported in the fibres above this geodesic.*

Set $V = (x, v)$ and $W = (y, w)$ in TM. Let $(x_t) = (\exp_x(t \exp_x^{-1}(y)))$ be the geodesic from x to y, and $\tau^{//}$ the parallel transport along (x_t). Then the curve

$$(X_t) = ((x_t, \tau_{0t}^{//}((\tau_{10}^{//}(w) - v)t + v))$$

is a geodesic with respect to the horizontal lift of p, joining V to W on TM.

Indeed, (a_1) is verified since (x_t) is a geodesic. For (b_1), computing the covariant derivative of (X_t) along (x_t) with formula (55) gives

$$\frac{\partial^{\mathcal{V}} X_t}{\partial t} = \tau_{0t}^{//}\ \frac{\partial}{\partial t}[\ \tau_{t0}^{//}\ (\tau_{0t}^{//}(\tau_{10}^{//}(w) - v)t + v)] = \tau_{0t}^{//}\ \frac{\partial}{\partial t}[(\tau_{10}^{//}(w) - v)t + v].$$

It follows

$$\frac{\partial^{\mathcal{V}} X_t}{\partial t} = \tau_{0t}^{//}(\tau_{10}^{//}(w) - v).$$

Consequently

$$\frac{\partial^{\mathcal{V}}}{\partial t}(x_t, \frac{\partial^{\mathcal{V}} X_t}{\partial t}) = \frac{\partial^{\mathcal{V}}}{\partial t}(x_t, \tau_{0t}^{//}(\tau_{10}^{//}(w) - v)).$$

But $\frac{\partial^{\mathcal{V}}}{\partial t}(x_t, \tau_{0t}^{//}) = \frac{\partial^{\mathcal{V}}}{\partial t}(x_t, u_t u_0^{-1}) = 0$ *by definition of the horizontal lift* (u_t) *of* (x_t)*. So* (b_1) *is verified.*

0.1.5 Interpolation and connection rules

The notions of interpolation and connection rules will be used to describe the jumps for a Stratonovich or an Itô integral.

Interpolation rule

Definition 0.9 (Cohen) *A* C^3 *mapping* $I : [0,1] \times U \times U \to U$ *is an interpolation rule if it satisfies, for all* s *in* $[0,1]$*, for all* x, y *in* U*,*

$$(i) \quad I(s,x,x) = x, \qquad (ii) \quad I(0,x,y) = x,\ I(1,x,y) = y, \qquad (iii) \quad \frac{\partial I(s,x,y)}{\partial y}|_{y=x} = \lambda(s) Id_{T_xU}.$$

Example 0.3 . Linear interpolation rule
If U *is a vector space, the linear interpolation*

$$I(s,x,y) = x + s(y-x)$$

is an interpolation rule on U*.*

Example 0.4 . Geodesic interpolation rule on M
If $U = M$ *is simply convex, the geodesic joining two points,* $c : [0,1] \times M \times M \to M$*, defined by*

$$\forall s \in [0,1], \forall x, y \in M,\ c(s,x,y) = \exp_x(s. \exp_x^{-1}(y)),$$

is an interpolation rule on M*, called the geodesic interpolation rule on* M*.*

Indeed, (i) *and* (ii)*are obvious. For* (iii)*, we have*

$$dc(s,x,.)(x) = d\exp_x\ (s. \exp_x^{-1}(x))(s.d\exp_x^{-1}\ (x)) = d\exp_x\ (0)(s.d\exp_x^{-1}\ (x))$$

Using proposition 0.7*, we get* $dc(s,x,.)(x) = s.Id_{T_xU}$ *and* (iii) *is proved.*

Example 0.5 . Geodesic interpolation rule on TM
If $U = TM$ *is simply convex, the geodesic* (X_t) *with respect to the horizontal lift of* p*, joining two points on* TM *(see example* 0.2*), yields in the same way an interpolation rule on* TM*, called the geodesic interpolation rule on* TM*.*

Connection rule

Definition 0.10 (Picard) *A C^2 mapping $\gamma : U \times U \to TU$, is a connection rule if it satisfies, for all x, y in U,*

$$(i) \quad \gamma(x,y) \in T_xU, \qquad (ii) \quad \gamma(x,x) = 0, \qquad (iii) \quad \frac{\partial \gamma(x,y)}{\partial y}|_{y=x} = Id_{T_xU}.$$

Note that, by the local inversion theorem, for all x, $\gamma(x,.)$ is a local diffeomorphism from a neighbourhood of x in U to a neighbourhood of 0_x in T_xU.

Example 0.6 . Linear connection rule
If U is a vector space, the linear map $\gamma(x,y) = y - x$ is a connection rule on U.

Example 0.7 . Geodesic connection rule on M
If $U = M$ is simply convex, the initial tangent vector to the geodesic joining the points x and y on M, $\gamma_0 : M \times M \to TM$, is defined by

$$\gamma_0(x,y) = \exp_x^{-1}(y)$$

It is a connection rule on M, called the geodesic connection rule.
Indeed, (i) is obvious, (ii) comes from $\exp_x^{-1}(x) = 0$. For (iii), we have $d\gamma_0(x,.)(x) = d\exp_x^{-1}(x) = Id_{T_xU}$, according to proposition 0.7.

Example 0.8 . Geodesic connection rule on TM
If $U = TM$ is simply convex, the tangent vector at time 0, $\dot{X}_0$, to the geodesic (X_t) joining V to W on TM, yields in the same way a connection rule on TM.

0.2 Reminders of Probability

In this section, we first summarize results about the stochastic calculus of order two on a manifold. Meyer and Schwartz proposed in [7] and [10] a formalism for continuous semimartingales. Then this formalism has been extended by Cohen in [1] to semimartingales with jumps.
Then we recall the formalism proposed by Norris in [8] for stochastic covariant calculus on manifolds.

0.2.1 Probabilistic assumptions and notations

- $(\Omega, \mathcal{F}, P)$ is a standard probability space, endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$, satisfying the usual hypothesis.
- Any process z on U is denoted by (z_t), where the time t is always assumed to be positive.

- Recall that a process is said to be "càdlàg" if it is right continuous and admits a limit on left.

  A U-valued process (z_t) is a continuous (resp. càdlàg) semimartingale on U if, for all C^2 function f on U, $(f(z_t))$ is a real continuous (resp. càdlàg) semimartingale. Note that, for all system of coordinates (z^i) at z, (z_t) is a semimartingale if and only if all (z_t^i) are.

- (z_t^c) denotes the continuous part of the càdlàg semimartingale (z_t)

- The notions of locally bounded and predictable processes on U will be used. We refer the reader to [3] and [1] for the precise definitions.

0.2.2 Stochastic continuous calculus of order two ($d\!\!I$)

To integrate on a manifold U, first order forms α are usually considered. If (z_t) is a C^1 curve on U, α is integrated against the tangent vector $\dot{z}_t \in T_{z_t}U$. If $\alpha = df$, with f a C^2 function on U, we get

$$\int_0^t df(z_s)\ \dot{z}_s\ ds = f(z_t) - f(z_0). \qquad (*)$$

The Itô formula in $\mathbb{R}^m$ shows that the Stratonovich infinitesimal variation ∂z_t of a semimartingale (z_t) integrates with the same rules as a tangent vector. Indeed, if f is a C^2 function on $\mathbb{R}^m$, we have

$$\int_0^t df(z_s)\ \partial z_s = f(z_t) - f(z_0). \qquad (**)$$

The analogy between the formulas $(*)$ and $(**)$ is called the transfer principle, which has been described by Elworthy and Malliavin. It is used to give meaning to a Stratonovich integral on U. The Itô calculus on U is then deduced from a Stratonovich to Itô conversion formula, for which we need a torsion-free connection on U. We will give these definitions in more details in part I.

The stochastic infinitesimal variations ∂z_t (Stratonovich) and dz_t (Itô) are not intrinsic. The Itô formula rather incites us to consider a second order stochastic infinitesimal variation, similar to a second order tangent vector on U,

$$d\!\!I z_t = dz_t^i\ \frac{\partial}{\partial z^i} + \frac{1}{2} d < z^i, z^j >_t\ \frac{\partial^2}{\partial z^i \partial z^j}.$$

Therefore, the integrands are second order forms Θ on U.
This second order formalism was developed by Meyer and Schwartz. It provides compact and geometrically intrinsic formulas. The stochastic integral of order two of Θ along (z_t) is defined in a unique way as follows (see [3]).

Definition 0.11 *Let (z_t) be a continuous semimartingale in U and (Θ_t) a τ^*U-valued predictable locally bounded process above (z_t). There exists a unique linear mapping $\Theta \to \int \Theta d\!\!I z$, such that*

$\int \Theta d\!\!I z$ is a real semimartingale, called the stochastic integral of Θ along (z_t), satisfying, for every smooth f and for every locally bounded predictable processes K and Θ,

$$\int d^2 f(z)\ d\!\!I z = f(z) - f(z_0),$$

$$\int K\Theta d\!\!I z = \int K\ d(\int \Theta d\!\!I z).$$

In an imbedding, we write

$$\int_0^t d^2 f(z_s) d\!\!I z_s = \int_0^t \frac{\partial f}{\partial z^i}(z_s) dz_s^i + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial z_s^i \partial z_s^j}(z_s) d < z^i, z^j >_s .$$

We will see in part I in which way the Stratonovich and Itô integrals appears as particular cases of integrals of order two.

0.2.3 Stochastic calculus of order two with jumps ($\overset{\triangle}{d\!\!I}$)

In [1], Cohen extends the formalism of calculus of order two to the case where the semimartingale can jump. The integrands θ are now in $\overset{\triangle}{\tau^*} U$, that is, for all z in U, θ_z is a real function on U, twice differentiable at z and such that $\theta_z(z) = 0$.
The stochastic integral of order two of θ along a càdlàg semimartingale (z_t) is defined in a unique and intrinsic way as follows.

Definition 0.12 (Cohen) *Let (z_t) be a càdlàg semimartingale on U and (θ_t) a $\overset{\triangle}{\tau^*} U$-valued predictable locally bounded process above (z_{t-}). There exists a unique linear mapping $\theta \to \int \theta \overset{\triangle}{d\!\!I} z$ such that $\int \theta \overset{\triangle}{d\!\!I} z$ is a real semimartingale, called the stochastic integral of θ along (z_t), which is null at 0 and satisfies, for every f in $C^2(U)$ and for every real locally bounded predictable process K,*

- $\forall t > 0,\ \theta_t(z) = f(z) - f(z_{t-}) \ \ \Rightarrow \ \ \int_0^t \theta_s \overset{\triangle}{d\!\!I} z_s = f(z_t) - f(z_0),$
- $\int K_s \theta_s \overset{\triangle}{d\!\!I} z_s = \int K_s d(\int \theta_u \overset{\triangle}{d\!\!I} z_u)_s,$
- $d^2\theta_t = 0 \ \ \Rightarrow \ \ \int_0^t \theta_s \overset{\triangle}{d\!\!I} z_s = \sum_{0\le s\le t} \theta_s(z_s).$

In an imbedding, we write

$$\int_0^t \theta_s \overset{\triangle}{d\!\!I} z_s = \int_0^t \frac{\partial \theta_s}{\partial z^i}(z_{s-}) dz_s^i + \frac{1}{2}\int_0^t \frac{\partial^2 \theta_s}{\partial z_s^i \partial z_s^j}(z_{s-}) d < z^{i^c}, z^{j^c} >_s$$

$$+\sum_{0\leq s\leq t}(\theta_s(z_s)-\frac{\partial\theta_s}{\partial z^i}(z_{s^-})\triangle z_s^i). \tag{56}$$

Note that we recover the change of variable formula for $\theta_t = f(z) - f(z_{t-})$, with $f \in C^2(V)$.

By the first and second points of the definition, the continuous case defined in the previous paragraph is recovered when the semimartingale is continuous, since we get

$$\int_0^t d^2\theta_s d\!I z_s = \int_0^t \frac{\partial\theta_s}{\partial z^i}(z_s)dz_s^i + \frac{1}{2}\int_0^t \frac{\partial^2\theta_s}{\partial z_s^i\partial z_s^j}(z_s)d < z^i, z^j >_s .$$

The third point describes the case of a finite number of jumps. In particular, the jump of the integral $\int \theta \overset{\triangle}{d\!I}\, z$ when the semimartingale (z_t) jumps from z_t^- to z_t is given by $\theta_t(z_t)$.
The map θ contains all the information : it describes the jumps of the integral and provides the drift and diffusion parts from its first and second derivatives. In part II, our approach will consist in starting from the jump of a semimartingale (z_t) on M and making it covariant. By the foregoing, we will be able to define a stochastic covariant integral of order two if we find the expression of one covariant jump θ of the stochastic integral along (z_t).

We will see in part I in which way the Stratonovich and Itô integrals can be seen as particular cases of integrals of order two.

0.3 Reminders on Stratonovich and Itô continuous covariant calculus

0.3.1 Norris's formalism

In [8], Norris applies the transfer principle to the expression (55) for the covariant derivative of $(Y_t) = ((x_t, y_t)) \in TM$, using the horizontal lift of the part (x_t) of (Y_t) on M. For a semimartingale, it is defined as follows.

Definition 0.13 *The horizontal lift of a continuous semimartingale (x_t), starting from a given $u_o \in GL(\mathbb{R}^m, T_{x_0}M)$, is the unique curve $((x_t, u_t))$ ($\forall t, u_t \in GL(\mathbb{R}^m, T_{x_t}M)$) satisfying the stochastic differential equation*

$$p_{(x_t,u_t)}(\partial x_t, \partial u_t) = 0, \quad u_0 = u_o. \tag{57}$$

(This equation means

$$\forall z \in \mathbb{R}^m, \quad p_{(x_t,u_t(z))}(\partial x_t, \partial u_t(z)) = 0, \quad u_0 = u_o.) \tag{58}$$

The parallel transport $\tau^{//}$ along (x_t) is defined, for all s, t, by the linear map

$$\tau_{st}^{//} = u_t u_s^{-1} : T_{x_s}M \to T_{x_t}M.$$

Then Norris proposes the following definitions.

Definition 0.14 (Norris) *Let p be a connection on M. For every continuous semimartingale $(Y_t) = ((x_t, y_t))$ on TM and every first order form α on M, the Stratonovich covariant integral of α along (Y_t) is defined by Norris as*

$$\int \alpha_{x_t} u_t \partial(u_t^{-1} y_t).$$

The Itô covariant integral of α along (Y_t) is defined by Norris as

$$\int \alpha_{x_t} u_t d(u_t^{-1} y_t).$$

0.3.2 Identification of the vertical space

Norris applies the transfer principle to formula (55), where the covariant derivative $\frac{\partial^{\mathcal{V}} Y_t}{\partial t}$ is seen as an element of the tangent space $T_{x_t}M$, but this covariant derivative belongs to $T_{y_t}(T_{x_t}M)$, according to definition (52). Actually, the equivalence between (52) and (55) contains an identification, specified in the proof of proposition 0.3 (we identified the differential $du_t(y_t)$ with u_t, using the fact that u_t is a linear map). It can also be written as an identification between the vertical space $T_{y_t}(T_{x_t}M)$ and the tangent space $T_{x_t}M$. We explain this identification here, in order to understand the consequences for the stochastic covariant calculus.

Recall that, for all $V = (x, v)$ in TM, the vertical space at V is $T_v(T_xM)$. Since T_xM is a vector space, the tangent space to T_xM at v can be identified with T_xM itself. Indeed, for every v in T_xM, $T_v(T_xM)$ is the set of all the derivatives at 0 of the vertical curves $\{\gamma(t) = v + tu\}_{u \in T_xM}$. By writing $\dot{\gamma}(0) = u$, where $\dot{\gamma}(0) \in T_v(T_xM)$ and $u \in T_xM$, we describe the identification i_V, depending on the point $V = (x, v)$ in TM, between $T_v(T_xM)$ and T_xM :

$$i_V : \dot{\gamma}(0) = u \in T_v(T_xM) \simeq u \in T_xM. \tag{59}$$

Without this identification, we should integrate a form on TM, instead of one on M, along the covariant derivative. Every form α in TM yields at each V in TM, a linear $\alpha_V : T_VTM \to \mathbb{R}$, but notice that only the part of α_V acting on the vertical space is involved for the covariant integration.

Therefore, for a C^1 curve (Y_t) on TM, we have the following.

- If α is a form on M, the covariant derivative $\frac{\partial^{\mathcal{V}} Y_t}{\partial t}$ is identified to an element of $T_{x_t}M$. Then, the covariant integral of α along (Y_t) is given by

$$\int \alpha_{x_t} \frac{\partial^{\mathcal{V}} Y_t}{\partial t} dt. \tag{60}$$

- If α is a form on TM, the covariant derivative $\frac{\partial^{\mathcal{V}} Y_t}{\partial t}$, belonging to $T_{y_t}(T_{x_t}M)$, is seen as an element of $T_{Y_t}TM$ by writing it $(O, \frac{\partial^{\mathcal{V}} Y_t}{\partial t})$ (where $0 \in T_{x_t}M$). Then the covariant integral of α along (Y_t) is given by
$$\int \alpha_{Y_t}(0, \frac{\partial^{\mathcal{V}} Y_t}{\partial t})dt. \tag{61}$$

The Stratonovich integral for a continuous semimartingale (Y_t) in TM is obtained by applying the transfer principle to (60) and (61), as follows.

- If α is a form on M, the covariant Stratonovich integral of α along (Y_t) is the one defined by Norris, given by
$$\int \alpha_{x_t} \partial^{\mathcal{V}} Y_t. \tag{62}$$
- If α is a form on TM, the covariant integral of α along (Y_t) is given by
$$\int \alpha_{Y_t}(0, \partial^{\mathcal{V}} Y_t). \tag{63}$$

Let us understand the difference of nature of these two integrals in the case where (Y_t) belongs to $\mathbb{R}^{2m}$, and p is the flat connection on $\mathbb{R}^m$. According to remark 0.7, we get the following.

The integral (62) yields
$$\int \alpha_{x_t} \partial^{\mathcal{V}} Y_t = \int (\alpha_{x_t})_i \partial y_t^i = \int (\alpha_{x_t})_i dy_t^i + \frac{1}{2} \int \frac{\partial \alpha_i}{\partial x^j}(x_t) d < x^i, y^j >_t,$$
where $\alpha_i = \alpha(\frac{\partial}{\partial x^i})$, $\frac{\partial}{\partial x^i} \in T_x M$.

The integral (63) yields
$$\int \alpha_{Y_t}(0, \partial^{\mathcal{V}} Y_t) = \int (\alpha_{Y_t})_i (0, \partial y_t^i) = \int (\alpha_{Y_t})_i dy_t^i + \frac{1}{2} \int \frac{\partial \alpha_i}{\partial y^j}(Y_t) d < y^i, y^j >_t,$$
where $\alpha_i = \alpha(0, \frac{\partial}{\partial v^i})$, $\frac{\partial}{\partial v^i} \in T_v(T_x M)$.

Note that the main difference between these two integrals comes from the quadratic variation terms.

The integral (63) is the one expected for a covariant integral when the manifold is $\mathbb{R}^m$ with the flat connection, since it corresponds to the Stratonovich integration of the semimartingale (y_t), part of (Y_t) in the fibre. Moreover, we can't write a canonical analoguous identification for the jumps and at the order two. Actually, in these cases, it requires a connection.
For these two reasons, we won't do the identification in the thesis. As a consequence, we will integrate forms on TM.

Part I

Stochastic covariant calculus with jumps

Chapter 1

Stratonovich covariant calculus with jumps

1.1 Reminders on Stratonovich calculus

1.1.1 On Stratonovich continuous calculus (∂)

The Stratonovich integral along a continuous semimartingale (z_t) is usually defined using the transfer principle. For any local chart on U, the Stratonovich integrals along the coordinates (z_t^i), semimartingales in $\mathbb{R}$, are well-defined between two stopping times T_i and T_{i+1} (respectively the first entrance time of (z_t) in the chart and the exiting time of (z_t) from the chart). Therefore, the Stratonovich integral of a form α on U along (z_t) is the sum of these integrals

$$\int \alpha_{z_t}\ \partial z_t = \sum_i \int_{T_i}^{T_{i+1}} (\alpha_{z_t})_i\ \partial z_t^i.$$

Note that it doesn't depend on the stopping times nor on the chosen atlas.

In [3], Emery presents the Stratonovich calculus on a manifold as a particular case of the stochastic calculus of order 2. It has the advantage of not using local charts. A map d_s is introduced, transforming the first order form α into a second order form. Let us recall its definition.

Definition 1.1 *The map d_s is the unique linear map from the set of first order forms on U to the set of second order ones that verifies, for every function f and every first order form α on U,*

$$d_s(df) = d^2 f \text{ and } d_s(f\alpha) = df.\alpha + f d_s\alpha. \tag{1.1}$$

In an imbedding, we write

$$d_s(\alpha) = \alpha_i d^2 z^i + \frac{\partial \alpha_i}{\partial z^j}(z) dz^i.dz^j.$$

See [3] for the proof of the existence and unicity of d_s.

Definition 1.2 (Emery) *For every continuous semimartingale (z_t) on U and every first order form α on U, the Stratonovich integral of α along (z_t) is defined by*

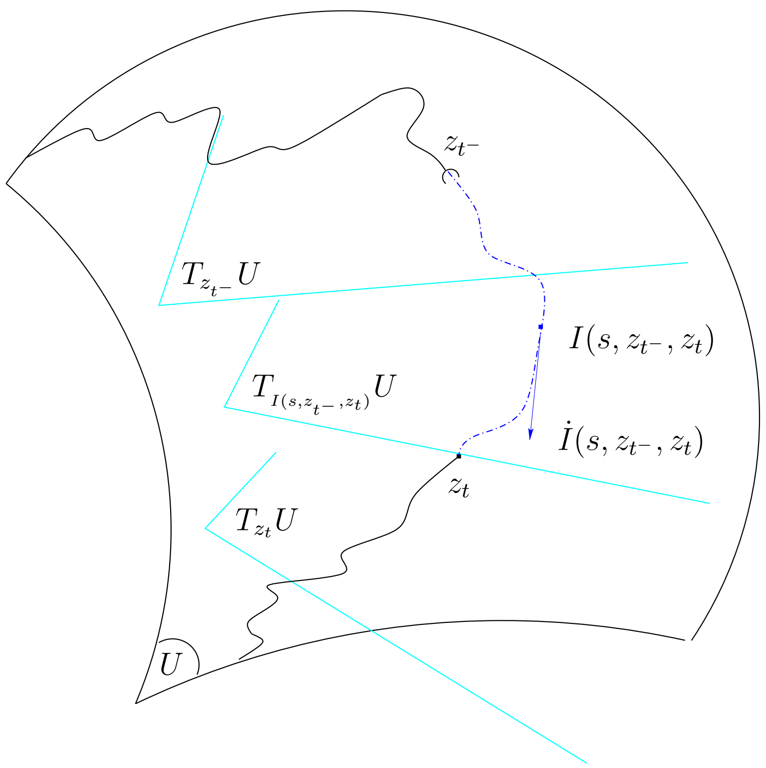
$$\int \alpha_{z_t}\ \partial z_t = \int (d_s\alpha)_{z_t}\ d\!I z_t.$$

In an imbedding, we write

$$\int \alpha_{z_t}\ \partial z_t = \int (\alpha_{z_t})_i dz_t^i + \frac{1}{2}\int \frac{\partial \alpha_i}{\partial z^j}(z_t) d < z^i, z^j >_t .$$

1.1.2 On Stratonovich calculus with jumps ($\overset{\triangle}{\partial}$)

In [1], Cohen defines a Stratonovich integral along a càdlàg semimartingale as a two order integral. The description of the jumps requires an interpolation rule I on U (see definition 0.9). When (z_t) jumps, its path is replaced by an interpolation curve between the points z_{t^-} and z_t on M. Therefore, the jump of the integral is given by the integration of the form α along this interpolation curve.



Definition 1.3 (Cohen) *Let I be an interpolation rule on U. For every càdlàg semimartingale (z_t) on U and every first order form α on U, the Stratonovich integral of α along (z_t) is defined*

by

$$\int \alpha_{z_{t-}} \overset{\triangle}{\partial} z_t = \int \overset{\triangle}{I} (\alpha)_{z_{t-}} \overset{\triangle}{dI} z_t,$$

where the jump of the integral, when (z_t) jumps from z_{t-} to z_t, is

$$\overset{\triangle}{I} (\alpha)_{z_{t-}}(z_t) = \int_0^1 \alpha_{I(s,z_{t-},z_t)} \ \dot{I}(s, z_{t-}, z_t) \ ds. \tag{1.2}$$

Note the link between $\overset{\triangle}{I}$ and d_s.

Proposition 1.1 (Cohen) *For all fixed z in U, the differential and the 2-jet at point $u = z$, of the map*

$$\overset{\triangle}{I} (\alpha)_z : u \in U \to \overset{\triangle}{I} (\alpha)_z(u) = \int_0^1 \alpha_{I(s,z,u)} \ \dot{I}(s, z, u) \ ds \in \mathbb{R},$$

are related with α as follows :

$$d \overset{\triangle}{I} (\alpha)_z(z) = \alpha_z, \qquad d^2 \overset{\triangle}{I} (\alpha)_z(z) = (d_s\alpha)_z.$$

By the following proposition, we can recover the known continuous case from this definition.

Proposition 1.2 (Cohen) *If (z_t) is continuous, $\int \alpha \overset{\triangle}{\partial} z$ is independent of the interpolation rule and it recovers the continuous Stratonovich integral*

$$\int \alpha \ \overset{\triangle}{\partial} \ z = \int \alpha \ \partial \ z,$$

that is

$$\int \alpha \ \overset{\triangle}{\partial} \ z = \int (d_s\alpha)_z \ dI z.$$

Proof : The continuous case is obtained by computing the 2-jet of the function describing the jumps. This 2-jet is given by proposition 1.1. See [1] for more details.□

1.2 Stratonovich covariant calculus

1.2.1 On Stratonovich continuous covariant calculus ($\partial^{\mathcal{V}}$)

Let p be a connection on M. According to the expression (53), the covariant derivative is the projection by p_{Y_t} of the tangent vector $\dot{Y}_t$ to the curve (Y_t). Therefore, we can consider that the covariant case is the image of the classical one by the projection p. With regard to the integration of a form α, it corresponds to the pull-back of α by p.

Definition 1.4 *The pull-back of the first order form α by the map p is the first order form $p^*(\alpha)$ on TM, defined by*

$$\forall V \in TM, \forall U \in T_V TM, \quad p^*(\alpha)_V(U) = \alpha_V(p_V(U)).$$

Remark 1.1 *Since the projection p_V is $T_v(T_xM)$-valued, only the part of $p^*(\alpha)$ acting on this vertical space is involved. The vector $p_V(U) \in T_v(T_xM)$ is seen as an element of T_VTM (we could have written $\alpha_V(O, p_V(U))$ (where $0 \in T_xM$)).*

We use the transfer principle to define a Stratonovich covariant integral as follows.

Definition 1.5 *Let p be a connection on M. For every continuous semimartingale $(Y_t) = ((x_t, y_t))$ on TM and every first order form α on TM, the Stratonovich covariant integral of α along (Y_t) is defined by*

$$\int \alpha_{Y_t} \partial^{\mathcal{V}} Y_t = \int p^*(\alpha)_{Y_t}\ \partial Y_t,$$

that is

$$\int \alpha_{Y_t} \partial^{\mathcal{V}} Y_t = \int d_s(p^*(\alpha))_{Y_t}\ dI Y_t.$$

In an imbedding, we write

$$\int \alpha_{Y_t} \partial^{\mathcal{V}} Y_t = \int\ (\alpha_{Y_t})_i [\partial y_t^i\ +\ \Gamma^i_{jk}(x_t)\ y_t^k\ \partial x_t^j\ ], \tag{1.3}$$

where $(\alpha_{Y_t})_i = \alpha_{Y_t}(\frac{\partial}{\partial y^i})$.

We will see in part III that our definition recovers Norris's definition, when doing the identification of the vertical space.

1.2.2 A Stratonovich covariant calculus with jumps ($\overset{\triangle}{\partial^{\mathcal{V}}}$)

On the one hand, the covariant calculus requires a connection p on M. On the other hand, the Stratonovich calculus with jumps in TM requires an interpolation rule on TM.

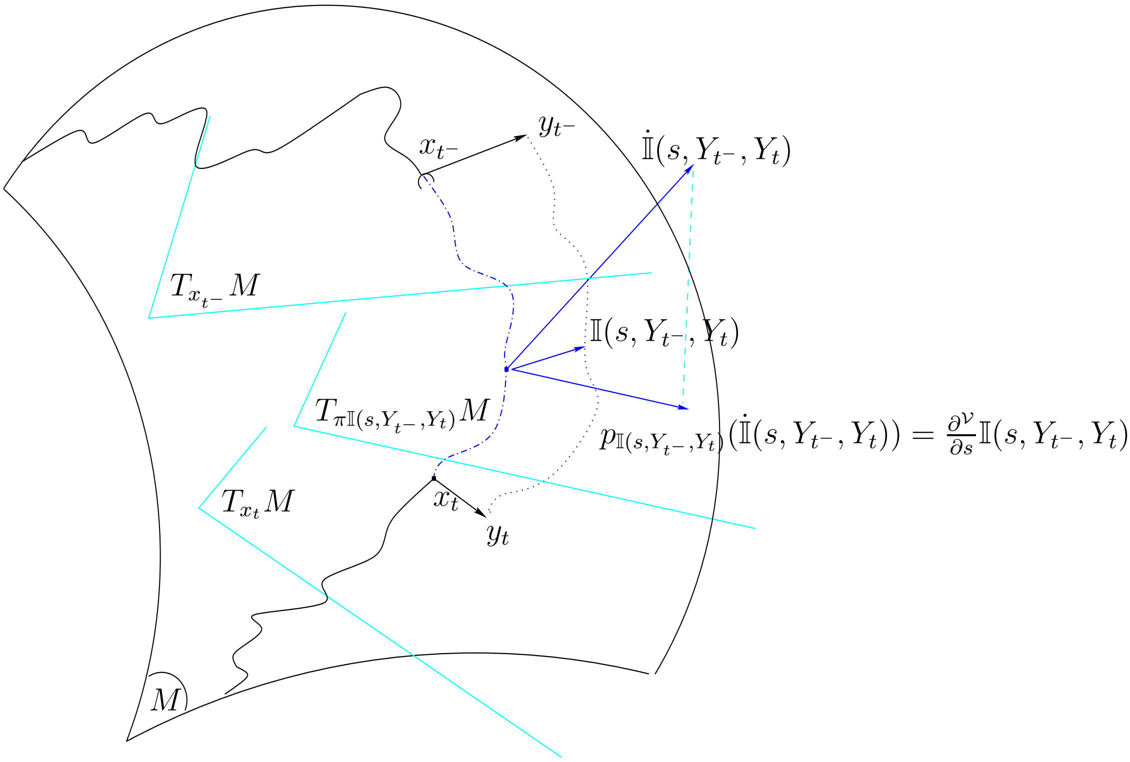
We give meaning to a Stratonovich covariant integral with jumps, by applying the results of paragraph 1.1.2, with the manifold $U = TM$, to the form $p^*\alpha$.

Definition 1.6 *Let p be a connection on M and $\mathbb{I}$ an interpolation rule on TM. For every càdlàg semimartingale $(Y_t) = ((x_t, y_t))$ on TM and every first order form α on TM, the Stratonovich covariant integral of α along (Y_t) is defined by*

$$\int \alpha_{y_{t-}}\ \overset{\triangle}{\partial^{\mathcal{V}}}\ Y_t = \int \overset{\triangle}{\mathbb{I}}\ (p^*\alpha)_{Y_{t-}}\ \overset{\triangle}{dI}\ Y_t,$$

where the jump of the integral, when (Y_t) jumps from Y_{t^-} to Y_t, is

$$\overset{\triangle}{\mathbb{I}}\ (p^*\alpha)_{Y_{t^-}}(Y_t) = \int_0^1 (p^*\alpha)_{\mathbb{I}(s, Y_{t^-}, Y_t)} \dot{\mathbb{I}}(s, Y_{t^-}, Y_t)\ ds. \tag{1.4}$$



Note that, using (53), formula (1.4) can be written

$$\overset{\triangle}{\mathbb{I}}\,(p^*\alpha)_{Y_{t^-}}(Y_t) = \int_0^1 \alpha_{\mathbb{I}(s,Y_{t^-},Y_t)}\,\frac{\partial^{\mathcal{V}}\mathbb{I}}{\partial s}(s, Y_{t^-}, Y_t)\,ds. \tag{1.5}$$

Example 1.1 *Let $\mathbb{I}$ be the geodesic interpolation rule on TM (see example 0.5 with example 0.2). It is given by*

$$\mathbb{I}(s, V, W) = (\ \exp_x(s\exp_x^{-1}y)\ ,\ \tau_{0s}^{//}[s(\tau_{yx}^{//}(w) - v) + v]\ ).$$

Then, we have

$$\overset{\triangle}{\mathbb{I}}\,(p^*\alpha)_{Y_{t^-}}(Y_t) = \int_0^1 \alpha_{\mathbb{I}(s,Y_{t^-},Y_t)}(0, \tau^{//}_{x_{t^-},\pi\mathbb{I}(s,Y_{t^-},Y_t)}(\tau^{//}_{x_t,x_{t^-}}y_t - y_{t^-}))ds \tag{1.6}$$

where $(\tau^{//}_{x_{t^-},\pi\mathbb{I}(s,Y_{t^-},Y_t)}) = (\tau^{//}_{0s})$ is the parallel transport along the geodesic

$$(\pi\mathbb{I}(s, Y_{t^-}, Y_t)) = (\exp_{x_{t^-}}(s\exp^{-1}_{x_{t^-}}(x_t))).$$

Indeed, using expression (55) for the covariant derivative, we get

$$\frac{\partial^{\mathcal{V}}\mathbb{I}}{\partial s}(s, Y_{t^-}, Y_t) = \tau^{//}_{0s}\frac{\partial}{\partial s}(\tau^{//}_{s0}\mathbb{I}(s, y_{t^-}, y_t)),$$

that is
$$\frac{\partial^{\mathcal{V}}\mathbb{I}}{\partial s}(s,Y_{t^-},Y_t)=\tau^{//}_{0s}\frac{\partial}{\partial s}(s(\tau^{//}_{x_t,x_{t^-}}(y_t)-y_{t^-})+y_{t^-})=\tau^{//}_{0s}(\tau^{//}_{x_t,x_{t^-}}(y_t)-y_{t^-}).$$

Then the jump (1.5) is expressed as in (1.6), where $\tau^{//}_{x_{t^-},\pi\mathbb{I}(s,Y_{t^-},Y_t)}(\tau^{//}_{x_t,x_{t^-}}y_t-y_{t^-})$ is seen as an element of $T_{\mathbb{I}(s,y_{t^-},y_t)}(T_{\pi\mathbb{I}(s,Y_{t^-},Y_t)}M)$ (where $\mathbb{I}(s,y_{t^-},y_t)$ is the part of $\mathbb{I}(s,Y_{t^-},Y_t)$ in the fibre), since it is a covariant derivative of $\mathbb{I}(s,Y_{t^-},Y_t)$.

Proposition 1.3 *If (Y_t) is a continuous semimartingale on TM, the covariant Stratonovich integral is independent of the choice of the interpolation rule $\mathbb{I}$ and it recovers the continuous Stratonovich covariant integral*

$$\int \alpha_{Y}\ \overset{\triangle}{\partial^{\mathcal{V}}}\ Y=\int p^*(\alpha)_Y\partial Y.$$

Proof : Just apply proposition 1.2 to the form $p^*(\alpha)$. □

Chapter 2

Itô covariant calculus with jumps

2.1 Reminders on Itô calculus

2.1.1 On Itô continuous calculus (d)

The Itô integral along a continuous semimartingale can be defined from a Stratonovich to Itô conversion formula. This formula requires that U is endowed with a connection, which is assumed to be torsion-free (see the part of paragraph 0.1.2 on the torsion).

In [3], Emery presents the Itô calculus on a manifold as a particular case of the stochastic calculus of order two. The connection induces a map G, which transforms the first order form α into a second order form.

Remark 2.1 *By abuse of language, we will also call G a connection.*

Definition 2.1 *G is the linear mapping from the set of first order forms on U to the set of second order ones defined, for every first order form α on U, by*

$$G(\alpha) = d_s\alpha - H(\nabla\alpha), \tag{2.1}$$

where

- *d_s is given by definition 1.1,*
- *H is the unique linear mapping from bilinear forms to second order ones such that its composition with the restriction to first order forms is null and that, for all forms α and β, we have $H(\alpha \otimes \beta) = \alpha.\beta$. In an imbedding, we write*

$$\nabla\alpha(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^i})(z) = \frac{\partial \alpha_i}{\partial z^j}(z) - (\alpha_z)_k \Gamma^k_{ij}(z),$$

and

$$H(\nabla\alpha)(z) = (\frac{\partial \alpha_i}{\partial z^j}(z) - (\alpha_z)_k \Gamma^k_{ij}(z)) dz^i.dz^j,$$

where (Γ^k_{ij}) are the Christoffel symbols of the connection G.

In an imbedding, we write

$$G(\alpha)_z = (\alpha_z)_i d^2 z^i + (\alpha_z)_k \Gamma^k_{ij}(z) dz^i.dz^j.$$

Definition 2.2 (Emery) *Let G be a connection on U. For every continuous semimartingale (z_t) on U and every first order form α on U, the Itô integral of α along (z_t) is defined by*

$$\int \alpha_{z_t}\ dz_t = \int (G\alpha)_{z_t}\ d\!\!I z_t.$$

Note that formula (2.1) is the Stratonovich to Itô conversion formula. It yields

$$\int G(\alpha)\ d\!\!I z = \int d_s\alpha\ d\!\!I z - \int H(\nabla\alpha)\ d\!\!I z,$$

that is

$$\int \alpha_{z_t} dz_t = \int \alpha_{z_t}\partial z_t - \frac{1}{2}\int \nabla\alpha(\partial z_t, \partial z_t). \tag{2.2}$$

In [8], Norris gives another but equivalent definition for the Itô integral, using the horizontal lift. Let us write it and verify the equivalence with the definition given by Emery.

Proposition 2.1 (Norris) *Let $(\bar{u}_t)$ be the horizontal lift of (z_t), starting from a given $\bar{u}_0$. Define the semimartingale $(\bar{z}_t)$ in $\mathbb{R}^m$ by*

$$\partial\bar{z}_t = \bar{u}_t^{-1}\partial z_t.$$

Then, we have

$$\int \alpha_{z_t} dz_t = \int (\alpha_{z_t}\bar{u}_t) d\bar{z}_t.$$

Proof : In [8], Norris expresses the covariant quadratic variation $\nabla\alpha(\partial z_t, \partial z_t)$, with the horizontal lift $\bar{u}_t$ of (z_t) and the semimartingale $(\bar{z}_t)$ in $\mathbb{R}^m$, as follows

$$\nabla\alpha(\partial z_t, \partial z_t) = d < \alpha_z\bar{u}, \bar{z} >_t . \tag{2.3}$$

Using the Stratonovich to Itô conversion formula in $\mathbb{R}^m$, we write

$$\int \alpha_{z_t}\bar{u}_t d\bar{z}_t = \int \alpha_{z_t}\bar{u}_t\partial\bar{z}_t - \frac{1}{2} d < \alpha_z\bar{u}, \bar{z} >_t .$$

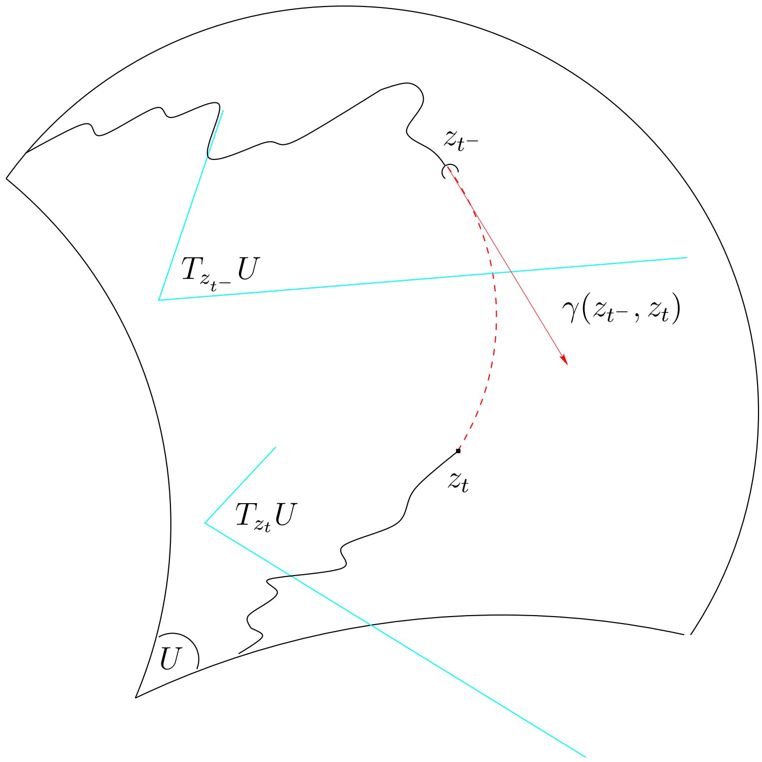
By (2.3) and the definition of $(\bar{z}_t)$ we get

$$\int \alpha_{z_t}\bar{u}_t d\bar{z}_t = \int \alpha_{z_t}\partial z_t - \frac{1}{2}\int \nabla\alpha(\partial z_t, \partial z_t),$$

Then, (2.2) implies that $\int \alpha_{z_t}\bar{u}_t d\bar{z}_t = \int \alpha_{z_t} dz_t$.□

2.1.2 On Itô calculus with jumps ($\overset{\triangle}{\partial}$)

In [1], Cohen defines an Itô integral along a càdlàg semimartingale as a two order integral. The description of the jumps requires a connection rule γ on U (see definition 0.10). The vector $\gamma(z_{t-}, z_t)$ represents the jump of (z_t) in $T_{z_{t-}}U$. Therefore, the jump of the integral is given by the application of the form α to this jump.



Definition 2.3 (Cohen) *Let γ be a connection rule on U. For every càdlàg semimartingale (z_t) on U and every first order form α on U, the Itô integral of α along (z_t) is defined by*

$$\int \alpha_{z_{t-}} \overset{\triangle}{d} z_t = \int \overset{\triangle}{\gamma} (\alpha)_{z_{t-}} \overset{\triangle}{dI} z_t,$$

where the jump of the integral, when (z_t) jumps from z_{t-} to z_t, is

$$\overset{\triangle}{\gamma} (\alpha)_{z_{t-}}(z_t) = \alpha_{z_{t-}} \circ \gamma(z_{t-}, z_t). \tag{2.4}$$

Note the link between $\overset{\triangle}{\gamma}$ and G.

Proposition 2.2 (Cohen) *For all fixed z in U, the differential and the 2-jet at point $u = z$, of the map*

$$\overset{\triangle}{\gamma} (\alpha)_z : u \in U \rightarrow \overset{\triangle}{\gamma} (\alpha)_z(u) = \alpha_z \circ \gamma(z, u) \in \mathbb{R},$$

are related with α as follows :

$$d \overset{\triangle}{\gamma} (\alpha)_z(z) = \alpha_{_z} \ , \qquad d^2 \overset{\triangle}{\gamma} (\alpha)_z(z) = G(\alpha)_{_z}.$$

By the following proposition, we can recover the known continuous case from this definition.

Proposition 2.3 (Cohen) *If (z_t) is continuous, $\int \alpha \overset{\triangle}{d} z$ only depends on the 2-jet of the connection rule γ, giving a connection G on U, defined for every first order form α on U, by*

$$\forall z \in U,\ G(\alpha)_z = \alpha_{_z} d^2\gamma(z,.)(z),$$

and it recovers the continuous Itô integral

$$\int \alpha \overset{\triangle}{d} z = \int \alpha dz,$$

that is

$$\int \alpha \overset{\triangle}{d} z = \int G(\alpha)_{_z} \ d\!\!I z.$$

Proof : The continuous case is obtained by computing the 2-jet of the function describing the jumps. This 2-jet is given by proposition 2.2. Moreover, the expression for the connection G comes proposition $B.1$ in Appendix B, with $f = \alpha$. Since α_z is linear, its 2-jet is also α_z, and we get

$$G(\alpha)_z = d^2(\overset{\triangle}{\gamma} (\alpha)_z)(z) = d^2(\alpha_z \circ \gamma(z,.))(z) = d^2(\gamma(z,.))(z)^*(\alpha_z) = \alpha_z \circ d^2(\gamma(z,.))(z)$$

See [1] for more details.□

2.2 Itô covariant calculus

2.2.1 On Itô continuous covariant calculus ($d^{\mathcal{V}}$)

A connection p is given on M to describe the covariant calculus. This covariant calculus concerns semimartingales in TM. Therefore, a connection $\mathbb{G}$ on TM is required to do Itô calculus, with *a priori* no link with the connection p on M.

We define an Itô covariant integral as follows.

Definition 2.4 *Let p be a connection on M and $\mathbb{G}$ a connection on TM. For every continuous semimartingale $(Y_t) = ((x_t, y_t))$ on TM and every first order form α on TM, the Itô covariant integral of α along (Y_t) is defined by*

$$\int \alpha_{_{Y_t}} d^{\mathcal{V}} Y_t = \int p^*(\alpha)_{_{Y_t}} \ dY_t,$$

that is

$$\int \alpha_{_{Y_t}} d^{\mathcal{V}} Y_t = \int \mathbb{G}(p^*(\alpha))_{_{Y_t}} \ d\!\!I Y_t.$$

We will see in part III that our definition recovers Norris's definition when doing the identification of the vertical space, and when considering a flat connection for $\mathbb{G}$ on TM.

2.2.2 An Itô covariant calculus with jumps ($\overset{\triangle}{d^{\mathcal{V}}}$)

On the one hand, the covariant calculus requires a connection p on M. On the other hand, the Itô calculus with jumps requires a connection rule on TM (yielding a connection on TM, which has *a priori* no link with p on M).

We give meaning to a covariant Itô integral with jumps by applying the results of paragraph 2.1.2 with the manifold $U = TM$, to the form $p^*\alpha$. Hence, we get the following definition.

Definition 2.5 *Let p a connection on M and $\boldsymbol{\gamma}$ a connection rule on TM. For every càdlàg semimartingale $(Y_t) = ((x_t, y_t))$ on TM and every first order form α on TM, the covariant Itô integral of α along (Y_t) is defined by*

$$\int \alpha_{Y_{t-}} \overset{\triangle}{d^{\mathcal{V}}} Y_t = \int \overset{\triangle}{\boldsymbol{\gamma}} (p^*\alpha)_{Y_{t-}} \overset{\triangle}{dI} Y_t,$$

where the jump of the integral, when (Y_t) jumps from Y_{t^-} to Y_t, is

$$\overset{\triangle}{\boldsymbol{\gamma}}(p^*\alpha)_{Y_{t-}}(Y_t) = (p^*\alpha)_{Y_{t-}} \circ \boldsymbol{\gamma}(Y_{t^-}, Y_t) = \alpha_{y_{t-}} \circ p_{Y_{t^-}}(\boldsymbol{\gamma}(Y_{t^-}, Y_t)). \tag{2.5}$$

Example 2.1 *Let $\boldsymbol{\gamma}$ be the geodesic connection rule on TM (see examples* 0.8 *and* 0.2*). It is given by*

$$\boldsymbol{\gamma}(V, W) = \dot{\mathbb{I}}(0, V, W),$$

where $\mathbb{I}(s, V, W)$ is the geodesic interpolation rule on TM. Then, we have

$$\overset{\triangle}{\boldsymbol{\gamma}}(p^*\alpha)_{Y_{t-}}(Y_t) = \alpha_{Y_{t-}}(0, \tau^{//}_{x_t, x_{t-}}(y_t) - y_{t^-}) \tag{2.6}$$

where $(\tau^{//}_{x_{t-}, \pi\mathbb{I}(s, Y_{t-}, Y_t)}) = (\tau^{//}_{0s})$ is the parallel transport along the geodesic

$$(\pi\mathbb{I}(s, Y_{t^-}, Y_t)) = (\exp_{x_{t-}}(s \exp^{-1}_{x_{t-}}(x_t)))$$

Indeed,

$$p_{Y_{t^-}}(\boldsymbol{\gamma}(Y_{t^-}, Y_t)) = p_{\mathbb{I}(0, Y_{t-}, Y_t)}(\dot{\mathbb{I}}(0, Y_{t^-}, Y_t)) = \frac{\partial^{\mathcal{V}}\mathbb{I}}{\partial s}(s, Y_{t^-}, Y_t))|_{s=0}.$$

The covariant derivative has been computed in example 1.1 as

$$\frac{\partial^{\mathcal{V}}\mathbb{I}}{\partial s}(s, Y_{t^-}, Y_t) = \tau^{//}_{0s}(\tau^{//}_{x_t, x_{t-}}(y_t) - y_{t^-}),$$

At time $s = 0$, we get

$$p_{Y_{t-}}(\boldsymbol{\gamma}(Y_{t^-}, Y_t)) = \tau^{//}_{x_t, x_{t-}}(y_t) - y_{t^-}.$$

Then the jump (2.5) is expressed as in (2.6), where $\tau^{//}_{x_t, x_{t-}} y_t - y_{t^-}$ is seen as an element of $T_{y_{t-}}(T_{x_{t-}} M)$, since it is a projection by $p_{Y_{t-}}$.

Proposition 2.4 *If (Y_t) is a continuous semimartingale on TM, the covariant Itô integral only depends on the 2-jet of the connection rule $\boldsymbol{\gamma}$, giving a connection $\mathbb{G}$ on TM, defined for every first order form α on TM, by*

$$\forall Y \in TM, \ \mathbb{G}(\alpha)_Y = \alpha_Y d^2 \overset{\triangle}{\boldsymbol{\gamma}} (Y, .)(Y),$$

and it recovers the continuous Itô covariant integral

$$\int \alpha_{Y_{t-}} \overset{\triangle}{d^{\mathcal{V}}} Y_t = \int \alpha_{Y_{t-}} d^{\mathcal{V}} Y_t.$$

Proof : Just apply proposition 2.3 to the form $p^*(\alpha)$.□

Chapter 3

Stochastic covariant calculus of order two

In this first part, we have described separately the case of the Stratonovich covariant integral and the one of the Itô covariant integral. In the case with jumps described by Cohen as for as in the continuous case described by Emery, the Stratonovich and the Itô integrals are particular cases of an integral of order two. We would like to include in the same way the covariant integrals in a covariant stochastic calculus of order two. In the continuous case, this formalism would give a continuous stochastic covariant calculus of order two, which has never, to our knowing, yet been defined. We propose one definition in this chapter, and explain how we recover as particular case the continuous covariant integrals of Stratonovich and Itô. For this, we need to specify a notion of connection of order two on M.

3.1 Connection of order two on M

We introduce the notion of connection of order two on M, as a way of decomposing at each V in TM the tangent bundle $\tau_V TM$ of order two in a second order vertical space and an horizontal one. In addition, we require that the restriction of a connection of order two on M to first order vectors is a connection of order one on M.

For all $V = (x, v)$ in TM, the fibre through V is T_xM. The tangent space of order two, $\tau_V TM$, to TM at V, contains a canonical subset $\tau_v(T_xM)$ (rigorously $d^2 i(V)(\tau_V(x, T_xM))$ with the inclusion $i : (x, T_xM) \hookrightarrow TM$). This space is called the vertical space of order two at V, and denoted by $\tilde{\mathcal{V}}_V TM$. A connection of order two on M is a way of choosing for each V a supplementary space $\tilde{\mathcal{H}}_V TM$ of this vertical space.

Definition 3.1 *A connection of order two on M is a smooth map $\bar{p} : V \in TM \to \bar{p}_V$ such that for all $V = (x, v)$ in TM,*

1. *$\bar{p}_V$ projects $\tau_V TM$ onto the vertical space of order two $\tilde{\mathcal{V}}_V TM = \tau_v(T_xM)$,*

2. $\forall \lambda \in \mathbb{R}$, $\bar{p}_{\bar{\lambda}V} \circ d^2\bar{\lambda}(V) = d^2\bar{\lambda}(V) \circ \bar{p}_V$ *where* $\bar{\lambda}(x,v) = (x, \lambda.v)$,
3. *the restriction of* $\bar{p}$ *to first order vectors is a connection* p *of order one on* M.

Expressions in coordinates

We give an analogue of proposition 0.1 for the existence of Christofell symbols. Actually, most of the connections of order two considered in this thesis are 2-jets of transports, for which the components can be expressed from Christoffel symbols, except for the two following maps, denoted with the specific symbols $\mathrm{I\!\Gamma}$ and C.

Proposition 3.1 *A smooth map* $\bar{p} : V \in TM \to \bar{p}_V$ *is a connection of order two on* M *if and only if, for all* $V = (x,v)$ *in* TM,

1. $\bar{p}_V$ *is a linear map such that, for every* $\mathbb{U}$ *in* $\tau_v T_x M$, *written* $\mathbb{U} = d^2v^i(\mathbb{U})\frac{\partial}{\partial v^i} + dv^i.dv^j(\mathbb{U})\frac{\partial^2}{\partial v^i \partial v^j}$,

$$\bar{p}_V(\mathbb{U}) = d^2v^i(\mathbb{U})\frac{\partial}{\partial v^i} + dv^i.dv^j(\mathbb{U})\frac{\partial^2}{\partial v^i \partial v^j}. \tag{3.1}$$

2. (a) *The map* $v \to d^2v^l(\bar{p}_V(\frac{\partial^2}{\partial x^i \partial v^j}))$ *is a constant. In particular, there exist smooth functions* (C^i_{jk}) *such that, for all* $V = (x,v)$ *in* TM,

$$d^2v^l(\bar{p}_V(\frac{\partial^2}{\partial x^i \partial v^j})) = C^l_{ij}(x). \tag{3.2}$$

   (b) *The maps* $v \to d^2v^l(\bar{p}_V(\frac{\partial^2}{\partial x^i \partial x^j}))$ *and* $v \to dv^l dv^k(\bar{p}_V(\frac{\partial^2}{\partial x^i \partial v^j}))$ *are linear. In particular there exist smooth functions* $(\mathrm{I\!\Gamma}^l_{ijk})$ *such that, for all* $V = (x,v)$ *in* TM,

$$d^2v^l(\bar{p}_V(\frac{\partial^2}{\partial x^i \partial x^j})) = \mathrm{I\!\Gamma}^l_{ijk}(x)v^k. \tag{3.3}$$

   Note the symmetry $\mathrm{I\!\Gamma}^l_{ijk}(x) = \mathrm{I\!\Gamma}^l_{jik}(x)$.

   (c) *The map* $v \to dv^l dv^k(\bar{p}_V(\frac{\partial^2}{\partial x^i \partial x^j}))$ *is quadratic.*

3. (a) *We have*

$$dv^l.dv^k(\bar{p}_V(\frac{\partial}{\partial x^i})) = 0$$

   (b) *The map* $v \to d^2v^l(\bar{p}_V(\frac{\partial}{\partial x^i}))$ *is linear. In particular there exist smooth functions* (Γ^l_{ik}) *such that, for all* $V = (x,v)$ *in* TM,

$$d^2v^l(\bar{p}_V(\frac{\partial}{\partial x^i})) = \Gamma^l_{ik}(x)v^k. \tag{3.4}$$

   These functions are the Christofell symbols of the restriction p *of* $\bar{p}$ *to first order vectors.*

Proof : See Appendix D

Remark 3.1 *If (z^i) and $(\bar{z}^\alpha)$ are local charts around z in a manifold U, the change of coordinates formula, for a second order vector $\mathbb{L}$ in $\tau_z U$ is (see [3])*

$$d^2\bar{z}^\alpha(\mathbb{L}) = d^2 z^i(\mathbb{L})\frac{\partial \bar{z}^\alpha}{\partial z^i} + dz^i.dz^j(\mathbb{L})\frac{\partial^2 \bar{z}^\alpha}{\partial z^i \partial z^j},$$

$$d\bar{z}^\alpha.d\bar{z}^\beta(\mathbb{L}) = dz^i.dz^j(\mathbb{L})\frac{\partial \bar{z}^\alpha}{\partial z^i}\frac{\partial \bar{z}^\beta}{\partial z^j},$$

so that, in general, second order vectors do not have an intrinsic first order part.

If $\bar{p}$ is a connection of order two, for every $\mathbb{L}$ in $\tau_V TM$, $\bar{p}(\mathbb{L})$ is a second order vector in $\tau_v(T_x M)$. Let us use coordinates on τM described in Appendix A, that is deduced from coordinates on M. Let (x^i) and $(\bar{x}^\alpha)$ be local charts around x in M. Let (v^i) and $(\bar{v}^\alpha)$ be local charts around v in $T_x M$, given by

$$v^i = dx^i(v), \ \bar{v}^\alpha = d\bar{x}^\alpha(v).$$

Then they are related by

$$\bar{v}^\alpha = \frac{\partial x^i}{\partial \bar{x}^\alpha} v^i,$$

and so

$$\frac{\partial^2 \bar{v}^\alpha}{\partial v^i \partial v^j} = 0.$$

Therefore, the change of coordinates formula, for the second order vector $\bar{p}(\mathbb{L})$ in $\tau_v(T_x M)$, becomes

$$d^2\bar{v}^\alpha(\mathbb{L}) = d^2 v^i(\mathbb{L})\frac{\partial \bar{v}^\alpha}{\partial v^i},$$

$$d\bar{v}^\alpha.d\bar{v}^\beta(\mathbb{L}) = dv^i.dv^j(\mathbb{L})\frac{\partial \bar{v}^\alpha}{\partial v^i}\frac{\partial \bar{v}^\beta}{\partial v^j}.$$

We conclude that, if the coordinates used on τM are deduced from coordinates on M, the second order vector $\bar{p}(\mathbb{L})$ has a first order part and a second order part, which doesn't depend on these coordinates on M.

3.2 Continuous covariant calculus of order two ($dI^{\mathcal{V}}$)

Let $\bar{p}$ be a connection of order two on M. Doing an analogy with the order one, we define the covariant calculus of order two as the image of the (non covariant) calculus of order two by the projection $\bar{p}$. With regard to the integration of a second order form Θ, it corresponds to the pull-back of Θ by $\bar{p}$.

Definition 3.2 *If Θ is a second order form on TM, the pull-back of Θ by the map $\bar{p}$ is the second order form $\bar{p}^*(\Theta)$ on TM defined by*

$$\forall V \in TM, \forall \mathbb{L} \in \tau M : \quad \bar{p}^*(\Theta)_V(\mathbb{L}) = \Theta_V \circ \bar{p}_V(\mathbb{L}).$$

Definition 3.3 *Let $\bar{p}$ be a connection of order two on M. For every continuous semimartingale (Y_t) in TM and every second order form Θ on TM, the integral*

$$\int \Theta dI^{\mathcal{V}} Y = \int \bar{p}^*(\Theta) dIY$$

is called the stochastic continuous covariant integral of Θ along (Y_t).

There are several ways to transform a connection of order one p into a connection of order 2 on M. The following example, $d_s p$, of a connection of order two, shows how to write the Stratonovich covariant integral as a covariant integral of order two. The Itô covariant integral can also be written as a covariant integral of order two, by using a connection $\tilde{p}$ of order two that will be studied in part II.

3.3 The connection of order two $d_s p$

Let p be a connection of order one on M. For all $i \in \{1, ..., m\}$, consider the map

$$p^i : V \to p^i_V = dv^i + \Gamma^i_{jk}(x) v^k dx^j : T_V TM \to \mathbb{R}$$

This is a first order form on TM, which can be transformed into a second order form on TM by the operator d_s of definition 1.1. In a way, the following definition extends this application d_s.

Definition 3.4 *Let p be a connection of order one on M. The map $d_s p$ is a map yielding for all V in TM a smooth map $d_s p_V : \tau_V TM \to \tau_v(T_x M)$, defined by*

$$\forall V \in TM, \forall \mathbb{L} \in \tau_V TM, \quad d_s p_V(\mathbb{L}) = (d_s p^i)_V(\mathbb{L}) \frac{\partial}{\partial v^i} + p^i_V . dv^j(\mathbb{L}) \frac{\partial^2}{\partial v^i \partial v^j}. \tag{3.5}$$

Remark 3.2 *For this definition, we use the coordinates $(V^i) = ((x^i, v^i))$ around $V = (x, v)$ on TM described in appendix A, that is coordinates deduced from coordinates (x^i) around x on M :*

$$V^i = (x^i, v^i) = (x^i, dx^i(v)).$$

Proposition 3.2 *We can write $d_s p$ as follows. For all V in TM, for all $\mathbb{L}$ in $\tau_V TM$,*

$$d_s p_V(\mathbb{L}) = [d^2 v^i(\mathbb{L}) + \Gamma^i_{ml}(x) v^l \, d^2 x^m \, (\mathbb{L}) + \frac{\partial \Gamma^i_{ml}}{\partial x^k}(x) v^l \, dx^m . dx^k(\mathbb{L}) + \Gamma^i_{mk}(x) \, dx^m . dv^k(\mathbb{L}) \,] \frac{\partial}{\partial v^i}$$

$$+ [\, dv^i . dv^j(\mathbb{L}) + \Gamma^i_{kl}(x) v^l \, dx^k . dv^j(\mathbb{L}) \,] \frac{\partial^2}{\partial v^i \partial v^j}. \tag{3.6}$$

Proof : By the definition of d_s, we have

$$(d_s p^i)_V = (p^i_V)_k d^2 V^k + \frac{\partial (p^i_V)_k}{\partial V^m} dV^k . dV^m.$$

Then we get

$$(d_s p^i)_V = \delta^i_k d^2 v^k + \Gamma^i_{kl}(x) v^l\ d^2 x^k + \frac{\partial \delta^i_k}{\partial v^m} dv^k.dv^m + \frac{\partial \delta^i_k}{\partial x^m} dv^k.dx^m + \frac{\partial(\Gamma^i_{kl}(x)v^l)}{\partial v^m} dx^k.dv^m + \frac{\partial(\Gamma^i_{kl}(x)v^l)}{\partial x^m} dx^k.dx^m.$$

It follows

$$(d_s p^i)_V = d^2 v^i + \Gamma^i_{kl}(x) v^l\ d^2 x^k + \Gamma^i_{km}(x) dx^k.dv^m + \frac{\partial \Gamma^i_{kl}(x)}{\partial x^m} v^l dx^k.dx^m.$$

Moreover, we get, for the second term,

$$p^i_V.dv^j = (dv^i + \Gamma^i_{kl}(x) v^l\ dx^k).dv^j = dv^i.dv^j + \Gamma^i_{kl}(x) v^l\ dx^k.dv^j.\square$$

The following proposition gives the properties that we need to write the continuous Stratonovich covariant integral as a covariant integral of order two.

Proposition 3.3 *$d_s p$ doesn't depend on the choice of the coordinates on M. It is a connection of order two, such that its restriction to first order vectors is p. Moreover, it satisfies, for all first order form α on TM,*

$$d_s(p^*(\alpha)) = (d_s p)^*(d_s \alpha).$$

Proof :

- First, we verify that $d_s p$ doesn't depend on the choice of the coordinates on M. Consider other coordinates $\bar{x}^\alpha$ on M and set $\bar{v}^\alpha = d\bar{x}^\alpha(v)$. The expression for $d_s p$ with new coordinates $(\bar{v}^\alpha)$ is

$$d_s p_V = d_s(d\bar{v}^\alpha(p_V)) \frac{\partial}{\partial \bar{v}^\alpha} + d\bar{v}^\alpha(p_V).d\bar{v}^\beta \frac{\partial^2}{\partial \bar{v}^\alpha \partial \bar{v}^\beta}.$$

  Using the change of coordinates formulas

$$\frac{\partial}{\partial \bar{v}^\alpha} = \frac{\partial v^i}{\partial \bar{v}^\alpha} \frac{\partial}{\partial v^i}\ ,\ \frac{\partial^2}{\partial v^i \partial v^j} = \frac{\partial \bar{v}^\alpha}{\partial v^i} \frac{\partial \bar{v}^\beta}{\partial v^j} \frac{\partial^2}{\partial \bar{v}^\alpha \partial \bar{v}^\beta}\ ,\ d\bar{v}^\alpha = \frac{\partial \bar{v}^\alpha}{\partial v^i} dv^i,$$

  we get

$$d_s p_V = d_s(\frac{\partial \bar{v}^\alpha}{\partial v^i} dv^i(p_V)) \frac{\partial}{\partial \bar{v}^\alpha} + \frac{\partial \bar{v}^\alpha}{\partial v^i} dv^i(p_V).\frac{\partial \bar{v}^\beta}{\partial v^j} dv^j \frac{\partial^2}{\partial \bar{v}^\alpha \partial \bar{v}^\beta}.$$

  By property (1.1) for the map d_s, we have

$$d_s(\frac{\partial \bar{v}^\alpha}{\partial v^i} dv^i(p_V)) = \frac{\partial^2 \bar{v}^\alpha}{\partial v^i \partial v^j} dv^j.dv^i(p_V) + \frac{\partial \bar{v}^\alpha}{\partial v^i}(d_s p^i)_V.$$

  Just as in remark 3.1, there exists a linear relation between the coordinates (v^i) and $(\bar{v}^\alpha)$, so that

$$\frac{\partial^2 \bar{v}^\alpha}{\partial v^i \partial v^j} = 0.$$

  It follows

$$d_s p_V = (d_s p^i)_V \frac{\partial \bar{v}^\alpha}{\partial v^i} \frac{\partial}{\partial \bar{v}^\alpha} + p^i_V.dv^j \frac{\partial \bar{v}^\alpha}{\partial v^i} \frac{\partial \bar{v}^\beta}{\partial v^j} \frac{\partial^2}{\partial \bar{v}^\alpha \partial \bar{v}^\beta},$$

that is

$$d_sp = d_s(p^i_V)\frac{\partial}{\partial v^i} + p^i_V.dv^j\frac{\partial^2}{\partial v^i \partial v^j}.$$

So d_sp doesn't depend on the choice of coordinates on M.

- Let us prove that d_sp is a connection of order two, using proposition 3.1 and the expression (3.6) for $d_sp(\mathbb{L})$. First note that the map $V \to d_sp_V$ is smooth. Moreover, for all V in TM, (3.6) implies the following.
  - d_sp_V is linear, and, for every $\mathbb{U} = d^2v^i(\mathbb{U})\frac{\partial}{\partial v^i} + dv^i.dv^j(\mathbb{U})\frac{\partial^2}{\partial v^i\partial v^j}$ in τ_vT_xM,

    $$d_sp_V(\mathbb{U}) = d^2v^i(\mathbb{U})d_sp_V(\frac{\partial}{\partial v^i}) + dv^i.dv^j(\mathbb{U})d_sp_V(\frac{\partial^2}{\partial v^i\partial v^j})$$

    But $d_sp_V(\frac{\partial}{\partial v^i}) = \frac{\partial}{\partial v^i}$ and $d_sp_V(\frac{\partial^2}{\partial v^i\partial v^j}) = \frac{\partial^2}{\partial v^i\partial v^j}$, so we get $d_sp_V(\mathbb{U}) = \mathbb{U}$, and the first point of proposition 3.1 is verified.
  - The map $v \to d^2v^i(d_sp_V)(\frac{\partial^2}{\partial x^m\partial v^k})$ corresponds to $v \to \Gamma^i_{mk}(x)$, so it is constant, and 2(a) of proposition 3.1 is verified.
  - The map $v \to d^2v^i(d_sp_V)(\frac{\partial^2}{\partial x^m\partial x^k})$ corresponds to $v \to \frac{\partial\Gamma^i_{ml}}{\partial x^k}(x)v^l$ and the map $v \to dv^idv^j(d_sp)_V(\frac{\partial^2}{\partial x^k\partial v^m}))$ to $v \to \Gamma^i_{kl}(x)v^l\delta^j_m$. So they are linear maps, and 2(b) of proposition 3.1 is verified.
  - The map $v \to dv^idv^j(d_sp)_V(\frac{\partial^2}{\partial x^k\partial x^m})$ is quadratic since it is null, then 2(c) of proposition 3.1 is verified.
  - We have $dv^i.dv^j(d_sp)_V(\frac{\partial}{\partial x^m}) = 0$, so 3(a) of proposition 3.1 is verified.
  - The map $v \to d^2v^i(d_sp)_V(\frac{\partial}{\partial x^m})$ corresponds to $\Gamma^i_{mk}(x)v^k$, so it is linear, and 3(b) of proposition 3.1 is verified. Moreover, since we recover the Christofell symbols of the connection p, the restriction of d_sp to first order vectors is the connection p.
- Let us show that for all first order form α on TM, we have

  $$d_s(p^*(\alpha)) = (d_sp)^*(d_s\alpha).$$

  On the one hand, we have, for every $\mathbb{L}$ in τ_VTM,

  $$d_s(p^*(\alpha))_V(\mathbb{L}) = d_s(\alpha(0,\frac{\partial}{\partial v^i})p^i)_V(\mathbb{L}).$$

  Recall that p^i is a first order form, so property (1.1) for the map d_s implies

  $$d_s(p^*(\alpha))(\mathbb{L}) = d(\alpha(0,\frac{\partial}{\partial v^i}))(V).p^i_V(\mathbb{L}) + \alpha(0,\frac{\partial}{\partial v^i})_V(d_sp^i)_V(\mathbb{L})$$

  Then

  $$d_s(p^*(\alpha))(\mathbb{L}) = \frac{\partial\alpha(0,\frac{\partial}{\partial v^i})}{\partial v^k}(V)dv^k.p^i_V(\mathbb{L}) + \alpha(0,\frac{\partial}{\partial v^i})_V(d_sp^i)_V(\mathbb{L}). \tag{3.7}$$

On the other hand, we have

$$(d_s p)^*(d_s\alpha)_V(\mathbb{L}) = (d_s\alpha)_V(d_s p_V(\mathbb{L})) = \alpha(\frac{\partial}{\partial V^i})_V dV^i(d_s p_V(\mathbb{L})) + \frac{\partial\alpha(\frac{\partial}{\partial V^i})}{\partial V^k}(V)dV^k.dV^i(d_s p_V(\mathbb{L}))$$

According to (3.6), we have

$$dV^i(d_s p_V)(\mathbb{L}) = dv^i(d_s p_V)(\mathbb{L}) = (d_s p^i)_V(\mathbb{L})$$

and

$$dV^k.dV^i(p_V)(\mathbb{L}) = dv^k.dv^i(p_V)(\mathbb{L}) = dv^k.p^i_V(\mathbb{L}).$$

Then we get

$$(d_s p)^*(d_s\alpha)_V(\mathbb{L}) = \alpha(0, \frac{\partial}{\partial v^i})_V dv^i(d_s p)_V(\mathbb{L}) + \frac{\partial\alpha(0, \frac{\partial}{\partial v^i})}{\partial v^k}(V)dv^k.dv^i(d_s p)_V(\mathbb{L}),$$

that is

$$(d_s p)^*(d_s\alpha)_V(\mathbb{L}) = \alpha(0, \frac{\partial}{\partial v^i})_V(d_s p^i)_V(\mathbb{L}) + \frac{\partial\alpha(0, \frac{\partial}{\partial v^i})}{\partial v^k}(V)dv^k.p^i_V(\mathbb{L}). \tag{3.8}$$

Comparing (3.7) and (3.8), we get the result.□

Theorem 1 *For every connection p of order one on M, there exists a connection of order two, dsp, such that*

- *the restriction of $d_s p$ to first order vectors is p,*
- *the Stratonovich covariant integral along a continuous semimartingale can be expressed as a two order covariant integral*

$$\int \alpha\partial^{\mathcal{V}} Y = \int (d_s p)^*(d_s\alpha)dIY.$$

In particular, we get an analogous formula than in definition 1.2

$$\int \alpha\partial^{\mathcal{V}} Y = \int d_s\alpha dI^{\mathcal{V}} Y,$$

where $dI^{\mathcal{V}}$ is defined with the connection $d_s p$.

Proof : Recall that, by definition 1.5, we have

$$\int \alpha\partial^{\mathcal{V}} Y = \int p^*(\alpha)\partial Y = \int d_s(p^*\alpha)dIY.$$

By proposition 3.3, $d_s p$ is a connection of order two with restriction p to first order vectors, and we have

$$\int d_s(p^*\alpha)dIY = \int (d_s p)^*(d_s\alpha)dIY.$$

Moreover, by definition 3.3, we can write

$$\int (d_s p)^*(d_s\alpha)dIY = \int d_s\alpha dI^{\mathcal{V}} Y.\square$$

An analogue of this theorem will be given in part II for the Itô covariant integral.

Part II

Stochastic calculus with covariant jumps

Chapter 4

Transport on M

We propose a formalism, called stochastic calculus with covariant jumps, which corresponds to the stochastic covariant calculus of order two described in the previous part when we integrate a continuous semimartingale. Therefore, we want to give an expression for a covariant jump. This requires a notion of transport. A well-known map bringing a vector from the tangent space T_xM to M at x to the tangent space T_yM to M at y is the parallel transport along a curve joining x to y. It has many properties, in particular it is a linear invertible map. Picard defined a more general linear transport in [9], and proposed an example constructed from a connection rule on M. We still extend the notion of transport as a way of mapping at each x in M the tangent bundle TM onto the tangent space T_xM. Moreover, a transport is allowed to depend on a tangent vector v based on x. In order to recover the stochastic covariant calculus of order two in the continuous case, the 2-jet of a transport has to yield a connection of order two on M (and so its differential yields a connection of order one on M). Thus, the definition of transport requires properties on its 2-jet. When the transport itself verifies similar properties, we get a stronger notion of transport, which is the one of Picard. Such a transport can be seen as an analogue to a linear connection for the jumps.

4.0.1 Definition

Definition 4.1 *A map τ is called a transport on M if it yields, for every $V = (x, v)$ in TM, a map $\tau_V : TM \to T_xM$ such that*

1. *$(V, W) \to \tau_V(W)$ is a C^2 map on a neighbourhood of the diagonal of $TM \times TM$,*
2. *the 2-jet $d^2\tau_V(V)$ of τ_V at V is a connection of order two on M.*

Recall that the second property means that for all $V = (x, v)$ in TM, τ_V satisfies

$(i)_2$ $d^2\tau_V(V) : \tau_V TM \to \tau_v(T_xM)$ is a projection onto the vertical space of order two $\tau_v(T_xM)$,

$(ii)_2$ $\forall \lambda \in \mathbb{R}$, $\; d^2\tau_{\bar{\lambda}V}(\bar{\lambda}V) \circ d^2\bar{\lambda}(V) = d^2\bar{\lambda}(V) \circ d^2\tau_V(V)$, where $\bar{\lambda}(x, v) = (x, \lambda.v)$,

$(iii)_2$ the differential $d\tau_V(V)$ of τ_V at V, which is the restriction of $d^2\tau_V(V)$ to first order vectors, is a connection of order one on M. This means that it satisfies, for all $V = (x, v)$ in TM,

$(i)_1$ $d\tau_V(V) : T_V TM \to T_v(T_xM)$ is a projection onto the vertical space $T_v(T_xM)$,

$(ii)_1$ $\forall\lambda \in \mathbb{R}$, $d\tau_{\bar{\lambda}V}(\bar{\lambda}V)\ d\bar{\lambda}(V) = \lambda\ d\tau_V(V)$, where $\bar{\lambda}(x, v) = (x, \lambda.v)$.

Note that, by requiring the differential $d\tau_V(V)$ to be $T_v(T_xM)$-valued in $(i)_1$, we impose on τ_V to verify

$$\forall V = (x, v) \in TM :\ \tau_V(V) = v. \tag{4.1}$$

4.0.2 Basic transport

The map τ_V may depend on the pair $V = (x, v)$. Actually, the usual notion of transport is such that, for every $V = (x, v)$ in TM, τ_V only depends on the projection point x of V on M. Such transports are called basic transports.

Definition 4.2 *A transport is said basic if*

$$\forall x \in M, \forall u, v \in T_xM,\ \forall W \in TM, \quad \tau_{(x,v)}(W) = \tau_{(x,u)}(W).$$

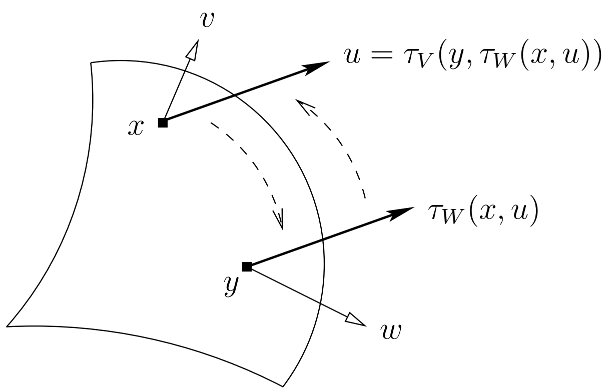
For a basic transport τ , we write $\tau_x(W)$ instead of $\tau_V(W)$.

The following lemma proposes a sufficient condition for τ to be basic. It is not a necessary condition and it implies much more. We see later that it yields the invertibility for the transport.

Proposition 4.1 *Every transport τ on M verifying*

$$\forall V = (x, v), W = (y, w) \in TM, \quad \tau_V(y, \tau_W(x, .)) = Id_{T_xM}, \tag{4.2}$$

is a basic transport.



Proof : For all $R = (y, r)$ and $U = (x, u)$ in TM, applying the property to $\tau_U(R)$ gives $\tau_V(y, \tau_W(x, \tau_U(R))) = \tau_U(R)$. By the same property $\tau_W(x, \tau_U(R))) = r$, so that $\tau_V(y, \tau_W(x, \tau_U(R))) = \tau_V(y, r) = \tau_V(R)$. Hence we get $\forall R \in TM, \forall v, u \in T_xM, \quad \tau_{(x,v)}(R) = \tau_{(x,u)}(R)$, that is saying that τ is basic. □

4.0.3 Idempotent transport

Let us come back to the properties $(i)_1$ for $d\tau_V(V)$ and $(i)_2$ for $d^2\tau_V(V)$. The part of τ which gives the projections onto $T_v(T_xM)$ and $\tau_v(T_xM)$ is the map $\tau_V(x,.) : T_xM \to T_xM$ whose differential and 2-jet at point $v \in T_xM$ yield the identity, that is

$$d\tau_V(x,.)(v) = Id_{T_v(T_xM)} \text{ and } d^2\tau_V(x,.)(v) = Id_{\tau_v(T_xM)}.$$

However, although we could expect this property of a transport, the definition of transport doesn't imply that the map $\tau_V(x,.)$ is the identity. We call this property the idempotence of the transport. Actually, it is verified for any basic transport. Otherwise, it is often required.

Definition 4.3 *A transport is said idempotent if, for all V in TM, the map $\tau_V(x,.)$ is the identity of T_xM, that is*

$$(i) \quad \forall V = (x,v) \in TM, \forall w \in T_xM, \ \tau_V(x,w) = w.$$

Remark 4.1 *Note that the term idempotent comes from the equivalence of this definition with the property $\tau_V \circ \tau_V = \tau_V$ when τ_V is an onto map. Indeed, for all $V = (x,v) \in TM$, since τ_V is $Im(\tau_V) = T_xM$-valued , we have $\tau_V \circ \tau_V = \tau_V|_{T_xM} \circ \tau_V = \tau_V(x,.) \circ \tau_V$, so it equals τ_V if and only if $\tau_V(x,.) = Id_{T_xM}$.*

Proposition 4.2 *Every basic transport is idempotent.*

Proof : For a basic transport, (4.1) is written

$$\forall x \in M, \forall v \in T_xM, \tau_x(x,v) = v,$$

what is exactly saying that τ is idempotent. □

The following proposition ensures that the property $(i)_2$ (and so $(i)_1$))of the definition of transport can be obtained as a consequence of the property (i) of idempotence.

Proposition 4.3 *If a map f yields, for all V in TM, a map $f_V : TM \to T_xM$ verifying $f_V(x,.) = Id_{T_xM}$, then the maps $df_V(V) : T_VTM \to T_v(T_xM)$ and $d^2f_V(V) : \tau_VTM \to \tau_v(T_xM)$ are projections.*

Proof : Set $V = (x,v) \in TM$. The property of idempotence $f_V \circ f_V = f_V$ implies that $d(f_V \circ f_V)(V) = df_V(V)$, that is $df_V(f_V(V)) \circ df_V(V) = df_V(V) \circ df_V(V) = df_V(V)$ and likewise $d^2f_V(V) \circ d^2f_V(V) = d^2f_V(V)$. Moreover, this two maps are linear. So they are projections.□

4.0.4 Linear transport

Most of the examples of transports have a property of linearity, defined as follows. Such transports yield a connection with strong properties.

Definition 4.4 *A transport τ is said to be linear if, for all $V = (x, v)$ in TM, the map $\tau_V : TM \to T_xM$ verifies*

$$(ii) \quad \forall\lambda \in \mathbb{R}, \forall W = (y, w) \in TM, \qquad \tau_{(x,\lambda v)}(y, \lambda w) = \lambda\tau_{(x,v)}(y, w). \tag{4.3}$$

The following proposition ensures that the property $(ii)_2$ of the definition of transport can be obtained as a consequence of the linearity. Recall that $\bar{\lambda}V = (x, \lambda v)$.

Proposition 4.4 *If a map f yields for all V in TM a map $f_V : TM \to T_xM$ such that $\forall\lambda \in \mathbb{R}, \forall W = (y, w) \in TM, \quad f_{\bar{\lambda}V}(\bar{\lambda}W) = \lambda f_V(W)$, then the maps $df_V(V)$ and $d^2f_V(V)$ verifies the properties $(ii)_1$ and $(ii)_2$ of linearity.*

Proof : Just compute the 2-jet of the both sides of the equality $f_{\bar{\lambda}V}(\bar{\lambda}W) = \lambda f_V(W)$. □

A transport verifying the property (4.3) is called linear, because of the following proposition. Note that this linearity is local and is both in v and w. By assuming more regularity on τ, we can get rid of this condition.

Proposition 4.5 *Any linear transport verifies, for all x and y nearby in M, for v_1, v_2 in T_xM, and w_1, w_2 in T_yM,*

$$\forall\lambda, \mu \in \mathbb{R}, \qquad \tau_{(x,\lambda v_1+\mu v_2)}(y, \lambda w_1 + \mu w_2) = \lambda\tau_{(x,v_1)}(y, w_1) + \mu\tau_{(x,v_2)}(y, w_2). \tag{4.4}$$

Moreover, if the map $(V, W) \to \tau_V(W)$ is C^2, we get this relation for all x and y in M.

Proof : For every x, y in M, consider the map $f_{x,y} : T_xM \times T_yM \to T_xM$ defined by $f_{x,y}(v, w) = \tau_{(x,v)}(y, w)$. From the first point of the definition of transport, $f_{x,y}$ is differentiable at $(0, 0)$ if x and y are nearby enough in the general case, and for all x and y if τ is smooth enough. Moreover, τ is a linear transport so we have $f_{x,y}(\lambda(v, w)) = \lambda f_{x,y}(v, w)$. Hence, the lemma 4 implies the property of linearity (4.4).□

Proposition 4.6 *Every linear transport τ is the sum of a basic linear transport and a linear map with respect to v. Precisely, τ is decomposed as follows*

$$\forall V = (x, v), W = (y, w) \in TM, \quad \tau_V(W) = \tau_{(x,0)}(W) + \tau_V(y, 0). \tag{4.5}$$

Proof : To get the decomposition (4.5), take $v_1 = w_2 = 0$ and $\lambda = \mu = 1$ in (4.4). The map $\tau_{(x,0)}(W)$ is clearly basic. Let show that $\tau_{(x,0)}(W) = \tau_V(W) - \tau_V(y,0)$ is a linear transport. It verifies $\tau_{(x,0)}(x,.) = \tau_V(x,.) - \tau_V(x,0) = Id_{T_xM} - 0 = Id_{T_xM}$ so it is idempotent and by proposition 4.3, the property $(i)_2$ for the transport is verified. Moreover, for all λ in $\mathbb{R}$, $\tau_{(x,\lambda 0)}(y,\lambda w) = \tau_{(x,\lambda v)}(y,\lambda w) - \tau_{(x,\lambda v)}(y,\lambda 0) = \lambda\tau_V(W) - \lambda\tau_V(y,0) = \lambda\tau_{(x,0)}(W)$, so by proposition 4.4. Then it is a linear transport on M.
Moreover, the map $v \to \tau_V(y,0)$ is linear since we have, by the property of linearity of τ, $\tau_{(x,\lambda v)}(y,0) = \tau_{(x,\lambda v)}(y,\lambda 0)) = \lambda\tau_V(y,0)$. □

The definition of linear transport implies $\tau_{(x,0)}(y,0) = 0$, for all x,y in M (take $\lambda = 0$). But τ_V doesn't necessary verify $\tau_V(y,0) = 0$ for all V. When it does, it characterizes basic linear transports.

Proposition 4.7 *A linear transport τ is basic if and only if $\tau_V(y,0) = 0$ for every V in TM and y in M.*

Proof : Being a basic transport is equivalent to satisfy $\forall V = (x,v), W \in TM : \quad \tau_{(x,v)}(W) = \tau_{(x,0)}(W)$ so by (4.5), it is equivalent for a linear transport to the property $\tau_V(y,0) = 0$. □

Proposition 4.8 *Every linear transport τ is idempotent.*

Proof : We have $\forall V = (x,v), W = (y,w) \in TM, \quad \tau_V(W) = \tau_{(x,0)}(W) + \tau_V(y,0)$ $\tilde{t}$ defined by $\tilde{t}_V(W) = \tau_{(x,0)}(W)$ is basic so idempotent by prop... Moreover, since $\tau_V(V) = v$ and $\tau_{(x,0)}(V) = v$, we get $v = \tau_V(V) = v + \tau_V(x,0)$ and then $\tau_V(x,0) = 0$. Finally, $\tau_{(x,v)}(x,w) = w + \tau_{(x,v}(y,0) = w$, so τ is idempotent. □

Note that, if τ is a basic transport such that $(x,y) \to \tau_y(x,.)$ is C^2, by proposition 4.5, a linear basic transport is such that for all x,y in M, the map $\tau_x(y,.)$ is linear, and also idempotent by the foregoing. Then the notion of linear basic transport is actually Picard's definition of transport.

4.0.5 Transport as a connection of order 0

To get a connection from τ_V, some conditions are required on its 2-jet only on the diagonal, and not expressed directly on the map τ_V. However, it could be interesting to understand the notion of transport through properties on τ_V and not only by computing its 2-jet at V. By propositions (4.3) and (4.4) , properties (i) and (ii) provide a transport. It yields a more simple way to verify if a map is a transport on M (in fact, we have already used this in the proof of proposition 4.6.)

Proposition 4.9 *Let f be a map on M yielding, for all $V = (x,v)$ in TM, a map $f_V : TM \to T_xM$, verifying*

- *$(V,W) \to f_V(W)$ is a C^2 map on a neighbourhood of the diagonal of $TM \times TM$,*

- *(i)* $\forall w \in T_xM, \quad f_V(x,w) = w,$
  (ii) $\forall \lambda \in \mathbb{R}, \ \forall W = (y,w) \in TM, \qquad f_{(x,\lambda v)}(y,\lambda w) = \lambda \tau_{(x,v)}(y,w),$

then f is a linear (idempotent) transport on M.

Proof : Just apply propositions 4.3 and 4.4. □

By these properties of idempotence and linearity, such a transport yields for all V in TM a projection map from TM onto T_xM. Then , if T_xM represents a vertical jump space at V, a linear transport can be seen as a way of projecting onto this vertical space, that is an analogue for the jumps case to a linear connection on M, which could be called a connection of order 0.

4.0.6 Examples

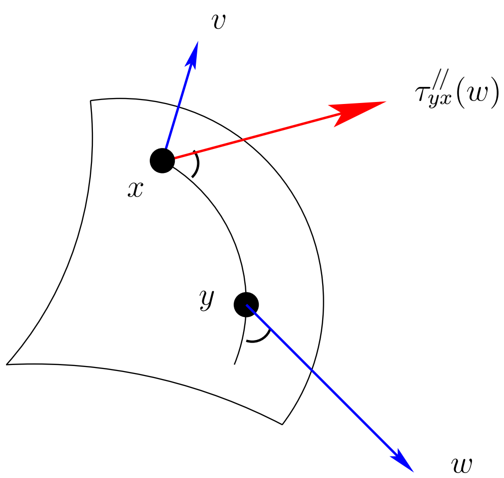
We now illustrate the notion of transport with some examples. To understand the properties we have studied (basic, idempotent, and linear transport), we first give the example of the parallel transport along a geodesic or an interpolation rule for which they are all verified. Then we present some more general examples, constructed from the parallel transport, for which one or more of these properties is not verified. Lastly, we see the transport defined by Picard, constructed with a connection rule, which also satisfies all the properties.
Through these examples, we see that a transport is often only locally defined. Indeed, although it is not defined along a curve, its construction often requires a way of joining two points on M. If M is not simply convex, this can only be locally done.
A way of solving this problem would be to consider a smooth map which coincides with τ_V on a neighbourhood of V, and vanishes elsewhere (by using a partition of the unity).
Actually, a transport will be used to describe the jumps of a semimartingale in TM. Then a locally defined transport would only describe the small jumps. Nevertheless, since there is a finite number of large jumps, each one can be described separately, and there is no loss of generality due to the local existence of a transport.

Example 4.1 . Parallel transport along a geodesic or an interpolation rule ($\tau^{//}$)
A connection is given on M, assumed to be simply convex. Let x and y be two points on M. We denote by $\tau^{//}_{yx} : T_yM \to T_xM$ the parallel transport along the geodesic joining x to y (or along $(I(s,y,x))_{s\in[0,1]}$, if I is an interpolation rule on M).

The following map is a basic linear transport on M, called the parallel transport along geodesics

$$\forall V = (x,v) \in TM, \ \forall W = (y,w) \in TM, \quad \tau_x(W) = \tau^{//}_{yx}(w)$$

Indeed, let us use proposition 4.9 to verify that τ is a linear transport. For all $V = (x,v)$ in TM, $\tau_x : TM \to T_xM$. The smoothness (C^2 in particular) of $(V,W) \to \tau_x(W)$ follows from the smoothness of $\tau^{//}$. Then the property $\tau^{//}_{xx} = Id_{T_xM}$ gives the idempotence (i) and the linearity of the parallel transport gives the linearity (ii). Moreover it is basic since $(y,w) \to \tau^{//}_{yx}(w)$ only depends on x.



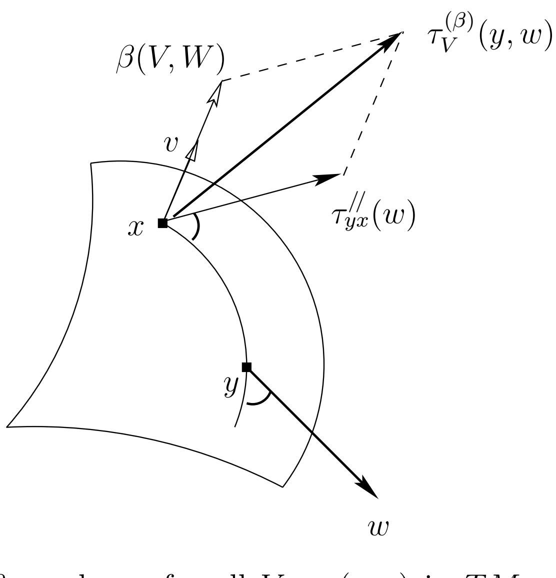
Example 4.2 *We propose more general examples, constructed with the parallel transport. Note that it can be replaced by any linear transport on M.*

Let β be a map on $TM \times TM$ such that

1. *β is smooth on a neighbourhood of the diagonal,*
2. *$\forall (V,W) \in TM \times TM,\ \beta(V,W) \in T_xM$,*
3. *$\beta(V,V) = 0$ and $d^2\beta(V,.)(V) = 0$.*

Then we get the following family of transports on M :

$$\forall V = (x,v) \in TM,\ \forall W = (y,w) \in TM, \quad \tau_V^{(\beta)}(W) = \tau^{//}_{yx}(w) + \beta(V,W). \tag{4.6}$$



Indeed, by property 1. for β, we have, for all $V = (x,v)$ in TM, $\tau_V : TM \to T_xM$. The smoothness (C^2 in particular) of $(V,W) \to \tau_V(W)$ on the diagonal follows from the smoothness

of $\tau^{//}$ and that of β. Let us compute the 2-jet of $\tau_V^{(\beta)}$ at V.
We have

$$d^2\tau_V^{(\beta)}(V) = d^2\tau_{.x}^{//}(.)(V) + d^2\beta(V,.)(V).$$

By 3. , $d^2\beta(V,.)(V) = 0$. Moreover, since $\tau^{//}$ is a transport, its 2-jet $d^2\tau_{.x}(.)(V)$ is a connection of order two on M. Consequently, $d^2\tau_V^{(\beta)}(V)$ is a connection of order two on M, then $\tau^{(\beta)}$ is a transport on M.

When $\beta \equiv 0$, we get the parallel transport, which is basic, linear, and so idempotent. To understand these properties, let us give some examples of β for which they are not all satisfied.

In the following examples, M is assumed to be a Riemmanian connected manifold with metric g. The length of a tangent vector v, $g(v,v)^{\frac{1}{2}}$, is denoted by $\|v\|$. The map defined for all v in T_xM by $w \in T_xM \to \|v-w\|^3 \in \mathbb{R}_+$ vanishes for $w = v$. It is twice differentiable at v and satisfies

$$d^2\|v - .\|^3(v) = 0 \tag{4.7}$$

A distance δ can be defined on M. For all x and y on M, $\delta(x,y)$ is the infinimum of the lengths of all piecewise differentiable curves of class C^1 joining x to y. The map defined for all x in M by $y \in M \to \delta^3(x,y) \in \mathbb{R}_+$ vanishes for $y = x$. It is twice differentiable at x and satisfies

$$d^2\delta^3(x,.)(x) = 0 \tag{4.8}$$

To show property 3. for β in the following examples, we use the following lemma

Lemma 1 *Consider two maps $f : E \to \mathbb{R}$ and $\phi : E \to E$ (where E is a differentiable manifold) such that, at point $z \in E$, we have $\phi(z) = z$ and $d^2f(z) = 0$. Then $d^2(f \circ \phi)(z) = 0$*

Proof : We use proposition $B.1$ of Appendix B.
If $d^2(f \circ \phi)(z) = 0$ and $\phi(z) = z$, then $\frac{\partial f}{\partial z^i}(\phi(z)) = \frac{\partial^2 f}{\partial z^i \partial z^j}(\phi(z)) = 0$ so $d^2(f \circ \phi)(z) = 0$ □

Example 4.3 *The map $\tau^{(\beta_1)}$, defined by*

$$\forall V = (x,v) \in TM,\ \forall W = (y,w) \in TM, \qquad \tau_V^{(\beta_1)}(W) = \tau_{yx}^{//}(w) + \|v - \tau_{yx}^{//}(w)\|^3 v,$$

is a non idempotent, non basic and non linear transport on M.

Indeed, assumptions 1 and 2 are immediate for

$$\beta_1(V,W) = \|v - \tau_{yx}^{//}(w)\|^3 v.$$

To get assumption 3 for β_1, we first see that it verifies

$$\beta_1(V,V) = \|v - \tau_{xx}^{//}(v)\|^3 v = \|v - v\|^3 v = 0,$$

Then, for every $V = (x,v) \in TM$, we apply lemma 1 to $f : w \to \|v-w\|^3$ and $\phi : w \to \tau_{yx}^{//}(w)$.
For every w in T_xM, we have

$$\beta_1((x,v),(x,w)) = \|v - w\|^2 v,$$

which is not 0 for $w \neq v$. So $\tau^{(\beta_1)}$ is non idempotent. By proposition **??**, it is also non basic. Moreover, it is non linear since the norm is not.

Example 4.4 *The map $\tau^{(\beta_2)}$, defined by*

$$\forall V = (x,v) \in TM,\ \forall W = (y,w) \in TM, \qquad \tau_V^{(\beta_2)}(W) = \tau_{yx}^{//}(w) + \|v - \tau_{yx}^{//}(w)\|^3 \exp_x^{-1}(y)$$

is an idempotent but non basic and non linear transport on M.

Indeed, likewise for β_1, we can show assumptions 1 and 2, and 3 for

$$\beta_2(V,W) = \|v - //_{yx}(w)\|^3 \exp_x^{-1}(y).$$

For all $w \in T_xM$, we have

$$\beta_2((x,v),(x,w)) = \|v - w\|^3 \exp_x^{-1}(x) = 0,$$

so $\tau^{(\beta_2)}$ is idempotent. It is non basic since

$$\beta_2((x,v),W) - \beta_2((x,u),W) = (\|v - \tau_{yx}^{//}(w)\|^3 - \|u - \tau_{yx}^{//}(w)\|^3) \exp_x^{-1}(y) \neq 0.$$

Moreover, it is non linear since the norm is not.

Example 4.5 *The map $\tau^{(\beta_3)}$, defined by*

$$\forall V = (x,v) \in TM,\ \forall W = (y,w) \in TM, \qquad \tau_V^{(\beta_3)}(W) = \tau_{yx}^{//}(w) + \delta^3(x,y) \exp_x^{-1}(y),$$

is a basic but non linear transport on M.

Indeed, likewise for β_1, we can show assumptions 1 and 2, and 3 for

$$\beta_3(V,W) = \delta^3(x,y) \exp_x^{-1}(y).$$

$\tau^{(\beta_3)}$ is basic since β_3 only depends on the points x and y. Moreover, it is non linear since β_3 is constant with respect to w and v.

Example 4.6 *The map $\tau^{(\beta_4)}$, defined by*

$$\forall V = (x,v) \in TM,\ \forall W = (y,w) \in TM, \qquad \tau_V^{(\beta_2)}(W) = \tau_{yx}^{//}(w) + \delta^3(x,y)v.$$

is a linear but non basic transport on M.
Note that $\beta_4(V,W) = \tau_V(x,0)$, so the expression for $\tau^{(\beta_4)}$ corresponds to the decomposition (4.5).

Indeed, likewise for β_1, we can show assumptions 1 and 2, and 3 for

$$\beta_4(V,W) = \delta^3(x,y)v.$$

$\tau^{(\beta_4)}$ is not basic since

$$\beta_4((x,v),W) - \beta_4((x,u),W) = \delta^3(x,y)(v-u) \neq 0.$$

However, it is linear since $v \to \beta_4(V,W)$ is linear and doesn't depend on w.

Example 4.7 . Transport from a connection rule ($\tau^{(\gamma)}$)

Let γ be a connection rule on M. For all $x \in M$, write γ_x for the map $\gamma(x,.) : M \to T_xM$ and suppose that for all y in M, its differential at point y is invertible. In [9], Picard defined a transport with γ as follows.

The following map is a linear basic transport on M, called the transport from the connection rule γ, and denoted by $\tau^{(\gamma)}$,

$$\forall V = (x, v) \in TM,\ \forall W = (y, w) \in TM, \qquad \tau_x^{(\gamma)}(W) = d\gamma_x(y)(w).$$

Indeed, we use proposition 4.9 to verify that $\tau^{(\gamma)}$ is a linear transport. For all x in M, $d\gamma_x(y) : T_yM \to T_{\gamma(x,y)}T_xM \simeq T_xM$, so $\forall V = (x, v) \in TM$, $\tau_x^{(\gamma)} : TM \to T_xM$. The C^2 property for $(V, W) \to \tau_x^{(\gamma)}(W)$ follows from that the connection rule is a C^2 mapping. Then the property $d\gamma_x(x) = Id_{T_xM}$ gives the idempotence (i) and the linearity of the differential gives the linearity (ii). Moreover it is basic since $(y, w) \to d\gamma_x(y)(w)$ only depends on x.

If $\gamma = \gamma_0$ is the geodesic connection rule, we obtain a transport which corresponds to the notion of Jacobi fields.

The transport from the geodesic connection rule γ_0 is :

$$\forall V = (x, v) \in TM,\ \forall W = (y, w) \in TM, \qquad \tau_x^{(\gamma_0)}(W) = d\exp_x^{-1}(y)(w).$$

Indeed, recall that the geodesic connection rule is given by $(\gamma_0)_x(y) = \exp_x^{-1}(y)$. Then $\tau_x^{(\gamma_0)}(W) = d(\gamma_0)_x(y)(w) = d\exp_x^{-1}(y)(w)$.

4.0.7 Invertible and strongly invertible transport

In this paragraph, we specify what we mean by an invertible transport. We will require this property in part IV to have unicity for the solution of a s.d.e. with covariant jumps, defined with a transport.

Definition 4.5 *A transport τ is said to be invertible if for all V in TM, for all y in M, the map $\tau_V(y,.) : T_yM \to T_xM$ is invertible.*

Actually, a local invertibility of the transport is often sufficient. The following proposition explains that a linear transport is always locally invertible, so that in several cases we won't need to require the invertibility of a transport.

Proposition 4.10 *If τ is a linear transport, it exists a neighbourhood N_x of x such that, for all y in N_x, $\tau_V(y,.) : T_yM \to T_xM$ is invertible.*

Proof :
Set $V = (x, v) \in TM$. Take y in the neighbourhood U_x of x in M on which the geodesic from x to y exists and is unique, and let $\tau_{yx}^{//}$ be the parallel transport from y to x along this geodesic. Note that the choice of the connection on M to define this parallel transport doesn't matter here. Consider the linear map $f_y^{(V)} = \tau_V(y, .) \circ \tau_{xy}^{//} : T_xM \to T_xM$. Thanks to the continuity of the determinant, the set of invertible linear maps from T_xM to T_xM is an open set around $f_x^{(V)} = \tau_V(x, .) \circ \tau_{xx}^{//} = Id_{T_xM}$. But the map $y \to f_y^{(V)}$ is continuous on U_x, since the parallel transport is in relation to the points on the basis, and according to the first point of the definition of transport for the smoothness of $\tau_V(y, .)$. So there is a neighbourhood $N_x \subseteq U_x$ of x in M such that for all y in N_x, $f_y^{(V)}$ is invertible. Now come back to the map $\tau_V(y, .)$ which is written as the composition of two invertible maps, namely for all y in N_x : $\tau_V(y, .) = f_y^{(V)} \circ (\tau_{xy}^{//})^{-1}$. Finally, $\tau_V(y, .)$ is invertible for all y in N_x.□

Let us now come back to proposition 4.1. The hypothesis implies the invertibility of the transport and we now get an equivalence.

Proposition 4.11 *If τ is a transport on M, the following properties are equivalent*

1. $\forall V = (x, v), W = (y, w) \in TM : \quad \tau_V(y, \tau_W(x, .)) = Id_{T_xM}$,
2. *τ is basic, invertible and verifies* $\tau_x(y, .)^{-1} = \tau_y(x, .)$.

Proof : Since 1. implies that τ is basic, according to proposition 4.1, we just write the property 1. with a basic transport : $\forall x, y \in M, \forall u \in T_xM : \tau_x(y, \tau_y(x, .)) = Id_{T_xM}$. This means that τ is invertible and such that $\tau_x(y, .)^{-1} = \tau_y(x, .)$.□

The two equivalent properties of proposition 4.11 will frequently be used. A transport satisfying these properties yields a connection with interesting properties. Let us give a name to this property.

Definition 4.6 *A transport τ is said to be strongly invertible if τ is basic, invertible and satisfies, for every x and y in M, $\tau_x(y, .)^{-1} = \tau_y(x, .)$.*

We may think about a link between the property of linearity and the one of strong invertibility. In fact, there is neither equivalence nor inclusion. Let us come back to our examples. They yield an example of non linear but strongly invertible transport and one of linear invertible but not strongly invertible transport.

Examples

- Example 4.1 revisited :
  The parallel transport along geodesics is strongly invertible, and we get

$$\forall x, y \in M, \; \tau_x(y, .)^{-1} = \tau_{xy}^{//}.$$

Indeed, if (c_t) denotes the geodesic from x to y, the geodesic from y to x is $(\tilde{c}_t) = (c_{1-t})$. Then the parallel transport along $\tilde{c}_t$ is $\tilde{\tau}_{0t}^{//} = \tau_{1\,1-t}^{//}$. Hence, $\tau_{yx}^{//} = \tilde{\tau}_{01}^{//} = \tau_{10}^{//} = (\tau_{01}^{//})^{-1} = (\tau_{xy}^{//})^{-1}$. So

τ is invertible and verifies $\tau_x(y,.)^{-1} = (\tau^{//}_{yx})^{-1} = \tau^{//}_{xy}$.

- Examples 4.3, 4.4, 4.5, and 4.6 revisited :
  The transports $\tau^{(\beta_3)}$ and $\tau^{(\beta_4)}$ are invertibles. For $i=3$ and $i=4$, we have

$$\forall V=(x,v)\in TM, \forall y\in M, \forall u\in T_xM, \quad \tau_V^{(\beta_i)}(y,.)^{-1}(u) = \tau^{//}_{yx}(u-\beta_i(V,(y,.)).$$

  Moreover, $\tau^{(\beta_3)}$ provides an example of non linear but strongly invertible transport.

  Indeed, for $i=3$, we have $\tau_x^{\beta_3}(y,w) = \tau^{//}_{yx}(w) + \delta^2(x,y)\exp_x^{-1}(y)$.
  $\tau_x^{\beta_3}(y,w) = u \Leftrightarrow \tau^{//}_{yx}(w) + \delta^2(x,y)\exp_x^{-1}(y) = u \Leftrightarrow w = \tau^{//}_{xy}(u) - \tau^{//}_{xy}(\delta^2(x,y)\ \exp_x^{-1}(y)) = \tau^{//}_{xy}(u) + \delta^2(x,y)\tau^{//}_{xy}(-\exp_x^{-1}(y))$. But $\tau^{//}_{xy}(-\exp_x^{-1}(y)) = \exp_y^{-1}(x)$, so we get $w = \tau^{//}_{xy}(u) + \delta^2(x,y)\exp_y^{-1}(x) = \tau_y^{\beta_3}(x,u)$. Then τ^{β_3} is invertible and $\tau_x^{\beta_3}(y,.)^{-1}(u) = \tau_y^{\beta_3}(x,u)$, so it is strongly invertible.
  For $i=4$, we have $\tau_x^{\beta_4}(y,w) = \tau^{//}_{yx}(w) + \delta^2(x,y)v$.
  $\tau_x^{\beta_4}(y,w) = u \Leftrightarrow \tau^{//}_{yx}(w) + \delta^2(x,y)v = u \Leftrightarrow w = \tau^{//}_{xy}(u - \delta^2(x,y)v)$. Then τ^{β_4} is invertible.

  With regard to the transports $\tau^{(\beta_1)}$ and $\tau^{(\beta_2)}$, note that the maps β_1 and β_2 itself contains w, so that we can't apply a similar proof to the above ones. In general, they are non invertibles transports.

- Example 4.7 revisited :
  The transport τ^γ is invertible (but non necessarily strongly invertible), and we get

$$\forall (x,y)\in M,\ \tau_x^\gamma(y,.)^{-1} = (d\gamma_x(y))^{-1}.$$

  Indeed, the expression for the inverse follows immediately from the expression of τ^γ. Note that we do not have necessarily $(d\gamma_x(y))^{-1} = d\gamma_y(x)$. For instance, for γ_0, we get $(d(\gamma_0)_x(y))^{-1} = (d\exp_x^{-1}(y))^{-1} = d\exp_x(\exp_x^{-1} y)$ which is not in general equal to $d\exp_y^{-1}(x) = d(\gamma_0)_y(x)$

Let us present an example of strongly invertible linear transport, different from the parallel transport along geodesics.

Example 4.8 *Let $k: M\times M\to M$ be a smooth map such that, for every x and y in M, $k(x,y)=k(y,x)$ and $k(x,x)=x$. Then the map, defined for every $V=(x,v)$ and $W=(y,w)$ in TM, by*

$$\tau_x(W) = \tau^{//}_{k(x,y),x}\tau^{//}_{y,k(x,y)}(w),$$

is a strongly invertible linear transport on M.

Note that, if $k(x,y)$ belongs to the geodesic joining x to y, then τ the parallel transport along geodesics. If $k(x,y)$ doesn't belong to the geodesic joining x to y, τ is linear since $\tau^{//}$ is, and τ is idempotent, since $k(x,x)=x$, then it is a transport, by proposition 4.9. Moreover, it is strongly invertible, since

$$\tau_x(y,w)=u \;\Leftrightarrow\; \tau^{//}_{k(x,y),x}\tau^{//}_{y,k(x,y)}(w)=u \;\Leftrightarrow\; w=\tau^{//}_{k(x,y),y}\tau^{//}_{x,k(x,y)}(u)$$

Since $k(x,y), = k(y,x)$, it follows

$$\tau_x(y,w)=u \;\Leftrightarrow\; w=\tau^{//}_{k(y,x),y}\tau^{//}_{x,k(y,x)}(u) \Leftrightarrow w=\tau_y(x,u)$$

4.0.8 Properties of the connection derived from a transport

Definition 4.7 *For every transport τ on M, define the maps p and $\tilde{p}$ by :*

$$p : V\in TM \to p_V = d\tau_V(V) : T_VTM \to T_v(T_xM)$$

$$\tilde{p} : V\in TM \to \tilde{p}_V = d^2\tau_V(V) : \tau_VTM \to \tau_v(T_xM)$$

Recall that, by the definition of τ, p is a connection of order one on M and $\tilde{p}$ is a connection of order two on M.

Expressions of p and $\tilde{p}$ in coordinates

Using the coordinates described in appendix A and B, we express the properties of connections of order one and two for the map τ_V and its partial derivatives. Then we give an expression for p and $\tilde{p}$ in coordinates.

Proposition 4.12 *Let (Γ^i_{jk}) be the Christoffel symbols of the connection p, and $(\mathrm{I\!\Gamma}^i_{jlk})$, (C^i_{jk}) be the symbols for $\tilde{p}$ (see proposition 3.1).*

The partial derivatives of τ_V at V are expressed as follows

$$(a)\;\; \frac{\partial\tau^i_V}{\partial x^j}(V)=\Gamma^i_{jk}(x)\;v^k, \qquad (b)\;\; \frac{\partial\tau^i_V}{\partial v^j}(V)=\delta^i_j, \qquad (c)\;\; \frac{\partial^2\tau^i_V}{\partial v^j\partial v^l}(V)=0,$$

$$(d)\;\; \frac{\partial^2\tau^i_V}{\partial x^j\partial x^l}(V)=\mathrm{I\!\Gamma}^i_{jlk}(x)v^k, \qquad (e)\;\; \frac{\partial^2\tau^i_V}{\partial x^j\partial v^k}(V)=C^i_{jk}(x).$$

Proof : On the one hand, we apply proposition 3.1 to connection $\tilde{p}$ of order two. On the other hand, for all $V=(x,v)$ in TM, we write the partial derivatives of τ_V with the maps p and $\tilde{p}$. Formula (a) comes from (3.4) and

$$\frac{\partial\tau^i_V}{\partial x^j}(V)=dv^i(p_V(\frac{\partial}{\partial x^j}))=d^2v^i(\tilde{p}_V(\frac{\partial}{\partial x^j})).$$

Computing the partial derivatives (b) and (c) with respect to v gives

$$dv^i(p_V(\frac{\partial}{\partial v^j})) = \frac{\partial \tau_V^i}{\partial v^j}(V).$$

But p_V is a projection onto $T_v(T_xM)$ so

$$p_V(\frac{\partial}{\partial v^j}) = \frac{\partial}{\partial v^j},$$

then we get (b), that is $\frac{\partial \tau_V^i}{\partial v^j}(V) = \delta_j^i$.
Now using the expression of the 2-jet in Appendix (B), we write

$$\tilde{p}_V(\frac{\partial^2}{\partial v^j \partial v^l}) = \frac{\partial^2 \tau_V^i}{\partial v^j \partial v^l}(V)\frac{\partial}{\partial v^i} + \frac{\partial \tau_V^i}{\partial v^j}(V)\frac{\partial \tau_V^k}{\partial v^l}(V)\frac{\partial^2}{\partial v^k \partial v^i},$$

which is, using (b),

$$\tilde{p}_V(\frac{\partial^2}{\partial v^j \partial v^l}) = \frac{\partial^2 \tau_V^i}{\partial v^j \partial v^l}(V)\frac{\partial}{\partial v^i} + \frac{\partial^2}{\partial v^j \partial v^l}.$$

But $\tilde{p}_V$ is a projection onto $\tau_v(T_xM)$ so

$$\tilde{p}_V(\frac{\partial^2}{\partial v^j \partial v^l}) = \frac{\partial^2}{\partial v^j \partial v^l}.$$

Then we get (c), that is $\frac{\partial^2 \tau_V^i}{\partial v^j \partial v^l}(V) = 0$.
Formula (d) comes from (3.3) and

$$\frac{\partial^2 \tau_V^i}{\partial x^j \partial x^l}(V) = d^2v^i(\tilde{p}_V(\frac{\partial^2}{\partial x^j \partial x^l})).$$

Formula (e) comes from (3.2) and

$$\frac{\partial^2 \tau_V^i}{\partial x^j \partial v^l}(V) = d^2v^i(\tilde{p}_V(\frac{\partial^2}{\partial x^j \partial v^l})).\Box$$

For instance, let us compute the Christoffel symbols for the parallel transport and for the transport from a connection rule γ.

Examples

- Example 4.1 revisited :
  Let q be the connection which permits to define the parallel transport $\tau^{//}$ along geodesics. Then the Christoffel symbols for p, derived from $\tau^{//}$, are the one of q. In other words, we have $d\tau_x^{//}(V) = q_V$.
  Indeed, we have to compute the Christoffel symbols for p, given in proposition 4.12 by

$$\Gamma_{jk}^i(x) = \frac{\partial(\tau_{yx}^{//})^i}{\partial y^j}(\frac{\partial}{\partial y^k})|_{y=x}.$$

Recall the expression (51) for the parallel transport in coordinates. Along a geodesic (x_s), satisfying $\dot{x}_s = \tau^{//}_{0s}\dot{x}_0$, it yields (for $t=0$)

$$\frac{\partial (\tau^{//}_{s0})^i_k}{\partial s} = (\tau^{//}_{s0})^i_r \tilde{\Gamma}^r_{lk}(x_s)\ (\tau^{//}_{0s})^l_m \dot{x}^m_0, \tag{4.9}$$

where $\tilde{\Gamma}^r_{lk}$ are the Christoffel symbols for q. Then, integrating (4.9) between 0 and 1 gives

$$(\tau^{//}_{10})^i_k = \delta^i_k + \int_0^1 (\tau^{//}_{s0})^i_r \tilde{\Gamma}^r_{lk}(x_s)\ (\tau^{//}_{0s})^l_m \dot{x}^m_0\ ds.$$

Note that $\tau^{//}_{10} = \tau^{//}_{yx}$ if $(x_s) = (\exp_x(s \exp^{-1}_x(y)))$ is the geodesic joining x to y. Then we get

$$\frac{\partial (\tau^{//}_{yx})^i_k}{\partial y^j} = \int_0^1 \frac{\partial}{\partial y^j}[(\tau^{//}_{s0})^i_r \tilde{\Gamma}^r_{lk}(x_s)\ (\tau^{//}_{0s})^l_m \dot{x}^m_0] ds$$

$$= \int_0^1 \frac{\partial}{\partial y^j}[(\tau^{//}_{s0})^i_r \tilde{\Gamma}^r_{lk}(x_s)\ (\tau^{//}_{0s})^l_m]\ ds\ \dot{x}^m_0 + \int_0^1 (\tau^{//}_{s0})^i_r \Gamma^r_{lk}(x_s)\ (\tau^{//}_{0s})^l_m\ ds \frac{\partial \dot{x}^m_0}{\partial y^j}$$

The map $\dot{x}_0 = \exp^{-1}_x(y)$ satisfies, by proposition 0.7, for $y = x$,

$$\dot{x}_0 = \exp^{-1}_x(x) = 0 \text{ and } \frac{\partial \dot{x}^m_0}{\partial y^j}|_{y=x} = \delta^m_j.$$

It follows

$$\frac{\partial (\tau^{//}_{yx})^i_k}{\partial y^j}|_{y=x} = \int_0^1 [\tau^{//}_{s0})^i_r \tilde{\Gamma}^r_{lk}(x_s)\ (\tau^{//}_{0s})^l_m]|_{y=x}\ ds\ \delta^m_j$$

For $y = x$, we get $\tau^{//}_{s0} = Id$, and $\tilde{\Gamma}^r_{lk}(x_s) = \tilde{\Gamma}^r_{lk}(x)$, then

$$\frac{\partial (\tau^{//}_{yx})^i_k}{\partial y^j}|_{y=x} = \int_0^1 \delta^i_r \tilde{\Gamma}^r_{lk}(x)\ \delta^l_m\ dt \delta^m_j,$$

that is

$$\Gamma^i_{jk}(x) = \tilde{\Gamma}^i_{jk}(x).$$

- Example 4.7 revisited :
  Let $\tau^{(\gamma)}$ be the transport from a connection rule γ. Then, the Christoffel symbols for p are given by

$$\Gamma^i_{jk}(x) = \frac{\partial^2 \gamma^i(x,y)}{\partial y^j \partial y^k}|_{y=x}.$$

  In particular, we get $\Gamma^i_{jk}(x) = \Gamma^i_{kj}(x)$, so $\tau^{(\gamma)}$ yields a torsion-free connection p.
  Indeed, we have to compute the Christoffel symbols for p, given in proposition 4.12 by

$$\Gamma^i_{jk}(x) = \frac{\partial (\tau^{(\gamma)}_x)^i}{\partial y^j}(y, \frac{\partial}{\partial y^k})|_{y=x}$$

Recall that $\tau_x^{(\gamma)}(y,w) = d\gamma_x(y)(w)$, so

$$(\tau_x^{(\gamma)})^i(y, \frac{\partial}{\partial y^k}) = (d\gamma_x(y))^i_k = \frac{\partial \gamma_x^i}{\partial y^k}(y).$$

Then we get, for $y = x$,

$$\Gamma^i_{jk}(x) = \frac{\partial}{\partial y^j}(\frac{\partial \gamma_x^i}{\partial y^k})|_{y=x} = \frac{\partial^2 \gamma^i(x,y)}{\partial y^j \partial y^k})|_{y=x}.$$

By the Schwartz lemma, we get the symmetry $\Gamma^i_{jk}(x) = \Gamma^i_{kj}(x)$.

We can now express p_V and $\tilde{p}_V$ in an imbedding as follows.

Proposition 4.13 *For every transport τ on M, the associated connections p and $\tilde{p}$ are given, for all $V = (x,v)$ in TM, by :*

1. $\forall \mathbb{W} \in T_V TM$:
$$p_V(\mathbb{W}) = [\ dv^i(\mathbb{W}) + \Gamma^i_{jk}(x) v^k\ dx^j\ (\mathbb{W})] \frac{\partial}{\partial v^i} \tag{4.10}$$
2. $\forall \mathbb{L} \in \tau_V TM$:

$$\tilde{p}_V(\mathbb{L}) \ = \ [\ d^2 v^i(\mathbb{L}) + \Gamma^i_{ml}(x) v^l\ d^2 x^m\ (\mathbb{L}) + I\!\Gamma^i_{mkl}(x) v^l\ dx^m.dx^k(\mathbb{L}) +\ 2\ C^i_{mk}(x)\ dx^m.dv^k(\mathbb{L})\ ] \frac{\partial}{\partial v^i}$$
$$+\ [dv^i.dv^j(\mathbb{L})\ +\ \Gamma^i_{kl}(x) v^l\ \Gamma^j_{mr}(x) v^r\ dx^m.dx^k(\mathbb{L})\ +\ 2\ \Gamma^i_{ml}(x) v^l\ dx^m.dv^j(\mathbb{L})\ ] \frac{\partial^2}{\partial v^i \partial v^j}$$

Proof : For $V = (x,v)$ in TM, we compute the 2-jet $\tilde{p}_V(\mathbb{L}) = d^2\tau_V(V)(\mathbb{L})$ of τ_V at V, using Appendix B. For all $\mathbb{L}$ in $\tau_V TM$, we get

$$\tilde{p}_V(\mathbb{L}) = [\frac{\partial \tau_V^i}{\partial V^m}(V) d^2 V^m(\mathbb{L}) + \frac{\partial^2 \tau_V^i}{\partial V^m \partial V^k}(V) dV^m.dV^k(\mathbb{L})] \frac{\partial}{\partial v^i} + [\frac{\partial \tau_V^i}{\partial V^m} \frac{\partial \tau_V^j}{\partial V^k} dV^m.dV^k(\mathbb{L})] \frac{\partial^2}{\partial v^i \partial v^j}. \tag{4.11}$$

The first term in (4.11),

$$\mathbb{L}(\tau_V^i) = \frac{\partial \tau_V^i}{\partial V^m}(V) d^2 V^m(\mathbb{L}) + \frac{\partial^2 \tau_V^i}{\partial V^m \partial V^k}(V) dV^m.dV^k(\mathbb{L}),$$

can be written

$$\mathbb{L}(\tau_V^i) = \frac{\partial \tau_V^i}{\partial v^m}(V) d^2 v^m(\mathbb{L}) + \frac{\partial \tau_V^i}{\partial x^m}(V) d^2 x^m(\mathbb{L}) + \frac{\partial^2 \tau_V^i}{\partial v^m \partial v^k}(V) dv^m.dv^k(\mathbb{L})$$
$$+ \frac{\partial^2 \tau_V^i}{\partial x^m \partial x^k}(V) dx^m.dx^k(\mathbb{L}) + 2 \frac{\partial^2 \tau_V^i}{\partial x^m \partial v^k}(V) dx^m.dv^k(\mathbb{L}).$$

Note that the factor '2' in the last term comes from the application of Schwartz's lemma to τ_V^i which is C^2 in a neighbourhood of V.
Then, by proposition 4.12, we get

$$\mathbb{L}(\tau_V^i) = \; d^2v^i(\mathbb{L}) + \Gamma^i_{ml}(x)v^l \; d^2x^m \; (\mathbb{L}) + \mathrm{I\Gamma}^i_{mkl}(x)v^l dx^m.dx^k(\mathbb{L}) + 2 \; C^i_{mk}(x) \; dx^m.dv^k(\mathbb{L}).$$

The second term in (4.11) can be written

$$\frac{\partial \tau_V^i}{\partial V^m}\frac{\partial \tau_V^j}{\partial V^k}dV^m.dV^k(\mathbb{L}) = \frac{\partial \tau_V^i}{\partial v^m}\frac{\partial \tau_V^j}{\partial v^k}dv^m.dv^k(\mathbb{L}) + \frac{\partial \tau_V^i}{\partial x^m}\frac{\partial \tau_V^j}{\partial x^k}dx^m.dx^k(\mathbb{L}) + 2\;\frac{\partial \tau_V^i}{\partial x^m}\frac{\partial \tau_V^j}{\partial v^k}dx^m.dv^k(\mathbb{L}).$$

Then, by proposition 4.12, we get

$$\frac{\partial \tau_V^i}{\partial V^m}\frac{\partial \tau_V^j}{\partial V^k}dV^m.dV^k(\mathbb{L}) = dv^i.dv^j(\mathbb{L}) + \Gamma^i_{kl}(x)v^l\Gamma^j_{mr}(x)v^r dx^k.dx^m(\mathbb{L}) + 2\;\Gamma^i_{ml}(x)v^l dx^m.dv^j(\mathbb{L}).$$

Adding the two terms, we find the expression of $\tilde{p}_V$ in coordinates. That of p_V is obtained by taking the part of $\tilde{p}_V$ acting on first order vectors. Of course it recovers the expression (46) for a connection p. $\square$

Note that, for $W \neq V$, the map $w \longrightarrow \frac{\partial \tau_V^i}{\partial y^j}(y,w)$ is not necessarily linear. Moreover, even if it is, one hasn't necessarily $\frac{\partial \tau_V^i}{\partial y^j}(W) = \Gamma^i_{jk}(x)v^k$. Therefore, in general, the coefficients $\mathrm{I\Gamma}^i_{jlk}(x)$ and $C^i_{jk}(x)$ can't be expressed with the Christoffel symbols and their derivatives. In other word, $\tilde{p}$, although it equals p when applied to first order vectors, can't be only deduced from p but really depends on the choice of τ. Two transports can give the same connection at the order one and different connections of order two, and hence different covariant calculus.

We study now the particular cases where $\tilde{p}$ can be only deduced from p. If τ is assumed to be linear or strongly invertible, we get the following interesting relations between the partial derivatives of τ.

The linearity implies that the constants $C^i_{jk}(x)$ are the Christoffel symbols.

Proposition 4.14 *For every basic linear transport τ, we have*

$$C^i_{jk}(x) = \Gamma^i_{jk}(x). \tag{4.12}$$

Proof : Recall that, by proposition 4.12, we have

$$C^i_{jk}(x) = \frac{\partial^2 \tau_V^i}{\partial x^j \partial v^k}(V)$$

that is

$$C^i_{jk}(x) = \frac{\partial^2 \tau_V^i(W)}{\partial y^j \partial w^k}|_{W=V}. \tag{4.13}$$

If τ is a linear basic transport, the map $w \to \tau_y(x,w)$ is linear, so that we have

$$\tau_x(W) = \tau_x(y, \frac{\partial}{\partial y^l})w^l. \tag{4.14}$$

It implies

$$\frac{\partial \tau_x^i}{\partial w^k}(W) = \tau_x^i(y, \frac{\partial}{\partial y^k}),$$

and then

$$\frac{\partial}{\partial y^j}(\frac{\partial \tau_x^i}{\partial w^k})(W) = \frac{\partial}{\partial y^j}(\tau_x^i(y, \frac{\partial}{\partial y^k})). \tag{4.15}$$

Then, for $y = x$, by (a) of proposition 4.12, (4.13) gives

$$C_{jk}^i(x) = \Gamma^i_{jk}(x).\square$$

Remark 4.2 *If τ is linear non basic, with the decomposition* (4.5)*, we write $\tau_V(W) = \tau_{(x,0)}(W) + \tau_V(y,0)$. We deduce*

$$\frac{\partial \tau_V^i}{\partial w^k}(W) = \frac{\partial}{\partial w^k}((\tau^i_{(x,0)}(y, \frac{\partial}{\partial y^l})w^l) = \tau^i_{(x,0)}(y, \frac{\partial}{\partial y^k}).$$

So

$$C_{jk}^i(x) = \frac{\partial}{\partial y^j}(\frac{\partial \tau_V^i}{\partial w^k})(W)|_{W=V} = \frac{\partial}{\partial y^j}(\tau^i_{(x,0)}(y, \frac{\partial}{\partial y^k}))|_{W=V}.$$

It follows

$$C_{jk}^i(x) = \frac{\partial}{\partial y^j}(\tau_V^i(y, \frac{\partial}{\partial y^k}))|_{W=V} - \frac{\partial \tau_V^i(y,0)}{\partial y^m}|_{y=x}.$$

Then we get, for a linear but non basic transport τ,

$$C_{mk}^i(x) = \Gamma^i_{mk}(x) - \frac{\partial \tau_V^i(y,0)}{\partial y^m}|_{y=x}.$$

Note that, by the above proposition, if $\tilde{\Gamma}^i_{jk}$ are the Christoffel symbols of the connection derived from the basic linear transport $\tau_{(x,0)}(W)$, we write also

$$C_{jk}^i(x) = \tilde{\Gamma}^i_{jk}(x).$$

The strongly invertibility permits to write the symbols $\Pi\Gamma^j_{klm}(x)$ with respect to the Christoffel symbols, and moreover gives another way of computing these Christoffel symbols. They can be obtained by derivation with respect to the point at which we transport.

Proposition 4.15 *For every strongly invertible transport τ, we get*

$$\frac{\partial \tau_y^i}{\partial y^j}|_{y=x}(x, \frac{\partial}{\partial x^k}) = -\Gamma^i_{jk}(x), \tag{4.16}$$

and

$$\Pi\Gamma^j_{klm}(x) = \frac{1}{2}[\frac{\partial \Gamma^j_{lm}}{\partial x^k}(x) + \frac{\partial \Gamma^j_{km}}{\partial x^l}(x) + \Gamma^j_{ki}(x)\Gamma^i_{lm}(x) + \Gamma^j_{li}(x)\Gamma^i_{km}(x)]. \tag{4.17}$$

Proof : If τ is strongly invertible, this property is written in coordinates

$$\tau_x^i(y, \tau_y(x, v)) = v^i.$$

Deriving this relation with respect to y^j yields

$$\frac{\partial}{\partial y^j}\tau_x^i(y, \tau_y(x, v)) + \frac{\partial}{\partial w^l}\tau_x^i(y, \tau_y(x, v))\ \frac{\partial \tau_y^l}{\partial y^j}(x, v) = 0. \tag{4.18}$$

For $W = V$, recall that we have $\tau_x(x, v) = v$ (τ is basic so idempotent by proposition 4.2), $\frac{\partial \tau_x^i}{\partial w^l}(y, w)|_{W=V} = \delta_l^i$ and $\frac{\partial}{\partial y^j}\tau_x^i(y, w)|_{W=V} = \Gamma_{jk}^i(x)v^k$. It follows

$$\Gamma_{jk}^i(x)v^k \ + \ \delta_l^i\ \frac{\partial \tau_y^i}{\partial y^j}|_{y=x}(x, v) = 0.$$

So we proved (4.16), that is

$$\frac{\partial \tau_y^i}{\partial y^j}|_{y=x}(x, \frac{\partial}{\partial x^k}) = -\Gamma_{jk}^i(x)v^k. \tag{4.19}$$

To prove (4.17), start with the partial derivative $\frac{\partial \Gamma_{lm}^j}{\partial x^k}(x)$. Note that $\Gamma_{lm}^j(x)$ can be seen as a map $f_{lm}^j(x, x)$ where $f_{lm}^j(z, y) = \frac{\partial \tau_z^j}{\partial y^l}(y, \frac{\partial}{\partial y^m})$. Hence we write

$$\frac{\partial \Gamma_{lm}^j}{\partial x^k}(x) = \frac{\partial f_{lm}^j(z, x)}{\partial z^k}|_{z=x} + \frac{\partial f_{lm}^j(x, y)}{\partial y^k}|_{y=x} = \frac{\partial^2}{\partial x^k \partial y^l}(\tau_x^j(y, \frac{\partial}{\partial y^m}))|_{y=x} + \frac{\partial^2 \tau_x^j}{\partial y^k \partial y^l}(x, \frac{\partial}{\partial y^m})|_{y=x}.$$

Recall that

$$\amalg_{klm}^j(x) = \frac{\partial^2 \tau_x^j}{\partial y^k \partial y^l}(y, \frac{\partial}{\partial y^m})|_{y=x},$$

so we get

$$\amalg_{klm}^j(x) = \frac{\partial \Gamma_{lm}^j}{\partial x^k}(x) - \frac{\partial^2}{\partial x^k \partial y^l}(\tau_x^j(y, \frac{\partial}{\partial y^m}))|_{y=x}. \tag{4.20}$$

Hence we need to compute the partial derivative of $\tau_x^j(y, .)$ with respect to x^k and y^l. We get this by derivating (4.18), with respect to x^k,

$$\frac{\partial^2}{\partial x^k \partial y^l}(\tau_x^j(y, \frac{\partial}{\partial x^i}))\ \tau_y^i(x, \frac{\partial}{\partial x^r}) + \frac{\partial}{\partial y^l}(\tau_x^j(y, \frac{\partial}{\partial x^i}))\ \frac{\partial}{\partial x^k}(\tau_y^i(x, \frac{\partial}{\partial x^r}))$$

$$+\frac{\partial \tau_x^j}{\partial x^k}(y, \frac{\partial}{\partial x^i})\ \frac{\partial \tau_y^i}{\partial y^l}(x, \frac{\partial}{\partial x^r}) + \tau_x^j(y, \frac{\partial}{\partial x^i})\ \frac{\partial^2}{\partial x^k \partial y^l}(\tau_y^i(x, \frac{\partial}{\partial x^r})) = 0.$$

Using (4.19), it yields, for $y = x$,

$$\frac{\partial^2}{\partial x^k \partial y^l}(\tau_x^j(y, \frac{\partial}{\partial x^i}))|_{y=x}\delta_r^i \ + \ \Gamma_{li}^j(x)\Gamma_{kr}^i(x) \ + \ (-\Gamma_{ki}^j(x))\ (-\Gamma_{lr}^i(x)) \ + \ \frac{\partial^2}{\partial x^k \partial y^l}(\tau_y^i(x, \frac{\partial}{\partial x^r}))|_{y=x}\delta_i^j = 0.$$

Note that the map $(x,y) \to \tau_x(y,.)$ is C^2 for x and y nearby, so that Schwartz lemma implies

$$\frac{\partial^2}{\partial x^k \partial y^l}(\tau_x^j(y,\frac{\partial}{\partial y^i}))|_{y=x} = \frac{\partial^2}{\partial y^l \partial x^k}(\tau_x^j(y,\frac{\partial}{\partial y^i}))|_{y=x}.$$

But $\frac{\partial^2}{\partial y^l \partial x^k}(\tau_x^j(y,\frac{\partial}{\partial y^i}))|_{y=x}$ is equivalently written $\frac{\partial^2}{\partial x^l \partial y^k}(\tau_y^j(x,\frac{\partial}{\partial x^i}))|_{y=x}$. Hence the following sum is computed

$$\frac{\partial^2}{\partial x^k \partial y^l}\tau_x^j(y,\frac{\partial}{\partial x^r})|_{y=x} + \frac{\partial^2}{\partial x^l \partial y^k}\tau_y^j(x,\frac{\partial}{\partial x^r})|_{y=x} = -[\Gamma^j_{li}(x)\Gamma^i_{kr}(x) + \Gamma^j_{ki}(x)\Gamma^i_{lr}(x)]. \quad (4.21)$$

To make this sum appear in (4.20), we use the symmetry of the symbols $\mathrm{I\!\Gamma}^j_{kli}(x) = \mathrm{I\!\Gamma}^j_{lki}(x)$. to write $\mathrm{I\!\Gamma}^j_{kli}(x) = \frac{1}{2}[\mathrm{I\!\Gamma}^j_{kli}(x) + \mathrm{I\!\Gamma}^j_{lki}(x)]$, and by (4.20), we obtain

$$\mathrm{I\!\Gamma}^j_{kli}(x) = \frac{1}{2}[\ \frac{\partial \Gamma^j_{lm}}{\partial x^k}(x) + \frac{\partial \Gamma^j_{km}}{\partial x^l}(x) - \frac{\partial^2}{\partial x^k \partial y^l}(\tau_x^j(y,\frac{\partial}{\partial y^m}))|_{y=x} - \frac{\partial^2}{\partial x^l \partial y^k}(\tau_x^j(y,\frac{\partial}{\partial y^m}))|_{y=x}\ ].$$

Now using (4.21), the result is proved.□

A consequence of the previous propositions is that the first and second partial derivatives of a linear strongly invertible transport τ only involve the Christoffel symbols and their partial derivatives. In other word, every linear and strongly invertible transport yields a connection $\tilde{p}$ only resulting from p.

Proposition 4.16 *For every linear and strongly invertible transport τ, $\tilde{p}$ is expressed as follows. For every $\mathbb{L}$ in $\tau_V TM$,*

$$\begin{aligned} \tilde{p}_V(\mathbb{L}) \ &= \ [\ d^2v^i(\mathbb{L}) \ + \ \Gamma^i_{ml}(x)v^l \ d^2x^m \ (\mathbb{L}) \\ &+ (\frac{\partial \Gamma^i_{lm}}{\partial x^k}(x) \ + \ \Gamma^i_{kj}(x)\Gamma^j_{lm}(x)) \ v^m \ dx^l.dx^k(\mathbb{L}) \ + \ 2\ \Gamma^i_{mk}(x)\ dx^m.dv^k(\mathbb{L})\ ]\frac{\partial}{\partial v^i} \\ &+ [dv^i.dv^j(\mathbb{L}) \ + \ \Gamma^i_{kl}(x)v^l\ \Gamma^j_{mr}(x)v^r\ dx^m.dx^k(\mathbb{L}) \ + \ 2\ \Gamma^i_{ml}(x)v^l\ dx^m.dv^j(\mathbb{L})\ ]\frac{\partial^2}{\partial v^i \partial v^j}. \end{aligned}$$

Proof : Use the results of propositions 4.13, 4.14 and 4.15. Since we do a sum, the term

$$\frac{1}{2}[\frac{\partial \Gamma^i_{lm}}{\partial x^k}(x) + \frac{\partial \Gamma^i_{km}}{\partial x^l}(x) + \Gamma^i_{kj}(x)\Gamma^j_{lm}(x) + \Gamma^i_{lj}(x)\Gamma^j_{km}(x)]v^m \ dx^l dx^k(\mathbb{L})$$

in proposition 4.15, becomes

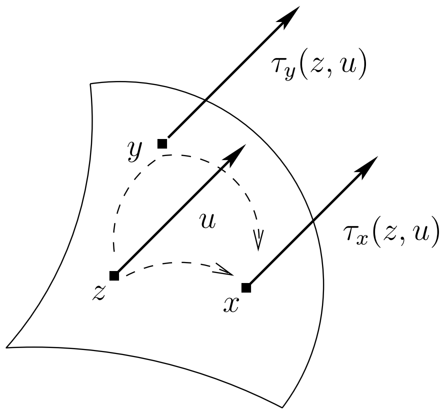
$$[\frac{\partial \Gamma^i_{lm}}{\partial x^k}(x) + \Gamma^i_{kj}(x)\Gamma^j_{lm}(x)]v^m \ dx^l.dx^k(\mathbb{L}).$$

Indeed, when we do the sum, we can write $\frac{\partial \Gamma^i_{lm}}{\partial x^k}(x)dx^l.dx^k(\mathbb{L}) = \frac{\partial \Gamma^i_{km}}{\partial x^l}(x)dx^k.dx^l(\mathbb{L})$ but $dx^k.dx^l(\mathbb{L}) = dx^l.dx^k(\mathbb{L})$ so that $\frac{1}{2}[\frac{\partial \Gamma^i_{lm}}{\partial x^k}(x) + \frac{\partial \Gamma^i_{km}}{\partial x^l}(x)]dx^l.dx^k(\mathbb{L}) = \frac{\partial \Gamma^i_{lm}}{\partial x^k}(x)dx^l.dx^k(\mathbb{L})$. The same goes for

$$\frac{1}{2}[\Gamma^i_{kj}(x)\Gamma^j_{lm}(x) + \Gamma^i_{lj}(x)\Gamma^j_{km}(x)]\ dx^l.dx^k(\mathbb{L}) = \Gamma^i_{kj}(x)\Gamma^j_{lm}(x)dx^l.dx^k(\mathbb{L}).\square$$

4.0.9 Curvature of a transport - Flat transport

Recall that the curvature of a connection is a way to measure the difference between a vector and its transport along an infinitesimal loop. We propose a notion of curvature for a basic linear transport τ. Given three points x, y, z on M, the curvature at point x of the transport τ represents the difference between a vector based at z transported to x, and the same vector transported to x by passing through y, that is from z to y and then from y to x.



Definition 4.8 *Let τ be a linear basic transport on M. We call curvature of τ the map r yielding, for every x in M, a map $r(x)$ defined as follows. For all y and z in M, $r(x)(y,z)$ is a linear map from T_xM to T_xM defined by*

$$\forall w \in T_xM : r(x)(y,z)(w) = \tau_x(y, \tau_y(z,w)) - \tau_x(z,w).$$

In coordinates, we write

$$r^i_j(x)(y,z) = \tau^i_x(y, \frac{\partial}{\partial y^k})\ \tau^k_y(z, \frac{\partial}{\partial z^j}) - \tau^i_x(z, \frac{\partial}{\partial z^j}), \tag{4.22}$$

with

$$dx^i r(x)(y,z)(\frac{\partial}{\partial x^j}) = r^i_j(x)(y,z).$$

Recall that the expression of the tensor of curvature R of the connection p deduced from τ is given by

$$R^i_{jml}(x) = dx^i R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^l}) = \frac{\partial \Gamma^i_{lj}}{\partial x^m}(x) - \frac{\partial \Gamma^i_{mj}}{\partial x^l}(x) + \Gamma^i_{mk}(x)\Gamma^k_{lj}(x) - \Gamma^i_{lk}(x)\Gamma^k_{mj}(x).$$

The following proposition ensures that the curvature for τ recovers the one for p.

Proposition 4.17 *Let τ be a transport on M, with curvature r. Then, the curvature R of the connection p, derived from τ, can be computed by*

$$R^i_{jml}(x) = [\ \frac{\partial}{\partial y^m}|_{y=x} \frac{\partial}{\partial z^l}|_{z=y}\ -\ \frac{\partial}{\partial y^l}|_{y=x} \frac{\partial}{\partial z^m}|_{z=y}\ ]\ (\ r^i_j(x)(y,z)\ ).$$

Proof : From (4.22) , we get :

$$\frac{\partial r^i_j}{\partial z^l}(x)(y,z) = \tau^i_x(y,\frac{\partial}{\partial y^k})\ \frac{\partial \tau^k_y}{\partial z^l}(z,\frac{\partial}{\partial z^j}) - \frac{\partial \tau^i_x}{\partial z^l}(z,\frac{\partial}{\partial z^j}).$$

Hence, for $z = y$, using (a) of proposition 4.12 for the expression of the Christoffel symbols of the connection p deduced from τ, we get

$$\frac{\partial r^i_j}{\partial z^l}(x)(y,z)|_{z=y} = \tau^i_x(y,\frac{\partial}{\partial y^k})\ \Gamma^k_{lj}(y) - \frac{\partial \tau^i_x}{\partial y^l}(y,\frac{\partial}{\partial y^j}).$$

Then, the derivative with respect to y gives

$$\frac{\partial}{\partial y^m}(\frac{\partial r^i_j}{\partial z^l}(x,y,z)|_{z=y}) = \frac{\partial \tau^i_x}{\partial y^m}(y,\frac{\partial}{\partial y^k})\ \Gamma^k_{lj}(y) + \tau^i_x(y,\frac{\partial}{\partial y^k})\ \frac{\partial \Gamma^k_{lj}}{\partial y^m}(y) - \frac{\partial^2 \tau^i_x}{\partial y^l \partial y^m}(y,\frac{\partial}{\partial y^j}).$$

Now, for $y = x$, using again (a) and also (d) of proposition 4.12, for the second derivative for τ , we get

$$\frac{\partial}{\partial y^m}|_{y=x}(\frac{\partial r^i_j}{\partial z^l}(x,y,z)|_{z=y}) = \Gamma^i_{mk}(x)\ \Gamma^k_{lj}(x) + \frac{\partial \Gamma^i_{lj}}{\partial y^i}(y) - \mathbb{\Gamma}^i_{lm}(x).$$

Recall that $\mathbb{\Gamma}^i_{lm}(x) = \mathbb{\Gamma}^i_{ml}(x)$. Thus, substracting the above term and the one obtained by changing m and l, the terms with $\mathbb{\Gamma}(x)$ disappear and we get exactly the expression for the curvature R.$\square$

Definition 4.9 *τ is called a flat transport if it is linear basic and such that*

$$\forall (x,y,z) \in M \times M \times M, \quad r(x)(y,z) = 0,$$

that is

$$\forall (x,y,z) \in M \times M \times M, \forall w \in T_zM : \quad \tau_x(y,\tau_y(z,w)) = \tau_x(z,w). \tag{4.23}$$

Remark 4.3 *Note that property* (4.23) *is different from the property yielding the strong invertibility in proposition* 4.11*, which was verified by the parallel transport along geodesics for example.* (4.23) *may be true along every smooth curve (x_t) on M, but not for all points. For instance, the parallel transport along a C^1 curve (x_t) verifies*

$$\tau^{//}_{ut}\tau^{//}_{su}(w) = \tau^{//}_{st}(w),$$

that is (4.23) *for all x_s, x_u, x_t points on the curve x_t. Then, the parallel transport along geodesics verifies $\tau^{//}_{yx}\tau^{//}_{zy}(w) = \tau^{//}_{zx}(w)$ if and only if y belongs to the geodesic from z to x, and not for every points x, y, z on M. As expected, the parallel transport is not flat in general.*

Note that, according to proposition 4.17, a flat transport τ yields a flat connection p, but the reverse can be false.

Example 4.9 *The map τ, defined in coordinates by $\tau_V(y,w) = w^i \frac{\partial}{\partial v^i}$ is a flat transport.*
In $\mathbb{R}^m$, this flat transport gives $\tau_V(W) = w$ since $\frac{\partial}{\partial v^i} = \frac{\partial}{\partial w^i}$.

Chapter 5

Stochastic calculus of order two with covariant jumps ($\overset{\triangle\mathcal{V}}{dI}$)

We use Cohen's results in [1], to define a stochastic integral with covariant jumps along a càdlàg semimartingale $(Y_t) = ((x_t, y_t))$. A transport τ on M allows to describe a covariant jump as follows. When (Y_t) jumps from Y_{t^-} to Y_t in TM, its corresponding covariant jump is a jump in $T_{x_{t^-}}M$, from y_{t^-} to $\tau_{Y_{t^-}}(Y_t)$. Therefore, the jump of the covariant integral of a $\overset{\triangle}{\tau^*}\, TM$-valued θ along (Y_t) is given by the function $\theta \circ \tau$, that is the pull-back of θ by τ.

Definition 5.1 *For every $\overset{\triangle}{\tau^*}\, TM$-valued form θ, the pull-back $\tau^*(\theta)$ of θ by τ is the $\overset{\triangle}{\tau^*}\, TM$-valued form defined by :*

$$\forall V \in TM, \forall W \in TM, \quad \tau^*(\theta)_V(W) = \theta_V(\tau_V(W))$$

Remark 5.1 *Since the transport τ_V is T_xM-valued, only the part of $\tau^*(\theta)$ acting on this space is involved. The vector $\tau_V(W) \in T_xM$ is seen as an element of TM (we could have written $\theta_V(x, \tau_V(W))$).*

Lemma 2 *Let $(Y_t) = ((x_t, y_t))$ be a càdlàg semimartingale in TM and (θ_t) a $\overset{\triangle}{\tau^*}\, TM$-valued predictable locally bounded process above (Y_{t^-}). Let τ be a transport on M.*
Then the process defined by

$$\tau^*(\theta)_t = \theta_t \circ \tau_{Y_{t^-}}$$

is a $\overset{\triangle}{\tau^*}\, TM$-valued predictable locally bounded process above (Y_{t^-}).

Proof : The process $(\tau^*(\theta)_t)$ is $\overset{\triangle}{\tau^*}\, TM$-valued since $\tau^*(\theta) \in \overset{\triangle}{\tau^*}\, TM$. Indeed, for all V in TM, $\tau^*(\theta)_V : TM \to \mathbb{R}$, it is twice differentiable at V and $\tau^*(\theta)_V(V) = \theta_{(x,v)}(x, \tau_V(V)) = \theta_V(V) = 0$
Moreover, it has the same properties (predictable and locally bounded above (Y_{t^-})) than (θ_t) since $\tau_{Y_{t^-}} : TM \to T_{x_t^-}M$ is a smooth map . □

Definition 5.2 *Let τ be a transport on M. For every càdlàg semimartingale $(Y_t) = ((x_t, y_t))$ in TM and every $\overset{\triangle}{\tau^*}$ TM-valued predictable locally bounded process (θ_t) above (Y_{t^-}), the stochastic integral with covariant jumps of (θ_t) along (Y_t) is defined by*

$$\int \theta \ \overset{\triangle\mathcal{V}}{d\!I} \ Y = \int \tau^*(\theta) \ \overset{\triangle}{d\!I} \ Y.$$

Using formula (56), one can write in an imbedding

$$\int \theta \ \overset{\triangle\mathcal{V}}{d\!I} \ Y = \int \frac{\partial \theta_s}{\partial v^i}(Y_{s^-}) \frac{\partial \tau^i_{Y_{s^-}}}{\partial Y^m}(Y_{s^-}) dY^m_s$$

$$+\frac{1}{2}\int [\frac{\partial \theta_s}{\partial v^i}(Y_{s^-}) \frac{\partial^2 \tau^i_{Y_{s^-}}}{\partial Y^m \partial Y^k}(Y_{s^-}) + \frac{\partial^2 \theta_s}{\partial v^i \partial v^j}(Y_{s^-}) \frac{\partial \tau^i_{Y_{s^-}}}{\partial Y^m}(Y_{s^-}) \frac{\partial \tau^j_{Y_{s^-}}}{\partial Y^k}(Y_{s^-})] d < (Y^m)^c, (Y^k)^c >_s$$

$$+ \sum_{0 \le s \le t} (\theta_s(\tau_{Y_{s^-}} Y_s) - \frac{\partial \theta_s}{\partial v^i}(Y_{s^-}) \frac{\partial \tau^i_{Y_{s^-}}}{\partial Y^m}(Y_{s^-}) \triangle Y^m_s).$$

For instance, if τ is the parallel transport along geodesics or any linear and strongly invertible transport on M, since the part without jumps is given by the 2-jet $\tilde{p}$, given in proposition 4.16, we get

$$\int \theta \ \overset{\triangle\mathcal{V}}{d\!I} \ Y = \int \frac{\partial \theta_s}{\partial v^i}(Y_{s^-})[\Gamma^i_{mj}(x_{s^-}) y^j_{s^-} \ dx^m_s \ + \ dy^i_s]$$

$$+ \ \frac{1}{2}\int [\ \frac{\partial \theta_s}{\partial v^i}(Y_{s^-})(\frac{\partial \Gamma^i_{mr}}{\partial x^k}(x_{s^-}) \ + \ \Gamma^i_{kl}(x_{s^-})\Gamma^l_{mr}(x_{s^-})) \ y^r_{s^-}$$

$$+ \ \frac{\partial^2 \theta_s}{\partial v^i \partial v^j}(Y_{s^-}) \Gamma^i_{mr}(x_{s^-}) \Gamma^j_{kl}(x_{s^-}) y^l_{s^-} y^r_{s^-} \ ] \ d < (x^m)^c, (x^k)^c >_s \tag{5.1}$$

$$+ \ \int [\ \frac{\partial \theta_s}{\partial v^i}(Y_{s^-}) \ + \ \frac{\partial^2 \theta_s}{\partial v^i \partial v^k}(Y_{s^-}) y^l_{s^-} \ ] \ \Gamma^i_{mk}(x_{s^-}) \ d < (x^m)^c, (y^k)^c >_s$$

$$+ \ \frac{1}{2}\int \frac{\partial^2 \theta_s}{\partial v^i \partial v^j}(Y_{s^-}) \ d < (y^i)^c, (y^j)^c >_s$$

$$+ \ \sum_{0 \le s \le t} (\theta_s(\tau^{//}_{x_s, x_{s^-}} y_s) - \frac{\partial \theta_s}{\partial v^i}(Y_{s^-}) \Gamma^i_{mk}(x_{s^-}) y^k_{s^-} \triangle x^m_s + \triangle y^i_s).$$

By the following proposition, our definition extends the covariant continuous calculus of order two, described in part I.

Proposition 5.1 *If (Y_t) is a continuous semimartingale on TM, then we recover*

$$\int \theta \ \overset{\triangle\mathcal{V}}{d\!I} \ Y = \int d^2\theta \ d\!I^{\mathcal{V}} Y,$$

where $d\!I^{\mathcal{V}}$ is defined with the connection $\tilde{p}$.

Proof : If (Y_t) is continuous, we get

$$\int \theta \, \overset{\triangle}{dI}^{\mathcal{V}} \, Y = \int \tau^*(\theta) \, \overset{\triangle}{dI} \, Y = \int d^2(\tau^*(\theta)) dI Y$$

We apply proposition $B.1$ in Appendix B, to $\phi = \tau_V$, and $f = \theta_V$, at point $W = V$. It gives

$$d^2(\theta_V \circ \tau_V)(V) = \tilde{p}_V^*(d^2\theta_V(V))$$

It follows

$$\int d^2(\tau^*(\theta)) dI Y = \int \tilde{p}^*(d^2\theta) dI Y$$

which is, by definition 3.3,

$$\int d^2\theta dI^{\mathcal{V}} Y.$$

Then we get the result. □

Chapter 6

Application : Stratonovich and Itô calculus with covariant jumps

6.1 A Stratonovich calculus with covariant jumps ($\overset{\triangle\mathcal{V}}{\partial}$)

We now apply the previous results to define a Stratonovich integral with covariant jumps. On the one hand, the covariant calculus is described by the notion of transport τ on M. On the other hand, the Stratonovich calculus with jumps in TM requires an interpolation rule on TM.

To get an integral with covariant jumps, we apply τ^* to the jump $\overset{\triangle}{\mathbb{I}}(\alpha)$.

A first definition
Let τ be a transport on M and $\mathbb{I}$ an interpolation rule on TM. For every càdlàg semimartingale $(Y_t) = ((x_t, y_t))$ on TM and every first order form α on TM, the Stratonovich integral with covariant jumps of α along (Y_t) is

$$\int \alpha_{Y_{t-}} \overset{\triangle\mathcal{V}}{\partial} Y_t = \int \overset{\triangle}{\mathbb{I}}(\alpha)_{Y_{t-}} \overset{\triangle\mathcal{V}}{dI} Y_t,$$

where the jump of the integral, when (Y_t) jumps from Y_{t-} to Y_t, is

$$\tau^*(\overset{\triangle}{\mathbb{I}}(\alpha))_{Y_{t-}}(Y_t) = \overset{\triangle}{\mathbb{I}}(\alpha)_{Y_{t-}}(\tau_{Y_{t-}}(Y_t)) = \int_0^1 \alpha_{\mathbb{I}(s,(x_{t-},y_{t-}),(x_{t-},\tau_{Y_{t-}}(Y_t)))} \dot{\mathbb{I}}(s,(x_{t-},y_{t-}),(x_{t-},\tau_{Y_{t-}}(Y_t)))ds \tag{6.1}$$

It is important to notice that this definition only involves the part of $\mathbb{I}$ in the fibre and not the one on M. The map $\mathbb{I}(s,(x,v),(x,u))$ can be seen as a map $(x, \mathbb{I}(s,v,u))$, with $\mathbb{I}(s,v,u) \in T_xM$ (we write it with an abuse of notation since it also depends on x and y.)

Then, it would be natural to consider for $\mathbb{I}(s,v,w)$ the linear interpolation rule

$$\mathbb{I}(s,v,w) = s(w-v)+v$$

In this case, we write

$$\mathbb{I}(s,(x,v),(x,w)) = (x, s(w-v)+v)$$

and then we get

$$\dot{\mathbb{I}}(s,(x,v),(x,w)) = (0, w-v)$$

where $w-v$ is seen as an element of $T_v(T_xM)$.

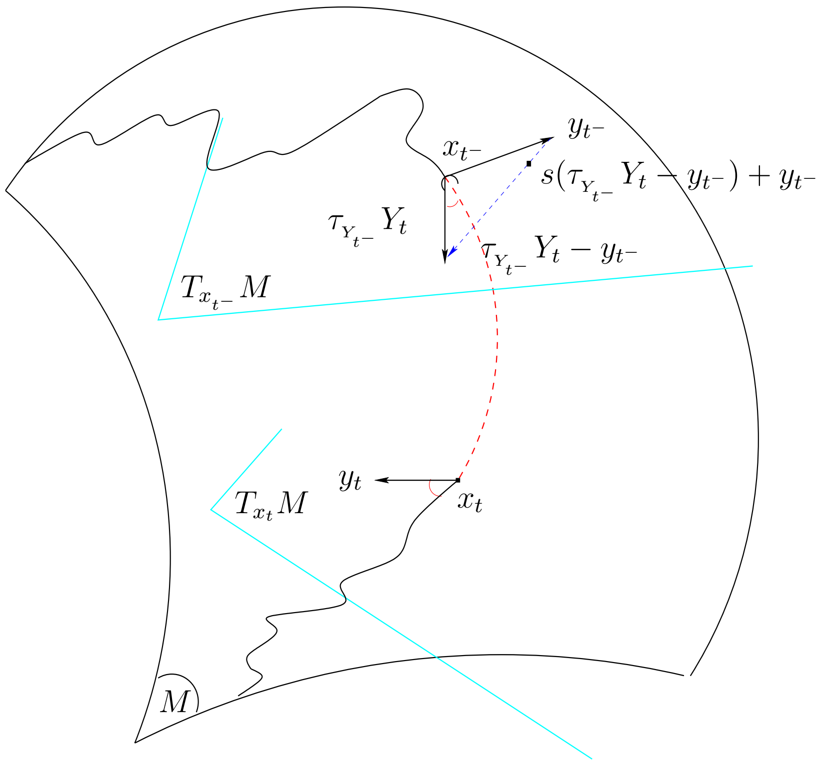
For this reason, the definition of the Stratonovich integral with covariant jumps doesn't really involves an interpolation rule. Then we will keep the following definition.

Definition 6.1 *Let τ be a transport on M. For every càdlàg semimartingale $(Y_t) = ((x_t, y_t))$ on TM and every first order form α on TM, the Stratonovich integral with covariant jumps of α along (Y_t) is*

$$\int \alpha_{Y_{t-}} \overset{\triangle\nu}{\partial} Y_t = \int \overset{\triangle}{\mathbb{I}}(\alpha)_{Y_{t-}} \overset{\triangle\nu}{dI} Y_t,$$

where the jump of the integral, when (Y_t) jumps from Y_{t-} to Y_t, is

$$\tau^*(\overset{\triangle}{\mathbb{I}}(\alpha))_{Y_{t-}}(Y_t) = \int_0^1 \alpha_{(x_{t-}, s[\tau_{Y_{t-}}(Y_t) - y_{t-}] + y_{t-})} ds \; (0, \tau_{Y_{t-}}(Y_t) - y_{t-}). \tag{6.2}$$



Proposition 6.1 *If (Y_t) is a continuous semimartingale on TM, the Stratonovich integral with covariant jumps gives*

$$\int \alpha \overset{\triangle\mathcal{V}}{\partial} Y = \int d_s\alpha \; d\!I^{\mathcal{V}}Y,$$

where $d\!I^{\mathcal{V}}$ is defined with the connection $\tilde{p}$.

Proof : Recall that the continuous case is obtained by computing the 2-jet of the jump of the integral on the diagonal. This jump is given by

$$\tau^*(\overset{\triangle}{\mathbb{I}}(\alpha))_V(W) = \overset{\triangle}{\mathbb{I}}(\alpha)_V(\tau_V(W)).$$

Using proposition B.1 in Appendix B, we get, for all $V=(x,v)$ and W in TM,

$$d^2(\tau^*(\overset{\triangle}{\mathbb{I}}(\alpha))_V)(W) = d^2(\overset{\triangle}{\mathbb{I}}(\alpha))_V(\tau_V(W)) \circ d^2\tau_V(W).$$

For $V=W$, since $\tau_V(V)=v$ and, by proposition 1.1, we have

$$d^2(\tau^*(\overset{\triangle}{\mathbb{I}}(\alpha))_V(V)) = (d_s\alpha)_V(\tilde{p}_V) = \tilde{p}^*(d_s\alpha)_V.$$

Hence we get an integral which is by definition $\int d_s\alpha \; d\!I^{\mathcal{V}}Y$, with the connection $\tilde{p}$.
Note that, if we consider the first definition, proposition 1.2 implies that the integral obtained in the continuous case doesn't depend on the choice of $\mathbb{I}$.□

6.2 An Itô calculus with covariant jumps ($\overset{\triangle\mathcal{V}}{d}$)

We now apply the previous results to define an Itô integral with covariant jumps. On the one hand, the covariant calculus is described by the notion of transport τ on M. On the other hand, the Itô calculus with jumps in TM requires a connection rule on TM.

To get an integral with covariant jumps, we apply τ^* to the jump $\overset{\triangle}{\boldsymbol{\gamma}}(\alpha)$.

A first definition
Let τ be a transport on M and $\boldsymbol{\gamma}$ a connection rule on TM. For every càdlàg semimartingale $(Y_t)=((x_t,y_t))$ on TM and every first order form α on TM, the Itô integral with covariant jumps of α along (Y_t) is

$$\int \alpha_{Y_{t-}} \overset{\triangle\mathcal{V}}{d} Y_t = \int \overset{\triangle}{\boldsymbol{\gamma}}(\alpha)_{Y_{t-}} \overset{\triangle\mathcal{V}}{d\!I} Y_t,$$

where the jump of the integral, when (Y_t) jumps from Y_{t-} to Y_t, is

$$\tau^*(\overset{\triangle}{\boldsymbol{\gamma}}(\alpha))_{Y_{t-}}(Y_t) = \overset{\triangle}{\boldsymbol{\gamma}}(\alpha)_{Y_{t-}}(\tau_{Y_{t-}}(Y_t)) = \alpha_{Y_{t-}} \circ \boldsymbol{\gamma}((x_{t-},y_{t-}),(x_{t-},\tau_{Y_{t-}}(Y_t))). \tag{6.3}$$

It is important to notice that this definition only involves the part of $\boldsymbol{\gamma}$ in the fibre and not the one on M. The map $\boldsymbol{\gamma}((x,v)(x,u))$ can be seen as a map $(0,\boldsymbol{\gamma}(v,u))$, with $\boldsymbol{\gamma}(v,u) \in T_xM$ (we write it with an abuse of notation since it also depends on x and y.)

Then, it would be natural to consider for $\boldsymbol{\gamma}(v,w)$ the linear connection rule

$$\boldsymbol{\gamma}(v,w) = w - v.$$

In this case, we write

$$\boldsymbol{\gamma}((x,v),(x,w)) = (0, w-v)$$

where $w-v$ is seen as an element of $T_v(T_xM)$.

For this reason, the definition of the Itô integral with covariant jumps doesn't really involves a connection rule. We recover the fact that it is enough to have one connection for the Itô covariant calculus, as in Norris's definition for the continuous case. One connection is p, the other one is a flat connection on the vertical space, derived from the linear connection rule :

$$\forall \alpha, \quad \mathbb{G}(\alpha)_Y = d^2\boldsymbol{\gamma}(Y,.)(Y)^*(d^2(\alpha_Y)) = \alpha_Y,$$

Then we will keep the following definition.

Definition 6.2 *Let τ be a transport on M. For every càdlàg semimartingale $(Y_t) = ((x_t, y_t))$ on TM and every first order form α on TM, the Itô integral with covariant jumps of α along (Y_t) is*

$$\int \alpha_{Y_{t-}} \overset{\triangle\mathcal{V}}{d} Y_t = \int \overset{\triangle}{\boldsymbol{\gamma}} (\alpha)_{Y_{t-}} \overset{\triangle\mathcal{V}}{dI} Y_t,$$

where the jump of the integral, when (Y_t) jumps from Y_{t^-} to Y_t, is

$$\tau^*(\overset{\triangle}{\boldsymbol{\gamma}} (\alpha))_{Y_{t^-}}(Y_t) = \alpha_{Y_{t^-}}(0, \tau_{Y_{t^-}}(Y_t) - y_{t^-}). \tag{6.4}$$

Proposition 6.2 *If (Y_t) is a continuous semimartingale on TM, the covariant Itô integral gives*

$$\int \alpha \overset{\triangle\mathcal{V}}{d} Y = \int \alpha dI^{\mathcal{V}} Y,$$

where $dI^{\mathcal{V}}$ is defined with the connection $\tilde{p}$.

Remark 6.1 *Recall that, according to remark* 3.1, *the connection of order two $\tilde{p}$, seen as a second order vector, has a first order part. Then $\tilde{p}^*(\alpha) = \alpha \circ \tilde{p}$ is the application of α to this first order part.*

Proof : Recall that the continuous case is obtained by computing the 2-jet of the jump of the integral, on the diagonal. This jump is given by

$$\tau^*(\overset{\triangle}{\boldsymbol{\gamma}} (\alpha))_V(W) = \overset{\triangle}{\boldsymbol{\gamma}} (\alpha)_V(\tau_V(W)).$$

Using proposition $B.1$ in Appendix B, we get, for all $V=(x,v)$ and W in TM,

$$d^2(\tau^*(\overset{\triangle}{\gamma}(\alpha))_V)(W) == d^2\overset{\triangle}{\gamma}(\alpha)_V(\tau_V(W))\circ d^2\tau_V(W).$$

Since $\gamma(V,W)=(0,w-v)$, we get $d^2\gamma(V,.)(W)=dw^2$. Then, for $V=W$, since $\tau_V(V)=v$ and, by proposition 2.2, we have

$$d^2\tau^*(\overset{\triangle}{\gamma}(\alpha))_V(V)=\tilde{p}_V=\tilde{p}^*(\mathbb{G}(\alpha))_V$$

Hence we get an integral which is by definition $\int \alpha d\mathcal{I}^{\mathcal{V}}Y$. □

An analogue to theorem 1 can now be given for the Itô integral, which is just a reformulation of the above proposition.

Theorem 2 *If $\mathbb{G}$ is a flat connection, for every connection p of order one on M, there exists a connection of order two, $\tilde{p}$,such that*

- *its restriction to first order vectors is p,*
- *the Itô covariant integral along a continuous semimartingale can be expressed as a two order covariant integral :*

$$\int \alpha d^{\mathcal{V}}Y=\int \tilde{p}^*(\alpha)d\mathcal{I}Y.$$

  In particular, we get an analogous formula than in definition 2.2

$$\int \alpha d^{\mathcal{V}}Y=\int \alpha d\mathcal{I}^{\mathcal{V}}Y,$$

  where $d\mathcal{I}^{\mathcal{V}}$ is defined with the connection $\tilde{p}$.

Proof : The map $\tilde{p}$ is, by definition, a connection of order two, with restriction p to first order vectors. The previous proposition gives

$$\int \alpha d^{\mathcal{V}}Y=\int \tilde{p}^*(\alpha)d\mathcal{I}Y.$$

Moreover, writing definition 2.2 with a flat $\mathbb{G}$ gives

$$\int \alpha d^{\mathcal{V}}Y=\int \alpha d\mathcal{I}^{\mathcal{V}}Y.\square$$

Note on the identification of the vertical space for the jumps

Recall that, when the vertical space $T_v(T_xM)$ is identified with T_xM, we integrate forms on M instead of forms on TM. We could express such an identification on the jumps, and then we should integrate a function θ on M instead of one on TM, as it is usual for a covariant integration. Such an identification would correspond for the jumps to a local diffeomorphism

$$\overset{\triangle}{\mathcal{E}}_V : T_xM \to M,$$

identifying T_xM with a neighbourhood of x in M, and such that its differential at V recovers an identification

$$d\overset{\triangle}{\mathcal{E}}_V(V) : T_v(T_xM) \to T_xM$$

at the order one, and its 2-jet yields an identification

$$d^2\overset{\triangle}{\mathcal{E}}_V(V) : \tau_v(T_xM) \to \tau_xM$$

at the order two.

The fundamental example for $\overset{\triangle}{\mathcal{E}}$ is given by an exponential map :

$$\forall V = (x, v) \in TM, \forall w \in T_xM, \overset{\triangle}{\mathcal{E}}_V(w) = \exp_x(w - v).$$

Indeed, according to Appendix C, it satisfies

$$d^2\overset{\triangle}{\mathcal{E}}_V(v) = d^2\exp_x(0) = Id_{\tau_v(T_xM),\tau_xM}$$

which contains also the identification

$$d\overset{\triangle}{\mathcal{E}}_V(v) = d\exp_x(0) = i_x(0) = Id_{T_v(T_xM),T_xM}$$

We understand that the identifications we can do for the jump and for the order two depend on a connection. In particular, $d\overset{\triangle}{\mathcal{E}}_V(V)$ also depends on a connection, so it is not exactly the canonical identification i_V described in the reminders.

Part III

Comparison of the two stochastic calculus : covariant with jumps or with covariant jumps

Introduction

In this part, we want to compare the two different approachs of parts I and II.

To define a Stratonovich and an Itô covariant integral, we integrated a first order form α along a càdlàg semimartingale (Y_t). We had to explain, on the one hand, how we transform α into a function on M describing the jump of the integral and, on the other hand, how we use the connection to pass to the covariant case. Actually, the two definitions correspond to the two possible orders for these steps :

- In part I, we have defined a **covariant stochastic calculus with jumps** (in the particular cases of Stratonovich and Itô calculus), using a connection p on M.
  We started with the continuous covariant calculus of Stratonovich and Itô given by the transfer principle and we derived the case with jumps from it. More precisely, we first understood that the covariant continuous case consists in integrating the form $p^*\alpha$ instead of α, then we defined the vertical jump of the integral as the image of $p^*\alpha$ by a smooth map $\overset{\triangle}{\mathbb{J}}: T^*TM \to \overset{\triangle}{\tau^*} TM$ ($\overset{\triangle}{\mathbb{J}} = \overset{\triangle}{\mathbb{I}}$ for Stratonovich, $\overset{\triangle}{\boldsymbol{\gamma}}$ for Itô).

- In part II, we have defined a **stochastic calculus with covariant jumps**, using a transport τ on M.
  We started with the stochastic calculus of order two with jumps described by Cohen. Then we composed the jump with a transport to get a stochastic covariant integral of order two with jumps. More precisely, we first defined the jump of the integral as the image of α by a smooth map $\overset{\triangle}{\mathbb{J}}: T^*TM \to \overset{\triangle}{\tau^*} TM$ ($\overset{\triangle}{\mathbb{J}} = \overset{\triangle}{\mathbb{I}}$ or $\overset{\triangle}{\boldsymbol{\gamma}}$), then we applied the pull-back τ^* of a transport τ.

Link between a connection and a transport

The main difference between these two approachs results in the difference between the given tools for the covariant calculus : the first definition uses a connection whereas a transport is required for the second definition. They can be linked as follows.

Proposition 6.3 *1. Every transport τ on M defines a connection p of order one on M by*

$p_V = d\tau_V(V)$. This connection yields a parallel transport $\tau^{//}$ along geodesics, but we should not necessarily have the equality $\tau_V(W) = \tau^{//}_{yx}(w)$.

$$\tau_V \quad \longrightarrow \quad d\tau_V(V) = p_V \quad \longrightarrow \quad \tau^{//}_x \neq \tau_V$$

2. *Every connection q of order one on M defines a parallel transport $\tau^{//}$ along geodesics. Then we recover q_V by $d\tau^{//}_x(V)$.*

$$q_V \quad \longrightarrow \quad \tau^{//}_x \quad \longrightarrow \quad d\tau^{//}_x(V) = q_V.$$

Proof : The first point comes from the definition of τ. Note that several transports give the same differential $p_V = d\tau_V(V)$ at V, so there is no unicity.
The second point has been proved when we computed the Christoffel symbols for the example 4.1 in part II.□

Thus, in general, there is not a real equivalence between a connection and a transport, except when we consider the parallel transport along geodesics.

We will also compare the continuous cases obtained from these integrals with jumps. For the Stratonovich integral, it consists in a comparison between the connections of order two $d_s p$ and $\tilde{p}$. $d_s p$ depends on p and its derivatives, whereas $\tilde{p}$ can depend on the second partial derivatives of the transport (see proposition 4.13). Recall that, when the transport is assumed to be linear and strongly invertible, $\tilde{p}$ can be deduced from p (see proposition 4.16). This is the case for the parallel transport along geodesics.

According to the foregoing, we will do the comparison under the following assumption :

Assumption (h_1)

- The transport τ of part II is the parallel transport along geodesics :

$$\forall V = (x, v) \in TM, W = (y, w) \in TM : \quad \tau_V(W) = \tau^{//}_{yx}(w).$$

- The connection p of part I is given by the differential of this transport :

$$p_V = d\tau_V(V).$$

The interpolation and connection rules

Recall that the covariant jumps of the Stratonovich and the Itô integrals of part II only involve the parts of the interpolation and connection rules in the tangent spaces, so we used linear interpolation and connection rules. Therefore, to compare these jumps to those of part I, we have to suppose that the part of the interpolation and connection rules of part I in the tangent spaces are linear too. For instance, we can consider the fundamental examples of the geodesic interpolation and connection rules.

Since the two definitions correspond to the order of composition of the covariant map (p or τ) and the jumping map $\overset{\triangle}{\mathbb{J}}$, their equality corresponds to a commutativity of these maps.

$$\begin{array}{ccccc} \alpha_V \ \in \ T^*_V TM & \xrightarrow{\overset{\triangle}{\mathbb{J}}} & & \overset{\triangle}{\mathbb{J}}(\alpha)_V \ \in \ \overset{\triangle}{\tau^*}_V TM & \\ \downarrow p^* & & & \downarrow \tau^* & \\ p^*(\alpha)_V \ \in \ T^*_V TM & \xrightarrow{\overset{\triangle}{\mathbb{J}}} & \overset{\triangle}{\mathbb{J}}(p^*(\alpha))_V & = \ \tau^*(\overset{\triangle}{\mathbb{J}}(\alpha))_V & \in \ \overset{\triangle}{\tau^*}_V TM \end{array} \tag{6.5}$$

For the Stratonovich integral, the equality of the jumps is written

$$\overset{\triangle}{\mathbb{I}}(p^*(\alpha)) \ = \ (\tau^{//})^*(\overset{\triangle}{\mathbb{I}}(\alpha)), \tag{6.6}$$

that is

$$\int_0^1 p^*(\alpha)_{\mathbb{I}(s,V,W)}\dot{\mathbb{I}}(s,V,W)ds \ = \int_0^1 \alpha_{\mathbb{I}(s,V,(x,\tau_V^{//}W))}\dot{\mathbb{I}}(s,V,(x,\tau_V^{//}W))ds \tag{6.7}$$

For the Itô integral, it is written

$$\overset{\triangle}{\boldsymbol{\gamma}}(p^*(\alpha)) \ = \ (\tau^{//})^*(\overset{\triangle}{\boldsymbol{\gamma}}(\alpha)), \tag{6.8}$$

that is

$$p^*(\alpha)\boldsymbol{\gamma}(V,W) \ = \ \alpha\boldsymbol{\gamma}(V,(x,\tau_V^{//}W)) \tag{6.9}$$

The given tools are the connection, represented by p or $\tau^{//}$, and on the interpolation and connection rules $\mathbb{I}$ and $\boldsymbol{\gamma}$. Let us write a commutativity on these given tools, to get the one for the jumps.

Definition 6.3 *(Commutativity between $\tau^{//}$ and $\mathbb{I}$)*
Let $\mathbb{I}$ be an interpolation rule on TM. We say that it commutes with $\tau^{//}$ if

$$\forall V=(x,v)\in TM, \forall W=(y,w)\in TM, \quad \tau^{//}_{\pi\mathbb{I}(s),x}(\mathbb{I}(s,v,w)) \ = \ \mathbb{I}(s,V,(x,\tau^{//}_{yx}(w))). \tag{6.10}$$

Example 6.1 *The fundamental example is the geodesic interpolation rule on TM (see examples* 0.5 *and* 0.2*).*

Indeed, recall that it is given by

$$\mathbb{I}(s,V,W) = (\ \exp_x(s\exp_x^{-1}y) \ , \ \tau^{//}_{0s}[s(\tau^{//}_{yx}(w)-v)+v] \ ),$$

where $(\tau^{//}_{x_{t-},\pi\mathbb{I}(s,Y_{t-},Y_t)}) = (\tau^{//}_{0s})$ is the parallel transport along the geodesic

$$(\pi\mathbb{I}(s,Y_{t^-},Y_t)) = (\exp_{x_{t^-}}(s\exp^{-1}_{x_{t^-}}(x_t)).$$

Then we have
$$\mathbb{I}(s,v,\tau^{//}_{10}(w)) = s(\tau^{//}_{10}(w)-v)+v = \tau^{//}_{s0}\mathbb{I}(s,v,w)).$$
Since $\tau^{//}$ is an invertible transport, it is equivalent to
$$\mathbb{I}(s,v,w) = \tau^{//}_{0s}\mathbb{I}(s,v,\tau^{//}_{10}(w)),$$
which means that $\mathbb{I}$ commutes with $\tau^{//}$.

Definition 6.4 *(Commutativity between $\tau^{//}$ and $\boldsymbol{\gamma}$)*
Let $\boldsymbol{\gamma}$ be a connection rule on TM. We say that it commutes with $\tau^{//}$ if :
$$\forall V=(x,v)\in TM, \forall W=(y,w)\in TM, \quad p_V(\boldsymbol{\gamma}(V,W)) = \boldsymbol{\gamma}(V,(x,\tau^{//}_{yx}(w))). \tag{6.11}$$

Example 6.2 *The fundamental example is the geodesic connection rule on TM (see examples* 0.8 *and* 0.2*).*

Indeed, recall that it is given by
$$\boldsymbol{\gamma}(V,W) = \dot{\mathbb{I}}(0,V,W),$$
where $\mathbb{I}$ is the geodesic interpolation rule on TM. By the foregoing, we have the relation
$$\tau^{//}_{\pi I(s),x}(\mathbb{I}(s,V,W)) = \mathbb{I}(s,v,\tau^{//}_{yx}(w))$$
Taking derivatives with respect to the time s, on both sides, yields
$$d\tau^{//}_{.x}(.)(\mathbb{I}(s,V,W))\dot{\mathbb{I}}(s,V,W) = \dot{\mathbb{I}}(s,v,\tau^{//}_{yx}(w))$$
At time $s=0$, it gives
$$d\tau^{//}_{.x}(.)(V)\dot{\mathbb{I}}(0,V,W) = \dot{\mathbb{I}}(0,v,\tau^{//}_{yx}(w)),$$
or equivalently
$$p_V \ \boldsymbol{\gamma}(V,W) = \boldsymbol{\gamma}(v,\tau^{//}_{yx}(w))),$$
which means that $\boldsymbol{\gamma}$ commutes with $\tau^{//}$.

According to the foregoing, we will do the comparison under the following assumption :

Assumption (h_2)

- The interpolation rules $\mathbb{I}$ in part I and II are the geodesic interpolation rule in TM :
$$\mathbb{I}(s,V,W) = (\ \exp_x(s\exp_x^{-1}y)\ ,\ \tau^{//}_{0s}[s(\tau^{//}_{10}(w)-v)+v]\ ).$$
- The connection rules $\boldsymbol{\gamma}$ in part I and II are the geodesic connection rule in TM :
$$\boldsymbol{\gamma}(V,W) = \dot{\mathbb{I}}(0,V,W).$$

We will see that the covariant Itô integrals of part I and II equal and that, on the contrary, there is a real difference of nature between the covariant Stratonovich integrals of part I and II.

Chapter 7

Comparison of the two Stratonovich covariant calculus

7.1 Comparison of the jumps

To compare the two definitions for the Stratonovich covariant integrals along a càdlàg semimartingale (Y_t) on TM, it is enough to compare the jumps of the integrals.

In part I, we have defined a **covariant Stratonovich integral with jumps** $\int \alpha \, \overset{\triangle}{\partial^{\mathcal{V}}} Y$. Under assumptions (h_1) and (h_2), its jump, when (Y_t) jumps from Y_{t^-} to Y_t, is given by

$$\overset{\triangle}{\mathbb{I}}\,(p^*\alpha)_{Y_{t^-}}(Y_t) = \int_0^1 \alpha_{\mathbb{I}(s,Y_{t^-},Y_t)}(0, \tau^{//}_{x_{t^-},\pi\mathbb{I}(s)}(\tau^{//}_{x_t,x_{t^-}} y_t - y_{t^-}))\, ds \tag{7.1}$$

(see example 1.1).

In part II, we have defined a **Stratonovich integral with covariant jumps** $\int \alpha \, \overset{\triangle\mathcal{V}}{\partial} Y$. Under assumptions (h_1) and (h_2), its jump, when (Y_t) jumps from Y_{t^-} to Y_t, is given by

$$(\tau^{//})^*(\overset{\triangle}{\mathbb{I}}\,(\alpha))_{Y_{t^-}}(Y_t) = \int_0^1 \alpha_{(x_{t^-},\mathbb{I}(s,Y_{t^-},(x_{t^-},\tau^{//}_{x_t,x_{t^-}} y_t)))}\, ds\ (0, \tau^{//}_{x_t,x_{t^-}} y_t - y_{t^-}). \tag{7.2}$$

Note that, since $\mathbb{I}$ and $\tau^{//}$ commutes, the second jump is equivalently written

$$(\tau^{//})^*(\overset{\triangle}{\mathbb{I}}\,(\alpha))_{Y_{t^-}}(Y_t) = \int_0^1 \alpha_{(x_{t^-},\tau^{//}_{\pi\mathbb{I}(s),x_{t^-}}\mathbb{I}(s,y_{t^-},y_t))}\, ds\ (0, \tau^{//}_{x_t,x_{t^-}} y_t - y_{t^-}). \tag{7.3}$$

In the first definition, the integration of α along $\mathbb{I}$ simultaneously uses the part of $\mathbb{I}$ on M and the one in the fibres $T_{\pi\mathbb{I}(s)}M$. In the second definition, (7.3) shows that the transport $\tau^{//}$ brings all at $\pi\mathbb{I}(0, Y_{t^-}, Y_t) = x_{t^-}$. Hence, α is integrated only along the part of $\mathbb{I}$ in the tangent space $T_{x_{t^-}}M$. If we change the part of $\mathbb{I}$ on M, the second definition won't change but the first one will.

As a consequence, the two Stratonovich integrals differ in general.

They can equal only if $\forall s, \pi\mathbb{I}(s) = \pi\mathbb{I}(0) = x_{t^-}$. It means that $x_t = x_{t^-}$. We will see in the last chapter of this part that it corresponds to a vertical jump for (Y_t).

Remark 7.1 *Comparing* (7.1) *with* (7.3)*, we can get the equality of the two jumps if the form* α *satisfies the particular property :*

$$\int_0^1 \alpha_{\mathbb{I}(s,Y_{t^-},Y_t)} \tau^{//}_{x_{t^-},\pi\mathbb{I}(s)} ds = \int_0^1 \alpha_{(x_{t^-},\tau^{//}_{\pi\mathbb{I}(s),x_{t^-}} \mathbb{I}(s,y_{t^-},y_t))} ds.$$

7.2 Comparison of the continuous Stratonovich integrals

Recall that, if the semimartingale (Y_t) is supposed to be continuous, we obtain, for the first integral,

$$\int \alpha \overset{\triangle}{\partial^{\mathcal{V}}} Y = \int (d_s p)^*(d_s\alpha) d\mathbb{I}Y$$

(see proposition 1.3 with definition 1.5), and, for the second one,

$$\int \alpha \overset{\triangle_{\mathcal{V}}}{\partial} Y = \int \tilde{p}^*(d_s\alpha) d\mathbb{I}Y$$

(see proposition 6.1).

Then the comparison consists in a comparison between the connections of order two $d_s p$ and $\tilde{p}$. Recall that assumption (h_1) allows to deduce $\tilde{p}$ from p.

On the one hand, $d_s p$ is obtained by considering the differential $d\tau_V(W)$ at point $W = V$ and then applying an analogue of the differentiation d_s to the application $V \to p_V = d\tau_V(V)$, which depends on V in two ways.
On the other hand, $\tilde{p}_V$ is obtained by computing the 2-jet $d^2\tau_V(W)$ of the application τ_V, with V fixed, and then taking it at point $W = V$.
Consequently, $d_s p$ is different from $\tilde{p}_V$, so that, in general, the covariant Stratonovich integrals of part I and II keep on being different in the continuous case.

We will see that the definition of part I can recover Norris's definition. With regard to the definition of part II, it doesn't recover Norris's one, and moreover, it is not a Stratonovich integral. Indeed, according to [3], a two order integral $\int \theta d\mathbb{I}Y$ is a Stratonovich integral if, by applying the application d_s to the first order part $R\theta$ of θ, we recover θ, that is, if we can write

$$\int \theta d\mathbb{I}Y = \int d_s(R\theta) d\mathbb{I}Y.$$

Therefore, the integral $\int \tilde{p}^*(d_s\alpha) d\mathbb{I}Y$ is not a Stratonovich integral, since $R(\tilde{p}^*(d_s\alpha)) = p^*(\alpha)$ but $d_s(p^*(\alpha))$ is $(d_s p)^*(d_s\alpha)$, and not $\tilde{p}^*(d_s\alpha)$.

7.2.1 Comparison with Norris formalism

The definition of part I recovers Norris's definition. Indeed, by the transfer principle, the Stratonovich integral obeys the same differential rules as the deterministic derivative with t. Then the equivalence between the first definition of Stratonovich covariant integral and Norris's definition is just an analogue of proposition 0.3.

Proposition 7.1 *When the semimartingale* (Y_t) *is continuous, the Stratonovich covariant integral with jumps of part I gives*

$$\int \alpha_{Y_{t-}} \overset{\triangle}{\partial^{\mathcal{V}}} Y_t = \int \alpha_{Y_t}(0, u_t\partial(u_t^{-1}y_t)).$$

With the identification of the vertical space, we recover Norris's definition

$$\int \alpha_{Y_{t-}} \overset{\triangle}{\partial^{\mathcal{V}}} Y_t = \int \alpha_{x_t} u_t\partial(u_t^{-1}y_t).$$

Proof : The proof is the same as the proof of proposition 0.3 in the C^1 case, replacing $\frac{\partial}{\partial t}$ by ∂ and using paragraph 0.3.2 for the identification of the vertical space. □

7.2.2 Comparison of covariant accelerations

In [4], Emery explains that the transfer principle for Stratonovich integrals consists, at the order two, in an analogy between the acceleration $\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t$ of a C^2 curve (Y_t) and the stochastic second order infinitesimal variation $d\!\!IY_t$ for a semimartingale (Y_t).

Let $(Y_t) = ((x_t, y_t))$ a curve of class C^2 in TM. The acceleration of Y_t is defined in [3]) by

$$\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t = d^2Y(\frac{\partial^2}{\partial t^2}), \tag{7.4}$$

given in coordinates by

$$\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t = \ddot{Y}_t^i\frac{\partial}{\partial V^i} + \dot{Y}_t^i\dot{Y}_t^j\frac{\partial^2}{\partial V^i\partial V^j}. \tag{7.5}$$

Using the coordinates on $T_{Y_t}TM$, we write also

$$\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t = \ddot{x}_t^i\frac{\partial}{\partial x^i} + \ddot{y}_t^i\frac{\partial}{\partial v^i} + \dot{x}_t^i\dot{x}_t^j\frac{\partial^2}{\partial x^i\partial x^j} + 2\dot{x}_t^i\dot{y}_t^j\frac{\partial^2}{\partial x^i\partial v^j} + \dot{y}_t^i\dot{y}_t^j\frac{\partial^2}{\partial v^i\partial v^j}. \tag{7.6}$$

Remark 7.2 *The notation* $\frac{\mathbf{d^2}}{\mathbf{dt^2}}$ *expresses that the second derivative with respect to the time is not the usual one, acting on coordinates. Formula* (7.5) *can be understood as follows*

$$\begin{aligned}\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t = \frac{\mathbf{d}}{\mathbf{dt}}[\frac{\partial Y_t^i}{\partial t}\frac{\partial}{\partial V^i}] &= \frac{\partial}{\partial t}(\frac{\partial Y_t^i}{\partial t})\frac{\partial}{\partial V^i} + \frac{\partial Y_t^i}{\partial t}\frac{\mathbf{d}}{\mathbf{dt}}[(\frac{\partial}{\partial V^i})_{Y_t}]\\ &= \frac{\partial^2 Y_t^i}{\partial t^2}\frac{\partial}{\partial V^i} + \frac{\partial Y_t^i}{\partial t}\frac{\partial Y_t^j}{\partial t}\frac{\partial^2}{\partial V^i\partial V^j}.\end{aligned}$$

A way of understanding the difference between the connections of order two $d_s p$ and $\tilde{p}$ is to compare the "covariant accelerations" they yield. Let us specify what we call a covariant acceleration.

Definition 7.1 *Let $\bar{p}$ be a connection of order two on M and (Y_t) be a C^2 curve on TM. We call*

$$\bar{p}_{Y_t}(\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t)$$

the covariant acceleration of (Y_t), given by the connection $\bar{p}$.

Remark 7.3 *We could write a covariant Stratonovich transfer principle, in the spirit of [4], consisting in an analogy between the integration of a covariant acceleration and the definition* 3.3 *for the covariant calculus of order two.*

Proposition 7.2 *Let $(Y_t) = ((x_t, y_t))$ be a C^2 curve on TM, and $\tau^{//}$ the parallel transport along the projection curve (x_t) of (Y_t) on M.*
Then, the covariant acceleration given by $d_s p$ is

$$d_s p(\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t) = \frac{\mathbf{d}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}), \tag{7.7}$$

also written

$$d_s p(\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t) = \frac{\mathbf{d}}{\mathbf{dt}}[\frac{\partial}{\partial s}(\tau^{//}_{st} y_s)|_{s=t}]. \tag{7.8}$$

The covariant acceleration given by $\tilde{p}$ is

$$\tilde{p}(\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t) = \frac{\mathbf{d}^{\mathcal{V}}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}), \tag{7.9}$$

also written

$$\tilde{p}(\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t) = [\frac{\mathbf{d^2}}{\mathbf{ds^2}}(\tau^{//}_{st} y_s)]|_{s=t}. \tag{7.10}$$

Proof :

- Let us explicit the term $\frac{\mathbf{d}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t})$ as

$$\frac{\mathbf{d}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}) = \frac{\mathbf{d}}{\mathbf{dt}}[(\frac{\partial^{\mathcal{V}} Y_t}{\partial t})^i \frac{\partial}{\partial v^i}] = \frac{\partial}{\partial t}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t})^i \frac{\partial}{\partial v^i} + (\frac{\partial^{\mathcal{V}} Y_t}{\partial t})^i \frac{\mathbf{d}}{\mathbf{dt}}[(\frac{\partial}{\partial v^i})_{y_t}].$$

  It follows

$$\frac{\mathbf{d}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}) == \frac{\partial}{\partial t}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t})^i \frac{\partial}{\partial v^i} + (\frac{\partial^{\mathcal{V}} Y_t}{\partial t})^i \dot{y}^j_t \frac{\partial^2}{\partial v^i \partial v^j},$$

  or equivalently, using the expression (54) for the covariant derivative,

$$\frac{\mathbf{d}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}) = \frac{\partial}{\partial t}[\frac{\partial}{\partial s}(\tau^{//}_{st} y_s)^i|_{s=t}]\frac{\partial}{\partial v^i} + \frac{\partial}{\partial s}(\tau^{//}_{st} y_s)^i|_{s=t}\dot{y}^j_t \frac{\partial^2}{\partial v^i \partial v^j},$$

which is the meaning of formula (7.8). Let us write it in coordinates, to compare it with the expression for $d_{s}p(\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t)$. Using the expression (53) for the covariant derivative with Christoffel symbols, it gives

$$\frac{\mathbf{d}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}) = \frac{\partial}{\partial t}\ (\ \dot{y}_t^i\ +\ \Gamma^i_{mj}(x_t)\ \dot{x}_t^m\ y_t^j)\ \frac{\partial}{\partial v^i}\ +\ (\ \dot{y}_t^i\ +\ \Gamma^i_{ml}(x_t)\ y_t^m\ \dot{x}_t^l)\ \dot{y}_t^j\ \frac{\partial^2}{\partial v^i \partial v^j}.$$

Then we get

$$\frac{\mathbf{d}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}) = [\ \ddot{y}_t^i\ +\ \frac{\partial \Gamma^i_{mj}}{\partial x^k}(x_t)\ y_t^j\ \dot{x}_t^m\ \dot{x}_t^k\ +\ \Gamma^i_{mj}(x_t)\ \ddot{x}_t^m\ y_t^j\ +\ \Gamma^i_{mk}(x_t)\ \dot{x}_t^m\ \dot{y}_t^k\ ]\ \frac{\partial}{\partial v^i}$$

$$+\ [\ \dot{y}_t^i\ \dot{y}_t^j\ +\ \Gamma^i_{ml}(x_t)\ y_t^m\ \dot{x}_t^l\ \dot{y}_t^j\ ]\ \frac{\partial^2}{\partial v^i \partial v^j}.$$

Comparing this expression with $d_{sp}(\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t)$, where $\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t$ is given in coordinates by (7.6) and d_{sp} by (3.6), we get the first point of the proposition.

- We give meaning to $\frac{\mathbf{d}^{\mathcal{V}}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t})$, using the expression (55) for the covariant derivative, as follows

$$\forall s,\ \ \frac{\mathbf{d}^{\mathcal{V}}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}) = \tau^{//}_{st}\frac{\mathbf{d}}{\mathbf{dt}}(\tau^{//}_{ts}\frac{\partial^{\mathcal{V}} Y_t}{\partial t}),$$

where $\tau^{//}$ is the parallel transport along (x_t), which is also the projection part of $\frac{\partial^{\mathcal{V}} Y_t}{\partial t}$ on M. Using (55) again, we get

$$\forall s,\ \ \frac{\mathbf{d}^{\mathcal{V}}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}) = \tau^{//}_{st}\frac{\mathbf{d}}{\mathbf{dt}}(\tau^{//}_{ts}\tau^{//}_{st}\frac{\partial}{\partial t}(\tau^{//}_{ts}y_t)) = \tau^{//}_{st}\frac{\mathbf{d^2}}{\mathbf{dt^2}}(\tau^{//}_{ts}y_t).$$

It follows

$$\frac{\mathbf{d}^{\mathcal{V}}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}) = (\frac{\mathbf{d^2}}{\mathbf{ds^2}}\tau^{//}_{st}y_s)_{|s=t},$$

that is

$$\frac{\mathbf{d}^{\mathcal{V}}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}) = (\frac{\partial^2\ \tau^{//}_{st}y_s}{\partial s^2})^i_{|s=t}\frac{\partial}{\partial v^i} + (\frac{\partial\ \tau^{//}_{st}y_s}{\partial s})^i_{|s=t}\ (\frac{\partial\ \tau^{//}_{st}y_s}{\partial s})^j_{|s=t}\frac{\partial^2}{\partial v^i \partial v^j}. \tag{7.11}$$

Let us write it in coordinates, to compare it with the expression for $\tilde{p}(\frac{\mathbf{d^2}}{\mathbf{dt^2}}Y_t)$. We have

$$(\frac{\partial\ \tau^{//}_{st}y_s}{\partial s})^i_{|s=t} = \frac{\partial^{\mathcal{V}} Y^i_t}{\partial t} =\ \dot{y}_t^i + \Gamma^i_{mk}(x_t)\ \dot{x}_t^m\ y_t^k. \tag{7.12}$$

It follows

$$(\frac{\partial\ \tau^{//}_{st}y_s}{\partial s})^i_{|s=t}\ (\frac{\partial\ \tau^{//}_{st}y_s}{\partial s})^j_{|s=t} = (\dot{y}_t^i + \Gamma^i_{mk}(x_t)\ \dot{x}_t^m\ y_t^k)(\dot{y}_t^j + \Gamma^j_{lr}(x_t)\ \dot{x}_t^l\ y_t^r)$$

$$= \dot{y}_t^i\ \dot{y}_t^j + \Gamma^i_{mk}(x_t)\ y_t^k\ \dot{x}_t^m\dot{y}_t^j + \Gamma^j_{lr}(x_t)\ y_t^r\ \dot{x}_t^l\ \dot{y}_t^i + \Gamma^i_{mk}(x_t)\ \Gamma^j_{lr}(x_t)\ y_t^k\ y_t^r\ \dot{x}_t^m\ \dot{x}_t^l.$$

Thus, the second term in (7.11) yields

$$(\frac{\partial\ \tau^{//}_{st}\, y_s}{\partial s})^i_{|s=t}\ (\frac{\partial\ \tau^{//}_{st}\, y_s}{\partial s})^j_{|s=t}\ \frac{\partial^2}{\partial v^i \partial v^j} = (\dot{y}^i_t\ \dot{y}^j_t + 2\ \Gamma^i_{mk}(x_t)\ y^k_t\ \dot{x}^m_t \dot{y}^j_t + \Gamma^i_{mk}(x_t)\ \Gamma^j_{lr}(x_t)\ y^k_t\ y^r_t\ \dot{x}^m_t\ \dot{x}^l_t)\frac{\partial^2}{\partial v^i \partial v^j}. \tag{7.13}$$

The first term in (7.11) involves the second derivative $\frac{\mathbf{d^2}}{\mathbf{ds^2}}(\tau^{//}_{st}\, y_s)^i$. It gives

$$\frac{\mathbf{d^2}}{\mathbf{dt^2}}(\tau^{//}_{st}\, y_s)^i = \frac{\partial}{\partial s}(\ \frac{\partial (\tau^{//}_{st})^i_k}{\partial s} y^k_s + (\tau^{//}_{st})^i_k \dot{y}^k_s\ ).$$

$\frac{\partial (\tau^{//}_{st})^i_k}{\partial s}$ is given by (51). Then, we get

$$\frac{\mathbf{d^2}}{\mathbf{dt^2}}(\tau^{//}_{st}\, y_s)^i = \frac{\partial}{\partial s}((\tau^{//}_{st})^i_r \Gamma^r_{lk}(x_s)\ \dot{x}^l_s y^k_s + (\tau^{//}_{st})^i_k \dot{y}^k_s)$$

$$= \frac{\partial (\tau^{//}_{st})^i_r}{\partial s}\Gamma^r_{lk}(x_s)\ \dot{x}^l_s y^k_s + (\tau^{//}_{st})^i_r[\ \frac{\partial \Gamma^r_{lk}}{\partial x^m}(x_s)\ \dot{x}^m_s \dot{x}^l_s y^k_s + \Gamma^r_{lk}(x_s)\ \ddot{x}^l_s y^k_s + \Gamma^r_{lk}(x_s)\ \dot{x}^l_s \dot{y}^k_s\ ] + \frac{\partial (\tau^{//}_{st})^i_k}{\partial s}\dot{y}^k_s + (\tau^{//}_{st})^i_k \ddot{y}^k_s$$

$$= \frac{\partial (\tau^{//}_{st})^i_r}{\partial s}[\ \Gamma^r_{lk}(x_s)\ \dot{x}^l_s y^k_s + \dot{y}^r_s\ ] + (\tau^{//}_{st})^i_r[\ \frac{\partial \Gamma^r_{lk}}{\partial x^m}(x_s)\ \dot{x}^m_s \dot{x}^l_s y^k_s + \Gamma^r_{lk}(x_s)\ \ddot{x}^l_s y^k_s + \Gamma^r_{lk}(x_s)\ \dot{x}^l_s \dot{y}^k_s\ + \ddot{y}^r_s]. \tag{7.14}$$

Using again (51), we deduce

$$\frac{\partial (\tau^{//}_{st})^i_r}{\partial s}[\ \Gamma^r_{lk}(x_s)\ \dot{x}^l_s y^k_s + \dot{y}^r_s\ ] = (\tau^{//}_{st})^i_n \Gamma^n_{rj}(x_s)\ \dot{x}^j_s[\ \Gamma^r_{lk}(x_s)\ \dot{x}^l_s y^k_s + \dot{y}^r_s\ ]. \tag{7.15}$$

Recall that, for $s = t$, we have $(\tau^{//}_{st})^i_n = \delta^i_n$. Then, by (7.14) and (7.15), we get

$$\frac{\mathbf{d^2}}{\mathbf{dt^2}}(\tau^{//}_{st}\, y_s)^i_{|s=t} = \Gamma^i_{rj}(x_t)\ \dot{x}^j_t[\ \Gamma^r_{lk}(x_t)\ \dot{x}^l_t y^k_t + \dot{y}^r_t\ ] + \frac{\partial \Gamma^i_{lk}}{\partial x^m}(x_t)\ \dot{x}^m_t \dot{x}^l_t y^k_t + \Gamma^i_{lk}(x_t)\ \ddot{x}^l_t y^k_t + \Gamma^i_{lk}(x_t)\ \dot{x}^l_t \dot{y}^k_t + \ddot{y}^r_t. \tag{7.16}$$

It follows

$$\frac{\mathbf{d}^{\mathcal{V}}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}) = \ddot{y}^i_t\ +\ \Gamma^i_{ml}(x_t) y^l_t\ \ddot{x}^m_t\ +\ [\frac{\partial \Gamma^i_{lm}}{\partial x^k}(x_t) + \Gamma^i_{kj}(x_t)\Gamma^j_{lm}(x_t)] y^m_t\ \dot{x}^l_t \dot{x}^k_t$$

$$+\ 2\ \Gamma^i_{mk}(x_t)\ \dot{x}^m_t \dot{y}^k_t\ ]\frac{\partial}{\partial v^i} +\ [\dot{y}^i_t \dot{y}^j_t\ +\ \Gamma^i_{kl}(x_t) y^l_t\ \Gamma^j_{mr}(x_t) y^r_t\ \dot{x}^m_t \dot{x}^k_t\ +\ 2\ \Gamma^i_{ml}(x_t) y^l_t\ \dot{x}^m_t \dot{y}^j_t\ ]\frac{\partial^2}{\partial v^i \partial v^j}.$$

Comparing this expression with $\tilde{p}(\frac{\mathbf{d^2}}{\mathbf{dt^2}} Y_t)$, where $\frac{\mathbf{d^2}}{\mathbf{dt^2}} Y_t$ is given in coordinates by (7.6) and $\tilde{p}$ by proposition 4.16, we get the second point of the proposition.□

Remark 7.4 *Note the link between the expression*

$$\tilde{p}(\frac{\mathbf{d^2}}{\mathbf{dt^2}} Y_t) = \frac{\mathbf{d}^{\mathcal{V}}}{\mathbf{dt}}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t})$$

and the horizontal lift of p, for which we described the geodesics in proposition 0.8. *We used for this, a second covariant derivative*

$$\frac{\partial^{\mathcal{V}}}{\partial t}(x_t, \frac{\partial^{\mathcal{V}} Y_t}{\partial t})$$

which has to be understood in coordinates, that is

$$\frac{\partial^{\mathcal{V}}}{\partial t}(x_t, \frac{\partial^{\mathcal{V}} Y_t}{\partial t}) = \frac{\partial^{\mathcal{V}}}{\partial t}[\frac{\partial^{\mathcal{V}} Y_t}{\partial t}^i \frac{\partial}{\partial v^i}] = [\frac{\partial}{\partial t}(\frac{\partial^{\mathcal{V}} Y_t}{\partial t}^i) + \Gamma^i_{jk}\dot{x}^j_t \frac{\partial^{\mathcal{V}} Y_t}{\partial t}^k] \frac{\partial}{\partial v^i}.$$

Actually, it corresponds to the first order part of $\tilde{p}(\frac{\mathbf{d^2}}{\mathbf{dt^2}} Y_t)$ *(recall that, according to remark* 3.1, *a connection of order two has a first order part which doesn't depend on the choice of the coordinates on* M*).*

Chapter 8

Comparison of the two Itô covariant calculus

8.1 Comparison of the jumps

To compare the two definitions given for the Itô covariant integral along a càdlàg semimartingale (Y_t) on TM, it is enough to compare the jumps of the integrals.

In part I, we have defined a **covariant Itô integral with jumps** $\int \alpha \, \overset{\triangle}{d^{\mathcal{V}}} Y$.
Its jump when (Y_t) jumps from Y_{t^-} to Y_t is given by

$$\overset{\triangle}{\boldsymbol{\gamma}} (p^*\alpha)_{Y_{t^-}}(Y_t) = \alpha_{y_{t^-}} p_{Y_{t^-}}(\boldsymbol{\gamma}(Y_{t^-}, Y_t)). \tag{8.1}$$

(see formula (2.5))

In fact, under assumptions (h_1) and (h_2), it yields

$$\overset{\triangle}{\boldsymbol{\gamma}} (p^*\alpha)_{Y_{t^-}}(Y_t) = \alpha_{Y_{t^-}}(0, \tau^{//}_{x_t, x_{t^-}}(y_t) - y_{t^-}) \tag{8.2}$$

(see example 2.1).

In part II, we have defined an **Itô integral with covariant jumps** $\int \alpha \, \overset{\triangle \mathcal{V}}{d} Y$.
Under assumptions (h_1) and (h_2), its jump when (Y_t) jumps from Y_{t^-} to Y_t is given by

$$(\tau^{//})^*(\overset{\triangle}{\boldsymbol{\gamma}} (\alpha))_{Y_{t^-}}(Y_t) = \alpha_{Y_{t^-}}(0, \tau^{//}_{x_t, x_{t^-}}(y_t) - y_{t^-}). \tag{8.3}$$

Writing the equality of the two jumps (8.1) and (8.3) is exactly writing the commutativity between $\tau^{//}$ and $\boldsymbol{\gamma}$. Under assumptions (h_1) and (h_2), for which we have this commutativity, we see that formulas (8.3) and (8.2) are the same :

$$\overset{\triangle}{\boldsymbol{\gamma}} (p^*\alpha)_{Y_{t^-}}(Y_t) = (\tau^{//})^*(\overset{\triangle}{\boldsymbol{\gamma}} (\alpha))_{Y_{t^-}}(Y_t) = \alpha_{Y_t}(0, \tau^{//}_{x_t, x_{t^-}} y_t - y_{t^-}). \tag{8.4}$$

Then, the Itô covariant integral with jumps of part I coincides with the Itô integral with covariant jumps of part II.

Remark 8.1 *Another way to find the jump of the Itô covariant integral* $\int \alpha \overset{\triangle}{d^{\mathcal{V}}} Y$*, following Norris's presentation with the horizontal lift, and Cohen's description in [1] for the jump of the horizontal lift, would be to write it as*

$$\alpha_{Y_{t^-}}(0, u_{t^-}\triangle(u_t^{-1}y_t)) = \alpha_{Y_{t^-}}(0, u_{t^-}\ (u_t^{-1}y_t - u_{t^-}^{-1}y_{t^-})) = \alpha_{Y_{t^-}}(0, (u_{t^-}u_t^{-1}y_t - y_{t^-})) = \alpha_{Y_{t^-}}(0, \tau^{//}_{x_t, x_t^-}y_t - y_{t^-}),$$

which is exactly formula (8.4).

8.2 Comparison of the continuous Itô integrals

Recall that, if the semimartingale (Y_t) is supposed to be continuous, we obtain, for the first integral,

$$\int \alpha \overset{\triangle}{d^{\mathcal{V}}} Y = \int \mathbb{G}(p^*\alpha) d\!\!I Y$$

(see proposition 2.4 with definition 2.4), and, for the second one,

$$\int \alpha \overset{\triangle_{\mathcal{V}}}{d} Y = \int \tilde{p}^*(\alpha) d\!\!I Y$$

(see proposition 6.2).

Under assumptions (h_1) and (h_2), since the case with jumps yields the equality of the two integrals, they also equal in the continuous case. Let us understand the consequences of assumptions (h_1) and (h_2) in the continuous case :

- Assumption (h_1) has consequences on the Itô integral of part II. By (h_1), the transport considered in part II is the parallel transport along geodesics, what allows to deduce the connection $\tilde{p}$ from p. It is given in coordinates by proposition 4.16. Moreover, by remark 7.4, the first order part of $\tilde{p}$ is related to the horizontal lift of p. Note that, according to prop 4.16, for the continuous case, (h_1) could be replaced by
  (h_1') : The transport τ of part II is linear and strongly invertible.
- Assumption (h_2) has consequences on the Itô integral of part I. By (h_2), we get that the connection $\mathbb{G}$ is the 2-jet of the geodesic connection rule, that is
  $$\mathbb{G}(\alpha)_V = \alpha_V d^2\boldsymbol{\gamma}(V,.)(V) = \alpha_V(d^2\exp_x^{-1}(x), d^2\boldsymbol{\gamma}(v,.)(v))$$
  where $\boldsymbol{\gamma}(v,w) = w - v$. We have $d^2\boldsymbol{\gamma}(v,.)(v) = d^2 v = Id_{T_v(T_xM)}$ and $d^2\exp_x^{-1}(x) = Id_{T_xM}$. Then
  $$\mathbb{G}(\alpha)_V = \alpha_V$$
  Hence, for the continuous case, (h_2) could be replaced by
  (h_2') : the connection rule $\boldsymbol{\gamma}$ of part I and II yields a flat connection $\mathbb{G}$.

Therefore, in both definitions, only one connection p is used, the one describing the covariant calculus, whereas the connection $\mathbb{G}$ required for the Itô calculus is a flat connection.

8.2.1 Comparison with Norris formalism

Just as our definitions, the definition 0.14 given by Norris only uses one connection for the covariant and the Itô calculus : the connection p defines the horizontal lift (u_t). Then, by (u_t), the semimartingale is brought in $\mathbb{R}^m$, where the Itô calculus is applied, so the second connection is the flat connection on $\mathbb{R}^m$.

Proposition 8.1 *Under assumption (h_2'), when (Y_t) is continuous, the Itô integral with covariant jumps of part II gives*

$$\int \alpha_{Y_{t-}} \overset{\triangle\mathcal{V}}{d} Y_t = \int \alpha_{Y_t}(0, u_t d(u_t^{-1} y_t)).$$

With the identification of the vertical space, we recover Norris's definition

$$\int \alpha_{Y_{t-}} \overset{\triangle\mathcal{V}}{d} Y_t = \int \alpha_{x_t} u_t d(u_t^{-1} y_t).$$

Moreover, under assumption (h_1'), the Itô covariant integral of part I also recovers Norris's definition.

Proof : On the one hand, for (Y_t) continuous, we have

$$\int \alpha_{Y_{t-}} \overset{\triangle\mathcal{V}}{d} Y_t = \int \tilde{p}^*_{Y_t}(\alpha) d\!I Y_t$$

If τ is the parallel transport along geodesics, $\tilde{p}$ is expressed in proposition 4.16. It yields

$$\begin{aligned}\int \tilde{p}^*_{Y_t}(\alpha) d\!I Y_t \;=\; & \int (\alpha_{Y_t})_i \,[\, dy_t^i + \Gamma^i_{ml}(x_t) y_t^l \, dx_t^m + \frac{1}{2}(\frac{\partial \Gamma^i_{lm}}{\partial x^k}(x_t) + \Gamma^i_{kj}(x_t)\Gamma^j_{lm}(x_t))\, y_t^m \, d < x^l, x^k >_t \\ & + \Gamma^i_{mk}(x_t)\; d < x^m . y^k >_t \;\;],\end{aligned} \tag{8.5}$$

where $\alpha_i = \alpha(0, \frac{\partial}{\partial v^i})$

On the other hand, by using the Stratonovich to Itô formula in $\mathbb{R}^m$, we have

$$\int \alpha_{Y_t}(0, u_t d(u_t^{-1} y_t)) = \int \alpha_{Y_t}(0, u_t \partial(u_t^{-1} y_t)) - \frac{1}{2} d < \alpha_Y u, u^{-1} y >_t \tag{8.6}$$

Recall that

$$\int \alpha_{Y_t}(0, u_t \partial(u_t^{-1} y_t) = \int \alpha_{Y_t} \partial^{\mathcal{V}} Y_t = \int (\alpha_{Y_t})_i (\partial y_t^i + \Gamma^i_{jk}(x_t) y_t^k \partial x_t^j),$$

that is, using the Stratonovich to Itô conversion formula in $\mathbb{R}$,

$$\int \alpha_{Y_t}(0, u_t \partial(u_t^{-1} y_t)) = \int (\alpha_{Y_t})_i dy_t^i + \frac{1}{2} \frac{\partial \alpha_i}{\partial y^j}(Y_t) d < y^i, y^j >_t \tag{8.7}$$

$$+(\alpha_{Y_t})_i\Gamma^i_{jk}(x_t)y^k_t dx^j_t + \frac{1}{2}\frac{\partial\alpha_i}{\partial y^j}(Y_t)\Gamma^i_{jk}(x_t)y^k_t d < y^i, x^j >_t$$

$$+\frac{1}{2}(\alpha_{Y_t})_i(\frac{\partial\Gamma^i_{jk}}{\partial x^l}(x_t)y^k_t d < x^l, x^j >_t +\Gamma^i_{jk}(x_t)d < y^k, x^j >_t)$$

Now, let us study the quadratic variation term of (8.6). We have

$$d(\alpha_{Y_t}u_t) = d\alpha_{Y_t}\ u_t + \alpha_{Y_t}du_t + \frac{1}{2}d < \alpha_Y, u >_t .$$

It implies

$$d < \alpha_Y u, u^{-1}y >_t = d < h, u^{-1}y >_t + + \alpha_{Y_t}d < u, u^{-1}y >_t,$$

where $dh_t = d\alpha_{Y_t}u_t$.
On the one hand, since $\partial u_t = \Gamma(x_t)\partial x_t u_t$ and $\partial(u_t^{-1}y_t) = u_t^{-1}\partial^{\mathcal{V}}Y_t$, we get

$$\alpha_{Y_t}d < u, u^{-1}y >_t = \alpha_{Y_t}\Gamma(x_t)dx_t.d^{\mathcal{V}}Y_t = (\alpha_{Y_t})_i\Gamma^i_{jk}(x_t)(d < x^k, y^j >_t +\Gamma^j_{mr}(x_t)y^m_t d < x^k, x^r >_t).$$

On the other hand, we have

$$d < h, u^{-1}y >_t = d\alpha_{Y_t}.d^{\mathcal{V}}Y_t = \frac{\partial\alpha_i}{\partial y^j}(Y_t)(d < y^i, y^j >_t +\Gamma^i_{jk}(x_t)y^k_t d < y^i, x^j >_t).$$

Then we get

$$-\frac{1}{2}d < \alpha_Y u, u^{-1}y >_t = -\frac{1}{2}\frac{\partial\alpha_i}{\partial y^j}(Y_t)(d < y^i, y^j >_t -\Gamma^i_{jk}(x_t)y^k_t d < y^i, x^j >_t) \tag{8.8}$$

$$-\frac{1}{2}(\alpha_{Y_t})_i\Gamma^i_{jk}(x_t)(d < x^k, y^j >_t +\Gamma^j_{mr}(x_t)y^m_t d < x^k, x^r >_t).$$

When we do the sum of (8.7) with (8.8), the terms with the partial derivatives for α_i disappear and then we get that (8.6) is exactly (8.5). Hence we get the result. When doing the identification of the vertical space, in the meaning of paragraph 0.3.2, we can recover Norris's definition. Moreover, since the Itô covariant integral with jumps equals the Itô integral with covariant jumps under assumption (h_1'), it also recovers Norris's definition. □

Chapter 9

Equality for vertical or horizontal semimartingales

We have introduced two covariant stochastic integrals along a semimartingale $(Y_t) = ((x_t, y_t))$. To call them "covariant", we expect of them that they correspond to the classical stochastic integration along (y_t) when (Y_t) is vertical, and that they vanish when (Y_t) is horizontal. Let us specify here these notions for a C^1 curve, and then for a semimartingale.

9.1 Vertical semimartingales on TM

Let (Y_t) be a C^1 curve on TM and p a connection on M. We say that (Y_t) is a vertical curve if , for every t, $\frac{\partial Y_t}{\partial t}$ belongs to the vertical space $T_{y_t}(T_{x_t}M)$ at Y_t.

This means that

$$\frac{\partial Y_t}{\partial t} = (0, \frac{\partial y_t}{\partial t}) \in T_{y_t}(T_{x_t}M)$$

The part $(\frac{\partial x_t}{\partial t})$ of $(\frac{\partial Y_t}{\partial t})$ vanishes, so the curve (x_t), projection of (Y_t) on M, is constant :

$$\forall t,\ x_t = x_0.$$

Note that the notion of vertical curve doesn't depend on the connection p. Nevertheless, if (Y_t) is vertical, we have

$$p_{Y_t}(\frac{\partial Y_t}{\partial t}) = \frac{\partial y_t}{\partial t}. \tag{9.1}$$

We extend naturally the notion for semimartingales in TM.
A càdlàg semimartingale $(Y_t) = ((x_t, y_t))$ on TM is said to be vertical if :

$$\forall t,\ x_t = x_0$$

9.2 Horizontal semimartingales on TM

Let (Y_t) be a C^1 curve on TM and p a connection on M. We say that (Y_t) is an horizontal curve if , for every t, $\frac{\partial Y_t}{\partial t}$ belongs to the horizontal space at Y_t, that is if

$$p_{Y_t}(\frac{\partial Y_t}{\partial t}) = 0. \tag{9.2}$$

Note that the notion of horizontal curve depends on the connection p.

To define a càdlàg horizontal semimartingale (Y_t) in TM, we write that it is solution of a covariant s.d.e. with jumps. This formalism is defined in part IV.

In this part, we will say that $(Y_t) = ((x_t, y_t))$ is horizontal if it satisfies

$$y_t = \tau^{//}_{x_{t^-},x_t} y_{t^-}, \text{ if } x_t \neq x_{t^-}$$

and

$$\partial^{\mathcal{V}} Y_t = 0 \text{ if } x_t = x_{t^-}.$$

9.3 Covariant integrals

Let us verify if, as expected, the covariant integration of a vertical semimartingale (Y_t) is the integration of (y_t) and the covariant integration of an horizontal semimartingale is null, as well as for the covariant integrals with jumps of part I and for the integrals with covariant jumps of part II.

Proposition 9.1 *If $(Y_t) = ((x_t, y_t))$ is a vertical càdlàg semimartingale on TM (or if M is a vector space with a flat connection p), we get*

$$\int \alpha \overset{\triangle\mathcal{V}}{\partial} Y = \int \alpha \overset{\triangle}{\partial^{\mathcal{V}}} Y = \int \alpha \overset{\triangle}{\partial} y,$$

$$\int \alpha \overset{\triangle\mathcal{V}}{d} Y = \int \alpha \overset{\triangle}{d^{\mathcal{V}}} Y = \int \alpha \overset{\triangle}{d} y.$$

Proof : If (Y_t) is a vertical semimartingale, then its part (x_t) on M, is a constant semimartingale. In particular we have $x_t = x_{t^-}$ for every jump of Y_t.

- With regard to the Stratonovich integral of part I, $x_t = x_{t^-}$ implies $\pi\mathbb{I}(s, Y_{t^-}, Y_t) = x_{t^-}$ for all s. It follows that the curve $(\mathbb{I}(s, Y_{t^-}, Y_t))$ is vertical, so that, by (9.1), its covariant derivative satisfies

  $$p_{\mathbb{I}(s,Y_{t^-},Y_t)}(\frac{\partial \mathbb{I}}{\partial s}(s, Y_{t^-}, Y_t)) = \frac{\partial \mathbb{I}}{\partial s}(s, y_{t^-}, y_t).$$

  Consequently

  $$\int \alpha \overset{\triangle}{\partial^{\mathcal{V}}} Y = \int \alpha \overset{\triangle}{\partial} y.$$

- With regard to the Itô integral of part I, $x_t = x_{t^-}$ implies $\pi\boldsymbol{\gamma}(Y_{t^-}, Y_t) = 0$. Then $\boldsymbol{\gamma}(Y_{t^-}, Y_t) = (0, \boldsymbol{\gamma}(y_{t^-}, y_t))$ is vertical and so we get

$$p_{Y_{t-}}(\boldsymbol{\gamma}(Y_{t^-}, Y_t)) = \boldsymbol{\gamma}(y_{t^-}, y_t).$$

It follows

$$\overset{\triangle}{\boldsymbol{\gamma}}(p^*\alpha)_{Y_{t-}}(Y_t) = \overset{\triangle}{\boldsymbol{\gamma}}(\alpha)_{y_{t-}}(y_t).$$

Consequently

$$\int \alpha \, \overset{\triangle}{d^{\mathcal{V}}} \, Y = \int \alpha \, \overset{\triangle}{\partial} \, y.$$

- With regard to the integrals of part II, $x_t = x_{t^-}$ implies

$$\tau_{Y_{t-}}(Y_t) = y_{t^-}.$$

Then, for all θ in τ^*TM,

$$(\tau^{//})^*(\theta)_{Y_{t-}}(Y_t) = \theta_{y_{t-}}(y_t).$$

and

$$\int \theta \, \overset{\triangle\mathcal{V}}{d\!I} \, Y = \int \theta \, \overset{\triangle}{d\!I} \, y.$$

Applying this to $\theta = \overset{\triangle}{\mathbb{I}}(\alpha)$ and $\theta = \overset{\triangle}{\boldsymbol{\gamma}}(\alpha)$, we get :

$$\int \alpha \, \overset{\triangle\mathcal{V}}{\partial} \, Y = \int \alpha \, \overset{\triangle}{\partial} \, y,$$

$$\int \alpha \, \overset{\triangle\mathcal{V}}{d} \, Y = \int \alpha \, \overset{\triangle}{d} \, y. \square$$

Proposition 9.2 *Let p be a connection on M. Under assumptions (h_1) and (h_2), if $(Y_t) = ((x_t, y_t))$ is an horizontal semimartingale on TM, all the covariant integrals vanish*

$$\int \alpha \, \overset{\triangle\mathcal{V}}{\partial} \, Y = \int \alpha \, \overset{\triangle}{\partial^{\mathcal{V}}} \, Y = \int \alpha \, \overset{\triangle\mathcal{V}}{d} \, Y = \int \alpha \, \overset{\triangle}{d^{\mathcal{V}}} \, Y = 0.$$

Proof : If (Y_t) is horizontal, we have $\tau_{Y_{t-}}(Y_t) = y_{t^-}$. The term $\tau_{Y_{t-}}(Y_t) - y_{t^-}$ is in factor in all the expressions of the integrals under the assumptions (h_1) and (h_2) (see (7.1), (7.3), (8.2) and (8.3)). Since it vanishes, the jumps and so the integrals vanish too. $\square$

As a consequence, when we integrate the particular classes of vertical and horizontal semimartingales, the covariant integrals of parts I and II equal.

Conclusion

The covariant stochastic calculus of part I and II both permit to recover the covariant calculus for a C^1 curve in TM. Moreover, they both recovers the vectorial integration when integrating a vertical semimartingale, and both vanish when integrating an horizontal semimartingale. The two calculus can recover the stochastic continuous covariant calculus of order two, involving a connection of order two, $d_s p$ for Stratonovich calculus, $\tilde{p}$ for Itô calculus. This extends in a way the Schwartz principle : like the classical stochastic calculus, the covariant stochastic calculus on manifolds is a two order calculus.

The stochastic covariant calculus with jumps of part I is well compatible with the Stratonovich and the Itô calculus, since it recovers Norris's formalism when integrating a continuous semimartingale. Nevertheless, it doesn't include these covariant Stratonovich and Itô calculus in a two order covariant formalism.

This two order formalism is given by the stochastic calculus with covariant jumps of part II. It is geometrically more significant since it describes a covariant jump on the manifold. This second part uses the notion of transport, which can be seen as a connection of order "0". Then we get a larger class of connections than connections of order one, with transports non necessarily linear, basic, idempotent, or invertible.
We saw that the calculus with covariant jumps is not well adapted to the Stratonovich calculus, but it is to the Itô calculus. In part II, we start with describing the covariant jump of a semimartingale, in order to describe the jump of a stochastic integral. This method is better adapted to the Itô integral. Indeed, in the non covariant case, the jump of an Itô integral is described from the one of the semimartingale, which is geometrically represented by a vector $\gamma(z_{t^-}, z_t)$ in $T_{z_{t^-}}U$. This is not the case for the Stratonovich calculus with jumps, for which only the jump of the integral is known, but doesn't come from a geometrical representation of the jump of the semimartingale itself.

Part IV

Covariant s.d.e. with jumps / s.d.e. with covariant jumps

Chapter 10

Reminders on s.d.e with jumps

We summarize here the results of [2] about stochastic differential equations with jumps.

Let U and W be two differential manifolds. Cohen defines a s.d.e driven by a càdlàg semimartingale (w_t) in W, with solution (z_t) in U, written as follows

$$\overset{\triangle}{dI}\ z = \phi(z, \overset{\triangle}{dI}\ w),$$

where $\phi(w_{t^-}, z_{t^-}, w_t)$ describes the jumps of the integral.

In the following, we will study s.d.e. with a punctual coefficient ϕ, defined as follows. However, note that our results are true with a more general ϕ (see [2]).

Definition 10.1 *Let $\phi : (W \times U) \times W \to U$ be a smooth map such that*

1. *$v \to \phi((w, z), v)$ is twice differentiable in a neighbourhood of the diagonal $\{((w, z), w), (w, z) \in W \times U\}$.*
2. *$\forall (w, z) \in W \times U : \quad \phi((w, z), w) = z$.*

Then the map $w \to \phi((w_{t^-}, z_{t^-}), w)$ is called a punctual coefficient. It is a previsible and locally bounded process above (w_{t^-}).

Definition 10.2 *Let (w_t) be a càdlàg semimartingale on W, ϕ a punctual coefficient above (w_{t^-}), and z_o a $\mathcal{F}_0$-measurable variable in U. We say that (z_t) is solution of the s.d.e with jumps*

$$\overset{\triangle}{dI}\ z = \phi(z, \overset{\triangle}{dI}\ w), \quad z_0 = z_o,$$

if (z_t) is a semimartingale in U, such that $z_0 = z_o$ and, for every θ in $\overset{\triangle}{\tau^*}\ U$, predictable and locally bounded above (z_{t^-}), we have

$$\int \theta \overset{\triangle}{dI}\ z = \int \phi(z)^*(\theta) \overset{\triangle}{dI}\ w, \quad z_0 = z_o,$$

where the process $(\phi(z)^*(\theta)_t)$ is defined by

$$\forall w \in W, \quad \phi(z)^*(\theta)_t(w) = \theta_t \circ \phi((w_{t^-}, z_{t^-}), w). \tag{10.1}$$

Proposition 10.1 *If ϕ is a punctual coefficient, of class C^3 on a neighbourhood of the diagonal $\{((w,z),w), (w,z) \in W \times U\}$, then there exists a unique solution (z_t) of the s.d.e with jumps*

$$\overset{\triangle}{dI}\, z = \phi(z, \overset{\triangle}{dI}\, w), \quad z_0 = z_o,$$

defined up to explosion (that is defined on $[0,T[$, where $T > 0$ is the previsible stopping time of explosion of (z_t)).

Moreover, in some imbeddings for W and U, we write

$$\begin{aligned} z_t^r &= z_o^r + \int_0^t \frac{\partial \phi^r}{\partial z^i}((w_{s^-}, z_{s^-}), w_{s^-}) dz_s^i + \frac{1}{2}\int_0^t \frac{\partial^2 \phi^r}{\partial z_s^i \partial z_s^j}((w_{s^-}, z_{s^-}), w_{s^-}) d < z^{i^c}, z^{j^c} >_s \\ &+ \sum_{0 \le s \le t} [\ \phi_s^r((w_{s^-}, z_{s^-}), w_s) - \frac{\partial \phi_s^r}{\partial z^i}((w_{s^-}, z_{s^-}), w_{s^-}) \triangle z_s^i\ ]. \end{aligned}$$

The map ϕ contains all the information : it describes the jumps of the solution and provides the drift and diffusion parts from its first and second derivatives.
Note that the second point of definition 10.1,

$$\phi((w_{t^-}, z_{t^-}), w_{t^-}) = z_{t^-},$$

means that the solution (z_t) jumps at the same times than (w_t).

We now give the application to Stratonovich and Itô s.d.e with jumps.

In the two following sections, a coefficient e is given, which is smooth and such that, for every (w,z) in $W \times U$, $e(w,z)$ is a linear map from $T_w W$ to $T_z U$.

10.1 Stratonovich s.d.e.

We consider a Stratonovich s.d.e. with jumps, driven by (w_t), with coefficient e. When the semimartingale (w_t) jumps, we interpolate it between the points w_{t^-} and w_t, with an interpolation rule I on W. Then, $\phi((w_{t^-}, z_{t^-}), w_t)$ is the solution at time $s = 1$ of the differential equation driven by $(I(s, w_{t^-}, w_t))_{s \in [0,1]}$, with coefficient e.

Definition 10.3 *Let I be an interpolation rule on W. For every càdlàg semimartingale (w_t) on W, for z_o in U, the Stratonovich s.d.e with jumps*

$$\overset{\triangle}{\partial}\, z_t \;=\; e(w_{t^-}, z_{t^-})\; \overset{\triangle}{\partial}\, w_t, \quad z_0 = z_o$$

is defined as equivalent to

$$\overset{\triangle}{dI}\, z = \phi(z, \overset{\triangle}{dI}\, w), \quad z_0 = z_o, \tag{10.2}$$

with

$$\phi((w_{t^-}, z_{t^-}), w_t) = \phi(1),$$

where $\phi(s)$ is the solution of the differential equation

$$\dot{\phi}(s) \;=\; e(I(s, w_{t^-}, w_t), \phi(s))\; \dot{I}(s, w_{t^-}, w_t), \quad \phi(0) = z_{t^-}. \tag{10.3}$$

By proposition 10.1, there exists a unique solution (z_t) in U of the s.d.e. (10.2), defined up to explosion.

Proposition 10.2 *When the semimartingale (w_t) is continuous, the s.d.e.*

$$\overset{\triangle}{\partial}\, z_t \;=\; e(w_{t^-}, z_{t^-})\; \overset{\triangle}{\partial}\, w_t$$

is equivalent to the continuous s.d.e.

$$\partial z_t = e(w_t, z_t)\partial w_t.$$

Proof : See [2].□

10.2 Itô s.d.e.

We consider an Itô s.d.e. with jumps, driven by (w_t), with coefficient e. When the semimartingale (w_t) jumps from w_{t^-} to w_t, we represent the jump in $T_{w_{t^-}}W$ by $\gamma^{(W)}(w_{t^-}, w_t)$, with a connection rule $\gamma^{(W)}$ on W, and then apply the coefficient e. We get the jump of the solution, represented in $T_{z_{t^-}}U$ by $\gamma^{(U)}(z_{t^-}, \phi((w_{t^-}, z_{t^-}), w_t))$, with a connection rule $\gamma^{(U)}$ on U. To have an expression for $\phi((w_{t^-}, z_{t^-}), w_t)$, this connection rule has to satisfy that, for every z in U, the map $\gamma^{(U)}(z, .)$ is invertible.

Definition 10.4 *Let $\gamma^{(W)}$ be a connection rule on W and $\gamma^{(U)}$ a connection rule on U, such that, for every z in U, the map $\gamma^{(U)}(z, .)$ is an invertible map. For every càdlàg semimartingale (w_t) on W, for z_o in U, the Itô s.d.e with jumps*

$$\overset{\triangle}{d}\, z_t \;=\; e(w_{t^-}, z_{t^-})\; \overset{\triangle}{d}\, w_t, \quad z_0 = z_o \tag{10.4}$$

is defined as equivalent to

$$\overset{\triangle}{dI}\ z = \phi(z, \overset{\triangle}{dI}\ w), \quad z_0 = z_o,$$

where $\phi((w_{t^-}, z_{t^-}), w_t)$ *satisfies*

$$\phi((w_{t^-}, z_{t^-}), w_t) \ = \ \gamma^{(U)}(z_{t^-}, .)^{-1}[e(w_{t^-}, z_{t^-})\ \gamma^{(W)}(w_{t^-}, w_t)]. \tag{10.5}$$

By proposition 10.1, there exists a unique solution (z_t) in U of the s.d.e. (10.4), defined up to explosion.

Remark 10.1 *For instance, we can consider the geodesic connection rule on* U*, given by*

$$\gamma^{(U)}(z, y) = \exp_z^{-1}(y),$$

for which $\gamma^{(U)}(z, .)^{-1}(u) = \exp_z(u)$*. The condition on* $\gamma^{(U)}$ *means that the inverse of the exponential map has to exist and to be invertible. Requiring this condition is more restrictive than requiring the existence of an exponential map on* U*. Therefore, instead of being given a connection rule* $\gamma^{(U)}$ *with the condition of invertibility, we could require to be given an exponential map* exp *on* U*. Then,* ϕ *would be defined in a unique way as*

$$\phi((w_{t^-}, z_{t^-}), w_t) = \exp_{z_{t^-}}[\ e(w_{t^-}, z_{t^-})\ \gamma^{(W)}(w_{t^-}, w_t)\ ].$$

Proposition 10.3 *When the semimartingale* (w_t) *is continuous, the s.d.e.*

$$\overset{\triangle}{d}\ z_t \ = \ e(w_{t^-}, z_{t^-})\ \ \overset{\triangle}{d}\ w_t$$

is equivalent to the continuous s.d.e.

$$dz_t = e(w_t, z_t)dw_t.$$

Proof : Let us give a proof for this proposition, since it is not given in [2].
Let α be a first order form in U. For every z in U, we define a first order form $\beta^{(z)}$ in W, by

$$\forall w \in W,\ \beta_w = \alpha_z \circ e(w, z) : T_wW \to \mathbb{R}.$$

Using the notations for the Itô jumps of part I, applying α to the relation (10.5) yields

$$\overset{\triangle}{\gamma}\ (\alpha)_{z_{t^-}}(\phi((w_{t^-}, z_{t^-}), w_t)) = \overset{\triangle}{\gamma}\ (\beta^{(z_{t^-})})_{w_{t^-}}(w_t),$$

that is, using (10.1),

$$\phi^*(\overset{\triangle}{\gamma}\ (\alpha))_t(w_t) = \overset{\triangle}{\gamma}\ (\beta^{(z_{t^-})})_{w_{t^-}}(w_t). \tag{10.6}$$

According to definition 10.2, if (z_t) is solution of (10.4), we have

$$\int \overset{\triangle}{\gamma}\ (\alpha)_{z_{t^-}}\ \overset{\triangle}{dI}\ z_t = \int \phi(z)^*(\overset{\triangle}{\gamma}\ (\alpha))_t\ \overset{\triangle}{dI}\ w_t. \tag{10.7}$$

When the semimartingale (w_t) is continuous, the solution (z_t) is also continuous. Then, by proposition 2.3, we get, on the one hand,

$$\int \overset{\triangle}{\gamma}\,(\alpha)_{z_{t-}}\,\overset{\triangle}{dI}\,z_t = \int \alpha_{z_t} dz_t,$$

and, on the other hand, using (10.6),

$$\int \phi(z)^*(\overset{\triangle}{\gamma}\,(\alpha))_t\,\overset{\triangle}{dI}\,w_t = \int \overset{\triangle}{\gamma}\,(\beta^{(z_{t-}})_{w_{t-}}\,\overset{\triangle}{dI}\,w_t = \int \alpha_{z_t} e(w_t, z_t) dw_t.$$

Then (10.7) shows that, for every form α on U, we have

$$\int \alpha_{z_t} dz_t = \int \alpha_{z_t} e(w_t, z_t) dw_t,$$

that is saying that (z_t) is solution of the continuous s.d.e.

$$dz_t = e(w_t, z_t) dw_t.\square$$

10.3 Discretisation theorem

We recall the discretisation theorem given by Cohen in [2].

Theorem 3 *Consider the sequence (Y_t^n), defined by*

- $Y_0^n = Y_o$,
- $Y_t^n = \phi((W_{T_n^k}, Y_{T_n^k}^n), W_t)$ *for* $t \in [T_k^n, T_{k+1}^n]$ *and* $k < k(n)$,
- $Y_t^n = \phi((W_{T_n^{k(n)}}, Y_{T_n^{k(n)}}^n), W_t)$ *if* $t \geq T_{k(n)}^n$,

for a subdivision $(\sigma^n)_{n\in\mathbb{N}} = ((T_n^k)_{k\leq k(n)})_{n\in\mathbb{N}}$ *converging to the Identity.*

Then the process (Y_t^n) converges uniformly on compacts in probability to the unique solution (Y_t), defined on $[0, T]$, of the s.d.e

$$\overset{\triangle}{dI}\,Y = \phi(Y, \overset{\triangle}{dI}\,W), \quad Y_0 = Y_o.$$

Chapter 11

On continuous covariant s.d.e

In the following chapters, we will study covariant stochastic differential equations. Therefore, we work with two manifolds N and M, and semimartingales on TN and TM.

From now on, we will consider a smooth coefficient e, such that, for every $W=(r,w)$ in TN and every $Y=(x,y)$ in TM, $e(W,Y)$ is a linear map from $T_w(T_rN)$ to $T_y(T_xM)$.

The following proposition shows that covariant s.d.e are well defined only if the projection part (πY_t) on M of the solution (Y_t) in TM is known.

Proposition 11.1 *Let $p^{(N)}$ and $p^{(M)}$ be connections of order one, respectively on N and M. Let (W_t) be a continuous semimartingale in TN, (x_t) a continuous semimartingale in M, and Y_o a $\mathcal{F}_0$- measurable variable in TM.*

We say that (Y_t) is solution of the Stratonovich covariant s.d.e

$$\partial^{\mathcal{V}} Y_t = e(W_t, Y_t)\partial^{\mathcal{V}} W_t, \quad Y_0 = Y_o \tag{11.1}$$

if (Y_t) is a continuous semimartingale in TM, such that $Y_0 = Y_o$ and, for every form α on TM, we have

$$\int \alpha_{Y_t} p^{(M)}_{Y_t} \partial Y_t = \int \alpha_{Y_t} e(W_t, Y_t) p^{(N)}_{W_t} \partial W_t, \quad Y_0 = Y_o.$$

There exists a unique solution $(Y_t) = ((x_t, y_t))$ of the Stratonovich covariant s.d.e (11.1)*, defined up to explosion.*

We say that (Y_t) is solution of the Itô covariant s.d.e

$$d^{\mathcal{V}} Y_t = e(W_t, Y_t) d^{\mathcal{V}} W_t, \quad Y_0 = Y_o \tag{11.2}$$

if (Y_t) is a continuous semimartingale in TM, such that $Y_0 = Y_o$ and, for every form α on TM, we have

$$\int \alpha_{Y_t} p^{(M)}_{Y_t} dY_t = \int \alpha_{Y_t} e(W_t, Y_t) p^{(N)}_{W_t} dW_t, \quad Y_0 = Y_o.$$

There exists a unique solution $(Y_t) = ((x_t, y_t))$ of the Itô covariant s.d.e (11.2), defined up to explosion.

To prove the unicity for the solution of a covariant s.d.e., we use the following lemma, and remark 0.2.

Lemma 3 *For every coefficient e as above and connections $p^{(N)}$ on N and $p^{(M)}$ on M, define the map f on the manifold $(M \times TN) \times TM$, as follows.*
For every $Z = (x, W)$ in $M \times TN$ and every $Y = (x, y)$ in TM, $f(Z, Y)$ is given by

$$\forall (U_1, U_2) \in T_Z(M \times TN) \simeq T_xM \times T_WTN, \quad f(Z, Y)(U_1, U_2) = e(W, Y)p_W^{(N)}(U_2) - p_Y^{(M)}(U_1, 0).$$

Then, f is a smooth map such that, for every $Z = (x, W)$ in $M \times TN$ and every $Y = (x, y)$ in TM, $f(Z, Y)$ is linear from $T_Z(M \times TN)$ to $T_y(T_xM)$.

Proof (of lemma) : The smoothness of the map f comes from that of $(W, Y) \to e(W, Y)$ and that of $V \to p_V$ for every connection p. Moreover, for every $Y = (x, y)$ in TM, the map

$$p_Y^{(M)}(., 0) : T_xM \times \{0\} \subset T_YTM \to T_y(T_xM) \quad (\text{with } 0 \in T_y(T_xM))$$

is linear. For every $Y = (x, y)$ in TM and $W = (r, w)$ in TN, the maps

$$p_W^{(N)} : T_WTN \to T_wT_rN$$

and

$$e(W, V) : T_w(T_rN) \to T_y(T_xM)$$

are linear, so

$$e(W, V) \circ p_W^{(N)} : T_WTN \to T_y(T_xM)$$

is linear. Then f is linear from $T_xM \times T_WTN$ to $T_y(T_xM)$.□

Proof (of proposition) : Let us write (11.1) as

$$p_{Y_t}^{(N)}(\partial Y_t) = e(W_t, Y_t)p_{W_t}^{(M)}(\partial W_t). \tag{11.3}$$

Using the linearity of $p_{Y_t}^{(N)}$, we have

$$p_{Y_t}^{(N)}(\partial Y_t) = p_{Y_t}^{(N)}(\partial x_t, 0) + p_{Y_t}^{(N)}(0, \partial y_t).$$

But $p_{Y_t}^{(N)}(0, \partial y_t) = \partial y_t$, since $\partial y_t \in T_{y_t}(T_{x_t}M)$, so we get

$$p_{Y_t}^{(N)}(\partial Y_t) = p_{Y_t}^{(N)}(\partial x_t, 0) + \partial y_t.$$

Then (11.3) is equivalent to

$$\partial y_t = e(W_t, Y_t)p_{W_t}^{(M)}(\partial W_t) - p_{Y_t}^{(N)}(\partial x_t, 0).$$

Set $(Z_t) = ((x_t, W_t))$, semimartingale in $N \times TM$. Then (11.3) is equivalent to the Stratonovich s.d.e.

$$\partial y_t = f(Z_t, Y_t)\partial Z_t,$$

where f is the coefficient given by lemma 3.
This s.d.e with linear coefficient f has a unique solution $(Y_t) = ((x_t, y_t))$ in TM, defined up to explosion, which is also the unique solution of (11.1).
In the same way, we show that (11.2) has a unique solution $(Y_t) = ((x_t, y_t))$ in TM, defined up to explosion. □

Remark 11.1 *When doing the identification of the vertical space, we can recover the Stratonovich and Itô covariant differential equations described by Norris in [8]. The coefficient e is then a smooth map yielding, for every r in N and every x in M, a linear map $e(r, x)$ from T_rN to T_xM. As a consequence, Norris's formalism also allows to consider covariant s.d.e. in the form*

$$\partial^{\mathcal{V}} Y_t = e(r_t, x_t)\partial r_t,$$

since ∂r_t belongs to $T_{r_t}N$. Our formalism of covariant s.d.e. doesn't include such s.d.e., since ∂r_t doesn't belongs to $T_y(T_xM)$.

In the following, we will be interrested in the case where (W_t) and so $(Y_t) = ((x_t, y_t))$ can jump. By the foregoing, we will also have to assume that the projection (x_t) of (Y_t) on M is known, so that we can write a covariant differential equation as a classical one, with the map f given by lemma 3. Actually, we understand that the driven semimartingale of a covariant differential equation is not only (W_t) in TN but more precisely $(Z_t) = ((x_t, W_t))$ in $M \times TN$. As a consequence, for the case with jumps, we will have to assume moreover that the semimartingales (x_t) on M and (W_t) on TN jumps at the same times.

Chapter 12

Stratonovich covariant s.d.e with jumps

12.1 Stratonovich covariant s.d.e with jumps

Using parts I, II, and the previous reminders, we can give meaning to Stratonovich covariant s.d.e with jumps and to Stratonovich s.d.e with covariant jumps. Following the conclusion of part III, we will only study Stratonovich covariant s.d.e with jumps :

$$\overset{\triangle}{\partial^{\mathcal{V}}} Y_t \;=\; e(W_{t^-}, Y_{t^-}) \; \overset{\triangle}{\partial^{\mathcal{V}}} W_t, \quad Y_0 = Y_o.$$

When (W_t) jumps from W_t^- to W_t, the solution (Y_t) jumps from Y_{t^-} to $\Phi_{\mathcal{S}}((W_{t^-}, Y_{t^-}), W_t)$. According to the reminders, we describe the solution (Y_t) by giving the map $\Phi_{\mathcal{S}}$.

Following the approach of part I, connections $p^{(N)}$ and $p^{(M)}$ are required, respectively on N and M, for the covariant calculus, and an interpolation rule $\mathbb{I}^{(N)}$ is required on TN to describe the jumps of (W_t). When the semimartingale (W_t) jumps, we interpolate it with $\mathbb{I}^{(N)}$ between the points W_{t^-} and W_t. Then, $\Phi_{\mathcal{S}}((W_{t^-}, Y_{t^-}), W_t)$ is the solution at time $s = 1$ of the covariant differential equation driven by $(\mathbb{I}^{(N)}(s, W_{t^-}, W_t))_{s\in[0,1]}$, with coefficient e. To get a unique solution of this covariant differential equation, the projection of the solution has to be known. It is given by $(I^{(M)}(s, x_{t^-}, x_t))_{s\in[0,1]}$, where $I^{(M)}$ is an interpolation rule on M.

Definition 12.1 *Let $p^{(N)}$ and $p^{(M)}$ be connections, respectively on N and M. Let $\mathbb{I}^{(N)}$ be a connection rule on N, and $I^{(M)}$ a connection rule on M.*
Let (W_t) be a càdlàg semimartingale in TN, (x_t) a càdlàg semimartingale in M, which jumps in M at the same times than (W_t) in TN, and Y_o a $\mathcal{F}_0$- measurable variable in TM.

We say that $(Y_t) = ((x_t, y_t))$ is solution of the Stratonovich covariant s.d.e. with jumps

$$\overset{\triangle}{\partial^{\mathcal{V}}} Y_t \;=\; e(W_{t^-}, Y_{t^-}) \; \overset{\triangle}{\partial^{\mathcal{V}}} W_t, \quad Y_0 = Y_o, \tag{12.1}$$

if (Y_t) is solution of the s.d.e with jumps

$$\overset{\triangle}{d\!I}\, Y = \Phi_{\mathcal{S}}(Y, \overset{\triangle}{d\!I}\, W), \quad Y_0 = Y_o, \tag{12.2}$$

with

$$\Phi_{\mathcal{S}}((W_{t^-}, Y_{t^-}), W_t) = \Phi_{\mathcal{S}}(1) = (x_t, \phi_{\mathcal{S}}(1))$$

where $(\Phi_{\mathcal{S}}(s))$ *is defined above* $(I^{(M)}(s, x_{t^-}, x_t))$*, as the solution of the differential covariant equation*

$$\frac{\partial^{\mathcal{V}} \Phi_{\mathcal{S}}(s)}{\partial s} \;=\; e(\mathbb{I}^{(N)}(s, W_{t^-}, W_t), \Phi_{\mathcal{S}}(s))\; \frac{\partial^{\mathcal{V}} \mathbb{I}^{(N)}(s, W_{t^-}, W_t)}{\partial s}, \quad \Phi_{\mathcal{S}}(0) = Y_{t^-}. \tag{12.3}$$

Note that the covariant derivatives in (12.3) are defined with the connections $p^{(N)}$ for $\mathbb{I}(s, W_{t^-}, W_t)$ and $p^{(M)}$ for $\Phi_{\mathcal{S}}(s)$.

Proposition 12.1 *The Stratonovich covariant s.d.e. with jumps* (12.1) *has a unique solution* $(Y_t) = ((x_t, y_t))$ *in* TM*, defined up to explosion.*

Proof : Since (12.1) is defined as equivalent to (12.2), let us show that (12.2) has a unique solution. First, let us see that $\Phi_{\mathcal{S}}$ satisfies $\Phi_{\mathcal{S}}((W_{t^-}, Y_{t^-}), W_{t^-}) = Y_{t^-}$. It is given by $\Phi_{\mathcal{S}}((W_{t^-}, Y_{t^-}), W_{t^-}) = \Phi_{\mathcal{S}}(1)$, where $\pi\Phi_{\mathcal{S}}(s) = I^{(M)}(s, x_{t^-}, x_t)$. We have $\mathbb{I}^{(N)}(s, W_{t^-}, W_{t^-}) = W_{t^-}$, so its covariant derivative vanishes, and then, since the map $e(W_{t^-}, \Phi_{\mathcal{S}}(s))$ is linear, $\Phi_{\mathcal{S}}(s)$ is solution of

$$\frac{\partial^{\mathcal{V}} \Phi_{\mathcal{S}}(s)}{\partial s} \;=\; 0,$$

that is $\Phi_{\mathcal{S}}(1)$ is horizontal. So we get

$$\Phi_{\mathcal{S}}(1) = (x_t, \tau^{//}_{01}(y_{t^-})),$$

where $\tau^{//}$ is the parallel transport along $(I^{(M)}(s, x_{t^-}, x_t))$. But (x_t) is assumed to satisfy $x_t = x_{t^-}$ if $W_t = W_{t^-}$. Then we get $\Phi_{\mathcal{S}}(1) = (x_{t^-}, y_{t^-})$, that is

$$\Phi_{\mathcal{S}}((W_{t^-}, Y_{t^-}), W_{t^-}) = Y_{t^-}.$$

Let us now verify that $\Phi_{\mathcal{I}_1}$ satisfies the conditions of proposition 10.1.
The differential covariant equation (12.3) is written

$$p^{(M)}_{\Phi_{\mathcal{S}}(s)}(\dot{\Phi}_{\mathcal{S}}(s)) = e(\mathbb{I}(s, W_{t^-}, W_t), \Phi_{\mathcal{S}}(s)) p^{(N)}_{\mathbb{I}(s, W_{t^-}, W_t)}(\dot{\mathbb{I}}(s, W_{t^-}, W_t)). \tag{12.4}$$

Writing $\Phi_{\mathcal{S}}(s) = (I(s, x_{t^-}, x_t), \phi_{\mathcal{S}}(s))$, it gives

$$p^{(M)}_{\Phi_{\mathcal{S}}(s)}(\dot{I}(s, x_{t^-}, x_t), \dot{\phi}_{\mathcal{S}}(s)) = e(\mathbb{I}(s, W_{t^-}, W_t), \Phi_{\mathcal{S}}(s)) p^{(N)}_{\mathbb{I}(s, W_{t^-}, W_t)}(\dot{\mathbb{I}}(s, W_{t^-}, W_t)).$$

Then it is equivalent to

$$\dot{\phi}_{\mathcal{S}}(s) = e(\mathbb{I}(s, W_{t^-}, W_t), \Phi_{\mathcal{S}}(s)) p^{(N)}_{\mathbb{I}(s, W_{t^-}, W_t)}(\dot{\mathbb{I}}(s, W_{t^-}, W_t)) - p^{(M)}_{\Phi_{\mathcal{S}}(s)}(\dot{I}(s, x_{t^-}, x_t), 0),$$

that is

$$\dot{\phi}_{\mathcal{S}}(s) \;=\; f(J(s, Z_{t^-}, Z_t), \Phi_{\mathcal{S}}(s))\; \dot{J}(s, Z_{t^-}, Z_t) \tag{12.5}$$

where

$$Z_t = (x_t, W_t) \in M \times TN,$$

J is the interpolation rule on $M \times TN$ given, for every $Z_1 = (x_1, W_1)$ and every $Z_2 = (x_2, W_2)$ in $M \times TN$, by

$$J(s, Z_1, Z_2) = (I(s, x_1, x_2), \dot{\mathbb{I}}(s, W_1, W_2)) \in M \times TN,$$

and f is the coefficient given by lemma 3.
Then we get a differential equation, similar to (10.3), which describes the jumps of the unique solution of a Stratonovich s.d.e.. Then, this solution is also the unique solution of (12.1).□

Proposition 12.2 *When the semimartingale* (W_t) *is continuous, the Stratonovich covariant s.d.e*

$$\overset{\triangle}{\partial^{\mathcal{V}}} Y_t = e(W_{t^-}, Y_{t^-}) \overset{\triangle}{\partial^{\mathcal{V}}} W_t$$

is equivalent to the continuous Stratonovich covariant s.d.e

$$\partial^{\mathcal{V}} Y_t = e(W_t, Y_t)\partial^{\mathcal{V}} W_t.$$

Proof : By the previous proof, we showed that the differential covariant equation (12.3) is equivalent to (12.5). By definition 10.3, it implies that the s.d.e (12.1) is equivalent to the Stratonovich s.d.e

$$\overset{\triangle}{\partial} y_t = f(Z_{t^-}, Y_{t^-}) \overset{\triangle}{\partial} Z_t.$$

When (W_t) is continuous, so are $(Y_t) = ((x_t, y_t))$ and $(Z_t) = ((x_t, W_t))$. By proposition 10.2, we get the continuous Stratonovich s.d.e.

$$\partial y_t = f(Z_t, Y_t)\partial Z_t,$$

which is equivalent to

$$\partial Y_t = e(W_t, Y_t)\partial W_t.\square$$

Application to $M = \mathbb{R}$

Assume that the manifold M is $\mathbb{R}$, endowed with the flat connection $p^{(M)}$. For every first order form α on TM, we consider the coefficient

$$\forall W = (r, w) \in TN, \forall Y = (x, y) \in \mathbb{R}^2, \quad e(W, Y) = \alpha_W(0, .) : T_w(T_r N) \to \mathbb{R}.$$

Then consider the particular s.d.e

$$\overset{\triangle}{\partial} y_t = \alpha_{W_{t^-}}(0, \overset{\triangle}{\partial^{\mathcal{V}}} W_t).$$

When (W_t) jumps, the jump of the solution (y_t) in $\mathbb{R}$ is those of the covariant integral $\int \alpha_{W_{t^-}} \overset{\triangle}{\partial^{\mathcal{V}}} W_t$, which should be the Stratonovich covariant integral of part I.

Proposition 12.3 *Let p be a connection on N and $\mathbb{I}$ an interpolation rule on TN. Then the solution of the Stratonovich covariant s.d.e. with jumps*

$$\overset{\triangle}{\partial}\, y_t = \alpha_{W_{t^-}}(0, \overset{\triangle}{\partial^{\mathcal{V}}}\, W_t) \tag{12.6}$$

is the Stratonovich covariant integral with jumps

$$\int \alpha_{W_t}\, \overset{\triangle}{\partial^{\mathcal{V}}}\, W_t$$

of definition 1.6.

Proof : It is enough to show the equality of the jumps of the solution (y_t) of (12.6) and those of the Stratonovich covariant integral.
First note that we can see (12.6) as a covariant s.d.e., since, with the flat connection $p^{(M)}$, we have $\partial^{\mathcal{V}} Y_t = \partial y_t$. Then, according to definition 12.1, the jump of the solution (y_t) of (12.6), when (W_t) jumps from W_{t^-} to W_t, is given by $\phi_{\mathcal{S}}((W_{t^-}, Y_{t^-}), W_t) - y_{t^-}$, with

$$\phi_{\mathcal{S}}((W_{t^-}, Y_{t^-}), W_t) = \phi_{\mathcal{S}}(1)$$

where $\phi_{\mathcal{S}}(s)$ is solution of the differential covariant equation

$$\frac{\partial \phi_{\mathcal{S}}(s)}{\partial s} \;=\; \alpha_{\mathbb{I}(s,W_{t^-},W_t)}(0, \frac{\partial^{\mathcal{V}} \mathbb{I}(s, W_{t^-}, W_t)}{\partial s}).$$

Then the jump of (y_t) is

$$\phi_{\mathcal{S}}(1) - y_{t^-} = \int_0^1 \alpha_{\mathbb{I}(s,W_{t^-},W_t)}(0, \frac{\partial^{\mathcal{V}} \mathbb{I}(s, W_{t^-}, W_t)}{\partial s}) = \mathbb{I}(p^*\alpha)_{W_{t^-}}(W_t),$$

which is also the jump of $\int \alpha_{W_t}\, \overset{\triangle}{\partial^{\mathcal{V}}}\, W_t$, according to definition 1.6.□

12.2 Application to the horizontal lift of a càdlàg semimartingale in M

We extend definition 0.5 for the horizontal lift of a càdlàg semimartingale as follows.

Definition 12.2 *Let p be a connection on M. Let (x_t) be a càdlàg semimartingale in M. The horizontal lift (u_t) of (x_t), starting from a given $u_o \in GL(\mathbb{R}^m, T_{x_0}M)$, is the unique curve $((x_t, u_t))$ in $GL(\mathbb{R}^m, TM)$ ($\forall t, u_t \in GL(\mathbb{R}^m, T_{x_t}M)$) satisfying the covariant s.d.e*

$$\overset{\triangle}{\partial^{\mathcal{V}}}\, (x_t, u_t) = 0\ , \quad u_0 = u_o. \tag{12.7}$$

(This equation means

$$\forall z \in \mathbb{R}^m, \quad \overset{\triangle}{\partial^{\mathcal{V}}}\, (x_t, u_t(z)) = 0.) \tag{12.8}$$

By the following proposition, we recover the description given by Cohen in [1] for the jumps of the horizontal lift.

Proposition 12.4 *Let I be an interpolation rule on M. When (x_t) jumps from x_{t^-} to x_t, the jump of the horizontal lift $((x_t, u_t))$, solution of (12.7), is a jump from (x_{t^-}, u_{t^-}) to*

$$(x_t, u_t) = (x_t, \tau^{//}_{x_{t^-}, x_t} u_{t^-}), \tag{12.9}$$

where $\tau^{//}$ is the parallel transport along $(I(s, x_{t^-}, x_t))_{s\in[0,1]}$.

Proof : We study the covariant s.d.e. $\partial^{\mathcal{V}} U_t = 0$, where $U_t = (x_t, u_t)$.
According to definition 12.1, taking $e \equiv 0$, the jump of (U_t), is given by

$$\Phi_{\mathcal{S}}((W_{t^-}, U_{t^-}), W_t) = \Phi_{\mathcal{S}}(1)$$

where $\pi\Phi_{\mathcal{S}}(s) = I(s, x_{t^-}, x_t)$, and $\Phi_{\mathcal{S}}(s)$ is solution of the differential covariant equation

$$\frac{\partial^{\mathcal{V}} \Phi_{\mathcal{S}}(s)}{\partial s} = O, \quad \Phi_{\mathcal{S}}(0) = u_{t^-}$$

By the expression (55) for the covariant derivative, writing $\Phi_{\mathcal{S}}(s) = (I(s, x_{t^-}, x_t), \phi_{\mathcal{S}}(s))$, we get

$$\tau^{//}_{0s} \frac{\partial \tau^{//}_{s0} \phi_{\mathcal{S}}(s)}{\partial s} = 0,$$

where $\tau^{//}$ is the parallel transport along $(I(s, , x_{t^-}, x_t))_{s\in[0,1]}$.
It follows that, for all s,

$$\tau^{//}_{s0} \phi_{\mathcal{S}}(s) = \tau^{//}_{00} \phi_{\mathcal{S}}(0)$$

Then, at time $s = 1$, we get

$$u_t = \phi_{\mathcal{S}}(1) = \tau^{//}_{01} \phi_{\mathcal{S}}(0) = \tau^{//}_{01} u_{t^-}.\square$$

The notion of horizontal semimartingale has been introduced in part III. We can now give a rigourous definition.

Definition 12.3 *Let $(Y_t) = ((x_t, y_t))$ be a càdlàg semimartingale in TM. We say that (Y_t) is horizontal if it is solution of the covariant s.d.e.*

$$\overset{\triangle}{\partial^{\mathcal{V}}} Y_t = 0.$$

Proposition 12.5 *Let I be an interpolation rule on M. When (x_t) jumps from x_{t^-} to x_t, the jump of the horizontal semimartingale (Y_t) is a jump from (x_{t^-}, y_{t^-}) to*

$$Y_t = (x_t, \tau^{//}_{x_{t^-}, x_t} y_{t^-}),$$

where $\tau^{//}$ is the parallel transport along $(I(s, x_{t^-}, x_t))_{s\in[0,1]}$.

Proof : The proof is the same as for the horizontal lift.$\square$

12.3 Discretisation of a Stratonovich covariant s.d.e.

For a Stratonovich covariant s.d.e. with jumps, theorem 3 gives the following. Note that we can take the same T_k^n for the semimartingales (W_t) and (x_t), since they are assumed to jump at the same times.

Theorem 4 *Consider the sequence* (Y_t^n), *defined by*

- $Y_0^n = Y_o$,
- *For* $t \in [T_k^n, T_{k+1}^n]$ *and* $k < k(n)$,
$$Y_t^n = \Phi_{\mathcal{S}}(1)$$
where $(\Phi_{\mathcal{S}}(s))$ *is defined above* $(I^{(M)}(s, x_{T_n^k}, x_t))$, *as the solution of the differential covariant equation*
$$\frac{\partial^{\mathcal{V}} \Phi_{\mathcal{S}}(s)}{\partial s} \;=\; e(\mathbb{I}(s, W_{T_n^k}, W_t), \Phi_{\mathcal{S}}(s))\; \frac{\partial^{\mathcal{V}} \mathbb{I}(s, W_{T_n^k}, W_t)}{\partial s}, \quad \Phi_{\mathcal{S}}(0) = Y_{T_n^k}^n,$$
- *For* $t \geq T_{k(n)}^n$,
$$Y_t^n = \Psi_{\mathcal{S}}(1)$$
where $(\Psi_{\mathcal{S}}(s))$ *is defined above* $(I^{(M)}(s, x_{T_n^{k(n)}}, x_t))$, *as the solution of the differential covariant equation*
$$\frac{\partial^{\mathcal{V}} \Psi_{\mathcal{S}}(s)}{\partial s} \;=\; e(\mathbb{I}(s, W_{T_n^{k(n)}}, W_t), \Psi_{\mathcal{S}}(s))\; \frac{\partial^{\mathcal{V}} \mathbb{I}(s, W_{T_n^{k(n)}}, W_t)}{\partial s}, \quad \Psi_{\mathcal{S}}(0) = Y_{T_n^{k(n)}}^n,$$

for a subdivision $(\sigma^n)_{n\in\mathbb{N}} = ((T_n^k)_{k\leq k(n)})_{n\in\mathbb{N}}$ *converging to the Identity.*

Then the process (Y_t^n) *converges uniformly on compacts in probability to the unique solution* $(Y_t) = ((x_t, y_t))$, *defined on* $[0, T]$, *of the Stratonovich covariant s.d.e*

$$\overset{\triangle}{\partial^{\mathcal{V}}}\, Y_t \;=\; e(W_{t^-}, Y_{t^-})\; \overset{\triangle}{\partial^{\mathcal{V}}}\, W_t, \quad Y_0 = Y_o.$$

Chapter 13

Itô covariant s.d.e with jumps / Itô s.d.e with covariant jumps

Using parts I, II, and the previous reminders, we can give meaning to Itô covariant s.d.e with jumps and to Itô s.d.e with covariant jumps.

13.1 Itô covariant s.d.e with jumps

Consider the Itô covariant s.d.e. with jumps

$$\overset{\triangle}{d^{\mathcal{V}}} Y_t \;=\; e(W_{t^-}, Y_{t^-})\; \overset{\triangle}{d^{\mathcal{V}}} W_t, \quad Y_0 = Y_o.$$

When (W_t) jumps from W_t^- to W_t, the solution (Y_t) jumps from Y_{t^-} to $\Phi_{\mathcal{I}_1}(W_t^-, Y_t^-, W_t)$. According to the reminders, we describe the solution (Y_t) by giving the map $\Phi_{\mathcal{I}_1}$.

Following the approach of part I, connections $p^{(N)}$ and $p^{(M)}$ are required, respectively on N and M, for the covariant calculus, and connection rules $\boldsymbol{\gamma}^{(N)}$ and $\boldsymbol{\gamma}^{(M)}$ are required, respectively on TN and TM, to describe the jumps.
Therefore, the covariant jump of (W_t) on N is given by

$$p^{(N)}_{W_{t^-}} \boldsymbol{\gamma}^{(N)}(W_{t^-}, W_t). \tag{13.1}$$

The covariant jump of the solution (Y_t) on TM is given, on the one hand, by

$$p^{(M)}_{Y_{t^-}} \boldsymbol{\gamma}^{(M)}(Y_{t^-}, \Phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_t)),$$

and one the other hand, as a consequence of (13.1), by

$$e(W_{t^-}, Y_{t^-})\; p^{(N)}_{W_{t^-}} \boldsymbol{\gamma}^{(N)}(W_{t^-}, W_t).$$

To get an expression for $\Phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_t))$, we have do the following assumption on the connection rule $\boldsymbol{\gamma}^{(M)}$:

(A). *For every $Y_1 = (x_1, y_1)$ and every $Y_2 = (x_2, y_2)$ in TM,*

$$\boldsymbol{\gamma}^{(M)}(Y_1, Y_2) = (\gamma^{(M)}(x_1, x_2), \boldsymbol{\gamma}^{(M)}(y_1, y_2)),$$

where $\gamma^{(M)}(x_1, x_2) \in T_{x_1}M$ and $\boldsymbol{\gamma}^{(M)}(y_1, y_2) \in T_{y_1}(T_{x_1}M)$ (note that it depends also on x_1 and x_2). Moreover, for every y in T_xM, the map $\boldsymbol{\gamma}^{(M)}(y, .) : T_xM \to T_y(T_xM)$ is assumed to be invertible.

Under this assumption, using the linearity of $p^{(M)}_{Y_{t^-}}$, we can write

$$p^{(M)}_{Y_{t^-}} \boldsymbol{\gamma}^{(M)}(Y_{t^-}, \phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_t)) = p^{(M)}_{Y_{t^-}}[\gamma^{(M)}(x_{t^-}, x_t), \boldsymbol{\gamma}^{(M)}(y_{t^-}, \phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_t))]$$

$$= p^{(M)}_{Y_{t^-}}(\gamma^{(M)}(x_{t^-}, x_t), 0) + \boldsymbol{\gamma}^{(M)}(y_{t^-}, \phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_t)),$$

with $\Phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_t)) = (x_t, \phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_t)))$.

Then we get the following definition.

Definition 13.1 *Let $p^{(N)}$, $p^{(M)}$ be connections, respectively on N and M. Let $\boldsymbol{\gamma}^{(N)}$ be a connection rule on TN and $\boldsymbol{\gamma}^{(M)}$ be a connection rule on TM, satisfying assumption (A).*
Let (W_t) be a càdlàg semimartingale in TN, (x_t) a càdlàg semimartingale in M, which jumps in M at the same times than (W_t) in TN, and Y_o a $\mathcal{F}_0$- measurable variable in TM.

We say that $(Y_t) = ((x_t, y_t))$ is solution of the Itô covariant s.d.e. with jumps

$$\overset{\triangle}{d}^{\mathcal{V}} Y_t \;=\; e(W_{t^-}, Y_{t^-}) \; \overset{\triangle}{d}^{\mathcal{V}} W_t, \quad Y_0 = Y_o, \tag{13.2}$$

if (Y_t) is solution of the s.d.e with jumps

$$\overset{\triangle}{d\mathcal{I}} \; Y = \Phi_{\mathcal{I}_1}(Y, \overset{\triangle}{d\mathcal{I}} \; W), \quad Y_0 = Y_o, \tag{13.3}$$

where

$$\Phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_t) = (\; x_t \;, \; \boldsymbol{\gamma}^{(M)}(y_{t^-}, .)^{-1}[\; e(W_{t^-}, Y_{t^-}) p^{(N)}_{W_{t^-}}(\boldsymbol{\gamma}^{(N)}(W_{t^-}, W_t)) - p^{(M)}_{Y_{t^-}}(\gamma^{(M)}(x_{t^-}, x_t), 0) \;] \;) \tag{13.4}$$

Proposition 13.1 *The Itô covariant s.d.e. with jumps* (13.2) *has a unique solution* $(Y_t) = ((x_t, y_t))$ *in TM, defined up to explosion.*

Proof : Since (13.2) is defined as equivalent to (13.3), let us show that (13.3) has a unique solution. First, let us see that $\Phi_{\mathcal{I}_1}$ satisfies $\Phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_{t^-}) = Y_{t^-}$. Recall that $x_t = x_{t^-}$ when $W_t = W_{t^-}$. Then we get

$$\Phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_{t^-}) = (x_{t^-}, \boldsymbol{\gamma}^{(M)}(y_{t^-}, .)^{-1}[e(W_{t^-}, Y_{t^-}) p^{(N)}_{W_{t^-}}(0) - p^{(M)}_{Y_{t^-}}(0, 0)]),$$

that is

$$\Phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_{t^-}) = (\ x_{t^-}\ ,\ \boldsymbol{\gamma}^{(M)}(y_{t^-}, .)^{-1}(0)) = Y_{t^-}.$$

Let us now verify that $\Phi_{\mathcal{I}_1}$ satisfies the conditions of proposition 10.1.
Equation (13.4) can be written

$$\phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_t) = (\ x_t\ ,\ \boldsymbol{\gamma}^{(M)}(y_{t^-}, .)^{-1}[f(Z_{t^-}, Y_{t^-})\gamma((Z_{t^-}, Z_t))]\ ), \tag{13.5}$$

where

$$Z_t = (x_t, W_t) \in M \times TN,$$

γ is the connection rule on $M \times TN$ given, for every $Z_1 = (x_1, W_1)$ and every $Z_2 = (x_2, W_2)$ in $M \times TN$, by

$$\gamma(Z_1, Z_2) = (\gamma^{(M)}(x_1, x_2), \boldsymbol{\gamma}^{(N)}(W_1, W_2)) \in M \times TN,$$

and f is the coefficient given by lemma 3.
Then we get an equation, similar to (10.5), which describes the jumps of the unique solution of an Itô s.d.e. Then, this solution is also the unique solution of (13.2).□

Proposition 13.2 *When the semimartingale* (W_t) *is continuous, the Itô covariant s.d.e*

$$\overset{\triangle}{d^{\mathcal{V}}}\ Y_t = e(W_{t^-}, Y_{t^-})\ \overset{\triangle}{d^{\mathcal{V}}}\ W_t$$

is equivalent to the continuous Itô covariant s.d.e

$$d^{\mathcal{V}} Y_t = e(W_t, Y_t) d^{\mathcal{V}} W_t.$$

Proof : By the previous proof, we showed that equation (13.4) is equivalent to (13.5). By definition 10.4, it implies that the s.d.e (13.2) is equivalent to the Itô s.d.e

$$\overset{\triangle}{d}\ y_t = f(Z_{t^-}, Y_{t^-})\ \overset{\triangle}{d}\ Z_t.$$

When (W_t) is continuous, so are $(Y_t) = ((x_t, y_t))$ and $(Z_t) = ((x_t, W_t))$. By proposition **??**, we get the continuous Itô s.d.e.

$$dy_t = f(Z_t, Y_t) dZ_t,$$

which is equivalent to

$$dY_t = e(W_t, Y_t) dW_t.\square$$

Application to $M = \mathbb{R}$

Assume that the manifold M is $\mathbb{R}$, endowed with the flat connection $p^{(M)}$, and the linear connection rule $\boldsymbol{\gamma}^{(M)}$. For every first order form α on TM, we consider the coefficient

$$\forall W = (r, w) \in TN, \forall Y = (x, y) \in \mathbb{R}^2,\quad e(W, Y) = \alpha_W(0, .) : T_w(T_r N) \to \mathbb{R}.$$

Then consider the particular s.d.e

$$\overset{\triangle}{d}\, y_t = \alpha_{W_{t^-}}(0, \overset{\triangle}{d^{\mathcal{V}}}\, W_t).$$

When (W_t) jumps, the jump of the solution (y_t) in $\mathbb{R}$ is those of the covariant integral $\int \alpha_{W_{t^-}} \overset{\triangle}{d^{\mathcal{V}}}\, W_t$, which should be the Itô covariant integral of part I.

Proposition 13.3 *Let p be a connection on N and $\boldsymbol{\gamma}$ a connection rule on TN. Then the solution of the Itô covariant s.d.e. with jumps*

$$\overset{\triangle}{d}\, y_t = \alpha_{W_{t^-}}(0, \overset{\triangle}{d^{\mathcal{V}}}\, W_t) \tag{13.6}$$

is the Itô covariant integral with jumps

$$\int \alpha_{W_{t^-}} \overset{\triangle}{d^{\mathcal{V}}}\, W_t$$

of definition 2.5.

Proof : It is enough to show the equality of the jumps of the solution (y_t) of (13.6) and those of the Stratonovich covariant integral.
First note that we can see (13.6) as a covariant s.d.e., since, with the flat connection $p^{(M)}$, we have $d^{\mathcal{V}} Y_t = dy_t$. Moreover, the Christoffel symbols of $p^{(M)}$ vanish everywhere and then, for every U in $T_x\mathbb{R}$, we get $p^{(M)}(U,0) = 0$ (recall the expression (46) for p).
Since $\boldsymbol{\gamma}^{(M)}$ is the linear connection rule, we have

$$\boldsymbol{\gamma}^{(M)}(Y_{t^-}, \Phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_t)) = (x_t - x_{t^-}, \Phi_{\mathcal{I}_1}((W_{t^-}, Y_{t^-}), W_t) - y_{t^-}).$$

Then, according to definition 13.1, the jump of the solution (y_t) of (13.6), when (W_t) jumps from W_{t^-} to W_t, is given by

$$\phi_{\mathcal{I}_1}((W_{t^-}, y_{t^-}), W_t) - y_{t^-} = \alpha_{W_t^-} p_{W_t^-} \boldsymbol{\gamma}(W_{t^-}, W_t),$$

which is also the jump of $\int \alpha_{W_t^-} \overset{\triangle}{d^{\mathcal{V}}}\, W_t$, according to definition 2.5.□

13.2 Itô s.d.e with covariant jumps

Consider the Itô s.d.e. with covariant jumps

$$\overset{\triangle \mathcal{V}}{d}\, Y_t \;=\; e(W_{t^-}, Y_{t^-})\; \overset{\triangle \mathcal{V}}{d}\, W_t, \quad Y_0 = Y_o.$$

When (W_t) jumps from W_t^- to W_t, the solution (Y_t) jumps from Y_{t^-} to $\Phi_{\mathcal{I}_2}((W_{t^-}, Y_{t^-}), W_t)$. According to the reminders, the solution (Y_t) will be defined if the map $\Phi_{\mathcal{I}_2}$ is.

Following the approach of part II, transports $\tau^{(N)}$ and $\tau^{(M)}$ are required, respectively on N and M.
Therefore, the covariant jump of (W_t) on N is given by

$$\tau^{(N)}_{W_{t^-}}(W_t) - w_{t^-}. \tag{13.7}$$

The covariant jump of the solution (Y_t) on TM is given, on the one hand, by

$$\tau^{(M)}_{Y_{t^-}}(\Phi_{\mathcal{I}_2}((W_{t^-}, Y_{t^-}), W_t))) - y_{t^-},$$

and one the other hand , as a consequence of (13.7), by

$$e(W_{t^-}, Y_{t^-})(\tau^{(N)}_{W_{t^-}}(W_t) - w_{t^-}).$$

To get an expression for $\Phi_{\mathcal{I}_2}((W_{t^-}, Y_{t^-}), W_t))$, we have to assume that the transport $\tau^{(M)}$ is invertible.

Then we get the following definition.

Definition 13.2 *Let $\tau^{(N)}$ be a transport on N and $\tau^{(M)}$ be an invertible transport on M. Let (W_t) be a càdlàg semimartingale in TN, (x_t) a càdlàg semimartingale in M, which jumps in M at the same times than (W_t) in TN, and Y_o a $\mathcal{F}_0$-measurable variable in TM.*

We say that $(Y_t) = ((x_t, y_t))$ is solution of the Itô s.d.e. with covariant jumps

$$\overset{\triangle\nu}{d}\ Y_t \;=\; e(W_{t^-}, Y_{t^-})\ \overset{\triangle\nu}{d}\ W_t, \quad Y_0 = Y_o, \tag{13.8}$$

if (Y_t) is solution of the s.d.e with jumps

$$\overset{\triangle}{dI}\ Y = \Phi_{\mathcal{I}_2}(Y, \overset{\triangle}{dI}\ W), \quad Y_0 = Y_o, \tag{13.9}$$

where

$$\Phi_{\mathcal{I}_2}((W_{t^-}, Y_{t^-}), W_t) = (\ x_t\ ,\ \tau^{(M)}_{Y_{t^-}}(x_t, .)^{-1}[y_{t^-} + e(W_{t^-}, Y_{t^-})(\tau^{(N)}_{W_{t^-}}(W_t) - w_{t^-})]\ ) \tag{13.10}$$

Proposition 13.4 *The Itô covariant s.d.e. with jumps* (13.8) *has a unique solution* $(Y_t) = ((x_t, y_t))$ *in TM, defined up to explosion.*

Proof : Since (13.8) is defined as equivalent to (13.9), let us show that (13.9) has a unique solution. First, let us see that $\Phi_{\mathcal{I}_2}$ satisfies $\Phi_{\mathcal{I}_2}((W_{t^-}, Y_{t^-}), W_{t^-}) = Y_{t^-}$. Recall that $x_t = x_{t^-}$ when

$W_t = W_{t^-}$. Moreover, recall that, by the properties of a transport, we have $\tau^{(N)}_{W_{t^-}}(W_{t^-}) = w_{t^-}$. Then we get

$$\Phi_{\mathcal{I}_2}((W_{t^-}, Y_{t^-}), W_{t^-}) = (x_{t^-}, \tau^{(M)}_{Y_{t^-}}(x_{t^-}, .)^{-1}[y_{t^-} + e(W_{t^-}, Y_{t^-})(0)],$$

that is

$$\Phi_{\mathcal{I}_2}((W_{t^-}, Y_{t^-}), W_{t^-}) = (x_{t^-}, \tau^{(M)}_{Y_{t^-}}(x_{t^-}, .)^{-1}(y_{t^-})) = (x_{t^-}, y_{t^-}) = Y_{t^-}.$$

Moreover, the map $V \to \tau_V$ is smooth for every transport τ, and the coefficient e is smooth, so the map $\Phi_{\mathcal{I}_2}$, defined, for every $W_1 = (r_1, w_1)$ and $W = (r, w)$ in TN, and every $Y_1 = (x_1, y_1)$ in TM, by

$$\Phi_{\mathcal{I}_2}((W_1, Y_1), W) = (\ x_1\ , \tau^{(M)}_{Y_1}(x_1, .)^{-1}(y_1 + e(W_1, Y_1)(\tau^{(N)}_{W_1}(W) - w_1))\ ),$$

is of class C^3 on a neighbourhood of the diagonal $\{((W, Y), W), (W, Y) \in TN \times TM\}$. Then, by proposition 10.1, there exist a unique solution of (13.8). □

Application to $M = \mathbb{R}$

Assume that the manifold M is $\mathbb{R}$, endowed with the flat transport $\tau^{(N)}$ defined, for every x in $\mathbb{R}$, by

$$\forall W = (y, w) \in \mathbb{R}^2, \quad \tau^{(N)}_x(W) = w.$$

For every first order form α on TM, we consider the coefficient

$$\forall W = (r, w) \in TN, \forall Y = (x, y) \in \mathbb{R}^2, \quad e(W, Y) = \alpha_W(0, .) : T_w(T_r N) \to \mathbb{R}.$$

Then consider the particular s.d.e

$$\overset{\triangle}{d}\ y_t = \alpha_{W_{t^-}}(0, \overset{\triangle\nu}{d}\ W_t).$$

When (W_t) jumps, the jump of the solution (y_t) in $\mathbb{R}$ is those of the covariant integral $\int \alpha_{W_{t^-}} \overset{\triangle\nu}{d}\ W_t$, which should be the Itô covariant integral of part II.

Proposition 13.5 *Let τ be a transport on N and $\boldsymbol{\gamma}$ a connection rule on TN. Then the solution of the Itô s.d.e. with covariant jumps*

$$\overset{\triangle}{d}\ y_t = \alpha_{W_{t^-}}(0, \overset{\triangle\nu}{d}\ W_t) \tag{13.11}$$

is the Itô integral with covariant jumps

$$\int \alpha_{W_{t^-}} \overset{\triangle\nu}{d}\ W_t$$

of definition 6.2.

Proof : It is enough to show the equality of the jumps of the solution (y_t) of (13.11) and those of the Stratonovich covariant integral.
First note that we can see (12.6) as a covariant s.d.e., since, with the flat transport $\tau^{(M)}$, we have $\tau_{Y_{t^-}}(Y_t) = y_t$. Morecover, we get $\tau^{(M)}_{Y_{t^-}}(x_t, .)^{-1}(u) = u$. Then, according to definition 13.2, the jump of the solution (y_t) of (13.11), when (W_t) jumps from W_{t^-} to W_t, is given by

$$\phi_{\mathcal{I}_2}((W_{t^-}, y_{t^-}), W_t) - y_{t^-} = \alpha_{W_{t^-}}(\tau^{(N)}_{W_{t^-}}(W_t) - w_{t^-}),$$

which is also the jump of $\int \alpha_{W_{t^-}} \overset{\triangle\nu}{d}\ W_t$, according to of definition 6.2.□

13.3 Discretisation of an Itô covariant s.d.e.

Let us write the discretisation theorem in the case described in part III, for which the two definitions coincide.

Proposition 13.6 *Let $(\tau^{//})^{(N)}$ and $(\tau^{//})^{(M)}$ be parallel transports along geodesics, respectively on N and M. Let $p^{(N)}$ be the connection on N, derived from $(\tau^{//})^{(N)}$, and $p^{(M)}$ be the connection on M, derived from $(\tau^{//})^{(M)}$. Let $\gamma^{(N)}$ and $\gamma^{(M)}$ be geodesics connection rules, respectively on TN and TM.*
Let $(W_t) = ((r_t, w_t))$ be a càdlàg semimartingale in TN, (x_t) a càdlàg semimartingale in M, and Y_o a $\mathcal{F}_0$- measurable variable in TM.

Then, the Itô covariant s.d.e with jumps (13.2) *and the Itô s.d.e with covariant jumps* (13.8) *are equivalent to*

$$\overset{\triangle}{dI}\ Y = \Phi_{\mathcal{I}}(Y, \overset{\triangle}{dI}\ W), \quad Y_0 = Y_o,$$

where $\Phi_{\mathcal{I}}((W_{t^-}, Y_{t^-}), W_t)$ is given by

$$\Phi_{\mathcal{I}}((W_{t^-}, Y_{t^-}), W_t) = (\ x_t\ ,\ (\tau^{//})^{(M)}_{x_{t^-}, x_t}[y_{t^-} + e(W_{t^-}, Y_{t^-})((\tau^{//})^{(N)}_{r_t, r_{t^-}}(w_t) - w_{t^-})]\ ). \tag{13.12}$$

Proof : By definition 13.2, we get exactly (13.12) when the transports are parallel transports along geodesics.
As definition 13.1 is concerned, we saw in part III that the geodesic connection rule and the parallel transport along geodesics commute. Then, writing $\Phi_{\mathcal{I}}((W_{t^-}, Y_{t^-}), W_t) = (x_t, \phi_{\mathcal{I}}((W_{t^-}, Y_{t^-}), W_t))$, we have the relations

$$p^{(M)}_{Y_{t^-}} \gamma^{(M)}(Y_{t^-}, \Phi_{\mathcal{I}}((W_{t^-}, Y_{t^-}), W_t)) = (\tau^{//})^{(M)}_{x_t, x_{t^-}}(\phi_{\mathcal{I}}((W_{t^-}, Y_{t^-}), W_t)) - y_{t^-}$$

and

$$p^{(N)}_{W_{t^-}} \gamma^{(N)}(W_{t^-}, W_{t^-}) = (\tau^{//})^{(N)}_{r_t, r_{t^-}}(w_t) - w_{t^-}.$$

Then, by inverting $(\tau^{//})^{(M)}_{x_t,.}$, definition 13.1 yields also (13.12). □

For such an Itô covariant s.d.e. with jumps, theorem 3 gives the following.

Theorem 5 *Consider the sequence (Y_t^n), defined by*

- $Y_0^n = Y_o$,
- *For* $t \in [T_k^n, T_{k+1}^n]$ *and* $k < k(n)$,

$$Y_t^n \;=\; (x_t \;,\; (\tau^{//})^{(M)}_{x_{T_n^k},x_t}[\; y^n_{T_n^k} + e(W_{T_n^k}, Y^n_{T_n^k})((\tau^{//})^{(N)}_{r_t,r_{T_n^k}}(w_t) - w_{T_n^k})\;]\;),$$

- *For* $t \geq T^n_{k(n)}$,

$$Y_t^n \;=\; (x_t \;,\; (\tau^{//})^{(M)}_{x_{T_n^{k(n)}},x_t}[\; y^n_{T_n^{k(n)}} + e(W_{T_n^{k(n)}}, Y^n_{T_n^{k(n)}})((\tau^{//})^{(N)}_{r_t,r_{T_n^{k(n)}}}(w_t) - w_{T_n^{k(n)}})\;]\;),$$

for a subdivision $(\sigma^n)_{n\in\mathbb{N}} = ((T_n^k)_{k\leq k(n)})_{n\in\mathbb{N}}$ *converging to the Identity.*

Then the process $(Y_t^n) = ((x_t, y_t^n))$ *converges uniformly on compacts in probability to the unique solution* $(Y_t) = ((x_t, y_t))$, *defined on* $[0, T]$, *of the Itô covariant s.d.e*

$$\overset{\triangle}{d}{}^{\mathcal{V}}\, Y_t \;=\; e(W_{t^-}, Y_{t^-})\;\; \overset{\triangle}{d}{}^{\mathcal{V}}\, W_t, \quad Y_0 = Y_o.$$

Appendix A

Coordinates on $T_V TM$ and $\tau_V TM$

We recall here the coordinates we use in the paper, which are standard coordinates on the spaces TM, τM, TTM, and τTM, arising from a given system of coordinates on M. For the tangent vector spaces of order two, we follow [3].

1. **Coordinates on M, T_xM, and τ_xM**

   Let x be a point of M, and $(x^i)_{i=1\ldots m}$ is a system of coordinates around x in M.

   The standard basis of T_xM is $(\frac{\partial}{\partial x^i})_{i=1\ldots m}$.
   The corresponding dual basis of T_x^*M is $(dx^i)_{i=1\ldots m}$.
   Then every u in T_xM is written

   $$u = dx^i(u)\frac{\partial}{\partial x^i}. \tag{A.1}$$

   A basis of τ_xM is given by $(\frac{\partial}{\partial x^i}, \frac{\partial^2}{\partial x^i \partial x^i}, \frac{\partial^2}{\partial x^i \partial x^j}(i<j))_{i,j=1\ldots m}$.
   The corresponding dual basis of τ_x^*M is $(d^2x^i, dx^i.dx^i, 2dx^i.dx^j(i<j))_{i,j=1\ldots m}$.
   Then every L in τ_xM is written

   $$L = d^2x^i(L)\frac{\partial}{\partial x^i} \; + \; dx^i.dx^j(L)\frac{\partial^2}{\partial x^i \partial x^j} \tag{A.2}$$

2. **Coordinates on $T_v(T_xM)$ and $\tau_v(T_xM)$**

   For every v in T_xM, $(v^i)_{i=1\ldots m} = (dx^i(v))_{i=1\ldots m}$ is a system of coordinates around v in T_xM.

   The standard basis of $T_v(T_xM)$ is $(\frac{\partial}{\partial v^i})_{i=1\ldots m}$.
   The corresponding dual basis of $T_v^*T_xM$ is $(dv^i)_{i=1\ldots m}$.
   Then every w in $T_v(T_xM)$ is written

   $$w = dv^i(w)\frac{\partial}{\partial v^i}. \tag{A.3}$$

A basis of $\tau_v(T_xM)$ is given by $(\frac{\partial}{\partial v^i}, \frac{\partial^2}{\partial v^i \partial v^i}, \frac{\partial^2}{\partial v^i \partial v^j}(i<j))_{i,j=1\ldots m}$.
The corresponding dual basis of $\tau_v^* T_x M$ is $(d^2v^i, dv^i.dv^i, 2dv^i.dv^j(i<j))_{i,j=1\ldots m}$.
Then every K in $\tau_v(T_xM)$ is written

$$K = d^2v^i(K)\frac{\partial}{\partial v^i} \; + \; dv^i.dv^j(K)\frac{\partial^2}{\partial v^i \partial v^j} \tag{A.4}$$

3. **Coordinates on TM, T_VTM and τ_VTM**

For every $V=(x,v)$ in TM, consider the systems $(x^i)_{i=1\ldots m}$ and $(v^i)_{i=1\ldots m}$ of coordinates around x and v described above.
Then $(V^i)_{i=1\ldots m} = ((x^i, v^i))_{i=1\ldots m}$ is a system of coordinates around V in TM.

The standard basis of T_VTM is $(\frac{\partial}{\partial V^i})_{i=1\ldots m} = ((\frac{\partial}{\partial x^i}, \frac{\partial}{\partial v^i}))_{i=1\ldots m}$.
The corresponding dual basis of T_V^*TM is $(dx^i, dv^i)_{i=1\ldots m}$.
Then every $\mathbb{U}$ in T_VTM is written

$$\mathbb{U} = dx^i(\mathbb{U})\frac{\partial}{\partial x^i} + dv^i(\mathbb{U})\frac{\partial}{\partial v^i}. \tag{A.5}$$

A basis of τ_VTM is given by

$$(\frac{\partial}{\partial V^i}, \frac{\partial^2}{\partial V^i \partial V^i}, \frac{\partial^2}{\partial V^i \partial V^j}(i<j))_{i,j=1\ldots m}$$

$$= (\frac{\partial}{\partial x^i}, \frac{\partial}{\partial v^i}, \frac{\partial^2}{\partial x^i \partial x^i}, \frac{\partial^2}{\partial x^i \partial x^j}(i<j), \frac{\partial^2}{\partial x^i \partial v^j}, \frac{\partial^2}{\partial v^i \partial v^j}(i<j))_{i,j=1\ldots m}.$$

Note that $\frac{\partial^2}{\partial x^i \partial v^j} = \frac{\partial^2}{\partial v^j \partial x^i}$ by the Schwartz lemma.

The corresponding dual basis of τ_V^*TM is given by

$$(d^2V^i, dV^i.dV^i, 2dV^i.dV^j(i<j)) = (d(dx^i,dv^i), (dx^i,dv^i).(dx^i,dv^i), 2(dx^i,dv^i).(dx^j,dv^j)(i<j))$$

$$= (d^2x^i, d^2v^i, dx^i.dx^i, 2dx^i.dx^j(i<j),$$

$$2dx^i.dv^i, 2dx^i.dv^j(i<j), 2dv^i.dx^j(i<j), dv^i.dv^i, 2dv^i.dv^j(i<j))_{i,j=1\ldots m}$$

Then every $\mathbb{L}$ in τ_VTM is written

$$\mathbb{L} \; = \; d^2x^i(\mathbb{L})\frac{\partial}{\partial x^i} + d^2v^i(\mathbb{L})\frac{\partial}{\partial v^i} + dx^i.dx^j(\mathbb{L})\frac{\partial^2}{\partial x^i \partial x^j} + 2\,dx^i.dv^j(\mathbb{L})\frac{\partial^2}{\partial x^i \partial v^j} + dv^i.dv^j(\mathbb{L})\frac{\partial^2}{\partial v^i \partial v^j}. \tag{A.6}$$

Appendix B

Differential and 2-jet

Let us recall here some expressions in coordinates for the differential and the 2-jet of a smooth map $\phi : E \to F$, where E and F are two differential manifolds.

We use the coordinates described in the previous appendix. Let z^i be coordinates on E and y^i coordinates on F.

Differential $d\phi$ of the map ϕ :

$$\forall z \in E, \forall w \in T_z E, d\phi(z)(\mathbb{L}) \in T_{\phi(z)}F$$

$$d\phi(z)(w) = \frac{\partial \phi^i}{\partial z^j}(z) dz^j(w) \frac{\partial}{\partial y^i}$$

2-jet $d^2\phi$ of the map ϕ :

$$\forall z \in E, \forall \mathbb{L} \in \tau_z E, d^2\phi(z)(\mathbb{L}) \in \tau_{\phi(z)}F$$

$$d^2\phi(z)(\mathbb{L}) = [\frac{\partial \phi^i}{\partial z^j}(z) dz^j(\mathbb{L}) + \frac{\partial^2 \phi^i}{\partial z^j \partial z^l}(z) dz^j dz^l(\mathbb{L})] \frac{\partial}{\partial y^i} + \frac{\partial \phi^i}{\partial z^j}(z) \frac{\partial \phi^k}{\partial z^l}(z) dz^j dz^k(\mathbb{L}) \frac{\partial^2}{\partial y^i \partial y^k}$$

Note that $d^2\phi(z)|_{T_z E} = d\phi(z)$

Let us also recall the following result of paragraph (6.19) in [3] :

Proposition B.1 *Let $f : F \to \mathbb{R}$ and $\phi : E \to F$ be smooth maps. Then the 2-jet of $f \circ \phi : E \to \mathbb{R}$ is given by*

$$d^2(f \circ \phi) = (d^2\phi)^*(d^2 f)$$

In an imbedding, we have

$$d^2(f \circ \phi)(z) = \frac{\partial f}{\partial z^i}(\phi(z)) d^2\phi^i(z) + \frac{\partial^2 f}{\partial z^i \partial z^j}(\phi(z)) d\phi^i . d\phi^j(z)$$

Appendix C

On the exponential map

Proposition C.1 *Let* exp *be an exponential map on M. For all x in M, $exp_x : T_xM \to M$ verifies*

$$d^2 \exp_x(0) = \tilde{i}_x(0) = Id_{\tau_x M}$$

It also implies

$$d \exp_x(0) = i_x(0) = Id_{T_x M}$$

Proof : For all x in M, let us compute the 2-jet $d^2 \exp_x(0) : \tau_0 T_x M \to \tau_{\exp_x(0)} M = \tau_x M$ at 0 of $\exp_x : T_x M \to M$, using normal coordinates at x. Since the 2-jet is intrinsic, the result won't depend on this choice.

Definition C.1 *(Normal coordinates) Given a linear frame $u = (u_1, u_2, ..., u_n)$ at x, the normal coordinate system $(x^1, x^2, ..., x^n)$ at x determined by u is the unique one such that the geodesic $\exp_x(tv)$ from x with initial tangent vector $v = v^i u_i$ is written in these coordinates : $\exp_x(tv)^i = tv^i$ When u is not precised, it is supposed to be the basis $(\frac{\partial}{\partial x^i})_{i=1...m}$ of $T_x M$*

If (x^i) is a system of normal coordinates at x and v a tangent vector of $T_x M$, the relation $\exp_x(tv)^i = tv^i$ at $t = 1$ gives $\exp_x(v)^i = v^i$. It implies

$$\frac{\partial \exp_x^i}{\partial v^k}(v) = \delta_{ki} \quad \text{and} \quad \frac{\partial^2 \exp_x^i}{\partial v^k \partial v^r}(v) = 0 \tag{C.1}$$

Let us use Appendix B. For every tangent vector of order two $\mathbb{L} = dv^i(\mathbb{L})\frac{\partial}{\partial v^i} + dv^i dv^j(\mathbb{L})\frac{\partial^2}{\partial v^i \partial v^j}$ in $\tau_v(T_x M)$, the 2-jet $d^2 \exp_x(v)(\mathbb{L})$ is given by

$$d^2 \exp_x(v)(\mathbb{L}) = dv^k(\mathbb{L}) \frac{\partial \exp_x^i}{\partial v^k}(v) \frac{\partial}{\partial \exp_x(v)^i}$$

$$+ \ dv^k dv^r(\mathbb{L}) \ [\ \frac{\partial^2 \exp_x^i}{\partial v^k \partial v^r}(v) \frac{\partial}{\partial \exp_x(v)^i} + \frac{\partial \exp_x^i}{\partial v^k}(v) \frac{\partial \exp_x^j}{\partial v^r}(v) \frac{\partial^2}{\partial \exp_x(v)^i \partial \exp_x(v)^j}]$$

By $(C.1)$, we get

$$d^2 \exp_x(v)(\mathbb{L}) = dv^i(\mathbb{L}) \frac{\partial}{\partial \exp_x(v)^i} + dv^i dv^j(\mathbb{L}) \frac{\partial^2}{\partial \exp_x(v)^i \partial \exp_x(v)^j}$$

For $v = 0$, $\exp_x(0) = x$ yields $\exp_x(v)^i = x^i$ and $dv^i = dx^i$, so that

$$d^2 \exp_x(0)(\mathbb{L}) = dx^i(\mathbb{L})\frac{\partial}{\partial x^i} + dx^i dx^j(\mathbb{L})\frac{\partial^2}{\partial x^i \partial x^j} = \mathbb{L}$$

Then

$$d^2 \exp_x(0) = Id_{\tau_x M}$$

Moreover, since $d\exp_x(0)$ is the restriction of $d^2 \exp_x(0)$ to first order vectors, we get

$$d\exp_x(0) = Id_{T_x M}\square$$

Appendix D

Some proofs

1. **Proof of remark** 0.1 :

Let us prove the equivalence between

$$\mathcal{H}_{\bar{\lambda}V}TM = d\bar{\lambda}(V)(\mathcal{H}_V TM) \tag{D.1}$$

and the second point of definition 0.2

$$\forall \lambda \in \mathbb{R}, \quad p_{\bar{\lambda}V} \circ d\bar{\lambda}(V) = \lambda p_V, \text{ where } \ \bar{\lambda}(V) = (x, \lambda.v) \tag{D.2}$$

Note that the vertical space $T_v(T_xM)$ verifies an analogous property than $(D.1)$. Indeed, since $d(\lambda v)^i(U) = \lambda dv^i(U)$, we have

$$T_{\bar{\lambda}V}(T_xM) = d\bar{\lambda}(V) \circ T_v(T_xM) \tag{D.3}$$

Let us first show that $(D.1)$ implies $(D.2)$.

- For every horizontal vector $U^{\mathcal{H}} \in \mathcal{H}_V TM$, we have $p_V(U^{\mathcal{H}}) = 0$ and so $\lambda p_V(U^{\mathcal{H}}) = 0$. $(D.1)$ implies that $d\bar{\lambda}(V)(U^{\mathcal{H}}) \in \mathcal{H}_{\bar{\lambda}V}TM$, what means that $p_{\bar{\lambda}V}(d\bar{\lambda}(V)(U^{\mathcal{H}})) = 0$. Then we get
$$p_{\bar{\lambda}V}(d\bar{\lambda}(V)(U^{\mathcal{H}})) = \lambda p_V(U^{\mathcal{H}}) = 0$$
so $(D.2)$ is proved for horizontal vectors.
- For every vertical vector $U^{\mathcal{V}} \in T_v(T_xM)$, , we have $p_V(U^{\mathcal{V}}) = U^{\mathcal{V}}$ so $\lambda p_V(U^{\mathcal{V}}) = \lambda U^{\mathcal{V}}$. But $\lambda U^{\mathcal{V}} = d\bar{\lambda}(V)(U^{\mathcal{V}})$ since $U^{\mathcal{V}} \in T_v(T_xM)$, so $\lambda p_V(U^{\mathcal{V}}) = d\bar{\lambda}(V)(U^{\mathcal{V}})$.
$(D.3)$ implies that $d\bar{\lambda}(V)(U^{\mathcal{V}}) \in T_{\bar{\lambda}V}(T_xM)$, what means that $p_{\bar{\lambda}V}(d\bar{\lambda}(V)(U^{\mathcal{V}})) = d\bar{\lambda}(V)(U^{\mathcal{V}})$.
The we get
$$p_{\bar{\lambda}V}(d\bar{\lambda}(V)(U^{\mathcal{V}})) = \lambda p_V(U^{\mathcal{V}}) = \lambda U^{\mathcal{V}}$$
so $(D.2)$ is proved for vertical vectors.

- Since the horizontal and the vertical spaces are supplementary spaces, every U in T_VTM can be decomposed as $U = U^{\mathcal{H}} + U^{\mathcal{V}}$. Then, using property (42) for p, the linearity of the map $d\bar{\lambda}(V)$, and the foregoing, we get $(D.2)$ for all U.

Reciprocally, let us show that $(D.2)$ implies $(D.1)$.
For every U in the horizontal space $\mathcal{H}_VTM$, we have $p_V(U) = 0$ and so, by $(D.2)$, $p_{\bar{\lambda}V}(d\bar{\lambda}(V)(U)) = 0$, that is $d\bar{\lambda}(V)(U) \in \mathcal{H}_{\bar{\lambda}V}TM$. Then we proved that $d\bar{\lambda}(V)(\mathcal{H}_VTM) \subset \mathcal{H}_{\bar{\lambda}V}TM$. Moreover, these two spaces have the same finite dimension, and the map $d\bar{\lambda}(V)$ is invertible, so we get $(D.1)$.□

2. **Proof of Proposition** 0.1 :

For $V = (x, v)$ in TM, we have $\bar{\lambda}V = (x, \lambda v)$. So we get, for all U in T_VTM,

$$d\bar{\lambda}(V)(U) = dx^i(U)\frac{\partial}{\partial x^i} + \lambda dv^i(U)\frac{\partial}{\partial v^i}$$

Then the second point of the definition 0.2 is equivalent to the following :

$$\lambda\ p_V(U) = p_{\bar{\lambda}V} \circ d\bar{\lambda}(U) \Leftrightarrow \lambda p_V(dx^i(U)\frac{\partial}{\partial x^i} + dv^i(U)\frac{\partial}{\partial v^i}) = p_{\bar{\lambda}V}(dx^i(U)\frac{\partial}{\partial x^i} + \lambda dv^i(U)\frac{\partial}{\partial v^i})$$

Using the linearity of p_V and (44), the above formula is equivalent to

$$\lambda dx^i(U)p_V(\frac{\partial}{\partial x^i}) + \lambda dv^i(U)\frac{\partial}{\partial v^i} = dx^i(U)p_{\bar{\lambda}V}(\frac{\partial}{\partial x^i}) + \lambda dv^i(U)\frac{\partial}{\partial v^i}$$

that is

$$\lambda p_V(\frac{\partial}{\partial x^i}) = p_{\bar{\lambda}V}(\frac{\partial}{\partial x^i})$$

So (45) is proved.
The existence of the functions Γ^i_{jk} is a consequence of (45) and the application of the following lemma

Lemma 4 *Let E and F be two vector spaces. If a map $f : E \to F$ is differentiable at* 0 *and satisfies $\forall\lambda \in \mathbb{R}, \forall v \in E, \quad f(\lambda v) = \lambda f(v)$, then it is a linear map.*

Proof : (of lemma) The property $f(\lambda v) = \lambda f(v)$ implies $f(0) = 0$ and $\forall v \in E, \quad df(0).v = \lim_{\lambda\to 0}(\frac{f(\lambda v)-f(0)}{\lambda}) = \lim_{\lambda\to 0}(\frac{\lambda f(v)}{\lambda}) = f(v)$. Hence $f = df(0)$ is linear.□

Set $\Gamma^i_j(x) : T_xM \to \mathbb{R}$ defined for all fixed x in M by $\Gamma^i_j(x)(v) = dx^i(p_{(x,v)}(\frac{\partial}{\partial x^j}))$. By the definition of p, $\Gamma^i_j(x)$ is differentiable and by (45) it satisfies $\forall\lambda \in \mathbb{R}, \ f(x)^i_j(\lambda v) = \lambda f(x)^i_j(v)$. Then by the lemma, it is a linear map and there exist functions (Γ^i_{jk}) defined for all v in T_xM by : $\Gamma^i_j(x)(v) = \Gamma^i_{jk}(x)v^k$. □

3. **Proof of Proposition** 3.1 :

Assertion 1. in the proposition express the fisrt point of the definition 3.1. Indeed, a projection on $T_v(T_xM)$ is a linear map such that $\bar{p}(\mathbb{U}) = \mathbb{U}$ if $\mathbb{U} \in T_v(T_xM)$.

By proposition 0.1, the third point of definition 3.1 implies the points $3(a)$ and $3(b)$ of proposition 3.1. Reciprocally, the point 1 (which implies in particular that $dv^l(\bar{p})(\frac{\partial}{\partial v^i}) = \delta^l_i$ and $dv^l dv^k(\bar{p})(\frac{\partial}{\partial v^i}) = 0$) and points $3(a)$ and $3(b)$ of proposition 3.1 implies, using again proposition 0.1, the third point of definition 3.1.

Let us prove the equivalence between the second point of the definition 3.1

$$\bar{p}_{\bar{\lambda}V} \circ d^2\bar{\lambda}(V)(\mathbb{L}) = d^2\bar{\lambda}(V) \circ \bar{p}_V(\mathbb{L}). \tag{D.4}$$

and the point 2 of proposition 3.1.
For $V = (x, v)$ in TM, we have $\bar{\lambda}V = (x, \lambda v)$. Thus, for all $\mathbb{L}$ in $\tau_V TM$:

$$d^2\bar{\lambda}(V)(\mathbb{L}) = dx^l(\mathbb{L})\frac{\partial}{\partial x^l} + \lambda dv^l(\mathbb{L})\frac{\partial}{\partial v^l} + dx^l dx^k(\mathbb{L})\frac{\partial^2}{\partial x^l \partial x^k}$$

$$+2\ \lambda\ dv^l dx^k(\mathbb{L})\frac{\partial^2}{\partial v^l \partial x^k} + \lambda^2 dv^l dv^k(\mathbb{L})\frac{\partial^2}{\partial v^l \partial v^k}. \tag{D.5}$$

Since the map $\bar{p}_V$ is $\tau_V T_xM$-valued, we have

$$dx^l(\bar{p}_V(\mathbb{L})) = dx^l dx^k(\bar{p}_V(\mathbb{L})) = dv^l dx^k(\bar{p}_V(\mathbb{L})) = 0.$$

Then $(D.5)$ implies :

$$d^2\bar{\lambda}(V)(\bar{p}_V(\mathbb{L})) = \lambda\ d^2v^l(\bar{p}_V(\mathbb{L}))\frac{\partial}{\partial v^l} + \lambda^2 dv^l dv^k(\bar{p}_V(\mathbb{L}))\frac{\partial^2}{\partial v^l \partial v^k}. \tag{D.6}$$

First, we apply $(D.4)$ to $\mathbb{L} = \frac{\partial^2}{\partial v^i \partial x^j}$.
On the one hand, we get, by $(D.5)$,

$$\bar{p}_{\bar{\lambda}V}(d^2\bar{\lambda}(V)(\frac{\partial^2}{\partial v^i \partial x^j})) = \bar{p}_{\bar{\lambda}V}(\lambda\frac{\partial^2}{\partial v^i \partial x^j}) = \lambda\bar{p}_{\bar{\lambda}V}(\frac{\partial^2}{\partial v^i \partial x^j}),$$

that is

$$\bar{p}_{\bar{\lambda}V}(d^2\bar{\lambda}(V)(\frac{\partial^2}{\partial v^i \partial x^j})) = \lambda d^2v^l\bar{p}_{\bar{\lambda}V}(\frac{\partial^2}{\partial v^i \partial x^j})\frac{\partial}{\partial v^l} + \lambda dv^l dv^k\bar{p}_{\bar{\lambda}V}(\frac{\partial^2}{\partial v^i \partial x^j})\frac{\partial^2}{\partial v^l \partial v^k} \tag{D.7}$$

On the other hand, we have, by $(D.6)$,

$$d^2\bar{\lambda}(V)(\bar{p}_V(\frac{\partial^2}{\partial v^i \partial x^j})) = \lambda\ d^2v^l(\bar{p}_V(\frac{\partial^2}{\partial v^i \partial x^j}))\frac{\partial}{\partial v^l} + \lambda^2 dv^l dv^k(\bar{p}_V(\frac{\partial^2}{\partial v^i \partial x^j}))\frac{\partial^2}{\partial v^l \partial v^k} \tag{D.8}$$

Then, according to $(D.7)$ and $(D.8)$, applying $(D.4)$ gives

$$d^2v^l\bar{p}_V(\frac{\partial^2}{\partial v^i\partial x^j}) = d^2v^l\bar{p}_{\bar{\lambda}V}(\frac{\partial^2}{\partial v^i\partial x^j}), \tag{D.9}$$

and

$$\lambda dv^l dv^k\bar{p}_V(\frac{\partial^2}{\partial v^i\partial x^j}) = dv^l dv^k\bar{p}_{\bar{\lambda}V}(\frac{\partial^2}{\partial v^i\partial x^j}). \tag{D.10}$$

Equation $(D.9)$ gives the point $2(a)$ of proposition 3.1. Equation $(D.10)$ with the application of lemma 4 yields the linearity of the map $v \to dv^l dv^k\bar{p}_V(\frac{\partial^2}{\partial v^i\partial x^j})$ of point $2(b)$.

Now apply $(D.4)$ to $\mathbb{L} = \frac{\partial^2}{\partial x^i\partial x^j}$.
On the one hand, we get, by $(D.5)$,

$$\bar{p}_{\bar{\lambda}V}(d^2\bar{\lambda}(V)(\frac{\partial^2}{\partial x^i\partial x^j})) = \bar{p}_{\bar{\lambda}V}(\frac{\partial^2}{\partial x^i\partial x^j}) \tag{D.11}$$

that is

$$\bar{p}_{\bar{\lambda}V}(d^2\bar{\lambda}(V)(\frac{\partial^2}{\partial x^i\partial x^j})) = d^2v^l\bar{p}_{\bar{\lambda}V}(\frac{\partial^2}{\partial x^i\partial x^j})\frac{\partial}{\partial v^l} + dv^l dv^k\bar{p}_{\bar{\lambda}V}(\frac{\partial^2}{\partial x^i\partial x^j})\frac{\partial^2}{\partial v^l\partial v^k} \tag{D.12}$$

On the other hand, we have, by $(D.6)$,

$$d^2\bar{\lambda}(V)(\bar{p}_V(\frac{\partial^2}{\partial x^i\partial x^j})) = \lambda\ d^2v^l(\bar{p}_V(\frac{\partial^2}{\partial x^i\partial x^j}))\frac{\partial}{\partial v^l} + \lambda^2 dv^l dv^k(\bar{p}_V(\frac{\partial^2}{\partial x^i\partial x^j}))\frac{\partial^2}{\partial v^l\partial v^k} \tag{D.13}$$

Then, according to $(D.12)$ and $(D.13)$, applying $(D.4)$ gives

$$\lambda d^2v^l\bar{p}_V(\frac{\partial^2}{\partial x^i\partial x^j}) = d^2v^l\bar{p}_{\bar{\lambda}V}(\frac{\partial^2}{\partial x^i\partial x^j}) \tag{D.14}$$

and

$$\lambda^2 dv^l dv^k\bar{p}_V(\frac{\partial^2}{\partial x^i\partial x^j}) = dv^l dv^k\bar{p}_{\bar{\lambda}V}(\frac{\partial^2}{\partial x^i\partial x^j}) \tag{D.15}$$

Equation $(D.14)$ with the application of lemma 4 yields the linearity of the map $v \to dv^l dv^k\bar{p}_V(\frac{\partial^2}{\partial x^i\partial x^j})$ of point $2(b)$. Note that the symmetry for the symbols $(\mathrm{I\!\Gamma}^l_{ijk})$ comes from those of $\frac{\partial^2}{\partial x^i\partial x^j}$ with $\frac{\partial^2}{\partial x^j\partial x^i}$.

Equation $(D.15)$ with the following lemma yields the point $2(c)$.

Lemma 5 *Let E and F be two vector spaces. If a map $f : E \to F$ is differentiable at* 0 *and satisfies $\forall\lambda \in \mathbb{R}, \forall v \in E,\quad f(\lambda^2 v) = \lambda^2 f(v)$, then it is a quadratic map.*

Proof : (of lemma) The property $f(\lambda^2 v) = \lambda^2 f(v)$ implies $f(0) = 0$ and $df(0)(v) = 0$. Moreover, the second differential for f, that we write $\partial^2 f(v,w)$, satisfies at point $(v,v) \in E \times E$,

$$\frac{1}{2}\partial^2 f(0)(v,v) = \lim_{\lambda \to 0}\left(\frac{f(\lambda v) - f(0) - df(0)(\lambda v)}{\lambda^2}\right) = \lim_{\lambda \to 0}\left(\frac{\lambda^2 f(v)}{\lambda^2}\right) = f(v).$$

Hence f is quadratic.$\square$

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