



Mécanique Quantique Matricielle et la Théorie des Cordes à Deux Dimensions dans des Fonds Non-triviaux

Serguei Y. Alexandrov

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Spécialité : **Physique Théorique**

présentée par

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pour obtenir le grade de

Docteur de l'Université Paris XI

Mécanique Quantique Matricielle et la Théorie des Cordes à Deux Dimensions dans des Fonds Non-triviaux

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Introduction

This thesis is devoted to application of the matrix model approach to non-critical string theory.

More than fifteen years have passed since matrix models were first applied to string theory. Although they have not helped to solve critical string and superstring theory, they have taught us many things about low-dimensional bosonic string theories. Matrix models have provided so powerful technique that a lot of results which were obtained in this framework are still inaccessible using the usual continuum approach. On the other hand, those results that were reproduced turned out to be in the excellent agreement with the results obtained by field theoretical methods.

One of the main subjects of interest in the early years of the matrix model approach was the $c = 1$ non-critical string theory which is equivalent to the two-dimensional critical string theory in the linear dilaton background. This background is the simplest one for the low-dimensional theories. It is flat and the dilaton field appearing in the low-energy target space description is just proportional to one of the spacetime coordinates.

In the framework of the matrix approach this string theory is described in terms of *Matrix Quantum Mechanics* (MQM). Already ten years ago MQM gave a complete solution of the 2D string theory. For example, the exact S -matrix of scattering processes was found and many correlation functions were explicitly calculated.

However, the linear dilaton background is only one of the possible backgrounds of 2D string theory. There are many other backgrounds including ones with a non-vanishing curvature which contain a dilatonic black hole. It was a puzzle during long time how to describe such backgrounds in terms of matrices. And only recently some progress was made in this direction.

In this thesis we try to develop the matrix model description of 2D string theory in non-trivial backgrounds. Our research covers several possibilities to deform the initial simple target space. In particular, we analyze winding and tachyon perturbations. We show how they are incorporated into Matrix Quantum Mechanics and study the result of their inclusion.

A remarkable feature of these perturbations is that they are exactly solvable. The reason is that the perturbed theory is described by Toda Lattice integrable hierarchy. This is the result obtained entirely within the matrix model framework. So far this integrability has not been observed in the continuum approach. On the other hand, in MQM it appears quite naturally being a generalization of the KP integrable structure of the $c < 1$ models. In this thesis we extensively use the Toda description because it allows to obtain many exact results.

We tried to make the thesis selfconsistent. Therefore, we give a long introduction into the subject. We begin by briefly reviewing the main concepts of string theory. We introduce

the Polyakov action for a bosonic string, the notion of the Weyl invariance and the anomaly associated with it. We show how the critical string theory emerges and explain how it is generalized to superstring theory avoiding to write explicit formulae. We mention also the modern view on superstrings which includes D-branes and dualities. After that we discuss the low-energy limit of bosonic string theories and possible string backgrounds. A special attention is paid to the linear dilaton background which appears in the discussion of non-critical strings. Finally, we present in detail 2D string theory both in the linear dilaton and perturbed backgrounds. We elucidate its degrees of freedom and how they can be used to perturb the theory.

The next chapter is an introduction to matrix models. We explain what the matrix models are and how they are related to various physical problems and to string theory, in particular. The relation is established through the sum over discretized surfaces and such important notions as the $1/N$ expansion and the double scaling limit are introduced. Then we consider the two simplest examples, the one- and the two-matrix model. They are used to present two of the several known methods to solve matrix models. First, the one-matrix model is solved in the large N -limit by the saddle point approach. Second, it is shown how to obtain the solution of the two-matrix model by the technique of orthogonal polynomials which works, in contrast to the first method, to all orders in perturbation theory. We finish this chapter giving an introduction to Toda hierarchy. The emphasis is done on its Lax formalism. Since the Toda integrable structure is the main tool of this thesis, the presentation is detailed and may look too technical. But this will be compensated by the power of this approach.

The third chapter deals with a particular matrix model — Matrix Quantum Mechanics. We show how it incorporates all features of 2D string theory. In particular, we identify the tachyon modes with collective excitations of the singlet sector of MQM and the winding modes of the compactified string theory with degrees of freedom propagating in the non-trivial representations of the $SU(N)$ global symmetry of MQM. We explain the free fermionic representation of the singlet sector and present its explicit solution both in the non-compactified and compactified cases. Its target space interpretation is elucidated with the help of the Das–Jevicki collective field theory.

Starting from the forth chapter, we turn to 2D string theory in non-trivial backgrounds and try to describe it in terms of perturbations of Matrix Quantum Mechanics. First, the winding perturbations of the compactified string theory are incorporated into the matrix framework. We review the work of Kazakov, Kostov and Kutasov where this was first done. In particular, we identify the perturbed partition function with a τ -function of Toda hierarchy showing that the introduced perturbations are integrable. The simplest case of the windings of the minimal charge is interpreted as a matrix model for the string theory in the black hole background. For this case we present explicit results for the free energy. Relying on these description, we explain our first work in this domain devoted to calculation of winding correlators in the theory with the simplest winding perturbation. This work is little bit technical. Therefore, we concentrate mainly on the conceptual issues.

The next chapter is about tachyon perturbations of 2D string theory in the MQM framework. It consists from three parts representing our three works. In the first one, we show how the tachyon perturbations should be introduced. Similarly to the case of windings, we find that the perturbations are integrable. In the quasiclassical limit we interpret them in terms of the time-dependent Fermi sea of fermions of the singlet sector. The second work

provides a thermodynamical interpretation to these perturbations. For the simplest case corresponding to the Sine–Liouville perturbation, we are able to find all thermodynamical characteristics of the system. However, many of the results do not have a good explanation and remain to be mysterious for us. In the third work we discuss the local and global structure of the string backgrounds corresponding to the perturbations introduced in the matrix model.

The sixth chapter is devoted to our fifth work where we establish an equivalence between the MQM description of tachyon perturbations and the so called Normal Matrix Model. We explain the basic features of the latter and its relation to various problems in physics and mathematics. The equivalence is interpreted as a kind of duality for which a mathematical as well as a physical sense can be given.

In the last chapter we formulate the problem discussed in our work on non-perturbative effects in matrix models and their relation to D-branes. This chapter is very short and we refer to the original paper for the exact statement of the obtained results and further details.

We attach the original text of our papers to the end of this thesis.

We would like to say several words about the presentation. We tried to do it in such a way that all the reported material would be connected by a continuous line of reasonings. Each result is supposed to be a more or less natural development of the previous ideas and results. Therefore, we tried to give a motivation for each step leading to something new. Also we explained various subtleties which occur sometimes and not always can be found in the published articles.

Finally, we tried to trace all the coefficients and signs and write all formulae in the once chosen normalization. Their discussion sometimes may seem to be too technical for the reader. But we hope he will forgive us because it is done to give the possibility to use this thesis as a source for correct equations in the presented domains.

Chapter I

String theory

String theory is now considered as the most promising candidate to describe the unification of all interactions and quantum gravity. It is a very wide subject of research possessing a very rich mathematical structure. In this chapter we will give just a brief review of the main ideas underlying string theory to understand its connection with our work. For a detailed introduction to string theory, we refer to the books [1, 2, 3].

1 Strings, fields and quantization

1.1 A little bit of history

String theory has a very interesting history in which one can find both the dark periods and remarkable breakthroughs of new ideas. In the beginning it appeared as an attempt to describe the strong interaction. In that time QCD was not yet known and there was no principle to explain a big tower of particles discovered in processes involving the strong interaction. Such a principle was suggested by Veneziano [4] in the so called *dual* models. He required that the sum of scattering amplitudes in s and t channels should coincide (see fig. I.1).

This requirement together with unitarity, locality and *etc.* was strong enough to fix completely the amplitudes. Thus, it was possible to find them explicitly for the simplest cases as well as to establish their general asymptotic properties. In particular, it was shown that the scattering amplitudes in dual models are much softer than the usual field theory amplitudes, so that the problems of field-theoretic divergences should be absent in these models.

Moreover, the found amplitudes coincided with scattering amplitudes of strings — objects extended in one dimension [5, 6, 7]. Actually, this is natural because for strings the property

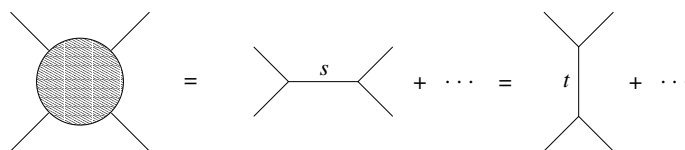


Fig. I.1: Scattering amplitudes in dual models.

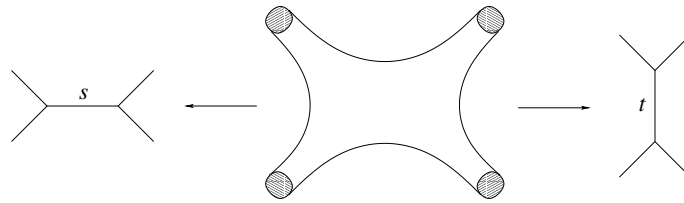


Fig. I.2: Scattering string amplitude can be seen in two ways.

of duality is evident: two channels can be seen as two degenerate limits of the same string configuration (fig. I.2). Also the absence of ultraviolet divergences got a natural explanation in this picture. In field theory the divergences appear due to a local nature of interactions related to the fact that the interacting objects are thought to be pointlike. When particles (pointlike objects) are replaced by strings the singularity is smoothed out over the string world sheet.

However, this nice idea was rejected by the discovery of QCD and description of all strongly interacting particles as composite states of fundamental quarks. Moreover, the exponential fall-off of string amplitudes turned out to be inconsistent with the observed power-like asymptotics. Thus, strings lost the initial reason to be related to fundamental physics.

But suddenly another reason was found. Each string possesses a spectrum of excitations. All of them can be interpreted as particles with different spins and masses. For a closed string, which can be thought just as a circle, the spectrum contains a massless mode of spin 2. But the graviton, quantum of gravitational interaction, has the same quantum numbers. Therefore, strings might be used to describe quantum gravity! If this is so, a theory based on strings should describe the world at the very microscopic level, such as the Planck scale, and should reproduce the standard model only in some low-energy limit.

This idea gave a completely new status to string theory. It became a candidate for the unified theory of all interactions including gravity. Since that time string theory has been developed into a rich theory and gave rise to a great number of new physical concepts. Let us have a look how it works.

1.2 String action

As is well known, the action for the relativistic particle is given by the length of its world line. Similarly, the string action is given by the area of its world sheet so that classical trajectories correspond to world sheets of minimal area. The standard expression for the area of a two-dimensional surface leads to the action [8, 9]

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int_{\Sigma} d\tau d\sigma \sqrt{-h}, \quad h = \det h_{ab}, \quad (\text{I.1})$$

which is called the Nambu–Goto action. Here α' is a constant of dimension of squared length. The matrix h_{ab} is the metric induced on the world sheet and can be represented as

$$h_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu, \quad (\text{I.2})$$

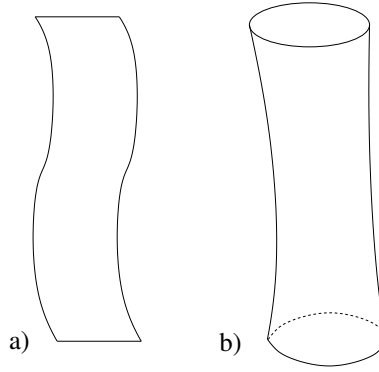


Fig. I.3: Open and closed strings.

where $X^\mu(\tau, \sigma)$ are coordinates of a point (τ, σ) on the world sheet in the spacetime where the string moves. Such a spacetime is called *target space* and $G_{\mu\nu}(X)$ is the metric there.

Due to the square root even in the flat target space the action (I.1) is highly non-linear. Fortunately, there is a much more simple formulation which is classically equivalent to the Nambu–Goto action. This is the Polyakov action [10]:

$$S_P = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d\tau d\sigma \sqrt{-h} G_{\mu\nu} h^{ab} \partial_a X^\mu \partial_b X^\nu. \quad (\text{I.3})$$

Here the world sheet metric is considered as a dynamical variable and the relation (I.2) appears only as a classical equation of motion. (More exactly, it is valid only up to some constant multiplier.) This means that we deal with a gravitational theory on the world sheet. We can even add the usual Einstein term

$$\chi = \frac{1}{4\pi} \int_{\Sigma} d\tau d\sigma \sqrt{-h} \mathcal{R}. \quad (\text{I.4})$$

In two dimensions $\sqrt{-h}\mathcal{R}$ is a total derivative. Therefore, χ depends only on the topology of the surface Σ , which one integrates over, and produces its Euler characteristic. In fact, any compact connected oriented two-dimensional surface can be represented as a sphere with g handles and b boundaries. In this case the Euler characteristic is

$$\chi = 2 - 2g - b. \quad (\text{I.5})$$

Thus, the full string action reads

$$S_P = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d\tau d\sigma \sqrt{-h} \left(G_{\mu\nu} h^{ab} \partial_a X^\mu \partial_b X^\nu + \alpha' \nu \mathcal{R} \right), \quad (\text{I.6})$$

where we introduced the coupling constant ν . In principle, one could add also a two-dimensional cosmological constant. However, in this case the action would not be equivalent to the Nambu–Goto action. Therefore, we leave this possibility aside.

To completely define the theory, one should also impose some boundary conditions on the fields $X^\mu(\tau, \sigma)$. There are two possible choices corresponding to two types of strings which one can consider. The first choice is to take Neumann boundary conditions $n^a \partial_a X^\mu = 0$ on $\partial\Sigma$, where n^a is the normal to the boundary. The presence of the boundary means that one considers an *open* string with two ends (fig. I.3a). Another possibility is given by periodic boundary conditions. The corresponding string is called *closed* and it is topologically equivalent to a circle (fig. I.3b).

1.3 String theory as two-dimensional gravity

The starting point to write the Polyakov action was to describe the movement of a string in a target space. However, it possesses also an additional interpretation. As we already mentioned, the two-dimensional metric h_{ab} in the Polyakov formulation is a dynamical variable. Besides, the action (I.6) is invariant under general coordinate transformations on the world sheet. Therefore, the Polyakov action can be equally considered as describing two-dimensional gravity coupled with matter fields X^μ . The matter fields in this case are usual scalars. The number of these scalars coincides with the dimension of the target space.

Thus, there are two *dual* points of view: target space and world sheet pictures. In the second one we can actually completely forget about strings and consider it as the problem of quantization of two-dimensional gravity in the presence of matter fields.

It is convenient to make the analytical continuation to the Euclidean signature on the world sheet $\tau \rightarrow -i\tau$. Then the path integral over two-dimensional metrics can be better defined, because the topologically non-trivial surfaces can have non-singular Euclidean metrics, whereas in the Minkowskian signature their metrics are always singular. In this way we arrive at a statistical problem for which the partition function is given by a sum over fluctuating two-dimensional surfaces and quantum fields on them¹

$$Z = \sum_{\text{surfaces } \Sigma} \int \mathcal{D}X_\mu e^{-S_P^{(E)}[X, \Sigma]}. \quad (\text{I.7})$$

The sum over surfaces should be understood as a sum over all possible topologies plus a functional integral over metrics. In two dimensions all topologies are classified. For example, for closed oriented surfaces the sum over topologies corresponds to the sum over genera g which is the number of handles attached to a sphere. In this case one gets

$$\sum_{\text{surfaces } \Sigma} = \sum_g \int \mathcal{D}\varrho(h_{ab}). \quad (\text{I.8})$$

On the contrary, the integral over metrics is yet to be defined. One way to do this is to discretize surfaces and to change the integral by the sum over discretizations. This way leads to matrix models discussed in the following chapters.

In string theory one usually follows another approach. It treats the two-dimensional diffeomorphism invariance as ordinary gauge symmetry. Then the standard Faddeev–Popov gauge fixing procedure is applied to make the path integral to be well defined. However, the Polyakov action possesses an additional feature which makes its quantization non-trivial.

1.4 Weyl invariance

The Polyakov action (I.6) is invariant under the local Weyl transformations

$$h_{ab} \longrightarrow e^{\phi} h_{ab}, \quad (\text{I.9})$$

where $\phi(\tau, \sigma)$ is any function on the world sheet. This symmetry is very crucial because it allows to exclude one more degree of freedom. Together with the diffeomorphism symmetry,

¹Note, that the Euclidean action $S_P^{(E)}$ differs by sign from the Minkowskian one.

it leads to the possibility to express at the classical level the world sheet metric in terms of derivatives of the spacetime coordinates as in (I.2). Thus, it is responsible for the equivalence of the Polyakov and Nambu–Goto actions.

However, the classical Weyl symmetry can be broken at the quantum level. The reason can be found in the non-invariance of the measure of integration over world sheet metrics. Due to the appearance of divergences the measure should be regularized. But there is no regularization preserving all symmetries including the conformal one.

The anomaly can be most easily seen analyzing the energy-momentum tensor T_{ab} . In any classical theory invariant under the Weyl transformations the trace of T_{ab} should be zero. Indeed, the energy-momentum tensor is defined by

$$T_{ab} = -\frac{2\pi}{\sqrt{-h}} \frac{\delta S}{\delta h^{ab}}. \quad (\text{I.10})$$

If the metric is varied along eq. (I.9) (ϕ should be taken infinitesimal), one gets

$$T_a^a = \frac{2\pi}{\sqrt{-h}} \frac{\delta S}{\delta \phi} = 0. \quad (\text{I.11})$$

However, in quantum theory T_{ab} should be replaced by a renormalized average of the quantum operator of the energy-momentum tensor. Since the renormalization in general breaks the Weyl invariance, the trace will not vanish anymore.

Let us restrict ourselves to the flat target space $G_{\mu\nu} = \eta_{\mu\nu}$. Then explicit calculations lead to the following anomaly

$$\langle T_a^a \rangle_{\text{ren}} = -\frac{c}{12} \mathcal{R}. \quad (\text{I.12})$$

To understand the origin of the coefficient c , we choose the flat gauge $h_{ab} = \delta_{ab}$. Then the Euclidean Polyakov action takes the following form

$$S_P^{(E)} = \nu \chi + \frac{1}{4\pi\alpha'} \int_{\Sigma} d\tau d\sigma \delta^{ab} \partial_a X^\mu \partial_b X_\mu. \quad (\text{I.13})$$

This action is still invariant under conformal transformations which preserve the flat metric. They are a special combination of the Weyl and diffeomorphism transformations of the initial action. Thus, the gauged fixed action (I.13) represents a particular case of conformal field theory (CFT). Each CFT is characterized by a number c , the so called *central charge*, which defines a quantum deformation of the algebra of generators of conformal transformations. It is this number that appears in the anomaly (I.12).

The central charge is determined by the field content of CFT. Each bosonic degree of freedom contributes 1 to the central charge, each fermionic degree of freedom gives 1/2, and ghost fields which have incorrect statistics give rise to negative values of c . In particular, the ghosts arising after a gauge fixation of the diffeomorphism symmetry contribute -26 . Thus, if strings propagate in the flat spacetime of dimension D , the central charge of CFT (I.13) is

$$c = D - 26. \quad (\text{I.14})$$

This gives the exact result for the Weyl anomaly. Thus, one of the gauge symmetries of the classical theory turns out to be broken. This effect can be seen also in another approaches

to string quantization. For example, in the framework of canonical quantization in the flat gauge one finds the breakdown of unitarity. Similarly, in the light-cone quantization one encounters the breakdown of global Lorentz symmetry in the target space. All this indicates that the Weyl symmetry is extremely important for the existence of a viable theory of strings.

2 Critical string theory

2.1 Critical bosonic strings

We concluded the previous section with the statement that to consistently quantize string theory we need to preserve the Weyl symmetry. How can this be done? The expression for the central charge (I.14) shows that it is sufficient to place strings into spacetime of dimension $D_{\text{cr}} = 26$ which is called *critical* dimension. Then there is no anomaly and quantum theory is well defined.

Of course, our real world is four-dimensional. But now the idea of Kaluza [11] and Klein [12] comes to save us. Namely, one supposes that extra 22 dimensions are compact and small enough to be invisible at the usual scales. One says that the initial spacetime is *compactified*. However, now one has to choose some compact space to be used in this compactification. It is clear that the effective four-dimensional physics crucially depends on this choice. But *a priori* there is no any preference and it seems to be impossible to find the right compactification.

Actually, the situation is worse. Among modes of the bosonic string, which are interpreted as fields in the target space, there is a mode with a negative squared mass that is a tachyon. Such modes lead to instabilities of the vacuum and can break the unitarity. Thus, the bosonic string theory in 26 dimensions is still a "bad" theory.

2.2 Superstrings

An attempt to cure the problem of the tachyon of bosonic strings has led to a new theory where the role of fundamental objects is played by superstrings. A superstring is a generalization of the ordinary bosonic string including also fermionic degrees of freedom. Its important feature is a supersymmetry. In fact, there are two formulations of superstring theory with the supersymmetry either in the target space or on the world sheet.

Green–Schwarz formulation

In the first formulation, developed by Green and Schwarz [13], to the fields X^μ one adds one or two sets of world sheet scalars θ^A . They transform as Majorana–Weyl spinors with respect to the global Lorentz symmetry in the target space. The number of spinors determines the number of supersymmetric charges so that there are two possibilities to have $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry. It is interesting that already at the classical level one gets some restrictions on possible dimensions D . It can be 3, 4, 6 or 10. However, the quantization selects only the last possibility which is the critical dimension for superstring theory.

In this formulation one has the explicit supersymmetry in the target space.² Due to this, the tachyon mode cannot be present in the spectrum of superstring and the spectrum starts with massless modes.

²Superstring can be interpreted as a string moving in a superspace.

RNS formulation

Unfortunately, the Green–Schwarz formalism is too complicated for real calculations. It is much more convenient to use another formulation with a supersymmetry on the world sheet [14, 15]. It represents a natural extension of CFT (I.13) being a two-dimensional super-conformal field theory (SCFT).³ In this case the additional degrees of freedom are world sheet fermions ψ^μ which form a vector under the global Lorentz transformations in the target space.

Since this theory is a particular case of conformal theories, the formula (I.12) for the conformal anomaly remains valid. Therefore, to find the critical dimension in this formalism, it is sufficient to calculate the central charge. Besides the fields discussed in the bosonic case, there are contributions to the central charge from the world sheet fermions and ghosts which arise after a gauge fixing of the local fermionic symmetry. This symmetry is a superpartner of the usual diffeomorphism symmetry and is a necessary part of supergravity. As was mentioned, each fermion gives the contribution $1/2$, whereas for the new superconformal ghosts it is 11 . As a result, one obtains

$$c = D - 26 + \frac{1}{2}D + 11 = \frac{3}{2}(D - 10). \quad (\text{I.15})$$

This confirms that the critical dimension for superstring theory is $D_{\text{cr}} = 10$.

To analyze the spectrum of this formulation, one should impose boundary conditions on ψ^μ . But now the number of possibilities is doubled with respect to the bosonic case. For example, since ψ^μ are fermions, for the closed string not only periodic, but also antiperiodic conditions can be chosen. This leads to the existence of two independent sectors called Ramond (R) and Neveu–Schwarz (NS) sectors. In each sector superstrings have different spectra of modes. In particular, from the target space point of view, R-sector describes fermions and NS-sector contains bosonic fields. But the latter suffers from the same problem as bosonic string theory — its lowest mode is a tachyon.

Is the fate of RNS formulation the same as that of the bosonic string theory in 26 dimensions? The answer is *not*. In fact, when one calculates string amplitudes of perturbation theory, one should sum over all possible spinor structures on the world sheet. This leads to a special projection of the spectrum, which is called Gliozzi–Scherk–Olive (GSO) projection [16]. It projects out the tachyon and several other modes. As a result, one ends up with a well defined theory.

Moreover, it can be checked that after the projection the theory possesses the global supersymmetry in the target space. This indicates that actually GS and RNS formulations are equivalent. This can be proven indeed and is related to some intriguing symmetries of superstring theory in 10 dimensions.

Consistent superstring theories

Once we have constructed general formalism, one can ask how many consistent theories of superstrings do exist? Is it unique or not?

³In fact, it is two-dimensional supergravity coupled with superconformal matter. Thus, in this formulation one has a supersymmetric generalization of the interpretation discussed in section 1.3.

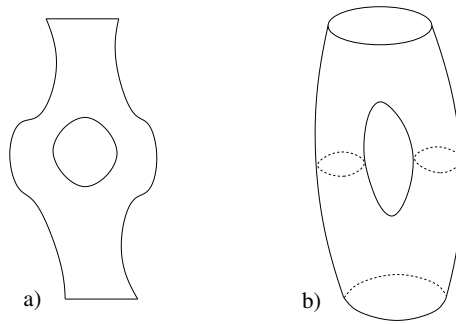


Fig. I.4: Interactions of open and closed strings.

At the classical level it is certainly not unique. One has open and closed, oriented and non-oriented, $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric string theories. Besides, in the open string case one can also introduce Yang–Mills gauge symmetry adding charges to the ends of strings. It is clear that the gauge group is not fixed anyhow. Finally, considering closed strings with $\mathcal{N} = 1$ supersymmetry, one can construct the so called heterotic strings where it is also possible to introduce a gauge group.

However, quantum theory in general suffers from anomalies arising at one and higher loops in string perturbation theory. The requirement of anomaly cancellation forces to restrict ourselves only to the gauge group $SO(32)$ in the open string case and $SO(32)$ or $E_8 \times E_8$ in the heterotic case [17]. Taking into account also restrictions on possible boundary conditions for fermionic degrees of freedom, one ends up with five consistent superstring theories. We give their list below:

- type IIA: $\mathcal{N} = 2$ oriented non-chiral closed strings;
- type IIB: $\mathcal{N} = 2$ oriented chiral closed strings;
- type I: $\mathcal{N} = 1$ non-oriented open strings with the gauge group $SO(32)$ + non-oriented closed strings;
- heterotic $SO(32)$: heterotic strings with the gauge group $SO(32)$;
- heterotic $E_8 \times E_8$: heterotic strings with the gauge group $E_8 \times E_8$.

2.3 Branes, dualities and M-theory

Since there are five consistent superstring theories, the resulting picture is not completely satisfactory. One should either choose a correct one among them or find a further unification. Besides, there is another problem. All string theories are defined only as asymptotic expansions in string coupling constant. This expansion is nothing else but the sum over genera of string world sheet in the closed case (see (I.8)) and over the number of boundaries in the open case. It is associated with *string loop expansion* since adding a handle (strip) can be interpreted as two subsequent interactions: a closed (open) string is emitted and then reabsorbed (fig. I.4).

Note, that from the action (I.13) it follows that each term in the partition function (I.7) is weighted by the factor $e^{-\nu\chi}$ which depends only on the topology of the world sheet. Due

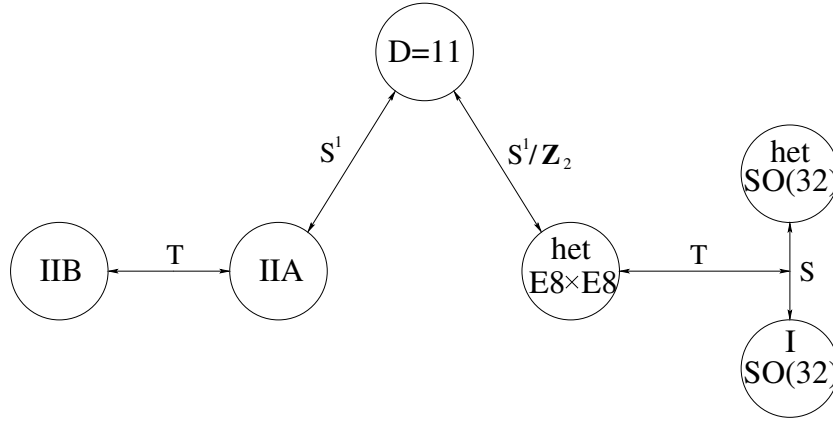


Fig. I.5: Chain of dualities relating all superstring theories.

to this one can associate $e^{2\nu}$ with each handle and e^ν with each strip. On the other hand, each interaction process should involve a coupling constant. Therefore, ν determines the closed and open string coupling constants

$$g_{\text{cl}} \sim e^\nu, \quad g_{\text{op}} \sim e^{\nu/2}. \quad (\text{I.16})$$

Since string theories are defined as asymptotic expansions, any finite value of ν leads to troubles. Besides, it looks like a free parameter and there is no way to fix its value.

A way to resolve both problems came from the discovery of a net of dualities relating different superstring theories. As a result, a picture was found where different theories appear as different vacua of a single (yet unknown) theory which got the name "M-theory". A generic point in its moduli space corresponds to an 11-dimensional vacuum. Therefore, one says that the unifying M-theory is 11 dimensional. In particular, it has a vacuum which is Lorentz invariant and described by 11-dimensional flat spacetime. It is shown in fig. I.5 as a circle labeled D=11.

Other superstring theories can be obtained by different compactifications of this special vacuum. Vacua with $\mathcal{N} = 2$ supersymmetry arise after compactification on a torus, whereas $\mathcal{N} = 1$ supersymmetry appears as a result of compactification on a cylinder. The known superstring theories are reproduced in some degenerate limits of the torus and cylinder. For example, when one of the radii of the torus is much larger than the other, so that one considers compactification on a circle, one gets the IIA theory. The small radius of the torus determines the string coupling constant. The IIB theory is obtained when the two radii both vanish and the corresponding string coupling is given by their ratio. Similarly, the heterotic and type I theories appear in the same limits for the radius and length of the cylinder.

This picture explains all existing relations between superstring theories, a part of which is shown in fig. I.5. The most known of them are given by T and S-dualities. The former relates compactified theories with inverse compactification radii and exchanges the windings around compactified dimension with the usual momentum modes in this direction. The latter duality says that the strong coupling limit of one theory is the weak coupling limit of another. It is important that T-duality has also a world sheet realization: it changes sign of the left modes on the string world sheet:

$$X_R \rightarrow X_R, \quad X_L \rightarrow -X_L. \quad (\text{I.17})$$

The above picture indicates that the string coupling constant is always determined by the background on which string theory is considered. Thus, it is not a free parameter but one of the moduli of the underlying M-theory.

It is worth to note that the realization of the dualities was possible only due to the discovery of new dynamical objects in string theory — D branes [18]. They appear in several ways. On the one hand, they are solitonic solutions of supergravity equations determining possible string backgrounds. On the other hand, they are objects where open strings can end. In this case Dirichlet boundary conditions are imposed on the fields propagating on the open string world sheet. Already at this point it is clear that such objects must present in the theory because the T-duality transformation (I.17) exchanges the Neumann and Dirichlet boundary conditions.

We stop our discussion of critical superstring theories here. We see that they allow for a nice unified picture of all interactions. However, the final theory remains to be hidden from us and we even do not know what principles should define it. Also a correct way to compactify extra dimensions to get the 4-dimensional physics is not yet found.

3 Low-energy limit and string backgrounds

3.1 General σ -model

In the previous section we discussed string theory in the flat spacetime. What changes if the target space is curved? We will concentrate here only on the bosonic theory. Adding fermions does not change much in the conclusions of this section.

In fact, we already defined an action for the string moving in a general spacetime. It is given by the σ -model (I.6) with an arbitrary $G_{\mu\nu}(X)$. On the other hand, one can think about a non-trivial spacetime metric as a coherent state of gravitons which appear in the closed string spectrum. Thus, the insertion of the metric $G_{\mu\nu}$ into the world sheet action is, roughly speaking, equivalent to summing of excitations of this mode.

But the graviton is only one of the massless modes of the string spectrum. For the closed string the spectrum contains also two other massless fields: the antisymmetric tensor $B_{\mu\nu}$ and the scalar dilaton Φ . There is no reason to turn on the first mode and to leave other modes non-excited. Therefore, it is more natural to write a generalization of (I.6) which includes also $B_{\mu\nu}$ and Φ . It is given by the most general world sheet action which is invariant under general coordinate transformations and renormalizable [19]:⁴

$$S_\sigma = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} \left[(h^{ab} G_{\mu\nu}(X) + i\epsilon^{ab} B_{\mu\nu}(X)) \partial_a X^\mu \partial_b X^\nu + \alpha' \mathcal{R}\Phi(X) \right], \quad (\text{I.18})$$

In contrast to the Polyakov action in flat spacetime, the action (I.18) is non-linear and represents an interacting theory. The couplings of this theory are coefficients of $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ of their expansion in X^μ . These coefficients are dimensionfull and the actual dimensionless couplings are their combinations with the parameter α' . This parameter has dimension of squared length and determines the *string scale*. It is clear that the perturbation expansion of the world sheet quantum field theory is an expansion in α' and, at the same time, it corresponds to the long-range or low-energy expansion in the target space. At large distances compared to the string scale, the internal structure of the string is not important and we should obtain an effective theory. This theory is nothing else but an effective field theory of massless string modes.

3.2 Weyl invariance and effective action

The effective theory, which appears in the low-energy limit, should be a theory for fields in the target space. On the other hand, from the world sheet point of view, these fields represent an infinite set of couplings of a two-dimensional quantum field theory. Therefore, equations of the effective theory should be some constraints on the couplings.

What are these constraints? The only condition, which is not imposed by hand, is that the σ -model (I.18) should define a consistent string theory. In particular, this means that the resulting quantum theory preserves the Weyl invariance. It is this requirement that gives the necessary equations on the target space fields.

With each field one can associate a β -function. The Weyl invariance requires the vanishing of all β -functions [20]. These are the conditions we were looking for. In the first order

⁴In the following, the world sheet metric is always implied to be Euclidean.

in α' one can find the following equations

$$\begin{aligned}\beta_{\mu\nu}^G &= R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\lambda\sigma} H_\nu{}^{\lambda\sigma} + O(\alpha') = 0, \\ \beta_{\mu\nu}^B &= -\frac{1}{2} \nabla_\lambda H_{\mu\nu}{}^\lambda + H_{\mu\nu}{}^\lambda \nabla_\lambda \Phi + O(\alpha') = 0, \\ \beta^\Phi &= \frac{D-26}{6\alpha'} - \frac{1}{4} R - \nabla^2 \Phi + (\nabla \Phi)^2 + \frac{1}{48} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + O(\alpha') = 0,\end{aligned}\tag{I.19}$$

where

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\mu}\tag{I.20}$$

is the field strength for the antisymmetric tensor $B_{\mu\nu}$.

A very non-trivial fact which, on the other hand, can be considered as a sign of consistency of the approach, is that the equations (I.19) can be derived from the spacetime action [19]

$$S_{\text{eff}} = \frac{1}{2} \int d^D X \sqrt{-G} e^{-2\Phi} \left[-\frac{2(D-26)}{3\alpha'} + R + 4(\nabla \Phi)^2 - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right].\tag{I.21}$$

All terms in this action are very natural representing the simplest Lagrangians for symmetric spin-2, scalar, and antisymmetric spin-2 fields. The first term plays the role of the cosmological constant. It is huge in the used approximation since it is proportional to α'^{-1} . But just in the critical dimension it vanishes identically.

The only non-standard thing is the presence of the factor $e^{-2\Phi}$ in front of the action. However, it can be removed by rescaling the metric. As a result, one gets the usual Einstein term what means that string theory reproduces Einstein gravity in the low-energy approximation.

3.3 Linear dilaton background

Any solution of the equations (I.19) defines a consistent string theory. In particular, among them one finds the simplest flat, constant dilaton background

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad B_{\mu\nu} = 0, \quad \Phi = \nu,\tag{I.22}$$

which is a solution of the equations of motion only in $D_{\text{cr}} = 26$ dimensions reproducing the condition we saw above.

There are also solutions which do not require any restriction on the dimension of space-time. To find them it is enough to choose a non-constant dilaton to cancel the first term in β^Φ . Strictly speaking, it is not completely satisfactory because the first term has another order in α' and, if we want to cancel it, one has to take into account contributions from the next orders. Nevertheless, there exist *exact* solutions which do not involve the higher orders. The most important solution is the so called *linear dilaton background*

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad B_{\mu\nu} = 0, \quad \Phi = l_\mu X^\mu,\tag{I.23}$$

where

$$l_\mu l^\mu = \frac{26-D}{6\alpha'}.\tag{I.24}$$

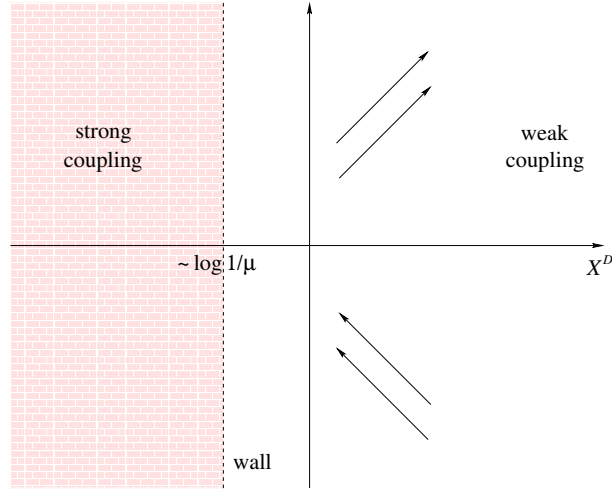


Fig. I.6: String propagation in the linear dilaton background in the presence of the tachyon mode. The non-vanishing tachyon produces a wall prohibiting the penetration into the region of a large coupling constant.

Note that the dilaton is a generalization of the coupling constant ν in (I.6). Therefore, from (I.16) it is clear that this is the dilaton that defines the string coupling constant which can now vary in spacetime

$$g_{cl} \sim e^{\Phi}. \quad (\text{I.25})$$

But then for the solution (I.23) there is a region where the coupling diverges and the string perturbation theory fails. This means that such background does not define a satisfactory string theory. However, there is a way to cure this problem.

3.4 Inclusion of tachyon

When we wrote the renormalizable σ -model (I.18), we actually missed one possible term which is a generalization of the two-dimensional cosmological constant

$$S_{\sigma}^T = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} T(X). \quad (\text{I.26})$$

From the target space point of view, it introduces a tachyon field which is the lowest mode of bosonic strings. One can repeat the analysis of section 3.2 and calculate the contributions of this term to the β -functions. Similarly to the massless modes, all of them can be deduced from the spacetime action which should be added to (I.21)

$$S_{\text{tach}} = -\frac{1}{2} \int d^D X \sqrt{-G} e^{-2\Phi} \left[(\nabla T)^2 - \frac{4}{\alpha'} T^2 \right]. \quad (\text{I.27})$$

Let us consider the tachyon as a field moving in the fixed linear dilaton background. Substituting (I.23) into the action (I.27), one obtains the following equation of motion

$$\partial^2 T - 2l^{\mu} \partial_{\mu} T + \frac{4}{\alpha'} T = 0. \quad (\text{I.28})$$

It is easy to find its general solution

$$T = \mu \exp(p_\mu X^\mu), \quad (p-l)^2 = \frac{2-D}{6\alpha'}. \quad (\text{I.29})$$

Together with (I.23), (I.29) defines a generalization of the linear dilaton background. Strictly speaking, it is not a solution of the equations of motion derived from the common action $S_{\text{eff}} + S_{\text{tach}}$. However, this action includes only the first order in α' , whereas, in general, as we discussed above, one should take into account higher order contributions. The necessity to do this is seen from the fact that the background fields (I.23) and (I.29) involve α' in a non-trivial way. The claim is that they give an *exact* string background. Indeed, in this background the complete σ -model action takes the form

$$S_\sigma^{\text{l.d.}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} \left[h^{ab} \partial_a X^\mu \partial_b X_\mu + \alpha' \mathcal{R} l_\mu X^\mu + \mu e^{p_\mu X^\mu} \right]. \quad (\text{I.30})$$

It can be checked that it represents an exact CFT and, consequently, defines a consistent string theory.

Why does the introduction of the non-vanishing tachyonic mode make the situation better? The reason is that this mode gives rise to an exponential potential, which suppresses the string propagation into the region where the coupling constant g_{cl} is large. It acts as an effective wall placed at $X^\mu \sim \frac{p^\mu}{p^2} \log(1/\mu)$. The resulting qualitative picture is shown in fig. I.6. Thus, we avoid the problem to consider strings at strong coupling.

4 Non-critical string theory

In the previous section we saw that, if to introduce non-vanishing expectation values for the dilaton and tachyon, it is possible to define consistent string theory not only in the spacetime of critical dimension $D_{\text{cr}} = 26$. Still one can ask the question: is there any sense for a theory where the conformal anomaly is not canceled? For example, if we look at the σ -model just as a statistical system of two-dimensional surfaces embedded into d -dimensional space and having some internal degrees of freedom, there is no reason for the system to be Weyl-invariant. Therefore, even in the presence of the Weyl anomaly, the system should possess some interpretation. It is called *non-critical string theory*.

When one uses the interpretation we just described, even at the classical level one can introduce terms breaking the Weyl invariance such as the world sheet cosmological constant. Then the conformal mode of the metric becomes a dynamical field and one should gauge fix only the world sheet diffeomorphisms. It can be done, for example, using the conformal gauge

$$h_{ab} = e^{\phi(\sigma)} \hat{h}_{ab}. \quad (\text{I.31})$$

As a result, one obtains an effective action where, besides the matter fields, there is a contribution depending on ϕ [10]. Let us work in the flat target space. Then, after a suitable rescaling of ϕ to get the right kinetic term, the action is written as

$$S_{\text{CFT}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\hat{h}} \left[\hat{h}^{ab} \partial_a X^\mu \partial_b X_\mu + \hat{h}^{ab} \partial_a \phi \partial_b \phi - \alpha' Q \hat{\mathcal{R}} \phi + \mu e^{\gamma\phi} + \text{ghosts} \right]. \quad (\text{I.32})$$

The second and third terms, which give dynamics to the conformal mode, come from the measure of integration over all fields due to its non-invariance under the Weyl transformations. The coefficient Q can be calculated from the conformal anomaly and is given by

$$Q = \sqrt{\frac{25-d}{6\alpha'}}. \quad (\text{I.33})$$

The coefficient γ is fixed by the condition that the theory should depend only on the full metric h_{ab} . This means that the effective action (I.32) should be invariant under the following Weyl transformations

$$\hat{h}_{ab}(\sigma) \longrightarrow e^{\rho(\sigma)} \hat{h}_{ab}(\sigma), \quad \phi(\sigma) \longrightarrow \phi(\sigma) - \rho(\sigma). \quad (\text{I.34})$$

This implies that the action (I.32) defines CFT. This is indeed the case only if

$$\gamma = -\frac{1}{\sqrt{6\alpha'}} \left(\sqrt{25-d} - \sqrt{1-d} \right). \quad (\text{I.35})$$

The CFT (I.32) is called Liouville theory coupled with $c = d$ matter. The conformal mode ϕ is the Liouville field.

The comparison of the two CFT actions (I.32) and (I.30) shows that they are equivalent if one takes $D = d + 1$, $p_\mu \sim l_\mu$ and identifies $X^D = \phi$. Then all coefficients also coincide as follows from (I.24), (I.29), (I.33) and (I.35). Thus, the conformal mode of the world sheet metric can be interpreted as an additional spacetime coordinate. With this interpretation non-critical string theory in the flat d -dimensional spacetime is seen as critical string theory in the $d + 1$ -dimensional linear dilaton background. The world sheet cosmological constant μ is identified with the amplitude of the tachyonic mode.

5 Two-dimensional string theory

In the following we will concentrate on the particular case of 2D bosonic string theory. It represents the main subject of this thesis. I hope to convince the reader that it has a very rich and interesting structure and, at the same time, it is integrable and allows for many detailed calculations.⁵ Thus, the two-dimensional case looks to be special and it is a particular realization of a very universal structure. It appears in the description of different physical and mathematical problems. We will return to this question in the last chapter of the thesis. Here we just mention two interpretations which, as we have already seen, are equivalent to the critical string theory.

From the point of view of non-critical strings, 2D string theory is a model of fluctuating two-dimensional surfaces embedded into 1-dimensional time. The second space coordinate arises from the metric on the surfaces.

Another possible interpretation of this system described in section 1.3 considers it as two-dimensional gravity coupled with the $c = 1$ matter. The total central charge vanishes since the Liouville field ϕ , arising due to the conformal anomaly, contributes $1 + 6\alpha'Q^2$, where Q is given in (I.33), and cancels the contribution of matter and ghosts.

5.1 Tachyon in two-dimensions

To see that the two-dimensional case is indeed very special, let us consider the effective action (I.27) for the tachyon field in the linear dilaton background

$$S_{\text{tach}} = -\frac{1}{2} \int d^D X e^{2Q\phi} \left[(\partial T)^2 - \frac{4}{\alpha'} T^2 \right], \quad (\text{I.36})$$

where ϕ is the target space coordinate coinciding with the gradient of the dilaton. According to the previous section, it can be considered as the conformal mode of the world sheet metric of non-critical strings. After the redefinition $T = e^{-Q\phi}\eta$, the tachyon action becomes an action of a scalar field in the flat spacetime

$$S_{\text{tach}} = -\frac{1}{2} \int d^D X \left[(\partial\eta)^2 + m_\eta^2 \eta^2 \right], \quad (\text{I.37})$$

where

$$m_\eta^2 = Q^2 - \frac{4}{\alpha'} = \frac{2-D}{6\alpha'} \quad (\text{I.38})$$

is the mass of this field. For $D > 2$ the field η has an imaginary mass being a real tachyon. However, for $D = 2$ it becomes massless. Although we will still call this mode "tachyon", strictly speaking, it represents a good massless field describing the stable vacuum of the two-dimensional bosonic string theory. As always, the appearance in the spectrum of the additional massless field indicates that the theory acquires some special properties.

In fact, the tachyon is the only field theoretic degree of freedom of strings in two dimensions. This is evident in the light cone gauge where there are physical excitations associated

⁵There is the so called $c = 1$ barrier which coincides with 2D string theory. Whereas string theories with $c \leq 1$ are solvable, we cannot say much about $c > 1$ cases.

with $D - 2$ transverse oscillations and the motion of the string center of mass. The former are absent in our case and the latter is identified with the tachyon field.

To find the full spectrum of states and the corresponding vertex operators, one should investigate the CFT (I.32) with one matter field X . The theory is well defined when the kinetic term for the X field enters with the $+$ sign so that X plays the role of a space coordinate. Thus, we will consider the following CFT

$$S_{\text{CFT}} = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{h}} \left[\hat{h}^{ab} \partial_a X \partial_b X + \hat{h}^{ab} \partial_a \phi \partial_b \phi - 2\hat{\mathcal{R}}\phi + \mu e^{-2\phi} + \text{ghosts} \right], \quad (\text{I.39})$$

where we chose $\alpha' = 1$ and took into account that in two dimensions $Q = 2$, $\gamma = -2$. This CFT describes the Euclidean target space. The Minkowskian version is defined by the analytical continuation $X \rightarrow it$.

The CFT (I.39) is a difficult interacting theory due to the presence of the Liouville term $\mu e^{-2\phi}$. Nevertheless, one can note that in the region $\phi \rightarrow \infty$ this interaction is negligible and the theory becomes free. Since the interaction is arbitrarily weak in the asymptotics, it cannot create or destroy states concentrated in this region. However, it removes from the spectrum all states concentrated at the opposite side of the Liouville direction. Therefore, it is sufficient to investigate the spectrum of the free theory with $\mu = 0$ and impose the so called *Seiberg bound* which truncates the spectrum by half [21].

The (asymptotic form of) vertex operators of the tachyon have already been found in (I.29). If $l_\mu = (0, -Q)$ and $p_\mu = (p_X, p_\phi)$, one obtains the equation

$$p_X^2 + (p_\phi + Q)^2 = 0 \quad (\text{I.40})$$

with the general solution ($Q = 2$)

$$p_X = ip, \quad p_\phi = -2 \pm |p|, \quad p \in \mathbf{R}. \quad (\text{I.41})$$

Imposing the Seiberg bound forbidding the operators growing at $\phi \rightarrow -\infty$, we have to choose the $+$ sign in (I.41). Thus, the tachyon vertex operators are

$$V_p = \int d^2\sigma e^{ipX} e^{(|p|-2)\phi}. \quad (\text{I.42})$$

Here p is the Euclidean momentum of the tachyon. When we go to the Minkowskian signature, the momentum should also be continued as follows

$$X \rightarrow it, \quad p \rightarrow -ik. \quad (\text{I.43})$$

As a result, the vertex operators take the form

$$\begin{aligned} V_k^- &= \int d^2\sigma e^{ik(t-\phi)} e^{-2\phi}, \\ V_k^+ &= \int d^2\sigma e^{-ik(t+\phi)} e^{-2\phi}, \end{aligned} \quad (\text{I.44})$$

where $k > 0$. The two types of operators describe outgoing right movers and incoming left movers, respectively. They are used to calculate the scattering of tachyons off the Liouville wall.

5.2 Discrete states

Although the tachyon is the only target space field in 2D string theory, there are also physical states which are remnants of the transverse excitations of the string in higher dimensions. They appear at special values of momenta and they are called *discrete states* [22, 23, 24, 25].

To define their vertex operators, we introduce the *chiral* fields

$$W_{j,m} = \mathcal{P}_{j,m}(\partial X, \partial^2 X, \dots) e^{2imX_L} e^{2(j-1)\phi_L}, \quad (\text{I.45})$$

$$\bar{W}_{j,m} = \mathcal{P}_{j,m}(\bar{\partial} X, \bar{\partial}^2 X, \dots) e^{2imX_R} e^{2(j-1)\phi_R}, \quad (\text{I.46})$$

where $j = 0, \frac{1}{2}, 1, \dots$, $m = -j, \dots, j$ and we used the decomposition of the world sheet fields into the chiral (left and right) components

$$X(\tau, \sigma) = X_R(\tau - i\sigma) + X_L(\tau + i\sigma) \quad (\text{I.47})$$

and similarly for ϕ . $\mathcal{P}_{j,m}$ are polynomials in the chiral derivatives of X . Their dimension is $j^2 - m^2$. Due to this, $\mathcal{P}_{j,\pm j} = 1$. For each fixed j , the set of operators $W_{j,m}$ forms an $\text{SU}(2)$ multiplet of spin j . Altogether, the operators (I.45) form $W_{1+\infty}$ algebra.

With the above definitions, the operators creating the discrete states are given by

$$V_{j,m} = \int d^2\sigma W_{j,m} \bar{W}_{j,m}, \quad (\text{I.48})$$

Thus, the discrete states appear at the following momenta

$$p_X = 2im, \quad p_\phi = 2(j-1). \quad (\text{I.49})$$

It is clear that the lowest and highest components $V_{j,\pm j}$ of each multiplet are just special cases of the vertex operators (I.42). The simplest non-trivial discrete state is the zero-momentum dilaton

$$V_{1,0} = \int d^2\sigma \partial X \bar{\partial} X. \quad (\text{I.50})$$

5.3 Compactification, winding modes and T-duality

So far we considered 2D string theory in the usual flat Euclidean or Minkowskian spacetime. The simplest thing which we can do with this spacetime is to compactify it. Since there is no translational invariance in the Liouville direction, it cannot be compactified. Therefore, we do compactification only for the Euclidean "time" coordinate X . We require

$$X \sim X + \beta, \quad \beta = 2\pi R, \quad (\text{I.51})$$

where R is the radius of the compactification. Because it is the time direction that is compactified, we expect the resulting Minkowskian theory be equivalent to a thermodynamical system at temperature $T = 1/\beta$.

The compactification restricts the allowed tachyon momenta to discrete values $p_n = n/R$ so that we have only a discrete set of vertex operators. Besides, depending on the radius, the compactification can create or destroy the discrete states. Whereas for rational values of the radius some discrete states are present in the spectrum, for general irrational radius there are no discrete states.

But the compactification also leads to the existence of new physical string states. They correspond to configurations where the string is wrapped around the compactified dimension. Such excitations are called *winding modes*. To describe these configurations in the CFT terms, one should use the decomposition (I.47) of the world sheet field X into the right and left moving components. Then the operators creating the winding modes, the *vortex* operators, are defined in terms of the dual field

$$\tilde{X}(\tau, \sigma) = X_R(\tau - i\sigma) - X_L(\tau + i\sigma). \quad (\text{I.52})$$

They also have a discrete spectrum, but with the inverse frequency: $q_m = mR$. In other respects they are similar to the vertex operators (I.42)

$$\tilde{V}_q = \int d^2\sigma e^{iq\tilde{X}} e^{(|q|-2)\phi}. \quad (\text{I.53})$$

The vertex and vortex operators are related by T-duality, which exchanges the radius of compactification $R \leftrightarrow 1/R$ and the world sheet fields corresponding to the compactified direction $X \leftrightarrow \tilde{X}$ (cf. (I.17)). Thus, from the CFT point of view it does not matter whether vertex or vortex operators are used to perturb the free theory. For example, the correlators of tachyons at the radius R should coincide with the correlators of windings at the radius $1/R$.⁶

Note that the self-dual radius $R = 1$ is distinguished by a higher symmetry of the system in this case. As we will see, its mathematical description is especially simple.

⁶In fact, one should also change the cosmological constant $\mu \rightarrow R\mu$ [26]. This change is equivalent to a constant shift of the dilaton which is necessary to preserve the invariance to all orders in the genus expansion.

6 2D string theory in non-trivial backgrounds

6.1 Curved backgrounds: Black hole

In the previous section we described the basic properties of string theory in two-dimensions in the linear dilaton background. In this thesis we will be interested in more general backgrounds. In the low-energy limit all of them can be described by an effective theory. Its action can be extracted from (I.21) and (I.27). Since there is no antisymmetric 3-tensor in two dimensions, the B -field does not contribute and we remain with the following action

$$S_{\text{eff}} = \frac{1}{2} \int d^2 X \sqrt{-G} e^{-2\Phi} \left[\frac{16}{\alpha'} + R + 4(\nabla\Phi)^2 - (\nabla T)^2 + \frac{4}{\alpha'} T^2 \right]. \quad (\text{I.54})$$

It is a model of dilaton gravity non-minimally coupled with a scalar field, the tachyon T . It is known to have solutions with non-vanishing curvature. Moreover, without the tachyon its general solution is well known and is written as [27] ($X^\mu = (t, r)$, $Q = 2/\sqrt{\alpha'}$)

$$ds^2 = -\left(1 - e^{-2Qr}\right) dt^2 + \frac{1}{1 - e^{-2Qr}} dr^2, \quad \Phi = \varphi_0 - Qr. \quad (\text{I.55})$$

In this form the solution resembles the radial part of the Schwarzschild metric for a spherically symmetric black hole. This is not a coincidence since the spacetime (I.55) does correspond to a two-dimensional black hole. At $r = -\infty$ the curvature has a singularity and at $r = 0$ the metric has a coordinate singularity corresponding to the black hole horizon. There is only one integration constant φ_0 which can be related to the mass of black hole

$$M_{\text{bh}} = 2Qe^{-2\varphi_0}. \quad (\text{I.56})$$

As the usual Schwarzschild black hole, this black hole emits the Hawking radiation at the temperature $T_H = \frac{Q}{2\pi}$ [28] and has a non-vanishing entropy [29, 30]. Thus, 2D string theory incorporates all problems of the black hole thermodynamics and represents a model to approach their solution. Compared to the quantum field theory analysis on curved spacetime, in string theory the situation is better since it is a well defined theory. Therefore, one can hope to solve the issues related to physics at Planck scale, such as microscopic description of the black hole entropy, which are inaccessible by the usual methods.

To accomplish this task, one needs to know the background not only in the low-energy limit but also at all scales. Remarkably, an exact CFT, which reduces in the leading order in α' to the world sheet string action in the black hole background (I.55), was constructed [31]. It is given by the so called $[\text{SL}(2, \mathbf{R})]_k/\text{U}(1)$ coset σ -model where k is the level of the representation of the current algebra. Relying on this CFT, the exact form of the background (I.55), which ensures the Weyl invariance in all orders in α' , was found [32]. We write it in the following form

$$ds^2 = -l^2(x)dt^2 + dx^2, \quad l(x) = \frac{(1-p)^{1/2} \tanh Qx}{(1-p \tanh^2 Qx)^{1/2}}, \quad (\text{I.57})$$

$$\Phi = \varphi_0 - \log \cosh Qx - \frac{1}{4} \log(1 - p \tanh^2 Qx), \quad (\text{I.58})$$

where p , Q and the level k are related by

$$p = \frac{2\alpha'Q^2}{1 + 2\alpha'Q^2}, \quad k = \frac{2}{p} = 2 + \frac{1}{\alpha'Q^2} \quad (\text{I.59})$$

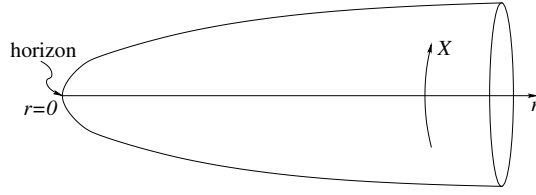


Fig. I.7: The Euclidean black hole.

so that in our case $p = 8/9$, $k = 9/4$. To establish the relation with the background (I.55), one should change the radial coordinate

$$Qr = \ln \left[\frac{\sqrt{1-p}}{1+\sqrt{1-p}} \left(\cosh Qx + \sqrt{\cosh^2 Qx + \frac{p}{1-p}} \right) \right] \quad (\text{I.60})$$

and take $p \rightarrow 0$ limit. This exact solution possesses the same properties as the approximate one. However, it is difficult to extract its quantitative thermodynamical characteristics such as mass, entropy, and free energy. The reason is that we do not know any action for which the metric (I.57) and the dilaton (I.58) give a solution.⁷ The existing attempts to derive these characteristics rely on some assumptions and lead to ambiguous results [34].

The form (I.57) of the solution is convenient for the continuation to the Euclidean metric. It is achieved by $t = -iX$ what changes sign of the first term. The resulting space can be represented by a smooth manifold if to take the time coordinate X be periodic with the period

$$\beta = \frac{2\pi}{Q\sqrt{1-p}}. \quad (\text{I.61})$$

The manifold looks as a cigar (fig. I.7) and the choice (I.61) ensures the absence of conical singularity at the tip. It is clear that this condition reproduces the Hawking temperature in the limit $p \rightarrow 0$ and generalizes it to all orders in α' . The function $l(x)$ multiplied by $R = \sqrt{\alpha'k}$ plays the role of the radius of the compactified dimension. It approaches the constant value R at infinity and vanishes at the tip so that this point represents the horizon of the Minkowskian black hole. Thus, the cigar describes only the exterior of the black hole.

Using the coset CFT, two and three-point correlators of tachyons and windings on the black hole background were calculated [32]. By T-duality they coincide with winding and tachyon correlators, respectively, on a dual spacetime, which is called *trumpet* and can be obtained replacing \cosh and \tanh in (I.57), (I.58) by \sinh and \coth . This dual spacetime describes a naked (without horizon) black hole of a negative mass [32]. In fact, it appears as a part of the global analytical continuation of the initial black hole spacetime.

6.2 Tachyon and winding condensation

In the CFT terms, string theory on the curved background considered above is obtained as a σ -model. If one chooses the dilaton as the radial coordinate, then the σ -model looks as CFT (I.39) where the kinetic term is coupled with the black hole metric $G_{\mu\nu}$ and there

⁷It is worth to mention the recent result that (I.57), (I.58) cannot be solution of any dilaton gravity model with only second derivatives [33].

is no Liouville exponential interaction. The change of the metric can be represented as a perturbation of the linear dilaton background by the gravitational vertex operator. Note that this operator creates one of the discrete states.

It is natural to consider also perturbations by another relevant operators existing in the initial CFT (I.39) defined in the linear dilaton background. First of all, these are tachyon vertex operators V_p (I.42). Besides, if we consider the Euclidean theory compactified on a circle, there exist vortex operators \tilde{V}_q (I.53). Thus, the both types of operators can be used to perturb the simplest CFT (I.39)

$$S = S_{\text{CFT}} + \sum_{n \neq 0} (t_n V_n + \tilde{t}_n \tilde{V}_n), \quad (\text{I.62})$$

where we took into account that tachyons and windings have discrete spectra in the compactified theory.

What backgrounds of 2D string theory do these perturbations correspond to? The couplings t_n introduce a non-vanishing vacuum expectation value of the tachyon. Thus, they simply change the background value of T . Note that, in contrast to the cosmological constant term $\mu e^{-2\phi}$, these tachyon condensates are time-dependent. For the couplings \tilde{t}_n we cannot give such a simple picture. The reason is that windings do not have a local target space interpretation. Therefore, it is not clear which local characteristics of the background change by the introduction of a condensate of winding modes.

A concrete proposal has been made for the simplest case $\tilde{t}_{\pm 1} \neq 0$, which is called *Sine-Liouville CFT*. In [35] it was suggested that this CFT is equivalent to the $\text{SL}(2, \mathbf{R})/\text{U}(1)$ σ -model describing string theory on the black hole background. This conjecture was justified by the coincidence of spectra of the two CFTs as well as of two and three-point correlators. Following this idea, it is natural to suppose that any general winding perturbation changes the target space metric.

Note, that the world sheet T-duality relates the CFT (I.62) with one set of couplings (t_n, \tilde{t}_n) and radius of compactification R to the similar CFT, where the couplings are exchanged (\tilde{t}_n, t_n) and the radius is inverse $1/R$. However, these two theories should not describe the same background because the target space interpretations of tachyons and windings are quite different. T-duality allows to relate their correlators, but it says nothing how their condensation changes the target space.

Chapter II

Matrix models

In this chapter we introduce a powerful mathematical technique, which allows to solve many physical problems. Its main feature is the use of matrices of a large size. Therefore, the models formulated using this technology are called *matrix models*. Sometimes a matrix formulation is not only a useful mathematical description of a physical system, but it also sheds light on its fundamental degrees of freedom.

We will be interested mostly in application of matrix models to string theory. However, in the beginning we should explain their relation to physics, their general properties, and basic methods to solve them (for an extensive review, see [36]). This is the goal of this chapter.

1 Matrix models in physics

Working with matrix models, one usually considers the situation when the size of matrices is very large. Moreover, these models imply integration over matrices or averaging over them taking all matrix elements as independent variables. This means that one deals with systems where some random processes are expected. Indeed, this is a typical behaviour for the systems described by matrix models.

Statistical physics

Historically, for the first time matrix models appeared in nuclear physics. It was discovered by Wigner [37] that the energy levels of large atomic nuclei are distributed according to the same law, which describes the spectrum of eigenvalues of one Hermitian matrix in the limit where the size of the matrix goes to infinity. Already this result showed the important feature of universality: it could be applied to any nucleus and did not depend on particular characteristics of this nucleus.

Following this idea, one can generalize the matrix description of statistics of energy levels to any system, which either has many degrees of freedom and is too complicated for an exact description, or possesses a random behaviour. A typical example of systems of the first type is given by mesoscopic physics, where one is interested basically only in macroscopic characteristics. The second possibility is realized, in particular, in chaotic systems.

Quantum chromodynamics

Another subject, where matrix models gave a new method of calculation, is particle physics. The idea goes back to the work of 't Hooft [38] where he suggested to use $1/N$ expansion for calculations in gauge theory with the gauge group $SU(N)$. Initially, he suggested this expansion for QCD as an alternative to the usual perturbative expansion, which is valid only in the weak coupling region and fails at low energies due to the confinement. However, in the case of QCD it is not well justified since the expansion parameter equals $1/3^2$ and is not very small.

Nevertheless, 't Hooft realized several important facts about the $1/N$ expansion. First, $SU(N)$ gauge theory can be considered as a model of $N \times N$ unitary matrices since the gauge fields are operators in the adjoint representation. Then the $1/N$ expansion corresponds to the limit of large matrices. Second, it coincides with the topological expansion where all Feynman diagrams are classified according to their topology, which one can associate if all lines in Feynman diagrams are considered as double lines. This gives the so called *fat graphs*. In the limit $N \rightarrow \infty$ only the *planar* diagrams survive. These are the diagrams which can be drawn on the 2-sphere without intersections. Thus, with each matrix model one can associate a diagrammatic expansion so that the size of matrices enters only as a prefactor for each diagram.

Although this idea has not led to a large progress in QCD, it gave rise to new developments, related with matrix models, in two-dimensional quantum gravity and string theory [39, 40, 41, 42, 43]. In turn, there is still a hope to find a connection between string theory and QCD relying on matrix models [44]. Besides, recently they were applied to describe supersymmetric gauge theories [45].

Quantum gravity and string theory

The common feature of two-dimensional quantum gravity and string theory is a sum over two-dimensional surfaces. It turns out that it also has a profound connection with matrix models ensuring their relevance for these two theories. We describe this connection in detail in the next section because it deserves a special attention. Here we just mention that the reformulation of string theory in terms of matrix models has lead to significant results in the low-dimensional cases such as 2D string theory. Unfortunately, this reformulation has not helped much in higher dimensions.

It is worth to note that there are matrix models of M-theory [46, 47], which is thought to be a unification of all string theories (see section I.2.3). (See also [48, 49, 50] for earlier attempts to describe the quantum mechanics of the supermembranes.) They claim to be fundamental non-perturbative and background independent formulations of Planck scale physics. However, they are based on the ideas different from the "old" matrix models of low-dimensional string theories.

Also matrix models appear in the so called *spin foam* approach to 3 and 4-dimensional quantum gravity [51, 52]. Similarly to models of M-theory, they give a non-perturbative and background independent formulation of quantum gravity but do not help with calculations.

2 Matrix models and random surfaces

2.1 Definition of one-matrix model

Now it is time to define what a matrix model is. In the most simple case of *one-matrix model* (1MM), one considers the following integral over $N \times N$ matrices

$$Z = \int dM \exp[-N \operatorname{tr} V(M)], \quad (\text{II.1})$$

where

$$V(M) = \sum_{k \geq 0} \frac{g_k}{k} M^k \quad (\text{II.2})$$

is a potential and the measure dM is understood as a product of the usual differentials of all independent matrix elements

$$dM = \prod_{i,j} dM_{ij}. \quad (\text{II.3})$$

The integral (II.1) can be interpreted as the partition function in the canonical ensemble of a statistical model. Also it appears as a generating function for the correlators of the operators $\operatorname{tr} M^k$, which are obtained differentiating Z with respect to the couplings g_k . For general couplings, the integral (II.1) is divergent and should be defined by analytical continuation.

Actually, one can impose some restrictions on the matrix M_{ij} which reflect the symmetry of the problem. Correspondingly, there exist 3 ensembles of random matrices:

- ensemble of hermitian matrices with the symmetry group $U(N)$;
- ensemble of real symmetric matrices with the symmetry group $O(N)$;
- ensemble of quaternionic matrices with the symmetry group $Sp(N)$.

We will consider only the first ensemble. Therefore, our systems will always possess the global $U(N)$ invariance under the transformations

$$M \longrightarrow \Omega^\dagger M \Omega \quad (\Omega^\dagger \Omega = I). \quad (\text{II.4})$$

In general, the integral (II.1) cannot be evaluated exactly and one has to use its perturbative expansion in coupling constants. Following the usual methodology of quantum field theories, each term in this expansion can be represented as a Feynman diagram. Its ingredients, propagator and vertices, can be extracted from the potential (II.2). The main difference with the case of a scalar field is that the propagator is represented by an oriented double line what reflects the index structure carried by matrices.

$$\begin{array}{c} j \\ i \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} k \\ l \end{array} = \frac{1}{Ng_2} \delta_{il} \delta_{jk}$$

The expression which is associated with the propagator can be obtained as an average of two matrices $\langle M_{ij} M_{kl} \rangle_0$ with respect to the Gaussian part of the potential. The coupling of indices corresponds to the usual matrix multiplication law. The vertices come from the

terms of the potential of third and higher powers. They are also composed from the double lines and are given by a product of Kronecker symbols.

$$= N \frac{g_k}{k} \delta_{j_1 i_2} \delta_{j_2 i_3} \cdots \delta_{j_k i_1}$$

Note that each loop in the diagrams gives the factor N coming from the sum over contracted indices. As a result, the partition function (II.1) is represented as

$$Z = \sum_{\text{diagrams}} \frac{1}{s} \left(\frac{1}{N g_2} \right)^E N^L \prod_k (-N g_k)^{n_k}, \quad (\text{II.5})$$

where the sum goes over all diagrams constructed from the drawn propagators and vertices (fat graphs) and we introduced the following notations:

- n_k is the number of vertices with k legs in the diagram,
- $V = \sum_k n_k$ is the total number of vertices,
- L is the number of loops,
- $E = \frac{1}{2} \sum_k k n_k$ is the number of propagators,
- s is the symmetry factor given by the order of the discrete group of symmetries of the diagram.

Thus, each diagram contributes to the partition function

$$\frac{1}{s} N^{V-E+L} g_2^{-E} \prod_k (-g_k)^{n_k}. \quad (\text{II.6})$$

2.2 Generalizations

Generalizations of the one-matrix model can be obtained by increasing the number of matrices. The simplest generalization is *two-matrix model* (2MM). In a general case it is defined by the following integral

$$Z = \int dA dB \exp[-NW(A, B)], \quad (\text{II.7})$$

where $W(A, B)$ is a potential invariant under the global unitary transformations

$$A \longrightarrow \Omega^\dagger A \Omega, \quad B \longrightarrow \Omega^\dagger B \Omega. \quad (\text{II.8})$$

The structure of this model is already much richer than the structure of the one-matrix model. It will be important for us in the study of 2D string theory.

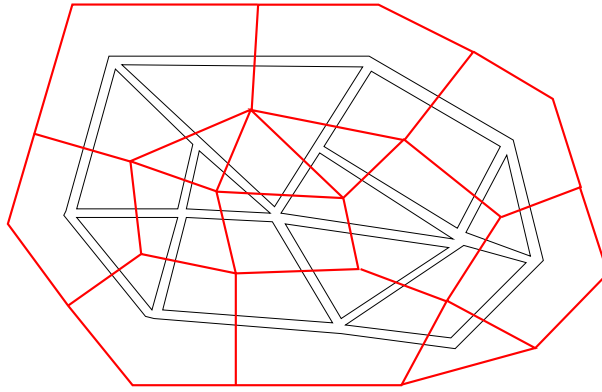


Fig. II.1: Duality between Feynman graphs and discretized surfaces.

Similarly, one can consider 3, 4, *etc.* matrix models. Their definition is the same as (II.7), where one requires the invariance of the potential under the simultaneous unitary transformation of all matrices. A popular choice for the potential is

$$W(A_1, \dots, A_n) = \sum_{k=1}^{n-1} c_k \operatorname{tr}(A_k A_{k+1}) - \sum_{k=1}^n \operatorname{tr} V_k(A_k). \quad (\text{II.9})$$

It represents a linear *matrix chain*. Other choices are also possible. For example, one can close the chain into a circle adding the term $\operatorname{tr}(A_n A_1)$. This crucially changes the properties of the model, since it loses integrability which is present in the case of the chain.

When the number of matrices increases to infinity, one can change the discrete index by a continuous argument. Then one considers a one-matrix integral, but the matrix is already a function. Interpreting the argument as a time variable, one obtains a quantum mechanical problem. It is called *matrix quantum mechanics* (MQM). The most part of the thesis is devoted to its investigation. Therefore, we will discuss it in detail in the next chapters.

Further generalizations include cases when one adds new arguments and discrete indices to matrices and combines them in different ways. One can even consider grassmanian, rectangular and other types of matrices. Also multitrace terms can be included into the potential.

2.3 Discretized surfaces

A remarkable fact, which allows to make contact between matrix models and two-dimensional quantum gravity and string theory, is that the matrix integral (II.1) can be interpreted as a sum over discretized surfaces [39, 40, 41]. Each Feynman diagram represented by a fat graph is dual to some triangulation of a two-dimensional surface as shown in fig. II.1. To construct the dual surface, one associates a k -polygon with each k -valent vertex and joins them along edges intersecting propagators of the Feynman diagram.

Note that the partition function is a sum over both connected and disconnected diagrams. Therefore, it gives rise to both connected and disconnected surfaces. If we are interested, as in quantum gravity, only in the connected surfaces, one should consider the free energy, which is the logarithm of the partition function, $F = \log Z$. Thus, taking into account (II.6),

the duality of matrix diagrams and discretized surfaces leads to the following representation

$$F = \sum_{\text{surfaces}} \frac{1}{s} N^{V-E+L} g_2^{-E} \prod_k (-g_k)^{n_k}. \quad (\text{II.10})$$

where we interpret:

- n_k is the number of k -polygons used in the discretization,
- $V = \sum_k n_k$ is the total number of polygons (faces),
- L is the number of vertices,
- $E = \frac{1}{2} \sum_k k n_k$ is the number of edges,
- s is the order of the group of automorphisms of the discretized surface.

It is clear that the relative numbers of k -polygons are controlled by the couplings g_k . For example, if one wants to use only triangles, one should choose the cubic potential in the corresponding matrix model.

The sum over discretizations (II.10) (more exactly, its continuum limit) can be considered as a definition of the sum over surfaces appearing in (I.8). Each discretization induces a curvature on the surface, which is concentrated at vertices where several polygons are joint to each other. For example, if at i th vertex there are $n_k^{(i)}$ k -polygons, the discrete counterpart of the curvature is

$$\mathcal{R}_i = 2\pi \frac{\left(2 - \sum_k \frac{k-2}{k} n_k^{(i)}\right)}{\sum_k \frac{1}{k} n_k^{(i)}}. \quad (\text{II.11})$$

It counts the deficit angle at the given vertex. In the limit of large number of vertices, the discretization approximates some continuous geometry. Varying discretization, one can approximate any continuous distribution of the curvature with any given accuracy.

Note, that the discretization encodes only the information about the curvature which is diffeomorphism invariant. Therefore, the sum over discretizations realizes already a gauge fixed version of the path integral over geometries. Due to this, one does not need to deal with ghosts and other problems related to gauge fixing.

If one considers generalizations of the one-matrix model, the dual surfaces will carry additional structures. For example, the Feynman diagrams of two-matrix model are drawn using two types of lines corresponding to two matrices. All vertices are constructed from the lines of a definite type since they come from the potential for either the first or the second matrix. Therefore, with each face of the dual discretization one can associate a discrete variable taking two values, say ± 1 . Summing over all structures, one obtains Ising model on a random lattice [53, 54]. Similarly, it is possible to get various fields living on two-dimensional dynamical surfaces or, in other words, coupled with two-dimensional gravity.

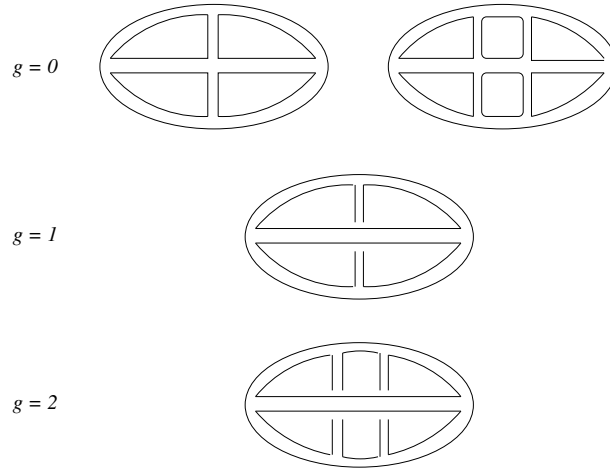


Fig. II.2: Diagrams of different genera.

2.4 Topological expansion

From (II.10) one concludes that each surface enters with the weight N^{V-E+L} . What is the meaning of the parameter N for surfaces? To answer this question, we note that the combination $V - E + L$ gives the Euler number $\chi = 2 - 2g$ of the surface. Indeed, the Euler number is defined as in (I.4). Under discretization the curvature turns into (II.11), the volume element at i th vertex becomes

$$\sqrt{h_i} = \sum_k n_k^{(i)} / k, \quad (\text{II.12})$$

and the integral over the surface is replaced by the sum over the vertices. Thus, the Euler number for the discretized surface is defined as

$$\chi = \frac{1}{4\pi} \sum_i \sqrt{h_i} \mathcal{R}_i = \frac{1}{2} \sum_i \left(2 - \sum_k \frac{k-2}{k} n_k^{(i)} \right) = L - \frac{1}{2} \sum_k (k-2) n_k = L - E + V. \quad (\text{II.13})$$

Due to this, we can split the sum over surfaces in (II.10) into the sum over topologies and the sum over surfaces of a given topology, which imitates the integral over metrics,

$$F = \sum_{g=0}^{\infty} N^{2-2g} F_g(g_k). \quad (\text{II.14})$$

Thus, N allows to distinguish surfaces of different topology. In the large N limit only surfaces of the spherical topology survive. Therefore, this limit is called also the *spherical limit*.

In terms of fat graphs of the matrix model, this classification by topology means the following. The diagrams appearing with the coefficient N^{2-2g} , which correspond to the surfaces of genus g , can be drawn on such surfaces without intersections. In particular, the leading diagrams coupled with N^2 are called *planar* and can be drawn on a 2-sphere or on a plane. For other diagrams, g can be interpreted as the minimal number of intersections which are needed to draw the diagram on a plane (see fig. II.2).

2.5 Continuum and double scaling limits

Our goal is to relate the matrix integral to the sum over Riemann surfaces. We have already completed the first step reducing the matrix integral to the sum over discretized surfaces. And the sum over topologies was automatically included. It remains only to extract a continuum limit.

To complete this second step, let us work for simplicity with the cubic potential

$$V(M) = \frac{1}{2}M^2 - \frac{\lambda}{3}M^3. \quad (\text{II.15})$$

Then (II.10) takes the form

$$F = \sum_{g=0}^{\infty} N^{\chi} \sum_{\substack{\text{genus } g \\ \text{triangulations}}} \frac{1}{s} \lambda^V. \quad (\text{II.16})$$

We compare this result with the partition function of two-dimensional quantum gravity

$$Z_{\text{QG}} = \sum_{g=0}^{\infty} \int \mathcal{D}\varrho(h_{ab}) e^{-\nu\chi - \mu A}, \quad (\text{II.17})$$

where g is the genus of the surface, $A = \int d^2\sigma \sqrt{h}$ is its area, and ν and μ are coupling constants. First of all, we see that one can identify $N = e^{-\nu}$. Then, if one assumes that all triangles have unit area, the total area is given by the number of triangles V . Due to this, (II.16) implies $\lambda = e^{-\mu}$.

However, this was a formal identification because one needs the coincidence of the partition function Z_{QG} and the free energy of the matrix model F . It is possible only in a continuum limit where we sum over the same set of continuous surfaces. In this limit the area of triangles used in the discretization should vanish. Since we fixed their area, in our case the continuum limit implies that the number of triangles should diverge $V \rightarrow \infty$.

In quantum theory one can speak only about expectation values. Hence we are interested in the behaviour of $\langle V \rangle$. In fact, for the spherical topology this quantity is dominated by non-universal contributions. To see the universal behaviour related to the continuum limit, one should consider more general correlation functions $\langle V^n \rangle$. From (II.16), for these quantities one obtains a simple expression

$$\langle V^n \rangle = \left(\lambda \frac{\partial}{\partial \lambda} \right)^n \log F. \quad (\text{II.18})$$

The typical form of the contribution of genus g to the free energy is

$$F_g \sim F_g^{(\text{reg})} + (\lambda_c - \lambda)^{(2-\gamma_{\text{str}})\chi/2}, \quad (\text{II.19})$$

where γ_{str} is the so called *string susceptibility*, which defines the critical behaviour, λ_c is some critical value of the coupling constant, and we took into account the non-universal contribution $F_g^{(\text{reg})}$. The latter leads to the fact that the expectation value $\langle V \rangle$ remains finite for $\chi > 0$. But for all $n > 1$, one finds

$$\frac{\langle V^n \rangle}{\langle V^{n-1} \rangle} \sim \frac{1}{\lambda_c - \lambda}. \quad (\text{II.20})$$

This shows that in the limit $\lambda \rightarrow \lambda_c$, the sum (II.16) is dominated by triangulations with large number of triangles. Thus, the continuum limit is obtained taking $\lambda \rightarrow \lambda_c$. One can renormalize the area and the couplings so that they remain finite in this limit.

Now we encounter the following problem. According to (II.19), the (universal part of the) free energy in the continuum limit either diverges or vanishes depending on the genus. On the other hand, in the natural limit $N \rightarrow \infty$ only the spherical contribution survives. How can one obtain contributions for all genera? It turns out that taking both limits not independently, but together in a correlated manner, one arrives at the desired result [55, 56, 57]. Indeed, we introduce the "renormalized" string coupling

$$\kappa^{-1} = N(\lambda_c - \lambda)^{(2-\gamma_{\text{str}})/2}, \quad (\text{II.21})$$

and consider the limit $N \rightarrow \infty$, $\lambda \rightarrow \lambda_c$, where κ is kept fixed. In this limit the free energy is written as an asymptotic expansion for $\kappa \rightarrow 0$

$$F = \sum_{g=0}^{\infty} \kappa^{-2+2g} f_g. \quad (\text{II.22})$$

Thus, the described limit allows to keep all genera in the expansion of the free energy. It is called the *double scaling limit*. In this limit the free energy of the one-matrix model reproduces the partition function of two-dimensional quantum gravity. This correspondence holds also for various correlators and is extended to other models. In particular, the double scaling limit of MQM, which we describe in the next chapter, gives 2D string theory.

The fact that nothing depends on the particular form of the potential, which is equivalent to the independence on the type of polygons used to discretize surfaces (triangles, quadrangles, *etc.*), is known as *universality* of matrix models. All of them can be splitted into classes of universality. Each class is associated with some continuum theory and characterized by the limiting behaviour of a matrix model near its critical point [58].

3 One-matrix model: saddle point approach

In the following two sections we review two basic methods to solve matrix models. This will be done relying on explicit examples of the simplest matrix models, 1MM and 2MM. This section deals with the first model, which is defined by the integral over one hermitian matrix

$$Z = \int dM \exp[-N \operatorname{tr} V(M)]. \quad (\text{II.23})$$

The potential was given in (II.2). Since the matrix is hermitian, its independent matrix elements are M_{ij} with $i \leq j$ and the diagonal elements are real. Therefore, the measure dM is given by

$$dM = \prod_i dM_{ii} \prod_{i < j} d\operatorname{Re} M_{ij} d\operatorname{Im} M_{ij}. \quad (\text{II.24})$$

3.1 Reduction to eigenvalues

Each hermitian matrix can be diagonalized by a unitary transformation

$$M = \Omega^\dagger x \Omega, \quad x = \operatorname{diag}(x_1, \dots, x_N), \quad \Omega^\dagger \Omega = I. \quad (\text{II.25})$$

Therefore, one can change variables from the matrix elements M_{ij} to the eigenvalues x_k and elements of the unitary matrix Ω diagonalizing M . This change produces a Jacobian. To find it, one considers a hermitian matrix which is obtained by an infinitesimal unitary transformation $\Omega = I + d\omega$ of the diagonal matrix x , where $d\omega$ is antisymmetric. In the first order in ω , one obtains

$$dM_{ij} \approx \delta_{ij} dx_j + [x, d\omega]_{ij} = \delta_{ij} dx_j + (x_i - x_j) d\omega_{ij}. \quad (\text{II.26})$$

This leads to the following result

$$dM = [d\Omega]_{SU(N)} \prod_{k=1}^N dx_k \Delta^2(x), \quad (\text{II.27})$$

where $[d\Omega]_{SU(N)}$ is the Haar measure on $SU(N)$ and

$$\Delta(x) = \prod_{i < j} (x_i - x_j) = \det_{i,j} x_i^{j-1} \quad (\text{II.28})$$

is the Vandermonde determinant.

Due to the $U(N)$ -invariance of the potential, after the substitution (II.25) into the integral (II.23) the unitary matrix Ω decouples and can be integrated out. As a result, one arrives at the following representation

$$Z = \operatorname{Vol}(SU(N)) \int \prod_{k=1}^N dx_k \Delta^2(x) \exp \left[-N \sum_{k=1}^N V(x_k) \right]. \quad (\text{II.29})$$

The volume of the $SU(N)$ group is a constant depending only on the matrix size N . It is not relevant for the statistics of eigenvalues, although it is important for the dependence of the free energy on N .

The representation (II.29) is very important because it reduces the problem involving N^2 degrees of freedom to the problem of only N eigenvalues. This can be considered as a "generalized integrability": models allowing such reduction are in a sense "integrable".

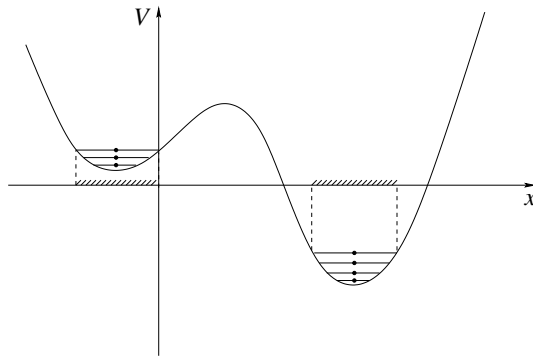


Fig. II.3: In the large N limit the eigenvalues fill finite intervals around the minima of the potential.

3.2 Saddle point equation

The integral (II.29) can also be presented in the form

$$Z = \int \prod_{k=1}^N dx_k e^{-N\mathcal{E}}, \quad \mathcal{E} = \sum_{k=1}^N V(x_k) - \frac{2}{N} \sum_{i<j} \log |x_i - x_j|, \quad (\text{II.30})$$

where we omitted the irrelevant constant factor. It describes a system of N particles interacting by the two-dimensional repulsive Coulomb law in the common potential $V(x)$. In the limit $N \rightarrow \infty$, one can apply the usual saddle point method to evaluate the integral (II.30) [59]. It says that the main contribution comes from configurations of the eigenvalues satisfying the classical equations of motion $\partial\mathcal{E}/\partial x_k = 0$. Thus, one obtains the following system of N algebraic equations

$$V'(x_k) = \frac{2}{N} \sum_{j \neq k} \frac{1}{x_k - x_j}. \quad (\text{II.31})$$

If we neglected the Coulomb force, all eigenvalues would sit at the minima of the potential $V(x)$. Due to the Coulomb repulsion they are spread around these minima and fill some finite intervals as shown in fig. II.3. In the large N limit their distribution is characterized by the density function defined as follows

$$\rho(x) = \frac{1}{N} \langle \text{tr} \delta(x - M) \rangle. \quad (\text{II.32})$$

The density contains an important information about the system. For example, the spherical limit of the free energy $F_0 = \lim_{N \rightarrow \infty} N^{-2} \log Z$ is given by

$$F_0 = - \int dx \rho(x) V(x) + \iint dx dy \rho(x) \rho(y) \log |x - y|. \quad (\text{II.33})$$

The system of equations (II.31) can be also rewritten as one integral equation for $\rho(x)$

$$V'(x) = 2\mathcal{P}.v. \int \frac{\rho(y)}{x - y} dy, \quad (\text{II.34})$$

where $\mathcal{P}.v.$ indicates the principal value of the integral. To solve this equation, we introduce the following function

$$\omega(z) = \int \frac{\rho(x)}{z-x} dx. \quad (\text{II.35})$$

Substitution of the definition (II.32) shows that it is the resolvent of the matrix M

$$\omega(z) = \frac{1}{N} \left\langle \text{tr} \frac{1}{z-M} \right\rangle. \quad (\text{II.36})$$

It is an analytical function on the whole complex plane except the intervals filled by the eigenvalues. At these intervals it has a discontinuity given by the density $\rho(x)$

$$\omega(x+i0) - \omega(x-i0) = -2\pi i \rho(x), \quad x \in \text{sup}[\rho]. \quad (\text{II.37})$$

On the other hand, the real part of the resolvent on the support of $\rho(x)$ coincides with the principal value integral as in (II.34). Thus, one obtains

$$\omega(x+i0) + \omega(x-i0) = V'(x), \quad x \in \text{sup}[\rho]. \quad (\text{II.38})$$

This equation is already simple enough to be solved explicitly.

3.3 One cut solution

In a general case the potential $V(x)$ has several minima and all of them can be filled by the eigenvalues. It means that the support of the density $\rho(x)$ will have several disconnected components. To understand the structure of the solution, let us consider the case when the support consists of only one interval (a, b) . Then the resolvent $\omega(z)$ should be an analytical function on the complex plane with one cut along this interval. On this cut the equations (II.37) and (II.38) must hold, They are sufficient to fix the general form of $\omega(z)$

$$\omega(z) = \frac{1}{2} \left(V'(z) - P(z) \sqrt{(z-a)(z-b)} \right). \quad (\text{II.39})$$

$P(z)$ is an analytical function which is fixed by the asymptotic condition

$$\omega(z) \underset{z \rightarrow \infty}{\sim} 1/z \quad (\text{II.40})$$

following from (II.35) and normalization of the density $\int \rho(x) dx = 1$. If $V(z)$ is a polynomial of degree n , $P(z)$ should be a polynomial of degree $n-2$. In particular, for the case of the Gaussian potential, it is a constant. The same asymptotic condition (II.40) fixes the boundaries of the cut, a and b .

The density of eigenvalues is found as the discontinuity of the resolvent (II.39) on the cut and is given by

$$\rho(x) = \frac{P(x)}{2\pi} \sqrt{(x-a)(b-x)}. \quad (\text{II.41})$$

If the potential is quadratic, then $P(x) = \text{const}$ and one obtains the famous semi-circle law of Wigner for the distribution of eigenvalues of random matrices in the Gaussian ensemble.

The free energy is obtained by substitution of (II.41) into (II.33). Note that the second term is one half of the first one due to (II.34). Therefore, one has

$$F_0 = -\frac{1}{2} \int_a^b dx \rho(x) V(x). \quad (\text{II.42})$$

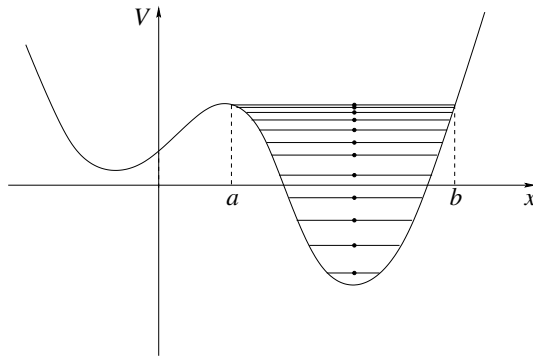


Fig. II.4: Critical point in one-matrix model.

3.4 Critical behaviour

The result (II.41) shows that near the end of the support the density of eigenvalues does not depend on the potential and always behaves as a square root. This is a manifestation of *universality* of matrix models mentioned in the end of section 2.

However, there are degenerate cases when this behaviour is violated. This can happen if the polynomial $P(x)$ vanishes at $x = a$ or $x = b$. For example, if a is a root of $M(x)$ of degree m , the density behaves as $\rho(x) \sim (x - a)^{m+1/2}$ near this point. It is clear that this situation is realized only for special values of the coupling constants g_k . They correspond to the critical points of the free energy discussed in connection with the continuum limit in paragraph 2.5. Indeed, one can show that near these configurations the spherical free energy (II.42) looks as (II.19) with $\chi = 2$ and $\gamma_{\text{str}} = -1/(m + 1)$ [58, 60].

Thus, each critical point with a given m defines a class of universality. All these classes correspond to continuum theories which are associated with special discrete series of CFTs living on dynamical surfaces. This discrete series is a part of the so called *minimal conformal theories* which possess only finite number of primary fields [61]. The minimal CFTs are characterized by two relatively prime integers p and q with the following central charge and string susceptibility

$$c = 1 - 6 \frac{(p - q)^2}{pq}, \quad \gamma_{\text{str}} = -\frac{2}{p + q - 1}. \quad (\text{II.43})$$

Our case is obtained when $p = 2m + 1$, $q = 2$. Thus, for $m = 1$ the central charge vanishes and we describe the pure two-dimensional quantum gravity. The critical points with other m describe the two-dimensional quantum gravity coupled to the matter characterized by the rational central charge $c = 1 - 3 \frac{(2m-1)^2}{2m+1}$.

From the matrix model point of view, there is a clear interpretation for the appearance of the critical points. Usually, in the large N limit the matrix eigenvalues fill consecutively the lowest energy levels in the well around a minimum of the effective potential. The critical behaviour arises when the highest filled level reaches an extremum of the effective potential. Generically, this happens as shown in fig. II.4. By fine tuning of the coupling constants, one can get more special configurations which will describe the critical behaviour with $m > 1$.

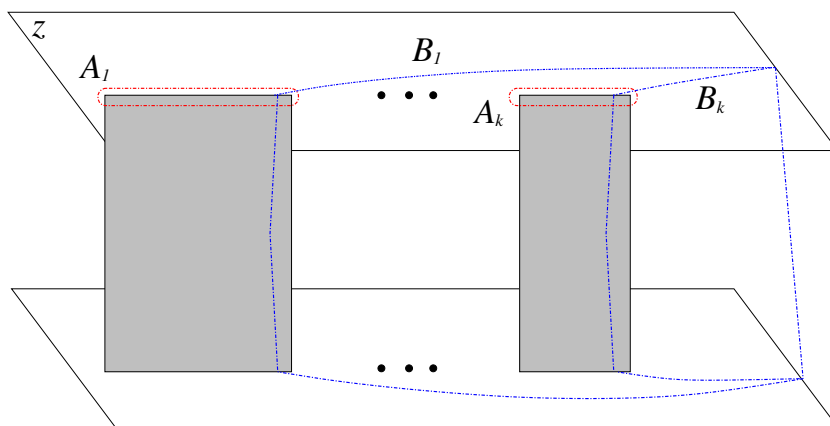


Fig. II.5: Riemann surface associated with the solution of 1MM.

3.5 General solution and complex curve

So far we considered the case where the density is concentrated on one interval. The general form of the solution can be obtained if we note that equation (II.38) can be rewritten as the following equation valid in the whole complex plane

$$y^2(z) = \tilde{Q}(z), \quad y = \omega(z) - \frac{1}{2}V'(z), \quad (\text{II.44})$$

where $\tilde{Q}(z)$ is some analytical function. Its form is fixed by the condition (II.40) which leads to $\tilde{Q}(z) = V'^2(z) + Q(z)$, where $Q(z)$ is a polynomial of degree $n - 2$. Thus, the solution reads

$$y(z) = \sqrt{V'^2(z) + Q(z)}. \quad (\text{II.45})$$

In a general case, the polynomial $\tilde{Q}(z)$ has $2(n - 1)$ roots so that the solution is

$$y(z) = \sqrt{\prod_{k=1}^{n-1} (z - a_k)(z - b_k)}, \quad (\text{II.46})$$

where we imply the ordering $a_1 < b_1 < a_2 < \dots < b_{n-1}$. The intervals (a_k, b_k) represent the support of the density. When a_k coincides with b_k , the corresponding interval collapses and we get a factor $(z - a_k)$ in front of the square root. When $n - 2$ intervals collapse, we return to the one cut solution (II.39). Note that at the formal level the eigenvalues can appear around each extremum, not only around minima, of the potential. However, the condensation of the eigenvalues around maxima is not physical and cannot be realized as a stable configuration.

Thus, we see that the solution of 1MM in the large N limit is completely determined by the resolvent $\omega(z)$ whose general structure is given in (II.45). It is an analytical function with at most $n - 1$ cuts having the square root structure. Therefore one can consider a Riemann surface associated with this function. It consists from two sheets joint by all cuts (fig. II.5). Similarly, it can be viewed as a genus n_c complex curve where n_c is the number of cuts.

On such curve there are $2n_c$ independent cycles. n_c compact cycles A_k go around cuts and n_c non-compact cycles B_k join cuts with infinity. The integrals of a holomorphic differential

along these cycles can be considered as the moduli of the curve. In our case the role of such differential is played by $y(z)dz$. From the definition (II.44) it is clear that the integrals along the cycles A_k give the relative numbers of eigenvalues in each cut

$$\frac{1}{2\pi i} \oint_{A_k} y(z)dz = \int_{a_k}^{b_k} \rho(x)dx \stackrel{\text{def}}{=} n_k. \quad (\text{II.47})$$

By definition $\sum_{k=1}^{n_c} n_k = 1$. The integrals along the cycles B_k can also be calculated and are given by the derivatives of the free energy [62, 63]

$$\int_{B_k} y(z)dz = \frac{\partial F_0}{\partial n_k}. \quad (\text{II.48})$$

Thus, the solution of the one-matrix model in the large N limit is encoded in the complex structure of the Riemann surface associated with the resolvent of the matrix.

4 Two-matrix model: method of orthogonal polynomials

In this section we consider the two-matrix model (II.7) which describes the Ising model on a random lattice. We restrict ourselves to the simplest case of the potential of the type (II.9). Thus, we are interested in the following integral over two hermitian matrices

$$Z = \int dA dB \exp \left[-N \operatorname{tr} \left(AB - V(A) - \tilde{V}(B) \right) \right], \quad (\text{II.49})$$

where the potentials $V(A)$ and $\tilde{V}(B)$ are some polynomials and the measures are the same as in (II.24). We will solve this model by the method of orthogonal polynomials. This approach can be also applied to 1MM where it looks even simpler. However, we would like to illustrate the basic features of 2MM and the technique of orthogonal polynomials is quite convenient for this.

4.1 Reduction to eigenvalues

Similarly to the one-matrix case, one can diagonalize the matrices and rewrite the integral in terms of their eigenvalues. However, now one has to use two unitary matrices, which are in general different, for the diagonalization

$$A = \Omega_A^\dagger x \Omega_A, \quad B = \Omega_B^\dagger y \Omega_B. \quad (\text{II.50})$$

At the same time, the action in (II.49) is invariant only under the common unitary transformation (II.8). Therefore, only one of the two unitary matrices is canceled. As a result, one arrives at the following representation

$$Z = C_N \int \prod_{k=1}^N dx_k dy_k e^{N(V(x_k) + \tilde{V}(y_k))} \Delta^2(x) \Delta^2(y) I(x, y), \quad (\text{II.51})$$

where

$$I(x, y) = \int [d\Omega]_{SU(N)} \exp \left[-N \operatorname{tr} \left(\Omega_A^\dagger x \Omega_B y \right) \right] \quad (\text{II.52})$$

and $\Omega = \Omega_A \Omega_B^\dagger$. The integral (II.52) is known as Itzykson–Zuber–Charish–Chandra integral and can be calculated explicitly [64]

$$I(x, y) = \tilde{C}_N \frac{\det e^{-N x_i y_j}}{\Delta(x) \Delta(y)}, \quad (\text{II.53})$$

where \tilde{C}_N is some constant. Substitution of this result into (II.51) leads to the cancellation of a half of the Vandermonde determinants. The remaining determinants make the integration measure antisymmetric under permutations x_k and y_k . Due to this antisymmetry, the determinant $\det e^{N x_i y_j}$ can be replaced by the the product of diagonal terms. The final result reads

$$Z = C'_N \int \prod_{k=1}^N d\mu(x_k, y_k) \Delta(x) \Delta(y), \quad (\text{II.54})$$

where we introduced the measure

$$d\mu(x, y) = dx dy e^{-N(x y - V(x) - \tilde{V}(y))}. \quad (\text{II.55})$$

4.2 Orthogonal polynomials

Let us introduce the system of polynomials orthogonal with respect to the measure (II.55)

$$\int d\mu(x, y) \Phi_n(x) \tilde{\Phi}_m(y) = \delta_{nm}. \quad (\text{II.56})$$

It is easy to check that they are given by the following expressions

$$\Phi_n(x) = \frac{1}{n! \sqrt{h_n}} \int \prod_{k=1}^n \frac{d\mu(x_k, y_k)}{h_{k-1}} \Delta(x) \Delta(y) \prod_{k=1}^n (x - x_k), \quad (\text{II.57})$$

$$\tilde{\Phi}_n(y) = \frac{1}{n! \sqrt{h_n}} \int \prod_{k=1}^n \frac{d\mu(x_k, y_k)}{h_{k-1}} \Delta(x) \Delta(y) \prod_{k=1}^n (y - y_k), \quad (\text{II.58})$$

where the coefficients h_n are fixed by the normalization condition in (II.56). They can be calculated recursively by the relation

$$h_n = \frac{1}{(n+1)!} \left(\prod_{k=1}^n h_{k-1} \right)^{-1} \int \prod_{k=1}^{n+1} d\mu(x_k, y_k) \Delta(x) \Delta(y). \quad (\text{II.59})$$

Due to this, the general form of the polynomials is the following

$$\Phi_n(x) = \frac{1}{\sqrt{h_n}} x^n + \sum_{k=0}^{n-1} c_{n,k} x^k, \quad (\text{II.60})$$

$$\tilde{\Phi}_n(y) = \frac{1}{\sqrt{h_n}} y^n + \sum_{k=0}^{n-1} d_{n,k} y^k. \quad (\text{II.61})$$

Using these polynomials, one can rewrite the partition function (II.54). Indeed, due to the antisymmetry the Vandermonde determinants (II.28) can be replaced by determinants of the orthogonal polynomials multiplied by the product of the normalization coefficients. Then one can apply the orthonormality relation (II.56) so that

$$Z = C'_N \left(\prod_{k=0}^{N-1} h_k \right) \int \prod_{k=1}^N d\mu(x_k, y_k) \det_{ij} (\Phi_{j-1}(x_i)) \det_{ij} (\tilde{\Phi}_{j-1}(y_i)) = C'_N N! \prod_{k=0}^{N-1} h_k. \quad (\text{II.62})$$

Thus, we reduced the problem of calculation of the partition function to the problem of finding the orthogonal polynomials and their normalization coefficients.

4.3 Recursion relations

To find the coefficients h_n , $c_{n,k}$ and $d_{n,k}$ of the orthogonal polynomials, one uses recursions relations which can be obtained for Φ_n and $\tilde{\Phi}_n$. They are derived using two pairs of conjugated operators which are the operators of multiplication and derivative [65]. We introduce them as the following matrices representing these operators in the basis of the orthogonal polynomials

$$x \Phi_n(x) = \sum_m X_{nm} \Phi_m(x), \quad \frac{1}{N} \frac{\partial}{\partial x} \Phi_n(x) = \sum_m P_{nm} \Phi_m(x), \quad (\text{II.63})$$

$$y \tilde{\Phi}_n(y) = \sum_m \tilde{\Phi}_m(y) Y_{mn}, \quad \frac{1}{N} \frac{\partial}{\partial y} \tilde{\Phi}_n(y) = \sum_m \tilde{\Phi}_m(y) Q_{mn}. \quad (\text{II.64})$$

Integrating by parts, one finds the following relations

$$P_{nm} = Y_{nm} - [V'(X)]_{nm}, \quad Q_{nm} = X_{nm} - [\tilde{V}'(Y)]_{nm}. \quad (\text{II.65})$$

If one takes the potentials $V(x)$ and $\tilde{V}(y)$ of degree p and q , correspondingly, the form of the orthogonal polynomials (II.60), (II.61) and relations (II.65) imply the following properties

$$\begin{aligned} X_{n,n+1} &= Y_{n+1,n} = \sqrt{h_{n+1}/h_n}, \\ X_{nm} &= 0, \quad m > n+1 \text{ and } m < n-q+1, \\ Y_{mn} &= 0, \quad m > n+1 \text{ and } m < n-p+1, \\ P_{n,n-1} &= Q_{n-1,n} = \frac{n}{N} \sqrt{h_{n-1}/h_n}, \\ P_{nm} &= Q_{nm} = 0, \quad m > n-1 \text{ and } m < n-(p-1)(q-1). \end{aligned} \quad (\text{II.66})$$

They indicate that it is more convenient to work with redefined indices

$$X_k(n/N) = X_{n,n-k}, \quad Y_k(n/N) = Y_{n-k,n} \quad (\text{II.67})$$

and similarly for P and Q . Then each set of functions can be organized into one operator as follows

$$\hat{X}(s) = \sum_{k=-1}^{q-1} X_k(s) \hat{\omega}^{-k}, \quad \hat{P}(s) = \sum_{k=1}^{(p-1)(q-1)} P_k(s) \hat{\omega}^{-k}, \quad (\text{II.68})$$

$$\hat{Y}(s) = \sum_{k=-1}^{p-1} \hat{\omega}^k Y_k(s), \quad \hat{Q}(s) = \sum_{k=1}^{(p-1)(q-1)} \hat{\omega}^k Q_k(s), \quad (\text{II.69})$$

where we denoted $s = n/N$ and introduced the shift operator $\hat{\omega} = e^{\frac{1}{N} \frac{\partial}{\partial s}}$. These operators satisfy

$$\hat{X}(n/N) \Phi_n(x) = x \Phi_n(x), \quad \hat{P}(n/N) \Phi_n(x) = \frac{1}{N} \frac{\partial}{\partial x} \Phi_n(x), \quad (\text{II.70})$$

$$\hat{Y}^\dagger(n/N) \tilde{\Phi}_n(x) = y \tilde{\Phi}_n(y), \quad \hat{Q}^\dagger(n/N) \tilde{\Phi}_n(y) = \frac{1}{N} \frac{\partial}{\partial y} \tilde{\Phi}_n(y), \quad (\text{II.71})$$

where the conjugation is defined as $(\hat{\omega} a(s))^\dagger = a(s) \hat{\omega}^{-1}$. With these definitions the relations (II.65) become

$$\hat{P}(s) = \hat{Y}(s) - V'(\hat{X}(s)), \quad \hat{Q}(s) = \hat{X}(s) - \tilde{V}'(\hat{Y}(s)). \quad (\text{II.72})$$

Substitution of the expansions (II.68) and (II.69) into (II.72) gives a system of finite-difference algebraic equations, which are obtained comparing the coefficients in front of powers of $\hat{\omega}$. Actually, one can restrict the attention only to negative powers. Then the reduced system contains only equations on the functions X_k and Y_k , because the left hand side of (II.72) is expanded only in positive powers of $\hat{\omega}$. This is a triangular system and for each given potential it can be solved by a recursive procedure. The free energy can be reproduced from the function $R_{n+1} \stackrel{\text{def}}{=} X_{-1}^2(n/N) = Y_{-1}^2(n/N)$. The representation (II.62) implies

$$Z = C'_N N! \prod_{k=0}^{N-1} R_k^{N-k}. \quad (\text{II.73})$$

This gives the solution for all genera.

In fact, to find the solution of (II.72), one has to use the perturbative expansion in $1/N$. Then the problem is reduced to a hierarchy of differential equations. The hierarchy appearing here is Toda lattice hierarchy which will be described in detail in the next section. The use of methods of Toda theory simplifies the problem and allows to find explicit differential, and even algebraic equations directly for the free energy.

4.4 Critical behaviour

We saw in section 3.4 that the multicritical points of the one-matrix model correspond to a one-parameter family of the minimal conformal theories. The two-matrix model is more general than the one-matrix model. Therefore, it can encompass a larger class of continuous models. It turns out that all minimal models (II.43) can be obtained by appropriately adjusting the matrix model potentials [65, 66].

There is an infinite set of critical potentials for each (p, q) point. The simplest one is when one of the potentials has degree p and the other has degree q . Their explicit form has been constructed in [66]. The key fact of the construction is that the momentum function $P(\omega, s)$ (up to an analytical piece) is identified with the resolvent (II.36) of the corresponding matrix X . Comparing with the one-matrix case, one concludes that the (p, q) point is obtained when the resolvent behaves near its singularity as $(x - x_c)^{p/q}$. Thus, it is sufficient to take the operators with the following behaviour at $\omega \rightarrow 1$

$$X - X_c \sim (\log \omega)^q, \quad P - P_c \sim (\log \omega)^p. \quad (\text{II.74})$$

Note that the singular asymptotics of Y and Q follow from (II.72) and lead to the dual (q, p) point. Then one can take some fixed $X(w)$ and $Y(w)$ with the necessary asymptotics at $\omega \rightarrow 1$ and $\omega \rightarrow \infty$ and solve the equations (II.72) with respect to the potentials $V(X)$ and $\tilde{V}(Y)$. The resulting explicit formulae can be found in [66].

4.5 Complex curve

As in the one-matrix model, the solution of 2MM in the large N limit can be represented in terms of a complex curve [67, 68]. However, there is a difference between these two cases. Whereas in the former case the curve coincides with the Riemann surface of the resolvent, in the latter case the origin of the curve is different. To illuminate it, let us consider how the solution of the model in the large N limit arises.

In this approximation one can apply the saddle point approach described in the previous section. It leads to the following two equations on the resolvents of matrices X and Y

$$y = V'(x) + \omega(x), \quad x = \tilde{V}'(y) + \tilde{\omega}(y). \quad (\text{II.75})$$

In fact, these equations are nothing else but the classical limit of the relations (II.72) obtained using the orthogonal polynomials. They coincide due to the identification mentioned above of the momentum operators P and Q with the resolvents ω and $\tilde{\omega}$, respectively.

The equations (II.75) can be considered as definitions of the multivalued analytical functions $y(x)$ and $x(y)$. It is clear that they must be mutually inverse. This is actually a

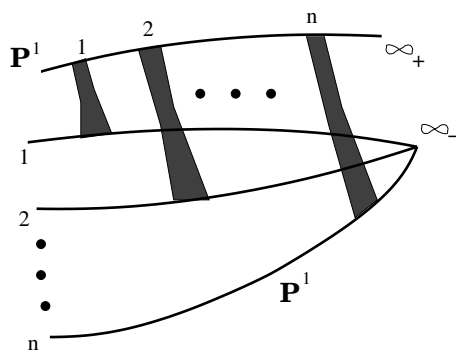


Fig. II.6: Generic curve of 2MM as a cover of x -plane. Each fat line consists from \tilde{n} cuts.

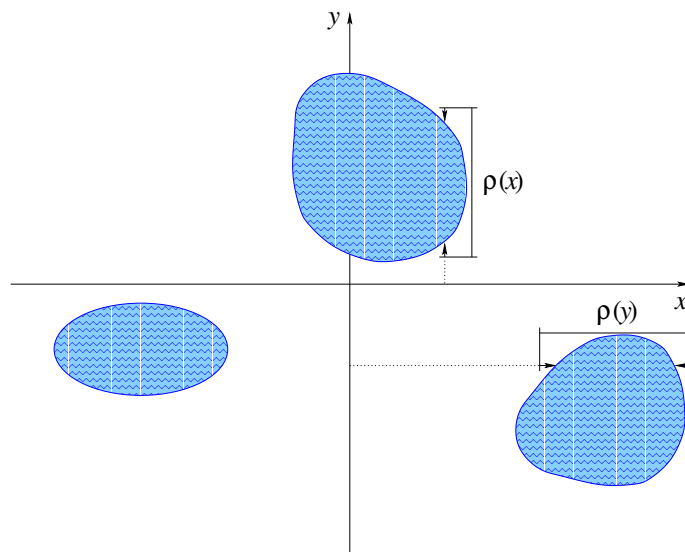


Fig. II.7: Eigenvalue plane of 2MM. The eigenvalues fill several spots which contain information about the density.

non-trivial restriction which, together with the asymptotic condition (II.40), fixes the resolvents and gives the solution. The complex curve, one should deal with, is the Riemann surface of one of these functions.

The general structure of this complex curve was studied in [68]. It was established that if the potentials are of degree $n + 1$ and $\tilde{n} + 1$, the maximum genus of the curve is $n\tilde{n} - 1$. In this most general case the Riemann surface is represented by $n + 1$ sheets. One of them is the "physical sheet" glued with each "unphysical" one along \tilde{n} cuts and all "unphysical sheets" join at infinity by the n th order branch point (see fig. II.6).

One can prove the analog of the formulae (II.47) and (II.48) [68]. We still integrate around two conjugated sets of independent cycles A_k and B_k on the curve. The role of the holomorphic differential is played again by $y(z)dz$ where $y(x)$ is a solution of (II.75).

As earlier, the cycles A_k surround the cuts of the resolvent on the physical sheet. This is the place where the eigenvalues of the matrix X live. However, in this picture it is not clear how to describe the distribution of eigenvalues of Y . Of course, it is enough to invert the function $y(x)$ to find it. But the arising picture is nevertheless non-symmetric with respect

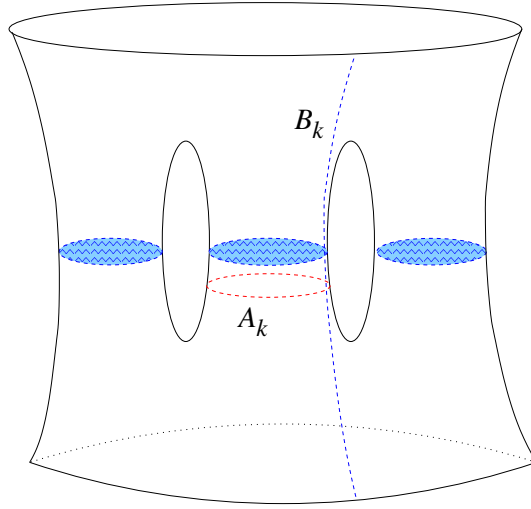


Fig. II.8: Complex curve associated with the solution of 2MM viewed as a "double" of (x, y) -plane.

to the exchange of x and y .

A more symmetric picture can be obtained considering the so called "double" [68]. To introduce this notion, note that since the two matrices X and Y are of the same size, with each eigenvalue x_i one can associate an eigenvalue y_i of the second matrix. Thus, one obtains N pairs (x_i, y_i) which can be put on a plane. In the large N limit, the eigenvalues are distributed continuously so that on the plane (x, y) their distribution appears as several disconnected regions. Each region corresponds to a cut of the resolvent. Its width in the y direction at a given point x is nothing else but the density $\rho(x)$ and *vice versa*. Thus, one arrives at a two-dimensional picture shown on fig. II.7. From this point of view, the functions $y(x)$ and $x(y)$, which are solutions of (II.75), determine the boundaries of the spots of eigenvalues. The fact that they are inverse means that they define the same boundary.

Now to define the double, we take two copies of the (x, y) plane, cut off the spots, and glue them along the boundaries. The resulting surface shown in fig. II.8 is smooth and can be considered as a genus $n_c - 1$ surface with two punctures (corresponding to two infinities) where n_c is the number of spots on the initial surface equal to the number of cuts of the resolvent $\omega(x)$.

All cycles A_k and B_k exist also in this picture and the integration formulae (II.47) and (II.48) along them hold as well. Note that the integrals along A_k cycles can be rewritten as two-dimensional integrals over the (x, y) plane with the density equal to 1 inside the spots and vanishing outside them. Such behaviour of the density is characteristic for fermionic systems. And indeed, 2MM can be interpreted as a system of free fermions.

4.6 Free fermion representation

Let us identify the functions

$$\psi_n(x) = \Phi_n(x)e^{NV(x)}, \quad \tilde{\psi}_n(y) = \tilde{\Phi}_n(y)e^{N\tilde{V}(y)}. \quad (\text{II.76})$$

with fermionic wave functions. The functions ψ_n and $\tilde{\psi}_n$ can be considered as two representations of the same state similarly to the coordinate and momentum representations. The two representations are related by a kind of Fourier transform and the scalar product between functions of different representations is given by the following integral

$$\langle \tilde{\psi} | \psi \rangle = \int dx dy e^{-Nxy} \tilde{\psi}(y) \psi(x). \quad (\text{II.77})$$

The second-quantized fermionic fields are defined as

$$\psi(x) = \sum_{n=0}^{\infty} a_n \psi_n(x), \quad \tilde{\psi}(y) = \sum_{n=0}^{\infty} a_n^\dagger \tilde{\psi}_n(y). \quad (\text{II.78})$$

Due to the orthonormality of the wave functions (II.76) with respect to the scalar product (II.77), the creation and annihilation operators satisfy the following anticommutation relations

$$\{a_n, a_m^\dagger\} = \delta_{n,m}. \quad (\text{II.79})$$

The fundamental state of N fermions is defined by

$$a_n |N\rangle = 0, \quad n \geq N, \quad a_n^\dagger |N\rangle = 0, \quad n < N. \quad (\text{II.80})$$

Its wave function can be represented by the Slater determinant. For example, in the x -representation it looks as

$$\Psi_N(x_1, \dots, x_N) = \det_{i,j} \psi_i(x_j). \quad (\text{II.81})$$

The key observation which establishes the equivalence of the two systems is that the correlators of matrix operators coincide with expectation values of the corresponding fermionic operators in the fundamental state (II.81):

$$\left\langle \prod_j \text{tr} B^{m_j} \prod_i \text{tr} A^{n_i} \right\rangle = \left\langle N | \prod_j \hat{y}^{m_j} \prod_i \hat{x}^{n_i} | N \right\rangle. \quad (\text{II.82})$$

Here a second-quantized operator $\hat{\mathcal{O}}(x, y)$ is defined as follows

$$\hat{\mathcal{O}}(x, y) = \int dx dy e^{-Nxy} \tilde{\psi}(y) \mathcal{O}(x, y) \psi(x). \quad (\text{II.83})$$

The proof of the statement (II.82) relies on the properties of the orthogonal polynomials. For example, for the one-point correlator we have

$$\begin{aligned} \langle \text{tr} A^n \rangle &\equiv Z^{-1} \int dA dB \text{tr} A^n e^{-N \text{tr} (AB - V(A) - \tilde{V}(B))} \\ &= \left(N! \prod_{k=0}^{N-1} h_k \right)^{-1} \int \prod_{k=1}^N d\mu(x_k, y_k) \Delta(x) \Delta(y) \sum_{i=1}^N x_i^n \\ &= \frac{1}{(N-1)!} \int \prod_{k=1}^N d\mu(x_k, y_k) \det_{ij} (\Phi_{j-1}(x_i)) \det_{ij} (\tilde{\Phi}_{j-1}(y_i)) x_1^n \\ &= \sum_{j=0}^{N-1} \int d\mu(x, y) \Phi_j(x) \tilde{\Phi}_j(y) x^n = \langle N | \hat{x}^n | N \rangle. \end{aligned} \quad (\text{II.84})$$

This equivalence supplies us with a powerful technique for calculations. For example, all correlators of the type (II.82) can be expressed through the two-point function

$$K_N(x, y) = \langle N | \tilde{\psi}(y) \psi(x) | N \rangle \quad (\text{II.85})$$

and it is sufficient to study this quantity. In general, the existence of the representation in terms of free fermions indicates that the system is integrable. This, in turn, is often related to the possibility to introduce orthogonal polynomials.

We will see that the similar structures appear in matrix quantum mechanics. Although MQM has much richer physics, it turns out to be quite similar to 2MM.

5 Toda lattice hierarchy

5.1 Integrable systems

One and two-matrix models considered above are examples of *integrable systems*. There exists a general theory of such systems. A system is considered as integrable if it has an infinite number of commuting Hamiltonians. Each Hamiltonian generates an evolution along some direction in the parameter space of the model. Their commutativity means that there is an infinite number of conserved quantities associated with them and, at least in principle, it is possible to describe any point in the parameter space.

Usually, the integrability implies the existence of a *hierarchy* of equations on some specific quantities characterizing the system. The hierarchical structure means that the equations can be solved one by one, so that the solution of the first equation should be substituted into the second one and *etc.* This recursive procedure allows to reproduce all information about the system.

The equations to be solved are most often of the finite-difference type. In other words, they describe a system on a lattice. Introducing a parameter measuring the spacing between nodes of the lattice, one can organize a perturbative expansion in this parameter. Then the finite-difference equations are replaced by an infinite set of partial differential equations. They also form a hierarchy and can be solved recursively. The equations appearing at the first level, corresponding to the vanishing spacing, describe a closed system which is considered as a continuum or classical limit of the initial one. In turn, starting with the classical system describing by a hierarchy of differential equations, one can construct its quantum deformation arriving at the full hierarchy.

The integrable systems can be classified according to the type of hierarchy which appears in their description. In this section we consider the so called *Toda hierarchy* [69]. It is general enough to include all integrable matrix models relevant for our work. In particular, it describes 2MM and some restriction of MQM, whereas 1MM corresponds to its certain reduction.

5.2 Lax formalism

There are several ways to introduce the Toda hierarchy. The most convenient for us is to use the so called *Lax formalism*.

Take two semi-infinite series¹

$$L = r(s)\hat{\omega} + \sum_{k=0}^{\infty} u_k(s)\hat{\omega}^{-k}, \quad \bar{L} = \hat{\omega}^{-1}r(s) + \sum_{k=0}^{\infty} \hat{\omega}^k \bar{u}_k(s), \quad (\text{II.86})$$

where s is a discrete variable labeling the nodes of an infinite lattice and $\hat{\omega} = e^{\hbar\partial/\partial s}$ is the shift operator in s . The Planck constant \hbar plays the role of the spacing parameter. The

¹In fact, the first coefficients in the expansions (II.86) can be chosen arbitrarily. Only their product has a sense and how it is distributed between the two operators can be considered as a choice of some gauge. In particular, often one uses the gauge where one of the coefficients equals 1 [70]. We use the symmetric gauge which agrees with the choice of the orthogonal polynomials (II.56) normalized to the Kronecker symbol in 2MM.

operators (II.86) are called Lax operators. The coefficients r , u_k and \bar{u}_k are also functions of two infinite sets of "times" $\{t_{\pm k}\}_{k=1}^{\infty}$. Each time variable gives rise to an evolution along its direction. This system represents Toda hierarchy if the evolution associated with each $t_{\pm k}$ is generated by Hamiltonians $H_{\pm k}$

$$\begin{aligned} \hbar \frac{\partial L}{\partial t_k} &= [H_k, L], & \hbar \frac{\partial \bar{L}}{\partial t_k} &= [H_k, \bar{L}], \\ \hbar \frac{\partial L}{\partial t_{-k}} &= [H_{-k}, L], & \hbar \frac{\partial \bar{L}}{\partial t_{-k}} &= [H_{-k}, \bar{L}], \end{aligned} \quad (\text{II.87})$$

which are expressed through the Lax operators (II.86) as follows²

$$H_k = (L^k)_+ + \frac{1}{2}(L^k)_0, \quad H_{-k} = (\bar{L}^k)_- + \frac{1}{2}(\bar{L}^k)_0, \quad (\text{II.88})$$

where the symbol $(\)_{\geq}$ means the positive (negative) part of the series in the shift operator $\hat{\omega}$ and $(\)_0$ denotes the constant part. Thus, Toda hierarchy is a collection of non-linear equations of the finite-difference type in s and differential with respect to t_k for the coefficients $r(s, t)$, $u_k(s, t)$ and $\bar{u}_k(s, t)$.

From the commutativity of the second derivatives, it is easy to obtain that the Lax–Sato equations (II.87) are equivalent to the zero-curvature condition for the Hamiltonians

$$\hbar \frac{\partial H_k}{\partial t_l} - \hbar \frac{\partial H_l}{\partial t_k} + [H_k, H_l] = 0. \quad (\text{II.89})$$

It shows that the system possesses an infinite set of commutative flows $\hbar \frac{\partial}{\partial t_k} - H_k$ and, therefore, Toda hierarchy is integrable.

One can get another equivalent formulation if one considers the following eigenvalue problem

$$x\Psi = L\Psi(x; s), \quad \hbar \frac{\partial \Psi}{\partial t_k} = H_k\Psi(x; s), \quad \hbar \frac{\partial \Psi}{\partial t_{-k}} = H_{-k}\Psi(x; s). \quad (\text{II.90})$$

The previous equations (II.87) and (II.89) appear as the integrability condition for (II.90). Indeed, differentiating the first equation, one reproduces the evolution law of the Lax operators (II.87) and the second and third equations lead to the zero-curvature condition (II.89). The eigenfunction Ψ is known as Baker–Akhiezer function. It is clear that it contains all information about the system.

Note that equations of Toda hierarchy allow a representation in terms of semi-infinite matrices. Then the Baker–Akhiezer function is a vector whose elements correspond to different values of the discrete variable s . The positive/negative/constant parts of the series in $\hat{\omega}$ are mapped to upper/lower/diagonal triangular parts of matrices.

The equations to be solved are either the equations (II.90) on Ψ or the Lax–Sato equations (II.87) on the coefficients of the Lax operators. Their hierarchic structure is reflected in the fact that one obtains a closed equation on the first coefficient $r(s, t)$ and its solution

² Sometimes the second set of Hamiltonians is defined with the opposite sign. This corresponds to the change of sign of t_{-k} . Doing both these replacements, one can establish the full correspondence with the works using this sign convention.

provides the necessary information for the following equations. This first equation is derived considering the Lax–Sato equations (II.87) for $k = 1$. They give

$$\hbar \frac{\partial \log r^2(s)}{\partial t_1} = u_0(s + \hbar) - u_0(s), \quad \hbar \frac{\partial \log r^2(s)}{\partial t_{-1}} = \bar{u}_0(s) - \bar{u}_0(s + \hbar), \quad (\text{II.91})$$

$$\hbar \frac{\partial \bar{u}_0(s)}{\partial t_1} = r^2(s) - r^2(s - \hbar), \quad \hbar \frac{\partial u_0(s)}{\partial t_{-1}} = r^2(s - \hbar) - r^2(s). \quad (\text{II.92})$$

Combining these relations, one finds the so called Toda equation

$$\hbar^2 \frac{\partial^2 \log r^2(s)}{\partial t_1 \partial t_{-1}} = 2r^2(s) - r^2(s + \hbar) - r^2(s - \hbar). \quad (\text{II.93})$$

Often it is convenient to introduce the following Orlov–Shulman operators [71]

$$\begin{aligned} M &= \sum_{k \geq 1} k t_k L^k + s + \sum_{k \geq 1} v_k L^{-k}, \\ \bar{M} &= - \sum_{k \geq 1} k t_{-k} \bar{L}^k + s - \sum_{k \geq 1} v_{-k} \bar{L}^{-k}. \end{aligned} \quad (\text{II.94})$$

The coefficients $v_{\pm k}$ are fixed by the condition on their commutators with the Lax operators

$$[L, M] = \hbar L, \quad [\bar{L}, \bar{M}] = -\hbar \bar{L}. \quad (\text{II.95})$$

The main application of these operators is that they can be considered as perturbations of the simple operators of multiplication by the discrete variable s . Indeed, if one requires that $v_{\pm k}$ vanish when all $t_{\pm k} = 0$, then in this limit $M = \bar{M} = s$. Similarly the Lax operators reduce to the shift operator. The perturbation leading to the general expansions (II.86) and (II.94) can be described by the dressing operators \mathcal{W} and $\bar{\mathcal{W}}$

$$\begin{aligned} L &= \mathcal{W} \hat{\omega} \mathcal{W}^{-1}, & M &= \mathcal{W} s \mathcal{W}^{-1}, \\ \bar{L} &= \bar{\mathcal{W}} \hat{\omega}^{-1} \bar{\mathcal{W}}^{-1}, & \bar{M} &= \bar{\mathcal{W}} s \bar{\mathcal{W}}^{-1}. \end{aligned} \quad (\text{II.96})$$

The commutation relations (II.95) are nothing else but the dressed version of the evident relation

$$[\hat{\omega}, s] = \hbar \hat{\omega}. \quad (\text{II.97})$$

To produce the expansions (II.86) and (II.94), the dressing operators should have the following general form

$$\begin{aligned} \mathcal{W} &= e^{\frac{1}{2\hbar} \phi} \left(1 + \sum_{k \geq 1} w_k \hat{\omega}^{-k} \right) \exp \left(\frac{1}{\hbar} \sum_{k \geq 1} t_k \hat{\omega}^k \right), \\ \bar{\mathcal{W}} &= e^{-\frac{1}{2\hbar} \phi} \left(1 + \sum_{k \geq 1} \bar{w}_k \hat{\omega}^k \right) \exp \left(\frac{1}{\hbar} \sum_{k \geq 1} t_{-k} \hat{\omega}^{-k} \right), \end{aligned} \quad (\text{II.98})$$

where the zero mode $\phi(s)$ is related to the coefficient $r(s)$ as

$$r^2(s) = e^{\frac{1}{\hbar}(\phi(s) - \phi(s + \hbar))}. \quad (\text{II.99})$$

However, the coefficients in this expansion are not arbitrary and the dressing operators should be subject of some additional condition. It can be understood considering evolution along

the times $t_{\pm k}$. Differentiating (II.96) with respect to $t_{\pm k}$, one finds the following expression of the Hamiltonians in terms of the dressing operators

$$\begin{aligned} H_k &= \hbar(\partial_{t_k} \mathcal{W}) \mathcal{W}^{-1}, & H_{-k} &= \hbar(\partial_{t_{-k}} \mathcal{W}) \mathcal{W}^{-1}, \\ \bar{H}_k &= \hbar(\partial_{t_k} \bar{\mathcal{W}}) \bar{\mathcal{W}}^{-1}, & \bar{H}_{-k} &= \hbar(\partial_{t_{-k}} \bar{\mathcal{W}}) \bar{\mathcal{W}}^{-1}, \end{aligned} \quad (\text{II.100})$$

Here $H_{\pm k}$ generate evolution of L and $\bar{H}_{\pm k}$ are Hamiltonians for \bar{L} . However, (II.87) implies that for both operators one should use the same Hamiltonian. This imposes the condition that $H_{\pm k} = \bar{H}_{\pm k}$ which relates two dressing operators. This condition can be rewritten in a more explicit way. Namely, it is equivalent to the requirement that $\mathcal{W}^{-1} \bar{\mathcal{W}}$ does not depend on times $t_{\pm k}$ [69, 72].

Studying the evolution laws of the Orlov–Shulman operators, one can find that [70]

$$\frac{\partial v_k}{\partial t_l} = \frac{\partial v_l}{\partial t_k}. \quad (\text{II.101})$$

It means that there exists a generating function $\tau_s[t]$ of all coefficients $v_{\pm k}$

$$v_k(s) = \hbar^2 \frac{\partial \log \tau_s[t]}{\partial t_k}. \quad (\text{II.102})$$

It is called τ -function of Toda hierarchy. It also allows to reproduce the zero mode ϕ and, consequently, the first coefficient in the expansion of the Lax operators

$$e^{\frac{1}{\hbar} \phi(s)} = \frac{\tau_s}{\tau_{s+\hbar}}, \quad r^2(s - \hbar) = \frac{\tau_{s+\hbar} \tau_{s-\hbar}}{\tau_s^2}. \quad (\text{II.103})$$

The τ -function is usually the main subject of interest in the systems described by Toda hierarchy. The reason is that it coincides with the partition function of the model. Then the coefficients v_k are the one-point correlators of the operators generating the commuting flows H_k . We will show a concrete realization of these ideas in the end of this section and in the next chapters.

5.3 Free fermion and boson representations

To establish an explicit connection with physical systems, it is sometimes convenient to use a representation of Toda hierarchy in terms of second-quantized free chiral fermions or bosons. The two representations are related by the usual bosonization procedure.

Fermionic picture

To define the fermionic representation, let us consider the chiral fermionic fields with the following expansion

$$\psi(z) = \sum_{r \in \mathbf{Z} + \frac{1}{2}} \psi_r z^{-r - \frac{1}{2}}, \quad \psi^*(z) = \sum_{r \in \mathbf{Z} + \frac{1}{2}} \psi_{-r}^* z^{-r - \frac{1}{2}} \quad (\text{II.104})$$

Their two-point function

$$\langle l | \psi(z) \psi^*(z') | l \rangle = \frac{(z'/z)^l}{z - z'} \quad (\text{II.105})$$

leads to the following commutation relations for the modes

$$\{\psi_r, \psi_s^*\} = \delta_{r,s}. \quad (\text{II.106})$$

The fermionic vacuum of charge l is defined by

$$\psi_r |l\rangle = 0, \quad r > l, \quad \psi_r^* |l\rangle = 0, \quad r < l. \quad (\text{II.107})$$

Also we need to introduce the current

$$J(z) = \psi^*(z)\psi(z) = \hat{p}z^{-1} + \sum_{n \neq 0} H_n z^{-n-1} \quad (\text{II.108})$$

whose components H_n are associated with the Hamiltonians generating the Toda flows. In terms of the fermionic modes they are represented as follows

$$H_n = \sum_{r \in \mathbf{Z} + \frac{1}{2}} \psi_{r-n}^* \psi_r \quad (\text{II.109})$$

and for any l they satisfy

$$H_n |l\rangle = \langle l | H_{-n} = 0, \quad n > 0. \quad (\text{II.110})$$

Finally, we introduce an operator of $GL(\infty)$ rotation

$$\mathbf{g} = \exp \left(\frac{1}{\hbar} \sum_{r,s \in \mathbf{Z} + \frac{1}{2}} A_{rs} \psi_r \psi_s^* \right). \quad (\text{II.111})$$

With these definitions the τ -function of Toda hierarchy is represented as the following vacuum expectation value

$$\tau_{lh}[t] = \langle l | e^{\frac{1}{\hbar} H_+[t]} \mathbf{g} e^{-\frac{1}{\hbar} H_-[t]} | l \rangle, \quad (\text{II.112})$$

where

$$H_+[t] = \sum_{k>0} t_k H_k, \quad H_-[t] = \sum_{k<0} t_k H_k. \quad (\text{II.113})$$

It is clear that each solution of Toda hierarchy is characterized in the unique way by the choice of the matrix A_{rs} .

Bosonic picture

The bosonic representation now follows from the bosonization formulae

$$\psi(z) =: e^{\varphi(z)} :, \quad \psi^*(z) =: e^{-\varphi(z)} :, \quad \partial\varphi(z) =: \psi^*(z)\psi(z) :. \quad (\text{II.114})$$

Since the last expression in (II.114) is the current (II.108), the Hamiltonians H_n appear now as the coefficients in the mode expansion of the free bosonic field

$$\varphi(z) = \hat{q} + \hat{p} \log z + \sum_{n \neq 0} \frac{1}{n} H_n z^{-n}. \quad (\text{II.115})$$

From (II.109) and (II.106) one finds the following commutation relations

$$[\hat{p}, \hat{q}] = 1, \quad [H_n, H_m] = n\delta_{m+n,0}, \quad (\text{II.116})$$

which lead to the two-point function of the free boson

$$\langle \varphi(z) \varphi(z') \rangle = \log(z - z'). \quad (\text{II.117})$$

The bosonic vacuum is defined by the Hamiltonians and characterized by the quantum number s , which is the eigenvalue of the momentum operator,

$$\hat{p}|s\rangle = s|s\rangle, \quad H_n|s\rangle = 0, \quad (n > 0). \quad (\text{II.118})$$

To rewrite the operator \mathbf{g} (II.111) in the bosonic terms, we introduce the vertex operators

$$V_\alpha(z) =: e^{\alpha\varphi(z)} :. \quad (\text{II.119})$$

Here the normal ordering is defined by putting all H_n , $n > 0$ to the right and H_n , $n < 0$ to the left. Besides $:\hat{q}\hat{p} := \hat{p}\hat{q} := \hat{q}\hat{p}$. Then the operator of $GL(\infty)$ rotation is given by

$$\mathbf{g} = \exp \left(-\frac{1}{4\pi^2\hbar} \oint dz \oint dw A(z, w) V_1(z) V_{-1}(w) \right), \quad (\text{II.120})$$

where

$$A(z, w) = \sum_{r,s} A_{rs} z^{r-\frac{1}{2}} w^{-s-\frac{1}{2}}. \quad (\text{II.121})$$

The substitution of (II.120) into (II.112) and replacing the fermionic vacuum $|l\rangle$ by the bosonic one $|s\rangle$ gives the bosonic representation of the τ -function $\tau_s[t]$.

Connection with the Lax formalism

The relation of these two representations with the objects considered in paragraph 5.2 is based on the realization of the Baker–Akhiezer function $\Psi(z; s)$ as the expectation value of the one-fermion field

$$\Psi(z; s) = \tau_s^{-1}[t] \left\langle \hbar^{-1} s | e^{\frac{1}{\hbar} H_+[t]} \psi(z) \mathbf{g} e^{-\frac{1}{\hbar} H_-[t]} | \hbar^{-1} s \right\rangle. \quad (\text{II.122})$$

One can show that it does satisfy the relations (II.90) with H_k defined as in (II.88) and the Lax operators having the form (II.86).

5.4 Hirota equations

The most explicit manifestation of the hierarchic structure of the Toda system is a set of equations on the τ -function, which can be obtained from the fermionic representation introduced above. One can show [73] that the ensemble of the τ -functions of the Toda hierarchy with different charges satisfies a set of bilinear equations. They are known as Hirota equations and can be written in a combined way as follows

$$\oint_{C_\infty} dz z^{l-l'} \exp \left(\frac{1}{\hbar} \sum_{k>0} (t_k - t'_k) z^k \right) \tau_l[t - \tilde{\zeta}_+] \tau_{l'}[t' + \tilde{\zeta}_+] = \\ \oint_{C_0} dz z^{l-l'} \exp \left(\frac{1}{\hbar} \sum_{k<0} (t_k - t'_k) z^k \right) \tau_{l+1}[t - \tilde{\zeta}_-] \tau_{l'-1}[t' + \tilde{\zeta}_-], \quad (\text{II.123})$$

where

$$\tilde{\zeta}_+/\hbar = (\dots, 0, 0, z^{-1}, z^{-2}/2, z^{-3}/3, \dots), \quad \tilde{\zeta}_-/\hbar = (\dots, z^3/3, z^2/2, z, 0, 0, \dots) \quad (\text{II.124})$$

and we omitted \hbar in the index of the τ -function. The proof of (II.123) relies on the representation (II.112) of the τ -function with \mathbf{g} taken from (II.111). The starting point is the fact that the following operator

$$\mathcal{C} = \sum_{r \in \mathbf{Z} + \frac{1}{2}} \psi_r^* \otimes \psi_r = \oint \frac{dz}{2\pi i} \psi^*(z) \otimes \psi(z) \quad (\text{II.125})$$

plays the role of the Casimir operator for the diagonal subgroup of $GL(\infty) \otimes GL(\infty)$. This means that it commutes with the tensor product of two \mathbf{g} operators

$$\mathcal{C}(\mathbf{g} \otimes \mathbf{g}) = (\mathbf{g} \otimes \mathbf{g})\mathcal{C}. \quad (\text{II.126})$$

Multiplying this relation by $\langle l+1 | e^{\frac{1}{\hbar}H_+[t]} \otimes \langle l'-1 | e^{\frac{1}{\hbar}H_+[t']}$ from the left and by $e^{-\frac{1}{\hbar}H_-[t]} | l \rangle \otimes e^{-\frac{1}{\hbar}H_-[t']} | l' \rangle$ from the right, one can commute the fermion operators until they hit the left (right) vacuum. The final result is obtained using the following relations

$$\langle l+1 | e^{\frac{1}{\hbar}H_+[t]} \psi^*(z) \mathbf{g} e^{-\frac{1}{\hbar}H_-[t]} | l \rangle = z^l \exp \left(\frac{1}{\hbar} \sum_{n>0} t_n z^n \right) \langle l | e^{\frac{1}{\hbar}H_+[t-\tilde{\zeta}_+]} \mathbf{g} e^{-\frac{1}{\hbar}H_-[t]} | l \rangle, \quad (\text{II.127})$$

$$\langle l+1 | e^{\frac{1}{\hbar}H_+[t]} \mathbf{g} \psi^*(z) e^{-\frac{1}{\hbar}H_-[t]} | l \rangle = z^l \exp \left(\frac{1}{\hbar} \sum_{n<0} t_n z^n \right) \langle l+1 | e^{\frac{1}{\hbar}H_+[t]} \mathbf{g} e^{-\frac{1}{\hbar}H_-[t-\tilde{\zeta}_-]} | l+1 \rangle, \quad (\text{II.128})$$

together with the similar relations for $\psi(z)$. They can be proven in two steps. First, one commutes the fermionic fields with the perturbing operators

$$\begin{aligned} e^{-\frac{1}{\hbar}H_{\pm}[t]} \psi^*(z) e^{\frac{1}{\hbar}H_{\pm}[t]} &= \exp \left(-\frac{1}{\hbar} \sum_{n>0} t_{\pm n} z^{\pm n} \right) \psi^*(z), \\ e^{\frac{1}{\hbar}H_{\pm}[t]} \psi(z) e^{-\frac{1}{\hbar}H_{\pm}[t]} &= \exp \left(-\frac{1}{\hbar} \sum_{n>0} t_{\pm n} z^{\pm n} \right) \psi(z). \end{aligned} \quad (\text{II.129})$$

After this it remains to show that, for example, $\psi^*(z) | l \rangle = z^l e^{\frac{1}{\hbar}H_-[\tilde{\zeta}_-]} | l+1 \rangle$. The easiest way to do it is to use the bosonization formulae (II.114) and (II.115).

The identities (II.123) can be rewritten in a more explicit form. For this we introduce the Schur polynomials p_j defined by

$$\sum_{k=0}^{\infty} p_k[t] x^k = \exp \left(\sum_{k=1}^{\infty} t_k x^k \right) \quad (\text{II.130})$$

and the following notations

$$y_{\pm} = (y_{\pm 1}, y_{\pm 2}, y_{\pm 3}, \dots), \quad (\text{II.131})$$

$$\tilde{D}_{\pm} = (D_{\pm 1}, D_{\pm 2}/2, D_{\pm 3}/3, \dots), \quad (\text{II.132})$$

where $D_{\pm n}$ represent the Hirota's bilinear operators

$$D_n f[t] \cdot g[t] = \frac{\partial}{\partial x} f(t_n + x) g(t_n - x) \Big|_{x=0}. \quad (\text{II.133})$$

Then identifying $y_n = \frac{1}{2\hbar}(t'_n - t_n)$, one obtains a hierarchy of partial differential equations

$$\begin{aligned} \sum_{j=0}^{\infty} p_{j+i}(-2y_+) p_j(\hbar \tilde{D}_+) \exp\left(\hbar \sum_{k \neq 0} y_k D_k\right) \tau_{l+i+1}[t] \cdot \tau_l[t] = \\ \sum_{j=0}^{\infty} p_{j-i}(-2y_-) p_j(\hbar \tilde{D}_-) \exp\left(\hbar \sum_{k \neq 0} y_k D_k\right) \tau_{l+i}[t] \cdot \tau_{l+1}[t]. \end{aligned} \quad (\text{II.134})$$

The Hirota equations lead to a triangular system of nonlinear difference-differential equations for the τ -function. Since the derivatives of the τ -function are identified with correlators, the Hirota equations are also equations for the correlators of operators generating the Toda flows. The first equation of the hierarchy is obtained by taking $i = -1$ and extracting the coefficient in front of y_{-1}

$$\hbar^2 \tau_l \frac{\partial^2 \tau_l}{\partial t_1 \partial t_{-1}} - \hbar^2 \frac{\partial \tau_l}{\partial t_1} \frac{\partial \tau_l}{\partial t_{-1}} + \tau_{l+1} \tau_{l-1} = 0. \quad (\text{II.135})$$

Rewriting this equations as

$$\hbar^2 \frac{\partial^2 \log \tau_l}{\partial t_1 \partial t_{-1}} + \frac{\tau_{l+1} \tau_{l-1}}{\tau_l^2} = 0, \quad (\text{II.136})$$

one can recognize the Toda equation (II.93) if one takes into account the identification (II.103).

5.5 String equation

We considered above the general structure of the Toda hierarchy. However, the equations of the hierarchy, for example, the Hirota equations (II.134), have many solutions. A particular solution is characterized by initial condition. The role of such condition can be played by the partition function of a non-perturbed system. If one requires that it should be equal to the τ -function at vanishing times and coincides with the full τ -function after the perturbation, the perturbed partition function can be found by means of the hierarchy equations with the given initial condition.

However, the Toda equations involve partial differential equations of high orders and require to know not only the τ -function at vanishing times but also its derivatives. Therefore, it is not always clear whether the non-perturbed function provides a sufficient initial information. Fortunately, there is another way to select a unique solution of Toda hierarchy. It uses some equations on the operators, usually, the Lax and Orlov–Shulman operators. The corresponding equations are called *string equations*.

The string equations cannot be arbitrary because they should preserve the structure of Toda hierarchy. For example, if they are given by two equations of the following type

$$\bar{L} = f(L, M), \quad \bar{M} = g(L, M), \quad (\text{II.137})$$

the operators defined by the functions f and g must satisfy

$$[f(\hat{\omega}, s), g(\hat{\omega}, s)] = -\hbar f. \quad (\text{II.138})$$

This condition appears since \bar{L} and \bar{M} commute in the same way.

The advantage of use of string equations is that they represent in a sense already a partially integrated version of the hierarchy equations. For example, as we will see, instead of differential equations of the second order, they produce algebraic and first order differential equations and make the problem of finding the τ -function much simpler.

5.6 Dispersionless limit

The classical limit of Toda hierarchy is obtained in the limit where the parameter measuring the lattice spacing vanishes. In our notations this parameter is the Planck constant \hbar . Putting it to zero, as usual, one replaces all operators by functions. In particular, as in the usual quantum mechanical systems, the classical limit of the derivative operator is a variable conjugated to the variable with respect to which one differentiates. In other words, one should consider the phase space consisting from s and ω , which is the classical limit of the shift operator. The Poisson structure on this phase space is defined by the Poisson bracket induced from (II.97)

$$\{\omega, s\} = \omega. \quad (\text{II.139})$$

All operators now become functions of s and ω and commutators are replaced by the corresponding Poisson brackets defined through (II.139).

All equations including the Lax–Sato equations (II.87), zero curvature condition (II.89), commutators with Orlov–Shulman operators (II.95) can be rewritten in the new terms. Thus, the general structure of Toda hierarchy is preserved although it becomes much simpler. The resulting structure is called *dispersionless Toda hierarchy* and the classical limit is also known as *dispersionless limit*.

As in the full theory, a solution of the dispersionless Toda hierarchy is completely characterized by a dispersionless τ -function. In fact, one should consider the free energy since it is the logarithm of the full τ -function that can be represented as a series in \hbar

$$\log \tau = \sum_{n \geq 0} \hbar^{-2+2n} F_n. \quad (\text{II.140})$$

Thus, the dispersionless limit is extracted as follows

$$F_0 = \lim_{\hbar \rightarrow 0} \hbar^2 \log \tau. \quad (\text{II.141})$$

The dispersionless free energy F_0 satisfies the classical limit of Hirota equations (II.134) and selected from all solutions by the same string equations (II.137) as in the quantum case.

Since the evolution along the times t_k is now generated by the Hamiltonians through the Poisson brackets, it can be seen as a canonical transformation in the phase space defined above. This fact is reflected also in the commutation relations (II.95). Since the Lax and Orlov–Shulman operators are dressed versions of ω and s , correspondingly, and have the same Poisson brackets, one can say that the dispersionless Toda hierarchy describe a canonical transformation from the canonical pair (ω, s) to (L, M) . The free energy F_0 plays the role of the generating function of this transformation.

5.7 2MM as τ -function of Toda hierarchy

In this paragraph we show how all abstract ideas described above get a realization in the two-matrix model. Namely, we identify the partition function (II.54) with a particular τ -function of Toda hierarchy. This can be done in two ways using either the fermionic representation or the Lax formalism and its connection with the orthogonal polynomials. However, the fermionic representation which arises in this case is not exactly the same as in paragraph 5.3, although it still gives a τ -function of Toda hierarchy. The difference is that one should use two types of fermions [74]. In fact, they can be reduced to the fermions appearing in the fermionic representation of 2MM presented in section 4.6. They differ only by the basis which is used in the mode expansions (II.104) and (II.78).

We will use the approach based on the Lax formalism. Following this way, one should identify the Lax operators in the matrix model and prove that they satisfy the Lax-Sato equations (II.87). Equivalently, one can obtain the Baker–Akhiezer function satisfying (II.90) where the Hamiltonians H_k are related to the Lax operators through (II.88).

First of all, the Lax operators coincide with the operators \hat{X} and \hat{Y} defined in (II.68) and (II.69). Due to the first equation in (II.66), the operators have the same expansion as required in (II.86) where the first coefficient is

$$r(n\hbar) = \sqrt{h_{n+1}/h_n}. \quad (\text{II.142})$$

The Baker–Akhiezer function is obtained as a semi-infinite vector constructed from the functions $\psi_n(x)$ (II.76)

$$\Psi(x; n\hbar) = \Phi_n(x) e^{NV(x)}. \quad (\text{II.143})$$

Due to (II.70), $L\Psi = x\Psi$. Thus, it remains to consider the evolution of Ψ in the coupling constants. We fix their normalization choosing the potentials as follows

$$V(x) = \sum_{n>0} t_n x^n, \quad \tilde{V}(x) = - \sum_{n>0} t_{-n} y^n. \quad (\text{II.144})$$

Then the differentiation of the orthogonality condition (II.56) with respect to the coupling constants leads to the following evolution laws

$$\begin{aligned} \frac{1}{N} \frac{\partial \Phi_n(x)}{\partial t_k} &= - \sum_{m=0}^{n-1} (X^k)_{nm} \Phi_m(x) - \frac{1}{2} (X^k)_{nn} \Phi_n(x), \\ \frac{1}{N} \frac{\partial \Phi_n(x)}{\partial t_{-k}} &= \sum_{m=0}^{n-1} (Y^k)_{nm} \Phi_m(x) + \frac{1}{2} (Y^k)_{nn} \Phi_n(x). \end{aligned} \quad (\text{II.145})$$

Using these relations one finds

$$\frac{1}{N} \frac{\partial \Psi}{\partial t_k} = H_k \Psi, \quad \frac{1}{N} \frac{\partial \Psi}{\partial t_{-k}} = H_{-k} \Psi, \quad (\text{II.146})$$

where $H_{\pm k}$ are defined as in (II.88). As a result, if one identifies $1/N$ with \hbar , one reproduces all equations for the Baker–Akhiezer function. This means that the dynamics of 2MM with respect to the coupling constants is governed by Toda hierarchy.

Combining (II.103) and (II.142), one finds

$$h_n = \frac{\tau_{n+1}}{\tau_n}. \quad (\text{II.147})$$

Then the representation (II.62) implies

$$Z(N) \sim \frac{\tau_N}{\tau_0}. \quad (\text{II.148})$$

The factor τ_0 does not depend on N and appears as a non-universal contribution to the free energy. Therefore, it can be neglected and, choosing the appropriate normalization, one can identify the partition function of 2MM with the τ -function of Toda hierarchy

$$Z(N) = \tau_N. \quad (\text{II.149})$$

Moreover, one can find string equations uniquely characterizing the τ -function. First, we note that the Orlov–Shulman operators (II.94) are given by

$$M = \hat{X} \left(V'(\hat{X}) + \hat{P} \right) - \hbar, \quad \bar{M} = \hat{Y} \left(\tilde{V}'(\hat{Y}) + \hat{Q} \right). \quad (\text{II.150})$$

Then the relations (II.72) imply

$$L\bar{L} = M + \hbar, \quad \bar{L}L = \bar{M}. \quad (\text{II.151})$$

Multiplying the first equation by L^{-1} from the left and by L from the right and taking into account (II.95), one obtains that

$$M = \bar{M}. \quad (\text{II.152})$$

This result together with the second equation in (II.151) gives one possible form of the string equations. It leads to the following functions f and g from (II.137)

$$f(\hat{\omega}, s) = s\hat{\omega}^{-1}, \quad g(\hat{\omega}, s) = s. \quad (\text{II.153})$$

It is easy to check that they satisfy the condition (II.138). Combining (II.151) and (II.152), one arrives at another very popular form of the string equation

$$[L, \bar{L}] = \hbar. \quad (\text{II.154})$$

The identification (II.148) allows to use the powerful machinery of Toda hierarchy to find the partition function of 2MM. For example, one can write the Toda equation (II.136) which, together with some initial condition, gives the dependence of $Z(N)$ on the first times $t_{\pm 1}$. In the dispersionless limit this equation simplifies to a partial differential equation and sometimes it becomes even an ordinary differential equation (for example, when it is known that the partition function depends only on the product of the coupling constants $t_1 t_{-1}$). Finally, the string equation (II.154) can replace the initial condition for the differential equations of the hierarchy and produce equations of lower orders.

Chapter III

Matrix Quantum Mechanics

Now we approach the main subject of the thesis which is *Matrix Quantum Mechanics*. This chapter is devoted to the introduction to this model and combines the ideas discussed in the previous two chapters. The reader will see how the technique of matrix models allows to solve difficult problems related to string theory.

1 Definition of the model and its interpretation

Matrix Quantum Mechanics is a natural generalization of the matrix chain model presented in section II.2.2. It is defined as an integral over hermitian $N \times N$ matrices whose components are functions of one real variable which is interpreted as "time". Thus, it represents the path integral formulation of a quantum mechanical system with N^2 degrees of freedom. We will choose the time to be Euclidean so that the matrix integral takes the following form

$$Z_N(g) = \int \mathcal{D}M(t) \exp \left[-N \operatorname{tr} \int dt \left(\frac{1}{2} \dot{M}^2 + V(M) \right) \right], \quad (\text{III.1})$$

where the potential $V(M)$ has the form as in (II.2). As it was required for all (hermitian) matrix models, this integral is invariant under the global unitary transformations

$$M(t) \longrightarrow \Omega^\dagger M(t) \Omega \quad (\Omega^\dagger \Omega = I). \quad (\text{III.2})$$

The range of integration over the time variable in (III.1) can be arbitrary depending on the problem we are interested in. In particular, it can be finite or infinite, and the possibility of a special interest is the case when the time is compact so that one considers MQM on a circle. The latter choice will be important later and now we will concentrate on the simplest case of the infinite time interval.

In section II.2 it was shown that the free energy of matrix models gives a sum over discretized two-dimensional surfaces. In particular, its special *double scaling limit* corresponds to the continuum limit for the discretization and reproduces the sum over continuous surfaces, which is the path integral for two-dimensional quantum gravity. For the case of the simple one-matrix integral (if we do not tune the potential to a multicritical point), the surfaces did not carry any additional structure, whereas we argued that the multi-matrix case should correspond to quantum gravity coupled to matter. Since the MQM integral goes

over a continuous set of matrices, we expect to obtain quantum gravity coupled with one scalar field. In turn, such a system can be interpreted as the sum over surfaces (or strings) embedded into one dimension [43].

Let us show how it works. As in section II.2, one can construct a Feynman expansion of the integral (III.1). It is the same as in the one-matrix case except that the propagator becomes time-dependent.

$$t_i^j \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} t_l^k = \frac{1}{Ng_2} \delta_{il} \delta_{jk} e^{-|t-t'|}$$

Then the expansion (II.10) is generalized to

$$F = \sum_{g=0}^{\infty} N^{2-2g} \sum_{\substack{\text{genus } g \text{ connected} \\ \text{diagrams}}} g_2^{-E} \prod_k (-g_k)^{n_k} \prod_{i=1}^V \int_{-\infty}^{\infty} dt_i \prod_{\langle ij \rangle} G(t_i - t_j), \quad (\text{III.3})$$

where $\langle ij \rangle$ denotes the edge connecting i th and j th vertices and $G(t) = e^{-|t|}$. As usual, each Feynman diagram is dual to some discretized surface and the sum (III.3) is interpreted as the sum over all discretizations. The new feature is the appearance of integrals over real variables t_i living at the vertices of the Feynman diagrams or at the centers of the faces of triangulated surfaces. They represent a discretization of the functional integral over a scalar field $t(\sigma)$. The action for the scalar field can be restored from the propagator $G(t)$. Its discretized version is given by

$$-\sum_{\langle ij \rangle} \log G(t_i - t_j). \quad (\text{III.4})$$

Taking into account the exact form of the propagator, in the continuum limit, where the lattice spacing goes to zero, one finds that the action becomes

$$\int d^2\sigma \sqrt{h} |h^{ab} \partial_a t \partial_b t|^{1/2}. \quad (\text{III.5})$$

It is not the standard action for a scalar field in two dimensions. The usual one would be obtained if we took the Gaussian propagator $G(t) = e^{-t^2}$. Does it mean that MQM does not describe two-dimensional gravity coupled with $c = 1$ matter?

In [43] it was argued that it does describe by the following reason. The usual scalar field propagator in the momentum space has the Gaussian form $G^{-1}(p) \sim e^{p^2}$. Its leading small momentum behaviour coincides with $G^{-1}(p) \sim 1 + p^2$ which is the momentum representation of the propagator for MQM. Thus, the replacement of one propagator by another affects only the short distance physics which is non-universal. The critical properties of the model surviving in the continuum limit should not depend on this choice. This suggestion was confirmed by a great number of calculations which showed full agreement of the results obtained by the CFT methods and in the framework of MQM.

Finally, we note that 2D gravity coupled with $c = 1$ matter can be interpreted as a non-critical string embedded in one dimension. The latter is equivalent to 2D critical string theory in the linear dilaton background. Thus Matrix Quantum Mechanics is an alternative description of 2D string theory. As it will be shown, it allows to manifest the integrability of this model and to address many questions inaccessible in the usual CFT formulation.

2 Singlet sector and free fermions

In this section we review the general structure of Matrix Quantum Mechanics and present its solution in the so called *singlet sector* of the Hilbert space [59]. The solution is relied on the interpretation of MQM as a quantum mechanical system of fermions. Therefore, we will consider t as a real Minkowskian time to have a good quantum mechanical description.

2.1 Hamiltonian analysis

To analyze the dynamics of MQM, as in the one-matrix case we change the variables from the matrix elements $M_{ij}(t)$ to the eigenvalues and the angular degrees of freedom

$$M(t) = \Omega^\dagger(t)x(t)\Omega(t), \quad x = \text{diag}(x_1, \dots, x_N), \quad \Omega^\dagger\Omega = I. \quad (\text{III.6})$$

Since the unitary matrix depends on time it is not canceled in the action of MQM. The kinetic term gives rise to an additional term

$$\text{tr} \dot{M}^2 = \text{tr} \dot{x}^2 + \text{tr} [x, \dot{\Omega}\Omega^\dagger]^2. \quad (\text{III.7})$$

The matrix $\dot{\Omega}\Omega^\dagger$ is anti-hermitian and can be considered as an element of the $su(n)$ algebra. Therefore, it can be decomposed in terms of the $SU(N)$ generators

$$\dot{\Omega}\Omega^\dagger = \sum_{i=1}^{N-1} \dot{\alpha}_i H_i + \frac{i}{\sqrt{2}} \sum_{i < j} (\dot{\beta}_{ij} T_{ij} + \dot{\gamma}_{ij} \tilde{T}_{ij}), \quad (\text{III.8})$$

where H_i are the diagonal generators of the Cartan subalgebra and the other generators are represented by the following matrices: $(T_{ij})_{kl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$ and $(\tilde{T}_{ij})_{kl} = -i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$. The MQM degrees of freedom are described now by x_i , α_i , β_{ij} and γ_{ij} . The Minkowskian action in terms of these variables takes the form

$$S_{\text{MQM}} = \int dt \left[\sum_{i=1}^N \left(\frac{1}{2} \dot{x}_i^2 - V(x_i) \right) + \frac{1}{2} \sum_{i < j} (x_i - x_j)^2 (\dot{\beta}_{ij}^2 + \dot{\gamma}_{ij}^2) \right]. \quad (\text{III.9})$$

We did not included the overall multiplier N into the action. It plays the role of the Planck constant so that in the following we denote $\hbar = 1/N$.

To understand the structure of the corresponding quantum theory we pass to the Hamiltonian formulation (see review [26]). It is clear that the Hamiltonian is given by

$$H_{\text{MQM}} = \sum_{i=1}^N \left(\frac{1}{2} p_i^2 + V(x_i) \right) + \frac{1}{2} \sum_{i < j} \frac{\Pi_{ij}^2 + \tilde{\Pi}_{ij}^2}{(x_i - x_j)^2}, \quad (\text{III.10})$$

where p_i , Π_{ij} and $\tilde{\Pi}_{ij}$ are momenta conjugated to x_i , β_{ij} and γ_{ij} , respectively. Besides, since the action (III.9) does not depend on α_i , we have the constraint that its momentum should vanish $\Pi_i = 0$.

In quantum mechanics all these quantities should be realized as operators. If we work in the coordinate representation, Π_{ij} and $\tilde{\Pi}_{ij}$ are the usual derivatives. But to find \hat{p}_i , one should take into account the Jacobian appearing in the path integral measure after the change of

variables (III.6). The Jacobian is the same as in (II.27). To see how it affects the momentum operator, we consider the scalar product in the Hilbert space of MQM. The measure of the scalar product in the coordinate representation coincides with the path integral measure and contains the same Jacobian. It can be easily understood because the change (III.6) can be done directly in the scalar product so that

$$\langle \Phi | \Phi' \rangle = \int dM \overline{\Phi(M)} \Phi'(M) = \int d\Omega \int \prod_{i=1}^N dx_i \Delta^2(x) \overline{\Phi(x, \Omega)} \Phi'(x, \Omega). \quad (\text{III.11})$$

Due to this the map to the momentum representation, where the measure is trivial, is given by

$$\Phi(p, \Omega) = \int \prod_{i=1}^N \left(dx_i e^{-\frac{i}{\hbar} p_i x_i} \right) \Delta(x) \Phi(x, \Omega). \quad (\text{III.12})$$

Then in the coordinate representation the momentum is realized as the following operator

$$\hat{p}_i = \frac{-i\hbar}{\Delta(x)} \frac{\partial}{\partial x_i} \Delta(x). \quad (\text{III.13})$$

As a result, we obtain that the Hamiltonian (III.10) is represented by

$$\hat{H}_{\text{MQM}} = \sum_{i=1}^N \left(-\frac{\hbar^2}{2\Delta(x)} \frac{\partial^2}{\partial x_i^2} \Delta(x) + NV(x_i) \right) + \frac{1}{2} \sum_{i < j} \frac{\hat{\Pi}_{ij}^2 + \hat{\tilde{\Pi}}_{ij}^2}{(x_i - x_j)^2}. \quad (\text{III.14})$$

The wave functions are characterized by the Schrödinger and constraint equations

$$i\hbar \frac{\partial \Phi(x, \Omega)}{\partial t} = \hat{H}_{\text{MQM}} \Phi(x, \Omega), \quad \hat{\Pi}_i \Phi(x, \Omega) = 0. \quad (\text{III.15})$$

2.2 Reduction to the singlet sector

Using the Hamiltonian derived in the previous paragraph, the partition function (III.1) can be rewritten as follows

$$Z_N = \text{Tr} e^{-T\hbar^{-1} \hat{H}_{\text{MQM}}}, \quad (\text{III.16})$$

where T is the time interval we are interested in. If one considers the sum over surfaces embedded in the infinite real line, the interval should also be infinite. In this limit only the ground state of the Hamiltonian contributes to the partition function and we have

$$F = \lim_{T \rightarrow \infty} \frac{\log Z_N}{T} = -E_0/\hbar. \quad (\text{III.17})$$

Thus, we should look for an eigenfunction of the Hamiltonian (III.14) which realizes its minimum. It is clear that the last term representing the angular degrees of freedom is positive definite and should annihilate this eigenfunction. To understand the sense of this condition, let us note that the angular argument Ω of the wave functions belongs to $\text{SU}(N)$. Hence, the wave functions are functions on the group and can be decomposed in its irreducible representations

$$\Phi(x, \Omega) = \sum_r \sum_{a,b=1}^{d_r} D_{ba}^{(r)}(\Omega) \Phi_{ab}^{(r)}(x), \quad (\text{III.18})$$

where r denotes an irreducible representation, d_r is its dimension and $D_{ba}^{(r)}(\Omega)$ is the representation matrix of an element $\Omega \in \text{SU}(N)$ in the representation r . The coefficients are functions of only the eigenvalues $x_i(t)$. On the other hand, the operators $\hat{\Pi}_{ij}$ and $\hat{\tilde{\Pi}}_{ij}$ are generators of the left rotations $\Omega \rightarrow U\Omega$. It is clear that in the sum (III.18) the only term remaining invariant under this transformation corresponds to the trivial, or *singlet*, representation. Thus, the condition $\hat{\Pi}_{ij}\Phi = \hat{\tilde{\Pi}}_{ij}\Phi = 0$ restricts us to the sector of the Hilbert space where the wave functions do not depend on the angular degrees of freedom $\Phi(x, \Omega) = \Phi^{(\text{sing})}(x)$.

In this singlet sector the Hamiltonian reduces to

$$\hat{H}_{\text{MQM}}^{(\text{sing})} = \sum_{i=1}^N \left(-\frac{\hbar^2}{2\Delta(x)} \frac{\partial^2}{\partial x_i^2} \Delta(x) + V(x_i) \right). \quad (\text{III.19})$$

Its form indicates that it is convenient to redefine the wave functions

$$\Psi^{(\text{sing})}(x) = \Delta(x) \Phi^{(\text{sing})}(x). \quad (\text{III.20})$$

In terms of these functions the Hamiltonian becomes the sum of the one particle Hamiltonians

$$\hat{H}_{\text{MQM}}^{(\text{sing})} = \sum_{i=1}^N \hat{h}_i, \quad \hat{h}_i = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial x_i^2} + V(x_i). \quad (\text{III.21})$$

Moreover, since a permutation of eigenvalues is also a unitary transformation, the singlet wave function $\Phi^{(\text{sing})}(x)$ should not change under such permutations and, therefore, it is symmetric. Then the redefined wave function $\Psi^{(\text{sing})}(x)$ is completely antisymmetric. Taking into account the result (III.21), we conclude that the problem involving N^2 bosonic degrees of freedom has been reduced to a system of N non-relativistic free fermions moving in the potential $V(x)$ [59]. This fact is at the heart of the integrability of MQM and represents an interesting and still not well understood equivalence between 2D critical string theory and free fermions.

2.3 Solution in the planar limit

According to the formula (III.17) and the fermionic interpretation found in the previous paragraph, one needs to find the ground state energy of the system of N non-interacting fermions. All states of such system are described by Slater determinants and characterized by the filled energy levels of the one-particle Hamiltonian \hat{h} (III.21)

$$\Psi_{n_1, \dots, n_N}(x) = \frac{1}{\sqrt{N!}} \det_{k,l} \psi_{n_k}(x_l), \quad (\text{III.22})$$

where $\psi_n(x)$ is the eigenfunction at the n th level

$$\hat{h}\psi_n(x) = \epsilon_n \psi_n(x). \quad (\text{III.23})$$

The ground state is obtained by filling the lowest N levels so that the corresponding energy is given by

$$E_0 = \sum_{n=1}^N \epsilon_n. \quad (\text{III.24})$$

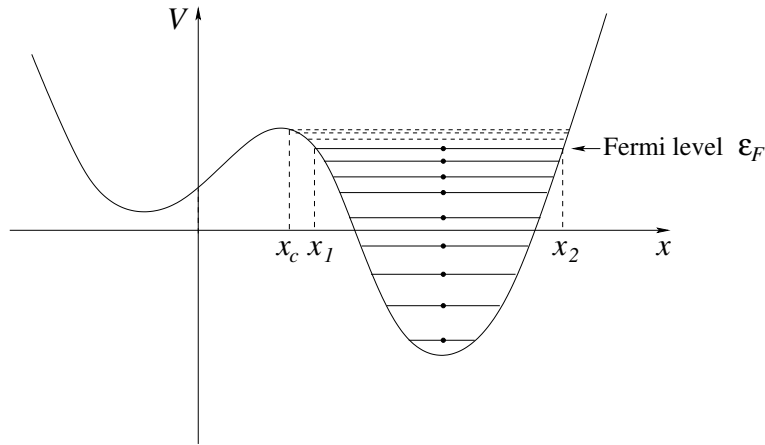


Fig. III.1: The ground state of the free fermionic system. The fermions fill the first N levels up to the Fermi energy. At the critical point where the Fermi level touches the top of the potential the energy levels condensate and the density diverges.

Note, that if the cubic potential is chosen, the system is non-stable. The same conclusion can be made for any unbounded potential. Therefore, strictly speaking, there is no ground state in such situation. However, we are interested only in the perturbative expansion in $1/N$, which corresponds to the expansion in the Planck constant. On the other hand, the amplitudes of tunneling from an unstable vacuum are exponentially suppressed as $\sim e^{-1/\hbar}$. Thus, these effects are not seen in the perturbation theory and we can simplify the life considering even unstable potentials forgetting about the instabilities. All what we need is to separate the perturbative effects from the non-perturbative ones.

As a result, we get the picture presented in fig. III.1. Let us consider this system in the large N limit. Since we identified $1/N$ with the Planck constant \hbar , $N \rightarrow \infty$ corresponds to the classical limit. In this approximation the energy becomes continuous and particles are characterized by their coordinates in the phase space. In our case the phase space is two-dimensional and each particle occupies the area $2\pi\hbar$. Moreover, due to the fermionic nature, two particles cannot take the same place. Thus, the total area occupied by N particles is 2π . Due to the Liouville theorem it is preserved in the time evolution. Therefore, the classical description of N free fermions is the same as that of an incompressible liquid.

For us it is important now only that the ground state corresponds to a configuration where the liquid fills a connected region (Fermi sea) with the boundary given by the following equation

$$h(x, p) = \frac{1}{2}p^2 + V(x) = \epsilon_F, \quad (\text{III.25})$$

where $\epsilon_F = \epsilon_N$ is the energy at the Fermi level. Then one can write

$$N = \iint \frac{dx dp}{2\pi\hbar} \theta(\epsilon_F - h(x, p)), \quad (\text{III.26})$$

$$E_0 = \iint \frac{dx dp}{2\pi\hbar} h(x, p) \theta(\epsilon_F - h(x, p)). \quad (\text{III.27})$$

Differentiation with respect to ϵ_F gives

$$\hbar \frac{\partial N}{\partial \epsilon_F} \stackrel{def}{=} \rho(\epsilon_F) = \iint \frac{dx dp}{2\pi} \delta(\epsilon_F - h(x, p)) = \frac{1}{\pi} \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(\epsilon_F - V(x))}}, \quad (\text{III.28})$$

$$\hbar \frac{\partial E_0}{\partial \epsilon_F} = \iint \frac{dx dp}{2\pi} h(x, p) \delta(\epsilon_F - h(x, p)) = \epsilon_F \rho(\epsilon_F), \quad (\text{III.29})$$

where x_1 and x_2 are turning points of the classical trajectory at the Fermi level. These equations determine the energy in an inexplicit way.

To find the energy in terms of the coupling constants, one should exclude the Fermi level ϵ_F by means of the normalization condition (III.26). For some simple potentials the integral in (III.26) can be calculated explicitly, but in general this cannot be done. However, the universal information related to the sum over continuous surfaces and 2D string theory is contained only in the singular part of the free energy. The singularity appears when the Fermi level reaches the top of the potential similarly to the one-matrix case (cf. figs. III.1 and II.4). Near this point the density diverges and shows together with the energy a non-analytical behaviour.

From this it is clear that the singular contribution to the integral (III.28) comes from the region of integration around the maxima of the potential. Generically, the maxima are of the quadratic type. Thus, up to analytical terms we have

$$\int_{x_1}^{x_2} \frac{dx}{\sqrt{2(\epsilon_F - V(x))}} \sim -\frac{1}{2} \log(\epsilon_c - \epsilon_F), \quad (\text{III.30})$$

where ϵ_c is the critical value of the Fermi level. Denoting $\epsilon = \epsilon_c - \epsilon_F$, one finds [43]

$$\rho(\epsilon) = -\frac{1}{2\pi} \log(\epsilon/\Lambda), \quad F_0 = \frac{1}{4\pi\hbar^2} \epsilon^2 \log(\epsilon/\Lambda), \quad (\text{III.31})$$

where we introduced a cut-off Λ related to the non-universal contributions.

2.4 Double scaling limit

In the previous paragraph we reproduced the free energy and the density of states in the planar limit. To find them in all orders in the genus expansion, one needs to consider the double scaling limit as it was explained in section II.2.5. For this one should correlate the large N limit with the limit where the coupling constants approach their critical values. In our case this means that one should introduce coordinates describing the region near the top of the potential. Let x_c is the coordinate of the maximum and $y = \frac{1}{\sqrt{\hbar}}(x - x_c)$. Then the potential takes the form

$$V(y) = \epsilon_c - \frac{\hbar}{2} y^2 + \frac{\hbar^{3/2} \lambda}{3} y^3 + \dots, \quad (\text{III.32})$$

where the dots denote the terms of higher orders in \hbar . The Schrödinger equation for the eigenfunction at the Fermi level can be rewritten as follows

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial y^2} - \frac{1}{2} y^2 + \frac{\hbar^{1/2} \lambda}{3} y^3 + \dots \right) \psi_N(y) = -\hbar^{-1} \epsilon \psi_N(y). \quad (\text{III.33})$$

It shows that it is natural to define the rescaled energy variable

$$\mu = \hbar^{-1}(\epsilon_c - \epsilon_F). \quad (\text{III.34})$$

This relation defines the double scaling limit of MQM, which is obtained as $N = \hbar^{-1} \rightarrow \infty$, $\epsilon_F \rightarrow \epsilon_c$ and keeping μ to be fixed [75, 76, 77, 78].

Note that the double scaling limit (III.34) differs from the naive limit expected from 1MM where we kept fixed the product of N and some power of $\lambda_c - \lambda$ (II.21). In our case the latter is renormalized in a non-trivial way. To get this renormalization, note that writing the relation (III.28) we actually decoupled N and \hbar . This means that in fact we rescaled the argument of the potential so that we moved the coupling constant from the potential to the coefficient in front of the action. For example, we can do this with the cubic coupling constant λ by rescaling $x \rightarrow x/\lambda$. In this normalization the overall coefficient should be multiplied by λ^{-2} what changes the relation between N and the Planck constant to $\hbar = \lambda^2/N$. Then for $\Delta = \frac{2\pi}{\hbar}(\lambda_c^2 - \lambda^2)$, (III.28) and (III.34) imply

$$\frac{\partial \Delta}{\partial \mu} = 2\pi\rho(\mu). \quad (\text{III.35})$$

Integrating this equation, one finds a complicated relation between two scaling variables. In the planar limit this relation reads

$$\Delta = -\mu \log(\mu/\Lambda). \quad (\text{III.36})$$

The remarkable property of the double scaling limit (III.34) is that it reduces the problem to the investigation of free fermions in the inverse oscillator potential $V_{\text{ds}}(x) = -\frac{1}{2}x^2$

$$-\frac{1}{2} \left(\frac{\partial^2}{\partial x^2} + x^2 \right) \psi_\epsilon(x) = \epsilon \psi_\epsilon(x), \quad (\text{III.37})$$

where we returned to the notations x and ϵ for already rescaled matrix eigenvalues and energy. All details of the initial potential disappear in this limit because after the rescaling the cubic and higher terms suppressed by positive powers of \hbar . This fact is the manifestation of the universality of MQM showing the independence of its results of the form of the potential.

The equation (III.37) for eigenfunctions has an explicit solution in terms of the parabolic cylinder functions. They have a complicated form and we do not give their explicit expressions. However, the density of states at the Fermi level can be calculated knowing only their asymptotics at large x . The density follows from the WKB quantization condition

$$\left(\Phi_{\epsilon_{n+1}} - \Phi_{\epsilon_n} \right) \Big|_{\sqrt{\Lambda}} = 2\pi, \quad (\text{III.38})$$

where Φ_ϵ is the phase of the wave function $\psi_\epsilon(x) = \frac{C}{\sqrt{x}} e^{i\Phi_\epsilon(x)}$ and the difference is calculated at the cut-off $x \sim \sqrt{\Lambda} \sim \sqrt{N}$. The asymptotic form of the phase is [79]

$$\Phi_\epsilon(x) \approx \frac{1}{2}x^2 + \epsilon \log x - \phi(\epsilon), \quad (\text{III.39})$$

$$\phi(\epsilon) = \frac{\pi}{4} - \frac{i}{2} \log \frac{\Gamma(\frac{1}{2} + i\epsilon)}{\Gamma(\frac{1}{2} - i\epsilon)}. \quad (\text{III.40})$$

In the WKB approximation the index n becomes continuous variable and the density of states is defined as its derivative

$$\rho(\epsilon) \stackrel{\text{def}}{=} \frac{\partial n}{\partial \epsilon} = \frac{1}{2\pi} \log \Lambda - \frac{1}{2\pi} \frac{d\phi}{d\epsilon} = \frac{1}{2\pi} \log \Lambda - \frac{1}{2\pi} \operatorname{Re} \psi\left(\frac{1}{2} + i\epsilon\right), \quad (\text{III.41})$$

where $\psi(\epsilon) = \frac{d}{d\epsilon} \log \Gamma(\epsilon)$. Neglecting the cut-off dependent term and expanding the digamma function in $1/\mu$ ($\mu = -\epsilon$), one finds the following result

$$\rho(\mu) = \frac{1}{2\pi} \left(-\log \mu + \sum_{n=1}^{\infty} (2^{2n-1} - 1) \frac{|B_{2n}|}{n} (2\mu)^{-2n} \right), \quad (\text{III.42})$$

where B_{2n} are Bernoulli numbers. Integrating (III.29), one obtains the expansion of the free energy

$$F(\mu) = \frac{1}{4\pi} \left(\mu^2 \log \mu - \frac{1}{12} \log \mu + \sum_{n=1}^{\infty} \frac{(2^{2n+1} - 1) |B_{2n+2}|}{4n(n+1)} (2\mu)^{-2n} \right). \quad (\text{III.43})$$

In fact, to compare this result with the genus expansion of the partition function of 2D string theory, one should reexpand (III.43) in terms of the renormalized string coupling. Its role, as usual, is played by $\kappa = \Delta^{-1}$ and its relation to μ is determined by (III.35). With $\rho(\mu)$ taken from (III.42), one can solve this equation with respect to $\Delta(\mu)$. Then it is sufficient to make substitution into (III.43) to get the following answer

$$F(\Delta) = \frac{1}{4\pi} \left(\frac{\Delta^2}{\log \Delta} - \frac{1}{12} \log \Delta + \sum_{n=1}^{\infty} \frac{(2^{2n+1} - 1) |B_{2n+2}|}{4n(n+1)(2n+1)} \left(\frac{2\Delta}{\log \Delta} \right)^{-2n} \right), \quad (\text{III.44})$$

where terms $O(\log^{-1} \Delta)$ were neglected because they contain the cut-off and vanish in the double scaling limit.

Some remarks related to the expansion (III.44) are in order. First, the coefficients associated with genus g grow as $(2g)!$. This behaviour is characteristic for closed string theories where the sum over genus- g surfaces exhibits the same growth. Besides, we observe a new feature in comparison with the one-matrix model. Although the couplings were renormalized, the sums over spherical and toroidal surfaces logarithmically diverge. This also can be explained in the context of the CFT approach. Finally, comparing (III.44) with (II.19), we find that the string susceptibility for MQM vanishes $\gamma_{\text{str}} = 0$. This is again in the excellent agreement with the continuum prediction of [80].

Thus, all results concerning the free energy of MQM coincide with the corresponding results for the partition function of 2D string theory. Therefore, it is tempting to claim that they are indeed equivalent theories. But we know that 2D string theory possesses dynamical degrees of freedom: the tachyon, winding modes and discrete states. To justify the equivalence further, we should show how all of them are realized in Matrix Quantum Mechanics.

3 Das–Jevicki collective field theory

The fermionic representation presented in the previous section gives a microscopic description of 2D string theory. As usual, the macroscopic description, which has a direct interpretation in terms of the target space fields of 2D string theory, is obtained as a theory of effective degrees of freedom. These degrees of freedom are collective excitations of the fermions of the singlet sector of MQM and identified with the tachyonic modes of 2D strings. Their dynamics is governed by a *collective field theory* [81], which in the given case was developed by Das and Jevicki [82]. This theory encodes all interactions of strings in two dimensions and, therefore, it gives an example of string field theory formulated directly in the target space.

3.1 Effective action for the collective field

A natural collective field in MQM is the density of eigenvalues

$$\varphi(x, t) = \text{tr } \delta(x - M(t)). \quad (\text{III.45})$$

In the double scaling limit, which implies $N \rightarrow \infty$, it becomes a continuous field. Its dynamics can be derived directly from the MQM action with the inverse oscillator potential [82]

$$S = \frac{1}{2} \int dt \text{tr} (\dot{M}^2 + M^2). \quad (\text{III.46})$$

However, it is much easier to use the Hamiltonian formulation. The Hamiltonian of MQM in the singlet representation is given by the energy of the Fermi sea similarly to the ground state energy (III.27). The difference is that now the Fermi sea can have an arbitrary profile which can differ from the trajectory of one fermion (III.25). Besides, to the expression (III.27) one should add a term which fixes the Fermi level and allows to vary the number of fermions. Otherwise the density would be subject of some normalization condition. Thus, the full double scaled Hamiltonian reads

$$H_{\text{coll}} = \iint_{\text{Fermi sea}} \frac{dx dp}{2\pi} (h(x, p) + \mu), \quad (\text{III.47})$$

where

$$h(x, p) = \frac{1}{2} p^2 + V(x), \quad V(x) = -\frac{1}{2} x^2. \quad (\text{III.48})$$

We restrict ourselves to the case where the boundary of the Fermi sea can be represented by two functions, say $p_+(x, t)$ and $p_-(x, t)$ satisfying the boundary condition $p_+(x_*, t) = p_-(x_*, t)$, where x_* is the leftmost point of the sea (fig. III.2a). It means that we forbid the situations shown on fig. III.2b. In the fermionic picture they do not cause any problems, but in the bosonic description they require a special attention.

In this restricted situation one can take the integral over the momentum in (III.47). The result is

$$H_{\text{coll}} = \int \frac{dx}{2\pi} \left(\frac{1}{6} (p_+^3 - p_-^3) + (V(x) + \mu)(p_+ - p_-) \right). \quad (\text{III.49})$$

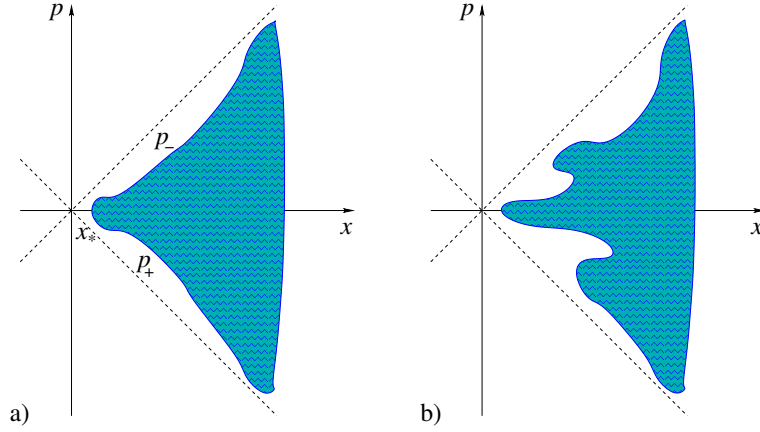


Fig. III.2: The Fermi sea of the singlet sector of MQM. The first picture shows the situation where the profile of the Fermi sea can be described by a two-valued function. The second picture presents a more general configuration.

It is clear that the difference of p_+ and p_- coincides with the density (III.45), whereas their sum plays the role of a conjugate variable. The right identification is the following [83]:

$$p_{\pm}(x, t) = \partial_x \Pi \pm \pi \varphi(x, t), \quad (\text{III.50})$$

where the equal-time Poisson brackets are defined as

$$\{\varphi(x), \Pi(y)\} = \delta(x - y). \quad (\text{III.51})$$

Substitution of (III.50) into (III.49) gives

$$H_{\text{coll}} = \int dx \left(\frac{1}{2} \varphi (\partial_x \Pi)^2 + \frac{\pi^2}{6} \varphi^3 + (V(x) + \mu) \varphi \right). \quad (\text{III.52})$$

One can exclude the momentum $\Pi(x, t)$ by means of the equation of motion

$$-\partial_x \Pi = \frac{1}{\varphi} \int dx \partial_t \varphi \quad (\text{III.53})$$

what leads to the following collective field theory action

$$S_{\text{coll}} = \int dt \int dx \left(\frac{1}{2\varphi} \left(\int dx \partial_t \varphi \right)^2 - \frac{\pi^2}{6} \varphi^3 - (V(x) + \mu) \varphi \right). \quad (\text{III.54})$$

This action can be considered as a background independent formulation of string field theory. It contains a cubic interaction and a linear tadpole term. The former describes the effect of splitting and joining strings and the latter represents a process of string annihilation into the vacuum. The important point is that the dynamical field $\varphi(x, t)$ is two-dimensional. The dimension additional to the time t appeared from the matrix eigenvalues. This shows that the target space of the corresponding string theory is also two-dimensional in agreement with our previous conclusion.

Another observation is that whereas the initial matrix model in the inverse oscillator potential was simple with the linear equations of motion

$$\ddot{M}(t) - M(t) = 0, \quad (\text{III.55})$$

the resulting collective field theory is non-linear. Thus, MQM provides a solution of a complicated non-linear theory through the transformation of variables (III.45). Nevertheless, the integrability of MQM is present also in the effective theory (III.54). Indeed, consider the equations of motion for the fields p_+ and p_- . The equations (III.50) and (III.51) imply the following Poisson brackets

$$\{p_{\pm}(x), p_{\pm}(y)\} = \mp 2\pi \partial_x \delta(x - y) \quad (\text{III.56})$$

so that the Hamiltonian (III.49) gives

$$\partial_t p_{\pm} + p_{\pm} \partial_x p_{\pm} + \partial_x V(x) = 0. \quad (\text{III.57})$$

This equation is a KdV type equation which is integrable. This indicates that the whole theory is also exactly solvable. In fact, one can write an infinite set of conserved commuting quantities [83]

$$H_n = \iint_{\text{Fermi sea}} \frac{dx dp}{2\pi} (p^2 - x^2)^n. \quad (\text{III.58})$$

It is easy to check that up to surface terms they satisfy

$$\{H_n, H_m\} = 0 \quad \text{and} \quad \frac{d}{dt} H_n = 0. \quad (\text{III.59})$$

The quantities H_n can be considered as Hamiltonians generating some perturbations. Since all of them are commuting, according to the definition given in section II.5.1, we conclude that the system is integrable.

3.2 Identification with the linear dilaton background

Now we choose a particular background of string theory. This will allow to identify the tachyon field and target space coordinates with the corresponding quantities of the collective field theory. In terms of the collective theory the choice of a background means to consider the perturbation theory around some classical solution of (III.54). We choose the solution describing the ground state of MQM. It is obtained from (III.25) and can be written as

$$\pi \varphi_0 = \pm p_{\pm} = \sqrt{x^2 - 2\mu}. \quad (\text{III.60})$$

This solution is distinguished by the fact that it is static and that the boundary of the Fermi sea coincides with a trajectory of one fermion.

Taking (III.60) as a background, we are interested in small fluctuations of the collective field around this background solution

$$\varphi(x, t) = \varphi_0(x) + \frac{1}{\sqrt{\pi}} \partial_x \eta(x, t). \quad (\text{III.61})$$

The dynamics of these fluctuations is described by the following action obtained by substitution of (III.61) into (III.54)

$$S_{\text{coll}} = \frac{1}{2} \int dt \int dx \left(\frac{(\partial_t \eta)^2}{(\pi \varphi_0 + \sqrt{\pi} \partial_x \eta)} - \pi \varphi_0 (\partial_x \eta)^2 - \frac{\sqrt{\pi}}{3} (\partial_x \eta)^3 \right). \quad (\text{III.62})$$

The expansion of the denominator in the first term gives rise to an infinite number of vertices of increasing order with the field η . A compact version of this interacting theory would be obtained if we worked with the Hamiltonian instead of the action. Then only cubic interaction terms would appear.

Let us consider the quadratic part of the action (III.62). It is given by

$$S_{(2)} = \frac{1}{2} \int dt \int dx \left(\frac{(\partial_t \eta)^2}{\pi \varphi_0} - (\pi \varphi_0) (\partial_x \eta)^2 \right). \quad (\text{III.63})$$

Thus, $\eta(x, t)$ can be interpreted as a massless field propagating in the background metric

$$g_{\mu\nu}^{(0)} = \begin{pmatrix} -\pi \varphi_0 & 0 \\ 0 & (\pi \varphi_0)^{-1} \end{pmatrix}. \quad (\text{III.64})$$

However, the non-trivial metric can be removed by a coordinate transformation. It is enough to introduce the time-of-flight coordinate

$$q(x) = \int^x \frac{dx}{\pi \varphi_0(x)}. \quad (\text{III.65})$$

The change of coordinate (III.65) brings the action to the form

$$S_{\text{coll}} = \frac{1}{2} \int dt \int dq \left((\partial_t \eta)^2 - (\partial_q \eta)^2 - \frac{1}{3\pi \sqrt{\pi} \varphi_0^2} \left((\partial_q \eta)^3 + 3(\partial_q \eta)(\partial_t \eta)^2 \right) + \dots \right), \quad (\text{III.66})$$

where we omitted the terms of higher orders in η . The action (III.66) describes a massless field in the flat Minkowski spacetime with a spatially dependent coupling constant

$$g_{\text{str}}(q) = \frac{1}{(\pi \varphi_0(q))^2}. \quad (\text{III.67})$$

Using the explicit formula (III.60) for the background solution, one can obtain

$$x(q) = \sqrt{2\mu} \cosh q, \quad p(q) = \sqrt{2\mu} \sinh q. \quad (\text{III.68})$$

Thus, the coupling constant behaves as

$$g_{\text{str}}(q) = \frac{1}{2\mu \sinh^2 q} \sim \frac{1}{\mu} e^{-2q}, \quad (\text{III.69})$$

where the asymptotics is given for $q \rightarrow \infty$.

Comparing (III.69) with (I.25), we see that the collective field theory action (III.66) describes 2D string theory in the linear dilaton background. In the asymptotic region of

large q the flat coordinates (t, q) can be identified with the coordinates of the target space of string theory coming from the $c = 1$ matter X and the Liouville field ϕ on the world sheet. The identification reads as follows

$$it \leftrightarrow X, \quad q \leftrightarrow \phi. \quad (\text{III.70})$$

Thus, the time of MQM and the time-of-flight coordinate, which is a function of the matrix eigenvalue variable, form the flat target space of the linear dilaton background. It is also clear that the two-dimensional massless collective field $\eta(q, t)$ coincides with the redefined tachyon $\eta = e^{2\phi}T$.

In fact, the above identification is valid only asymptotically. When we go to the region of small q , one should include into account the Liouville exponent $\mu e^{-2\phi}$ in the CFT action (I.39). The tachyon field can be considered as a wave function describing the lowest eigenstate of the Hamiltonian of this CFT. Therefore, instead of Klein–Gordon equation, it should satisfy the Liouville equation

$$\left(\partial_X^2 + \partial_\phi^2 + 4\partial_\phi + 4 - \mu e^{-2\phi}\right) T(\phi, X) = 0. \quad (\text{III.71})$$

Hence the Liouville mode does not coincide exactly with the collective field coordinate q . The correct identification is obtained as follows [84]. We return to the eigenvalue variable x and consider its conjugated momentum $p = -i\partial/\partial x$. The Fourier transform of the Klein–Gordon equation written in the metric (III.64) gives

$$\left(\partial_t^2 - \sqrt{x^2 - 2\mu} \partial_x \sqrt{x^2 - 2\mu} \partial_x\right) \eta(x, t) = 0 \Rightarrow \left(\partial_t^2 - (p\partial_p)^2 - 2p\partial_p - 1 - 2\mu p^2\right) \tilde{\eta}(p, t) = 0. \quad (\text{III.72})$$

where $\tilde{\eta}$ is the Fourier image of η . Finally, the change of variables

$$it = X, \quad ip = \frac{1}{\sqrt{2}} e^{-\phi} \quad (\text{III.73})$$

together with $T(\phi, X) \sim p^3 \tilde{\eta}(p, t)$ brings (III.72) to the Liouville equation (III.71). This shows that, more precisely, the Liouville coordinate is identified with the logarithm of the momentum conjugated to the matrix eigenvalue.

The meaning of this rule becomes more clear after realizing that the Fourier transform with an imaginary momentum of the collective field φ is the Wilson loop operator

$$W(l, t) = \text{tr} \left(e^{-lM} \right) = \int dx e^{-lx} \varphi(x, t). \quad (\text{III.74})$$

This operator inserts a loop of the length l into the world sheet. Therefore, it has a direct geometrical interpretation and its parameter l is related to the scale of the metric, which is governed by the Liouville mode ϕ . Thus, it is quite natural that l and ϕ are identified through (III.73) where one should take $ip = l$. The substitution of the expansion (III.61) gives

$$W(l, t) = W_0 + \frac{1}{\sqrt{\pi}} \int dx e^{-lx} \partial_x \eta(x, t) = W_0 + \frac{l}{\sqrt{\pi}} \tilde{\eta}(-il, t), \quad (\text{III.75})$$

where

$$W_0 = \frac{\sqrt{2\mu}}{l} K_1(\sqrt{2\mu}l) \quad (\text{III.76})$$

is the genus zero one-point function of the density. Using this representation, it is easy to check that the Wilson loop operator satisfies the following Wheeler–DeWitt equation [85]

$$\left(\partial_t^2 - (l\partial_l)^2 + 2\mu l^2\right) W(l, t) = 0. \quad (\text{III.77})$$

Thus, in the l -representation it is the Wilson loop operator that is the analog of the free field for which the derivative terms have standard form. Therefore, we should identify the field η from section I.5.1 with W rather than with $\tilde{\eta}$ defined in (III.72). The precise relation between the tachyon field and the Wilson loop operator is the following

$$T(\phi, X) = e^{-2\phi} W(l(\phi), -iX) = e^{-2\phi} W_0 + e^{-2\phi} \int_0^\infty dq \exp\left[-\sqrt{\mu} e^{-\phi} \cosh q\right] \partial_q \eta. \quad (\text{III.78})$$

The integral transformation (III.78) expresses solutions of the non-linear Liouville equation through solutions of the Klein–Gordon equation. This reduces the problem of calculating the tachyon scattering amplitudes in the linear dilaton background to calculation of the S -matrix for the collective field theory of the Klein–Gordon field η . As we saw above, this theory is integrable. Therefore, the scattering problem in 2D string theory can be exactly solved. Before to show that, we should introduce the operators creating the asymptotic states, *i.e.*, the tachyon vertex operators.

3.3 Vertex operators and correlation functions

The vertex operators of the tachyon field were constructed in section I.5.1. Their Minkowskian form is given by (I.44) and describes the left and right movers. Note that the representation for the operators was written only in the asymptotic region $\phi \rightarrow \infty$ where the Liouville potential can be ignored. Therefore, we can use the simple identification (III.70) to relate the matrix model quantities with the target space objects. Then one should find operators that behave as left and right movers in the space of t and q .

First, the t -dependence of a matrix model operator is completely determined by the inverse oscillator potential, which leads to the following simple Heisenberg equation

$$\frac{\partial}{\partial t} \hat{A}(t) = i \left[\frac{1}{2} (\hat{p}^2 - \hat{x}^2), \hat{A}(t) \right]. \quad (\text{III.79})$$

Its solution is conveniently represented in the basis of the chiral operators

$$\hat{x}_\pm(t) \stackrel{\text{def}}{=} \frac{\hat{x}(t) \pm \hat{p}(t)}{\sqrt{2}} = \hat{x}_\pm(0) e^{\pm t}. \quad (\text{III.80})$$

Therefore, the time-independent operators, which should be used in the Schrödinger representation, are $e^{\mp t} \hat{x}_\pm(t)$. This suggests that the vertex operators can be constructed from powers of x_\pm . Indeed, it was argued [83] that their matrix model realization is given by

$$T_n^\pm = e^{\pm nt} \text{tr} (M \mp P)^n. \quad (\text{III.81})$$

To justify further this choice, let us consider the collective field representation of the operators (III.81)

$$T_n^\pm = e^{\pm nt} \int \frac{dx}{2\pi} \int_{p_-(x,t)}^{p_+(x,t)} dp (x \mp p)^n = \frac{e^{\pm nt}}{n+1} \int \frac{dx}{2\pi} (x \mp p)^{n+1} \Big|_{p_-}^{p_+}, \quad (\text{III.82})$$

where p_\pm are second quantized fields satisfying equations (III.57). We shift these fields by the classical solution

$$p_\pm(x, t) = \pm \pi \varphi_0(x) + \frac{\alpha_\mp(x, t)}{\pi \varphi_0(x)}. \quad (\text{III.83})$$

Then the linearized equations of motion for the quantum corrections α_\pm coincide with the conditions for chiral fields

$$(\partial_t \mp \partial_q) \alpha_\pm = 0 \Rightarrow \alpha_\pm = \alpha_\pm(t \pm q). \quad (\text{III.84})$$

In the asymptotics $q \rightarrow \infty$, $\pi \varphi_0 \approx x \approx \frac{1}{2} e^q$. Therefore, in the leading approximation the operators (III.82) read

$$T_n^\pm \approx \int \frac{dq}{2\pi} e^{n(q \pm t)} \alpha_\pm. \quad (\text{III.85})$$

The integral extracts the operator creating the component of the chiral field which behaves as $e^{n(q \pm t)}$. This shows that the matrix operators (III.81) do possess the necessary properties. The factor $e^{-2\phi}$, which is present in the definition of the vertex operators (I.44) and absent in the matrix case, can be seen as coming from the measure of integration over the world sheet or, equivalently, as a result of the redefinition of the tachyon field (III.78). In fact, it is automatically restored by the matrix model.

The matrix operators (III.81) correspond to the Minkowskian vertex operators with imaginary momenta $k = in$. After the continuation of time to the Euclidean region $t \rightarrow -iX$, they also can be considered as vertex operators with Euclidean momenta $p = \mp n$. To realize other momenta, one should analytically continue from this discrete set to the whole complex plane. In particular, the vertex operators of Minkowskian real momenta are obtained as

$$V_k^\pm \sim T_{-ik}^\pm = e^{\mp ikt} \text{tr} (M \mp P)^{-ik}. \quad (\text{III.86})$$

Using these operators, one can construct and calculate scattering amplitudes of tachyons. The result has been obtained from both the collective field theory formalism [86, 87, 88, 89] and the fermionic representation [26, 90, 91, 92, 79]. Moreover, the generating functional for all S -matrix elements has been constructed [92]. It takes an especially transparent form when the S -matrix is represented as a composition of three processes: fermionization of incoming tachyon modes, scattering in the free fermion theory and reverse bosonization of the scattered fermions

$$S_{TT} = \iota_{f \rightarrow b} \circ S_{FF} \circ \iota_{b \rightarrow f}. \quad (\text{III.87})$$

The fermionic S -matrix S_{FF} was explicitly calculated from the properties of the parabolic cylinder functions [91]. We do not give more details since these results will be reproduced in much simpler way from the formalism which we develop in the next chapters.

We restrict ourselves to two remarks. The first one is that in those cases where the scattering amplitudes in 2D string theory can be calculated by the CFT methods, the results coincided with the corresponding calculations in MQM [93]. The only thing to be done to ensure the complete agreement is a local redefinition of the vertex operators. It turns out that the exact relation between the tachyon operators (I.44) and their matrix model realization (III.81) include the so called *leg-factors* [94]

$$V_k^+ = \frac{\Gamma(-ik)}{\Gamma(ik)} T_{-ik}^+, \quad V_k^- = \frac{\Gamma(ik)}{\Gamma(-ik)} T_{-ik}^-. \quad (\text{III.88})$$

This redefinition is not surprising because the matrix model gives only a discrete approximation to the local vertex operators and in the continuum limit the operators can be renormalized. Therefore, one should expect the appearance of such leg-factors in any matrix/string correspondence. Note that the Minkowskian leg-factors (III.88) are pure phases. Thus, they represent a unitary transformation and do not affect the amplitudes. However, they are relevant for the correct spacetime physics, in particular, for the gravitational scattering of tachyons [89]. In fact, the leg-factors can be associated with a field redefinition given by the integral transformation (III.78). Written in the momentum space for q , it gives rise to additional factors for the left and right components whose ratio produces the leg-factor.

The second remark is that in the case when the Euclidean momenta of the incoming and outgoing tachyons belong to an equally spaced lattice (as in a compactified theory), the generating functional for S -matrix elements has been shown to coincide with a τ -function of Toda hierarchy [92]. However, this fact has not been used to address other problems like scattering in presence of a tachyon condensate. We will show that with some additional information added, it allows to solve many interesting questions related to 2D string theory in non-trivial backgrounds.

3.4 Discrete states and chiral ring

Finally, we show how the discrete states of 2D string theory appear in MQM. They are created by a natural generalization of the matrix operators (III.81) [95]

$$T_{n,\bar{n}} = e^{(\bar{n}-n)t} \text{tr} ((M+P)^n (M-P)^{\bar{n}}) \quad (\text{III.89})$$

which have the following Euclidean momenta

$$p_X = i(n - \bar{n}), \quad p_\phi = n + \bar{n} - 2. \quad (\text{III.90})$$

Comparing with the momenta of the discrete states (I.49), one concludes that

$$m = \frac{n - \bar{n}}{2}, \quad j = \frac{n + \bar{n}}{2}. \quad (\text{III.91})$$

Taking into account that n and \bar{n} are integers, one finds that the so defined pair (j, m) spans all discrete states.

It is remarkable that the collective field theory approach allows to unveil the presence of a large symmetry group [95, 84, 25]. Indeed, the operators (III.89) are realized as

$$T_{j,m} = e^{-2mt} \int \frac{dx}{2\pi} \int_{p_-}^{p_+} dp (x+p)^{j+m} (x-p)^{j-m}, \quad (\text{III.92})$$

where we changed indices of the operator from n, \bar{n} to j, m . One can check that they obey the commutation relations of w_∞ algebra

$$\{T_{j_1, m_1}, T_{j_2, m_2}\} = 4i(j_1 m_2 - j_2 m_1) T_{j_1 + j_2 - 1, m_1 + m_2}. \quad (\text{III.93})$$

In particular, the operators $T_{j, m}$ are eigenstates of the Hamiltonian $H = -\frac{1}{2}T_{1,0}$

$$\{H, T_{j, m}\} = -2imT_{j, m} \quad (\text{III.94})$$

what means that they generate its spectrum. When we replace the Poisson brackets of the classical collective field theory by the quantum commutators, the w_∞ algebra is promoted to a $W_{1+\infty}$ algebra.

Each generator (III.89) gives rise to an element of the ground ring of the $c = 1$ CFT [25] which plays an important role in many physical problems. First, we introduce the so called *chiral ground ring*. It consists of chiral ghost number zero, conformal spin zero operators \mathcal{O}_{JM} which are closed under the operator product $\mathcal{O} \cdot \mathcal{O}' \sim \mathcal{O}''$ up to BRST commutators. The entire chiral ring can be generated from the basic operators

$$\begin{aligned} \mathcal{O}_{0,0} &= 1, \\ y &\stackrel{\text{def}}{=} \mathcal{O}_{\frac{1}{2}, \frac{1}{2}} = (cb + i\partial X - \partial\phi) e^{iX+\phi}, \\ w &\stackrel{\text{def}}{=} \mathcal{O}_{\frac{1}{2}, -\frac{1}{2}} = (cb - i\partial X - \partial\phi) e^{-iX+\phi}. \end{aligned} \quad (\text{III.95})$$

The *ground ring* is constructed from products of the chiral and antichiral operators. We consider the case of the theory compactified at the self-dual radius $R = 1$ in the absence of the cosmological constant μ . Then the ground ring contains the following operators

$$\mathcal{V}_{j, m, \bar{m}} = \mathcal{O}_{j, m} \bar{\mathcal{O}}_{j, \bar{m}}. \quad (\text{III.96})$$

The ring has four generators

$$a_1 = y\bar{y}, \quad a_2 = w\bar{w}, \quad a_3 = y\bar{w}, \quad a_4 = w\bar{y}. \quad (\text{III.97})$$

These generators obey one obvious relation which determines the ground ring of the $c = 1$ theory

$$a_1 a_2 - a_3 a_4 = 0. \quad (\text{III.98})$$

It has been shown [25] that the symmetry algebra mentioned above is realized on this ground ring as the algebra of diffeomorphisms of the three dimensional cone (III.98) preserving the volume form

$$\Theta = \frac{da_1 da_2 da_3}{a_3}. \quad (\text{III.99})$$

Furthermore, it was argued that the inclusion of perturbations by marginal operators deforms the ground ring to

$$a_1 a_2 - a_3 a_4 = M(a_1, a_2), \quad (\text{III.100})$$

where M is an arbitrary function. In particular, to introduce the cosmological constant, one should take M to be constant. As a result, one removes the conic singularity and obtains a smooth manifold

$$a_1 a_2 - a_3 a_4 = \mu. \quad (\text{III.101})$$

In the limit of uncompactified theory only the generators a_1 and a_2 survive. The symmetry of volume preserving diffeomorphisms is reduced to some abelian transformations plus area preserving diffeomorphisms of the plane (a_1, a_2) that leave fixed the curve

$$a_1 a_2 = \mu \tag{III.102}$$

or its deformation according to (III.100). The suggestion was to identify the plane (a_1, a_2) with the eigenvalue phase space of MQM. Namely, one has the following relations

$$a_1 = x + p, \quad a_2 = x - p. \tag{III.103}$$

Then it is clear that equation (III.102) corresponds to the Fermi surface of MQM. The mentioned abelian transformations are associated with time translations. And the operators (III.89) are identified with $a_1^n a_2^{\bar{n}}$.

The sense of the operators a_3 and a_4 existing in the compactified theory is also known. They correspond to the winding modes of strings which we are going to consider in the next section. The relation (III.101) satisfied by these operators is very important because it shows how to describe the theory containing both the tachyon and winding modes. However, it has not been yet understood how to obtain this relation directly from MQM.

4 Compact target space and winding modes in MQM

In the previous section we showed that the modes of 2D string theory in the linear dilation background are described by the collective excitations of the singlet sector of Matrix Quantum Mechanics. If we compactify the target space of string theory, then there appear additional states — windings of strings around the compactified dimension. In this section we demonstrate how to describe these modes in the matrix language.

4.1 Circle embedding and duality

In matrix models the role of the target space of non-critical string theory is played by the parameter space of matrices. Therefore, to describe 2D string theory with one compactified dimension one should consider MQM on a circle. The partition function of such matrix model is given by the integral (III.1) where the integration is over a finite Euclidean time interval $t \in [0, \beta]$ with the two ends identified. The length of the interval is $\beta = 2\pi R$ where R is the radius of the circle. The identification $t \sim t + \beta$ requires to impose the periodic boundary condition $M(0) = M(\beta)$. Thus, one gets

$$Z_N(R, g) = \int_{M(0)=M(\beta)} \mathcal{D}M(t) \exp \left[-N \operatorname{tr} \int_0^\beta dt \left(\frac{1}{2} \dot{M}^2 + V(M) \right) \right]. \quad (\text{III.104})$$

As usual, one can write the Feynman expansion of this integral. It is the same as in (III.3) with the only difference that the propagator should be replaced by the periodic one

$$G(t) = \sum_{m=-\infty}^{\infty} e^{-|t+m\beta|}. \quad (\text{III.105})$$

We see that for large β the term $m = 0$ dominates and we return to the uncompactified case. But for finite β the sum should be retained and this leads to important phenomena related with the appearance of vortices on the discretized world sheet [96, 97, 98, 99].

We mentioned in section I.5.3 that the $c = 1$ string theory compactified at radius R is T-dual to the same theory at radius $1/R$. Does this duality appear in the sum over discretized surfaces regularizing the sum over continuous geometries? To answer this question, we perform the duality transformation of the Feynman expansion of the matrix integral (III.104)

$$F = \sum_{g=0}^{\infty} N^{2-2g} \sum_{\substack{\text{connected} \\ \text{diagrams } \Gamma_g}} \lambda^V \prod_{i=1}^V \int_0^\beta dt_i \prod_{\langle ij \rangle} \sum_{m_{ij}=-\infty}^{\infty} e^{-|t_i - t_j + \beta m_{ij}|}, \quad (\text{III.106})$$

where we have chosen for simplicity the cubic potential (II.15). The duality transformation is obtained applying the Poisson formula to the propagator (III.105)

$$G(t_i - t_j) = \frac{1}{\beta} \sum_{k_{ij}=-\infty}^{\infty} e^{i \frac{2\pi}{\beta} k_{ij} (t_i - t_j)} \tilde{G}(k_{ij}) = \frac{1}{\beta} \sum_{k_{ij}=-\infty}^{\infty} e^{i \frac{2\pi}{\beta} k_{ij} (t_i - t_j)} \frac{2}{1 + \left(\frac{2\pi}{\beta} k_{ij} \right)^2}. \quad (\text{III.107})$$

The substitution of (III.107) into (III.106) allows to integrate over t_i which gives the momentum conservation constraint at each vertex

$$k_{ij_1} + k_{ij_2} + k_{ij_3} = 0. \quad (\text{III.108})$$

This reduces the number of independent variables from E to $E - V + 1$. (One additional degree of freedom appears due to the zero mode which is canceled in all $t_i - t_j$.) By virtue of the Euler theorem this equals to $L - 1 + 2g$. According to this, we attach a momentum p_I to each elementary loop (face) of the graph (one of p_I is fixed) and define remaining $2g$ variables as momenta l_a running along independent non-contractable loops. Thus, one arrives at the following representation

$$F = \sum_{g=0}^{\infty} \left(\frac{N\beta}{\lambda^2} \right)^{2-2g} \sum_{\substack{\text{connected} \\ \text{diagrams } \Gamma}} \left(\frac{\lambda^2}{\beta} \right)^L \left(\prod_{I=1}^{L-1} \sum_{p_I=-\infty}^{\infty} \right) \left(\prod_{a=1}^{2g} \sum_{l_a=-\infty}^{\infty} \right) \prod_{\langle IJ \rangle} \tilde{G} \left(p_I - p_J + \sum_{a=1}^{2g} l_a \epsilon_{IJ}^a \right), \quad (\text{III.109})$$

where the sum goes over the dual graphs (triangulations) with L dual vertices and we introduced the matrix ϵ_{IJ}^a equal ± 1 when a dual edge $\langle IJ \rangle$ crosses an edge belonging to a th non-contractable cycle (the sign depends on the mutual orientation) and zero otherwise.

The transformation (III.107) changes $R \rightarrow 1/R$ which is seen from the form of the propagators. But the result (III.109) does not seem to be dual to the original representation (III.106). Actually, at the spherical level, instead of describing a compact target space of the inverse radius, it corresponds to the embedding into the discretized real line with lattice spacing $1/R$ [26]. This is natural because the variables p_I live in the momentum space of the initial theory which is discrete.

Thus, even in the continuum limit, the sum over discretized surfaces embedded in a circle cannot be identical to its continuum analog. The reason is that it possesses additional degrees of freedom which are ignored in the naive continuum limit. These are the vortex configurations. Indeed, in the continuum geometry the simplest vortex of winding number n is described by the field $X(\theta) = nR\theta$ where θ is the azimuth angle. However, this is a singular configuration and should be disregarded. In contrast, on a lattice the singularity is absent and such configurations are included into the statistical sum. For example, in the notations of (III.106) the number of vortices associated with a face I is given by

$$w_I = \sum_{\langle ij \rangle \in I} m_{ij}. \quad (\text{III.110})$$

It is clear that it coincides with the number of times the string is wrapped around the circle. In other words, the vortices are world sheet realizations of windings in the target space. Thus, MQM with compactified time intrinsically contains winding string configurations. But just due to this fact, it fails to reproduce the partition function of compactified 2D string theory.

It is clear that to obtain the sum over continuous surfaces possessing selfduality one should somehow exclude the vortices. This can be done restricting the sum over m_{ij} in (III.106). The distribution m_{ij} can be seen as an abelian gauge field defined on links of a graph. Then the quantity (III.110) is its field strength. We want that this strength vanishes. With this condition only the "pure gauge" configurations of m_{ij} are admissible. They are

represented as

$$m_{ij} = m_i - m_j + \sum_{a=1}^{2g} \tilde{\epsilon}_{ij}^a \tilde{l}_a, \quad (\text{III.111})$$

where integers \tilde{l}_a are associated with non-contractable loops of the dual graph and $\tilde{\epsilon}_{ij}^a$ is the matrix dual to ϵ_{IJ}^a . If we change the sum over all m_{ij} by the sum over these pure gauge configurations, the free energy (III.106) is rewritten as follows

$$\tilde{F} = \beta \sum_{g=0}^{\infty} N^{2-2g} \sum_{\substack{\text{connected} \\ \text{diagrams } \Gamma_g}} \lambda^V \prod_{i=1}^{V-1} \int_{-\infty}^{\infty} dt_i \left(\prod_{a=1}^{2g} \sum_{\tilde{l}_a=-\infty}^{\infty} \right) \prod_{\langle ij \rangle} \exp \left[-|t_i - t_j + \beta \sum_{a=1}^{2g} \tilde{\epsilon}_{ij}^a \tilde{l}_a| \right], \quad (\text{III.112})$$

where the overall factor β arises from the integration over the zero mode. The sum over m_i resulted in the extension of the integrals over t_i to the whole line. Due to this the dual transformation gives rise to integrals over momenta rather than discrete sums. Repeating the steps which led to (III.109) and renormalizing the momenta, one obtains

$$\begin{aligned} \tilde{F} = & \frac{2\pi}{R} \sum_{g=0}^{\infty} \left(\frac{N\beta}{\lambda^2} \right)^{2-2g} \sum_{\substack{\text{connected} \\ \text{diagrams } \tilde{\Gamma}}} \left(\frac{\lambda^2}{4\pi^2} \right)^L \left(\prod_{I=1}^{L-1} \int_{-\infty}^{\infty} dp_I \right) \times \\ & \times \left(\prod_{a=1}^{2g} \sum_{l_a=-\infty}^{\infty} \right) \prod_{\langle IJ \rangle} \frac{2}{1 + \frac{1}{4\pi^2} \left(p_I - p_J + \frac{2\pi}{R} \sum_{a=1}^{2g} l_a \epsilon_{IJ}^a \right)^2}. \end{aligned} \quad (\text{III.113})$$

The only essential difference between two representation (III.112) and (III.113) is the propagator. However, the universality of the continuum limit implies that the results in the macroscopic scale do not depend on it. Moreover, if we choose the Gaussian propagator, which follows from the usual Polyakov action, its Fourier transform coincides with the original one. Due to this one can neglect this discrepancy. Then the two representations are dual to each other with the following matching of the arguments

$$R \rightarrow 1/R, \quad N \rightarrow RN. \quad (\text{III.114})$$

Thus, the exclusion of vortices allowed to make the sum over discretized surfaces selfdual and they are those degrees of freedom that are responsible for the breaking of this duality in the full matrix integral.

We succeeded to identify and eliminate the vortices in the sum over discretized surfaces. How can this be done directly in the compactified Matrix Quantum Mechanics defined by the integral (III.104)? In other words, what matrix degrees of freedom describe the vortices? Let us see how MQM in the Hamiltonian formulation changes after compactification. The partition function is represented by the trace in the Hilbert space of the theory of the evolution operator as in (III.16)

$$Z_N(R) = \text{Tr} e^{-\frac{\beta}{\hbar} \hat{H}_{\text{MQM}}}. \quad (\text{III.115})$$

Now the time interval coincides with β which can also be considered as the inverse temperature. It is finite so that one should consider the finite temperature partition function. This

fact drastically complicates the problem because one should take into account the contributions of all states and not only the ground state. In particular, all representations of the $SU(N)$ global symmetry group come to the game. The states associated with these representations are those additional states which appear in the compactified theory. Therefore, it is natural to expect that they correspond to vortices on the discretized world sheet or windings of the string [99, 100].

The first check of this expectation which can be done is to verify the duality (III.114). Since we expect vortices only in the non-singlet representations of $SU(N)$, to exclude them one should restrict oneself to the singlet sector as in the uncompactified theory. Then we still have a powerful description in terms of free fermions. In the double scaling limit, we are interested in, the fermions move in the inverse oscillator potential. Thus we return to the problem stated by equation (III.37). However, the presence of a finite temperature gives rise to a big difference with the previous situation. Due to the thermal fluctuations, the Fermi surface cannot be defined in the ensemble with fixed number of particles N . The solution is to pass to the grand canonical ensemble with the following partition function

$$\mathcal{Z}(\mu, R) = \sum_{N=0}^{\infty} e^{-\beta\mu N} Z_N(R). \quad (\text{III.116})$$

The chemical potential μ is exactly the Fermi level which is considered now as the basic variable while N becomes an operator. The grand canonical free energy $\mathcal{F} = \log \mathcal{Z}$ can be expressed through the density of states. In the singlet sector the relation reads as follows

$$\mathcal{F}^{(\text{sing})} = \int_{-\infty}^{\infty} d\epsilon \rho(\epsilon) \log \left(1 + e^{-\beta(\epsilon+\mu)} \right), \quad (\text{III.117})$$

where the energy ϵ is rescaled by the Planck constant to be the same order as μ . The compactification does not affect the density which is therefore given by equation (III.41). There is a nice integral representation of this formula

$$\rho(\epsilon) = \frac{1}{2\pi} \text{Re} \int_{\Lambda^{-1}}^{\infty} d\tau \frac{e^{i\epsilon\tau}}{2 \sinh \frac{\tau}{2}} \Rightarrow \frac{\partial \rho(\epsilon)}{\partial \epsilon} = -\frac{1}{2\pi} \text{Im} \int_0^{\infty} d\tau e^{i\epsilon\tau} \frac{\tau/2}{\sinh \tau/2}. \quad (\text{III.118})$$

Taking the first derivative makes the integral well defined and allows to remove the cut-off. By the same reason, let us consider the third derivative of the free energy (III.117) with respect to μ . It can be written as

$$\frac{\partial^3 \mathcal{F}^{(\text{sing})}}{\partial \mu^3} = -\beta \int_{-\infty}^{\infty} d\epsilon \frac{\partial^2 \rho}{\partial \epsilon^2} \frac{1}{1 + e^{\beta(\epsilon+\mu)}}. \quad (\text{III.119})$$

Then the substitution of (III.118) and taking the integral over ϵ by residues closing the contour in the upper half plane gives [99]

$$\frac{\partial^3 \mathcal{F}^{(\text{sing})}}{\partial \mu^3} = \frac{\beta}{2\pi} \text{Im} \int_0^{\infty} d\tau e^{-i\mu\tau} \frac{\frac{\tau}{2}}{\sinh \frac{\tau}{2}} \frac{\frac{\pi\tau}{\beta}}{\sinh \frac{\pi\tau}{\beta}}. \quad (\text{III.120})$$

This representation possesses the explicit duality symmetry

$$R \rightarrow 1/R, \quad \mu \rightarrow R\mu, \quad (\text{III.121})$$

where one should take into account that $\partial_\mu^3 \mathcal{F}^{(\text{sing})} \rightarrow R^{-3} \partial_\mu^3 \mathcal{F}^{(\text{sing})}$. Thus, the grand canonical free energy of the singlet sector of MQM compactified on a circle is indeed selfdual. This fact can be verified also from the expansion of the free energy in $1/\mu$. The result reads [99]

$$\mathcal{F}^{(\text{sing})}(\mu, R) = -\frac{R}{2} \mu^2 \log \mu - \frac{1}{24} \left(R + \frac{1}{R} \right) \log \mu + \sum_{n=1}^{\infty} f_n^{(\text{sing})}(R) (\mu \sqrt{R})^{-2n}, \quad (\text{III.122})$$

where the coefficients are selfdual finite series in R

$$f_n^{(\text{sing})}(R) = \frac{2^{-2n-1} (2n-2)!}{n-1} \sum_{k=0}^n |2^{2k} - 2| |2^{2(n-k)} - 2| \frac{|B_{2k}| |B_{2(n-k)}|}{(2k)! [2(n-k)]!} R^{n-2k}. \quad (\text{III.123})$$

To return to the canonical ensemble, one should take the Laplace transform of $\mathcal{F}(\mu)$. But already analyzing equation (III.116), one can conclude that the canonical free energy is also selfdual. This is because $\beta N \mu$ is selfdual under the simultaneous change of R , μ and N according to (III.121) and (III.114). The same result can be obtained by the direct calculation which shows that the expansion of the canonical free energy in $\frac{1}{\Delta} \log \Delta$ differs from the expansion of the grand canonical one in $1/\mu$ only by the sign of the first term [99, 26]. These results confirm the expectation that the singlet sector of MQM does not contain vortices and that the latters are described by higher $\text{SU}(N)$ representations.

4.2 MQM in arbitrary representation: Hamiltonian analysis

We succeeded to describe the partition function of the compactified MQM in the singlet sector which does not contain winding excitations of the corresponding string theory. Also it is possible to calculate correlation functions of the tachyon modes in this case [101]. However, if we want to understand the dynamics of windings, we should study MQM in the non-trivial $\text{SU}(N)$ representations [102].

The dynamics in the sector of the Hilbert space corresponding to an irreducible representation r is described by the projection of the Hamiltonian (III.14) on this subspace. As in the case of the singlet representation, it is convenient to redefine the wave function, which is now represented as a matrix, by the Vandermonde determinant

$$\Psi_{ab}^{(r)}(x) = \Delta(x) \Phi_{ab}^{(r)}(x). \quad (\text{III.124})$$

Then the action of the Hamiltonian \hat{H}_{MQM} on the wave functions $\Psi_{ab}^{(r)}$ is given by the following matrix-differential operator

$$\hat{H}_{ab}^{(r)} = \sum_{c,d=1}^{d_r} P_{ac}^{(r)} \left[\delta_{cd} \sum_{i=1}^N \left(-\frac{\hbar^2}{2} \frac{\partial^2}{\partial x_i^2} + V(x_i) \right) + \frac{\hbar^2}{4} \sum_{i \neq j} \frac{(Q_{ij}^{(r)})_{cd}}{(x_i - x_j)^2} \right] P_{db}^{(r)}, \quad (\text{III.125})$$

where

$$Q_{ij}^{(r)} \equiv \tau_{ij}^{(r)} \tau_{ji}^{(r)} + \tau_{ji}^{(r)} \tau_{ij}^{(r)} \quad (\text{III.126})$$

and we introduced the representation matrix of $u(N)$ generators $\tau_{ij} \in u(N)$: $(\tau_{ij}^{(r)})_{ab} = D_{ab}^R(\tau_{ij})$, which satisfies

$$[\tau_{ij}, \tau_{kl}] = \delta_{jk}\tau_{il} - \delta_{il}\tau_{kj}. \quad (\text{III.127})$$

$P^{(r)}$ is the projector to the subspace consisting of the wave functions that satisfy

$$(\tau_{mm}^{(r)} \Psi^{(r)})_{ab} = \sum_{c=1}^{d_r} (\tau_{mm}^{(r)})_{ac} \Psi_{cb}^{(r)} = 0, \quad m = 1, \dots, N. \quad (\text{III.128})$$

The explicit form of $P^{(r)}$ is given by

$$P^{(r)} = \int_0^{2\pi} \prod_{m=1}^N \frac{d\theta_m}{2\pi} e^{i \sum_{m=1}^N \theta_m \tau_{mm}^{(r)}}. \quad (\text{III.129})$$

The Hilbert structure is induced by the scalar product (III.11) which has the following decomposition

$$\langle \Psi | \Psi' \rangle = \sum_r \frac{1}{d_r} \int \prod_{i=1}^N dx_i \sum_{a,b=1}^{d_r} \overline{\Psi_{ab}^{(r)}(x)} \Psi'_{ab}(x). \quad (\text{III.130})$$

This shows that the full Hilbert space is indeed a direct sum of the Hilbert spaces corresponding to irreducible representations of $SU(N)$. In the each subspace the scalar product is given by the corresponding term in the sum (III.130).

Note that in the Schrödinger equation

$$i\hbar \frac{\partial \Psi_{ab}^{(r)}}{\partial t} = \sum_{c=1}^{d_r} \hat{H}_{ac}^{(r)} \Psi_{cb}^{(r)}. \quad (\text{III.131})$$

the last index b is totally free and thus can be neglected. In other words, one should consider the eigenvalue problem given by the equation (III.131) with the constraint (III.128) where the matrix $\Psi_{ab}^{(r)}$ is replaced by the vector $\Psi_a^{(r)}$ and each solution is degenerate with multiplicity d_r .

Example: adjoint representation

Let us consider how the above construction works on the simplest non-trivial example of the adjoint representation. The representation space in this case is spanned by $|\tau_{ij}\rangle$. The $u(N)$ generators act on these states as

$$\tau_{ij} |\tau_{mn}\rangle \equiv [|\tau_{ij}, \tau_{mn}\rangle] = \delta_{jm} |\tau_{in}\rangle - \delta_{in} |\tau_{mj}\rangle \quad (\text{III.132})$$

what means that their representation matrices are given by

$$(\tau_{ij}^{(\text{adj})})_{kl,mn} = \delta_{ik} \delta_{jm} \delta_{ln} - \delta_{in} \delta_{jl} \delta_{km}. \quad (\text{III.133})$$

The operator $Q_{ij}^{(\text{adj})}$ and the projector $P^{(\text{adj})}$ are found to be

$$(Q_{ij}^{(\text{adj})})_{kl,mn} = (\delta_{ik} + \delta_{jl}) \delta_{km} \delta_{ln} - (\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}) \delta_{kl} \delta_{mn} + (i \leftrightarrow j), \quad (\text{III.134})$$

$$P_{kl,mn}^{(\text{adj})} = \delta_{kl} \delta_{nk} \delta_{ln}. \quad (\text{III.135})$$

The projector (III.135) leads to that only the diagonal components of the adjoint wave function survive

$$\Psi_{kl}^{(\text{adj})} = \sum_{m,n=1}^N P_{kl,mn}^{(\text{adj})} \Psi_{mn}^{(\text{adj})} = \delta_{kl} \Psi_{kk}^{(\text{adj})}. \quad (\text{III.136})$$

Due to this it is natural to introduce the functions $\psi_k^{(\text{adj})} = \Psi_{kk}^{(\text{adj})}$ on which the operator $Q_{ij}^{(\text{adj})}$ simplifies further

$$\left(Q_{ij}^{(\text{adj})} \right)_{kk,mm} = 2 [(\delta_{ik} + \delta_{jk})\delta_{km} - (\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})]. \quad (\text{III.137})$$

As a result, one obtains a set of N coupled equations on $\psi_k^{(\text{adj})}(x, t)$

$$\left\{ i \frac{\partial}{\partial t} + \sum_{i=1}^N \left(\frac{1}{2N} \frac{\partial^2}{\partial x_i^2} - NV(x_i) \right) \right\} \psi_k^{(\text{adj})}(x, t) - \frac{1}{N} \sum_{l(\neq k)} \frac{\psi_k^{(\text{adj})} - \psi_l^{(\text{adj})}}{(x_k - x_l)^2} = 0. \quad (\text{III.138})$$

This shows that instead by a system of free fermions the adjoint representation is described by an interacting system and we lose the integrability that allows to solve exactly the singlet case.

4.3 MQM in arbitrary representation: partition function

Since the full Hilbert space is decomposed into the direct sum, the partition function (III.115) of the compactified MQM can be represented as the sum of contributions from different representations

$$Z_N(R) = \sum_r d_r Z_N^{(r)}(R) = \sum_r d_r \text{Tr}_{(r)} e^{-\frac{\beta}{\hbar} \hat{H}^{(r)}}, \quad (\text{III.139})$$

where the Hamiltonian is defined in (III.125) and the trace is over the subspace of the r th irreducible representation. An approach to calculate the partition functions $Z_N^{(r)}$ was developed in [102]. It is based on the introduction of a new object, the so called *twisted partition function*. It is obtained by rotating the final state by a unitary transformation with respect to the initial one

$$Z_N(\Omega) = \text{Tr} \left(e^{-\frac{\beta}{\hbar} \hat{H}_{\text{MQM}}} \hat{\Theta}(\Omega) \right), \quad (\text{III.140})$$

where $\hat{\Theta}(\Omega)$ is the rotation operator. The partition functions in a given $\text{SU}(N)$ representation can be obtained by projecting the twisted partition function with help of the corresponding character $\chi^{(r)}(\Omega)$

$$Z_N^{(r)} = \int [d\Omega]_{\text{SU}(N)} \chi^{(r)}(\Omega) Z_N(\Omega). \quad (\text{III.141})$$

The characters for different representations are orthogonal to each other

$$\int [d\Omega]_{\text{SU}(N)} \chi^{(r_1)}(\Omega^\dagger) \chi^{(r_2)}(\Omega \cdot U) = \delta_{r_1, r_2} \chi^{(r_1)}(U) \quad (\text{III.142})$$

and are given by the Weyl formula

$$\chi^{(r)}(\Omega) \stackrel{\text{def}}{=} \text{tr} [D^{(r)}(\Omega)] = \frac{\det_{i,j} (e^{i\lambda_i \theta_j})}{\Delta(e^{i\theta})}, \quad (\text{III.143})$$

where $z_i = e^{i\theta_i}$ are eigenvalues of Ω , Δ is the Vandermonde determinant and the ordered set of integers $l_1 > l_2 > \dots > l_N$ is defined in terms of the components of the highest weight $\{m_k\}$: $l_i = m_i + N - i$ of the given representation.

Thus, the twisted partition function plays the role of the generating functional for the set of partition functions $Z_N^{(r)}$. The characters for the unitary group are well known objects. Therefore, the main problem is to find $Z_N(\Omega)$. It was done in the double scaling limit where the potential $V(x)$ becomes the potential of the inverse oscillator. Then it can be related by analytical continuation to the usual harmonic oscillator potential where the twisted partition function can be trivially found. The derivation is especially simple when one uses the matrix Green function defined by the following initial value problem

$$\left\{ \frac{\partial}{\partial \beta} - \frac{1}{2} \text{tr} \left(\frac{\partial^2}{\partial M^2} - \omega^2 M^2 \right) \right\} G(\beta, M, M') = 0, \quad G(0, M, M') = \delta^{(N^2)}(M - M'). \quad (\text{III.144})$$

The solution is well known to be

$$G(\beta, M, M') = \left(\frac{\omega}{2\pi \sinh(\omega\beta)} \right)^{\frac{1}{2}N^2} \exp \left[-\frac{\omega}{2} \coth(\omega\beta) \text{tr}(M^2 + M'^2) + \frac{\omega}{\sinh(\omega\beta)} \text{tr}(MM') \right]. \quad (\text{III.145})$$

It is clear that the twisted partition function is obtained from this Green function as follows

$$Z_N(\Omega) = \int dM G(\beta, M, \Omega^\dagger M \Omega). \quad (\text{III.146})$$

One can easily perform the simple Gaussian integration over M and find the following result

$$Z_N(\Omega; \omega) = 2^{-\frac{1}{2}N^2} \left(2 \sinh^2 \frac{\omega\beta}{2} \right)^{-N/2} \prod_{i>j} \frac{1}{\cosh(\omega\beta) - \cos(\theta_i - \theta_j)}. \quad (\text{III.147})$$

The answer for the inverse oscillator is obtained by the analytical continuation to the imaginary frequency $\omega \rightarrow i$. It can be also represented (up to $(-1)^{N/2}$) in the following form

$$Z_N(\Omega) = q^{\frac{1}{2}N^2} \prod_{i,j=1}^N \frac{1}{1 - q e^{i(\theta_i - \theta_j)}}. \quad (\text{III.148})$$

where $q = e^{i\beta}$. Remarkably, the partition function (III.148) depends only on the eigenvalues of the twisting matrix Ω . Due to this, the integral (III.141) is rewritten as follows

$$Z_N^{(r)} = \frac{1}{N!} \int_0^{2\pi} \prod_{k=1}^N \frac{d\theta_k}{2\pi} |\Delta(e^{i\theta})|^2 \chi^{(r)}(e^{i\theta}) Z_N(\theta). \quad (\text{III.149})$$

In fact, it is not evident that the analytical continuation of the results obtained for the usual oscillator gives the correct answers for the inverse oscillator. The latter is complicated by the necessity to introduce a cut-off since otherwise it would represent an unstable system. Because of that the analytical continuation should be performed in a way that avoids the problems related with arising divergences. In this respect, the presented derivation is not rigorous. However, the validity of the final result (III.148) was confirmed by the reasonable physical conclusions which were derived relying on it. Besides, in [102] an alternative derivation based on the density of states, which are eigenstates of the inverse oscillator Hamiltonian with the twisted boundary conditions, was presented. It led to the same formula as in (III.148).

4.4 Non-trivial $SU(N)$ representations and windings

The technique developed in the previous paragraph allows to study the compactified MQM in the non-trivial representations in detail. In particular, it was shown [102] that the partition function associated with some representation of $SU(N)$ corresponds to the sum over surfaces in the presence of pairs of vortices and anti-vortices of charge defined by this representation. In terms of 2D string theory this means that $\log Z_N^{(r)}$ gives the partition function of strings among which there are strings wrapped around the compactified dimension n times in one direction and the same number of strings wrapped n times in the opposite one.

This result can be established considering the diagrammatic expansion of $Z_N^{(r)}$. The expansion is found using a very important fact that the non-trivial representations are associated with correlators of matrix operators which are traces of matrices taken in different moments of time. For example, the two-point correlators describe the propagation of states belonging to the adjoint representation

$$\left\langle \text{tr} \left(e^{\alpha_1 M(0)} e^{\alpha_2 M(\beta)} \right) \right\rangle = \sum_{i,j=1}^N \left\langle 0 | e^{\alpha_1 x_i} \left(e^{-\frac{\beta}{\hbar} \hat{H}^{(\text{adj})}} \right)_{ij} e^{\alpha_2 x_j} | 0 \right\rangle. \quad (\text{III.150})$$

The diagrammatic expansion for the correlators is known and it gives an expansion for the partition functions $Z_N^{(r)}$ after a suitable identification of the legs of the Feynman graphs.

Another physical consequence of the previous analysis is that there is a large energy gap between the singlet and the adjoint representations. It was found to be [100, 102]

$$\delta = \mathcal{F}^{(\text{adj})} - \mathcal{F}^{(\text{sing})} \underset{\mu \rightarrow \infty}{\sim} -\frac{\beta}{2\pi} \log(\mu/\Lambda). \quad (\text{III.151})$$

Due to this gap the contribution of few vortices to the partition function is negligible and they seem to be suppressed. However, the vortices have a large entropy related to the degeneracy factor d_r in the sum (III.139). For the adjoint representation it equals $d_{(\text{adj})} = N^2 - 1$. Therefore, there is a competition between the two factors. It leads to the existence of a phase transition when the radius of compactification becomes sufficiently small [100]. Indeed, from (III.139) one finds

$$Z_N(R) \approx Z_N^{(\text{sing})}(R) \left(1 + d_{(\text{adj})} e^{-\delta} + \dots \right) \approx \exp \left[F^{(\text{sing})} + \text{const} \cdot N^2 (\mu/\Lambda)^R \right]. \quad (\text{III.152})$$

Since $\Lambda \sim N$, we see that for large radii the second term in the exponent is very small and is irrelevant with respect to the first one. However, at $R_c = 2$ the situation changes and now the contribution of entropy dominates. Physically this means that the vortex-antivortex pairs become dynamically more preferable and populate densely the string world sheet. This effect is called the vortex condensation and the change of behaviour at R_c is known as the Berezinski–Kosterlitz–Thouless phase transition [96, 97]. For radii $R < R_c$ MQM does not describe anymore the $c = 1$ CFT. Instead it describes $c = 0$ theory corresponding to the pure two-dimensional gravity. This fact can be easily understood from the MQM point of view because at very small radii we expect the usual dimension reduction. The dimension reduction of MQM is the simple one-matrix model which is known to describe pure gravity as it was shown in section II.2.

Of course, this phase transition is seen also in the continuum formalism. There it is related to the fact whether the operator creating vortex-antivortex pairs is relevant or not. These operators were introduced in (I.53). The simplest such operator has the form

$$\int d^2\sigma e^{(R-2)\phi} \cos(R\tilde{X}). \quad (\text{III.153})$$

It is relevant if it decreases in the asymptotics $\phi \rightarrow \infty$. This happens when $R < 2$ exactly as it was predicted from the matrix model. This gives one more evidence that we correctly identified the winding modes of string theory with the states of MQM arising in the non-trivial $SU(N)$ representations.

* * *

We conclude that Matrix Quantum Mechanics successfully describes 2D string theory in the linear dilaton background. All excitations of 2D string theory were identified with appropriate degrees of freedom of MQM and all continuum results were reproduced by the matrix model technique. Moreover, MQM in the singlet sector represents an integrable system which allowed to exactly solve the corresponding (tachyon) sector of string theory.

Once the string physics in the linear dilaton background has been understood and solved, it is natural to turn our attention to other backgrounds. We have in our hands two main tools to obtain new backgrounds: to consider either a tachyon or winding condensation since their vertex operators are well known. But the most interesting problem is string theory in curved backgrounds. We know that 2D string theory does possess such a background which describes the two-dimensional dilatonic black hole. Therefore, we expect that the Matrix Quantum Mechanics is also able to describe it and may be to provide its exact solution.

Chapter IV

Winding perturbations of MQM

1 Introduction of winding modes

1.1 The role of the twisted partition function

In this chapter we consider one of the two possibilities to change the string background in Matrix Quantum Mechanics. We introduce non-perturbative sources of windings which perturb the theory and give rise to a winding condensation. From the previous chapter we know that the winding modes of string theory are described in MQM by the non-trivial $SU(N)$ representations. However, the problem is that we do not have a control on them. Namely, we do not know how to introduce a portion of windings of charge 1, another portion of windings of charge 2, *etc.* In other words one should have an analog of the couplings \tilde{t}_n which are associated with the perturbations by the vortex operators in the CFT (I.62). Such couplings would allow to construct a generating functional for all correlators of windings.

Actually, we already encountered one generating functional of quantities related to windings. This was the twisted partition function (III.140) which generated the partition functions of MQM in different representations. It turns out that this is the object we are looking for because it can be considered as the generating functional of vortex operators with couplings being the moments of the twisting matrix. In the following two sections we review the work [103] where this fact has been established and exploited to get a matrix model for 2D string theory on the black hole background.

By definition the twisted partition function describes MQM with twisted boundary condition. Therefore, it can be represented by the following matrix integral

$$Z_N(\Omega) = \int_{M(\beta)=\Omega^\dagger M(0)\Omega} \mathcal{D}M(t) \exp \left[-N \operatorname{tr} \int_0^\beta dt \left(\frac{1}{2} \dot{M}^2 + V(M) \right) \right]. \quad (\text{IV.1})$$

Hence it has the usual representation as the sum over Feynman diagrams of the matrix model or as the sum over discretized two-dimensional surfaces embedded in one-dimensional space. Since the target space is compactified, we expect to obtain something like the expansion given in (III.106). However, the presence of the twisting matrix introduces new ingredients.

The only thing which can change is the propagator

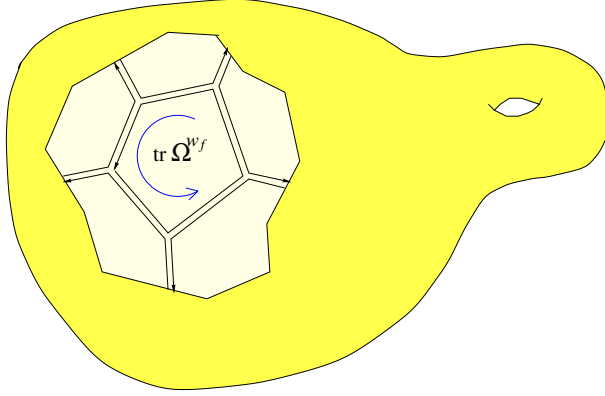


Fig. IV.1: A piece of a discretized world sheet. The twisted boundary condition associates with each loop of the dual graph a moment of the twisting matrix corresponding to the winding number of the loop.

$$G_{ij,kl}(t, t') = t_i^j \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} t_l^k = \langle M_{ij}(t) M_{kl}(t') \rangle_0.$$

As it is seen from its definition through the two-point correlator, it should satisfy the periodic boundary condition

$$G_{ij,kl}(t + \beta, t') = \Omega_{ii'}^\dagger G_{i'j',kl}(t, t') \Omega_{jj'}. \quad (\text{IV.2})$$

Due to this, the propagator is given by a generalization of the simple periodic solution (III.105). In contrast to previous cases, its index structure is not anymore described by the Kronecker symbols but by the twisting matrix

$$G_{ij,kl}(t, t') = \frac{1}{N} \sum_{m=-\infty}^{\infty} (\Omega^m)_{il} (\Omega^{-m})_{jk} e^{-|t-t'+m\beta|}. \quad (\text{IV.3})$$

As a result, the contraction of indices along each loop in a given graph gives the factor $\text{tr } \Omega^{w_I}$ where the integer field w_I was defined in (III.110). It is equal to the winding number of the loop around the target space circle (see fig. IV.1). We will call this field *vorticity*.

Thus, the only effect of the introduction of the twisted boundary condition is that the factors N , which were earlier associated with each loop, are replaced by the factors $\text{tr } \Omega^{w_I}$. The rest remains the same as in (III.106). In particular, one has the sum over distributions m_{ij} . This sum can be splitted into the sum over vorticities w_I and the sum over the "pure gauge" configurations (III.111). The latter can be removed at the cost of extending the integrals over time to the whole real axis. We conclude that each term in the resulting sum is characterized by a graph with particular distributions of times t_i at vertices and vorticity at faces (loops) w_I . It enters the sum with the following coefficient

$$N^{2-2g} \lambda^V \prod_{I=1}^L \frac{\text{tr } \Omega^{w_I}}{N} \prod_{\langle ij \rangle} e^{-|t_i - t_j + \beta m_{ij}|}, \quad (\text{IV.4})$$

As usual, in the double scaling limit the sum over discretizations becomes the sum over continuous geometries and the twisted partition function (IV.1) can be interpreted as the

partition function of 2D string theory including the sum over vortex insertions. The vortices of charge m are coupled to the m th moment of the twisting matrix so that the moments control the probability to find vortices of a given vorticity.

To use this interpretation to extract some information concerning a particular vortex configuration (for instance, to study the dependence of the theory perturbed by vortices of charge 1 of the coupling constant), one should express the twisted partition function as a function of moments $s_m = \text{tr } \Omega^m$. However, it turns out to be a quite difficult problem. Therefore, one should find an alternative way to describe the perturbed system.

1.2 Vortex couplings in MQM

The main problem with the twisted partition function is that its natural argument is a matrix. Moreover, in the large N limit its size goes to infinity. (Although $Z_N(\Omega)$ depends actually only on the eigenvalues, as it was shown in section III.4.3, this does not help much.) On the other hand, usually one integrates over matrices of a large size. For example, the partition functions of MQM in different representations were represented as integrals over the twisting matrix (III.141). The measure of the integration was given by the characters of irreducible representations. But the characters are not related to the coupling constants of vortices in an explicit way.

We can generalize this construction and integrate the twisted partition function with an arbitrary measure. Then the vortex coupling constants will be associated with some parameters of the measure. Thus, the problem can be reformulated as follows: what choice of the measure gives the most convenient parameterization of the generating functional of vortices?

The answer is as simple as it can be. Indeed, as usual, we require from the measure the invariance under the unitary transformations

$$\Omega \rightarrow U^\dagger \Omega U \quad (U^\dagger U = I). \quad (\text{IV.5})$$

Then, as in the one-matrix model, the most natural choice of the measure is given by the exponential of a potential. Since the matrix is unitary, in contrast to 1MM, now both the positive and negative powers of the twisting matrix are allowed. Thus, we define the following functional [103]

$$Z_N[\lambda] = \int [d\Omega]_{SU(N)} \exp \left(\sum_{n \neq 0} \lambda_n \text{tr } \Omega^n \right) Z_N(\Omega). \quad (\text{IV.6})$$

Note that the parameters λ_n are coupled exactly to the moments s_n of the twisting matrix playing the role of fugacities of vortices. Therefore, the functional (IV.6) is nothing else but the Legendre transform of the twisted partition function considered as a function of the moments.

This statement can be formulated more rigorously with help of the following identity

$$\int [d\Omega]_{SU(N)} \exp \left(\sum_{n \neq 0} \lambda_n \text{tr } \Omega^n \right) = \exp \left(\sum_{n > 0} n \lambda_n \lambda_{-n} \right), \quad (\text{IV.7})$$

which is valid up to non-perturbative terms $O(e^{-N})$ provided the couplings do not grow linearly in N . This property of integrals over the unitary groups shows that in the large N

limit the moments s_n can be considered as independent variables and the measure $[d\Omega]_{SU(N)}$ is expressed in a very simple way through them. As a result, the generating function (IV.6) is written as

$$Z_N[\lambda] = \int \prod_{n \neq 0} \frac{ds_n}{\sqrt{\pi}} e^{\sum_{n \neq 0} (\lambda_n s_n - \frac{1}{2|n|} s_n s_{-n})} Z_N[s]. \quad (\text{IV.8})$$

The relation (IV.7) is a generating equation for integrals of products of moments. Among them the following relation is most important for us

$$\int [d\Omega]_{SU(N)} \text{tr } \Omega^n \text{tr } \Omega^m = |n| \delta_{n+m,0}. \quad (\text{IV.9})$$

It helps to elucidate the sense of the couplings λ_n which is hidden in the diagrammatic representation of $Z_N[\lambda]$. This representation follows from the expansion of the twisted partition function if to perform the integration over Ω . From (IV.9) one concludes that there will be three kinds of contributions. The first one is a trivial factor given by the r.h.s of (IV.7). It comes from the coupling of two moments from the measure in (IV.6). The second contribution arises when a moment from the measure is coupled with the factor $\frac{1}{N} \text{tr } \Omega^{w_I}$ in (IV.4) associated with a vortex of vorticity w_I . It results in substitution of the coupling λ_{w_I} in place of the trace of the twisting matrix. Thus, whenever it appears, λ_n is always associated with a vortex of winding number n . Hence, it plays the role of the coupling constant of the operator creating the vortices. Finally, there is the third contribution related with the coupling of two moments from (IV.4). However, it was argued that it vanishes in the double scaling limit [103].

To summarize, the double scaling limit of the free energy of (IV.6) coincides with the partition function of the $c = 1$ theory perturbed by vortex operators with the coupling constants proportional to λ_n . Thus, we obtain a matrix model realization of the CFT (I.62) with $t_n = 0$. Since the couplings λ_n are explicitly introduced from the very beginning, it is much easier to work with $Z_N[\lambda]$ than with the twisted partition function. Moreover, it turns out that in terms of λ 's the system becomes integrable.

1.3 The partition function as τ -function of Toda hierarchy

To reveal the relation of the partition function (IV.6) to integrable systems, one should do two things. First, one should pass to the grand canonical ensemble

$$\mathcal{Z}_\mu[\lambda] = \sum_{N=0}^{\infty} e^{-\beta\mu N} Z_N[\lambda]. \quad (\text{IV.10})$$

One can observe that $-\beta\mu$ plays the role of the "zero time" λ_0 which appears if one includes $n = 0$ into the sum in (IV.6). Therefore, the necessity to use the grand canonical ensemble goes in parallel with the change from Ω to λ 's and it is natural in this context.

Second, we use the result (III.148) for the twisted partition function found in the double scaling limit [102]. Combining (III.148), (IV.6) and (IV.10) and integrating out the angular part of the twisting matrix, one obtains

$$\mathcal{Z}_\mu[\lambda] = \sum_{N=0}^{\infty} \frac{e^{-\beta\mu N}}{N!} \oint \prod_{k=1}^N \left(\frac{dz_k}{2\pi i z_k} \frac{e^{u(z_k)}}{q^{1/2} - q^{-1/2}} \right) \prod_{i \neq j} \frac{z_i - z_j}{q^{1/2} z_i - q^{-1/2} z_j}, \quad (\text{IV.11})$$

where $q = e^{i\beta}$, z_k are eigenvalues of Ω , and $u(z) = \sum_n \lambda_n z^n$ is a potential associated with the perturbation. Initially, the eigenvalues belonged to the unit circle. But due to the holomorphicity, the integrals in (IV.11) can be understood as contour integrals around $z = 0$. Finally, using the Cauchy identity

$$\frac{\Delta(x)\Delta(y)}{\prod_{i,j}(x_i - y_j)} = \det_{i,j} \left(\frac{1}{x_i - y_j} \right), \quad (\text{IV.12})$$

we rewrite the product of different factors as a determinant what gives the following representation for the grand canonical partition function of 2D string theory perturbed by vortices

$$\mathcal{Z}_\mu[\lambda] = \sum_{N=0}^{\infty} \frac{e^{-\beta\mu N}}{N!} \oint \prod_{k=1}^N \frac{dz_k}{2\pi i} \det_{i,j} \left(\frac{\exp \left[\frac{1}{2}(u(z_i) + u(z_j)) \right]}{q^{1/2}z_i - q^{-1/2}z_j} \right). \quad (\text{IV.13})$$

Relying on this representation one can prove that the grand canonical partition function coincides with a τ -function of Toda hierarchy [103]. In this case it is convenient to establish the equivalence with the fermionic representation of τ -function (II.111). We claim that if one chooses the matrix determining the operator of $GL(\infty)$ rotation as follows

$$A_{rs} = \delta_{r,s} q^{i\mu+r}, \quad (\text{IV.14})$$

the τ -function is given by

$$\tau_l[t] = e^{-\sum_{n>0} nt_n t_{-n}} \mathcal{Z}_{\mu-il}[\lambda], \quad (\text{IV.15})$$

where the coupling constants are related to the Toda times through

$$\lambda_n = 2i \sin(\pi n R) t_n. \quad (\text{IV.16})$$

Indeed, with the matrix (IV.14) the operator \mathbf{g} is written as

$$\mathbf{g} \equiv \exp(q^{i\mu} \hat{A}) = \exp \left(e^{-\beta\mu} \oint \frac{dz}{2\pi i} \psi(q^{-1/2}z) \psi^*(q^{1/2}z) \right). \quad (\text{IV.17})$$

The expansion of the exponent gives the sum over N and factors $e^{-\beta\mu N}$ in (IV.13). Then in the N th term of the expansion one should commute the exponents $e^{H_\pm[t]}$ associated with perturbations between each other and with \hat{A}^N . The former commutation gives rise to the trivial factor appearing in (IV.15). It is to be compared with the similar contribution to $Z_N[\lambda]$ coming from (IV.7). The commutator with \hat{A}^N is found using the relations (II.129). As a result, one obtains

$$\oint \prod_{k=1}^N \left(\frac{dz_k}{2\pi i} \exp \left[\sum_{n \neq 0} (q^{n/2} - q^{-n/2}) t_n z_k^n \right] \right) \left\langle l \left| \prod_{k=1}^N \psi(q^{-1/2}z_k) \psi^*(q^{1/2}z_k) \right| l \right\rangle. \quad (\text{IV.18})$$

Comparing this expression with (IV.13) we see the necessity to redefine the coupling constants according to (IV.16) to match the potentials. Finally, the quantum average in the vacuum of charge l produces the same determinant as in (IV.13) and additional factor q^{lN} (see (II.105)). The latter leads to the shift of μ shown in (IV.15).

Actually, this result is not unexpected because, as we mentioned in the end of section III.3.3, a similar result has been obtained for the generating functional of tachyon correlators. The tachyon and winding perturbations are related by T-duality. Therefore, both the tachyon and winding perturbations of 2D string theory should be described by the same τ -function with T -dual parameters.

Nevertheless, it is remarkable that one can obtain an explicit matrix representation of this τ -function which can be directly interpreted in terms of discretized surfaces with vortices. Therefore, one can use the powerful matrix technique to solve some problems which may be inaccessible even by methods of integrable systems. For example, while the tachyon and winding perturbations are integrable when they are introduced separately, the integrability disappears as only both of them are present. In such situation the Toda hierarchy does not work anymore, but the matrix description is still valid.

The Toda description can be used exploiting its hierarchy of equations. To characterize their unique solution we should provide either a string equation or an initial condition. The string equation can be found in principle [104] (and we will show how it appears in the dual picture of tachyon perturbations), but it is not so evident. In contrast, it is clear that the initial condition is given by the partition function with vanishing coupling constants, *i.e.*, without vortices. It corresponds to the partition function of the compactified MQM in the singlet sector. It is well known and its expansion is given by (III.122). Thus, we have the necessary information to use the equations of Toda hierarchy.

These equations are of the finite-difference type. Therefore, usually one represents them as a series of partial differential equations. We associated this expansion in section II.5 with an expansion in the Planck constant which is the parameter measuring the lattice spacing. What is this parameter in our case? On the string theory side one has the genus expansion. The only possibility is to identify these two expansions. From (III.122) we see that the parameter playing the role of the string coupling constant, which is the parameter of the genus expansion, is $g_{\text{cl}} \sim \mu^{-1}$. Thus, one concludes that the role of the spacing parameter is played by μ^{-1} and to get the dispersionless limit of Toda hierarchy one should investigate the limit of large μ . Note that this conclusion is in the complete agreement with the consequence of (IV.15) that μ is associated with the discrete charge of τ -function.

2 Matrix model of a black hole

2.1 Black hole background from windings

Due to the result of the previous section, the Toda hierarchy provides us with equations on the free energy as a function of μ and the coupling constants. Any result found for finite couplings λ_n would already correspond to some result in string theory with a non-vanishing condensate of winding modes. In section I.6.2 we discussed that such winding condensates do not have a local target space interpretation. In other words, there is no special field in the string spectrum describing them. Therefore, the effect of winding condensation should be seen on another characteristics of the background: dilaton and metric. Thus, it is likely that considering MQM with non-vanishing λ_n , we actually describe 2D string theory in a curved background.

Let us consider the simplest case when only $\lambda_{\pm 1} \neq 0$. Without lack of generality one can take them equal $\lambda_1 = \lambda_{-1} \sim \lambda$. Then the corresponding string theory is described by the so called *Sine-Liouville CFT*

$$S_{\text{SL}} = \frac{1}{4\pi} \int d^2\sigma \left[(\partial X)^2 + (\partial\phi)^2 - Q\hat{\mathcal{R}}\phi + \mu e^{\gamma\phi} + \lambda e^{b\phi} \cos(R\tilde{X}) \right], \quad (\text{IV.19})$$

where \tilde{X} is T-dual to the field $X(\sigma)$ which is compactified at radius R . The requirement that the perturbations are given by marginal operators, leads to the following conditions on the parameters

$$\gamma = -Q, \quad b = R - Q. \quad (\text{IV.20})$$

The central charge of this theory is $c = 2 + 6Q^2$. Therefore, to get $c = 26$, as always in matrix models, one should take $Q = 2$.

We have already mentioned that it was suggested [35] that the theory (IV.19) is dual to 2D string theory in the black hole background described in section I.6.1. Actually, the exact statement is the following. The coset CFT $[\text{SL}(2, \mathbf{R})]_k / \text{U}(1)$ at the level k is equivalent to the CFT (IV.19) with the vanishing cosmological constant μ and fixed the radius and the charge Q

$$R = \sqrt{k}, \quad Q = \frac{1}{\sqrt{k-2}}. \quad (\text{IV.21})$$

The condition on R comes from the matching of the asymptotic radii of the cigar geometry describing the Euclidean black hole and of the cylindrical target space of the Sine-Liouville CFT. Then the value of Q is fixed within the coset CFT (I.59).

Due to these restrictions on the parameters, there is only one point in the parameter space of two models where they intersect. It corresponds to the following choice

$$Q = 2, \quad R = 3/2, \quad \mu = 0. \quad (\text{IV.22})$$

Since for these values of the parameters the Sine-Liouville CFT can be obtained as a matrix model constructed in the previous section, on the one hand, and is dual to the coset CFT, on the other hand, in this point we have a matrix model description of string theory in the black hole background [103]. The string partition function is given by the free energy of (IV.6) or its grand canonical counterpart (IV.10) where one puts μ and all couplings except $\lambda_{\pm 1}$ to zero as well as $R = 3/2$.

This remarkable correspondence opens the possibility to study the black hole physics using the matrix model methods. Of course, the most interesting questions are related to the thermodynamics of black holes. In particular, any theory of quantum gravity should be able to explain the microscopic origin of the black hole entropy and to resolve the information paradox. One might hope that the matrix model will allow to identify the fundamental degrees of freedom of this system to solve both these problems.

2.2 Results for the free energy

Before to address the question about the entropy, it is much more easy to get some information about another thermodynamical quantity — the free energy. The grand canonical free energy is given by the logarithm of the partition function (IV.10) and, hence, by the logarithm of the τ -function. Thus, one can use the integrable structure of Toda hierarchy to find it.

The problem is that we cannot work directly in the black hole point of the parameter space. Indeed, it puts μ to zero whereas the dispersionless limit of Toda hierarchy, which allows to write differential equations on the free energy, requires to consider large μ . The solution is to study the theory with a large non-vanishing μ treating λ as a perturbation. Then, one should try to make an analytical continuation to the opposite region of small μ . In the end, one should also fix the radius of the compactification.

In fact, in the matrix model it is very natural to turn on the cosmological constant μ and to consider an arbitrary radius R . The values (IV.22) (except $Q = 2$) are not distinguished anyhow. Even $\mu = 0$ is not the most preferable choice because there is another value which is associated with a critical point where the theory acquires some special properties (see below). Moreover, to analyze thermodynamical issues, one should be able to vary the temperature of the system what means for the black hole to vary the radius R . Thus, it is strange that only at the values (IV.22) MQM describes a black hole. What do other values correspond to? It is not clear at the moment, but it would be quite natural that they describe some deformation of the initial black hole background. Therefore, we will keep μ and R arbitrary in the most of calculations.

Let us use the Toda integrable structure to find the free energy $\mathcal{F}(\mu, \lambda) = \log \mathcal{Z}_\mu(\lambda)$ where $t_1 t_{-1} = \lambda^2$ and all other couplings vanish. Due to the winding number conservation, it depends only on the product of two couplings and not on them separately. The identification (IV.15) allows to conclude that the evolution along the first times is governed by the Toda equation (II.136). Since the shift of the discrete charge l is equivalent to an imaginary shift of μ , in terms of the free energy the Toda equation becomes

$$\frac{\partial^2 \mathcal{F}(\mu, \lambda)}{\partial t_1 \partial t_{-1}} + \exp [\mathcal{F}(\mu + i, \lambda) + \mathcal{F}(\mu - i, \lambda) - 2\mathcal{F}(\mu, \lambda)] = 1. \quad (\text{IV.23})$$

Rewriting the finite shifts of μ as the result of action of a differential operator, one obtains

$$\frac{1}{4} \lambda^{-1} \partial_\lambda \lambda \partial_\lambda \mathcal{F}(\mu, \lambda) + \exp \left[-4 \sin^2 \left(\frac{1}{2} \frac{\partial}{\partial \mu} \right) \mathcal{F}(\mu, \lambda) \right] = 1. \quad (\text{IV.24})$$

The main feature of this equation is that it is compatible with the scaling

$$\lambda \sim \mu^{\frac{2-R}{2}} \quad (\text{IV.25})$$

which can be read off from the Sine–Liouville action (IV.19). Due to this, the free energy can be found order by order in its genus expansion which has the usual form of the expansion in μ^{-2}

$$\mathcal{F}(\mu, \lambda) = \lambda^2 + \mu^2 \left[-\frac{R}{2} \log \mu + \tilde{f}_0(\zeta) \right] + \left[-\frac{R + R^{-1}}{24} \log \mu + \tilde{f}_1(\zeta) \right] + \sum_{g=2}^{\infty} \mu^{2-2g} \tilde{f}_g(\zeta), \quad (\text{IV.26})$$

where $\zeta = (R - 1)\lambda^2 \mu^{R-2}$ is a dimensionless parameter. The first term is not universal and can be ignored. It is intended to cancel 1 in the r.h.s. of (IV.24). The coefficients $\tilde{f}_g(\zeta)$ are smooth functions near $\zeta = 0$. The initial condition given by (III.122) fixes them at the origin: $\tilde{f}_0(0) = \tilde{f}_1(0) = 0$ and $\tilde{f}_g(0) = R^{1-g} f_g^{(\text{sing})}(R)$ with $f_g^{(\text{sing})}(R)$ from (III.123).

It is clear that one can redefine the coefficients in such a way that the genus expansion will be associated with an expansion in λ . More precisely, if one introduces the following scaling variables

$$w = \mu \xi, \quad \xi = (\lambda \sqrt{R - 1})^{-\frac{2}{2-R}}, \quad (\text{IV.27})$$

the genus expansion of the free energy reads

$$\mathcal{F}(\mu, \lambda) = \lambda^2 + \xi^{-2} \left[\frac{R}{2} w^2 \log \xi + f_0(w) \right] + \left[\frac{R + R^{-1}}{24} \log \xi + f_1(w) \right] + \sum_{g=2}^{\infty} \xi^{2g-2} f_g(w). \quad (\text{IV.28})$$

Thus, the string coupling constant is identified as $g_{\text{cl}} \sim \xi$. This is a simple consequence of the scaling (IV.25). It is clear that the dimensionless parameters ζ and w are inverse to each other: $\zeta = w^{R-2}$. We included the factor $(R - 1)$ in the definition of the scaling variables for convenience as it will become clear from the following formulae. This implies that $R > 1$. It is the region we are interested in because it contains the black hole radius. However, the final result can be presented in the form avoiding this restriction.

Plugging (IV.28) into (IV.24), one obtains a system of ordinary differential equations for $f_g(w)$. Each equation is associated with a definite genus. At the spherical level there is a closed non-linear equation for $f_0(w)$

$$\frac{R - 1}{(2 - R)^2} (w \partial_w - 2)^2 f_0(w) + e^{-\partial_w^2 f_0(w)} = 0. \quad (\text{IV.29})$$

Its solution is formulated as a non-linear algebraic equation for $X_0 = \partial_w^2 f_0$ [103]

$$w = e^{-\frac{1}{R} X_0} - e^{-\frac{R-1}{R} X_0}. \quad (\text{IV.30})$$

In terms of the solution of this equation, the spherical free energy itself is represented as

$$\mathcal{F}_0(\mu, \lambda) = \frac{1}{2} \mu^2 (R \log \xi + X_0) + \xi^{-2} \left(\frac{3}{4} R e^{-\frac{2}{R} X_0} - \frac{R^2 - R + 1}{R - 1} e^{-X_0} + \frac{3}{4} \frac{R}{R - 1} e^{-2 \frac{R-1}{R} X_0} \right). \quad (\text{IV.31})$$

Let us rewrite the equation (IV.30) in terms of the susceptibility $\chi = \partial_\mu^2 \mathcal{F}$, more precisely, in terms of its spherical part

$$\chi_0 = R \log \xi + X_0. \quad (\text{IV.32})$$

The result can be written in the following form

$$\mu e^{\frac{1}{R}\chi_0} + (R-1)\lambda^2 e^{\frac{2-R}{R}\chi_0} = 1. \quad (\text{IV.33})$$

From this it is already clear why we included the factor $(R-1)$ in the scaling variables. Note that this form of the answer is not restricted to $R > 1$ and it is valid for all radii. However, it shows that the limit $\mu \rightarrow 0$ exists only for $R > 1$. Otherwise the susceptibility becomes imaginary. For $R > 1$, a critical point where the equation (IV.33) does not have real solutions anymore also exists and it is given by

$$\mu_c = -(2-R)(R-1)^{\frac{R}{2-R}} \lambda^{\frac{2}{2-R}}. \quad (\text{IV.34})$$

This critical value of the cosmological constant was found previously by Hsu and Kutasov [105]. The result (IV.34) shows that the vanishing value of μ is inaccessible also for $R > 2$. Actually, in this region the situation is even more dramatic because ξ becomes an increasing function of λ and the genus expansion breaks down in the limit of large λ . This is related to the fact that the vortex perturbation in (IV.19) is not marginal for $R > 2$ and is non-renormalizable because it grows in the weak coupling region. Thus, the analytical continuation to the black hole point $\mu = 0$ is possible only in the finite interval of radii $1 < R < 2$. Fortunately, the needed value $R = 3/2$ belongs to this interval and the proposal survives this possible obstruction.

The equation (IV.33) can be used to extract expansion of the spherical free energy either in λ or in μ . In particular, the former expansion reproduces the $2n$ -point correlators of vortex operators

$$\langle \tilde{V}_R^n \tilde{V}_{-R}^n \rangle_0 = -n! \mu^2 R^{2n+1} \left((1-R)\mu^{R-2} \right)^n \frac{\Gamma(n(2-R)-2)}{\Gamma(n(1-R)+1)}. \quad (\text{IV.35})$$

For small values of n they have been found and for other values conjectured by Moore in [106]. These correlators should coincide with the coefficients in the λ -expansion of \mathcal{F}_0 because they can be organized into the partition function as follows

$$\mathcal{F}(\mu, \lambda) = \left\langle e^{\tilde{\lambda} \tilde{V}_R + \tilde{\lambda} \tilde{V}_{-R}} \right\rangle_{\text{gr.c.}} = \mathcal{F}(\mu, 0) + \sum_{n=1}^{\infty} \frac{\tilde{\lambda}^{2n}}{(n!)^2} \langle \tilde{V}_R^n \tilde{V}_{-R}^n \rangle_{\text{gr.c.}}, \quad (\text{IV.36})$$

where the expectation value is evaluated in the grand canonical ensemble. The comparison shows that the correlators do coincide if one identifies $\lambda = R\tilde{\lambda}$.¹ This indicates that the correct relation between the Toda times t_n and the CFT coupling constants \tilde{t}_n in (I.62) is the following

$$t_n = iR\tilde{t}_n, \quad t_{-n} = -iR\tilde{t}_{-n}, \quad (n > 0). \quad (\text{IV.37})$$

The appearance of the factor R will be clear when we consider the dual system with tachyon perturbations.

The black hole limit $\mu = 0$ corresponds to $X_0 = 0$. Then the free energy (IV.31) becomes

$$\mathcal{F}_0(0, \lambda) = -\frac{(2-R)^2}{4(R-1)} (\sqrt{R-1}\lambda)^{\frac{4}{2-R}}. \quad (\text{IV.38})$$

¹In fact, the correlators (IV.35) differ by sign from the coefficients in the expansion of \mathcal{F}_0 . This is related to that \mathcal{F}_0 is the grand canonical free energy whereas in paper [106] the canonical ensemble was considered.

Note that in the point $R = 3/2$ the free energy is proportional to an integer power of the coupling constant $\sim \lambda^8$. Therefore, the spherical contribution seems to be non-universal. However, for general R it is not so and it is crucial for thermodynamical issues.

At the next levels the equations obtained from (IV.24) form a triangular system so that the equation for $f_g(w)$ is linear with respect to this function and contains all functions of lower genera as a necessary input [103]

$$\left(\frac{R-1}{(2-R)^2} (w\partial_w + 2g-2)^2 - e^{-X_0} \partial_w^2 \right) f_g = - \left[\xi^{2-2g} \exp \left(-4 \sin^2 \left(\frac{\xi}{2} \partial_w \right) \sum_{k=0}^{g-1} \xi^{2k-2} f_k \right) \right]_0, \quad (\text{IV.39})$$

where $[\dots]_0$ means the terms of zero order in ξ -expansion. Up to now, only the solution for the genus $g = 1$ has been obtained [103]

$$\mathcal{F}_1(\mu, \lambda) = \frac{R+R^{-1}}{24} \left(\log \xi + \frac{1}{R} X_0 \right) - \frac{1}{24} \log \left(1 - (R-1) e^{\frac{2-R}{R} X_0} \right). \quad (\text{IV.40})$$

For the genus $g = 2$ the differential operator of the second order contains 4 singular points and the solution cannot be presented in terms of hypergeometric functions [107].

2.3 Thermodynamical issues

An attempt to analyze thermodynamics of the black hole relying on the result (IV.38) was done in [103] and [34]. However, no definite conclusions have been obtained. First of all, it is not clear whether the free energy of the black hole vanishes or not. The "old" analysis in the framework of dilaton gravity predicts that it should vanish [29, 30]. However, the matrix model leads to the opposite conclusion. One can argue [103] that since for $R = 3/2$ the leading term is non-universal, it can be thrown away giving the vanishing free energy. But if the matrix model realizes string theory in a black hole background for any R , this would be quite unnatural.

Moreover, even in the framework of the dilaton gravity the issue is not clear. The value of the free energy depends on a subtraction procedure which is to be done to regularize diverging answers. There is a natural reparameterization invariant procedure which leads to a non-vanishing free energy [34] in contradiction with the previous results. However, in this case it is not clear how to get the correct expressions for the mass and entropy.

The related problem which prevents to clarify the situation is what quantity should be associated with the temperature. At the first glance this is the inverse radius. In particular, if one follows this idea and uses the reparameterization invariant subtraction procedure, one arrives at reasonable results but the mass of the black hole differs by factor 2 from the standard expression (I.56) [108]. Note that the possibility to get this additional factor was emphasized in [109]. It is related to the definition of energy in dilaton gravity.

However, from the string point of view the radius is always fixed. Therefore, in the analysis of the black hole thermodynamics, the actual variations of the temperature were associated with the position of a "wall" which is introduced to define the subtraction [29, 30, 34]. But there is no corresponding quantity in the matrix model.

In the next chapter, relying on the analysis of a dual system, we will argue that it is R^{-1} that should be considered as the temperature. Also we will shed some light on the puzzle with the free energy.

3 Correlators of windings

After this long introduction, finally we arrived at the point where we start to discuss the new results of this thesis. The first of these results concerns correlators of winding operators in the presence of winding condensate. According to the proposal of [103] reviewed in the previous section, they give the correlators of winding modes in the black hole background. The calculation of these correlators represents the next step in exploring the Toda integrable structure describing the winding sector of 2D string theory. For the one- and two-point correlators in the spherical approximation this task has been fulfilled in the work [110] (Article I).

Due to the identification (IV.15), the generating functional of all correlators of vortices is the τ -function of Toda hierarchy. For the Sine–Liouville theory where only the first couplings $\lambda_{\pm 1}$ are non-vanishing, the correlators are defined as follows

$$\mathcal{K}_{i_1 \dots i_n} = \frac{\partial^n}{\partial \lambda_{i_1} \dots \partial \lambda_{i_n}} \log \tau_0 \Big|_{\lambda_{\pm 2} = \lambda_{\pm 3} = \dots = 0}, \quad (\text{IV.41})$$

where the coupling constants λ_n are related to the Toda times t_n by (IV.16). Whereas to find the free energy it was enough to establish the evolution law along the first times, to find the correlators one should know how the τ -function depends on all Toda times, at least near $t_n = 0$.

The evolution law for the first times $t_{\pm 1}$ was determined by the Toda equation. It is the first equation in the hierarchy of bilinear differential Hirota equations (II.134). The idea of [110] was to use these equations to find the correlators.

The first step was to identify the necessary equations because not the whole hierarchy is relevant for the problem. After that we observe that the extracted equations are of the finite-difference type. To reduce them to differential equations, one should plug in the ansatz (IV.28), where now the coefficients f_g are functions of all dimensionless parameters: w and $s = (s_{\pm 2}, s_{\pm 3}, \dots)$. The first parameter was defined in (IV.27) and the parameters s_n are related to higher times

$$s_n = i \left(-\frac{t_{-1}}{t_1} \right)^{n/2} \xi^{\Delta[t_n] t_n}, \quad (\text{IV.42})$$

where $\Delta[t_n]$ is the dimension of the coupling with respect to μ

$$\Delta[t_n] = 1 - \frac{R|n|}{2}. \quad (\text{IV.43})$$

The spherical approximation corresponds to the dispersionless limit of the hierarchy. It is obtained as $\xi \rightarrow 0$. Thus, extracting the first term in the small ξ expansion, we found the spherical approximation of the initial equations. In principle, they could mix different correlators and of different genera. However, it turned out that in the spherical limit the situation is quite simple. In the equations we have chosen only the spherical part of the two-point correlators survives. We succeeded to rewrite the resulting equations as equations on the generating functions of these correlators. There are two such functions: for correlators with vorticities of the same and opposite signs

$$F^{\pm}(x, y) = \sum_{n, m=0}^{\infty} x^n y^m \tilde{X}_{\pm n, \pm m}^{\pm}, \quad (\text{IV.44})$$

$$G(x, y) = \sum_{n,m=1}^{\infty} x^n y^m \tilde{X}_{n,-m}, \quad (\text{IV.45})$$

where

$$\tilde{X}_{0,m}^{\pm} := \mp i \frac{1}{|n|} \frac{\partial^2}{\partial t_n \partial \mu} \mathcal{F}_0, \quad n \neq 0, \quad (\text{IV.46})$$

$$\tilde{X}_{n,m}^{\pm} := 2 \frac{1}{|n||m|} \frac{\partial^2}{\partial t_n \partial t_m} \mathcal{F}_0, \quad n, m \neq 0. \quad (\text{IV.47})$$

The difference between F^+ and F^- is not essential and it disappears when one chooses $t_1 = -t_{-1}$ ($\lambda_1 = \lambda_{-1}$). Since the dependence of μ is completely known and the cosmological constant plays a distinguished role, the quantities $\tilde{X}_{0,m}^{\pm}$ can be actually considered as one-point correlators. We define their generating function as

$$h(x) = F(x, 0). \quad (\text{IV.48})$$

The equations for the generating functions $F(x, y)$ and $G(x, y)$ have been explicitly solved in terms of the generating function of one-point correlators

$$F(x, y) = \log \left[\frac{4xy}{(x-y)^2} \text{sh}^2 \left(\frac{1}{2} (h(x) - h(y) + \log \frac{x}{y}) \right) \right], \quad (\text{IV.49})$$

$$G(x, y) = 2 \log \left(1 - Axy e^{h^+(x) + h^-(y)} \right), \quad (\text{IV.50})$$

where

$$A = \exp \left(-\partial_{\mu}^2 \mathcal{F}_0 \right) = \xi^{-R} e^{-X_0}. \quad (\text{IV.51})$$

These solutions are universal in the sense that they are valid for any system describing by Toda hierarchy. In other words, their form does not depend on the potential or another initial input. All dependence of particular characteristics of the model enters through the one-point correlators and the free energy. In a little bit different form these solutions appeared in [67, 111] and resemble the equations for the two-point correlators in 2MM found in [66].

The main difficulty is to find the one-point correlators. The Hirota equations are not sufficient to accomplish this task. One needs to provide an additional input. It comes from the fact that we know the dependence of the free energy of the first times $t_{\pm 1}$ and of the cosmological constant μ . Due to this one can write a relation between the one-point correlators entering $h(x)$ and two-point correlators of the kind $\tilde{X}_{n,\pm 1}$. (Roughly speaking, one should integrate over μ and differentiate with respect to $t_{\pm 1}$.) The latter are generated by two functions $\partial_y F(x, 0)$ and $\partial_y G(x, 0)$. As a result, we arrive at the following two equations

$$\partial_y F(x, 0) = \hat{K}^{(+)} h(x) + \tilde{X}_{1,0}, \quad (\text{IV.52})$$

$$\partial_y G(x, 0) = \hat{K}^{(-)} h(x), \quad (\text{IV.53})$$

where $K^{(\pm)}$ are linear integral-differential operators. The operators are found from the explicit expressions for the free energy and the scaling variables. On the other hand, the generating functions F and G are known in terms of $h(x)$ from (IV.49) and (IV.50). Substituting them into (IV.52) and (IV.53), we obtain two equations for one function $h(x)$.

We succeeded to solve the found equations. As it must be, they turned out to be compatible. The solution was represented in terms of the following algebraic equation

$$e^{\frac{1}{R}h} - ze^h = 1, \quad (\text{IV.54})$$

where $z = x \frac{\xi^{-R/2}}{\sqrt{R-1}} e^{-\frac{R-1}{R}X_0} = x\lambda e^{-\frac{R-1}{R}X_0}$. Note that if we take different $t_{\pm 1}$, one would have two equations for h^{\pm} with the parameters z^{\pm} where λ is replaced by $t_{\mp 1}$, respectively.

The equation (IV.54) represents the main result of Article I. It was used to find the explicit expressions for the correlators which are given by the coefficients of the expansion of the generating functions $h(x)$, $F(x, y)$ and $G(x, y)$. They are expressed as functions of X_0 and ξ . The former contains all the non-trivial dependence of the coupling constants μ and λ , whereas the latter provides the correct scaling. The results for the black hole point $\mu = 0$ are obtained when one considers the limit $X_0 = 0$.

In principle, with each index n of the correlators one should associate also the factor $(2 \sin(\pi n R))^{-1}$ coming from the change of couplings (IV.16). Exactly at the black hole radius $R = 3/2$, it becomes singular for even n . Besides, it leads to negative answers and breaks the interpretation of the correlators as probabilities to find vortices of a given vorticity. Most probably, we should not attach these factors to the correlators because they are part of the leg-factors which always appear when comparing the matrix model and CFT results (see section III.3.3). An additional evidence in favour of this is that one does not attach this factor to $\lambda = \sqrt{t_1 t_{-1}}$ and, nevertheless, one finds agreement for the free energy with the results of [106] and [105].

The possibility to introduce the leg-factors makes difficult to compare our results with ones obtained in the Sine-Liouville [35, 112] or coset CFT [32]. There only the two-point correlators of opposite and equal by modulo vorticities have been calculated. They cannot be identified with our quantity $\mathcal{K}_{n,-n}$ while the normalization of the vortex operators in the matrix model with respect to the vortex operators in CFT is not established. This could be done with help of the one-point correlators but they are not known in the CFT approach. It was argued that they should vanish [113]. But this contradicts to our result (IV.54).

Actually, the situation might be even more complicated. In principle, the relation between the matrix model and CFT vortex operators may involve a mixing of operators where not only primary operators appear. Probably, after the correct identification, one-point correlators will vanish in the matrix model too. But we do not see any reason why this should be so.

Let us mention that from the CFT side some three-point correlators are also accessible [35]. However, we did not study the corresponding problem in the matrix model yet. The calculation of these correlators would provide already sufficient information to make the comparison between the two theories.

Note that the situation is that complicated because the two theories have only one intersection point, so that the correlators to be compared are just numbers. We would be in a better position if we are able to extend the correspondence of [35] to arbitrary radius or cosmological constant, for example. Such extension may give answers to many questions which are not understood until now.

Chapter V

Tachyon perturbations of MQM

In this chapter we consider the second way to obtain a non-trivial background in 2D string theory, which is to perturb it by tachyon sources. Of course, the results should be T-dual to ones obtained by a winding condensation. However, although expected, the T-duality of tachyons and windings is not evident in Matrix Quantum Mechanics. We saw in chapter III that they appear in a quite different way. And this is a remarkable fact that the two so different pictures do agree. It gives one more evidence that MQM provides the correct description of 2D string theory and their equivalence can be extended to include the perturbations of both types.

Besides, the target space interpretations of the winding and tachyon perturbations are different. Therefore, although they are described by the same mathematical structure (Toda hierarchy), the physics is not the same. In particular, as we will show, it is impossible to get a curved background using tachyon perturbations, whereas the winding condensate was conjectured to correspond to the black hole background. At the same time, the interpretation in terms of free fermions allows to obtain a more detailed information about both the structure of the target space and thermodynamical properties of tachyonic backgrounds.

Since the introduction of tachyon modes does not require a compactification of the time direction, as the winding modes do, we are not forced to work with Euclidean theory. In turn, the free fermionic representation is naturally formulated in spacetime of Minkowskian signature. Therefore, in this chapter we will work with the real Minkowskian time t . Nevertheless, we are especially interested in the case when the tachyonic momenta are restricted to values of the Euclidean theory compactified at radius R . It is this case that should be dual to the situation considered in the previous chapter and we expect to find that it is exactly integrable.

1 Tachyon perturbations as profiles of Fermi sea

First of all, one should understand how to introduce the tachyon sources in Matrix Quantum Mechanics. The first idea is just to follow the CFT approach: to add the vertex operators realized in terms of matrices to the MQM Hamiltonian. However, this idea fails by several reasons. The first one is that although the matrix realization of the vertex operators is well known (III.81), this form of the operators is valid only in the asymptotic region. When approaching the Liouville wall, they are renormalized in a complicated way. Thus, we cannot

write a Hamiltonian that determines the dynamics everywhere. The second reason is that such perturbations introduced into the Hamiltonian disappear in the double scaling limit where only the quadratic part of the potential is relevant. This is especially obvious for the spectrum of perturbations corresponding to the self-dual radius of compactification $R = 1$. Then the perturbing terms do not differ from the terms of the usual potential which are all inessential.

The last argument shows that one should do something with the system directly in the double scaling limit. There the system is universal being always described by the inverse oscillator potential. This means that *we should change not the system but its state*. Indeed, we established the correspondence with the linear dilaton background only for the ground state of the singlet sector of MQM. In particular, in the spherical approximation this state is described by the stationary Fermi sea (III.60). The propagation of small perturbations above this ground state was associated with the scattering of tachyons [89]. Therefore, it is natural to expect that a tachyon condensation is obtained when one considers excited states associated with non-perturbative deformations of the Fermi sea. This idea has got a concrete realization in the work [114] (Article II).

1.1 Tachyons in the light-cone representation

A state containing a non-perturbative source of particles is, usually, a coherent state obtained by action of the exponent of the operator creating the particles on a ground state. Of course, the easiest way to describe such states is to work in the representation where the creation operator is diagonal. In our case, there are two creation operators of right and left moving tachyons. They are associated with powers of the following matrix operators

$$X_{\pm} = \frac{M \pm P}{\sqrt{2}}. \quad (\text{V.1})$$

Their eigenvalues x_{\pm} can be considered as light-cone like variables but defined in the phase space of the theory rather than in the target space. Thus, we see that to describe the tachyon perturbations, one should work in the light-cone representation of MQM.

In Article II we constructed such light-cone representation. It turned out that MQM looks very simply in this formulation. In particular, the one-particle Hamiltonian

$$H_0 = -\frac{1}{2}(\hat{x}_+ \hat{x}_- + \hat{x}_- \hat{x}_+) \quad (\text{V.2})$$

is represented by a linear differential operator of the first order even for the higher $SU(N)$ representations! Due to this, one can write the general solution of the Schrödinger equation and the eigenfunctions of the light-cone Hamiltonian are given by the simple power functions

$$\psi_{\pm}^E(x_{\pm}) = \frac{1}{\sqrt{2\pi}} x_{\pm}^{\pm iE - \frac{1}{2}}. \quad (\text{V.3})$$

In fact, we see that there are two light-cone representations, right and left. *All the non-trivial dynamics of fermions in the inverse oscillator potential is hidden in the relation between these two representations.* This relation is just the usual unitary transformation

between two quantum mechanical representations. In the given case it is described by the Fourier transform

$$[\hat{S}\psi_+](x_-) = \int dx_+ K(x_-, x_+) \psi_+(x_+). \quad (\text{V.4})$$

The exact form of the kernel $K(x_-, x_+)$ depends on the non-perturbative definition of the model. There are two possible definitions [91]. In the first model the domain of definition of the wave functions is restricted to the positive half-lines $x_{\pm} > 0$, whereas in the second one it coincides with the whole line. This corresponds to that either we consider fermions from one side of the inverse oscillator potential or from the both sides. The two sides of the potential are connected only by tunneling processes which are non-perturbative in the cosmological constant being proportional to $e^{-\pi\mu}$. Therefore, the choice of the model does not affect the perturbative (genus) expansion of the free energy. We prefer to work with the first model avoiding the doubling of fermions.

Calculating the transformation (V.4) on the eigenfunctions (V.3), one finds that it is diagonal

$$[\hat{S}^{\pm 1}\psi_{\pm}^E](x_{\mp}) = \mathcal{R}(\pm E)\psi_{\mp}^E(x_{\mp}), \quad (\text{V.5})$$

where in the first model the coefficient is given by

$$\mathcal{R}(E) = \sqrt{\frac{2}{\pi}} \cosh\left(\frac{\pi}{2}(i/2 - E)\right) \Gamma(iE + 1/2). \quad (\text{V.6})$$

The coefficient is nothing else but the scattering coefficient of fermions off the inverse oscillator potential. It can be seen as fermionic S -matrix in the energy representation. Because $\mathcal{R}(E)$ is a pure phase the S -matrix is unitary. Thus, one gets a non-perturbatively defined formulation of the double scaled MQM or, in other terms, of 2D string theory.

Note that the fermionic S -matrix has been calculated in [91] from properties of the parabolic cylinder functions defined by equation (III.37). In our case it appears from the usual Fourier transformation and it does not involve a solution of a complicated differential equation. Thus, the light-cone representation does crucially simplify the problem of scattering in 2D string theory.

But the main advantage of the light-cone representation becomes clear when one considers the tachyon perturbations. As we discussed, they should be introduced as coherent states of tachyons. Following this idea, we consider the one-fermion wave functions of the form

$$\Psi_{\pm}^E(x_{\pm}) = e^{\mp i\varphi_{\pm}(x_{\pm}; E)} \psi_{\pm}^E(x_{\pm}), \quad (\text{V.7})$$

where the expansion of the phase $\varphi_{\pm}(x_{\pm}; E)$ in powers of x_{\pm} in the asymptotics $x_{\pm} \rightarrow \infty$ is fixed and gives the spectrum of tachyons. The exact form of φ_{\pm} is determined by the condition that S -matrix (V.4) remains diagonal on the perturbed wave functions

$$\hat{S} \Psi_{+}^E = \Psi_{-}^E. \quad (\text{V.8})$$

what means that Ψ_{+}^E and Ψ_{-}^E are two representations of the same physical state.

Two remarks are in order. First, the wave functions (V.7) are not eigenfunctions of the Hamiltonian (V.2). Nevertheless, they can be promoted to solutions of the Schrödinger equation by replacement $x_{\pm} \rightarrow e^{\mp t} x_{\pm}$ and multiplying by the overall factor $e^{\mp \frac{1}{2}t}$. As we will

show, this leads to a time-dependent Fermi sea and the corresponding string background. But the energy of the whole system remains constant. In principle, one can introduce a perturbed Hamiltonian with respect to which Ψ_{\pm}^E would be eigenfunctions [114]. However, such Hamiltonian has nothing to do with the physical time evolution. Nevertheless, it contains all information about the perturbation and may be a useful tool to investigate the perturbed system.

The second remark is that introducing the tachyon perturbations according to (V.7), we change the Hilbert space of the system. Indeed, such states cannot be created by an operator acting in the initial Hilbert space formed by the non-perturbed eigenfunctions (V.3). Roughly speaking, this is so because the scalar product of a perturbed state (V.7) with an eigenfunction (V.3) diverges. Therefore, in contrast to the infinitesimal perturbations, the coherent states (V.7) are not elements of the Hilbert space associated with the fermionic ground state. Thus, our perturbation is intrinsically non-perturbative and we arrive at the following picture. With each tachyon background one can associate a Hilbert space. Its elements describe propagation of small tachyon perturbations over this background. But the Hilbert spaces associated with different backgrounds are not related to each other.

An explicit characteristics of these backgrounds can be obtained in the quasiclassical limit $\mu \rightarrow \infty$, which is identified with the spherical approximation of string theory. In this limit the state of the system of free fermions is described as an incompressible Fermi liquid and, consequently, it is enough to define the region of the phase space filled by fermions to determine completely the state. Assuming that the filled region is connected, the necessary data are given by one curve representing the boundary of the Fermi sea. In a general case the curve is defined by a multivalued function. For example, for the ground state in the coordinates (x, p) it is given by the two-valued function $p(x)$ (III.60).

In Article II we showed that for the perturbation (V.7) the profile of the Fermi sea is determined by the consistency condition of the following two equations

$$x_+ x_- = M_{\pm}(x_{\pm}) \equiv \mu + x_{\pm} \partial \varphi_{\pm}(x_{\pm}; -\mu). \quad (\text{V.9})$$

Each of the two equations defines a curve in the phase space. One of them is most naturally written as $x_+ = x_+(x_-)$ where the function $x_+(x_-)$ is single-valued in the asymptotic region $x_- \rightarrow \infty$. The other, in turn, is determined by the function $x_-(x_+)$ with the same properties in the asymptotics $x_+ \rightarrow \infty$. These curves should coincide, which means that the functions $x_+(x_-)$ and $x_-(x_+)$ are mutually inverse. This condition imposes a restriction on the perturbing phases φ_{\pm} . It allows to restore the full phases from their asymptotics at infinity. The resulting curve coincides with the boundary of the Fermi sea.

Thus, we see that the tachyon perturbations are associated with changes of the asymptotic form of the Fermi sea of free fermions of the singlet sector of MQM. Given the asymptotics, the exact form can be found with help of equations (V.9) which express the matching condition of in-coming and out-going tachyons. Note that the replacement $x_{\pm} \rightarrow e^{\mp t} x_{\pm}$ does lead to a time-dependent Fermi sea.

1.2 Toda description of tachyon perturbations

Up to now, we considered perturbations of tachyons of arbitrary momenta. Let us restrict ourselves to the case which is the most interesting for us: when the momenta are imaginary

and form an equally spaced lattice as in the compactified Euclidean theory or as in the presence of a finite temperature. Thus, the perturbations to be studied are given by the phases

$$\varphi_{\pm}(x_{\pm}; E) = V_{\pm}(x_{\pm}) + \frac{1}{2}\phi(E) + v_{\pm}(x_{\pm}; E), \quad (\text{V.10})$$

where the asymptotic part has the following form

$$V_{\pm}(x_{\pm}) = \sum_{k \geq 1} t_{\pm k} x_{\pm}^{k/R}. \quad (\text{V.11})$$

The rest contains the zero mode $\phi(E)$ and the part v_{\pm} vanishing at infinity. They are to be found from the compatibility condition (V.9) and expressed through the parameters of the potentials (V.11). These parameters are the parameter R measuring the spacing of the momentum lattice, which plays the role of the compactification radius in the corresponding Euclidean theory, and the coupling constants $t_{\pm n}$ of the tachyons.

In Article II we demonstrated that with each coupling t_n one can associate a flow generated by some operator H_n . These operators are commuting in the sense of (II.89). Moreover, they have the same structure as the Hamiltonians from the Lax formalism of Toda hierarchy. Namely, one can introduce the analogs of the two Lax operators L_{\pm} and the Hamiltonians H_n are expressed through them similarly to (II.88)

$$H_{\pm n} = \pm (L_{\pm}^{n/R})_{>} \pm \frac{1}{2} (L_{\pm}^{n/R})_0, \quad n > 0, \quad (\text{V.12})$$

Thus, the perturbations generated by (V.11) are integrable and described by Toda hierarchy.

This result has been proven by explicit construction of the representation of all operators of the Lax formalism of section II.5.2. The crucial fact for this construction is that in the basis of the non-perturbed functions (V.3) the operators of multiplication by x_{\pm} coincide with the energy shift operator

$$\hat{x}_{\pm} \psi_{\pm}^E(x_{\pm}) = \hat{\omega}^{\pm 1} \psi_{\pm}^E(x_{\pm}), \quad \hat{\omega} = e^{-i\partial_E}. \quad (\text{V.13})$$

Due to this property the perturbed wave functions (V.7) can be obtained from the non-perturbed ones by action of some operators \mathcal{W}_{\pm} in the energy space

$$\Psi_{\pm}^E \equiv e^{\mp i\varphi_{\pm}(x_{\pm})} \psi_{\pm}^E = \mathcal{W}_{\pm} \psi_{\pm}^E. \quad (\text{V.14})$$

The operators \mathcal{W}_{\pm} are constructed from $\hat{\omega}$ and can be represented as series in $\hat{\omega}^{1/R}$. This shows that if one starts from a wave function of a given energy, for instance $E = -\mu$, then the perturbed function is a linear combination of states with energies $-\mu + in/R$. Of course, they do not belong to the initial Hilbert space, but this is not important for the construction. The important fact is that only a discrete set of energies appears. Therefore, one can identify this set of imaginary energies (shifted by $-\mu$ which plays the role of the Fermi level) with the discrete lattice $\hbar s$ of the Lax formalism.

Then it is easy to recognize the operators \mathcal{W}_{\pm} as the dressing operators (II.98). The coupling constants $t_{\pm n}$ play the role of the Toda times. And the wave function $\Psi_{+}^E(x_{+})$ appears as the Baker–Akhiezer function (II.90). Since the Lax operators act on the Baker–Akhiezer function as the simple multiplication operators, in our case they are just represented

by \hat{x}_\pm . Their expansion in terms of $\hat{\omega}$ is given by the representation of \hat{x}_\pm in the basis of $\Psi_\pm^{-\mu+i\eta/R}$ and can be obtained by dressing $\hat{\omega}^{\pm 1}$. Similarly, the Orlov–Shulman operators (II.94) are the dressed version of the energy operator $-\hat{E}$

$$L_\pm = \mathcal{W}_\pm \hat{\omega}^{\pm 1} \mathcal{W}_\pm^{-1}, \quad (\text{V.15})$$

$$M_\pm = -\mathcal{W}_\pm \hat{E} \mathcal{W}_\pm^{-1}. \quad (\text{V.16})$$

All the relations determining the Toda structure can be easily established. The only difficult place is the connection between the right and left representations. In section II.5.2 we mentioned that to define Toda hierarchy, the dressing operators must be subject of some condition. Namely, $\mathcal{W}_+^{-1} \mathcal{W}_-$ should not depend on the Toda times $t_{\pm n}$. It turns out that this condition is exactly equivalent to the requirement (V.8) which we imposed on the perturbations. Indeed, in terms of the dressing operators it is written as

$$\mathcal{W}_- = \mathcal{W}_+ \hat{\mathcal{R}}, \quad (\text{V.17})$$

where $\hat{\mathcal{R}}$ is the operator corresponding to (V.6). This operator is independent of the couplings, so that the necessary condition is fulfilled.

Besides, this framework provides us with the string equations in a very easy way. They are just consequences of the trivial relations¹

$$[\hat{x}_+, \hat{x}_-] = -i, \quad \hat{x}_+ \hat{x}_- + \hat{x}_- \hat{x}_+ = -E. \quad (\text{V.18})$$

Relying on the defining equation (V.17), it was shown that one obtains the following string equations

$$L_+ L_- = M_+ + i/2, \quad L_- L_+ = M_- - i/2, \quad M_- = M_+. \quad (\text{V.19})$$

They resemble the string equations of the two-matrix model (II.151) and (II.152). Similarly to that case, only two of them are independent.

Actually, we should say that L_\pm and M_\pm are not exactly the operators that appear in the Lax formalism of Toda hierarchy. The reason is that they are series in $\hat{\omega}^{1/R}$ whereas the standard definition of the Lax operators (II.86) involves series in $\hat{\omega}$. This is because our shift operator is R th power of the standard one what follows from the relation between s and the energy E . Therefore, the standard Lax and Orlov–Shulman operators should be defined as follows

$$L = L_+^{1/R}, \quad \bar{L} = L_-^{1/R}, \quad M = R M_+, \quad \bar{M} = R M_-. \quad (\text{V.20})$$

In terms of these Lax operators the string equation (V.19) is rewritten as

$$[L^R, \bar{L}^R] = i. \quad (\text{V.21})$$

This equation was first derived in [104] for the dual Toda hierarchy, where $R \rightarrow 1/R$, describing the winding perturbations of MQM considered in the previous chapter.² Thus, indeed there is a duality between windings and tachyons: the both perturbations are described by the same integrable structure with the dual parameters. Finally, the standard Toda times are related with the coupling constants as $t_{\pm n} \rightarrow \pm t_{\pm n}$ (see footnote 2 on page 53) and the Planck constant is pure imaginary $\hbar = i$.

¹We neglect the subtleties related with the necessity to insert the S -matrix operator passing from the right to left representation and back. The exact formulae can be found in Article II.

²Before, only the string equation with integer R appeared in the literature [115, 116].

1.3 Dispersionless limit and interpretation of the Lax formalism

Especially simple and transparent formulation is obtained in the dispersionless limit of Toda hierarchy. In this limit the state of the fermionic system is described by the Fermi sea in the phase space of free fermions. Therefore, we expect that the Toda hierarchy governs the dynamics of this sea.

Indeed, as we know from section II.5.6, the shift operator becomes a classical variable and together with the lattice parameter defines the symplectic form

$$\{\omega, E\} = \omega. \quad (\text{V.22})$$

Remembering that the Lax operators L_{\pm} coincide with the light-cone variables x_{\pm} , the classical limit of the string equations reads

$$\{x_-, x_+\} = 1, \quad (\text{V.23})$$

$$x_+ x_- = M_{\pm}(x_{\pm}) = \frac{1}{R} \sum_{k \geq 1} k t_{\pm k} x_{\pm}^{k/R} + \mu + \frac{1}{R} \sum_{k \geq 1} v_{\pm k} x_{\pm}^{-k/R}. \quad (\text{V.24})$$

The first equation is nothing else but the usual symplectic form on the phase space of MQM. The second equation coincides with the compatibility condition (V.9) where the explicit form of the perturbing phase was substituted. Thus, this is the equation that determines the exact form of the Fermi sea. We note also that for the self-dual case $R = 1$, the deformation described by (V.24) is similar to the deformation of the ground ring (III.100) suggested by Witten in [25].

We conclude that for the tachyon perturbations all the ingredients of the Lax formalism have a clear interpretation in terms of free fermions:

- the discrete lattice on which the Toda hierarchy is defined is the set of energies given by the sum of the Fermi level $-\mu$ and Euclidean momenta of the compactified theory;
- the Lax operators are the light-cone coordinates in the phase space of the free fermions;
- the Orlov–Shulman operators define asymptotics of the profile equation describing deformations of the Fermi level;
- the Baker–Akhiezer function is the perturbed one-fermion function;
- the first string equation describes the canonical transformation from the light-cone coordinates x_{\pm} to the energy E and $\log \omega$.
- the second string equation is the equation for the profile of the Fermi sea.

1.4 Exact solution of the Sine–Liouville theory

As in the case of the winding perturbations, the Toda integrable structure can be applied to find the exact solution of the Sine–Liouville theory dual to the one considered in (IV.19). But in the present case we possess a more powerful tool to extract the solution: the string

equation. In the dispersionless limit, it allows to avoid any differential equations and gives the solution quite directly. The result is the following [114]

$$x_{\pm} = e^{-\frac{1}{2R}\chi}\omega^{\pm 1}(1 + a_{\pm}\omega^{\mp\frac{1}{R}}), \quad (\text{V.25})$$

$$\mu e^{\frac{1}{R}\chi} - \frac{1}{R^2} \left(1 - \frac{1}{R}\right) t_1 t_{-1} e^{\frac{2R-1}{R^2}\chi} = 1, \quad a_{\pm} = \frac{t_{\pm 1}}{R} e^{\frac{R-1/2}{R^2}\chi}, \quad (\text{V.26})$$

where $\chi = \partial_{\mu}^2 \log \tau$.³ This solution reproduces both the free energy and the one-point correlators. The former is expressed through the solution of the first equation in (V.26) and the latter is contained in (V.25) where one should identify

$$\omega(x_{\pm}) = e^{\frac{1}{2R}\chi} x_{\pm} e^{-Rh_{\pm}(x_{\pm})}. \quad (\text{V.27})$$

Here $h_{\pm}(x_{\pm})$ are the generating functions of the one-point correlators similar to (IV.48).

Comparing (V.26) and (V.25) with (IV.33) and (IV.54), respectively, one finds that the former equations are obtained from the latter by the following duality transformation⁴

$$R \rightarrow 1/R, \quad \mu \rightarrow R\mu, \quad t_n \rightarrow R^{-nR/2} t_n. \quad (\text{V.28})$$

Note that it would be more natural to transform the couplings as $t_n \rightarrow R^{1-nR/2} t_n$ adding additional factor R . Then the scaling parameter $\lambda^2 \mu^{R-2}$ would not change. Moreover, in the CFT formulation this factor is naturally associated with the Liouville factor $e^{-2\phi}$ of each marginal operator. Its absence in our case is related to that we identified the winding couplings in incorrect way. The correct couplings \tilde{t}_n are given in (IV.37). Thus, whereas the couplings of the tachyon perturbations are exactly the Toda times, for the winding perturbations they differ by factor R .

The explicit solution (V.25) determines the boundary of the Fermi sea, describing a condensate of tachyons of momenta $\pm 1/R$, in a parametric form. For each set of parameters one can draw the corresponding curve. This geometric representation gives also an interpretation to the (T-dual of the) critical point (IV.34). At this point the Fermi sea forms a spike and the quasiclassical approximation breaks down. Some pictures of the typical configurations of the Fermi sea can be found in Article II.

It is interesting that only the case $t_{\pm 1} > 0$ has a good interpretation. In all other cases at some moment of time the Fermi sea begins to penetrate into the region $x < 0$. This corresponds to the transfusion of the fermions through the top of the potential to the other side. Such processes are forbidden at the perturbative level.

This can be understood for $t_1 t_{-1} < 0$ because the corresponding CFT is not unitary and one can expect some problems. On the other hand, the case $t_{\pm 1} < 0$ is well defined from the Euclidean CFT point of view, as is the case of positive couplings, because they differ just by a shift of the Euclidean time. It is likely that the problem is intrinsically related to the Minkowskian signature. In terms of the Minkowskian time the potential is given by $\cosh(t/R)$ and it is crucial with which sign it appears in the action. The two possibilities lead to quite different pictures as it is clear for our solution.

³From now on, we omit the index 0 indicating the spherical approximation.

⁴Performing this transformation, one should take into account that the susceptibility transforms as $\chi \rightarrow R^{-2}\chi - R^{-1} \log R$. This result can be established, for example, from (IV.32).

2 Thermodynamics of tachyon perturbations

2.1 MQM partition function as τ -function

In the previous section we showed that the tachyon perturbations with momenta as in the compactified Euclidean theory are described by the constrained Toda hierarchy. Hence, they are characterized by a τ -function. What is the physical interpretation of this τ -function? In [92] it was identified as the generating functional for scattering amplitudes of tachyons. On the other hand, since the tachyon spectrum coincides with that of the theory at finite temperature, it is tempting to think that the theory possesses a thermodynamical interpretation. Then the τ -function could be seen as the partition function of the model.

Although expected, the existence of the thermodynamical interpretation is not guaranteed because the system is formulated in the Minkowskian time and except the coincidence of the spectra there is no reference to a temperature. However, we will show that it does exist at least for the case of the Sine-Liouville perturbation [117]. In the following we will accept this point of view and will show that the τ -function is indeed the grand canonical partition function at temperature $T = 1/\beta$. In the spherical limit this was done in [114, 117] and we will present that derivation in the next paragraph. Here we prove the statement to all orders in perturbation theory following the paper [118] (Article IV) which will be discussed in chapter VI.

The grand canonical partition function is defined as follows

$$\mathcal{Z}(\mu, t) = \exp \left[\int_{-\infty}^{\infty} dE \rho(E) \log \left(1 + e^{-\frac{1}{\hbar} \beta(\mu + E)} \right) \right], \quad (\text{V.29})$$

where $\rho(E)$ is the density of states. It can be found by confining the system in a box of size $\sim \sqrt{\Lambda}$ similarly as it was done in section III.2.4. The difference is that now we work in the light cone representation. Therefore, one should generalize the quantization condition (III.38). The generalization is given by

$$[\hat{S}\Psi](\sqrt{\Lambda}) = \Psi(\sqrt{\Lambda}) \quad (\text{V.30})$$

so that one identifies the scattered state with the initial one at the wall. Then from the explicit form of the perturbed wave function (V.7) with (V.10) one finds [118]

$$\rho(E) = \frac{\log \Lambda}{2\pi} - \frac{1}{2\pi} \frac{d\phi(E)}{dE}. \quad (\text{V.31})$$

Integrating by parts in (V.29), closing the contour in the upper half plane and taking the integral by residues of the thermal factor, one obtains [118]

$$\mathcal{Z}(\mu, t) = \prod_{n \geq 0} \exp \left[i\phi \left(i\hbar \frac{n + \frac{1}{2}}{R} - \mu \right) \right]. \quad (\text{V.32})$$

On the other hand, the zero mode of the perturbing phase is actually equal to the zero mode of the dressing operators (II.98). Hence it is expressed through the τ -function as in

(II.103). Since the shift in the discrete parameter s is equivalent to an imaginary shift of the chemical potential μ , this formula reads

$$e^{\frac{i}{\hbar}\phi(-\mu)} = \frac{\tau_0\left(\mu + i\frac{\hbar}{2R}\right)}{\tau_0\left(\mu - i\frac{\hbar}{2R}\right)}. \quad (\text{V.33})$$

Comparing this result with (V.32), one concludes that

$$\mathcal{Z}(\mu, t) = \tau_0(\mu, t). \quad (\text{V.34})$$

2.2 Integration over the Fermi sea: free energy and energy

As always, in the dispersionless limit one can give to all formulae clear geometrical interpretation. Moreover, this limit allows to obtain additional information about the system. Namely, in this paragraph we show how such thermodynamical quantities as the free energy and the energy of 2D string theory perturbed by tachyon sources are restored from the Fermi sea of the singlet sector of MQM. This was done in the work [117] (Article III).

Since in the classical limit the profile of the Fermi sea uniquely determines the state of the free fermion system, it encodes all the interesting information. Indeed, in the full quantum theory the expectation value of an operator is given by its quantum average in the given state. Its classical counterpart is the integral over the phase space of the corresponding function multiplied by the density of states. For the free fermions the density of states equals 1 or 0 depending on either the phase space point is occupied or not. As a result, the classical value of an observable \mathcal{O} is represented by its integral over the Fermi sea

$$\langle \mathcal{O} \rangle = \frac{1}{2\pi} \int \int_{\text{Fermi sea}} dx_+ dx_- \mathcal{O}(x_+, x_-). \quad (\text{V.35})$$

Now the description in terms of the dispersionless Toda hierarchy comes into the game. We saw that the tachyon perturbation give rise to the canonical transformation from x_{\pm} to the set of variables $(E, \log \omega)$. The explicit map is given by (V.25) for the simplest case of the Sine-Liouville theory and it is easily extended to the general case of a potential of degree n . Due to this the measure in (V.35) can be rewritten in terms of E and ω . As a result, one arrive at the following formula

$$\partial_{\mu} \langle \mathcal{O} \rangle = -\frac{1}{2\pi} \int_{\omega_-(\mu)}^{\omega_+(\mu)} \frac{d\omega}{\omega} \mathcal{O}(x_+(\omega, \mu), x_-(\omega, \mu)). \quad (\text{V.36})$$

The limits of integration are defined by the cut-off Λ . How they can be found is explained in detail in Article III.

The result (V.36) is rather general. We applied it to the main two observables which are the energy and the number of particles. They were already defined in (III.27) and (III.26) for the case of the ground state. In a more general case they are written as integrals (V.35)

with $\mathcal{O} = -x_+ x_-$ and $\mathcal{O} = 1$, correspondingly. In these two cases the integral (V.36) is easily calculated leading to

$$\partial_\mu N = -\frac{1}{2\pi} \log \Lambda - \frac{1}{2\pi R} \chi, \quad (\text{V.37})$$

$$\partial_\mu E = \frac{1}{2\pi} e^{-\frac{1}{R}\chi} \left\{ (1 + a_+ a_-) \left(\log \Lambda + \frac{1}{R} \chi \right) - 2a_+ a_- \right\} + \frac{R}{2\pi} (t_1 + t_{-1}) \Lambda^{1/2R}, \quad (\text{V.38})$$

where the second equation is valid for the Sine–Liouville perturbation. Using (V.26), one can integrate (V.38) and obtain an expression for the energy of the system [117].

To reproduce the free energy, note that in the grand canonical ensemble it is related to the number of particles through

$$N = \partial \mathcal{F} / \partial \mu, \quad (\text{V.39})$$

where we changed our notations assuming the usual thermodynamical definition of the free energy at the temperature T

$$\mathcal{F} = -T \log \mathcal{Z}. \quad (\text{V.40})$$

Comparing (V.39) and (V.37), one obtains the expected result

$$\mathcal{F} = -\frac{1}{\beta} \log \tau. \quad (\text{V.41})$$

In particular, the temperature is associated with $1/\beta$. Integrating the equation for the susceptibility χ , one can get an explicit representation for the free energy. It coincides with the T-dual transform of (IV.31).

2.3 Thermodynamical interpretation

In Article III we proved that the derived expressions for the energy and free energy allow a thermodynamical interpretation. This is not trivial because there are two definitions of the macroscopic energy. One of them is the sum of microscopic energies of the individual particles which is expressed as an integral over the phase space. We used this definition in the previous paragraph. Another definition follows from the first law of thermodynamics which relates the energy, free energy and entropy

$$S = \beta(E - F). \quad (\text{V.42})$$

It allows to express the energy and entropy as derivatives of the free energy with respect to the temperature

$$E = \frac{\partial(\beta F)}{\partial \beta}, \quad S = -\frac{\partial F}{\partial T}. \quad (\text{V.43})$$

To have a consistent thermodynamical interpretation, the two definitions must give the same result. It is a very non-trivial check, although very simple from the technical point of view. All that we need is to differentiate the free energy, which is given through the solution of the algebraic equation (V.26), and check that the result coincides with the one obtained integrating (V.38).

However, there are two important subtleties. The first one is that the first law is formulated in terms of the canonical free energy F rather than for its grand canonical counterpart \mathcal{F} . This is in agreement with the fact that it is the canonical free energy that is interpreted as the partition function of string theory. Thus, it is F that carries an information about properties of the string background. Due to this one should pass to the canonical ensemble

$$F = \mathcal{F} - \mu \frac{\partial \mathcal{F}}{\partial \mu}. \quad (\text{V.44})$$

If the first subtlety answers to the question "*what to differentiate?*", the second one concerns the problem "*how to differentiate?*". The problem is what parameters should be held fixed when one differentiates with respect to the temperature. First, this may be either μ or N . It is important to make the correct choice because they are non-trivial functions of each other. Since we are working in the canonical ensemble, it is natural to take N as independent variable. Besides, one should correctly identify the coupling λ and the cut-off Λ . Their definition can involve R and, thus, contribute to the result.

It turns out [117] that the coupling and cut-off that we have chosen are already the correct ones. By direct calculation it was shown that the thermodynamical relations (V.43), where the derivatives are taken with N , λ and Λ fixed, are indeed fulfilled. Thus, the 2D string theory perturbed by tachyons of momenta $\pm i/R$ in Minkowskian spacetime has a consistent interpretation as a thermodynamical system at temperature $T = 1/(2\pi R)$.

This result also answers to the question risen in section IV.2.3: what variable should be associated with the temperature? Our analysis definitely says that this is the parameter R . We do not need to introduce such notion as "temperature at the wall" [29, 30, 34]. The differentiation is done directly with respect to the compactification radius. Also this supports the idea that in the dual picture a black hole background should exist for any compactification radius, at least in the interval $1 < R < 2$.

The first law of thermodynamics (V.42) or the relations (V.43) allow to calculate the entropy. It vanishes in the absence of perturbations when $\lambda = 0$ but it is a complicated function in the general case. It would be quite interesting to understand the microscopic origin of this entropy. In other words, we would like to find the microscopic degrees of freedom giving rise to non-vanishing entropy. However, we have not found the solution yet. The problem is that a state of the system is uniquely characterized by the profile of the Fermi sea and there is only one profile described by our solution for each state. The only possibility which we found to obtain different microscopic states is to associate them with different positions in time of the same Fermi sea. Although the Fermi sea is time-dependent, all macroscopic thermodynamical quantities do not depend on time. Thus, different microscopic states would define the same macroscopic state. This idea is supported also by the fact that the entropy vanishes only if the Fermi sea is stationary. However, we have not succeeded to get the correct result for the entropy from this picture.

It is tempting to claim that the obtained thermodynamical quantities describe after the duality transformation (V.28) the thermodynamics of winding perturbations and their string backgrounds. This is, of course, true for the free energy. However, it is not clear whether the energy and entropy are dual in the two systems. For example, it is not understood even how to define the energy of a winding condensate. Nevertheless, if we assume that our results can be related to the backgrounds generated by windings, this gives a plausible picture. For

example, the non-vanishing entropy can be explained with the existence of a black hole.

Our results concern arbitrary radius R and cosmological constant μ . When the latter goes to zero, which corresponds to the black hole point according to [35], the situation becomes little bit special. Namely, the logarithmic term $\xi^{-2} \log(\Lambda \xi)$ in the free energy disappears, whereas it is present in the energy and entropy. This term is the leading one. Therefore, both the energy and entropy are much larger than the free energy. This can explain the puzzle that, on the one hand, the dilaton gravity predicts vanishing free energy and, on the other hand, the matrix model gives a non-vanishing result. Our approach shows that it does not vanish but it is negligible in comparison with other quantities so that in the main approximation the law $S = \beta E$ is valid.

3 Backgrounds of 2D string theory

We introduced the tachyon perturbations as one of the ways to change the background of 2D string theory. How exactly does the background corresponding to non-vanishing t_n look? Any background has two types of characteristics, local and global. The local structure is captured by values of the background fields which are the metric, dilaton and tachyon. But also backgrounds can differ by their global structure. Of course, the introduction of tachyon couplings in the way described in section 1 changes the value of the tachyon field. But may be it can change also some other properties of the background? We tried to answer this question in the work [119] (Article V).

3.1 Local properties

First, let us analyze the local properties of the background obtained by a perturbation of the type (V.7). We will restrict ourselves to the spherical approximation. Then the background is uniquely determined by the profile of the Fermi sea. Thus, we should find the background fields using a given profile as a starting point.

As we could see above, the fermionic representation used to introduce the perturbations is very convenient for the solution of many problems. However, it is not suited to discuss the target space phenomena. For these purposes we should use a collective field theory. We have already given an introduction to such theory constructed by Das and Jevicki (see section III.3). It turns out that it can be used to extract an information about the string background.

In fact, we have already done this for the simplest case of the linear dilaton background which was obtained from the ground state of the system of free fermions of the singlet sector of MQM. In particular, we identified the tachyon field with fluctuations of the matrix model density of eigenvalues around the ground state. We expect that this identification holds to be true also in more complicated cases. All that we need to obtain another background is to replace the background value of the density by a new function obtained from the exact form of the deformed profile of the Fermi sea. In this way one can find an effective action for the tachyon. Then our idea was to compare this action with the similar action obtained as a low-energy limit of string theory discussed in section I.3.

One should say that our analysis uses a simplified version of the relation between the Das–Jevicki collective theory and 2D string theory. The exact relation requires identification of the tachyon with the loop operator rather than with the density (see (III.78)). The former is the Fourier transform of the latter, so that the relation between the density and the string tachyon is non-local. For the simplest case of the ground state, the analysis can be nevertheless completed. For the time-dependent backgrounds, we are going to consider, the situation becomes much more complicated. We do not know how to find the exact relation in this case. Instead we neglect the Liouville and sine-Liouville terms in the equation for the tachyon field and identify this field with the fluctuations of the density. And it is already a remarkable fact that in this approximation the problem can be solved exactly.

First, we obtain an effective action for the density fluctuations from the Das–Jevicki theory. For this it is sufficient to make the substitution (III.61) into the background independent effective action (III.54). However, in contrast to the previous case, the background

value of the density φ_0 now depends on the time t . This time-dependence leads to additional terms. Extracting only the quadratic term of the expansion in the fluctuations η , one finds the following result

$$S_{(2)} = \frac{1}{2} \int dt \int \frac{dx}{\varphi_0} \left[(\partial_t \eta)^2 - 2 \frac{\int dx \partial_t \varphi_0}{\varphi_0} \partial_t \eta \partial_x \eta - \left(\varphi_0^2 - \left(\frac{\int dx \partial_t \varphi_0}{\varphi_0} \right)^2 \right) (\partial_x \eta)^2 \right]. \quad (\text{V.45})$$

The crucial property of this action is that for any $\varphi_0(x, t)$ the determinant of the matrix coupled to the derivatives of η equals -1 . Besides, there are no terms without derivatives. As a result, this quadratic part can be represented as the usual action for a massless scalar field in a curved metric $g_{\mu\nu}$. The metric in the coordinates (t, x) is fixed by (V.45) up to a conformal factor. For example, we can choose it to coincide with the matrix we were talking about, so that $\det g = -1$.

Then, we compared the action for η to the low-energy effective action for the tachyon field (I.27) restricted to two dimensions. They coincide if one makes the usual identification $T = e^\Phi \eta$ and requires the following property

$$m_\eta^2 = (\nabla \Phi)^2 - \nabla^2 \Phi - 4\alpha'^{-1} = 0, \quad (\text{V.46})$$

which ensures that η is a massless field. This condition appears as an additional constraint for the equations of motion on the background fields. Its appearance is directly related to the restriction on the form of the action coming from the Das–Jevicki formalism. In particular, if the determinant of the matrix in (V.45) was arbitrary, we would not have this condition.

In Article V we argued that the constraint (V.46) selects a unique solution of the equations of motion similarly to an initial condition. This solution is just the usual linear dilaton background (I.23). Of course, it is valid only in the leading order in α' . If we take into account the next orders, the tachyon field should be modified as it was discussed in section I.3.4. However, we cannot expect that the dilaton or the metric are modified too. Thus, the introduction of arbitrary tachyon perturbation cannot modify the local structure of the target space. It always remains flat.

Note that this result illustrates that the T-dual theories on the world sheet are not the same in the target space. Whereas the CFT perturbed by windings looks similar to that of perturbed by tachyons, the former was supposed to correspond to the black hole background and the latter lives always in the target space of the vanishing curvature.

3.2 Global properties

Although the local structure of the background obtained by tachyon perturbations is trivial, we still have a possibility to find a non-trivial global structure. Our approach relied again on the effective action (V.45). We know from the previous paragraph that the target space metric is flat. Hence, locally one can always find coordinates where it takes the standard Minkowski form $\eta_{\mu\nu} = \text{diag}(-1, 1)$. However, the image of the initial (t, x) -plane under the coordinate transformation can be only a subspace of the plane of the new flat coordinates. If we identify this subspace as the physical region where the resulting 2D string theory lives, the global structure of the target space will be non-trivial. Depending on boundary conditions, either boundaries will appear or a compactification will take place.

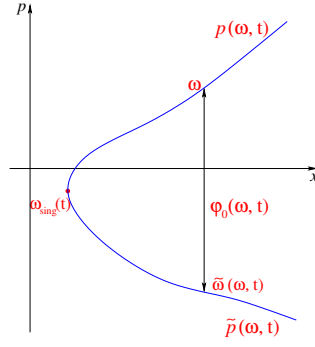


Fig. V.1: The Fermi sea of the perturbed MQM. Its boundary is defined by a two-valued function with two branches parameterized by $p(\omega, t)$ and $\tilde{p}(\omega, t)$. The background field φ_0 coincides with the width of the Fermi sea.

Thus, the global structure of the target space is determined by the transformation to the flat coordinates. Up to a conformal factor it can be deduced from the action (V.45). Unfortunately, we are not able to find the transformation for arbitrary $\varphi_0(x, t)$ which is a solution of the classical equations followed from (III.54). Nevertheless, this task was accomplished for the integrable perturbations generated by the potential (V.11) [119].

In this case we have the description in terms of Toda hierarchy which allows to find the exact form of the profile of the Fermi sea. It is given by a generalization of (V.25). For the perturbing potentials of degree n , it is represented in the following parametric form

$$\begin{aligned} p(\omega, t) &= \sum_{k=0}^n a_k \sinh \left[\left(1 - \frac{k}{R}\right) \omega + \frac{k}{R} t + \alpha_k \right], \\ x(\omega, t) &= \sum_{k=0}^n a_k \cosh \left[\left(1 - \frac{k}{R}\right) \omega + \frac{k}{R} t + \alpha_k \right], \end{aligned} \quad (\text{V.47})$$

where ω is now the logarithm of the shift operator from the previous sections. The function $p(x)$ is two valued. The density $\varphi_0(x, t)$ is given by the difference of the two branches. However, it is more convenient for us to work in the ω -parameterization. Then the first branch is given by $p(\omega, t)$ in (V.47), whereas the second one is obtained as $\tilde{p}(\omega, t) = p(\tilde{\omega}(\omega, t), t)$ where $\tilde{\omega}(\omega, t)$ is defined as a parameter mirror to ω (see fig. V.1). In terms of these functions the quadratic part of the effective action (V.45) is rewritten as follows

$$S_{(2)} = \int dt \int \frac{dx}{p - \tilde{p}} \left[(\partial_t \eta)^2 + (p + \tilde{p}) \partial_t \eta \partial_x \eta + p \tilde{p} (\partial_x \eta)^2 \right]. \quad (\text{V.48})$$

In Article V we showed the following coordinate transformation brings the action (V.48) to the standard form with the kinetic term written in the Minkowski metric $\eta_{\mu\nu}$

$$\tau = t - \frac{\omega + \tilde{\omega}}{2}, \quad q = \frac{\omega - \tilde{\omega}}{2}. \quad (\text{V.49})$$

The change of coordinates (V.49) is remarkably simple and has a transparent interpretation. It associates the target space light-cone coordinates $\tau \pm q$ with the parameters of incoming and outgoing tachyons, $t - \omega$ and $t - \tilde{\omega}$. For the ground state, when the perturbation is absent, the mirror parameter is $\tilde{\omega} = -\omega$ so that we return to the simple situation described in section III.3.2.

The coordinates (V.49) are not necessarily the ones where the metric is trivial. The action in two dimensions possesses the remaining gauge symmetry allowing to make the conformal transformations

$$(\tau, q) \longrightarrow (u(t - \omega), v(t - \tilde{\omega})). \quad (\text{V.50})$$

We were not able to fix the functions u and v . However, we argued that since the conformal maps preserve the causal structure, some conclusions can be made neglecting this ambiguity.

Once the transformation (V.49) was found, it is possible to study the exact form of the resulting target space. This was done for the simplest case of the Sine–Liouville perturbation when $n = 1$ in (V.47). The result crucially depends on the parameter R . There are 3 different regions. When $R \geq 1$ nothing special happens and the target space coincides with the whole plane. For $1/2 < R < 1$ the target space is deformed so that asymptotically it can be considered as two conic regions $|\tau/q| < 2R - 1$. When we approach the origin we find that two cones are smoothly glued along a finite interval of length

$$\Delta = 2R \log \left[\frac{R}{1 - R} \frac{a_0}{a_1} \right]. \quad (\text{V.51})$$

For $R < 1/2$ the picture is similar to the previous case but the time and space coordinates are exchanged so that the picture should be rotated by 90° .

It is interesting that this target space exhibits the same singularity as the one we found in the analysis of the free energy and the perturbed Fermi sea. The singularity appears when the length Δ vanishes so that the two conic regions separate from each other. The corresponding critical value of μ is exactly the same as (IV.34) after the duality transformation (V.28).

The obtained conic spacetimes possess boundaries. Therefore, some boundary conditions should be imposed on the fields propagating there. We discussed them and their physical interpretation in detail in Article V. We proposed three possible boundary conditions: the vanishing boundary conditions, periodic and twisted periodic ones. Which conditions are relevant for each case remained an unsolved problem.

Another unsolved problem is related to the thermodynamical interpretation. We saw that the perturbed 2D string theory in the Minkowskian spacetime can be considered as a theory at finite temperature $T = 1/(2\pi R)$. It would be very interesting to reproduce this result from the analysis of quantum field theory in the target space obtained above. It is well known that boundaries or a compactification with varying radius can give rise to a thermal spectrum of observed particles [120]. Therefore, our picture seems to be reasonable. However, the problem is technically difficult and complicated by the ambiguity in the conformal transformation (V.50).

The obtained results shed light on some properties of 2D string theory with tachyon condensation, but they are not related (at least directly) to the most interesting problem of 2D string theory in curved backgrounds. The latter should be obtained by winding perturbations. Although from the CFT point of view they are not much different from the tachyon perturbations, in the MQM framework their description is much more complicated. For the tachyon modes there is a powerful free fermion representation and, as a consequence, the Das–Jevicki collective field theory. It is this formulation that allowed us to get some information about the string target space. For windings there is no analog of this formalism and how to show, for example, that the winding perturbations of MQM correspond to the black hole background is still an open problem.

Chapter VI

MQM and Normal Matrix Model

In this chapter we present the next work included in this thesis [118] (Article IV). It stays a little bit aside the main line of our investigation. Nevertheless it opens one more aspect of 2D string theory in non-trivial backgrounds and its Matrix Quantum Mechanical formulation, in particular. This work establishes an equivalence of the tachyonic perturbations of MQM described in the previous chapter with the so called Normal Matrix Model (NMM). This model appears in the study of various physical and mathematical problems. Therefore, before to discuss our results, we will briefly describe the main features of NMM and the related issues.

1 Normal matrix model and its applications

1.1 Definition of the model

The Normal Matrix Model was first introduced in [121, 122]. It is a statistical model of random complex matrices which commute with their conjugates

$$[Z, Z^\dagger] = 0. \quad (\text{VI.1})$$

As usual, its partition function is represented as the matrix integral

$$Z_N = \int d\nu(Z, Z^\dagger) \exp \left[-\frac{1}{\hbar} W(Z, Z^\dagger) \right], \quad (\text{VI.2})$$

where the measure $d\nu$ is a restriction of the usual measure on the space of all $N \times N$ complex matrices to those that satisfy the relation (VI.1). For our purposes we introduced explicitly the Planck constant at the place of N . They are supposed to be related in the large N limit which is obtained as $N \rightarrow \infty$, $\hbar \rightarrow 0$ with $\hbar N$ fixed.

The potentials which can be considered are quite general. We will be interested especially in the following type of potentials

$$W_R(Z, Z^\dagger) = \text{tr} (ZZ^\dagger)^R - \hbar\gamma \text{tr} \log(ZZ^\dagger) - \text{tr} V(Z) - \text{tr} \bar{V}(Z^\dagger). \quad (\text{VI.3})$$

where

$$V(Z) = \sum_{k \geq 1} t_k Z^k, \quad \bar{V} = \sum_{k \geq 1} t_{-k} Z^{\dagger k}. \quad (\text{VI.4})$$

The probability measure with such a potential depends on several parameters. These are R , γ and two sets of t_n and t_{-n} . The latter are considered as coupling constants because the dependence on them of the partition function contains an information about correlators of the matrix operators $\text{tr } Z^k$. The other two parameters R and γ are just real numbers characterizing the particular model.

The partition function (VI.2) resembles the two-matrix model studied in section II.4. Similarly to that model, one can do the reduction to eigenvalues. The difference with respect to 2MM is that now the eigenvalues are complex numbers and the measure takes the form

$$d\nu(Z, Z^\dagger) = \frac{1}{N!} [d\Omega]_{SU(N)} \prod_{k=1}^N d^2 z_k |\Delta(z)|^2. \quad (\text{VI.5})$$

Thus, instead of two integrals over real lines one has one integral over the complex plane. Despite this difference, one can still introduce the orthogonal polynomials and the related fermionic representation. Then, repeating the arguments of section II.5.7, it is easy to prove that the partition function (VI.2) is a τ -function of Toda hierarchy as well. For the case of $R = 1$ this was demonstrated in [67] and for generic R the proof can be found in appendix A of Article IV. In fact, for the particular case $R = 1$ (and $\gamma = 0$) it was proven [68] that NMM and 2MM coincide in the sense that they possess the same free energy as function of the coupling constants. Nevertheless, their interpretation remains different.

The eigenvalue distribution of NMM in the large N limit is also similar to the picture arising in 2MM and shown on fig. II.7. The eigenvalues fill some compact spots on a two-dimensional plane. The only difference is that earlier this was the plane formed by real eigenvalues of the two matrices, and now this is the complex z -plane. Therefore, the width of the spots does not have anymore a direct interpretation in terms of densities. Instead, the density inside the spots for a generic potential can be non-trivial. For example, for the potential (VI.3) it is given by

$$\rho(z, \bar{z}) = \frac{1}{\pi} \partial_z \partial_{\bar{z}} W_R(z, \bar{z}) = \frac{R^2}{\pi} (z\bar{z})^{R-1}. \quad (\text{VI.6})$$

For $R = 1$ where NMM reduces to 2MM, we return to the constant density.

1.2 Applications

Recently, the Normal Matrix Model found various physical applications [123, 124]. Most remarkably, it describes phenomena whose characteristic scale differs by a factor of 10^9 . Whereas some of these phenomena are purely classical, another ones are purely quantum.

Quantum Hall effect

First, we mention the relation of NMM to the Quantum Hall effect [125]. There one considers electrons on a plane in a strong magnetic field B . The spectrum of such system consists from Landau levels. Even if the magnetic field is not uniform, the lowest level is highly degenerate. The degeneracy is given by the integer part of the total magnetic flux $\frac{1}{2\pi\hbar} \int B(z) d^2 z$, and the one-particle wave functions at this level have the following form

$$\psi_n(z) = P_n(z) \exp\left(-\frac{W(z)}{2\hbar}\right). \quad (\text{VI.7})$$

Here $W(z)$ is related to the magnetic field through $B(z) = \frac{1}{2}\Delta W(z)$ and $P_n(z)$ are holomorphic polynomials of degree n with the first coefficient normalized to 1.

Usually, one is interested in situations when all states at the lowest level are occupied. Then the wave function of N electrons is the Slater determinant of the one-particle wave functions (VI.7). Hence, it can be represented as

$$\Psi_N(z_1, \dots, z_N) = \frac{1}{\sqrt{N!}} \Delta(z) \exp \left(-\frac{1}{2\hbar} \sum_{k=1}^N W(z_k) \right). \quad (\text{VI.8})$$

Its norm coincides with the probability measure of NMM. Therefore, the partition function (VI.2) appears in this picture as a normalization factor of the N -particle wave function

$$Z_N = \int \prod_{k=1}^N d^2 z_k |\Psi_N(z_1, \dots, z_N)|^2. \quad (\text{VI.9})$$

Similarly, the density of electrons can be identified with the eigenvalue density and the same is true for their correlation functions.

Due to this identification, in the quasiclassical limit the study of eigenvalue spots is equivalent to the study of electronic droplets. In particular, varying the matrix model potential one can investigate how the shape of the droplets changes with the magnetic field. At the same time, varying the parameter $\hbar N$ one examines its evolution with increasing the number of electrons.

Note that although we discussed the semiclassical regime, the system remains intrinsically quantum. The reason is that all electrons under consideration occupy the same lowest level, whereas the usual classical limit implies that higher energy levels are most important.

Laplacian growth and interface dynamics

It was shown [125] that when one increases the number of electrons the semiclassical droplets from the previous paragraph evolve according to the so called Darcy's law which is also known as Laplacian growth. It states that the normal velocity of the boundary of a droplet occupying a simply connected domain \mathcal{D} is proportional to the gradient of a scalar function

$$\frac{1}{\hbar} \frac{\delta \vec{n}}{\delta N} \sim \vec{\nabla} \varphi(z), \quad (\text{VI.10})$$

which is harmonic outside the droplet and vanishes at its boundary

$$\Delta \varphi(z, \bar{z}) = 0, \quad z \in \mathbf{C} \setminus \mathcal{D}, \quad \varphi(z, \bar{z}) = 0, \quad z \in \partial \mathcal{D}. \quad (\text{VI.11})$$

In the matrix model this function appears as the following correlator

$$\varphi(z, \bar{z}) = \hbar \left\langle \text{tr} \left(\log(z - Z)(z - Z^\dagger) \right) \right\rangle. \quad (\text{VI.12})$$

It turns out that exactly the same law governs the dynamics of viscous flows. This phenomenon appears when an incompressible fluid with negligible viscosity is injected into a viscous fluid. In this case the harmonic function φ has a concrete physical meaning. It is identified with the pressure in the viscous fluid $\varphi = -P$. In fact, the Darcy's law is only

an approximation to a real evolution. Whereas the condition $P = 0$ in the incompressible fluid is reasonable, the vanishing of the pressure at the interface is valid only when the surface tension can be neglected. This approximation fails to be true when the curvature of the boundary becomes large. Then the dynamics is unstable and the incompressible fluid develops many fingers so that its shape looks as a fractal. This is known as the Saffman–Taylor fingering.

NMM provides a mathematical framework for the description of this phenomenon. From the previous discussion it is clear that it describes the interface dynamics in the large N limit. In this approximation the singularity corresponding to the described instabilities arises when the eigenvalue droplet forms a spike which we encountered already in the study of the Fermi sea of MQM (section V.1.4). We know that at this point the quasiclassical approximation is not valid anymore. But the full quantum description still exists. Therefore, it is natural to expect that the fingering, which is a feature of the Laplacian growth, can be captured by including next orders of the $1/N$ expansion of NMM.

Complex analysis

The Darcy’s law (VI.10) shows that there is a relation between NMM and several problems of complex analysis. Indeed, on the one hand, it can be derived from NMM as the evolution law of eigenvalue droplets and, on the other hand, it gives rise to a problem to find a harmonic function given by the domain in the complex plane. The latter problem appears in different contexts such as the conformal mapping problem, the Dirichlet boundary problem, and the 2D inverse potential problem [111]. For instance, if we fix the asymptotics of $\varphi(z, \bar{z})$ requiring that

$$\varphi(z, \bar{z}) \underset{z \rightarrow \infty}{\sim} \log |z|, \quad (\text{VI.13})$$

the solution of (VI.11) is unique and given by the holomorphic function $\omega(z)$

$$\varphi(z, \bar{z}) = \log |\omega(z)|, \quad (\text{VI.14})$$

which maps the domain \mathcal{D} onto the exterior of the unit circle and has infinity as a fixed point. To find such a function is the content of the conformal mapping problem.

Further, the Dirichlet boundary problem, which is to find a harmonic function $f(z, \bar{z})$ in the exterior domain given a function $g(z)$ on the boundary of \mathcal{D} , is solved in terms of the above defined $\omega(z)$. The solution is given by

$$f(z, \bar{z}) = -\frac{1}{\pi i} \oint_{\partial \mathcal{D}} g(\zeta) \partial_{\bar{\zeta}} G(z, \zeta) d\zeta, \quad (\text{VI.15})$$

where the Green function is

$$G(z, \zeta) = \log \left| \frac{\omega(z) - \omega(\zeta)}{\omega(z)\overline{\omega(\zeta)} - 1} \right|. \quad (\text{VI.16})$$

In turn, $\omega(z)$ is obtained as the holomorphic part of $G(z, \infty)$.

Finally, the inverse potential problem can be formulated as follows. Let the domain \mathcal{D} is filled by a charge spread with some density. The charge creates an electrostatic potential which is characterized by two functions φ_+ and φ_- defined in the interior and exterior

domains, respectively. They and their derivatives are continuous at the boundary $\partial\mathcal{D}$. The problem is to restore the form of the charged domain given one of these functions. At the same time, when both of them are known the task can be trivially accomplished. Therefore, the problem is equivalent to the question how to restore φ_- from φ_+ . Its relation to the Dirichlet boundary problem becomes now evident because, since φ_- is harmonic, it is given (up to a logarithmic singularity at infinity) by the formula (VI.15) with $g = \varphi_+$.

Thus, we see that the Normal Matrix Model provides a unified description for all these mathematical and physical problems. The main lesson which we learn from this is that all of them possess a hidden integrable structure revealed in NMM as Toda integrable hierarchy.

2 Dual formulation of compactified MQM

2.1 Tachyon perturbations of MQM as Normal Matrix Model

In section V.1 we showed how to introduce tachyon perturbations into the Matrix Quantum Mechanical description of 2D string theory. Although it is still the matrix model framework, we have done it in an unusual way. Instead to deform the matrix model potential, we have reduced MQM to the singlet sector and changed there the one-fermion wave functions. One can ask: can the resulting partition function be represented directly as a matrix integral?

In fact, the tachyon perturbations of MQM are quite similar to the perturbations of the two-matrix model. First, they are both described by Toda hierarchy. Second, the phase space of MQM looks as the eigenvalue (x, y) plane of 2MM. In the former case the fermions associated with the time dependent eigenvalues fill the non-compact Fermi sea, whereas in the latter case they fill some spots. If the 2MM potential is unstable like the inverse oscillator potential $-x_+x_-$, the spots will be non-compact as well. Thus, it is tempting to identify the two pictures.

Of course, MQM is much richer theory than 2MM and one may wonder how such different theories could be equivalent. The answer is that we have restricted ourselves just to a little sector of MQM. First, we use the restriction to the singlet sector and, second, we are interested only in the scattering processes. This explains why only two matrices appear. They are associated with in-coming and out-going states or, in other words, with $M(-\infty)$ and $M(\infty)$.

But it is easy to guess that the idea to identify MQM perturbed by tachyons with 2MM does not work. Indeed, the partition function of 2MM is a τ -function of Toda hierarchy when it is considered in the canonical ensemble. Therefore, one should find a representation of the grand canonical partition function of MQM which has a form of a canonical one. Such a representation does exist and is given by (V.32). But it implies a discrete equally spaced energy spectrum. It is evident for the system without perturbations where the partition function is a product of \mathcal{R} -factors (V.6) corresponding to $E_n = -\mu + i\hbar \frac{n+\frac{1}{2}}{R}$. However, it is difficult to obtain such a spectrum from two hermitian matrices. Moreover, one can show that their diagonalization would produce Vandermonde determinants of monomials of incorrect powers.

All these problems are resolved if one considers another model of two matrices which is the NMM. In the work [118] (Article IV) we proved that the grand canonical partition function of MQM with tachyon perturbations coincides with a certain analytical continuation of the canonical partition function of NMM. Thus, NMM can be regarded as a new realization of 2D string theory perturbed by tachyons.

Our proof is based on the fact that the two partition functions are τ -functions of Toda hierarchy. Therefore, it is enough to show that they are actually the same τ -function. This fact follows from the coincidence of either string equations or the initial conditions given by the non-perturbed partition functions. Finally, one should correctly identify the parameters of the two models.

We suggested two ways to identify the parameters. First, let us consider NMM given by the integral (VI.2) with the potential (VI.3) where $\gamma = \frac{1}{2}(R-1) + \frac{\alpha}{\hbar}$. We denote its partition

function by $\mathcal{Z}_h^{\text{NMM}}(N, t, \alpha)$. Then there is the following identification

$$\mathcal{Z}_h(\mu, t) = \lim_{N \rightarrow \infty} \mathcal{Z}_{i\hbar}^{\text{NMM}}(N, t, R\mu - i\hbar N). \quad (\text{VI.17})$$

We proved this result by direct comparison of two non-perturbed partition functions. Their coincidence is a consequence of the fact that the normalization coefficients (II.59) of the orthogonal polynomials in NMM coincide with the reflection coefficients (V.6)

$$h_n(\alpha) \sim \mathcal{R} \left(-i\hbar \frac{n + \frac{1}{2}}{R} - i \frac{\alpha}{R} \right). \quad (\text{VI.18})$$

Note that although the identification (VI.17) involves a large N limit, the relation is valid to all orders in the genus expansion. This is because N enters non-trivially through the parameter α . Actually, N appears always in a combination with μ . This is needed because the discrete charge of the τ -function is identified either with $i\mu$ or with N .

This fact hints that there should exist another way to match the two models in which these parameters are directly identified with each other. It is given by the second model proposed in Article IV

$$\mathcal{Z}_h(\mu, t) = \mathcal{Z}_{i\hbar}^{\text{NMM}} \left(-\frac{i}{\hbar} R\mu, t, 0 \right). \quad (\text{VI.19})$$

The equivalence of the two partition functions is proven in the same way as earlier. This second model is simpler than the first one because it does not involve the large N limit and one can compare the $1/N$ expansion of NMM directly with the $1/\mu$ expansion of MQM.

2.2 Geometrical description in the classical limit and duality

The relation of the perturbed MQM and NMM is a kind of duality. Most explicitly, this is seen in the classical limit where the both models have a geometrical description in terms of incompressible liquids. In the case of MQM, it describes the Fermi sea in the phase space parameterized by two real coordinates x_{\pm} , whereas in the case of NMM the liquid corresponds to the compact eigenvalue spots on the complex z -plane. Thus, the first conclusion is that the variables of one model are obtained from the variables of the other by an analytical continuation. The exact relation is the following

$$x_+ \leftrightarrow z^R, \quad x_- \leftrightarrow \bar{z}^R. \quad (\text{VI.20})$$

It relates all correlators in the two models if simultaneously one substitutes $\hbar \rightarrow i\hbar$ and $N = -iR\mu/\hbar$.

In particular, the analytical continuation (VI.20) replaces the non-compact Fermi sea of MQM by a compact eigenvalue spot of NMM. We have already discussed in the context of MQM that the profile of the Fermi sea is determined by the solution $x_-(x_+)$ (or or its inverse $x_+(x_-)$ depending on what asymptotics is considered) of the string equation (V.24). The same is true for the boundary of the NMM spot. Since the two models coincide, the string equations are also the same up to the change (VI.20). Nevertheless, they define different profiles. The MQM equation gives a non-compact curve and the NMM equation leads to a compact one. For example, when all $t_n = 0$ the two equations read

$$x_+ x_- = \mu, \quad (z\bar{z})^R = \hbar N/R \quad (\text{VI.21})$$

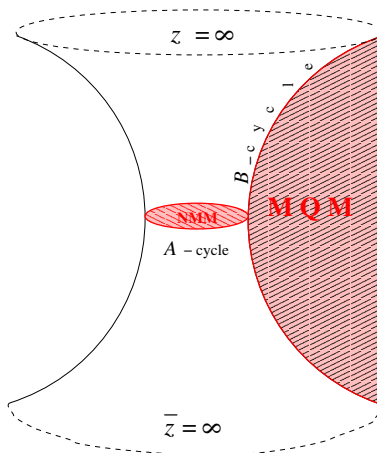


Fig. VI.1: Complex curve associated with both models, NMM and MQM. The regions filled by eigenvalues of the two models coincide with two real sections of the curve. The duality exchanges the A and B cycles which bound the filled regions.

and describe a hyperbola and a circle, correspondingly. This explicitly shows how the analytical continuation relates the Fermi seas of the two models.

A more transparent way to present this relation is to consider a complex curve associated with the solution in the classical limit. We showed in section II.4.5 how to construct such a curve for the two-matrix model. NMM does not differ from 2MM in this respect and the construction can be repeated in our case. The only problem is that for generic R the potential (VI.3) involves infinite branch singular point. Therefore, the curve given by the Riemann surface of the function $\bar{z}(z)$, which describes the shape of the eigenvalue spots, is not rational anymore and the results of [68] cannot be applied. However, the complex curve constructed as a "double" still exists and has the same structure as for the simple $R = 1$ case shown in fig. II.8.

In Article IV we considered the situation when there is only one simply connected domain filled by eigenvalues of the normal matrix. This restriction is due to the dual Fermi sea of MQM being simply connected. If we give up this restriction, it would correspond to excitations of MQM which break the Fermi sea to several components. They represent a very interesting issue to study but we have not considered them yet.

When there is only one spot, one gets the curve shown in fig. VI.1. It is convenient to think about it as a curve embedded into \mathbf{C}^2 . Let the coordinates of \mathbf{C}^2 are parameterized by z and \bar{z} . Then the embedding is defined by the function $\bar{z}(z)$ (or its inverse $z(\bar{z})$) which is a solution of the string equation. But we know that when z and \bar{z} are considered as complex conjugated this function defines also the boundary of the eigenvalue spot. At the same time if one takes z and \bar{z} to be real, the same function gives the profile of the Fermi sea of MQM. Thus, the two models are associated with two real sections of the same complex curve. Its intersection with the planes

$$z^* = \bar{z} \quad \text{and} \quad z^* = z, \quad \bar{z}^* = \bar{z} \quad (\text{VI.22})$$

coincides with the boundary of the region filled by eigenvalues of either NMM or MQM, respectively.

Moreover, the integrals over the conjugated cycles A and B , which give the moduli of the curve, also exhibit duality. To discuss them, one should define a holomorphic differential to be integrated along the cycles. In Article IV we derived it using the interpretation of the large N limit of NMM in terms of the inverse potential problem discussed in the previous section. It supplies us with the notion of electrostatic potential of a charged domain. It appears to be very natural in our context. For instance, the condition of the continuity of its derivatives at the boundary coincides with the string equation. But its main application is the construction of the holomorphic differential which is given by

$$\Phi \stackrel{\text{def}}{=} \begin{cases} \Phi_+(z) = \varphi(z) - \frac{1}{2}(z\bar{z}(z))^R & \text{in the north hemisphere,} \\ \Phi_-(\bar{z}) = -\bar{\varphi}(\bar{z}) + \frac{1}{2}(z(\bar{z})\bar{z})^R & \text{in the south hemisphere,} \end{cases} \quad (\text{VI.23})$$

where $\varphi(z)$ and $\bar{\varphi}(\bar{z})$ are the holomorphic and antiholomorphic parts of the potential. With this definition and using the relation of the zero mode of the electrostatic potential to the τ -function, one can prove the standard formulae

$$\frac{1}{2\pi i} \oint_A d\Phi = \hbar N = R\mu, \quad \int_B d\Phi = \hbar \frac{\partial}{\partial N} \log \mathcal{Z}^{\text{NMM}} = -\frac{1}{R} \frac{\partial \mathcal{F}}{\partial \mu}. \quad (\text{VI.24})$$

As the relations (VI.24) are written, the duality between MQM and NMM is not seen. It becomes evident if to remember that the free energies are taken in different ensembles. If one changes the ensemble, the cycles are exchanged. For example, for the canonical free energy of MQM defined as $F = \mathcal{F} + \hbar R\mu M$, where $M = -\frac{1}{\hbar R} \frac{\partial \mathcal{F}}{\partial \mu}$ is the number of eigenvalues, the relations (VI.24) take the following form

$$\frac{1}{2\pi i} \oint_A d\Phi = \frac{1}{\hbar} \frac{\partial F}{\partial M}, \quad \int_B d\Phi = \hbar M. \quad (\text{VI.25})$$

Thus, the duality between the two models is interpreted as the duality with respect to the exchange of the conjugated cycles on the complex curve.

We see that the fact that one compares the grand canonical partition function of one model with the canonical one of the other is very crucial. Actually, such kind of dualities can be interpreted as an electric-magnetic duality which replaces a gauge coupling constant by its inverse [126]. This idea may get a concrete realization in supersymmetric gauge theories. Their relation to matrix models was established recently in [45, 127]. This may open a new window for application of matrix models already in critical string theories.

Chapter VII

Non-perturbative effects in matrix models and D-branes

In all previous chapters we considered matrix models as a tool to produce perturbative expansions of two-dimensional gravity coupled to matter and of string theory in various backgrounds. However, it is well known that these theories exhibit non-perturbative effects which play a very important role. In particular, they are responsible for the appearance of D-branes and dualities between different string theories (see section I.2.3). Are matrix models able to capture such phenomena?

We saw that in the continuum limit matrix models can have several non-perturbative completions. For example, in MQM one can place a wall either at the top of the potential or symmetrically at large distances from both sides of it. In any case, the non-perturbative definition is always related to particularities of the regularization which is done putting a cut-off. Therefore, it is non-universal.

Nevertheless, it turns out that the non-perturbative completion of the perturbative results of matrix models is highly restricted. Actually, we expect that it is determined, roughly speaking, up to a coefficient. For example, if we consider the perturbative expansion of the free energy

$$F_{\text{pert}} = \sum_{g=0}^{\infty} g_{\text{str}}^{2g-2} f_g, \quad (\text{VII.1})$$

the general form of the leading non-perturbative corrections is the following

$$F_{\text{non-pert}} \sim C g_{\text{str}}^{f_A} e^{-\frac{f_D}{g_{\text{str}}}}, \quad (\text{VII.2})$$

where f_A and f_D can be found from the asymptotic behaviour of the coefficients f_g as the genus grows. The overall constant C is undetermined and reflects the non-universality of the non-perturbative effects.

On the other hand, the matrix model free energy should reproduce the partition function of the corresponding string theory. More precisely, its perturbative part describes the partition function of closed strings. At the same time, the non-perturbative corrections are associated with open strings with ends living on a D-brane. A particular source of non-perturbative terms of order $e^{-1/g_{\text{str}}}$ was identified with D-instantons — D-branes localized in spacetime [128].

The fact that the perturbative expansion contains already some information about non-perturbative terms has a clear explanation in string theory. In this language it means that closed string amplitudes can be simultaneously interpreted as open string amplitudes. In particular, the leading term in the exponent of (VII.2) should be given by a closed string diagram with one hole, which is a disk for the spherical approximation [128]. The boundary of the disk lives on a D-instanton implying the Dirichlet boundary conditions for all string coordinates.

Thus, in the two approaches one has more or less clear qualitative picture of how the non-perturbative corrections to the partition function arise. However, to find them explicitly, one should understand which D-instantons are to be taken into account and how to calculate the corrections using these D-instantons. This is not a hard problem in the critical string theory, whereas it remained unsolved for a long time for non-critical strings. The obstacle was that the D-instantons can be localized in the Liouville direction only in the strong coupling region. As a result, the perturbative expansion breaks down together with the description of these D-instantons.

A clue came with the work of A. and Al. Zamolodchikov [129] where the necessary D-branes were constructed in Liouville field theory. This opened the possibility to study non-perturbative effects in non-critical strings and to compare them with the matrix model results. In fact, such results existed only for some class of minimal ($c < 1$) models [130] described in the matrix approach, for example, using 2MM (see section II.4).

In the work [131] (Article VI) the matrix model results on non-perturbative corrections were extended to the case of the $c = 1$ string theory perturbed by windings. This was done using the Matrix Quantum Mechanical description of section IV.2. Namely, the leading non-perturbative contribution described by f_D in (VII.2) was calculated. In our case this coefficient is already a function of a dimensionless parameter composed from μ and the Sine–Liouville coupling λ . Also It was verified that in all cases, including both the minimal unitary models and the small coupling limit of the considered $c = 1$ string, these results can be reproduced from conformal field theory calculations where some set of D-branes is appropriately chosen.

Note that the matrix model analysis gives much more information than one can verify by the CFT methods. In particular, it gives some predictions for D-branes sitting at the tip of the cigar describing the Euclidean black hole background of two-dimensional string theory. Unfortunately, such non-perturbative effects are still inaccessible from the string theory side.

We do not give a detailed explanation of this paper here because it would require to introduce a lot of new notions and would take a lot of space. Therefore, we refer directly to Article VI.

Conclusion

We conclude this thesis by summarizing the main results achieved here and giving the list of the main problems, which either were not solved or not addressed at all, although their understanding would shed light on important physical issues.

1 Results of the thesis

- The two- and one-point correlators of winding modes at the spherical level in the compactified Matrix Quantum Mechanics in the presence of a non-vanishing winding condensate (Sine-Liouville perturbation) have been calculated (Article I).
- It has been shown how the tachyon perturbations can be incorporated into MQM. They are realized by changing the Hilbert space of the one-fermion wave functions of the singlet sector of MQM in such way that the asymptotics of the phases contains the perturbing potential. At the quasiclassical level these perturbations are equivalent to non-perturbative deformations of the Fermi sea which becomes time-dependent. The equation determining the exact form of the Fermi sea has been derived (Article II).
- When the perturbation contains only tachyons of discrete momenta as in the compactified Euclidean theory, it is integrable and described by the constrained Toda hierarchy. Using the Toda structure, the exact solution of the theory with the Sine-Liouville perturbation has been found (Article II). The grand canonical partition function of MQM has been identified as a τ -function of Toda hierarchy (Article IV).
- For the Sine-Liouville perturbation the energy, free energy and entropy have been calculated. It has been shown that they satisfy the standard thermodynamical relations what proves the interpretation of the parameter R of the perturbations in the Minkowski spacetime as temperature of the system (Article III).
- It has been proven that the tachyon perturbations cannot change the local structure (metric and dilaton) of the string target space. Nevertheless, they can change its global structure leading either to the appearance of boundaries or to a compactification (Article V).
- MQM with tachyon perturbations with equidistant spectrum has been proven to be equivalent to certain analytical continuation of the Normal Matrix Model. They coincide at the level of the partition functions and all correlators. In the quasiclassical limit this equivalence has been interpreted as a duality which exchanges the conjugated cycles of a complex curve associated with the solution of the two models. Physically this duality is of the electric-magnetic type (S-duality) (Article IV).

Conclusion

- The leading non-perturbative corrections to the partition function of 2D string theory perturbed by a source of winding modes have been found using its MQM description. In particular, from this result some predictions for the non-perturbative effects of string theory in the black hole background have been extracted (Article VI).
- The matrix model results concerning non-perturbative corrections to the partition function of the $c < 1$ unitary minimal models and the $c = 1$ string theory have been verified from the string theory side where they arise from amplitudes of open strings attached to D-instantons. Whenever this check was possible it showed excellent agreement of the matrix model and CFT calculations (Article VI).

2 Unsolved problems

- The first problem is the disagreement of the calculated (non-zero) one-point correlators with the CFT result that they should vanish. The most reasonable scenario is the existence of an operator mixing which includes also some of the discrete states. However, if this is indeed the case, by comparing with the CFT result one can only find the coefficients of this mixing. But it was not yet understood how to check this coincidence independently.
- Whereas we have succeeded to find the correlators of windings in presence of a winding condensate and to describe the T-dual picture of a tachyon condensate, we failed to calculate tachyon correlators in the theory perturbed by windings and *vice versa*. The reason is that the integrability seems to be lost when the two types of perturbations are included. Therefore, the problem is not solvable anymore by the present technique.
- On this way it would be helpful to find a matrix model incorporating both these perturbations. Of course, MQM does this, but we mean to represent them directly in terms of a matrix integral with a deformed potential. Such representation for windings was constructed as a unitary one-matrix integral, whereas for tachyon perturbations this task is accomplished by Normal Matrix Model. However, there is no matrix integral which was proven to describe both perturbations simultaneously.

Nevertheless, we hope that such matrix model exists. For example, in the CFT framework at the self-dual radius of compactification there is a nice description which includes both the winding and tachyon modes. It is realized in terms of a ground ring found by Witten. A similar structure should arise in the matrix model approach.

In fact, in the end of Article II we proposed a 3-matrix model which is supposed to incorporate both tachyon and winding perturbations. However, the status of this model is not clear up to now. The reason to believe that it works is based on the expectation that in the case when only one type of the perturbations is present, the matrix integral gives the corresponding τ -function of MQM. This is obvious for windings, but it is difficult to prove this statement for tachyons. It is not clear whether these are technical difficulties or they have a more deep origin.

- Studying the Das–Jevicki collective field theory, we saw that the discrete states are naturally included into the MQM description together with the tachyon modes. However, we realized only how to introduce a non-vanishing condensate of tachyons. We did not address the question how the discrete states can also be incorporated into the picture where they appear as a kind of perturbations of the Fermi sea.
- Also we did not consider seriously how the perturbed Fermi sea consisting from several simply connected components can be analyzed. Although a qualitative picture is clear, the exact mathematical description is not known yet. In particular, it would be interesting to generalize the duality of MQM and NMM to this multicomponent case.
- The next unsolved problem is related to the results obtained for the global structure of the target space corresponding to the tachyon perturbations. We discovered that it can be non-trivial. Is it indeed the case? What are the boundary conditions? What is the physics associated with them? All these questions have no answers up to now.

Conclusion

Also we should mention that these results were obtained in an approximation. The exact relation of the perturbed MQM to the target space physics should be found in the momentum representation. How does this change the results is an open question.

- Although it seems to be reasonable that the obtained non-trivial global structure of the target space can give rise to a finite temperature, this has not been demonstrated explicitly. This is related to a set of technical problems. However, the integrability of the system, which has already led to a number of miraculous coincidences, allows to hope that these problems can be overcome.
- One of the main unsolved problems is how to find the string background obtained by the winding condensation. In particular, one should reproduce the black hole target space metric for the simplest Sine–Liouville perturbation. Unfortunately, this has not been done. In principle, some information about the metric should be contained in the mixed correlators already mentioned here. But they have neither been calculated.

For the case of tachyon perturbations, the crucial role in establishing the connection with the target space physics is played by the collective field theory of Das and Jevicki. There is no analogous theory for windings. Its construction could lead to a real breakthrough in this problem.

- The thermodynamics represents one of the most interesting issues because we hope to describe the black hole physics. We have succeeded to analyze it in detail for the tachyon perturbations and even to find the entropy. However, we do not know yet how to identify the degrees of freedom giving rise to the entropy. Another way to approach this problem would be to consider the winding perturbations. But it is also unclear how to extract thermodynamical quantities from the dynamics of windings.
- All our results imply that it is very natural to consider the theory where all parameters like μ , λ and R are kept arbitrary. At the same time, from the CFT side a progress has been made only either for $\lambda = 0$ (the $c = 1$ CFT coupled with Liouville theory) or for $\mu = 0$, $R = 3/2$ (the Sine–Liouville theory coupled with gravity at the black hole radius). This is a serious obstacle for the comparison of the matrix model and CFT approaches.

In particular, we observe that from the matrix model point of view the values of the parameters corresponding to the black hole background of string theory are not distinguished anyhow. Therefore, we suppose that for other values the corresponding string background should have a similar structure. But the explicit form of this more general background has not yet been found.

- Finally, it is still a puzzle where the Toda integrable structure is hidden in the CFT corresponding to the perturbed MQM. In this CFT there are some infinite symmetries indicating the presence of such structure. But this happens only at the self-dual radius, whereas MQM does not give any restrictions on R . Probably the answer is in the operator mixing mentioned above because the disagreement in the one-point correlators found by two approaches cannot be occasional. Until this problem is solved, the understanding of the relation between both approaches will be incomplete.

We see that many unsolved problems wait for their solution. This shows that, in spite of the significant progress, 2D string theory and Matrix Quantum Mechanics continue to be a rich field for future research. Moreover, new unexpected relations with other domains of theoretical physics were recently discovered. And maybe some manifestations of the universal structure that describes these theories are not discovered yet and will appear in the nearest future.

Article I

Correlators in 2D string theory with vortex condensation

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Correlators in 2D string theory with vortex condensation

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We calculate one- and two-point correlators of winding operators in the matrix model of 2D string theory compactified on a circle, recently proposed for the description of string dynamics on the 2D black hole background.

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1 Introduction

The matrix quantum mechanics (MQM) approach to the 2D string theory has been proven very effective, in particular, in the case of the compactification on a circle of a radius R [1, 2]. Recently, it was shown [3] that this approach is capable to capture such important phenomena as the emergence of a black hole background and condensation of winding modes (vortices on the world sheet).

From the statistical-mechanical point of view such a model describes a system of planar rotators (XY -model) on a two-dimensional surface with fluctuating metric. In this sense it is a natural generalization of its analogue on the flat 2D space studied long ago by Berezinski, Kosterlitz and Thouless (BKT). The role of the fluctuating surfaces is played by the planar graphs of the MQM and the vortices appear due to the compactification of the time direction.

As it is usual for the matrix model approach, it enables us to apply the powerful framework of classical integrability to the computations of physical quantities, such as partition functions of various genera or correlators of physical operators.

The appropriate MQM model for the compactified 2D string is the inverted matrix oscillator with twisted boundary conditions [2]. The partition function of such a system as a function of couplings of the potential for the twist angles appears to be a τ -function of the Toda lattice hierarchy [4, 3]. As such, it satisfies an infinite set of partial difference-differential Hirota equations. The couplings play the role of “times” of commuting flows in the Toda hierarchy. As was shown in [3] they are at the same time the couplings of vortex operators of various vorticity charges. The tree approximation for the string theory corresponds to the dispersionless limit of Toda equations.

In this paper we will use Hirota equations to calculate all one-point and two-point correlators of vortex operators with arbitrary vorticity charges for the compactified 2D string in the tree-like approximation in the regime of the radius of compactification $\frac{1}{2}R_{KT} < R < R_{KT}$ (R_{KT} is the BKT radius), when condensation of vortices of the lowest charge $n = \pm 1$ takes place. Unlike the standard BKT model in flat 2D space, where the Debye screening in the vortex plasma destroys the long range correlations, the BKT vortices in the fluctuating 2D metric can be described as a conformal matter with central charge $c = 1$ coupled to the metric.

An important particular case of this model, for $R = \frac{3}{4}R_{KT}$, claimed in [3] to describe the 2D string in the dilatonic black hole background, due to the conjecture of [5] based on the results of [7, 6]. This conjecture establishes a kind of a weak-strong duality between two CFT’s: the dilatonic black hole and the so called Sine-Liouville theory. We will try to compare our results with a few calculations of correlators in the continuum CFT performed in [7, 6, 5] and discuss the difficulties of the direct comparison.

In the next section we briefly describe the basic ideas of the MQM approach to the 2D string theory with vortex excitations; the Toda integrable structure of the model allows to formulate the Hirota equations describing the dynamics of vortices. In section 3, using Hirota equations the two point correlators of arbitrary vorticities are expressed through the one point correlators. In section 4 the one point correlators are explicitly calculated. The explicit expressions for the two-point correlators can be found in section 5. Section 6 is devoted to the comparison with the results of continuous (Liouville) approach and to the conclusions.

2 Toda hierarchy for the compactified 2D string

2.1 Matrix model of 2D string and coupling to the windings

In this section we recall the main results of the paper [3] leading to the Toda hierarchy description for the compactified 2D string theory.

This theory is represented by the Polyakov string action:

$$S(X, g) = \frac{1}{4\pi} \int d^2\sigma \sqrt{\det g} [g^{ab} \partial_a X \partial_b X + \mu'], \quad (2.1)$$

where g_{ab} is a two dimensional fluctuating world sheet metric and the only bosonic field $X(\sigma)$ is compactified on a circle of a radius R :

$$X(\sigma) \sim X(\sigma) + 2\pi R. \quad (2.2)$$

Such a field admits in general Berezinski-Kosterlitz-Thouless vortices (windings) in its configurations. If we denote by $X^{n_{\pm 1}, n_{\pm 2}, \dots}(\sigma)$ a configuration of the bosonic field containing $n_{\pm 1}$ vortices of vorticity charge ± 1 , $n_{\pm 2}$ vortices of vorticity charge ± 2 , etc, the partition function of the theory in this sector will be

$$Z_{n_{\pm 1}, n_{\pm 2}, \dots} = \int Dg(\sigma) \int DX^{n_{\pm 1}, n_{\pm 2}, \dots}(\sigma) e^{-S(X, g)}. \quad (2.3)$$

Obviously, only the total vorticity $\sum_{k=-\infty}^{\infty} k n_k$ is conserved by topological reasons, but we will distinguish configurations with different individual vortices as point-like objects with vorticities k , to give a statistical-mechanical meaning to the system.

Instead of fixing the numbers of vortices n_k we can also introduce the partition function with fixed fugacities $\lambda'_{\pm 1}, \lambda'_{\pm 2}, \dots$ of the vortices with given charges:

$$Z_{\lambda'_{\pm 1}, \lambda'_{\pm 2}, \dots} = \sum_{n_{\pm 1}, n_{\pm 2}, \dots} \lambda'^{n_{\pm 1}}_{\pm 1} \lambda'^{n_{\pm 2}}_{\pm 2} \dots Z_{n_{\pm 1}, n_{\pm 2}, \dots} \quad (2.4)$$

It is well known that in the case of flat 2D space ($g_{ab} = \delta_{ab}$) such a system can be described in a dual, coulomb gas picture, in terms of a 2D field theory of one massless scalar field $X(\sigma)$ (not necessarily compactified) perturbed by vortex operators built out of this field. In the case of fluctuating metric this scalar field gets coupled to the Liouville field. The action (at least in the small coupling regime $\phi \rightarrow \infty$) looks as

$$S = \frac{1}{4\pi} \int d^2\sigma [(\partial X)^2 + (\partial \phi)^2 - 2\hat{\mathcal{R}}\phi + \mu' \phi e^{-2\phi} + \sum_{n \neq 0} \lambda'_n e^{(|n|R-2)\phi} e^{inR\tilde{X}}], \quad (2.5)$$

where $X = X_L + X_R$, $\tilde{X} = X_L - X_R$ (X_L and X_R are left and right movers). The total curvature is normalized as $\int d^2\sigma \sqrt{g} \hat{\mathcal{R}} = 4\pi(2-2h)$, where h is the genus of the surface. Note that we chose the central charge $c = 26$ and the Liouville “charges” of vortex operators to make them marginal. The compactification radius R remains a free parameter of the theory. If we choose the couplings as

$$\lambda'_n = \frac{\lambda'}{2} (\delta_{n,1} + \delta_{n,-1}), \quad (2.6)$$

the model will turn into the Sine-Gordon theory coupled to 2D gravity:

$$S_{SGG} = \frac{1}{4\pi} \int d^2\sigma [(\partial X)^2 + (\partial\phi)^2 - 2\hat{R}\phi + \mu'\phi e^{-2\phi} + \lambda' e^{(R-2)\phi} \cos R\tilde{X}]. \quad (2.7)$$

As its counterparts on the flat space, this theory is supposed to be in the same universality class as the compactified 2D string theory (2.1).

In particular case of vanishing cosmological constant μ' and $R = 3/2$ (here $R_{KT} = 2$) the CFT (2.7) was conjectured [5] to be dual to the coset CFT describing the string theory on the dilatonic black hole background [8, 9] (see [3] for the details).

The matrix model version of this theory is represented by the partition function of the MQM on a “time” circle of the radius R

$$Z_N[\lambda, g] = \int [d\Omega]_{SU(N)} \exp \left(\sum_{n \in \mathbb{Z}} \lambda_n \text{tr } \Omega^n \right) Z_N(\Omega, g), \quad (2.8)$$

where

$$Z_N(\Omega, g) = \int_{M(2\pi R) = \Omega^\dagger M(0)\Omega} \mathcal{D}M(X) e^{-\text{tr} \int_0^{2\pi R} dX [\frac{1}{2}(\partial_X M)^2 + V(M)]}. \quad (2.9)$$

We take the twisted boundary conditions for the hermitian matrix field $M(X)$, where the twist matrix Ω can be chosen without lack of generality as a Cartan element of the $SU(N)$ group $\Omega = \text{diag}(z_1, z_2, \dots, z_N)$. The matrix potential is, for example, $V(M) = \frac{1}{2}M^2 - \frac{q}{3\sqrt{N}}M^3$. The couplings λ_n will play the same role here as λ'_n in (2.5) although the exact relation between them depends on the regularization procedure (the shape of the potential $V(M)$).

Following the usual logic of the “old” matrix models, the Feynman expansion of this model can be shown (see [3] for the details) to describe the compactified lattice scalar field x_i living in the vertices $i = 1, 2, 3, \dots$ of ϕ^3 type planar graphs. This field can have vortex configurations: the vortices of charges $n = \pm 1, \pm 2, \dots$ occur on the faces of planar graphs and are weighted with the factors λ_n . So the model (2.8) represents a lattice analogue of the continuous partition function for the models (2.1), (2.4) or (2.5). The sum over planar graphs represents the functional integral over two dimensional metrics of the world sheet and due to the standard ‘t Hooft argument $1/N$ expansion goes over the powers $1/N^{2-2h}$ where h is a genus of the world sheet.

This model has a long history. In [10] some estimates of the matrix approach were compared to the predictions of (2.7) and appeared to be in full agreement. In [2] the model (2.1) was formulated in terms of a twisted inverted matrix oscillator (2.9). The Toda hierarchy description of the model (for the usual stable oscillatorial potential) was proposed in [4]. In the paper [11] the dual version of the matrix model of [1, 10], the Sine-Gordon model on random graphs, was used to find (or rather correctly conjecture) the free energy of (2.5) as a function of the cosmological constant μ' , Sine-Gordon coupling λ' and radius R .

The twisted inverted matrix oscillator of [2] appeared to be very useful for the identification of the free energy of the whole theory (2.5) and of the corresponding matrix model (2.8) with the τ -function of the Toda chain hierarchy of integrable PDF’s [3]. To see how this description emerges we pass to the grand canonical partition function:

$$\mathcal{Z}_\mu[\lambda, g] = \sum_{N=0}^{\infty} e^{-2\pi R \mu N} Z_N[\lambda, g]. \quad (2.10)$$

The chemical potential μ happens to be the cosmological constant similar to $\tilde{\mu}$ appearing in (2.7) (see [12] and [2] for the details). Then following the double scaling prescription for the $c = 1$ matrix model [13] corresponding to the continuous limit for the lattice world sheets we should send in (2.8) $N \rightarrow \infty$ and $g \rightarrow g_{crit}$ in such a way that at the saddle point of the sum over N in (2.10)

$$\mu = \frac{1}{2\pi R} \frac{\partial}{\partial N} \log Z_N[\lambda, g]$$

remains fixed. This amounts to shifting $M \rightarrow M + \frac{\sqrt{N}}{g}$ and neglecting the cubic term in the potential $V(M)$, or rather treating it as a cut-off wall at $M \rightarrow \infty$, which gives $V(M) \simeq -\frac{1}{2}M^2$. Then the integral [2] can be calculated exactly, giving:

$$Z_N[\lambda] = \frac{1}{N!} \oint \prod_{k=1}^N \frac{dz_k}{2\pi i z_k} \frac{e^{2u(z_k)}}{(q^{1/2} - q^{-1/2})} \prod_{j \neq j'}^N \frac{z_j - z_{j'}}{q^{1/2} z_j - q^{-1/2} z_{j'}}. \quad (2.11)$$

where $u(z) = \frac{1}{2} \sum_n \lambda_n z^n$, $q = e^{2\pi i R}$.

For the identification with τ -function it is convenient to redefine the vortex couplings as

$$t_n = \frac{\lambda_n}{q^{n/2} - q^{-n/2}}. \quad (2.12)$$

Then plugging (2.11) into (2.10) we recognize in it the τ -function of Toda lattice hierarchy [14, 15]

$$\tau_l[t] = e^{-\sum_n n t_n t_{-n}} \sum_{N=0}^{\infty} (q^l e^{-2\pi R \mu})^N Z_N[t] = e^{-\sum_n n t_n t_{-n}} \mathcal{Z}_{\mu-il}[t], \quad (2.13)$$

where the charge l giving an extra “lattice” dimension to the Toda equations appears to be an imaginary integer shift of the cosmological constant μ . The couplings t_n of vortices, together with $t_0 = \mu$ turn out to be the “times” of commuting flows of the hierarchy. Due to this the whole sector of the theory describing the dynamics of winding modes of the model (2.1) is completely solvable and the calculations of particular physical quantities, such as the correlators of vortex operators, are greatly simplified. In particular, the τ function (2.13) represents the generating function for such correlators in the theory (2.7):

$$\mathcal{K}_{i_1 \dots i_n} = \frac{\partial^n}{\partial \lambda_{i_1} \dots \partial \lambda_{i_n}} \log \tau_0 \Big|_{\lambda_{\pm 2} = \lambda_{\pm 3} = \dots = 0}. \quad (2.14)$$

with μ and $\lambda_{\pm 1}$ fixed.

Using the Hirota equations of Toda hierarchy we will calculate in this paper the one-point and two-point functions of vortex operators of arbitrary vorticities in the spherical approximation.

2.2 Hirota equations

One can show [16] that the ensemble of the τ -functions of the Toda hierarchy with different charges satisfies a set of bilinear partial differential equations known as Hirota equations

which are written as follows (the derivatives are taken with respect to t_n rather than λ_n)

$$\begin{aligned} \sum_{j=0}^{\infty} p_{j+i}(-2y_+)p_j(\tilde{D}_+) \exp\left(\sum_{k \neq 0} y_k D_k\right) \tau_{i+l+1}[t] \cdot \tau_l[t] = \\ \sum_{j=0}^{\infty} p_{j-i}(-2y_-)p_j(\tilde{D}_-) \exp\left(\sum_{k \neq 0} y_k D_k\right) \tau_{i+l}[t] \cdot \tau_{l+1}[t], \end{aligned} \quad (2.15)$$

where

$$y_{\pm} = (y_{\pm 1}, y_{\pm 2}, y_{\pm 3}, \dots), \quad (2.16)$$

$$\tilde{D}_{\pm} = (D_{\pm 1}, D_{\pm 2}/2, D_{\pm 3}/3, \dots) \quad (2.17)$$

represent the Hirota's bilinear operators

$$D_n f[t] \cdot g[t] = \frac{\partial}{\partial x} f(t_n + x) g(t_n - x) \Big|_{x=0}, \quad (2.18)$$

and p_j are Schur polynomials defined by

$$\sum_{k=0}^{\infty} p_k[t] x^k = \exp\left(\sum_{k=1}^{\infty} t_n x^n\right). \quad (2.19)$$

Since the τ -function of the Toda hierarchy is related to the free energy by

$$\tau_s[\mu, t] = \exp(\mathcal{F}(\mu - is)), \quad (2.20)$$

the Hirota equations lead to a triangular system of nonlinear difference-differential equations for the free energy of the model (2.10)-(2.11). Due to (2.14) it can be actually thought of as a system of equations for the correlators which we are interested in.

2.3 Scaling of winding operators

It turns out that there is a scale in the model which corresponds to the scale given by the string coupling constant in the string theory. Since $g_s \sim e^{\langle \phi \rangle}$ it can be associated with the Liouville field ϕ or the cosmological constant μ' coupled with it. All couplings λ'_n have definite dimensions with respect to this scale. In the Toda hierarchy the counterparts of μ' and λ'_n are μ and t_n correspondingly. As one can immediately see from (2.5) by shifting the zero mode of the Liouville field, the corresponding dimensions of the couplings t_n with respect to the rescaling of the cosmological constant μ are

$$\Delta[t_n] = 1 - \frac{R|n|}{2}. \quad (2.21)$$

This scaling leads to an expansion for the free energy which can be interpreted as its genus expansion

$$\mathcal{F} = \sum_{h=0}^{\infty} \mathcal{F}_h, \quad (2.22)$$

$$\mathcal{F}_0 = \xi^{-2} f_0(w; s) + \frac{R}{2} \mu^2 \log \xi, \quad (2.23)$$

$$\mathcal{F}_0 = f_1(w; s) + \frac{R + R^{-1}}{24} \log \xi, \quad (2.24)$$

$$\mathcal{F}_h = \xi^{2h-2} f_h(w; s), \quad h > 1, \quad (2.25)$$

where the quantities f_h are the functions of the dimensionless parameters

$$s = (s_{\pm 2}, s_{\pm 3}, \dots), \quad s_n = i \left(-\frac{t_{-1}}{t_1} \right)^{n/2} \xi^{\Delta[t_n]} t_n, \quad (2.26)$$

$$w = \mu \xi, \quad \text{with} \quad \xi = (\lambda \sqrt{R-1})^{-\frac{2}{2-R}} = (t_1 t_{-1} (R-1))^{-\frac{1}{2-R}}. \quad (2.27)$$

Remarkably, the Toda equations (2.15) are compatible with this scaling.

The spherical limit of the string partition function is described by the dispersionless Toda hierarchy which can be obtained taking $\xi \rightarrow 0$. In this limit the explicit expression for the partition function with vanishing s_n has been conjectured in [11] and proven in [3]. The result formulated in terms of $X_0(w) = \partial_w^2 f_0(w)$ reads

$$w = e^{-\frac{1}{R} X_0} - e^{-\frac{R-1}{R} X_0} \quad (2.28)$$

or, integrating X_0 twice over w

$$\mathcal{F}_0 = \frac{1}{2} \mu^2 (R \log \xi + X_0) + \xi^{-2} \left(\frac{3}{4} \frac{R}{R-1} e^{-2\frac{R-1}{R} X_0} + \frac{3}{4} R e^{-\frac{2}{R} X_0} - \frac{R^2 - R + 1}{R-1} e^{-X_0} \right). \quad (2.29)$$

This result will serve as an input for the following calculation of the correlators in the spherical limit.

3 Two-point correlators from the Toda hierarchy

3.1 Equation for the generating function of two-point correlators

To find equations for the two-point correlators in the spherical limit let us take in (2.15) $i = 0$ and the coefficient in front of $y_n y_m$, $n, m > 0$. Then we obtain the following equation

$$\begin{aligned} \left[4p_{n+m}(\tilde{D}_+) - 2p_n(\tilde{D}_+)D_m - 2p_m(\tilde{D}_+)D_n + D_n D_m \right] \tau_{s+1} \cdot \tau_s &= D_n D_m \tau_s \cdot \tau_{s+1}, \\ \left[2p_{n+m}(\tilde{D}_+) - p_n(\tilde{D}_+)D_m - p_m(\tilde{D}_+)D_n \right] \tau_{s+1} \cdot \tau_s &= 0. \end{aligned} \quad (3.1)$$

This equation is valid also at $m = 0$ or $n = 0$ if we set $D_0 \equiv 1$. Such an equation can be obtained from the coefficient in front of y_n in (2.15).

If we multiply the equation (3.1) by $x^n y^m$ and sum over all n and m we obtain

$$\begin{aligned} \left[2 \sum_{n,m=0}^{\infty} x^n y^m p_{n+m}(\tilde{D}_+) - \sum_{m=0}^{\infty} y^m D_m \exp \left\{ \sum_{n=1}^{\infty} x^n \tilde{D}_n \right\} \right. \\ \left. - \sum_{n=0}^{\infty} x^n D_n \exp \left\{ \sum_{m=1}^{\infty} y^m \tilde{D}_m \right\} \right] \tau_{s+1} \cdot \tau_s = 0. \end{aligned} \quad (3.2)$$

Furthermore, it is easy to see that the first term can be rewritten as follows:

$$\frac{2}{x-y} \left[x e^{\sum_{k=1}^{\infty} x^k \tilde{D}_k} - y e^{\sum_{k=1}^{\infty} y^k \tilde{D}_k} \right] \tau_{s+1} \cdot \tau_s. \quad (3.3)$$

It is easy to check the following remarkable formula

$$f(D)e^F \cdot e^G = e^F e^G f(D + (\partial F \cdot 1 - 1 \cdot \partial G)), \quad (3.4)$$

which allows to express τ -functions in the equation (3.2), (3.3) through the derivatives of the free energy:

$$\begin{aligned} & \frac{2}{x-y} \left(\sum_{k=1}^{\infty} x^k (\tilde{D}_k + \tilde{X}_k) - y \sum_{k=1}^{\infty} y^k (\tilde{D}_k + \tilde{X}_k) \right) = \\ & \left(1 + \sum_{m=1}^{\infty} y^m (D_m + X_m) \right) \sum_{k=1}^{\infty} x^k (\tilde{D}_k + \tilde{X}_k) + \left(1 + \sum_{n=1}^{\infty} x^n (D_n + X_n) \right) \sum_{k=1}^{\infty} y^k (\tilde{D}_k + \tilde{X}_k), \quad (3.5) \end{aligned}$$

where we have introduced the notation $\tilde{X}_k = \frac{1}{|k|} X_k$ and¹

$$X_k = \partial_k \mathcal{F}_{\frac{1}{2}} \cdot 1 - 1 \cdot \partial_k \mathcal{F}_{-\frac{1}{2}}, \quad k \neq 0. \quad (3.6)$$

The 1's in the right hand side of (3.5) arise from the definition of D_0 .

We show in Appendix A that in the dispersionless limit only the second derivatives of the free energy survive. This means that we can put the second and higher derivatives of X_k to zero: $D^n X_k = 0$, $n > 1$. Besides, using (2.20) we obtain in this limit

$$\tilde{X}_n = -i \frac{1}{|n|} \frac{\partial^2}{\partial t_n \partial \mu} \mathcal{F}_0 := \tilde{X}_{0,n}, \quad (3.7)$$

$$\tilde{D}_n \tilde{X}_m = 2 \frac{1}{|n||m|} \frac{\partial^2}{\partial t_n \partial t_m} \mathcal{F}_0 := \tilde{X}_{n,m}. \quad (3.8)$$

Thus the fields $\tilde{X}_{n,m}$ correspond (up to a numerical factor) to the two-point correlators, whereas $\tilde{X}_{0,n}$ give the one-point ones. We also define the generating function of all \tilde{X} 's

$$F(x, y) = \sum_{n,m=0}^{\infty} x^n y^m \tilde{X}_{n,m}, \quad (3.9)$$

where we have taken $\tilde{X}_{0,0} = 0$.

With these definitions, trivial manipulations which can be found in Appendix B, lead to the following equation for the generating function

$$\frac{x+y}{x-y} \left(e^{\frac{1}{2}F(x,x)} - e^{\frac{1}{2}F(y,y)} \right) = x \partial_x F(x, y) e^{\frac{1}{2}F(y,y)} + y \partial_y F(x, y) e^{\frac{1}{2}F(x,x)}. \quad (3.10)$$

¹We can shift the index s of the free energy $\mathcal{F}_s = \log \tau_s$ since due to (2.20) the resulting equation depends only on the difference of the indices of two τ -functions. This index should not be mixed with the index h indicating the genus. In the following only the spherical free energy \mathcal{F}_0 will appear.

3.2 Solution in terms of one-point correlators

It turns out that the differential equation (3.10) is solvable. First of all, change the variables to

$$\begin{aligned} d\eta &= e^{\frac{1}{2}F(x,x)} \frac{dx}{x} + e^{\frac{1}{2}F(y,y)} \frac{dy}{y}, \\ d\zeta &= e^{\frac{1}{2}F(x,x)} \frac{dx}{x} - e^{\frac{1}{2}F(y,y)} \frac{dy}{y}. \end{aligned} \quad (3.11)$$

It is easy to see that

$$\zeta = \int_y^x \frac{dz}{z} e^{\frac{1}{2}F(z,z)} = \log \frac{x}{y} + h(x) - h(y), \quad (3.12)$$

where

$$h(x) = \int_0^x \frac{dz}{z} \left(e^{\frac{1}{2}F(z,z)} - 1 \right) \quad (3.13)$$

is a regular function at $x = 0$.

In terms of the new variables the equation reads

$$\partial_\eta F(x, y) = \frac{1}{2} \frac{x+y}{x-y} \left(e^{-\frac{1}{2}F(y,y)} - e^{-\frac{1}{2}F(x,x)} \right) = \partial_\eta \log \frac{xy}{(x-y)^2}. \quad (3.14)$$

Thus

$$F(x, y) = \log \frac{xy}{(x-y)^2} + g(\zeta), \quad (3.15)$$

where the function $g(\zeta)$ is fixed by the requirement for $F(x, y)$ to be analytical in both arguments.

The analyticity condition together with (3.12) leads to the following requirements on $g(\zeta)$: i) $g(\zeta) \underset{\zeta \rightarrow \pm\infty}{\sim} \pm\zeta$; ii) $g(\zeta) \underset{\zeta \rightarrow 0}{\sim} \log \zeta^2$. There is only one function satisfying both of them, which is given by

$$g(\zeta) = \log \left(e^\zeta + e^{-\zeta} - 2 \right) = \log \left(4 \operatorname{sh}^2 \left(\frac{\zeta}{2} \right) \right). \quad (3.16)$$

Taking together (3.15), (3.16) and (3.12) we obtain the solution for the generating function of the two-point correlators in terms of a still unknown regular function $h(x)$

$$F(x, y) = \log \left[\frac{4xy}{(x-y)^2} \operatorname{sh}^2 \left(\frac{1}{2} (h(x) - h(y) + \log \frac{x}{y}) \right) \right]. \quad (3.17)$$

From (3.17) it is easy to see that $h(x)$ is nothing else than the generating function for the one-point correlators since $h(x) = F(x, 0)$.

3.3 Two-point correlators with vorticities of opposite signs

Up to now we considered the two-point correlators for vorticities of positive sign only. Now we generalize the results for other cases.

First of all, the case of both negative signs can be directly obtained from the previous one. The corresponding equation which is derived from the coefficient in front of $y_{-n}y_{-m}$ in the Hirota equation looks as (3.1)

$$\left[2p_{n+m}(\tilde{D}_-) - p_n(\tilde{D}_-)D_{-m} - p_m(\tilde{D}_-)D_{-n}\right] \tau_s \cdot \tau_{s+1} = 0. \quad (3.18)$$

The only difference is that τ_s and τ_{s+1} are exchanged. Due to this there is an additional minus in the definition of $\tilde{X}_{0,-n}$. So we should distinguish two types of correlators

$$\tilde{X}_{0,n}^\pm := \mp i \frac{1}{|n|} \frac{\partial^2}{\partial t_n \partial \mu} \mathcal{F}_0, \quad n \neq 0, \quad (3.19)$$

$$\tilde{X}_{n,m}^\pm := 2 \frac{1}{|n||m|} \frac{\partial^2}{\partial t_n \partial t_m} \mathcal{F}_0, \quad n, m \neq 0. \quad (3.20)$$

With this definition, the generating functions of the correlators for positive and negative vorticities are

$$F^\pm(x, y) = \sum_{n,m=0}^{\infty} x^n y^m \tilde{X}_{\pm n, \pm m}^\pm, \quad (3.21)$$

Despite of this minus all equations for the case of negative vorticities being written in terms of $\tilde{X}_{n,m}^-$ are the same as for the case of positive ones. As a result $F^-(x, y)$ is given by the same solution (3.17) as $F^+(x, y)$. The only difference between them is the generating function of the one-point correlators h which should be replaced in (3.17) by h^\pm correspondingly.

To find the correlators with vorticities of opposite signs, take the coefficient in the Hirota equation in front of $y_n y_{-m}$ and set $i = 1$. Then we obtain the following equation

$$\begin{aligned} -2p_n(\tilde{D}_+)D_{-m}\tau_{s+2} \cdot \tau_s &= \left[-2p_m(\tilde{D}_-)D_n + \tilde{D}_{-1}D_{-m}D_n\right] \tau_{s+1} \cdot \tau_{s+1}, \\ p_n(\tilde{D}_+)D_{-m}\tau_{s+2} \cdot \tau_s &= p_m(\tilde{D}_-)D_n\tau_{s+1} \cdot \tau_{s+1}. \end{aligned} \quad (3.22)$$

The last term in the first line vanishes since it always gives rise to $D_k(\mathcal{F}_{s+1} \cdot 1 - 1 \cdot \mathcal{F}_{s+1}) = 0$. In general, the one-point correlators appearing from $\tau_{s+k} \cdot \tau_s$ are always supplied with the coefficient k what follows from (2.20). Due to this in the right hand side there are no one-point correlators whereas in the left hand side they enter with the coefficient 2. This equation is valid also for $m = 0$ if we take $D_0 \equiv -2$ instead of the previous choice. However, for $n = 0$ it is never fulfilled.

The arguments similar to those of Section 1 lead to the following equation for the generating function

$$A \left[y \partial_y (G(x, y) - 2h^-(y)) - 2 \right] e^{\frac{1}{2}F^+(x,x)+h^+(x)} = \frac{1}{y} \partial_x G(x, y) e^{\frac{1}{2}F^-(y,y)-h^-(y)}, \quad (3.23)$$

where we have introduced

$$A = \exp \left(-\partial_\mu^2 \mathcal{F}_0 \right) = \xi^{-R} e^{-\partial_w^2 f_0}, \quad (3.24)$$

$$G(x, y) = \sum_{n,m=1}^{\infty} x^n y^m \tilde{X}_{n,-m}. \quad (3.25)$$

Using (3.13) we can rewrite (3.23) as follows

$$\left(\left[A \partial_x (x e^{h^+(x)}) \right]^{-1} \partial_x + \left[\partial_y \left(\frac{1}{y} e^{-h^-(y)} \right) \right]^{-1} \partial_y \right) G(x, y) = -2y e^{h^-(y)}. \quad (3.26)$$

It is clear that the solution of this equation looks as

$$G(x, y) = 2 \log \left(y e^{h^-(y)} \right) + f \left(\frac{1}{y} e^{-h^-(y)} - A x e^{h^+(x)} \right). \quad (3.27)$$

The unknown function f is determined by the analyticity of $G(x, y)$. Requiring cancelation of the logarithmic singularity at $y = 0$ we obtain that $f(z) = 2 \log z$. As a result we find the generating function

$$G(x, y) = 2 \log \left(1 - A x y e^{h^+(x) + h^-(y)} \right). \quad (3.28)$$

Let us note that the equations (3.17) and (3.28) resemble the equations for two-point correlators of the two-matrix model found in [17].

4 One-point correlators

4.1 Equation for the generating function

Now we should work out an equation for the one-point correlators. We can always put $t_1 = -t_{-1}$ (corresponding to $\lambda_1 = \lambda_{-1}$, see (2.12)). From the equations written below for positive vorticities it is clear that in this case $h^\pm(x)$ coincide with each other. For this reason we omit in the following the inessential sign label in $h(x)$. The case of general t 's can be obtained from the previous one by the substitution $h^\pm(x) = h \left((-t_{-1}/t_1)^{\pm 1/2} x \right)$. It can be easily seen from (2.26) and is due to the total vorticity conservation. Accordingly,

$$F^\pm(x, y) = F \left((-t_{-1}/t_1)^{\pm 1/2} x, (-t_{-1}/t_1)^{\pm 1/2} y \right), \quad (4.1)$$

$$G(x, y) = G_{t_1 = -t_{-1}} \left((-t_{-1}/t_1)^{1/2} x, (-t_{-1}/t_1)^{-1/2} y \right). \quad (4.2)$$

The equation we are looking for arises from two facts: i) we know the dependence of the free energy on $t_{\pm 1}$ and μ (2.28), (2.29); ii) the system of equations for the correlators is triangular. Due to this we can express $\tilde{X}_{\pm 1, n}$ through $\tilde{X}_{0, n}$ by a linear integral-differential operator. Namely, in terms of the generating functions we can write

$$\partial_y F^+(x, 0) = \hat{K}^{(+)} h(x) + \tilde{X}_{1, 0}^+, \quad (4.3)$$

$$\partial_y G(x, 0) = \hat{K}^{(-)} h(x), \quad (4.4)$$

where the operators $\hat{K}^{(\pm)}$ are to be found. The last term in the first line is explicitly added since in our notations $F^+(0, 0) = \tilde{X}_{0, 0}^+ = 0$. Moreover, we can rewrite these equations in terms of the generating function for the one-point correlators only. Indeed, from (3.17) and (3.28) we obtain

$$\hat{K}^{(+)} h(x) = \frac{2}{x} \left(1 - e^{-h(x)} \right) - 2 \tilde{X}_{0, 1}^+, \quad (4.5)$$

$$\hat{K}^{(-)} h(x) = -2 A x e^{h(x)}. \quad (4.6)$$

To find the operators $\hat{K}^{(\pm)}$ we compare the derivatives of the free energy with respect to μ and $t_{\pm 1}$. From (2.23), (2.26) and (2.27) we obtain for $|n| > 1$

$$\left. \frac{\partial}{\partial \mu} \left(\frac{\partial}{\partial t_n} \mathcal{F}_0 \right) \right|_{s_k=0} = i \xi^{-\frac{|n|R}{2}} \frac{\partial}{\partial w} \left(\frac{\partial}{\partial s_n} f_0(w; s) \right) \Big|_{s_k=0}, \quad (4.7)$$

$$\left. \frac{\partial}{\partial t_{\pm 1}} \left(\frac{\partial}{\partial t_n} \mathcal{F}_0 \right) \right|_{s_k=0} = \mp \frac{\xi^{-\frac{(|n|+1)R}{2}}}{2-R} \left(w \frac{\partial}{\partial w} - \left(1 + \frac{R|n|}{2} \pm \frac{(R-2)n}{2} \right) \right) \left(\frac{\partial}{\partial s_n} f_0(y; s) \right) \Big|_{s_k=0} \quad (4.8)$$

These relations together with (3.7) and (3.8) imply that the operators are given by

$$\hat{K}^{(+)} = -a \left[w - (1 + (R-1)x) \frac{\partial}{\partial x} \int^w dw \right], \quad (4.9)$$

$$\hat{K}^{(-)} = a \left[w - (1 + x) \frac{\partial}{\partial x} \int^w dw \right], \quad (4.10)$$

where $a = 2 \frac{\sqrt{R-1}}{2-R} \xi^{-\frac{R}{2}}$. This expression is also valid for the case $n = \pm 1$ where the derivatives of the free energy contain additional terms appearing from the logarithmic term in (2.23).

Inserting (4.9) and (4.10) into (4.5) and (4.6) and taking the derivative with respect to w we obtain two differential equations for the generating function of the one-point correlators

$$\left[-a \left((R-1)x \frac{\partial}{\partial x} - w \frac{\partial}{\partial w} \right) + \frac{2}{x} e^{-h(x;w)} \frac{\partial}{\partial w} \right] h(x; w) = 2 \frac{\partial}{\partial w} \tilde{X}_{0,1}^+, \quad (4.11)$$

$$\left[a \left(x \frac{\partial}{\partial x} - w \frac{\partial}{\partial w} \right) - 2Ax e^{h(x;w)} \frac{\partial}{\partial w} \right] h(x; w) = 2x e^{h(x;w)} \frac{\partial}{\partial w} A. \quad (4.12)$$

4.2 Solution

Let us take advantage of having two equations for one quantity. Using (4.11) and (4.12) we can exclude the derivative with respect to x . The resulting equation is

$$\left(a(2-R)w + 2 \frac{e^{-h}}{x} - 2(R-1)Ax e^h \right) \partial_w h = 2 \partial_w \tilde{X}_{0,1}^+ + 2(R-1) \partial_w Ax e^h. \quad (4.13)$$

From (3.19), (2.23) and (2.29) one can obtain

$$\begin{aligned} \tilde{X}_{0,1}^+ &= -i \frac{\partial^2}{\partial \mu \partial t_1} \mathcal{F}_0 \Big|_{s_k=0} = \frac{\sqrt{R-1}}{2-R} \xi^{-R/2} ((w \partial_w - 1) \partial_w f_0(w) + R w) \\ &= \frac{R}{\sqrt{R-1}} \xi^{-R/2} e^{-\frac{R-1}{R} X_0}. \end{aligned} \quad (4.14)$$

Noting that the equation (4.13) is homogeneous in the derivative ∂_w we can change it by ∂_{X_0} . Then after the substitution of the definitions of all entries the common multiplier

$$2\sqrt{R-1} \xi^{-R/2} \left(e^{-\frac{R-1}{R} X_0} + \sqrt{R-1} \xi^{-R/2} x e^{h-X_0} \right)$$

can be canceled. As a result we obtain the following simple equation

$$\left(1 - \frac{\xi^{R/2}}{x\sqrt{R-1}} e^{\frac{R-1}{R}X_0} e^{-h}\right) \partial_{X_0} h = 1. \quad (4.15)$$

This equation can be trivially integrated giving after the proper choice of the integration constant

$$e^{\frac{1}{R}h} - ze^h = 1, \quad (4.16)$$

where $z = \frac{\xi^{-R/2}}{\sqrt{R-1}} x e^{-\frac{R-1}{R}X_0}$. As it is easy to check this solution satisfies both equations (4.11) and (4.12).

4.3 One-point correlators of fixed vorticities

Explicit expressions for the one-point correlators are given by the coefficients of the expansion in x by the generating function $h(x)$ satisfying equation (4.16). To find them we can use equation 5.2.13.30 in [18]

$$\frac{1}{b} s^b = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma(na+b)}{\Gamma(n(a-1)+b+1)}, \quad (4.17)$$

$$s = 1 + zs^a. \quad (4.18)$$

Taking $s = e^{\frac{1}{R}h}$, $a = R$ we obtain for $b = kR$

$$e^{kh} = kR \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma((n+k)R)}{\Gamma((n+k)R-n+1)}. \quad (4.19)$$

Also the limit $b \rightarrow 0$ gives

$$h = R \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{\Gamma(nR)}{\Gamma(n(R-1)+1)}. \quad (4.20)$$

This means that the derivative with respect to μ of the one-point correlator is

$$\frac{\partial^2}{\partial \mu \partial t_n} \mathcal{F}_0 = \frac{i\Gamma(nR+1)}{n!\Gamma(n(R-1)+1)} \frac{\xi^{-nR/2}}{(R-1)^{n/2}} e^{-n\frac{R-1}{R}X_0}. \quad (4.21)$$

After integration over μ and taking into account the relation (2.12) we obtain the following one-point correlators of operators of vorticity n in the spherical limit

$$\mathcal{K}_n = \frac{1}{2 \sin \pi n R} \frac{\Gamma(nR+1)}{n!\Gamma(n(R-1)+1)} \frac{\xi^{-\frac{nR+2}{2}}}{(R-1)^{n/2}} \left(\frac{e^{-\frac{n(R-1)+1}{R}X_0}}{n(R-1)+1} - \frac{e^{-(n+1)\frac{R-1}{R}X_0}}{n+1} \right). \quad (4.22)$$

(Here the lower limit of the integration is chosen to be $X_0 = +\infty$. Only this choice reproduces $\partial_{\lambda_1} \mathcal{F}_0$ from (2.29).) In particular, in the limit $\mu \rightarrow 0$ ($X_0 \rightarrow 0$) corresponding to string theory on the black hole background (in fact, only the point $R = 3/2$ is supposed to describe the conventional black hole) we find

$$\mathcal{K}_n = \frac{1}{2 \sin \pi n R} \frac{\xi^{-\frac{nR+2}{2}}}{(R-1)^{n/2}} \frac{(2-R)n\Gamma(nR+1)}{(n+1)!\Gamma(n(R-1)+2)}. \quad (4.23)$$

5 Results for two-point correlators

5.1 Correlators with vorticities of opposite signs

These correlators are given by the expansion coefficients of the generating function (3.28). This function can be easily expanded in e^h for which the series (4.19) can be used. This leads to the following expansion

$$\begin{aligned} G(x, y) &= -2 \sum_{k=1}^{\infty} \frac{B^k}{k} z^k(x) z^k(y) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n^k C_m^k z^n(x) z^m(y) \\ &= -2 \sum_{n,m=1}^{\infty} z^n(x) z^m(y) \sum_{k=1}^{\min(n,m)} \frac{B^k}{k} C_{n-k}^k C_{m-k}^k, \end{aligned} \quad (5.1)$$

where

$$B = A(R-1) \xi^R e^{2 \frac{R-1}{R} X_0} = (R-1) e^{\frac{R-2}{R} X_0}, \quad (5.2)$$

$$C_n^k = \frac{kR}{n!} \frac{\Gamma((n+k)R)}{\Gamma(n(R-1) + kR + 1)}. \quad (5.3)$$

The two-point correlators appear to be

$$\begin{aligned} \mathcal{K}_{n,-m} &= -\frac{\Gamma(nR+1) \Gamma(mR+1)}{2 \sin \pi nR \ 2 \sin \pi mR} \frac{\xi^{-(n+m)R/2}}{(R-1)^{(n+m)/2}} e^{-(n+m) \frac{R-1}{R} X_0} \times \\ &\times \sum_{k=1}^{\min(n,m)} \frac{k(R-1)^k e^{k \frac{R-2}{R} X_0}}{(n-k)!(m-k)! \Gamma(n(R-1) + k + 1) \Gamma(m(R-1) + k + 1)}. \end{aligned} \quad (5.4)$$

To get the result in the black hole limit it is enough to set $X_0 = 0$ in the above expression.

It is worth to give also the result for the ratios of correlators which do not depend on possible leg-factors — wave function renormalisations of the operators (see for ex. [12]):

$$\frac{\mathcal{K}_{n,-m}}{\mathcal{K}_n \mathcal{K}_{-m}} = -\frac{\xi^2}{(2-R)^2 n m} \sum_{k=1}^{\min(n,m)} \frac{k(R-1)^k (n+1)!(m+1)! \Gamma(n(R-1) + 2) \Gamma(m(R-1) + 2)}{(n-k)!(m-k)! \Gamma(n(R-1) + k + 1) \Gamma(m(R-1) + k + 1)}. \quad (5.5)$$

Since they can in principle be compared with CFT calculations in the black hole limit we restrict ourselves only to this particular case.

5.2 Correlators with vorticities of same signs

To investigate this case we should expand (3.17) in x and y . The expansion in the first argument gives for $n > 0$

$$\begin{aligned} \partial_x^n F(0, y) &= \frac{2}{y^n} \left(\frac{1}{n} - \sum_{k=1}^n \frac{2}{k} z^{n-k}(y) e^{-kh(y)} C_{n-k}^k \right) - \tilde{X}_{0,n} \\ &= -2 \frac{\xi^{-nR/2}}{(R-1)^{n/2}} e^{-n \frac{R-1}{R} X_0} \sum_{m=0}^{\infty} z^m(y) \sum_{k=1}^n \frac{1}{k} C_{n-k}^k C_{m+k}^{-k} - \tilde{X}_{0,n} \\ &+ \frac{2}{y^n} \left(\frac{1}{n} - \sum_{m=1}^n z^{n-m}(y) \sum_{k=m}^n \frac{1}{k} C_{n-k}^k C_{k-m}^{-k} \right). \end{aligned} \quad (5.6)$$

The last term in the right hand side is singular and we know from the analyticity of $F(x, y)$ that it should vanish. The remaining part gives the values of the two-point correlators

$$\begin{aligned} \mathcal{K}_{n,m} &= -\frac{\Gamma(nR+1)}{2 \sin \pi nR} \frac{\Gamma(mR+1)}{2 \sin \pi mR} \frac{\xi^{-(n+m)R/2}}{(R-1)^{(n+m)/2}} e^{-(n+m)\frac{R-1}{R}X_0} \times \\ &\times \sum_{k=1}^n \frac{k}{(n-k)!(m+k)!\Gamma(n(R-1)+k+1)\Gamma(m(R-1)-k+1)}. \end{aligned} \quad (5.7)$$

As in the previous case the black hole limit of (5.7) is obtained vanishing X_0 . The normalized correlators are

$$\frac{\mathcal{K}_{n,m}}{\mathcal{K}_n \mathcal{K}_m} = -\frac{\xi^2}{(2-R)^2 n m} \sum_{k=1}^n \frac{k(n+1)!(m+1)!\Gamma(n(R-1)+2)\Gamma(m(R-1)+2)}{(n-k)!(m+k)!\Gamma(n(R-1)+k+1)\Gamma(m(R-1)-k+1)}. \quad (5.8)$$

It is clear from the definition that (5.7) as well as (5.8) should be symmetric in m and n . However, our results do not possess this symmetry explicitly. The reason is that the expansion of (3.17) has been done in a nonsymmetric way. It may be possible to symmetrize them by means of identities between Γ -functions like

$$\sum_{k=m}^n \frac{1}{k} C_{n-k}^k C_{k-m}^{-k} = 0, \quad 1 \leq m < n. \quad (5.9)$$

This identity follows from vanishing of the last term in (5.6).

6 Discussion

The main result of this paper is the calculation of one- and two-point vorticity (winding) correlators given by the generating functions (3.17), (3.28) and (4.16). What kind of physics do they describe?

We can look at our model from two points of view: as a statistical-mechanical system describing a gas of vortices and as a string theory.

Statistical mechanical picture

In the statistical mechanical picture we interpret the planar graphs of the MQM (2.9) as random dynamical lattices populated by Berezinski-Kosterlitz-Thouless (BKZ) vortices.

Our one point correlator \mathcal{K}_n (4.22) should be in principle proportional to the probability to find in the system a vortex of a given vorticity n . The probability should be a positive quantity. However, we see that the explicit expressions (4.22)-(4.23) contain the sign-changing factors $\frac{1}{\sin \pi R n}$. Their origin is not completely clear to us but we see two possible explanations for them:

i) They appeared as a result of wave function renormalisation in the continuous (double scaling) limit and should be absorbed into the “leg-factors”. The rest of the one-point correlator is strictly positive in the interval $1 < R < 2$ which we pretend to describe. This point of view is supported by the fact that these factors appear as the same wave-function renormalisations also in the two point correlators (5.4),(5.7). Since their origin is completely

due to the change of variables (2.12) these factors appear to be a general feature for all multipoint correlators.

ii) The sign-changing might be a consequence of the fact that our approach based on the Hirota equations (describing directly the double scaling limit) does not keep track of non-universal terms due to the ultraviolet (lattice) cut-off in the system. Such UV divergences are well known in the Coulomb gas description of the BKT system: they usually correspond to the proper energy of a vortex. In that case the correlators we calculated above are only the universal parts of the full correlators containing also big positive non-universal contributions. These universal parts cannot be interpreted as full probabilities and have no reason to be positively defined.

To prove or reject these explanations we have to see what happens with the individual vortices in the process of taking the double scaling limit, from planar graphs picture to the inverted oscillator description. It is not an easy, although not hopeless, task which we leave for the future.

String theory picture

The MQM describes also the two-dimensional bosonic string theory in compact imaginary time (i.e. at finite temperature). Due to the FZZ [5] conjecture, at the vanishing cosmological constant it can be also interpreted as a string theory in the Euclidean 2D black hole background (at least at $R = 3/2$) [3]. Our one-point correlators should contain information about the amplitudes of emission of winding modes by the black hole, whereas the two-point correlators describe the S-matrix of scattering of the winding states from the tip of the cigar (or from the Sine-Liouville wall). However, the exact correspondence is lacking due to the absence of the proper normalization of the operators. In [5] and [6] some two- and three-point correlators were computed in the CFT approach to this theory. It would be interesting to compare their results with the correlators calculated in this paper from the MQM approach. However there are immediate obstacles to this comparison.

First of all, these authors do not give any results for the one point functions of vortices. In the conformal theory such functions are normally zero since the vortex operators have the dimension one. But in the string theory we calculate the averages of a type $\langle \int d^2\sigma \hat{V}_n(\sigma) \rangle$ integrated over the parameterization space. They are already quantities of zero dimension, and the formal integration leads to the ambiguity $0 * \infty$ which should in general give a finite result. Another possible reason for vanishing of the one-point correlator could be the additional infinite W-symmetry found in [7] for the CFT (2.7) (at $R = 3/2$, $\mu' = 0$).² The generators of this symmetry do not commute with the vortex operators \hat{V}_n , $n \neq \pm 1$. Hence its vacuum average should be zero, unless there is a singlet component under this symmetry in it. Our results suggest the presence of such a singlet component.

Note that the one-point functions were calculated [6] in a non-conformal field theory with the Lagrangian:

$$L = \frac{1}{4\pi} \left[(\partial X)^2 + (\partial \phi)^2 - 2\hat{R}\phi + m \left(e^{-\frac{1}{2}\phi} + e^{\frac{1}{2}\phi} \right) \cos\left(\frac{3}{2}X\right) \right]. \quad (6.1)$$

²We thank A.Zamolodchikov for this comment

This theory coincides with the CFT (2.7) (again at $R = 3/2$ and $\mu' = 0$) in the limit $m \rightarrow 0$, $\phi_0 \rightarrow -\infty$, with $\lambda = me^{-\frac{1}{2}\phi_0}$ fixed, where ϕ_0 is a shift of the zero mode of the Liouville field $\phi(z)$. So we can try to perform this limit in the calculated one-point functions directly. As a result we obtain $\mathcal{K}_n^{(m)} \sim m^2 \lambda^{3n-4}$. The coefficient we omitted is given by a complicated integral which we cannot perform explicitly. It is important that it does not depend on the couplings and is purely numerical. We see that, remarkably, the vanishing mass parameter enters in a constant power which is tempting to associate with the measure $d^2\sigma$ of integration. Moreover, the scaling in λ is the same as in (4.23) (at $R = 3/2$) up to the n -independent factor λ^{-8} . All these n -independent factors disappear if we consider the correlators normalized with respect to $\mathcal{K}_1^{(m)}$ which is definitely nonzero. They behave like $\sim \lambda^{3(n-1)}$ what coincides with the MQM result.

The two point correlators for the CFT (2.7) with $\mu' = 0$ are calculated in [6] only in the case of vortex operators of opposite and equal by modulo vorticities. They should be in principle compared to our $\mathcal{K}_{n,-n}$ correlators from (5.4). However, we should compare only properly normalized quantities, knowing that the matrix model correlators generally differ from the CFT correlators by the leg-factors. To fix the normalization we have to compare the quantities similar to (5.5) and (5.8) which are not available in the CFT approach.

A lot of interesting physical information is contained in multipoint correlators (3-point and more) which we did not consider in this paper and some of which are calculated in the CFT approach of [5, 19]. The direct calculations from the Hirota equations look very tedious. Hopefully the collective field theory approach of [20] can help on this way.

A lot is to be done to learn how to extract information about the black hole physics from the correlators. First of all, we have only calculated some vortex (winding modes) correlators in our MQM approach. The vertex (momentum modes, or tachyon) correlators are unavailable in the Toda hierarchy approach used here and do not seem to fit any known integrable structure in the MQM, although they are in principle calculable directly from the MQM in the large N limit. On the other hand, they bear even a more important information about the system than the winding correlators: they describe the S -matrix of scattering of tachyons off the black hole and can be used to “see” the black hole background explicitly. The one-point tachyon correlators may give information about the black hole radiation. In general, the MQM approach seems to be a good chance to work out a truly microscopic picture for the 2D black hole physics.

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Appendix A.

Let us investigate which terms will survive in the dispersionless limit in the equation (3.1). This limit corresponds to $\xi \longrightarrow 0$. It is clear that all terms have the form

$$\left[\prod_{j=1}^l D_{k_j} \right] \tau_{s+1} \cdot \tau_s \quad (\text{A.1})$$

with $n = \sum_{j=1}^l k_j$ fixed. When we rewrite them in terms of the free energy $\mathcal{F}_s = \log \tau_s$ we obtain a set of terms of a kind

$$e^{\mathcal{F}_s + \mathcal{F}_{s+1}} \prod_{j=1}^p \left(\left[\prod_{i=1}^{q_j} \frac{\partial}{\partial t_{k_{j_i}}} \right] (\mathcal{F}_{s+1} + (-1)^{q_j} \mathcal{F}_s) \right), \quad (\text{A.2})$$

where $\sum_{j=1}^p q_j = l$. In the dispersionless limit (2.22), (2.27) and (2.20) imply

$$\mathcal{F}_{s+1} + \mathcal{F}_s \sim 2\mathcal{F}_0, \quad (\text{A.3})$$

$$\mathcal{F}_{s+1} - \mathcal{F}_s \sim -i\xi \frac{\partial}{\partial w} \mathcal{F}_0, \quad (\text{A.4})$$

The main contribution in the dispersionless limit comes from the terms with the minimal degree of ξ . The total degree of ξ for a given term is

$$\sum_{j=1}^l \Delta[t_{k_j}] - 2p + r = l - 2p + r - \frac{nR}{2}, \quad (\text{A.5})$$

where r is the number of odd q_j . It is easy to see that the minimum is achieved when all $q_j = 1$ or 2 and it equals $-\frac{nR}{2}$ what is independent on all particular parameters and hence is similar for all terms in the equation. It means that only two- and one-point correlators survive.

Appendix B.

Combine together all terms with \tilde{D}_k and separately with \tilde{X}_k in the exponents of the eq. (3.5). Then due to vanishing of $\tilde{D}^2 \tilde{X}_k$ we can apply the formula $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ which is valid for operators having c-number commutator. This gives

$$\begin{aligned} & \frac{2}{x-y} \left(x e^{\sum_{k=1}^{\infty} x^k \tilde{X}_k} e^{\frac{1}{2} \sum_{k,l=1}^{\infty} x^{k+l} \tilde{X}_{k,l}} - y e^{\sum_{k=1}^{\infty} y^k \tilde{X}_k} e^{\frac{1}{2} \sum_{k,l=1}^{\infty} y^{k+l} \tilde{X}_{k,l}} \right) = \\ & \left(1 + \sum_{m=1}^{\infty} y^m (D_m + X_m) \right) e^{\sum_{k=1}^{\infty} x^k \tilde{X}_k} e^{\frac{1}{2} \sum_{k,l=1}^{\infty} x^{k+l} \tilde{X}_{k,l}} + \left(1 + \sum_{n=1}^{\infty} x^n (D_n + X_n) \right) e^{\sum_{k=1}^{\infty} y^k \tilde{X}_k} e^{\frac{1}{2} \sum_{k,l=1}^{\infty} y^{k+l} \tilde{X}_{k,l}} \end{aligned} \quad (\text{B.1})$$

The right hand side of this equation can be rewritten as follows

$$\begin{aligned} & \left(1 + \sum_{m=1}^{\infty} m y^m \left(\tilde{X}_m + \sum_{n=1}^{\infty} x^n \tilde{X}_{n,m}\right)\right) e^{\sum_{n=1}^{\infty} x^n \tilde{X}_n + \frac{1}{2} \sum_{k,l=1}^{\infty} x^{k+l} \tilde{X}_{k,l}} + \\ & \left(1 + \sum_{n=1}^{\infty} n x^n \left(\tilde{X}_n + \sum_{m=1}^{\infty} y^m \tilde{X}_{n,m}\right)\right) e^{\sum_{m=1}^{\infty} y^m \tilde{X}_m + \frac{1}{2} \sum_{k,l=1}^{\infty} y^{k+l} \tilde{X}_{k,l}}. \end{aligned} \quad (\text{B.2})$$

Due to (3.7) we obtain

$$\begin{aligned} & \frac{x+y}{x-y} \left(e^{\frac{1}{2} \sum_{k,l=0}^{\infty} x^{k+l} \tilde{X}_{k,l}} - e^{\frac{1}{2} \sum_{k,l=0}^{\infty} y^{k+l} \tilde{X}_{k,l}} \right) = \\ & \left(\sum_{n,m=0}^{\infty} m x^n y^m \tilde{X}_{n,m} \right) e^{\frac{1}{2} \sum_{k,l=0}^{\infty} x^{k+l} \tilde{X}_{k,l}} + \left(\sum_{n,m=0}^{\infty} n x^n y^m \tilde{X}_{n,m} \right) e^{\frac{1}{2} \sum_{k,l=0}^{\infty} y^{k+l} \tilde{X}_{k,l}} = 0, \end{aligned} \quad (\text{B.3})$$

what can be written in terms of the generating function (3.9) as in (3.10).

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Article II

Time-dependent backgrounds of 2D string theory

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Time-dependent backgrounds of 2D string theory

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We study possible backgrounds of 2D string theory using its equivalence with a system of fermions in upside-down harmonic potential. Each background corresponds to a certain profile of the Fermi sea, which can be considered as a deformation of the hyperbolic profile characterizing the linear dilaton background. Such a perturbation is generated by a set of commuting flows, which form a Toda Lattice integrable structure. The flows are associated with all possible left and right moving tachyon states, which in the compactified theory have discrete spectrum. The simplest nontrivial background describes the Sine-Liouville string theory. Our methods can be also applied to the study of 2D droplets of electrons in a strong magnetic field.

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1 Introduction

One of the most important problems of the modern string theory is a search for new nontrivial backgrounds and the study of the underlying string dynamics. In most of the cases the target space metric of such backgrounds is curved and often it incorporates the black hole singularities. In the superstring theories, the supersymmetry allows for some interesting nontrivial solutions which are stable and exact. But the string theory on such backgrounds is usually an extremely complicated sigma-model, very difficult even to formulate it explicitly, not to mention studying quantitatively its dynamics.

The two-dimensional bosonic string theory is a rare case of sigma-model where such a dynamics is integrable, at least for some particular backgrounds, including the dilatonic black hole background. A physically transparent way to study the perturbative (one loop) string theory around such a background is provided by the CFT approach. However, once we want to achieve more ambitious quantitative goals, especially in analyzing higher loops or multipoint correlators, we have to address ourselves to the matrix model approach to the 2D string theory proposed in [1] in the form of the matrix quantum mechanics (MQM) of [2]. The string theory has been constructed as the collective field theory (the Das-Jevicki-Sakita theory), in which the only excitation, the massless "tachyon", was related to the eigenvalue density of the matrix field.

In the framework of MQM it is difficult to formulate directly a string theory in a nontrivial background metric since the operators perturbing the metric do not have a simple realization. However, we can perturb the theory by other operators, like tachyon or winding operators. We hope that such a perturbation can also produce a curved metric but in an indirect way. An example of such a duality was given in [3] where the 2D black hole background is induced by a winding mode perturbation.

The 2D string theory in the simplest, translational invariant background (the linear dilaton background) is described by the singlet sector of MQM with inversed quadratic potential. In the singlet sector of the Hilbert space, the eigenvalues have Fermi statistics, and the problem of calculating the S -matrix of the string theory tachyon becomes a rather standard problem in a quantum theory of a one-dimensional nonrelativistic free fermionic field. The tree-level S -matrix can be extracted by considering the propagation of "pulses" along the Fermi surface and their reflection off the "Liouville wall" [4, 5].

The very existence of a formulation in terms of free fermions indicates that the 2D string theory should be integrable. For example, the fermionic picture allows to calculate the exact S -matrix elements. Each S -matrix element can be associated with a single fermionic loop with a number of external lines [6]. One can then expect that the theory is also solvable in a nontrivial, time-dependent background generated by a finite tachyonic source. Dijkgraaf, Moore and Plesser [7] demonstrated that this is indeed the case when the allowed momenta form a lattice as in the case of the compactified Euclidean theory. In [7] it has been shown that the string theory compactified at any radius R possesses the integrable structure of the Toda lattice hierarchy. However, this important observation had not been, until recently, really exploited. The Toda structure is too general, and it becomes really of use only if accompanied by an initial condition or a constraint (a "string equation"), which eliminates all the solutions but one. Thus the first results for a nontrivial background in case of a general radius were obtained as a perturbative expansion in the tachyon source [8].

Recently, the Toda integrable structure of the compactified Euclidean 2D string theory was rediscovered [9, 10] and used to find the explicit solution of the theory [10, 11]. Finally, the string equation at an arbitrary compactification radius was found in [12]. These papers studied the T-dual formulation of the string theory, where instead of the discrete spectrum of tachyon excitations the winding modes of the string around the compactification circle (Berezinski-Kosterlitz-Thouless vortices) were used. These modes appear in the non-singlet sectors of MQM [13] and can be generated by imposing twisted periodic boundary conditions [14].

In this paper we return to the study of the Toda integrable structure of tachyon excitations of 2D string theory originated in [7]. It describes special perturbations of the ground state within the singlet sector of the MQM. We will construct the Lax operators as operators in the Hilbert space of the singlet sector of MQM. We will be able to find an interpretation of the Toda spectral parameters in terms of the coordinates of the two-dimensional target phase space and interpret the solutions of the Toda hierarchy in terms of the shape of the Fermi sea. In particular, we find the explicit shape of the Fermi sea for the Sine-Liouville string theory.

We can give two different interpretations of our problem. The first one is that of a 2D string theory in Minkowski space in a non-stationary background. The simplest, time-independent ground state of the theory is characterized by a condensation of the constant tachyon mode, which is controlled by the cosmological constant μ . Here we will study more general, time-dependent backgrounds characterized by a set of coupling constants $t_{\pm n}$ associated with non-trivial tachyon modes with purely imaginary energies $E_n = in/R$, where R is a real number. Since the incoming/outgoing tachyons with imaginary energies have wave functions exponentially decreasing/increasing with time, such a ground state will contain only incoming tachyons in the far past and only outgoing tachyons in the far future. In other words, the right and left vacua are replaced by coherent states depending correspondingly on the couplings t_n and t_{-n} ($n = 1, 2, \dots$), which will be identified with the “times” of the Toda hierarchy. The coherent states of bosons modify the asymptotics of the profile of the Fermi sea at far past and future, without changing the number of fermions. The flow of the fermionic liquid is no more stationary, but its time dependence happens to be quite trivial, and the profiles of the Fermi sea at different moments of time are related by Lorentz boosts.

The second interpretation of our analysis is that we consider perturbations of an Euclidean 2D string theory in which the Euclidean time is compactified at some length $\beta = 2\pi R$. Together with the cosmological term, we allow all possible vertex operators with momenta n/R , $n \in \mathbf{Z}$. The simplest case of such a perturbation is the Sine-Liouville theory. This case will be considered in detail and the shape of the Fermi sea produced by the Sine-Liouville perturbation will be found explicitly. The solution we have found exhibits interesting thermodynamical properties, which may be relevant to the thermodynamics of the string theory on the dilatonic 2D black hole background.

Let us mention also another possible application of our analysis: the fermionic system that appears in MQM is similar to the problem of two-dimensional fermions in a strong transverse magnetic field [15]. The electrons filling the first Landau level form stationary droplets of Fermi liquid, similar to that of the eigenvalues in the phase space of MQM. The form of such droplets is also described by the Toda hierarchy [16]. Our problem might be related to the situation in which two such droplets are about to touch, which can happen at

a saddle point of the effective potential [17].

The paper is organized as follows. In section 2 we will remind the CFT formulation of the 2D string theory. In section 3 the MQM in the inverted harmonic potential in the “light-cone” phase space variables is formulated and the free fermion structure of its singlet sector is revealed. In section 4 the one particle wave functions are studied. In section 5, after the description of the fermionic ground state, tachyonic perturbations are introduced and the equations defining the corresponding time-dependent profile of the Fermi sea are derived. In section 6 the Toda integrable structure of the perturbations restricted to a lattice of Euclidean momenta is derived directly from the Schroedinger picture for the free fermions. In section 7 we recover the solution of the Sine-Liouville model and describe explicitly its Fermi sea profile. In section 8 we reproduce the free energy of the perturbed system from the profile of the Fermi sea. The section 9 is devoted to conclusions and in section 10 we discuss some problems and future perspectives. In particular, we propose a 3-matrix model describing the 2D string theory perturbed by both tachyon and winding modes. In the appendix the one particle wave functions of the type II model, defined on both sides of the quadratic potential, in the “light cone” formalism are presented.

2 Tachyon and winding modes in 2D Euclidean string theory

The 2D string theory is defined by Polyakov Euclidean string action

$$S(X, g) = \frac{1}{4\pi} \int d^2\sigma \sqrt{\det g} [g^{ab} \partial_a X \partial_b X + \mu], \quad (2.1)$$

where the bosonic field $X(\sigma)$ describes the embedding of the string into the Euclidean time dimension and g^{ab} is a world sheet metric. In the conformal gauge $g^{ab} = e^{\phi(\sigma)} \hat{g}^{ab}$, where \hat{g}^{ab} is a background metric, the dilaton field ϕ becomes dynamical due to the conformal anomaly and the world-sheet CFT action takes the familiar Liouville form

$$S_0 = \frac{1}{4\pi} \int_{\text{world sheet}} d^2\sigma [(\partial X)^2 + (\partial\phi)^2 - 2\hat{\mathcal{R}}\phi + \mu e^{-2\phi} + \text{ghosts}]. \quad (2.2)$$

This action describes the unperturbed linear dilaton string background corresponding to the flat 2D target space (X, ϕ) .

In the target space this theory possesses only one propagating degree of freedom which corresponds to tachyon field. If the Euclidean time is compactified to a circle of radius R , $X(\sigma) \equiv X(\sigma) + 2\pi R$, the spectrum of admissible momenta is discrete: $p_n = n/R$, $n \in \mathbf{Z}$. In this case there is also a discrete spectrum of vortex operators, describing the winding modes (Kosterlitz-Thouless vortices) on the world sheet. A vortex of charge $q_m = mR$ corresponds to a discontinuity $2\pi mR$ of the time variable around a point on the world sheet.¹ The explicit expressions of the vertex operators V_p and the vortex operators \tilde{V}_q in terms of the position field $X = X_R + X_L$ and its dual $\tilde{X} = X_R - X_L$ are

$$V_p = \frac{\Gamma(|p|)}{\Gamma(-|p|)} \int d^2\sigma e^{ipX} e^{(|p|-2)\phi}, \quad (2.3)$$

¹At rational values of R there exist additional physical operators (similar to discrete states at the self-dual radius $R = 1$) containing the derivatives of fields $X(\sigma)$ and $\phi(\sigma)$, which we will ignore in this paper.

$$\tilde{V}_q = \frac{\Gamma(|q|)}{\Gamma(-|q|)} \int d^2\sigma e^{iq\tilde{X}} e^{(|q|-2)\phi}. \quad (2.4)$$

Any background of the compactified 2D string theory can be obtained (at least in the case of irrational R) by a perturbation of the action (2.2) with both vertex and vortex operators

$$S = S_0 + \sum_{n \neq 0} (t_n V_n + \tilde{t}_n \tilde{V}_n). \quad (2.5)$$

Such a theory possesses the T-duality symmetry of the non-perturbed theory $X \leftrightarrow \tilde{X}$, $R \rightarrow 1/R$, $\mu \rightarrow R\mu$ [18] if one also exchanges the couplings as $t_n \leftrightarrow \tilde{t}_n$. Another general feature of the theory (2.5) is the existence of the physical scaling of various couplings, including the string coupling g_s , with respect to the cosmological coupling μ . It can be found from the zero mode shift of the dilaton $\phi \rightarrow \phi + \phi_0$. In this way we obtain

$$t_n \sim \mu^{1-\frac{1}{2}|n|/R}, \quad \tilde{t}_n \sim \mu^{1-\frac{1}{2}R|n|}, \quad g_s \sim \mu^{-1}. \quad (2.6)$$

The scaling (2.6) allows to conclude that at the compactification radius lower than the Berezinski-Kosterlitz-Thouless radius $R_{KT} = 1/2$ all vertex operators are irrelevant. In the interval $1/2 < R < 1$ the only relevant momenta are $p = \pm 1/R$. The theory perturbed by such operators looks as the Sine-Gordon theory coupled to 2d gravity (“Sine-Liouville” theory):

$$S_{SG} = \frac{1}{4\pi} \int d^2\sigma [(\partial X)^2 + (\partial\phi)^2 - 2\hat{\mathcal{R}}\phi + \mu e^{-2\phi} + \lambda e^{(\frac{1}{R}-2)\phi} \cos(X/R)]. \quad (2.7)$$

It was conjectured by [3] that at $R = 2/3$ and $\mu = 0$ the “Sine-Liouville” is dual to the $\left[\frac{SL(2,C)}{SU(2) \times U(1)}\right]_k$ coset model with central charge $c = \frac{3k}{k-2} - 1 = 26$, which describes the 2D string theory in the black hole (“cigar”) background.

If we go to the Minkowski space, the perturbation (2.5) is made by tachyons with purely imaginary momenta. In this case there are more general perturbations, generated by tachyons with any real energies. In the next section we will introduce the matrix formulation of the string theory in Minkowski space using light-cone coordinates.

3 MQM in light cone formulation

The 2D string theory in Minkowski time appears as the collective field theory for the large- N limit of MQM in the inverted oscillator potential [1, 19, 20, 21, 22]. The matrix Hamiltonian is

$$H = \frac{1}{2} \text{Tr} (P^2 - M^2), \quad (3.1)$$

where $P = -i\partial/\partial M$ and M_{ij} is an $N \times N$ hermitian matrix variable. The cosmological constant μ in (2.2) is introduced as a “chemical potential” coupled to the size of the matrix N , which should be considered as a dynamical variable. In this formulation the time coordinate of the string target space coincides with the MQM time (or its Euclidean analogue $X = it$ in the compactified version of the MQM) and that of the Liouville field is related to the spectral variable of the random matrix.

The tachyon modes of the string theory are represented by the asymptotic states of the collective theory. The scattering operators with real energy E and describing right- and left-moving waves, respectively, are given by [5]

$$T_-^E = e^{-iEt} \text{Tr} (M + P)^{iE}, \quad T_+^E = e^{-iEt} \text{Tr} (M - P)^{-iE}. \quad (3.2)$$

These operators can be used to construct the in- and out-states of scattering theory. Namely, for an in-state, one needs a left-moving wave while the out-state is necessarily given by a right-moving one. The vertex operators (2.3) are tachyons with purely imaginary energies and are therefore represented by the following chiral operators in the MQM

$$V_p \rightarrow \begin{cases} e^{-pt} \text{Tr} (M + P)^{|p|}, & p > 0, \\ e^{-pt} \text{Tr} (M - P)^{|p|}, & p < 0. \end{cases} \quad (3.3)$$

Since we are interested in perturbations with the chiral operators (3.3), it is natural to perform a canonical transformation to light-cone variables

$$X_{\pm} = \frac{M \pm P}{\sqrt{2}} \quad (3.4)$$

and write the matrix Hamiltonian as

$$H_0 = -\frac{1}{2} \text{Tr} (X_+ X_- + X_- X_+), \quad (3.5)$$

where the matrix operators X_{\pm} obey the canonical commutation relation

$$[(X_+)_j^i, (X_-)_l^k] = -i \delta_l^i \delta_j^k. \quad (3.6)$$

Define the right and left Hilbert spaces as the spaces of functions of X_+ and X_- only, with the scalar product

$$\langle \Phi_{\pm} | \Phi'_{\pm} \rangle = \int d^{N^2} X_{\pm} \overline{\Phi_{\pm}(X_{\pm})} \Phi'_{\pm}(X_{\pm}). \quad (3.7)$$

The operator of coordinate in the right Hilbert space is the momentum operator in the left one and the wave functions in the X_+ and X_- representations are related by a Fourier transform. The second-order Schrödinger equation associated with the Hamiltonian (3.1) becomes a first-order one when written in the \pm -representations

$$\partial_t \Phi_{\pm}(X_{\pm}, t) = \mp \text{Tr} \left(X_{\pm} \frac{\partial}{\partial X_{\pm}} + \frac{N}{2} \right) \Phi_{\pm}(X_{\pm}, t), \quad (3.8)$$

whose general solution is

$$\Phi_{\pm}(X_{\pm}, t) = e^{\mp \frac{1}{2} N^2 t} \Phi_{\pm}^{(0)}(e^{\mp t} X_{\pm}). \quad (3.9)$$

The Hilbert space decomposes into a direct sum of eigenspaces labeled by the irreducible representations of $SU(N)$, which are invariant under the action of the Hamiltonian (3.5). The functions $\vec{\Phi}_{\pm}^{(r)} = \{\Phi_{\pm}^{(r,J)}\}_{J=1}^{\dim(r)}$ belonging to given irreducible representation r transform as

$$\Phi_{\pm}^{(r,I)}(\Omega^\dagger X_{\pm} \Omega) = \sum_J D_{IJ}^{(r)}(\Omega) \Phi_{\pm}^{(r,J)}(X_{\pm}), \quad (3.10)$$

where $D_{IJ}^{(r)}$ is the group matrix element in representation r and I, J are the representation indices. The wave functions transforming according to given irreducible representation depend only on the N eigenvalues $x_{\pm,1}, \dots, x_{\pm,N}$ and the Hamiltonian (3.5) reduces to its radial part

$$H_0 = \mp i \sum_k (x_{\pm,k} \partial_{x_{\pm,k}} + \frac{N}{2}). \quad (3.11)$$

A potential advantage of the light-cone approach is that the Hamiltonian (3.11) does not contain any angular part, which is not the case for the standard one (3.1), whose angular term induces a Calogero-like interaction.

In the scalar product (3.7), the angular part can also be integrated out, leaving only the trace over the representation indices:

$$\langle \vec{\Phi}_{\pm}^{(r)} | \vec{\Phi}_{\pm}'^{(r)} \rangle = \sum_J \int \prod_k dx_{\pm,k} \Delta^2(x_{\pm}) \overline{\Phi_{\pm}^{(r,J)}(x_{\pm})} \Phi_{\pm}'^{(r,J)}(x_{\pm}), \quad (3.12)$$

where $\Delta(x_{\pm})$ is the Vandermonde determinant. If we define

$$\vec{\Psi}_{\pm}^{(r)}(x_{\pm}) = \Delta(x_{\pm}) \vec{\Phi}_{\pm}^{(r)}(x_{\pm}), \quad (3.13)$$

these determinants disappear from the scalar product. The Hamiltonian for the new functions $\vec{\Psi}_{\pm}^{(r)}(x_{\pm})$ takes the same form as (3.11) but with a different constant term:

$$H_0 = \mp i \sum_k (x_{\pm,k} \partial_{x_{\pm,k}} + 1/2). \quad (3.14)$$

In the singlet representation, the wave function $\Psi_{\pm}(x_{\pm}) \equiv \Psi_{\pm}^{(\text{singlet})}(x_{\pm})$ is a completely antisymmetric scalar function. The scalar product in the singlet representation is given by

$$\langle \Psi_{\pm} | \Psi'_{\pm} \rangle = \int \prod_k dx_{\pm,k} \overline{\Psi_{\pm}(x_{\pm})} \Psi'_{\pm}(x_{\pm}). \quad (3.15)$$

Thus the singlet sector describes a system of N free fermions. The singlet eigenfunctions of the Hamiltonian (3.14) are represented by Slater determinants of one-particle eigenfunctions discussed in the next section.

It is known [13, 14, 10] that unlike singlet, which is free of vortices, the adjoint representation contains string states with a vortex-antivortex pair, and higher representations contain higher number of such pairs. In what follows we will concentrate on the fermionic system describing the singlet sector of the matrix model. We will start from the properties of the ground state of the model, representing the unperturbed 2D string background and then go over to the perturbed fermionic states describing the (time-dependent) backgrounds characterized by nonzero expectation values of some vertex operators.

4 Eigenfunctions and fermionic scattering

To study the system of non-interacting fermions we have to start with one-particle eigenfunctions. The one-particle Hamiltonian in the light-cone variables of the previous section can be written as

$$H_0 = -\frac{1}{2}(\hat{x}_+ \hat{x}_- + \hat{x}_- \hat{x}_+), \quad (4.1)$$

where x_{\pm} turn out to be canonically conjugated variables

$$[\hat{x}_+, \hat{x}_-] = -i. \quad (4.2)$$

We can work either in x_+ or in x_- representation, where the theory is defined in terms of fermionic fields $\psi_{\pm}(x_{\pm})$ respectively. General solutions of the Schrödinger equation with the Hamiltonian (4.1) written in these representations take the form

$$\psi_{\pm}(x_{\pm}, t) = x_{\pm}^{-1/2} f_{\pm}(e^{\mp t} x_{\pm}). \quad (4.3)$$

There are two versions of the theory, referred in [6] as theories of type I and II. In the theory of type I the eigenvalues λ_k of the original matrix field M are restricted to the positive half-line. The theory of type II is defined on the whole real axis λ . Such a string theory splits, at the perturbative level, into two disconnected string theories of type I. Here we will consider the fermion eigenfunctions in the theory of type I. The theory of type II will be considered in Appendix A.

In the light cone formalism it is natural to replace the restriction $\lambda > 0$ with $x_{\pm} > 0$, which again does not affect the perturbative behavior. In this case the solutions with a given energy are $\psi_{\pm}^E(x_{\pm}, t) = e^{-iEt} \psi_{\pm}^E(x_{\pm})$ with

$$\psi_{\pm}^E(x_{\pm}) = \frac{1}{\sqrt{2\pi}} x_{\pm}^{\pm iE - \frac{1}{2}}. \quad (4.4)$$

The functions (4.4) with E real form two complete systems of δ -function normalized orthonormal states

$$\langle \psi_{\pm}^E | \psi_{\pm}^{E'} \rangle \equiv \int_0^{\infty} dx_{\pm} \overline{\psi_{\pm}^E(x_{\pm})} \psi_{\pm}^{E'}(x_{\pm}) = \delta(E - E'), \quad (4.5)$$

$$\int_{-\infty}^{\infty} dE \overline{\psi_{\pm}^E(x_{\pm})} \psi_{\pm}^E(x'_{\pm}) = \delta(x_{\pm} - x'_{\pm}). \quad (4.6)$$

As any two representations of a quantum mechanical system, the x_+ and x_- representations are related by a unitary operator \hat{S} , which in our case is nothing but the Fourier transformation on the half-line. The latter is defined by the integral

$$[\hat{S}\psi_+](x_-) = \int_0^{\infty} dx_+ K(x_-, x_+) \psi_+(x_+), \quad (4.7)$$

where there are two choices for the kernel:

$$K(x_-, x_+) = \sqrt{\frac{2}{\pi}} \cos(x_- x_+) \quad \text{or} \quad K(x_-, x_+) = i\sqrt{\frac{2}{\pi}} \sin(x_- x_+). \quad (4.8)$$

The sine and the cosine kernels describe two possible theories, which differ only on the non-perturbative level.² Let us choose the cosine kernel in (4.8) and evaluate the action of the

²The fact that there are two choices for the kernel can be explained as follows. In order to define the theory of type I for the original second-order Hamiltonian (3.1), we should fix the boundary condition at $\lambda = 0$, and there are two linearly independent boundary conditions.

\hat{S} -operator on the eigenfunctions. This integral is essentially the defining integral for the Γ -function:

$$[\hat{S}^{\pm 1} \psi_{\pm}^E](x_{\mp}) = \frac{1}{\pi} \int_0^{\infty} dx_{\pm} \cos(x_{+} x_{-}) x_{\pm}^{\pm iE - \frac{1}{2}} = \mathcal{R}(\pm E) \psi_{\mp}^E(x_{\mp}), \quad (4.9)$$

where

$$\mathcal{R}(E) = \sqrt{\frac{2}{\pi}} \cosh\left(\frac{\pi}{2}(i/2 - E)\right) \Gamma(iE + 1/2). \quad (4.10)$$

(The sine kernel would give the same result, but with \cosh replaced by \sinh). The factor $\mathcal{R}(E)$ is a pure phase

$$\overline{\mathcal{R}(E)} \mathcal{R}(E) = \mathcal{R}(-E) \mathcal{R}(E) = 1, \quad (4.11)$$

which proves the unitarity of the operator \hat{S} . The operator \hat{S} relates the incoming and the outgoing waves and therefore can be interpreted as the fermionic scattering matrix. The factor $\mathcal{R}(E)$ is identical to the reflection coefficient (the "bounce factor" of [6]), characterizing the scattering off the upside-down oscillator potential. In the standard scattering picture, the reflection coefficient is extracted by comparing the incoming and outgoing waves in the large- λ asymptotics of the exact eigenfunction of the inversed oscillator Hamiltonian $H = -\frac{1}{2}(\partial_{\lambda}^2 + \lambda^2)$.

It follows from the above discussion that the scattering amplitude between two arbitrary in and out states is given by the integral with the Fourier kernel (4.8)

$$\begin{aligned} \langle \psi_{-} | \hat{S} \psi_{+} \rangle &= \langle \hat{S}^{-1} \psi_{-} | \psi_{+} \rangle = \langle \psi_{-} | K | \psi_{+} \rangle \\ \langle \psi_{-} | K | \psi_{+} \rangle &\equiv \int_0^{\infty} dx_{+} dx_{-} \overline{\psi_{-}(x_{-})} K(x_{-}, x_{+}) \psi_{+}(x_{+}). \end{aligned} \quad (4.12)$$

The integral (4.12) can be interpreted as a scalar product between the in and out states. By (4.5) and (4.9) one finds that the in and out eigenfunctions satisfy the orthogonality relation

$$\langle \psi_{-}^E | K | \psi_{+}^{E'} \rangle = \mathcal{R}(E) \delta(E - E'). \quad (4.13)$$

5 String theory backgrounds as profiles of Fermi sea

The ground state of the MQM is obtained by filling all energy levels up to some fixed Fermi energy which we choose to be $E_F = -\mu$. Quasiclassically every energy level corresponds to a certain trajectory in the phase space of x_{+}, x_{-} variables. The Fermi sea can be viewed as a stack of all classical trajectories with $E < E_F$ and the ground state is completely characterized by the curve representing the trajectory of the fermion with highest energy E_F . For the Hamiltonian (4.1) all trajectories are hyperboles $x_{+} x_{-} = -E$ and the profile of the Fermi sea is given by

$$x_{+} x_{-} = \mu. \quad (5.1)$$

In the quasiclassical limit the phase-space density of fermions is either 0 or 1. Then the low lying collective excitations are represented by deformations of the Fermi surface, *i.e.*, is the boundary of the region in the phase space filled by fermions. At any moment of time such deformation can be obtained by replacing the constant μ on the right hand side of (5.1) with a function of x_{+} and x_{-}

$$x_{+} x_{-} = M(x_{+}, x_{-}). \quad (5.2)$$

In contrast to the ground state which is stationary, an excited state given by a generic function M leads to a time dependent profile. However, this dependence is quite trivial: since the Fermi surface consists of free fermions each moving according to its classical trajectory $x_{\pm}(t) = e^{\pm t}x_{\pm}(0)$, where $x_{\pm}(0)$ is the initial data, we can always replace (5.2) by the equation for the initial shape. So the evolution of a shape in time is simply its homogeneous extension with the factor e^t along the x_+ axis and a homogeneous contraction with the same factor along the x_- axis.

Below we will find equations that determine the shape of the Fermi sea for a generic perturbation with tachyon operators. Our analysis is in the spirit of the Polchinski's derivation of the tree-level S -matrix [4], but we will consider finite, and not only infinitesimal perturbation.

In terms of the Fermi liquid, the incoming and the outgoing states are defined by the asymptotics of the profile of the Fermi sea at $x_+ \gg x_-$ and $x_- \gg x_+$, correspondingly. If we want to consider such a perturbation as a new fermion ground state, we should change the one-fermion wave functions. The new wave functions are related to the old ones by a phase factor

$$\Psi_{\pm}^E(x_{\pm}) = e^{\mp i\varphi_{\pm}(x_{\pm}; E)} \psi_{\pm}^E(x_{\pm}), \quad (5.3)$$

whose asymptotics at large x_{\pm} characterizes the incoming/outgoing tachyon state. We split the phase into three terms

$$\varphi_{\pm}(x_{\pm}; E) = V_{\pm}(x_{\pm}) + \frac{1}{2}\phi(E) + v_{\pm}(x_{\pm}; E), \quad (5.4)$$

where the potentials V_{\pm} are fixed smooth functions vanishing at $x_{\pm} = 0$, while the term v_{\pm} vanishing at infinity and the constant ϕ are to be determined. Thus, the potentials V_{\pm} fix unambiguously the perturbation. The time evolution of these states with the original Hamiltonian (4.1) is determined by the eq. (4.3). To see that the state (5.3) contains incoming and outgoing tachyons, it is enough to note that it arises as a coherent state of vertex operators (3.3) acting on the unperturbed wave function (4.4).

Since the functions Ψ_+^E and Ψ_-^E should describe the same one-fermion state, the Fourier transform (4.7) should be diagonal in their basis. We fix the zero mode ϕ so that

$$\hat{S} \Psi_+^E = \Psi_-^E. \quad (5.5)$$

This condition can also be expressed as the orthonormality of in and out eigenfunctions (5.3)

$$\langle \Psi_-^{E-} | K | \Psi_+^{E+} \rangle = \delta(E_+ - E_-), \quad (5.6)$$

with respect to the scalar product (4.12). This requirement fixes the exact form of the wave functions. Let us look at this problem in the quasiclassical limit $E_{\pm} \rightarrow \infty$. The scalar product in (5.6) is written as

$$\langle \Psi_-^{E-} | \Psi_+^{E+} \rangle = \frac{1}{\pi\sqrt{2\pi}} \int_0^{\infty} \frac{dx_+ dx_-}{\sqrt{x_+ x_-}} \cos(x_+ x_-) e^{-i\varphi_+(x_+) - i\varphi_-(x_-)} x_-^{iE_-} x_+^{iE_+}. \quad (5.7)$$

At the quasiclassical level it can be evaluated by the saddle point approximation. One obtains two equations for the saddle point³

$$x_+x_- = -E_{\pm} + x_{\pm}\partial\varphi_{\pm}(x_{\pm}). \quad (5.8)$$

Generically, the two equations (5.8) define two different curves in the x_+-x_- plane, and their compatibility can render at most a discrete set of saddle points. However, if the two solutions define the same curve, we obtain a whole line of saddle points, which implies the existence of a zero mode in the double integral (5.7). The contribution of the zero mode is infinite, amounting to the δ -functional orthogonality relation. Thus, the orthonormality condition is reduced to the compatibility of two equations (5.8) at $E_+ = E_-$.

For example, in the absence of perturbations ($\varphi_{\pm} = 0$), the saddle point equations $x_+x_- = -E_{\pm}$ are inconsistent, unless $E_+ = E_-$. For equal energies they coincide leading to a zero mode in the double integral (4.13). The resulting saddle point equation gives the classical hyperbolic trajectory in the phase space of an individual fermion. If $E_+ \neq E_-$, the integrand of (5.7) is a rapidly oscillating function and the integral is zero.

The equations (5.8) and the requirement of their compatibility can be obtained from another point of view. The perturbed wave function (5.3) can be interpreted as an eigenstate of a new, perturbed Hamiltonian. Indeed, the functions (5.3) are not eigenstates of the original Hamiltonian (4.1) in \pm -representations $H_0^{\pm} = \mp i(x_{\pm}\partial_{x_{\pm}} + 1/2)$, but for each given E they are evidently eigenstates of the operators

$$H_{\pm}^E = H_0^{\pm} + x_{\pm}\partial\varphi_{\pm}(x_{\pm}; E), \quad (5.9)$$

where φ_{\pm} contain so far unknown functions $v_{\pm}(x_{\pm}; E)$. However, the operators H_{\pm}^E depend on the energy E through these functions. Therefore, they can not be considered as Hamiltonians. But one can define the Hamiltonians as solutions of the equations

$$H_{\pm} = H_0^{\pm} + x_{\pm}\partial\varphi_{\pm}(x_{\pm}; H_{\pm}). \quad (5.10)$$

Then all functions (5.3) are their eigenstates.

The orthonormality condition (5.6) can be equivalently rewritten as the condition that the Hamiltonians H_{\pm} define the action of the same self-adjoint operator H in the \pm -representations. In the quasiclassical limit, this is equivalent to the coincidence of the phase space trajectories associated with H_+ and H_- :

$$H_+(x_+, x_-) = E \Leftrightarrow H_-(x_+, x_-) = E. \quad (5.11)$$

This condition is equivalent to the compatibility of two equations (5.8).

Note that with respect to the time τ defined by the new Hamiltonian, the time dependence of the states characterized by the wave functions (5.3) is given by $e^{-iE\tau}$. This corresponds to a stationary flow of the Fermi liquid. The profile of the Fermi sea coincides with the classical trajectory of the fermion with the highest energy $E_F = -\mu$. Its equation can be written in two forms similar to (5.2) which should be consistent

$$x_+x_- = M_{\pm}(x_{\pm}) \equiv \mu + x_{\pm}\partial\varphi_{\pm}(x_{\pm}; -\mu). \quad (5.12)$$

³We assume that only one exponent of $\cos(x_+x_-)$ gives a contribution. The result will be shortly justified from another point of view.

These equations are the non-compact version of the equations arising in the conformal map problem which is a semiclassical description of compact Fermi droplet [16]. In that case the potential must be an entire function which is not required in our case. The functional equation (5.12) contains all the information of the tachyon interactions in the tree level string theory. To proceed further with our analysis, we should concretize the form of the perturbing potentials V_{\pm} . We will show in the next section that the perturbations produced by vertex operators are integrable.

6 Integrable flows associated with vertex operators

6.1 Lax formalism, Toda Lattice structure and string equations

Now we restrict ourselves to the time dependent coherent states made of tachyons with discrete Euclidean momenta $p_n = n/R$. This spectrum of momenta arises when the system compactified on a circle of length $2\pi R$, or heated to the temperature $T = 1/(2\pi R)$. Such perturbations are described by the potentials of the following form

$$V_{\pm}(x_{\pm}) = \sum_{k \geq 1} t_{\pm k} x_{\pm}^{k/R}. \quad (6.1)$$

In this section we will show that such a deformation is exactly solvable, being generated by a system of commuting flows H_n associated with the coupling constants $t_{\pm n}$. The associated integrable structure is that of a constrained Toda Lattice hierarchy. The method is very similar to the standard Lax formalism of Toda theory, but we will not assume that the reader is familiar with this subject. It is based on rewriting all operators in the energy representation, in which the Fourier transformation \hat{S} is diagonal as we required in the previous section. The energy E will play the role of the coordinate along the lattice formed by the allowed energies $E_n = ip_n$ of the tachyons.

Let us start with the operators $\hat{x}_{\pm} = x_{\pm}$ and $\hat{\partial}_{\mp} = \frac{\partial}{\partial x_{\mp}}$, which are related as

$$-i\hat{\partial}_{-} = \hat{S}\hat{x}_{+}\hat{S}^{-1}, \quad i\hat{\partial}_{+} = \hat{S}^{-1}\hat{x}_{-}\hat{S}. \quad (6.2)$$

The Heisenberg commutation relation $[\hat{\partial}_{\pm}, \hat{x}_{\pm}] = 1$ leads to the operator identity

$$\hat{x}_{-}\hat{S}\hat{x}_{+}\hat{S}^{-1} - \hat{S}\hat{x}_{+}\hat{S}^{-1}\hat{x}_{-} = i. \quad (6.3)$$

Further, the Hamiltonian (4.1) in the x_{\pm} -representation $H_0^{\pm} = \mp i(x_{\pm}\partial_{x_{\pm}} + 1/2)$ is expressed in terms of \hat{x}_{\pm} and \hat{S} as

$$H_0^{-} = -\frac{1}{2}(\hat{x}_{-}\hat{S}\hat{x}_{+}\hat{S}^{-1} + \hat{S}\hat{x}_{+}\hat{S}^{-1}\hat{x}_{-}) = \hat{S}H_0^{+}\hat{S}^{-1}. \quad (6.4)$$

It follows from the identities

$$\begin{aligned} H_0^{\pm}\psi_{\pm}^E(x_{\pm}) &= E\psi_{\pm}^E(x_{\pm}), \\ x_{\pm}\psi_{\pm}^E(x_{\pm}) &= \psi_{\pm}^{E \mp i}(x_{\pm}), \\ \hat{S}^{\pm 1}\psi_{\pm}^E(x_{\pm}) &= \mathcal{R}^{\pm 1}(E)\psi_{\mp}^E(x_{\mp}) \end{aligned} \quad (6.5)$$

that the operators H_0^\pm , \hat{S} and \hat{x}_\pm are represented in the basis of the non-perturbed wave functions (4.4) by

$$H_0^\pm = \hat{E}, \quad \hat{x}_\pm = \hat{\omega}^{\pm 1}, \quad \hat{S} = \mathcal{R}(\hat{E})_\iota \equiv \hat{\mathcal{R}}_\iota, \quad (6.6)$$

where $\iota\psi_\pm^E = \psi_\mp^E$ and $\hat{\omega}$ is the shift operator

$$\hat{\omega} = e^{-i\partial_E}. \quad (6.7)$$

In order to have a closed operator algebra, we should also define the action of the operators $\hat{S}^{\pm 1}$ on the functions $\psi_\pm^{E \mp ni}(x_\pm) = x_\pm^n \psi_\pm^E(x_\pm)$. To satisfy the identities (6.2), we should define $\hat{S}^{\pm 1}$ by the cosine kernel for n even and by the sine kernel for n odd. Therefore $\hat{S} \psi_\pm^{E \mp i}(x_\pm) = \hat{\mathcal{R}}' \psi_\pm^{E \mp i}(x_\pm)$ where $\hat{\mathcal{R}}' = \mathcal{R}'(\hat{E})$ is given by (4.10) with cosh replaced by sinh.

The algebraic relations (6.3) and (6.4) are transformed into algebraic relations among the operators \hat{E} , $\hat{\omega}$ and $\hat{\mathcal{R}}$ in the E -space⁴

$$\begin{aligned} \hat{\mathcal{R}}^{-1} \hat{\omega} \hat{\mathcal{R}}' \hat{\omega}^{-1} &= -\hat{E} + i/2, \\ \hat{\omega}^{-1} \hat{\mathcal{R}}'^{-1} \hat{\omega} \hat{\mathcal{R}} &= -\hat{E} - i/2. \end{aligned} \quad (6.8)$$

This is equivalent, according to the remark above, to the functional constraint

$$\mathcal{R}(E) = -(i/2 + E) \mathcal{R}'(E + i), \quad (6.9)$$

which is evidently satisfied. To simplify the further discussion, in the rest of this section we will identify the functions $\mathcal{R}(E)$ and $\mathcal{R}'(E)$, thus neglecting the non-perturbative terms $\sim e^{\pi E}$. Note however that all statements made below can be proved without this identification.

Our aim is to extend the E -space representation for the basis of wave functions (4.4) perturbed by a phase factor $e^{\pm i\varphi_\pm}$ as in eqs. (5.3)-(5.4), with V_\pm given by (6.1). We assume that the phase φ_\pm can be expanded, for sufficiently large x_\pm as a Laurent series

$$\varphi_\pm(x_\pm) = \sum_{k \geq 1} t_{\pm k} x_\pm^{k/R} + \frac{1}{2} \phi - \sum_{k \geq 1} \frac{1}{k} v_{\pm k} x_\pm^{-k/R}. \quad (6.10)$$

This assumption will be justified by the subsequent analysis. With the special form (6.10) of the perturbing phase, there exist unitary “dressing operators” \mathcal{W}_\pm acting in the E -space, which transform the “bare” wave functions ψ_\pm^E into the “dressed” ones

$$\Psi_\pm^E \equiv e^{\mp i\varphi_\pm(x_\pm)} \psi_\pm^E = \mathcal{W}_\pm \psi_\pm^E. \quad (6.11)$$

It is evident from (6.10) and (6.6) that the dressing operators can be expressed as series in the shift operator $\hat{\omega}$ with E -dependent coefficients

$$\mathcal{W}_\pm = e^{\mp i\phi/2} \left(1 + \sum_{k \geq 1} w_{\pm k} \hat{\omega}^{\mp k/R} \right) e^{\mp i \sum_{k \geq 1} t_{\pm k} \hat{\omega}^{\pm k/R}}. \quad (6.12)$$

In the basis of the perturbed wave functions (6.11), the operators of canonical coordinates \hat{x}_\pm are represented by two Lax operators

$$L_\pm = \mathcal{W}_\pm \hat{\omega}^{\pm 1} \mathcal{W}_\pm^{-1} = e^{\mp i\phi/2} \hat{\omega}^{\pm 1} e^{\pm i\phi/2} \left(1 + \sum_{k \geq 1} a_{\pm k} \hat{\omega}^{\mp k/R} \right) \quad (6.13)$$

⁴Note that we should inverse the order of the operators.

and the Hamiltonians H_0^\pm are represented (up to a change of sign) by the so-called Orlov-Shulman operators

$$M_\pm = -\mathcal{W}_\pm \hat{E} \mathcal{W}_\pm^{-1} = \frac{1}{R} \sum_{k \geq 1} k t_{\pm k} L_\pm^{k/R} - \hat{E} + \frac{1}{R} \sum_{k \geq 1} v_{\pm k} L_\pm^{-k/R}. \quad (6.14)$$

In the above formulas all coefficients are functions of E and $\vec{t} = \{t_{\pm k}\}_{k=1}^\infty$. The Lax and Orlov-Shulman operators act on the wave functions as

$$L_\pm \Psi_\pm^E(x_\pm) = x_\pm \Psi_\pm^E(x_\pm), \quad (6.15)$$

$$M_\pm \Psi_\pm^E(x_\pm) = \left(\frac{1}{R} \sum_{k \geq 1} k t_{\pm k} x_\pm^{k/R} - E + \frac{1}{R} \sum_{k \geq 1} v_{\pm k} x_\pm^{-k/R} \right) \Psi_\pm^E(x_\pm) \quad (6.16)$$

and satisfy the Lax equations

$$[L_\pm, M_\pm] = \pm i L_\pm, \quad (6.17)$$

which are the dressed version of the relation $[\hat{\omega}^{\pm 1}, -\hat{E}] = \pm i \hat{\omega}^{\pm 1}$.

The structure of constrained Toda hierarchy follows from the requirement (5.5), which means that the action of the \hat{S} operator on the perturbed wave functions is totally compensated by the dressing operators:

$$\mathcal{W}_- = \mathcal{W}_+ \hat{\mathcal{R}}. \quad (6.18)$$

The condition (6.18) defines both the Toda structure and a constraint which plays the role of an initial condition for the PDE of the Toda hierarchy.

The Toda structure implies that the tachyon operators generating the perturbation are represented in the E -space by an infinite set of commuting flows. To show this, we evaluate the variations of the Lax operators with respect to the coupling constants t_n . From the definition (6.13) we have

$$i \partial_{t_n} L_\pm = [H_n, L_\pm], \quad (6.19)$$

where the operators H_n are related to the dressing operators as

$$H_n = i(\partial_{t_n} \mathcal{W}_+) \mathcal{W}_+^{-1} = i(\partial_{t_n} \mathcal{W}_-) \mathcal{W}_-^{-1}. \quad (6.20)$$

The two representations of the flows H_n are equivalent by virtue of the relation (6.18). It is important that $\hat{\mathcal{R}}$ does not depend on t_n 's. A more explicit expression in terms of the Lax operators is derived by the following standard argument. Let us consider the case $n > 0$. From the explicit form of the dressing operators (6.12) it is clear that $H_n = \pm W_+ \hat{\omega}^{n/R} W_+^{-1} +$ negative powers of $\hat{\omega}^{1/R}$. The variation of t_n will change only the coefficients of the expansions (6.13) of the Lax operators, preserving their general form. But it is clear that if the expansion of H_n contained negative powers of $\hat{\omega}^{1/R}$, its commutator with L_- would create extra powers $\hat{\omega}^{-1-k/R}$. Therefore

$$H_{\pm n} = \pm (L_\pm^{n/R})_{\geq} \pm \frac{1}{2} (L_\pm^{n/R})_0, \quad n > 0, \quad (6.21)$$

where the symbol $()_{\geq}$ means the positive (negative) parts of the series in the shift operator $\hat{\omega}$ and $()_0$ means the constant part. By a similar argument one shows that the Lax equations (6.19) are equivalent to the zero-curvature conditions

$$i \partial_{t_m} H_n - i \partial_{t_n} H_m - [H_m, H_n] = 0. \quad (6.22)$$

The equations (6.19) and (6.21) ensure that the perturbations related with the couplings t_n are described by the Toda integrable structure.

The Toda structure leads to an infinite set of PDE's for the coefficients w_n of the dressing operators, the first of which is the Toda equation for the zero mode of the dressing operators

$$i \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_{-1}} \phi = e^{i\phi(E) - i\phi(E+i/R)} - e^{i\phi(E-i/R) - i\phi(E)}. \quad (6.23)$$

The uniqueness of the solution is assured by appropriate boundary conditions or additional constraints, known also as string equations. In our case the string equation can be obtained by dressing the equation (6.8) using the formula (6.18), which leads to the following relation between the Lax and Orlov-Shulman operators

$$L_+ L_- = M_+ + i/2, \quad L_- L_+ = M_- - i/2. \quad (6.24)$$

Similarly, the identity $\hat{\mathcal{R}}\hat{E} = \hat{E}\hat{\mathcal{R}}$ implies

$$M_- = M_+. \quad (6.25)$$

Thus, the perturbations of the one-fermion wave functions of the form (6.11) are described by a constrained Toda lattice hierarchy. In the T-dual formulation the same result was proved recently using the standard definition of the Toda Lax operators [12]. The standard Lax and Orlov-Shulman operators are related to L_{\pm} and M_{\pm} by

$$L = L_+^{1/R}, \quad \bar{L} = L_-^{1/R}, \quad M = R M_+, \quad \bar{M} = R M_-. \quad (6.26)$$

This operators satisfy R -dependent constraints [12]

$$L^R \bar{L}^R = \frac{1}{R} M + i/2, \quad \bar{L}^R L^R = \frac{1}{R} \bar{M} - i/2, \quad M = \bar{M}. \quad (6.27)$$

(The string equations for integer values of R have been discussed previously in [23, 24, 25, 26].)

The integrability allows to find the unknown coefficients in the Laurent expansion of the phase (6.10). All the information about them is contained in the so called τ -function, which is the generating function for the vector fields corresponding to the flows H_n . Its existence follows from the zero-curvature condition (6.22) and can be proved in the standard way [27]. The coefficients are related to the τ -function as follows

$$v_n = \frac{\partial \log \tau}{\partial t_n}, \quad \phi(E) = i \log \frac{\tau(E + i/2R)}{\tau(E - i/2R)}. \quad (6.28)$$

The τ -function can thus be considered as the generating function for the perturbations by vertex operators. As we will show in sec. 8, the logarithm of the τ -function gives the free energy of the Euclidean theory compactified at radius R .

6.2 The dispersionless (quasiclassical) limit

Let us consider the quasiclassical limit $E \rightarrow -\infty$ when all allowed energies are large and negative. In this limit the integrable structure described above reduces to the dispersionless Toda hierarchy [28, 29, 27], where the operators \hat{E} and $\hat{\omega}$ can be considered as a pair of classical canonical variables with Poisson bracket

$$\{\omega, E\} = \omega. \quad (6.29)$$

Similarly, all operators become c -functions of these variables. The Lax operators can be identified, by eq. (6.15), with the phase space coordinates x_{\pm} . The functions $x_{\pm}(\omega, E)$ are given by eq.

$$x_{\pm}(\omega, E) = e^{-\frac{1}{2R}\chi} \omega^{\pm 1} \left(1 + \sum_{k \geq 1} a_{\pm k}(E) \omega^{\mp k/R} \right), \quad (6.30)$$

where we have introduced the “string susceptibility”

$$\chi = -R \partial_E \phi = \partial_E^2 \log \tau. \quad (6.31)$$

The equations (6.24) and (6.25) in the quasiclassical limit give

$$\{x_-, x_+\} = 1, \quad (6.32)$$

$$x_+ x_- = M_+ = M_-. \quad (6.33)$$

It follows from the first equation, (6.32), that eq. (6.30) defines a canonical transformation relating the phase-space coordinates x_+, x_- to $\log \omega$ and E . Moreover, the dressing procedure itself can be interpreted as a canonical transformation between x_+, x_- and the “bare” coordinates

$$\omega_+ = \sqrt{-E} \omega, \quad \omega_- = \sqrt{-E}/\omega. \quad (6.34)$$

The second equation, (6.33), which is a deformation of the relation

$$\omega_+ \omega_- = -E \quad (6.35)$$

satisfied by the “bare” variables, yields the form of the perturbed one-fermion trajectories.

Let us consider the highest trajectory that defines the shape of the Fermi sea. The variable E then should be interpreted as the Fermi energy, which is equal to (minus) the chemical potential, $E_F = -\mu$. Taking into account the expression for M_{\pm} through x_{\pm} given by the r.h.s. of (6.16), we can expand the right-hand side either in x_+ or in x_-

$$x_+ x_- = \frac{1}{R} \sum_{k \geq 1} k t_{\pm k} x_{\pm}^{k/R} + \mu + \frac{1}{R} \sum_{k \geq 1} v_{\pm k} x_{\pm}^{-k/R}. \quad (6.36)$$

(These expansions are of course convergent only for sufficiently large values of x_{\pm} .) Eq. (6.36), which is the dispersionless string equation, is a particular case of the relation (5.12), which was derived for an arbitrary potential. In this way we found a direct geometrical interpretation for the dispersionless string equation in terms of the profile of the Fermi sea.

A simple procedure to calculate the coefficients $a_{\pm k}$ has been suggested in [12]. First, note that if all $t_{\pm k}$ with $k > n$ vanish, the sum in (6.30) can be restricted to $k \leq n$. Then it

is enough to substitute the expressions (6.30), with $E = -\mu$, in the profile equations (6.36) and compare the coefficients in front of $\omega^{\pm k/R}$. The result will give the generating function of one point correlators, which is contained in the inverse function $\omega(x_{\pm})$. In the next section we will apply this method to calculate the profile of the Fermi sea in the important case of the Sine-Liouville string theory.

7 Solution for Sine-Gordon coupled to gravity

The simplest nontrivial string theory with time-dependent background is the Sine-Gordon theory coupled to gravity known also as Sine-Liouville theory. It is obtained by perturbing with the lowest couplings t_1 and t_{-1} . It can be easily seen that in this case the expansion (6.30) consists of only two terms

$$x_{\pm} = e^{-\frac{1}{2R}\chi} \omega^{\pm 1} (1 + a_{\pm} \omega^{\mp \frac{1}{R}}). \quad (7.1)$$

The procedure described in [12] gives the following result for the susceptibility χ and the coefficients a_{\pm} :

$$\mu e^{\frac{1}{R}\chi} - \frac{1}{R^2} \left(1 - \frac{1}{R}\right) t_1 t_{-1} e^{\frac{2R-1}{R^2}\chi} = 1, \quad (7.2)$$

$$a_{\pm} = \frac{t_{\mp 1}}{R} e^{\frac{R-1/2}{R^2}\chi}. \quad (7.3)$$

The first equation in (7.2) for the susceptibility $\chi = \partial_{\mu}^2 \log \tau$ was found in [10]⁵ and it was shown to reproduce the Moore's expansion of the free energy $F = \log \tau$ [8]. The equation (7.1) with a_{\pm} given in (7.2) was first found in [11] in the form

$$e^{Rh_{\pm}} - z_{\pm} e^{h_{\pm}} = 1, \quad z_{\pm} = e^{-\chi/2R^2} a_{\pm} x_{\pm}^{-1/R}, \quad (7.4)$$

where $h_{\pm}(x_{\pm})$ is the generating function of the one-point correlators

$$h_{\pm}(x_{\pm}) = \sum_{n=1}^{\infty} \frac{1}{n} x_{\pm}^{-n/R} \frac{\partial^2 F}{\partial \mu \partial t_{\pm n}}. \quad (7.5)$$

It is related to $\omega(x_{\pm})$ as follows

$$\omega(x_{\pm}) = e^{\frac{1}{2R}\chi} x_{\pm} e^{-Rh_{\pm}(x_{\pm})}. \quad (7.6)$$

The equation (7.2) for the susceptibility imposes some restrictions on the allowed values of the parameters $t_{\pm 1}$ and μ . Let us consider the interval of the radii $\frac{1}{2} < R < 1$, which is the most interesting from the point of view of the conjectured duality between the Sine-Liouville theory and the 2D string theory in the black hole background at $R = 2/3$ [3]. In this interval, if we choose $t_1 t_{-1} < 0$, a real solution for χ exists only for $\mu > 0$. On the other hand, if $t_1 t_{-1} > 0$, the allowed interval is $\mu > \mu_c$, where the critical value is negative:

$$\mu_c = - \left(2 - \frac{1}{R}\right) \left(\frac{1}{R} - 1\right)^{\frac{1}{2R-1}} \left(\frac{t_1 t_{-1}}{R}\right)^{\frac{R}{2R-1}}. \quad (7.7)$$

⁵Since the papers [10, 11, 12, 30] considered the case of vortex perturbations, the duality transformation $R \rightarrow 1/R$, $\mu \rightarrow R\mu$, $t_n \rightarrow R^{-nR/2} t_n$ should be performed before comparing the formulas.

Let us note that (7.7) was interpreted in [31] as a critical point of the pure 2D gravity type.

The solution (7.1)-(7.2) allows us to find the explicit shape of the Fermi sea in the phase space. In the model of the type I considered so far there is an additional restriction for the admissible values of parameters since the boundary of the Fermi sea should be entirely in the positive quadrant $x_{\pm} > 0$. This forces us to take $t_{\pm 1} > 0$. The general situation is shown on Fig. 1a. The unperturbed profile corresponds to the hyperbola asymptotically approaching the x_{\pm} axis, whereas the perturbed curves deviate from the axes by a power law. We see that there is a critical value of μ , where the contour forms a spike. It coincides with the critical point $\mu = \mu_c$ given by (7.7). At this point the quasiclassical description breaks down. On the Fig. 1b, the physical time evolution of a profile is demonstrated.

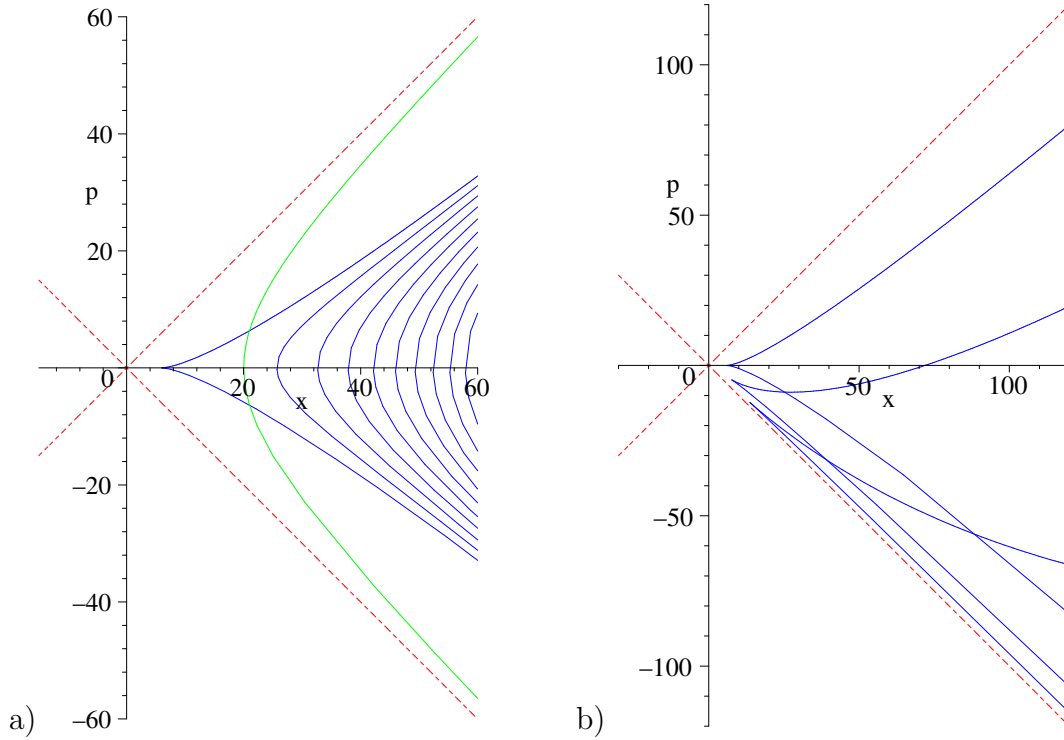


Fig. 1: Profiles of the Fermi sea ($x_{\pm} = x \pm p$) in the theory of the type I at $R = 2/3$. Fig. 1a contains several profiles corresponding to $t_1 = t_{-1} = 2$ and values of μ starting from $\mu_c = -1$ with a step 40. For comparison, we also drew the unperturbed profile for $\mu = 100$. Fig. 1b shows three moments of the time evolution of the critical profile at $\mu = -1$.

In the same way one can find the solution in the classical limit of the theory of the type II described in Appendix A. In this case we can introduce two pairs of perturbing potentials describing the asymptotics of the wave functions at $x_{\pm} \rightarrow \infty$ and $x_{\pm} \rightarrow -\infty$. For sufficiently large μ the Fermi sea consists of two connected components and the theory decomposes into two theories of type I. However, in contrast to the previous case, there are no restrictions on the signs of the coupling constants. When μ decreases, the two Fermi seas merge together at some critical value μ^* . This leads to interesting (for example, from the point of view of the Hall effect) phenomena, which we intend to discuss elsewhere. Here we will only mention that, depending on the choice of couplings, it can happen that for some interval of μ around the point μ^* , the Toda description is not applicable.

8 Free energy of the perturbed background

As we mentioned, the perturbations considered above appear in a theory at the finite temperature $T = 1/\beta$, $\beta = 2\pi R$. The free energy per unit volume \mathcal{F} is related to the partition function \mathcal{Z} by

$$\mathcal{Z} = e^{-\beta\mathcal{F}}. \quad (8.1)$$

The free energy is considered as a function of the chemical potential μ , so that the number of fermions is given by

$$N = \partial\mathcal{F}/\partial\mu. \quad (8.2)$$

We have seen that any perturbation can be characterized by the profile of the Fermi sea. For a generic profile, the number of fermions is given by the volume of the domain in the phase space occupied by the Fermi liquid

$$N = \frac{1}{2\pi} \int \int_{\text{Fermi sea}} dx_+ dx_-. \quad (8.3)$$

It is implied that the integral is regularized by introducing a cut-off at distance $\sqrt{\Lambda}$. For the unperturbed ground state (5.1), one reproduces from (8.2) and (8.3) the well known result for the (universal part of the) free energy

$$\mathcal{F} = \frac{1}{4\pi} \mu^2 \log \mu. \quad (8.4)$$

The authors of [7] identify the partition function (8.1) for the tachyon backgrounds studied in sect. 6 with the τ -function of the Toda hierarchy, defined by (6.28). In this section we will reproduce this statement by direct integration over the Fermi sea, adjusting the above derivation of (8.4) to the case of a general profile of the form (6.36).

Since the change of variables described by eq. (6.30) is canonical (see eq. (6.32)), one can rewrite the integral (8.3) as

$$N = \frac{1}{2\pi} \int_{-\mu}^{\omega_+(E)} dE \int_{\omega_-(E)}^{\omega_+(E)} \frac{d\omega}{\omega}. \quad (8.5)$$

The limits of integration over ω are determined by the cut-off.⁶ If we put it at the distance $x_{\pm} = \sqrt{\Lambda}$, they can be found from the equations

$$x_{\pm}(\omega_{\pm}(E), E) = \sqrt{\Lambda}. \quad (8.6)$$

Taking the derivative with respect to μ , we obtain

$$\partial_{\mu} N = -\frac{1}{2\pi} \log \frac{\omega_+(-\mu)}{\omega_-(-\mu)}. \quad (8.7)$$

⁶The integral over E is also bounded from below, but we do not need to specify the boundary explicitly.

It is enough to keep only the leading order in the cut-off Λ for the boundary values ω_{\pm} . From (8.6) and (6.30) we find to this order

$$\omega_{\pm} = \left(\Lambda e^{\chi/R} \right)^{\pm 1/2}. \quad (8.8)$$

Combining (8.7) and (8.8), we find

$$\partial_{\mu} N = -\frac{1}{2\pi} \log \Lambda - \frac{1}{2\pi R} \chi. \quad (8.9)$$

Taking into account the relation (8.2) between the free energy and N , we see that

$$\mathcal{F} = -\frac{1}{\beta} \log \tau, \quad \beta = 2\pi R. \quad (8.10)$$

As a result, if we compactify the theory at the time circle of the length β , our free energy coincides with the logarithm of the Toda τ -function.

The perturbed flow of the Fermi liquid is non-stationary with respect to the physical time t associated with the original Hamiltonian H_0 . In sec. 5 we constructed a new Hamiltonian, which preserves the shape of the Fermi sea. The latter coincides with the phase-space trajectory (5.11) at $E = -\mu$. Using the fact that $\partial_{\mu} H(x_+, x_-) = 0$, one can check that

$$\mathcal{F} = \langle H(x_+, x_-) + \mu \rangle. \quad (8.11)$$

Note that the function $H(x_+, x_-)$ is obtained by solving the profile equations (6.36) with respect to μ . One can see, looking at the large x_{\pm} asymptotics of the profile equations, that it can be written in the following compact form:

$$H(x_+, x_-) = H_0 + \sum_{k \geq 1} \frac{k}{R} (t_k H_k + t_{-k} H_{-k}). \quad (8.12)$$

Here the Hamiltonians $H_n(x_+, x_-; \vec{t})$ are given by equations (6.21), with ω and E expressed as functions of x_+ and x_- through eq. (6.30).

9 Conclusions

In this paper we studied the $2D$ string theory in the presence of arbitrarily strong tachyonic perturbations. In the matrix model language, we dealt with the singlet sector of the MQM described by a system of free fermions in inverted quadratic potential. In the quasiclassical limit the state of the system is described by the shape of the domain in the phase space occupied by the fermionic liquid (the profile of the Fermi sea). In this limit we formulated functional equations, which determine the shape of the Fermi sea, given its asymptotics at two infinities. There is some analogy between our problem and the problem of conformal maps studied in [32, 33, 34, 35].

This allows to calculate various physical quantities related to the perturbed shape, such as the free energy of the compactified Euclidean theory, in terms of the parameters defining the asymptotic shapes.

For a particular case of perturbations generated by vertex operators (in- and out-going with discrete equally spaced imaginary energies), the system solves the Toda lattice hierarchy where the Toda “times” correspond to the couplings characterizing the asymptotics of the profile of the Fermi sea. Using the “light cone” representation in the phase space, we developed the Lax formalism of the constrained Toda hierarchy, where the constraint coincides with the string equation found recently in [12]. In the dispersionless limit, the Lax formalism reproduces the functional equations for the shape of the perturbed Fermi sea and allows to give a direct physical meaning to the Toda mathematical quantities. In particular, we identified the two spectral parameters of the Toda system with the two chiral coordinates x_{\pm} in the target phase space.

Let us list the other possible applications of our formalism:

- Calculation of the tachyon scattering matrix for 2D string theory in nontrivial backgrounds.
- Analysis of the systems with more complicated than Sine-Liouville backgrounds. In particular, the quasiclassical analysis of the sections 4 and 7 are not limited to any lattice of tachyon charges and can be used for a mixture of any commensurate or non-commensurate charges. It provides a way of studying the whole space of possible tachyonic backgrounds of the 2D string theory.
- Investigation of the thermodynamics of the 2D string theory in Sine-Liouville type tachyonic backgrounds characterized by a lattice of discrete Matsubara energies [30, 36]. This is supposed to be the thermodynamics of the dilatonic black hole, according to the conjecture of [3].
- The type 2 theory can be used to model the situation where either two quantum Hall droplets approach each other or one droplet is about to split in two.

10 Problems and proposals

We list also some unsolved problems, which could be approached by our formalism:

- To find the metric of the target space for the Sine-Liouville string theory.
- To construct the discrete states of the 2D string theory (at any rational compactification radius) in the framework of our formalism.
- A few important unsolved problems concern the relation between the MQM and CFT formulations of the 2D string theory. We still don’t have a precise mapping of the vertex and vortex operators of the MQM to the corresponding operators of CFT, in spite of the useful suggestions of the papers [37, 4]. As the result, we did not manage to match the Sine-Liouville/Black Hole correlators calculated by [3, 38, 39] from the CFT approach with the corresponding correlators found in [11] from the Toda approach to the MQM. Another related question is how can the target space integrable structure be seen in the CFT formulation of 2D string theory.
- Description and quantitative analysis of the mixed (vertex&vortex) perturbations in the 2D string theory by means of the MQM. We hope that the “light-cone” formalism might be appropriate to study such perturbations, in spite of the absence of integrability.⁷

⁷An interesting proposal to describe nontrivial string backgrounds on the CFT side was given long ago in [40] and elaborated in [41, 42]. It suggested to parameterize the moduli space of the 2D string theory perturbed in both sectors, by a conifold. This structure may be hidden in the 3 matrix model proposed

Finally, keeping in mind this interesting problem, let us propose here a 3-matrix model whose grand canonical partition function $\mathcal{Z}(\mu, t_n, \tilde{t}_n) = \sum_{N=0}^{\infty} e^{-2\pi R \mu N} Z_N(t_n, \tilde{t}_n)$ is the generating function of correlators for both types of perturbations: the parameters \tilde{t}_n are the couplings of the vortex operators, in the same way as t_n 's are the couplings of the tachyon (vertex) operators throughout this paper. As usual, the chemical potential μ plays the role of the string coupling. The partition function at fixed N is represented by the integral over two hermitian matrices X_{\pm} and one unitary matrix, the holonomy factor Ω around the circle

$$Z_N(t_n, \tilde{t}_n) = \int [\mathcal{D}X_+]_{\{t_n\}} [\mathcal{D}X_-]_{\{t_{-n}\}} [d\Omega]_{\{\tilde{t}_{\pm n}\}} e^{\text{Tr}(X_- X_+ - q X_- \Omega X_+ \Omega^{-1})}, \quad (10.1)$$

where we denoted $q = e^{i2\pi R}$ and introduced the following matrix integration measures

$$[dX_{\pm}]_{\{t_{\pm n}\}} = dX_{\pm} e^{\sum_{n>0} t_{\pm n} \text{Tr} X_{\pm}^{n/R}}, \quad (10.2)$$

$$[d\Omega]_{\{\tilde{t}_{\pm n}\}} = [d\Omega]_{SU(N)} e^{\sum_{n \neq 0} \tilde{t}_n \text{Tr} \Omega^n}. \quad (10.3)$$

This 3-matrix integral is nothing but the Euclidean partition function of the upside-down matrix oscillator on a circle of length $2\pi R$ with twisted periodic boundary conditions $X_{\pm}(\beta) = \Omega^{\dagger} X_{\pm}(0) \Omega$, needed for the introduction of the winding modes (vortices) in the MQM model of the 2D string theory [14]. It is trivial to show that in particular case of all $t_n = 0$ this 3-matrix model reduces to the model with only vortex perturbations of the paper [10]. It can be also shown [36] that in the case of all $\tilde{t}_n = 0$ it reduces to the Euclidean version of the model with only tachyonic perturbations studied in this paper. The crucial point here is that the sources for tachyon perturbations are introduced in a single point on the time circle, similarly to the perturbation imposing the initial conditions on the wave function (5.3). The model (10.1) reduces the problem of the study of the backgrounds of the compactified 2D string theory with arbitrary tachyon and winding sources, to a three-matrix integral.

We learned from R. Dijkgraaf and C. Vafa that they discovered a similar Toda structure in the $c = 1$ type theory arising in connection to the topological strings.

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Appendix A. Theory of type II

In the theory of type II, the fermions are defined on the whole real line. In this case one should introduce two sets of functions describing, in the quasiclassical limit, fermions at

below.

different sides of the potential. They are defined on right (left) semi-axis by

$$\psi_{\pm, >}^E(x_{\pm}) = \frac{1}{\sqrt{2\pi}} \frac{x_{\pm}^{\pm iE - \frac{1}{2}}}{\sqrt{1 + e^{2\pi E}}} \quad (x_{\pm} > 0), \quad (\text{A.1})$$

$$\psi_{\pm, <}^E(x_{\pm}) = \frac{1}{\sqrt{2\pi}} \frac{(-x_{\pm})^{\pm iE - \frac{1}{2}}}{\sqrt{1 + e^{2\pi E}}} \quad (x_{\pm} < 0) \quad (\text{A.2})$$

and the continuation to the other semi-axis is performed according to the rule $x_{\pm} \rightarrow e^{\mp \pi i} x_{\pm}$. This gives, for $x_{\pm} > 0$

$$\psi_{\pm, >}^E(-x_{\pm}) = \pm i e^{\pi E} \psi_{\pm, >}^E(x_{\pm}), \quad (\text{A.3})$$

$$\psi_{\pm, <}^E(x_{\pm}) = \pm i e^{\pi E} \psi_{\pm, <}^E(-x_{\pm}). \quad (\text{A.4})$$

The functions (A.1) satisfy the orthonormality and completeness conditions

$$\langle \psi_{\pm, a}^E | \psi_{\pm, b}^{E'} \rangle = \delta^{ab} \delta(E - E'), \quad (\text{A.5})$$

$$\int_{-\infty}^{\infty} dE \sum_a \overline{\psi_{\pm, a}^E(x_{\pm})} \psi_{\pm, a}^E(x'_{\pm}) = \delta(x_{\pm} - x'_{\pm}), \quad (\text{A.6})$$

where the indices $a, b = (>, <)$ label the right and left wave functions and the scalar product is defined as in (4.5) but with the integral over the whole real axis.

The unitary operator relating x_{\pm} representations is given by the Fourier kernel on the whole line $K(x_{-}, x_{+}) = \frac{1}{\sqrt{2\pi}} e^{ix_{+}x_{-}}$

$$[\hat{S}\psi_{+}](x_{-}) = \int_{-\infty}^{\infty} dx_{+} K(x_{-}, x_{+}) \psi_{+}(x_{+}). \quad (\text{A.7})$$

It acts on the eigenfunctions (A.1) by the following matrix

$$[\hat{S}^{\pm 1} \psi_{\pm, a}^E](x_{\mp}) = \sum_b [S^{\pm 1}(E)]^{ab} \psi_{\pm, b}^E(x_{\mp}), \quad S(E) = \mathcal{R}(E) \begin{pmatrix} 1 & -ie^{\pi E} \\ -ie^{\pi E} & 1 \end{pmatrix}, \quad (\text{A.8})$$

where

$$\mathcal{R}(E) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\pi}{2}(E - i/2)} \Gamma(iE + 1/2). \quad (\text{A.9})$$

In this case the reflection coefficient $\mathcal{R}(E)$ is not a pure phase because of the tunneling through the potential described by the off-diagonal elements of the S -matrix. Nevertheless, the whole S -matrix is unitary

$$S^{\dagger}(E)S(E) = 1. \quad (\text{A.10})$$

The scalar product between left and right states is given by

$$\langle \psi_{-} | K | \psi_{+} \rangle = \int_{-\infty}^{\infty} dx_{+} dx_{-} \overline{\psi_{-}(x_{-})} K(x_{-}, x_{+}) \psi_{+}(x_{+}). \quad (\text{A.11})$$

It is clear that the matrix of scalar products of states (A.1) coincides with $S(E)\delta(E - E')$. The corresponding completeness condition is

$$\int_{-\infty}^{\infty} dE \sum_{a, b} \overline{\psi_{-, a}^E(x_{-})} (S^{-1})_{ab} \psi_{+, b}^E(x_{+}) = \frac{1}{\sqrt{2\pi}} e^{-ix_{+}x_{-}}. \quad (\text{A.12})$$

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Article III

Thermodynamics of 2D string theory

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Thermodynamics of 2D string theory

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We calculate the free energy, energy and entropy in the matrix quantum mechanical formulation of 2D string theory in a background strongly perturbed by tachyons with the imaginary Minkowskian momentum $\pm i/R$ (“Sine-Liouville” theory). The system shows a thermodynamical behaviour corresponding to the temperature $T = 1/(2\pi R)$. We show that the microscopically calculated energy of the system satisfies the usual thermodynamical relations and leads to a non-zero entropy.

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1 Introduction

Nontrivial backgrounds in the string theory often have a thermodynamical behaviour corresponding to a certain temperature and a classically big entropy. Typical examples are the black holes where the thermodynamics manifests itself by the Hawking radiation at a certain temperature T_H and the Bekenstein-Hawking entropy. The latter is expressed through the classical parameters of the system (the area of the horizon) and satisfies the standard thermodynamical relations with the free energy and energy (mass of the black hole).¹

In this paper we will try to demonstrate that the thermodynamical behaviour with a temperature T is a rather natural phenomenon in the 2-dimensional string theory in specific backgrounds created by tachyon sources with Euclidean momenta corresponding to the Matsubara frequencies $2\pi k/T$, $k = 1, 2, \dots$. For $k = 1$ this system is T-dual to the so called “Sine-Liouville” theory conjectured to describe 2D string theory on the Euclidean black hole background [3]. The latter can be obtained introducing a vortex source with the Euclidean time compactified on a radius $R = 1/2\pi T$. It was studied in [4] and further in [5, 6] in the matrix quantum mechanical (MQM) formulation using the integrability properties.

We will use here the MQM in the singlet sector to study this system in Minkowskian time, perturbed by a source of “tachyons” with imaginary momenta. This approach, elaborated in the paper [7], using some early ideas of [8], extensively relies on the classical Toda integrability and the representation of the system in terms of free fermions arising from the eigenvalues of the matrix field. We will demonstrate that the microscopically defined energy of the system, calculated as the energy of (perturbed) Fermi sea, coincides with its thermodynamical counterpart E calculated as the derivative of the free energy with respect to the temperature. The free energy itself can be found from the Fermi sea due to its relation with the number of particles $N = \partial\mathcal{F}/\partial\mu$.

This coincidence of the microscopic and thermodynamical energies looks satisfactory, though a little bit mysterious to us. We could not figure out here where the entropy, also following from this derivation, comes from since the system looks like a single classical state of the moving Fermi liquid. The formulas become especially simple in the black hole limit proposed in [4], where the energy and entropy dominate with respect to the free energy. On the other hand, the entropy naturally disappears in the opposite limit of the trivial linear dilaton background where the perturbation source is absent.

The entropy seems to be more visible in the T-dual formulation of the theory based on the vortex perturbation in the compact Euclidean time where we deal with the higher representations of the $SU(N)$ symmetry of the MQM [9]. However the direct counting of the nonsinglet states appears to be technically a difficult problem. But we still think that our observation could shed light on the origin of the entropy of the black hole type solutions in the string theory.

The paper is organized as follows. In the next section we review the fermionic representation of $c = 1$ string theory. In section 3 we show how one can describe the tachyon perturbations in this formalism. Section 4 presents the explicit solution for the particular case of the Sine-Liouville theory. In section 5 we evaluate the grand canonical free energy and energy of the system using the techniques of integration over the Fermi sea. In section 6

¹For review of black hole physics see [1, 2].

we demonstrate that the quantities found in the previous section satisfy the thermodynamical relations and give rise to a nonvanishing entropy. The parameter R of the perturbations can be interpreted as the inverse temperature. Finally, we discuss our results and remaining problems.

2 The $c = 1$ string theory as the collective theory of the Fermi sea

The $c = 1$ string theory can be formulated in terms of collective excitations of the Fermi sea of the upside-down harmonic oscillator [10, 11, 12, 13]. The tree-level tachyon dynamics of the string theory is contained in the semiclassical limit of the ensemble of fermions, in which the fermionic density in the phase space is either one or zero. Each state of the string theory corresponds to a particular configuration of the incompressible Fermi liquid [14, 15, 16].

In the quasi-classical limit the motion of the fermionic liquid is determined by the classical trajectories of its individual particles in the phase space. It is governed by the one-particle Hamiltonian

$$H_0 = \frac{1}{2} (p^2 - x^2), \quad (2.1)$$

where the coordinate of the fermion x stands for an eigenvalue of the matrix field and $p = -i\frac{\partial}{\partial x}$ is its conjugated momentum. We introduce the chiral variables

$$x_{\pm}(t) = \frac{x(t) \pm p(t)}{\sqrt{2}}, \quad (2.2)$$

and define the Poisson bracket as $\{f, g\} = \frac{\partial f}{\partial x_-} \frac{dg}{\partial x_+} - \frac{\partial f}{\partial x_+} \frac{dg}{\partial x_-}$. In these variables $H_0 = -x_+ x_-$ and the equations of motion have a simple solution

$$x_{\pm}(t) = e^{\pm t} x_{\pm}, \quad (2.3)$$

where the initial values z_{\pm} parameterize the points of the Fermi sea. Each trajectory represents a hyperbole $H_0(p, x) = E$. The state of the system is completely characterized by the profile of the Fermi sea that is a curve in the phase space which bounds the region filled by fermions. For the ground state, the Fermi sea is made by the classical trajectories with $E < -\mu$, and the profile is given by the hyperbole

$$x_+ x_- = \mu. \quad (2.4)$$

The state is stationary since the Fermi surface coincides with one of the classical trajectories and thus the form of the Fermi sea is preserved in time. For an arbitrary state of the Fermi sea, the Fermi surface can be defined more generally by

$$x_+ x_- = M(x_+, x_-). \quad (2.5)$$

It is clear from (2.3) that a generic function M in (2.5) leads to a time-dependent profile. However, this dependence is completely defined by (2.3) and it is of little interest to us. We can always replace (2.5) by the equation for the initial values x_{\pm} .

For a given profile, we define the energy of the system and the number of fermions as the following integrals over the Fermi sea

$$E_0 = \frac{1}{2\pi} \int \int_{\text{Fermi sea}} dx dp H_0(x, p), \quad N = \frac{1}{2\pi} \int \int_{\text{Fermi sea}} dx dp. \quad (2.6)$$

It is implied that the integrals are bound by a cut-off at a distance $\sqrt{2\Lambda}$. For example, we can restrict the integration to $0 < z_{\pm} < \sqrt{\Lambda}$. For the ground state (2.4), dropping non-universal terms proportional to the cut-off one reproduces the well known result [17]

$$E_0(\mu) = -\frac{1}{4\pi} \mu^2 \log(\mu/\Lambda), \quad N(\mu) = \frac{1}{2\pi} \mu \log(\mu/\Lambda). \quad (2.7)$$

Given the number of fermions, one can also introduce the grand canonical free energy through the relation

$$N = \partial \mathcal{F} / \partial \mu, \quad (2.8)$$

where \mathcal{F} is given in terms of the partition function for a time interval T by

$$\mathcal{F} = -\log \mathcal{Z} / T. \quad (2.9)$$

For the case of the ground state, one finds from (2.7) that the (universal part of the) grand canonical free energy is related to the energy of the fermions as

$$\mathcal{F} = E_0 + \mu N = \frac{1}{4\pi} \mu^2 \log(\mu/\Lambda). \quad (2.10)$$

3 The profile of the Fermi sea for time dependent tachyon backgrounds

The ground state (2.4) describes the simplest, linear dilaton background of the bosonic string. Now we would like to study more general backgrounds characterized by condensation of tachyons with nonzero momenta. Such backgrounds correspond to time-dependent profiles (2.5), characterized by the function $M(x_+, x_-)$. We restrict ourselves to the states that contain only tachyon excitations whose momenta belong to a discrete lattice

$$p_n = in/R, \quad n \in \mathbf{Z}. \quad (3.1)$$

They look as Matsubara frequencies corresponding to the temperature $T = 1/(2\pi R)$, or to the compactification in the Euclidean time with the radius R . We will perturb the system in the same manner as in [7]. It has been shown that at the quasiclassical level the perturbations can be completely characterized by the asymptotics of the profile of the Fermi sea at $x_+ \gg x_-$ and $x_- \gg x_+$. Accordingly, the equation for the profile can be written in two forms which should be compatible with each other [7]

$$x_+ x_- = M_{\pm}(x_{\pm}) = \frac{1}{R} \sum_{k=1}^n k t_{\pm k} x_{\pm}^{k/R} + \mu + \frac{1}{R} \sum_{k=1}^{\infty} v_{\pm k} x_{\pm}^{-k/R}, \quad (3.2)$$

where $t_{\pm k}$ are coupling constants associated with the perturbing operators and defining the asymptotics of the Fermi sea for $z_+ \gg z_-$ and $z_- \gg z_+$ correspondingly. $v_{\pm k}$ are the coefficients to be found, which completely fix the form of the profile.

It was shown in [7] that such perturbations are described by the dispersionless limit of a constrained Toda hierarchy. The phase space coordinates x_{\pm} play the role of the Lax operators, $t_{\pm k}$ are the Toda times, and eq. (3.2) appears as the constraint (string equation) of the hierarchy [6].

The integrability allows to find the explicit solution for the profile. It is written in the parametric form as follows

$$x_{\pm}(\omega, \mu) = e^{-\frac{1}{2R}\chi(\mu)} \omega^{\pm 1} \left(1 + \sum_{k=1}^n a_{\pm k}(\mu) \omega^{\mp k/R} \right), \quad (3.3)$$

where χ is the so called string susceptibility related to the Toda τ -function as $\chi = \partial_{\mu}^2 \log \tau$. The latter is the generating function for the coefficients $v_{\pm k}$ in (3.2)

$$v_{\pm k} = \partial_{t_{\pm k}} \log \tau, \quad (3.4)$$

which can be identified with one-point vertex correlators [5, 7]. Moreover, the change of variables $x_{\pm} \rightarrow \log \omega, \mu$ turns out to be canonical, *i.e.*,

$$\{\mu, \log \omega\} = 1. \quad (3.5)$$

To find the coefficients $a_{\pm k}$ it is enough to use a simple procedure suggested in [6]. One should substitute the expressions (3.3) in the profile equations (3.2) and compare the coefficients in front of $\omega^{\pm k/R}$.

4 Solution for Sine-Gordon coupled to gravity

Let us restrict ourselves to the case $n = 1$ corresponding to the Sine-Gordon theory coupled to gravity or the so called Sine-Liouville theory. In this case there are only $t_{\pm 1}$ coupling constants and the equation (3.3) takes the form [5]

$$x_{\pm} = e^{-\frac{1}{2R}\chi} \omega^{\pm 1} (1 + a_{\pm} \omega^{\mp \frac{1}{R}}), \quad (4.1)$$

where χ can be found from [4, 18]

$$\mu e^{\frac{1}{R}\chi} + \frac{1}{R^2} \left(\frac{1}{R} - 1 \right) t_1 t_{-1} e^{\frac{2R-1}{R^2}\chi} = 1, \quad a_{\pm} = \frac{t_{\mp 1}}{R} e^{\frac{2R-1}{2R^2}\chi}. \quad (4.2)$$

Assuming that $\frac{1}{2} < R < 1$ (which means that the corresponding compactification radius is between the Kosterlitz-Thouless and the self-dual one), the first equation for the susceptibility can also be rewritten in terms of scaling variables:

$$w = \mu \xi, \quad \xi = \left(\frac{\lambda}{R} \sqrt{1/R - 1} \right)^{-\frac{2R}{2R-1}}, \quad (4.3)$$

where $\lambda = \sqrt{t_1 t_{-1}}$. The result reads

$$\begin{aligned} \chi &= R \log \xi + X(w), \\ w &= e^{-\frac{1}{R}X} - e^{-\frac{R-1}{R}X}. \end{aligned} \quad (4.4)$$

5 Free energy and energy of the perturbed background

Let us consider the average over the Fermi sea of an observable \mathcal{O} defined on the phase space

$$\langle \mathcal{O} \rangle = \frac{1}{2\pi} \int_{\text{Fermi sea}} \int dx_+ dx_- \mathcal{O}(x_+, x_-). \quad (5.1)$$

Here the boundary of the integration is defined by the profile equation (3.2). Since the change of variables described by eq. (3.3) is canonical (see eq. (3.5)), one can write

$$\langle \mathcal{O} \rangle = \frac{1}{2\pi} \int_{\mu} ds \int_{\omega_-(s)}^{\omega_+(s)} \frac{d\omega}{\omega} \mathcal{O}(x_+(\omega, s), x_-(\omega, s)). \quad (5.2)$$

The limits of integration over ω are defined by the cut-off. We choose it as two walls at $x_+ = \sqrt{\Lambda}$ and $x_- = \sqrt{\Lambda}$. Then the limits can be found from the equations

$$x_{\pm}(\omega_{\pm}(s), s) = \sqrt{\Lambda}. \quad (5.3)$$

First, consider the case of $\mathcal{O} = 1$. Then the integral (5.1) gives the number of fermions in the Fermi sea, i.e., $N = \langle 1 \rangle$. Taking the derivative with respect to μ , we obtain

$$\partial_{\mu} N = -\frac{1}{2\pi} \log \frac{\omega_+(\mu)}{\omega_-(\mu)}. \quad (5.4)$$

In this case it is enough to restrict ourselves to the main order in the cut-off Λ for the boundary values ω_{\pm} . From (5.3) and (3.3) we find to this order

$$\omega_{\pm} = \omega_{(0)}^{\pm 1}, \quad \omega_{(0)} = \sqrt{\Lambda e^{\chi/R}}. \quad (5.5)$$

Combining (5.4) and (5.5), we find

$$\partial_{\mu} N = -\frac{1}{2\pi} \log \Lambda - \frac{1}{2\pi R} \chi. \quad (5.6)$$

Taking into account the relation (2.8), we find that the free energy coincides with the logarithm of the τ -function and $\beta = 2\pi R$ represents the time interval at which the theory is compactified

$$\mathcal{F} = -\frac{1}{\beta} \log \tau. \quad (5.7)$$

This derivation of the grand canonical free energy was done in [7]. However, using the general formula (5.2), we can also find the energy of the system, at least for the case of the Sine-Liouville theory. Since it is given by the integral of the free Hamiltonian (2.6), from (5.2) and (4.1) one obtains

$$\partial_{\mu} E = -\partial_{\mu} \langle x_+ x_- \rangle = \frac{1}{2\pi} e^{-\frac{1}{R}\chi} \left\{ (1 + a_+ a_-) \log \omega - R a_+ \omega^{-1/R} + R a_- \omega^{1/R} \right\} \Big|_{\omega_-}^{\omega_+}. \quad (5.8)$$

To find the final expression one needs to know the limits ω_{\pm} in the second order in the cut-off. It is easy to show from (5.3) that eq. (5.5) should be changed by

$$\omega_{\pm} = \omega_{(0)}^{\pm 1} (1 \mp a_{\pm} \omega_{(0)}^{-1/R}). \quad (5.9)$$

Substituting these limits into (5.8), one finds

$$\partial_{\mu} E = \frac{1}{2\pi} e^{-\frac{1}{R}\chi} \left\{ (1 + a_+ a_-) \left(\log \Lambda + \frac{1}{R}\chi \right) - 2a_+ a_- \right\} + \frac{R}{2\pi} (t_1 + t_{-1}) \Lambda^{1/2R}. \quad (5.10)$$

Integrating over μ and using $a_+ a_- = \frac{R}{1-R} e^{\frac{2R-1}{R^2}\chi}$, we get

$$\begin{aligned} 2\pi E &= \xi^{-2} \left(\frac{1}{2R} e^{-\frac{2}{R}\chi} + \frac{2R-1}{R(1-R)} e^{-\chi} - \frac{1}{2(1-R)} e^{-2\frac{R-1}{R}\chi} \right) (\chi + R \log \Lambda) \\ &+ \xi^{-2} \left(\frac{1}{4} e^{-\frac{2}{R}\chi} - \frac{R}{1-R} e^{-\chi} + \frac{R(4-5R)}{4(1-R)^2} e^{-2\frac{R-1}{R}\chi} \right) + R(t_1 + t_{-1}) \mu \Lambda^{1/2R}. \end{aligned} \quad (5.11)$$

We observe that the last term is non-universal since it does not contain a singularity at $\mu = 0$. However, $\log \Lambda$ enters in a non-trivial way. It is combined with a non-trivial function of μ and ξ .

6 Thermodynamics of the system

We found above the energy of the Sine-Liouville theory (5.11) and also restored the free energy in the grand canonical ensemble for a general perturbation. In the Sine-Liouville case one can integrate the equation (4.4) to get the explicit expression [4]

$$2\pi \mathcal{F} = -\frac{1}{2R} \mu^2 (\chi + R \log \Lambda) - \xi^{-2} \left(\frac{3}{4} e^{-\frac{2}{R}\chi} - \frac{R^2 - R + 1}{1-R} e^{-\chi} + \frac{3R}{4(1-R)} e^{-2\frac{R-1}{R}\chi} \right). \quad (6.1)$$

Now we are going to establish thermodynamical properties of our system. It is expected to possess them since the spectrum of perturbations corresponds to the Matsubara frequencies typical for a system at a nonzero temperature. First of all, note that in the standard thermodynamical relations the free energy appears in the canonical, rather than in grand canonical ensemble. Therefore we define

$$F = \mathcal{F} - \mu \frac{\partial \mathcal{F}}{\partial \mu}. \quad (6.2)$$

It is easy to check that it can be written as

$$\begin{aligned} 2\pi F &= \frac{1}{R} \int^{\mu} s \chi(s) ds = \frac{1}{2R} \mu^2 (\chi + R \log \Lambda) \\ &+ \xi^{-2} \left(\frac{1}{4} e^{-\frac{2}{R}\chi} - R e^{-\chi} + \frac{R}{4(1-R)} e^{-2\frac{R-1}{R}\chi} \right). \end{aligned} \quad (6.3)$$

Following the standard thermodynamical relations, let us also introduce the entropy as a difference of the energy and the free energy:

$$S = \beta(E - F). \quad (6.4)$$

One finds from (5.11) and (6.3)

$$\begin{aligned}
S &= \xi^{-2} \left(\frac{R}{1-R} e^{-X} - \frac{1}{2(1-R)} e^{-2\frac{R-1}{R}X} \right) (\chi + R \log \Lambda) \\
&+ \xi^{-2} \left(-\frac{R^3}{1-R} e^{-X} + \frac{R^2(3-4R)}{4(1-R)^2} e^{-2\frac{R-1}{R}X} \right) + \frac{R^2}{2\pi} (t_1 + t_{-1}) \mu \Lambda^{1/2R}.
\end{aligned} \tag{6.5}$$

Thus, all quantities are supplied by a thermodynamical interpretation. The role of the temperature is played by $T = \beta^{-1} = 1/2\pi R$. However, to have a consistent picture, the following (equivalent) thermodynamical relations should be satisfied:

$$\begin{aligned}
S &= -\frac{\partial F}{\partial T} = 2\pi R^2 \frac{\partial F}{\partial R}, \\
E &= \frac{\partial(\beta F)}{\partial \beta} = \frac{\partial(RF)}{\partial R}.
\end{aligned} \tag{6.6}$$

In these relations it is important which parameters are hold to be fixed when we evaluate the derivatives with respect to R . Since we use the free energy in the canonical ensemble, one should fix the number of fermions N and the coupling constant λ but not the chemical potential μ .

Another subtle point is that *a priori* the relation of our initial parameters to the corresponding parameters which are to be fixed may contain a dependence on the inverse temperature R . Therefore, in general we should redefine

$$\begin{aligned}
\xi &\longrightarrow \xi = (a(R)\lambda)^{-\frac{2}{2-R}}, \\
\Lambda &\longrightarrow b(R)\Lambda, \\
F &\longrightarrow F + \frac{1}{2}c(R)\mu^2.
\end{aligned} \tag{6.7}$$

We find the functions $a(R)$, $b(R)$ and $c(R)$ from the condition (6.6), where N , λ and Λ are fixed. We emphasize that it looks rather non-trivial to us that it is possible to find such functions of R that this condition is satisfied. It can be considered as a remarkable coincidence indicating that the functions F , E and S do have the interpretation of thermodynamical quantities as free energy, energy and entropy correspondingly. In appendix A it is shown that the parameters should be taken as

$$a(R) = \left(\frac{1-R}{R^3} \right)^{1/2}, \quad b(R) = 1, \quad c(R) = 0. \tag{6.8}$$

It is worth noting several features of this solution. First of all, the plausible property is that we do not redefine the cut-off and the free energy. Moreover, the constant λ coincides with its initial definition (4.3) what probably indicates that λ is correctly identified with the corresponding coupling constant in the conformal Sine-Liouville theory. Another important consequence of the choice (6.8) is that the entropy (6.5) is proportional to λ^2 and therefore vanishes in the absence of perturbations. It appears to be non-zero only when the background of the string theory is described by a time dependent Fermi sea. However, a puzzle still remains how to calculate the entropy in a similar way as the energy was evaluated, that is by directly identifying the corresponding degrees of freedom which give rise to the same macroscopic state and, hence, the entropy.

6.1 The "Black hole" limit

Let us restrict ourselves to the limit $\mu \rightarrow 0$. If we perturbed the 2D string theory by the vortex, instead of the tachyon, modes it would correspond to the black hole limit [3, 4]. In this limit we have $X = 0$. As a result, we obtain from (6.3), (5.11) and (6.5) for all three thermodynamical quantities:

$$\begin{aligned} 2\pi F &= \frac{(2R-1)^2}{4(1-R)} \tilde{\lambda}^{\frac{4R}{2R-1}}, \\ 2\pi E &= \frac{2R-1}{2R(1-R)} \left(\log(\Lambda\xi) - \frac{R}{2(1-R)} \right) \tilde{\lambda}^{\frac{4R}{2R-1}}, \\ S &= \frac{2R-1}{2(1-R)} \left(\log(\Lambda\xi) - \frac{R^2(3-2R)}{2(1-R)} \right) \tilde{\lambda}^{\frac{4R}{2R-1}}, \end{aligned} \quad (6.9)$$

where $\tilde{\lambda} = a(R)\lambda$. It is worth to note an interesting feature of this solution. Since $\log(\Lambda\xi) \gg 1$, we get $2\pi E \approx S$ and in comparison with these quantities F is negligible. This is to be compared with the result of [19] that the analysis of the dilatonic gravity derived from 2D string theory leads to the vanishing free energy and to the equal energy and entropy. We see that it can be true only in a limit, but the free energy can never be exactly zero since it is its derivative with respect to the temperature that produces all other thermodynamical quantities.

For the special case of $R = 2/3$ which is T-dual to the stringy black hole model, we find:

$$\begin{aligned} \beta F &= \frac{1}{18} \tilde{\lambda}^8, \\ \beta E &= \left(\frac{1}{2} \log(\Lambda\xi) - \frac{1}{2} \right) \tilde{\lambda}^8, \\ S &= \left(\frac{1}{2} \log(\Lambda\xi) - \frac{5}{9} \right) \tilde{\lambda}^8. \end{aligned} \quad (6.10)$$

For the dual compactification radius $R = 3/2$, all quantities change their sign:

$$\begin{aligned} \beta F &= -3\tilde{\lambda}^3, \\ \beta E &= -(2\log(\Lambda\xi) - 3) \tilde{\lambda}^3, \\ S &= -2\log(\Lambda\xi) \tilde{\lambda}^3. \end{aligned} \quad (6.11)$$

This might be interpreted as a result for the background dual to the black hole of a negative mass, which does appear in the analysis of [20].

7 Conclusions and problems

The main puzzle of the black hole physics is to find the states corresponding to the entropy of a classical gravitating system. In our case, instead of a black hole, the role of such system is played by a nontrivial tachyonic background. However, it manifests the similar properties as the usual dilatonic black hole, which appears as a T-dual perturbation of the initial

flat spacetime. Therefore, it is not surprising that the studied background also possesses a nonvanishing entropy.

In our case we have a little more hope to track the origin of the entropy, than, for example, in the study of the Schwarzschild black hole, since we are dealing with the well defined string theory and can identify its basic degrees of freedom. It allowed us in this paper to find microscopically the energy and free energy of the “Sine-Liouville” background. The entropy was calculated not independently, but followed from the basic thermodynamical relations. Our main result is the coincidence of this microscopic and the thermodynamical energies. The latter is calculated as a derivative of the free energy with respect to the temperature. Note that the “temperature” appears from the very beginning as a parameter of the tachyonic perturbation, and not as a macroscopic characteristic of a heated system. That is why the thermodynamical interpretation comes as a surprise.

An analogous situation can be seen in the Unruh effect [21]. There the temperature seen by the accelerating observer is defined by the value of the acceleration, which does not *a priori* have any relation to thermodynamics. In fact, in the latter case the temperature can be found from the analysis of the quantum field theory on the Rindler space which is the spacetime seen by the accelerating observer. The same should be true for our case. It should be possible to reproduce the global structure of the tachyonic background from the matrix model solution, and then the thermodynamical interpretation should appear from the analysis of the spectrum of particles detected by some natural observer [22].

One might try to find a hint to the thermodynamical behaviour of our system in the very similar results for the T-dual system of [4] where instead of tachyons we have the perturbation by vortices. There the Euclidean time is compactified from the very beginning and the Gibbs ensemble is thus explicitly introduced with R as a temperature parameter. On the other hand, the coincidence of microscopic energies in two mutually dual formulations (up to duality changes in parameters, like $R \rightarrow 1/R$) is far from obvious. Even how to find the energy in this dual case is not clear.

In conclusion, we hope that our observation will help to find a microscopic approach to the calculation of entropy at least in the 2D string theory with nontrivial backgrounds. Together with the plausible conjecture that the dual system describes the 2D dilatonic black hole, it could open a way for solving the puzzle of the microscopic origin of entropy in the black hole physics.

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Appendix A. Calculation of $\partial F/\partial R$

In this appendix we calculate the derivative of the free energy (6.3) with respect to R . First, note that due to (6.2) and (2.8), we have

$$\left(\frac{\partial F}{\partial R}\right)_{N,\lambda,\Lambda} = \left(\frac{\partial \mathcal{F}}{\partial R}\right)_{\mu,\lambda,\Lambda} = \left(\frac{\partial \mathcal{F}}{\partial R}\right)_{\mu,\xi,\Lambda} + \left(\frac{\partial \xi}{\partial R}\right)_\lambda \left(\frac{\partial \mathcal{F}}{\partial \xi}\right)_{\mu,\Lambda,R}. \quad (\text{A.1})$$

From (6.1) and (6.7) one can obtain

$$2\pi\xi \left(\frac{\partial \mathcal{F}}{\partial \xi}\right)_{\mu,\Lambda,R} = -\xi^{-2} \frac{R(2R-1)}{1-R} \left(e^{-X} - \frac{1}{2R} e^{-2\frac{R-1}{R}X}\right), \quad (\text{A.2})$$

$$\xi^{-1} \left(\frac{\partial \xi}{\partial R}\right)_\lambda = -\frac{1}{R(2R-1)} \left(\log \xi + 2R^2 \frac{d \log a(R)}{dR}\right). \quad (\text{A.3})$$

The most complicated contribution comes from the first term in (A.1). From (6.1), (6.7) and (4.4) one finds

$$\begin{aligned} 2\pi \left(\frac{\partial \mathcal{F}}{\partial R}\right)_{\mu,\xi,\Lambda} &= \xi^{-2} \left(\frac{1}{1-R} e^{-X} - \frac{1}{2R(1-R)} e^{-2\frac{R-1}{R}X} \right) \frac{X}{R} \\ &+ \xi^{-2} \left(\frac{R(2-R)}{(1-R)^2} e^{-X} - \frac{3}{4(1-R)^2} e^{-2\frac{R-1}{R}X} \right) \\ &- \frac{1}{2}\mu^2 \left(\frac{d \log b(R)}{dR} + \frac{dc(R)}{dR} \right). \end{aligned} \quad (\text{A.4})$$

Putting the results (A.2), (A.3) and (A.4) together, we obtain

$$\begin{aligned} 2\pi \left(\frac{\partial F}{\partial R}\right)_{N,\lambda,\Lambda} &= \frac{1}{R^2} \xi^{-2} \left(\frac{R}{1-R} e^{-X} - \frac{1}{2(1-R)} e^{-2\frac{R-1}{R}X} \right) \chi \\ &+ \xi^{-2} \left(\frac{R(2-R)}{(1-R)^2} e^{-X} - \frac{3}{4(1-R)^2} e^{-2\frac{R-1}{R}X} \right) \\ &+ \xi^{-2} \left(\frac{2R^2}{1-R} e^{-X} - \frac{R}{1-R} e^{-2\frac{R-1}{R}X} \right) \frac{d \log a(R)}{dR} \\ &- \frac{1}{2}\mu^2 \left(\frac{d \log b(R)}{dR} + \frac{dc(R)}{dR} \right). \end{aligned} \quad (\text{A.5})$$

We observe that the term proportional to the susceptibility χ coincides with the corresponding term in the entropy (6.5) what can be considered as the main miracle of the present derivation. To get the coincidence of other terms, we obtain a system of three equations, from which only two are independent:

$$\begin{aligned} \frac{d \log b(R)}{dR} + \frac{dc(R)}{dR} &= 0, \\ 2 \frac{d \log a(R)}{dR} - \frac{1}{R^2} \log b(R) &= \frac{2R-3}{R(1-R)}. \end{aligned} \quad (\text{A.6})$$

Since we have only two equations on three functions, there is an ambiguity in the solution. It can be fixed if we require that the entropy vanishes without perturbations. Then we come to the following result

$$a(R) = \left(\frac{1-R}{R^3} \right)^{1/2}, \quad \log b(R) = 0, \quad c(R) = 0, \quad (\text{A.7})$$

which ensures the thermodynamical relations (6.6). Note that in fact they are fulfilled only up to terms depending on the cut-off $\log \Lambda$. Such terms can not be reproduced by differentiating with respect to the temperature, but they are considered as non-universal in our approach and thus can be dropped.

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Article IV

2D String Theory as Normal Matrix Model

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2D String Theory as Normal Matrix Model

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We show that the $c = 1$ bosonic string theory at finite temperature has two matrix-model realizations related by a kind of duality transformation. The first realization is the standard one given by the compactified matrix quantum mechanics in the inverted oscillator potential. The second realization, which we derive here, is given by the normal matrix model. Both matrix models exhibit the Toda integrable structure and are associated with two dual cycles (a compact and a non-compact one) of a complex curve with the topology of a sphere with two punctures. The equivalence of the two matrix models holds for an arbitrary tachyon perturbation and in all orders in the string coupling constant.

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1 Introduction

The $c = 1$ string theory¹ has been originally constructed in the early 90's as the theory of random surfaces embedded into a one-dimensional spacetime [2]. Since then it became clear that this is only one of the realizations of a more universal structure, which reappeared in various mathematical and physical problems. Most recently, it was used for the description of the 2D black hole [3, 4, 5, 6] in the context of the topological string theories on Calabi-Yau manifolds with vanishing cycles [7] and $\mathcal{N} = 1$ SYM theories [8].

The $c = 1$ string theory can be constructed as the collective field theory for a one-dimensional $N \times N$ hermitian matrix field theory known also as *Matrix Quantum Mechanics* (MQM). This construction represents the simplest example of the strings/matrix correspondence. The collective excitations in the singlet sector of MQM are massless "tachyons" with various momenta, while the non-singlet sectors contain also winding modes.

The singlet sector of MQM can be reduced to a system of N nonrelativistic fermions in the upside-down gaussian potential. Thus the elementary excitations of the $c = 1$ string can be represented as collective excitations of free fermions near the Fermi level. The tree-level string-theory S -matrix can be extracted by considering the propagation of infinitesimal "pulses" along the Fermi sea and their reflection off the "Liouville wall" [9, 10]. The (time-dependent) string backgrounds are associated with the possible profiles of the Fermi sea [11].

An important property of the $c = 1$ string theory is its integrability. The latter has been discovered by Dijkgraaf, Moore and Plesser [12] in studying the properties of the tachyon scattering amplitudes. It was demonstrated in [12] that the partition function of the $c = 1$ string theory in the case when the allowed momenta form a lattice, as in the case of the compactified Euclidean theory, is a tau function of Toda hierarchy [13]. The operators associated with the momentum modes in the string theory have been interpreted in [12] as Toda flows. A special case represents the theory compactified at the self-dual radius $R = 1$. It is equivalent to a topological theory that computes the Euler characteristic of the moduli space of Riemann surfaces [14]. When $R = 1$ and only in this case, the partition function of the string theory has alternative realization as a Kontsevich-type model [12, 15].

In this paper we will show that there is another realization of the $c = 1$ string theory by the so-called *Normal Matrix Model* (NMM). This is a complex $N \times N$ matrix model in which the integration measure is restricted to matrices that commute with their hermitian conjugates. The normal matrix model has been studied in the recent years in the context of the laplacian growth problem, the integrable structure of conformal maps, and the quantum Hall droplets [16, 17, 18, 19, 20, 21]. It has been noticed that both matrix models, the singlet sector of MQM and NMM, have similar properties. Both models can be reduced to systems of nonrelativistic fermions and possess Toda integrable structure. There is however an essential difference: the normal matrix model describes a compact droplet of Fermi liquid while the Fermi sea of the MQM is non-compact. Furthermore, the perturbations of NMM are introduced by a matrix potential while these of MQM are introduced by means of time-dependent asymptotic states. In this paper we will show that nevertheless these two models are equivalent.

¹As a good review we recommend [1].

The exact statement is that the matrix quantum mechanics compactified at radius R is equivalent to the normal matrix model defined by the probability distribution function $\exp[-\frac{1}{\hbar}W_R(Z, Z^\dagger)]$, where

$$W_R(Z, Z^\dagger) = \text{tr} (ZZ^\dagger)^R - \hbar \frac{R-1}{2} \text{tr} \log(ZZ^\dagger) - \sum_{k \geq 1} t_k \text{tr} Z^k - \sum_{k \geq 1} t_{-k} \text{tr} Z^{\dagger k}. \quad (1.1)$$

More precisely, the grand canonical partition function of MQM is identical to the canonical partition function of NMM at the perturbative level, by which we mean that the genus expansions of the two free energies coincide. This will be proved in two steps. First, we will show that the non-perturbed partition functions are equal. Then we will use the fact that both partition functions are τ -functions of Toda lattice hierarchy, which implies that they also coincide in presence of an arbitrary perturbation.

We also give a unified geometrical description of the two models in terms of a complex curve with the topology of a sphere with two punctures. This complex curve is analog of the Riemann surfaces arising in the case of hermitian matrix models. However, for generic R it is not an algebraic curve. The curve has two dual non-contractible cycles, which determine the boundaries of the supports of the eigenvalue distributions for the two models. The normal matrix model is associated with the compact cycle while the MQM is associated with the non-compact cycle connecting the two punctures. We will construct a globally defined one-form whose integrals along the two cycles give the number of eigenvalues N and the derivative of the free energy with respect to N .

The paper is organized as follows. In the next section we will remind the realization of the $c = 1$ string theory as a fermionic system with chiral perturbations worked out in [11]. We will stress on the calculation of the free energy, which will be needed to establish the equivalence with the NMM. In Sect. 3 we construct the NMM having the same partition function. In Sect. 4 we consider the quasiclassical limit and give a unified geometrical description of the two models.

2 MQM, free fermions and integrability

2.1 Eigenfunctions and fermionic scattering

In absence of winding modes, the 2D string theory is described by the singlet sector of Matrix Quantum Mechanics in the double scaling limit with Hamiltonian

$$\hat{\mathbf{H}}_0 = \frac{1}{2} \text{tr} \left(-\hbar^2 \frac{\partial^2}{\partial M^2} - M^2 \right). \quad (2.1)$$

The radial part of this Hamiltonian is expressed in terms of the eigenvalues x_1, \dots, x_N of the matrix M . The wave functions in the $SU(N)$ -singlet sector are completely antisymmetric and thus describe a system of N nonrelativistic fermions in the inverse gaussian potential. The dynamics of the fermions is governed by the Hamiltonian

$$\hat{\mathbf{H}}_0 = \frac{1}{2} \sum_{i=1}^N (\hat{p}_i^2 - \hat{x}_i^2), \quad (2.2)$$

where p_i are the momenta conjugated to the fermionic coordinates x_i .

To describe the incoming and outgoing tachyonic states, it is convenient to introduce the "light-cone" coordinates in the phase space

$$\hat{x}_\pm = \frac{\hat{x} \pm \hat{p}}{\sqrt{2}} \quad (2.3)$$

satisfying the canonical commutation relations

$$[\hat{x}_+, \hat{x}_-] = -i\hbar. \quad (2.4)$$

In these variables the one-particle Hamiltonian takes the form

$$\hat{H}_0 = -\frac{1}{2}(\hat{x}_+ \hat{x}_- + \hat{x}_- \hat{x}_+). \quad (2.5)$$

We can work either in x_+ or in x_- representation, where the theory is defined in terms of fermionic fields $\psi_\pm(x_\pm)$ respectively. The solutions with a given energy are $\psi_\pm^E(x_\pm, t) = e^{-\frac{i}{\hbar}Et} \psi_\pm^E(x_\pm)$ with

$$\psi_\pm^E(x_\pm) = \frac{1}{\sqrt{2\pi\hbar}} e^{\mp \frac{i}{2\hbar}\phi_0} x_\pm^{\pm \frac{i}{\hbar}E - \frac{1}{2}}, \quad (2.6)$$

where the phase factor $\phi_0(E)$ will be determined below. These solutions form two complete systems of δ -function normalized orthonormal states under the condition that the domain of the definition of the wave functions is a semi-axis.² The left and right representations are related to each other by a unitary operator which is the Fourier transform on the half-line

$$[\hat{S}\psi_+](x_-) = \int_0^\infty dx_+ S(x_-, x_+) \psi_+(x_+). \quad (2.7)$$

The integration kernel $S(x_-, x_+)$ is either a sine or a cosine depending on the boundary conditions at the origin. For definiteness, let us choose the cosine kernel

$$S(x_-, x_+) = \sqrt{\frac{2}{\pi\hbar}} \cos\left(\frac{1}{\hbar}x_+x_-\right). \quad (2.8)$$

Then the operator \hat{S} is diagonal on the eigenfunctions (2.6) with given energy

$$[\hat{S}\psi_+^E](x_-) = e^{-\frac{i}{\hbar}\phi_0} \mathcal{R}(E) \psi_-^E(x_-), \quad (2.9)$$

where

$$\mathcal{R}(E) = \hbar^{iE} \sqrt{\frac{2}{\pi}} \cosh\left(\frac{\pi}{2} \left(\frac{1}{\hbar}E - i/2\right)\right) \Gamma\left(\frac{i}{\hbar}E + 1/2\right). \quad (2.10)$$

²In order to completely define the theory, we should define the interval where the phase-space coordinates x_\pm are allowed to take their values. One possibility is to allow the eigenvalues to take any real value. The whole real axis corresponds to a theory whose Fermi sea consists of two disconnected components on both sides of the maximum of the potential. To avoid the technicalities related to the tunneling phenomena between the two Fermi seas, we will define the theory by restricting the eigenvalues to the positive real axis. The difference between the two choices is seen only at the non-perturbative level.

Since the operator \hat{S} is unitary, i.e.

$$\overline{\mathcal{R}(E)}\mathcal{R}(E) = \mathcal{R}(-E)\mathcal{R}(E) = 1, \quad (2.11)$$

it can be absorbed into the function $\phi_0(E)$. This fixes the phase of the eigenfunctions as

$$\phi_0(E) = -i\hbar \log \mathcal{R}(E). \quad (2.12)$$

In fact, the operator \hat{S} appears in our formalism as the fermionic S -matrix describing the scattering off the inverse oscillator potential and the factor $\mathcal{R}(E)$ gives the fermionic reflection coefficient.

Let us introduce the scalar product between left and right states as follows

$$\langle \psi_- | \hat{S} | \psi_+ \rangle = \iint_0^\infty dx_+ dx_- \overline{\psi_-}(x_-) S(x_-, x_+) \psi_+(x_+). \quad (2.13)$$

Then the in- and out-eigenfunctions are orthonormal with respect to this scalar product

$$\langle \psi_-^E | \hat{S} | \psi_+^{E'} \rangle = \delta(E - E'). \quad (2.14)$$

It is actually this relation that defines the phase factor containing all information about the scattering.

The ground state of the Hamiltonian (2.5) can be constructed from the wave functions (2.6) and describes the linear dilaton background of string theory [10]. The tachyon perturbations can be introduced by changing the asymptotics of the wave functions at $x_\pm \rightarrow \infty$ to

$$\Psi_\pm^E(x_\pm) \sim e^{\mp \frac{i}{\hbar} \sum t_{\pm k} x_\pm^{k/R}} x_\pm^{\pm \frac{i}{\hbar} E - \frac{1}{2}}. \quad (2.15)$$

The exact phase contains also a constant mode $\mp \frac{1}{2\hbar} \phi(E)$ as in (2.6) and negative powers of $x_\pm^{1/R}$ which are determined by the orthonormality condition

$$\langle \Psi_-^E | \hat{S} | \Psi_+^{E'} \rangle = \delta(E - E'). \quad (2.16)$$

The constant mode $\phi(E)$ of the phase of the fermion wave function contains all essential information about the perturbed system.

2.2 Cut-off prescription and density of states

To find the density of states, we introduce a cut-off Λ by confining the phase space to a periodic box

$$x_\pm + 2\sqrt{\Lambda} \equiv x_\pm, \quad (2.17)$$

which can be interpreted as putting a reflecting wall at distance $\sqrt{\Lambda}$. This means that at the points $x_+ = \sqrt{\Lambda}$ and $x_- = \sqrt{\Lambda}$ the reflected wave function coincides with the incoming one

$$[\hat{S}\Psi](\sqrt{\Lambda}) = \Psi(\sqrt{\Lambda}). \quad (2.18)$$

Applied to the wave functions (2.15), this condition gives an equation for the admissible energies

$$e^{\frac{i}{\hbar} \phi(E)} = e^{-\frac{i}{\hbar} V(\Lambda)} \Lambda^{\frac{i}{\hbar} E}, \quad V(\Lambda) = \sum (t_k + t_{-k}) \Lambda^{k/2R}, \quad (2.19)$$

where we neglected the negative powers of Λ . It is satisfied by a discrete set of energies E_n defined by the relation

$$E_n \log \Lambda - \phi(E_n) = 2\pi\hbar n + V(\Lambda), \quad n \in \mathbf{Z}. \quad (2.20)$$

Taking the limit $\Lambda \rightarrow \infty$ we find the density of states (in units of \hbar)

$$\rho(E) = \frac{\log \Lambda}{2\pi} - \frac{1}{2\pi} \frac{d\phi(E)}{dE}. \quad (2.21)$$

2.3 The partition function of the compactified 2D string theory

Knowing the density of states, we can calculate the grand canonical partition function \mathcal{Z} of the quantum-mechanical system compactified at Euclidean time $\beta = 2\pi R$. If μ is the chemical potential, then the free energy $\mathcal{F} = \hbar^2 \log \mathcal{Z}$ of the ensemble of free fermions is given by

$$\mathcal{F}(\mu, t) = \hbar \int_{-\infty}^{\infty} dE \rho(E) \log \left[1 + e^{-\frac{1}{\hbar}\beta(\mu+E)} \right]. \quad (2.22)$$

Integrating by parts and dropping out the Λ -dependent non-universal piece, we obtain

$$\mathcal{F}(\mu, t) = -\frac{\beta}{2\pi} \int_{-\infty}^{\infty} dE \frac{\phi(E)}{1 + e^{\frac{1}{\hbar}\beta(\mu+E)}}. \quad (2.23)$$

We close the contour of integration in the upper half plane and take the integral as a sum of residues of the thermal factor³

$$\mathcal{F}(\mu, t) = i\hbar \sum_{n \geq 0} \phi \left(i\hbar \frac{n + \frac{1}{2}}{R} - \mu \right). \quad (2.24)$$

We will be interested only in the perturbative expansion in $1/\mu$ and neglect the non-perturbative terms $\sim e^{-\frac{\pi}{\hbar}\mu}$, as well as non-universal terms in the free energy (regular in μ). Therefore, for the case of zero tachyon couplings we can retain only the Γ -function in the reflection factor (2.10) and choose the phase ϕ_0 as

$$\phi_0(E) = -i\hbar \log \Gamma \left(\frac{i}{\hbar} E + 1/2 \right). \quad (2.25)$$

It is known [12] that the tachyon scattering data for MQM are generated by a τ -function of Toda lattice hierarchy where the coupling constants t_k play the role of the Toda times. The τ -function is related to the constant mode $\phi(E)$ by [11]

$$e^{\frac{i}{\hbar}\phi(-\mu)} = \frac{\tau_0 \left(\mu + i\frac{\hbar}{2R} \right)}{\tau_0 \left(\mu - i\frac{\hbar}{2R} \right)}. \quad (2.26)$$

Taking into account (2.24) we conclude that the partition function of the perturbed MQM is equal to the τ -function:

$$\mathcal{Z}(\mu, t) = \tau_0(\mu, t). \quad (2.27)$$

³We neglect the nonperturbative terms associated with the cuts of the function $\phi(E)$.

The discrete space parameter $s \in \mathbf{Z}$ along the Toda chain is related to the chemical potential μ . More precisely, s corresponds to an imaginary shift of μ [6, 11]:

$$\tau_s(\mu, t) = \tau_0 \left(\mu + i\hbar \frac{s}{R}, t \right). \quad (2.28)$$

This fact was used to rewrite the Toda equations as difference equations in μ rather than in the discrete parameter s . It will be also used below to prove the equivalence of 2D string theory to the normal matrix model.

3 2D string theory as normal matrix model

In this section we will show that the partition function of 2D string theory with tachyonic perturbations can be rewritten as a normal matrix model. It is related to the original matrix quantum mechanics by a kind of duality transformation. In fact, we will give two slightly different normal matrix models fulfilling this goal.

Consider the following matrix integral

$$\mathcal{Z}_\hbar^{\text{NMM}}(N, t, \alpha) = \int_{[Z, Z^\dagger]=0} dZ dZ^\dagger \left[\det(ZZ^\dagger) \right]^{(R-1)/2 + \frac{\alpha}{\hbar}} e^{-\frac{1}{\hbar} \text{tr} [(ZZ^\dagger)^R - V_+(Z) - V_-(Z^\dagger)]}, \quad (3.1)$$

where the integral goes over all complex matrices satisfying $[Z, Z^\dagger] = 0$ and the potentials are given by

$$V_\pm(z) = \sum_{k \geq 1} t_{\pm k} z^k. \quad (3.2)$$

We made the dependence on the Planck constant explicit because later we will need to analytically continue in \hbar . The integral (3.1) defines a τ -function of Toda hierarchy [20]. We remind the proof of this statement in Appendix A, where we also derive the string equation specifying the unique solution of Toda equations. This string equation coincides (up to change $\hbar \rightarrow i\hbar$) with that of the hierarchy describing the 2D string theory [6, 11]. Therefore, if we identify correctly the parameters of the two models, the τ -functions should also be the same. Namely, we should relate the chemical potential μ of 2D string theory to the size of matrices N and the parameter α of the normal matrix model. There are two possibilities to make such identification.

3.1 Model I

The first possibility is realized taking a large N limit of the matrix integral (3.1). Namely, we will prove that the full perturbed partition function of MQM is given by the large N limit of the partition function (3.1) with $\alpha = R\mu - \hbar N$ and a subsequent analytical continuation $\hbar \rightarrow i\hbar$

$$\mathcal{Z}_\hbar(\mu, t) = \lim_{N \rightarrow \infty} \mathcal{Z}_{i\hbar}^{\text{NMM}}(N, t, R\mu - i\hbar N). \quad (3.3)$$

The necessity to change the Planck constant by the imaginary one follows from the comparison of the string equations of two models as was discussed above. More generally, the

τ -function (2.28) for arbitrary s is obtained from the partition function (3.1) in the following way

$$\tau_{s,\hbar}(\mu, t) = \lim_{N \rightarrow \infty} \mathcal{Z}_{i\hbar}^{\text{NMM}}(N + s, t, R\mu - i\hbar N). \quad (3.4)$$

Let us stress that in spite of the large N limit taken here, we obtain as a result the full (and not only dispersionless) partition function of the $c = 1$ string theory.

To prove this statement we will use the fact that both partition functions are τ -functions of the Toda lattice hierarchy with times $t_{\pm k}$, $k = 1, 2, \dots$. Since the unperturbed τ -function provides the necessary boundary conditions for the unique solution of Toda equations, it is sufficient to show that the unperturbed partition functions of the two models coincide and the s -parameters of two τ -functions are identical.

The integral (3.1) reduces to the product of the normalization coefficients of the orthogonal polynomials, eq. (A.5). From the definition of the scalar product (A.4) it follows that when all $t_{\pm k} = 0$ the orthogonal polynomials are simple monomials and the normalization factors h_n are given by

$$h_n(0, \alpha) = \frac{1}{2\pi i} \int_{\mathbf{C}} d^2 z e^{-\frac{1}{\hbar}(z\bar{z})^R} (z\bar{z})^{(R-1)/2 + \frac{\alpha}{\hbar} + n}. \quad (3.5)$$

Up to constant factors and powers of \hbar , we find

$$h_n(0, \alpha) \sim \Gamma\left(\frac{\alpha}{\hbar R} + \frac{n + \frac{1}{2}}{R} + \frac{1}{2}\right). \quad (3.6)$$

We see that the result coincides with the reflection coefficient given in (2.25). Therefore, taking the analytical continuation of parameters as in (3.3), we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \hbar^2 \log \mathcal{Z}_{i\hbar}^{\text{NMM}}(N, 0, R\mu - i\hbar N) &= \lim_{N \rightarrow \infty} i\hbar \sum_{n=0}^{N-1} \phi_0 \left(i\hbar \frac{N - n - \frac{1}{2}}{R} - \mu \right) \\ &= i\hbar \sum_{n=0}^{\infty} \phi_0 \left(i\hbar \frac{n + \frac{1}{2}}{R} - \mu \right). \end{aligned} \quad (3.7)$$

This coincides with the unperturbed free energy $\mathcal{F}(\mu, 0)$ from (2.24), what proves (3.3) for all $t_{\pm k} = 0$.

Thus, it remains to show that the s -parameter of the τ -function describing the normal matrix model is associated with μ . From (A.17) we see that it coincides with N . Hence it trivially follows from (3.4) that

$$\tau_{s,\hbar}(\mu, t) = \lim_{N \rightarrow \infty} \mathcal{Z}_{i\hbar}^{\text{NMM}} \left(N, t, R \left(\mu + i\hbar \frac{s}{R} \right) - i\hbar N \right) = \tau_{0,\hbar} \left(\mu + i\hbar \frac{s}{R}, t \right), \quad (3.8)$$

what means that the τ -function defined in (3.4) possesses the characteristic property (2.28).

3.2 Model II

In fact, one can even simplify the representation (3.3) what will provide us with the second realization of 2D string theory in terms of NMM. Let us consider the matrix model (3.1)

for $\alpha = 0$. We claim that the partition function $Z_{i\hbar}(N, t, 0)$ in the canonical ensemble, analytically continued to imaginary Planck constant and

$$N = -\frac{i}{\hbar} R\mu, \quad (3.9)$$

coincides with the partition function of 2D string theory in the grand canonical ensemble:⁴

$$\mathcal{Z}_{i\hbar}(\mu, t) = \mathcal{Z}_{i\hbar}^{\text{NMM}}(-\frac{i}{\hbar} R\mu, t). \quad (3.10)$$

Indeed, as for the first model, they are both given by τ -functions of Toda hierarchy. After the identification (3.9), the s -parameters of these τ -functions are identical due to (A.17) and (2.28). Therefore, it only remains to show that the integral $Z_{i\hbar}(N, 0)$ without potential is equal to the unperturbed partition function (2.24). In this case the method of orthogonal polynomials gives

$$\mathcal{Z}_{i\hbar}^{\text{NMM}}(N, 0) = \prod_{n=0}^{N-1} \mathcal{R} \left(-i\hbar \frac{n + \frac{1}{2}}{R} \right), \quad (3.11)$$

where the \mathcal{R} factors are the same as in MQM, as was shown in (3.6). Then we write

$$\mathcal{Z}_{i\hbar}^{\text{NMM}}(N, 0) = \Xi(0)/\Xi(N), \quad \text{where} \quad \Xi(N) = \prod_{n=N}^{\infty} \mathcal{R} \left(-i\hbar(n + \frac{1}{2})/R \right). \quad (3.12)$$

$\Xi(0)$ is a constant and can be neglected, whereas $\Xi(N)$ can be rewritten as

$$\Xi(N) = \prod_{n=0}^{\infty} \mathcal{R} \left(-i\hbar N/R - i\hbar(n + \frac{1}{2})/R \right). \quad (3.13)$$

Taking into account the unitarity relation (2.11) and substituting N from (3.9), we obtain

$$\mathcal{Z}_{i\hbar}^{\text{NMM}}(-\frac{i}{\hbar} R\mu, 0) \sim \Xi^{-1}(-\frac{i}{\hbar} R\mu) = \prod_{n=0}^{\infty} \mathcal{R} \left(\mu + i\hbar(n + \frac{1}{2})/R \right). \quad (3.14)$$

Thus $\hbar^2 \log \mathcal{Z}_{i\hbar}^{\text{NMM}}(-\frac{i}{\hbar} R\mu, 0)$ coincides with the free energy (2.24) with all $t_k = 0$. Note that the difference in the sign of μ does not matter since the free energy is an even function of μ (up non-universal terms).⁵ Since the two partition functions are both solutions of the Toda hierarchy, the fact that they coincide at $t_k = 0$ implies that they coincide for arbitrary perturbation.

We conclude that the grand canonical partition function of the $c = 1$ string theory equals the canonical partition function of the normal matrix model (3.1) for $\alpha = 0$ and $N = -\frac{i}{\hbar} R\mu$. It is also clear that we can identify the operators of tachyons in two models

$$\text{Tr } X_+^{n/R} \leftrightarrow \text{Tr } Z^n, \quad \text{Tr } X_-^{n/R} \leftrightarrow \text{Tr } (Z^\dagger)^n. \quad (3.15)$$

⁴From now on we will omit the last argument of \mathcal{Z}^{NMM} corresponding to the vanishing parameter α .

⁵Moreover, this sign can be correctly reproduced from (2.23) if closing the integration contour in the lower half plane.

4 Geometrical meaning of the duality between the two models

The quasiclassical limit of most of the solvable matrix models has a nice geometrical interpretation. Namely, the free energy in this limit can be parameterized in terms of the periods of a holomorphic 1-form around the cycles of an analytical curve [22, 23, 24, 25]. Recently this geometrical picture appeared also in the context of the topological strings on singular Calabi-Yau manifolds and supersymmetric gauge theories [26, 27, 28].

In this section we will show that both MQM and NMM have in the quasiclassical limit a similar geometrical interpretation in terms of a one-dimensional complex curve, which is topologically a sphere with two punctures. Here by complex curve we understand a complex manifold with punctures and given behavior of the functions on this manifold at the punctures. Each of the two matrix models corresponds to a particular real section of this curve coinciding with one of its two non-contractible cycles, and the duality between them is realized by the exchange of the cycles of the curve.

4.1 The dispersionless limit

The quasiclassical limit $\hbar \rightarrow 0$ corresponds to the dispersionless limit of the Toda hierarchy where it has a description in terms of a classical dynamical system. The Lax operators are considered as c -functions of the two canonically conjugated coordinates μ and ω , where ω is the quasiclassical limit of the shift operator $\hat{\omega} = e^{\partial/\partial s}$. Moreover, the two Lax operators can be expressed as series in ω . The solutions of the dispersionless hierarchy correspond to canonical transformations in the phase space of the dynamical system. In the case of MQM this is the transformation from the coordinates μ and $\log \omega$ to x_+ and x_- .

The particular solution of the Toda hierarchy that appears in our problem also satisfies the dispersionless string equation. In the case of MQM this is nothing but the equation of the profile of the Fermi sea in the phase space [11]

$$x_+ x_- = \frac{1}{R} \sum_{k \geq 1} k t_{\pm k} x_{\pm}^{k/R} + \mu + \frac{1}{R} \sum_{k \geq 1} v_{\pm k} x_{\pm}^{-k/R}. \quad (4.1)$$

To get the string equation for NMM it is enough to make the substitution following (3.15)

$$x_+ \leftrightarrow z^R, \quad x_- \leftrightarrow \bar{z}^R \quad (4.2)$$

and $\mu = \hbar N/R$ as explained in the previous section. (We took into account that \hbar from (3.9) should be replaced here by $-i\hbar$ so that the factor i is canceled.) Then the string equation describes the contour γ bounding the region \mathcal{D} in the complex z -plane filled by the eigenvalues of the normal matrix.⁶

4.2 NMM in terms of electrostatic potential

In this section we will introduce a function of the spectral variables z and \bar{z} , which plays a central role in the NMM integrable structure. This function has several interpretations in

⁶Here we consider the case when the eigenvalues are distributed in a simply connected domain.

the quasiclassical limit. First, it can be viewed as the generating function for the canonical transformation mentioned above, which maps variables μ and $\log \omega$ to the variables z and \bar{z} . Second, it gives the phase of the fermion wave function at the Fermi level after the identification (4.2).

There is also a third, electrostatic interpretation, which is geometrically the most explicit and which we will follow in this section.⁷ According to this interpretation, the eigenvalues distributed in the domain \mathcal{D} can be considered to form a charged liquid with the density

$$\rho(z, \bar{z}) = \frac{1}{\pi} \partial_z \partial_{\bar{z}} W_R(z, \bar{z}) = \frac{R^2}{\pi} (z\bar{z})^{R-1}. \quad (4.3)$$

Let $\varphi(z, \bar{z})$ be the potential of the charged eigenvalue liquid, which is a harmonic function outside the domain \mathcal{D} and it is a solution of the Laplace equation inside the domain

$$\varphi(z, \bar{z}) = \begin{cases} \varphi(z) + \bar{\varphi}(\bar{z}), & z \notin \mathcal{D}, \\ (z\bar{z})^R, & z \in \mathcal{D}. \end{cases} \quad (4.4)$$

To fix completely the potential, we should also impose the asymptotics of the potential at infinity. This asymptotics is determined by the coupling constants $t_{\pm n}$, $n = 1, 2, \dots$ and can be considered as the result of placing a dipole, quadrupole etc. charges at infinity.

The solution of this electrostatic problem is obtained as follows. The continuity of the potential $\varphi(z, \bar{z})$ and its first derivatives leads to the following conditions to be satisfied on the boundary $\gamma = \partial\mathcal{D}$

$$\varphi(z) + \bar{\varphi}(\bar{z}) = (z\bar{z})^R, \quad (4.5)$$

$$z \partial_z \varphi(z) = \bar{z} \partial_{\bar{z}} \bar{\varphi}(\bar{z}) = R z^R \bar{z}^R. \quad (4.6)$$

Each of two equations (4.6) can be interpreted as an equation for the contour γ . Since we obtain two equations for one curve, they should be compatible. This imposes a restriction on the holomorphic functions $\varphi(z)$ and $\bar{\varphi}(\bar{z})$. The solutions for these chiral parts of the potential can be found comparing (4.6) with the string equation (4.1). In this way we have

$$\begin{aligned} \varphi(z) &= \hbar N \log z + \frac{1}{2} \phi + \sum_{k \geq 1} t_k z^k - \sum_{k \geq 1} \frac{1}{k} v_k z^{-k}, \\ \bar{\varphi}(\bar{z}) &= \hbar N \log \bar{z} + \frac{1}{2} \phi + \sum_{k \geq 1} t_{-k} \bar{z}^k - \sum_{k \geq 1} \frac{1}{k} v_{-k} \bar{z}^{-k}. \end{aligned} \quad (4.7)$$

The zero mode ϕ is fixed by the condition (4.5). However, it is easier to use another interpretation. As we mentioned above, the holomorphic functions $\varphi(z)$ and $\bar{\varphi}(\bar{z})$ coincide with the phases of the fermionic wave functions and in particular their constant modes are the same. Then, taking the limit $\hbar \rightarrow 0$ of (2.26), in the variables of NMM we find

$$\phi = -\hbar \frac{\partial}{\partial N} \log \mathcal{Z}^{\text{NMM}}. \quad (4.8)$$

⁷See also [29] for another interesting interpretation which appears useful in string field theory.

4.3 The complex curve

Instead of considering the function $\varphi(z, \bar{z})$ harmonic in the exterior domain $\bar{\mathcal{D}} = \mathbf{C}/\mathcal{D}$, we can introduce one holomorphic function Φ defined in the double cover of $\bar{\mathcal{D}}$. The doubly covered domain $\bar{\mathcal{D}}$ is topologically equivalent to a two-sphere with two punctures (at $z = \infty$ and $\bar{z} = \infty$), which we identify with the north and the south poles. It can be covered by two coordinate patches associated with the north and south hemispheres and parameterized by z and \bar{z} . They induce a natural complex structure so that the sphere can be viewed as a complex manifold. The patches overlap in a ring containing the contour γ and the transition function between them $\bar{z}(z)$ (or $z(\bar{z})$) is defined through the string equation (4.6).

The string equation can also be considered as a defining equation for the manifold. We start with the complex plane \mathbf{C}^2 with flat coordinates z and \bar{z} . Then the manifold is defined as solution of equation (4.6), thus providing its natural embedding in \mathbf{C}^2 .

To completely define the complex curve, we should also specify the singular behavior at the punctures of the analytic functions on this curve. We fix it to be the same as the one of the holomorphic parts of the potential (4.7).

We define on the complex curve the following holomorphic field

$$\Phi \stackrel{\text{def}}{=} \begin{cases} \Phi_+(z) = \varphi(z) - \frac{1}{2}(z\bar{z}(z))^R & \text{in the north hemisphere,} \\ \Phi_-(\bar{z}) = -\bar{\varphi}(\bar{z}) + \frac{1}{2}(z(\bar{z})\bar{z})^R & \text{in the south hemisphere.} \end{cases} \quad (4.9)$$

Its analyticity follows from equation (4.5), which now holds on the entire curve, since z and \bar{z} are no more considered as conjugated to each other.

The field Φ gives rise to a closed (but not exact) holomorphic 1-form $d\Phi$, globally defined on the complex curve. Actually, this is the unique globally defined 1-form with given singular behavior at the two punctures. Therefore, we can characterize the curve by periods of this 1-form around two conjugated cycles A and B defined as follows. The cycle A goes along the ring where both parameterizations overlap and is homotopic to the contour γ . The cycle B is a path going from the puncture at $z = \infty$ to the puncture at $\bar{z} = \infty$.

The integral around the cycle A is easy to calculate using (4.7) and (4.6) by picking up the pole

$$\frac{1}{2\pi i} \oint_A d\Phi = \frac{1}{2\pi i} \oint_\gamma d\Phi_+(z) = \frac{1}{2\pi i} \oint_{\gamma^{-1}} d\Phi_-(\bar{z}) = \hbar N, \quad (4.10)$$

where we indicated that in \bar{z} -coordinates the contour should be reversed.

To find the integral along the non-compact cycle B , one should introduce a regularization by cutting it at $z = \sqrt{\Lambda}$ and $\bar{z} = \sqrt{\Lambda}$. Then we obtain

$$\int_B d\Phi = \int_{\sqrt{\Lambda}}^{z_0} d\Phi_+(z) + \int_{\bar{z}(z_0)}^{\sqrt{\Lambda}} d\Phi_-(\bar{z}) = \Phi_-(\sqrt{\Lambda}) - \Phi_+(\sqrt{\Lambda}), \quad (4.11)$$

where z_0 is any point on the cycle and we used (4.5). Taking into account the definition (4.9) and the explicit solution (4.7), we see that the result contains the part vanishing at $\Lambda \rightarrow \infty$, the diverging part which does not depend on N and, as non-universal, can be neglected, and the contribution of the zero mode ϕ . Since ϕ enters Φ_+ and Φ_- with different signs, strictly speaking, it is not a zero mode for Φ . As a result, it does not disappear from the integral, but is doubled. Finally, using (4.8) we get

$$\int_B d\Phi = \hbar \frac{\partial}{\partial N} \log \mathcal{Z}^{\text{NMM}}. \quad (4.12)$$

In Appendix B we derive this formula using the eigenvalue transfer procedure similar to one used in [28], or in [25] in a similar case of the two matrix model equivalent to our $R = 1$ NMM. Thus, the free energy of NMM can be reproduced from the monodromy of the holomorphic 1-form around the non-compact cycle on the punctured sphere.

Since the string equations coincide, it is clear that the solution of MQM is described by the same analytical curve and holomorphic 1-form. To write the period integrals in terms of MQM variables, it is enough to change the coordinates according to (4.2) and $\hbar N = R\mu$ in all formulas. In this way we find⁸

$$\frac{1}{2\pi i} \oint_A d\Phi = R\mu, \quad \int_B d\Phi = -\frac{1}{R} \frac{\partial \mathcal{F}}{\partial \mu}. \quad (4.13)$$

The last integral can be also obtained from the integral over the Fermi sea. Indeed, we can choose a point on the contour of the Fermi sea and split the integral, which actually calculates the area of the sea, into three parts

$$\frac{1}{R} \frac{\partial \mathcal{F}}{\partial \mu} = - \int_{\text{F.s.}} dx_+ dx_- = \int_{x_0}^{\sqrt{\Lambda}} x_-(x_+) dx_+ + \int_{x_-(x_0)}^{\sqrt{\Lambda}} x_+(x_-) dx_- + x_0 x_-(x_0). \quad (4.14)$$

Since $\partial_+ \varphi = x_-(x_+)$ and $\partial_- \bar{\varphi} = x_+(x_-)$ (see (4.6)), the last expression is equal to the integral (4.11) of $d\Phi$ around the (reversed) cycle B . This derivation gives an independent check that the free energies of NMM and MQM do coincide.

4.4 Duality

We found that the solutions of both models are described by the same complex curve. The curve is characterized by a pair of conjugated cycles: the compact cycle A encircling one of the punctures and the non-compact cycle B connecting the two punctures. The parameter of the free energy μ or N and the derivative of the free energy itself are given by the integrals of the unique globally defined holomorphic 1-form along the A and B cycles, correspondingly.

Now one can ask: what is the difference between the two models? Is it seen at the level of the curve? In fact, both models, NMM and MQM can be associated with two different real sections of the complex curve. Indeed, in NMM the variables are conjugated to each other, whereas in MQM they are real. Therefore, let us take the interpretation where the curve is embedded into \mathbf{C}^2 and consider its intersection with two planes. The first plane is defined by the condition $z^* = \bar{z}$ and the second one is given by $z^* = z$, $\bar{z}^* = \bar{z}$. In the former case we get the cycle A , whereas in the latter case the intersection coincides with the cycle B (see fig. 1). One can think about the planes as the place where the eigenvalues of NMM and MQM, correspondingly, live (with the density given by (4.3)). Then the intersections describe the contours of the regions filled by the eigenvalues. We see that for NMM it is given by the compact cycle A , and for MQM the Fermi sea is bounded by the cycle B of the curve.

⁸To get the second integral from (4.12), one should take into account that also $\hbar \rightarrow i\hbar$.

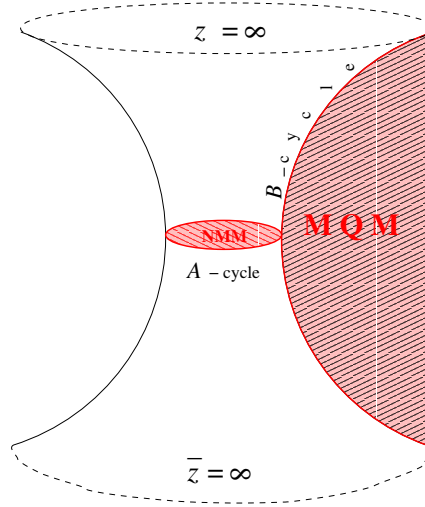


Fig. 1: Symbolical representation of the complex curve and the two real sections along the cycles A and B . The filled regions symbolize the Fermi seas of the two matrix models.

Therefore, the duality between the two models can be interpreted as the duality exchanging the cycles of the complex curve describing the solution. Under this exchange the real variables of one model go to the complex conjugated variables of another model, and the grand canonical free energy is replaced by the canonical one.

This duality can be seen even more explicitly if one rewrites the relations (4.13) in terms of the canonical free energy of MQM defined as $F = \mathcal{F} + \hbar R \mu M$, where $M = -\frac{1}{\hbar R} \frac{\partial \mathcal{F}}{\partial \mu}$ is the number of eigenvalues. Then they take the following form

$$\frac{1}{2\pi i} \oint_A d\Phi = \frac{1}{\hbar} \frac{\partial F}{\partial M}, \quad \int_B d\Phi = \hbar M. \quad (4.15)$$

They have exactly the same form as (4.10) and (4.12) provided we change the cycles. In particular, the numbers of eigenvalues are given in the two models as integrals around the dual cycles.

5 Conclusions and problems

We have shown that the tachyonic sector of the compactified $c = 1$ string theory is described in terms of a normal matrix model. This matrix model is related by duality to the traditional representation in terms of Matrix Quantum Mechanics. More precisely, the canonical partition function of NMM is equal to the grand canonical partition function of MQM, provided we identify the number of eigenvalues N of NMM with the chemical potential μ in MQM. This holds for any compactification radius. Moreover, the duality between the two matrix models has a nice geometrical interpretation. In the quasiclassical limit the solutions of the two models can be described in terms of a complex curve with the topology of a sphere with two punctures. The Matrix Quantum Mechanics and the Normal Matrix Model are associated with two real sections of this curve which are given by the two non-contractible cycles on the punctured sphere. The duality acts by exchanging the two cycles.

The duality we have observed relates a fermionic problem with non-compact Fermi sea and continuous spectrum to a problem with compact Fermi sea and discrete spectrum. The solutions of the two problems are obtained from each other by analytic continuation. Thus, the reason for the duality can be seen in the common analytic structure of both problems. We also emphasize that the duality relates the partition functions in the canonical and the grand canonical ensembles, which are related by Legendre transform. This is natural regarding that it exchanges the cycles of the complex curve.

Let us mention two among the many unsolved problems related to the duality between these two models. First, it would be interesting to generalize the above analysis to the case when the eigenvalues of NMM form two or more disconnected droplets. For $R = 1$ this is the analog of the multicut solutions of the two-matrix model, considered recently in [25]. In terms of MQM this situation would mean the appearance of new, compact components of the Fermi sea. The corresponding complex curve has the topology of a sphere with two punctures and a number of handles.

Second, we would like to generalize the correspondence between the two matrix models in such a way that it incorporates also the winding modes. For this we should understand how to introduce the winding modes in the Normal Matrix Model. Whereas in MQM they appear when we relax the projection to the singlet sector, in NMM this could happen when we relax the normality condition $[Z, \bar{Z}] = 0$.

Up to now there were two suggestions how to include both the tachyon and winding modes within a single matrix model. In [11], a 3-matrix model was proposed with interacting two hermitian matrices and one unitary matrix. This model can be seen as a particular reduction of Euclidean compactified MQM. The model correctly describes the cases of only tachyon or only winding perturbations, though its validity in the general case is still to be proven. For the particular case of the self-dual radius and the multiples of it, a 4-matrix model was recently proposed in [8]. It is based on the old observation of [30] about the geometry of the ground ring of $c = 1$ string theory. It would be interesting to find the relation of this model to our approach, at least in the above-mentioned particular cases. The general understanding of this important problem is still missing.

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Appendix A. Toda description of the perturbed system

Here we show that the partition function (3.1) is a τ -function of the Toda lattice hierarchy (for details see [31]). We will do it following the standard method of orthogonal polynomials. First, we write the integral (3.1) as integral over the eigenvalues z_1, \dots, z_N

$$\mathcal{Z}_h^{\text{NMM}}(N, t, \alpha) = \frac{1}{N!} \int \prod_{k=1}^N w_\alpha(z_k, \bar{z}_k) \Delta(z) \Delta(\bar{z}) \quad (\text{A.1})$$

with measure

$$w_\alpha(z, \bar{z}) = \frac{d^2 z}{2\pi i} e^{-\frac{1}{\hbar} [(z\bar{z})^R - V_+(z) - V_-(\bar{z})]} (z\bar{z})^{(R-1)/2 + \frac{\alpha}{\hbar}}. \quad (\text{A.2})$$

Then we introduce a set of bi-orthogonal polynomials

$$\Phi_+^n(z) = \frac{1}{n! \sqrt{h_n}} \int_{\mathbf{C}} \prod_{k=1}^n \frac{w_\alpha(z_k, \bar{z}_k)}{h_{k-1}(t, \alpha)} \Delta(z) \Delta(\bar{z}) \prod_{k=1}^n (z - z_k), \quad (\text{A.3})$$

and similarly for $\Phi_-^n(\bar{z})$. Here $\Delta(z) = \prod_{k < l} (z_k - z_l)$ is the Vandermonde determinant and normalization factors h_k are determined by the orthonormality condition

$$\langle \Phi_-^n | \Phi_+^m \rangle_w \equiv \int_{\mathbf{C}} w_\alpha(z, \bar{z}) \Phi_-^n(\bar{z}) \Phi_+^m(z) = \delta_{n-m}. \quad (\text{A.4})$$

Then the partition function (A.1) reduces to the product of the normalization factors

$$\mathcal{Z}_h^{\text{NMM}}(N, t, \alpha) = \prod_{n=0}^{N-1} h_n(t, \alpha). \quad (\text{A.5})$$

The operators of multiplication by z and \bar{z} are represented in the basis of orthogonal polynomials by the infinite matrices

$$z \Phi_+^n(z) = \sum_m L_{nm} \Phi_+^m(z), \quad \bar{z} \Phi_-^n(\bar{z}) = \sum_m \Phi_-^m(\bar{z}) \bar{L}_{mn}. \quad (\text{A.6})$$

with

$$\begin{aligned} L_{n,n+1} &= \bar{L}_{n+1,n} = \sqrt{h_{n+1}/h_n}, \\ L_{nm} &= \bar{L}_{mn} = 0, \quad m > n + 1. \end{aligned} \quad (\text{A.7})$$

Differentiation of the orthogonality relation (A.4) with respect to coupling constants t_k gives

$$\begin{aligned} \hbar \frac{\partial \Phi_+^n(z)}{\partial t_k} &= - \sum_{m=0}^{n-1} (L^k)_{nm} \Phi_+^m(z) - \frac{1}{2} (L^k)_{nn} \Phi_+^n(z), \\ \hbar \frac{\partial \Phi_+^n(z)}{\partial t_{-k}} &= - \sum_{m=0}^{n-1} (\bar{L}^k)_{nm} \Phi_+^m(z) - \frac{1}{2} (L^k)_{nn} \Phi_+^n(z) \end{aligned} \quad (\text{A.8})$$

and similarly for $\Phi_-^n(\bar{z})$. Let us define now a wave function which is a vector $\Psi = \{\Psi_n\}$ with elements

$$\Psi_n(t; z) = \Phi_+^n(z) e^{\frac{1}{\hbar} V_+(z)}. \quad (\text{A.9})$$

Then the equations (A.8) lead to the following eigenvalue problem

$$z \Psi = L \Psi, \quad \hbar \frac{\partial \Psi}{\partial t_k} = H_k \Psi, \quad \hbar \frac{\partial \Psi}{\partial t_{-k}} = H_{-k} \Psi, \quad (\text{A.10})$$

where we introduced the Hamiltonians

$$H_k = (L^k)_+ + \frac{1}{2} (L^k)_0, \quad H_{-k} = -(\bar{L}^k)_- - \frac{1}{2} (\bar{L}^k)_0. \quad (\text{A.11})$$

Here the subscripts $0/-/+$ denote diagonal/lower/upper triangular parts of the matrix. From the commutativity of the second derivatives, it is easy to find the Zakharov-Shabat zero-curvature condition

$$\hbar \frac{\partial H_k}{\partial t_l} - \hbar \frac{\partial H_l}{\partial t_k} + [H_k, H_l] = 0. \quad (\text{A.12})$$

It means that the Hamiltonians generate commuting flows and the perturbed system is described by the Toda Lattice hierarchy.

In addition, one can obtain the string equation for the hierarchy. It follows from the Ward identity

$$z \frac{\partial \Psi}{\partial z} = \frac{R}{\hbar} \bar{L}^R L^R \Psi - \left[(R+1)/2 + \frac{\alpha}{\hbar} \right] \Psi \quad (\text{A.13})$$

and can be written as

$$[L^R, \bar{L}^R] = \hbar. \quad (\text{A.14})$$

Here we should understand the operators L^R as analytical continuation of the operators in integer powers.

The Toda structure leads to an infinite set of PDE's for the coefficients of the operators L and \bar{L} . The first of these equations can be written for the normalization factors and is known as Toda equation

$$\hbar^2 \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_{-1}} \log h_n = \frac{h_n}{h_{n-1}} - \frac{h_{n+1}}{h_n}. \quad (\text{A.15})$$

The quantity h_n is known to be related to the τ -function of the Toda hierarchy by

$$h_n = \frac{\tau_{n+1}^{\text{NMM}}}{\tau_n^{\text{NMM}}}. \quad (\text{A.16})$$

Taking into account (A.5), this gives for the partition function

$$\mathcal{Z}_\hbar^{\text{NMM}}(N, t, \alpha) = \tau_{N, \hbar}^{\text{NMM}}(t, \alpha). \quad (\text{A.17})$$

Appendix B. Derivation of the formula for the cycle B

The formula (4.12) can be obtained by means of the procedure of transfer of an eigenvalue from the point $(z_1, \bar{z}_1 = \bar{z}(z_1))$ belonging to the boundary γ of the spot to ∞ (or rather to the cut-off point $\sqrt{\Lambda}$), worked out for the case $R = 1$ in [25]. On the double cover described in subsection 4.3, z_1 and \bar{z}_1 are considered as complex coordinates of the point in two patches. We find from (A.1) that the free energy in the large N limit changes during this transfer as follows:

$$\hbar \frac{\partial}{\partial N} \log \mathcal{Z}^{\text{NMM}} = -(z_1 \bar{z}_1)^R + V_+(z_1) + V_-(\bar{z}_1) + \hbar \sum_{m=2}^N \log[(z_1 - z_m)(\bar{z}_1 - \bar{z}_m)], \quad (\text{B.1})$$

where all z_m 's are taken at their saddle point values. Here we took into account that the determinant in the matrix integral (3.1) does not contribute in the quasiclassical limit. In this limit the integral (A.1) leads to the following saddle point equations [25]

$$Rz^R \bar{z}^{R-1} - V'_-(\bar{z}) = \bar{G}(\bar{z}), \quad Rz^{R-1} \bar{z}^R - V'_+(z) = G(z), \quad (\text{B.2})$$

where $G(z)$ and $G(\bar{z})$ are the resolvents of the one dimensional distributions of z_k 's and \bar{z}_k 's, respectively.⁹ They have the following asymptotics at large z and \bar{z} :

$$G(z) \longrightarrow \frac{\hbar N}{z}, \quad \bar{G}(\bar{z}) \longrightarrow \frac{\hbar N}{\bar{z}}. \quad (\text{B.3})$$

Rewriting the sum in (B.1) as an integral with the measure given by the eigenvalue density and then expressing it through the resolvents, we find

$$\hbar \frac{\partial}{\partial N} \log \mathcal{Z}^{\text{NMM}} = -(z_1 \bar{z}_1)^R + V_+(z_1) + V_-(\bar{z}_1) + \oint \frac{dz}{2\pi i} G(z) \log(z_1 - z) + \oint \frac{d\bar{z}}{2\pi i} \bar{G}(\bar{z}) \log(\bar{z}_1 - \bar{z}), \quad (\text{B.4})$$

where the contours of integration encircle the whole support of the distribution of eigenvalues on the physical sheets of the functions $\bar{z}(z)$ and $z(\bar{z})$, respectively.¹⁰

Due to (B.2) we do not have any singularities at $z \rightarrow \infty$ ($\bar{z} \rightarrow \infty$) except poles. Blowing up the contour we pick up the logarithmic cut and obtain

$$\hbar \frac{\partial}{\partial N} \log \mathcal{Z}^{\text{NMM}} = -(z_1 \bar{z}_1)^R - R \int_{z_1}^{\sqrt{\Lambda}} \frac{dz}{z} (z \bar{z}(z))^R - R \int_{\bar{z}_1}^{\sqrt{\Lambda}} \frac{d\bar{z}}{\bar{z}} (z(\bar{z}) \bar{z})^R. \quad (\text{B.5})$$

The first term in the r.h.s. can be regrouped with the other two terms, giving (up to a nonuniversal term $\sim \Lambda^R$)

$$\hbar \frac{\partial}{\partial N} \log \mathcal{Z}^{\text{NMM}} = \int_{\sqrt{\Lambda}}^{z_1} \left[R(z \bar{z}(z))^R \frac{dz}{z} - \frac{1}{2} d(z \bar{z})^R \right] - \int_{\bar{z}_1}^{\sqrt{\Lambda}} \left[R(z(\bar{z}) \bar{z})^R \frac{d\bar{z}}{\bar{z}} - \frac{1}{2} d(z \bar{z})^R \right], \quad (\text{B.6})$$

which immediately yields (4.11) if we take into account (4.6) and the definition (4.9). Note that (B.2) is essentially the same as (4.6) if one uses the explicit expansion (4.7) for the holomorphic parts of the potential. Therefore, they define the same functions $\bar{z}(z)$ and $z(\bar{z})$. On the complex curve described in the subsection 4.3, the formula (B.6) reduces to the period integral (4.12) of the holomorphic differential $d\Phi$.

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⁹We view $\bar{z}(z)$ and $z(\bar{z})$ as analytical functions similarly to the interpretation of subsection 4.3.

¹⁰Note that the point of infinite branching at the origin related to the singularities of z^R and \bar{z}^R should not appear on the physical sheets.

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Article V

Backgrounds of 2D string theory from matrix model

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Backgrounds of 2D string theory from matrix model

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In the Matrix Quantum Mechanical formulation of 2D string theory it is possible to introduce arbitrary tachyonic perturbations. In the case when the tachyonic momenta form a lattice, the theory is known to be integrable and, therefore, it can be used to describe the corresponding string theory. We study the backgrounds of string theory obtained from these matrix model solutions. They are found to be flat but the perturbations can change the global structure of the target space. They can lead either to a compactification, or to the presence of boundaries depending on the choice of boundary conditions. Thus, we argue that the tachyon perturbations have a dual description in terms of the unperturbed theory in spacetime with a non-trivial global structure.

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1 Introduction

The two-dimensional string theory has proven to be a model possessing a very rich structure [1, 2, 3]. It allows to address various questions inaccessible in higher dimensional string theories as well as it appears as an equivalent description of topological strings and supersymmetric gauge theories [4, 5, 6]. Perhaps, its main feature is the integrability. Many problems formulated in its framework, such as calculation of S -matrix, were found to be exactly solvable.

However, this integrability is hidden in the usual CFT formulation and becomes explicit only when one uses a matrix model description of string theory. In this case the model, one has to consider, is the Matrix Quantum Mechanics (MQM) [7]. It was successfully used to describe 2D string theory in the simplest linear dilaton background as well as to incorporate perturbations. In two dimensions there are basically two types of possible perturbations: tachyon and winding modes. The former always present in the theory as momentum modes of a massless field which, following tradition, is called “tachyon”. The latter appear when one considers Euclidean theory compactified on a time circle.

There are two ways to change the background of string theory: either to consider strings propagating in a non-trivial target space or to introduce the perturbations mentioned above. In the first case one arrives at a complicated sigma-model. Not many examples are known when such a model turns out to be solvable. Besides, it is extremely difficult to construct a matrix model realization of a general sigma-model since not much known about matrix operators explicitly perturbing the metric of the target space. Thus, we lose the possibility to use the powerful matrix model machinery to tackle our problems.

On the other hand, following the second way, we find that the integrability of the theory in the trivial background is preserved by the perturbations. Namely, when we restrict our attention only to windings or to tachyons with discrete momenta, they are described by Toda integrable hierarchy [8, 9, 10]. Therefore, the backgrounds of string theory obtained by these perturbations represent a good laboratory for investigation of different problems.

One of the such interesting problems is the black hole thermodynamics. As it is well known, 2D string theory possesses an exact background which incorporates the black hole singularity [11, 12]. From the target space point of view, it is described just by one of the dilaton models, the so-called CHGS model [13]. Its thermodynamical properties have been extensively studied (see, for example, the review on dilaton models [14]). However, to achieve the microscopical statistical description, one should deal with the full string theory so that one has to obtain the complete solution of the corresponding sigma-model.

Fortunately, there exists an alternative description of string theory in this background. It has been conjectured that the black hole background can be created also by introduction of the winding modes [15]. Moreover, a matrix model incorporating these perturbations has been proposed [9]. Thus, we get a possibility to describe dynamics of string theory in non-trivial backgrounds using the usual perturbations of MQM.

In this paper we consider tachyon perturbations of 2D string theory which are T-dual to perturbations by winding modes. We saw that the latter are supposed to create curvature of the target space. Do tachyon perturbations can also change the structure of the target space? The paper is devoted to addressing this question.

First of all, we explain why we study tachyon and not winding perturbations, whereas

these are the windings that have the most interesting physical applications like the physics of black holes. The main reason is that it is much easier to study the tachyon modes from the MQM point of view. In this case we have a powerful description in terms of free fermions on the one hand [16] and the Das–Jevicki collective field theory on the other [17]. The latter has a direct target space interpretation what allows to identify matrix model quantities with characteristics of the string background. However, this description is absent when the winding modes are included what is possibly related to the absence of a local target space interpretation of windings.

Tachyon modes do have such local interpretation and, therefore, the background, which they create, is characterized simply by a (time-dependent) condensate of these modes. However, we are looking for another description which would be dual to the one with the tachyon condensate. We expect it to be similar to the dual description of the winding perturbations, that is it should replace the time-dependent tachyon condensate by a change of the structure of the target space.

As one of justifications of why such dual description should exist, we mention that it has been shown that the tachyon perturbations lead to a finite temperature as if we had a black hole [18]. In fact, to give rise a temperature, it is not necessary for the target space to contain a black hole singularity. The thermal description can arise in very different situations including flat spacetimes, but in any case it should be different from the usual Minkowski space [19].

In this paper we show that comparison of the Das–Jevicki collective theory with the low-energy effective action for the string background leads to the picture we sketched above. Instead of the tachyon condensate we get the target space with a non-trivial global structure. Non-triviality can appear either as boundaries or as a compactification. Locally spacetime remains the same as it was before the perturbation.

The paper is organized as follows. In the next section we briefly review 2D string theory in the CFT formulation. In section 3 the Das–Jevicki collective field theory for the singlet sector of MQM is analyzed. We derive an effective action for the quantum correction to the eigenvalue density of MQM. In section 4 this correction and its action are identified with the tachyon field of string theory and the low-energy effective action for the string background, correspondingly. The identification imposes a condition which leaves us with the linear dilaton background. However, there is still a freedom to have a non-trivial global structure which is investigated in section 5. The case of the simplest Sine–Liouville perturbation is considered in detail.

2 Tachyon and winding modes in 2D Euclidean string theory

2D string theory is defined by the Polyakov action

$$S_P = \frac{1}{4\pi} \int d^2\sigma \sqrt{h} [h^{ab} \partial_a X \partial_b X + \mu], \quad (2.1)$$

where the bosonic field $X(\sigma)$ describes the embedding of the string into the Euclidean time dimension and h^{ab} is a world sheet metric. In the conformal gauge $h^{ab} = e^{\phi(\sigma)} \hat{h}^{ab}$, where

\hat{h}^{ab} is a background metric, the conformal mode ϕ becomes dynamical due to the conformal anomaly and the world-sheet CFT action takes the familiar Liouville form

$$S_0 = \frac{1}{4\pi} \int d^2\sigma [(\partial X)^2 + (\partial\phi)^2 - 2\hat{\mathcal{R}}\phi + \mu e^{-2\phi} + \text{ghosts}]. \quad (2.2)$$

This action is known to describe the unperturbed linear dilaton string background corresponding to the flat 2D target space parameterized by the coordinates (X, ϕ) [20].

In two dimensions string theory possesses only one field-theoretic degree of freedom, which is called ‘‘tachyon’’ since it corresponds to the tachyon mode of the bosonic string in 26 dimensions. However, in the present case it is massless. The vertex operators for the tachyons of momentum p are written as

$$V_p = \frac{\Gamma(|p|)}{\Gamma(-|p|)} \int d^2\sigma e^{ipX} e^{(|p|-2)\phi}. \quad (2.3)$$

If we consider the Euclidean theory compactified to a time circle of radius R , the momentum is allowed to take only discrete values $p_n = n/R$, $n \in \mathbf{Z}$. In this case there are also winding modes of the string around the compactified dimension which also have a representation in the CFT (2.2) as vortex operators \tilde{V}_q .

Both tachyons and windings can be used to perturb the simplest theory (2.2)

$$S = S_0 + \sum_{n \neq 0} (t_n V_n + \tilde{t}_n \tilde{V}_n). \quad (2.4)$$

How do these perturbations look from the target space point of view? The presence of non-vanishing couplings t_n (together with the cosmological constant μ) means that we consider a string propagation in the background with a non-trivial vacuum expectation value of the tachyon. In particular, the introduction of momentum modes makes it to be time-dependent. On the other hand, the winding modes, which are defined in terms of the T-dual world sheet field $\tilde{X} = X_R - X_L$, where X_R and X_L are the right and left-moving components of the initial field $X(\sigma)$, have no obvious interpretation in terms of massless background fields. Since windings are related to global properties of strings, they should correspond to some non-local quantities in the low-energy limit. Their condensation can result in change of the background we started with.

A particular proposal for the structure of the resulting background has been made for the simplest case of $\tilde{t}_{\pm 1} \neq 0$ [15]. The CFT with the lowest winding perturbation has been conjectured to be dual to the WZW $\text{SL}(2, \mathbf{R})/\text{U}(1)$ sigma-model, which describes the 2D string theory on the black hole background [11, 12, 21]. In [10] it was suggested that any perturbation of type (2.4) should have a dual description as string theory in some non-trivial background, which is determined by the two sets of couplings t_n and \tilde{t}_n . Thus, not only for winding, but also for tachyon perturbations we expect to find such a description.

Note, that the two sets of perturbations are related by T-duality on the world sheet [1]. However, this does not mean that the corresponding string backgrounds should be the same. This is because the momentum and winding modes have different target space interpretations and it is not clear how to carry over this T-duality from the world sheet to the target space.

3 Das–Jevicki collective field theory

The 2D string theory can be described using Matrix Quantum Mechanics in the double scaling limit (for review see[1]). The matrix model Lagrangian corresponding to the unperturbed CFT (2.2) is

$$L = \text{Tr} \left(\frac{1}{2} \dot{M}^2 - u(M) \right), \quad (3.1)$$

where $u(M) = -\frac{1}{2}M^2$ is the inverse oscillator potential. It is known also how to introduce tachyons and windings into this model [10, 22, 23, 9]. In the following we restrict ourselves to the tachyon perturbations. In this case it is enough to consider only the singlet sector of MQM, where the wave function is a completely antisymmetric function of the matrix eigenvalues x_i . Due to this the system has an equivalent description in terms of free fermions in the potential $u(M)$.

The fermionic description is very powerful and allows to calculate many interesting quantities in the theory. However, to make contact with the target space description of string theory, another representation has proven to be quite useful. It is obtained by a bosonization procedure and leads to a collective field theory developed by Das and Jevicki [17, 3]. The role of the collective field is played by the density of matrix eigenvalues

$$\varphi(x, t) = \text{Tr} \delta(x - M(t)). \quad (3.2)$$

If to introduce the conjugated field $\Pi(x)$ such that

$$\{\varphi(x), \Pi(y)\} = \delta(x, y), \quad (3.3)$$

the effective action for the collective theory is written as

$$\begin{aligned} S_{\text{coll}} &= \int dt \left[\int dx (\Pi \partial_t \varphi) - H_{\text{coll}} \right], \\ \text{where } H_{\text{coll}} &= \int dx \left(\frac{1}{2} \varphi (\partial_x \Pi)^2 + \frac{\pi^2}{6} \varphi^3 + (u(x) + \mu) \varphi \right) \end{aligned} \quad (3.4)$$

and we added the chemical potential μ to the action. The classical equations of motion read

$$-\partial_x \Pi = \frac{1}{\varphi} \int dx \partial_t \varphi, \quad (3.5)$$

$$-\partial_t \Pi = \frac{1}{2} (\partial_x \Pi)^2 + \frac{\pi^2}{2} \varphi^2 + (u(x) + \mu). \quad (3.6)$$

Due to the first equation one can exclude the momenta and pass to the Lagrangian form of the action

$$S_{\text{coll}} = \int dt \int dx \left(\frac{1}{2\varphi} \left(\int dx \partial_t \varphi \right)^2 - \frac{\pi^2}{6} \varphi^3 - (u(x) + \mu) \varphi \right) \quad (3.7)$$

with the following equation for the classical field

$$\partial_t \left(\frac{\int dx \partial_t \varphi}{\varphi} \right) = \frac{1}{2} \partial_x \left(\frac{\int dx \partial_t \varphi}{\varphi} \right)^2 + \frac{\pi^2}{2} \partial_x \varphi^2 + \partial_x u(x). \quad (3.8)$$

Let us consider a function $\frac{1}{\pi}\varphi_0(x, t)$, which is a solution of the classical equation of motion (3.8). We expand the collective field around the background given by this solution

$$\varphi(x, t) = \frac{1}{\pi}\varphi_0(x, t) + \frac{1}{\sqrt{\pi}}\partial_x\eta(x, t). \quad (3.9)$$

The action takes the form

$$S_{\text{coll}} = S_{(0)} + S_{(1)} + S_{(2)} + \cdots, \quad (3.10)$$

where $S_{(0)}$ is a constant, $S_{(1)}$ vanishes due to the equation (3.8), and $S_{(2)}$ is given by

$$S_{(2)} = \frac{1}{2} \int dt \int \frac{dx}{\varphi_0} \left[(\partial_t \eta)^2 - 2 \frac{\int dx \partial_t \varphi_0}{\varphi_0} \partial_t \eta \partial_x \eta - \left(\varphi_0^2 - \left(\frac{\int dx \partial_t \varphi_0}{\varphi_0} \right)^2 \right) (\partial_x \eta)^2 \right]. \quad (3.11)$$

We want to use the quadratic part of the action to extract information about the background of string theory corresponding to the solution $\varphi_0(x, t)$. For that we will identify the quantum correction η with the tachyon field and interpret (3.11) as the kinetic term for the tachyon in the low-energy string effective action. Therefore, we will need the properties of the matrix standing in front of derivatives of η . The crucial property is that for any φ_0 its determinant is equal to -1 . As a result, the action can be represented in the following form

$$S_{(2)} = -\frac{1}{2} \int dt \int dx \sqrt{-g} g^{\mu\nu} \partial_\mu \eta \partial_\nu \eta. \quad (3.12)$$

In a more general case we would have to introduce a dilaton dependent factor coupled with the kinetic term.

The action (3.12) is conformal invariant. Therefore, one can always choose coordinates where the metric takes the usual Minkowski form $\eta_{\mu\nu} = \text{diag}(-1, 1)$. Unfortunately, we do not know the explicit form of this transformation for arbitrary solution φ_0 . Note, however, that one can bring the metric to another standard form which is of the Schwarzschild type. Indeed, it is easy to check that if one changes the x coordinate to

$$y(x, t) = \int \varphi_0(x, t) dx, \quad (3.13)$$

the metric becomes

$$g_{\mu\nu} = \begin{pmatrix} -\varphi_0^2 & 0 \\ 0 & \varphi_0^{-2} \end{pmatrix}. \quad (3.14)$$

4 Low-energy effective action and background of 2D string theory

Now we turn to the string theory side of the problem. In a general background the string theory is defined by the following σ -model action

$$S_\sigma = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \left[G_{\mu\nu}(X) h^{ab} \partial_a X^\mu \partial_b X^\nu + \alpha' \hat{\mathcal{R}} \Phi(X) + T(X) \right], \quad (4.1)$$

where $X(\sigma)$ is a two-dimensional field on the world sheet. As it is well known the condition that the conformal invariance of this theory is unbroken reduces to some equations on the

background fields: target space metric $G_{\mu\nu}$, dilaton Φ and tachyon T . These equations can be obtained from the effective action which in the leading order in α' is given by [24]

$$S_{\text{eff}} = \frac{1}{2} \int d^2 X \sqrt{-G} e^{-2\Phi} \left[\frac{16}{\alpha'} + R + 4(\nabla\Phi)^2 - (\nabla T)^2 + \frac{4}{\alpha'} T^2 \right]. \quad (4.2)$$

Here the covariant derivative ∇_μ is defined with respect to the metric $G_{\mu\nu}$ and R is its curvature. The first term comes from the central charge and in D dimensions looks as $\frac{2(26-D)}{3\alpha'}$ disappearing in the critical case. The equations of motion following from the action (4.2) are equivalent to

$$\begin{aligned} \beta_{\mu\nu}^G &= R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \nabla_\mu T \nabla_\nu T = 0, \\ \frac{4}{\alpha'} \beta^\Phi &= -\frac{16}{\alpha'} - R + 4(\nabla\Phi)^2 - 4\nabla^2\Phi + (\nabla T)^2 - \frac{4}{\alpha'} T^2 = 0, \\ \beta^T &= -2\nabla^2 T + 4\nabla\Phi \cdot \nabla T - \frac{8}{\alpha'} T = 0. \end{aligned} \quad (4.3)$$

Let us concentrate on the tachyonic part of the action (4.2). We want to relate it to the action (3.12) for the collective theory obtained from Matrix Quantum Mechanics. In particular, the tachyon field T and quantum fluctuation of density η should be correctly identified with each other. We see that the main difference between (4.2) and (3.12) is the presence of the dilaton dependent factor $e^{-2\Phi}$. To remove this factor, we redefine the tachyon as $T = e^\Phi \eta$. Then the part of the action quadratic in the tachyon field reads

$$S_{\text{tach}} = -\frac{1}{2} \int d^2 X \sqrt{-G} \left[(\nabla\eta)^2 + m_\eta^2 \eta^2 \right], \quad (4.4)$$

where

$$m_\eta^2 = (\nabla\Phi)^2 - \nabla^2\Phi - 4\alpha'^{-1}. \quad (4.5)$$

We denoted the new tachyon by the same letter as the matrix model field since they should be identical. However, comparing two actions (4.4) and (3.12), we see that for this identification to be true, one must require the vanishing of the tachyonic mass

$$m_\eta^2 = 0. \quad (4.6)$$

This equation plays the role of an additional condition for the equations of motion (4.3) and allows to select the unique solution. It is easy to check that the linear dilaton background

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad \Phi = -\frac{2}{\sqrt{\alpha'}} X^1, \quad \eta = 0 \quad (4.7)$$

satisfies all these equations. Besides, we argue that it is the unique solution of the equations (4.3) and (4.6). It means that for any solution of MQM restricted to the singlet sector we always get the simplest linear dilaton background. Since the theory (4.2) is a model of two-dimensional dilaton gravity with non-minimal coupled scalar field, it is not exactly solvable [14]. Therefore, we are not able to prove our statement rigorously. However, in Appendix A

we show that there are no solutions which can be represented as an expansion around the linear dilaton background (4.7).¹

Thus, any tachyon perturbation can not modify the local structure of spacetime. However, the found solution does not forbid modifications of the global structure. There are, in principle, two possibilities to change it. Either something can happen with the topology, for example, a spontaneous compactification, or there can appear boundaries. We argue that just these possibilities are indeed realized in our case.

To understand why this can happen, note that in the initial coordinates (t, x) of MQM the metric is non-trivial, although flat. We can represent it as

$$G_{\mu\nu} = e^{2\rho} g_{\mu\nu}, \quad (4.8)$$

where $g_{\mu\nu}$ is the matrix coupled with the derivatives of η in (3.11), whose determinant equals -1 . It takes the usual form $\eta_{\mu\nu}$ only after a coordinate transformation. Under this transformation, the image of the (t, x) -plane is, in general, not the whole plane, but some its subspace. Depending on boundary conditions, it will give rise either to a compactification, or to the appearance of boundaries.

5 Target space of string theory with integrable tachyon perturbations

In this section we analyze the structure of the target space of 2D string theory in the presence of tachyon perturbations with discrete momenta. Such perturbations correspond to the admissible perturbations of the Euclidean theory described in section 2. Nevertheless, our analysis is carried out in spacetime of the Minkowskian signature.

The case, we consider here, was shown to be exactly integrable and described by Toda Lattice hierarchy [10]. Therefore, it is possible to find explicitly the classical solution φ_0 , playing the role of the background field in the Das–Jevicki collective theory. Moreover, as we will show, the integrability manifests itself at the next steps of the derivation, what allows to complete the analysis of this case.

5.1 Background field from matrix model solution

First, we show how the classical solution φ_0 determining the string background can be extracted from MQM. As it was noted, it represents the density of eigenvalues. The latter is most easily found from the exact form of the Fermi sea formed by fermions of the MQM singlet sector. To see how this works, let us introduce the left and right moving chiral fields

¹Of course, this result is valid only for the equations of motion obtained in the leading order in α' . For example, the background described by the CFT (2.2) or even (2.4) with $\tilde{t}_n = 0$ corresponds to the linear dilaton background with a non-vanishing tachyon field. Such background does not satisfy the equations (4.3). However, it was suggested that it satisfies the exact equations which include all corrections in α' [2]. We are interested in the existence of a dual description where the non-vanishing tachyon vacuum transforms into expectation values of other fields. Our results show that the curvature of the target space and the dilaton can not be changed in the leading order in α' .

[3]:

$$p_{\pm}(x, t) = \partial_x \Pi \pm \pi \varphi(x, t). \quad (5.1)$$

They have the following Poisson brackets

$$\{p_{\pm}(x), p_{\pm}(y)\} = \mp 2\pi \partial_x \delta(x - y) \quad (5.2)$$

and bring the Hamiltonian to the form

$$H_{\text{coll}} = \int \frac{dx}{2\pi} \left(\frac{1}{6} (p_+^3 - p_-^3) + (u(x) + \mu)(p_+ - p_-) \right). \quad (5.3)$$

Using (5.2), it is easy to show that they should satisfy the Hopf equation

$$\partial_t p_{\pm} + p_{\pm} \partial_x p_{\pm} + \partial_x u(x) = 0. \quad (5.4)$$

This equation is of integrable type and much easier to solve than eq. (3.8). The key observation, which establishes the contact with MQM, is that in the quasiclassical limit $p_{\pm}(x, t)$ determine the upper and lower boundaries of the Fermi sea in the phase space. Then the density is given by their difference what agrees with the definition (5.1).

For general integrable tachyon perturbations, the solution for the profile of the Fermi sea has been found in [10]. If we introduce the sources for tachyons with momenta $p_k = k/R$, $k = 1, \dots, n$, it can be represented in the following parametric form

$$\begin{aligned} p(\omega, t) &= \sum_{k=0}^n a_k \sinh \left[\left(1 - \frac{k}{R} \right) \omega + \frac{k}{R} t + \alpha_k \right], \\ x(\omega, t) &= \sum_{k=0}^n a_k \cosh \left[\left(1 - \frac{k}{R} \right) \omega + \frac{k}{R} t + \alpha_k \right]. \end{aligned} \quad (5.5)$$

Here a_k are coefficients which can be found in terms of the coupling constants t_k of the tachyon operators and the chemical potential μ which fixes the Fermi level. Actually, they depend only on μ and the products $\lambda_k^2 = t_k t_{-k}$. Instead, the “phases” α_k are determined by ratios of the coupling constants. (Two of them can be excluded by constant shifts of ω and t .) It is easy to check that $p(x, t)$ indeed satisfies (5.4) with $u(x) = -\frac{1}{2}x^2$.

To get from this solution φ_0 , note that $p(x)$ is a two-valued function so that $p_{\pm}(x)$ can be identified with its two branches. As functions of ω they appear as follows. Let us define a “mirror” parameter $\tilde{\omega}(\omega, t)$ such that (see fig. 1)

$$x(\tilde{\omega}(\omega, t), t) = x(\omega, t), \quad \tilde{\omega} \neq \omega. \quad (5.6)$$

Then p_+ can be identified with p and p_- with $p(\tilde{\omega}(\omega, t), t)$. The solution for the background field is given again in the parametric form

$$\varphi_0(\omega, t) = \frac{1}{2} (p(\omega, t) - \tilde{p}(\omega, t)), \quad (5.7)$$

where we denoted $\tilde{p}(\omega, t) = p(\tilde{\omega}(\omega, t), t)$ and ω is related to x by eq. (5.5). Due to (3.5), (5.1) and (5.7), the effective action (3.11) can now be rewritten as

$$S_{(2)} = \int dt \int \frac{dx}{p - \tilde{p}} \left[(\partial_t \eta)^2 + (p + \tilde{p}) \partial_t \eta \partial_x \eta + p \tilde{p} (\partial_x \eta)^2 \right]. \quad (5.8)$$

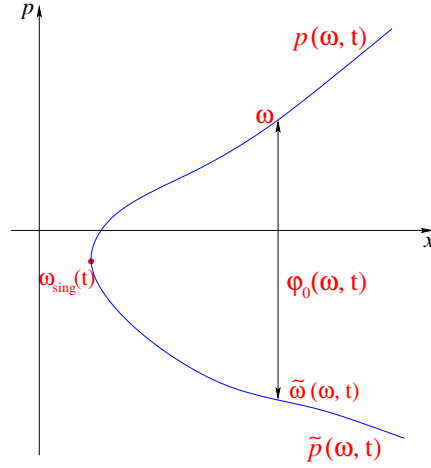


Fig. V.1: The Fermi sea of the perturbed MQM. Its boundary is defined by a two-valued function with two branches parameterized by $p(\omega, t)$ and $\tilde{p}(\omega, t)$. The background field φ_0 coincides with the width of the Fermi sea.

5.2 Flat coordinates

We know from section 4 that any background described by the Das-Jevicki effective action leads to the flat target space metric. However, as we discussed above, the global structure of the target space can differ from that of the usual Minkowski space. To find whether this is the case, one should investigate the coordinate transformation mapping the target space metric into the standard form $\eta_{\mu\nu}$.

In general, it is quite a difficult problem to find such a map explicitly. Remarkably, in our case given by the solution (5.5), this task can be accomplished. Indeed, we show in Appendix B that the change of coordinates from (t, x) to (τ, q) with

$$\tau = t - \frac{\omega + \tilde{\omega}}{2}, \quad q = \frac{\omega - \tilde{\omega}}{2} \quad (5.9)$$

brings the action (5.8) to the standard form

$$S_{(2)} = \pm \frac{1}{2} \int d\tau \int dq \left[(\partial_\tau \eta)^2 - (\partial_q \eta)^2 \right], \quad (5.10)$$

where the sign is determined by the sign of the Jacobian of the coordinate transformation (5.9). The Jacobian is defined by the function D in (B.2) and can be represented in the form

$$2D = \frac{2(p - x_t)(\tilde{p} - \tilde{x}_t)}{\tilde{p} - p}, \quad (5.11)$$

where $x_t \equiv \partial_t x(\omega, t)$ and $\tilde{x}_t \equiv x_t(\tilde{\omega}, t)$ given explicitly in (B.4).

The zeros of the Jacobian show where the map (5.9) is degenerate. All multipliers in (5.11) vanish at the line defined by the condition $\tilde{\omega}(\omega, t) = \omega$ or $\partial x / \partial \omega = 0$, which corresponds to the most left point of the Fermi sea where two branches of $p(x)$ meet each other (see fig. 1). In terms of the flat coordinates (5.9), this line is given in the following parametric form

$$\tau = t - \omega_{\text{sing}}(t), \quad q = 0, \quad (5.12)$$

where $\omega_{\text{sing}}(t)$ is a solution of the above condition. Also it defines the limiting value of the coordinate x , which should always be larger than $x_{\text{sing}}(t) = x(\omega_{\text{sing}}(t), t)$. However, the spacetime can be analytically continued through this line. This analytical continuation corresponds to that in all integrals x should run from ∞ to x_{sing} and return back with simultaneous interchanging the roles of p and \tilde{p} . This shows that it is more natural to consider the plane of (t, ω) as the starting point rather than (t, x) . The former is a two-sheet cover of the physical region of the latter. ω appears as a parameter along fermionic trajectories and it should not be restricted to a half-line. Thus, we incorporate the whole scattering picture and glue two copies of the resulting flat spacetime together along the line (5.12).

Unfortunately, there is an ambiguity in the choice of coordinates where the action has the form (5.10). Any conformal change of variables (5.9)

$$(\tau, q) \longrightarrow (u(t - \omega), v(t - \tilde{\omega})) \quad (5.13)$$

leaves the action unchanged. Due to this we can find the form of the resulting spacetime only up to the conformal map. However, one can reduce the ambiguity imposing some conditions on the map (5.13). First of all, it should be well defined on the subspace which is the image of the (t, ω) -plane under the map (5.9), that is the functions u and v should not have singularities there. The next condition is the triviality of (5.13) in the absence of perturbations since in this limit (5.9) reduces to the well known transformation giving the flat Minkowski space [3]. Besides, we would like to retain the symmetry $(t, \omega) \leftrightarrow (-t, -\omega)$ which is explicit when there are no phases α_k in (5.5). It reflects the fact the our spacetime is glued from two identical copies (see discussion in the previous paragraph). This leads to the condition that u and v are odd functions.

As a byproduct of our analysis we obtain also the conformal factor $e^{2\rho}$ appearing in (4.8). It should be found, in principle, from the condition of vanishing curvature for the metric $G_{\mu\nu}$. Instead, we get it as the Jacobian of the transformation to the coordinates where $G_{\mu\nu} = \eta_{\mu\nu}$. If u and v from (5.13) are such coordinates, than $dt dx \sqrt{-G} = du dv$ and, since $\sqrt{-G} = e^{2\rho}$, we conclude from (5.11) that

$$e^{2\rho} = \frac{\tilde{p} - p}{2(p - x_t)(\tilde{p} - \tilde{x}_t)} u' v'. \quad (5.14)$$

The conformal factor becomes singular on the line (5.12) discussed above. However, this is only a coordinate singularity since it disappears after our coordinate transformation.

5.3 Structure of the target space

In this subsection we investigate the concrete form of the target space which is obtained as the image under the coordinate transformation trivializing the metric. From the previous subsection it follows that this transformation is a superposition of the map (5.9) and a conformal change of coordinates (5.13). Since the latter remains unknown, we will analyze only the image of the map (5.9). It will be sufficient to get a general picture because conformal transformations do not change the causal structure of spacetime.

We restrict our analysis to the case of the simplest non-trivial perturbation with $n = 1$ in (5.5). It corresponds to the conformal Sine-Liouville theory. At $R = 2/3$ this theory is

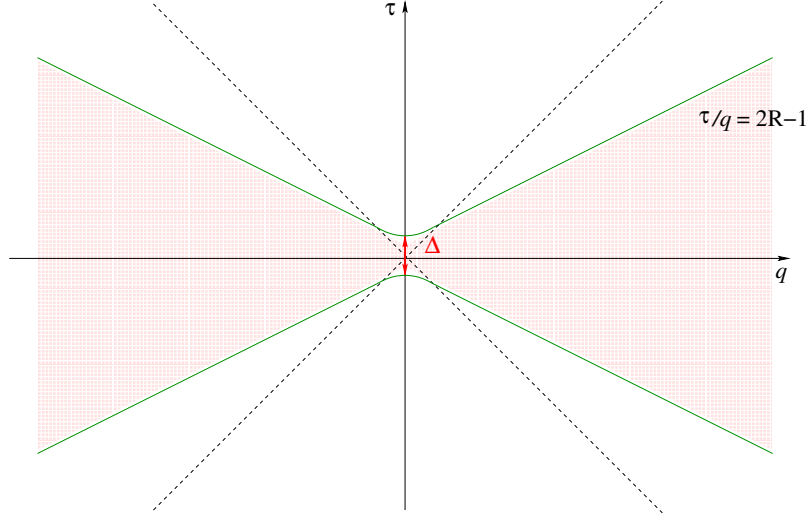


Fig. V.2: Target space of the perturbed theory for the case $R < 1$.

T-dual to the CFT suggested to describe string theory on the black hole background [15]. We also assume that the coefficients a_k in (5.5) are positive what corresponds to the positivity of the coupling constants $t_{\pm 1}$. It is the case which was considered in [9, 25, 10, 18, 26].

To find the image, we investigate the change of coordinates at infinity of the (t, ω) plane which plays the role of the boundary of the initial spacetime. In other words, we want to find where the boundary is mapped to. Thus we assume t and ω to be large and look at the dependence of $\tilde{\omega}$ of their ratio. At the asymptotics $\tilde{\omega}(\omega, t)$ can be easily found, but the answer depends non-continuously on t/ω as well as on the parameter R . There are several different cases which we summarize in Appendix C.

$R \geq 1$: From the results presented in Appendix C it follows that in this case the image of the map (5.9) is the entire Minkowski spacetime. Thus, we do not see any effect of the tachyon perturbation on the structure of the target space.

$1/2 < R < 1$: In this case the resulting spacetime looks like two conic regions bounded by lines $\tau = \pm(2R - 1)q \pm R \log \frac{a_0}{a_1}$ (fig. 2). Near the origin the boundaries are not anymore the straight lines but defined by the solution of the following irrational equation

$$a_0 e^{\tilde{\xi}} + a_1 e^{(1-1/R)\tilde{\xi}} = a_0 e^{\xi} + a_1 e^{(1-1/R)\xi}, \quad \tilde{\xi} \neq \xi. \quad (5.15)$$

It allows to represent the boundary in the parametric form

$$\tau = -\frac{1}{2}(\xi + \tilde{\xi}(\xi)), \quad q = \frac{1}{2}(\xi - \tilde{\xi}(\xi)). \quad (5.16)$$

Two conic regions are glued along some finite interval belonging to the τ -axis. It is just that line where the change of coordinates (5.9) is degenerate. We can easily find its length. As it was discussed in the previous subsection, the interval is defined by eq. (5.12). We need to find the limiting value of $\tau(t)$ when $t \rightarrow \infty$. In this limit the condition for $\omega_{\text{sing}}(t)$ can be written as

$$2\frac{\partial x}{\partial \omega} \approx a_0 e^{\omega_{\text{sing}}} + \left(1 - \frac{1}{R}\right) a_1 e^{(1-1/R)\omega_{\text{sing}} + \frac{1}{R}t} = 0 \Rightarrow \omega_{\text{sing}}(t) = t - \frac{1}{2}\Delta, \quad (5.17)$$

where

$$\Delta = 2R \log \left[\frac{R}{1-R} \frac{a_0}{a_1} \right]. \quad (5.18)$$

Thus $\tau(t) \xrightarrow[t \rightarrow \infty]{} \Delta/2$ and we conclude that Δ is the length we were looking for.

From this result it is easy to understand what happens when we switch off the perturbation. This corresponds to the limit $a_1 \rightarrow 0$. Then the interval Δ (the minimal distance between two boundaries) logarithmically diverges and the boundaries go away to infinity. In this way we recover the entire Minkowski space. The similar picture emerges in the limit $R \rightarrow 1$ when we return to the previous case $R \geq 1$.

It is interesting to look at the opposite limit where the length Δ vanishes. It happens when $a_1 = \frac{Ra_0}{1-R}$. The parameters are related to the coupling constant $\lambda = \sqrt{t_1 t_{-1}}$ as follows [10]

$$a_0 = \sqrt{2} e^{-\frac{1}{2R}\chi}, \quad a_1 = \frac{\sqrt{2}\lambda}{R} e^{\frac{R-1}{2R^2}\chi}, \quad (5.19)$$

where $\chi = \partial^2 \mathcal{F} / \partial \mu^2$ is the second derivative of the grand canonical free energy. This quantity is defined by the equation [9]

$$\mu e^{\frac{1}{R}\chi} - \left(1 - \frac{1}{R}\right) \frac{\lambda^2}{R^2} e^{\frac{2R-1}{R^2}\chi} = 1. \quad (5.20)$$

Taking into account (5.19) and (5.20), the condition of vanishing Δ leads to the following critical point

$$\mu_c = - \left(2 - \frac{1}{R}\right) \left(\frac{1}{R} - 1\right)^{\frac{1}{2R-1}} \left(\frac{\lambda}{R}\right)^{\frac{2R}{2R-1}}. \quad (5.21)$$

In [10] it was shown that at this point the Fermi sea forms a pinch and beyond it the solution for the free energy does not exist anymore. Also this point coincides with the critical point of Hsu and Kutasov [27], who interpreted it as a critical point of the pure 2D gravity type.

$R = 1/2$: This case can be analyzed using the result (C.1). We conclude that the spacetime takes the form of a strip bounded by lines $\tau = \pm \frac{1}{2} \log \frac{a_0}{a_1}$ what agrees also with (5.18).

$R < 1/2$: This interval of values of the parameter R was splitted in Appendix C into two cases. However, they both lead to the same picture. Moreover, it coincides with one analyzed for $1/2 < R < 1$ and shown in fig. 2. Nevertheless, there is an essential distinction with respect to that case. Using explicit expressions for the new coordinates from Appendix C, it is easy to check that the Jacobian of the transformation (5.9) is negative for $R < 1/2$. Therefore, one has to choose the minus sign in front of the action in (5.10). Due to this, the time and space coordinates are exchanged, so that τ and q are associated with space and time directions, correspondingly. As a result, the picture at fig. 2 should now be rotated by 90° .

5.4 Boundary conditions and global structure

We found that the introduction of tachyon perturbations in MQM is equivalent to consider string theory in the flat spacetime of the form described in the previous subsection (or its

conformal transform). The main feature of this spacetime is the presence of boundaries. Therefore, to define the theory completely, we should impose some boundary conditions on the fields propagating there. We do not see any rigorous way to derive them. Nevertheless, we discuss the most natural choices for the boundary conditions and their physical interpretation.

Vanishing boundary conditions

The most natural choice is to choose the vanishing boundary conditions. It can be justified by that the boundary on fig. 2 came from infinity on the (t, ω) plane of the matrix model. In the usual analysis of a quantum field theory one demands that all fields disappear at infinity. Therefore, it seems to be natural to impose the same condition in our case also.

The presence of boundaries with vanishing conditions on them can be interpreted as if one placed the system between two moving mirrors. However, we note that such interpretation is reliable only for $R < 1/2$ where the boundaries are timelike. In the most interesting case $R \in (1/2, 1)$ we would have to suppose that mirrors move faster than light.

Nevertheless, this interpretation opens an interesting possibility. It is known that moving mirrors cause particle creation [19]. In particular, the resulting spectrum can be thermal so that the system turns out to be at a finite temperature. On the other hand, in [18] it was shown that the Sine-Liouville tachyon perturbation gives rise to the thermal description at $T = 1/(2\pi R)$. Therefore, it would be extremely interesting to reproduce this result from the exact form of the spacetime obtained here. In our case the particle creation is a quite expected effect because two cones lead to the existence of two natural vacua associated with the right and left cone, correspondingly. When the modes, defining one of the vacua, propagate from one cone to another, they disperse on the hole and, in general, will lead to appearance of particles with respect to the second vacuum. The concrete form of the particle spectrum depends crucially on the exact behaviour near the origin since the creation happens only when the mirror is accelerated [28, 29]. Unfortunately, in this region the form of the boundary is known only inexplicitly through the equation (5.15). Besides, the situation is complicated by the possibility that one needs to make a conformal transform (5.13) to arrive to the flat coordinates. Usually, such transform can change thermal properties [14]. Therefore, it remains unsolved problem to check whether the particle creation in the found spacetime reproduces the result obtained in the MQM framework.

Periodic conditions

Although the vanishing boundary conditions are very natural, it is questionable to apply them in the case of spacelike boundaries. For example, this would lead to a discrete spectrum at finite distance from the origin although the spacelike slices are non-compact. In general, the situation when one encounters a boundary in time is very strange.

Therefore, in this case it could be more natural to choose periodic boundary conditions where one identifies the upper boundary on fig. 2 with the lower one. As a result, the spacetime is compactified. For $R \in (1/2, 1)$ the compact dimension is timelike so that we find closed timelike curves. Of course, their existence is a very bad feature. But it might be better than to have spacelike boundaries. For $R < 1/2$ the periodic boundary conditions can also be applied. Then the spacetime has the topology of a cylinder with a spacelike compact dimension and we do not encounter the above mentioned problems. In this case it is also expected that the change of radius of the cylinder in time will cause particle creation and

can give rise to a temperature.

Note that the picture obtained in this case resembles the orbifold construction of [30], where the Minkowski spacetime was factored out by action of a discrete subgroup of boosts. It leads to four cones, joint at the origin, corresponding to the causal cones in the Minkowski spacetime. If one tries to interpret fig. 2 with the periodic boundary conditions following this analogy, one concludes that it corresponds to two cones of [30] (spacelike or timelike, depending on the value of R) with the resolved singularity because the cones are joint not at a point but along some finite interval.

Twisted periodic conditions

There is also another possible choice for the boundary conditions which comes from comparison with Matrix Quantum Mechanics. It is very unlikely that this possibility can be realized but we mention it for completeness. In MQM to define the density of states and, consequently, the free energy, one should place the system in a box and also impose some boundary conditions on the wave functions of fermions of the singlet sector of MQM. They relate the scattered (out-going) wave function to the in-coming one [10]. Thus one can say that they identify two infinities $t, \omega \rightarrow \infty$ and $t, \omega \rightarrow -\infty$. Being applied to our picture, these conditions mean that one should identify the left lower boundary with the right upper one and *vice-versa*. In this way we arrive at the compactified spacetime but where it was “twisted” before the compactification. The topology of the resulting spacetime is that of the Möbius sheet. Note that these boundary conditions have an advantage that they are invariant under the conformal map (5.13) if one required from it to preserve the symmetry $(t, \omega) \leftrightarrow (-t, -\omega)$. However, the fact that they lead to a unorientd spacetime rises doubts on their credibility.

We stop our discussion at this point. We are not able to choose the right boundary conditions. Due to this, we are not able also to establish the effect produced by tachyon perturbations on the target space: either it is the appearance of boundaries or a compactification. But the form of the obtained spacetime indicates that one of these modifications of the global structure does take place.

6 Conclusions

We analyzed the proposal that the tachyon perturbations of 2D string theory in the linear dilation background have a dual description in a non-trivial target space. Comparing the Das–Jevicki collective field theory with the low-energy effective action of string theory, we found that the tachyon perturbations can not change the local structure of spacetime so that we always remain in the linear dilaton background. However, the relation between the flat coordinates, where the background metric has the standard Minkowski form, and the coordinates of the collective theory coming from Matrix Quantum Mechanics is affected by the perturbation. As a result, MQM describes only a part of the Minkowski spacetime. This introduces boundaries for the target space. But its real structure depends on the boundary conditions. If they are chosen to be the periodic conditions, we get a compactified target space. Unfortunately, we were not able to choose the correct boundary conditions and, therefore, we can not say what the final form of the target space is.

One could ask about the physical meaning of the boundaries because the spacetime can be continued through them to get the entire Minkowski spacetime. However, when we work with the tachyon perturbations in the framework of Matrix Quantum Mechanics, we can not access the parts of spacetime obtained by this continuation. The situation can be compared with that of the Unruh effect [31], when an accelerated observer “sees” only the Rindler cone of the Minkowski spacetime. Thus, we can think of the tachyon perturbations as if they place the system into another reference frame. However, there is a crucial difference between the Unruh effect and our situation. In the former case, the boundaries are lightlike and considered as event horizons, whereas our boundaries are spacelike or timelike and one should find for them a physical interpretation. In the context of string theory it is tempting to think about them as branes. Note that spacelike branes (S-branes) have also been introduced in string theory [32], so that the existence of spacelike boundaries should not be an obstacle for this interpretation. Recently, it was also suggested a relation between S-branes and thermodynamics [33].

May be one of the most interesting problems, which is left unsolved in the paper, is to show that the field-theoretic analysis on the found target space leads to a thermal description with the temperature $1/(2\pi R)$. This would confirm the result obtained in the MQM framework [18] and would be a non-trivial consistency check. Also, if the periodic boundary conditions are realized, it would be interesting to elaborate a connection with the work [30]. This might help to understand how to resolve the orbifold singularity of the type considered there. In [30] the resolution was found to be impossible.

Finally, the main problem, which we did not consider here, is to find the string background corresponding to winding perturbations. The results obtained in this paper could be useful to approach the answer if one can realize the world sheet T-duality directly in the target space. Unfortunately, when the duality transformation does not act in the direction of a Killing vector, it is still an open problem.

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Appendix A. Perturbative solution of dilaton gravity

We are interested in the solution of the system of equations (4.3) and (4.6). After the substitution $T = e^\Phi \eta$, they can be rewritten as

$$\begin{aligned}
 -\frac{1}{2}e^{-\Phi}\beta^T - m_\eta^2\eta &= \nabla^2\eta = 0, \\
 m_\eta^2 &= (\nabla\Phi)^2 - \nabla^2\Phi - 4\alpha'^{-1} = 0, \\
 4\alpha'^{-1}\beta^\Phi + \beta_{\mu\nu}^G G^{\mu\nu} - 4m_\eta^2 &= 2\nabla^2\Phi - 4\alpha'^{-1}e^{2\Phi}\eta^2 = 0, \\
 \beta_{\mu\nu}^G &= R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \nabla_\mu(e^\Phi\eta)\nabla_\nu(e^\Phi\eta) = 0.
 \end{aligned} \tag{A.1}$$

Let us choose the conformal gauge

$$G_{\mu\nu} = e^{2\rho}\eta_{\mu\nu}. \tag{A.2}$$

Then in the light-cone coordinates $X_{\pm} = \frac{X^0 \pm X^1}{\sqrt{2}}$ the equations take the following form

$$\begin{aligned}
\partial_+ \partial_- \eta &= 0, \\
\partial_+ \partial_- \Phi - \partial_+ \Phi \partial_- \Phi &= 2\alpha'^{-1} e^{2\rho}, \\
\partial_+ \partial_- \Phi &= -\alpha'^{-1} e^{2(\Phi+\rho)} \eta^2, \\
\partial_+ \partial_- \rho - \partial_+ \partial_- \Phi &= -\frac{1}{2} \partial_+ (e^\Phi \eta) \partial_- (e^\Phi \eta), \\
\partial_+ \partial_+ \Phi &= \frac{1}{2} \left(\partial_+ (e^\Phi \eta) \right)^2, \\
\partial_- \partial_- \Phi &= \frac{1}{2} \left(\partial_- (e^\Phi \eta) \right)^2,
\end{aligned} \tag{A.3}$$

where the last three equations are produced by the last equation in (A.1).

First of all, it is easy to see that if $\eta = 0$, (4.7) is the unique solution. To generalize this statement, let us consider a solution which can be represented as an expansion around the linear dilaton background (4.7). We use the following ansatz

$$\rho = \sum_{k>0} \epsilon^{k+1} \rho_k, \quad \Phi = -\frac{2}{\sqrt{\alpha'}} X^1 + \sum_{k>0} \epsilon^{k+1} \phi_k, \quad \eta = \sum_{k>0} \epsilon^k \eta_k. \tag{A.4}$$

In the first order in ϵ we find

$$\begin{aligned}
(a) \quad \eta_1 &= f_1(X_+) + g_1(X_-), \\
(b) \quad \partial_+ \partial_- \phi_1 - \sqrt{\frac{2}{\alpha'}} (\partial_+ - \partial_-) \phi_1 &= \frac{4}{\alpha'} \rho_1, \\
(c) \quad \partial_+ \partial_- \phi_1 &= -\frac{1}{\alpha'} e^{-\frac{4}{\sqrt{\alpha'}} X^1} \eta_1^2, \\
(d) \quad \partial_+ \partial_- (\rho_1 - \phi_1) &= -\frac{1}{2} \partial_+ \left(e^{-\frac{2}{\sqrt{\alpha'}} X^1} \eta_1 \right) \partial_- \left(e^{-\frac{2}{\sqrt{\alpha'}} X^1} \eta_1 \right), \\
(e) \quad \partial_+ \partial_+ \phi_1 &= \frac{1}{2} \left(\partial_+ \left(e^{-\frac{2}{\sqrt{\alpha'}} X^1} \eta_1 \right) \right)^2, \\
(f) \quad \partial_- \partial_- \phi_1 &= \frac{1}{2} \left(\partial_- \left(e^{-\frac{2}{\sqrt{\alpha'}} X^1} \eta_1 \right) \right)^2.
\end{aligned} \tag{A.5}$$

Using equations (b) and (c), one can check that (d) is fulfilled identically. On the other hand, for (c), (e) and (f) to be consistent, some integrability conditions must hold. They are obtained comparing the derivatives of these equations. Taking into account (a), we get the following conditions on $f_1(X_+)$ and $g_1(X_-)$

$$\begin{aligned}
(f' - g') \left(f' - \sqrt{\frac{2}{\alpha'}} (f + g) \right) &= 0, \\
(f' - g') \left(g' + \sqrt{\frac{2}{\alpha'}} (f + g) \right) &= 0.
\end{aligned} \tag{A.6}$$

The only solution of (A.6) is obtained when $f' = g'$. Then we find

$$\rho_1 = \frac{\alpha' c_1^2}{32} e^{-\frac{4}{\sqrt{\alpha'}} X^1}, \quad \phi_1 = \frac{1}{8} \left((c_1 X^0 + d_1)^2 + \frac{\alpha' c_1^2}{4} \right) e^{-\frac{4}{\sqrt{\alpha'}} X^1}, \quad \eta_1 = c_1 X^0 + d_1, \tag{A.7}$$

where c_1 and d_1 are some constants. In the second order we get the similar system of equations as (A.5), where one should change one of η_1 in terms quadratic in tachyon by $2\eta_2$. Then we obtain

$$\begin{aligned}\rho_2 &= \frac{\alpha' c_1 c_2}{16} e^{-\frac{4}{\sqrt{\alpha'}} X^1}, \quad \phi_2 = \frac{1}{4} \left((c_1 X^0 + d_1)(c_2 X^0 + d_2) + \frac{\alpha' c_1 c_2}{4} \right) e^{-\frac{4}{\sqrt{\alpha'}} X^1}, \\ \eta_2 &= c_2 X^0 + d_2.\end{aligned}\tag{A.8}$$

But already in the next order in ϵ we encounter inconsistencies which can not be overcome. We must choose the vanishing tachyon and all other perturbations. Thus, we conclude that the only solution of the system (A.1), which can be represented in the form (A.4), is the linear dilaton background.

Appendix B. Change to the flat coordinates

In this appendix we demonstrate that the quadratic part of the effective action for the tachyon field (5.8) in the coordinates (5.9) takes the form (5.10).

First of all, for generic functions x , p and $\tilde{\omega}$ of ω and t the passage to the variables (5.9) leads to the action

$$S_{(2)} = \frac{1}{2} \int d\tau \int dq \frac{|D|}{p - \tilde{p}} \left[A(\partial_\tau \eta)^2 - 2B\partial_\tau \eta \partial_q \eta - C(\partial_q \eta)^2 \right], \tag{B.1}$$

where

$$\begin{aligned}A &= \left[2 - \partial_t \tilde{\omega} - (\partial_\omega x)^{-1} (1 + \partial_\omega \tilde{\omega}) (p - \partial_t x) \right] \left[2 - \partial_t \tilde{\omega} - (\partial_\omega x)^{-1} (1 + \partial_\omega \tilde{\omega}) (\tilde{p} - \partial_t x) \right], \\ B &= \partial_t \tilde{\omega} (2 - \partial_t \tilde{\omega}) - (\partial_\omega x)^{-1} (p + \tilde{p} - 2\partial_t x) (1 - \partial_\omega \tilde{\omega} + \partial_t \tilde{\omega} \partial_\omega \tilde{\omega}) \\ &\quad + (\partial_\omega x)^{-2} (1 - \partial_\omega \tilde{\omega}) (1 + \partial_\omega \tilde{\omega}) (p - \partial_t x) (\tilde{p} - \partial_t x), \\ C &= \left[\partial_t \tilde{\omega} - (\partial_\omega x)^{-1} (1 - \partial_\omega \tilde{\omega}) (p - \partial_t x) \right] \left[\partial_t \tilde{\omega} - (\partial_\omega x)^{-1} (1 - \partial_\omega \tilde{\omega}) (\tilde{p} - \partial_t x) \right], \\ D &= \frac{\partial_\omega x}{1 - (\partial_\omega \tilde{\omega} + \partial_t \tilde{\omega})}.\end{aligned}\tag{B.2}$$

From the explicit form (5.5) of $p(\omega, t)$ and $x(\omega, t)$ it is easy to check the following properties

$$\frac{\partial x}{\partial t} = p - \frac{\partial x}{\partial \omega}, \quad \frac{\partial \tilde{\omega}}{\partial \omega} = \frac{\partial x / \partial \omega}{\tilde{p} - \tilde{x}_t}, \quad \frac{\partial \tilde{\omega}}{\partial t} = \frac{p - \tilde{x}_t}{\tilde{p} - \tilde{x}_t} - \frac{\partial \tilde{\omega}}{\partial \omega} \tag{B.3}$$

with

$$\tilde{x}_t \equiv \partial_t x(\tilde{\omega}, t) = \frac{1}{R} \sum_{k=1}^n k a_k \sinh \left[\left(1 - \frac{k}{R} \right) \tilde{\omega} + \frac{k}{R} t \right]. \tag{B.4}$$

These properties are enough to show that $DA = DC = p - \tilde{p}$ and $B = 0$ what gives the action (5.10).

Appendix C. Asymptotic values of the coordinates

In this appendix we display the results for $\tilde{\omega}$, τ and q as functions of t and ω which are obtained in the asymptotics $t, \omega \rightarrow \infty$ in the case of Sine-Liouville theory ($n = 1$). The results depend on the ratio t/ω as well as on the radius parameter. There are four different cases dependent of the value of R . We summarize them in four tables. In each table we distinguish several regions of t/ω . For each region we give the asymptotic expressions for the coordinates, the corresponding interval of values of τ/q , and the coefficients in front of the constant term $\log \frac{a_0}{a_1}$, which should be added to $\tilde{\omega}$. The latter are given for $\omega > 0$. For $\omega < 0$ one should change their sign.

$R > 1$

t/ω	$(-1, 1)$	$(1, \infty) \cup (-\infty, 1 - 2R)$	$(1 - 2R, -1)$
$\tilde{\omega}$	$-\omega$	$-\left(1 - \frac{1}{R}\right)\omega - \frac{1}{R}t$	$-\frac{1}{R-1}(R\omega + t)$
τ	t	$\left(1 + \frac{1}{2R}\right)t - \frac{1}{2R}\omega$	$\frac{1}{2(R-1)}((2R-1)t + \omega)$
q	ω	$\left(1 - \frac{1}{2R}\right)\omega + \frac{1}{2R}t$	$\frac{1}{2(R-1)}((2R-1)\omega + t)$
τ/q	$(-1, 1)$	$(1, 2R+1) \cup (2R+1, \infty)$	$(-\infty, -1)$
$\log \frac{a_0}{a_1}$	0	± 1	$-\frac{R}{R-1}$

$1/2 < R < 1$

t/ω	$(1 - 2R, 2R - 1)$	$(2R - 1, 1)$	$(1, \infty) \cup (-\infty, 1 - 2R)$
$\tilde{\omega}$	$-\omega$	$-\frac{1}{1-R}(R\omega - t)$	$-\left(\frac{1}{R} - 1\right)\omega + \frac{1}{R}t$
τ	t	$\frac{2R-1}{2(1-R)}(\omega - t)$	$\left(1 - \frac{1}{2R}\right)(t - \omega)$
q	ω	$\frac{1}{2(1-R)}(\omega - t)$	$\frac{1}{2R}(\omega - t)$
τ/q	$(1 - 2R, 2R - 1)$	$\approx (2R - 1)$	$\approx (1 - 2R)$
$\log \frac{a_0}{a_1}$	0	$-\frac{R}{1-R}$	∓ 1

$1/3 < R < 1/2$

t/ω	$(\frac{(1-R)(1-2R)}{1-3R}, 1 - 2R)$	$(1 - 2R, 1)$	$(1, \infty) \cup (-\infty, \frac{(1-R)(1-2R)}{1-3R})$
$\tilde{\omega}$	$-\omega + \frac{2}{1-R}t$	$-\frac{1}{1-R}(R\omega - t)$	$-\left(\frac{1}{R} - 1\right)\omega + \frac{1}{R}t$
τ	$-\frac{R}{1-R}t$	$\frac{2R-1}{2(1-R)}(\omega - t)$	$\left(1 - \frac{1}{2R}\right)(t - \omega)$
q	$\omega - \frac{1}{1-R}t$	$\frac{1}{2(1-R)}(\omega - t)$	$\frac{1}{2R}(\omega - t)$
τ/q	$(2R - 1, 1 - 2R)$	$\approx (2R - 1)$	$\approx (1 - 2R)$
$\log \frac{a_0}{a_1}$	0	$-\frac{R}{1-R}$	∓ 1

$R < 1/3$

t/ω	$(\frac{(1-R)(1-2R)}{1-3R}, \infty) \cup (-\infty, 1 - 2R)$	$(1 - 2R, 1)$	$(1, \frac{(1-R)(1-2R)}{1-3R})$
$\tilde{\omega}$	$-\omega + \frac{2}{1-R}t$	$-\frac{1}{1-R}(R\omega - t)$	$-\left(\frac{1}{R} - 1\right)\omega + \frac{1}{R}t$
τ	$-\frac{R}{1-R}t$	$\frac{2R-1}{2(1-R)}(\omega - t)$	$\left(1 - \frac{1}{2R}\right)(t - \omega)$
q	$\omega - \frac{1}{1-R}t$	$\frac{1}{2(1-R)}(\omega - t)$	$\frac{1}{2R}(\omega - t)$
τ/q	$(R, 1 - 2R) \cup (2R - 1, R)$	$\approx (2R - 1)$	$\approx (1 - 2R)$
$\log \frac{a_0}{a_1}$	0	$-\frac{R}{1-R}$	-1

Finally, there are also two critical cases $R = 1$ and $R = \frac{1}{2}$. In these cases one can find explicitly the transformation of coordinates (5.9) on the entire plane. The results are

$$\begin{array}{ll}
 R = 1 & R = 1/2 \\
 \tilde{\omega} = -\omega & \tilde{\omega} = -\omega + \log \frac{a_0 + a_1 e^{2t}}{a_0 + a_1 e^{-2t}} \\
 \tau = t & \tau = t - \frac{1}{2} \log \frac{a_0 + a_1 e^{2t}}{a_0 + a_1 e^{-2t}} \\
 q = \omega & q = \omega - \frac{1}{2} \log \frac{a_0 + a_1 e^{2t}}{a_0 + a_1 e^{-2t}}
 \end{array} \tag{C.1}$$

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Article VI

Non-perturbative effects in matrix models and D-branes

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Non-Perturbative Effects in Matrix Models and D-branes

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The large order growth of string perturbation theory in $c \leq 1$ conformal field theory coupled to world sheet gravity implies the presence of $O(e^{-\frac{1}{gs}})$ non-perturbative effects, whose leading behavior can be calculated in the matrix model approach. E. Martinec recently proposed that the same effects should be reproduced by studying certain localized D-branes in Liouville Field Theory, which were constructed by A. and Al. Zamolodchikov. We discuss this correspondence in a number of different cases: unitary minimal models coupled to Liouville, where we compare the continuum analysis to the matrix model results of Eynard and Zinn-Justin, and compact $c = 1$ CFT coupled to Liouville in the presence of a condensate of winding modes, where we derive the matrix model prediction and compare it to Liouville theory. In both cases we find agreement between the two approaches. The $c = 1$ analysis also leads to predictions about properties of D-branes localized in the vicinity of the tip of the cigar in $SL(2)/U(1)$ CFT with $c = 26$.

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1 Introduction

In the late 1980's and early 1990's it was pointed out that low dimensional models of non-critical string theory, corresponding to $c \leq 1$ conformal field theories coupled to world sheet quantum gravity, provide interesting toy models in which one can hope to study non-perturbative effects in string theory in a controlled setting. In particular, in [1, 2, 3] it was shown that certain large N matrix models in a double scaling limit allow one to efficiently compute amplitudes in these theories to all orders in string perturbation theory. The general structure of the perturbative amplitudes is

$$\mathcal{A} \simeq \sum_{h=0}^{\infty} g_s^{2h-2} c_h , \quad (1.1)$$

where the genus h contribution c_h can be computed from the matrix model.

The question of the non-perturbative completion of these perturbative results was raised already in the original papers on the double scaling limit and was studied further in [4, 5, 6] and other papers. The matrix model results for the coefficients c_h showed that the latter grow with the genus like $c_h \simeq a^h (2h)!$; a was found to be positive for unitary models, so that the amplitudes are not Borel summable. Standard results on asymptotic series suggest that in this situation the leading non-perturbative ambiguities are of order $\mathcal{A}_{\text{np}} \sim g_s^{f_A} \exp(-f_D/g_s)$, where f_A and f_D are computable from the large order behavior of the perturbative series.

In another significant development, S. Shenker [7] pointed out that the $(2h)!$ large order behavior exhibited by low-dimensional non-critical strings is in fact expected to be generic in string theory, and thus the $\exp(-1/g_s)$ non-perturbative effects are very general as well. A natural source of such effects was found by J. Polchinski [8], who suggested that when going beyond perturbation theory in closed string theory, one has to include contributions of Riemann surfaces with holes, with boundary conditions corresponding to D-instantons – D-branes that are localized in spacetime and have finite action (or disk partition sum). It was shown in [8] that the combinatorics of summing over configurations containing multiple disconnected disks with these boundary conditions is such that the contribution of a single disk has to be exponentiated, leading to a non-perturbative contribution

$$\mathcal{A}_D \sim e^{Z_{\text{disk}}} . \quad (1.2)$$

Here Z_{disk} is the disk partition sum of the D-instanton; it goes like $Z_{\text{disk}} \sim 1/g_s$, as required for Shenker's arguments.

The idea that D-instantons provide an important source of non-perturbative effects in string theory proved to be very fruitful in many contexts, but remarkably the original setting in which these effects were first discussed, $c \leq 1$ string theory, remained mysterious for a long time.¹ The main problem was identifying the localized branes that could lead to such effects. Indeed, if one naively tries to construct a localized D-brane in Liouville theory, since the energy of the brane goes like $1/g_s$, and the string coupling depends on the Liouville field ϕ , $g_s \simeq \exp(Q\phi)$, it seems that the D-brane will feel a force pushing it to the strong coupling region $\phi \rightarrow \infty$, where it becomes light, and leads to breakdown of perturbation

¹But, see [9, 10] for some discussions.

theory. On general grounds one would expect the Liouville potential, $\mu \exp(2b\phi)$ to regularize the problem, but no explicit construction of localized D-branes with sensible properties was known.

Progress came in a beautiful paper by A. and Al. Zamolodchikov [11], who constructed the necessary D-branes in Liouville field theory, and analyzed some of their properties. They showed that these branes correspond to a boundary state obtained by quantization of a classical solution for which $\phi \rightarrow \infty$ on the boundary of the world sheet. Nevertheless, correlation functions of bulk and boundary operators on the disk with these boundary conditions are sensible and well behaved.

The open string spectrum corresponding to these D-branes is particularly simple. For the simplest brane, labeled as the $(1, 1)$ brane in [11], one finds in the open string sector only the conformal block of the identity. Thus, these branes are very natural candidates for the role of D-instantons in non-critical string theory. Indeed, in a recent paper [12], E. Martinec proposed that the branes of [11] are the source of the leading non-perturbative effects observed in the matrix model (see [13, 14] for related recent work on non-perturbative physics in $c \leq 1$ string theory).

The main purpose of this paper is to study the leading non-perturbative effects in the matrix model solutions of various $c \leq 1$ models coupled to Liouville and, when possible, to compare them to the effects of D-instantons in the continuum formalism, obtained by using the D-branes of [11].

For $c < 1$, the matrix model analysis was done in [4, 5, 6]. We construct the appropriate D-instantons in the continuum approach, and show that they give the same leading non-perturbative contributions as those found in [6] (for the unitary minimal models). This case was discussed in [12]; our results do not agree with those of [12], since we use a different set of Liouville \times matter branes to construct the D-instantons.

For $c = 1$, we study a compact scalar field $x \sim x + 2\pi R$, with a winding mode perturbation $\delta\mathcal{L} = \lambda \cos R(x_L - x_R)$, coupled to world sheet gravity. We derive the leading non-perturbative effects as a function of λ from the matrix model, and compare a small subset of these results to those obtained by using the branes of [11]. In all cases where a comparison can be made, the two approaches agree. We also use the $c = 1$ results to obtain matrix model predictions for non-perturbative effects in Sine-Liouville and the two dimensional Euclidean black hole backgrounds.

The paper is organized as follows. In section 2 we discuss unitary $c < 1$ minimal models coupled to gravity. We briefly review the matrix model analysis of [6], and then discuss the form of the relevant D-instantons in the continuum approach. We point out that the quantity calculated in [6] is very natural from the continuum point of view, compute it and show that the continuum result agrees with the matrix model one.

In section 3 we discuss the (T-dual of the) Sine-Gordon model coupled to gravity. Using the results of [15], we derive a differential equation that governs the non-perturbative effects as a function of the Sine-Gordon coupling λ . We solve this equation, and use the solution to obtain information about the leading non-perturbative effects in Sine-Liouville theory, and for a particular value of R ($= 3/2$ of the self dual radius), for the Euclidean two dimensional black hole (or cigar) with $c = 26$. We also perform a few continuum (Liouville) calculations, which reproduce some aspects of the matrix model analysis.

In section 4 we conclude and comment on our results. A few appendices contain useful

technical results.

2 Non-perturbative effects in unitary minimal models coupled to gravity

In this section we study the leading non-perturbative effects in minimal models coupled to gravity. We restrict the discussion to the case of unitary minimal models, since the matrix model analysis for this case was already done in [6]. It should be straightforward to generalize our results to the non-unitary minimal model case.

Minimal models (for a review see e.g. [16]) are labeled by two integers (p, p') . The unitary case corresponds to $p' = p + 1$. The central charge of the $(p, p + 1)$ model is given by

$$c = 1 - \frac{6}{p(p+1)} . \quad (2.1)$$

The minimal model contains a finite number of primaries of the Virasoro algebra labeled by two integers, (m, n) , whose dimensions are

$$h_{m,n} = \frac{[(p+1)m - pn]^2 - 1}{4p(p+1)} , \quad (2.2)$$

where the labels (m, n) run over the range $m = 1, 2, \dots, p-1$, $n = 1, 2, \dots, p$, and one identifies the pairs (m, n) and $(p-m, p+1-n)$.

The coupling to gravity leads in conformal gauge to the appearance of the Liouville field, ϕ , which is governed by the action

$$S_L = \int \frac{d^2\sigma}{4\pi} \left((\partial\phi)^2 + Q\hat{R}\phi + \mu_L e^{2b\phi} \right) . \quad (2.3)$$

The central charge of the Liouville model is given by

$$c = 1 + 6Q^2 \quad (2.4)$$

and the parameter b is related to Q via the relation

$$Q = b + \frac{1}{b} . \quad (2.5)$$

In general, b and Q are determined by the requirement that the total central charge of matter and Liouville is equal to 26. In our case, (2.1) and (2.4) imply that

$$b = \sqrt{\frac{p}{p+1}} . \quad (2.6)$$

An important class of conformal primaries in Liouville theory corresponds to the operators

$$V_\alpha(\phi) = e^{2\alpha\phi} \quad (2.7)$$

whose scaling dimension is given by $\Delta_\alpha = \bar{\Delta}_\alpha = \alpha(Q - \alpha)$. The Liouville interaction in (2.3) is $\delta\mathcal{L} = \mu_L V_b$.

The partition sum and correlation functions of the minimal models (2.1) coupled to gravity were computed using matrix models. We start this section by briefly reviewing the analysis of the leading non-perturbative effects in these models due to [6]. In the next subsection we will discuss the D-instantons that give rise to these effects.

2.1 Matrix model results

We start with the simplest case, $q = 2$, corresponding to pure gravity. The partition sum of the model is given by the solution of the Painleve-I equation for the second derivative of the partition function $F(\mu)$, $u(\mu) = -\partial_\mu^2 F(\mu)$:

$$u^2(\mu) - \frac{1}{6}u''(\mu) = \mu . \quad (2.8)$$

We denoted the cosmological constant in the matrix model by μ ; it differs from the Liouville cosmological constant in (2.3), μ_L , by a multiplicative factor.

String perturbation theory is in this case an expansion in even powers of $g_s = \mu^{-5/4}$:

$$u(\mu) = \mu^{1/2} \sum_{h=0}^{\infty} c_h \mu^{-5h/2} , \quad (2.9)$$

where $c_0 = 1, c_1 = -1/48, \dots$, and $c_h \underset{h \rightarrow \infty}{\sim} -a^{2h} \Gamma(2h - 1/2)$, with $a = 5/8\sqrt{3}$. The series (2.9) is asymptotic, and hence non-perturbatively ambiguous. The size of the leading non-perturbative ambiguities can be estimated as follows. Suppose u and \tilde{u} are two solutions of (2.8) which share the asymptotic behavior (2.9). Then, the difference between them, $\varepsilon = \tilde{u} - u$, is exponentially small in the limit $\mu \rightarrow \infty$, and we can treat it perturbatively. Plugging $\tilde{u} = u + \varepsilon$ into (2.8), and expanding to first order in ε , we find that

$$\varepsilon'' = 12u\varepsilon \quad (2.10)$$

which can be written for large μ as

$$\frac{\varepsilon'}{\varepsilon} = r\sqrt{u} + b\frac{u'}{u} + \dots \quad (2.11)$$

with $r = -2\sqrt{3}$, $b = -1/4$. Using (2.9), $u = \sqrt{\mu} + \dots$, one finds that

$$\varepsilon \propto \mu^{-\frac{1}{8}} e^{-\frac{8\sqrt{3}}{5}\mu^{\frac{5}{4}}} . \quad (2.12)$$

The constant of proportionality in (2.12) is a free parameter of the solution, and cannot be determined solely from the string equation (2.8) without further physical input.

The authors of [6] generalized the analysis above to the case of $(p, p+1)$ minimal models (2.1) coupled to gravity. They found it convenient to parameterize the results in the same way as in the $c = 0$ case, (2.11), *i.e.* to define the quantity r ,

$$\frac{\varepsilon'}{\varepsilon} = r\sqrt{u} + \dots , \quad (2.13)$$

where ε is again the leading non-perturbative ambiguity in $u = -F''$. They found that for general p there is in fact a whole sequence of different solutions for r labeled by two integers (m, n) which vary over the same range as the Kac indices in eq. (2.2). The result for $r_{m,n}$ was found to be:

$$r_{m,n} = -4 \sin \frac{\pi m}{p} \sin \frac{\pi n}{p+1} . \quad (2.14)$$

Our main purpose in the rest of this section is to reproduce the result (2.14) from Liouville theory.

2.2 Liouville analysis

As discussed in the introduction, general considerations suggest that the leading non-perturbative effects in string theory should be due to contributions of world sheets with holes, with boundary conditions corresponding to localized D-branes. The first question that we need to address is which D-branes should be considered for this analysis.

The minimal model part of the background can be thought of as a finite collection of points. All D-branes corresponding to it are localized and therefore should contribute to the non-perturbative effects. Minimal model D-branes are well understood [17]. They are in one to one correspondence with primaries of the Virasoro algebra (2.2). For our purposes, the main property that will be important is the disk partition sum (or boundary entropy) corresponding to the (m, n) brane, which is given by

$$Z_{m,n} = \left(\frac{8}{p(p+1)} \right)^{1/4} \frac{\sin \frac{\pi m}{p} \sin \frac{\pi n}{p+1}}{\left(\sin \frac{\pi}{p} \sin \frac{\pi}{p+1} \right)^{1/2}}. \quad (2.15)$$

What about the Liouville part of the background? The authors of [11] introduced an infinite sequence of localized D-branes, labeled by two integers (m', n') . Which of these branes should we take in evaluating instanton effects?

The analysis of [11] shows that open strings stretched between the (m', n') and (m'', n'') Liouville branes belong to one of a finite number of degenerate representations of the Virasoro algebra with central charge (2.4). The precise set of degenerate representations that arises depends on m', n', m'', n'' . Degenerate representations at $c > 25$ occur at negative values of world sheet scaling dimension, except for the simplest degenerate operator, 1, whose dimension is zero. One finds [11] that in all sectors of open strings, except those corresponding to $m' = n' = m'' = n'' = 1$ there are negative dimension operators. It is thus natural to conjecture that the only stable D-instantons correspond to the case $(m', n') = (1, 1)$, and we will assume this in the analysis below.

To recapitulate, the D-instantons that give rise to the non-perturbative effects in $c < 1$ minimal models coupled to gravity have the form: $(1, 1)$ brane in Liouville \times (m, n) brane in the minimal model. We next show that these D-branes give rise to the correct leading non-perturbative effects (2.14).

In order to compare to the matrix model results we should in principle use eq. (1.2), and evaluate the disk partition sum of the Liouville \times minimal model D-brane. It turns out, however, much more convenient to compare directly the quantity r (2.13) which appears naturally in the matrix model analysis. The basic point is that this quantity is a natural object to consider in the continuum approach as well.

Indeed, from the continuum point of view, r is the ratio

$$r = \frac{\frac{\partial}{\partial \mu} \log \varepsilon}{\sqrt{u}} = \frac{\frac{\partial}{\partial \mu_L} Z_{\text{disc}}}{\sqrt{-\partial_{\mu_L}^2 F}}, \quad (2.16)$$

where in the numerator we used the fact that $\log \varepsilon$ is the disk partition sum corresponding to the D-instanton (see (1.2)). Thus we see that r is the ratio between the one point function of the cosmological constant operator V_b on the disk, and the square root of its two point

function on the sphere. This is a very natural object to consider since it is known in general in CFT that n point functions on the disk behave like the square roots of $2n$ point functions on the sphere. In particular, for the purpose of computing r , we do not have to worry about the multiplicative factor relating μ and μ_L , as it drops out in the ratio; r is a pure number. We next compute it using the results of [11].

We start with the numerator in (2.16). We have

$$\frac{\partial}{\partial \mu_L} Z_{\text{disk}} = Z_{m,n} \times \langle V_b \rangle_{(1,1)} , \quad (2.17)$$

where we used the fact that the contribution of the minimal model is simply the disk partition sum (2.15), and the second factor is the one point function of the cosmological constant operator (2.3) on the disk with the boundary conditions corresponding to the $(1,1)$ D-brane of [11].

$Z_{m,n}$ is given by eq. (2.15). The one point function of V_b can be computed as follows.² The annulus partition sum corresponding to open strings ending on the $(1,1)$ brane of [11] is given in the open string channel by

$$Z_{1,1}(t) = \frac{q^{-\frac{Q^2}{4}}(1-q)}{\eta(q)} , \quad (2.18)$$

where $q = \exp(-2\pi t)$ is the modulus of the annulus, and $\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta function. Performing the modular transformation to the closed string channel, $t' = 1/t$, one finds [11]:

$$Z_{1,1}(t') = \int_{-\infty}^{\infty} dP \Psi_{1,1}(P) \Psi_{1,1}(-P) \chi_P(q') , \quad (2.19)$$

where $\Psi_{1,1}(P)$ is given in appendix A, $q' = \exp(-2\pi t') = \exp(-2\pi/t)$, and

$$\chi_P(q) = \frac{q^{P^2}}{\eta(q)} \quad (2.20)$$

is a non-degenerate Virasoro character. $\Psi_{1,1}(P)$ can be interpreted as an overlap between the $(1,1)$ boundary state, $B_{1,1}$, and the state with Liouville momentum P ,

$$\Psi_{1,1}(P) = \langle B_{1,1} | P \rangle . \quad (2.21)$$

Therefore, $\Psi_{1,1}(P)$ is proportional to the one point function on the disk, with $(1,1)$ boundary conditions, of the Liouville operator V_α , with

$$\alpha = \frac{Q}{2} + iP . \quad (2.22)$$

The proportionality constant is a pure number (independent of P and Q). We will not attempt to calculate this number precisely here, since we can deduce it by matching any one

²We are grateful to Al. Zamolodchikov for very useful discussions of this issue.

of the matrix model predictions, and then use it in all other calculations, but we will next mention a few contributions to it, for illustrative purposes.

First, in (2.19) the P integral runs over the whole real line, but $|P\rangle$ and $|-P\rangle$ are the same state, due to reflection from the Liouville wall, so we should replace $\int_{-\infty}^{\infty} dP = 2 \int_0^{\infty} dP$. Hence, the physical wavefunctions which should appear in (2.21) are $\sqrt{2}\Psi_{1,1}$. Furthermore, the state $|P\rangle$ in (2.21) is normalized as $\langle P|P'\rangle = \delta(P - P')$, for consistency with (2.19). On the other hand, the state created by acting with the Liouville exponential V_α (2.7), with α given by (2.22), is normalized to $\pi\delta(P - P')$. Finally, when computing $\langle V_\alpha(0)\rangle_{\text{disk}}$, the part of the $SL(2, \mathbf{R})$ symmetry of the disk related to rotations around the origin is unfixed. This gives an extra factor of $1/2\pi$ in the one point function. Altogether, one finds

$$\langle V_\alpha(0)\rangle_{(1,1)} = C \cdot \sqrt{2} \cdot \sqrt{\pi} \cdot \frac{1}{2\pi} \cdot \Psi_{1,1} \quad (2.23)$$

where α is given by (2.22) and C is an undetermined constant which we will fix by comparing to matrix model results (we will find that $C = 2$).

We can now plug in (2.23) into (2.17) and, using the form of $\Psi_{1,1}$ given in (A3), find

$$\langle V_b(0)\rangle_{(1,1)} = -C \frac{2^{1/4} \sqrt{\pi} [\pi\mu_L \gamma(b^2)]^{\frac{1}{2}(1/b^2-1)}}{b\Gamma(1-b^2)\Gamma(1/b^2)} . \quad (2.24)$$

We next move on to the denominator of (2.16). This is given by the two point function $\langle V_b V_b \rangle_{\text{sphere}}$. This quantity was calculated in [18, 19]. It is convenient to first compute the three point function $\langle V_b V_b V_b \rangle_{\text{sphere}}$ and then integrate once, to avoid certain subtle questions regarding the fixing of the $SL(2, C)$ Conformal Killing Group of the sphere. One has (see appendix A):

$$\langle V_b V_b V_b \rangle_{\text{sphere}} = b^{-1} [\pi\mu_L]^{1/b^2-2} [\gamma(b^2)]^{1/b^2} \gamma(2 - 1/b^2) . \quad (2.25)$$

Integrating once w.r.t. $-\mu_L$ we find

$$\langle V_b V_b \rangle_{\text{sphere}} = \frac{1/b^2 - 1}{\pi b} [\pi\mu_L \gamma(b^2)]^{1/b^2-1} \gamma(b^2) \gamma(1 - 1/b^2) . \quad (2.26)$$

We are now ready to compute r , (2.16). Plugging in (2.15), (2.24) and (2.26) into (2.16), we find

$$r_{m,n} = -2C \sin \frac{\pi m}{p} \sin \frac{\pi n}{p+1} , \quad (2.27)$$

which agrees with the matrix model result (2.14) if we set $C = 2$. Since C is independent of m , n and p , we can fix it by matching to any one case, and then use it in all others. We conclude that the Liouville analysis gives the same results as the matrix model one.

3 Non-perturbative effects in compactified $c = 1$ string theory

In this section we discuss the leading non-perturbative effects in (Euclidean) two dimensional string theory. The world sheet description contains a field x , compactified on a circle of radius

R , $x \simeq x + 2\pi R$, coupled to gravity. The conformal gauge Lagrangian describing this system is

$$\mathcal{L} = \frac{1}{4\pi} \left[(\partial x)^2 + (\partial \phi)^2 + 2\hat{R}\phi + \mu_L \phi e^{2\phi} + \lambda e^{(2-R)\phi} \cos[R(x_L - x_R)] \right]. \quad (3.1)$$

where we chose $\alpha' = 1$, and perturbed the $c = 1$ model by a Sine-Gordon type interaction, $\lambda \cos[R(x_L - x_R)]$, which gives rise to the last term in (3.1) after coupling to gravity. The perturbation is relevant for $R < 2$, and we will restrict to that case in the discussion below.

We will next describe the matrix model predictions for the leading non-perturbative effects corresponding to (3.1), and then verify some of them in Liouville theory.

3.1 Matrix model analysis

The model (3.1) was solved using matrix model techniques in [15], extending some earlier results [20, 21, 22, 23, 24]. In particular, it was shown in [15] that the Legendre transform \mathcal{F} of the string partition sum F satisfies the Toda differential equation:

$$\frac{1}{4} \lambda^{-1} \partial_\lambda \lambda \partial_\lambda \mathcal{F}(\mu, \lambda) + \exp \left[-4 \sin^2 \left(\frac{1}{2} \frac{\partial}{\partial \mu} \right) \mathcal{F}(\mu, \lambda) \right] = 1. \quad (3.2)$$

The initial condition for this equation is supplied by the unperturbed $c = 1$ string theory on a circle

$$\begin{aligned} \mathcal{F}(\mu, 0) &= \frac{R}{4} \operatorname{Re} \int_{\Lambda^{-1}}^\infty \frac{ds}{s} \frac{e^{-i\mu s}}{\sinh \frac{s}{2} \sinh \frac{s}{2R}} \\ &= -\frac{R}{2} \mu^2 \log \frac{\mu}{\Lambda} - \frac{1}{24} (R + \frac{1}{R}) \log \frac{\mu}{\Lambda} + R \sum_{h=2}^\infty \mu^{2-2h} c_h(R) + O(e^{-2\pi\mu}) + O(e^{-2\pi R\mu}), \end{aligned} \quad (3.3)$$

where the genus h term $c_h(R)$ is a known polynomial in $1/R$. The function $\mathcal{F}(\lambda, \mu)$ has a genus expansion which follows from (3.2). The string partition sum F is obtained from \mathcal{F} by flipping the sign of the genus zero term [25].

We will need below some features of the solution of equations (3.2), (3.3), which we review next. In order to study the solution of (3.2), (3.3), it is convenient to introduce the parameters

$$y = \mu \xi, \quad \xi = (\lambda \sqrt{R-1})^{-\frac{2}{2-R}}. \quad (3.4)$$

This parameterization is useful when λ is large, so that the last term in (3.1) sets the scale, and one can study the cosmological term perturbatively in μ . As explained in [15], this can only be done for $R > 1$. This is the physical origin of the branch cut in the definition of ξ in (3.4). In the regime where λ sets the scale, the genus expansion of the string partition sum is an expansion in even powers of ξ :

$$\mathcal{F}(\mu, \lambda) = \lambda^2 + \xi^{-2} \left[\frac{R}{2} y^2 \log \xi + f_0(y) \right] + \left[\frac{R + R^{-1}}{24} \log \xi + f_1(y) \right] + \sum_{h=2}^\infty \xi^{2h-2} f_h(y). \quad (3.5)$$

The partition sum on the sphere, $\mathcal{F}_0(\mu, \lambda)$, is given by the second term on the r.h.s. of (3.5). One can show [15] that at this order, the full Toda equation (3.2) reduces to an algebraic equation for $X = \partial_y^2 f_0$

$$y = e^{-\frac{1}{R}X} - e^{-\frac{R-1}{R}X}. \quad (3.6)$$

This leads to the following perturbative expansion of $\mathcal{F}_0(\mu, \lambda)$ in powers of λ :

$$\mathcal{F}_0(\mu, \lambda) = \frac{R}{2}\mu^2 \log \frac{\Lambda}{\mu} + R\mu^2 \sum_{n=1}^{\infty} \frac{1}{n!} \left((1-R)\mu^{R-2}\lambda^2 \right)^n \frac{\Gamma(n(2-R)-2)}{\Gamma(n(1-R)+1)}, \quad (3.7)$$

which agrees with the original (conjectured) result for this quantity in [20]. The genus $h \geq 1$ terms in the expansion (3.5) can be computed in principle by substituting (3.5) into (3.2), but only f_0 and f_1 are known in closed form. Nevertheless, we will see below that the large h behavior of f_h can be obtained directly from the differential equation (3.2), by computing the leading non-perturbative corrections to (3.5). We next turn to these corrections.

At $\lambda = 0$, there are two types of non-perturbative corrections to the $1/\mu$ expansion (3.3), associated with the poles of the integrand in that equation. These occur at $s = 2\pi i k$ and $s = 2\pi R i k$, $k \in \mathbf{Z}$, and give rise to the non-perturbative effects indicated on the second line of (3.3), $\exp(-2\pi\mu k)$ and $\exp(-2\pi R\mu k)$, respectively.

At finite λ , the situation is more interesting. The series of non-perturbative corrections

$$\Delta\mathcal{F} = \sum_{n=1}^{\infty} C_n e^{-2\pi n\mu} \quad (3.8)$$

gives rise to an exact solution of the *full* equation (3.2) [15]. Thus, the corresponding instantons are insensitive to the presence of the sine-Gordon perturbation! We will return to this interesting fact, and explain its interpretation in Liouville theory, in the next subsection.

The second type of corrections, which starts at $\lambda = 0$ like $\Delta\mathcal{F} = e^{-2\pi R\mu}$, does not solve the full equation (3.2), and gets λ dependent corrections. To study these corrections one can proceed in a similar way to that described in section 2.1.

Let \mathcal{F} and $\tilde{\mathcal{F}} = \mathcal{F} + \varepsilon$ be two solutions of the Toda equation (3.2). The linearized equation for ε reads

$$\frac{1}{4}\lambda^{-1}\partial_{\lambda}\lambda\partial_{\lambda}\varepsilon(\mu, \lambda) - 4e^{-\partial_{\mu}^2\mathcal{F}_0(\mu, \lambda)}\sin^2\left(\frac{1}{2}\frac{\partial}{\partial\mu}\right)\varepsilon(\mu, \lambda) = 0, \quad (3.9)$$

where in the exponential in the second term we approximated

$$4\sin^2\left(\frac{1}{2}\frac{\partial}{\partial\mu}\right)\mathcal{F}(\mu, \lambda) \simeq \partial_{\mu}^2\mathcal{F}_0 = R\log\xi + X(y). \quad (3.10)$$

This is similar to the fact that in the discussion of section 2, one can replace u in eq. (2.10) by its limit as $\mu \rightarrow \infty$. After the change of variables from (λ, μ) to (y, μ) , equation (3.9) can be written as

$$\alpha\frac{y^2}{\mu^2}(y\partial_y)^2\varepsilon(\mu, y) - 4e^{-X(y)}\sin^2\left(\frac{1}{2}\left(\frac{\partial}{\partial\mu} + \frac{y}{\mu}\frac{\partial}{\partial y}\right)\right)\varepsilon(\mu, y) = 0, \quad (3.11)$$

where $\alpha \equiv \frac{R-1}{(2-R)^2}$.

Since we know that ε is non-perturbatively small in the $\mu \rightarrow \infty$ limit, we use the following ansatz for it:

$$\varepsilon(\mu, y) = P(\mu, y)e^{-\mu f(y)}. \quad (3.12)$$

Here P is a power-like prefactor in g_s , and $f(y)$ is the function we are interested in (the analogue of r in the minimal models of section 2). Substituting (3.12) into (3.11) and keeping

only the leading terms in the $\mu \rightarrow \infty$ limit, one finds the following first order differential equation:

$$\sqrt{\alpha} e^{\frac{1}{2}X(y)}(1 - y\partial_y)g(y) = \pm \sin[\partial_y g(y)] , \quad (3.13)$$

where we introduced

$$g(y) = \frac{1}{2}yf(y) . \quad (3.14)$$

The \pm in (3.13) is due to the fact that one in fact finds the square of this equation.

Equation (3.13) is a first order differential equation in y , and to solve it we need to specify boundary conditions. As discussed earlier for the perturbative series, it is natural to specify these boundary conditions at $\lambda \rightarrow 0$, or $y \rightarrow \infty$ (see (3.4)). We saw in the discussion of (3.3) that there are two solutions, $f(y \rightarrow \infty) \rightarrow 2\pi$ or $2\pi R$. This implies via (3.14) that $g(y \rightarrow \infty) \simeq \pi y$ or πRy . We already saw that $g(y) = \pi y$ gives an exact solution of (3.2), and this is true for (3.13) as well (as it should be). Thus, to study non-trivial non-perturbative effects, we must take the other boundary condition,

$$g(y \rightarrow \infty) \simeq \pi Ry . \quad (3.15)$$

Interestingly, eq. (3.13) is exactly solvable. We outline the solution in Appendix B. For the initial condition (3.15), the solution can be written as

$$g(y) = y\phi(y) \pm \frac{1}{\sqrt{\alpha}}e^{-\frac{1}{2}X(y)} \sin \phi(y) , \quad (3.16)$$

where $\phi(y) = \partial_y g$ satisfies the equation

$$e^{\frac{2-R}{2R}X(y)} = \pm \sqrt{R-1} \frac{\sin\left(\frac{1}{R}\phi\right)}{\sin\left(\frac{R-1}{R}\phi\right)} . \quad (3.17)$$

The solution with the minus sign in (3.17) can be shown to be unphysical (see below). We will thus use the solution with a plus sign.

Equations (3.16), (3.17) are the main result of this subsection. We next discuss some features of the corresponding non-perturbative effects.

Consider first the situation for small λ , or large y . The first three terms in the expansion of $\phi(y)$ are

$$\phi(y) \approx \pi R \pm \frac{R \sin(\pi R)}{\sqrt{R-1}} y^{-\frac{2-R}{2}} + \frac{R}{2} \sin(2\pi R) y^{-(2-R)} . \quad (3.18)$$

This gives the following result for $f(y)$ (3.12):

$$\begin{aligned} f(y) &= 2\pi R \pm \frac{4 \sin(\pi R)}{\sqrt{R-1}} y^{-\frac{2-R}{2}} + \frac{R \sin(2\pi R)}{R-1} y^{-(2-R)} + O(y^{-3(2-R)/2}) \\ &= 2\pi R \pm 4 \sin(\pi R) \mu^{-\frac{2-R}{2}} \lambda + R \sin(2\pi R) \mu^{-(2-R)} \lambda^2 + O(\lambda^3) . \end{aligned} \quad (3.19)$$

We see that for large y , the expansion parameter is $y^{-\frac{(2-R)}{2}} \sim \lambda$, as one would expect.

Another interesting limit is $\mu \rightarrow 0$ at fixed λ , *i.e.* $y \rightarrow 0$, which leads to the Sine-Liouville model, (3.1) with $\mu_L = 0$. For $R = 3/2$, this model is equivalent to the Euclidean black hole

background [26, 15]. As we see in (3.6), in this limit $X \rightarrow 0$. The first two terms in the expansion of ϕ around this point are

$$\phi(y) = \phi_0 + \frac{R}{2} \left((R-1) \cot \left(\frac{R-1}{R} \phi_0 \right) - \cot \left(\frac{1}{R} \phi_0 \right) \right)^{-1} y + O(y^2) , \quad (3.20)$$

where ϕ_0 is defined by the equation

$$\frac{\sin \left(\frac{1}{R} \phi_0 \right)}{\sin \left(\frac{R-1}{R} \phi_0 \right)} = \pm \frac{1}{\sqrt{R-1}} . \quad (3.21)$$

The function $f(y)$ is given in this limit by the expansion

$$f(y) = \pm \frac{2(2-R)}{y\sqrt{R-1}} \sin \phi_0 + 2\phi_0 + O(y) . \quad (3.22)$$

Note that the behavior of f as $y \rightarrow 0$, $f \sim 1/y$, leads to a smooth limit as $\mu \rightarrow 0$ at fixed λ . The non-perturbative effect (3.12) goes like $\exp(-\mu f(y))$, so that as $y \rightarrow 0$ the argument of the exponential goes like $\mu/y = 1/\xi$, and all dependence on μ disappears.

For $R = 3/2$, which corresponds to the Euclidean black hole, the equations simplify. One can explicitly find ϕ_0 because (3.21) gives

$$\cos \frac{\phi_0}{3} = \pm \frac{1}{\sqrt{2}} \Rightarrow \phi_0 = \frac{3\pi}{4} \text{ or } \phi_0 = \frac{9\pi}{4} . \quad (3.23)$$

As a result, one finds at this value of the radius a simple result

$$\mu f(y) = \pm \frac{\mu}{y} + \frac{(6 \mp 3)\pi}{2} \mu + \dots = \pm \frac{1}{2} \lambda^4 + \frac{(6 \mp 3)\pi}{2} \mu + \dots . \quad (3.24)$$

In fact, for the particular case $R = 3/2$ one can find $f(y)$ exactly. The result is given in Appendix B. Note that the solution with the minus sign leads to a growing exponential, $e^{\frac{1}{2}\lambda^4}$. Therefore, it is not physical, as mentioned above.

Another nice consistency check on our solution is the RG flow from $c = 1$ to $c = 0$ CFT coupled to gravity, implicit in (3.1). Before coupling to gravity, the Sine-Gordon model associated to (3.1) describes the following RG flow. In the UV, the Sine-Gordon coupling effectively goes to zero, and one approaches the standard CFT of a compact scalar field. In the IR, the potential given by the Sine-Gordon interaction gives a world sheet mass to x , and the model approaches a trivial $c = 0$ fixed point. As shown in [20, 22], this RG flow manifests itself after coupling to gravity in the dependence of the physics on μ . Large μ corresponds to the UV limit; in it, all correlators approach those of the $c = 1$ theory coupled to gravity. Decreasing μ corresponds in this language to the flow to the IR, with the $c = 0$ behavior recovered as μ approaches a critical value μ_c , which is given by:

$$\mu_c = -(2-R)(R-1)^{\frac{R}{2-R}} \lambda^{\frac{2}{2-R}} . \quad (3.25)$$

The non-perturbative contributions to the partition sum computed in this section must follow a similar pattern. In particular, $f(y)$ must exhibit a singularity as $y \rightarrow y_c$, with

$$y_c = -(2-R)(R-1)^{\frac{R-1}{2-R}} \quad (3.26)$$

and furthermore, the behavior of f near this singularity should reproduce the non-perturbative effects of the $c = 0$ model coupled to gravity discussed in section 2.

The singularity at $y = y_c$ corresponds to a critical point of (3.6), near which the relation between y and X degenerates:

$$\frac{y_c - y}{y_c} \simeq \frac{R-1}{2R^2}(X - X_c)^2 + O((X - X_c)^3) . \quad (3.27)$$

Solving for the critical point, one finds that

$$e^{-\frac{2-R}{2R}X_c} = \sqrt{R-1} . \quad (3.28)$$

The corresponding y_c is indeed given by (3.26). We can now solve for $f(y)$, near $y = y_c$. Substituting (3.28) in (3.17) we find

$$\frac{\sin\left(\frac{1}{R}\phi\right)}{\sin\left(\frac{R-1}{R}\phi\right)} = \frac{1}{R-1} . \quad (3.29)$$

Thus, the $c = 0$ critical point corresponds to $\phi \rightarrow 0$.³ The first two terms in the expansion of ϕ around the singularity are

$$\phi(y) = \sqrt{3}(X_c - X)^{1/2} - \frac{\sqrt{3}(R^2 - 2R + 2)}{20R^2}(X_c - X)^{3/2} + O((X_c - X)^{5/2}) . \quad (3.30)$$

Substituting this in (3.16) one finds

$$g(y) = -y_c \frac{2\sqrt{3}(R-1)}{5R^2}(X_c - X)^{5/2} + O((X_c - X)^{5/2}) \quad (3.31)$$

or, using (3.14):

$$f(y) \approx -\frac{8\sqrt{3}}{5} \left(\frac{2R^2}{R-1} \right)^{1/4} \left(\frac{\mu_c - \mu}{\mu_c} \right)^{5/4} . \quad (3.32)$$

The power of $\mu - \mu_c$ is precisely right to describe a leading non-perturbative effect in pure gravity. It is interesting to compare the coefficient in (3.32) to what is expected in pure gravity. It is most convenient to do this by again computing r (2.11). u is computed by evaluating the leading singular term as $\mu \rightarrow \mu_c$ in $\partial_\mu^2 \mathcal{F}_0 = R \log \chi + X(y)$. One finds

$$r = -2\sqrt{3} \left(\frac{2R^2}{R-1} \right)^{1/4} \left(\frac{\mu_c - \mu}{\mu_c} \right)^{1/4} (X_c - X)^{-1/2} = -2\sqrt{3} \quad (3.33)$$

in agreement with the result (2.14) for pure gravity. This provides another non-trivial consistency check of our solution.

³Note that if we chose the minus sign in (3.17), we would find a more complicated solution for ϕ . One can show that it would lead to the wrong critical behavior. This is an additional check of the fact that the physical solution corresponds to the plus sign in (3.17).

3.2 Liouville analysis

In this subsection we will discuss the Liouville interpretation of the matrix model results presented in the previous subsection. Consider first the unperturbed $c = 1$ theory corresponding to $\lambda = 0$ in (3.1). As we saw, in the matrix model analysis one finds two different types of leading non-perturbative effects (see (3.3)), $\exp(-2\pi\mu)$, and $\exp(-2\pi R\mu)$. It is not difficult to guess the origin of these non-perturbative effects from the Liouville point of view. The $\exp(-2\pi\mu)$ contribution is due to a D-brane that corresponds to a $(1, 1)$ Liouville brane \times a Dirichlet brane in the $c = 1$ CFT, *i.e.* a brane located at a point on the circle parameterized by x . Similarly, the $\exp(-2\pi R\mu)$ term comes from a brane wrapped around the x circle.

This identification can be verified in the same way as we did for minimal models in section 2. To avoid normalization issues, one can again calculate the quantity r (2.16). The matrix model prediction for the Neumann brane⁴ is

$$r = -\frac{2\pi\sqrt{R}}{\sqrt{\log \frac{\Lambda}{\mu}}}, \quad (3.34)$$

where we used the sphere partition function $\mathcal{F}_0(\mu, 0)$ of the unperturbed compactified $c = 1$ string given by the first term in (3.7).

The CFT calculation is similar to that performed in the $c < 1$ case. The partition function of $c = 1$ CFT on a disk with Neumann boundary conditions is well known; the calculation is reviewed in Appendix C. One finds

$$Z_{\text{Neumann}} = 2^{-1/4}\sqrt{R}. \quad (3.35)$$

The disk amplitude corresponding to the $(1, 1)$ Liouville brane, and the two-point function on the sphere are computed using equations (2.24) and (2.26), in the limit $b \rightarrow 1$. The limit is actually singular, but computing everything for generic b and taking the limit at the end of the calculation leads to sensible, finite results. The leading behavior of (2.24) as $b \rightarrow 1$ is

$$\langle V_b(0) \rangle_{(1,1)} \approx -\frac{2^{5/4}\sqrt{\pi}}{\Gamma(1-b^2)}, \quad (3.36)$$

with the constant C in (2.24) chosen to be equal to 2, as in the minimal model analysis of equations (2.23) – (2.27). The two point function (2.26) approaches

$$\partial_{\mu_L}^2 \mathcal{F}_0 \simeq -\frac{\log \mu_L}{\pi\Gamma^2(1-b^2)}. \quad (3.37)$$

Substituting (3.35), (3.36) and (3.37) into (2.16) gives the result (3.34). This provides a non-trivial check of the statement in section 2, that the constant C in eq. (2.23) is a pure number, independent of all the parameters of the model.

The agreement of (3.34) with the Liouville analysis supports the identification of the Neumann D-branes as the source of the non-perturbative effects $\exp(-2\pi R\mu)$. A similar

⁴Similar formulae can be written for the Dirichlet brane.

analysis leads to the same conclusion regarding the relation between the Dirichlet $c = 1$ branes and the non-perturbative effects $\exp(-2\pi\mu)$ (the two kinds of branes are related by T-duality).

Having understood the structure of the unperturbed theory, we next turn to the theory with generic λ . In the matrix model we found that the non-perturbative effects associated with the Dirichlet brane localized on the x circle are in fact independent of λ (see (3.8) and the subsequent discussion). How can we understand this statement from the Liouville point of view?

The statement that (3.8) is an exact solution of the Toda equation (3.2) corresponds in the continuum formulation to the claim that the disk partition sum of the model (3.1), with $(1, 1)$ boundary conditions for Liouville, and Dirichlet boundary conditions for the matter field x is independent of λ . In other words, all n -point functions of the operator given by the last term in (3.1) on the disk vanish

$$\left\langle \left(\int d^2z e^{(2-R)\phi} \cos[R(x_L - x_R)] \right)^n \right\rangle_{(1,1) \times \text{Dirichlet}} = 0. \quad (3.38)$$

Is it reasonable to expect (3.38) to be valid from the world sheet point of view? For odd n (3.38) is trivially zero because of winding number conservation. Indeed, the Dirichlet boundary state for x breaks translation invariance, but preserves winding number. The perturbation in (3.1) carries winding number, and for odd n all terms in (3.38) have non-zero winding number. Thus, the correlator vanishes.

For even n one has to work harder, but it is still reasonable to expect the amplitude to vanish in this case. Indeed, consider the T-dual statement to (3.38), that the n point functions of the momentum mode $\cos(x/R)$, on the disk with $(1, 1) \times$ Neumann boundary conditions, vanish. This is reasonable since the operator whose correlation functions are being computed localizes x at the minima of the cosine, while the D-brane on which the string ends is smeared over the whole circle. It might be possible to make this argument precise by using the fact that in this case the D-instanton preserves a different symmetry from that preserved by the perturbed theory, and thus it should not contribute to the non-perturbative effects.

To summarize, the matrix model analysis predicts that (3.38) is valid. We will not attempt to prove this assertion here from the Liouville point of view (it would be nice to verify it even for the simplest case, $n = 2$), and instead move on to discuss the non-perturbative effects due to the localized branes on the x circle.

The non-trivial solution of eq. (3.13) given by (3.16), (3.17) should correspond from the Liouville point of view to the disk partition sum of the $(1, 1)$ Liouville brane which is wrapped around the x circle. The prediction is that

$$\left\langle \left(\int d^2z e^{(2-R)\phi} \cos[R(x_L - x_R)] \right)^n \right\rangle_{(1,1) \times \text{Neumann}} \quad (3.39)$$

are the coefficients in the expansion of $f(y)$ (3.12) in a power series in λ , the first terms of which are given by (3.19). It would be very nice to verify this prediction directly using Liouville theory, but in general this seems hard given the present state of the art. A simple check that can be performed using results in [11] is to compare the order λ term in (3.19) with the $n = 1$ correlator (3.39). We next compare the two.

Like in the other cases studied earlier, it is convenient to define a dimensionless quantity, ρ , given by the ratio of the one point function on the disk (3.39) and the square root of the appropriate two point function on the sphere,

$$\rho = \frac{\frac{\partial}{\partial \lambda} \log \varepsilon}{\sqrt{-\partial_\lambda^2 \mathcal{F}_0}} \Big|_{\lambda=0} . \quad (3.40)$$

The matrix model result (3.19), (3.7), for this quantity is

$$\rho = - \frac{\mu \frac{\partial}{\partial \lambda} f}{\sqrt{-\partial_\lambda^2 \mathcal{F}_0}} \Big|_{\lambda=0} = -2\sqrt{2} \sin(\pi R) . \quad (3.41)$$

In the Liouville description, ρ is given by

$$\rho = \frac{B_{\mathcal{T}} \langle V_{b-\frac{R}{2}} \rangle_{(1,1)}}{\sqrt{-\langle \mathcal{T}^2 \rangle}} . \quad (3.42)$$

Here, $\mathcal{T} = \cos R(x_L - x_R) V_{b-\frac{R}{2}}$. $B_{\mathcal{T}}$ is the one point function of $\cos R(x_L - x_R)$ on the disk. It has the same value as (3.35) (see appendix C)

$$B_{\mathcal{T}} = 2^{-1/4} \sqrt{R} . \quad (3.43)$$

The one-point function of the operator of $V_{b-\frac{R}{2}}$ is related to the wavefunction (A.3) with momentum $iP = b - R/2 - Q/2$ and is given by

$$\langle V_{b-\frac{R}{2}}(0) \rangle_{(1,1)} = - \frac{2^{5/4} \sqrt{\pi} [\pi \mu_L \gamma(b^2)]^{\frac{1}{2}(1/b^2-1+R/b)}}{b \Gamma(1-b^2+Rb) \Gamma(1/b^2+R/b)} . \quad (3.44)$$

The two-point function of \mathcal{T} on the sphere is computed as above from the three point function (A.10). One finds

$$\langle \mathcal{T}^2 \rangle = \frac{\left(\frac{1}{b^2} + \frac{R}{b} - 1\right)}{2\pi b} \left[\pi \mu_L \gamma(b^2)\right]^{\frac{1}{b^2} + \frac{R}{b} - 1} \gamma(b^2 - Rb) \gamma\left(1 - \frac{1}{b^2} - \frac{R}{b}\right) . \quad (3.45)$$

Substituting these results into (3.42) leads, in the limit $b = 1$, to (3.41). We see that the Liouville results are again in complete agreement with the corresponding matrix model calculation.

4 Discussion

In this paper we studied non-perturbative effects in $c \leq 1$ non-critical string theory, using both matrix model and continuum (Liouville) methods. We showed that the matrix model results correspond in the continuum approach to D-instanton effects due to localized branes in Liouville theory, the $(1, 1)$ branes constructed in [11]. Our work is motivated by the recent proposal of [12], but the details are different.

For $c < 1$ unitary minimal models, we showed that the contribution of these branes reproduces the matrix model results of [6]. For $c = 1$ we used the results of [15] to derive the leading non-perturbative effects in the matrix model description of the Sine-Gordon model coupled to gravity, as a function of the compactification radius and the Sine-Gordon coupling, and reproduced some of these predictions using the Liouville branes of [11].

The matrix model analysis led in addition to a number of predictions regarding the properties of D-instantons in $c = 1$ string theory. In particular, we showed that an infinite number of correlation functions given in eq. (3.38) must vanish, and the correlation functions (3.39) are given by the expansion of $f(y)$ (3.14) in a power series in λ , the first few terms of which appear in (3.19).

Another interesting set of matrix model predictions concerns the limit $\mu \rightarrow 0$, in which one finds the Sine-Liouville model, (3.1) with $\mu_L = 0$. The expansion of $f(y)$ around $y = 0$, (3.22), provides information about the disk partition sum and correlation functions of closed string operators in the presence of D-branes in this model. For the particular case $R = 3/2$, these predictions apply also to the $SL(2)/U(1)$ (cigar) background corresponding to a Euclidean two dimensional black hole. In the cigar, the D-branes in question are wrapped around the angular direction of the cigar, and are localized near the tip. Our results give rise to predictions for correlation functions on the disk with these boundary conditions.

Our analysis also resolves a previously open problem, of describing world sheet gravity on the disk with the boundary conditions described in [11] using the matrix model approach. It opens the possibility of studying gravity on AdS_2 using matrix model techniques.

An interesting open problem is how to go beyond the calculation of the leading non-perturbative contributions to the amplitudes (1.1), and study the perturbative expansion around the D-instantons discussed in this paper. In the matrix model approach one finds that the leading non-perturbative terms behave like

$$\mathcal{A}_{1-inst.} \sim C g_s^{f_A} e^{-\frac{f_D}{g_s}}, \quad (4.1)$$

where f_D was discussed in this paper, and f_A is known as well.⁵ The constant C is ambiguous in the matrix model.

In the continuum Liouville description, f_D has been computed in this paper, and it would be interesting to compute f_A and understand whether C is ambiguous or can be determined. In order to study these issues, one has to understand the perturbative expansion about the D-instantons discussed in this paper. In particular, it is natural to expect that f_A will arise from the annulus with boundary conditions corresponding to the $(1, 1)$ brane (this gives the correct scaling with g_s).

The discussion of this paper has some interesting higher dimensional generalizations. For example, if one replaces Liouville by $N = 2$ Liouville, and the minimal models by $N = 2$ minimal models, one finds the background

$$\frac{SL(2)_k}{U(1)} \times \frac{SU(2)_k}{U(1)} \quad (4.2)$$

⁵For the partition sum one finds a simple universal behaviour. For the minimal models, in all cases that we are aware of, $F_{\text{non-pert}} \sim g_s^{1/2} e^{-\frac{f_D}{g_s}}$ (see [27]). For the $c = 1$ model, equation (3.8) suggests that $\mathcal{F}_{\text{non-pert}} \sim g_s^0 e^{-\frac{f_D}{g_s}}$.

describing $NS5$ -branes spread out on a circle [28, 29]. The D-instantons analogous to those studied in this paper correspond to Euclidean $D0$ -branes stretched between pairs of five-branes. From the point of view of the geometry (4.2), they are described by D-branes living near the tip of the cigar, which are $N = 2$ superconformal generalizations of the $(1, 1)$ brane of [11]. In particular, the spectrum of open strings ending on them corresponds to the $N = 2$ superconformal block of the identity.

Like in our case, the D-instantons are obtained by tensoring the above $N = 2$ Liouville branes with the different D-branes in the $N = 2$ minimal model. Understanding the non-perturbative effects due to these instantons is an interesting open problem. Studying the branes discussed in this paper in more detail is a very useful warmup exercise for the fivebrane problem. It might suggest hints for finding an analog of the matrix model for fivebranes.

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Appendix A. Correlation functions in Liouville theory.

In this appendix we summarize some results on Liouville field theory, from [11, 18, 19], which are used in the text. The one-point function of the operator (2.7) in Liouville theory on the disk with boundary conditions corresponding to the $(1, 1)$ brane is given by

$$U(\alpha) = \frac{[\pi\mu_L\gamma(b^2)]^{-\alpha/b} Q\Gamma(bQ)\Gamma(Q/b)}{(Q-2\alpha)\Gamma(b(Q-2\alpha))\Gamma((Q-2\alpha)/b)} , \quad (\text{A.1})$$

where

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} . \quad (\text{A.2})$$

$U(\alpha)$ is the normalized one-point function, *i.e.* the one point function divided by the disk partition function. As discussed in the text, the unnormalized one point function of V_α is related to the wavefunction with momentum $iP = \alpha - Q/2$ defined by (2.19), which is given by

$$\Psi_{1,1}(P) = \frac{2^{3/4} 2\pi i P [\pi\mu_L\gamma(b^2)]^{-iP/b}}{\Gamma(1-2ibP)\Gamma(1-2iP/b)} . \quad (\text{A.3})$$

The three-point function on the sphere is (suppressing the standard dependence on the world sheet positions of the vertex operators)

$$\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} \rangle_{\text{sphere}} = \left[\pi\mu_L\gamma(b^2)b^{2-2b^2} \right]^{(Q-\sum\alpha_i)/b} \times \frac{\Upsilon_0\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_3+\alpha_1-\alpha_2)} , \quad (\text{A.4})$$

§B Solution of the equation for non-perturbative corrections in the $c = 1$ model.

where

$$\Upsilon(x) = \exp \left\{ \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left(\frac{Q}{2} - x \right) \frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right] \right\} \quad (\text{A.5})$$

and

$$\Upsilon_0 \stackrel{\text{def}}{=} \left. \frac{d\Upsilon(x)}{dx} \right|_{x=0} = \Upsilon(b) . \quad (\text{A.6})$$

An important property of the function $\Upsilon(x)$ is the set of recursion relations it satisfies,

$$\begin{aligned} \Upsilon(x+b) &= \gamma(bx) b^{1-2bx} \Upsilon(x) , \\ \Upsilon(x+1/b) &= \gamma(x/b) b^{2x/b-1} \Upsilon(x) . \end{aligned} \quad (\text{A.7})$$

Special cases of these relations that are used in the text are:

$$\begin{aligned} \frac{\Upsilon(2b)}{\Upsilon(b)} &= \gamma(b^2) b^{1-2b^2} , \\ \frac{\Upsilon(2b)}{\Upsilon(2b-1/b)} &= \gamma(2-1/b^2) b^{3-2/b^2} , \\ \frac{\Upsilon(2b-R)}{\Upsilon(b-R)} &= \gamma(b(b-R)) b^{1-2b^2+2bR} , \\ \frac{\Upsilon(2b-R)}{\Upsilon(2b-1/b-R)} &= \gamma(2-1/b^2-R/b) b^{3-2/b^2-2R/b} . \end{aligned} \quad (\text{A.8})$$

Using these relations one finds:

$$\begin{aligned} \langle V_b V_b V_b \rangle_{\text{sphere}} &= \left[\pi \mu_L \gamma(b^2) b^{2-2b^2} \right]^{1/b^2-2} \frac{\Upsilon_0}{\Upsilon(2b-1/b)} \left(\frac{\Upsilon(2b)}{\Upsilon(b)} \right)^3 \\ &= b^{-1} [\pi \mu_L]^{1/b^2-2} [\gamma(b^2)]^{1/b^2} \gamma(2-1/b^2) , \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned} \langle V_{b-\frac{R}{2}} V_{b-\frac{R}{2}} V_b \rangle_{\text{sphere}} &= \left[\pi \mu_L \gamma(b^2) b^{2-2b^2} \right]^{\frac{1}{b^2}-2+\frac{R}{b}} \frac{\Upsilon_0 \Upsilon(2b) \Upsilon^2(2b-R)}{\Upsilon(2b-1/b-R) \Upsilon(b-R) \Upsilon^2(b)} \\ &= b^{-1} [\pi \mu_L \gamma(b^2)]^{\frac{1}{b^2}+\frac{R}{b}-1} \gamma(b^2) \gamma(b(b-R)) \gamma\left(2-\frac{1}{b^2}-\frac{R}{b}\right) \end{aligned} \quad (\text{A.10})$$

Appendix B. Solution of the equation for non-perturbative corrections in the $c = 1$ model.

In this appendix we present the solution of equation (3.13). First, differentiate equation (3.13) once w.r.t. y . The result is

$$\left[2\sqrt{\alpha} \sinh \left(\frac{2-R}{2R} X \right) \mp \cos(\partial_y g) \right] \partial_y^2 g = \frac{\sqrt{\alpha}}{2} e^{\frac{1}{2}X} \partial_y X (y \partial_y - 1) g = \mp \frac{1}{2} \partial_y X \sin(\partial_y g) . \quad (\text{B.1})$$

Define

$$h(X) = \pm \cos(\partial_y g(y)) . \quad (\text{B.2})$$

In terms of h , equation (B.1) reads

$$\left[h - 2\sqrt{\alpha} \sinh \left(\frac{2-R}{2R} X \right) \right] \partial_X h = \frac{1}{2} (h^2 - 1) . \quad (\text{B.3})$$

Changing variables to

$$z = \exp\left(\frac{2-R}{2R}X\right) \quad (\text{B.4})$$

and considering z as a function of h , one arrives at a Riccati type equation

$$\partial_h z = a(h)z^2 + b(h)z + c(h) , \quad (\text{B.5})$$

where

$$a(h) = -\frac{\sqrt{R-1}}{R} \frac{1}{h^2-1}, \quad b(h) = \frac{2-R}{R} \frac{h}{h^2-1}, \quad c(h) = \frac{\sqrt{R-1}}{R} \frac{1}{h^2-1} . \quad (\text{B.6})$$

The general solution of (B.5) is easily written if a particular solution is known. One can check that

$$z_0(h) = -\sqrt{R-1} \left(h + \sqrt{h^2-1} \right) \quad (\text{B.7})$$

is a solution. Then after the substitution

$$z(h) = z_0(h) + \frac{1}{w(h)} , \quad (\text{B.8})$$

equation (B.5) takes the form

$$\partial_h w = -(2az_0 + b)w - a , \quad (\text{B.9})$$

whose general solution is given by

$$w(h) = -\frac{1}{d(h)} \left(C + \int dh a(h) d(h) \right) \quad \text{where} \quad d(h) = \exp \left(\int dh (2az_0 + b) \right) . \quad (\text{B.10})$$

The integrals can be calculated explicitly. First, one finds

$$d(h) = \sqrt{h^2-1} \exp \left[2 \frac{R-1}{R} \log \left(h + \sqrt{h^2-1} \right) \right] . \quad (\text{B.11})$$

The subsequent substitution into w gives

$$w(h) = -\frac{1}{d(h)} \left(C - \frac{1}{2\sqrt{R-1}} \exp \left[2 \frac{R-1}{R} \log \left(h + \sqrt{h^2-1} \right) \right] \right) . \quad (\text{B.12})$$

Taking together (B.8), (B.7) and (B.12), one obtains

$$z(h) = -\sqrt{R-1} \left\{ h + \sqrt{h^2-1} \frac{C' \exp \left[-2 \frac{R-1}{R} \log \left(h + \sqrt{h^2-1} \right) \right] + 1}{C' \exp \left[-2 \frac{R-1}{R} \log \left(h + \sqrt{h^2-1} \right) \right] - 1} \right\} , \quad (\text{B.13})$$

where we redefined the integration constant $C' = 2\sqrt{R-1}C$.

The solution (B.13) is written in a form valid for $|h| > 1$. We are interested in $|h| < 1$ due to the definition (B.2). Therefore, one should replace $\sqrt{h^2-1} \rightarrow i\sqrt{1-h^2}$ in (B.13).

The integration constant C' must be chosen such that $z(h)$ is real. This implies that C' is a phase, $C' = e^{2i\psi}$. Denoting

$$h = \pm \cos \phi, \quad (\phi = \partial_y g(y)) , \quad (\text{B.14})$$

one finds⁶

$$z(\phi) = \mp \sqrt{R-1} \left\{ \cos \phi + \sin \phi \tan \left(\frac{R-1}{R} \phi + \psi \right) \right\} . \quad (\text{B.15})$$

Finally, we rewrite (B.15) as

$$z(\phi) = \mp \sqrt{R-1} \frac{\cos \left(\frac{1}{R} \phi - \psi \right)}{\cos \left(\frac{R-1}{R} \phi + \psi \right)} . \quad (\text{B.16})$$

The initial condition (3.15) means that $z(\pi R) = 0$, and leads to

$$\psi = \frac{\pi}{2} . \quad (\text{B.17})$$

This reduces (B.16) to (3.17). In terms of z and ϕ , our initial function g is written as follows

$$g(\phi) = y(\phi)\phi \pm \frac{1}{\sqrt{\alpha}} e^{-\frac{1}{2}X(\phi)} \sin \phi = z^{-\frac{R}{2-R}} \left((z^{-1} - z)\phi \pm \frac{1}{\sqrt{\alpha}} \sin \phi \right) . \quad (\text{B.18})$$

For the particular case $R = 3/2$ the solution can be written explicitly. Then (3.17) and (3.6) give

$$e^{\frac{1}{6}X} = \pm \sqrt{2} \cos \frac{\phi}{3} , \quad (\text{B.19})$$

$$e^{-1/3X} = \frac{1}{2} (1 + \sqrt{1+4y}) \quad (\text{B.20})$$

and (3.16) leads to

$$g(y) = 3y \arccos \left[\pm \left(1 + \sqrt{1+4y} \right)^{-1/2} \right] \pm \frac{1}{4} (1+4y)^{1/4} (3 - \sqrt{1+4y}) . \quad (\text{B.21})$$

Appendix C. Disk one-point functions in $c = 1$ CFT

We want to calculate the one-point function $B_{\mathcal{T}} = \langle \cos[R(x_L - x_R)] \rangle$ of the winding one operator on the disk with Neumann boundary conditions for x . This is the same as $\langle \cos(x/R) \rangle$ with Dirichlet boundary conditions for x , which lives on a circle of radius $1/R$.

A simple way to calculate $B_{\mathcal{T}}$ is to study the annulus partition function with Neumann boundary conditions, as a function of the modulus τ in the open string channel,

$$Z_{ann} = \text{Tr } q^{L_0 - \frac{1}{24}} , \quad (\text{C.1})$$

⁶The ambiguity to replace $\phi \rightarrow \phi + 2\pi n$ is captured by the integration constant ψ .

where $q = e^{2\pi i\tau}$ and $L_0 = p^2 + N - 1$. The momentum p takes values in $\frac{1}{R}\mathbf{Z}$. A standard calculation gives

$$Z_{ann} = \frac{1}{\eta(q)} \sum_{n \in \mathbf{Z}} q^{(n/R)^2} = \frac{\theta_3(0|\frac{2\tau}{R^2})}{\eta(\tau)}, \quad (\text{C.2})$$

where $\eta(q) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)$ and $\theta_3(0|\tau) = \sum_{n \in \mathbf{Z}} q^{\frac{1}{2}n^2}$.

In the closed string channel the same annulus partition function can be represented as a sum over winding modes exchanged between two D-branes. Performing the modular transformation $\tilde{\tau} = -1/\tau$, and using the modular properties of the elliptic functions,

$$\begin{aligned} \eta(\tau) &= (-i\tau)^{-1/2} \eta(-1/\tau), \\ \theta_3(0|\tau) &= (-i\tau)^{-1/2} \theta_3(0|-1/\tau), \end{aligned} \quad (\text{C.3})$$

we find the following expansion in the closed string channel

$$Z_{ann} = \frac{R}{\sqrt{2}} \frac{1}{\eta(\tilde{q})} \sum_{n \in \mathbf{Z}} \tilde{q}^{\frac{1}{2}(nR)^2}, \quad (\text{C.4})$$

where $\tilde{q} = \exp(-2\pi i/\tau)$. The disk partition sum (3.35) is the square root of the $n = 0$ term in (C.4). The one point function (3.43) is the square root of the $n = 1$ term. Both are equal to

$$Z_{\text{Neumann}} = B_T = 2^{-\frac{1}{4}} \sqrt{R}. \quad (\text{C.5})$$

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Mécanique quantique matricielle et la théorie des cordes à deux dimensions dans des fonds non-triviaux

La théorie des cordes est le candidat le plus promettant pour la théorie unissant toutes les interactions en incluant la gravitation. Elle a la dynamique très compliquée. C'est pourquoi c'est utile d'étudier ses simplifications. Une de celles-ci est la théorie des cordes non-critiques qui peut être définie dans les dimensions inférieures. Le cas particulièrement intéressant est la théorie des cordes à deux dimensions. D'une part elle a la structure très riche et d'autre part elle est résoluble exactement. La solution complète de la théorie des cordes à deux dimensions dans le fond le plus simple du dilaton linéaire a été obtenue en utilisant sa représentation comme la mécanique quantique matricielle. Ce modèle de matrices fournit une technique très puissante et découvre l'intégrabilité cachée dans la formulation habituelle de CFT.

Cette thèse prolonge la formulation de la théorie des cordes à deux dimensions par des modèles de matrices dans des fonds non-triviaux. Nous montrons comment les perturbations changeants le fond sont incorporés à la mécanique quantique matricielle. Les perturbations sont intégrables et dirigées par la hiérarchie de Toda. Cette intégrabilité est utilisée pour extraire l'information divers sur le système perturbé: les fonctions des corrélations, le comportement thermodynamique, la structure de l'espace-temps. Les résultats concernant ces et autres questions, comme des effets non-perturbatifs dans la théorie des cordes non-critiques, sont présentés dans cette thèse.

Matrix quantum mechanics and two-dimensional string theory in non-trivial backgrounds

String theory is the most promising candidate for the theory unifying all interactions including gravity. It has an extremely difficult dynamics. Therefore, it is useful to study some its simplifications. One of them is non-critical string theory which can be defined in low dimensions. A particular interesting case is 2D string theory. On the one hand, it has a very rich structure and, on the other hand, it is solvable. A complete solution of 2D string theory in the simplest linear dilaton background was obtained using its representation as Matrix Quantum Mechanics. This matrix model provides a very powerful technique and reveals the integrability hidden in the usual CFT formulation.

This thesis extends the matrix model description of 2D string theory to non-trivial backgrounds. We show how perturbations changing the background are incorporated into Matrix Quantum Mechanics. The perturbations are integrable and governed by Toda Lattice hierarchy. This integrability is used to extract various information about the perturbed system: correlation functions, thermodynamical behaviour, structure of the target space. The results concerning these and some other issues, like non-perturbative effects in non-critical string theory, are presented in the thesis.