A contribution to the theory of graph homomorphisms and colorings

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in memory of Borojethu (eldest paternal uncle) -
late Prof. Subir Kumar Sen

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Résumé

Dans cette thèse, nous considérons des questions relatives aux homomorphismes de quatre types distincts de graphes : les graphes orientés, les graphes orientables, les graphes 2-arête colorés et les graphes signés. Pour chacun des ces quatre types, nous cherchons à déterminer le nombre chromatique, le nombre de clique relatif et le nombre de clique absolu pour différentes familles de graphes planaires : les graphes planaires extérieurs, les graphes planaires extérieurs de maille fixée, les graphes planaires et les graphes planaires de maille fixée. Nous étudions également les étiquetages "2-dipath" et "L(p,q)" des graphes orientés et considérons les catégories des graphes orientables et des graphes signés. Nous étudions enfin les différentes relations pouvant exister entre ces quatre types d'homomorphismes de graphes.

Keywords: graphes orientés, graphes orientables, graphes 2-arête colorés, graphes signés, homomorphismes

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Abstract

An oriented graph is a directed graph with no cycle of length at most two. A homomorphism of an oriented graph to another oriented graph is an arc preserving vertex mapping. To push a vertex is to switch the direction of the arcs incident to it. An orientable graph is an equivalence class of oriented graph with respect to the push operation. An orientable graph $[\overrightarrow{G}]$ admits a homomorphism to an orientable graph $[\overrightarrow{H}]$ if an element of $[\overrightarrow{G}]$ admits a homomorphism to an element of $[\overrightarrow{H}]$.

A signified graph (G, Σ) is a graph whose edges are assigned either a positive sign or a negative sign, while Σ denotes the set of edges with negative signs assigned to them. A homomorphism of a signified graph to another signified graph is a vertex mapping such that the image of a positive edge is a positive edge and the image of a negative edge is a negative edge. A signed graph $[G, \Sigma]$ admits a homomorphism to a signed graph $[H, \Lambda]$ if an element of $[G, \Sigma]$ admits a homomorphism to an element of $[H, \Lambda]$.

The oriented chromatic number of an oriented graph $\overrightarrow{G}$ is the minimum order of an oriented graph $\overrightarrow{H}$ such that $\overrightarrow{G}$ admits a homomorphism to $\overrightarrow{H}$. A set R of vertices of an oriented graph $\overrightarrow{G}$ is an oriented relative clique if no two vertices of R can have the same image under any homomorphism. The oriented relative clique number of an oriented graph $\overrightarrow{G}$ is the maximum order of an oriented relative clique of $\overrightarrow{G}$. An oriented clique or an oclique is an oriented graph whose oriented chromatic number is equal to its order. The oriented absolute clique number of an oriented graph $\overrightarrow{G}$ is the maximum order of an oclique contained in $\overrightarrow{G}$ as a subgraph.

The chromatic number, the relative chromatic number and the absolute chromatic number for orientable graphs, signified graphs and signed graphs are defined similarly.

In this thesis we study the chromatic number, the relative clique number and the absolute clique number of the above mentioned four types of graphs. We specifically study these three parameters for the family of outerplanar graphs, of outerplanar graphs with given girth, of planar graphs and of planar graphs with given girth. We also try to investigate the relation between the four types of graphs and prove some results regarding that.

In this thesis, we provide tight bounds for the absolute clique number of these families in all these four settings. We provide improved bounds for relative clique numbers for the same. For some of the cases we manage to provide improved bounds for the chromatic number as well.

One of the most difficult results that we prove here is that the oriented absolute clique number of the family of planar graphs is at most 15. This result settles a conjecture made by Klostermeyer and MacGillivray in 2003. Using the same technique we manage to prove similar results for orientable planar graphs and signified planar graphs.

We also prove that the signed chromatic number of triangle-free planar graphs is at most 25 using the discharging method. This also implies that the signified chromatic number of triangle-free planar graphs is at most 50 improving the previous upper bound.

We also study the 2-dipath and oriented $L(p, q)$-labeling (labeling with a condition for distance one and two) for several families of planar graphs.

It was not known if the categorical product of orientable graphs and of signed graphs exists. We prove both the existence and also provide formulas to construct them.

Finally, we propose some conjectures and mention some future directions of works to conclude the thesis.

Keywords: oriented graphs, orientable graphs, signified graphs, signed graphs, homomorphisms

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Contents

Chapter 1
Introduction

"How many colors do you need to color a planar map?" This simple sounding question, asked by Francis Guthrie while coloring the map of England, gave rise to a popular topic of discrete mathematics called graph coloring. He postulated the four color conjecture, noting that four colors were sufficient to color the map so that no regions sharing a common border received the same color. Guthrie's brother passed on the question to his mathematics teacher Augustus de Morgan at University College, who mentioned it in a letter to William Hamilton in 1852. Arthur Cayley raised the problem at a meeting of the London Mathematical Society in 1879. The same year, Alfred Kempe published a paper that claimed to establish the result, and for a decade the four color problem was considered solved. For his accomplishment Kempe was elected a Fellow of the Royal Society and later President of the London Mathematical Society.

In 1890, Heawood pointed out that Kempe's proof was wrong. However, in that paper he proved the five color theorem, saying that every planar map can be colored with no more than five colors, using ideas of Kempe. In the following century, a vast amount of work and theories were developed to reduce the number of colors to four, until the Four-Color Conjecture was finally positively settled in 1976 by Appel and Haken. The proof went back to the ideas of Heawood and Kempe and largely disregarded the intervening developments. The proof of the Four-Color Theorem is also noteworthy for being the first major computer-aided proof. Later some simplifications of the proof were done but each of them involves a computer-check and uses the idea of Appel and Haken.

By the time of Appel and Haken, researchers had reformulated the problem in terms of graphs, instead of maps, using the notion of proper vertex coloring of graphs and chromatic numbers. Basically, the graph theoretic formulation of the Four-Color Theorem said that the chromatic number of a planar graph is at most 4. Though the question asked by Guthrie gave rise to a particular type of graph coloring, namely proper vertex coloring, naturally many variations of graph colorings have been defined and studied for over a century by several researchers, turning graph colorings into a very rich theory underlined by many great results.

Nowadays, a popular way of studying proper graph coloring is through graph homomorphisms, which, in a way, is a more general concept. It is possible to define proper graph colorings through graph homomorphisms on different types of graphs and study problems analogous to the Four-Color Theorem in different graph theoretic settings.

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In this thesis, we study homomorphisms of oriented graphs (directed graphs without directed cycles of length 1 or 2), orientable graphs (particular equivalence classes of oriented graphs), signified graphs (undirected graphs with positive or negative signs assigned to the edges) and signed graphs (particular equivalence classes of signified graphs). One of our main interests is to study chromatic numbers and clique numbers, defined using homomorpisms, for the family and some sub-families of planar graphs of the above kinds. There will be some related problems that we consider as well.

Untill now, from what we have discussed, it can be understood that the problem of determining the chromatic number of graphs is difficult when we consider the family of planar graphs. In fact, speaking in a more mathematical way, we can take a leaf from the theory of computational complexity (which we will not define in this thesis as we do not use it anywhere other than here to motivate and justify our works) and say that "given a graph, it is an NP-complete problem to determine if it has a k-coloring or not for $k \geq 3$". In general, it is NP-hard to determine the chromatic number of a given graph. There are many polynomial algorithms to determine if a graph is planar or not. So, basically, the Four-Color Theorem answers a question which is NP-hard in general for a class of graphs that are easy to recognize.

Now, we know that the notion of proper vertex coloring can be captured by the notion of graph homomorphisms. A substantial amount of research has been done, and is being done, on vertex coloring of graphs. But what about its counter-part, vertex coloring of directed graphs? A definition of vertex coloring of directed graphs can be given using homomorphisms (that is, arc preserving vertex mappings) of directed graphs.

* * *

The formal definition of oriented coloring and oriented chromatic number will be given in Chapter 3. We also define "oriented clique numbers" with respect to oriented colorings. It turns out that we need to define two different parameters, namely, the oriented relative clique number and the oriented absolute clique number, to capture the parameter analogous to the clique number for simple graphs. The notion analogous to cliques is called ocliques for oriented graphs. The ocliques are basically the oriented graphs whose chromatic number is the same as their order.

It was proved by Klostermeyer and MacGillivray that, given an oriented graph, it is an NP-complete problem to determine if it has a k-coloring or not for $k \geq 4$ while Culus and Demange showed that the problem is difficult even for bipartite graphs (note that the classical vertex coloring problem is easy for simple bipartite graphs).

It has been proved by Bensmail, Duvignau and Kirgizov that given a simple graph, it is NP-hard to determine if any orientation of it is an oclique or not. Also to determine the oriented relative clique number or the oriented absolute clique number of an oriented graph is an NP-hard problem (this easily follows from the fact that to determine the clique number of a graph is NP-hard).

We address the problem of determining the oriented chromatic number, the oriented relative clique number and the oriented absolute clique number for the family of planar graphs and some of its subfamilies. The subfamilies that we consider are mainly the family $\mathcal{O}_g$ of outerplanar graphs with girth (the girth of a graph is the length of its shortest cycle) at least g, and the family $\mathcal{P}_g$ of planar graphs with girth at least g for $g \geq 3$. That is, we are trying to find results analogous to the Four-Color Theorem and the Grötzsch Theorem (triangle-free planar graphs have chromatic number at most 3) from classical graph coloring in the domain of oriented coloring. For now, there is not even a conjecture regarding what the oriented chromatic number of the family of planar graphs could be, let alone a conjecture analogous to the Hadwiger's conjecture (K_k-minor-free graphs have chromatic number at most $k-1$) for oriented graphs. The oriented chromatic number for these families have been well studied and the existing upper and lower bounds are difficult to improve. We provide improved bounds for the other two parameters of these families.

In particular, we settle a conjecture made by Klostermeyer and MacGillivray in 2002 by proving that there is no planar oclique of order more than 15. In fact, in a joint work with Sopena, we have improved the above mentioned result and proved a uniqueness property as well. We could not improve the existing bound for the oriented relative clique number of the family $\mathcal{P}_3$ but we do improve it for the family $\mathcal{P}_4$. In fact, we provide tight bounds for the oriented absolute clique number of the families $\mathcal{O}_g$ and $\mathcal{P}_g$ for all $g \geq 3$ while we provide tight bounds for the oriented relative clique number of the families $\mathcal{O}_g$ and $\mathcal{P}_j$ for all $g \geq 3$ and for all $j \geq 5$.

In the same chapter, we also prove results related to the $L(p,q)$-labeling span of some planar families. For this, we use homomorphisms of oriented graphs as a tool. The definitions of labeling is given and discussed in Chapter 3, Section 3.4, in details.

In the next chapter, we consider a particular equivalence relation, called push relation, on oriented graphs. Klostermeyer and MacGillivray had defined and studied homomorphisms of orientable graphs, which are basically equivalence classes of oriented graphs with respect to the push operation, and proved that the problem of determining the orientable chromatic number (defined using homomorphisms) of an orientable graph is NP-hard in general.

We will define the orientable relative clique number and the orientable absolute clique number similar to parameters defined for oriented graphs and study them for the families $\mathcal{O}_g$ and $\mathcal{P}_g$ for all $g \geq 3$. We also study and provide improved upper and lower bounds for the orientable chromatic number for the above mentioned families of graphs. Interestingly, some of the results will imply improved bounds for $L(2, 1)$-labeling of some oriented planar families.

It has been observed and remarked that the chromatic number of 2-edge-colored graphs or signified graphs seems to have some relation with the oriented chromatic number of graphs. While there is no result supporting this speculation, in practice it has been observed that usually the same kind of technique can be used to prove similar bounds for these two chromatic numbers. Due to this similarity, we first tried to investigate if there is some general relation between the two types of chromatic numbers. To our surprise, we ended up constructing examples of (underlying) graphs with the two chromatic numbers arbitrarily higher or lower from each other in Chapter 5.

Then, in a joint work with Bensmail, we defined signified relative clique number and signified absolute clique number and tried to adopt the techniques used to prove bounds for the oriented chromatic number, the oriented relative clique number and the oriented absolute clique number of several planar families. This idea worked and hence we ended up proving many results similar to the ones proved for oriented graphs.

Like we have considered the chromatic number and clique number problems for orientable graphs, we also consider the similar parameters for signed graphs, which are equivalence class of signified graphs with respect to the resigning relation in Chapter 6. Homomorphisms of signed graphs have been recently studied by Naserasr, Rollová and Sopena where they showed how the notion of homomorphism of signed graphs can capture the notion of classical graph coloring. They, in fact, restated some major important theorems and conjectures, such as the Four-Color Theorem and Hadwiger's conjecture, using the notion of signed graph homomorphisms and extended some of them to achieve stronger results or conjectures.

In order to make this theory richer, they asked some basic questions regarding the signed chromatic number, the signed relative clique number and the signed absolute clique number of some families of graphs. In particular, questions related to the families we studied for the other three kinds of graphs were asked too. In a joint work with Ochem and Pinlou, we tried to answer them and improved some of the existing lower and upper bounds including the upper bound of 40 for the signed chromatic number of planar graphs and the upper bound of 25 for the signed chromatic number of triangle free planar graphs.

Among the conjectures extended by Naserasr, Rollová and Sopena, a conjecture by Naserasr was extended as well. The extended version of this conjecture is proved to be equivalent to a conjecture by Seymour regarding edge coloring of regular planar multigraphs (this conjecture, if proved true, will generalize the Four-Color Theorem). In this thesis, we prove a result with supporting evidence in favor of the conjecture. This is a joint work with Naserasr and Qiang. The result is discussed in details in Chapter 6, Section 6.4.

In this thesis, we will see that we can consider each of the class of all oriented graphs, the class of all orientable graphs, the class of all signified graphs and the class of all signed graphs as a category. Also, we will show that the category of orientable graphs is actually isomorphic to a subcategory of the category of oriented graphs while the category of signed graphs is isomorphic to a subcategory of the category of signified graphs.

It was unknown whether categorical product exists for signed graphs. In a joint effort with Naserasr and Sopena we showed that such a product indeed exists and, in fact, provided a formula for the product of two signed graphs. The same argument worked for orientable graphs as well. To prove the existence of these products we used the known fact that the categorical product exists for oriented (resp. signified) graphs and the obsevation that the categorical product exists in the subcategory of oriented (resp. signified) graphs which is isomorphic to the category of orientable (resp. signed) graphs.

Some open problems and discussion regarding possible future work will be discussed in Chapter 7.

* * *

The organization of thesis is as follows. Some basic definitions and results, mainly regarding classical graph colorings and simple graphs, are given in Chapter 2. In Chapter 3 we discuss results related to oriented graphs and in Chapter 4 we discuss results related to orientable graphs. Then in Chapter 5 we discuss results related to signified graphs and in Chapter 6 we discuss results related to signed graphs. The definitions and notations regarding oriented graphs, orientable graphs, signified graphs and signed graphs will be given in the begining of their respective chapters. Some definitions and notations, that are only needed for specific proofs, are stated under the heading of that particular proof. Finally, in Chapter 7, some open problems and possible future direction of works will be discussed to conclude the thesis.

Chapter 2
Preliminaries

IN this chapter we provide the basic definitions and fix some notations, mainly regarding undirected graphs, for the rest of the thesis. The definitions that will be used specifically for a chapter are not given here as they will be mentioned in the begining of their respective chapters. While defining something we will use *italics* for the defined item.

Graphs

A *graph* or an *undirected graph* is an ordered pair $G = (V(G), E(G))$ with a set of *vertices* $V(G)$ and a set of *edges* $E(G)$, where an edge is an unordered pair of vertices (uv, or equivalently vu). A *loop* is an edge between a vertex and itself. A *simple graph* is a graph without loops. Sometimes we allow the set of edges $E(G)$ to be a multiset and the repeated elements in it are called *multiple edges* or *multi-edges*. In this case G is called a *multigraph*.

Two vertices $u, v \in V(G)$ are adjacent if they are an edge of G, that is, $uv \in E(G)$. Adjacent vertices are *neighbors* and the set of neighbors of a vertex v in a graph G is denoted by $N_G(v)$ (or $N(v)$ when there is no chance of confusion). The *degree* $d_G(v)$ (or $d(v)$ when there is no chance of confusion) of a vertex v in a graph G is the number of neighbors of v in G. A *d-regular* graph is a graph in which every vertex has degree d.

The *order* of a graph is the cardinality of its set of vertices while the *size* of a graph is the cardinality of its set of edges. The order of a graph G is denoted by $|V(G)|$ or $|G|$.

A *digraph* or *directed graph* is an ordered pair $\overrightarrow{G} = (V(\overrightarrow{G}), A(\overrightarrow{G}))$ with a set of *vertices* $V(\overrightarrow{G})$ and a set of *arcs* $A(\overrightarrow{G})$, where an *arc* is an ordered pair of *vertices*. Two arcs having two different ordered pairs with the same pair of vertices (for example, $\overrightarrow{xy}$ and $\overrightarrow{yx}$) are *opposite arcs*. A *loop* is an arc from a vertex to itself. An *oriented graph* is a directed graph without loops or opposite arcs. The underlying graph $und(\overrightarrow{G})$ or G of the directed graph $\overrightarrow{G}$ is obtained by replacing each arc by an edge (that is, replacing $\overrightarrow{uv}$ by uv).

The *order* of a digraph is the cardinality of its set of vertices while the *size* of a digraph is the cardinality of its set of arcs.

A *k-edge-colored graph* $(G) = (V(G), E_1(G), E_2(G), ..., E_k(G))$ is an ordered $(k+1)$-tuple with a set of *vertices* $V(G)$ and k different types of set of *edges*, where an edge is an unordered pair of vertices of type i for some $i \in \{1, 2, ..., k\}$ and $E_i(G)$ denotes the set of edges of type i. The underlying graph $und(G)$ (or simply G) of a k-edge-colored graph $(G) = (V(G), E_1(G), E_2(G), ..., E_k(G))$ is the graph with the set of vertices $V(G)$ and the set of edges $E(G) = E_1(G) \cup E_2(G) \cup .... \cup E_k(G)$. A 2-edge-colored graph is also known as a *signified graph* and is alternatively defined in Chapter 5.

Note that an 1-edge-colored graph is just an undirected graph.

Subgraphs and minors

A subgraph H of a graph G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We use the notation $H \subseteq G$ for H is a subgraph of G.

The induced subgraph $G[V']$ of G, induced by the vertex set $V' \subseteq V(G)$ is the graph $G[V']$ with $V(G[V']) = V'$ and $E(G[V']) = \{uv \in E(G) \mid u, v \in V'\}$.

Let F be a set of graphs. A *F-free* graph is a graph that does not contain any graph from the set F as a subgraph. Here F is a set of *forbidden subgraphs*.

In a graph G we can remove one edge and identify the vertices of that edge to get another graph. This process is called *edge contraction*. A graph H obtained by zero or more edge contractions on a subgraph of a graph G is a *minor* of the graph G. Let M be a set of graphs.

A M-*minor free* graph is a graph that does not contain any graph from the set M as a minor. Here M is a set of *forbidden minors*.

Some particular graphs

An *independent set of vertices* of a graph is a subset of the vertex set that induces a subgraph with no edges.

A *path* of length k or a k-path is a graph with vertices $u_0, u_1, ...u_k$ and edges $u_0u_1, u_1u_2, ..., u_{k-1}$ u_k and is denoted by $u_0u_1...u_k$. A path of odd length is an *odd-path* and a path of even length is an *even-path*.

A *connected* graph is a graph in which any pair of vertices is connected by a path inside the graph.

A graph H obtained by replacing some edges of a graph G each by a path is a *subdivision* of G.

A *cycle* of length k or a k-cycle is a graph with vertices $u_0, ..., u_{k-1}$ and edges $u_0u_1, u_1u_2, ...,$ $u_{k-2}u_{k-2}, u_{k-1}u_0$ and is denoted by $u_0u_1...u_{k-1}$. A 3-cycle is a *triangle*. A cycle of odd length is an *odd-cycle* and a cycle of even length is an *even-cycle*.

A *tree* is a graph with exactly one path between any two vertices. In other words, any connected cycle-free graph is a *tree*. A *star* is a tree with a vertex u that is adjacent to all the other vertices (every vertex except u of the graph has degree 1). A *forest* is a disjoint union of trees.

A *complete graph* on n vertices is a simple graph on n vertices such that there is an edge between any two vertices. We denote it by K_n. A *tournament* on n vertices is an oriented graph such that its underlying graph is K_n.

A *k-partite graph* is a graph whose vertex set can be written as the disjoint union of k independent sets. These k independent sets are called *parts* of the graph. This partition may not be unique. A *complete k-partite graph* is a simple k-partite graph with all the edges between two different parts. A 2-partite graph is a *bipartite graph*. Moreover, we denote a *complete bipartite graph* with parts of size r and s by $K_{r,s}$.

The *complement* of a graph G is the graph $\bar{G}$ with $V(\bar{G}) = V(G)$ such that two distinct vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G.

The projective cube $Proj_n$ of dimention n is the graph with set of vertices $V(Proj_n) = \{u = (u_1, u_2, ..., u_n) | u_i \in \{0, 1\}$ for $i = 1, 2, ..., n\}$ and with set of edges $E(Proj_n) = \{uv | u_i \neq v_i$ for exactly one $i \in \{1, 2, ..., n\}\} \cup \{uv | u_i \neq v_i$ for all $i \in \{1, 2, ..., n\}\}$.

Planar graphs

To draw a graph we can draw some points in the plane representing the vertices of the graph and join the points corresponding to the adjacent vertices with a line. These lines represent the edges. We can draw a digraph in a similar way by replacing the lines by arrows pointing towards the point corresponding to the successor. Note that these drawings are not unique.

A *planar graph* is a graph that can be drawn in the plane in such way that its edges do not intersect. Such a drawing of a planar graph is a *planar embedding*. Note that, in a planar embedding of a planar graph, the edges of the graph divide the plane into different connected components. Each of these connected components, including the outer unbounded component, is a *face* of the planar graph.

A planar graph is *triangulated* if every face is surrounded by exactly three edges. *Triangulating* a planar graph means to add some edges to obtain a triangulated planar graph. Notice that it is not possible to add more edges between the vertices of a triangulated planar graph, keeping the graph planar.

The oldest mathematical formula related to planar graphs is probably the Euler's formula.

Theorem 2.1 (Euler's formula [60])**.** *If a finite, connected planar graph has v vertices, e edges and f faces, then $v - e + f = 2$.*

The polish mathematician Kuratowski provided a characterization of planar graphs in terms of forbidden subgraphs, known as Kuratowski's theorem.

Theorem 2.2 (Kuratowski's theorem [60]). *A finite graph is planar if and only if it does not contain a subdivision of the complete graph K_5 or of the complete bipartite graph $K_{3,3}$ as a subgraph.*

Instead of dealing with subdivisions, Wagner stated the result, equivalent to Kuratowski's theorem, using the concept of forbidden minors.

Theorem 2.3 (Wagner's theorem [60]). *A finite graph is planar if and only if it does not contain the complete graph K_5 or the complete bipartite graph $K_{3,3}$ as a minor.*

An important subfamily of the family of planar graphs is the family of outerplanar graphs. An *outerplanar graph* is a planar graph with a planar embedding where every vertex of the graph is on the same face. Such a planar embedding of an outerplanar graph is an *outerplanar embedding*.

There is a similar characterization for outerplanar graphs as well.

Theorem 2.4. *[60] A finite graph is outerplanar if and only if it does not contain the complete graph K_4 or the complete bipartite graph $K_{2,3}$ as a minor.*

The *girth* of a graph is the length of the shortest cycle contained in it as a subgraph. While determining graph parameters for the family of planar graphs sometimes is a difficult problem to handle, researchers tend to solve the similar problems on easier subfamilies, such as, the family of planar graphs with girth at least g for $g \geq 4$ (in particular, graphs with girth at least 4 are also known as *triangle-free graphs*), the family of outerplanar graphs, the family of outerplanar graphs with girth at least g for $g \geq 4$, etc. In this thesis, these are the major planar subfamilies that we consider.

The *average degree* of a graph G is $ad(G) = 2|E(G)|/|V(G)|$, while the *maximum average degree* of a graph G is $mad(G) = max\{ad(H)|H \subseteq G\}$. We use maximum average degree as a tool to solve problems for families of planar graphs with girth at least g due to the following result:

Theorem 2.5. *[7] If G is a planar graph with girth at least g then $mad(G) < 2g/(g-2)$.*

Using this result, we prove results for graphs with bounded maximum average degree which implies results for planar graphs with girth restrictions.

Some families of planar graphs and their notations

We denote by $\mathcal{O}_g$ the family of outerplanar graphs with girth at least g and by $\mathcal{P}_g$ the family of planar graphs with girth at least g. Note that $\mathcal{O}_3$, in fact, denotes the family $\mathcal{O}$ of all outerplanar graphs and $\mathcal{P}_3$ denotes the family $\mathcal{P}$ of all planar graphs.

Clique number and Chromatic number

A *clique* of order k or a k-clique of a graph G is a complete graph K_k contained in G as a subgraph. The *clique number* $\omega(G)$ of a graph G is the maximum order of a clique of G.

A *proper vertex k-coloring* or *vertex k-coloring* of a graph G is a mapping ϕ from the set of vertices $V(G)$ to a set of cardinality k such that for any two adjacent vertices u, v we have $\phi(u) \neq \phi(v)$. Usually, $\{1, 2, ..., k\}$ is chosen for the set of cardinality k for convenience. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G has a proper vertex k-coloring.

It is easy to note that for any graph G we have $\omega(G) \leq \chi(G)$.

It is not very difficult to see that the clique number of a planar graph is at most 4 (as the complete graph K_5 is not planar). But the question, "what the chromatic number of a planar graph is upper bounded by?" is actually the graph theoretic formulation of the question "how many colors do you need to color a map?" mentioned in the introduction. As one can imagine, the answer, even though the same as the upper bound of the clique number, is far from easy and is the famous Four-Color Theorem.

Theorem 2.6 (Four-Color Theorem [60]). ***[Appel and Haken 1976]*** *The chromatic number of a planar graph is at most 4.*

Even though the Four-Color Theorem was very difficult to prove, if we restrict the problem to the family of outerplanar graphs or to the family of planar graphs with girth restrictions, the problem becomes relatively easier to handle. It is easy to prove that the chromatic number of an outerplanar graph is at most 3 [60], which is also the upper bound for the clique number of an outerplanar graph.

In two of the above instances, the clique number and the chromatic number coincide. But in general, the chromatic number of a graph can be arbitrarily higher than its clique number, as proved by Erdös. In fact, Erdös and many other researchers have constructed examples of graphs with arbitrary high girth as well as arbitrary high chromatic number.

This made the study of the chromatic number of graphs with higher girth interesting. For planar graphs with girth restriction, we have the following result.

Theorem 2.7 (Grötzsch's theorem [60]). *The chromatic number of a triangle-free planar graph is at most 3.*

One of the significant early attempts to solve the Four-Color Conjecture was on 1880 (unsolved at that time) by Tait using edge coloring. Before mentioning it, we will define a few more things.

A *proper k-edge-coloring* or *k-edge-coloring* of a graph G is a mapping ϕ from the set of edges $E(G)$ to a set of cardinality k such that, for any *two incident* edges uv and vw, we have $\phi(uv) \neq \phi(vw)$. The *chromatic index* $\chi'(G)$ of a graph G is the smallest integer k such that G has a proper edge k-coloring.

A *bridgeless* graph is a graph which remains connected after removing any edge from it.

Initially Tait proposed a conjecture stronger than the Four-Color Conjecture. This conjecture was disproved by Tutte. After that a weaker version of the conjecture, equivalent to the Four-Color Conjecture was proposed. This conjecture is known as Tait's conjecture. As the conjecture is equivalent to the Four-Color Conjecture, we can state it as a theorem now because the Four-Color Theorem has been proved.

Theorem 2.8 (Tait's conjecture [60]). *Every bridgeless 3-regular planar graph has a 3-edge-coloring.*

Later, on 1973, Seymour generalized Tait's conjecture and stated the following:

Conjecture 2.9 (Seymour 1973 [13]). *Every d-regular planar multigraph M has a d-edge-coloring, provided that for every odd set $X \subseteq V(M)$ of vertices, there are at least d edges between the vertices of X and the vertices of $V(M) \setminus X$.*

For $d = 3$ the above conjecture is precisely Tait's conjecture, hence, equivalent to the Four-Color Theorem. So, this conjecture generalizes the Four-Color Theorem. It has been proved for $d \leq 8$ by different researchers [13].

The most popular generalization of the Four-Color Theorem is probably Hadwiger's conjecture.

Conjecture 2.10 (Hadwiger's conjecture [60]). *Every K_k-minor free graph has chromatic number at most $(k-1)$.*

While the case $k = 5$ clearly implies the Four-Color Theorem, it has been shown by Wagner that it is actually equivalent to the Four-Color Theorem. This conjecture has been proved for $k \leq 6$ by different researchers [52].

There are some concepts that generalize the concept of coloring. For instance, graph coloring problem can be viewed as partitioning a graph into independent sets. In other words, the vertices that receive same color induce an independent set. But there is no condition on the type of graph induced by the vertices that received different colors. An extension of graph coloring is by putting such a condition.

An *acyclic k-coloring* ϕ of a graph G is a proper graph coloring of G where any subgraph of G colored by at most two colors does not have a cycle. The result analogous to that of the Four-Color Theorem for vertex coloring is the celebrated result by Borodin which is the following.

Theorem 2.11 (Borodin [4]). *Any planar graph has an acyclic 5-coloring.*

Another generalization of graph coloring is graph labelings. A special such labeling is the $L(p, q)$-labeling which we will define and discuss about in Chapter 3.

Graph homomorphisms

A popular way of formulating proper graph coloring is through graph homomorphisms. Indeed, graph homomorphism is a more general concept.

A *homomorphism* of a graph G to a graph H is a mapping $\phi : V(G) \to V(H)$ such that for every edge $uv \in E(G)$ we have $\phi(u)\phi(v) \in E(H)$. We denote the existence of a homomorphism of a graph G to a graph H by $G \to H$ and H is called the *target graph*. We also use the term H *bounds* G to mean that $G \to H$. If every graph from a family $\mathcal{F}$ of graphs admit a homomorphism to a particular graph H then H is a *universal target* or a *universal bound* of $\mathcal{F}$. An *isomorphism* of a graph G to a graph H is a bijective homomorphism of G to H whose inverse is also a homomorphism.

Notice that a homomorphism $G \to K_k$ actually corresponds to a vertex k-coloring of G. Hence an alternative definition of the chromatic number of a graph G is the smallest k for which G admits a homomorphism to the complete graph K_k.

Hence, according to the Four-Color Theorem, the complete graph K_4 is a universal bound for the family of planar graphs while according to Grötzsch's Theorem the triangle K_3 is a universal bound for the family of triangle-free planar graphs. Note that the graph K_4 is actually planar while the graph K_3 is not in the family of triangle-free planar graphs. Naturally, the question occurs to mind that if there is a triangle-free planar graph that bounds the family of triangle-free planar graphs. Regarding this question by Nešetřil, Naserasr has proposed a general conjecture, discussed below, that captures this question.

We can even ask general questions like, "what is the smallest (in terms of order) graph with property $\mathcal{P}$ which is a universal bound of the family of graphs $\mathcal{F}$?" This question can be modified to correspond to different coloring and labeling problems.

The most important theorem proved in this context is the one by Nešetřil and Ossona De Mendez. For any set X of graphs, let $Forb_h(X)$ denote the set of graphs that admit no homomorphism from a member of X, and $Forb_m(X)$ denote the set of graphs that admit no member of X as a minor. Then we have:

Theorem 2.12 (Nešetřil and Ossona De Mendez [23])**.** *For every set of graphs M and every set of connected graphs H, the class $Forb_m(M) \cap Forb_h(H)$ is bounded by a graph in $Forb_h(H)$.*

Finding a bound as in Theorem 2.12 with the smallest possible number of vertices proves to be a very difficult question in general. For the simplest case of $M = H = \{K_n\}$, finding the smallest bound in terms of number of vertices will, in particular, solve Hadwiger's conjecture.

For the case $M = \{K_5, K_{3,3}\}$ and $H = \{C_{2k-1}\}$ it is conjectured by Naserasr that the projective cube $Proj_{2k}$ of dimension $2k$ is a solution [38]. Note that the conjecture actually claims that every planar graph with no odd-cycle of length at most $2k + 1$ admits a homomorphism to the projective cube $Proj_{2k}$ of dimension $2k$. The *odd-girth* of a graph is the length of the shortest odd-cycle contained in it as as subgraph. Hence a restatement of the conjecture is as follows:

Conjecture 2.13. *Every planar graph with odd girth $2k + 1$ admits a homomorphism to the projective cube $Proj_{2k}$ of dimension $2k$.*

This conjecture for each k is equivalent to Conjecture 2.9 by Seymour for $d = 2k + 1$. Hence, Conjecture 2.13 by Naserasr captures Conjecture 2.9 by Seymour for the odd numbers.

Conjecture 2.13 has been extended (see Chapter 6, Section 6.4, Conjecture 6.36) by Naserasr, Rollová and Sopena using the notion of consistent signed graphs that captures all the cases of Seymour's conjecture [39]. In this thesis, we will prove a supporting result to that conjecture in Chapter 6, Section 6.4.

Some other graphs

A graph is *self-complementary* if it is isomorphic to its complement.

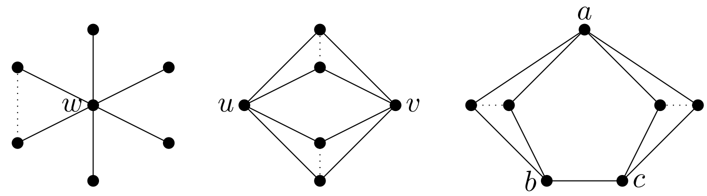


Figure 2.1: List of all triangle-free planar graphs with diameter 2 (Plesník (1975)).

A graph G is *vertex-transitive* if for each pair of vertices $u, v \in V(G)$ we have an isomorphism ϕ of G to G such that $\phi(u) = v$.

A graph G is *edge-transitive* if for each pair of edges $uv, wx \in E(G)$ we have an isomorphism ϕ of G to G such that $\phi(u) = w$ and $\phi(v) = x$.

A *strongly regular graph* G with parameters (v, k, λ, μ) is a k-regular graph of order v such that every pair of adjacent vertices have exactly λ common neighbors and every pair of non-adjacent vertices have exactly μ common neighbors.

Some other graph parameters

We define some graph parameters, not related to coloring, and some results related to those parameter which will be used later.

The *distance* $d_G(x, y)$ (or $d(x, y)$ when there is no confusion) between two vertices x and y of a graph G is the smallest length of a path connecting x and y. The *diameter* $diam(G)$ of a graph G is the maximum distance between pairs of vertices of the graph.

Theorem 2.14. *[24] The triangle-free graphs with diameter 2 are precisely the graphs listed in Fig. 2.1.*

The graphs depicted in Fig. 2.1 are the stars, the complete bipartite graphs $K_{2,n}$ for some natural number n, and the graph obtained by adding copies of two non-adjacent vertices of the 5-cycle.

A vertex subset D is a *dominating* set of a graph G if every vertex of G is either in D or adjacent to a vertex of D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G.

Theorem 2.15. *[18] Any planar graph with diameter 2 has domination number at most 2, except for a particular planar graph on 9 vertices (depicted in Fig. 2.2) which has domination number 3.*

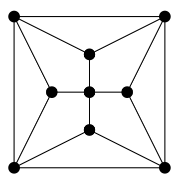


Figure 2.2: The unique planar graph with diameter 2 and domination number 3.

We will use these definitions and results to prove some results in the upcoming chapters. As we will use these definitions and results several times, we are defining and stating them here.

Category

Now we will state the formal definition of category and functors which will be used later.

A *category* $\mathcal{C}$ consists of

– a class $ob(\mathcal{C})$ of *objects*

– a class $hom(\mathcal{C})$ of *morphisms* between objects, where a morphism ϕ is from a unique object A to a unique object B. The class of all morphisms from an object A to an object B is denoted by $hom_{\mathcal{C}}(A, B)$.

– a binary operation *composition* $\circ : hom_{\mathcal{C}}(A, B) \times hom_{\mathcal{C}}(B, C) \to hom_{\mathcal{C}}(A, C)$, (that is, for every $\alpha, \beta \in hom_{\mathcal{C}}(A, B) \times hom_{\mathcal{C}}(B, C)$ we have $\beta \circ \alpha \in hom_{\mathcal{C}}(A, C)$) for every three objects A, B, C such that the following axioms hold:

  ▷ for every triplet $(\alpha, \beta, \gamma) \in hom_{\mathcal{C}}(A, B) \times hom_{\mathcal{C}}(B, C) \times hom_{\mathcal{C}}(C, D)$ we have, $(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$ (*associativity* property).

  ▷ for every $A \in ob(\mathcal{C})$ there exists an *identity morphism* $1_A \in hom_{\mathcal{C}}(A, A)$ such that, for every $\phi \in hom_{\mathcal{C}}(A, A)$ we have, $1_A \circ \phi = \phi \circ 1_A = \phi$ (*identity* property).

A *subcategory* $\mathcal{S}$ of a category $\mathcal{C}$ is a category with $ob(\mathcal{S}) \subseteq ob(\mathcal{C})$ and for any two objects $A, B \in ob(\mathcal{S})$ we have $hom_{\mathcal{S}}(A, B) \subseteq hom_{\mathcal{C}}(A, B)$.

An object $(A \oplus B)$ of $\mathcal{C}$ is the *coproduct* of two objects $A, B \in \mathcal{C}$ if it satisfies the following universal property:

– there exist morphisms $\phi_1 : A \to A \oplus B, \phi_2 : B \to A \oplus B$ such that for every object C and every pair of morphisms $\psi_1 : A \to C, \psi_2 : B \to C$ there exists a unique morphism $\varphi : A \oplus B \to C$ such that the following diagram commutes:

$$\begin{tikzcd}
& C & \\
A \arrow[ur, "\psi_1"] \arrow[r, "\phi_1"'] & A \oplus B \arrow[u, dashed, "\exists ! \varphi"] & B \arrow[ul, "\psi_2"'] \arrow[l, "\phi_2"]
\end{tikzcd}$$

An object $A \times B$ of $\mathcal{C}$ is the *product* of two objects $A, B \in \mathcal{C}$ if it satisfies the following universal property:

– there exist morphisms $\phi_1 : A \times B \to A, \phi_2 : A \times B \to B$ such that for every object C and every pair of morphisms $\psi_1 : C \to A, \psi_2 : C \to B$ there exists a unique morphism $\varphi : C \to A \times B$ such that the following diagram commutes:

$$\begin{tikzcd}
A & A \times B \arrow[l, "\phi_1"'] \arrow[r, "\phi_2"] & B \\
& C \arrow[ul, "\psi_1"] \arrow[u, dashed, "\exists ! \varphi"] \arrow[ur, "\psi_2"'] &
\end{tikzcd}$$

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* F of $\mathcal{C}$ to $\mathcal{D}$ is a mapping such that

– for each $A \in ob(\mathcal{C})$ we have $F(A) \in ob(\mathcal{D})$,

– for each $\phi \in hom_{\mathcal{C}}(A, B)$ we have $F(\phi) \in hom_{\mathcal{D}}(F(A), F(B))$ and the following holds:

  ▷ $F(1_A) = 1_{F(A)}$ for all $A \in ob(\mathcal{C})$,

  ▷ $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ for all $(\phi, \psi) \in hom_{\mathcal{C}}(B, C) \times hom_{\mathcal{C}}(A, B)$ where $A, B, C \in ob(\mathcal{C})$.

Two categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic* if there is a functor F of $\mathcal{C}$ to $\mathcal{D}$ such that

– $F : ob(\mathcal{C}) \to ob(\mathcal{D})$ is a bijection,

– $F : hom_{\mathcal{C}}(A, B) \to hom_{\mathcal{D}}(F(A), F(B))$ is a bijection for all $A, B \in \mathcal{C}$.

It is not difficult to check that the class of all graphs can be regarded as a category with homomorphisms of graphs as the morphisms of the category with isomorphic graphs being considered as the same object.

Overview

We have given the basic definitions, fixed some notations and have recalled some famous results here. We have also mentioned some other results which will be used in this thesis. The definitions, specific to a chapter, will be given in the beginning of the chapter. The main objective of the thesis is to present results regarding homomorphisms of four different types of graphs, namely, oriented graphs (Chapter 3), orientable graphs (Chapter 4), signified graphs (Chapter 5) and signed graphs (Chapter 6).

We have four main chapters in this thesis, each dedicated to each of these types of graphs. We will consider the problems of determining the chromatic number and the clique number, defined using homomorphisms, of these different types of graphs. We will consider some related questions regarding the $L(p, q)$-labeling of oriented graphs and the bound of planar consistent signed graphs. Some categorical aspects of these types of graphs will be discussed as well. We have tried our best to provide a comparative study of homomorphisms in these four different settings.

Chapter 3
Oriented graphs

In this chapter we deal with oriented graphs. Our main focus is to present some results regarding oriented colorings and oriented $L(p, q)$-labelings. While oriented coloring is a well-studied topic, oriented $L(p, q)$-labeling is quite new.

The organization of the chapter is as follows. In Section 3.1 we give the basic definitions and notations related to oriented graphs and homomorphisms of oriented graphs. Then in Section 3.2 we give definitions of some oriented graphs which we will use later in this thesis. After that we present our main results regarding oriented coloring in Section 3.3 and our main results regarding oriented $L(p, q)$-labeling in Section 3.4. Finally, we conclude this chapter in Section 3.5.

The proof of Theorem 3.21(a) stated in Section 3.3 appeared in [55] while the other parts of the same theorem appeared as an extended abstract in Bordeaux Graph Workshop BGW'2012 [54]. The rest of the results proved in Section 3.3 is a joint work with Sopena and is an article in process. Most of the results proved in Section 3.4 are part of an article which is accepted and in press for publication [53].

3.1 Preliminaries

An *oriented graph* is a directed graph with no cycle of length 1 or 2. By replacing each edge of a simple graph G with an arc (ordered pair of vertices) we obtain an oriented graph $\overrightarrow{G}$; $\overrightarrow{G}$ is an *orientation* of G and G is the *underlying graph* of $\overrightarrow{G}$. We denote by $V(\overrightarrow{G})$ and $A(\overrightarrow{G})$ respectively the set of vertices and arcs of $\overrightarrow{G}$. An arc (u, v) (where u and v are vertices) is denoted by $\overrightarrow{uv}$.

The set of all adjacent vertices of a vertex v in an oriented graph $\overrightarrow{G}$ is called its set of *neighbors* and is denoted by $N_{\overrightarrow{G}}(v)$ (or $N(v)$ when there is no chance of confusion). If there is an arc $\overrightarrow{uv}$, then u is an *in-neighbor* of v and v is an *out-neighbor* of u. The set of all in-neighbors and the set of all out-neighbors of v are denoted by $N^-_{\overrightarrow{G}}(v)$ (or $N^-(v)$ when there is no chance of confusion) and $N^+_{\overrightarrow{G}}(v)$ (or $N^+(v)$ when there is no chance of confusion) respectively. The *degree* of a vertex v in an oriented graph $\overrightarrow{G}$, denoted by $d_{\overrightarrow{G}}(v)$ (or $d(v)$ when there is no chance of confusion), is the number of neighbors of v in $\overrightarrow{G}$. Naturally, the *in-degree* (resp. *out-degree*) of a vertex v in an oriented graph $\overrightarrow{G}$, denoted by $d^-_{\overrightarrow{G}}(v)$ (resp. $d^+_{\overrightarrow{G}}(v)$) (or $d^-(v)$ (resp. $d^+(v)$) when there is no chance of confusion), is the number of in-neighbors (resp. out-neighbors) of v in $\overrightarrow{G}$. The *order* $|\overrightarrow{G}|$ of an oriented graph $\overrightarrow{G}$ is the cardinality of its set of vertices $V(\overrightarrow{G})$.

Two vertices u and v of an oriented graph *agree* on a third vertex w of that graph if $w \in N^\alpha(u) \cap N^\alpha(v)$ for some $\alpha \in \{+, -\}$. Two vertices u and v of an oriented graph *disagree* on a third vertex w of that graph if $w \in N^\alpha(u) \cap N^\beta(v)$ for some $\{\alpha, \beta\} = \{+, -\}$.

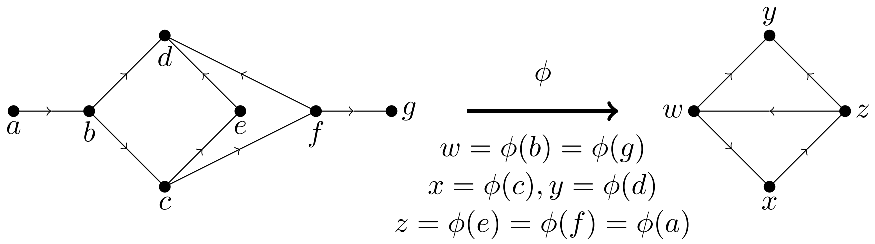


Figure 3.1: Oriented graph homomorphism.

A *directed path* of length k or a k*-dipath* from v_0 to v_k an oriented graph with vertices $v_0, v_1, ..., v_k$ and arcs $\overrightarrow{v_0v_1}, \overrightarrow{v_1v_2}, ..., \overrightarrow{v_{k-1}v_k}$ where v_0 and v_k are the *terminal* vertices and $v_1, ..., v_{k-1}$ are *internal* vertices. A *directed cycle* of length k or a directed k-cycle is an oriented graph with vertices $v_1, v_2, ..., v_k$ and arcs $\overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, ..., \overrightarrow{v_{k-1}v_k}$ and $\overrightarrow{v_kv_1}$. The directed distance $\overrightarrow{d}_{\overrightarrow{G}}(u, v)$ (or $\overrightarrow{d}(u, v)$ when there is no chance of confusion) between two vertices u and v of an oriented graph $\overrightarrow{G}$ is the smallest length of a directed path of $\overrightarrow{G}$ from u to v. A 2-dipath with arcs $\overrightarrow{uv}$ and $\overrightarrow{vw}$ is denoted by $\overrightarrow{uvw}$. More generally, a 2-dipath with terminal vertices u, w and internal vertex v is denoted by uvw (this denotes, either the 2-dipath $\overrightarrow{uvw}$ or the 2-dipath $\overrightarrow{wvu}$).

Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be two oriented graphs. A homomorphism of $\overrightarrow{G}$ to $\overrightarrow{H}$ is a mapping $\phi : V(\overrightarrow{G}) \to V(\overrightarrow{H})$ which preserves the arcs, that is, $uv \in A(\overrightarrow{G})$ implies $\phi(u)\phi(v) \in A(\overrightarrow{H})$. We write $\overrightarrow{G} \to \overrightarrow{H}$ whenever there exists a homomorphism of $\overrightarrow{G}$ to $\overrightarrow{H}$ and say that $\overrightarrow{H}$ *bounds* $\overrightarrow{G}$. A bijective homomorphism whose inverse is also a homomorphism is an *isomorphism.* If two oriented graphs admit oriented homomorphisms to each other then they are *homomorphically equivalent* oriented graphs.

Example 3.1. *A sample homomorphism of oriented graphs is given in Fig. 3.1.*

Different coloring and labeling problems on oriented graphs are solved using homomorphisms.

3.2 Some oriented graphs and their properties

In this section we define some families of oriented graphs and discuss some of their properties which will be helpful for proving some of the results discussed in this thesis.

For any prime $p \equiv 3 \pmod 4$ and for any positive integer n the *Paley tournament* [32] $\overrightarrow{P}_q$ of order $q = p^n$ is the oriented graph with set of vertices $\{0, 1, 2, ..., q-1\}$ and set of arcs $\{\overrightarrow{xy} \mid y - x \pmod p$ is a non-zero square$\}$. As $-1 \pmod p$ is a not a square , either $(x - y)$ or $(y - x)$ (but not both) is a square for all $x, y \in F_q$. Hence $\overrightarrow{P}_q$ is a tournament.

The *Paley plus graph* $\overrightarrow{P}^+_q$ is the oriented graph obtained from the oriented Paley graph $\overrightarrow{P}_q$ by adding a new vertex ∞ which is an in-neighbor of every other vertex.

For any prime $p \equiv 3 \pmod 4$ and for any positive integer n the *Tromp graph* [32] $\overrightarrow{T}_{2q+2}$ of order $(2q + 2)$, where $q = p^n$, is the oriented graph with the set of vertices and the set of arcs as follows:

$$\begin{aligned} V(\overrightarrow{T}_{2q+2}) &= \{0, 1, ..., (q-1)\} \cup \{0', 1', ..., (q-1)'\} \cup \{\infty, \infty'\} \\ A(\overrightarrow{T}_{2q+2}) &= \{\overrightarrow{ij}, \overrightarrow{i'j'}, \overrightarrow{j'i}, \overrightarrow{ji'} \mid i, j \in \{0, 1, ..., (q-1)\} \text{ and} \\ &\qquad (j - i) \pmod p \text{ is a non-zero square}\} \\ &\qquad \cup \{\overrightarrow{i\infty}, \overrightarrow{\infty i'}, \overrightarrow{i'\infty'}, \overrightarrow{\infty' i} \mid i \in \{0, 1, ..., (q-1)\}\}. \end{aligned}$$

Intuitively, in $\overrightarrow{T}_{2q+2}$ there are two vertices ∞, ∞' such that $N^+(\infty) = N^-(\infty')$ and $N^+(\infty) = N^-(\infty')$ with each of the sets $N^+(\infty)$ and $N^-(\infty)$ inducing a Paley tournament $\overrightarrow{P}_q$. Also, if $\overrightarrow{ij}$ is an arc in the $\overrightarrow{P}_q$ induced by $N^+(\infty)$ and $\overrightarrow{i'j'}$ is the corresponding arc of the $\overrightarrow{P}_q$ induced by

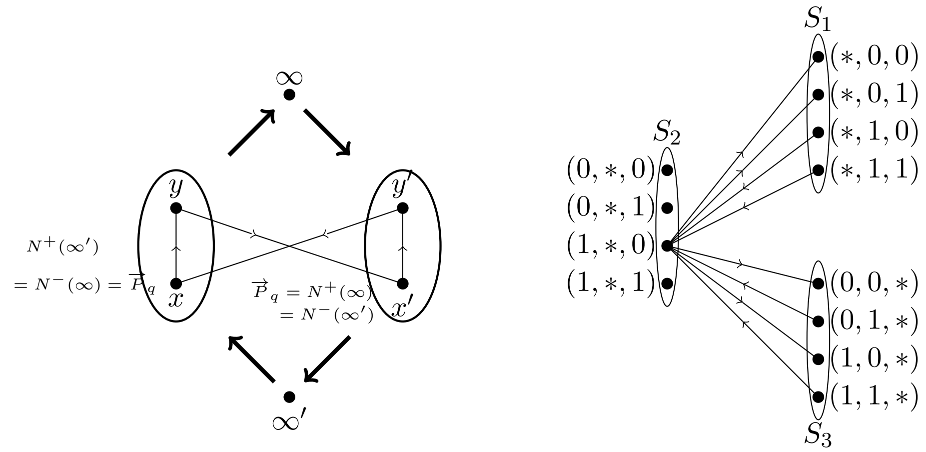


Figure 3.2: (a) The Tromp graph $\vec{T}_{2q+2}$ (thick arrow means all the arcs are to/from ∞ or ∞' from/to the vertices inside the ellipse. (b) Adjacency of a vertex of the Zielonka graph $\vec{Z}_3$.)

$N^-(\infty)$, then we also have the "reversed" arcs $\overrightarrow{ji'}$ and $\overrightarrow{j'i}$. For pictorial help, check Fig. 3.2 (a). Note that $\overrightarrow{T}_{2q+2}$ is a complete $(q+1)$-partite graph with all parts of size two. For further details about this graph, see Marshall's paper [32].

For any positive integer k the *Zielonka graph* [57] $\overrightarrow{Z}_k$ of order $k \times 2^{k-1}$ is the oriented graph with set of vertices $V(\overrightarrow{Z}_k) = \cup_{i=1,2,...,k} S_i$ where

$$S_i = \{x = (x^1, ..., x^k) | x^j \in \{0, 1\} \text{ for } j \neq i \text{ and } x^i = *\}$$

and set of arcs

$$A(Z_k) = \{\overrightarrow{xy} \mid x = (x^1, ..., x^k) \in S_i, y = (y^1, ..., y^k) \in S_j \text{ and}$$
$$\text{either } x^j = y^i \text{ and } i < j \text{ or } x^j \neq y^i \text{ and } i > j\}.$$

Note that Z_k is a complete k-partite graph with all parts of size 2^{k-1}.

Example 3.2. *Adjacency of a vertex of the Zielonka graph Z_3 is depicted in Fig. 3.2(b).*

A *pattern* Q of length k is a word $Q = q_0q_1...q_{k-1}$ with $q_i \in \{+, -\}$ for every i, $0 \leq i \leq k-1$. A *Q-walk* in a directed graph or digraph $\overrightarrow{G}$ is a walk $P = x_0x_1.....x_k$ such that for every i, $0 \leq i \leq k-1$, $x_ix_{i+1} \in A(\overrightarrow{G})$ if $q_i = +$ and $x_{i+1}x_i \in A(G)$ otherwise. For $X \subseteq V(\overrightarrow{G})$ we denote by $N^Q(X)$ the set of all vertices y such that there exists a Q-walk going from some vertex $x \in X$ to y. We then say that a digraph $\overrightarrow{G}$ is *k-nice* if for every pattern Q of length k and for every vertex $x \in V(\overrightarrow{G})$ we have $N^Q(\{x\}) = V(\overrightarrow{G})$. In other words, a digraph is k-nice if for all pairs of vertices x, y (allowing $x = y$) there is a k-walk from x to y for each of the 2^k possible oriented patterns. Observe that if a digraph G is k-nice for some k, then it is k'-nice for every $k' \geq k$.

Example 3.3. *Consider the graph $\overrightarrow{B}$ (Fig 3.3). One can check that it is a 4-nice graph.*

Now we state a theorem from [22] (recall that $\mathcal{P}_g$ denotes the class of planar graphs with girth at least g).

Theorem 3.4 (Hell *et al.* 1997)**.** *Let N_k be a k-nice oriented graph, $k \geq 3$. Every oriented graph whose underlying graph is in $\mathcal{P}_{5k-4}$ admits a homomorphism to N_k.*

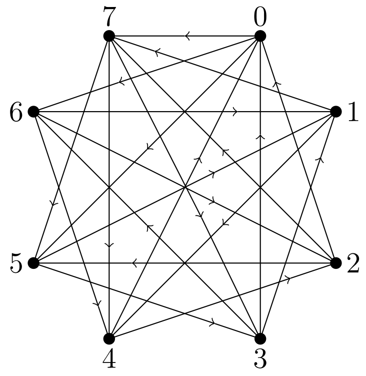
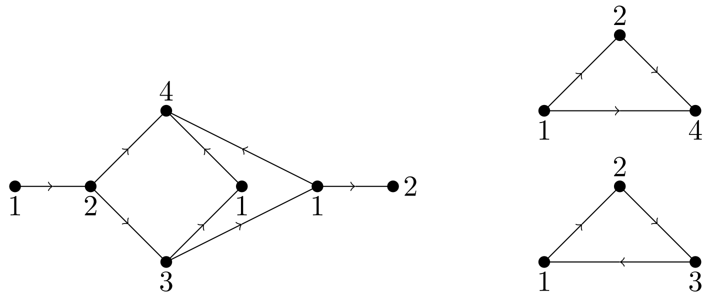


Figure 3.3: $\vec{B}$ is a 4-nice graph.



(a) (b)

Figure 3.4: Oriented coloring.

3.3 Oriented coloring

Colorings of oriented graphs first appeared in the work of Courcelle [14] on the monoadic second order logic of graphs. Since then it has been considered by many researchers, following the work of Raspaud and Sopena [51] on oriented colorings of planar graphs.

An *oriented k-coloring* [58] of an oriented graph $\overrightarrow{G}$ is a mapping ϕ from the vertex set $V(\overrightarrow{G})$ to the set $\{1, 2, ..., k\}$ such that,

- (i) $\phi(u) \neq \phi(v)$ whenever u and v are adjacent and
- (ii) if $\overrightarrow{uv}$ and $\overrightarrow{wx}$ are two arcs in $\overrightarrow{G}$, then $\phi(u) = \phi(x)$ implies $\phi(v) \neq \phi(w)$.

The *oriented chromatic number* $\chi_o(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the smallest integer k for which $\overrightarrow{G}$ has an oriented k-coloring.

Alternatively, we can define oriented chromatic number by defining homomorphisms of oriented graphs. The *oriented chromatic number* $\chi_o(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the minimum order (number of vertices) of an oriented graph $\overrightarrow{H}$ such that $\overrightarrow{G}$ admits a homomorphism to $\overrightarrow{H}$.

The oriented chromatic number $\chi_o(G)$ of an undirected graph G is the maximum of the oriented chromatic numbers of all the oriented graphs with underlying graph G. The oriented chromatic number $\chi_o(\mathcal{F})$ of a family $\mathcal{F}$ of graphs is the maximum of the oriented chromatic numbers of the graphs from the family $\mathcal{F}$.

Example 3.5. *We give an oriented 4-coloring of the graph in Fig. 3.4 (a) whereas, in Fig. 3.1 we showed that the same graph admits a homomorphism to an oriented graph of order 4. These two facts essentially means the same thing. Also note that we cannot provide an oriented 3-coloring of this graph, hence it has oriented chromatic number 4.*

Example 3.6. *Consider the oriented coloring of the disjoint union of two graphs from Fig. 3.4 (b). Note that if we oriented color the two graphs from Fig. 3.4 (b) (both are orientations of a triangle)*

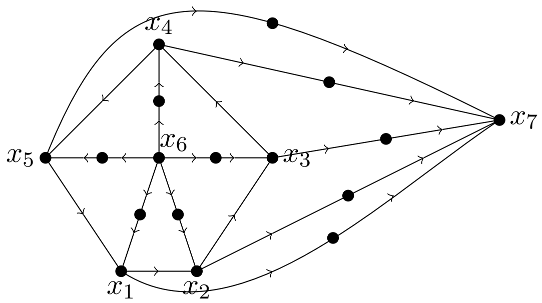


Figure 3.5: $\vec{B}$ is an oriented planar graph with girth 5.

individually, as each of them has three vertices which are pairwise adjacent, we can easily show that each of these graphs has oriented chromatic number 3. But when we consider the disjoint union of these two graphs as one oriented graph, we are not able to provide an oriented 3-coloring of the graph. In this case, we will need at least 4 colors to provide an oriented coloring of the graph. In Fig. 3.4 (b) we have given an oriented 4-coloring of the graph.

Notice that the terminal vertices of a 2-dipath must receive distinct colors in every oriented coloring because of the second condition of the definition. In fact, for providing an oriented coloring of an oriented graph, only the pairs of vertices which are either adjacent or connected by a 2-dipath must receive distinct colors (that is, for every other type of pair of vertices there exists an oriented coloring which assigns the same color to the pair of vertices). Motivated by this observation, the following definitions were proposed.

An *oriented relative clique* of an oriented graph $\overrightarrow{G}$ is a set $R \subseteq V(\overrightarrow{G})$ of vertices such that any two vertices from R are at directed distance at most 2 in $\overrightarrow{G}$. The *oriented relative clique number* $\omega_{ro}(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the maximum order of an oriented relative clique of $\overrightarrow{G}$. The oriented relative clique number of an oriented graph is otherwise known as the *2-dipath chromatic number* [35] [61] of an oriented graph. The term oriented relative clique is given by following the term used in [40] for a similar definition for signed graphs.

An *oriented absolute clique* or simply *oclique*, a term coined by Klostermeyer and MacGillivray [27], is an oriented graph $\overrightarrow{G}$ for which $\chi_o(\overrightarrow{G}) = |V(\overrightarrow{G})|$. Note that ocliques can hence be characterized as those oriented graphs whose any two distinct vertices are at directed distance at most 2, that is, either adjacent or connected by a 2-dipath. Note that an oriented graph with an oclique of order n as a subgraph has oriented chromatic number and oriented relative clique number at least n. The *oriented absolute clique number* $\omega_{ao}(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the maximum order of an oclique contained in $\overrightarrow{G}$ as a subgraph.

The oriented relative clique number $\omega_{ro}(G)$ (resp. oriented absolute clique number $\omega_{ao}(G)$) of a simple graph G is the maximum of the oriented relative clique numbers (resp. oriented absolute clique numbers) of all the oriented graphs with underlying graph G. The oriented relative clique number $\omega_{ro}(\mathcal{F})$ (resp. oriented absolute clique number $\omega_{ao}(\mathcal{F})$) of a family $\mathcal{F}$ of graphs is the maximum of the oriented relative clique numbers (resp. oriented absolute clique numbers) of the graphs from the family $\mathcal{F}$.

From the definitions, clearly we have the following:

Lemma 3.7. *For any oriented graph $\overrightarrow{G}$ we have, $\omega_{ao}(\overrightarrow{G}) \leq \omega_{ro}(\overrightarrow{G}) \leq \chi_o(\overrightarrow{G})$.*

Corollary 3.8. *For any oclique $\overrightarrow{O}$ we have $\omega_{ao}(\overrightarrow{O}) = \omega_{ro}(\overrightarrow{O}) = \chi_o(\overrightarrow{O}) = |V(\overrightarrow{O})|$.*

Example 3.9. *Note that the vertices $x_1, ..., x_5$ of the graph $\overrightarrow{B}$ depicted in Fig. 3.5 induce an oclique of order 5 and there is no oclique of order more than 5 in $\overrightarrow{B}$.*

Now, if we try to provide an oriented coloring of the graph $\overrightarrow{B}$, the vertices $x_1, ..., x_5$ will receive distinct colors as they are vertices of an oclique. But note that the vertex x_6 must receceive a color distinct from the color received by $x_1, ..., x_5$ even though these six vertices do not induce *an oclique in* $\overrightarrow{B}$*.*

It is easy to show that for the graph $\overrightarrow{B}$ *depicted in Fig. 3.5 we have,* $\omega_{ao}(\overrightarrow{B}) = 5$, $\omega_{ro}(\overrightarrow{G}) = 6$ *and* $\chi_o(\overrightarrow{G}) = 7$*. This is an example of a graph for which each inequality of Lemma 3.7 is strict.*

Note that the above defined three graph parameters respect homomorphisms of oriented graphs in the sense of the following result:

Lemma 3.10. *Let* $\overrightarrow{G} \to \overrightarrow{H}$*. Then* $\chi_o(\overrightarrow{G}) \leq \chi_o(\overrightarrow{H})$, $\omega_{ro}(\overrightarrow{G}) \leq \omega_{ro}(\overrightarrow{H})$ *and* $\omega_{ao}(\overrightarrow{G}) \leq \omega_{ao}(\overrightarrow{H})$*.*

Determining the oriented chromatic number, the relative clique number and the absolute clique number of different families of graphs are challenging problems in the domain of oriented coloring. A usual technique for obtaining the upper bound is to prove that every graph in the family of graphs in question admits a homomorphism to a particular oriented graph, that is to find an oriented graph that bounds every graph of that family. Such a graph is called a *universal bound* of that family of graphs. Note that not every family of graphs have a universal bound of order equal to the oriented chromatic number of the family.

For example, the family of all oriented graphs on 3 vertices have oriented chromatic number 3 as each graph in the family is clearly oriented 3-colorable. As previously discussed, the two oriented graphs on 3 vertices depicted in Fig. 3.4 (their disjoint union is not 3-colorable) cannot admit a homomorphism to a single oriented graph on 3 vertices.

If we consider the set of all oriented graphs to be a category with objects being the oriented graphs and morphisms being the oriented homomorphisms then we clearly have the proposition stated below. This proposition is just a restatement of a proposition by Sopena [57] regarding *complete* family of graphs (a family of graphs is complete if given two graphs in the family there is a third graph in it that contains both the given graphs as subgraphs) in the language of categories.

Proposition 3.11. *For any family* $\mathcal{F}$ *of oriented graphs that also contains the categorical co-products of the graphs from the family, there exists a universal bound of* $\mathcal{F}$ *on* $\chi_o(\mathcal{F})$ *vertices.*

Observe that a categorical co-product (unique up to homomorphic equivalence) of oriented graphs is simply the oriented graph obtained by taking the disjoint union of the oriented graphs. The families of planar graphs, outerplanar graphs, planar graphs with given girth and outerplanar graphs with given girth are each of the type that we mentioned in the above theorem.

3.3.1 Oriented chromatic number

One of the first major results proved related to oriented chromatic number is the following theorem by Raspaud and Sopena [51].

Theorem 3.12. *Every graph with acyclic chromatic number at most* k *has oriented chromatic number at most* $k.2^{k-1}$*.*

To prove this theorem, Raspaud and Sopena showed that for every graph G with acyclic chromatic number at most k, any oriented graph with underlying graph G admits a homomorphism to the Zielonka graph Z_k.

Ochem [44] proved that this bound is tight for every $k \geq 3$. Conversely, Kostochka, Sopena and Zhu [29] proved that every undirected simple graph with bounded oriented chromatic number has bounded acyclic chromatic number.

Another general bound, for graphs with bounded degree, was proved by Kostochka, Sopena and Zhu [29].

Theorem 3.13. *Every oriented graph with maximum degree* Δ *has oriented chromatic number at most* $2^{(\Delta+1)}.\Delta^2$*. Also, for every* $\Delta \geq 2$*, there exists an oriented graph with maximum degree* Δ *and oriented chromatic number at least* $2^{\Delta/2}$*.*

Some specific families of graphs have also been studied. The most studied family of graphs for oriented chromatic number is the family of planar graphs and some sub-families of planar graphs, such as outerplanar graphs, planar graphs with given girth, outerplanar graphs with given girth etc.

The known bounds, which are tight, for the oriented chromatic number of outerplanar graphs and outerplanar graphs with given girth are listed in the following theorem. The relevent references are given after the results. Recall that $\mathcal{O}_g$ denotes the family of outerplanar graphs with girth at least g.

Theorem 3.14.

(a) $\chi_o(\mathcal{O}_3) = 7$. *[57]*

(b) $\chi_o(\mathcal{O}_4) = 6$. *[47]*

(c) $\chi_o(\mathcal{O}_k) = 5$ *for* $k \geq 5$. *[47]*

To prove Theorem 3.14(a) Sopena [57] showed that every oriented outerplanar graph admits a homomorphism to the Paley tournament $\overrightarrow{P}_7$ of order 7. This proves that the oriented chromatic number of outerplanar graphs is bounded above by 7. He also constructed an outerplanar oclique of order 7 to complete the proof. This proof by Sopena [57] also proves Theorem 3.17(a) and Theorem 3.20(a) clearly. Similarly, to prove the upper bounds in the other two results, Pinlou and Sopena [47] found oriented graphs that are universal bounds on 6 and 5 vertices for the family of oriented outerplanar graphs with girth at least 4 and 5, respectively. They also constructed examples to prove the lower bounds.

The known bounds for the oriented chromatic number of planar graphs and planar graphs with given girth are listed in the following theorem. The reference for lower bound is given first in each case. Recall that $\mathcal{P}_g$ denotes the family of planar graphs with girth at least g.

Theorem 3.15.

(a) $18 \leq \chi_o(\mathcal{P}_3) \leq 80$. *[33] [51]*

(b) $11 \leq \chi_o(\mathcal{P}_4) \leq 40$. *[43] [45]*

(c) $7 \leq \chi_o(\mathcal{P}_5) \leq 16$. *[34] [46]*

(d) $7 \leq \chi_o(\mathcal{P}_6) \leq 11$. *[34] [7]*

(e) $6 \leq \chi_o(\mathcal{P}_7) \leq 7$. *[42] [5]*

(f) $5 \leq \chi_o(\mathcal{P}_8) \leq 7$. *[42] [5]*

(g) $5 \leq \chi_o(\mathcal{P}_k) \leq 6$ *for* $9 \leq k \leq 11$. *[42] [31]*

(h) $\chi_o(\mathcal{P}_k) = 5$ *for* $k \geq 12$. *[42] [6]*

To prove the lower bound of Theorem 3.15(a) Marshall [33] showed that there is no oriented graph on 17 vertices that bounds the family of planar graphs (this suffice due to Proposition 3.11). The proof is quiet novel and complicated. To prove the upper bound of Theorem 3.15(a) Raspaud and Sopena [51] used Theorem 3.12 and the famous theorem of Borodin [4] (see Chapter 2, Theorem 2.11) that proves that every planar graph has an acyclic 5-coloring, to show that every oriented planar graph admits a homomorphism to the Zielonka graph $\overrightarrow{Z}_5$ of order 5.

For proving the lower bound of Theorem 3.15(b), Ochem used an oriented planar graph of girth 4 with oriented chromatic number 10 to construct a bigger graph which does not admit a homomorphism to any oriented graph on 10 vertices. Ochem used a computer check to prove his result. However, it is easy to note that the oriented planar graph of girth 4 with oriented chromatic number 10 used by Ochem for his proof also has oriented relative clique number 10 which actually proves the lower bound of Theorem 3.18(b).

To prove the lower bounds of Theorem 3.15(c) and (d) Marshall showed that no oriented graph of order 6 bounds the family of oriented planar graphs with girth 6 (this suffice due to Proposition 3.11). The proof is highly non-trivial, while our example of the graph depicted in Fig. 3.5 independently proves Theorem 3.15(c) without much difficulty. For the upper bound of

Theorem 3.15(c) Pinlou proved that the Tromp graph $\overrightarrow{T}_{16}$ bounds the family of oriented planar graphs with girth at least 5.

All the other upper bounds are proved using oriented graphs that bound certain families of graphs while the lower bounds are proved by constructing examples.

3.3.2 Oriented relative clique number

The bounds in the above theorem are difficult to narrow. Hence it is natural to consider "easier" parameters, such as the oriented relative clique number and the oriented absolute clique number and try to figure out the bounds in these cases. In fact, we managed to obtain tight bounds for the oriented absolute clique number of the families of planar graphs and of planar graphs with given girth. After that we tried the "more difficult" problem of determining the oriented relative clique number of the families of planar graphs and of planar graphs with given girth. Before listing the best bounds obtained for the above mentioned problem, we present a general bound for the oriented relative clique number of graphs with maximum degree Δ.

Proposition 3.16. *Every oriented graph with maximum degree Δ has oriented relative clique number at most $\Delta^2 + 1$.*

Proof. Let $\overrightarrow{G}$ be an oriented graph with maximum degree Δ. Let R be a relative clique of maximum order in $\overrightarrow{G}$. Let $v \in R$ be a vertex. Now, v can have at most Δ adjacent vertices and each of these vertices can have at most $(\Delta - 1)$ adjacent vertices excluding v. Hence, there can be at most $\Delta.(\Delta - 1)$ vertices of $\overrightarrow{G}$ at directed distance 2 from v. As every vertex, other than v, in R is at most at directed distance 2 from v, we have,

$$\begin{aligned}
|R| &\leq |\{v\}| + |\{\text{vertices adjacent to } v\}| \\
&\qquad + |\{\text{vertices at directed distance 2 from } v\}| \\
&\leq 1 + \Delta + \Delta.(\Delta - 1) \\
&= \Delta^2 + 1.
\end{aligned}$$

Hence, we are done. □

We consider the problem of determining the oriented relative clique number of the families of outerplanar graphs and of outerplanar graphs with given girth. We list the related results below.

Theorem 3.17.

(a) $\omega_{ro}(\mathcal{O}_3) = 7$.

(b) $\omega_{ro}(\mathcal{O}_4) = 5$.

(c) $\omega_{ro}(\mathcal{O}_5) = 5$.

(d) $\omega_{ro}(\mathcal{O}_k) = 3$ *for* $k \geq 6$.

Finally, we list the bounds for the oriented relative clique number of the families of planar graphs and of planar graphs with given girth below.

Theorem 3.18.

(a) $15 \leq \omega_{ro}(\mathcal{P}_3) \leq 80$.

(b) $10 \leq \omega_{ro}(\mathcal{P}_4) \leq 26$.

(c) $\omega_{ro}(\mathcal{P}_5) = 6$.

(d) $\omega_{ro}(\mathcal{P}_6) = 4$.

(e) $\omega_{ro}(\mathcal{P}_k) = 3$ *for* $k \geq 7$.

Proof of Theorem 3.17

(a) The example constructed by Sopena [57] to prove the lower bound of Theorem 3.14(a) is an outerplanar oclique of order 7. Hence, our result directly follows from Theorem 3.14(a), using Lemma 3.7.

(b) The lower bound follows from the fact that the directed 5-cycle is an oclique.

Let $\overrightarrow{G}$ be a triangle-free outerplanar oriented graph of minimum order with $\omega_{ro}(\overrightarrow{G}) > 5$. Let R be a relative clique of maximum order of $\overrightarrow{G}$ and let $S = V(\overrightarrow{G}) \setminus R$.

Claim 1: For any $v \in V(\overrightarrow{G})$ we have, $|N^{\alpha}(v) \cap R| \leq 2$ for $\alpha \in \{+, -\}$.

Proof of Claim 1: Let $v \in V(\overrightarrow{G})$ and $N^{\alpha}(v) \cap R = \{v_1, ..., v_k\}$ with $k \geq 3$. Fix an outerplanar embedding $\overrightarrow{G}$. Assume without loss of generality that the vertices $v_1, v_2, ..., v_k$ are arranged around v in a clockwise order in the embedding. Clearly $v_1, v_2, ..., v_k$ are pairwise non-adjacent vertices as the graph $\overrightarrow{G}$ is triangle-free. Now, as $v_1, v_2, v_3 \in R$, pairwise they should be at directed distance at most 2. Let, v_1 and v_2 be connected by a 2-dipath using the internal vertex v_{12}. Now note that v_3 is non-adjacent to v_{12} as otherwise v, v_{12}, v_1, v_2, v_3 together will induce a complete bipartite subgraph $K_{2,3}$ in $\overrightarrow{G}$, which is not possible as $\overrightarrow{G}$ is an outerplanar graph. With a similar arguement, we can conclude that there are three distinct vertices v_{12}, v_{23} and v_{13} such that for $1 \leq i < j \leq 3$, v_i and v_j are connected by a 2-dipath using the internal vertex v_{ij}. Now the complete bipartite graph $K_{2,3}$ is a minor of the graph induced by the vertices $v, v_{12}, v_{23}, v_{13}, v_1, v_2, v_3$, which is a contradiction as $\overrightarrow{G}$ is outerplanar. $\Diamond$

Let $v \in R$ then as $\overrightarrow{G}$ is connected (because of minimality), clearly $d(v) \geq 1$.

Assume, $d(v) = 1$ and without loss of generality assume that $N(v) = N^+(v) = \{u\}$. Then every $w \in R \setminus \{u, v\}$ will be connected to v by a 2-dipath with the internal vertex u. As $|R| \geq 6$ we have, $|N^+(u) \cap R| \geq 4$ which is a contradiction by Claim 1. Hence, $d(v) \geq 2$.

Assume, $d(v) = 2$ and $N(v) = \{u_1, u_2\}$. Then every $w \in R \setminus \{u_1, u_2, v\}$ will disagree with v either on u_1 or on u_2.

Case 1: Let $u_1, u_2 \in S$. Then $|R \setminus \{u, v\}| \geq 5$. Hence, at least three vertices of R disagree with v either on u_1 or on u_2 which contradicts Claim 1. Hence, at least one of u_1, u_2 is in R.

Case 2: First assume, without loss of generality, $u_1 \in S$. Then by Case 1, $u_2 \in R$. Hence we will have at least two vertices $w_1, w_2 \in R$ that disagree with v on u_1 as $|R \setminus \{v, u_1, u_2\}| \geq 4$.

Now assume, $u_1, u_2 \in R$. Then $|R \setminus \{v, u_1, u_2\}| \geq 3$ and hence Claim 1 implies that we will have at least two vertices $w_1, w_2 \in R$ that disagree with v either on u_1 or on u_2.

So, we can assume, without loss of generality, that $u_2, w_1, w_2 \in R$ and that w_1, w_2 disagree with v on u_1. As $\overrightarrow{G}$ is triangle-free, w_1, w_2 are non-adjacent. Now, w_1 and w_2 cannot be both adjacent to u_2 as $\overrightarrow{G}$ is outerplanar and the edges w_1u_2 and w_2u_2 will create the complete bipartite graph $K_{2,3}$ which is a contradiction. Hence, w_1 and w_2 are both adjacent to some vertex $w \notin \{u_1, u_2, v\}$ to have directed distance at most 2 between themselves. We also require $\overrightarrow{d}(u_2, w_1) \leq 2$ and $\overrightarrow{d}(u_2, w_2) \leq 2$. We cannot have the edge u_2w as it creates a subdivision of the complete bipartite graph $K_{2,3}$ which is a contradiction. If we try to connect w_i and u_2 through an edge or by a 2-dipath with internal vertex y_i ($y_1 = y_2$ is possible) for $i \in \{1, 2\}$, we will create a subdivision of the complete graph K_4 in $\overrightarrow{G}$ which is a contradiction.

So, the above case analysis implies that $d(v) \geq 3$ for any $v \in R$.

Now note that, for any $z \in S$, we have $d(z) \geq 2$ as otherwise the vertex z does not help connecting any two (or more) vertices of R by a 2-dipath with the internal vertex being itself (that is, z) and hence can be deleted to get an oriented outerplanar triange-free graph with the relative oriented chromatic number equal to that of $\overrightarrow{G}$ but with order less that $\overrightarrow{G}$ which contradicts the minimality of $\overrightarrow{G}$. Now, a degree 2 vertex z of S must connect two vertices of R by a 2-dipath with the internal vertex being itself (that is, z).

Now delete each degree 2 vertex of S and connect its neighbors with an edge (there was no edge between them as $\overrightarrow{G}$ is triangle-free). Note that this graph is also outerplanar (may not be triangle-free) and that vertices of the new graph have the same degree in the new graph as they had in $\overrightarrow{G}$. We know that every outerplanar graph has at least one vertex of degree 2. That vertex has to be a vertex from R by the construction of the new graph. Hence, there was a vertex of degree 2 in $\overrightarrow{G}$ that belonged to R. But this gives a contradiction.

(c) Note that a directed cycle of length 5 is an oclique. Hence, using Theorem 3.14(c), our result follows using Lemma 3.7.

(d) The 2-dipath is an oclique of order 3 and contains no cycle. This gives us the lower bound.

For the upper bound, assume $\overrightarrow{G}$ is an oriented outerplanar graph with girth at least 6 such that $\omega_{ro}(\overrightarrow{G}) > 3$. Let R be a relative oclique of maximum order in $\overrightarrow{G}$. Now, it is easy to check that it is not possible to pairwise connect any four vertices of R with an arc or a 2-dipath without creating a cycle of length at most 5 or a subdivision of K_4 or a subdivision of $K_{2,3}$. □

Proof of Theorem 3.18

(a) The upper bound follows from Theorem 3.15(a) and the lower bound follows from Theorem 3.21(a), which proves $\omega_{ao}(\mathcal{P}_3) = 15$, using Lemma 3.7.

(b) The lower bound follows from the triangle-free planar oclique of order 10 constructed by Ochem [43] for proving the lower bound of Theorem 3.15(b).

Let $\overrightarrow{G}$ be a triangle-free planar oriented graph of minimum order with $\omega_{ro}(\overrightarrow{G}) > 26$. Let R be a relative clique of maximum order of $\overrightarrow{G}$ and let $S = V(\overrightarrow{G}) \setminus R$.

Claim 1: For any $v \in V(\overrightarrow{G})$ we have, $|N^{\alpha}(v) \cap R| \leq 4$ for $\alpha \in \{+, -\}$.

Proof of claim 1: Let $v \in V(\overrightarrow{G})$ and $N^{\alpha}(v) \cap R = \{v_1, ..., v_k\}$ with $k \geq 5$. Fix a planar embedding $\overrightarrow{G}$. Assume, without loss of generality, that the vertices $v_1, v_2, ..., v_k$ are arranged around v in a clockwise order in the embedding. Clearly $v_1, v_2, ..., v_k$ are pairwise non-adjacent vertices as the graph $\overrightarrow{G}$ is triangle-free. Now, as $v_1, ..., v_5 \in R$, pairwise they should be at directed distance at most 2. Hence, each pair of vertices v_i and v_j must be connected by a 2-dipath using an internal vertex $v_{ij} \neq v$ (it is not necessary to have these vertices all distinct from each other), for $1 \leq i < j \leq k$.

Suppose $v_{15} \neq v_{24}$. Now, without loss of generality, also suppose that the 2-dipath is $\overrightarrow{v_2 v_{24} v_4}$. Now, we must have $v_{24} = v_{13}$ and $v_{24} = v_{35}$ to keep the graph planar. Also we will have $v_{24} = v_{14}$ and $v_{24} = v_{25}$ to keep the graph planar.

Now, $v_{24} = v_{14}$ implies the arc $\overrightarrow{v_1 v_{24}}$. This and $v_{24} = v_{13}$ imples the arc $\overrightarrow{v_{24} v_3}$.

Similarly, $v_{24} = v_{25}$ implies the arc $\overrightarrow{v_{24} v_5}$. This and $v_{24} = v_{35}$ imples the arc $\overrightarrow{v_3 v_{24}}$.

But $\overrightarrow{G}$ is an oriented graph, hence cannot have both the arcs $\overrightarrow{v_{24} v_3}$ and $\overrightarrow{v_3 v_{24}}$. Hence, we must have $v_{15} = v_{24}$. This will force $v_{15} = v_{24} = v_{13} = v_{35}$. Now, without loss of generality, suppose that the 2-dipath connecting v_1 and v_5 is $\overrightarrow{v_1 v_{15} v_5}$. This will imply the arcs $\overrightarrow{v_{15} v_3}$ and $\overrightarrow{v_3 v_{15}}$ to connect v_3 by 2-dipath with v_1 and v_5 respectively. But $\overrightarrow{G}$ is an oriented graph, hence cannot have both the arcs $\overrightarrow{v_{15} v_3}$ and $\overrightarrow{v_3 v_{15}}$. ◊

Now note that, for any $z \in S$, we have $d(z) \geq 2$ as otherwise the vertex z does not help connecting any two (or more) vertices of R by a 2-dipath with the internal vertex being itself (that is, z) and hence can be deleted to get an oriented planar triange-free graph with the relative oriented chromatic number equal to that of $\overrightarrow{G}$ but with order less that $\overrightarrow{G}$ which contradicts the minimality of $\overrightarrow{G}$. Now, a vertex z of S must connect at least two vertices of R by a 2-dipath with the internal vertex being itself (that is, z).

Now for each vertex $z \in S$ with $d(z) \leq 5$, assume that the neighbors of z are $v_1, v_2, ..., v_k$. Fix a planar embedding of $\overrightarrow{G}$ and assume that the neighbors of z are arranged in a clockwise order around z. Now delete the vertex z and add the edges v_1v_2, v_2v_3, ..., $v_{k-1}v_k$, v_kv_1 (for $d(z) = 2$ add only one edge v_1v_2) to obtain a new graph.

Note that this new graph is also planar and the degree of each vertex in the new graph, which were in $\overrightarrow{G}$ also, is as much as the degree of the corresponding vertex in $\overrightarrow{G}$. Hence, there is a vertex v in the new graph, which belongs to R and has degree at most 5.

Hence, there will be a neighbor u of v and at least 5 vertices of $R \setminus (N(v) \cup \{v\})$ which are connected to v with a 2-dipath with internal vertex u which is a contradiction to Claim 1. So, we can conclude that $\omega_{ro}(\overrightarrow{G}) \leq 26$.

(c) The lower bound follows from the oriented graph depicted in Fig. 3.5. As previously discussed, this graph has oriented relative clique number 6.

We will now prove the upper bound by contradiction.

Assume that $\overrightarrow{G}$ is an oriented planar graph with girth at least 5 of minimum order with $\omega_{ro}(\overrightarrow{G}) > 6$. Moreover assume that $\overrightarrow{G}$ is such that, if we delete any arc of $\overrightarrow{G}$, the new graph will have oriented relative chromatic number at most 6. Let R be an oriented relative clique of maximum order of $\overrightarrow{G}$ and let $S = V(\overrightarrow{G}) \setminus R$. Note that S induces an independent set of $\overrightarrow{G}$ as deleting any arc between two vertices of S will not decrease the oriented relative clique number of the graph $\overrightarrow{G}$.

First note that, for any $z \in S$, we have $d(z) \geq 2$ as otherwise the vertex z does not help connecting any two (or more) vertices of R by a 2-dipath with the internal vertex being itself (that is, z) and hence can be deleted to get an oriented planar graph with girth at least 5 and with relative oriented chromatic number equal to that of $\overrightarrow{G}$ but with order less that $\overrightarrow{G}$ which contradicts the minimality of $\overrightarrow{G}$.

Also, for any $z \in V(\overrightarrow{G})$, we must have $|N(z) \cap R| \leq 2$. If not then we will have $|N^{\alpha}(z) \cap R| \geq 2$ for some $\alpha \in \{+, -\}$. Now to have directed distance at most 2 between two vertices of $N^{\alpha}(z) \cap R$, there should be a 2-dipath joining the two vertices. This will create a cycle of length 4, which is a contradiction. Hence for any $z \in S$ we have, $d(z) = 2$ and that z must be an internal vertex of a 2-dipath with two terminal vertices from R.

As oriented forests have an oriented 3-coloring [58], $\overrightarrow{G}$ must have a cycle. Also, $\overrightarrow{G}$ must contain a cycle of length 5 with edges ab, bc, cd, de and ea, using Theorem 3.18(c) and (d) whose proofs are independent from this proof. As S is an independent set in $\overrightarrow{G}$, we can have at most 2 vertices (which should be non-adjacent) of the cycle from S.

First assume that all the vertices of the cycle are from R. Then any $w \in R \setminus \{a, b, c, d, e\}$ is not adjacent to any of a, b, c, d, e as otherwise we will have a vertex with at least three neighbors from R which is not possible. Hence, w is connected to each of a, b, c, d, e with a 2-dipath with internal vertices from S. Now, we have at least two such vertices in R, say, w and u as $|R \setminus \{a, b, c, d, e\}| \geq 2$. Now, w and u are either adjacent or connected by a 2-dipath with the internal vertex different from a, b, c, d, e. Note that this will create a subdivision of the complete graph K_5 which is a contradiction as $\overrightarrow{G}$ is planar.

Hence, R does not induce any cycle of length 5 in $\overrightarrow{G}$.

Now assume, there is a cycle of length 5, with edges ab, bc, cd, de and ea, in $\overrightarrow{G}$ with four vertices a, b, c, d from R. But, at most one vertex from $R \setminus \{a, b, c, d\}$ can be adjacent to a and at most one vertex from $R \setminus \{a, b, c, d\}$ can be adjacent to d, while no vertex from $R \setminus \{a, b, c, d\}$ can be adjacent to b or c because of the degree restrictions proved before. As $|R \setminus \{a, b, c, d\}| \geq 3$ there is a $w \in R$ such that w is non-adjacent to each of a, b, c, d, e. Now, w is connected to each of a, b, c, d with a 2-dipath.

Now assume that at least one internal vertex is from R. Without loss of generality we can assume that the internal vertex u connecting w and a by a 2-dipath is from R. Now, u cannot be adjacent to d, b or c (this will create a cycle of length less than 5). Hence, u must be connected by a 2-dipath with c. But this creates a subdivision of the complete bipartite graph $K_{3,3}$ which contradicts the planarity of $\overrightarrow{G}$.

Hence, all the internal vertices connecting w to of a, b, c, d with 2-dipaths are from S.

If there is another vertex $u \neq w$ in $R \setminus \{a, b, c, d\}$ such that u is neither adjacent to a nor to d then u must be connected to a, b, c, d by 2-paths with internal vertices from S. Also, w and u will be either adjacent or connected by a 2-dipath. This will create a subdivision of the complete graph K_5 which is a contradiction as $\overrightarrow{G}$ is planar. Hence, any vertex $u \in R \setminus \{a, b, c, d, w\}$ must be adjacent either to a or to d. Without loss of generality we may assume that u is adjacent to a. Now, u cannot be adjacent to w, d, b or c (this will create a cycle of length less than 5). Hence, u must be connected by a 2-dipath with w, c and d. But this creates a subdivision of the complete graph K_5 which contradicts the planarity of $\overrightarrow{G}$.

Hence, there is no cycle of length 5 in $\overrightarrow{G}$ with four vertices of it from R.

So, there is a cycle of length 5 in $\overrightarrow{G}$, with edges ab, bc, cd, de and ea, in $\overrightarrow{G}$ with three vertices, without loss of generality, a, c, d (no two vertices from S can be adjacent) from R. At most two vertices from $R \setminus \{a, c, d\}$ can be adjacent to a. Hence there is at least two vertices, say, $u, v \in R \setminus \{a, c, d\}$ which are non-adjacent to a. The vertices u, v cannot be adjacent to c or d as well, because otherwise it will create a cycle of length less than 4 to have $\overrightarrow{d}(t, t') \leq 2$ for $t \in \{u, v\}$ and $t' \in \{c, d\}$. Hence, both u and v are connected to a, c, d by 2-dipaths. Also, u and v are either adjacent or connected by a 2-dipath. But then this creates a subdivision of the complete graph K_5 which contradicts the planarity of $\overrightarrow{G}$.

(d) The lower bound follows from the oriented graph obtained by connecting a vertex with 2-dipaths to three independent vertices of a directed 6-cycle. It is easy to check that this oriented graph has girth 6 and oriented relative clique number 4.

We will now prove the upper bound by contradiction.

Assume that $\overrightarrow{G}$ is a planar oriented graph with girth at least 6 of minimum order with $\omega_{ro}(\overrightarrow{G}) > 5$. Moreover assume $\overrightarrow{G}$ is such that, if we delete any arc of $\overrightarrow{G}$, the new graph will have oriented relative chromatic number at most 5. Let R be an oriented relative clique of maximum order of $\overrightarrow{G}$. Let, $S = V(\overrightarrow{G}) \setminus R$. Note that S induces an independent set of $\overrightarrow{G}$ as deleting any arc between two vertices of S will not decrease the oriented relative clique number of the graph $\overrightarrow{G}$.

First note that, for any $z \in S$, we have $d(z) \geq 2$ as otherwise the vertex z does not help connecting any two (or more) vertices of R by a 2-dipath with the internal vertex being itself (that is, z) and hence can be deleted to get an oriented planar graph with girth at least 6 and with relative oriented chromatic number equal to that of $\overrightarrow{G}$ but with order less that $\overrightarrow{G}$ which contradicts the minimality of $\overrightarrow{G}$.

Also, for any $z \in V(\overrightarrow{G})$, we must have $|N(z) \cap R| \leq 2$. If not, then we will have $|N^{\alpha}(z) \cap R| \geq 2$ for some $\alpha \in \{+, -\}$. Now to have directed distance at most 2 between two vertices of $N^{\alpha}(z) \cap R$, there should be a 2-dipath joining the two vertices. This will create a cycle of length 4, which is a contradiction. Hence for any $z \in S$ we have, $d(z) = 2$ and that z must be an internal vertex of a 2-dipath with two terminal vertices from R.

As every oriented forest admit an oriented 3-coloring [58], $\overrightarrow{G}$ must have a cycle. Also, $\overrightarrow{G}$ must contain a cycle $abcdef$ of length 6 (that is, the edges of the cycle are ab, bc, cd, de, ef and fa) using Theorem 3.18(d) whose proofs are independent from this proof. As S is an independent set in $\overrightarrow{G}$, we can have at most three vertices (which should be non-adjacent) of the cycle from S.

As the vertices of S are non-adjacent, without loss of generality, we can assume that $a, c, e \in R$. If another vertex of R is in the cycle, then, without loss of generality, we can assume it to be d. If so, then d and a must be connected by a 2-dipath or be adjacent which creates a cycle of length less than 6, hence is not possible. Therefore, exactly three vertices of the cycle are from R.

Now, for any $w \in R$, we have w connected by a 2-dipath to each of a, c, e. If we have two such vertices w and x, which are adjacent or connected by a 2-dipath, it will create a subdivision of the complete graph K_5 in $\overrightarrow{G}$ which contradicts the planarity of $\overrightarrow{G}$.

(e) The lower bound follows from the fact that a 2-dipath is an oclique of order 3.

It is easy to check that it is not possible to construct an oriented graph with girth at least 7 in which at least 4 vertices are at directed distance at most 2 keeping the graph planar. $\square$

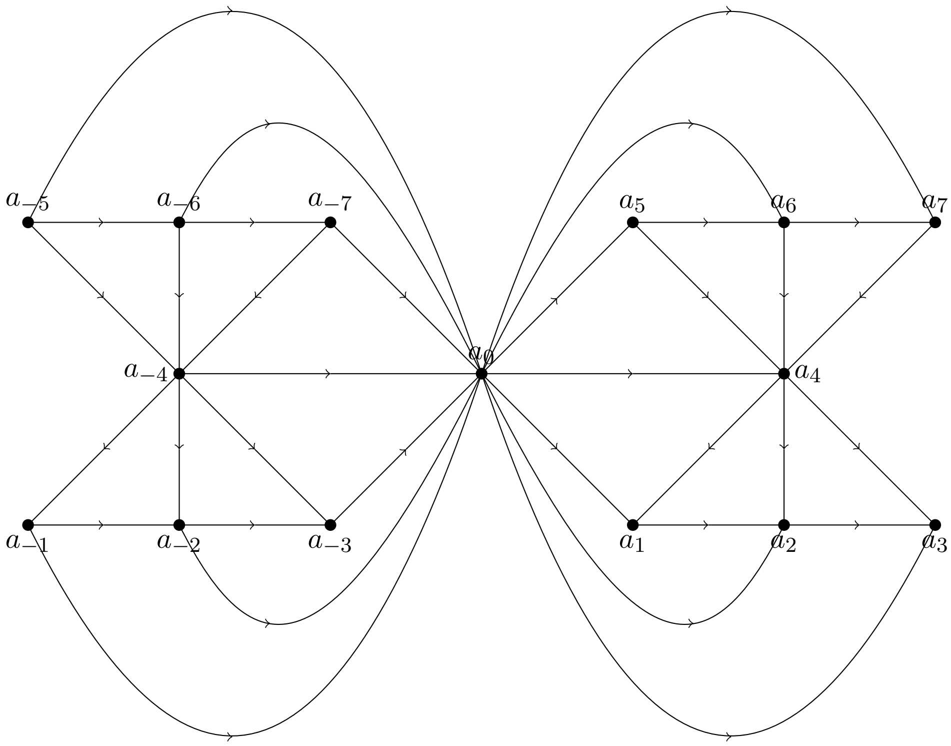


Figure 3.6: Planar oclique of order 15.

3.3.3 Oriented absolute clique number

The questions related to the oriented absolute clique number have been first asked by Klostermeyer and MacGillivray [27] in 2002. In their paper they asked: "what is the maximum order of a planar oclique?" which is equivalent to asking "what is the oriented absolute clique number of planar graphs?" In order to find the answer to this question, Sopena [59] found a planar oclique of order 15 while Klostermeyer and MacGillivray [27] showed that there is no planar oclique of order more than 36 and conjectured that the maximum order of a planar oclique is 15. Later in 2011, we positively settled that conjecture [55] and will state it as Theorem 3.21(a) in this section. Later, in a joint effort with Sopena, we proved the following stronger result which implies Theorem 3.21(a) and proves a uniqueness property of planar ocliques of maximum order.

Theorem 3.19. *A planar oclique has order at most 15 and every planar oclique of order 15 contains the planar oclique $\overrightarrow{P}$ depicted in Fig. 3.6 as a subgraph.*

The similar question for planar ocliques with given girth is also of interest and was asked by Klostermeyer and MacGillivray [27]. We answered these questions and provided tight bounds for the problem. Before stating the results concerning the oriented absolute chromatic number for the families of planar graphs and of planar graphs with given girth, we state the similar results for the families of outerplanar graphs and of outerplanar graphs with given girth.

Theorem 3.20.

(a) $\omega_{ao}(\mathcal{O}_3) = 7$.

(b) $\omega_{ao}(\mathcal{O}_4) = 5$.

(c) $\omega_{ao}(\mathcal{O}_5) = 5$.

(d) $\omega_{ao}(\mathcal{O}_k) = 3$ *for* $k \geq 6$.

The proof of this theorem clearly follows from the proof of Theorem 3.14 and Theorem 3.17.

Theorem 3.21.

(a) $\omega_{ao}(\mathcal{P}_3) = 15$.

(b) $\omega_{ao}(\mathcal{P}_4) = 6$.

(c) $\omega_{ao}(\mathcal{P}_5) = 5$.

(d) $\omega_{ao}(\mathcal{P}_k) = 3$ *for* $k \geq 6$.

Proof of Theorem 3.19

Goddard and Henning [18] (see Chapter 2, Theorem 2.15) proved that every planar graph of diameter 2 has domination number at most 2 except for a particular graph on nine vertices.

Let $\overrightarrow{B}$ be a planar oclique dominated by the vertex v. Sopena [58] showed that any oriented outerplanar graph has an oriented 7-coloring. Hence let c be an oriented 7-coloring of the oriented outerplanar graph obtained from $\overrightarrow{B}$ by deleting the vertex v. Now for $u \in N^{\alpha}(v)$ let us assign the color $(c(u), \alpha)$ to u for $\alpha \in \{+, -\}$ and the color 0 to v. It is easy to check that this is an oriented 15-coloring of $\overrightarrow{B}$. Hence any planar oclique dominated by one vertex has order at most 15.

Lemma 3.22. *Let $\overrightarrow{H}$ be a planar oclique of order 15 dominated by one vertex. Then $\overrightarrow{H}$ contains the planar oclique depicted in Fig. 3.6 as a subgraph.*

Proof. Suppose $\overrightarrow{H}$ is a triangulated planar oclique of order 15 dominated by one vertex v. Now, note that $N^{\alpha}(v)$ is an oriented relative clique in $\overrightarrow{H}[N(v)]$ (that is, the oriented subgraph of $\overrightarrow{H}$ induced by the neighbors of v which is actually the oriented graph obtained by deleting the vertex v from $\overrightarrow{H}$) for any $\alpha \in \{+, -\}$. Also note that $\overrightarrow{H}[N(v)]$ is an outerplanar graph. Hence by Theorem 3.17(a) we have $|N^{\alpha}(v)| \leq 7$ for any $\alpha \in \{+, -\}$. But we also have,

$$|N^{+}(v)| + |N^{-}(v)| = 14.$$

Hence we have,

$$|N^{+}(v)| = |N^{-}(v)| = 7.$$

Now assume $N(v) = \{x_1, x_2, ..., x_{14}\}$. Moreover fix a planar embedding of $\overrightarrow{H}$ and without loss of generality assume that the vertices $x_1, x_2, ..., x_{14}$ are arranged in a clockwise order around v. The triangulation of $\overrightarrow{H}$ forces the edges $x_1x_2, x_2x_3, ..., x_{13}x_{14}, x_{14}x_1$. Now we know from the above discussion that there should be two disjoint oriented relative cliques $N^{+}(v)$ and $N^{-}(v)$, each of order 7, in the outerplanar graph $\overrightarrow{H}[N(v)]$. We already have the cycle $x_1x_2...x_{14}$ forced in the outerplanar graph $\overrightarrow{H}[N(v)]$. We will now prove some more structural properties of $\overrightarrow{H}[N(v)]$.

As $\overrightarrow{H}[N(v)]$ is an outerplanar graph, it must have at least two vertices of degree at most 2. As every vertex of the graph is part of a cycle, there is no vertex of degree at most 1. Hence there is at least two vertices of degree exactly 2 in $\overrightarrow{H}[N(v)]$.

Without loss of generality assume that $d_{\overrightarrow{H}[N(v)]}(x_2) = 2$ and $x_2 \in N^{\alpha}(v)$ for some fixed $\alpha \in \{+, -\}$. By triangulation we must have the edge x_1x_3. Now the vertices of $N^{\alpha}(v) \setminus \{x_1, x_2, x_3\}$ must be connected to x_2 by 2-dipaths with internal vertex either x_1 or x_3.

Let four vertices of $N^{\alpha}(v) \setminus \{x_1, x_2, x_3\}$ be connected to x_2 by 2-dipaths with internal vertex x_1. Then there will be two vertices, among the above mentioned four vertices, at directed distance at most 3 which is a contradiction. So at most three vertices of $N^{\alpha}(v) \setminus \{x_1, x_2, x_3\}$ can be connected to x_2 by 2-dipaths with internal vertex x_1. Similarly we can show that at most

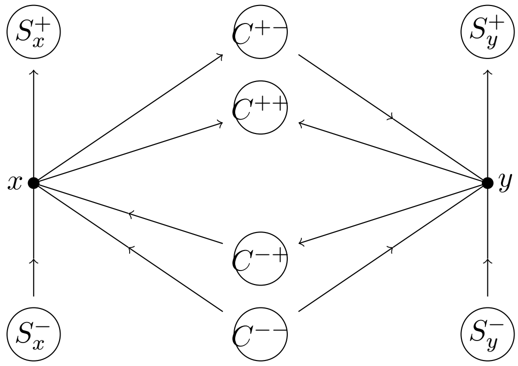


Figure 3.7: Structure of $\vec{G}$ (not a planar embedding)

three vertices of $N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ can be connected to x_2 by 2-dipaths with internal vertex x_3.

Now suppose there are at least two vertices $x_i, x_j \in N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ that are connected to x_2 by 2-dipaths with internal vertex x_1 and there are at least two vertices $x_k, x_l \in N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ that are connected to x_2 by 2-dipaths with internal vertex x_3.

Notice that, as the graph $\overrightarrow{H}$ is planar, with the given planar embedding of $\overrightarrow{H}$ we must have $i, j > k, l$. Now, without loss of generality, we can assume that $i > j$ and $k > l$. Now it will be impossible to have directed distance at most 2 between x_i and x_l keeping the graph $\overrightarrow{H}$ planar. So, at least one of the vertices between x_1, x_3 must be the internal vertex of at most one 2-dipath connecting a vertex of $N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ to x_2.

Now if at least one vertex between x_1 and x_3 is from $N^{\overline{\alpha}}(v)$, then we will have,

$$|N^\alpha(v) \setminus \{x_1, x_2, x_3\}| \geq 5.$$

But then by the above discussion we will have a contradiction (there will be at least two vertices of $N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ connected by 2-dipaths with internal vertex x_1 and at least two vertices of $N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ connected by 2-dipaths with internal vertex x_3).

Hence we must have $x_1, x_3 \in N^\alpha(v)$. Without loss of generality we have three vertices $x_i, x_j, x_k \in N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ connected by 2-dipaths with internal vertex x_1 and 1 vertex $x_l \in N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ connected by a 2-dipath with internal vertex x_3. Without loss of generality we can assume $i > j > k > l$.

Now to have $\overrightarrow{d}(x_i, x_s) \leq 2$ for $s \in \{2, 3, l\}$, the vertices x_2, x_3, x_l must disagree with x_i on x_1. Also to have $\overrightarrow{d}(x_i, x_k)$, $\overrightarrow{d}(x_2, x_l) \leq 2$ we must have the 2-dipaths $x_i x_j x_k$ and $x_2 x_3 x_l$. But then the induced oriented graph $\overrightarrow{H}[N^\alpha(v)]$ contains the oriented graph induced by $N^\alpha(a_0)$ of the planar oclique depicted in Fig. 3.6.

Further notice that no vertex of $N^\alpha(v)$, other than x_2, has degree 2 in $\overrightarrow{H}[N(v)]$. Hence we can infer that a vertex of $N^\alpha(v)$ has degree 2 in $\overrightarrow{H}[N(v)]$. That will imply that the induced oriented graph $\overrightarrow{H}[N^{\overline{\alpha}}(v)]$ contains the oriented graph induced by $N^{\overline{\alpha}}(a_0)$ of the planar oclique depicted in Fig. 3.6.

Hence the planar oclique depicted in Fig. 3.6 is a subgraph of $\overrightarrow{H}$. It is easy to check that, regardless of the choice of $\overrightarrow{H}$ (it is a triangulation of the planar oclique depicted in Fig. 3.6), if we delete one arc of the oriented subgraph, isomorphic to the planar oclique depicted in Fig. 3.6, of $\overrightarrow{H}$, the oriented graph $\overrightarrow{H}$ does not longer remain an oclique.

This proves the lemma. □

Now, to prove Theorem 3.19, it will be enough to prove that every planar oclique of order at least 15 must have domination number 1. In other words, it will be enough to prove that any planar oclique with domination number 2 must have order at most 14. More precisely, we need to prove the following lemma.

Lemma 3.23. *Let $\overrightarrow{H}$ be a planar oclique with domination number 2. Then $|\overrightarrow{H}| \leq 14$.*

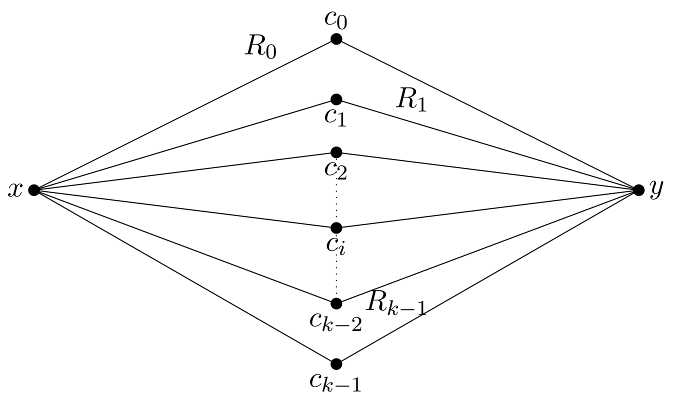


Figure 3.8: A planar embedding of $und(\vec{H})$

Let $\overrightarrow{G}$ be a planar oclique with $|\overrightarrow{G}| > 14$. Assume that $\overrightarrow{G}$ is triangulated and has domination number 2.

We define the partial order $\prec$ for the set of all dominating sets of order 2 of $\overrightarrow{G}$ as follows: for any two dominating sets $D = \{x, y\}$ and $D' = \{x', y'\}$ of order 2 of $\overrightarrow{G}$, $D' \prec D$ if and only if $|N(x') \cap N(y')| < |N(x) \cap N(y)|$.

Let $D = \{x, y\}$ be a maximal dominating set of order 2 of $\overrightarrow{G}$ with respect to $\prec$. Also for the rest of this thesis, $t, t', \alpha, \overline{\alpha}, \beta, \overline{\beta}$ are variables satisfying $\{t, t'\} = \{x, y\}$ and $\{\alpha, \overline{\alpha}\} = \{\beta, \overline{\beta}\} = \{+, -\}$.

Now, we fix the following notations (Fig: 3.7):

$$C = N(x) \cap N(y),\ C^{\alpha\beta} = N^{\alpha}(x) \cap N^{\beta}(y),\ C_t^{\alpha} = N^{\alpha}(t) \cap C,$$
$$S_t = N(t) \setminus C,\ S_t^{\alpha} = S_t \cap N^{\alpha}(t),\ S = S_x \cup S_y.$$

Hence we have,

$$15 \leq |\overrightarrow{G}| = |D| + |C| + |S|. \tag{3.1}$$

Let $\overrightarrow{H}$ be the oriented graph obtained from the induced subgraph $\overrightarrow{G}[D \cup C]$ of $\overrightarrow{G}$ by deleting all the arcs between the vertices of D and all the arcs between the vertices of C. Note that it is possible to extend the planar embedding of $und(\overrightarrow{H})$ given in Fig 3.8 to a planar embedding of $und(\overrightarrow{G})$ for some particular ordering of the elements of, say $C = \{c_0, c_1, ..., c_{k-1}\}$.

Notice that $und(\overrightarrow{H})$ has k faces, namely the unbounded face F_0 and the faces F_i bounded by edges $xc_{i-1}, c_{i-1}y, yc_i, c_ix$ for $i \in \{1, ..., k-1\}$. Geometrically, $und(\overrightarrow{H})$ divides the plane into k connected components. The *region* R_i of $\overrightarrow{G}$ is the i^{th} connected component (corresponding to the face F_i) of the plane. *Boundary points* of a region R_i are c_{i-1} and c_i for $i \in \{1, ..., k-1\}$ and, c_0 and c_{k-1} for $i = 0$. Two regions are *adjacent* if they have at least one common boundary point (hence, a region is adjacent to itself).

Now for the different possible values of $|C|$, we want to show that $und(\overrightarrow{H})$ cannot be extended to a planar oclique of order at least 15. Note that, for extending $und(\overrightarrow{H})$ to $\overrightarrow{G}$, we can add new vertices only from S. Any vertex $v \in S$ will be inside one of the regions R_i. If there is at least one vertex of S in a region R_i, then R_i is *non-empty* and *empty* otherwise. In fact, when there is no chance of confusion, R_i might represent the set of vertices of S contained in the region R_i.

As any two distinct non-adjacent vertices of $\overrightarrow{G}$ must be connected by a 2-dipath, we have the following three lemmas:

Lemma 3.24. *(a) If $(u, v) \in S_x \times S_y$ or $(u, v) \in S_t^{\alpha} \times S_t^{\alpha}$, then u and v are in adjacent regions.*

(b) If $(u, c) \in S_t^{\alpha} \times C_t^{\alpha}$, then c is a boundary point of a region adjacent to the region containing u.

Lemma 3.25. *Let R, R^1, R^2 be three distinct regions such that R is adjacent to R^i with common boundary point c^i while the other boundary points of R^i is $\overline{c^i}$ for all $i \in \{1, 2\}$. If $v \in S_t^{\alpha} \cap R$ and*

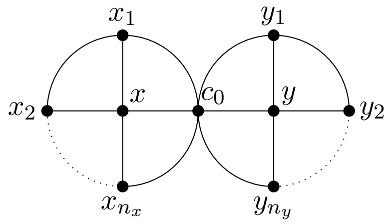


Figure 3.9: For $|C| = 1$ while x and y are non-adjacent

$u^i \in ((S_t^\alpha \cup S_{t'}) \cap R^i) \cup (\{\overline{c^i}\} \cap C_t^\alpha)$, then v disagrees with u^i on c^i, where $i \in \{1, 2\}$. If both u^1 and u^2 exist, then $|S_t^\alpha \cap R| \leq 1$.

Lemma 3.26. *For any arc $\overrightarrow{uv}$ in $\overrightarrow{G}$, we have $|N^\alpha(u) \cap N^\beta(v)| \leq 3$.*

Now we ask the question "How small $|C|$ can be?" and try to prove possible lower bounds of $|C|$. The first result regarding the lower bound of $|C|$ is proved below.

Lemma 3.27. $|C| \geq 2$.

Proof. We know that x and y are either connected by a 2-dipath or by an arc. If x and y are adjacent, then as $\overrightarrow{G}$ is triangulated, we have $|C| \geq 2$. If x and y are non-adjacent, then $|C| \geq 1$. Hence it is enough to show that we cannot have $|C| = 1$ while x and y are non-adjacent.

If $|C| = 1$ and x and y are non-adjacent, then the triangulation will force the configuration depicted in Fig 3.9 as a subgraph of $und(\overrightarrow{G})$, where $C = \{c_o\}$, $S_x = \{x_1, ..., x_{n_x}\}$ and $S_y = \{y_1, ..., y_{n_y}\}$. Without loss of generality we may assume $|S_y| \geq |S_x|$. Then by equation (3.1) we have,

$$n_y = |S_y| \geq \lceil (15 - 2 - 1)/2 \rceil = 6.$$

Clearly $n_x \geq 3$ as otherwise $\{c_0, y\}$ is a dominating set with at least two common neighbors $\{y_1, y_{n_y}\}$ which contradicts the maximality of D.

For $n_x = 3$, we know that c_0 is not adjacent to x_2 as otherwise $\{c_0, y\}$ is a dominating set with at least two common neighbors $\{y_1, y_{n_y}\}$ contradicting the maximality of D. But then x_2 should be adjacent to y_i for some $i \in \{1, ..., n_y\}$ as otherwise $d(x_2, y) > 2$. Now the triangulation will force x_2 and y_i to have at least two common neighbors. Also x_2 cannot be adjacent to y_j for any $j \neq i$, as it will create a dominating set $\{x_2, y\}$ with at least two common neighbors $\{y_i, y_j\}$ contradicting the maximality of D. Hence, x_2 and y_i are adjacent to both x_1 and x_3. Note that t_{ℓ_t} and t_{ℓ_t+k} are adjacent if and only if $k = 1$, as otherwise $d(t_{\ell_t+1}, t') > 2$ for $1 \leq \ell_t < \ell_t + k \leq n_t$. In this case, by equation (3.1) we have,

$$n_y = |S_y| \geq 15 - 2 - 1 - 3 = 9.$$

Assume $i \geq 5$. Hence, c_0 is adjacent to y_j for all $j = 1, 2, 3$, as otherwise $d(y_j, x_3) > 2$. This implies $d(y_2, x_2) > 2$, a contradiction. Similarly $i < 5$ will also force a contradiction. Hence $n_x \geq 4$.

For $n_x = 4$, c_0 cannot be adjacent to both x_3 and $x_{n_x-2} = x_2$ as it creates a dominating set $\{c_0, y\}$ with at least two common neighbors $\{y_1, y_{n_y}\}$ contradicting the maximality of D. For $n_x \geq 5$, c_0 is adjacent to x_3 implies, either for all $i \geq 3$ or for all $i \leq 3$, x_i is adjacent to c_0, as otherwise $d(x_i, y) > 2$. Either of these cases will force c_0 to become adjacent to y_j, as otherwise we will have either $d(x_1, y_j) > 2$ or $d(x_{n_x}, y_j) > 2$ for all $j \in \{1, 2, ..., n_y\}$. But then we will have a dominating set $\{c_0, x\}$ with at least two common vertices contradicting the maximality of D. Hence for $n_x \geq 5$, c_0 is not adjacent to x_3. Similarly we can show, for $n_x \geq 5$, that c_0 is not adjacent to either x_3 or x_{n_x-2}.

So, for $n_x \geq 4$, without loss of generality we can assume that c_0 is not adjacent to x_3. We know that $d(y_1, x_3) \leq 2$. We have already noted that t_{l_t} and t_{l_t+k} are adjacent if and only if

$k = 1$ for any $0 \leq l_t < l_t + k \leq n_t$. Hence to have $d(y_1, x_3) \leq 2$, we must have one of the following edges: y_1x_2, y_1x_3, y_1x_4 or y_2x_3. The first edge will imply the edges x_2y_j as otherwise $d(x_1, y_j) > 2$ for all $j = 3, 4, 5$. These three edges will imply $d(x_4, y_3) > 2$. Hence we do not have y_1x_2.

The other three edges, assuming we cannot have y_1x_2, will force the edges x_2c_0 and x_1c_0 for having $d(x_2, y) \leq 2$ and $d(x_1, y) \leq 2$. This will imply $d(x_1, y_4) > 2$, a contradiction. Hence we cannot have the other three edges also.

Hence we are done. □

Now we will prove that, for $2 \leq |C| \leq 5$, at most one region of $\overrightarrow{G}$ can be non-empty. Later, using this result, we will improve the lower bound of $|C|$.

Lemma 3.28. *If $2 \leq |C| \leq 5$, then at most one region of $\overrightarrow{G}$ is non-empty.*

Proof. For pictorial help one can look at Fig 3.8. For $|C| = 2$, if x and y are adjacent, then the region that contains the edge xy is empty, as otherwise triangulation will force x and y to have a common neighbor other than c_0 and c_1. So for the rest of the proof we can assume x and y to be non-adjacent for $|C| = 2$.

Step 0: First we shall show that it is not possible to have either $S_x = \emptyset$ or $S_y = \emptyset$ and have at least two non-empty regions. Without loss of generality assume that $S_x = \emptyset$. Then x and y are non-adjacent, as otherwise y will be a dominating vertex which is not possible.

For $|C| = 2$, if both $S_y \cap R_0$ and $S_y \cap R_1$ are non-empty, then triangulation will force, either multiple edges c_0c_1 (one in each region) or a common neighbor of x, y other than c_0, c_1, a contradiction.

For $|C| = 3, 4$ and 5, triangulation implies the edges $c_0c_1, ..., c_{k-2}c_{k-1}, c_{k-1}c_0$. Hence every $v \in S_y$ must be connected to x by a 2-dipath through c_i for some $i \in \{1, 2, ..., k-1\}$. Now assume $|S_y^\alpha| \geq |S_y^{\overline{\alpha}}|$ for some $\alpha \in \{+, -\}$. Then by equation (3.1) we have,

$$|S_y^\alpha| \geq \lceil (15 - 2 - 5)/2 \rceil = 4.$$

Now by Lemma 3.24, we know that the vertices of S_y^α will be contained in two adjacent regions for $|C| = 4, 5$. For $|C| = 3$, $S_y^\alpha \cap R_i$ for all $i \in \{0, 1, 3\}$ implies $|S_y^\alpha| \leq 3$ by Lemma 3.25. Hence, without loss of generality, we may assume $S_y^\alpha \subseteq R_1 \cup R_2$. If both $S_y^\alpha \cap R_1$ and $S_y^\alpha \cap R_2$ are non-empty, then by Lemma 3.25, each vertex of $S_y^\alpha \cap R_1$ disagrees with each vertex of $S_y^\alpha \cap R_2$ on c_1. Then $\{c_1, y\}$ becomes a dominating set with at least six common neighbors (c_0, c_2 and four vertices from S_y^α) which contradicts the maximality of D.

Hence, all the vertices of S_y^α must be contained in one region, say R_1. Then each of them should be connected to x by a 2-dipath with internal vertex either c_0 or c_1. However, the vertices that are connected to x by a 2-dipath with internal vertex c_0 should have directed distance at most 2 with the vertices connected to x by a 2-dipath with internal vertex c_1. It is not possible to connect them unless they are all adjacent to either c_0 or c_1. But then it will contradict the maximality of D.

Hence both S_x and S_y are non-empty.

Step 1: Now we will prove that at most four sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, for all $t \in \{x, y\}$ and $i \in \{0, 1, ..., k-1\}$. It is trivial for $|C| = 2$. For $|C| = 4$ and 5, the statement follows from Lemma 3.24. For $|C| = 3$, we consider the following two cases:

(i) Assume $S_t \cap R_i \neq \emptyset$ for all $t \in \{x, y\}$ and for all $i \in \{0, 1, 2\}$. Then by Lemma 3.25 we have, $|S_t \cap R_i| \leq 1$ for all $t \in \{x, y\}$ and for all $i \in \{0, 1, 2\}$. Then by equation (3.1) we have,

$$15 \leq |\overrightarrow{G}| = 2 + 3 + 4 = 9.$$

This is a contradiction.

(ii) Assume that five out of the six sets $S_t \cap R_i$ are non-empty and the other one is empty, where $t \in \{x, y\}$ and $i \in \{0, 1, 2\}$. Without loss of generality we can assume $S_x \cap R_0 = \emptyset$. By Lemma 3.25 we have $|S_t \cap R_i| \leq 1$ for all $(t, i) \in \{(x, 1), (x, 2), (y, 0)\}$. In particular, $|S_x| \leq 2$.

Now, all verticex of $S_t \cap R_i$ is adjacent to c_1, for being at directed distance at most 2 from each other, by Lemma 3.25. That means, every vertex of S_x is adjacent to c_1. Hence, there can be at most three vertices in $(S_y \cap R_1) \cup (S_y \cap R_2)$ as otherwise the dominating set $\{c_1, y\}$ will contradict the maximality of D. Hence, $|S_y| \leq 4$.

Therefore by equation (3.1) we have,

$$15 \leq |\overrightarrow{G}| = 2 + 3 + (2 + 4) = 11.$$

This is a contradiction.

Hence at most four sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, where $t \in \{x, y\}$ and $i \in \{0, 1, ..., k-1\}$.

Step 2: Now assume that exactly four sets out of the sets $S_t \cap R_i$ are non-empty, for all $t \in \{x, y\}$ and $i \in \{0, ..., k-1\}$. Without loss of generality we have the following three cases (by Lemma 3.24):

(i) Assume the four non-empty sets are $S_x \cap R_1, S_y \cap R_0, S_y \cap R_1$ and $S_y \cap R_2$ (only possible for $|C| \geq 3$). We have the edges $c_0 c_{k-1}$ and $c_1 c_2$ by triangulation. Lemma 3.25 implies that $S_x \cap R_1 = \{x_1\}$ and that the vertices of $S_y \cap R_0$ and the vertices of $S_y \cap R_2$ disagree with x_1 on c_0 and c_1 respectively. Hence by Lemma 3.26, we have $|S_y \cap R_0|, |S_y \cap R_2| \leq 3$.

For $|C| = 3$, if every vertex from $S_y \cap R_1$ is adjacent to either c_0 or c_1, then $\{c_0, c_1\}$ will be a dominating set with at least four common neighbors $\{x, y, x_1, c_2\}$ contradicting the maximality of D. If not, then triangulation will force x_1 to be adjacent to at least two vertices y_1, y_2 (say) from S_y. But then $\{x_1, y\}$ will become a dominating set with at least four common neighbors $\{y_1, y_2, c_0, c_1\}$ contradicting the maximality of D.

For $|C| = 4$ and 5, Lemma 3.24 implies that vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_2$ disagree with each other on y. Now by Lemma 3.25, any vertex of $S_y \cap R_1$ is adjacent to either c_0 (if it agrees with the vertices of $S_y \cap R_0$ on y) or c_1 (if it agrees with the vertices of $S_y \cap R_2$ on y). Also vertices of $S_y \cap R_0$ and $S_y \cap R_2$ are connected to x_1 by a 2-dipath through c_0 and c_1 respectively.

Now by equation (3.1) we have,

$$|S_y| \geq (15 - 2 - 5 - 1) = 7.$$

Hence, without loss of generality, at least four vertices y_1, y_2, y_3, y_4 of S_y are adjacent to c_0. Hence $\{c_0, y\}$ is a dominating set with at least five common neighbors $\{y_1, y_2, y_3, y_4, c_{k-1}\}$ contradicting the maximality of D for $|C| = 4$.

For $|C| = 5$, each vertex of $S_y \cap R_1$ disagree with c_3 by Lemma 3.24 and hence without loss of generality are all adjacent to c_0. Now $|S_y \cap R_2| \leq 3$ and $|S_y| \geq 8$ implies $|S_y \cap (R_0 \cup R_1)| \geq 5$. But every vertex of $S_y \cap (R_0 \cup R_1)$ and c_4 are adjacent to c_0. Hence $\{c_0, y\}$ is a dominating set with at least six common neighbors, contradicting the maximality of D for $|C| = 5$.

(ii) Assume the four non-empty sets are $S_x \cap R_0, S_x \cap R_1, S_y \cap R_0$ and $S_y \cap R_1$. For $|C| = 2$ every vertex in S is adjacent to either c_0 or c_1 (by Lemma 3.25). So, $\{c_0, c_1\}$ is a dominating set. Hence no vertex $w \in S$ can be adjacent to both c_0 and c_1 because otherwise $\{c_0, c_1\}$ will be a dominating set with at least three common neighbors $\{x, y, w\}$ contradicting the maximality of D. By equation (3.1) we have,

$$|S| \geq 15 - 2 - 2 = 11.$$

Hence, without loss of generality, we may assume $|S_x \cap R_0| \geq 3$. Assume $\{x_1, x_2, x_3\} \subseteq S_x \cap R_0$. Now all vertices of $S_x \cap R_0$ must be adjacent to c_0 (or c_1), as otherwise it will force all vertices of $S_y \cap R_1$ to be adjacent to both c_0 and c_1 (by Lemma 3.25). Without loss of generality assume all vertices of $S_x \cap R_0$ are adjacent to c_0. Then all $w \in S_y$ will be adjacent to c_0, as otherwise $d(w, x_i) > 2$ for some $i \in \{1, 2, 3\}$. But then $\{c_0, x\}$ will be a dominating set with at least three common vertices $\{x_1, x_2, x_3\}$ contradicting the maximality of D.

For $|C| = 3, 4$, every vertex of S will be adjacent to c_0 (by Lemma 3.25). By equation (3.1) we have,

$$|S| \geq (15 - 2 - 4) = 9.$$

Hence, without loss of generality, $|S_x| \geq 5$. Hence $\{c_o, x\}$ is a dominating set with at least five common neighbors $S_x \cup \{y\}$ contradicting the maximality of D for $|C| = 3, 4$.

For $|C| = 5$, every vertex of $S_t \cap R_i$ disagree with c_{i+2} on t and hence $|S_t \cap R_i| \leq 3$ for $i \in \{0, 1\}$ by Lemma 3.24. Assume, $|S_x \cap R_0| = 3$ and $S_x \cap R_0 = \{x_1, x_2, x_3\}$. Now assume without loss of generality that $c_2 \in N^{\alpha}(x)$. Hence, we must have $\{x_1, x_2, x_3\} \subseteq N^{\overline{\alpha}}(x)$.

Note that x_1, x_2, x_3 must agree on c_0 in order to be at directed distance at most 2 with the vertices of $S_y \cap R_1$. Further assume that $\{x_1, x_2, x_3\} \subseteq N^{\beta}(c_0)$. But then as all the three vertices $\{x_1, x_2, x_3\}$ are adjacent to both x and c_0, the only way each of them can be at directed distance 2 with c_3 is by a 2-dipath with internal vertex x. Hence we have $c_3 \in N^{\alpha}(x)$. This implies that $x_4 \in N^{\overline{\alpha}}(x)$ for any vertex $x_4 \in S_x \cap R_1$. But then the vertices of $S_x \cap R_1$ must disagree with vertices of $S_x \cap R_0$ on c_0 making it impossible for the vertices of $S_1 \cap R_0$ to be at directed distance at most 2 with x_1, x_2, x_3 and with the vertices of $S_x \cap R_1$. Hence we must have $|S_x \cap R_0| \leq 2$.

Similarly we can prove $|S_t \cap R_i| \leq 2$ for $i \in \{0, 1\}$.

Now we will show that it is not possible to have $|S_t \cap R_i| = 2$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$.

Suppose we have $|S_t \cap R_i| = 2$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$. Then clearly, the vertices of $S_t \cap R_i$ disagree with c_{i+2} and c_{i+3} on t. Hence, the vertices of $S_t \cap R_0$ agree with the vertices of $S_t \cap R_1$ on t. Therefore, the vertices of $S_t \cap R_0$ must disagree with the vertices of $S_t \cap R_1$ on c_0.

Then it will not be possible to have both the vertices of $S_x \cap R_0$ at directed distance at most 2 with all the four vertices of S_y.

Therefore, we have $|S| \leq 7$. Hence by equation (3.1) we have,

$$15 \leq |\overrightarrow{G}| \leq 2 + 5 + 7 = 14.$$

This is a contradiction. Hence we are done.

(iii) Assume the four non-empty sets are $S_x \cap R_1, S_x \cap R_2, S_y \cap R_0$ and $S_y \cap R_1$ (only possible for $|C| = 3$). Now Lemma 3.25 implies that every vertex of $(S_x \cap R_1) \cup (S_y \cap R_0)$ is adjacent to c_0 and every vertex of $(S_x \cap R_2) \cup (S_y \cap R_1)$ is adjacent to c_1.

Moreover triangulation forces the edges c_0c_2 and c_1c_2. Triangulation also forces some vertex $v_1 \in S_y \cap R_1$ to be adjacent to c_0. This will create the dominating set $\{c_0, c_1\}$ with at least four common neighbors $\{x, y, v_1, c_2\}$ contradicting the maximality of D.

Hence at most three sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, where $t \in \{x, y\}$ and $i \in \{0, 1, ..., k - 1\}$.

Step 3: Now assume that exactly three sets out of the sets $S_t \cap R_i$ are non-empty, where $t \in \{x, y\}$ and $i \in \{0, ..., k - 1\}$. Without loss of generality we have the following two cases (by Lemma 3.24):

(i) Assume the three non-empty sets are $S_x \cap R_0, S_y \cap R_0$ and $S_y \cap R_1$. Triangulation implies the edge c_0c_1 inside the region R_1.

For $|C| = 2$, there exists $u \in S_y \cup R_1$ such that u is adjacent to both c_0 and c_1 by triangulation. Now if $|S_y \cup R_1| \geq 2$, then some other vertex $v \in S_y \cup R_1$ must be adjacent to either c_0 or c_1. Without loss of generality we may assume that v is adjacent to c_0. Then every $w \in S_x \cap R_0$ will be adjacent to c_0 to have $d(v, w) \leq 2$. But then $\{c_0, y\}$ will be a dominating set with at least three common neighbors $\{c_1, u, v\}$ contradicting the maximality of D.

So we must have $|S_y \cup R_1| = 1$. Now let us assume that $S_y \cup R_1 = \{u\}$. Then any $w \in S_x \cap R_0$ is adjacent to either c_0 or c_1. If $|S_x| \geq 5$, then without loss of generality we can assume that at least three vertices of S_x are adjacent to c_0. Now to have at most distance 2 with all those three vertices, every vertex of S_y will be adjacent to c_0. This will create the dominating set $\{c_0, x\}$ with at least three common neighbors contradicting the maximality of D.

Also $|S_x| = 1$ clearly creates the dominating set $\{c_0, y\}$ (as x_1 is adjacent to c_0 by triangulation) with at least three common neighbors (a vertex from $S_y \cap R_0$ by triangulation, u and c_1) contradicting the maximality of D.

For $2 \leq |S_x| \leq 4$, c_0 (or c_1) can be adjacent to at most two vertices of $S_y \cap R_0$ because otherwise there will be one vertex $v \in S_y \cap R_0$ which will force c_0 (or c_1) to be adjacent to all vertices of $w \in S_x$ in order to satisfy $d(v, w) \leq 2$ and create a dominating set $\{c_0, y\}$ that contradicts the maximality of D.

Also, not all vertices of S_x can is adjacent to c_0 (or c_1) as otherwise $\{c_o, y\}$ (or $\{c_1, y\}$) will be a dominating set with at least three common neighbors (u, c_1 (or c_0) and a vertex from $S_y \cap R_0$) contradicting the maximality of D.

Note that, by equation (3.1), we have,

$$|S_y \cap R_0| \geq 10 - S_x.$$

Assume $S_x = \{x_1, ..., x_n\}$ with triangulation forcing the edges c_0x_1, x_1x_2, ..., $x_{n-1}x_n$, x_nc_1 for $n \in \{2, 3, 4\}$.

For $|S_x| = 2$, at most four vertices of $S_y \cap R_0$ can be adjacent to c_0 or c_1. Hence there will be at least four vertices of $S_y \cap R_0$ each connected to x by a 2-dipath through x_1 or x_2. Without loss of generality x_1 will be adjacent to at least 2 vertices of S_y and hence $\{x_1, y\}$ will be a dominating set contradicting the maximality of D.

For $|S_x| = 3$, without loss of generality assume that x_2 is adjacent to c_0. To satisfy $d(x_1, v) \leq 2$ for all $v \in S_y \cap R_0$, at least four vertices of S_y will be adjacent connected to x_1 by a 2-dipath through x_2 (as, according to previous discussions, at most two vertices of S_y can be adjacent to c_0). This will create the dominating set $\{x_2, y\}$ contradicting the maximality of D.

For $|S_x| = 4$ we have x_2c_0 and x_3c_1 as otherwise at least three vertices of S_x will be adjacent to either c_0 or c_1 which is not possible (because it forces all vertices of S_y to be adjacent to c_0 or c_1). Now each vertex $v \in S_y \cap R_0$ must be adjacent to either c_0 or x_2 (to satisfy $d(v, x_1) \leq 2$) and also to either c_1 or x_3 (to satisfy $d(v, x_4) \leq 2$) which is not possible to do keeping the graph planar.

For $|C| = 3, 4, 5$ by Lemma 3.25, each vertex of S_x disagree with each vertex of $S_y \cap R_1$ on c_0. We also have the edge x_1c_2 for some $x_1 \in S_x$ by triangulation. Now by equation (3.1) we have,

$$|S| \geq (15 - 2 - |C|) = 13 - |C|.$$

Hence $|S_x| \leq 2$ for $|C| = 3, 4$, as otherwise every vertex $u \in S_y$ will be adjacent to c_0 creating a dominating set $\{c_0, t\}$ with at least $(|C| + 1)$ common neighbors $S_t \cup \{c_1\}$ for some $t \in \{x, y\}$ contradicting the maximality of D. For $|C| = 5$, as every vertex in $S_x \cap R_0$ agree with each other on x (as they all must disagree with c_2 on x) and on c_0 (as they all disagree with vertices of $S_y \cap R_1$ on c_0). So, by Lemma 3.26, we have $|S_x \cap R_0| \leq 3$. But if $|S_x \cap R_0| = 3$ then every vertex of S_y will be adjacent to c_0 creating a dominating set $\{c_0, y\}$ with at least six common neighbors $S_y \cup \{c_1\}$ contradicting the maximality of D.

Hence $|S_x| \leq 2$ for $|C| = 3, 4$ and 5.

Now for $|C| = 3$, we can assume x and y are non-adjacent as otherwise $\{c_0, y\}$ will be a dominating set with at least four common neighbors (x, c_1 and, two other vertices each from the sets $S_y \cap R_0$, $S_y \cap R_1$ by triangulation) contradicting the maximality of D. Hence triangulation will imply the edge c_1c_2. Now for $|S_x| \leq 2$, either $\{c_0, c_2\}$ is a dominating set with at least four common neighbors $\{x, y, c_1, x_1\}$ contradicting the maximality of D or x_1 is adjacent to at least two vertices $y_1, y_2 \in S_y \cap R_0$ creating a dominating set $\{x_1, y\}$ (the other vertex in S_x must be adjacent to x_1 by triangulation) with at least four common neighbors $\{y_1, y_2, c_0, c_2\}$ contradicting the maximality of D.

For $|C| = 4$ we have $|S_y \cap R_1| \leq 2$ as otherwise we will have the dominating set $\{c_0, y\}$ with at least five common neighbors (c_1, vertices of $S_y \cap R_1$ and one vertex of $S_y \cap R_0$ by triangulation) contradicting the maximality of D. Now by equation (3.1) we have,

$$\begin{aligned}|S_y \cap R_0| &\geq (15 - |D| - |C| - |S_x| - |S_y \cap R_1|)\\ &\geq (15 - 2 - 4 - 2 - 2) = 5.\end{aligned}$$

Now, at most two vertices of $S_y \cap R_0$ can be adjacent to c_0 as otherwise $\{c_0, y\}$ will be a dominating set with at least five common neighbors (c_1, vertices of $S_y \cap R_0$ and one vertex of $S_y \cap R_1$ by triangulation) contradicting the maximality of D.

Also by triangulation in R_3 we either have the edge xy or have the edge c_2c_3. But, if we have the edge xy, then $|S_y \cap R_1| = 1$ as otherwise the dominating set $\{c_0, y\}$ will contradict the maximality of D. Hence, by triangulation, and to have directed distance at most 2 with the vertices of S_x, each vertex of $S_y \cap R_0$ will be adjacent either to c_3 or to x_1. This will create a dominating set $\{x_1, y\}$ or $\{c_3, y\}$ that contradicts the maximality of D. Hence, we do not have the edge xy (not even in other regions) and have the edge c_2c_3.

For $|S_x| \leq 2$, the vertices of $S_y \cap R_0$ will be adjacent to either c_3 or c_0 or x_1 to have directed distance at most 2 with x. But then triangulation will force at least one vertex of $S_y \cap R_0$ to be common neighbor of c_3 and x_1 and another vertex of $S_y \cap R_0$ to be common neighbor of c_3 and x_1 or the edge c_0c_3. It is not difficult to check, casewise, (drawing a picture for individual cases will help in understanding the scenario) that one of the sets $\{c_0, y\}$, $\{c_3, y\}$ or $\{x_1, y\}$ will be a dominating set contradicting the maximality of D.

For $|C| = 5$ by Lemma 3.24, each vertex of $S_y \cap R_i$ must disagree with c_{i+2} on y. If vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_1$ agree with each other on y, then they must disagree with each other on c_0 which implies $|S_y \cap R_i| \leq 3$ for all $i \in \{0, 1\}$. If vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_1$ disagree with each other on y, then vertices of $S_y \cap R_i$ must agree with c_{3-i} on y. Then, by Lemma 3.25, each vertex of $S_y \cap R_i$ must be connected to c_{3-i} by a 2-dipath through c_{4-3i} which implies $|S_y \cap R_i| \leq 3$ for all $i \in \{0, 1\}$.

Assume, we have $|S_y \cap R_0| = 3$ and $|S_y \cap R_1| = 3$. Then each vertex of $S_y \cap R_i$ must disagree with both c_{i+2} and c_{i+3} on y. This will imply that the vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_1$ disagree with each other on c_0. Now there will be no way to have directed distance at most 2 between a vertex of S_x and all the six vertices of S_y.

Hence we must have $|S_y| \leq 5$. Then by equation (3.1) we have,

$$15 \leq |\overrightarrow{G}| \leq 2 + 5 + (2 + 5) = 14.$$

This is a contradiction. This concludes this particular subcase.

(ii) Assume the three non-empty sets are $S_x \cap R_1, S_y \cap R_0$ and $S_y \cap R_2$ (only possible for $|C| \geq 3$). By Lemma 3.25, we have $S_x = \{x_1\}$ and the fact that each vertex of $S_y \cap R_i$ disagrees with $c_{i^2/4}$ on x_1 for $i \in \{0, 2\}$. Triangulation implies the edges x_1c_0, x_1c_1, $c_{k-1}c_0$, c_0c_1 and c_1c_2.

For $|C| = 3$, $\{c_0, c_1\}$ is a dominating set with at least four common neighbors $\{x, y, c_2, x_1\}$ contradicting the maximality of D. For $|C| = 4$ and 5 we have, every vertex of $S_y \cap R_0$ disagree with every vertex of $S_y \cap R_2$ on y. Hence, by Lemma 3.26, we have $|S_y \cap R_i| \leq 3$ for all $i \in \{0, 2\}$. Hence by equation (3.1) we have

$$\begin{aligned}15 \leq |\overrightarrow{G}| &= |D| + |C| + |S|\\ &\leq [2 + 5 + (1 + 3 + 3)] = 14.\end{aligned}$$

This is a contradiction.

Step 4: Hence at most two sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, where $t \in \{x, y\}$ and $i \in \{0, 1, ..., k-1\}$.

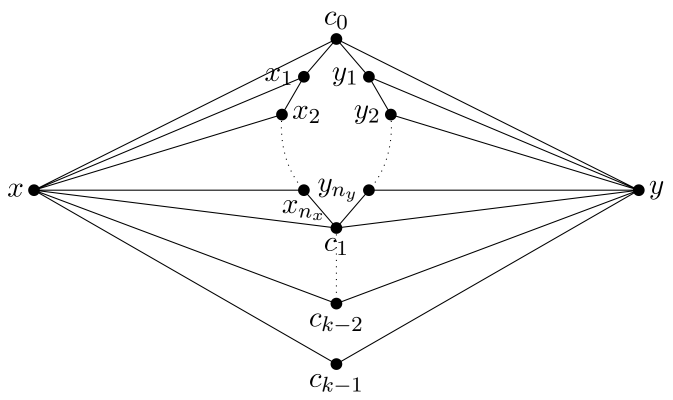


Figure 3.10: The only non-empty region is R_1

Now assume that exactly two sets out of the sets $S_t \cap R_i$ are non-empty, where $t \in \{x, y\}$ and $i \in \{0, ..., k-1\}$, yet there are two non-empty regions. Without loss of generality assume that the two non-empty sets are $S_x \cap R_0$ and $S_y \cap R_1$. Triangulation will force x and y to have a common neighbor other than c_0 and c_1 for $|C| = 2$ which is a contradiction.

For $|C| = 3, 4, 5$ triangulation implies the edges $c_{k-1}c_0$ and c_0c_1. By Lemma 3.25, we know that each vertex of S is adjacent to c_0. By equation (3.1) we have,

$$|S| \geq (16 - 2 - 5) = 9.$$

Hence, without loss of generality, we may assume $|S_x| \geq 4$. Then $\{c_0, x\}$ will be a dominating set with at least six common neighbors $S_x \cup \{c_{k-1}, c_1\}$ contradicting the maximality of D.

Hence we are done. □

The lemma proved above was one of the key steps to prove the theorem. Now we will improve the lower bound of $|C|$.

Lemma 3.29. $|C| \geq 6$.

Proof. For $|C| = 2, 3, 4, 5$ without loss of generality by Lemma 3.28, we may assume R_1 to be the only non-empty region. Then triangulation will force the configuration depicted in Fig 3.10 as a subgraph of $und(\overrightarrow{G})$, where $C = \{c_o, ..., c_{k-1}\}$, $S_x = \{x_1, ..., x_{n_x}\}$ and $S_y = \{y_1, ..., y_{n_y}\}$. Without loss of generality we may assume,

$$|S_y| = n_y \geq n_x = |S_x|.$$

Then by equation (3.1) we have,

$$n_y = |S_y| \geq (15 - 2 - |C| - |S_x|) = 13 - |C| - |S_x|. \tag{3.2}$$

First of all assume $n_x = 0$. Then x is non-adjacent to y as otherwise y will dominate the whole graph. So we have the edges $c_0c_1, c_1c_2, ..., c_{k-1}c_0$ by triangulation. Then by equation 3.2 we have,

$$|S_y| \geq 13 - 5 = 8.$$

Now to have $\overrightarrow{d}(x, y_i) \leq 2$, every y_i must be connected to x by a 2-dipath with internal vertex either c_0 or c_1. Hence at least four vertices of S_y must be adjacent to either c_0 or c_1. Note that c_0 is also adjacent to c_{k-1}, c_1 and that c_1 is also adjacent to c_0, c_2. So, the dominating set $\{c_0, y\}$ or $\{c_1, y\}$ will contradict the maximality of D. Hence $n_x \geq 1$.

Claim 1: $|C| = 5$ is not possible.

Proof of claim 1: Assume that $|C| = 5$. Then by equation 3.2 we have,

$$|S_y| \geq 13 - 5 - n_x = 8 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 4$. Now every vertex of S_y disagree with c_3 on y. They also must disagree with y on c_2 as otherwise all of them will be connected to c_2 by 2-dipaths with internal vertex c_1 and imply $\overrightarrow{d}(y_1, y_4) > 2$. For similar reason, the vertices of S_y must disagree with c_4 on y.

Moreover, the edge c_0c_1 does not exist because it will force each vertex of S_y to be connected to vertices of S_x by 2-dipaths with internal vertex either c_0 or c_1. In fact, for $n_x \geq 2$, as not all vertices of S_x can be adjacent to both c_0 and c_1, every vertex of S_y will be connected to the vertices of S_x by 2-dipaths with internal vertex being exactly one of c_0, c_1 implying $\overrightarrow{d}(y_1, y_4) > 2$. For $n_x = 1$, as $n_y \geq 7$, at least four vertices of S_y will be connected to the vertices of S_x by 2-dipaths with internal vertex being exactly one of c_0, c_1 implying $\overrightarrow{d}(y_i, y_{i+3}) > 2$ for some $i \in \{1, 2, ..., n_y\}$. Hence the edge c_0c_1 does not exist.

Also, if we have the edge y_1y_4 and without loss of generality assume the edge y_1y_3 by triangulation, then every vertex of S_x must be connected to y_2 by 2-dipaths with internal vertex y_1. In this case $\{y_1, y\}$ is a dominating set with at least n_y common neighbors (c_0 and $n_y - 1$ common neighbors from S_y). Hence, to avoid contradicting the maximality of D, we must have $n_y \leq 5$. Then we must also have $n_x \geq 3$. But then, as every vertex of S_x agree on c_0 and on x (as they all disagree with c_3 on x), they must disagree with c_1, c_2 and c_4 to have directed distance at most 2 with them. Also the vertices of S_y must disagree on c_1 to have directed distance at most 2 with it. Hence the vertex c_4 and c_1 agree with each other on x and y. Hence we have $\overrightarrow{d}(c_4, c_1) > 2$ as the edge c_0c_1 does not exist. This is a contradiction. Hence we do not have the edge y_1y_4.

Therefore, y_1 and y_4 must be connected by a 2-dipath with an internal vertex x_j from S_x for some $j \in \{1, 2, .., n_x\}$. As we cannot have the edge y_1y_4, this will imply that every vertex of $S \setminus \{x_j\}$ will be adjacent to x_j to be at directed distance at most 2 from each other. Then we can arrive to a contradiction exactly like the case described in the paragraph above.

This proves the claim. $\Diamond$

Claim 2: $|C| = 4$ is not possible.

Proof of claim 2: Assume that $|C| = 4$. Then by equation 3.2 we have,

$$|S_y| \geq 13 - 4 - n_x = 9 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 5$.

Now we will show that every vertex of S_y disagree with c_2 and c_3 on y. First note that no vertex can agree with both c_2 and c_3 on y as otherwise it must be adjacent to both c_0 and c_1 which is impossible as $n_y \geq 5$. So, if the claim is not true, then some vertices of S_y will agree with c_2 on y and the other vertices of S_y will agree with c_3 on y.

Also at most three vertices of S_y can agree with c_2 (or c_3) on y. So, $n_y \leq 6$. Hence, $n_x \geq 3$.

Now, three vertices agree on, say, c_2, then they will all disagree with c_2 on c_1 and every vertex (there are at least three such vertices) of S_x will disagree with those three vertices on c_1. Then, to have directed distance at most 2 with the vertices of S_x, the other vertices (there are at least two such vertices) of S_y should be adjacent to c_1 which is not possible as they are already connected to c_3 with 2-dipaths with internal vertex c_0.

The rest of the proof is similar to the proof Claim 1. Using similar arguments it is possible to show that the edge c_0c_1 does not exist, the edge y_1y_4 does not exist and it is not possible to have a 2-dipath with internal vertex from S_x connecting y_1 and y_4. $\Diamond$

Proof of claim 3: Assume that $|C| = 3$. Then by equation 3.2 we have,

$$|S_y| \geq 13 - 3 - n_x = 10 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 5$.

First note that it is not possible to have the edge c_0c_1 as this will force some three vertices of S_y to be connected to vertices of S_x by 2-dipaths with internal vertex c_0 (or c_1) making $\{c_0, y\}$ (or $\{c_1, y\}$) a dominating set that contradicts the maximality of D.

For $n_x \geq 7$, there are at least 4 vertices in S_y that agree with each other on y. We need to have directed distance at most 2 between them. Let those four vertices be y_i, y_j, y_k, y_l with $i > j > k > l$.

Now assume we have the edge y_iy_l. Then every vertex of S_x will be adjacent to either y_i or y_l. Without loss of generality assume that every vertex of S_x is adjacent to y_i. But then $\{y_i, y\}$ will be a dominating set with at least 4 common neighbors contradicting the maximality of D. Hence $n_y \leq 6$. Therefore we must have $n_x \geq 4$.

For $n_y = 5, 6$, one can show that these cases are not possible without creating a dominating set that contradicts the maximality of D. If one just tries to have directed distance at most 2 between the vertices of S, the proof will follow. The proof of this part is also similar to the ones done before and, though a bit tedious, is not difficult to check. $\Diamond$

Proof of claim 4: Assume that $|C| = 2$. Then by equation 3.2 we have,

$$|S_y| \geq 13 - 2 - n_x = 11 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 6$.

This is actually the easiest of the four claims. The case $n_y \geq 7$ can be argued as in the previous proof. For $n_y = 6$, we must have $n_x \geq 5$. If one just tries to have directed distance at most 2 between the vertices of S, the proof will follow. The proof of this part is also similar to the ones done beforeand, though a bit tedious, is not difficult to check. $\Diamond$

This completes the proof of the lemma. $\square$

So, now we have proved that the value of $|C|$ is at least 6. This is an answer to our question "how small $|C|$ can be?". Now we will ask the question “How big $|C|$ can be?” and try to provide upper bounds for the value of $|C|$. The following lemma will help us to do so.

Lemma 3.30. *If $|C| \geq 6$, then the following holds:*

(a) $|C^{\alpha\beta}| \leq 3$, $|C_t^{\alpha}| \leq 6$, $|C| \leq 12$. Moreover, if $|C^{\alpha\beta}| = 3$, then $\overrightarrow{G}[C^{\alpha\beta}]$ is a 2-dipath.

(b) $|C_t^{\alpha}| \geq 5$ (respectively $4, 3, 2, 1, 0$) implies $|S_t^{\alpha}| \leq 0$ (respectively $1, 3, 4, 5, 6$).

Proof. **(a)** If $|C^{\alpha\beta}| \geq 4$, then there will be two vertices $u, v \in C^{\alpha\beta}$ with $d(u, v) > 2$ which is a contradiction. Hence we have the first inequality which implies the other two.

Also if $|C^{\alpha\beta}| = 3$, then the only way to connect the two non-adjacent vertices u, v of $C^{\alpha\beta}$ is to connected them with a 2-dipath through the other vertex (other than u, v) of $C^{\alpha\beta}$.

(b) Lemma 3.24(b) implies that if all elements of C_t^{α} do not belong to the set of four boundary points of any three consecutive regions (like R, R^1, R^2 in Lemma 3.25), then $|S_t^{\alpha}| = 0$. Hence we have $|C_t^{\alpha}| \geq 5$ implies $|S_t^{\alpha}| \leq 0$.

By Lemma 3.25, if all the elements of C_t^{α} belong to the set of four boundary points $c^1, c^2, \overline{c^1}, \overline{c^2}$ of three consecutive regions R, R^1, R^2 (like in Lemma 3.25) and contains both $\overline{c^1}, \overline{c^2}$, then $|S_t^{\alpha}| \leq 1$. Also $S_t^{\alpha} \subseteq R$ by Lemma 3.25. Hence we have,

$$|C_t^{\alpha}| \geq 4 \text{ implies } |S_t^{\alpha}| \leq 1.$$

Now assume that all the elements of C_t^{α} belongs to the set of three boundary points $c^1, c^2, \overline{c^1}$ of two adjacent regions R, R^1 (like in Lemma 3.25) and contains both $\overline{c^1}, c^2$. Then by Lemma 3.24, $v \in S_t^{\alpha}$ implies v is in R or R^1.

Now if both $S_t^{\alpha} \cap R$ and $S_t^{\alpha} \cap R^1$ are non-empty, then each vertex of $(S_t^{\alpha} \cap R) \cup \{c^2\}$ disagrees with each vertex of $(S_t^{\alpha} \cap R^1) \cup \{\overline{c^1}\}$ on c^1 (by Lemma 3.25).

Hence by Lemma 3.26 we have,

$$|(S_t^\alpha \cap R) \cup \{\overline{c^1}\}|, |(S_t^\alpha \cap R^1) \cup \{c^2\}| \leq 3.$$

This clearly implies,

$$|S_t^\alpha \cap R|, |S_t^\alpha \cap R^1| \leq 2 \text{ and } |S_t^\alpha| \leq 4.$$

Now suppose we have $|S_t^\alpha| = 4$ and hence also $|S_t^\alpha \cap R|, |S_t^\alpha \cap R^1| = 2$. Then $S_{t'} = \emptyset$ as the only way for a vertex of $S_{t'}$ to have directed distance at most 2 with every vertex of S_t is by being connected by a 2-dipath with internal vertex c_1, which is impossible as the vertices of $S_t^\alpha \cap R$ disagree with the vertices $S_t^\alpha \cap R^1$ on c_1.

In fact, for the same reason, it is impossible to have directed distance at most 2 between all the vertices of S_t and t' unless we have the edge tt' (that is the edge xy). But then the edge tt' makes t a vertex that dominates the whole graph contradicting the domination number of the graph being 2. Therefore, it is not possible to have $|S_t^\alpha| = 4$. Hence we have $|S_t^\alpha| = 3$ in this case.

Also if one of $S_t^\alpha \cap R$ and $S_t^\alpha \cap R^1$ is empty then we must have $|S_t^\alpha| \leq 3$ by Lemma 3.25 and 3.26.

Hence we have

$$|C_t^\alpha| \geq 3 \text{ implies } |S_t^\alpha| \leq 3.$$

Let $R, R^1, R^2, c^1, c^2, \overline{c^1}, \overline{c^2}$ be like in Lemma 3.25 and assume $C_t^\alpha = \{c^1, c^2\}$. By Lemma 3.24, $v \in S_t^\alpha$ implies v is in R, R^1 or R^2 and also that both $S_t^\alpha \cap R^1$ and $S_t^\alpha \cap R^2$ can not be non-empty. Hence, without loss of generality, assume $S_t^\alpha \cap R^2 = \emptyset$.

Then by Lemma 3.25, vertices of $S_t^\alpha \cap R^1$ disagree with vertices of $(S_t^\alpha \cap R) \cup \{c^2\}$ on c^1. Hence by Lemma 3.26 we have,

$$|S_t^\alpha \cap R^1|, |(S_t^\alpha \cap R) \cup \{c^2\}| \leq 3.$$

This implies $|S_t^\alpha| \leq 5$.

Now if $S_t^\alpha \cap R^1 = \emptyset$, then we have $S_t^\alpha = S_t^\alpha \cap R$. Let $|S_t^\alpha \cap R| \geq 6$. Now consider the induced graph $\overrightarrow{O} = \overrightarrow{G}[(S \cap R) \cup \{c^1, c^2\}]$. In this graph the vertices of $(S_t^\alpha \cap R) \cup \{c^1, c^2\}$ are pairwise at directed distance at most 2. Hence $\chi_o(\overrightarrow{O}) \geq 8$. But this is a contradiction as $\overrightarrow{O}$ is an outerplanar graph and every outerplanar graph has an oriented 7-coloring [58]. Hence,

$$|C_t^\alpha| \geq 2 \text{ implies } |S_t^\alpha| \leq 5.$$

Now suppose we have $|S_t^\alpha| = 5$. Then we must have $S_{t'} = \emptyset$ as otherwise it is not possible to have directed distance at most 2 between the vertices of S.

We also do not have the edge xy as it will contradict the domination number of the graph being 2 (t will dominate the graph). So, by triangulation we have the edges c^1c^2 and $\overline{c^1}c^1$. So, each vertex of S_t must be connected to t' with a 2-dipath with internal vertices from $\{\overline{c^1}, c^1, c^2\}$. But then it will not be possible to have directed distance at most 2 between the five vertices of S_t.

Hence,

$$|C_t^\alpha| \geq 2 \text{ implies } |S_t^\alpha| \leq 4.$$

In general S_t^α is contained in two distinct adjacent regions by Lemma 3.24. Without loss of generality assume $S_t^\alpha \subseteq R_1 \cup R_2$. If both $S_t^\alpha \cap R_1$ and $S_t^\alpha \cap R_2$ are non-empty, then by

Lemma 3.25 we know that vertices of $S_t^\alpha \cap R_1$ disagree with vertices of $S_t^\alpha \cap R_2$ on c_1. Hence $|S_t^\alpha \cap R_1|, |S_t^\alpha \cap R_2| \leq 3$ which implies $|S_t^\alpha| \leq 6$.

Now assume only one of the two sets $S_t^\alpha \cap R_1$ and $S_t^\alpha \cap R_2$ is non-empty. Without loss of generality assume $S_t^\alpha \cap R_1 \neq \emptyset$. If $c_0, c_1 \notin C_t^\alpha$ and $|C_t^\alpha| = 1$, then we have $|S_t^\alpha \cap R_1| \leq 3$ by Lemma 3.25 and 3.26. In the induced outerplanar graph $\overrightarrow{O} = \overrightarrow{G}[(S \cap R_1) \cup \{c_1, c_2\}]$ vertices of $S_t^\alpha \cup (c_t^\alpha \cap \{c_1, c_2\})$ are pairwise at directed distance at most 2.

Hence $7 \geq \chi_o(\overrightarrow{O}) \geq |S_t^\alpha \cup (c_t^\alpha \cap \{c_1, c_2\})|$. Therefore,

$$|C_t^\alpha| \geq 1 \text{ (respectively 0) implies } |S_t^\alpha| \leq 6 \text{ (respectively 7).}$$

Now, when both the equalities hold, we must have $S_{t'} = \emptyset$ as otherwise $C_t^\alpha \cup S_t \cup S_{t'}$ will contain an oriented outerplanar graph with oriented chromatic number at least 8, which is not possible, in order to have all the vertices of S at directed distance at most 2.

Now, $S_{t'} = \emptyset$ will imply that the edge xy is not there as otherwise t will dominate the whole graph. Hence, each vertex of S_t must be connected to t' by a 2-dipath with internal vertex c_i for some $i \in \{0, 1, 2\}$. But this will force $|S_t| \leq 5$ as otherwise the vertices of S_t will no longer be at directed distance at most 2 from each other.

Hence,

$$|C_t^\alpha| \geq 1 \text{ (respectively 0) implies } |S_t^\alpha| \leq 5 \text{ (respectively 6).}$$

Hence we are done. □

Now we will prove that the value of $|C|$ can be at most 5 which contradicts our previously proven lower bound of $|C|$. That actually proves Lemma 3.23.

Lemma 3.31. $|C| \leq 5$.

Proof. Without loss of generality we can suppose $|C_x^\alpha| \geq |C_y^\beta| \geq |C_y^{\overline{\beta}}| \geq |C_x^{\overline{\alpha}}|$ (the last inequality is forced). We know that $|C| \leq 12$ and $|C_x^\alpha| \leq 6$ (Lemma 3.30(a)). So it is enough to show that $|S| \leq 12 - |C|$ for all possible values of $(|C|, |C_x^\alpha|, |C_y^\beta|)$ since it contradicts (3.1).

For $(|C|, |C_x^\alpha|, |C_y^\beta|) = (12, 6, 6)$, $(11, 6, 6)$, $(10, 6, 6)$, $(10, 6, 5)$, $(10, 5, 5)$, $(9, 5, 5)$, $(8, 4, 4)$ we have $|S| \leq 12 - |C|$ using Lemma 3.30(b).

For $(|C|, |C_x^\alpha|, |C_y^\beta|) = (8, 6, 6)$, $(7, 6, 6)$, $(7, 6, 5)$,$(6, 6, 6)$, $(6, 6, 5)$, $(6, 6, 4)$, $(6, 5, 5)$ we are forced to have,

$$|C^{\alpha\beta}| > 3.$$

This is a contradiction by Lemma 3.30(a).

So, $(|C|, |C_x^\alpha|, |C_y^\beta|) \neq (12, 6, 6)$, $(11, 6, 6)$, $(10, 6, 6)$, $(10, 6, 5)$, $(10, 5, 5)$, $(9, 5, 5)$, $(8, 4, 4)$, $(8, 6, 6)$, $(7, 6, 6)$, $(7, 6, 5)$,$(6, 6, 6)$, $(6, 6, 5)$, $(6, 6, 4)$, $(6, 5, 5)$.

We will be done if we prove that $(|C|, |C_x^\alpha|, |C_y^\beta|)$ cannot take the other possible values also. That leaves us checking a lot of cases. We will check just a few cases and observe that the other cases can be checked using similar logic.

Case 1: Assume $(|C|, |C_x^\alpha|, |C_y^\beta|) = (9, 6, 6)$.

Then we are forced to have, $|C^{\alpha\beta}| = |C^{\alpha\overline{\beta}}| = |C^{\overline{\alpha}\beta}| = 3$ in order to satisfy the first inequality of Lemma 3.30(a). So $\overrightarrow{G}[C^{\alpha\beta}]$, $\overrightarrow{G}[C^{\alpha\overline{\beta}}]$ and $\overrightarrow{G}[C^{\overline{\alpha}\beta}]$ are 2-dipaths by Lemma 3.30(a). Without loss of generality we can assume $C^{\alpha\overline{\beta}} = \{c_0, c_1, c_2\}$ and $C^{\overline{\alpha}\beta} = \{c_3, c_4, c_5\}$. Hence by Lemma 3.24 we have $u \in R_1 \cup R_2$ and $v \in R_4 \cup R_5$ for any $(u, v) \in S_y^{\overline{\beta}} \times S_x^{\overline{\alpha}}$. Hence by Lemma 3.24, either $S_y^{\overline{\beta}}$ or $S_x^{\overline{\alpha}}$ is empty. Without loss of generality assume $S_y^{\overline{\beta}} = \emptyset$. Therefore we have, $|S| = |S_x| = |S_x^{\overline{\alpha}}| \leq 3$ (by Lemma 3.30(b)). So this case is not possible.

Case 2: Assume $(|C|, |C_x^\alpha|, |C_y^\beta|) = (7, 6, 4)$.

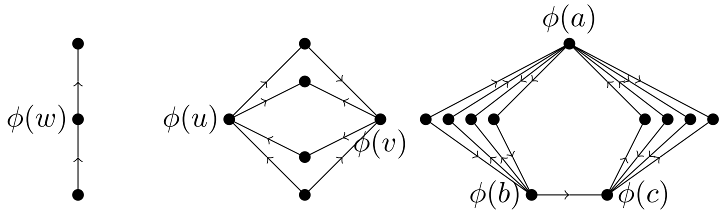


Figure 3.11: Planar ocliques with girth at least 4

So, without loss of generality, we can assume that $\overrightarrow{G}[C^{\alpha\beta}]$ and $\overrightarrow{G}[C^{\alpha\overline{\beta}}]$ are 2-dipaths and, $C^{\alpha\beta} = \{c_0, c_1, c_2\}$, $C^{\alpha\overline{\beta}} = \{c_3, c_4, c_5\}$ and $C^{\overline{\alpha}\beta} = \{c_6\}$.

By Lemma 3.30 we have $|S_x| \leq 5$ and $|S_y| \leq 3 + 1 = 4$. So we are done if either $S_x = \emptyset$ or $S_y = \emptyset$.

So assume both S_x and S_y are non-empty. First assume that $S_y^{\beta} \neq \emptyset$. Then by Lemma 3.24 we have $S_y^{\beta} \subseteq R_5$, $S_x^{\overline{\alpha}} \subseteq R_5 \cup R_6$ and hence $S_y^{\overline{\beta}} = \emptyset$. By Lemma 3.25, vertices of S_y^{β} and vertices of $S_x^{\overline{\alpha}} \cap R_5$ must disagree with c_6 on c_5 while disagreeing with each other on c_5, which is not possible. Hence, $S_x^{\overline{\alpha}} \cap R_5 = \emptyset$. Also $|S_x^{\overline{\alpha}} \cap R_6| \leq 3$ as they all disagree on c_5 with the vertex of S_y^{β}. So $|S| \leq 4$ when $S_y^{\beta} \neq \emptyset$.

Now assume $S_y^{\beta} = \emptyset$ hence $S_y^{\overline{\beta}} \neq \emptyset$. Then by Lemma 3.24 we have $S_y^{\overline{\beta}} \subseteq R_1 \cup R_2$, $S_x^{\overline{\alpha}} \subseteq R_0 \cup R_1$ and hence $S_y^{\beta} = \emptyset$. Assume $S_y^{\overline{\beta}} \cap R_2 = \emptyset$ as otherwise vertices of $S_x^{\overline{\alpha}}$ will be adjacent to both c_0 and c_1 (to be connected to c_6 and vertices of $S_y^{\overline{\beta}} \cap R_2$ by a 2-dipath) implying $|S_x^{\overline{\alpha}}| \leq 1$ implying $|S| \leq 5$. If $S_x^{\overline{\alpha}} \cap R_0 \neq \emptyset$, then $|S_y^{\overline{\beta}} \cap R_1| = 1$, $|S_y^{\overline{\alpha}} \cap R_1| \leq 1$ and $|S_y^{\overline{\alpha}} \cap R_0| \leq 3$ by Lemma 3.25 and hence $|S| \leq 5$. If $S_x^{\overline{\alpha}} \cap R_0 = \emptyset$ then we have $|S_y^{\overline{\beta}} \cap R_1| \leq 2$, $|S_y^{\overline{\alpha}} \cap R_1| \leq 3$ and hence $|S| \leq 5$. So this case is not possible.

Similarly one can handle the other cases. □

Therefore, from the above lemmas, we learnt that every planar oclique of order at least 15 is dominated by a single vertex. Moreover, we also have proved that a planar oclique dominated by one vertex can have order at most 15.

Hence, there is no planar oclique of order more than 15. We also proved that every oclique of order 15 must contain the planar oclique depicted in Fig. 3.6 as a subgraph.

This concludes the proof of Theorem 3.19. □

Proof of Theorem 3.21

(a) The proof follows directly from Theorem 3.19.

(b) In 1975, Plesník [24] characterized and listed all triangle-free planar graphs with diameter 2 in Theorem 2.14. They are precisely the graphs depicted in Fig. 2.1 (see Chapter 2). Now note that any orientation of those graphs from Fig. 2.1 admits a homomorphism to the graphs depicted in Fig. 3.11 respectively (that is, the first oriented graph depicted in Fig. 3.11 is a universal bound for the first family of graphs described in Fig. 2.1; the second ... etc.).

To prove the homomorphisms, we map the vertices w, u, v, a from Fig. 2.1 to the corresponding vertices $\phi(w), \phi(u), \phi(v), \phi(a)$ in Fig 3.11 respectively. The vertices b and c are mapped to the vertices $\phi(b)$ (or $\phi(c)$) and $\phi(c)$ (or $\phi(b)$) depending on the orientation of the edge bc. Without loss of generality we can assume the edge to be oriented as $\overrightarrow{bc}$ and assume that the vertices b, c map to the vertices $\phi(b), \phi(c)$ respectively.

Now to complete the first homomorphism, map the vertices of $N^{\alpha}(w)$ to the unique vertex in $N^{\alpha}(\phi(w))$ for $\alpha \in \{+, -\}$.

To complete the second homomorphism, map the vertices of $N^{\alpha}(u) \cap N^{\beta}(u)$ to the unique vertex in $N^{\alpha}(\phi(u)) \cap N^{\beta}(\phi(v))$ for $\alpha, \beta \in \{+, -\}$.

To complete the third homomorphism, map the vertices of $N^{\alpha}(a) \cap N^{\beta}(t)$ to the unique vertex in $N^{\alpha}(\phi(a)) \cap N^{\beta}(\phi(t))$ for $\alpha, \beta \in \{+, -\}$ and $t \in \{b, c\}$.

Now note that the first two oriented graphs depicted in Fig. 3.11 are ocliques of order 3 and 6 respectively, while the third graph is not an oclique but clearly has oriented relative chromatic number 5.

Hence, there is no triangle-free planar oclique of order more than 6. Also, the only example of a trianlge-free oclique of order 6 is the second graph depicted in Fig. 3.11.

(c) From the proof above, we know that there is no triangle-free planar oclique of order more than 6 and the only example of a trianlge-free oclique of order 6 is the second graph depicted in Fig. 3.11, which is a graph with girth 4. Hence, there is no planar oclique with girth at least 5 on more than 5 vertices while the directed cycle of length 5 is clearly a planar oclique with girth 5.

(d) The 2-dipath is an oclique of order 3. From Plesník's characterization, the rest of the proof follows easily. □

3.4 $L(p, q)$-labeling of oriented graphs

To distinguish between close and very close transmitters in a wireless communication system, Griggs and Yeh [20] proposed a variation of the Frequency Assignment Problem (or simply FAP) by introducing the $L(2, 1)$-labeling which was generalized by Georges and Mauro [17] as follows.

For any two positive integers p and q, a *k-$L(p, q)$-labeling* of a graph G is a mapping ℓ from the vertex set $V(G)$ to the set $\{0, 1, ..., k\}$ such that

- $| \ell(u) - \ell(v) | \geq p$ if u and v are at distance 1 in G,

- $| \ell(u) - \ell(v) | \geq q$ if u and v are at distance 2 in G.

The *$L(p, q)$-span* $\lambda_{p,q}(G)$ of a graph G is defined as $min\{k \mid G$ has a k-$L(p, q)$-labeling $\}$. For a family $\mathcal{F}$ of graphs, $\lambda_{p,q}(\mathcal{F}) = max\{\lambda_{p,q}(H) \mid H \in \mathcal{F}\}$.

A common feature of graph theoretic models for FAP is that communication is assumed to be possible in both directions (duplex) between two radio transmitters and, therefore, these models are based on undirected graphs. But in reality, modelling FAP with directed or oriented graphs could be interesting as pointed by Aardal *et al* [1] in their survey.

There are two different oriented versions of $L(p, q)$-labeling, namely 2-dipath $L(p, q)$-labeling, introduced by Chang *et al* [12], and oriented $L(p, q)$-labeling, introduced by Gonçalves, Raspaud and Shalu [19].

A *2-dipath k-$L(p, q)$-labeling* of an oriented graph $\overrightarrow{G}$ is a mapping ℓ from the vertex set $V(\overrightarrow{G})$ to the set $\{0, 1, ..., k\}$ such that

- $| \ell(u) - \ell(v) | \geq p$ if u and v are adjacent in $\overrightarrow{G}$,

- $| \ell(u) - \ell(v) | \geq q$ if u and v are connected by a 2-dipath in $\overrightarrow{G}$.

The *2-dipath span* $\overrightarrow{\lambda}_{p,q}(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is defined as $min\{k \mid \overrightarrow{G}$ has a 2-dipath k-$L(p, q)$-labeling $\}$. The *2-dipath span* $\overrightarrow{\lambda}_{p,q}(G)$ of an undirected graph G is defined as $max\{\overrightarrow{\lambda}_{p,q}(\overrightarrow{G}) \mid \overrightarrow{G}$ is an orientation of $G\}$. The *2-dipath span* $\overrightarrow{\lambda}_{p,q}(\mathcal{F})$ of a family $\mathcal{F}$ of (oriented or undirected) graphs is defined as $max\{\overrightarrow{\lambda}_{p,q}(H) \mid H \in \mathcal{F}\}$.

An *oriented k-$L(p, q)$-labeling* of an oriented graph $\overrightarrow{G}$ is a mapping ℓ from the vertex set $V(\overrightarrow{G})$ to the set $\{0, 1, ..., k\}$ such that

- ℓ is a 2-dipath k-$L(p, q)$-labeling of G,

- if $\overrightarrow{xy}$ and $\overrightarrow{uv}$ are two arcs in $\overrightarrow{G}$, then $\ell(x) = \ell(v)$ implies $\ell(y) \neq \ell(u)$.

The *oriented spans* $\lambda^o_{p,q}(\overrightarrow{G})$, $\lambda^o_{p,q}(G)$ and $\lambda^o_{p,q}(\mathcal{F})$ are defined similarly as 2-dipath spans.

Example 3.32. *The oriented graph depicted in Fig. 3.12(a) admits a 2-dipath $(p + q)$-$L(p, q)$-labeling. The same labeling is also an oriented $(p+q)$-$L(p, q)$-labeling of the graph. This labeling is optimal for $p \geq q$. Hence, for $p \geq q$, the graph has both 2-dipath and oriented $L(p, q)$-labeling span equal to $(p + q)$.*

Example 3.33. *Consider the 2-dipath and oriented $L(p, q)$-labeling of the disjoint union of the two oriented 3-cycle depicted in Fig. 3.12(b) for $p \geq q$. Note that it is possible to provide a 2-dipath $2p$-$L(p, q)$-labeling of the graph by labeling each triangle by the labels $0, p$ and $2p$. But the*

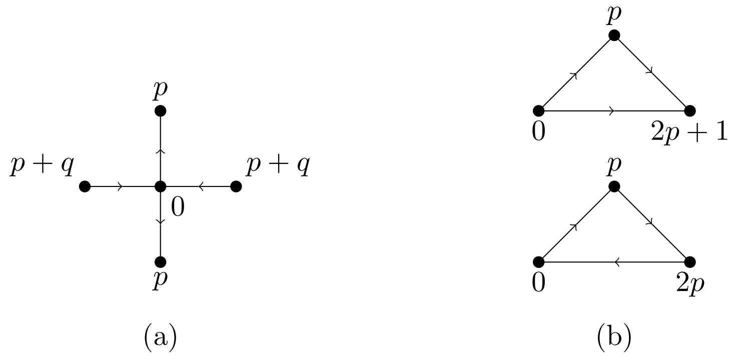


Figure 3.12: 2-dipath and oriented $L(p,q)$-labeling for $p \geq q$.

same is not possible for oriented $L(p,q)$-labeling. Hence, the graph has 2-dipath $L(p,q)$-labeling span $2p$ and oriented $L(p,q)$-labeling span $2p+1$.

From the definitions, the following lemma is immediate:

Lemma 3.34. *For every (undirected or oriented) graph G and every $p, q > 0$, $\overrightarrow{\lambda}_{p,q}(G) \leq \lambda^o_{p,q}(G)$.*

Also from these definitions we easily get the following:

Lemma 3.35. *If there is a homomorphism $f : \overrightarrow{G} \longrightarrow \overrightarrow{H}$ then $\overrightarrow{\lambda}_{p,q}(\overrightarrow{G}) \leq \overrightarrow{\lambda}_{p,q}(\overrightarrow{H})$ and $\lambda^o_{p,q}(\overrightarrow{G}) \leq \lambda^o_{p,q}(\overrightarrow{H})$, for every $p, q > 0$.*

The additional condition in oriented $L(p,q)$-labeling ensures that any oriented $L(p,q)$-labeling is an oriented coloring as well. Note that any oriented k-$L(p,q)$-labeling is an oriented $(k+1)$-coloring but a 2-dipath k-$L(p,q)$-labeling is not necessarily an oriented $(k+1)$-coloring.

Now we prove a general upper bound on oriented $L(p,q)$-span of multipartite graphs.

Theorem 3.36. *For every k-partite oriented graph $\overrightarrow{G}$, where $k \geq 3$, we have*

$$\overrightarrow{\lambda}_{p,q}(\overrightarrow{G}) \leq \lambda^o_{p,q}(\overrightarrow{G}) \leq \mid V(\overrightarrow{G}) \mid q + k(max(p,q) - q) - max(p,q).$$

In particular for $p \geq q$, both the equalities hold if $\overrightarrow{G}$ is a complete k-partite oclique.

Proof. Let, $K = K_{n_1,n_2,\ldots,n_k}$ be the complete k-partite graph with the parts being $V_1, V_2, \ldots V_k$ with $\mid V_i \mid = n_i$ for all $i = 1, 2, \ldots, k$. Also, let the vertices of V_i be denoted by $\{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$.

Let $\overrightarrow{K}$ be any orientation of K. Now, consider the labeling L of $\overrightarrow{K}$ given by

$L(v_{ij}) = (\sum_{t<i}(n_t - 1)q) + (j-1)q + (i-1)max(p,q)$, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$.

For any i, v_{ir} and v_{is} ($r \neq s$) cannot be connected by an arc but can be connected by a 2-dipath. While for any v_{ir} and v_{js}, $i \neq j$, can be connected by either an arc or a 2-dipath.

Then we have the following,

$$
\begin{aligned}
| L(v_{ir}) - L(v_{is}) | &= | [(\sum_{t<i}(n_t - 1)q) + (r-1)q + (i-1)max(p,q)] \\
&\quad - [(\sum_{t<i}(n_t - 1)q) + (s-1)q + (i-1)max(p,q)] | \\
&= |(r-s)q| \geq q, \text{ for } r \neq s.
\end{aligned}
$$

and (without loss of generality we assume that $j < i$),

$$
\begin{aligned}
| L(v_{ir}) - L(v_{js}) | &= |[(\sum_{t<i}(n_t - 1)q) + (r-1)q + (i-1)max(p,q)] \\
&\quad - [(\sum_{t<j}(n_t - 1)q) + (s-1)q + (j-1)max(p,q)]| \\
&= |(\sum_{j<t<i}(n_t - 1)q) + (n_j - 1)q + (r-1)q \\
&\qquad\qquad - (s-1)q + (i-j)max(p,q)| \\
&\qquad \text{(without loss of generality we assume } i > j) \\
&= |(\sum_{j<t<i}(n_t - 1)q) + (n_j - s)q + (r-1)q + (i-j)max(p,q)| \\
&\qquad\qquad (n_j \geq s \text{ as } v_{js} \in V_j) \\
&\geq max(p,q).
\end{aligned}
$$

As all vertices have different labels, L is an oriented coloring of $\overrightarrow{K}$.
Hence we have,

$$
\begin{aligned}
\overrightarrow{\lambda}_{p,q}(K) \leq \lambda^o_{p,q}(K) &\leq \sum_{t=1}^{k-1}(n_t - 1)q + (n_k - 1)q + (k-1)max(p,q) \\
&= | V(K) | q + k(max(p,q) - q) - max(p,q).
\end{aligned}
$$

Now as any oriented k-partite graph $\overrightarrow{G}$ is a subgraph of some orientation of the complete k-partite graph K, using Lemma 3.34 and Lemma 3.35 the theorem follows.

In particular, if $\overrightarrow{K}$ is an oclique, then any two vertices are at distance at most 2. Moreover, if $\overrightarrow{K}$ is also an orientation of the complete k-partite graph, then any two vertices from different parts, are adjacent. Hence both the equalities hold for $p \geq q$. □

We have the following corollaries of Theorem 3.36:

Corollary 3.37. $\overrightarrow{\lambda}_{2,1}(T_{2q+2}) = \lambda^o(T_{2q+2}) = 3q + 1$.

Corollary 3.38. $\overrightarrow{\lambda}_{2,1}(Z_k) = \lambda^o(Z_k) = k(2^{k-1} + 1) - 2$.

The 2-dipath and oriented $L(1,1)$-labeling span corresponds to the oriented relative clique number and the oriented chromatic number respectively. Apart from these two, the most frequently studied $L(p,q)$-labeling problem is for $(p,q) = (2,1)$ (both undirected and oriented versions).

In this section, we mainly focus on studying 2-dipath and oriented $L(2,1)$-span of some families of planar graphs. We will state and proof the results in the following.

For the family of planar graphs and for the family of planar graphs with given girth, we prove the following result.

Theorem 3.39.

(a) $18 \leq \overrightarrow{\lambda}_{2,1}(\mathcal{P}_3) \leq \lambda^o_{2,1}(\mathcal{P}_3) \leq 83$.

(b) $9 \le \overrightarrow{\lambda}_{2,1}(\mathcal{P}_4) \le \lambda^o_{2,1}(\mathcal{P}_4) \le 58$.

(c) $6 \le \overrightarrow{\lambda}_{2,1}(\mathcal{P}_5) \le \lambda^o_{2,1}(\mathcal{P}_5) \le 22$.

(d) $4 \le \overrightarrow{\lambda}_{2,1}(\mathcal{P}_6) \le \lambda^o_{2,1}(\mathcal{P}_6) \le 19$.

(e) $4 \le \overrightarrow{\lambda}_{2,1}(\mathcal{P}_7) \le \lambda^o_{2,1}(\mathcal{P}_7) \le 19$.

(f) $4 \le \overrightarrow{\lambda}_{2,1}(\mathcal{P}_k) \le \lambda^o_{2,1}(\mathcal{P}_k) \le 10$ *for* $8 \le k \le 15$.

(g) $4 \le \overrightarrow{\lambda}_{2,1}(\mathcal{P}_k) \le \lambda^o_{2,1}(\mathcal{P}_k) \le 7$ *for* $k \ge 16$.

Theorem 3.39(c) disproves the conjecture $\overrightarrow{\lambda}_{2,1}(\mathcal{P}_5) \le 5$ proposed by Calamoneri and Sinaimeri [11] and Theorem 3.39(f,g) improve the previous bounds $\overrightarrow{\lambda}_{2,1}(\mathcal{P}_{11}) \le 12$ and $\overrightarrow{\lambda}_{2,1}(\mathcal{P}_{16}) \le 8$ given by the same authors [11].

For the family $\mathcal{O}$ of outerplanar graphs, we prove the following:

Theorem 3.40. $9 \le \overrightarrow{\lambda}_{2,1}(\mathcal{O}) \le \lambda^o_{2,1}(\mathcal{O}) \le 10$.

We are not able to provide exact results for the family of outerplanar graphs. We also consider a planar superfamily and a planar subfamily of it, namely the family $\mathcal{T}_2$ of partial 2-trees and the family $\mathcal{C}$ of cacti. For both these families we managed to give exact results. In fact, we prove the following general result for the family $\mathcal{T}_k$ of partial k-trees:

Theorem 3.41.

(a) $\overrightarrow{\lambda}_{2,1}(\mathcal{T}_2) = \lambda^o_{2,1}(\mathcal{T}_2) = 10$.

(b) $\overrightarrow{\lambda}_{2,1}(\mathcal{T}_3) \le \lambda^o_{2,1}(\mathcal{T}_3) \le 22$.

(c) $\overrightarrow{\lambda}_{2,1}(\mathcal{T}_k) \le \lambda^o_{2,1}(\mathcal{T}_k) \le (k+1)(2^k+1) - 2$.

In [11] Calamoneri and Sinaimeri proved that $6 \le \overrightarrow{\lambda}_{2,1}(\mathcal{C}) \le 8$. We improve this result as follows:

Theorem 3.42. $\overrightarrow{\lambda}_{2,1}(\mathcal{C}) = \lambda^o_{2,1}(\mathcal{C}) = 7$.

Proof of Theorem 3.39

(a) Raspaud and Sopena [51] showed that every oriented planar graph admits a homomorphism to the Zielonka graph $\overrightarrow{Z}_5$. Hence, using Lemma 3.35 and Corollary 3.38, we get the upper bound.

For proving the lower bound, we will construct an example. Recall the oriented planar graph depicted in Fig. 3.6 and call it $\overrightarrow{H}_1$. Now we add twelve common neighbors of a_0 and a_i for each $i \in \{-7, -6, ..., -1, 1, 2, ..., 7\}$. The new arcs are oriented in such a way that for each such pair a_0, a_i of vertices, we have exactly three common neigbors, among the twelve new ones, in $N^\alpha(a_0) \cap N^\beta(a_i)$ and add two arcs between the new three vertices of $N^\alpha(a_0) \cap N^\beta(a_i)$ to obtain a 2-dipath for any $\alpha, \beta \in \{+, -\}$. This new oriented graph is $\overrightarrow{H}_2$. Note that $\overrightarrow{H}_2$ is planar. We regard $\overrightarrow{H}_2$ as an extension of $\overrightarrow{H}_1$, hence the names of the vertices, as presented in Fig. 3.6, remains the same.

First we show that it is impossible to have a 2-dipath 17-$L(2,1)$-labeling f of $\overrightarrow{H}_2$ with $f(a_0) = 1$. We will prove this by contradiction. Hence, assume that f is a 2-dipath 17-$L(2,1)$-labeling of $\overrightarrow{H}_2$. Notice that the graph $\overrightarrow{H}_2$ is dominated by the vertex a_0. So, the labels $\{0, 1, 2\}$ cannot be used for labeling any of the vertices, except a_0 of $\overrightarrow{H}_2$. Note that we cannot use the same label, used to label a vertex of $N^+(a_0)$, to label a vertex of $N^-(a_0)$ as those two vertices are connected by a 2-dipath through internal vertex a_0. As both $N^+(a_0)$ and $N^-(a_0)$ contains an oclique of order 7, we need at least seven labels for each of them.

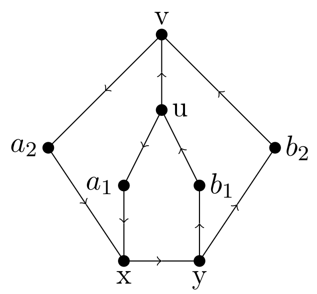


Figure 3.13: $\vec{F}$ is an oriented planar graph with girth 5.

In total, we have $\{3, 4, 5, ..., 17\}$ labels, that is, fifteen labels, available to label the vertices of $N(a_0)$. Hence, we need to use exactly seven labels for either the vertices of $N^+(a_0)$ or the vertices of $N^-(a_0)$. Without loss of generality assume that exaclty seven labels are used for labeling the vertices of $N^+(a_0)$.

In particular, $f(a_i) \neq f(a_j)$ for $i \neq j$ and $i, j \in \{-7, -6, ..., -1, 1, 2, ..., 7\}$. Now, we cannot use the same label, used to label a vertex of $N^+(a_0) \cap N^+(a_i)$, to label a vertex of $N^+(a_0) \cap N^-(a_i)$ for $i \in \{-7, -6, ..., -1, 1, 2, ..., 7\}$. Both these sets contain a 2-dipath and we need to use three different labels to label a 2-dipath. So, for each label $f(a_i)$, we must have six labels, none of them belonging to $\{f(a_i) - 1, f(a_i), f(a_i) + 1\}$, from the set of labels used to label the vertices of $N^+(a_0)$ for $i \in \{-7, -6, ..., -1, 1, 2, ..., 7\}$.

This will force the set of seven labels, used to label the vertices of $N^+(a_0)$, to contain two labels $f(a_{-i}) - 1$ and $f(a_{-i}) + 1$ for some $i \in \{1, 2, ..., 7\}$. Now, it is not possible to use the labels $f(a_{-i}) - 1$ and $f(a_{-i}) + 1$ to label neighbors of a_{-i}. So, we can use at most five labels from the set of labels used to label the vertices of $N^+(a_0)$ to label neighbors of a_{-i}. On the other hand, as noticed earlier, we need at least six labels from the set of labels used to label the vertices of $N^+(a_0)$ to label the vertices from the sets $N^+(a_0) \cap N^+(a_{-i})$ and $N^+(a_0) \cap N^-(a_{-i})$. This is a contradiction.

Hence, it is not possible to provide a 2-dipath 17-$L(2, 1)$-labeling of $\overrightarrow{H}_2$ with $f(a_0) = 1$. Similarly we can show that it is not possible to provide a 2-dipath 17-$L(2, 1)$-labeling of $\overrightarrow{H}_2$ with $f(a_0) = 16$.

Now consider the graph obtained by gluing $\overrightarrow{H_1}$ on each vertex of a planar oclique of order at least 7 by identifying that vertex with the vertex a_0 of $\overrightarrow{H}_1$. Call this new graph $\overrightarrow{H_0}$. Note that $\overrightarrow{H}_0$ is planar. Notice that, for any 2-dipath 17-$L(2, 1)$-labeling of a planar oclique of order at least 7 (for example, $\overrightarrow{H_1}$ is such an oclique), we need to use at least seven labels. Therefore, to label the oclique, we must use some label $j \notin \{0, 1, 2, 15, 16, 17\}$ to label some vertex v of it. Then we cannot use the labels $\{j-1, j, j+1\}$ to label the fourteen other vertices of the $\overrightarrow{H}_1$ glued to v. But these fourteen vertices are part of an oclique, hence must receive fourteen different labels. Now to label those fourteen vertices, we must use either 1 or 16 as labels.

Now, construct the oriented graph $\overrightarrow{H_3}$ by gluing a copy of $\overrightarrow{H_2}$ on each vertex of $\overrightarrow{H_0}$ by identifying that vertex with the vertex a_0 of $\overrightarrow{H_2}$. Note that $\overrightarrow{H}_2$ is planar. Now, if we try provide a 2-dipath 17-$L(2, 1)$-labeling of this graph, we will have a induced $\overrightarrow{H}_2$ inside it, with a 2-dipath 17-$L(2, 1)$-labeling f of it with $f(a_0) = 1$ or 16. This is a contradiction.

(b) The lower bound follows from the lower bound of Theorem 3.18(b).

To prove the upper bound of Theorem 3.15(b) Ochem and Pinlou [45] showed that the Tromp graph $\overrightarrow{T}_{40}$ is a universal bound for the family of oriented planar graphs with girth at least 4. Now the upper bound easily follow using Lemma 3.35 and Corollary 3.37. This completes the proof.

(c) In Theorem 3.15(c) Pinlou [46] proved that every planar graph of girth at least 5 admits a homomorphism to the Tromp graph $\overrightarrow{T}_{16}$. Then, using Lemma 3.35 and Corollary 3.37, we get the upper bound.

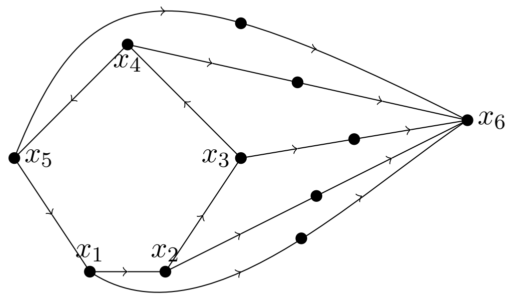


Figure 3.14: $\vec{E}$ is an oriented planar graph with girth 5.

To prove the lower bound, we first show that it is impossible to have a 2-dipath 5-$L(2,1)$-labeling f of the graph $\overrightarrow{F}$, depicted in Figure 3.13, with $\{f(x), f(y)\} = \{3,5\}$.

Notice that, if $f(x) = 3$ and $f(y) = 5$, then $f(a_1) \in \{0,1\}$ and $f(b_1) \in \{0,1,2\}$. This implies $f(u) = 4$. Similarly, we have $f(v) = 4$ which is not possible as u, v are adjacent. The case $f(x) = 5$ and $f(y) = 3$ is similar.

The oriented planar graph $\overrightarrow{E}$, depicted in Figure 3.14, has girth 5. Moreover, the vertices $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ will get pairwise different labels for any 2-dipath $L(2,1)$-labeling since they are pairwise connected by a 2-dipath. Consider a 2-dipath 5-$L(2,1)$-labeling g of $\overrightarrow{E}$ such that $g(x_6) = 0$. Then we have $\{g(x_1), g(x_2), g(x_3), g(x_4), g(x_5)\} = \{1,2,3,4,5\}$. Hence there exists an arc $\overrightarrow{wz} \in A(\overrightarrow{G}[\{x_1, x_2, x_3, x_4, x_5\}])$ such that $\{f(w), f(z)\} = \{3,5\}$.

Now on each of the five vertices $x \in \{x_1, x_2, x_3, x_4, x_5\}$ of $\overrightarrow{E}$, we glue a copy of $\overrightarrow{E}$ by identifying x with the vertex x_6 of the copy. Call this graph $\overrightarrow{G}$.

Note that $\overrightarrow{G}$ is a planar graph with girth 5 such that for any 2-dipath 5-$L(2,1)$-labeling ℓ of $\overrightarrow{G}$, there is $\overrightarrow{wz} \in A(\overrightarrow{G})$ with $\{\ell(w), \ell(z)\} = \{3,5\}$. We can then construct a new graph $\overrightarrow{H}$ by gluing the oriented graph $\overrightarrow{F}$ (Fig. 3.13) on each arc of $\overrightarrow{G}$ by identifying that arc with the arc $\overrightarrow{xy}$ of $\overrightarrow{F}$. Clearly, $\overrightarrow{H}$ is also a planar graph with girth 5 which does not have any 2-dipath 5-$L(2,1)$-labeling. Hence we get the lower bound.

(d), (e) It is easy to observe that the directed path of length 5 has 2-dipath $L(2,1)$-span 4. Hence the lower bounds.

Let $R(\overrightarrow{P}_7)$ be the oriented graph obtained by deleting the vertices ∞ and ∞' from the Tromp graph $\overrightarrow{T}_{16}$. Later in this thesis, in Theorem 4.17(d) (Chapter 4, Section 4.2.1), we prove a result equivalent to proving that every planar graph with girth at least 6 admits a homomorphism to the oriented graph $R(\overrightarrow{P}_7)$. It is easy to check that $R(\overrightarrow{P}_7)$ admits an oriented 5-$L(2,1)$-labeling. Hence, using Lemma 3.35 and Lemma 3.34, we have the upper bounds. This completes the proof.

(f), (g) It is easy to observe that the directed path of length 5 has 2-dipath $L(2,1)$-span 4. Hence the lower bounds.

Later in this thesis, in Theorem 4.17(d) (Chapter 4, Section 4.2.1), we prove a result equivalent to proving that every planar graph with girth at least 6 admits a homomorphism to the oriented graph $\overrightarrow{T}_8$. Also, one can check that the graph $\overrightarrow{B}$ (Fig 3.3) is 4-nice (we have verified it using a computer-check). Then, using Theorem 3.4, Lemma 3.35, Lemma 3.34 and Corollary 3.37, we have the results.

Proof of Theorem 3.41

To prove the following lemma we use the same technique as the one used to prove that every oriented outerplanar graph has oriented chromatic number at most 7 in [58].

Lemma 3.43. *Every oriented partial 2-tree $\overrightarrow{D}$ admits a homomorphism to the Tromp graph T_8.*

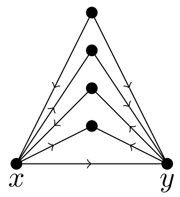


Figure 3.15: The oriented 2-tree D_1

Proof. It is possible to check that for every $u, v \in V(T_8)$ and every $\alpha, \beta \in \{+, -\}$, there exists $w_{\alpha\beta} \in N^\alpha(u) \cap N^\beta(v)$.

Let $\overrightarrow{G}$ be a minimal (with respect to the number of vertices) counterexample to the lemma. Without loss of generality we may assume that $\overrightarrow{G}$ is a 2-tree. Since $\overrightarrow{G}$ is a 2-tree, $\overrightarrow{G}$ must have a vertex x of degree 2. Let $N(x) = \{x_1, x_2\}$. Now, by removing the vertex x from $\overrightarrow{G}$ and adding an arc between x_1 and x_2 (if there was not already one), we get a 2-tree that admits a homomorphism to T_8 (because of the minimality of $\overrightarrow{G}$). Using the property of T_8 stated in the begining of the proof, clearly this homomorphism can be extended to a homomorphism of $\overrightarrow{G}$ to T_8, a contradiction. □

The next result is to prove the lower bound.

Lemma 3.44. *There exists an oriented 2-tree D_{13} for which $\overrightarrow{\lambda}_{2,1}(D_{13}) \geq 10$.*

Proof. First, we will describe a family of oriented 2-trees by induction. We start with the oriented 2-tree D_1 (Fig: 3.15). By induction, we construct a graph D_{i+1} by gluing D_1 on each arc of D_i by identifying that arc with the arc $\overrightarrow{xy}$ of D_1. Note that every so-obtained graph D_i is a 2-tree.

Assume that f is a 9-$L(2,1)$-labeling of D_{13}.

Step 0: Notice that, in each copy of D_1, all the vertices should get different labels and for any vertex $v \in N(x) \cap N(y)$ we have, $|f(t) - f(v)| \geq 2$ for $t = x, y$.

Step 1: If we restrict f to D_1 then there is a vertex $v_1 \in N(x) \cap N(y)$ such that $f(v_1) \notin \{0, 9\}$. Similarly, if we restrict f to D_2, we can find a vertex $v_2 \in N(x) \cap N(v_1)$ such that $f(v_2) \notin \{0, 9\}$.

Step 2: Now, if we restrict f to D_3, we can find a $v_3 \in N(v_1) \cap N(v_2)$ such that $f(v_3) \notin \{0, 9\}$. So we have $\{f(v_1), f(v_2), f(v_3)\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$ with no two of $\{f(v_1), f(v_2), f(v_3)\}$ being consecutive numbers, since $\{v_1, v_2, v_3\}$ are pairwise adjacent vertices. Hence there exists $i, j \in \{1, 2, 3\}$ such that $\{f(v_i) - 1, f(v_i), f(v_i) + 1\} \cap \{f(v_j) - 1, f(v_j), f(v_j) + 1\} = \emptyset$ and $f(v_i) < f(v_j)$.

Step 3: In D_4 there exists $v_{\alpha\beta} \in N^\alpha(v_i) \cap N^\beta(v_j)$ for all $\alpha, \beta \in \{+, -\}$. Notice that the vertices $\{v_{++}, v_{+-}, v_{-+}, v_{--}\}$ will be labeled by the four remaining labels different from the labels $\{f(v_i) - 1, f(v_i), f(v_i) + 1, f(v_j) - 1, f(v_j), f(v_j) + 1\}$.

Step 4: Now we want to show that there is a vertex in D_5 that receives label 1 or 8. If $f(t) \in \{1, 8\}$ for some $t \in \{v_i, v_j, v_{++}, v_{+-}, v_{-+}, v_{--}\}$, we are done.

If not, then we can conclude that $f(v_i) = 2$, $f(v_j) = 7$ since any other possible choice of labels (other than 1 or 8) for v_i, v_j will force at least one of the labels among $\{f(v_{++}), f(v_{+-}), f(v_{-+}), f(v_{--})\}$ to be equal to 1 or 8. This will imply $\{f(v_{++}), f(v_{+-}), f(v_{-+}), f(v_{--})\} = \{0, 4, 5, 9\}$. Choose v_4 from the set $\{v_{++}, v_{+-}, v_{-+}, v_{--}\}$ such that $f(v_4) = 5$. Then in D_5, there is a vertex $v_5 \in N(v_i) \cap N(v_4)$ with $f(v_5) = 8$.

Hence in D_5, there exists a vertex v_6 with $f(v_6) \in \{1, 8\}$.

Step 5: Now we want to show that there is a vertex in D_7 that receives label 1. If $f(v_6) = 1$, we are done.

If not, then $f(v_6) = 8$. This implies that, in D_6, there exists $t \in N(v_6)$ such that $f(t) \in \{1, 4, 5\}$, since we need to use at least five distinct labels from $\{0, 1, 2, 3, 4, 5, 6\}$ to label all vertices of $N(v_6)$. If $f(t) = 1$, we are done. otherwise in D_7, we can find some $s \in N(v_6) \cap N(t)$ such that $f(s) = 1$.

Hence in D_7 we can find a vertex a with $f(a) = 1$.

Step 6: Now we want to show that in D_9 there is a vertex $b \in N(a)$ with $f(b) = 8$.

Now, in D_8, there are at least five vertices in $N(a)$ which receive pairwise different labels. Therefore, for some $t \in N(a)$, we will have $f(t) \in \{8, 4, 5\}$. If $f(t) = 8$, we are done. Otherwise, in D_9, we can find $s \in N(a) \cap N(t)$ with $f(s) = 8$.

Hence, in D_9, there is a pair of adjacent vertices a and b with $f(a) = 1$ and $f(b) = 8$.

Step 7: Therefore, in D_{10}, there will be a copy of D_1 with vertices $\{a, b\}$ corresponding to the vertices $\{x, y\}$ of D_1 (as in Fig 3.15).

Step 8: Now notice that, in D_{12}, there are $u_i \in N(a)$ such that $f(u_i) = i$ for all $i \in \{3, 4, ..., 9\}$. Hence, in D_{13}, there are $u_i^{\alpha\beta} \in N^{\alpha}(a) \cap N^{\beta}(u_i)$ for all $\alpha, \beta \in \{+, -\}$.

Step 9: Note that it is not possible to have $p \in N^+(a)$ and $q \in N^-(a)$ with $f(p) = f(q)$. Hence the function $F_a(i) = \alpha$ if $t \in N^{\alpha}(a)$ and $f(t) = i$ for $i \in \{3, 4, ..., 9\}$ and $\alpha \in \{+, -\}$, is well defined. Intuitively, the function F_a is a function indicating whether a label is used for an in-neighbor of a or for an out-neighbor of a.

Step 10: Note that for each $i \in \{3, 4, ..., 9\}$, $F_a(f(u_i^{++})) = F_a(f(u_i^{+-})) = +$ and $F_a(f(u_i^{-+})) = F_a(f(u_i^{--})) = -$.

Also, notice that $\{f(u_i^{\alpha\beta}) | \alpha, \beta \in \{+, -\}\} = \{3, 4, ..., 9\} \setminus \{i-1, i, i+1\}$ for each $i \in \{4, 5, 6, 7, 8\}$.

We will use the two above observations repeatedly in the following.

Step 11: Let $\{\gamma, \overline{\gamma}\} = \{+, -\}$. Without loss of generality assume that $F_a(3) = \gamma$.

Claim: $F_a(6) = \gamma$.

Proof of the claim: If possible, let $F_a(6) = \overline{\gamma}$. Now $\{f(u_8^{\alpha\beta}) | \alpha, \beta \in \{+, -\}\} = \{3, 4, 5, 6\}$. So two of $F_a(3), F_a(4), F_a(5), F_a(6)$ will be $+$ and the other two will be $-$. But we already have $F_a(3) = \gamma$ and $F_a(6) = \overline{\gamma}$. Hence, $\{F_a(4), F_a(5)\} = \{\gamma, \overline{\gamma}\}$. Similarly, we have $\{f(u_7^{\alpha\beta}) | \alpha, \beta \in \{+, -\}\} = \{3, 4, 5, 9\}$. This will force $F_a(9) = \overline{\gamma}$. After that we have $\{f(u_4^{\alpha\beta}) | \alpha, \beta \in \{+, -\}\} = \{6, 7, 8, 9\}$ which forces $F_a(7) = F_a(8) = \gamma$. Now we also have $\{f(u_5^{\alpha\beta}) | \alpha, \beta \in \{+, -\}\} = \{3, 7, 8, 9\}$. But $F_a(3) = F_a(7) = F_a(8) = \gamma$, a contradiction. Hence, $F_a(6) = \gamma$.

Step 12: Now $\{f(u_8^{\alpha\beta}) | \alpha, \beta \in \{+, -\}\} = \{3, 4, 5, 6\}$ implies $F_a(4) = F_a(5) = \overline{\gamma}$. Similarly, $\{f(u_7^{\alpha\beta}) | \alpha, \beta \in \{+, -\}\} = \{3, 4, 5, 9\}$ implies $F_a(9) = \gamma$. Lastly $\{f(u_4^{\alpha\beta}) | \alpha, \beta \in \{+, -\}\} = \{6, 7, 8, 9\}$ implies $F_a(7) = F_a(8) = \overline{\gamma}$.

Hence, we got the full description of F_a (depending on the value of γ).

Step 13: Similarly, we can define a function F_b (one can imitate the previous steps, or just use symmetry). As $f(a) = 1$, $f(b) = 8$ and $F_a(8) = \overline{\gamma}$, we have $F_b(1) = \gamma$. Now, by symmetry we get $F_b(1) = F_b(2) = F_b(4) = F_b(5) = \gamma$ and $F_b(0) = F_b(3) = F_b(6) = \overline{\gamma}$.

Step 14: Therefore, $F_a(l) \neq F_b(l)$ for all such l on which both the functions are defined. But we have $F_a(f(u_8^{++})) = F_b(f(u_8^{++})) = +$. This is a contradiction.

Hence, we are done. □

We are now able to prove Theorem 3.41.

Proof of Theorem 3.41(a) The proof follows from Lemma 3.43, Lemma 3.44, Lemma 3.35 and Corollary 3.37.

Proof of Theorem 3.41(b), (c) From [57] we know that any partial 3-tree admits a homomorphism to the tromp graph $\overrightarrow{T}_{16}$ and that any partial k-tree admits a homomorphism to the Zielonka graph $\overrightarrow{Z}_{k+1}$. Hence the proof follows using Lemma 3.35 and Corollaries 3.37 and 3.38. □

Proof of theorem 3.40

Every outerplanar graph is also a partial 2-tree. So, the upper bound follows from Theorem 3.41.

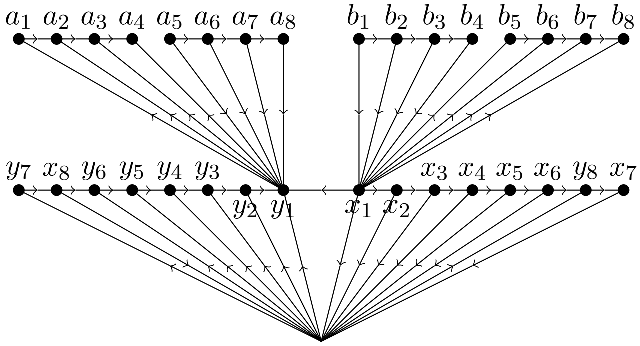


Figure 3.16: The oriented outerplanar graph $\vec{O}$

To prove the lower bound, we will construct an oriented outerplanar graph $\overrightarrow{O^*}$ with $\overrightarrow{\lambda}_{2,1}(\overrightarrow{O^*}) \geq 9$. This will complete the proof.

First, we show that the outerplanar graph $\overrightarrow{O}$ (Figure 3.16) has no 2-dipath 8-$L(2,1)$-labeling if v gets label 1.

Let f be a 2-dipath 8-$L(2,1)$-labeling of $\overrightarrow{O}$ such that $f(v) = 1$. This implies $f(t) \notin \{0, 1, 2\}$ for $t \in \{x_1, x_2, ..., x_8, y_1, ..., y_8\}$ and $f(x_i) \neq f(y_j)$ for any $i, j = 1, 2, ..., 8$.

Clearly, we need at least three distinct labels for each of the sets $\{x_i \mid i = 1, ..., 8\}$ and $\{y_i \mid i = 1, ..., 8\}$. Also, if we use exactly three labels for either of these sets, then those three labels should have pairwise difference at least 2.

To satisfy the above conditions, by symmetry, we may assume without loss of generality that we use labels $\{3, 5, 7\}$ for $\{x_1, ..., x_8\}$ and $\{4, 6, 8\}$ for $\{y_1, ..., y_8\}$.

Now, with these assumptions, the following conditions are forced:

(a) $f(b_i) \notin \{f(x_1) - 1, f(x_1), f(x_1) + 1\}$ for $i = 1, 2, ..., 8$.

(b) $f(b_i) \neq f(b_j)$ for $i \in \{1, 2, 3, 4\}$ and $j \in \{5, 6, 7, 8\}$.

(c) $f(b_i) \notin \{1, f(x_2), f(y_1)\}$ for $i = 1, 2, 3, 4$.

(d) $f(x_i) = f(x_{i+3})$ for all $i = 1, 2, 3$.

(e) we need at least three distinct labels for either of the sets $\{b_1, ..., b_4\}$ and $\{b_5, ..., b_8\}$ for $i = 1, ..., 8$. Also, if we use exactly three labels for either of these sets, then those three labels should have mutual difference at least 2.

There are three cases to consider.

Case 1: If $f(x_1) = 7$, then $f(y_1) = 4$ and $f(x_2) = 3$ or 5.
Then $\{f(b_1), f(b_2), f(b_3), f(b_4)\} = \{0, 2, 5\}$ (by *(a), (c), (e)*).
This implies $\{f(b_5), f(b_6), f(b_7), f(b_8)\} = \{1, 3, 4\}$ (by *(a), (b)*) which contradicts *(e)*.

Case 2: If $f(x_1) = 5$, then $f(y_1) = 8$ and $f(x_2) = 3$ or 7.
Then $\{f(b_1), f(b_2), f(b_3), f(b_4)\} = \{0, 2, 7\}$ (by *(a), (c), (e)*).
Hence $f(x_2) = 3$. This implies $f(x_3) = 7$. Therefore, $f(x_6) = 7$ (by *(d)*).
Now, the only possibility is to have $f(y_8) = 4$ which will force $f(x_7) = 7$ since $f(x_7) \in \{3, 5, 7\}$. But x_6 and x_7 cannot have same labels since they are connected by a 2-dipath through y_8. This is a contradiction.

Case 3: If $f(x_1) = 3$, then $f(y_1) = 6$ or 8 and $f(x_2) = 5$ or 7.
Then $\{f(b_1), f(b_2), f(b_3), f(b_4)\} = \{0, 5, 8\}$ (by *(a), (c), (e)*).
This implies $\{f(b_5), f(b_6), f(b_7), f(b_8)\} = \{1, 6, 7\}$ (by *(a), (b)*) which contradicts *(e)*.

Hence, we do not have a 8-$L(2,1)$-labeling f of $\overrightarrow{O}$ such that $f(v) = 1$. By symmetry, we can say that we do not have a 8-$L(2,1)$-labeling f of $\overrightarrow{O}$ such that $f(v) = 7$.

Now define $S = V(\overrightarrow{O}) \setminus \{x_2, x_7, x_8, y_2, y_7, y_8\}$ and let $\overrightarrow{G} = \overrightarrow{O}[S]$.

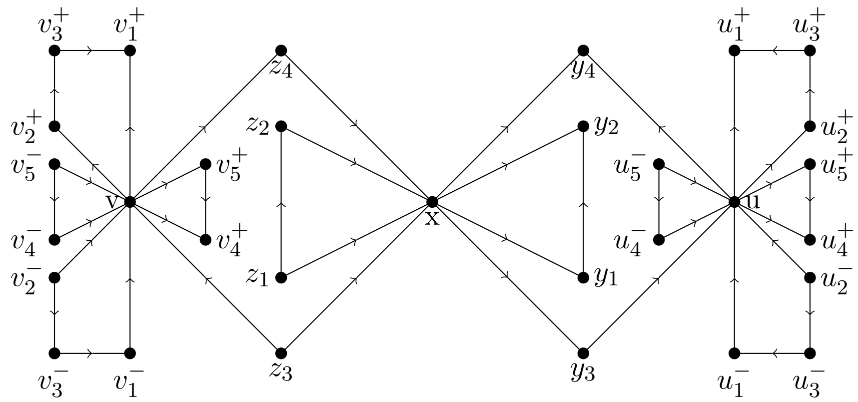


Figure 3.17: The oriented cactus $\vec{H}$.

Notice that if we try to 2-dipath 8-$L(2,1)$-label $\overrightarrow{G}$, then we need to use three different labels for the vertices v, x_1 and y_1. One of these three vertices should have a label $l \notin \{0, 8\}$. To label the neighbors of that vertex, we clearly need at least six labels other than $l-1, l$ and $l+1$. So, we have to use all the remaining six labels and whatever the value of l may be, we necessarily use label 1 or 7 to 2-dipath 8-$L(2,1)$-label $\overrightarrow{G}$.

Now, we construct a new graph $\overrightarrow{O^*}$ by gluing a copy of $\overrightarrow{O}$ on each vertex of $\overrightarrow{G}$ by identifying that vertex of $\overrightarrow{G}$ with the vertex v of $\overrightarrow{O}$.

Note that $\overrightarrow{O^*}$ is an outerplanar graph that cannot admit a 2-dipath 8-$L(2,1)$-labeling, which proves the theorem. □

Proof of Theorem 3.42

Lemma 3.45. *There exists an oriented cactus $\overrightarrow{C}$ with $\overrightarrow{\lambda}_{2,1}(C) \geq 7$.*

Proof. Let $\overrightarrow{H}$ be the oriented cactus depicted in Figure 3.17. We first show that there is no 6-$L(2,1)$-labeling f of $\overrightarrow{H}$ with $f(x) = 2$. Assume to the contrary that such a labeling f exists.

The assumption implies that $f(t) \notin \{1, 2, 3\}$ for $t \in \{z_1, z_2, z_3, z_4, y_1, y_2, y_3, y_4\}$.

Also we have, $f(z_i) \neq f(y_j)$ for $i, j = 1, 2, 3, 4$ and, for $t \in \{y, z\}$,
$\mid f(t_1) - f(t_2) \mid \geq 2$ and $f(t_3) \neq f(t_4)$.

This will force either $\{f(z_3), f(z_4)\} = \{0, 5\}$ or $\{f(y_3), f(y_4)\} = \{0, 5\}$. But these two cases are symmetric. So, without loss of generality, we can assume $\{f(z_3), f(z_4)\} = \{0, 5\}$.

Again, by symmetry, we can assume $f(z_3) = 0$ and $f(z_4) = 5$. This will force $f(v) = 3$. Then $f(t) \notin \{2, 3, 4\}$ for $t \in \{v_1^+, v_2^+, v_4^+, v_5^+, v_1^-, v_2^-, v_4^-, v_5^-\}$.

Similarly as before, we have $f(v_i^+) \neq f(v_j^-)$ for $i, j = 1, 2, 4, 5$ and, for $t \in \{v^+, v^-\}$, $\mid f(t_4) - f(t_5) \mid \geq 2$ and $f(t_1) \neq f(t_2)$.

Moreover, $f(v_i^+) \neq f(z_3) = 0$ and $f(v_i^-) \neq f(z_4) = 5$ for $i = 1, 2, 4, 5$.

This forces $\{f(v_1^+), f(v_2^+)\} = \{5, 1\}$.

Then no label is available for v_3^+, a contradiction.

Hence, we do not have a 6-$L(2,1)$-labeling f of $\overrightarrow{H}$ such that $f(x) = 2$.

Let $\overrightarrow{G}$ be a graph obtained by gluing a copy of the induced subgraph $\overrightarrow{H}[x, y_1, y_2, z_1, z_2]$ on each vertex of the directed 5-cycle $\overrightarrow{C_5}$ by identifying each vertex of $\overrightarrow{C_5}$ with the vertex x of $\overrightarrow{H}[x, y_1, y_2, z_1, z_2]$. Clearly, $\overrightarrow{G}$ is a cactus.

Now, if we 2-dipath 6-$L(2,1)$-label $\overrightarrow{G}$, we need to use at least five labels for the $\overrightarrow{C_5}$ inside it. If 2 is not among those five labels, then at least one of $\{4, 5\}$ is among those five labels. Now, the $\overrightarrow{H}[x, y_1, y_2, z_1, z_2]$ glued with the vertex that got label 4 (or 5) clearly must use label 2. Hence, for any 2-dipath 6-$L(2,1)$-labeling of the cactus $\overrightarrow{G}$, we need to use 2 as one of the labels.

Now, we construct a new graph $\overrightarrow{C}$ by gluing a copy of $\overrightarrow{H}$ on each vertex of $\overrightarrow{G}$ by identifying that vertex of $\overrightarrow{G}$ with the vertex x of $\overrightarrow{H}$.

Note that $\overrightarrow{C}$ is a cactus that cannot admit a 2-dipath 6-$L(2,1)$-labeling. This completes the proof. □

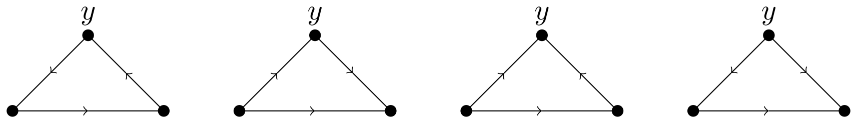


Figure 3.18: Four different oriented 3-cycles with respect to the vertex y.

Let $\overrightarrow{B}$ be the oriented graph depicted in Figure 3.3. Then we have:

Lemma 3.46. *Let $\overrightarrow{O}$ be an oriented cycle. Given any $x \in V(\overrightarrow{O})$ and $y \in V(\overrightarrow{B})$, there exists a homomorphism $h : \overrightarrow{O} \longrightarrow \overrightarrow{B}$ such that $h(x) = y$.*

Proof. We know that $\overrightarrow{B}$ is 4-nice. Hence it is enough to show that for any oriented 3-cycle $\overrightarrow{T}$ and given any $x \in V(\overrightarrow{T})$ and $y \in V(\overrightarrow{B})$, there exists a homomorphism $h : \overrightarrow{O} \longrightarrow \overrightarrow{B}$ such that $h(x) = y$. In other words, we need to show that for each $y \in V(\overrightarrow{B})$, the 3-cycles in Figure 3.18 are subgraphs of $\overrightarrow{B}$, which can easily be checked. □

Lemma 3.47. *Every oriented cactus $\overrightarrow{C}$ admits a homomorphism to $\overrightarrow{B}$.*

Proof. Let $\overrightarrow{G}$ be a minimal counterexample to Lemma 3.47.

If there is a degree one vertex v in $\overrightarrow{G}$ such that $v \in N^+(u)$ (or $v \in N^-(u)$) for some $u \in V(\overrightarrow{G})$, then $\overrightarrow{G}[V(\overrightarrow{G}) \setminus \{v\}]$ is also a cactus. As $\overrightarrow{G}$ is a minimal counterexample, there is a homomorphism f from $\overrightarrow{G}[V(\overrightarrow{G} \setminus \{v\}]$ to $\overrightarrow{B}$. Now, since all the vertices of $\overrightarrow{B}$ have at least one in-neighbor and one out-neighbor, we can extend the homomorphism f to a homomorphism of $\overrightarrow{G}$ to $\overrightarrow{B}$ by mapping v to any vertex $x \in N^+(f(u))$ (or $x \in N^-(f(u))$). This is a contradiction. Hence there cannot be a degree one vertex in $\overrightarrow{G}$.

No vertex of degree one in $\overrightarrow{G}$ implies at least one cycle $\overrightarrow{C} \subseteq \overrightarrow{G}$ such that exactly one vertex z of the cycle $\overrightarrow{C}$ has degree greater than 2 (since, by Lemma 3.46, $\overrightarrow{G}$ cannot be a cycle).

Now, $\overrightarrow{G}[V(\overrightarrow{G}) \setminus \{V(\overrightarrow{C}) \setminus \{z\}\}]$ is a cactus and, since $\overrightarrow{G}$ is a minimal counterexample, there is a homomorphism f from $\overrightarrow{G}[V(\overrightarrow{G}) \setminus \{V(\overrightarrow{C}) \setminus \{z\}\}]$ to $\overrightarrow{B}$. By Lemma 3.46, we can extend f to a homomorphism of $\overrightarrow{G}$ to $\overrightarrow{B}$, a contradiction. This completes the proof. □

We are now able to prove Theorem 3.42.

The proof of Theorem 3.42 follows from Lemma 3.45, Lemma 3.47, Lemma 3.35 and the fact that $\lambda^o_{2,1}(\overrightarrow{B}) = 7$ (from Fig: 3.3). □

3.5 Conclusion

In this chapter we mainly studied oriented colorings and oriented $L(p,q)$-labelings of some families of planar graphs. Our main focus remained on the families $\mathcal{O}_g$ (family of outerplanar graphs with girth at least g) and $\mathcal{P}_g$ (family of planar graphs with girth at least g) for $g \geq 3$.

Concerning oriented colorings we focused on determining the oriented chromatic number, the oriented relative clique number and the oriented absolute clique number of the above mentioned families.

The oriented chromatic number of these families is a well studied problem. The existing bounds for this parameter are tight for the family of outerplanar graphs with girth at least g and the family of planar graphs with girth at least k for all $g \geq 3$ and for all $k \geq 12$. So, the bounds are not tight for the family of planar graphs with girth at least g for $3 \leq g \leq 11$. These bounds seem difficult to improve.

Though we could not improve any of these existing bounds, we want to comment that it might be possible to construct a planar graph with girth at least 5 with oriented chromatic number at least 8 using the graph depicted in Fig. 3.5 with techniques similar to those used for proving the lower bound of Theorem 3.39(c). This would improve the lower bound of $\chi_o(\mathcal{P}_5)$. Of course, even though we have a rough idea of how this might work, we are yet to do it.

For the other two parameters, that is the oriented relative clique number and the oriented absolute clique number, we provided improved bounds, and mostly tight bounds in that matter.

In fact, we provided tight bounds for $\omega_{ao}(\mathcal{O}_g)$ and for $\omega_{ao}(\mathcal{P}_g)$ for all $g \geq 3$. We also provided tight bounds for $\omega_{ro}(\mathcal{O}_g)$ for all $g \geq 3$ and for $\omega_{ro}(\mathcal{P}_k)$ for all $k \geq 5$. Even though we could not provide exact bounds for $\omega_{ro}(\mathcal{P}_3)$ and for $\omega_{ro}(\mathcal{P}_4)$, we have an intuition about what the answer could be.

So we propose the following conjectures.

Conjecture 3.48. $\omega_{ro}(\mathcal{P}_3) = 15$.

Notice that we already know $\omega_{ro}(\mathcal{P}_3) \geq 15$. So we already know that the lower bound of the conjecture is tight. Right now the upper bound for $\omega_{ro}(\mathcal{P}_3)$ is 80. Even though we do not have any idea to prove Conjecture 3.48, we want to comment that the upper bound of $\omega_{ro}(\mathcal{P}_3)$ might be improved using a similar idea used by Klostermeyer and Macgillivray [27] to prove $\omega_{ao}(\mathcal{P}_3) \leq 36$.

Conjecture 3.49. $\omega_{ro}(\mathcal{P}_4) = 10$.

Notice that we already know $\omega_{ro}(\mathcal{P}_4) \geq 10$. So we already know that the lower bound of the conjecture is tight.

Notice that, by proving $\omega_{ao}(\mathcal{P}_3) = 15$, we actually proved that the largest (in terms of number of vertices) planar oclique is of order 15. We can ask the similar question for graphs on other surfaces, such as the torus or, in general, the surfaces with genus k for some $k \geq 1$.

We showed that the only minimal (with respect to subgraph inclusion) planar oclique of order 15 is the oriented graph depicted in Fig. 3.6. We think that it is possible to extend the proof of the above mentioned result to list out all the minimal planar ocliques. The task, though tedious, is possible to do with a little modification of the proof.

We also studied two different $L(p,q)$-labeling problems, namely, the 2-dipath $L(p,q)$-labeling problem and the oriented $L(p,q)$-labeling problem, on oriented graphs. We tried to determine the two corresponding labeling spans for some planar families. We mainly focused on the particular case $(p,q) = (2,1)$.

However, we want to remark that studying the oriented chromatic number and the oriented relative clique number of graphs is actually equivalent to studying the oriented $L(1,1)$-labeling span and the 2-dipath $L(1,1)$-labeling span, respectively. It will be interesting to study $L(p,q)$-labeling of oriented graphs for other values of (p,q) as well.

Finally, trying to bound all these parameters for specific families of graphs is a big pool of problems in the domain of oriented coloring and oriented $L(p,q)$-labeling.

Chapter 4
Orientable graphs

In this chapter we deal with orientable graphs. Our main focus is to present some results regarding orientable colorings.

Vertex pushing, that is, reversing the orientation of the incident arcs of a vertex, of directed graphs has been studied by Fisher and Ryan [16], who studied the operation in tournaments and counted the number of non-isomorphic equivalence classes of tournaments that the push operation induces. Klostermeyer [25] [26] studied the operation in powers of graphs with particular emphasis on graphs whose all orientations can be pushed to have a Hamilton cycle. Additional works on push operation has been done by Mosesian [37], Pretzel [50] [49] [48] and MacGillivray and Wood [30].

The push operation was used on oriented graphs by Ochem and Pinlou [45] while proving the upper bounds of the oriented chromatic number for the families of triangle-free planar graphs and of 2-outerplanar graphs. Finally, Klostermeyer and MacGillivray [28] considered the push operation on oriented graphs to define equivalence classes of oriented graphs and studied homomorphisms between them. These graphs were named "push graphs". We, in this thesis, propose the name "orientable graphs" instead as it readily suggests a relation with oriented graphs.

There is a similarity in the way orientable graphs and signed graphs (see Chapter 6) are defined. Hence, we speculate that the study of homomorphisms of orientable graphs will help develop a rich theory similar to the one introduced and studied by Naserasr, Rollová and Sopena [40] for signed graphs.

The organization of the chapter is as follows. In Section 4.1 we give the basic definitions and notations related to orientable graphs and homomorphisms of orientable graphs. Then we present our main results regarding orientable coloring in Section 4.2. In Section 4.3 we discuss some categorical aspects which is a joint work with Naserasr and Sopena. Finally, we conclude this chapter in Section 4.4.

The proof of Theorem 4.17 and Theorem 4.22(a) proved in Section 4.2 appeared as a poster at the Eurcomb 2013 Conference [56]. The rest of the results proved in Section 4.2 is an article in process.

4.1 Preliminaries

Recall the definitions and notations related to oriented graphs from Chapter 3. To *push* a vertex v of an oriented graph $\overrightarrow{G}$ is to change the orientations of all the arcs (that is, to replace the arc $\overrightarrow{uv}$ by $\overrightarrow{vu}$) incident with v. Two oriented graphs $\overrightarrow{G}^1$ and $\overrightarrow{G}^2$ are in a *push relation* if it is possible to obtain $\overrightarrow{G}^2$ by pushing some vertices of $\overrightarrow{G}^1$. Note that this push relation is in fact an equivalence relation. A *push graph* or an *orientable graph* $[\overrightarrow{G}]$ is an equivalance class of oriented

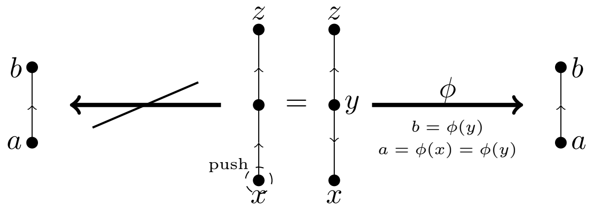


Figure 4.1: Orientable graph homomorphism.

graphs (where $\overrightarrow{G}^1$ is an element of the equivalence class) with respect to the above mentioned relation. An element $\overrightarrow{G}^1$ of the equivalence class $[\overrightarrow{G}]$ is a *presentation* of $[\overrightarrow{G}]$. We use the notation $\overrightarrow{G}^1 \in [\overrightarrow{G}]$ for $\overrightarrow{G}^1$ is a presentation of $[\overrightarrow{G}]$.

Note that the graphs having a push relation have the same underlying graph. Hence, we can define the *underlying graph* of an orientable graph $[\overrightarrow{G}]$ by the underlying graph of any presentation of it and denote it by G. The *order* of an orientable graph is the number of vertices of its underlying graph, hence can be denoted by $|V(\overrightarrow{G})|$ or $|V(G)|$. Intuitively, we can treat an orientable graph as an oriented graph whose arcs, incedent to a vertex, are able to switch directions. For a fixed presentation of an orientable graph, we can use the notations defined for oriented graphs. Given any oriented graph $\overrightarrow{G}$ we can consider the orientable graph $[\overrightarrow{G}]$.

Note that two oriented trees whose underlying graphs are isomorphic to the same tree are equivalent with respect to the push relation. Given a connected graph G with n vertices and m edges the minimum number of edges we need to delete in order to obtain a tree from G is exactly $m-n+1$ (it follows from the fact that a connected tree on n vertices has $n-1$ edges). So, given a graph G with n vertices, m edges and c connected components the minimum number of edges we need to delete in order to obtain a tree from G is exactly $m-n+c$.

Now let G be a graph with n vertices, m edges and c connected components. Also let T be a forest obtained from G by deleting $m-n+c$ edges. Now fix an orientation $\overrightarrow{T}$ of T. Clearly every orientable graph whose underlying graph is G has a presentation which contains $\overrightarrow{T}$ as a subgraph. The following result follows from counting the number of distinct ways we can orient the deleted edges of G.

Proposition 4.1. *If an undirected graph G has n vertices, m edges and c connected components, then there are 2^{m-n+c} distinct orientable graphs with underlying graph G.*

Some of the "distinct" orientable graphs can actually be isomorphic. The distinction in the above theorem works considering fixed labels for the vertices of G. So up to isomorphism, there are at most 2^{m-n+c} distinct orientable graphs with underlying graph G. Babai and Cameron had studied automorphism of orientable graphs [3].

In [28], Klostermeyer and MacGillivray introduced homomorphisms of orientable graphs. An orientable graph $[\overrightarrow{G}]$ admits a homomorphism ϕ to an orientable graph $[\overrightarrow{H}]$ if there are presentations $\overrightarrow{G}^1 \in [\overrightarrow{G}]$ and $\overrightarrow{H}^1 \in [\overrightarrow{H}]$ such that ϕ is a homomorphism of $\overrightarrow{G}^1$ to $\overrightarrow{H}^1$. We write $[\overrightarrow{G}] \to [\overrightarrow{H}]$ whenever there exists a homomorphism of $[\overrightarrow{G}]$ to $[\overrightarrow{H}]$ and say that $[\overrightarrow{H}]$ *bounds* $[\overrightarrow{G}]$. A bijective homomorphism whose inverse is also a homomorphism is an *isomorphism.* If two orientable graphs admit homomorphisms to each other then they are *homomorphically equivalent* orientable graphs.

Lemma 4.2. *If $[\overrightarrow{G}]$ admits a homomorphism to $[\overrightarrow{H}]$, then for any presentation $\overrightarrow{H}^1$ of $[\overrightarrow{H}]$ there exists a presentation $\overrightarrow{G}^1$ of $[\overrightarrow{G}]$ such that $\overrightarrow{G}^1$ admits a homomorphism to $\overrightarrow{H}^1$.*

Proof. Let ϕ be a homomorphism of $[\overrightarrow{G}]$ to $[\overrightarrow{H}]$. This implies ϕ is a homomorphism of $\overrightarrow{G}^1$ to $\overrightarrow{H}^1$, for some presentation $\overrightarrow{G}^1$ and $\overrightarrow{H}^1$ of $[\overrightarrow{G}]$ and $[\overrightarrow{H}]$ respectively. Now, let $\overrightarrow{H}^2$ be any presentation of $[\overrightarrow{H}]$.

Let $X = \{x_1, x_2, ..., x_k\}$ be the set of vertices we need to push to obtain $\overrightarrow{H}^2$ from $\overrightarrow{H}^1$. Now push the vertices of the set $\{v \in V(G) | \phi(v) \in X\}$ to obtain the presentation $\overrightarrow{G}^2 \in [\overrightarrow{G}]$ from $\overrightarrow{G}^1$.

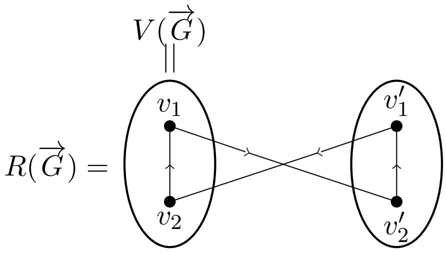


Figure 4.2: The anti-twined graph $R(\overrightarrow{G})$ of $[\overrightarrow{G}]$.

Clearly, ϕ is a homomorphism of $\overrightarrow{G}^2$ to $\overrightarrow{H}^2$. □

The above lemma allows us to study homomorphisms of an orientable graph $[\overrightarrow{G}]$ to an oriented graph $\overrightarrow{H}$.

Example 4.3. *A sample homomorphism of orientable graphs is given in Fig. 4.1. Note that we need to push one of the vertices to obtain the homomorphism.*

Let $[\overrightarrow{G}]$ be an orientable graph with vertex set $V(G) = \{v_1, v_2, ..., v_k\}$ and $\overrightarrow{G}^1 \in [\overrightarrow{G}]$. Then the *anti-twined graph* $R(\overrightarrow{G})$ of $[\overrightarrow{G}]$ is the oriented graph with the set of vertices and the set of arcs as the following:

$$V(R(\overrightarrow{G})) = \{v_1, v_2, ..., v_k\} \cup \{v'_1, v'_2, ..., v'_k\}$$
$$A(R(\overrightarrow{G})) = \{\overrightarrow{v_iv_j}, \overrightarrow{v'_iv'_j}, \overrightarrow{v_jv'_i}, \overrightarrow{v'_jv_i} \mid \overrightarrow{v_iv_j} \in A(\overrightarrow{G}^1)\}.$$

Intuitively, $R(\overrightarrow{G})$ is the graph obtained from $[\overrightarrow{G}]$ by adding and pushing a twin vertex v'_i for each of the vertices v_i of $\overrightarrow{G}^1$. Observe that $R(\overrightarrow{G})$ is well defined, that is, for any presentation of $[\overrightarrow{G}]$, we will get the same oriented graph $R(\overrightarrow{G})$. The anti-twined graph $R(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}^1$ was defined and used by Klostermeyer and MacGillivray in [28].

Example 4.4. *For better understanding of the definition of an anti-twined graph see Fig. 4.2 where we have pictorially presented the construction of the anti-twined graph of an orientable graph* $[\overrightarrow{G}]$.

Now suppose the orientable graph $[\overrightarrow{G}]$ admits a homomorphism ϕ to the orientable graph $[\overrightarrow{H}]$. Also suppose that the presentations $\overrightarrow{G}^1 \in [\overrightarrow{G}]$ and $\overrightarrow{H}^1 \in [\overrightarrow{H}]$ are such that ϕ is a homomorphism of $\overrightarrow{G}^1$ to $\overrightarrow{H}^1$. The *anti-twined homomorphism* $R(\phi)$ of ϕ is the homomorphism of $R(\overrightarrow{G})$ to $R(\overrightarrow{H})$ such that we have,

$$R(\phi)(v) = \phi(v) \text{ and } R(\phi)(v') = \phi(v)' \text{ for } v \in V(\overrightarrow{G}).$$

Observe that $R(\phi)$ is indeed a homomorphism. Also $R(\phi)$ is well defined, that is, it does not depend on the choice of the presentations $\overrightarrow{G}^1 \in [\overrightarrow{G}]$ and $\overrightarrow{H}^1 \in [\overrightarrow{H}]$ such that ϕ is a homomorphism of $\overrightarrow{G}^1$ to $\overrightarrow{H}^1$.

The following result, proved in [28], directly follows from the above definitions.

Proposition 4.5. *Given two orientable graphs* $[\overrightarrow{G}]$ *and* $[\overrightarrow{H}]$, *we have* $[\overrightarrow{G}] \to [\overrightarrow{H}]$ *if and only if* $R(\overrightarrow{G}) \to R(\overrightarrow{H})$.

Proof. First assume that $[\overrightarrow{G}]$ admits a homomorphism ϕ to $[\overrightarrow{H}]$. Then according to the above discussion $R(\phi)$ is a homomorphism of $R(\overrightarrow{G})$ to $R(\overrightarrow{H})$.

Now assume that $R(\overrightarrow{G}) \to R(\overrightarrow{H})$. Also we have $\overrightarrow{G} \to R(\overrightarrow{G})$ by inclusion. Now consider the orientable graphs $[R(\overrightarrow{G})]$ and $[R(\overrightarrow{H})]$. Then we must have orientable homomorphisms $[\overrightarrow{G}] \to$

$[R(\overrightarrow{G})]$ and $[R(\overrightarrow{G})] \to [R(\overrightarrow{H})]$. By composing the above two homomorphisms we have an orientable homomorphism $[\overrightarrow{G}] \to [R(\overrightarrow{H})]$.

Now consider the presentation $\overrightarrow{H} \in [\overrightarrow{H}]$. Consider the function $\psi : V(R(\overrightarrow{H})) \to \overrightarrow{H}$ defined as the following:

$$\psi(x) = \psi(x') = x \text{ for } x \in V(\overrightarrow{H}).$$

It is easy to check that ψ is a homomorphism of $R(\overrightarrow{H})$ to $\overrightarrow{H}$. Hence, there exists an orientable homomorphism $[R(\overrightarrow{H})] \to [\overrightarrow{H}]$. By composing it with a homomorpism $[\overrightarrow{G}] \to [R(\overrightarrow{H})]$ we will obtain a homomorphism of $[\overrightarrow{G}]$ to $[\overrightarrow{H}]$. □

A *splitable oriented graph* $\overrightarrow{S}$ is an oriented graph isomorphic to the anti-twined graph $R(\overrightarrow{T})$ of some oriented graph $\overrightarrow{T}$. The oriented graph $\overrightarrow{T}$ is the *split graph* of $\overrightarrow{S}$.

Similarly, a *splitable oriented homomorphism* is a homomorphism ψ of a splitable oriented graph $\overrightarrow{S_1} = R(\overrightarrow{T_1})$ to a splitable oriented graph $\overrightarrow{S_2} = R(\overrightarrow{T_2})$ such that we have $\psi = R(\phi)$ for some orientable homomorphism ϕ of $[\overrightarrow{T_1}]$ to $[\overrightarrow{T_2}]$.

Notice that the set of vertices of any splitable oriented graph $\overrightarrow{S}$ can be partitioned into two equal parts and a 1-1 correspondence between the vertices of those two parts can be established in a way that the corresponding vertices are not adjacent and they disagree with each other on all their common neighbors, while they have the same set of neigbors. The oriented induced subgraph on one of those partitions is the split graph of the splitable graph in question. Hence we have the following result.

Lemma 4.6. *An oriented graph $\overrightarrow{S}$ is splitable if and only if it is possible to partition the set of vertices $V(\overrightarrow{S})$ into two equal parts V_1 and V_2 with a bijection $f : V_1 \to V_2$ such that for all $u \in V_1$ we have $N^+(u) = N^-(f(u))$ and $N^-(u) = N^+(f(u))$.*

For example, for each vertex u in a Tromp graph or a Zielonka graph (see Chapter 3, Section 3.2) there is a unique vertex v such that we have $N^+(u) = N^-(v)$ and $N^-(u) = N^+(v)$. So, it is easy to see that these graphs are splitable oriented graphs.

Using the notion of splitable graphs we now state the following useful result.

Lemma 4.7. *Let $\overrightarrow{S} = R(\overrightarrow{T})$ be a splitable graph. Then $\overrightarrow{G} \to \overrightarrow{S}$ if and only if $[\overrightarrow{G}] \to \overrightarrow{T}$.*

Proof. Let $\overrightarrow{S} = R(\overrightarrow{T})$ be a splitable graph. Assume that $\overrightarrow{G} \to \overrightarrow{S}$. This implies $[\overrightarrow{G}] \to [\overrightarrow{S}]$. Now consider the following function ψ from $V(R(\overrightarrow{T}))$ to $V(\overrightarrow{T})$:

$$\psi(x) = \psi(x') = x \text{ for } x \in V(\overrightarrow{T}).$$

It is easy to check that ψ is a homomorphism of $R(\overrightarrow{T})$ to $\overrightarrow{T}$. This implies that there exists an orientable homomorphism $[R(\overrightarrow{T})] \to [\overrightarrow{T}]$. By composing this homomorphism with the homomorphism $[\overrightarrow{G}] \to [\overrightarrow{S}]$ we obtain a homomorphism $[\overrightarrow{G}] \to \overrightarrow{T}$. This proofs the "only if" part.

For proving the "if" part assume $[\overrightarrow{G}] \to \overrightarrow{T}$. Then by Proposition 4.5 we have $R(\overrightarrow{G}) \to \overrightarrow{S}$. By composing this homomorphism with the inclusion homomorphism of $\overrightarrow{G}$ to $R(\overrightarrow{G})$ we will be done. □

We will use this lemma several times in this chapter.

4.2 Orientable coloring

Colorings of orientable graphs first appeared in the work of Klostermeyer and MacGillivray [28] (they called it push coloring) and is a recent field of research.

An *orientable k-coloring* of an orientable graph $[\overrightarrow{G}]$ is a vertex coloring which is an oriented k-coloring of a presentation of the graph. The *orientable chromatic number* $\chi_{[o]}([\overrightarrow{G}])$ of an

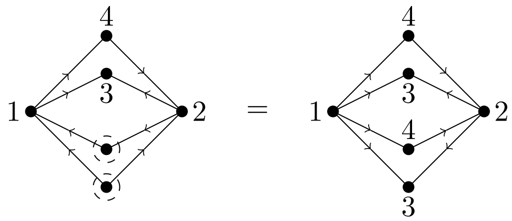


Figure 4.3: Orientable coloring (we push the vertices marked with dashed circles of the graph in the left).

orientable graph $[\overrightarrow{G}]$ is the minimum of the oriented chromatic numbers of the elements of the equivalence class $[\overrightarrow{G}]$.

Alternatively, the orientable chromatic number $\chi_{[o]}([\overrightarrow{G}])$ of the orientable graph $[\overrightarrow{G}]$ is the minimum order of an orientable graph $[\overrightarrow{H}]$ such that $[\overrightarrow{G}]$ admits a homomorphism to $[\overrightarrow{H}]$.

By virtue of Lemma 4.2, we can equivalently define the orientable chromatic number $\chi_{[o]}([\overrightarrow{G}])$ of an orientable graph $[\overrightarrow{G}]$ by the minimum order of an oriented graph $\overrightarrow{H}$ such that $[\overrightarrow{G}]$ admits a homomorphism to $\overrightarrow{H}$.

The orientable chromatic number $\chi_{[o]}(G)$ of an undirected graph G is the maximum of the oriented chromatic numbers of all the orientable graphs with underlying graph G. The orientable chromatic number $\chi_{[o]}(\mathcal{F})$ of a family $\mathcal{F}$ of graphs is the maximum of the orientable chromatic numbers of the graphs from the family $\mathcal{F}$.

Example 4.8. *Fig 4.3 depicts an orientable 4-coloring of an orientable graph. One can easily check that the orientable chromatic number of this graph is 4.*

Now, take an oriented cycle of length 4 with arcs $\overrightarrow{ab}, \overrightarrow{bc}, \overrightarrow{cd}, \overrightarrow{ad}$. Note that all the oriented graphs which are in push relation with it are isomorphic to it. We name this special 4-cycle an *unbalanced 4-cycle*. Notice that the non-adjacent vertices of an unbalanced 4-cycle always get different colors as they are always connected with a 2-dipath, no matter which vertex of the graph you push. This is in fact a necessary and sufficient condition for two non-adjacent vertices to receive two distinct colors under an orientable coloring.

A *relative orientable clique* of an orientable graph $[\overrightarrow{G}]$ is a set $R \subseteq V(\overrightarrow{G})$ of vertices such that any two vertices from R are either adjacent or part of an unbalanced 4-cycle. The *orientable relative clique number* $\omega_{[ro]}([\overrightarrow{G}])$ of an orientable graph $[\overrightarrow{G}]$ is the maximum order of an orientable relative clique of $[\overrightarrow{G}]$.

An *orientable clique*, or simply an *[o]-clique*, is an orientable graph $[\overrightarrow{G}]$ for which $\chi_{[o]}([\overrightarrow{G}]) = |V(\overrightarrow{G})|$. Note that [o]-cliques can hence be characterized as those orientable graphs whose any two distinct vertices are either adjacent or part of an unbalanced 4-cycle (for any presentation). Note that an orientable graph with an [o]-clique of order n as a subgraph has orientable chromatic number and orientable relative clique number at least n. The *orientable absolute clique number* $\omega_{[ao]}([\overrightarrow{G}])$ of an orientable graph $[\overrightarrow{G}]$ is the maximum order of an [o]-clique contained in $[\overrightarrow{G}]$ as a subgraph.

The orientable relative clique number $\omega_{[ro]}(G)$ (resp. orientable absolute clique number $\omega_{[ao]}(G)$) of a simple graph G is the maximum of the orientable relative clique numbers (resp. orientable absolute clique numbers) of all the orientable graphs with underlying graph G. The orientable relative clique number $\omega_{[ro]}(\mathcal{F})$ (resp. orientable absolute clique number $\omega_{[ao]}(\mathcal{F})$) of a family $\mathcal{F}$ of graphs is the maximum of the orientable relative clique numbers (resp. orientable absolute clique numbers) of the graphs from the family $\mathcal{F}$.

From the definitions, clearly we have the following:

Lemma 4.9. *For any orientable graph $[\overrightarrow{G}]$ we have, $\omega_{[ao]}([\overrightarrow{G}]) \leq \omega_{[ro]}(\overrightarrow{G}) \leq \chi_{[o]}([\overrightarrow{G}])$.*

Corollary 4.10. *For any [o]-clique $[\overrightarrow{O}]$ we have, $\omega_{[ao]}([\overrightarrow{O}]) = \omega_{[ro]}(\overrightarrow{O}) = \chi_{[o]}([\overrightarrow{O}])$.*

Example 4.11. *Consider the oriented graph $\overrightarrow{B}^+$ obtained by adding a new vertex ∞ to the oriented graph $\overrightarrow{B}$ depicted in Fig. 3.5 (see Chapter 3, Section 3.3) in such a way that we have $N^+(\infty) = V(\overrightarrow{B} \setminus \{\infty\}$. It is easy to check that $\omega_{[ao]}([\overrightarrow{B}^+]) = 6$, $\omega_{[ro]}([\overrightarrow{B}^+]) = 7$ and $\chi_{[o]}([\overrightarrow{B}^+]) = 8$. This is an example of a graph for which each inequality of the above theorem is strict.*

Note that the above defined three graph parameters respect homomorphisms of orientable graphs in the sense of the following result.

Lemma 4.12. *Let $[\overrightarrow{G}]$ and $[\overrightarrow{H}]$ be two orientable graphs. If $[\overrightarrow{G}] \to [\overrightarrow{H}]$, then $\chi_{[o]}([\overrightarrow{G}]) \leq \chi_{[o]}([\overrightarrow{H}])$, $\omega_{[ro]}([\overrightarrow{G}]) \leq \omega_{[ro]}([\overrightarrow{H}])$ and $\omega_{[ao]}([\overrightarrow{G}]) \leq \omega_{[ao]}([\overrightarrow{H}])$.*

A usual technique for obtaining an upper bound for the three graph parameters of orientable graphs defined in this section is to prove that every graph in the family of graphs in question admits a homomorphism to a particular orientable graph, that is, to find an orientable graph that bounds every graph of that family. Such a graph is called a *universal bound* of that family of graphs. Note that not every family of graphs have a universal bound of order equal to its orientable chromatic number (the family of all orientable graphs of order 4 is such an example).

If we consider the set of all orientable graphs to be a category with objects being the orientable graphs and morphisms being the orientable homomorphisms then we clearly have the following:

Theorem 4.13. *For any family $\mathcal{F}$ of orientable graphs that also contains the categorical co-products of the graphs from the family, there exists a universal bound of $\mathcal{F}$ on $\chi_{[o]}(\mathcal{F})$ vertices.*

Observe that a categorical co-product (unique up to homomorphic equivalence) of orientable graphs is simply the orientable graph obtained by taking the disjoint union of the orientable graphs. The families of planar graphs, outerplanar graphs, planar graphs with given girth and outerplanar graphs with given girth are each of the type that we mentioned in the above theorem.

4.2.1 Orientable chromatic number

The first result proved by Klostermeyer and MacGillivray [28] on orientable chromatic number is the following relation between oriented chromatic numbers and orientable chromatic numbers.

Proposition 4.14. *For any oriented graph $\overrightarrow{G}$, we have $\chi_{[o]}([\overrightarrow{G}]) \leq \chi_o(\overrightarrow{G}) \leq 2\chi_{[o]}([\overrightarrow{G}])$.*

In the first relation, equality holds for any oriented graph whose underlying graph is a complete graph, while in the second relation, equality holds for splitable graphs.

One of the main general results we can prove using Lemma 4.7 follows easily from the fact that Zielonka graphs are splitable.

Theorem 4.15. *Every graph with acyclic chromatic number at most k has orientable chromatic number at most $k.2^{k-2}$.*

As the bound in Theorem 3.12 is tight for $k \geq 3$, it is tight for $k \geq 3$ in the above theorem too by Proposition 4.5.

Now we list the bounds for the orientable chromatic number of the families of outerplanar graphs and of outerplanar graphs with given girth. The relevant references are given beside the results. Recall that $\mathcal{O}_g$ denotes the family of outerplanar graphs with girth at least g.

Theorem 4.16.

(a) $\chi_{[o]}(\mathcal{O}_k) = 4$ *for* $k = 3, 4$. *[28]*

(b) $\chi_{[o]}(\mathcal{O}_k) = 3$ *for* $k \geq 5$.

Now we list the bounds for the orientable chromatic number of the families of planar graphs and of planar graphs with given girth. Recall that $\mathcal{P}_g$ denotes the family of planar graphs with girth at least g.

Theorem 4.17.

(a) $9 \leq \chi_{[o]}(\mathcal{P}_3) \leq 40$.

(b) $6 \leq \chi_{[o]}(\mathcal{P}_4) \leq 20$.

(c) $4 \leq \chi_{[o]}(\mathcal{P}_5) \leq 8$.

(d) $4 \leq \chi_{[o]}(\mathcal{P}_6) \leq 7$.

(e) $\chi_{[o]}(\mathcal{P}_8) = 4$.

(f) $3 \leq \chi_{[o]}(\mathcal{P}_k) \leq 4$ *for* $9 \leq k \leq 20$.

(g) $\chi_{[o]}(\mathcal{P}_k) = 3$ *for* $k \geq 21$.

We prove the lower bound of part (a) of the above theorem by constructing an example. Even though the lower bound immediately follows from Marshall's [33] proof of $18 \leq \chi_o(\mathcal{P}_3)$, our proof is independent of that and is simpler.

Proof of Theorem 4.16

(a) Klostermeyer and MacGillivray [28] proved that the orientable chromatic number of the family of outerplanar graphs is at most 4. It is easy to note that the bound is tight, even for the family of outerplanar graphs with girth at least 4, as the unbalanced 4-cycle has orientable chromatic number 4.

(b) In [47], Pinlou and Sopena showed that every outerplanar graph with girth at least k and minimum degree at least 2 contains a face of length $l \geq k$ with at least $(l-2)$ consecutive vertices of degree 2.

Now, to prove Theorem 4.16(b), we will show that every orientable outerplanar graph of girth at least 5 admits a homomorphism to the directed 3-cycle $\overrightarrow{C_3}$.

Let $[\overrightarrow{H}]$ be a minimal (with respect to inclusion as a subgraph) orientable outerplanar graph with girth at least 5 having no homomorphism to $\overrightarrow{C_3}$.

(i) Suppose that $[\overrightarrow{H}]$ contains a vertex u of degree 1. Then, due to the minimality of $[\overrightarrow{H}]$, the orientable outerplanar graph obtained by deleting the vertex u from $[\overrightarrow{H}]$ (which has girth at least 5) admits a homomorphism to $\overrightarrow{C_3}$. Since every vertex of $\overrightarrow{C_3}$ has in-degree and out-degree equal to 1, the homomorphism can easily be extended to obtain a homomorphism of $[\overrightarrow{H}]$ to $\overrightarrow{C_3}$, a contradiction.

(ii) Suppose that $[\overrightarrow{H}]$ contains a face $ux_1x_2...x_{l-2}v$ of length $l \geq 5$ with at least $(l-2)$ consecutive vertices $x_1, x_2, ..., x_{l-2}$ of degree 2. Then, due to the minimality of $[\overrightarrow{H}]$, the orientable outerplanar graph $[\overrightarrow{H'}]$ obtained by deleting the vertices $x_1, x_2, ..., x_{l-2}$ from $[\overrightarrow{H}]$ (which has girth at least 5) admits a homomorphism ϕ to $\overrightarrow{C_3}$. Now, let $\overrightarrow{H'}^1$ be a presentation of $[\overrightarrow{H'}]$ with $\phi : \overrightarrow{H'}^1 \to \overrightarrow{C_3}$. Note that the vertices u and v are adjacent in $\overrightarrow{H'}^1$. Hence, $\phi(u) \neq \phi(v)$.

It is possible to check that (a bit tidious, but not difficult, by case analysis), given any oriented path of length $m \geq 4$, with edges $a_1a_2, a_2a_3, ..., a_{m-1}a_m$ and a mapping $\psi : \{a_1, a_m\} \to V(\overrightarrow{C_3})$ with $\psi(a_1) \neq \psi(a_m)$, it is possible to push the vertices a_i for $i \in \{2, .., m-1\}$ to obtain an oriented path and extend the mapping ψ to a homomorphism of that oriented path to $\overrightarrow{C_3}$.

Hence, by the above observation, we can extend the homomorphism of $[\overrightarrow{H'}]$ to $\overrightarrow{C_3}$ to a homomorphism of $[\overrightarrow{H}]$ to $\overrightarrow{C_3}$, a contradiction.

Hence, every orientable outerplanar graph with girth at least 5 admits a homomorphism to $\overrightarrow{C_3}$. Of course, any cycle of odd length has orientable chromatic number at least 3. Hence the bound is tight. □

Proof of Theorem 4.17(a),(b) and (c)

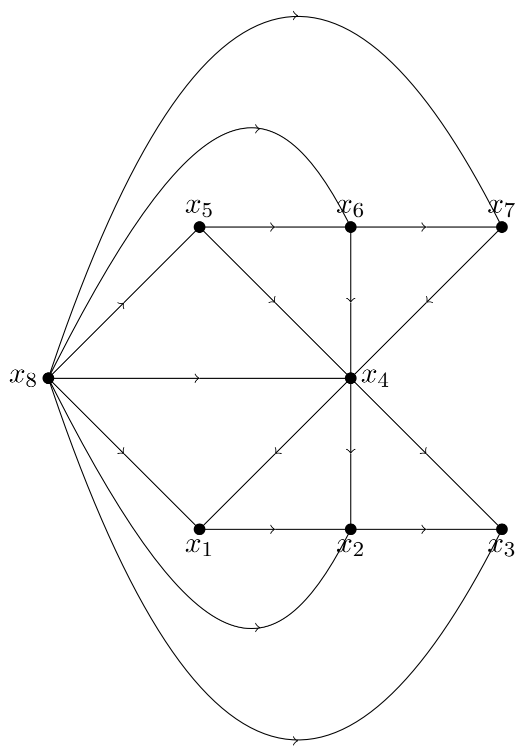


Figure 4.4: The planar [o]-clique $\vec{B}_0$ of order 8.

(a) To prove the upper bound we use Theorem 4.15 and the theorem of Borodin [4] (see Chapter 2, Theorem 2.11) that states that every planar graph has an acyclic 5-coloring.

For the lower bound, we show that there is no universal bound of the family of orientable planar graphs on 8 veritces.

The planar graph from Fig 4.4 is an orientable planar [o]-clique on 8 vertices. Now assume that $\overrightarrow{H}$ is an oriented graph of order 8 to which every orientable planar graph admits a homomorphism.

Then we construct an oriented planar graph $\overrightarrow{B_1}$ by gluing a copy of the planar [o]-clique $\overrightarrow{B_0}$ (Fig. 4.4) to each vertex of $\overrightarrow{B_0}$ by identifying the vertex with the vertex x_8 of $\overrightarrow{B_0}$. After that we construct another oriented planar graph $\overrightarrow{B_2}$ by gluing a copy of the planar [o]-clique $\overrightarrow{B_0}$ to each arc of $\overrightarrow{B_1}$ by identifying the arc with the arc $\overrightarrow{x_8x_4}$ of $\overrightarrow{B_0}$.

By assumption $[\overrightarrow{B_2}]$ admits a homomorphism to $\overrightarrow{H}$. Now, each vertex of the inital [o]-clique $\overrightarrow{B_0}$ dominates an [o]-clique of order 8 in $[\overrightarrow{B_2}]$. So, in particular, we must have the degree of each vertex in $\overrightarrow{H}$ at least 7. Hence, $\overrightarrow{H}$ is a tournament.

Now note that as $[\overrightarrow{B_2}]$ admits a homomorphism to $\overrightarrow{H}$, for each arc $\overrightarrow{uv}$ of $\overrightarrow{H}$ there are at least three vertices on which u and v agree and there are at least three vertices on which u and v disagree. No matter how you push the vertices, this will be true. But as $\overrightarrow{H}$ has order 8, for each arc $\overrightarrow{uv}$, the oriented graph $\overrightarrow{H}$ must have exactly three vertices on which u and v agree and exactly three vertices on which u and v disagree.

We now construct a planar graph with two vertices x and y and a directed 5-cycle with vertices a, b, c, d, e such that $N^+(x) = \{a, b, c, d, e\}$, $N^+(y) = \{a, b, c, d\}$ and $N^+(y) = \{e\}$. Since this is a planar graph, it admits a homomorphism to $\overrightarrow{H}$. The images of x and y are different vertices (they are part of an unbalanced 4-cycle) and have at least 4 vertices on which they either agree or disagree (those vertices are images of a, b, c, d which have pairwise distinct images as they are pairwise either adjacent or part of an unbalanced 4-cycle) which contradicts the above discussion.

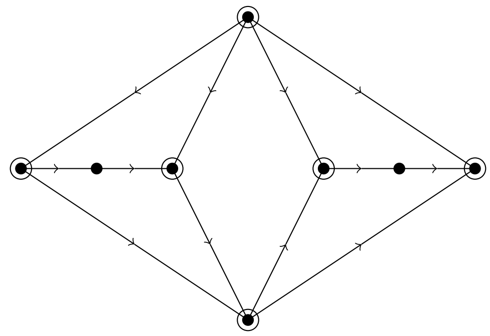


Figure 4.5: An orientable triangle-free planar graph with orientable relative clique number 6 (the six vertices with black circles around them are part of a relative clique).

Hence there is no universal bound of the family of orientable planar graphs on 8 vertices. □

(b) Ochem and Pinlou [45] showed that the Tromp graph $\overrightarrow{T}_{40}$ bounds the family of oriented planar graphs with girth at least 4. Note that the Tromp graph $\overrightarrow{T}_{40}$ is a splitable oriented graph whose split graph is the graph $\overrightarrow{P}^{+}_{19}$ on 20 vertices. Hence, by Lemma 4.7, the upper bound follows.

The lower bound follows from the example in Fig. 4.5.

(c) The lower bound follows from the fact that every cycle of odd length has orinetable chromatic number 3.

Pinlou [46] showed that the Tromp graph $\overrightarrow{T}_{16}$ bounds the family of oriented planar graphs with girth at least 5. Note that the Tromp graph $\overrightarrow{T}_{16}$ is a splitable oriented graph whose split graph is the graph $\overrightarrow{P}^{+}_{7}$ on 8 vertices. Hence, by Lemma 4.7, the upper bound follows. □

Proof of Theorem 4.17(d),(e)

To prove Theorem 4.17(d) and (e) we use the discharging method. We first provide a (small) set of *forbidden confgurations*, that is a set of graphs that a minimal counterexample $[\overrightarrow{H}]$ to our claim cannot contain as subgraphs. We will then assume that every vertex v in $[\overrightarrow{H}]$ is valued by its degree $deg(v)$ and define a *discharging procedure* which specifies some transfer of values among the vertices in $[\overrightarrow{H}]$, keeping the sum of all the values constant. We will then get a contradiction by considering the *modernized degree* $deg^*(v)$ of every vertex v, that is the value obtained by v owing to the discharging procedure.

Drawing conventions: In all the figures depicting forbidden configurations, we will draw vertices with prescribed degrees as 'square vertices' and vertices with unbounded degree as 'circular vertices'. All the neighbors of square vertices are drawn. Unless otherwise specified, two or more circular vertices may coincide in a single vertex, provided that they do not share a common square neighbor.

(d) The lower bound follows from the lower bound of Theorem 4.17(e) whose proof is given independently of this proof.

For proving the upper bound we will show that every orientable $[\overrightarrow{G}]$ with maximum average degree less than 3 admits a homomorphism to the Paley tournament $\overrightarrow{P}_7$ (see Chapter 3, Section 3.2). We will use the discharging method for our proof.

Observation 1: Given a pair of distinct vertices u, v of $\overrightarrow{P}_7$, the set $N^{\alpha}_{\overrightarrow{P}_7} \cap N^{\beta}_{\overrightarrow{P}_7}$ is non-empty for any $\alpha, \beta \in \{+, -\}$.

Observation 2: It directly follows from Observation 1 that $N^{++}(\{v\}) = N^{--}(\{v\}) = V(\overrightarrow{P}_7) \setminus \{v\}$ and $N^{+-}(\{v\}) = N^{-+}(\{v\}) = V(\overrightarrow{P}_7)$ for all $v \in V(\overrightarrow{P}_7)$ (recall the definition of $N^{Q}(X)$ for a pattern Q and a set of vertices X from Chapter 3, Section 3.2).

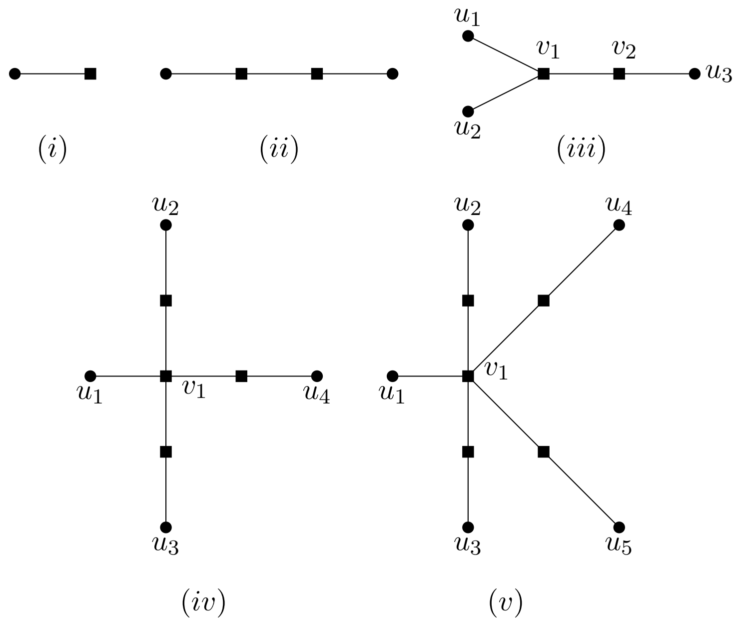


Figure 4.6: Forbidden configurations for Theorem 4.17(d).

First assume that $[\overrightarrow{H}]$ is a mimimal (with respect to the number of vertices) orientable graph with maximum average degree less than 3 that does not admit a homomorphism to the Paley tournament $\overrightarrow{P}_7$.

First we will show that $[\overrightarrow{H}]$ does not contain any of the configurations depicted in Fig. 4.6.

(i) Obvious since every vertex of $\overrightarrow{P}_7$ has degree at least one.

(ii) Directly follows from Observation 2.

(iii) Consider the orientable graph $[\overrightarrow{H'}]$ obtained by deleting the vertex v_2 of $[\overrightarrow{H}]$. Therefore, there exists a presentation $\overrightarrow{H'}^1 \in [\overrightarrow{H'}]$ such that $\overrightarrow{H'}^1$ admits a homomorphism f' to $\overrightarrow{P}_7$.

Now consider a presentation $\overrightarrow{H}^1 \in [\overrightarrow{H}]$ that contains $\overrightarrow{H'}^1$ as a subgraph and is such that the vertices v_1 and u_3 agree with each other on v_2 (such a presentation is possible to obtain by pushing v_1 if needed). Now we can extend f' to a homomorphism (we might need to change the value of $f'(v_1)$ in the extension) of $\overrightarrow{H}^1$ to $\overrightarrow{P}_7$ by Observation 1 and Observation 2.

(iv) Consider the orientable graph $[\overrightarrow{H'}]$ obtained by deleting the square vertex of degree 3 from $[\overrightarrow{H}]$. Therefore, there exists a presentation $\overrightarrow{H'}^1 \in [\overrightarrow{H'}]$ such that $\overrightarrow{H'}^1$ admits a homomorphism f' to $\overrightarrow{P}_7$.

Now choose a vertex $x \in V(\overrightarrow{P}_7) \setminus \{f'(u_1), f'(u_2), f'(u_3), f'(u_4)\}$. Suppose that $x \in N^{\alpha}_{\overrightarrow{P}_7}(f'(u_1))$.

Now consider the presentation $\overrightarrow{H}^1 \in [\overrightarrow{H}]$ that contains $\overrightarrow{H'}^1$ as a subgraph and is such that $v_1 \in N^{\alpha}_{\overrightarrow{H}^1}(u_1)$ (such a presentation is possible to obtain by pushing v_1 if needed).

Now we can extend f' to a homomorphism of f of $\overrightarrow{H}^1$ to $\overrightarrow{P}_7$ with $f(v_1) = x$ using Observation 2.

(v) Consider the orientable graph $[\overrightarrow{H'}]$ obtained by deleting the square vertex of degree 3 from $[\overrightarrow{H}]$. Therefore, there exists a presentation $\overrightarrow{H'}^1 \in [\overrightarrow{H'}]$ such that $\overrightarrow{H'}^1$ admits a homomorphism f' to $\overrightarrow{P}_7$.

Now choose a vertex $x \in V(\overrightarrow{P}_7) \setminus \{f'(u_1), f'(u_2), f'(u_3), f'(u_4), f'(u_5)\}$. Suppose that $x \in N^{\alpha}_{\overrightarrow{P}_7}(f'(u_1))$.

Now consider the presentation $\overrightarrow{H}^1 \in [\overrightarrow{H}]$ that contains $\overrightarrow{H'}^1$ as a subgraph and is such that $v_1 \in N^{\alpha}_{\overrightarrow{H}^1}(u_1)$ (such a presentation is possible to obtain by pushing v_1 if needed).

Now we can extend f' to a homomorphism of f of $\overrightarrow{H}^1$ to $\overrightarrow{P}_7$ with $f(v_1) = x$ using Observation 2.

We now use the following discharging procedure: each vertex of degree at least 4 gives 1/2 to each of its neighbors with degree 2.

Let us check that the modernized degree $deg^*(v)$ of each vertex v is at least 3, which contradicts the assumption $mad(H) < 3$. We consider the possible cases for the old degree $deg(v)$ of v:

(i) $deg(v) = 1$: there is no such vertex in $[\overrightarrow{H}]$ by (i).

(ii) $deg(v) = 2$: by (ii) and (iii), its neighbors have degree at least 4. Therefore, it receives exactly $2 \times 1/2 = 1$, and thus $deg^*(v) = 2 + 1 = 3$.

(iii) $deg(v) = 3$: by (iii), it neither receives nor gives away anything. Therefore, we have $deg^*(v) = 3$.

(iv) $deg(v) = 4$: by (iv), it gives away at most $2 \times 1/2 = 1$. Therefore, we have $deg^*(v) \geq 4 - 1 = 3$.

(v) $deg(v) = 5$: by (v), it gives away at most $3 \times 1/2 = 3/2$. Therefore, we have $deg^*(v) \geq 5 - 3/2 = 7/2 > 3$.

(vi) $deg(v) = k \geq 6$: it gives away at most $k \times 1/2 = k/2$. Therefore, we have $deg^*(v) \geq k - k/2 = k/2 \geq 6/2 = 3$.

Therefore, every vertex of $[\overrightarrow{H}]$ gets a modernized degree at least 3. Hence, every orientable graph with maximum average degree less than 3 admits a homomorphism to $\overrightarrow{P}_7$. Hence our theorem is proved using Theorem 2.5 (see Chapter 2).

(e) Take the directed 9-cycle $\overrightarrow{C_9}$. Now construct the graph $\overrightarrow{H}$ by taking $\overrightarrow{C_9}$ and a new vertex v and then connecting each vertex of $\overrightarrow{C_9}$ to v by two distinct paths of length 4 (one of them directed and the other with three forward arcs and one backward arc). Now consider the orientable graph $[\overrightarrow{H}]$. If $\chi_{[o]} \leq 3$, then there must be a presentation of $[\overrightarrow{H}]$ that admits a homomorphism to the directed 3-cycle $\overrightarrow{C_3}$.

Notice that for any presentation $\overrightarrow{H} \in [\overrightarrow{H}]$ we will have one 4-path, with either three forward arcs and one backward arc or with three backward arcs and one forward arc, connecting v to each vertex of the 9-cycle.

Observe that the 4-path $x_0x_1...x_4$ with three forward arcs and one backward arc does not admit a homomorphism with x_0 and x_4 mapped to the same vertex of $\overrightarrow{C_3}$. Now let f be a homomorphism of $\overrightarrow{H}$ to $\overrightarrow{C_3}$. Then, because of the above observation, $f(v) \neq f(u)$ for every vertex u from the 9-cycle. But we know that the 9-cycle has orientable chromatic number equal to 3. That means f must be onto on the vertices of $\overrightarrow{C_3}$ when restricted to the 9-cycle. Hence $f(v) \notin V(\overrightarrow{C_3})$. This is a contradiction.

Hence we have the lower bound.

For proving the upper bound we will show that every orientable $[\overrightarrow{G}]$ with maximum average degree less than 8/3 admits a homomorphism to the Paley plus graph $\overrightarrow{P}^+_3$. We will use the discharging method for our proof.

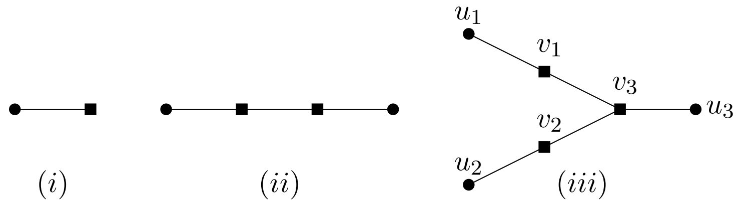


Figure 4.7: Forbidden configurations for Theorem 4.17(e).

Observation 3: It is easy to check that $N^{++}(\{v\}) \cup N^{--}(\{v\}) = V(\overrightarrow{P}_3^+) \setminus \{v\}$ and $N^{+-}(\{v\}) \cup N^{-+}(\{v\}) = V(\overrightarrow{P}_3^+)$ for all $v \in V(\overrightarrow{P}_3^+)$ (recall the definition of $N^Q(X)$ for a pattern Q and a set of vertices X from Chapter 3, Section 3.2).

First assume that $[\overrightarrow{H}]$ is a mimimal (with respect to the number of vertices) orientable graph with maximum average degree less than 8/3 that does not admit a homomorphism to $\overrightarrow{P}_3^+$.

First we will show that $[\overrightarrow{H}]$ does not contain any of the configuration depicted in Fig. 4.7.

(i) Obvious since every vertex of $\overrightarrow{P}_3^+$ has degree at least one.

(ii) Directly follows from Observation 3.

(iii) Consider the orientable graph $[\overrightarrow{H'}]$ obtained by deleting all the square vertex of degree 3 from $[\overrightarrow{H}]$. Therefore, there exists a presentation $\overrightarrow{H'}^1 \in [\overrightarrow{H'}]$ such that $\overrightarrow{H'}^1$ admits a homomorphism f' to $\overrightarrow{P}_7$.

Now choose a vertex $x \in V(\overrightarrow{P}_3^+) \setminus \{f'(u_1), f'(u_2), f'(u_3)\}$. Suppose that $x \in N^{\alpha}_{\overrightarrow{P}_3^+}(f'(u_3))$.

Now consider the presentation $\overrightarrow{H}^1 \in [\overrightarrow{H}]$ that contains $\overrightarrow{H'}^1$ as a subgraph and is such that $v_3 \in N^{\alpha}_{\overrightarrow{H}^1}(u_3)$ (such a presentation is possible to obtain by pushing v_3 if needed).

Now we can extend f' to a homomorphism of f of $\overrightarrow{H}^1$ to $\overrightarrow{P}_3^+$ (by pushing the vertices v_1 and v_2 if needed) with $f(v_1) = x$ using Observation 3.

We now use the following discharging procedure: each vertex of degree at least 3 gives 1/3 to each of its neighbors with degree 2.

Let us check that the modernized degree $deg^*(v)$ of each vertex v is at least 8/3 which contradicts the assumption $mad(H) < 8/3$. We consider the possible cases for the old degree $deg(v)$ of v:

(i) $deg(v) = 1$: there is no such vertex in $[\overrightarrow{H}]$ by (i).

(ii) $deg(v) = 2$: by (ii), both its neighbors have degree at least 3. Therefore, it receives exactly $2 \times 1/3 = 2/3$, and thus $deg^*(v) = 2 + 2/3 = 8/3$.

(iii) $deg(v) = 3$: by (iii), gives away at most 1/3. Therefore, we have $deg^*(v) = 3 - 1/3 = 8/3$.

(iv) $deg(v) = k \geq 4$: it gives away at most $k \times 1/3 = k/2$. Therefore, we have $deg^*(v) \geq k - k/3 = 2k/3 \geq 8/3$.

Therefore, every vertex of $[\overrightarrow{H}]$ gets a modernized degree at least 8/3. Hence, every orientable graph with maximum average degree less than 8/3 admits a homomorphism to $\overrightarrow{P}_3^+$. Hence our theorem is proved using Theorem 2.5 (see Chapter 2). □

Proof of Theorem 4.17(f),(g)

(f) The lower bound follows from the fact that every cycle of odd length has orientable chromatic number 3.

The upper bound follows from the upper bound proved in part (e) of this theorem.

(g) The lower bound follows from the fact that every cycle of odd length has orientable chromatic number 3.

Consider the anti-twined graph $R(\overrightarrow{C_3})$ of the directed 3-cycle $\overrightarrow{C_3}$. It is possible to check that the oriented graph $R(\overrightarrow{C_3})$ is 5-nice. Hence by Theorem 3.4 (see Chapter 3, Section 3.2) we know that every oriented planar graph with girth at least 21 admits a homomorphism to the oriented graph $R(\overrightarrow{C_3})$. Now Lemma 4.7 implies that every orientable planar graph with girth at least 21 admits a homomorphism to the directed 3-cycle $\overrightarrow{C_3}$. Hence, we have the upper bound. □

4.2.2 Orientable relative clique number

As we have done for oriented colorings, naturally we also define and consider the orientable relative clique number for some planar families of graphs. Before listing those bounds, we will present a general bound for the oriented relative clique number of graphs with maximum degree Δ.

Proposition 4.18. *Every orientable graph with maximum degree Δ has orientable relative clique number at most* $\frac{\Delta(\Delta+1)}{2}+1$.

Proof. Let $[\overrightarrow{G}]$ be an orientable graph with maximum degree Δ. Let R be a relative clique of maximum order in of $[\overrightarrow{G}]$. Let $v \in R$ be a vertex. Now, v has Δ adjacent vertices and each of these vertices can have at most $(\Delta-1)$ adjacent vertices excluding v. But, if a vertex u at distance 2 from v is in R, then it has to be part of an unbalanced 4-cycle of which also v is part of. For that u needs to be adjacent to at least two neighbors of v. There are at most $\Delta.(\Delta-1)$ edges between the neighbors of v and the vertices at distance 2 from v. Now, there are $(|R|-\Delta-1)$ vertices of R that are each adjacent to at least two neighbors of v. Hence we have,

$$\begin{aligned}
2(|R|-\Delta-1) \leq \Delta.(\Delta-1) &\Rightarrow 2|R|-2\Delta-2 \leq \Delta^2-\Delta \\
&\Rightarrow 2|R| \leq \Delta^2+\Delta+2 \\
&\Rightarrow |R| \leq \frac{\Delta.(\Delta+1)}{2}+1
\end{aligned}$$

Hence, we are done. □

We consider the problem of determining the orientable relative clique number of the families of outerplanar graphs and of outerplanar graphs with given girth. We list the related results below.

Theorem 4.19.

(a) $\omega_{[ro]}(\mathcal{O}_k) = 4$ *for* $k = 3, 4$.

(b) $\omega_{[ro]}(\mathcal{O}_k) = 2$ *for* $k \geq 5$.

The proof of the above theorem directly follows from Theorem 4.16.

Now we list the best bounds for the orientable relative clique number of the families of planar graphs and of planar graphs with given girth.

Theorem 4.20.

(a) $8 \leq \omega_{[ro]}(\mathcal{P}_3) \leq 40$.

(b) $6 \leq \omega_{[ro]}(\mathcal{P}_4) \leq 17$.

(c) $\omega_{[ro]}(\mathcal{P}_k) = 2$ *for* $k \geq 5$.

Proof of Theorem 4.20

(a) The upper bound follows from Theorem 4.17(a) while the lower bound follows from Fig. 4.4.

(b) Let $[\overrightarrow{G}]$ be a triangle-free planar orientable graph of minimum order with $\omega_{[ro]}([\overrightarrow{G}]) > 17$. Let R be a relative clique of maximum order of $[\overrightarrow{G}]$ and let $S = V(\overrightarrow{G}) \setminus R$.

Claim 1: For any $v \in V(\overrightarrow{G})$ we have, $|N(v) \cap R| \leq 4$.

Proof of Claim 1: Push the neighbors of v in such a way that we obtain a presentation of $[\overrightarrow{G}]$ with $N(v) = N^+(v)$. Now, as $[\overrightarrow{G}]$ is triangle-free, each pair of neighbors of v must be part of an unbalanced 4-cycle. For a pair of neighbors of v to be part of an unbalanced 4-cycle, we must have a directed 2-path joining them. Now by the proof of Claim 1 of the proof of Theorem 3.18(b), we know this is not possible. Hence, our claim is proved. $\Diamond$

Now note that for any $z \in S$, we have $d(z) \geq 2$ as otherwise z can be deleted to get an orientable triange-free planar graph whose relative orientable chromatic number is equal to that of $[\overrightarrow{G}]$ but with order less than $[\overrightarrow{G}]$, which contradicts the minimality of $[\overrightarrow{G}]$. Now, a vertex z of S must connect at least two vertices of R the internal vertex being itself (that is, z).

Now for each vertex $z \in S$ with $d(z) \leq 5$, assume that the neighbors of z are $v_1, v_2, ..., v_k$. Fix a planar embedding of $[\overrightarrow{G}]$ and assume that the neighbors of z are arranged in a clockwise order around z. Now delete the vertex z and add the edges v_1v_2, v_2v_3, ..., $v_{k-1}v_k$, v_kv1 (for $d(z) = 2$ add only one edge v_1v_2) to obtain a new graph.

Note that this new graph is also planar (may not be triangle-free) and the degree of each vertex in the new graph, which were in $[\overrightarrow{G}]$ also, is as much as the degree of the vertex in $[\overrightarrow{G}]$. Hence, there is a vertex v in the new graph, which belongs to R, with degree at most 5.

As each vertex from $R \setminus N(v) \cup \{v\}$ is adjacent to at least two neighbors of v for being part of the same unbalanced 4-cycle with v, there will be a neighbor u of v and at least five neighbors from $R \setminus N(v) \cup \{v\}$ which is a contradiction to Claim 1. So, we can conclude that $\omega_{[ro]}([\overrightarrow{G}]) \leq 17$.

(c) As any two non-adjacent vertices of an orientable relative clique must be part of an unbalanced 4-cycle, the vertices of an orientable relave clique in a planar graph with girth at least 5 must be all adjacent to each other. So, it is not possible to have a planar orientable relative clique with girth at least 5 of order more than 2. Hence the upper bound.

An arc has oriented relative clique number 2. Hence the lower bound. $\square$

4.2.3 Orientable absolute clique number

As we have done for oriented colorings, we also define and consider the orientable absolute clique number for some planar families of graphs. First we list the results regarding the orientable absolute clique number of the families of outerplanar graphs and of outerplanar graphs with given girth.

Theorem 4.21.

(a) $\omega_{[ao]}(\mathcal{O}_k) = 4$ *for* $k = 3, 4$.

(b) $\omega_{[ao]}(\mathcal{O}_k) = 2$ *for* $k \geq 5$.

The above result follows directly from Theorem 4.19.

Now we list the results regarding the orientable absolute clique number of the families of planar graphs and of planar graphs with given girth.

Theorem 4.22.

(a) $\omega_{[ao]}(\mathcal{P}_3) = 8$.

(b) $\omega_{[ao]}(\mathcal{P}_4) = 4$.

(c) $\omega_{[ao]}(\mathcal{P}_k) = 2$ *for* $k \geq 5$.

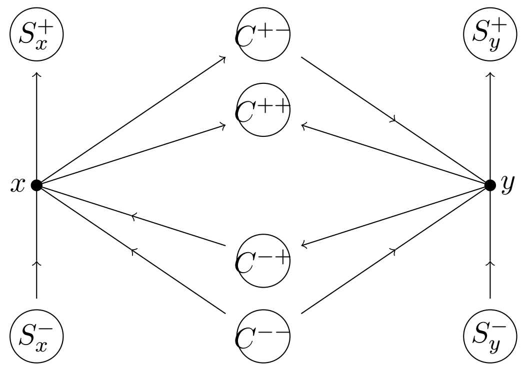


Figure 4.8: Structure of $\vec{G}$ (not a planar embedding)

The parts (b) and (c) of the above theorem easily follow from the list of triangle-free planar graphs with diameter 2 given by Plesník (see Chapter 2, Theorem 2.14). We do prove part (a) of the theorem.

Proof of Theorem 4.22(a)

The lower bound follows from the example of the planar [o]-clique of order 8 depicted in Fig. 4.4.

Now we will prove the upper bound.

We know that an [o]-clique is an orientable graph with each pair of non-adjacent vertices in the same unbalanced 4-cycle. So, each pair of non-adjacent vertices of an [o]-clique is connected by two distinct 2-paths (one of them is directed while the other is not). In particular, an [o]-clique has diameter at most 2.

Goddard and Henning (see Chapter 2, Theorem 2.15) showed that every planar graph with diameter 2 has domination number at most 2 except for a particular graph on nine vertices (see Fig. 2.2). It is easy to find a pair of vertices in this graph which are not connected by two distinct 2-paths. Therefore, it is not an [o]-clique. Hence, planar [o]-cliques must have domination number at most 2.

Let $[\overrightarrow{B}]$ be a planar [o]-clique dominated by the vertex v. Fix the presentation $\overrightarrow{B} \in [\overrightarrow{B}]$ such that we have $N^+_{\overrightarrow{B}}(v) = N_{\overrightarrow{B}}(v)$ (it is possible to obtain such a presentation by pushing all the in-neighbors of v from any presentation of $[\overrightarrow{B}]$). Note that every vertex of $N_{\overrightarrow{B}}(v)$ is connected by a 2-path with internal vertex v which is not a 2-dipath. So, each pair of non-adjacent vertices from $N_{\overrightarrow{B}}(v)$ must be connected by a 2-dipath with its internal vertex from $N_{\overrightarrow{B}}(v)$. Therefore, the oriented induced subgraph $\overrightarrow{B}[N(v)]$ is an oclique. Notice that $\overrightarrow{B}[N(v)]$ is also an outerplanar graph. Now Sopena [58] showed that any oriented outerplanar graph has an oriented 7-coloring. Therefore, $[\overrightarrow{B}]$ has order at most 8.

To prove Theorem 4.22 it will be enough to prove that any planar [o]-clique with domination number 2 must have order at most 8. More precisely, we need to prove the following lemma.

Lemma 4.23. *Let $[\overrightarrow{H}]$ be a planar [o]-clique with domination number 2. Then $|V(\overrightarrow{H})| \leq 8$.*

Let $[\overrightarrow{G}]$ be a planar [o]-clique with $|V(\overrightarrow{G})| > 8$. Assume that $[\overrightarrow{G}]$ is triangulated and has domination number 2. Now fix the presentation $\overrightarrow{G} \in [\overrightarrow{G}]$ and without loss of generality assume that we have $N^+_{\overrightarrow{G}}(v) = N_{\overrightarrow{G}}(v)$ (it is possible to obtain such a presentation by pushing all the in-neighbors of v from any presentation of $[\overrightarrow{G}]$).

We define the partial order $\prec$ for the set of all dominating sets of order 2 of $\overrightarrow{G}$ as follows: for any two dominating sets $D = \{x, y\}$ and $D' = \{x', y'\}$ of order 2 of $\overrightarrow{G}$, $D' \prec D$ if and only if $|N_{\overrightarrow{G}}(x') \cap N_{\overrightarrow{G}}(y')| < |N_{\overrightarrow{G}}(x) \cap N_{\overrightarrow{G}}(y)|$.

Let $D = \{x, y\}$ be a maximal dominating set of order 2 of $\overrightarrow{G}$ with respect to $\prec$. Also for the rest of this section, $t, t', \alpha, \overline{\alpha}, \beta, \overline{\beta}$ are variables satisfying $\{t, t'\} = \{x, y\}$ and $\{\alpha, \overline{\alpha}\} = \{\beta, \overline{\beta}\} = \{+, -\}$.

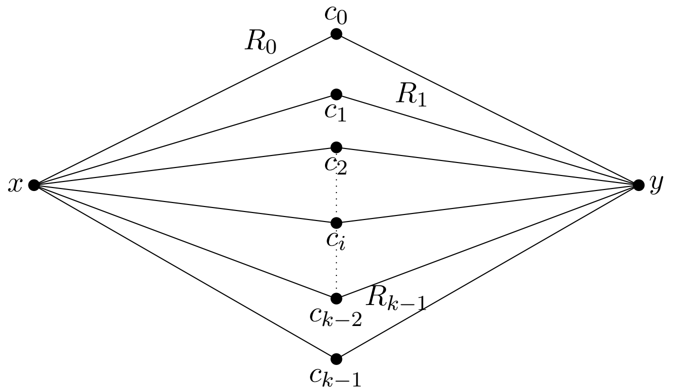


Figure 4.9: A planar embedding of $und(\vec{H})$

Now, we fix the following notations (Fig: 4.8):

$$C = N_{\overrightarrow{G}}(x) \cap N_{\overrightarrow{G}}(y),\ C^{\alpha\beta} = N^{\alpha}_{\overrightarrow{G}}(x) \cap N^{\beta}_{\overrightarrow{G}}(y),\ C^{\alpha}_t = N^{\alpha}_{\overrightarrow{G}}(t) \cap C,$$
$$S_t = N_{\overrightarrow{G}}(t) \setminus C,\ S^{\alpha}_t = S_t \cap N^{\alpha}_{\overrightarrow{G}}(t) \text{ and } S = S_x \cup S_y.$$

Hence we have,

$$9 \leq |\overrightarrow{G}| = |D| + |C| + |S|. \tag{4.1}$$

Let $\overrightarrow{H}$ be the oriented graph obtained from the induced subgraph $\overrightarrow{G}[D \cup C]$ of $\overrightarrow{G}$ by deleting all the arcs between the vertices of D and all the arcs between the vertices of C. Note that it is possible to extend the planar embedding of $und(\overrightarrow{H})$ given in Fig 4.9 to a planar embedding of $und(\overrightarrow{G})$ for some particular ordering of the elements of, say $C = \{c_0, c_1, ..., c_{k-1}\}$.

Notice that $und(\overrightarrow{H})$ has k faces, namely the unbounded face F_0 and the faces F_i bounded by edges $xc_{i-1}, c_{i-1}y, yc_i, c_ix$ for $i \in \{1, ..., k-1\}$. Geometrically, $und(\overrightarrow{H})$ divides the plane into k connected components. The *region* R_i of $\overrightarrow{G}$ is the i^{th} connected component (corresponding to the face F_i) of the plane. *Boundary points* of a region R_i are c_{i-1} and c_i for $i \in \{1, ..., k-1\}$ and, c_0 and c_{k-1} for $i = 0$. Two regions are *adjacent* if they have at least one common boundary point (hence, a region is adjacent to itself).

Now for the different possible values of $|C|$, we want to show that $und(\overrightarrow{H})$ cannot be extended to a planar [o]-clique of order at least 9. Note that for extending $und(\overrightarrow{H})$ to $\overrightarrow{G}$ we can add new vertices only from S. Any vertex $v \in S$ will be inside one of the regions R_i. If there is at least one vertex of S in a region R_i, then R_i is *non-empty* and *empty* otherwise. In fact, when there is no chance of confusion, R_i might represent the set of vertices of S contained in the region R_i.

First we will ask the question that "How small $|C|$ can be?" and prove the following lower bound of $|C|$.

Lemma 4.24. $|C| \geq 3$.

Proof. We know that x and y are either connected by two distinct 2-paths or by an arc. So, if x and y are non-adjacent, then we have $|C| \geq 2$. If x and y are adjacent, then the triangulation of $\overrightarrow{G}$ implies $|C| \geq 2$. Hence we have

$$|C| \geq 2.$$

To complete the proof we need to show that $|C| \neq 2$. We will prove by contradiction. Therefore, assume that $|C| = 2$. To get a contradiction to our assumption, by equation 4.1, it will be enough to show

$$|S| \leq 4.$$

Note that, if we have $S_t = \emptyset$, then triangulation will force either mutiple edges c_0c_1 (one in R_0 and one in R_1) or the edge xy making x a dominating vertex. Both contradicts our assumption. Hence we do not have $S_t = \emptyset$ for any $t \in \{x, y\}$.

First assume that all the four sets $S_t \cap R_i \neq \emptyset$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$. In this case, to have two distinct 2-paths connecting a vertex $u \in S_t \cap R_0$ and a vertex $v \in S'_t \cap R_1$, both u and v must be adjacent to both c_0 and c_1. For that we must have $|S_t \cap R_i| \leq 1$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$. This implies,

$$|S| \leq 4.$$

Hence we cannot have all the four sets $S_t \cap R_i \neq \emptyset$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$.

Now assume that we have exactly three non-empty sets among the four sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$. Without loss of generality we can assume that the three non-empty sets are $S_x \cap R_0$, $S_x \cap R_1$ and $S_y \cap R_0$. Hence by triangulation we have the edge c_0c_1 inside R_1. Now to have two distinct 2-paths connecting a vertex $u \in S_y \cap R_0$ and a vertex $v \in S_x \cap R_1$, both u and v must be adjacent to both c_0 and c_1. For that we must have,

$$|S_x \cap R_1| \leq 1 \text{ and } |S_y \cap R_0| \leq 1.$$

By triangulation, there is at least one vertex in $|S_x \cap R_0|$ adjacent to c_0. Now we have the dominating set $\{x, c_0\}$ with at least three common neighbors (c_1, a vertex from $S_x \cap R_0$ and a vertex from $S_x \cap R_1$) contradicting the maximality of D. Hence we cannot have three of the four sets $S_t \cap R_i$ non-empty for all $(t, i) \in \{x, y\} \times \{0, 1\}$.

Now assume that we have at most two non-empty sets among the four sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$. As we cannot have $S_t = \emptyset$ for any $t \in \{x, y\}$, we must have exactly two non-empty sets of the form $S_x \cap R_i$ and $S_y \cap R_j$ for some $i, j \in \{0, 1\}$.

If $i \neq j$, then to have two distinct 2-paths connecting a vertex $u \in S_x \cap R_i$ and a vertex $v \in S_y \cap R_j$, both u and v must be adjacent to both c_0 and c_1. For that we must have,

$$|S_x \cap R_1| \leq 1 \text{ and } |S_y \cap R_0| \leq 1.$$

This will imply,

$$|S| \leq 2.$$

So, we can assume $i = j$.

Therefore, assume without loss of generality that exactly the sets $S_x \cap R_0$ and $S_y \cap R_0$ are the two non-empty sets among the four sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$.

For this case, assume that $S_x = \{x_1, x_2, \ldots, x_{n_x}\}$ and $S_y = \{y_1, y_2, ..., y_{n_y}\}$. Without loss of generality also assume that we have the edges $c_0x_1, x_1x_2, .. ., x_{n_x-1}x_{n_x}, x_{n_x}c_1$ and the edges $c_0y_1, y_1y_2, ..., y_{n_y-1}y_{n_y}, y_{n_y}c_1$ by triangulation. Furthermore, we can assume $n_x \geq n_y$ without loss of generality.

Assume $n_y = 1$. So, to have $|S| \geq 5$ we should have $n_x \geq 4$. Note that we cannot have the edge xy as otherwise $\{y_1, x\}$ will be a dominating set with at least three common neighbors $\{c_0, c_1, y\}$ contradicting the maximality of D. Hence, we have the edge c_0c_1 inside R_1 by triangulation. Note that the vertex $x_2 \in S_x$ must be adjacent to either c_0 or c_1 or y_1 to have two distinct 2-paths connecting it to y. This will create a dominating set $\{c_0, x\}$ or $\{c_1, x\}$ or $\{y_1, x\}$ with at least three common neighbors $\{x_1, x_2, c_1\}$ or $\{x_{n_x}, x_2, c_0\}$ or $\{x_2, c_0, c_1\}$ respectively. This will contradict the maximality of D. Therefore,

$$n_y \geq 2.$$

Now assume that we have the edge x_2c_0. Then to have two distinct 2-paths connecting a vertex $w \in S_y$ to x_1 we will have y adjacent to both x_2 and c_0. That means, each vertex of S_y will be adjacent to both x_2 and c_0. But this is not possible keeping the graph planar as $n_y \geq 2$. So, there is no edge between c_0 and x_2. By similar arguments, we can show that every t_i is non-adjacent to c_0 for $i \in \{2, 3, ..., n_t\}$ and every t_j is non-adjacent to c_1 for $i \in \{1, 2, ..., n_t - 1\}$ for all $t \in \{x, y\}$. With a similar argument we can also show that the edge t_it_{i+k} for $1 \leq i < i+k \leq n_t$ does not exist unless $k = 1$ for any $t \in \{x, y\}$.

Now notice that $n_x \geq 3$ by equation 4.1 and the assumption that $n_x \geq n_y$. By triangulation we must have the edge x_2y_i for some $i \in \{1, 2, ..., n_y\}$. Then to have two distinct 2-paths between x_1 and y_j for $j \in \{i+1, ..., n_y\}$ and to have two distinct 2-paths between x_3 and y_l for $l \in \{1, ..., i-1\}$ we must have every vertex of S_y adjacent to x_2.

If $n_y \geq 3$, then we cannot have two distinct 2-paths between the non-adjacent vertices x_1 and y_3. So we must have

$$n_y = 2.$$

Now to have two distinct 2-paths between the non-adjacent vertices x_1 and y_2 we must have the edge x_1y_1. This creates the dominating set $\{x, y_1\}$ with at least three common neighbors $\{c_0, x_1, x_2\}$ contradicting the maximality of D. Therefore, it is not possible to have $|C| = 2$.

Hence we are done. □

Now we will ask the question that "How big $|C|$ can be?" and prove the following upper bound of $|C|$.

Lemma 4.25. $|C| \leq 3$.

Proof. First assume that $|C| \geq 7$. Recall that $C \subseteq N^{+}_{\overrightarrow{G}}(x)$. So, we must have

$$|C \cap N^{\alpha}_{\overrightarrow{G}}(y)| \geq 4.$$

Further assume that $C \cap N^{\alpha}_{\overrightarrow{G}}(y) = \{c_{i_1}, c_{i_2}, ..., c_{i_m}\}$ for some $m \geq 4$ and $i_1 < i_2 < ... < i_m$. Now without loss of generality we may assume that $i_1 = 1$. So, each c_{i_l} must be either adjacent or connected by a 2-dipath to c_{i_1} for $l \in \{i_2, ..., i_m\}$.

For that we must have $\{c_{i_2}, ..., c_{i_m}\} \subseteq \{c_2, c_3, c_{m-1}, c_m\}$. If $c_3 \in \{c_{i_2}, ..., c_{i_m}\}$ then we have $c_{m-1}, c_m \notin \{c_{i_2}, ..., c_{i_m}\}$ as there is no way to connect c_3 by a 2-dipath to either c_{m-1} or c_m. But this will imply $\{c_{i_2}, ..., c_{i_m}\} \subseteq \{c_2, c_3\}$ which is a contradiction as $m \geq 4$. Hence, $c_3 \notin \{c_{i_2}, ..., c_{i_m}\}$. Similarly we can show that $c_{m-1} \notin \{c_{i_2}, ..., c_{i_m}\}$. But this will imply $\{c_{i_2}, ..., c_{i_m}\} \subseteq \{c_2, c_m\}$ which is a contradiction as $m \geq 4$.

Hence we must have

$$|C| \leq 6.$$

Now assume $|C| \geq 5$ and $S \neq \emptyset$. Then, without loss of generality, assume a vertex $v \in S_x \cap R_0$. Notice that it is not possible to have two distinct 2-paths connecting the non-adjacent vertices v and c_2. So, we have $|S| = 0$. Then by equation 4.1 we have

$$9 \leq |G| \leq 2 + 6 + 0 = 8.$$

This is a contradiction. Hence we must have

$$|C| \leq 4.$$

Now assume $|C| = 4$ and $S \neq \emptyset$. Then, without loss of generality, assume that there is a vertex $v \in S_x \cap R_0$. To have two distinct 2-paths connecting v to c_1 and c_2 we must have the edges vc_0 and vc_3. Hence we have $|S_x \cap R_0| \leq 1$. In fact with a similar argument we can show that

$$|S_t \cap R_i| \leq 1 \text{ for all } (t, i) \in \{x, y\} \times \{0, 1, 2, 3\}. \tag{4.2}$$

Also note that if we have a vertex $v \in S_x \cap R_0$, then it is not possible to have any vertex in $S_y \cap R_i$ for $i \in \{1, 2, 3\}$ and in $S_x \cap R_2$. So basically, only adjacent regions can be non-empty.

Hence, at most two of the eight sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1, 2, 3\}$ can be non-empty. Then equation 4.2 implies $|S| \leq 2$. Then by equation 4.1 we have

$$9 \leq |G| \leq 2 + 4 + 2 = 8.$$

This is a contradiction. Hence we must have

$$|C| \leq 3.$$

Hence we are done. □

Therefore, the only possible value for $|C|$ is 3. To prove Theorem 4.22(a) we will show that $|C| = 3$ is not possible in the following lemma.

Lemma 4.26. $|C| \neq 3$.

Proof. We will prove this lemma by contradiction. So, assume that $|C| = 3$. Also, without loss of generality, assume that $S_x \geq S_y$.

Note that by equation 4.1 we have $|S| \geq 5$. Hence we have $|S_x| \geq 3$.

First assume that $S_y = \emptyset$. Hence we do not have the edge xy as otherwise x will dominate the whole graph. Now note that any two regions are adjacent for $|C| = 3$. The vertices from different regions must be adjacent to their unique common boundary point to have two distinct 2-paths connecting them.

Hence, if we have all the regions non-empty, then we will have the vertices of each region adjacent to both the boundary points of that region. This will imply

$$|S_x \cap R_i| \leq 1 \text{ for all } i \in \{0, 1, 2\}.$$

This will imply $|S| \leq 3$ and contradict our assumption. Hence, it is not possible to have all the three regions non-empty when $S_y = \emptyset$.

If we have exactly two regions, say R_0 and R_1, non-empty, then every vertex of S_x must be adjacent to c_0 to create two distinct 2-paths between the vertices of $S_x \cap R_0$ and the vertices of $S_x \cap R_1$. This will create a dominating set $\{c_0, x\}$ with at least four common neighbors contradicting the maximality of D. Hence, we can have at most one region non-empty when $S_y = \emptyset$.

Now assume that exactly one region, say R_1, is non-empty. Then each vertex of S_x must be adjacent to either c_0 or c_1 to have two distinct 2-paths connecting it to c_2. Then, without loss of generality, we will have at least three vertices of S_x adjacent to c_0. This will create a dominating set $\{c_0, x\}$ with at least four common neighbors (three vertices from S_x and c_2 because of triangulation) contradicting the maximality of D.

Hence $S_y \neq \emptyset$.

Now assume, without loss of generality, that $S_x \cap R_0 \neq \emptyset$. This implies $S_y \cap R_1 = \emptyset$ and $S_y \cap R_2 = \emptyset$ as it is not possible to have two distinct 2-paths between the vertices of S_x from one region and the vertices of S_y from a different region. But we also know that $S_y \neq \emptyset$. Hence we must have $S_y \cap R_0 \neq \emptyset$.

Now assume that $S_x = \{x_1, x_2, \ldots , x_{n_x}\}$ and $S_y = \{y_1, y_2, ..., y_{n_y}\}$. Without loss of generality also assume that we have the edges $c_0x_1, x_1x_2, .., x_{n_x-1}x_{n_x}$, $x_{n_x}c_1$ and the edges $c_0y_1, y_1y_2, ...,$ $y_{n_y-1}y_{n_y}, y_{n_y}c_1$ by triangulation.

Now x_2 must be adjacent to either c_0 or c_2 for having two distinct 2-paths connecting x_2 and c_1. Without loss of generality assume that x_2 is adjacent to c_0. Now each vertex of S_y must be adjacent to both x_2 and c_0 to have two distinct 2-paths connecting it to x_1. But this contradicts the planarity of $\overrightarrow{G}$ unless we have $n_y = 1$. Therefore,

$$n_y = 1.$$

If $n_y = 1$, then $n_x \geq 4$ by equation 4.1. If we have the edge xy (say, inside region R_2), then each vertex of S_x must be adjacent to c_0 to have two distinct 2-paths connecting it to c_1 creating the dominating set $\{c_0, x\}$ with at least four common neighbors (the vertices of S_x) contradicting the maximality of D. Hence we do not have the edge xy.

Therefore, by triangulation, we must have the edges c_0c_1 and c_1c_2. Now it is not possible to have more than two vertices of S_x adjacent to c_0 as it will create the dominating set $\{c_0, x\}$ contradicting the maximality of D. Similarly, it is not possible to have more than two vertices of S_x adjacent to c_2. But as $|S_x| \geq 4$, by triangulation we must have (to avoid having more than two vertices from S_x adjacent to c_0 or c_2) at least two vertices of S_x adjacent to y_1. This will create the dominating set $\{y_1, x\}$ with at least four common neighbors (c_0, c_2 and two vertices of S_x) contradicting the maximality of D.

Hence it is not possible to have $|C| = 3$. Hence we are done. □

The above three lemmas prove that for no value of $|C|$ it is possible to have a planar [o]-clique with domination number 2 and order at least 9. Whereas, we already showed that every planar [o]-clique with domination number not equal to 2 has order at most 8.

Therefore, we have proved Theorem 4.22(a). □

4.3 Categorical aspects

First note that the family of all oriented graphs and the family of all orientable graphs can be regarded as categories with the morphisms of the category being the homomorphisms of graphs. Notice that isomorphic graphs represent the same object in the category. Let $\mathcal{C}_O$ and $\mathcal{C}_{[O]}$ denote the category of oriented graphs and the category of orientable graphs respectively.

We want to show that $\mathcal{C}_{[O]}$ is isomorphic to a subcategory of $\mathcal{C}_O$.

Consider the subcategory $\mathcal{C}_{O_s}$ of $\mathcal{C}_O$ with $ob(\mathcal{C}_{O_s})$ being the class of splitable oriented graphs and $hom_{\mathcal{C}_{O_s}}(\overrightarrow{A}, \overrightarrow{B})$ being the set of splitable oriented homomorphisms for any $\overrightarrow{A}, \overrightarrow{B} \in \mathcal{C}_{O_s}$.

Note that the function R, defined in Section 4.1, acts as a functor of $\mathcal{C}_{[O]}$ to $\mathcal{C}_{O_s}$.

In fact, is it not difficult to notice that the functor R gives an isomorphism of $\mathcal{C}_{[O]}$ to $\mathcal{C}_{O_s}$. Therefore, we have the following result.

Proposition 4.27. *The two categories $\mathcal{C}_{[O]}$ and $\mathcal{C}_{O_s}$ are isomorphic categories.*

It was not known if a categorical product existed for orientable graphs or not. Whereas, it is a well known fact that a categorical product exists for oriented graphs and coincides with the cartesian product of it [23].

The product $\overrightarrow{A} \times \overrightarrow{B}$ of the oriented graphs $\overrightarrow{A}$ and $\overrightarrow{B}$ has the set of vertices and the set of arcs given as follows:

$$V(\overrightarrow{A} \times \overrightarrow{B}) = \{(u,v) | (u,v) \in V(\overrightarrow{A}) \times V(\overrightarrow{B})\},$$
$$A(\overrightarrow{A} \times \overrightarrow{B}) = \{\overrightarrow{(u,v)(w,x)} | (\overrightarrow{uw}, \overrightarrow{vx}) \in A(\overrightarrow{A}) \times A(\overrightarrow{B}).$$

Hence, we know that given any two splitable oriented graphs $R(\overrightarrow{A})$ and $R(\overrightarrow{B})$ there exists a categorical product $R(\overrightarrow{A}) \times R(\overrightarrow{B})$ of them. Note that the product $R(\overrightarrow{A}) \times R(\overrightarrow{B})$ is an oriented graph though it is not ensured if it is also a splitable oriented graph or not. In the following we will prove that the product indeed is a splitable oriented graph.

Lemma 4.28. *The cross product of two splitable oriented graphs is also a splitable oriented graph.*

Proof. Let $R(\overrightarrow{A})$ and $R(\overrightarrow{B})$ be two splitable oriented graphs with $V(\overrightarrow{A}) = \{a_1, a_2, ..., a_n\}$ and $V(\overrightarrow{B}) = \{b_1, b_2, ..., b_m\}$.

Then the cross product $R(\overrightarrow{A}) \times R(\overrightarrow{B})$ of $R(\overrightarrow{A})$ and $R(\overrightarrow{B})$ is an oriented graph with the set of vertices and the set of arcs given as follows:

$$\begin{aligned} V(R(\overrightarrow{A}) \times R(\overrightarrow{B})) &= \{(a_i, b_j), (a_i, b'_j), (a'_i, b_j), (a'_i, b'_j) | \text{ for} \\ &\qquad i = 1, 2, ...n \text{ and } j = 1, 2, ..., m\} \\ A(R(\overrightarrow{A}) \times R(\overrightarrow{B})) &= \{\overrightarrow{(u,v)(w,x)} | \overrightarrow{uw} \in A(R(\overrightarrow{A})) \\ &\qquad \text{and } \overrightarrow{vx} \in A(R(\overrightarrow{B}))\} \end{aligned}$$

Now let us partition $V(R(\overrightarrow{A}) \times R(\overrightarrow{B}))$ into two equal parts $V_1 = \{(a_i, b_j), (a_i, b'_j)$ and $V_2 = \{(a'_i, b'_j), (a'_i, b_j)$. Moreover, define the function $f : V_1 \to V_2$ as follows:

$$f(a_i, b_j) = (a'_i, b'_j) \text{ and } f(a_i, b'_j) = (a'_i, b_j).$$

Note that f is a bijection and satisfies the conditions of Lemma 4.6. Therefore, the product $R(\overrightarrow{A}) \times R(\overrightarrow{B})$ is indeed a splitable oriented graph. □

Let $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{C}$ be three oriented graphs. Let ψ_1 be a homomorphism of $\overrightarrow{C}$ to $\overrightarrow{A}$ and ψ_2 be a homomorphism of $\overrightarrow{C}$ to $\overrightarrow{B}$. Then the following diagram must commute.

$$\begin{tikzcd} \overrightarrow{A} & \overrightarrow{A} \times \overrightarrow{B} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \overrightarrow{B} \\ & \overrightarrow{C} \arrow[ul, "f_1"] \arrow[u, dashed, "\exists! \, f"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

For oriented graphs, the product $\overrightarrow{A} \times \overrightarrow{B}$ is the cross product of $\overrightarrow{A}$ and $\overrightarrow{B}$ while the homomorphisms π_1, π_2 and f are defined as

$$\pi_1(a,b) = a \text{ and } \pi_2(a,b) = b \text{ for all } (a,b) \in V(\overrightarrow{A} \times \overrightarrow{B}).$$
$$f(u) = (f_1(u), f_2(u)) \text{ for all } u \in V(\overrightarrow{C}).$$

Now notice that if $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{C}$ are all splitable oriented graphs and the homomorphisms f_1 and f_2 are splitable oriented homomorphisms, then the product $\overrightarrow{A} \times \overrightarrow{B}$ is a splitable oriented graph (by Lemma 4.28) and the homomorphisms π_1, π_2 and f are splitable oriented homomorphisms (it is easy to check). Hence the categorical product (of countable objects) exists in $\mathcal{C}_{O_s}$.

$$
\begin{tikzcd}
& R(\overrightarrow{A}) \times R(\overrightarrow{B}) \arrow[d, equal] & \\
R(\overrightarrow{A}) & R(\overrightarrow{P}) \arrow[l, "R(\phi_1)"'] \arrow[r, "R(\phi_2)"] & R(\overrightarrow{B}) \\
& R(\overrightarrow{C}) \arrow[ul, "R(\psi_1)"] \arrow[ur, "R(\psi_2)"'] \arrow[u, dashed, "\exists!\ R(\varphi)"] & \\
& \overrightarrow{A} \times \overrightarrow{B} \arrow[u, "R"'] \arrow[d, equal] & \\
\overrightarrow{A} & \overrightarrow{P} \arrow[l, "\phi_1"'] \arrow[r, "\phi_2"] & \overrightarrow{B} \\
& \overrightarrow{C} \arrow[ul, "\psi_1"] \arrow[ur, "\psi_2"'] \arrow[u, dashed, "\exists!\ \varphi"] &
\end{tikzcd}
$$

Figure 4.10: The functor R.

Hence by Fig. 4.10 we know that the categorical product (of countable objects) also exists in $\mathcal{C}_{[O]}$.

Formula

Now we will provide a formula for obtaining the categorical product of two orientable graphs $[\overrightarrow{A}]$ and $[\overrightarrow{B}]$.

Let $\overrightarrow{A}^1 \in [\overrightarrow{A}]$ and $\overrightarrow{B}^1 \in [\overrightarrow{B}]$ be such that $V(\overrightarrow{A}) = \{a_1, a_2, ..., a_n\}$ and $V(\overrightarrow{B}) = \{b_1, b_2, ..., b_m\}$. Then the categorical product $[\overrightarrow{A}] \times [\overrightarrow{B}]$ of $[\overrightarrow{A}]$ and $[\overrightarrow{B}]$ has the set of vertices and the set of arcs given as follows:

$$
\begin{aligned}
V([\overrightarrow{A}] \times [\overrightarrow{B}]) &= \{(a_i, b_j), (a_i, b'_j) |\text{ for } i = 1, 2, ...n \text{ and } j = 1, 2, ..., m\} \\
A([\overrightarrow{A}] \times [\overrightarrow{B}]) &= \{\overrightarrow{(a_i, b_j)(a_k, b_l)}, \overrightarrow{(a_i, b'_j)(a_k, b'_l)}, \overrightarrow{(a_i, b_l)(a_k, b'_j)}, \\
&\qquad \overrightarrow{(a_i, b'_l)(a_k, b_j)} | \overrightarrow{a_i a_k} \in A(\overrightarrow{A}^1) \text{ and } \overrightarrow{b_j b_l} \in A(\overrightarrow{B}^1)\}
\end{aligned}
$$

Notice that the formula is independent of the choice of the presentations $\overrightarrow{A}^1 \in [\overrightarrow{A}]$ and $\overrightarrow{B}^1 \in [\overrightarrow{B}]$. An interesting point to note is that the product has $2nm$ vertices whereas usually one would expect the product to have mn vertices.

4.4 Conclusion

Klostermeyer and MacGillivray [28] introduced the orientable chromatic number and asked several questions regarding them. One of their main questions was to determine the orientable chromatic number of planar graphs. They determined the orientable chromatic number for the family of outerplanar graphs in [28].

Here we considered the problem for the families of outerplanar graphs with given girth, of planar graphs and of planar graphs with given girth. The topic is new and not much works have been done on it before. So, all the results are new (that is, none of them are improvements of previous known bounds). We also introduced the notion of the orientable relative clique number and the orientable absolute clique number mimiking the case of oriented graphs.

We provided upper bounds of $\chi_{[o]}(\mathcal{P}_g)$ for all $g \geq 3$. It turns out that the target graphs used to prove the best known upper bounds for the oriented chromatic number of the family of planar graphs with girth at least g are splitable graphs for $g = 3, 4, 5$. So, in these cases an upper bound directly follows from the results regarding oriented colorings. This also means that improving these upper bounds for orientable graphs will improve the upper bounds for oriented graphs, which we know to be difficult. So, improving these bounds is also difficult.

For $g \geq 6$ we proved upper bounds for $\chi_{[o]}(\mathcal{P}_g)$ independently of the results from oriented colorings. In particular, for proving $\chi_{[o]}(\mathcal{P}_6) \leq 7$ and $\chi_{[o]}(\mathcal{P}_8) \leq 4$ we used discharging methods and actually proved the following two stronger results.

Theorem 4.29.

(a) If $mad(G) < 3$, then $\chi_{[o]}(G) \leq 7$.

(b) If $mad(G) < 8/3$, then $\chi_{[o]}(G) \leq 4$.

We showed that any planar graph with girth at least 21 admits an orientable homomorphism to the directed 3-cycle $\overrightarrow{C}_3$. It seems that it would be possible to prove a similar result for planar graphs with girth g for some $g < 21$. So, the question is: “what is the smallest g such that every planar graph with girth at least g admits an oriented homomorphism to the directed 3-cycle $\overrightarrow{C}_3$?”.

For the other two parameters, that is, the orientable relative clique number and the orientable absolute clique number, we proved mostly tight bounds.

In fact, we provided tight bounds for $\omega_{[ao]}(\mathcal{O}_g)$ and for $\omega_{[ao]}(\mathcal{P}_g)$ for all $g \geq 3$. We also provided tight bounds for $\omega_{[ro]}(\mathcal{O}_g)$ for all $g \geq 3$ and for $\omega_{[ro]}(\mathcal{P}_k)$ for all $k \geq 5$.

Given a graph G we clearly have $\omega_{[ao]}(G) \leq \omega_{ao}(G)$ and $\omega_{[ro]}(G) \leq \omega_{ro}(G)$. Is it possible to obtain a better relation between these parameters?

We also proved the existence of categorical products for orientable graphs and provided a formula for it. For proving this we showed that the category of orientable graphs is isomorphic to a subcategory of the category of oriented graphs. One might try to obtain more categorical relations between the two categories.

Finally, we would like to remark that the study of homomorphisms of orientable graphs is a fresh field of research and is worth persuing. We hope that a sound theory can be developed using the notion of orientable homomorphisms that will capture both theories of classical colorings and of oriented colorings using ideas similar to the one used for developing the theory of signed homomorphisms, recently, by Naserasr, Rollová and Sopena [40].

Chapter 5
Signified graphs

In this chapter we study homomorphisms of signified graphs which are also known as 2-edge-colored graphs. Despite the fact that the term 2-edge-colored graph is more popular and well known, we, in this thesis, prefer to use the term signified graphs as it readily gives a hint of the graph having a relation with signed graphs.

The term signified graphs has been proposed by Naserasr, Rollová and Sopena [40] while studying particular equivalence class of them which they called signed graphs (see Chapter 6 for details). An important point to note is that Harary [21] and Zaslavsky [62] initially used the term signed graph for 2-edge-colored graphs. Brewster studied the vertex coloring of edge-colored graphs in his thesis [8]. Due to the notion of signed graph homomorphisms, the term signed graph was used for the equivalence class by Naserasr, Rollová and Sopena [40].

Homomorphisms of signified graphs were studied by Nešetřil and Raspaud [41], Alon and Marshall [2] and Montejano *et al.* [36]. It was observed that there seems to be some connection between signified colorings of graphs and oriented colorings of graphs as several results in these two settings appear to be the same (or similar) and could be proved using similar techniques.

Our primary idea was to study oriented colorings which we did in Chapter 3. We proved some results in the domain of oriented colorings and wanted to check whether similar results can be proved in the domain of signified colorings or not. To our surprise, most results had a similar "signified version" and could be proved using the same proof techniques, with little adaptations. This made us wonder if there is any relation between the oriented chromatic number and the signified chromatic number of undirected simple graphs. We, in fact, managed to prove on the contrary that no such relation exists in general.

In Section 5.1 we give the basic definitions and notations needed. We define some important signified graphs in Section 5.2 which will be used later. In Section 5.3 we present our main results about signified colorings and then conclude the chapter in Section 5.4. In this chapter we will present our main results regarding signified colorings which are joint work with Bensmail and is an article in process.

5.1 Preliminaries

A *signified graph* (G, Σ) is a graph G with an assignment of *positive* (black lines used to denote them in the figures) and *negative* (black "dashed" lines used to denote them in the figures) signs to its edges where Σ is the set of negative edges and G is its underlying graph. We denote the set of positive edges by Σ^c. When the set of negative edges Σ is understood, we can denote the signified graph (G, Σ) by (G). In general, the set of vertices and the set of edges of the signified graph (G, Σ) are denoted by $V(G)$ and $E(G)$. Two adjacent edges $uv \in \Sigma$ and $vw \in \Sigma^c$ are

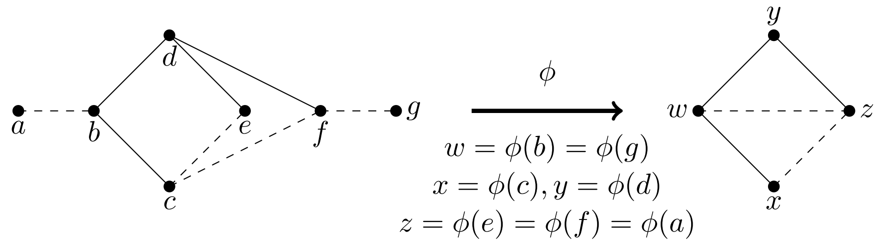


Figure 5.1: Signified graph homomorphism.

together called an *unbalanced 2-path* with *terminal* vertices u, w and *internal* vertex v. In such cases, we say that the vertices u and w are connected by an unbalanced 2-path.

The set of all adjacent vertices of a vertex v in a signified graph (G, Σ) is called its set of *neighbors* and is denoted by $N_{(G,\Sigma)}(v)$ (or $N(v)$ when there is no chance of confusion). If $uv \in \Sigma$, then v is a positive-neighbor of u and if $uv \in \Sigma^c$, then v is a negative-neighbor of u. The set of all positive-neighbors and the set of all negative-neighbors of v are denoted by $N^+_{(G,\Sigma)}(v)$ (or $N^+(v)$ when there is no chance of confusion) and $N^-_{(G,\Sigma)}(v)$ (or $N^-(v)$ when there is no chance of confusion) respectively. The *degree* of a vertex v in a signified graph (G, Σ), denoted by $d_G(v)$ (or $d(v)$ when there is no chance of confusion), is the number of neighbors of v in (G, Σ). Naturally, the *positive-degree* (resp. *negative-degree*) of a vertex v in a signified graph (G, Σ), denoted by $d^+_{(G,\Sigma)}(v)$ (resp. $d^-_{(G,\Sigma)}(v)$) (or $d^+(v)$ (resp. $d^-(v)$) when there is no chance of confusion), is the number of positive-neighbors (resp. negative-neighbors) of v in (G, Σ). The *order* $|(G, \Sigma)|$ of a signified graph (G, Σ) is the cardinality of its set of vertices $V(G)$.

Two vertices u and v of a signified graph *agree* on a third vertex w of that graph if $w \in N^\alpha(u) \cap N^\alpha(v)$ for some $\alpha \in \{+, -\}$. Two vertices u and v of a signified graph *disagree* on a third vertex w of that graph if $w \in N^\alpha(u) \cap N^\beta(v)$ for some $\{\alpha, \beta\} = \{+, -\}$.

If two vertices u, v of a signified graph (G, Σ) are adjacent then they have unbalanced distance 1 denoted by $Ud_{(G,\Sigma)}(u, v) = 1$ (or $Ud(u, v) = 1$ when there is no chance of confusion) and if two vertices of a signified graph are connected by an unbalanced 2-path then they have unbalanced distance 2 denoted by $Ud_{(G,\Sigma)}(u, v) = 2$ (or $Ud(u, v) = 2$ when there is no chance of confusion). We are only defining unbalanced distance 1 and 2 here as it serves our purpose and as we are not entirely sure about what the best general definition could be. For consistancy, let us fix a convention for pair of vertices u, v which are neither adjacent nor connected by an unbalanced 2-path by saying they are at unbalanced distance "infinity" denoted by $Ud_{(G,\Sigma)}(x, y) = \infty$ (or $Ud(x, y) = \infty$ when there is no chance of confusion).

Given two signified graphs (G, Σ) and (H, Λ), ϕ is a *homomorphism* of (G, Σ) to (H, Λ) if $\phi : V(G) \longrightarrow V(H)$ is a mapping such that every edge of (G, Σ) is mapped to an edge of the same sign of (H, Λ). We write $(G, \Sigma) \to (H, \Lambda)$ whenever there exists a homomorphism of (G, Σ) to (H, Λ). A bijective homomorphism whose inverse is also a homomorphism is an *isomorphism.* If two signified graphs admit signified homomorphisms to each other then they are *homomorphically equivalent* signified graphs.

Example 5.1. *A sample homomorphism of signified graphs is given in Fig. 5.1. Note that it will also be a homomorphism if we have $\phi(g) = x$ instead of $\phi(g) = w$.*

This example is similar to the one described to explain oriented homomorphisms in Chapter 3, Section 3.1. While noting the clear similarity between the two examples, note that here we actually can have two different homomorphisms by making a choice for $\phi(g)$ as indicated above. Whereas, there is only one homomorphism possible in the oriented case.

Throughout this chapter we will encounter results, similar to results concerning oriented colorings, proved by adapting the proof of its oriented counterpart. This will show how similar the two types of graphs are. Nevertheless, in Section 5.3 we will point out some differences between the two families of graphs.

5.2 Some signified graphs

For any prime power $q \equiv 1 \pmod 4$, there is a unique (up to isomosphism) finite field F_q of order q. The *Paley graph* P_q is the undirected graph with set of vertices $V(P_q) = F_q$ and the set of edges $E(P_q) = \{xy \mid y - x$ is a non-zero square in $F_q\}$. Also, -1 is a square in F_q so $(x - y)$ is a square if and only if $(y - x)$ is a square. That is why the definition of edge is consistent. We also know that a Paley graph P_q is self-complementary, edge transitive [63] and strongly regular with parameters $(q, (q-1)/2, (q-5)/4, (q-1)/4)$ [10].

For any prime power $q \equiv 1 \pmod 4$ the *signified Paley graph* [36] (SP_q, N) is the signified graph with set of vertices F_q and the set of negative edges $N = \{xy \mid y - x \pmod p$ is a non-square$\}$. Intuitively, we replace the edges and non-edges of a Paley graph P_q by positive and negative edges respectively to get a signified Paley graph (SP_q).

The *signified Paley plus graph* (SP_q^+) is the signified graph obtained from the signified Paley graph (SP_q) by adding a new vertex ∞ which is adjacent to every other vertex with a positive edge.

For any prime $p \equiv 1 \pmod 4$ and for any positive integer n the *signified Tromp graph* [36] (ST_{2q+2}, Σ) of order $(2q + 2)$, where $q = p^n$, is the signified graph with the set of vertices, the set of negative edges and the set of positive edges as the following:

$$V(ST_{2q+2}) = \{0, 1, ..., (q-1)\} \cup \{0', 1', ..., (q-1)'\} \cup \{\infty, \infty'\}.$$

$$\begin{aligned}\Sigma = \{ij, i'j', \mid i, j \in \{0, 1, ..., (q-1)\} \\ \text{and } (j-i) \pmod p \text{ is a non-zero square }\} \\ \cup \{ij' \mid i, j \in \{0, 1, ..., (q-1)\} \\ \text{and } (j-i) \pmod p \text{ is not a non-zero square }\} \\ \cup \{i\infty, i'\infty' \mid i \in \{0, 1, ..., (q-1)\}\}.\end{aligned}$$

$$\begin{aligned}\Sigma^c = \{ij, i'j', \mid i, j \in \{0, 1, ..., (q-1)\} \\ \text{and } (j-i) \pmod p \text{ is not a non-zero square }\} \\ \cup \{ij' \mid i, j \in \{0, 1, ..., (q-1)\} \\ \text{and } (j-i) \pmod p \text{ is a non-zero square }\} \\ \cup \{i\infty', i'\infty \mid i \in \{0, 1, ..., (q-1)\}\}.\end{aligned}$$

Intuitively, in (ST_{2q+2}, Σ) there are two vertices ∞, ∞' such that $N^+(\infty) = N^-(\infty')$ and $N^+(\infty) = N^-(\infty')$ with each of the sets $N^+(\infty)$ and $N^-(\infty)$ inducing a signified Paley graph (SP_q). Also, if the edge ij is a positive edge in the (SP_q) induced by $N^+(\infty)$ and $i'j'$ is the corresponding positive edge of the (SP_q) induced by $N^-(\infty)$, then the edges ij' and $i'j$ are negative. For pictorial help, check Fig. 5.2 (a). Note that (ST_{2q+2}) is a complete $(q+1)$-partite graph with all parts of order two.

For any positive integer k the *signified Zielonka graph* [2] (SZ_k, Σ) of order $k \times 2^{k-1}$ is the signified graph with set of vertices $V(SZ_k) = \cup_{i=1,2,...,k} S_i$, where $S_i = \{x = (x^1, ..., x^k) | x^j \in \{0, 1\}$ for $j \neq i$ and $x^i = *\}$ while the set of negative edges $\Sigma = \{xy \mid x = (x^1, ..., x^k) \in S_i, y = (y^1, ..., y^k) \in S_j$ and $x^j \neq y^i\}$ and the set of positive edges $\Sigma^c = \{xy \mid x = (x^1, ..., x^k) \in S_i, y = (y^1, ..., y^k) \in S_j$ and $x^j = y^i \neq *\}$. For pictorial help, check Fig. 5.2 (b). Note that (SZ_k) is a complete k-partite graph with all parts of size 2^{k-1}.

5.3 Signified coloring

An *signified k-coloring* [2] of an signified graph (G, Σ) is a mapping ϕ from the vertex set $V((G, \Sigma))$ to the set $\{1, 2, ..., k\}$ such that,

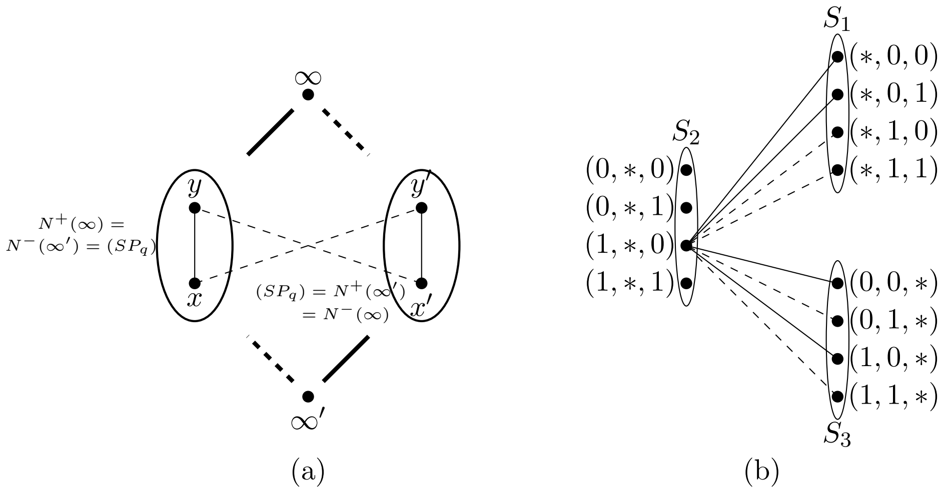


Figure 5.2: (a) The signified Tromp graph (ST_{2q+2}). (thick edges refer to positive edges between the vertex ∞ or ∞' and all vertices inside the ellipse while thick dashed edges refer to negative edges between the vertex ∞ or ∞' and all vertices inside the ellipse (b) Adjacency of a vertex of the signified Zielonka graph (SZ_3).)

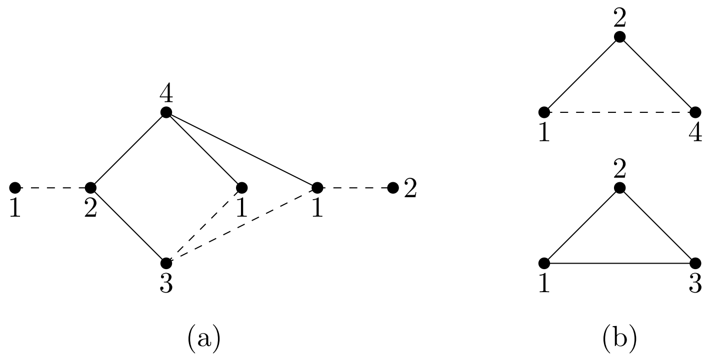


Figure 5.3: Signified coloring.

- (i) $\phi(u) \neq \phi(v)$ whenever u and v are adjacent and
- (ii) if uv is a positive edge and wx is a negative edge of (G, Σ), then $\phi(u) = \phi(w)$ implies $\phi(v) \neq \phi(x)$.

The *signified chromatic number* $\chi_s((G, \Sigma))$ of a signified graph (G, Σ) is the smallest integer k for which (G, Σ) has a signified k-coloring.

Alternatively, we can define the signified chromatic number using homomorphisms of signified graphs. The *signified chromatic number* $\chi_s((G, \Sigma))$ of a signified graph (G, Σ) is the minimum order of a signified graph (H, Λ) such that (G, Σ) admits a homomorphism to (H, Λ).

The signified chromatic number $\chi_s(G)$ of an undirected graph G is the maximum of the signified chromatic numbers of all the signified graphs with underlying graph G. The signified chromatic number $\chi_s(\mathcal{F})$ of a family $\mathcal{F}$ of graphs is the maximum of the signified chromatic numbers of the graphs from the family $\mathcal{F}$.

Example 5.2. *We give a signified 4-coloring of the graph in Fig. 5.3 (a) whereas, in Fig. 5.1 we showed that the same graph admits a homomorphism to a signified graph of order 4. These two facts essentially means the same thing. Also note that we cannot provide an oriented 3-coloring of this graph, hence it has oriented chromatic number 4.*

Example 5.3. *The signified coloring of the disjoint union of the two graphs depicted in Fig. 5.3 (b) is interesting. Note that if we signified color the two graphs from Fig. 5.3 (b) (both have triangles as underlying graphs but with different set of negative edges) individually, as each of them has three vertices which are pairwise adjacent, we can easily show that each of these graphs has signified chromatic number 3. But when we consider the disjoint union of these two graphs as one*

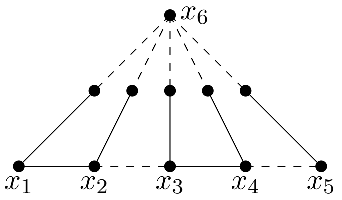


Figure 5.4: (B, Λ) is a signified planar graph with girth 5.

signified graph, we are not able to signified 3-color the graph. In this case, we will need at least 4 colors for a signified coloring of the graph. In Fig. 5.3 (b) we have given a signified 4-coloring of the graph.

Notice that the terminal vertices of an unbalanced 2-path must receive distinct colors in every signified coloring because of the second condition of the definition. In fact, for providing a signified coloring of a signified graph, only the pairs of vertices which are either adjacent or connected by a 2-dipath, must receive distinct colors (that is, for every other type of pair of vertices there exists a signified coloring which assigns the same color to the pair of vertices). Motivated by this observation, the following definitions are proposed.

A *relative clique* of a signified graph (G, Σ) is a set $R \subseteq V((G, \Sigma))$ of vertices such that any two vertices from R are either adjacent or connected by an unbalanced 2-path. The *relative clique number* $\omega_{rs}((G, \Sigma))$ of a signified graph (G, Σ) is the maximum order of a signified relative clique of (G, Σ). The term relative clique and the definition is given by following the similar term and definition used in [40] for signed graphs.

A *signified clique* or an *sclique* is a signified graph (G, Σ) for which $\chi_s((G, \Sigma)) = |V((G, \Sigma))|$. Note that scliques can hence be characterized as those signified graphs whose any two distinct vertices are either adjacent or connected by an unbalanced 2-path. Note that a signified graph with an sclique of order n as a subgraph has signified chromatic number at least n. The *signified absolute clique number* $\omega_{as}((G, \Sigma))$ of a signified graph (G, Σ) is the maximum order of an sclique contained in (G, Σ) as a subgraph.

The relative clique number $\omega_{rs}(G)$ (resp. absolute clique number $\omega_{as}(G)$) of a simple graph G is the maximum of the relative clique numbers (resp. absolute clique numbers) of all the signified graphs with underlying graph G. The relative clique number $\omega_{rs}(\mathcal{F})$ (resp. absolute clique number $\omega_{as}(\mathcal{F})$) of a family $\mathcal{F}$ of graphs is the maximum of the relative clique numbers (resp. absolute clique numbers) of the graphs from the family $\mathcal{F}$.

From the definitions, clearly we have the following:

Lemma 5.4. *For any signified graph (G, Σ) we have, $\omega_{as}((G, \Sigma)) \leq \omega_{rs}((G, \Sigma)) \leq \chi_s((G, \Sigma))$.*

Corollary 5.5. *For any sclique (O, Λ) we have $\omega_{as}((O, \Lambda)) = \omega_{rs}((O, \Lambda)) = \chi_s((O, \Lambda)) = |V(G)|$.*

Example 5.6. *The vertices x_1, x_2, x_3 of the graph (B, Λ) depicted in Fig. 5.4 induce an sclique of order 3 and there is no sclique of order more than 3 in (B, Λ). Now, if we try to provide a signified coloring of the graph (B, Λ), the vertices x_1, x_2, x_3 will receive distinct colors as they are vertices of an sclique. But note that the vertex x_6 must receieve a color distinct from the colors received by x_1, x_2, x_3 even though these four vertices do not induce an sclique in (B, Λ). It is easy to show that for the graph (B, Λ) depicted in Fig. 5.4 we have, $\omega_{as}((B, \Lambda)) = 3$, $\omega_{rs}((B, \Lambda)) = 4$ and $\chi_s((B, \Lambda)) = 5$. This is an example of a graph for which each inequality of lemma 5.4 is strict.*

From the above example we can observe that it is not possible to provide a signified 3-coloring of the 5-path with adjacent edges having alternative signs (that is, the path $x_1x_2x_3x_4x_5$ from Fig. 5.4). It is easy to check that the signified 5-path $x_1x_2x_3x_4x_5$ has signified chromatic number 4. Whereas, any oriented tree admits an oriented 3-coloring. So, any orientation of a 5-path will admit an oriented 3-coloring. Therefore, if we consider the undirected 5-path $Path_5$, then $\chi_s(Path_5) - \chi_o(Path_5) = 1$.

Example 5.7. *On the other hand, it is easy to check that every signified 5-cycle admits a signified 4-coloring. In fact, the signified 5-cycle with 2 non-adjacent negative edges and 3 positive edges has signified chromatic number 4. So, we have* $\chi_s(C_5) = 4$. *But we know that the directed 5-cycle has oriented chromatic number 5. So, we have* $\chi_s(C_5) - \chi_o(C_5) = -1$.

So, we can have either of the two chromatic numbers higher than the other for the same undirected graph. Motivated by the above examples we have the following result.

Proposition 5.8. *Given any integer* n*, there exists an undirected graph* G *such that* $\chi_s(G) - \chi_o(G) = n$.

Proof. All the complete graphs have their oriented chromatic number equal to their signified chromatic number. So, we need to prove $\chi_s(G) - \chi_o(G) = n$ for non-zero integers n.

Let A and B be two undirected graphs. Let $A + B$ be the undirected graph obtained by taking disjoint copies of A and B and adding a new vertex ∞ adjacent to all the vertices of A and B. So, $A+B$ is the undirected graph dominated by the vertex ∞ and $N(\infty)$ is the disjoint union of A and B.

It is easy to observe that

$$\chi_s(A+B) \leq \chi_s(A) + \chi_s(B) + 1.$$

Let $\Sigma_A \subseteq E(A)$ and $\Sigma_B \subseteq E(B)$ be such that we have $\chi_s((A,\Sigma_A)) = \chi_s(A)$ and $\chi_s((B,\Sigma_B)) = \chi_s(B)$. Now choose $\Sigma_{A+B} \subseteq E(A+B)$ such that,

$$\Sigma_{A+B} = \Sigma_A \cup \Sigma_B \cup \{\infty b | b \in V(B)\}.$$

Clearly, the vertex ∞ must receive a color different from any other vertex in the graph in any signified coloring. Also, due to the choice of Σ_{A+B}, the signified graph induced by $N^+(\infty)$ is isomorphic to (A,Σ_A) and the signified graph induced by $N^-(\infty)$ is isomorphic to (B,Σ_B). Note that the vertices of $N^+(\infty)$ must receive colors different from the colors received by the vertices of $N^-(\infty)$ in any signified coloring. Therefore,

$$\begin{aligned}\chi_s((A+B,\Sigma_{A+B})) &\geq \chi_s((A,\Sigma_A)) + \chi_s((B,\Sigma_B)) + 1\\ &= \chi_s(A) + \chi_s(B) + 1.\end{aligned}$$

This implies

$$\chi_s(A+B) = \chi_s(A) + \chi_s(B) + 1. \tag{5.1}$$

Similarly, consider orientations $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{A+B}$ of A, B and $A+B$ respectively such that $\chi_o(\overrightarrow{A}) = \chi_o(A)$, $\chi_o(\overrightarrow{B}) = \chi_o(B)$ and that the oriented graphs induced by $N^+(\infty)$ and $N^-(\infty)$ in $\overrightarrow{A+B}$ are isomorphic to $\overrightarrow{A}$ and $\overrightarrow{B}$ respectively.

With arguments similar to what we gave for proving equation 5.1 we have,

$$\chi_o(A+B) = \chi_o(A) + \chi_o(B) + 1. \tag{5.2}$$

Let H be an undirected graph. Then we define, by induction, the graph $H_k = H + H_{k-1}$ for $k \geq 2$ where $H_1 = H$. Note that,

$$\chi_s(H_k) = k \times \chi_s(H) + (k-1) \text{ and } \chi_o(H_k) = k \times \chi_o(H) + (k-1).$$

The above two equations implies

$$\chi_s(H_k) - \chi_o(H_k) = k \times (\chi_s(H) - \chi_o(H)). \tag{5.3}$$

Taking $H = Path_5$ and $H = C_5$ in equation 5.3 we have the proof of our result as $\chi_s(Path_5) - \chi_o(Path_5) = 1$ and $\chi_s(C_5) - \chi_o(C_5) = -1$ as observed earlier. □

Note that the above defined three graph parameters respect homomorphisms of signified graphs in the sense of the following result.

Lemma 5.9. *Let $(G, \Sigma) \to (H, \Lambda)$. Then $\chi_s((G, \Sigma)) \leq \chi_s((H, \Lambda))$, $\omega_{rs}((G, \Sigma)) \leq \omega_{rs}((H, \Lambda))$ and $\omega_{as}((G, \Sigma)) \leq \omega_{as}((H, \Lambda))$.*

Determining the signified chromatic number, the signified relative clique number and the signified absolute clique number of different families of graphs are challenging problems in the domain of signified coloring. A usual technique for obtaining the upper bound for these three graph parameters is to prove that every graph in the family of graphs in question admits a homomorphism to a particular signified graph. Such a graph is called a *universal bound* of that family of graphs. Note that not every family of graphs have a universal bound of order equal to the signified chromatic number of the family.

Example 5.10. *Note that the family of all signified graphs on 3 vertices have signified chromatic number 3 as each graph in the family is clearly signified 3-colorable. But, as previously discussed, the two signified graphs on 3 vertices depicted in Fig. 5.3 (their disjoint union is not 3-colorable) cannot admit a homomorphism to a single signified graph on 3 vertices.*

If we consider the set of all signified graphs to be a category with objects being the signified graphs and the morphisms being the signified homomorphisms then we clearly have the following:

Theorem 5.11. *For any family $\mathcal{F}$ of signified graphs that also contains the categorical co-products of the graphs from the family, there exists a universal bound of $\mathcal{F}$ on $\chi_s(\mathcal{F})$ vertices.*

Observe that the categorical co-product (unique up to homomorphic equivalence) of signified graphs is simply the signified graph obtained by taking the disjoint union of the signified graphs. The families of planar graphs, outerplanar graphs, planar graphs with given girth and outerplanar graphs with given girth are each of the type that we mentioned in the above theorem.

5.3.1 Signified chromatic number

One of the general results proved related to signified chromatic number is the following regarding graphs that admits an acyclic k-coloring.

Theorem 5.12. *Every graph with acyclic chromatic number at most k has signified chromatic number at most $k.2^{k-1}$.*

This theorem was proved by Alon and Marshall [2] and later generalized by Nešetřil and Raspaud [41], both in a more general setting. Fabila-Monroy *et. al.* [15] proved that the above mentioned bound is tight for $k \geq 3$.

Some specific families of graphs have also been studied in the domain of signified homomorphism. The most studied family of graphs for signified chromatic number is the family of planar graphs and some sub-families of planar graphs, such as outerplanar graphs, planar graphs with given girth, outerplanar graphs with given girth etc.

The known bounds, which are tight, for the signified chromatic number of outerplanar graphs and outerplanar graphs with given girth are listed in the following theorem.

Theorem 5.13.

(a) $\chi_s(\mathcal{O}_3) = 9$. *[36]*

(b) $\chi_s(\mathcal{O}_k) = 5$ *for* $k \geq 4$. *[36]*

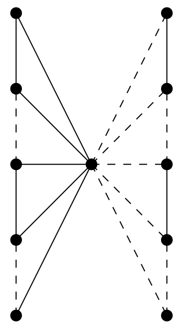


Figure 5.5: A signified outerplanar graph with signified chromatic number 9.

In the above theorem, the lower bounds were achieved by constructing suitable examples. The upper bounds were achieved by showing that the signified Paley graphs SP_9 and SP_5 bound the family of signified outerplanar graphs with girth 3 and 4 respectively, by Montejano *et al.* [36].

The known bounds for the signified chromatic number of planar graphs and planar graphs with given girth are listed in the following theorem. The relevent references are given after the results. We will prove part (*b*) of the theorem.

Theorem 5.14.

(a) $19 \leq \chi_s(\mathcal{P}_3) \leq 80$. *[41][2]*

(b) $10 \leq \chi_s(\mathcal{P}_4) \leq 50$. *(proved in this thesis)*

(c) $5 \leq \chi_s(\mathcal{P}_5) \leq 20$. *[36]*

(d) $5 \leq \chi_s(\mathcal{P}_6) \leq 12$. *[36]*

(e) $5 \leq \chi_s(\mathcal{P}_7) \leq 10$. *[36]*

(f) $5 \leq \chi_s(\mathcal{P}_8) \leq 8$. *[36]*

(g) $\chi_s(\mathcal{P}_k) = 5$ *for* $k \geq 14$. *[36]*

The upper bound of Theorem 5.14(a) follows from Theorem 5.12. Theorem 5.14(c)-(g) was proved by Montejano *et al.* [36]. We will prove the lower bound of Theorem 5.14(a) and both bounds of Theorem 5.14(b) in the following.

Proof of Theorem 5.14

(a) (proof for the lower bound) For the lower bound we use the example of the outerplanar signified graph with signified chromatic number 9 depicted in Fig. 5.5.

We take two copies of that graph, say, (A, Σ) and (A', Σ'). We take a vertex v and put positive edges between all the vertices of (A, Σ) and v and put negative edges between all the vertices of (A', Σ') and v. This new graph is planar. Now if we try to provide a signified coloring of this new graph, as v is adjacent to every other vertex, it will get a color distinct from every other vertex. Also, as each vertex of (A, Σ) is connected with an unbalanced 2-path with each vertex of (A', Σ'), they will receive distinct colors. Now, we know that we need 9 colors to provide a signified coloring of each of the graphs (A, Σ) and (A', Σ'). Hence we need in total $9+9+1 = 19$ colors to provide a signified coloring of the whole graph. Hence the lower bound.

(b) The lower bound follows from the example depicted in Fig. 5.6.

Let $R(SP_{25})$ be the signified graph obtained by deleting the vertices ∞ and ∞' from the signified Tromp graph (ST_{52}). In Chapter 6, Section 6.3.1 while proving Theorem 6.23 we will prove a fact equivalent to proving that every signified triangle-free planar graph admits a homomorphism to the signified graph $R(SP_{25})$ of order 50. The upper bound follows from that. This completes the proof. $\square$

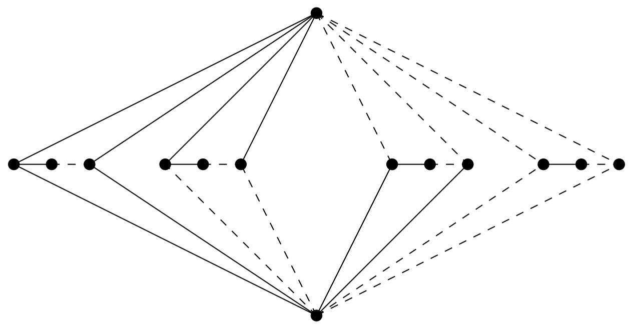


Figure 5.6: A signified planar graph with girth 4 and signified chromatic number 10.

5.3.2 Signified relative clique number

Like in the case of oriented graphs, we consider some parameters "easier" to handle than the signified chromatic number, such as the signified relative clique number and the signified absolute clique number, and try to figure out the bounds in these cases. First we present a general bound for the signified relative clique number of graphs with maximum degree Δ.

Proposition 5.15. *Every signified graph with maximum degree Δ has signified relative clique number at most $\Delta^2 + 1$.*

Proof. Let (G, Σ) be an signified graph with maximum degree Δ. Let R be a relative clique of maximum order in (G, Σ). Let $v \in R$ be a vertex. Now, v can have at most Δ adjacent vertices and each of these vertices can have at most $(\Delta - 1)$ adjacent vertices excluding v. Hence, there can be at most $\Delta.(\Delta - 1)$ vertices of (G, Σ) at unbalanced distance 2 from v. As every vertex, other than v, in R is at most at unbalanced distance 2 from v, we have,

$$\begin{aligned} |R| &\leq |\{v\}| + |\{\text{vertices adjacent to } v\}| \\ &\qquad + |\{\text{vertices at unbalanced distance 2 from } v\}| \\ &\leq 1 + \Delta + \Delta.(\Delta - 1) \\ &= \Delta^2 + 1. \end{aligned}$$

Hence, we are done. □

We consider the problem of determining the signified relative clique number for the families of outerplanar graphs and of outerplanar graphs with given girth as well. We list the related results below.

Theorem 5.16.

(a) $\omega_{rs}(\mathcal{O}_3) = 7$.

(b) $\omega_{rs}(\mathcal{O}_4) = 5$.

(c) $\omega_{rs}(\mathcal{O}_5) = 4$.

(d) $\omega_{rs}(\mathcal{O}_k) = 3$ *for* $k \geq 6$.

Finally, we list the bounds for the signified relative clique number for the families of planar graphs and of planar graphs with given girth below.

Theorem 5.17.

(a) $15 \leq \omega_{rs}(\mathcal{P}_3) \leq 80$.

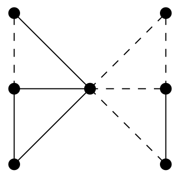


Figure 5.7: A signified outerplanar graph with signified relative clique number 7.

(b) $10 \leq \omega_{rs}(\mathcal{P}_4) \leq 27$.

(c) $\omega_{rs}(\mathcal{P}_5) = 5$.

(d) $\omega_{rs}(\mathcal{P}_6) = 4$.

(e) $\omega_{rs}(\mathcal{P}_k) = 3$ *for* $k \geq 7$.

Proof of Theorem 5.16

(a) The lower bound follows from Fig. 5.7.

Now we prove the upper bound.

Assume that (G, Σ) is a signified outerplanar graph of minimum order with $\omega_{rs}((G, \Sigma)) > 7$. Moreover, assume (G, Σ) is such that if we delete any edge of (G, Σ), it will no longer have signified relative clique number greater than 7.

Let R be a relative clique of maximum order in (G, Σ) and let $S = V(G) \setminus R$. Note that S induces an independent set of (G, Σ) as deleting any edge between two vertices of S will not decrease the signified relative clique number of the graph (G, Σ).

First note that, for any $z \in S$, we have $d(z) \geq 2$ as otherwise the vertex z does not help connecting any two (or more) vertices of R by an unbalanced 2-path with the internal vertex being itself (that is, z) and hence can be deleted to get a signifieded planar graph with relative signified chromatic number equal to that of (G, Σ) but with order less than (G, Σ), which contradicts the minimality of (G, Σ).

In fact, any $z \in S$ with $d(z) = 2$ must be the internal vertex of an unbalanced 2-path that connects two vertices of R. But, we can replace that unbalanced 2-path by an edge and obtain another signified outerplanar graph to contradict the minimality of (G, Σ). Hence, $d(z) \geq 3$ for all $z \in S$.

Claim 1: For any $v \in V(G)$ we have, $|N^{\alpha}(v) \cap R| \leq 3$ for $\alpha \in \{+, -\}$.

Proof of claim 1: Let $v \in V(G)$ and $N^{\alpha}(v) \cap R = \{v_1, ..., v_k\}$ with $k \geq 4$. Fix an outerplanar embedding of (G, Σ). Assume, without loss of generality, that the vertices $v_1, v_2, ..., v_k$ are arranged around v in a clockwise order in the embedding. Now, to have unbalanced distance at most 2 between the vertices v_1 and v_3, we must have an unbalanced 2-path connecting them with internal vertex either v_2 or v_4 as anything else will contradict the fixed embedding of the graph (G, Σ) being outerplanar. Without loss of generality we can assume that v_1 and v_3 are connected by an unbalanced 2-path with internal vertex v_2. Similarly, we can also assume that v_2 and v_4 are connected by an unbalanced 2-path with internal vertex v_3. So, we have the edges v_1v_2, v_2v_3, v_3v_4 in our graph. Now, to have $Ud(v_1, v_4) \leq 2$ we must have either the edge v_1v_4 or an unbalanced 2-path connecting v_1 and v_4 both of which contradicts the embedding being outerplanar. Hence we have proved the claim. $\Diamond$

As (G, Σ) is an outerplanar graph, there exists a vertex $x \in V(G)$ with $d(x) \leq 2$. By the above discussion we know that $x \in R$. Clearly $d(x) \neq 0$ as otherwise it will contradict the minimality of (G, Σ).

If $d(x) = 1$ and $N(x) = N^{\alpha}(x) = \{w\}$ for some $\alpha \in \{+, -\}$, then every vertex from $R \setminus \{x, w\}$ is connected to x by an unbalanced 2-path with internal vertex w. This contradicts Claim 1 as $|R \setminus \{x, w\}| \geq 6$. Hence, $d(x) = 2$.

Assume that $N(x) = \{w, z\}$. Now as $|R \setminus \{x, w, z\}| \geq 5$, without loss of generality, we can assume that at least three vertices of R are connected to x by unbalanced 2-paths with internal

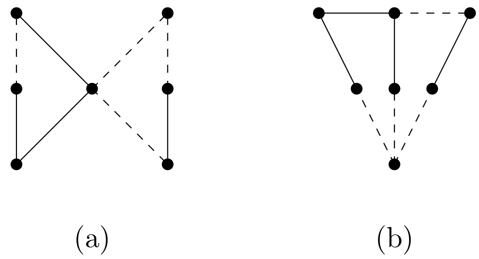


Figure 5.8: (a) A signified outerplanar graph with girth 4 and signified relative clique number 5. (b) A signified outerplanar graph with girth 5 and signified relative clique number 4.

vertex w and at least two vertices of R are connected to x by an unbalanced 2-path with internal vertex z. It is easy to note that it is not possible to have distance (we do not even need to use unbalanced distance in this case) at most 2 among these five vertices keeping the graph outerplanar. So, this is a contradiction.

(b) This result follows from Theorem 5.13(b) and the signified outerplanar graph with signified relative clique number 5 depicted in Fig. 5.8(a).

(c) Let (G, Σ) be a signified outerplanar graph, with girth at least 5, of minimum order with $\omega_{rs}((G, \Sigma)) > 4$. Moreover assume (G, Σ) is such that, if we delete any edge of (G, Σ), it will no longer have $\omega_{rs}((G, \Sigma)) > 4$.

Let R be a relative clique of maximum order in (G, Σ) and let $S = V(G) \setminus R$. Note that S induces an independent set of $\overrightarrow{G}$ as deleting any edge between two vertices of S will not decrease the signified relative clique number of the graph (G, Σ).

Claim 1: For any $v \in V(G)$ we have $|N(v) \cap R| \leq 2$ and $|N^{\alpha}(v) \cap R| \leq 1$ for some $\alpha \in \{+, -\}$.

Proof of claim 1: Let $v \in V(G)$ and $|N(v) \cap R| \geq 3$. Then we must also have $|N^{\alpha}(v) \cap R| \geq 2$ for some $\alpha \in \{+, -\}$. Any two vertices from $N^{\alpha}(v) \cap R$ must be either adjacent or connected by an unbalanced 2-path which will create a cycle of length less than 5, hence is a contradiction. So, the claim is true. $\Diamond$

First note that, for any $z \in S$, we have $d(z) \geq 2$ as otherwise the vertex z does not help connecting any two (or more) vertices of R by an unbalanced 2-path with the internal vertex being itself (that is, z) and hence can be deleted to get a signified planar graph with girth at least 5 and relative signified chromatic number equal to that of (G, Σ) but with order less that (G, Σ) which contradicts the minimality of (G, Σ).

In fact, any $z \in S$ with $d(z) = 2$ must be the internal vertex of an unbalanced 2-path that connects two vertices of R. But, we can replace that unbalanced 2-path by an edge and obtain another signified outerplanar graph (not necessarily of girth at least 5) without decreasing the degree of the remaining vertices of the graph. As there is at least one vertex of degree at most 2 in an outerplanar graph, we can conclude that there is a vertex of degree at most 2 in (G, Σ) from R.

Now, let $x \in R$ be such that, $d(x) \leq 2$. Clearly, $d(x) \neq 0$. If $d(x) = 1$ with w being its neighbor, then each vertex from $R \setminus \{x, w\}$ will be connected to x by an unbalanced 2-path with internal vertex w, which contradicts Claim 1.

Hence, $d(x) = 2$. Assume that $N(v) = \{w, z\}$. Then each vertex from $R \setminus \{x, w\}$ will be connected to x by an unbalanced 2-path with internal vertex either w or z. If both w and z are not from R, then we have at least three vertices in $R \setminus \{x, w\}$, which will imply $|N^{\alpha}(t) \cap R| \geq 2$ for some $t \in \{w, z\}$ and for some $\alpha \in \{+, -\}$, which contradicts Claim 1.

So, we have $w, z \in R$ and exactly one vertex w' connected to x by an unbalanced 2-path with internal vertex w and exactly one vertex z' connected to x by an unbalanced 2-path with internal vertex z. We need to have distance at most 2 between the pairs of vertices w', z' and w, z' and w', z which is not possible keeping the graph outerplanar. This is a contradiction.

(d) The lower bound follows from the fact that an unbalanced 2-path is an sclique of order 3.

It is easy to check that it is not possible to construct a signified graph with girth at least 6 in which at least 4 vertices are at unbalanced distance at most 2 keeping the graph outerplanar. This completes the proof. □

Proof of Theorem 5.17

(a) The upper bound follows from Theorem 5.14(a).

The lower bound follows from the planar sclique of order 15 described later in Section 5.3.3 to prove the lower bound of Theorem 5.19(a).

(b) The lower bound follows from the example in Fig. 5.6.

The proof of the upper bound is similar to the proof of Theorem 3.18(b) (see Chapter 3).

We omit the rest of the proof (see Appendix).

(c) Assume that (G, Σ) is a signified planar graph with girth at least 5 of minimum order with $\omega_{rs}((G, \Sigma)) > 6$. Moreover, assume (G, Σ) is such that, if we delete any edge of (G, Σ), it will no longer have signified relative clique number greater than 6.

Let R be a relative clique of maximum order in (G, Σ) and let $S = V(G) \setminus R$. Note that S induces an independent set of (G, Σ) as deleting any edge between two vertices of S will not decrease the signified relative clique number of the graph (G, Σ).

First note that, for any $z \in S$, we have $d(z) \geq 2$ as otherwise the vertex z does not help connecting any two (or more) vertices of R by an unbalanced 2-path with the internal vertex being itself (that is, z) and hence can be deleted to get a signified planar graph with girth at least 5 and with relative signified chromatic number equal to that of (G, Σ) but with order less than (G, Σ) which contradicts the minimality of (G, Σ).

Also, for any $z \in V(G)$, we must have $N(z) \cap R \leq 2$. If not, then we will have $|N^{\alpha}(z) \cap R| \geq 2$ for some $\alpha \in \{+, -\}$. Now to have unbalanced distance at most 2 between two vertices of $N^{\alpha}(z) \cap R$, there should be an edge or an unbalanced 2-path connecting the two vertices. This will create a cycle of length less than 5, which is a contradiction. Hence for any $z \in S$ we have, $d(z) = 2$ and that z must be an internal vertex of an unbalanced 2-path with two terminal vertices from R.

The graph (G, Σ) must contain a 5-cycle $abcde$ because of Theorem 3.18(d) and (e) whose proofs are independent from this proof. As S is an independent set in $\overrightarrow{G}$, we can have at most two vertices (should be non-adjacent) of the cycle from S.

First assume that all the vertices of the cycle is from R. Then there are at least two adjacent edges in the cycle that have the same sign. Assume these edges are ab and bc. Now, a and c are either adjacent or connected by an unbalanced 2-path, hence creating a cycle of length less than 5 which is a contradiction. Hence, R does not induce any cycle of length 5 in (G, Σ).

Now assume that the 5-cycle has four vertices a, b, c, d from R. But, at most one vertex from $R \setminus \{a, b, c, d\}$ can be adjacent to a and at most one vertex from $R \setminus \{a, b, c, d\}$ can be adjacent to d, while no vertex from $R \setminus \{a, b, c, d\}$ can be adjacent to b or c because of the degree restrictions proved before.

For any $w \in R$ such that, w is non-adjacent to each of a, b, c, d, we must have w connected to each of a, b, c, d with an unbalanced 2-path.

Now, we have $|R \setminus \{a, b, c, d\}| \geq 2$. Assume that $u, v \in R \setminus \{a, b, c, d\}$.

Assume first that both vertices are non-adjacent to a, b, c, d. Hence, each of them is connected to each of a, b, c, d with unbalanced 2-paths. They are also either adjacent or connected by an unbalanced 2-path to each other. But to achieve all these connections, either we need to create a cycles of length less than 5 or contradict the planarity of the graph.

Now assume that exactly one vertex, say v, is non-adjacent to a, b, c, d. Hence, v is connected to each of a, b, c, d with unbalanced 2-paths. Assume, without loss of generality, that u is adjacent to a. The vertices v and u are also either adjacent or connected by an unbalanced 2-path to each other. Now u is connected by an unbalanced 2-path to c (the edge uc creates a cycle of length

4). But to achieve all these connections, either we need to create a cycles of length less than 5 or contradict the planarity of the graph.

Now assume, without loss of generality, that u is adjacent to a and v is adjacent to d. They are also either adjacent or connected by an unbalanced 2-path to each other. Then u is connected by an unbalanced 2-path to c (the edge uc creates a cycle of length 4) and v is connected by an unbalanced 2-path to b (the edge vb creates a cycle of length 4). But to achieve all these connections, either we need to create a cycle of length less than 5 or contradict the planarity of the graph.

Now assume that the 5-cycle has three vertices a, c, d from R and two vertices b, e from S. Assume $u, v \in R \setminus \{a, c, d\}$ and u and v are adjacent to c and d respectively.

To avoid creating a cycle of length less than 5, u and v must be connected by an unbalanced 2-path with internal vertex, say, w. But then $cuwvda$ is a cycle with four vertices from R. This is not possible. Hence, at least one vertex among c and d must be non-adjacent to every vertex of $R \setminus \{a, c, d\}$. Let us assume, without loss of generality, that c is the vertex.

Then every vertex of $R \setminus \{a, c, d\}$ must be connected to c by unbalanced 2-paths. Now, they can no longer be adjacent to d as well because that will create a 4 cycle. But the vertices of $R \setminus \{a, c, d\}$ (there are at least three such vertices) must be at unbalanced distance at most 2 to the vertex a also. That is not possible to achieve without creating a cycle of length less than 5 or contradicting the planarity of the graph.

(d) Consider the signified 6-cycle $x_1x_2x_3x_4x_5x_6$ with the set of negative edges $\{x_1x_2, x_3x_4, x_5x_6\}$. Obtain the signified graph by connecting a new vertex with unbalanced 2-paths to the vertices x_1, x_3, x_5 of the signified 6-cycle described above.

It is easy to check that this signified graph has girth 6 and signified relative clique number 4.

The proof of the upper bound is similar to the proof of Theorem 3.18(d).

We omit the rest of the proof (see Appendix).

(e) The lower bound follows from the fact that an unbalanced 2-path is an sclique of order 3.

It is easy to check that it is not possible to construct a signified graph with girth at least 7 in which at least four vertices are at unbalanced distance at most 2 keeping the graph planar. This completes the proof. □

5.3.3 Signified absolute clique number

As we have done for oriented colorings and orientable colorings, we define the signified absolute clique number and consider the problem of determining the parameter for some families of planar graphs. First we list the results regarding the signified absolute clique number of the families of outerplanar graphs and of outerplanar graphs with given girth.

Theorem 5.18.

(a) $\omega_{as}(\mathcal{O}_3) = 7$.

(b) $\omega_{as}(\mathcal{O}_4) = 4$.

(c) $\omega_{as}(\mathcal{O}_k) = 3$ *for* $k \geq 5$.

The proof of Theorem 5.18(a) and (b) easily follows from the proof of Theorem 5.16(a) and the proof of Theorem 5.19(b) respectively, while the proof of Theorem 5.18(c) is trivial.

Now we list the results regarding the signified absolute clique number of the families of planar graphs and of planar graphs with given girth.

Theorem 5.19.

(a) $\omega_{as}(\mathcal{P}_3) = 15$.

(b) $\omega_{as}(\mathcal{P}_4) = 6$.

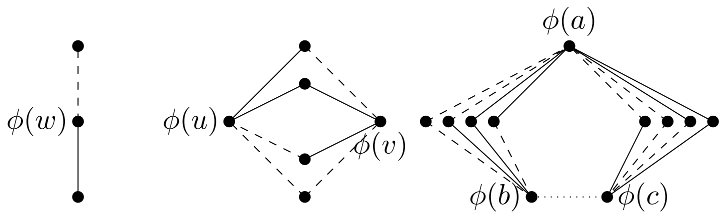


Figure 5.9: Planar signified graphs with diameter 2 and girth at least 4 (the dotted edge bc refers to an edge which can be negative or positve as per necessity).

(c) $\omega_{as}(\mathcal{P}_k) = 3$ *for* $k \geq 5$.

Proof of Theorem 5.19

(a) Take two copies of the signified outerplanar sclique depicted in Fig. 5.7 and a vertex ∞. Join ∞ and the vertices of the first copy of the signified outerplanar sclique depicted in Fig. 5.7 with positive edges. Join ∞ and the vertices of the second copy of the signified outerplanar sclique depicted in Fig. 5.7 with negative edges. This so-obtained graph is an sclique of order 15. Note that the graph is also planar. This proves the lower bound.

For proving the upper bound, first consider an sclique (H) with domination number 1. Suppose (H) is dominated by the vertex v. As (H) is an sclique, the set of vertices $N^+(v)$ are part of a relative clique in the signified outerplanar graph $((H[N(v)])$. Therefore, by Theorem 5.16(a) we have $|N^+(v)| \leq 7$.

Similarly we have $|N^-(v)| \leq 7$. Hence, $|N(v)| \leq 14$. This implies that the order of the graph (H) is at most 15.

Goddard and Henning [18] (see Chapter 2, Theorem 2.15) proved that every planar graph of diameter 2 has domination number at most 2 except for a particular graph on nine vertices.

Hence, to prove the theorem, it will be enough to prove that any planar sclique with domination number 2 must have order at most 15. More precisely, we need to prove the following result.

Lemma 5.20. *Let* (H) *be a planar sclique with domination number 2. Then* $|V((H)| \leq 15$.

The above lemma can be proved using similar arguments used for proving Lemma 3.23. The only changes we need to make are to consider unbalanced instead of directed distance and use the concept of unbalanced 2-path instead of 2-dipath.

This proof is in fact a bit easier because of the relaxed upper bound in the statement of the claim (15 instead of 14).

We omit the rest of the proof (see Appendix).

(b) In 1975, Plesník [24] characterized and listed all triangle-free planar graphs with diameter 2. They are precisely the graphs depicted in Fig. 2.1 (see Chapter 2, Theorem 2.14). Now note that any signified graph with the graphs from Fig. 2.1 as underlying graphs admits a homomorphism to the graphs depicted in Fig. 5.9 respectively (that is, the first signified graph depicted in Fig. 5.9 is a universal bound for the first family of graphs described in Fig. 2.1; the second ... etc.).

To prove the homomorphisms we map the vertices w, u, v, a, b, c from Fig. 2.1 to the corresponding vertices $\phi(w), \phi(u), \phi(v), \phi(a), \phi(b), \phi(c)$ in Fig 5.9 respectively. Choose the sign of the edge $\phi(b)\phi(c)$ the same as the sign of the edge bc.

Now to complete the first homomorphism, map the vertices of $N^\alpha(w)$ to the unique vertex in $N^\alpha(\phi(w))$ for $\alpha \in \{+,-\}$.

To complete the second homomorphism, map the vertices of $N^\alpha(u) \cap N^\beta(u)$ to the unique vertex in $N^\alpha(\phi(u)) \cap N^\beta(\phi(v))$ for $\alpha, \beta \in \{+,-\}$.

To complete the third homomorphism, map the vertices of $N^\alpha(a) \cap N^\beta(t)$ to the unique vertex in $N^\alpha(\phi(a)) \cap N^\beta(\phi(t))$ for $\alpha, \beta \in \{+,-\}$ and $t \in \{b, c\}$.

Now note that the first two signified graphs depicted in Fig. 5.9 are scliques of order 3 and 6 respectively, while the third graph is not an sclique but clearly has signified relative clique number 5.

Hence, there is no triangle-free planar sclique of order more than 6. Also, the only example of a triangle-free sclique of order 6 is the second graph depicted in Fig. 5.9.

Hence we are done.

(c) It is easy to check that the upper bound and the lower bound follows from the fact that the unbalanced 2-path is an sclique of order 3. □

5.4 Conclusion

In this chapter we mainly studied signified colorings of some families of planar graphs. Our main focus remained on the families $\mathcal{O}_g$ (family of outerplanar graphs with girth at least g) and $\mathcal{P}_g$ (family of planar graphs with girth at least g) for $g \geq 3$.

Concerning signified coloring we focused on determining the signified chromatic number, the signified relative clique number and the signified absolute clique number of the above mentioned families.

There was no known upper bound for the signified chromatic number for the family of planar graphs with girth at least 4 apart from the upper bound 80 of $\chi_s(\mathcal{P}_3)$. In this chapter we improved the upper bound of $\chi_s(\mathcal{P}_4)$ to 50.

We, in fact, defined two other parameters, that is, the signified relative clique number and the signified absolute clique number, mimicking the case of oriented coloring. We provided upper and lower bounds, most of them being tight, for some planar families.

In fact, we provided tight bounds for $\omega_{as}(\mathcal{O}_g)$ and for $\omega_{as}(\mathcal{P}_g)$ for all $g \geq 3$. We also provided tight bounds for $\omega_{rs}(\mathcal{O}_g)$ for all $g \geq 3$ and for $\omega_{rs}(\mathcal{P}_k)$ for all $k \geq 5$. Even though we could not provide exact bounds for $\omega_{ro}(\mathcal{P}_4)$, we have an intuition about what the answer could be.

So we propose the following conjecture.

Conjecture 5.21. $\omega_{rs}(\mathcal{P}_4) = 10$.

We think it is possible to mimic the proof of Theorem 3.19 to prove that the only minimal (with respect to subgraph inclusion) planar sclique of order 15 is the signified planar sclique described to prove the lower bound of Theorem 5.19(a). It should alo be possible to extend the proof and list out all the minimal planar scliques.

Nevertheless, despite Proposition 5.8, it is still interesting to seek for similarities between oriented and signified coloring. It might be possible to prove similar results regarding the relation between oriented and signified clique numbers as well.

We think it is possible to find specific families of graphs and prove some relation between the two chromatic numbers of those families (without knowing the chromatic numbers). For instance, can we prove some relation between $\chi_o(\mathcal{P}_3)$ and $\chi_s(\mathcal{P}_3)$ without determining those values?

Finally, the problem of bounding these three parameters for specific families of graphs is an interesting general problem in this topic.

Chapter 6
Signed graphs

In this chapter we deal with signed graphs. Our main focus is to present some results regarding signed colorings.

Signed homomorphisms have been introduced and studied recently by Naserasr, Rollová and Sopena in [40]. The theory captures a number of well known conjectures which can be reformulated using the definition of signed homomorphisms. Here we mention some definitions introduced in [40] and study signed homomorphisms for the class of planar graphs with different girth.

An important point to note is that Harary [21] and Zaslavsky [62] initially used the term signed graph for 2-edge-colored graphs. Due to the notion of signed graph homomorphisms, the term signed graph was used for the equivalence class by Naserasr, Rollová and Sopena [40].

The organization of the chapter is as follows. In Section 6.1 we give the basic definitions and notations related to signed graphs and homomorphisms of signed graphs. Then we present our main results regarding signed colorings in Section 6.3. In Section 6.4 we prove some results regarding bounds of planar consistent signed graphs. In Section 6.5 we discuss some categorical aspects which is a joint work with Naserasr and Sopena. Finally, we conclude this chapter in Section 4.4.

Section 6.2 and Section 6.3.1 is a joint work with Ochem and Pinlou and is an article in process. Section 6.4 is a joint work with Naserasr and Sun and is an article in process.

6.1 Preliminaries

Recall the definitions and notations related to signified graphs from Chapter 5. To *resign* a vertex v of a signified graph (G, Σ) is to change the signs of the edges incident to v. Two signified graphs (G, Σ) and (G, Σ') are in a *resign relation* if we can obtain (G, Σ') by resigning some vertices of (G, Σ). Note that the resign relation is an equivalence relation. A *signed graph* $[G, \Sigma]$ is an equivalance class of signified graphs (where (G, Σ) is an element of the equivalence class) with respect to resign relation. Any element of the equivalence class $[G, \Sigma]$ is a *presentation* of it. We use the notation $(G, \Sigma) \in [G, \Sigma]$ for (G, Σ) is a presentation of $[G, \Sigma]$.

We might use $[G]$ for a signed graph when the set of its negative edges is understood, while G is its *underlying graph* (note that every presentation of a particular signed graph has the same underlying graph).

The *order* of a signed graph is the number of vertices of its underlying graph and hence can be denoted by $|V(G)|$. Intuitively, we can treat a signed graph as a signified graph whose vertices are able to change the signs of all the edges incident to them. For a fixed presentation of a signed

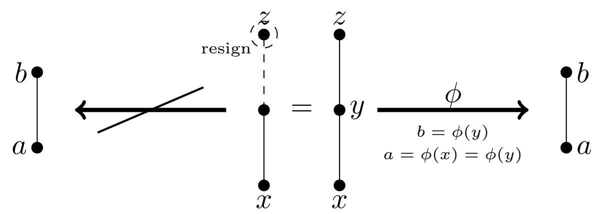


Figure 6.1: Signed graph homomorphism.

graph, we can use the notations defined for signified graphs. Given any signified graph (G, Σ) we can consider the signed graph $[G, \Sigma]$.

It is easy to check the following result from [40]:

Proposition 6.1. *If an undirected graph G has n vertices, m edges and c connected components, then there are 2^{m-n+c} distinct signed graphs with underlying graph G.*

An *unbalanced cycle* of length k, or an *unbalanced k-cycle* for short, of a signed graph $[G, \Sigma]$ is a cycle of length k in (G) which has an odd number of negative edges for some $(G) \in [G]$. Notice that this definition is independent of the choice of the presentation of $[G]$. Similarly, a *balanced cycle* of length k, or a *balanced k-cycle* for short, of a signed graph $[G, \Sigma]$ is a cycle of length k in (G) which has an even number of negative edges for some $(G) \in [G]$.

The first major result in the theory of signed graphs is the following by Zaslavsky [62].

Theorem 6.2. *Two signified graphs are presentation of the same signed graph if and only if they have the same set of unbalanced cycles.*

In other words, the set of unbalanced cycles uniquely determines a signed graph. The *unbalanced-girth* of a signed graph is the shortest length of its unbalanced cycles.

In [40], Naserasr, Rollová and Sopena introduced homomorphisms of signed graphs. Given two signed graphs $[G, \Sigma]$ and $[H, \Lambda]$, we say there is a *homomorphism* ϕ of $[G, \Sigma]$ to $[H, \Lambda]$ if ϕ is a homomorphism of a presentation $(G, \Sigma') \in [G, \Sigma]$ to a presentation $(H, \Lambda') \in [H, \Lambda]$. We write $[G, \Sigma] \to [H, \Lambda]$ whenever there exists a homomorphism of $[G, \Sigma]$ to $[H, \Lambda]$ and say that $[H, \Lambda]$ *bounds* $[G, \Sigma]$. A bijective homomorphism whose inverse is also a homomorphism is an *isomorphism*. If two signed graphs admit homomorphisms to each other then they are *homomorphically equivalent* signed graphs.

Lemma 6.3. *If $[G, \Sigma]$ admits a homomorphism to $[H, \Lambda]$, then for any presentation (H, Λ') of $[H, \Lambda]$ there exists a presentation (G, Σ') of $[G, \Sigma]$ such that (G, Σ') admits a homomorphism to (H, Λ').*

Proof. Let ϕ be a homomorphism of $[G, \Sigma]$ to $[H, \Lambda]$. This implies ϕ is a homomorphism of (G, Σ_1) to (H, Λ_1), for some presentation (G, Σ_1) and (H, Λ_1) of $[G, \Sigma]$ and $[H, \Lambda]$ respectively. Now, let (H, Λ') be any presentation of $[H, \Lambda]$.

Let $X = \{x_1, x_2, ..., x_k\}$ be the set of vertices we need to resign to obtain (H, Λ') from (H, Λ_1). Now resign the vertices of the set $\{v \in V(G) \mid \phi(v) \in X\}$ to obtain the presentation (G, Σ') of $[G, \Sigma]$ from (G, Σ_1). Clearly, ϕ is a homomorphism of (G, Σ') to (H, Λ'). □

The above lemma allows us to study homomorphisms of a signed graph $[G]$ to a signified graph (H).

Example 6.4. *A sample homomorphism of signed graphs is given in Fig. 6.1. Note that in the example we need to resign one of the vertices to obtain the homomorphism.*

Let $[G, \Sigma]$ be a signed graph with vertex set $V(G) = \{v_1, v_2, ..., v_k\}$ and $(G, \Sigma) \in [G, \Sigma]$. Then its *anti-twined graph* $(R(G), R(\Sigma))$ is the signified graph with the set of vertices, the set of edges and the set of positive edges as follows:

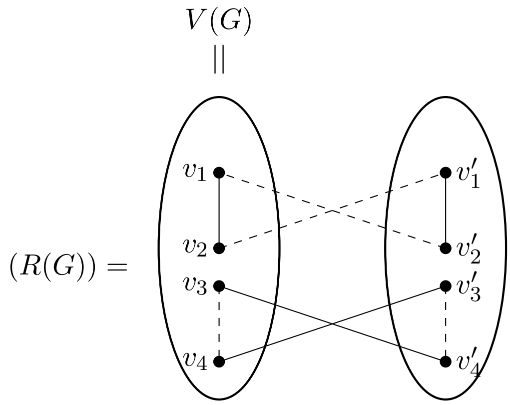


Figure 6.2: The anti-twined graph $(R(G))$ of $[G]$.

$$
\begin{aligned}
V(R(G)) &= \{v_1, v_2, ..., v_k\} \cup \{v'_1, v'_2, ..., v'_k\}.\\
E(R(G)) &= \{v_iv_j, v'_iv'_j, v_iv'_j \mid v_iv_j \in E(G)\}\\
R(\Sigma) &= \{v_iv_j, v'_iv'_j, v_iv'_k \mid v_iv_j \in \Sigma \text{ and } v_iv_k \in \Sigma^c\}.
\end{aligned}
$$

Intuitively, $(R(G))$ is the graph obtained from (G) by adding and resigning a twin v'_i for each of the vertices v_i of (G). Observe that $(R(G), R(\Sigma))$ is well defined, that is, for any presentation $(G, \Sigma) \in [G, \Sigma]$ we will get the same signified graph $(R(G), R(\Sigma))$. The anti-twined graph $(R(G), R(\Sigma))$ of a signified graph $(R(G), R(\Sigma))$ was defined in a more general setting by Brewster and Graves in [9].

Example 6.5. *For better understanding of the definition of an anti-twined graph see Fig. 6.2 where we have pictorially presented the construction of the anti-twined graph of a signed graph* $[G]$.

Now suppose the signed graph $[G, \Sigma]$ admits a homomorphism ϕ to the signed graph $[H, \Lambda]$. Also suppose that the presentations $(G, \Sigma) \in [G, \Sigma]$ and $(H, \Lambda) \in [H, \Lambda]$ are such that ϕ is a homomorphism of (G, Σ) to (H, Λ). The *anti-twined homomorphism* $R(\phi)$ of ϕ is the homomorphism of $(R(G))$ to $(R(H))$ such that we have,

$$R(\phi)(v) = \phi(v) \text{ and } R(\phi)(v') = \phi(v)' \text{ for } v \in V(G).$$

Observe that $R(\phi)$ is indeed a homomorphism. Also $R(\phi)$ is well defined, that is, it does not depend on the choice of the presentations $(G, \Sigma) \in [G, \Sigma]$ and $(H, \Lambda) \in [H, \Lambda]$ such that ϕ is a homomorphism of (G) to (H).

The following result follows from the above definitions.

Proposition 6.6. *Given two signed graphs* $[G]$ *and* $[H]$, *we have* $[G] \to [H]$ *if and only if* $(R(G)) \to (R(H))$.

The above result can be proved similarly as we proved Proposition 4.5 in Chapter 4.

A *splitable signified graph* (S) is a signified graph isomorphic to the signified graph $(R(T))$ of some signed graph $[T]$. The signified graph (T) is the *split* graph of (S).

Similarly, a *splitable signified homomorphism* is a homomorphism ψ of a splitable signified graph $(S_1) = (R(T_1))$ to a splitable signified graph $(S_2) = (R(T_2))$ such that we have $\psi = R(\phi)$ for some signed homomorphism ϕ of $[T_1]$ to $[T_2]$.

Notice that the set of vertices of any splitable signified graph (S) can be partitioned into two equal parts and a 1-1 correspondence between the vertices of those two parts can be established in a way that the corresponding vertices are not adjacent and they disagree with each other on all their common neighbors, while they have the same set of neigbors. The induced signified subgraph on one of those partitions is isomorphic to the split graph of the splitable graph in question. Hence we have the following result.

Lemma 6.7. *A signified graph (S) is splitable if and only if it is possible to partition the set of vertices $V(S)$ into two equal parts V_1 and V_2 with a bijecttion $f : V_1 \to V_2$ such that for all $u \in V_1$ we have $N^+(u) = N^-(f(u))$ and $N^-(u) = N^+(f(u))$.*

For example, for each vertex u in a signified Tromp graph or a signified Zielonka graph (see Chapter 5, Section 5.2) there is a unique vertex v such that we have $N^+(u) = N^-(v)$ and $N^-(u) = N^+(v)$. So, it is easy to see that these graphs are splitable signified graphs.

Using the notion of splitable graphs we now state the following useful result.

Lemma 6.8. *Let $(S) = (R(T))$ be a splitable graph. Then $(G) \to (S)$ if and only if $[G] \to (T)$ for any $(T) \in [T]$.*

The above result can be proved similarly as we proved Proposition 4.7 in Chapter 4.

We will use this lemma several times in this chapter.

6.2 Some signed graphs and their properties

While studying homomorphisms of signed graphs, to get upper bounds for the signed chromatic number of a family $\mathcal{F}$ of graphs, we will try to find a target signified graph (T) such that every graphs of $\mathcal{F}$ admits a homomorphism to (T).

Recall the definitions of Paley graphs and signified Paley graphs from Chapter 5, Section 5.2. Also note that the Paley graph P_q is self-complementary, edge transitive [63] and strongly regular with parameters $(q, (q-1)/2, (q-5)/4, (q-1)/4)$ [10].

A *k-type-vector* is a sequence $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k) \in \{+, -\}^k$ of k signs (that is, $+$ or $-$) while its *conjugate* is another k-type-vector $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2, ..., \bar{\alpha}_k)$ with $\{\alpha_i, \bar{\alpha}_i\} = \{+, -\}$ for all $i \in \{1, 2, ..., k\}$.

Given a sequence of k vertices $X_k = (v_1, v_2, ..., v_k)$ of a signified graph (T), that induces a clique in T, a vertex $u \in V(T)$ is an *α-successor* of X_k if for every $i \in \{1, 2, ..., k\}$, we have $u \in N^{\alpha_i}(v_i)$. Also, u is called an $\tilde{\alpha}$-successor of X_k if it is an α-successor or an $\bar{\alpha}$-successor of X_k. The set of α-successors of X_k is denoted by $N^{\alpha}(X_k)$ and the set of $\tilde{\alpha}$-successors of X_k is denoted by $N^{\tilde{\alpha}}(X_k)$.

A signified graph (T) has property $P_{k,l}$ (resp., property $\tilde{P}_{k,l}$) if, $|N^{\alpha}(X_k)| \geq l$ (resp., $|N^{\tilde{\alpha}}(X_k)| \geq l$) for any sequence X_k of k vertices of (T) which induces a clique in T and for any k-type-vector α.

Lemma 6.9.

(a) For $q \equiv 1 \pmod 4$ and $q \geq 5$, (SP_q) has property $P_{1,q-1/2}$, $\tilde{P}_{1,q-1}$ and (SP_q^+) has property $\tilde{P}_{1,q}$.

(b) For $q \equiv 1 \pmod 4$ and $q \geq 9$, (SP_q) has property $P_{2,q-5/4}$, $\tilde{P}_{2,q-3/2}$ and (SP_q^+) has property $\tilde{P}_{2,q-2/2}$.

(c) For $q \equiv 1 \pmod 4$ and $q \geq 9$, (SP_q) has property $\tilde{P}_{3,q-9/4}$ and (SP_q^+) has property $\tilde{P}_{3,q-5/4}$.

Proof. **(a)** Any vertex of the Paley graph P_q has $(q-1)/2$ neighbors and hence, $(q-1)/2$ non-neighbors (as it has q vertices). So, any vertex in the signified Paley graph (SP_q) has $(q-1)/2$ distinct $(+)$-neighbors and $(q-1)/2$ distinct $(-)$-neighbors. Hence, (SP_q) has property $P_{1,q-1/2}$.

The signified graphs (SP_q) and (SP_q^+) have order q and $(q+1)$ respectively and any two vertices in each of them are joined either by a positive edge or a negative edge. Therefore, clearly, (SP_q) has property $\tilde{P}_{1,q-1}$ and (SP_q^+) has property $\tilde{P}_{1,q}$.

(b) The Paley graph P_q is a strongly regular graph with parameters $(q, (q-1)/2, (q-5)/4, (q-1)/4)$. Hence, for any edge uv of P_q there are exactly $(q-5)/4$ common neighbors of u and v. Note that in the complement $\bar{P}_q$ of P_q, uv is a non-edge. As P_q is self-complementary, $\bar{P}_q$ is also a strongly regular graph with parameters $(q, (q-1)/2, (q-5)/4, (q-1)/4)$. Hence, there are exactly $(q-1)/4$ common neighbors of u and v in $\bar{P}_q$. Therefore, there are exactly $(q-1)/4$ common non-neighbors of u and v in P_q.

Now assume that there are k_1 vertices of P_q that are neighbors of u but non-neighbors of v and k_2 vertices of P_q that are non-neighbors of u but neighbors of v. Now P_q is strongly regular with parameters $(q, (q-1)/2, (q-5)/4, (q-1)/4)$. So we have,

$$\begin{aligned}
(q-1)/2 &= |\{\text{neighbors of } u\}| \\
&= |\{v\}| + |\{\text{common neighbors of } u \text{ and } v\}| \\
&\qquad\qquad + |\{\text{neighbors of } u \text{ that are non-neighbors of } v\}| \\
&= 1 + (q-5)/4 + k_1 \\
&= (q-1)/4 + k_1 \\
\Rightarrow \quad k_1 &= (q-1)/2 - (q-1)/4 \\
&= (q-1)/4.
\end{aligned}$$

Hence we have, $k_1 = (q-1)/4$. Similarly by counting the neighbors of v we can show that $k_2 = (q-1)/4$.

Therefore, any for edge uv in P_q we have,

$$\begin{aligned}
&|\{\text{common neighbors of } u \text{ and } v\}| \\
&\qquad = (q-5)/4 \\
&|\{\text{common non-neighbors of } u \text{ and } v\}| \\
&\qquad = |\{\text{neighbors of } u \text{ that are non-neighbors of } v\}| \\
&\qquad = |\{\text{non-neighbors of } u \text{ that are neighbors of } v\}| \\
&\qquad = (q-1)/4
\end{aligned}$$

Now take the complement $\bar{P}_q$ of P_q. For any non-edge uv in $\bar{P}_q$ (as uv was any edge in P_q) we have,

$$\begin{aligned}
&|\{\text{common non-neighbors of } u \text{ and } v\}| \\
&\qquad = (q-5)/4 \\
&|\{\text{common neighbors of } u \text{ and } v\}| \\
&\qquad = |\{\text{non-neighbors of } u \text{ that are neighbors of } v\}| \\
&\qquad = |\{\text{neighbors of } u \text{ that are non-neighbors of } v\}| \\
&\qquad = (q-1)/4
\end{aligned}$$

As P_q is self-complementary, any non-edge of P_q will have the same property as above.

Now note that we obtain the signified Paley graph (SP_q) from the Paley graph P_q by replacing its edges by positive edges and its non-edges by negative edges. So, from the above discussion, for any pair of distinct vertices $\{x, y\}$ of (SP_q) and for any 2-type-vector α we have,

$$|N^{\alpha}(\{x, y\})| \geq (q-5)/4.$$

This proves that (SP_q) has property $P_{2,q-5/4}$.

From the above discussion we learn that, for any two distinct vertices $\{x, y\}$ of (SP_q) we have,

$$\begin{aligned}
&|N^{\alpha}(\{x, y\})| = (q-1)/4 \text{ for } \alpha \in \{(+,-), (-,+)\}. \\
&|N^{\beta}(\{x, y\})| = (q-5)/4 \text{ and } |N^{\beta}(\{x, y\})| \geq (q-1)/4 \\
&\qquad\qquad\qquad \text{for some } \beta = (\beta_1, \beta_2) \in \{(+,+), (-,-)\} \\
&\qquad\qquad\qquad\qquad \text{while the sign of the edge } xy \text{ is } \beta_1.
\end{aligned}$$

Therefore, for any two distinct vertices $\{x, y\}$ of (SP_q) we have,

$$|N^{\tilde{\alpha}}(\{x, y\})| = (q-1)/2 \text{ for } \alpha \in \{(+,-),(-,+)\}.$$
$$|N^{\tilde{\beta}}(\{x, y\})| = (q-3)/2 \text{ for } \beta \in \{(+,+),(-,-)\}.$$

This proves Lemma 6.11 (stated later in this section) and shows that (SP_q) has property $\tilde{P}_{2,q-3/2}$.

Now we will show that the signified graph (SP_q^+) has property $\tilde{P}_{2,q-1/2}$. We know that there is this vertex ∞ of (SP_q^+) and by deleting ∞ from (SP_q^+) we obtain the signified Paley graph (SP_q).

Let $u \in V(SP_q^+) \setminus \{\infty\}$. We know by Lemma 6.9(a) that u has $(q-1)/2$ distinct $(+)$-neighbors and $(q-1)/2$ distinct $(-)$-neighbors among $V(SP_q^+) \setminus \{\infty, u\}$. As every other vertex is adjacent to ∞ with a positive edge in (SP_q^+) we have,

$$|N^{\alpha}(\{\infty, u\})| = (q-1)/2 \text{ for } \alpha \in \{(+,+),(+,-)\}.$$

Hence, for any 2-type-vector α, we have,

$$|N^{\tilde{\alpha}}(\{\infty, u\})| = (q-1)/2.$$

Now let $v \in V(SP_q^+) \setminus \{\infty, u\}$. Then, from the above discussion, we know that $\{u, v\}$ has exactly $(q-1)/2$ distinct $\tilde{\alpha}$-neighbors for $\alpha \in \{(+,-),(-,+)\}$ and exactly $(q-3)/2$ distinct $\tilde{\beta}$-neighbors for $\beta \in \{(+,+),(-,-)\}$ in $V(SP_q^+) \setminus \{\infty, u, v\}$. Note that ∞ is a $(+,+)$-neighbor of $\{u, v\}$. Hence in (SP_q^+) we have,

$$|N^{\tilde{\alpha}}(\{u, v\})| = (q-1)/2 \text{ for } \beta \in \{(+,+),(-,-)\}.$$

This shows that (SP_q^+) has property $\tilde{P}_{2,q-1/2}$.

(c) First we will show that (SP_q^+) has property $\tilde{P}_{3,q-4/5}$. Let ∞, u and v be three distinct vertices of (SP_q^+).

Let $\gamma \in \{+,-\}$ be the sign of the edge uv. Then from the proof of Lemma 6.9(b) we know that $\{u, v\}$ has exactly $(q-5)/4$ distinct (γ,γ)-neighbors and exactly $(q-1)/4$ distinct β-neighbors in $V(SP_q^+) \setminus \{\infty, u, v\}$ for any 2-type-vector $\beta \neq (\gamma,\gamma)$. Hence in (SP_q^+), for any 3-type-vector α, we have,

$$|N^{\tilde{\alpha}}(\{\infty, u, v\})| \geq (q-5)/4.$$

Now recall that $V(SP_q) = F_q$. Also we know that there exists $t \in F_q$ such that $F_q = \{0, t, t^2, ..., t^{q-1} = 1\}$. Fix such a t for the rest of this proof.

So, the set of squares of F_q is the set $N^+ = \{t^2, t^4, ..., t^{q-1}\}$ of even powers of t while the set of non-squares of F_q is the set $N^- = \{t, t^3, ..., t^{q-2}\}$ of odd powers of t. Now by the definition of the signified Paley plus graph (SP_q^+) we have, $N^+_{(SP_q^+)}(0) = N^+ \cup \{\infty\}$ and $N^-_{(SP_q^+)}(0) = N^-$.

Now let (P) be the signified graph obtained from (SP_q^+) by resigning the vertices of the set N^-. Now define the mapping ϕ from $V(SP_q^+)$ to $V(P)$ as follows:

$$\phi(x) = \begin{cases} 0, & \text{if } x = \infty, \\ \infty, & \text{if } x = 0, \\ t^{-i}, & \text{if } x = t^i. \end{cases} \tag{6.1}$$

Claim 1: The mapping ϕ is an isomorphism of (SP_q^+) to (P).

Proof of the Claim 1: It is easy to check that the mapping ϕ is a bijection. Now we will show that it is also a homomorphism. For that we need to show that each edge uv of (SP_q^+) and $\phi(u)\phi(v)$ of (P) have the same sign.

First of all note that exactly the set N^- of negative neighbors of 0 in (SP_q^+) has been resigned to obtain (P). Hence in (P), the edge $0u$ is positive for any $u \in V(P) \setminus \{0\}$.

Hence, each edge ∞v of (SP_q^+) and $\phi(\infty)\phi(v)$ (which is basically the edge $0\phi(v)$ as $\phi(\infty) = 0$) of (P) are both positive for any $v \in V(SP_q^+) \setminus \{\infty\}$.

Now note that $\phi(u) \in N^+$ if and only if $u \in N^+$ and $\phi(u) \in N^-$ if and only if $u \in N^-$ because $t^{-i} = t^{q-1-i}$ is a square in F_q if and only if t^i is a square in F_q (as $(q-1-i)$ is even if and only if i is even).

Hence, each edge $0v$ of (SP_q^+) and the edge $\phi(0)\phi(v)$ (which is the edge $\infty\phi(v)$ of (P) basically) are both positive for each $v \in N^+ \cup \{\infty\}$ and are both negative for each $v \in N^-$.

Also note that the edge $\infty 0$ is positive in (SP_q^+) while its image $\phi(\infty)\phi(0)$ (which is the edge 0∞) is positive as well.

Therefore, now we are left with checking the edges of the form uv for all $u, v \in V(SP_q^+) \setminus \{0, \infty\} = N^+ \cup N^-$. Without loss of generality assume that $u = t^i$ and $v = t^j$. So we have,

$$\begin{aligned} f(t^i) - f(t^j) &= t^{-i} - t^{-j} \\ &= (t^j - t^i)/t^{i+j} \end{aligned}$$

We know that in F_q, -1 is a square and the product of two squares is a square, the product of two non-squares is a square, whereas the product of a square and a non-square is a non-square in F_q.

Now assume that $u = t^i$ and $v = t^j$ are both from N^+. Note that i and j are both even as $u, v \in N^+$. Now, uv is positive in (SP_q^+),

if and only if $(t^i - t^j)$ is a square in F_q,

if and only if $(t^j - t^i)$ is a square in F_q (becasue -1 is a square and product of two squares is a square),

if and only if $(t^j - t^i)/t^{i+j}$ is a square in F_q (becasue t^{i+j} is a square as $(i + j)$ is even),

if and only if $\phi(u)\phi(v)$ is positive in (SP_q^+),

if and only if $\phi(u)\phi(v)$ is positive in (P) (as $\phi(u), \phi(v) \in N^+$, neither of them have been resigned to obtain (P) from (SP_q^+)).

Similarly assume that $u = t^i$ and $v = t^j$ are both from N^-. Note that i and j are both odd as $u, v \in N^-$. Now, uv is positive in (SP_q^+),

if and only if $\phi(u)\phi(v)$ is positive in (SP_q^+) (by similar arguments as the above case),

if and only if $\phi(u)\phi(v)$ is positive in (P) (as $\phi(u), \phi(v) \in N^-$, both of them have been resigned to obtain (P) from (SP_q^+)).

Now assume that $u = t^i \in N^+$ and $v = t^j \in N^-$. Note that i is even and j is odd. Now, uv is positive in (SP_q^+),

if and only if $(t^j - t^i)$ is a square in F_q (similar arguements as before),

if and only if $(t^j - t^i)/t^{i+j}$ is a non-square in F_q (t^{i+j} is a non-square as $(i + j)$ is odd),

if and only if $\phi(u)\phi(v)$ is negative in (SP_q^+),

if and only if $\phi(u)\phi(v)$ is positive in (P) (as $\phi(u) \in N^+$ and $\phi(v) \in N^-$, only one of them have been resigned to obtain (P) from (SP_q^+) and therefore the edge have changed its sign).

Hence the mapping ϕ is indeed an isomorphism of (SP_q^+) to (P). $\Diamond$

Now let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be any 3-type-vector. Fix two distinct vertices $x_1, x_2, x_3 \in V(SP_q^+) \setminus \{\infty\}$. Now define,

$$\begin{aligned} \beta_i &= \bar{\alpha}_i \text{ if } x_i \in N^-, \\ &= \alpha_i \text{ otherwise, for all } i \in \{1, 2\}. \end{aligned}$$

Now define the 3-type-vector $\beta = (\beta_1, \beta_2, \alpha_3)$. Now we will show that $|N^{\tilde{\alpha}}(\{x_1, x_2, x_3\})| \geq (q-5)/4$ in (SP_q^+). To prove that we can assume, without loss of generality, that $x_3 = 0$ because the Paley graph P_q is vertex transitive. So, basically we are left with showing $|N^{\tilde{\alpha}}(\{x_1, x_2, 0\})| \geq (q-5)/4$ in (SP_q^+).

Claim 2: A vertex $v \in N^{\tilde{\alpha}}(\{x_1, x_2, 0\})$ in (SP_q^+) if and only if $v \in N^{\tilde{\beta}}(\{\phi(x_1),\ \phi(x_2), \infty\})$ in (SP_q^+).

Proof of the Claim 2: As ϕ is an isomorphism we have, $v \in N^{\alpha}(\{x_1, x_2, 0\})$ (resp., $v \in N^{\bar{\alpha}}(\{x_1, x_2, 0\})$) in (SP_q^+),

if and only if $v \in N^{\alpha}(\{\phi(x_1), \phi(x_2), \infty\})$ (resp., $v \in N^{\bar{\alpha}}(\{\phi(x_1), \phi(x_2), \infty\})$ in (P).

Recall that we obtained (P) from (SP_q^+) by resigning the set N^- of vertices. Hence we will obtain (SP_q^+) from (P) by resigning the set N^- of vertices.

Hence, if $\phi(v) \in V(P) \backslash N^-$, then $\phi(v)$ is an α-neighbor (or an $\bar{\alpha}$-neighbor) of $\{\phi(x_1), \phi(x_2), \infty\}$ in (P),

if and only if $\phi(v)$ is a β-neighbor (or a $\bar{\beta}$-neighbor) of $\{\phi(x_1), \phi(x_2), \infty\}$ in (SP_q^+) (as $\phi(v)$ has not been resigned to obtain (P) from (SP_q^+)).

Similarly, if $\phi(v) \in N^-$, then $\phi(v)$ is an α-neighbor (or an $\bar{\alpha}$-neighbor) of $\{\phi(x_1), \phi(x_2), \infty\}$ in (P),

if and only if $\phi(v)$ is a $\bar{\beta}$-neighbor (or a β-neighbor) of $\{\phi(x_1), \phi(x_2), \infty\}$ in (SP_q^+) (as $\phi(v)$ has been resigned to obtain (P) from (SP_q^+)).

This proves the claim. $\diamondsuit$

Now as any three distinct vertices $\{\infty, u, v\}$ have at least $(q-5)/4$ distinct $\tilde{\alpha}$-neighbors in (SP_q^+) for any 3-type-vector α, we have proved that (SP_q^+) has property $\tilde{P}_{3,q-5/4}$ using the above claim.

Let $\{x, y, z\}$ be three distinct vertices from $V(SP_q^+) \setminus \{\infty\}$. We know by the above, that $\{x, y, z\}$ have at least $(q-5)/4$ distinct $\tilde{\alpha}$-neighbors in (SP_q^+) for any 3-type-vector α.

Note that ∞ is a $(+,+,+)$-neighbor of $\{x, y, z\}$. Hence, $\{x, y, z\}$ have at least $(q-5)/4$ distinct $\tilde{\alpha}$-neighbors in (SP_q) for any 3-type-vector $\alpha \notin \{(+,+,+), (-,-,-)\}$.

Also, we are sure that there is at least $(q-9)/4$ (excluding ∞) distinct $\tilde{\beta}$-neighbors in (P_q) for any $\beta \in \{(+,+,+), (-,-,-)\}$.

Therefore, (P_q) has property $\tilde{P}_{3,q-9/4}$.

Similarly, we can show that there is at most $(q-5)/4$ distinct $\tilde{\beta}$-neighbors in (P_q) for any $\beta \in \{(+,+,+), (-,-,-)\}$. This will prove Lemma 6.12. $\square$

From the above proof we readily get the following three results.

Lemma 6.10. *For $q \equiv 1$ (mod 4), $q \geq 5$ and for any vertex $v \in V(SP_q)$,*

$$\cup_{u \in N^{\alpha_1}(v)} N^{\alpha_2}(u) = \begin{cases} V(SP_q) \setminus \{v\}, & \text{for } (\alpha_1, \alpha_2) \in \{(+,-), (-,+)\}, \\ V(SP_q), & \text{for } (\alpha_1, \alpha_2) \in \{(+,+), (-,-)\}. \end{cases}$$

Lemma 6.11. *For $q \equiv 1$ (mod 4), $q \geq 9$ and for any two distinct vertices u and v of (SP_q),*

$$|N^{\tilde{\alpha}}(\{u, v\})| = \begin{cases} (q-1)/2, & \text{for } \alpha \in \{(+,-), (-,+)\}, \\ (q-3)/2, & \text{for } \alpha \in \{(+,+), (-,-)\}. \end{cases}$$

Lemma 6.12. *For $q \equiv 1$ (mod 4), $q \geq 9$ and for any three distinct vertices u, v and w of (SP_q),*

$$|N^{\tilde{\alpha}}(\{u, v, w\})| \leq \begin{cases} (q-1)/4, & \text{for any 3-type-vector } \alpha, \\ (q-5)/4, & \text{for } \alpha \in \{(+,+,+), (-,-,-)\}. \end{cases}$$

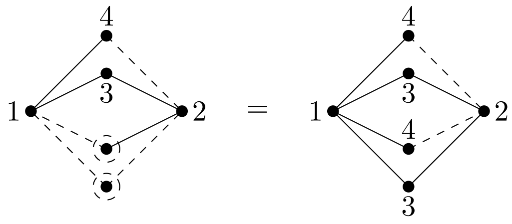


Figure 6.3: Signed coloring (we resign the vertices marked with dashed circles of the graph in the left.

6.3 Signed coloring

Colorings of signed graphs here correspond to colorings associated with homomorphisms of signed graphs defined, discussed and studied by Naserasr, Rollová and Sopena [40]. Even though the definition is based on homomorphisms of signed graphs, we first will define it without using homomorphism.

A *signed k-coloring* of a signed graph $[G]$ is a vertex coloring which is a signified k-coloring of a presentation of the graph. The *signed chromatic number* $\chi_{[s]}([G])$ of a signed graph $[G]$ is the minimum of the signified chromatic numbers of the elements of the equivalence class $[G]$.

Alternatively, the signed chromatic number $\chi_{[s]}([G])$ of the signed graph $[G]$ is the minimum order of a signed graph $[H]$ such that $[G]$ admits a homomorphism to $[H]$.

By virtue of Lemma 6.3, we can equivalently define the signed chromatic number $\chi_{[s]}([G])$ of a signed graph $[G]$ by the minimum order of a signified graph (H) such that $[G]$ admits a homomorphism to (H).

The signed chromatic number $\chi_{[s]}(G)$ of a simple graph G is the maximum of the signed chromatic numbers of all the signed graphs with underlying graph G. The signed chromatic number $\chi_{[s]}(\mathcal{F})$ of a family $\mathcal{F}$ of graphs is the maximum of the signed chromatic numbers of the graphs from the family $\mathcal{F}$.

Example 6.13. *Check the signed 4-coloring provided in Fig. 6.3.*

Notice that the non-adjacent vertices of an unbalanced 4-cycle always gets different colors as they are always connected with an unbalanced 2-path, no matter which vertex of the graph you resign. This is in fact the necessary and sufficient condition for two non-adjacent vertices to receive two distinct colors under a signed coloring.

A *relative signed clique* of a signed graph $[G]$ is a set $R \subseteq V(G)$ of vertices such that any two vertices from R are either adjacent or part of an unbalanced 4-cycle. The *signed relative clique number* $\omega_{[rs]}([G])$ of a signed graph $[G]$ is the maximum order of a signed relative clique of $[G]$.

A *signed clique* or simply an *[s]-clique* is a signed graph $[G]$ for which $\chi_{[s]}([G]) = |V(G)|$. Note that [s]-cliques can hence be characterized as those signed graphs whose any two distinct vertices are either adjacent or part of an unbalanced 4-cycle. Note that a signed graph with an [s]-clique of order n as a subgraph has signed chromatic number and signed relative clique number at least n. The *signed absolute clique number* $\omega_{as}([G])$ of a signed graph $[G]$ is the maximum order of an [s]-clique contained in $[G]$ as a subgraph.

The signed relative clique number $\omega_{[rs]}(G)$ (resp. signed absolute clique number $\omega_{[as]}(G)$) of a simple graph G is the maximum of the signed relative clique numbers (resp. signed absolute clique numbers) of all the signed graphs with underlying graph G. The signed relative clique number $\omega_{[rs]}(\mathcal{F})$ (resp. signed absolute clique number $\omega_{[as]}(\mathcal{F})$) of a family $\mathcal{F}$ of graphs is the maximum of the signed relative clique numbers (resp. signed absolute clique numbers) of the graphs from the family $\mathcal{F}$.

From the definitions, clearly we have the following:

Lemma 6.14. *For any signed graph* $[G]$ *we have,* $\omega_{[as]}([G]) \leq \omega_{[rs]}([G]) \leq \chi_{[s]}([G])$.

Corollary 6.15. *For any [s]-clique* $[O]$ *we have,* $\omega_{[as]}([O]) = \omega_{[rs]}([O]) = \chi_{[s]}([O])$.

Example 6.16. *Consider the signified graph (B^+, Λ') obtained by adding a new vertex ∞ to the signified graph (B, Λ) depicted in Fig. 5.4 (see Chapter 5, Section 5.3) in such a way that we have $N^+(\infty) = V(B \setminus \{\infty\}$. It is easy to check that $\omega_{[as]}([B^+, \Lambda']) = 4$, $\omega_{[rs]}([B^+, \Lambda']) = 5$ and $\chi_{[s]}([B^+, \Lambda']) = 6$. This is an example of a graph for which each inequality of the above theorem is strict.*

Note that the above defined three graph parameters respect homomorphisms of signed graphs in the sense of the following result.

Lemma 6.17. *Let $[G]$ and $[H]$ be two signed graphs. If $[G] \to [H]$, then $\chi_{[s]}([G]) \leq \chi_{[s]}([H])$, $\omega_{[rs]}([G]) \leq \omega_{[rs]}([H])$ and $\omega_{[as]}([G]) \leq \omega_{[as]}([H])$.*

A usual technique for obtaining an upper bound for the three graph parameters of signed graphs, defined in this section, is to prove that every graph in the family of graphs in question admits a homomorphism to a particular signed graph, that is to find a signed graph that bounds every graph of that family. Such a graph is called a *universal bound* of that family of graphs. Note that not every family of graphs have a universal bound of order equal to its signed chromatic number.

Example 6.18. *For example, the family of all signed graphs on 3 vertices has signed chromatic number 3 as each graph in the family is clearly signed 3-colorable. But it is easy to show that a balanced 3-cycle and an unbalanced 3-cycle cannot admit a homomorphism to a single signed graph on 3 vertices.*

If we consider the set of all signed graphs to be a category with objects being the signed graphs and morphisms being the signed homomorphisms, then we clearly have the following:

Theorem 6.19. *For any family $\mathcal{F}$ of signed graphs that also contains the categorical co-products of the graphs from the family, there exists a universal bound of $\mathcal{F}$ on $\chi_{[s]}(\mathcal{F})$ vertices.*

Observe that a categorical co-product (unique up to homomorphic equivalence) of signed graphs is simply the signed graph obtained by taking the disjoint union of the signed graphs. The families of planar graphs, outerplanar graphs, planar graphs with given girth and outerplanar graphs with given girth are each of the type that we mentioned in the above theorem.

6.3.1 Signed chromatic number

The first result on signed chromatic number is the following relation between the signed chromatic number and the signified chromatic number which follows from Proposition 6.20.

Proposition 6.20. *For any signified graph (G), we have $\chi_{[s]}([G]) \leq \chi_s((G)) \leq 2\chi_{[s]}([G])$.*

In the first relation, equality holds for any signified graph whose underlying graph is a complete graph, while in the second relation, equality holds for splitable signified graphs.

One of the main general results we can prove using Lemma 6.8 follows easily from the fact that the signified Zielonka graphs are splitable.

Theorem 6.21. *Every graph with acyclic chromatic number at most k has signed chromatic number at most $k.2^{k-2}$.*

As the bound in Theorem 5.12 is tight for $k \geq 3$, it is tight for $k \geq 3$ in the above theorem too by Proposition 6.20.

Now we list the bounds for the signed chromatic number of the families of outerplanar graphs and of outerplanar graphs with given girth. The relevant references are given beside the results. Recall that $\mathcal{O}_g$ denotes the family of outerplanar graphs with girth at least g.

Theorem 6.22.

(a) $\chi_{[s]}(\mathcal{O}_3) = 5$. *[40]*

(b) $\chi_{[s]}(\mathcal{O}_k) = 4$ *for* $k \geq 4$.

Naserasr, Rollová and Sopena [40] proved that the signed chromatic number of the family of outerplanar graphs is at most 5 and that this bound is tight. That proves part (a) of the result. We will prove the other part in this section.

Now we list the bounds for the signed chromatic number of the families of planar graphs and of planar graphs with given girth. Recall that $\mathcal{P}_g$ denotes the family of outerplanar graphs with girth at least g.

Theorem 6.23.

(a) $10 \leq \chi_{[s]}(\mathcal{P}_3) \leq 40$.

(b) $6 \leq \chi_{[s]}(\mathcal{P}_4) \leq 25$.

(c) $4 \leq \chi_{[s]}(\mathcal{P}_5) \leq 10$.

(d) $4 \leq \chi_{[s]}(\mathcal{P}_6) \leq 6$.

(e) $\chi_{[s]}(\mathcal{P}_k) = 4$ *for* $k \geq 8$.

Part (e) of the above theorem is proved by Charpentier, Naserasr and Sopena and is currently part of an article in process.

The above theorem improves the previous upper bound of 48 for $\chi_{[s]}(\mathcal{P}_3)$ proved by Naserasr, Rollová and Sopena [40]. We could not improve the lower bound of 10, but we do prove the following result:

Theorem 6.24. *If there exists a signed graph of order 10 to which every planar signed graph admits a homomorphism, then that signed graph must be the signed Paley plus graph* $[SP_9^+]$.

Proof of Theorem 6.22(b)

(b) The lower bound follows from the fact that every unbalanced cycle of even length has signed chromatic number equal to 4 (shown in [40]).

In [47], Pinlou and Sopena showed that every outerplanar graph with girth at least k and minimum degree at least 2 contains a face of length $l \geq k$ with at least $(l-2)$ consecutive vertices of degree 2.

Now, to prove Theorem 6.22(b), we will show that every signed outerplanar graph with girth at least 4 admits a homomorphism to the signified graph (K_4, Λ) with $|\Lambda| = 1$, that is the signified graph on the complete graph K_4 having only one negative edge.

Let $[H]$ be a minimal (with respect to inclusion as a subgraph) signed outerplanar graph with girth 4 having no homomorphism to (K_4, Λ).

(i) Suppose that $[H]$ contains a vertex u of degree 1. Then, due to the minimality of $[H]$, the signed outerplanar graph obtained by deleting the vertex u from $[H]$ (which has girth at least 4) admits a homomorphism to (K_4, Λ). Since every vertex of (K_4, Λ) has at least one positive edge, the homomorphism can easily be extended to obtain a homomorphism of $[H]$ to (K_4, Λ) by resigning (if needed) u.

(ii) Suppose that $[H]$ contains a face of length $l \geq 5$ with at least $(l-2)$ consecutive vertices $x_1, x_2, ..., x_{l-2}$ of degree 2. Then, due to the minimality of $[H]$, the signed outerplanar graph $[H']$ obtained by deleting the vertices $x_1, x_2, ..., x_{l-2}$ from $[H]$ (which has girth at least 4) admits a homomorphism ϕ to (K_4, Λ). Now, let (H') be a presentation of $[H]$ with $\phi : (H') \to (K_4, \Lambda)$.

It is possible to check (a bit tidious, but not difficult, by case analysis), that given any signified path $a_0a_1...a_m$ of length $m \geq 3$ and a mapping $\psi : \{a_1, a_m\} \to V(K_4, \Lambda)$ (where $\psi(a_1)$ can be equal to $\psi(a_m)$), it is possible to resign the vertices a_i for $i \in \{1, .., m-1\}$ to obtain a signified path and extend the mapping ψ to a homomorphism of that signified path to (K_4, Λ). Note that it is enough to check the case $m = 3$ as the cases $m > 3$ are implied by it.

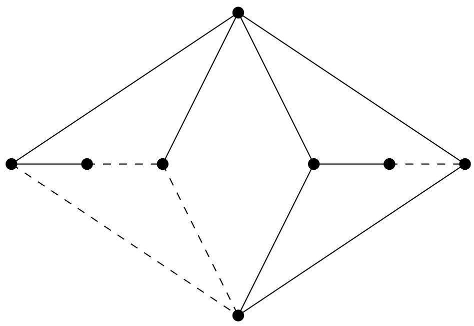


Figure 6.4: A signed triangle-free planar graph with signed relative clique number 6.

Hence, by the above observation, we can extend the homomorphism of $[H']$ to (K_4, Λ) to a homomorphism of $[H]$ to (K_4, Λ).

This is a contradiction. Hence, every signed outerplanar graph with girth at least 4 admit a homomorphism to (K_4, Λ). □

Proof of Theorem 6.23(a),(c) and (d)

(a) To prove the upper bound we use Theorem 6.21 and the theorem of Borodin [4] (see Chapter 2, Theorem 2.11) that states that every planar graph has an acyclic 5-coloring.

The lower bound was proved by Naserasr, Rollová and Sopena [40].

(c) The lower bound follows from the fact that every unbalanced cycle of even length has signed chromatic number equal to 4 (shown in [40]).

Montejano *et al.* [36] showed that the signified Tromp graph (ST_{20}) bounds the family of signified planar graphs with girth at least 5. Note that the signified Tromp graph (ST_{20}) is a splitable signified graph whose split graph is the graph (SP_9^+) on 10 vertices. Hence, by Lemma 6.8, the upper bound follows.

(d) The lower bound follows from the fact that every unbalanced cycle of even length has signed chromatic number equal to 4 (shown in [40]).

Montejano *et al.* [36] showed that the signified Tromp graph (ST_{12}) bounds the family of signified planar graphs with girth at least 6. Note that the signified Tromp graph (ST_{12}) is a splitable signified graph whose split graph is the graph (SP_5^+) on 6 vertices. Hence, by Lemma 6.8, the upper bound follows. □

Proof of Theorem 6.23(b)

The lower bound follows from the example depicted in Fig. 6.4 as the six vertices with degree at least 3 are part of a relative signed clique.

For proving the upper bound we will use a discharging method to show that every triangle-free planar signed graphs admit a homomorphism to the signified Paley graph (SP_{25}).

First let us define the partial order $\prec$. Let $n_3(G)$ be the number of vertices in G with degree at most 3. For any two signed graphs $[G_1]$ and $[G_2]$, we let $[G_1] \prec [G_2]$ if and only if at least one of the following conditions hold:

(i) $\mid V(G_1) \mid < \mid V(G_2) \mid$ and $n_3(G_1) \leq n_3(G_2)$.

(ii) $n_3(G_1) < n_3(G_2)$.

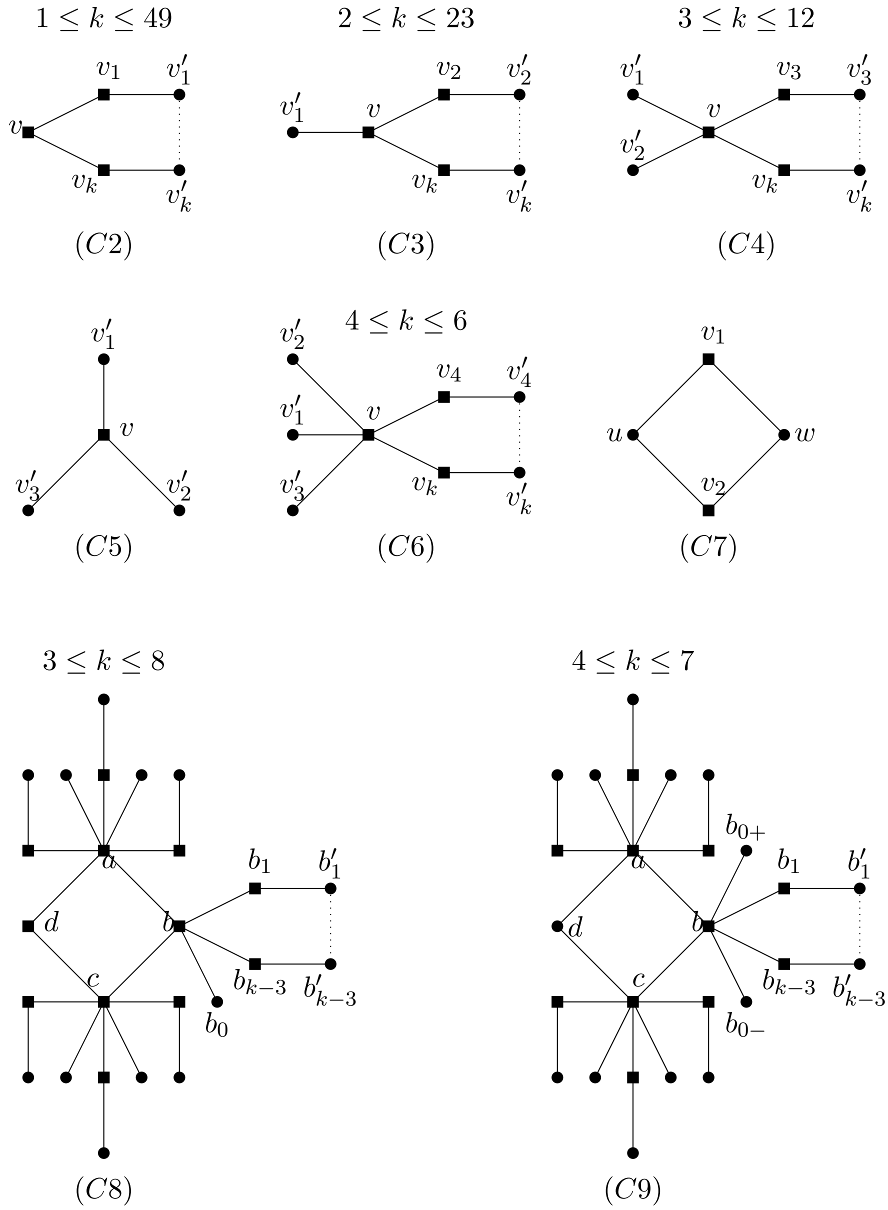


Figure 6.5: Forbidden configurations for Theorem 6.23(b).

Note that the partial order $\prec$ is well-defined and is an extension of the induced subgraph poset.

Let $[H]$ be a minimal (with repect to $\prec$) triangle-free, planar signed graph (that is, the underlying undirected graph H is triangle-free planar) that does not admit a homomorphism to the signified Paley graph (SP_{25}). We first prove that the underlying undirected graph H of $[H]$ does not contain the set of configurations listed in Lemma 6.25, 6.26 and 6.27 (depicted in Fig. 6.5).

Then, using a discharging procedure, we will show that each signed triangle-free planar graph contains at least one of these configurations, contradicting the fact that H is a triangle-free planar graph.

For convenience, we will refer to a vertex of degree exactly (respectively, at most, at least) k by a *k-vertex (respectively, $^{\geq}k$-vertex, $^{\leq}k$-vertex).* Also, we will refer to a neighbor of degree exactly (respectively, at most, at least) k of a vertex v by a *k-neighbor (respectively, $^{\geq}k$-neighbor, $^{\leq}k$-neighbor)* of v.

Assume a fixed planar embedding of H is given. A *weak* 7*-vertex* u in H is a 7-vertex adjacent to four 2-vertices $v_1, ..., v_4$ and three $^{\geq}3$-vertices w_1, w_2, w_3 in such a way that the sequence of neighbors of v appear as $v_1, w_1, v_2, w_2, v_3, w_3, v_4$ (clockwise or counterclockwise).

The drawing conventions for a configuration (Ck) contained in a signed graph $[H]$ are the following. If u and v are two vertices of (Ck), then they are adjacent in the underlying graph H if and only if they are adjacent in (Ck). Moreover, the neighbors of a vertex drawn as a square (called a square vertex) in H are exactly its neighbors in (Ck), whereas a vertex drawn as a circle (called a circular vertex) may have neighbors outside of Ck. Two or more circular vertices in (Ck) may coincide in a single vertex in H, provided they do not share a square vertex as common neighbor. Finally, an edge will represent an edge in the underlying graph H of the discussed signed graph $[H]$, hence can represent either a positive or a negative edge. Configurations $(C2)$-$(C9)$ are depicted in Fig. 6.5.

Proving that configurations $(C1)$-$(C7)$ are not contained in H as subgraphs is relatively easier and is proved in the following lemma. For proving that configurations $(C8)$ and $(C9)$ are not contained in H as subgraphs, we state and prove two different lemmas as the proofs are somewhat more difficult.

Lemma 6.25. *The graph H does not contain the following configurations (depicted in Fig. 6.5):*

$(C1)$ *a $^{\leq}1$-vertex;*

$(C2)$ *a k-vertex adjacent to k 2-vertices for $1 \leq k \leq 49$;*

$(C3)$ *a k-vertex adjacent to $(k-1)$ 2-vertices for $2 \leq k \leq 23$;*

$(C4)$ *a k-vertex adjacent to $(k-2)$ 2-vertices for $3 \leq k \leq 12$;*

$(C5)$ *a 3-vertex;*

$(C6)$ *a k-vertex adjacent to $(k-3)$ 2-vertices for $4 \leq k \leq 6$;*

$(C7)$ *two vertices u and v linked by two distinct 2-paths, both paths having a 2-vertex as internal vertex.*

Proof. For each configuration, we suppose that $[H]$ contains (that is, the same that H contains the configuration) it and we consider a triangle-free reduction $[H']$ such that $[H'] \prec [H]$. Therefore, by minimality of $[H]$, $[H']$ admits a homomorphism f to (SP_{25}). We will then show that we can choose f so that it can be extended to a homomorphism of $[H]$ to (SP_{25}) contradicting the fact that $[H]$ is a counterexample. Also, for the remainder of the proof, if $[H]$ contains a configuration, then $[H^*]$ will denote the graph obtained from $[H]$ by removing all the square vertices from the configuration. Also we will assume that (H^*) is the presentation of $[H^*]$ for which f is a homomorphism of (H^*) to (SP_{25}) and (H^*) can be obtained by deleting the square vertices from the presentation (H) of $[H]$.

Proof of configuration $(C1)$***:*** Obvious since every vertex of (SP_{25}) has degree at least 1.

Proof of configuration $(C2)$***:*** Suppose $[H]$ contains the configuration depicted in Fig. 6.5 $(C2)$ and f is a homomorphism of $[H^*]$ to (SP_{25}).

Note that the edges vv_i and $v_iv'_i$ can have either the same signs or different signs in (H) for each $i \in \{1, 2, ..., 49\}$. Now, if we resign the vertex v then the edges vv_i and $v_iv'_i$ having same signs will change to have different signs and vice versa.

Now we resign v in (H), if needed, to obtain the presentation $(H_1) \in [H]$ such that vv_i and $v_iv'_i$ have different signs for at most $\lfloor 49/2 \rfloor = 24$ instances where $i \in \{1, 2, ..., 49\}$.

Let $F = \{f(v'_i) \mid vv_i \text{ and } v_iv'_i \text{ have different signs in } (H_1)\}$. As $| F | \leq 24$, we can extend f to a homomorphism of $[H]$ to (SP_{25}) (with (H_1) being the suitable presentation) using lemma 6.10, by selecting $f(v)$ from the set $V(SP_{25}) \setminus F$ and choosing a feasible value for $f(v_i)$ using property $P_{2,5}$ of (SP_{25}) for $i \in \{1, 2, ..., 49\}$.

Proof of configuration $(C3)$***:*** Suppose $[H]$ contains the configuration depicted in Fig. 6.5 $(C3)$ and f is a homomorphism of $[H^*]$ to (SP_{25}).

Let $F = \{f(v_1), f(v'_1), f(v'_2), ..., f(v'_k)\}$. As $| F | \leq 24$, it is possible to find $u \in V(SP_{25}) \setminus F$.

Resign v in (H) to obtain a presentation $(H_1) \in [H]$ such that the edge v_1v in (H_1) has the same sign as the edge $uf(v_1)$ of (SP_{25}). Then we can fix $f(v) = u$ and choose a feasible value for

$f(v_i)$ using property $P_{2,5}$ of (SP_{25}) for $i \in \{1, 2, ..., 23\}$ such that f extends to a homomorphism of (H_1) to (SP_{25}).

Proof of configuration $(C4)$**:** Suppose $[H]$ contains the configuration depicted in Fig. 6.5 $(C4)$ and f is a homomorphism of $[H']$ to (SP_{25}) where $[H']$ is the signed graph obtained from $[H]$ by deleting the vertices $\{v_3, ..., v_k\}$. Without loss of generality we may assume (H') to be the presentation that admits a homomorphism f to (SP_{25}) and (H') can be obtained by deleting the vertices $\{v_3, ..., v_k\}$ from the presentation (H) of $[H]$.

Let $F = \{f(v'_3), f(v'_4), ..., f(v'_k)\}$ and v be an α-neighbor of (v_1, v_2) in (H). Then by lemma 6.9(b) we have,

$$|N^{\tilde{\alpha}}(f(v_1), f(v_2))| \geq 11.$$

Now, as $\mid F \mid \leq 10$, it is possible to find $u \in N^{\tilde{\alpha}}(f(v_1), f(v_2)) \setminus F$. Replace the value of $f(v)$ with u keeping f a homomorphism (resign v, if needed, to find a suitable presentation) and choose a feasible value for $f(v_i)$ using property $P_{2,5}$ of (SP_{25}) for $i \in \{3, 4, ..., 12\}$.

Proof of configuration $(C5)$**:** Suppose that $[H]$ contains the configuration depicted in Fig. 6.5 $(C5)$ and (H) be any presentation of it.

Let (H') be the signified graph obtained from (H) by deleting the vertex v and adding, for every $1 \leq i < j \leq 3$, a new vertex v_{ij} and the edges v_iv_{ij} and $v_{ij}v_j$ having the same sign as the sign of the edges v_iv and v_jv respectively in (H) denoted by α_i and α_j respectively.

As configuration $(C1)$-$(C4)$ are forbidden in H by the above, we have $d_H(v_i) \geq 3$ for $i \in \{1, 2, 3\}$. Hence, $n_3(H) < n_3(H')$. Therefore, $[H'] \prec [H]$. Clearly, H' is triangle-free. Hence, there is a homomorphism f of $[H']$ to (SP_{25}). Without loss of generality we may assume (H') to be the presentation that admits a homomorphism f to (SP_{25}).

Now by lemma 6.25 we can find an $\tilde{\alpha}$-neighbor u of $(f(v_1), f(v_2), f(v_3))$ in (SP_{25}) with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Now fix $f(v) = u$. It is easy to note that f restricted to $V(H)$ is a homomorphism of $[H]$ (resign v of (H), if needed, to get a suitable presentation) to (SP_{25}).

Proof of configuration $(C6)$**:** Let H' be the graph obtained by deleting the vertices $v_4, ..., v_k$ from H. Suppose $[H]$ contains the configuration depicted in Fig. 6.5 $(C6)$ and f is a homomorphism of $[H']$ to (SP_{25}).

Without loss of generality we may assume (H') to be the presentation that admits a homomorphism f to (SP_{25}) and (H') can be obtained by deleting the vertices $\{v_4, ..., v_k\}$ from the presentation (H) of $[H]$.

Let $F = \{f(v'_4), ..., f(v'_k)\}$ and v be an α-neighbor of (v_1, v_2, v_3) in (H). Then by lemma 6.9(b) we have,

$$|N^{\tilde{\alpha}}(f(v_1), f(v_2), f(v_3))| \geq 4.$$

Now, as $\mid F \mid \leq 3$, it is possible to find $u \in N^{\tilde{\alpha}}(f(v_1), f(v_2), f(v_3)) \setminus F$. Replace the value of $f(v)$ with u keeping f a homomorphism of $[H]$ (resign v of (H), if needed, to get a suitable presentation) to (SP_{25}) and choose a feasible value for $f(v_i)$ using property $P_{2,5}$ of (SP_{25}) for $i \in \{4, 5, ..., 7\}$.

Proof of configuration $(C7)$**:** Suppose $[H]$ contains the configuration depicted in Fig. 6.5 $(C7)$.

If u and w have no common neighbor other than v_1 and v_2 then consider $[H']$ to be the graph obtained from $[H^*]$ by adding the edge uw. Clearly $[H'] \prec [H]$ as we have $\mid V(H') \mid = (\mid V(H) \mid -2)$ and $n_3(H') \leq n_3(H)$. Also H' is triangle-free as u and v do not have any common neighbor in H'. Therefore, $[H']$ admits a homomorphism f to (SP_{25}) from which we can easily find a homomorphism of $[H]$ to (SP_{25}) using property $\tilde{P}_{2,11}$ of (SP_{25}).

Now suppose u and w have at least one common neighbor v_3 other than v_1 and v_2. Note that, irrespective of the presentation of $[H]$, there is $i \neq j \neq 3$ such that both v_i and v_j are $\tilde{\alpha}$-neighbors of $\{u, w\}$ for some 2-type-vector (depending on the presentation) α.

Now consider $[H']$ to be the graph obtained from $[H]$ by deleting the vertex v_j. Clearly $[H'] \prec [H]$ as H' is a subgraph of H. Therefore, $[H']$ admits a homomorphism f to (SP_{25}).

Without loss of generality we may assume that (H') is a presentation that admits a homomorphism f to (SP_{25}) and that (H') can be obtained by deleting the vertex v_j from the presentation (H) of $[H]$.

Assume that v_i is a β-neighbor of $\{u, w\}$ in (H). Now, we resign v_j in (H) to obtain the presentation (H_1) of $[H]$ such that v_j is also a β-neighbor of $\{u, w\}$ like v_i.

Now, fix $f(v_j) = f(v_i)$ and thereby extend f to a homomorphism of (H_1) to (SP_{25}). □

In the following we will prove that configuration $(C8)$ is not contained in H as subgraph.

Lemma 6.26. *The graph H does not contain the following configuration (depicted in Fig. 6.5):*

$(C8)$ *a 4-face $abcd$ such that d is 2-vertex, a and c are weak 7-vertices, and b is a k-vertex adjacent to $(k-3)$ 2-vertices for $3 \leq k \leq 8$.*

Proof. Suppose $[H]$ contains the configuration depicted in Fig. 6.5$(C8)$.

Consider $[H']$ to be the graph obtained from $[H]$ by deleting the vertex d. Clearly $[H'] \prec [H]$ as H' is a subgraph of H. Therefore, $[H']$ admits a homomorphism f to (SP_{25}). Without loss of generality we may assume that (H') is a presentation that admits a homomorphism f to (SP_{25}) and that (H') can be obtained by deleting the vertex d from the presentation (H) of $[H]$. Also assume that (H^*) is the presentation of $[H^*]$ obtained by deleting the square vertices of (H) and that f^* is the homomorphism of (H^*) to (SP_{25}) obtained by restricting f.

We want to show that f^* can be extended to a homomorphism of $[H]$ (by resigning the square vertices of (H) we can obtain a suitable presentation) to (SP_{25}). We will assume the contrary and prove by contradiction.

Let a_1 and a_2 be the two $^{\geq}3$-neighbors, other than b, of the vertex a. Now obtain the presentation $(H_1) \in [H]$ by resigning the vertices a, b and c of (H), if needed, in such a way that the edges a_1a, ab and bc are all positive in (H_1).

Note that we still get (H^*) if we remove the square vertices from (H_1). Now let the three 2-neighbors of a (other than d) be a_3, a_4 and a_5. Also, let the neighbor of a_i, other than a, be a'_i for $i \in \{3, 4, 5\}$. Assume that a is an α_a-neighbor of (a_1, a_2) in (H_1) where $\alpha_a = (\alpha_{a_1}, \alpha_{a_2})$ (note that $\alpha_{a_1} = +$).

Now define the following sets in (SP_{25}):

$$S_{a+} = N^{\alpha_a}(f^*(a_1), f^*(a_2)) \setminus \{f(a'_3), f(a'_4), f(a'_5)\}.$$
$$S_{a-} = N^{\bar{\alpha}_a}(f^*(a_1), f^*(a_2)) \setminus \{f(a'_3), f(a'_4), f(a'_5)\}.$$
$$S_a = S_{a+} \cup S_{a-}.$$

Similarly, we can define $c_1, c_2, c_3, c_4, c_5, c'_3, c'_4, c'_5, \alpha_c, \alpha_{c_1}, \alpha_{c_2}, S_{c+}, S_{c-}$ and S_c.

Claim 1: We have $f^*(a_1) \neq f^*(a_2)$ and $f^*(c_1) \neq f^*(c_2)$.

Proof of Claim 1: First we will show that $f^*(a_1) \neq f^*(a_2)$.

Assume that $f^*(a_1) = f^*(a_2)$. Then $f(a_1) = f(a_2)$ in (H') as well. As f is a homomorphism, a_1a and a_2a have the same sign in (H') as the sign of the edge $f(a_1)f(a)$ in (SP_{25}).

Now let a be a β-neighbor of (a_1, b) in (H') for some 2-type-vector β. Now, as (SP_{25}) has property $P_{2,5}$ by Lemma 6.9(b)), we have

$$|N^{\beta}(f(a_1), f(b))| \geq 5.$$

Hence, we can choose some $u \in N^{\beta}(f(a_1), f(b)) \setminus \{f(a'_3), f(a'_4), f(a'_5), f(c)\}$.

Now fix $\phi(a) = u$ and $\phi(x) = f(x)$ for all $x \in V(H') \setminus \{a, a_3, a_4, a_5, d\}$. Note that ϕ can be extended to a homomorphism of $[H]$ (with (H) being the suitable presentation) to (SP_{25}) using property $P_{2,5}$ of (SP_{25}). But this is a contradiction. Hence, $f^*(a_1) \neq f^*(a_2)$.

Similarly, we can show that $f^*(c_1) \neq f^*(c_2)$. ◊

Let b_0 be the $^{\geq}3$-neighbor, other than a and c, of the vertex b. Now let $b_1, b_2, ..., b_{k-3}$ be the $(k-3)$ 2-neighbors of b and let the neighbor of b_i, other than b, be b'_i for $i \in \{1, 2, ..., k-3\}$. Assume that bb_0 is a an edge with the sign α_b.

Define the following sets in (SP_{25}):

$$\begin{aligned} S_{b+} &= N^{\alpha_b}(f^*(b_0)) \setminus \{f(b'_1), f(b'_2), ..., f(b'_{k-3})\}. \\ S_{b-} &= N^{\bar{\alpha_b}}(f^*(b_0)) \setminus \{f(b'_1), f(b'_2), ..., f(b'_{k-3})\}. \\ S_b &= S_{b+} \cup S_{b-}. \end{aligned}$$

As $k \leq 8$, we have $|S_b| \geq 19$ (by Lemma 6.9(a)). Also, using Lemma 6.9(b), we have $|S_a|, |S_c| \geq 8$.

Claim 2: For every $u \in S_b \setminus \{f^*(a_1), f^*(a_2)\}$, there exists a $w \in S_a$ such that the homomorphism f^* can be extended with $f^*(a) = w$ and $f^*(b) = u$.

Proof of Claim 2: Assume the contrary. If $u \in S_{b+}$, then every vertex $w_1 \in S_{a+} \setminus \{u\}$ is a negative neighbor of u (otherwise f^* can be extended with $f(a) = w_1$ and $f(b) = u$) and every vertex of $w_2 \in S_{a-} \setminus \{u\}$ is a positive neighbor of u (otherwise f^* can be extended with $f(a) = w_2$ and $f(b) = u$).

Basically then each vertex of $S_a \setminus \{u\}$ will be a $\tilde{\gamma}$-neighbor of $(f^*(a_1), f^*(a_2),\ u)$ in (SP_{25}) for $\gamma = (\alpha_1, \alpha_2, -)$. By Lemma 6.12 we know that,

$$|N^{\tilde{\gamma}}(f^*(a_1), f^*(a_2), u)| \leq 6 \text{ in } (SP_{25}).$$

This is a contradiction as we have,

$$\begin{gathered} S_a \setminus \{u\} \subseteq N^{\tilde{\gamma}}(f^*(a_1), f^*(a_2), u) \text{ and} \\ |S_a \setminus \{u\}| \geq 7. \end{gathered}$$

Hence the claim is true for $u \in S_{b+}$.

Similarly, we can prove the claim for $u \in S_{b-}$. $\Diamond$

Similar to the above claim we can also prove that for every $u \in S_b \setminus \{f^*(c_1), f^*(c_2)\}$, there exists $z \in S_c$ such that the homomorphism f^* can be extended with $f^*(c) = z$ and $f^*(b) = u$.

Hence, combining the two statements we get the following claim.

Claim 3: For every $u \in S_b \setminus \{f^*(a_1), f^*(a_2),\ f^*(c_1), f^*(c_2)\}$, there exists a $w \in S_a$ and a $z \in S_c$ such that the homomorphism f^* can be extended with $f^*(b) = u$, $f^*(a) = w$ and $f^*(c) = z$.

Notice that, if in this extention $w \neq v$, then we can extend the homomorphism to obtain a homomorphism of $[H]$ using property $P_{2,5}$ of (SP_{25}).

Therefore, as $w = z$, we also have the uniqueness of w and z. Hence, for every $u \in S_b \setminus \{f^*(a_1), f^*(a_2), f^*(c_1), f^*(c_2)\}$ there exists exactly one $w \in S_a$ and exactly one $z \in S_c$ such that $w = z$ and the homomorphism f^* can be extended with $f^*(b) = u$ and $f^*(a) = f^*(c) = w$ $(= z)$.

Claim 4: If $u \in S_b \setminus \{f^*(a_1), f^*(a_2), f^*(c_1), f^*(c_2)\}$, then $u \in S_a \cap S_c$.

Proof of Claim 3: Assume that $u \in S_b \setminus \{f^*(a_1), f^*(a_2), f^*(c_1), f^*(c_2)\}$ and $u \notin S_a$. We know by Claim 2 that there is exactly one vertex $w \in S_a$ such that the homomorphism f^* can be extended with $f^*(b) = u$ and $f^*(a) = w$. That means if $u \in S_{b+}$, then each vertex of $S_a \setminus \{w\}$ will be a $\tilde{\gamma}$-neighbor of $(f^*(a_1), f^*(a_2), u)$ in (SP_{25}) for $\gamma = (\alpha_1, \alpha_2, -)$.

By Lemma 6.12 we know that

$$|N^{\tilde{\gamma}}(f^*(a_1), f^*(a_2), u)| \leq 6 \text{ in } (SP_{25}).$$

This is a contradiction as

$$S_a \setminus \{w\} \subseteq N^{\tilde{\gamma}}(f^*(a_1), f^*(a_2), u) \text{ and}$$
$$|S_a \setminus \{w\}| \geq 7.$$

We arrive to a similar contradiction if $u \in S_{b-}$. $\Diamond$

Hence, by Claim 4 we have shown that $S_b \setminus \{f^*(a_1), f^*(a_2), f^*(c_1) \subseteq S_a$. But we have

$$|S_b \setminus \{f^*(a_1), f^*(a_2), f^*(c_1)| \geq 19 \text{ and } |S_a| \leq 12.$$

This is clearly a contradiction. Hence, H does not contain the configuration ($C8$). $\square$

In the following we will prove that configuration ($C9$) is not contained in H as subgraph.

Lemma 6.27. *The graph H does not contain the following configuration (depicted in Fig. 6.5):*

($C9$) *a 4-face $abcd$ such that d is 2-vertex, a and c are weak 7-vertices, and b is a k-vertex adjacent to $(k-4)$ 2-vertices for $4 \leq k \leq 7$.*

Proof. Suppose $[H]$ contains the configuration depicted in Fig. 6.5.

Consider $[H']$ to be the graph obtained from $[H]$ by deleting the vertex d. Clearly $[H'] \prec [H]$ as H' is a subgraph of H. Therefore, $[H']$ admits a homomorphism f to (SP_{25}).

Without loss of generality we may assume that (H') is a presentation that admits a homomorphism f to (SP_{25}) and that (H') can be obtained by deleting the vertex d from the presentation (H) of $[H]$. Also assume that (H^*) is the presentation of $[H^*]$ obtained by deleting the square vertices of (H) and that f^* is the homomorphism of (H^*) to (SP_{25}) obtained by restricting f.

We want to show that f^* can be extended to a homomorphism of $[H]$ (by resigning the square vertices of (H) we can obtain a suitable presentation) to (SP_{25}). We will assume the contrary and prove by contradiction.

We define (H_1), $a_1, a_2, a_3, a_4, a_5, a'_3, a'_4, a'_5, \alpha_a, \alpha_{a_1}, \alpha_{a_2}, S_{a+}, S_{a-}, S_a, c_1, c_2, c_3, c_4, c_5, c'_3, c'_4, c'_5,$ $\alpha_c, \alpha_{c_1}, \alpha_{c_2}, S_{c+}, S_{c-}$ and S_c similarly as in the proof of Lemma 6.26.

Let b_{0+} and b_{0-} be the two $^{\geq}3$-neighbors, other than a and c, of the vertex b. Now let the $(k-4)$ 2-neighbors of b be $b_1, b_2, ..., b_{k-4}$ and let the neighbor of b_i, other than b, be b'_i for $i \in \{1, 2, ..., k-4\}$. Assume that b is an α_b-neighbor of $(f^*(b_{0+}), f^*(b_{0-}))$ for the 2-type-vector $\alpha_b = (\alpha_{b+}, \alpha_{b-})$.

Define the following sets in (SP_{25}):

$$S_{b+} = N^{\alpha_b}(f^*(b_{0+}), f^*(b_{0-})) \setminus \{f(b'_1), f(b'_2), ..., f(b'_{k-4})\}.$$
$$S_{b-} = N^{\bar{\alpha_b}}(f^*(b_{0+}), f^*(b_{0-})) \setminus \{f(b'_1), f(b'_2), ..., f(b'_{k-4})\}.$$
$$S_b = S_{b+} \cup S_{b-}.$$

Claim 1: We have $f^*(b_{0+}) \neq f^*(b_{0-})$.

Proof of Claim 1: Let $f^*(b_{0+}) = f^*(b_{0-})$. Then $f(b_{0+}) = f(b_{0-})$ in (H') as well. As, f is a homomorphism, $b_{0+}b$ and $b_{0-}b$ has the same sign in (H') as the sign of the edge $f(b_{0+})f(b)$ in (SP_{25}).

Moreover, even if we resign the square vertices of (H'), the two edges $b_{0+}b$ and $b_{0-}b$ still have the same sign (that is, if the sign changes, it changes for both the edges simultaniously). Hence, this case can be reduced to configuration ($C8$) as $b_{0+}b$ and $b_{0-}b$ have identical signs even if we resign the square vertices of (H'). We have shown in the proof of Lemma 6.26 that f^* can be extended to a homomorphism of $[H]$ to (P_{25}) and by resigning the square vertices of (H) we can obtain a suitable presentation. Hence, $f^*(b_{0+}) \neq f^*(b_{0-})$. $\Diamond$

As $k \leq 3$ we have, by Lemma 6.9(b), $|S_a|, |S_b|, |S_c| \geq 8$.

Claim 2: We have $f^*(a_1) \neq f^*(a_2)$ and $f^*(c_1) \neq f^*(c_2)$.

Proof of Claim 2: This claim can be proved in a way similar to the proof of Claim 1 of Lemma 6.26. $\Diamond$

Define the subset $C_a \subseteq S_b$ to be the set of vertices $u \in S_b$ such that there exists $w \in S_a$ and the homomorphism f^* can be extended with $f^*(a) = w$ and $f^*(b) = u$. Similarly define C_c and let $C = C_a \cap C_c$.

Claim 3: We have $S_b \setminus \{f^*(a_1), f^*(a_2)\} \subseteq C_a$, $S_b \setminus \{f^*(c_1), f^*(c_2)\} \subseteq C_c$ and $S_b \setminus \{f^*(a_1), f^*(a_2), f^*(c_1), f^*(c_2)\} \subseteq C$.

Proof of Claim 3: To prove $S_b \setminus \{f^*(a_1), f^*(a_2)\} \subseteq C_a$ we can just imitate the proof of Claim 2 of Lemma 6.26. For the rest, we can imitate the discussion following the proof of Claim 2 in Lemma 6.26. $\Diamond$

Claim 4: For every $u \in C$, there exist exactly one $w \in S_a$ and exactly one $z \in S_c$ such that $w = z$ and the homomorphism f^* can be extended with $f^*(b) = u$ and $f^*(a) = f^*(c) = w$ $(= z)$.

Proof of Claim 4: For every $u \in C$, there exists $w \in S_a$ and $z \in S_c$ such that the homomorphism f^* can be extended with $f^*(b) = u$ and $f^*(a) = w$ and $f^*(c) = z$ is true by Claim 3. Notice that if in this extention $w = f^*(a) \neq f^*(c) = z$, then we can extend the homomorphism to obtain a homomorphism of $[H]$ using property $P_{2,5}$ of (SP_{25}). Therefore, as $w = z$, we also have the uniqueness of w and z. $\Diamond$

Claim 5: We have $f^*(a_1), f^*(a_2), f^*(c_1), f^*(c_2) \notin C$.

Proof of Claim 5: Assume $f^*(a_1) \in C$. Hence by Claim 4, there exists exactly one $w \in S_a$ and exactly one $z \in S_c$ such that $w = z$ and the homomorphism f^* can be extended with $f^*(b) = f^*(a_1)$ and $f^*(a) = f^*(c) = w$. If $f^*(a_1) \in S_{b+}$, then each vertex of the set $S_a \setminus \{w\}$ will be a $\tilde{\gamma}$-neighbor of $(f^*(a_1), f^*(a_2), f^*(a_1))$ in (SP_{25}) for $\gamma = (+, \alpha_2, -)$ (as $\alpha_1 = +$) which is a contradiction. If $f^*(a_1) \in S_{b+}$, then w will be a $(+)$-neighbor (resp., $(-)$-neighbor) of $f^*(a_1)$ for $w \in S_{a+}$ (resp., $w \in S_{a-}$) as the homomorphism f^* can be extended with $f^*(b) = f^*(a_1)$ and $f^*(a) = w$. Hence, w is a $\tilde{\gamma}$-neighbor of $(f^*(a_1), f^*(a_2), f^*(a_1))$ in (SP_{25}) for $\gamma = (+, \alpha_2, -)$ (as $\alpha_1 = +$) which is again a contradiction. Hence, $f^*(a_1) \notin C$. Similarly, we can show that $f^*(a_2), f^*(c_1), f^*(c_2) \notin C$. $\Diamond$

Claim 6: We have $C \subseteq S_a \cap S_c$.

Proof of Claim 6: Assume that $u \in C$ and $u \notin S_a$. We know by Claim 4 that there is exactly one vertex $w \in S_a$ such that the homomorphism f^* can be extended with $f^*(b) = u$ and $f^*(a) = w$. That means if $u \in S_{b+}$, then each vertex of $S_a \setminus \{w\}$ will be a $\tilde{\gamma}$-neighbor of $(f^*(a_1), f^*(a_2), u)$ in (SP_{25}) for $\gamma = (\alpha_1, \alpha_2, -)$. By Lemma 6.12 we know that

$$|N^{\tilde{\gamma}}(f^*(a_1), f^*(a_2), u)| \leq 6 \text{ in } (SP_{25}).$$

This is a contradiction as $S_a \setminus \{w\} \subseteq N^{\tilde{\gamma}}(f^*(a_1), f^*(a_2), u)$ and $|S_a \setminus \{w\}| \geq 7$. We arrive to a similar contradiction if $u \in S_{b-}$. Hence the claim is proved. $\Diamond$

Claim 7: We have $|S_a|, |S_c| = 8$.

Proof of Claim 7: Assume that $|S_a| \geq 9$. Then we have, for some $u \in C$, exactly one $w \in S_a$ such that we can extend the homomorphism f^* with $f^*(b) = u$ and $f^*(a) = w$. Hence, $S_a \setminus \{w\} \subseteq |N^{\tilde{\gamma}}(f^*(a_1), f^*(a_2), u)|$ for some 3-type-vector γ. This is a contradiction as $|N^{\tilde{\gamma}}(f^*(a_1), f^*(a_2), u)| \leq 6$ by Lemma 6.12 and $|S_a \setminus \{w\}| \geq 7$. So, $|S_a| \leq 8$. But we know already that $|S_a| \geq 8$. Hence, $|S_a| = 8$. Similarly, we can show that $|S_c| = 8$. $\Diamond$

Claim 8: We have $\alpha_a = (+,+)$ and $\alpha_c \in \{(+,+), (-,-)\}$.

Proof of Claim 8: Now we have, $|S_a| = 8$. Hence, $|N^{\alpha_a}(f^*(a_1), f^*(a_2))| \leq 11$. So, by Lemma 6.12, we have $\alpha_a \in \{(+,+), (-,-)\}$. Similarly we have $\alpha_c \in \{(+,+), (-,-)\}$. Now, $\alpha_a = (+,+)$ because $\alpha_1 = +$. $\Diamond$

Claim 9: We have $C \subseteq S_{b+}$ and $\alpha_c = (+,+)$.

Proof of Claim 9: Assume $u \in C$ such that $u \notin S_{b+}$. Hence, $y \in S_{b-}$. By Claim 4 there exists exactly one $w \in S_a$ such that the homomorphism f^* can be extended with $f^*(b) = u$ and $f^*(a) = w$. Hence, $S_a \setminus \{u, w\} \subseteq N^{\tilde{\gamma}}(f^*(a_1), f^*(a_2), u)$ for $\gamma = (+,+,+)$. Hence, we arrive to a contradiction as $|N^{\tilde{\gamma}}(f^*(a_1), f^*(a_2), u)| \leq 5$ by Lemma 6.12 and $|S_a \setminus \{u, w\}| \geq 6$. Hence, $C \subseteq S_{b+}$ as $\alpha_a = (+,+)$. With similar logic, $\alpha_c = (-,-)$ will imply $C \subseteq S_{b-}$. Hence, $\alpha_c \neq (-,-)$. Hence, $\alpha_c = (+,+)$. ♢

Claim 10: We have, $S_{b-} = S_b \cap \{f^*(a_1), f^*(a_2), f^*(c_1), f^*(c_2)\}$.

Proof of Claim 10: By Claim 3 and Claim 9 we already know that $S_{b-} \subseteq S_b \cap \{f^*(a_1), f^*(a_2), f^*(c_1), f^*(c_2)\}$. Now assume that $f^*(a_1) \in S_{b+}$. But then for any $w \in S_a$, we can extend the homomorphism f^* with $f^*(b) = f^*(a_1)$ and $f^*(a) = w$ (that the extension is a homomorphism can be checked using the above claims). Hence, $f^*(a_1) \in C_a$. If $f^*(a_1) \in \{f^*(c_1), f^*(c_2)\}$, then similarly $f^*(a_1) \in C_c$. If $f^*(a_1) \notin \{f^*(c_1), f^*(c_2)\}$, then by Claim 3, $f^*(a_1) \in C_c$. Hence, $f^*(a_1) \in C_a \cap C_c = C$. But this contradicts Claim 5. So, $f^*(a_1) \notin S_{b+}$. Therefore, if $f^*(a_1) \in S_b$, then $f^*(a_1) \in S_{b-}$. Similar statement is true for $f^*(a_2), f^*(c_1)$ and $f^*(c_2)$. So we have $S_b \cap \{f^*(a_1), f^*(a_2), f^*(c_1), f^*(c_2)\} \subseteq S_b$. ♢

Claim 11: If $x \in S_a \cap S_c$, then $x \in S_{a+}$ if and only if $x \in S_{c+}$.

Proof of Claim 11: Assume that $x \in S_a \cap S_c$ and $x \in S_{a+}$. But yet let $x \in S_{c-}$. If the vertex x is such that it is possible to extend the homomorphism f^* with $f^*(a) = x$ and $f^*(b) = u$ for some $u \in C$. Then by the definition of C and Claim 4 we can further extend f^* with $f^*(c) = f^*(a) = x$. But this extension is no longer a homomorphism because to have $f^*(c) = f^*(a) = x$, we need to resign the vertex c of (H_1), while keeping the vertex a as it is (as $x \in S_{a+} \cap S_{c-}$), to obtain a suitable presentation needed for the homomorphism (the edges ab and bc will have different signs whereas the edges $f^*(a)f^*(b)$ and $f^*(c)f^*(b)$ have the same sign).

Now assume x is such that it is not possible to extend the homomorphism f^* with either $f^*(a) = x$ and $f^*(b) = u$ or $f^*(c) = x$ and $f^*(b) = u$ for any $u \in C$. Then, as it is not possible to extend the homomorphism f^* with $f^*(a) = x$ and $f^*(b) = u$ for any $u \in C$, we have $C \subseteq N^{(-)}(x)$ (because, $x \in S_{a+}$). On the other hand, as it is not possible to extend the homomorphism f^* with $f^*(c) = x$ and $f^*(b) = u$ for any $u \in C$, we have $C \subseteq N^{(+)}(x)$. This is a contradiction. Hence, $x \in S_a \cap S_c$ and $x \in S_{a+}$ implies $x \in S_{c+}$. Similarly, we can prove that $x \in S_a \cap S_c$ and $x \in S_{c+}$ implies $x \in S_{a+}$. ♢

Claim 12: We have $\{f^*(a_1), f^*(a_2)\} = \{f^*(c_1), f^*(c_2)\}$.

Proof of Claim 12: Note that $C \subseteq N^{\tilde{\gamma}}(X)$ where $X = \{f^*(a_1), f^*(a_2), f^*(c_1), f^*(c_2)\}$ and $\gamma = (+, ..., +)$ is a $|X|$-type-vector by Claim 8, Claim 9 and Claim 11. If $|X| \geq 3$, then $|C| \leq 5$ by Lemma 6.12. If $|C| \leq 5$, then $|S_b \setminus C| \geq 3$ as $|S_b| \geq 8$. Now by Claim 3 we have $S_b \setminus C \subseteq X \cap S_{b-}$. Hence $C \cup \{b_{0+}, b_{0-}\} \subseteq N^{\tilde{\gamma}'}(S_b \setminus C)$ where $\gamma = (+, ..., +)$ is a $|S_b \setminus C|$-type-vector. As $|C \cup \{b_{0+}, b_{0-}\}| \geq 6$ (because $|C| \geq 4$ by Claim 3 and as $|S_b| \geq 8$), this is a contradiction to Lemma 6.12. Hence, $|X| \leq 2$. Then by Claim 1 and Claim 2 we have $\{f^*(a_1), f^*(a_2)\} = \{f^*(c_1), f^*(c_2)\}$. ♢

We have $S_{b-} = \{f^*(a_1), f^*(a_2)\}$ by Claim 10 as by Lemma 6.9(b) we already know $|S_{b-}| \geq 2$. By the same lemma we know that $|S_{b+}| \leq 6$. But as $|S_b| \geq 8$, we have $|S_{b+}| = 6$. Hence $\alpha_b \in \{(+,+), (-,-)\}$. Also we know already that $S_{b+} = C$. Hence $|C| = 6$.

Let $w \in C$. Then there exists $u \in S_b$ such that we can extend the homomorphism f^* with $f^*(b) = u$ and $f^*(a) = w$, as otherwise $S_b \setminus \{u\} \subseteq N^{\tilde{\gamma}}(f^*(b_{0+}), f^*(b_{0-}), w)$ which leads us to a contradiction by Lemma 6.12.

Hence, by Claim 4, there is a bijection λ from C to C such that we can extend the homomorphism f^* with $f^*(b) = \lambda(u)$ and $f^*(a) = u$ for each $u \in C$.

Now assume $w_1, w_2 \in C$ such that $w_i \in S_{a_i}$ for $i \in \{1, 2\}$. Then $S_b \setminus \{w_i, \lambda(w_i)\} \subseteq N^{\tilde{\gamma}_i}(f^*(b_{0+}), f^*(b_{0-}), w_i)$ for $\gamma_1 = (\alpha_{b+}, \alpha_{b-}, -)$ and $\gamma_2 = (\alpha_{b+}, \alpha_{b-}, +)$. Now we know by Lemma 6.12 that either $|N^{\tilde{\gamma}_1}(f^*(b_{0+}), f^*(b_{0-}), w_1)| \leq 5$ or $|N^{\tilde{\gamma}_2}(f^*(b_{0+}), f^*(b_{0-}), w_2)| \leq 5$. This is a contradiction as $|S_b \setminus \{w_i, \lambda(w_i)\}| \geq 6$. Hence, by Claim 11, either $C \subseteq S_{a+} \cap S_{c+}$ or $C \subseteq S_{a-} \cap S_{c-}$.

If $\alpha_b = (+,+)$, then $C \subseteq S_{a+} \cap S_{c+}$. Then $C \subseteq N^{\tilde{\gamma}'}(f^*(b_{0+}), f^*(b_{0-}), f^*(a_1))$ for $\gamma' = (+,+,+)$. Also, if $\alpha_b = (-,-)$, then $C \subseteq S_{a-} \cap S_{c-}$. Then $C \subseteq N^{\tilde{\gamma}'}(f^*(b_{0+}), f^*(b_{0-}), f^*(a_1))$ for $\gamma' = (+,+,+)$. This is a contradiction by Lemma 6.12 and the fact that $|C| \geq 6$. Therefore, the homomorphism f^* can be extended to a homomorphism of $[H]$ to (SP_{25}). $\square$

To complete the proof of Theorem 6.23(b), we use a discharging procedure. We define the weight function ω by

$$\omega(x) = d(x) - 4 \text{ for every } x \in V(H) \cup F(H).$$

Since H is a plane graph, we have by Euler's formula (see Chapter 2, Theorem 2.1) ($|V(H)| - |A(H)| + |F(H)| = 2$):

$$\sum_{x \in V(H)} \omega(x) + \sum_{x \in F(H)} \omega(x) = \sum_{x \in V(H)} (d(x) - 4) + \sum_{x \in F(H)} (d(x) - 4) = -8 < 0$$

In what follows, we will define discharging rules $(R1)$, $(R2)$, and $(R3)$ and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is not modified by the discharging rules. Nevertheless, we can show that $\omega^*(x) \geq 0$ for every $x \in V(H) \cup F(H)$. This leads to the following obvious contradiction:

$$0 > \sum_{x \in V(H)} \omega(x) + \sum_{x \in F(H)} \omega(x) \leq \sum_{x \in V(H)} \omega^*(x) + \sum_{x \in F(H)} \omega^*(x) \geq 0$$

The discharging rules are defined as follows:

$(R1)$ Each $^{\geq}4$-vertex gives 1 to each of its 2-neighbors.

$(R2)$ Each $^{\geq}5$-face $f = ...axb$ such that a and b are 2-vertices gives 1 (resp. 1/2) to x if x is a weak 7-vertex (resp. is not a weak 7-vertex).

$(R3)$ Each $^{\geq}5$-face $f = ...awxyb$ such that a, b, x are 2-vertices and w, y are weak 7-vertices either receives 1/2 from the vertex z if $wxyz$ is a 4-face, or receives 1 from the $^{\geq}5$-face $f' = ...cwxyd$ if c, d are $^{\geq}4$-vertices.

First we will calculate the value of the new weight function for every vertex of the graph and prove the following.

Lemma 6.28. *For all vertices v, $\omega^*(v) \geq 0$.*

Proof. In the following, $d_{\geq 4}(v)$ denotes the number of neighbors of v with degree at least 4. Similarly, $d_2(v)$ denotes the number of neighbors of v with degree exactly 2.

Then it is clear that, for every vertex v of $[H]$, we have $d(v) = d_{\geq 4}(v) + d_2(v)$ since $[H]$ contains neither vertices of degree at most 1 by $(C1)$, nor 3-vertices by $(C5)$.

Let v be a k-vertex of $[H]$. Therefore, $k = d_{\geq 4}(v) + d_2(v)$. Recall that the initial charge of v is $\omega(v) = k - 4$. If $k = 2$, then v receives 2×1 by $(R1)$; hence, $\omega^*(v) = \omega(v) + 2 = 0$.

In the remainder of this section assume $k \geq 4$.

- if $d_{\geq 4} = 0$, then $d_2(v) = k \geq 50$ by $(C2)$. By $(R1)$, v gives $k \times 1$. By $(C7)$, v is incident to k $^{\geq}5$-faces, and therefore v receives $k \times 1/2$ by $(R2)$. Hence, $\omega^*(v) = \omega(v) - k + k/2 \geq 21$.
- if $d_{\geq 4} = 1$, then $d_2(v) = k - 1 \geq 23$ by $(C3)$. By $(R1)$, v gives $(k-1) \times 1$. By (C7), v is incident with $(k-2)$ $^{\geq}5$-faces, and therefore v receives $(k-2) \times 1/2$ by $(R2)$. Moreover, v is adjacent to at most one weak 7-vertex and therefore $(R3)$ does not apply. Hence, $\omega^*(v) = \omega(v) - (k-1) + (k-2)/2 \geq 8$.

- if $d_{\geq 4} = 2$, then $d_2(v) = k - 2 \geq 11$ by ($C3$). By ($R1$), v gives $(k-2) \times 1$. By (C7), v is incident with $(k-3)$ $^{\geq}$5-faces, and therefore v receives at least $(k-3) \times \frac{1}{2}$ by ($R2$). Moreover, v is adjacent to at most two consecutive weak 7-vertices, therefore gives away at most 1/2 by ($R3$). Hence, $\omega^*(v) = \omega(v) - (k-2) + (k-4)/2 - \frac{1}{2} \geq 2$.

- if $d_{\geq 4} = 3$, then $d_2(v) = k - 3 \geq 4$ by ($C5$) and ($C7$). So, $k \geq 7$. In each subcase, by ($R1$), v gives $(k-3) \times 1$.

  ▷ Suppose that the three $^{\geq}$4-neighbors of v are consecutive. By ($C7$), v is incident with $(k-4)$ $^{\geq}$5-faces, each of which gives $\frac{1}{2}$ to v by ($R2$). Moreover, v is adjacent to at most 3 consecutive weak 7-vertices, therefore gives away at most $2 \times \frac{1}{2}$ by ($R3$). If $k \leq 8$, then $d_2(v) \leq 5$, therefore by ($C8$) the discharging rule ($R3$) does not apply to v. Hence, for $k \leq 8$, $\omega^*(v) = \omega(v) - (k-3) + (k-4)/2 \geq \frac{1}{2}$. If $k \geq 9$, then $\omega^*(v) = \omega(v) - (k-3) + (k-4)/2 - 2 \times \frac{1}{2} \geq \frac{1}{2}$.

  ▷ Suppose that two $^{\geq}$4-neighbors of v are consecutive. By ($C7$), v is incident with $(k-5)$ $^{\geq}$5-faces, each of which gives $\frac{1}{2}$ to v by ($R2$). Moreover, v is adjacent to at most 2 consecutive weak 7-vertices, therefore gives away at most $\frac{1}{2}$ by ($R3$). If $k \leq 8$, then $d_2(v) \leq 5$, therefore by ($C8$) the discharging rule ($R3$) does not apply to v. Hence, for $k \leq 8$, $\omega^*(v) = \omega(v) - (k-3) + (k-5)/2 \geq 0$. If $k \geq 9$, then $\omega^*(v) = \omega(v) - (k-3) + (k-5)/2 - \frac{1}{2} \geq \frac{1}{2}$.

  ▷ Suppose that none of the $^{\geq}$4-neighbors of v are consecutive. Hence ($R3$) does not apply. By ($C7$), v is incident with $(k-6)$ $^{\geq}$5-faces, each of which gives $\frac{1}{2}$ to v if $k \geq 8$ or gives 1 to v if $k = 7$ (that is, if v is a weak 7-vertex) by ($R2$). Hence, for $k = 7$, $\omega^*(v) = \omega(v) - (k-3) + (k-6) = 0$ and for $k \geq 8$, $\omega^*(v) = \omega(v) - (k-3) + (k-6)/2 \geq 0$.

- if $d_{\geq 4} = 4$, then $d_2(v) = k - 44$. By ($R1$), v gives $(k-4) \times 1$. Suppose ($R3$) does not apply. Then $\omega^*(v) \geq \omega(v) - (k-4) = 0$. Now suppose that ($R3$) applies. Note that it applies at most twice (otherwise there will be a weak 7-vertex with three consecutive 2-neighbors). Moreover, by ($C9$), we have $d_2(v) \geq 4$, that implies $k \geq 8$.

  ▷ Suppose first that ($R3$) applies only once; then v gives $\frac{1}{2}$ to the corresponding $^{\geq}$5-face. Moreover, by ($R2$), v receives at least $\frac{k-7}{2}$. Hence, $\omega^*(v) = \omega(v) - (k-4) + (k-7)/2 - \frac{1}{2} \geq 0$.

  ▷ Suppose now that ($R3$) applies twice; then v gives $2 \times \frac{1}{2}$ to the corresponding $^{\geq}$5-faces. Moreover, by ($R2$), v receives at least $\frac{k-6}{2}$. Hence, $\omega^*(v) = \omega(v) - (k-4) + (k-6)/2 - 2 \times \frac{1}{2} \geq 0$.

- Suppose finally that $d_{\geq 4}(v) \geq 5$. By ($C1$), v gives $(k - d_{\geq 4}(v)) \times 1$. Moreover, by ($R3$), v gives at most $\lfloor \frac{d_{\geq 4}(v)}{2} \rfloor \times \frac{1}{2}$. Hence, $\omega^*(v) \geq \omega(v) - (k - d_{\geq 4}(v)) - (\lfloor \frac{d_{\geq 4}(v)}{2}) \times \frac{1}{2} \geq 0$.

Thus, for every $v \in V(H)$, we have $\omega^*(v) \geq 0$. □

Now we will calculate the value of the new weight function for every face of the graph and prove the following.

Lemma 6.29. *For all faces f, $\omega^*(f) \geq 0$.*

Proof. Let f be a k-face of H. Since H is triangle-free, we have $k \geq 4$. Recall that the initial charge of f is $\omega(f) = k - 4$.

- If $k = 4$, then no discharging rule applies. Hence, $\omega^*(f) = \omega(f) = 0$.

- If $k = 5$, then f is incident with at most two 2-vertices by ($C3$).

  ▷ If f has no incident 2-vertices, then $\omega^*(v) \geq \omega(f) = 1$.

- ▷ If f is incident with one 2-vertex, then only $(R3)$ may apply at most once and thus $\omega^*(v) \geq \omega(f) - 1 = 0$.
- ▷ If f is adjacent to two 2-vertices x and z, then f gives at most 1 to the common neighbor of x and z by $(R2)$. $(R3)$ does not apply in this case. Hence $\omega^*(v) \geq \omega(f) - 1 = 0$.

– If $k = 6$, then f is incident with at most three 2-vertices by $(C3)$.

- ▷ If f has no incident 2-vertices, then $\omega^*(v) \geq \omega(f) = 2$.
- ▷ If f is incident with one 2-vertex, then only $(R3)$ may apply at most once and thus $\omega^*(v) \geq \omega(f) - 1 = 1$.
- ▷ If f is adjacent to two 2-vertices x and z, and if x and z have a common neighbor, then f gives at most 1 to that common neighbor by $(R2)$. $(R3)$ does not apply in this case. Hence, $\omega^*(v) \geq \omega(f) - 1 = 1$. If x and z has no common neighbor, then $(R2)$ does not apply. But $(R3)$ may apply at most twice. Hence $\omega^*(f) \geq \omega(f) - 2 \times 1 = 0$.
- ▷ Finally, suppose that f is adjacent to three 2-vertices.
  - ◇ If f is incident with at most one weak 7-vertex, then f gives at most $(1 \times 1) + (2 \times \frac{1}{2}) = 2$ by $(R2)$. Hence, $\omega^*(f) \geq \omega(f) - 2 \times 1 = 0$.
  - ◇ If f is incident with two weak 7-vertices, then f gives $(2 \times 1) + (1 \times \frac{1}{2}) = \frac{5}{2}$ by $(R2)$. Moreover, f receives at least $\frac{1}{2}$ by $(R3)$. Hence, $\omega^*(f) \geq \omega(f) - \frac{5}{2} + \frac{1}{2} = 0$.
  - ◇ If f is incident with three weak 7-vertices, then f gives 3×1 by $(R2)$. Moreover, f receives at least $3 \times \frac{1}{2}$ by $(R3)$. Hence, $\omega^*(f) \geq \omega(f) - 3 \times 1 + 3 \times \frac{1}{2} = \frac{1}{2}$.

– Suppose finally that $k \geq 7$, and assume that $(R2)$ applies n times and $(R3)$ applies m times. It is clear that f gives weights by $(R2)$ to at most $\lfloor \frac{k}{2} \rfloor$ vertices; hence $n \leq \lfloor \frac{k}{2} \rfloor$. Moreover, we can easily check that $2n+3m \leq k$. With these constraints, we have $n+m = \frac{n+2n+3m}{3} \leq \frac{\lfloor \frac{k}{2} \rfloor + k}{3}$, which implies that $n + m \leq k - 4$ when $k \geq 7$. Hence, $\omega^*(v) \geq \omega(f) - n - m \geq 0$.

Thus, for every $f \in F(H)$, we have $\omega^*(f) \geq 0$. □

This completes the proof. □

Proof of Theorem 6.24

The signified planar graph (B_0) from Fig 6.6 is a presentation of the signed planar graph $[B_0]$ with signed chromatic number 10 (one can check [40] for the proof). To prove that the signed graph $[B_0]$ has signed chromatic number at least 10, Naserasr, Rollová and Sopena [40] showed that for any signed coloring we need to use at least 4 colors for each of the sets $\{x_1, x_2, ..., x_5\}$ and $\{y_1, y_2, ..., y_5\}$ of vertices of the graph while u and v receives two distinct colors different from the colors received by every other vertices.

Note that the graph induced by the vertices $u, v, x_1, x_2, x_3, y_1, y_2, y_3$ is an [s]-clique on 8 vertices. Now assume that (H) is a signified graph of order 10 to which every signed planar graph admits a homomorphism.

Now construct the signified planar graph (B_1) by gluing a copy of the signified planar graph (B_0) (see Fig. 6.6) to each vertex of (B_0) by identifying the vertex with the vertex v of (B_0). After that construct another signified planar graph (B_2) by gluing a copy of (B_0) to each positive edge of (B_1) by identifying the positive edge with the edge uv of (B_0).

Now note that $[B_2]$ admits a homomorphism to (H). Now, each vertex of the inital graph $[B_0]$ has nine neighbors which will have distinct images under the homomorphism. So, in particular, we must have the degree of each vertex in (H) at least 9. Hence, the underlying graph H is a complete graph.

Now note that, as $[B_2]$ admits a homomorphism to (H), for each edge $ab \in E(H)$ (regardless of the sign of the edge ab, as a and b must be the images of two adjacent vertices that has a (B_0) glued to them by identifying the edge with uv) there are at least four vertices, which got mapped to four different vertices under the homomorphism f, that a and b agree on and there are at least four vertices, which got mapped to four different vertices under the homomorphism f, that a and b disagree on with each other. No matter how you resign the vertices, this will be true. But as (H) has order 10, for each edge $ab \in (H)$, the signified graph (H) must have exactly four vertices on which a and b agree and exactly four vertices on which a and b disagree.

Now, without loss of generality, assume that the vertices of (H) are $z_0, z_1, ..., z_9$. Recall that, by Lemma 6.3, any presentation of $[H]$ will bound all the planar signed graphs. Assume, without loss of generality (H) is the presentation in which all the vertices are adjacent to z_9 with a positive edge (this presentation which can be obtained by resigning the negative neighbors of z_9).

Now every other vertex agrees on exactly four vertices with z_9 and disagrees on exactly four vertices with z_9. Hence, the signified induced by the neighbors of z_9 has positve degree 4 for every vertex. Assume that for z_0, without loss of generality, $N^+(z_0) = \{z_1, z_2, z_3, z_6\}$ and $N^-(z_0) = \{z_4, z_5, z_7, z_8\}$.

Now, if z_1 has no positive neighbor from $\{z_2, z_3, z_6\}$, then it must have exactly three positive neighbors from $\{z_4, z_5, z_7, z_8\}$ as it must have exactly four positive neighbors from $N(z_9)$. But then z_0 and z_1 will disagree on at least six vertices, three from $\{z_2, z_3, z_6\}$ and three from $\{z_4, z_5, z_7, z_8\}$. So, it is not possible for z_1 to have no positive neighbor from $\{z_2, z_3, z_6\}$.

Now assume z_1 has at least two positive neighbors from $\{z_2, z_3, z_6\}$. Then z_1 will have at most one positive neighbor from $\{z_4, z_5, z_7, z_8\}$. Hence, z_0 and z_1 will agree on at least five vertices, two from $\{z_2, z_3, z_6\}$ and three from $\{z_4, z_5, z_7, z_8\}$.

Hence, z_1 can have exactly one positive neighbor from $\{z_2, z_3, z_6\}$. Similarly arguing about z_2, z_3, z_6, we can show that the signified graph induced by $\{z_1, z_2, z_3, z_6\}$ has exactly two disjoint positive edges, say z_1z_2 and z_3z_6 and all the other edges negative.

With a similar argument, we can conclude that the signified graph induced by $\{z_4, z_5, z_7, z_8\}$ has exactly two disjoint negative edges, say z_4z_8 and z_5z_7 and all the other edges positive.

Now, z_1 has z_0 and z_2 as its positive neighbors. It must have exactly two positive neighbors from $\{z_4, z_5, z_7, z_8\}$. Without loss of generality assume that z_4 and z_7 are positive neighbors of z_1.

Now note that z_1 and z_2 agrees on four vertices z_9, z_0, z_3, z_6. Hence, they must disagree on $\{z_4, z_5, z_7, z_8\}$. So, z_4 and z_7 are negative neighbors of z_2 while z_5 and z_8 are positive neighbors of z_2.

Now, z_3 also must have exactly two positive neighbors among $\{z_4, z_5, z_7, z_8\}$. But it should also agree on exactly two vertices from $\{z_4, z_5, z_7, z_8\}$ with z_1 (as it agrees on exactly two vertices z_0, z_9 with z_1 outside $\{z_4, z_5, z_7, z_8\}$). This forces z_3 to have exactly one positive neighbor from $\{z_4, z_7\}$ and exactly one negative neighbor among $\{z_5, z_8\}$. Without loss of generality assume that z_4, z_5 are the positive neighbors and $_7, z_8$ are the negative neighbors.

Also note that z_6 must disagree with z_3 on the vertices $\{z_4, z_5, z_7, z_8\}$. Hence, we constructed the whole graph.

It can be shown that this graph is isomorphic to the signified Paley plus graph (SP_9^+) on 10 vertices. That ends the proof.

We are not explicitly showing the isomorphism. But one can easily observe that the graph induced by the positive edges between the vertices $\{z_0, z_1, ..., z_8\}$ (that is, all vertices except z_9) is the undirected graph $C_3 \times C_3$ obtained by the product of two 3-cycles. The signified graph obtained from replacing the non-edges of $C_3 \times C_3$ is basically the signified Paley graph SP_9 (this is not difficult to notice). □

6.3.2 Signed relative clique number

The signed relative clique number was defined by Naserasr, Rollová and Sopena [40]. They studied signed chromatic numbers and signed absolute clique numbers of planar graphs but did not study their signed relative clique numbers. Here we study that for some planar families.

Before listing those bounds, we will present a general bound for the signed relative clique number of graphs with maximum degree Δ.

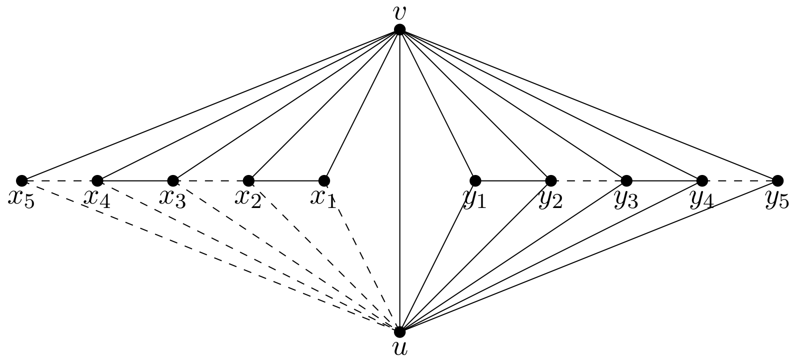


Figure 6.6: A presentation (B_0) of a signed planar graph with signed chromatic number 10.

Proposition 6.30. *Every signed graph with maximum degree* Δ *has signed relative clique number at most* $\dfrac{\Delta(\Delta+1)}{2}+1$.

Proof. Let $[G]$ be a signed graph with maximum degree Δ. Let R be a relative clique of maximum order in of $[G]$. Let $v \in R$ be a vertex. Now, v has Δ adjacent vertices and each of these vertices can have at most $(\Delta-1)$ adjacent vertices excluding v. But, if a vertex u at distance 2 from v is in R, then it has to be part of an unbalanced 4-cycle of which also v is part of. For that u needs to be adjacent to at least two neighbors of v. There are at most $\Delta.(\Delta-1)$ edges between the neighbors of v and the vertices at distance 2 from v. Now, there are $(|R|-\Delta-1)$ vertices of R that are each adjacent to at least two neighbors of v. Hence we have,

$$\begin{aligned} 2(|R|-\Delta-1) \leq \Delta.(\Delta-1) &\Rightarrow 2|R|-2\Delta-2 \leq \Delta^2-\Delta \\ &\Rightarrow 2|R| \leq \Delta^2+\Delta+2 \\ &\Rightarrow |R| \leq \frac{\Delta.(\Delta+1)}{2}+1 \end{aligned}$$

Hence, we are done. □

We consider the problem of determining the signed relative clique number for the families of outerplanar graphs and of outerplanar graphs with given girth. We list the related results below.

Theorem 6.31.

(a) $\omega_{[rs]}(\mathcal{O}_k) = 4$ *for* $k = 3, 4$.

(b) $\omega_{[rs]}(\mathcal{O}_k) = 2$ *for* $k \geq 5$.

The proof of the above theorem is similar to the proof of Theorem 5.16 (see Chapter 5, Section 5.3). We omit the proof (see Appendix).

Now we consider the problem of determining the signed relative clique number for the families of planar graphs and of planar graphs with given girth.

Theorem 6.32.

(a) $8 \leq \omega_{[rs]}(\mathcal{P}_3) \leq 40$.

(b) $6 \leq \omega_{[rs]}(\mathcal{P}_4) \leq 17$.

(c) $\omega_{[rs]}(\mathcal{P}_k) = 2$ *for* $k \geq 5$.

The proof of the above theorem is similar to the proof of Theorem 4.20 (see Chapter 4, Section 4.2). We omit the proof (see Appendix).

6.3.3 Signed absolute clique number

The signed absolute clique number was defined by Naserasr, Rollová and Sopena [40]. They studied this parameter for the family of planar graphs and provided tight bounds for it.

We list the signed absolute clique number for the families of outerplanar graphs and of outerplanar graphs with given girth.

Theorem 6.33.

(a) $\omega_{[as]}(\mathcal{O}_k) = 4$ *for* $k = 3, 4$.

(b) $\omega_{[as]}(\mathcal{O}_k) = 2$ *for* $k \geq 5$.

The proof of the above result directly follows from Theorem 6.31.

We now list the signed absolute clique number for the families of planar graphs with given girth.

Theorem 6.34.

(a) $\omega_{[as]}(\mathcal{P}_3) = 8$. *[40]*

(b) $\omega_{[as]}(\mathcal{P}_4) = 4$.

(c) $\omega_{[as]}(\mathcal{P}_k) = 2$ *for* $k \geq 5$.

The proof of part (a) was given by Naserasr, Rollová and Sopena [40].

The parts (b) and (c) of the above theorem easily follow from the list of triangle-free planar graphs with diameter 2 given by Plesnik in Theorem 2.14.

6.4 Bounding planar consistent graphs

A *consistent* signed graph is a signed graph in which every balanced cycle is of even length and all unbalanced cycles are of the same parity. Thus there are the following two types of consistent signed graphs:

(i) *Odd signed graphs:* An odd signed graph $[G, \Sigma]$ is a signed graph with all unbalanced cycles having odd length while all cycles of even length are balanced. It can be shown (using Theorem 6.2) that in this case we have $(G, E(G)) \in [G, \Sigma]$ as a presentation (that is, a signified graph with all edges negative). It is easy to notice that any signed graph with a presentation with all edges negative is an odd signed graph. Therefore, a signed graph is an odd signed graph if and only if it has a presentation with all edges negative.

(ii) *Even signed graphs (or bipartite signed graphs):* An even signed graph or a bipartite signed graph $[G, \Sigma]$ is a signed graph with all cycles (both balanced and unbalanced) having even length. In this case the underlying graph G is bipartite. Moreover, every signed graph with its underlying graph a bipartite graph is an even signed graph. Therefore, a signed graph is an even signed graph if and only if its underlying graph is bipartite.

It is easy to prove that if $(G, \Sigma) \to (H, \Lambda)$, then the unbalanced-girth of (G, Σ) is at least as much as the unbalanced-girth of (H, Λ).

Recall the definition of the projective cube $Proj_n$ of dimention n (see Chapter 2). The signified projective cube $(SProj_n, \Sigma)$ of dimention n is the signified graph with underlying graph $Proj_n$ and the set of negative edges $\Sigma = \{uv | u_i \neq v_i \text{ for all } i \in \{1, 2, ..., n\}$ where $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)\}$. We refer to [39] for some properties of signed projective cubes and for a proof of the following result.

Theorem 6.35. *The signed projective cube of dimension* k *is a consistent signed graph and has unbalanced-girth* $k + 1$.

Therefore, the signed projective cube $[SProj_{2k-1}]$ is an even signed graph with unbalanced girth $2k$ and the signed projective cube $[SProj_{2k}]$ is an odd signed graph with unbalanced girth $2k+1$. It follows that if a signed graph admits a homomorphism to an odd (respectively, even) signed projective cube, it must be an odd (respectively, even) signed graph.

Recall Theorem 2.12 from Chapter 2 and the discussion that followed. There we mentioned a conjecture by Naserasr (see Chapter 2, Conjecture 2.13) which claims that every planar graph with odd-girth $2k+1$ admits a homomorphism to the projective cube $Proj_{2k}$ of dimension $2k$.

Recall the conjecture by Seymour from Chapter 2 (see Conjecture 2.9). Naserasr showed that his conjecture for each k is equivalent to Conjecture 2.9 by Seymour for $d = 2k+1$. But this way, Naserasr's conjecture only captures Seymour's conjecture for odd values of d.

Recently Naserasr, Rollová and Sopena generalized Naserasr's conjecture using the notion of consistent signed graphs and proved that it captures Seymour's conjecture (Conjecture 2.9) for every value of d in [39].

Conjecture 6.36. *Given $k \geq 2$, every planar consistent signed graph of unbalanced-girth $k+1$ admits a homomorphism to $[SProj_k]$.*

In [39] Naserasr, Rollová and Sopena showed that this conjecture, for each $k \geq 2$, is equivalent to Conjecture 2.9 by Seymour for $d = k+1$.

The conjecture is formed of two parts. For even values of d it is to say that every planar odd signed graph with unbalanced-girth $k+1$ admits a homomorphism to $[SProj_k]$. This is equivalent to saying that every planar graph with odd-girth $k+1$ admits a homomorphism to $Proj_k$, which is basically Conjecture 2.13 by Naserasr.

Since $Proj_2$ is isomorphic to K_4, the very first case of this conjecture (that is, for $k = 2$) is the Four-Color Theorem. For $k = 4$, it is proved in [38] that $Proj_4$, known as the *Clebsch graph*, is the optimal bound (that is, a bound of minimum order).

For odd values of k it says that every planar even signed graph with unbalanced-girth $k+1$ admits a homomorphism to $[SProj_k]$. That is saying that every planar bipartite signed graph with unbalanced-girth $k+1$ admits a homomorphism to $[SProj_k]$.

In [40], Conjecture 6.36 has been proved for $k = 3$. The result is stronger than the Four-Color Theorem and the proof was done using the Four-Color Theorem.

For the family of planar odd signed graphs with unbalanced girth $k+1$ (where k is odd), Theorem 2.12 confirms the existence of a universal bound (this is an easy observation as this case corresponds to Conjecture 2.13 by Naserasr), no such analogus result for odd values of k is known. While Conjecture 6.36 proposes $[SProj_k]$ as a candidate for a universal bound of the family of planar consistent signed graphs (that is, for both odd and even signed graphs) with unbalanced girth $k+1$, we will prove in this section that if the conjecture is true then $[SProj_k]$ must be a minimum bound with respect to, both, number of vertices and number of edges.

Theorem 6.37. *Let $[B_k]$ be a universal bound for the family of planar consistent signed graphs with unbalanced-girth $k+1$, for all $k \geq 2$. Then $|V(B_k)| \geq 2^k$.*

Note that, as $[SProj_k]$ is a k-regular graph on 2^k vertices, if Conjecture 6.36 is true, then it must be a minimum bound with respect to, both, number of vertices and number of edges.

To prove Theorem 6.37, we consider two cases, namely, when k is even and when k is odd. We prove two results (Theorem 6.38 and Theorem 6.41) in the following which, together, will directly imply Theorem 6.37.

Odd signed graphs

First we consider the case of odd signed graphs, that is, the case when k is even. We know that this case corresponds to Conjecture 2.13 by Naserasr.

Given an undirected graph G and an integer k we define the k-th *walk-power* of G, denoted $G^{(k)}$ to be the graph whose vertex set $V(G^{(k)}) = V(G)$ with two vertcies x and y being adjacent if there is a *walk* of length k connecting x and y in G. This graph is loopless only if k is odd and G has odd-girth at least $k+2$. If ϕ is a homomorphism of G to H, then it can easily be checked that ϕ is also a homomorphism of $G^{(k)}$ to $H^{(k)}$. Thus to prove our result we will prove the following.

Theorem 6.38. *There is a planar graph G of odd-girth $2k+1$ with $\omega(G^{(2k-1)}) = 2^{2k}$.*

To prove the theorem we will construct an example of such a graph. This construction is based on the following local construction.

Lemma 6.39. *Let G be a graph obtained by subdividing the edges of K_4 such that in a planar embedding of G each of the four faces of G is a cycle of length $2k+1$. Then $G^{(2k-1)}$ is isomorphic to K_{4k}.*

Proof. Let a, b, c and d be the original vertices of the K_4 from which G is constructed. Let t_{uv} be the length of the path joining u and v in G for $u, v \in \{a, b, c, d\}$.

Then we have

$$\begin{aligned} t_{ab} + t_{bc} + t_{ca} &= t_{ab} + t_{bd} + t_{ad} \\ &= t_{ac} + t_{cd} + t_{ad} \\ &= t_{bc} + t_{cd} + t_{bd} \\ &= 2k+1 \end{aligned} \tag{6.2}$$

From equation 6.2 we have

$$t_{uv} = t_{wx} \text{ for } \{u, v, w, x\} = \{a, b, c, d\}. \tag{6.3}$$

Let u and v be a pair of vertices of G. If they are both vertices of a particular facial cycle of G, then there is a walk of length $2k-1$ connecting them since the facial cycle is of length $2k+1$.

If there is no facial cycle of G containing both u and v, then we may assume, without loss of generality, that u is a vertex of the path obtained by subdividing ab and v is a vertex of the path obtained by subdividing cd.

Suppose that u has partitioned the path obtained by subdividing ab to paths of length t'_{ab} (length of the path connecting a and u) and t''_{ab} (length of the path connecting u and b) and that v has partitioned the path obtained by subdividing cd into paths of length t'_{cd} (length of the path connecting c and v) and t''_{cd} (length of the path connecting v and d).

Note that by equation 6.3 we have

$$\begin{aligned} t'_{ab} + t''_{ab} &= t'_{cd} + t''_{cd} \\ &= t_{ab} = t_{cd}. \end{aligned} \tag{6.4}$$

If $t_{ab} = t_{cd}$ is even, then t'_{ab} and t''_{ab} have the same parity and t'_{cd} and t''_{cd} have the same parity. Moreover, this will imply that t_{ac} and t_{ad} have different parities.

On the other hand, if $t_{ab} = t_{cd}$ is odd, then t'_{ab} and t''_{ab} have different parities and t'_{cd} and t''_{cd} have different parities. Moreover, this will imply that t_{ac} and t_{ad} have the same parity.

Now the path connecting u, v that contains the vertices a, c is of length $t'_{ab} + t_{ac} + t'_{cd}$ and the path connecting u, v that contains the vertices b, d is of length $t''_{ab} + t_{bd} + t''_{cd}$. Clearly, $t'_{ab} + t_{ac} + t'_{cd}$ and $t''_{ab} + t_{bd} + t''_{cd}$ have the same parity irrespective of the parity of $t_{ab} = t_{cd}$. Now note that

$$\begin{aligned} t'_{ab} + t_{ac} + t'_{cd} + t''_{ab} + t_{bd} + t''_{cd} &= t_{ab} + t_{ac} + t_{cd} + t_{bd} \text{ (by equation A.5)} \\ &= 2(t_{ab} + t_{ac}) \text{ (by equation 6.3)} \\ &= 2(t_{ab} + t_{ac} + t_{bc}) - 2t_{bc} \\ &= 4k + 2 - 2t_{bc} \text{ (by equation 6.2)} \\ &\leq 4k \text{ (as } t_{bc} \geq 1) \end{aligned}$$

Hence we have $min\{(t'_{ab} + t_{ac} + t'_{cd}), (t'_{ab} + t_{bd} + t'_{cd})\} \leq 2k$. Similarly, we can show that $min\{(t'_{ab} + t_{ad} + t''_{cd}), (t''_{ab} + t_{bc} + t'_{cd})\} \leq 2k$.

But note that $min\{(t'_{ab}+t_{ac}+t'_{cd}),(t''_{ab}+t_{bd}+t''_{cd})\}$ and $min\{(t'_{ab}+t_{ad}+t''_{cd}),(t''_{ab}+t_{bc}+t'_{cd})\}$ have different parities irrespective of the parity of $t_{ab}=t_{cd}$. Therefore, there is a walk of length $2k-1$ from u to v. □

Proof of Theorem 6.38

Consider a K_4 on four vertices a,b,c and d. Let G_1 be a subdivision of this K_4 where edges ab and cd each are subdivided into $2k-1$ edges. Thus G_1 is a subdivision of K_4 in which all the four faces are cycles of length $2k+1$. Hence by Lemma 6.39 we have

$$\omega(G_1^{(2k-1)}) = |V(G_1)| = 4k.$$

In the following we build a sequence of graphs G_i, for $i \in \{1,2,\cdots,2k-2\}$, such that each G_{i+1} contains G_i as a subgraph, G_{i+1} is planar and of odd-girth $2k+1$ and such that $\omega(G_{i+1}^{(2k-1)}) > \omega(G_i^{(2k-1)})$. At the final step we will have,

$$\omega(G_{2k-2}^{(2k-1)}) \geq 2^{2k}.$$

We start with the following partial construction. Suppose G_i is built and let $P = uv_1v_2\cdots v_rw$ be a maximal thread, that is, a path P connecting u and w such that all v_j's are of degree 2 in G_i but u and w are of degree at least 3, of G_i for $j \in \{1,2,...,r\}$. Furthermore, assume that P is either part of a path of length $2k-1$ connecting a and b or part of a path of length $2k-1$ connecting c and d.

Since P is a thread, if we add the new edge uw in G_i, the resulting graph will still be planar. So we add such an edge and subdivide it r times to obtain the new thread $P' = uv'_1v'_2\cdots v'_rw$. Consider a planar drawing of the graph in which P and P' form a facial cycle of length $2r$. In the face PP' connect v_1 and v'_r by a new edge. Subdivide this new edge $2k-r-1$ times (that is, into $2k-r$ edges), so that each of the facial cycles containing the new thread is of length $2k+1$.

Let G'_i be the resulting graph. We first note that G'_i is also of odd-girth $2k+1$. Now suppose a maximal clique W of $G_i^{(2k-1)}$ contains v_j of the thread P. Then we claim that $W \cup v'_j$ is also a clique of $G_i'^{(2k-1)}$.

To prove this let x be any vertex of W. If x is not in P, then consider a walk of length $(2k-1)$ from v_j to x. Each time this walk uses a part of P replace it with the corresponding part from P' and this would give a walk of length $2k-1$ connecting a to v'_j.

If $x \in P$, then, without loss of generality, assume that P is part of a path of length $2k-1$ connecting a and b. Consider the subgraph induced by this path together with c, P' and the $v_1...v'_r$ thread we added to build G'_i. This induced subgraph is a subdivision of K_4 in which all the faces are cycles of length $2k+1$. Thus, by Lemma 6.39 there is a walk of length $2k-1$ connecting x and v'_j. In particular if all vertices of P are in W, then $W \cup \{v'_1, v'_2, \cdots, v'_r\}$ is a clique in $G_i'^{(2k-1)}$.

Now we describe our general construction. At first we have G_1 on $4k$ vertices and two maximal threads. By Lemma 6.39 all the vertices of these two threads are parts of the unique clique of order $4k$ in $G_1^{(2k-1)}$. We apply the previously mentioned construction on both threads to build G_2 which will have four maximal threads each of length $2k-2$ (we are only considering maximal threads that are part of a path of length $2k-1$ connecting a,b or c,d). There is a clique of order $4k+2(2k-2)$ in $G_2^{(2k-1)}$, and there are four maximal threads of length $2k-2$, each a part of a path of length $2k-1$ either connecting a and b or c and d.

Continuing this construction, in general, there is a clique W_i of $G_i^{(2k-1)}$ $(2 \leq i \leq 2k-1)$ which is of order $4k + \sum_{j=1}^{i-1} 2^j(2k-j-1)$ and there are 2^i maximal threads of length $2k-i$ which are part of a path of length $2k-1$ connecting a,b or c,d.

Note that G_i at each step is a planar graph of odd-girth $2k+1$. The clique W_{2k-2} of $G_{2k-2}^{(2k-1)}$ has order equal to

$$\begin{aligned}
4k + \sum_{j=1}^{2k-2} 2^j(2k-j-1) &= 4k + (2k-1)\sum_{j=1}^{2k-2} 2^j - 2\sum_{j=1}^{2k-2} j2^{j-1} \\
&= 4k + [(2k-1)(2^{2k-1}-2)] - \\
&\quad 2[(1-2^{2k-1}) - (-1)(2k-1)2^{2k-2}] \\
&= 4k + [k2^{2k} - 4k - 2^{2k-1} + 2] - \\
&\quad [2 - 2^{2k} + k2^{2k} - 2^{2k-1}] \\
&= 2^{2k}.
\end{aligned}$$

This completes the proof. □

Even signed graphs

Recall that even signed graphs are simply signed graphs whose underlying graph is bipartite.

To prove our result we first introduce the following notations. Given a signed graph $[G, \Sigma]$ and an integer r we define $[G, \Sigma]^{(r)}$ to be a graph (not signed) whose vertices are vertices of G where two vertices u and v are adjacent if either they are connected by a path of odd length or for any given presentation $(G, \Sigma) \in [G, \Sigma]$, there are two paths P_1 and P_2 of even length, not necessarily disjoint, connecting u, v such that P_1 and P_2 are both of length at most r and one of them, say P_1, has an odd number of edges in Σ and the other one, P_2, has an even number of edges in Σ.

Note that, while resigning at internal vertcies of P_1 or P_2 does not change the parity of their number of negative edges, resigning at u or v will switch the role of P_1 and P_2. Hence $[G, \Sigma]^{(r)}$ is well defined.

The point of this definition is that if a homomorphism of $[G, \Sigma]$ to $[H, \Lambda]$ identifies u and v, then (H, Σ) must contain either an odd-cycle or an unbalanced even-cycle of length at most r. In other words, if $[H, \Lambda]$ is an even signed graph of unbalanced-girth $k > r$, then no such pair of vertices can be identified. Thus to prove our claim we will build a planar even signed graph $[G, \Sigma]$ of unbalanced-girth $2k$ for which $[G, \Sigma]^{(2k-2)}$ has a clique of order 2^{2k-1}. To this end we start with the following lemma which is the bipartite analogue of Lemma 6.39.

Lemma 6.40. *Let $[G, \Sigma]$ be a planar signed graph whose underlying graph G is a subdivision of K_4 in such a way that each of the four facial cycles of $[G, \Sigma]$ is an unbalanced cycle of length $2k$. Then $[G, \Sigma]^{(2k-2)}$ is isomorphic to $K_{2^{2k-1}}$.*

The proof of this lemma is similar to the proof of Lemma 6.39. We omit this proof (see Appendix).

Theorem 6.41. *There exist a planar signed bipartite graph $[G, \Sigma]$ with unbalanced-girth $2k$ such that $\omega([G, \Sigma]^{(2k-2)}) \geq 2^{2k-1}$.*

This theorem can be proved using Lemma 6.40 similarly as we proved Theorem 6.38 using Lemma 6.39. We omit this proof (see Appendix).

6.5 Categorical aspects

First note that the family of all signified graphs and the family of all signed graphs can be regarded as categories with the morphisms of the category being the homomorphisms of graphs. Notice that isomorphic graphs represent the same object in the category. Let $\mathcal{C}_S$ and $\mathcal{C}_{[S]}$ denote the category of signified graphs and the category of signed graphs respectively.

We want to show that $\mathcal{C}_{[S]}$ is isomorphic to a subcategory of $\mathcal{C}_S$.

Consider the subcategory $\mathcal{C}_{S_s}$ of $\mathcal{C}_S$ with $ob(\mathcal{C}_{S_s})$ being the class of splitable signified graphs and $hom_{\mathcal{C}_{S_s}}((A, \Sigma), (B, \Lambda))$ is the set of splitable signified homomorphisms for any $(A, \Sigma), (B, \Lambda) \in \mathcal{C}_{S_s}$.

Note that the function R, defined in Section 6.1, acts as a functor of $\mathcal{C}_{[S]}$ to $\mathcal{C}_{S_s}$.

In fact, is it not difficult to notice that the functor R gives an isomorphism of $\mathcal{C}_{[S]}$ to $\mathcal{C}_{S_s}$. Therefore, we have the following result.

Proposition 6.42. *The two categories $\mathcal{C}_{[S]}$ and $\mathcal{C}_{S_s}$ are isomorphic categories.*

It was not known if a categorical product existed for signed graphs or not. Whereas, it was known that a categorical product exists for signified graphs and coincides with the cartesian product of it.

Let (G, Σ) and (H, Λ) be two signified graphs. Then their categorical product $(P, \Omega) = (G) \times (H)$ is the signified graph with the set of vertices, the set of negative edges and the set of positive edges given as follows:

$$\begin{aligned} V(P) &= \{(u, v) \in V(G) \times V(H)\}, \\ \Omega &= \{(u, v)(w, x) | (uw, vx) \in \Sigma \times \Lambda\}, \\ \Omega^c &= \{(u, v)(w, x) | (uw, vx) \in \Sigma^c \times \Lambda^c\}. \end{aligned}$$

Hence, we know that given any two splitable signified graphs $(R(A), R(\Sigma))$ and $(R(B), R(\Lambda))$ there exists a categorical product $(R(A), R(\Sigma)) \times (R(B), R(\Lambda))$ of them. Note that the product $(R(A), R(\Sigma)) \times (R(B), R(\Lambda))$ is a signified graph though it is not ensured if it is also a splitable signified graph or not. In the following we will prove that the product indeed is a splitable signified graph.

Lemma 6.43. *The cartesian product of two splitable signified graphs is also a splitable signified graph.*

Proof. Let $(R(A), R(\Sigma))$ and $(R(B), R(\Lambda))$ be two splitable signified graphs with $V(A) = \{a_1, a_2, ..., a_n\}$ and $V(B) = \{b_1, b_2, ..., b_m\}$.

Then the cartesian product $(P, \Omega) = (R(A), R(\Sigma)) \times (R(B), R(\Lambda))$ of $(R(A), R(\Sigma))$ and $(R(B), R(\Lambda))$ is a signified graph with the set of vertices, the set of negative edges and the set of positive edges given as follows:

$$\begin{aligned} V(P) = \{&(a_i, b_j), (a_i, b'_j), (a'_i, b_j), (a'_i, b'_j) | \text{ for} \\ &i = 1, 2, ...n \text{ and } j = 1, 2, ..., m\}, \\ \Omega = \{&(u, v)(w, x) | uw \in R(\Sigma) \\ &\text{and } vx \in R(\Lambda)\}, \\ \Omega^c = \{&(u, v)(w, x) | uw \in R(\Sigma)^c \\ &\text{and } vx \in R(\Lambda)^c\}. \end{aligned}$$

Now let us partition $V(R(A)) \times (R(B))$ into two equal parts $V_1 = \{(a_i, b_j), (a_i, b'_j)$ and $V_2 = \{(a'_i, b'_j), (a'_i, b_j)$. Moreover, define the function $f : V_1 \to V_2$ as

$$f(a_i, b_j) = (a'_i, b'_j) \text{ and } f(a_i, b'_j) = (a'_i, b_j).$$

Note that f is a bijection and satisfies the conditions of Lemma 6.7. Therefore, the product $(R(A), R(\Sigma)) \times (R(B), R(\Lambda))$ is indeed a splitable signified graph. □

Let (A, Σ), (B, Λ) and (C, Ω) be three signified graphs. Let ψ_1 be a homomorphism of (C, Ω) to (A, Σ) and ψ_2 be a homomorphism of (C, Ω) to (B, Λ). Then the following diagram must commute.

For signified graphs, the product $(A) \times (B)$ is the cartesian product of (A) and (B) while the homomorphisms π_1, π_2 and f are defined as

$$\begin{tikzcd}
(A) & (A)\times(B) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & (B) \\
 & (C) \arrow[ul, "f_1"] \arrow[ur, "f_2"'] \arrow[u, dashed, "\exists!\ f"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (R(A))\times(R(B)) \arrow[d, phantom, "||"] & \\
(R(A)) & (R(P)) \arrow[l, "R(\phi_1)"'] \arrow[r, "R(\phi_2)"] & (R(B)) \\
 & (R(C)) \arrow[ul, "R(\psi_1)"] \arrow[ur, "R(\psi_2)"'] \arrow[u, dashed, "\exists!\ R(\varphi)"] & \\
 & {} \arrow[u, thick, "R"'] & \\
 & {[A]\times[B]} \arrow[d, phantom, "||"] & \\
{[A]} & {[P]} \arrow[l, "\phi_1"'] \arrow[r, "\phi_2"] & {[B]} \\
 & {[C]} \arrow[ul, "\psi_1"] \arrow[ur, "\psi_2"'] \arrow[u, dashed, "\exists!\ \varphi"] &
\end{tikzcd}$$

Figure 6.7: The functor R.

$$\pi_1(a,b) = a \text{ and } \pi_2(a,b) = b \text{ for all } (a,b) \in V((A)\times(B)).$$
$$f(u) = (f_1(u), f_2(u)) \text{ for all } u \in V(C). \tag{6.5}$$

Now notice that if (A), (B) and (C) are all splitable signified graphs and the homomorphisms f_1 and f_2 are splitable signified homomorphisms, then the product $(A)\times(B)$ is a splitable signified graph (by Lemma 6.43) and the homomorphisms π_1, π_2 and f are splitable signified homomorphisms (it is easy to check). Hence the categorical product (of countable objects) exists in $\mathcal{C}_{S_s}$.

Hence by Fig. 6.7 we know that the categorical product (of countable objects) also exists in $\mathcal{C}_{[S]}$.

Formula

Now we will provide a formula for obtaining the categorical product of two signed graphs $[A,\Sigma]$ and $[B,\Lambda]$.

Let $(A,\Sigma) \in [A,\Sigma]$ and $(B,\Lambda) \in [B,\Lambda]$ be such that $V(A) = \{a_1, a_2, ..., a_n\}$ and $V(B) = \{b_1, b_2, ..., b_m\}$. Then the categorical product $[P,\Omega] = [A]\times[B]$ of $[A]$ and $[B]$ has the set of vertices, the set of negative edges and the set of positive edges

$$\begin{aligned}
V(P) &= \{(a_i, b_j), (a_i, b'_j) | \text{ for } i = 1, 2, ...n \text{ and } j = 1, 2, ..., m\}, \\
\Omega &= \{(a_i, b_j)(a_k, b_l), (a_i, b'_j)(a_k, b'_l), (a_i, b_j)(a_k, b'_{l'}), \\
&\qquad (a_i, b'_l)(a_k, b_j) | (a_i a_k, b_j b_l, b_j b_{l'}) \in \Sigma \times \Lambda \times \Lambda^c\}, \\
\Omega^c &= \{(a_i, b_j)(a_k, b_l), (a_i, b'_j)(a_k, b'_l), (a_i, b_j)(a_k, b'_{l'}), \\
&\qquad (a_i, b'_l)(a_k, b_j) | (a_i a_k, b_j b_l, b_j b_{l'}) \in \Sigma^c \times \Lambda^c \times \Lambda\}.
\end{aligned}$$

Notice that the formula is independent of the choice of the presentations $(A) \in [A]$ and $(B) \in [B]$. An interesting point to note is that the product has $2nm$ vertices whereas usually one would expect the product to have mn vertices.

6.6 Conclusion

Naserasr, Rollová and Sopena [40] introduced and studied homomorphisms of signed graphs and asked several questions regarding them. They captured several existing theories related to graph colorings using the notion of signed graphs. Apart from extending existing results and conjectures, they also introduced three parameters, namely, the signed chromatic number, the signed relative clique number and the signed absolute clique number, regarding homomorphisms of signed graphs.

Here we considered the problem of determining lower and upper bounds for those three parameters for the families of outerplanar graphs and of planar graphs.

We provided upper bounds of $\chi_{[s]}(\mathcal{P}_g)$ for all $g \geq 3$. It turns out that the target graphs used to prove the best known upper bounds for the signified chromatic number of the family of planar graphs with girth at least g are splitable graphs for $g = 3, 5, 6$. So, in these cases an upper bound directly followed from the results regarding signified colorings. This also means that improving these upper bounds for signed graphs will improve the upper bounds for signified graphs. So, improving these bounds is also difficult.

We improved the existing upper bound from 48 down to 40 for the signed chromatic number of the family of planar graphs. We could not improve the existing lower bound of 10 for the same family. However, we proved that if there exists a signed graph of order 10 which bounds the family of planar graphs, then it must be isomorphic to $[SP_9^+]$. We tried to find examples of planar signed graphs that do not admit a homomorphism to $[SP_9^+]$ in order to improve the lower bound but failed to do so. This naturally makes us wonder about the following question:

Question 6.44. *Is $[SP_9^+]$ a universal bound for the family of planar graphs?*

We proved the upper bound of 25 for $\chi_{[s]}(\mathcal{P}_4)$. The proof based on discharging method is non-trivial and is inspired by a proof of a similar result concerning the oriented chromatic number by Ochem and Pinlou [45]. An important thing to notice here is that Ochem and Pinlou used a computer for reducing two of their configurations needed for discharging. In this thesis, we have reduced those two configurations using some argument using field theory. We guess that something similar can be done for the oriented case as well thus providing a proof without the use of computer.

For the other two parameters, that is, the signed relative clique number and the signed absolute clique number, we mostly provided tight bounds.

In fact, we provided tight bounds for $\omega_{[as]}(\mathcal{O}_g)$ and for $\omega_{[as]}(\mathcal{P}_k)$ for all $g \geq 3$ and $k \geq 4$. A tight bound for $\omega_{[as]}(\mathcal{P}_k)$ was already proved by Naserasr, Rollová and Sopena [40]. We also provided tight bounds for $\omega_{[rs]}(\mathcal{O}_g)$ for all $g \geq 3$ and for $\omega_{[rs]}(\mathcal{P}_k)$ for all $k \geq 5$.

Given a graph G we clearly have $\omega_{[as]}(G) \leq \omega_{as}(G)$ and $\omega_{[rs]}(G) \leq \omega_{rs}(G)$. Is it possible to obtain a better relation between these parameters?

We also proved that any consistent signed graphs with unbalanced girth $k+1$ that bounds the family of planar consistent signed graphs with unbalanced girth $k+1$ must have order at least 2^k. We want to comment that with some more work we might be able to prove that the only consistent signed graph of order 2^k and with unbalanced girth $k+1$ that bounds the family of planar consistent signed graph with unbalanced girth $k+1$ is the signed projective cube $[SProj_k]$ of dimension k.

We also proved the existence of categorical products for signed graphs and provided a formula for it. For proving this we showed that the category of signed graphs is isomorphic to a subcategory of the category of signified graphs. One might try to obtain more categorical relations between the two categories.

Finally, we would like to remark that the theory of signed homomorphisms is a rich theory and has the potential of becoming a popular area of research in near future. Especially, consistent signed graphs are of interest and specific works regarding them might yield major improvements in the domain of graph colorings in general.

Chapter 7
Conclusion

IN this thesis we considered problems related to homomorphisms of four different types of graphs, namely oriented graphs, orientable graphs, signified graphs and signed graphs. For each of these types, the problems of determining the chromatic number, the relative clique number and the absolute clique number was considered.

We tried to determine the value for each of these parameters for different planar families, mainly the families of outerplanar graphs, of outerplanar graphs with given girth, of planar graphs and of planar graphs with given girth, in these four settings.

Given any family $\mathcal{F}$ of graphs one can consider the problem of determining upper and lower bounds for one of these parameters for $\mathcal{F}$. That offers us a rich collection of challenging open problems. However, here we will mention some specific interesting problems which we would like to investigate in the near future.

Question 7.1.

(a) *Can we find a universal bound of minimum order for the family of oriented graphs whose underlying graph is the complete graph K_n?*

(b) *Can we find a universal bound of minimum order for the family of signified graphs whose underlying graph is the complete graph K_n?*

We can ask a similar question for orientable and signed graphs as well but there is another interesting question which, probably, should be asked before that.

Question 7.2.

(a) *How many non-isomorphic orientable graphs, whose underlying graphs are each isomorphic to the complete graph K_n, are there?*

(b) *How many non-isomorphic signed graphs, whose underlying graphs are each isomorphic to the complete graph K_n, are there?*

We provided tight bounds for the absolute clique number of the above mentioned families in these four settings. In particular, we proved that the maximum order of a planar oclique (or sclique) is 15. Basically, we proved that the maximum order of an oclique (or an sclique) drawn on a surface of genus 0 is 15. Now we want to ask the following general question.

Question 7.3. *What is the maximum order of an oclique (or an sclique) drawn on a surface of genus k (≥ 1)?*

Also, we can think of generalizing the idea of scliques to k-edge-colored graphs and define k-edge-colored cliques. Then we can consider the problem of determining the maximum order of a planar k-edge-colored clique.

A graph G is a *perfect graph* if for every induced subgraph H of G we have $\omega(H) = \chi(H)$. We do not have a notion of perfect graph for oriented or signified graphs. We wonder if it is possible to introduce a notion of perfectness using the definitions of chromatic numbers, relative clique numbers and absolute clique numbers for oriented or signified graphs.

Both oriented and signified graphs are similar in the sense that a vertex can have two different kinds of neighbors in each of these types. While trying to find upper and lower bounds for the above mentioned parameters for oriented and signified graphs we observed striking similarities in the techniques used for proving the bounds. It lead us to wonder if they have some underlying relation between them. To our surprise, we ended up constructing examples of graphs which can have their oriented chromatic number and signified chromatic number arbitrarily different from each other. We think it is possible to prove similar results for the other two parameters as well.

Orientable and signed graphs are defined by considering equivalence classes of oriented and signified graphs. The equivalence relation, in each case is by switching the two different kinds of neighbors of a vertex. As in the case of oriented and signified graphs, we observed that similar techniques can be used for proving bounds regarding the parameters of orientable and signed graphs.

Unlike the previous case, we could not find an example of a graph which has higher orientable chromatic number than signed chromatic number. So, finding such an example or proving that the signed chromatic number is higher than the orientable chromatic number for each graph will be a good problem to consider regarding these two types of graphs. However, we did observe that the orientable chromatic number of an outerplanar graph is at most 4, while the signed chromatic number of an outerplanar graph is at most 5 and both the bounds are tight. To be precise, we would like to answer the following question.

Question 7.4. *Given any graph G, can we show that $\chi_{[o]}(G) \leq \chi_{[s]}(G)$?*

The following result directly follows from the fact that the bounds are tight for $k \geq 3$ in Theorem 3.12, 4.15, 5.12 and 6.21:

Proposition 7.5. *Let $\mathcal{A}_k$ denote the family of graphs with acyclic chromatic number at most k. Then for $k \geq 3$,*

$$\chi_o(\mathcal{A}_k) = \chi_s(\mathcal{A}_k) \text{ and } \chi_{[o]}(\mathcal{A}_k) = \chi_{[s]}(\mathcal{A}_k).$$

Naturally we want to know if there are more such families for which the chromatic numbers coincide. In particular, can we ask if something similar for the family of planar graphs holds. Regarding relations between the orientable chromatic number and the signed chromatic number of graphs, we would like to state the following conjecture.

Conjecture 7.6. *Let G be a bipartite graph. Then $\chi_{[o]}(G) \leq \chi_{[s]}(G)$.*

For the 4-cycle C_4 we know that $\chi_{[o]}(C_4) = \chi_{[s]}(C_4) = 4$. Hence the conjecture, if true, is tight. We know from Theorem 4.17 that an even cycle of length at least 22 has orientable chromatic number at most 3, whereas we also know that a signed unbalanced 22-cycle has signed chromatic number equal to 4. Therefore, we cannot replace the inequality with equality in the above conjecture.

For oriented graphs, we also studied the problem regarding 2-dipath and oriented $L(p,q)$-labeling of graphs and provided upper and lower bounds for several planar families. Considering the problem for other families of graphs can be an interesting field of research.

We proved the existence of categorical products for orientable and signed graphs. It will be interesting to discover other categorical aspects of these two types of graphs. Also, we wonder if there are any other categorical relations, other than the ones proved in this thesis, between these four different types of graphs.

Finally, we would like to mention that it might be possible to adapt the techniques used for the theory of signed homomorphisms developed by Naserasr, Rollová and Sopena for orientable graphs. To do that, trying to figure out what the analogue of consistent signed graphs could be in the notion of orientable graphs may be the first step.

Bibliography

[1] K. I. Aardal, S. P. Van Hoesel, A. M. Koster, C. Mannino, and A. Sassano. Models and solution techniques for frequency assignment problems. *Annals of Operations Research*, 153(1):79–129, 2007.

[2] N. Alon and T. H. Marshall. Homomorphisms of edge-colored graphs and Coxeter groups. *Journal of Algebraic Combinatorics*, 8(1):5–13, 1998.

[3] L. Babai and P. J. Cameron. Automorphisms and enumeration of switching classes of tournaments. *Journal of Combinatorics*, 7(1):R38–R38, 2001.

[4] O. V. Borodin. On acyclic colorings of planar graphs. *Discrete Mathematics*, 25(3):211–236, 1979.

[5] O. V. Borodin and A. O. Ivanova. An oriented 7-colouring of planar graphs with girth at least 7. *Sibirskie Ehlektronnye Matematicheskie Izvestiya [electronic only]*, 2:222–229, 2005.

[6] O. V. Borodin, A. O. Ivanova, and A. V. Kostochka. Oriented 5-coloring of sparse plane graphs. *Journal of Applied and Industrial Mathematics*, 1(1):9–17, 2007.

[7] O. V. Borodin, A. V. Kostochka, J. Nešetřil, A. Raspaud, and É. Sopena. On the maximum average degree and the oriented chromatic number of a graph. *Discrete Mathematics*, 206(1–3):77–89, 1999.

[8] R. Brewster. *Vertex colourings of edge-coloured graphs*. PhD thesis, Simon Fraser University, Canada, 1993.

[9] R. C. Brewster and T. Graves. Edge-switching homomorphisms of edge-coloured graphs. *Discrete Mathematics*, 309(18):5540–5546, 2009.

[10] A. E. Brouwer and W. H. Haemers. Strongly regular graphs. In *Spectra of Graphs*, pages 115–149. Springer, 2012.

[11] T. Calamoneri and B. Sinaimeri. L(2, 1)-labeling of oriented planar graphs. *Discrete Applied Mathematics*, 161(12):1719–1725, 2013.

[12] G. J. Chang, J.-J. Chen, D. Kuo, and S.-C. Liaw. Distance-two labelings of digraphs. *Discrete Applied Mathematics*, 155(8):1007–1013, 2007.

[13] M. Chudnovsky, K. Edwards, and P. D. Seymour. Edge-colouring eight-regular planar graphs. *CoRR*, abs/1209.1176, 2012.

[14] B. Courcelle. The monadic second order logic of graphs vi: On several representations of graphs by relational structures. *Discrete Applied Mathematics*, 54(2):117–149, 1994.

[15] R. Fabila-Monroy, D. Flores, C. Huemer, and A. Montejano. Lower bounds for the colored mixed chromatic number of some classes of graphs. *Comment. Math. Univ. Carolinae*, 49(4):637–645, 2008.

[16] D. C. Fisher and J. Ryan. Tournament games and positive tournaments. *Journal of Graph Theory*, 19(2):217–236, 1995.

[17] J. P. Georges and D. W. Mauro. Generalized vertex labelings with a condition at distance two. *Congressus Numerantium*, 109:141–159, 1995.

[18] W. Goddard and M. A. Henning. Domination in planar graphs with small diameter. *Journal of Graph Theory*, 40:1–25, May 2002.

[19] D. Gonçalves, A. Raspaud, M. Shalu, et al. On oriented labelling parameters. *Series Machine Perception and Artificial Intelligence*, 66:33–45, 2006.

[20] J. R. Griggs and R. K. Yeh. Labelling graphs with a condition at distance 2. *SIAM Journal on Discrete Mathematics*, 5(4):586–595, 1992.

[21] F. Harary. On the notion of balance of a signed graph. *The Michigan Mathematical Journal*, 2(2):143–146, 1953.

[22] P. Hell, A. V. Kostochka, A. Raspaud, and É. Sopena. On nice graphs. *Discrete Mathematics*, 234(1–3):39–51, 2001.

[23] P. Hell and J. Nešetřil. *Graphs and Homomorphisms*. Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, 2004.

[24] S.-J. Kim and D. B. West. Triangle-free planar graphs with minimum degree 3 have radius at least 3. *Discussiones Mathematicae Graph Theory*, 28(3):563–566, 2008.

[25] W. F. Klostermeyer. Pushing vertices and orienting edges. *Ars Combinatoria*, 51:65–76, 1999.

[26] W. F. Klostermeyer et al. Hamiltonicity and reversing arcs in digraphs. *Journal of Graph Theory*, 28(1):13–30, 1998.

[27] W. F. Klostermeyer and G. MacGillivray. Analogs of cliques for oriented coloring. *Discussiones Mathematicae Graph Theory*, 24(3):373–388, 2004.

[28] W. F. Klostermeyer and G. MacGillivray. Homomorphisms and oriented colorings of equivalence classes of oriented graphs. *Discrete Mathematics*, 274(1–3):161–172, 2004.

[29] A. V. Kostochka, É. Sopena, and X. Zhu. Acyclic and oriented chromatic numbers of graphs. *Journal of Graph Theory*, 24:331–340, 1997.

[30] G. MacGillivray and K. L. B. Wood. Re-orienting tournaments by pushing vertices. *Ars Combinatoria*, 57, 2000.

[31] T. H. Marshall. An oriented 6-coloring of planar graphs with girth at least 9. (preprint).

[32] T. H. Marshall. Homomorphism bounds for oriented planar graphs. *Journal of Graph Theory*, 55:175–190, July 2007.

[33] T. H. Marshall. On oriented graphs with certain extension properties. (in press). *Ars Combinatoria*, 2012.

[34] T. H. Marshall. Homomorphism bounds for oriented planar graphs of given minimum girth. *Graphs and Combinatorics*, 29(5):1489–1499, 2013.

[35] C. Min and W. Weifan. The 2-dipath chromatic number of Halin graphs. *Information Processing Letters*, 99(2):47–53, July 2006.

[36] A. Montejano, P. Ochem, A. Pinlou, A. Raspaud, and É. Sopena. Homomorphisms of 2-edge-colored graphs. *Discrete Applied Mathematics*, 158(12):1365–1379, 2010.

[37] K. M. Mosesian. Strongly basable graphs (russian). *Akad. Nauk. Armian. SSR Dokl.*, 54:134–138, 1972.

[38] R. Naserasr. Mapping planar graphs into projective cubes. *Journal of Graph Theory*, 74:249–259, November 2013.

[39] R. Naserasr, E. Rollová, and É. Sopena. Homomorphisms of planar signed graphs to signed projective cubes. *Discrete Mathematics & Theoretical Computer Science*, 15(3):1–12, 2013.

[40] R. Naserasr, E. Rollová, and É. Sopena. Homomorphisms of signed graphs. (preprint). 2013.

[41] J. Nešetřil and A. Raspaud. Colored homomorphisms of colored mixed graphs. *Journal of Combinatorial Theory, Series B*, 80(1):147–155, 2000.

[42] J. Nešetřil, A. Raspaud, and É. Sopena. Colorings and girth of oriented planar graphs. *Discrete Mathematics*, 165–166(0):519–530, 1997.

[43] P. Ochem. Oriented colorings of triangle-free planar graphs. *Information Processing Letters*, 92(2):71–76, 2004.

[44] P. Ochem. Negative results on acyclic improper colorings. In *Proc. European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2005, DMTCS proc. AE*, pages 357–362, 2005.

[45] P. Ochem and A. Pinlou. Oriented coloring of triangle-free planar graphs and 2-outerplanar graphs. *Graphs and Combinatorics*, pages 1–15, 2013.

[46] A. Pinlou. An oriented coloring of planar graphs with girth at least five. *Discrete Mathematics*, 309(8):2108–2118, 2009.

[47] A. Pinlou and É. Sopena. Oriented vertex and arc colorings of outerplanar graphs. *Information Processing Letters*, 100(3):97–104, 2006.

[48] O. Pretzel. On graphs that can be oriented as diagrams of ordered sets. *Order*, 2:25–40, 1985.

[49] O. Pretzel. On reordering graphs by pushing down maximal vertices. *Order*, 3:135–153, 1986.

[50] O. Pretzel. Orientations and edge functions on graphs. *Surveys in Combinatorics, London Math. Soc. Lecture Notes 66*, pages 161–185, 1991.

[51] A. Raspaud and É. Sopena. Good and semi-strong colorings of oriented planar graphs. *Information Processing Letters*, 51(4):171–174, 1994.

[52] N. Robertson, P. Seymour, and R. Thomas. Hadwiger's conjecture for K_6-free graphs. *Combinatorica*, 13(3):279–361, 1993.

[53] S. Sen. L(2,1)-labelings of some families of oriented planar graphs. (to appear). *Discussiones Mathematicae Graph Theory.*

[54] S. Sen. Largest planar ocliques. In *Bordeaux Graph Workshop*, 2012.

[55] S. Sen. Maximum order of a planar oclique is 15. In *Proc. International Workshop On Combinatorial Algorithms, IWOCA 2012, Combinatorial Algorithms*, volume 7643 of *Lecture Notes in Computer Science*, pages 130–142. Springer Berlin Heidelberg, 2012.

[56] S. Sen. On push chromatic number of planar graphs and planar p-cliques. In *Proc. European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2013*, 2013.

[57] É. Sopena. The chromatic number of oriented graphs. *Journal of Graph Theory*, 25:191–205, 1997.

[58] É. Sopena. Oriented graph coloring. *Discrete Mathematics*, 229(1–3):359–369, 2001.

[59] É. Sopena. There exist oriented planar graphs with oriented chromatic number at least sixteen. *Information Processing Letters*, 81(6):309–312, 2002.

[60] D. B. West. *Introduction to Graph Theory (2nd Edition)*. Prentice Hall, 2001.

[61] K. Young. 2-dipath and proper 2-dipath colourings. Master's thesis, University of Victoria, Canada, 2011.

[62] T. Zaslavsky. Signed graphs. *Discrete Applied Mathematics*, 4(1):47–74, 1982.

[63] H. Zhang. Self-complementary symmetric graphs. *Journal of graph theory*, 16(1):1–5, 1992.

Appendix A

Appendix: omitted proofs

In this appendix, we gather some proofs that are of minor relevance or repetitive.

A.1 Proof of Theorem 5.17(b) and (d) (upper bounds)

(b) (proof of the upper bound)

Let (G, Σ) be a triangle-free planar signified graph of minimum order with $\omega_{rs}((G, \Sigma)) > 26$. Let R be a signified relative clique of maximum order of (G, Σ) and let $S = V(G) \setminus R$.

Claim 1: For any $v \in V(G)$ we have, $|N^{\alpha}(v) \cap R| \leq 4$ for $\alpha \in \{+, -\}$.

Proof of claim 1: Let $v \in V(G)$ and $N^{\alpha}(v) \cap R = \{v_1, ..., v_k\}$ with $k \geq 5$. Fix a planar embedding (G, Σ). Assume, without loss of generality, that the vertices $v_1, v_2, ..., v_k$ are arranged around v in a clockwise order in the embedding. Clearly $v_1, v_2, ..., v_k$ are pairwise non-adjacent vertices as the graph (G, Σ) is triangle-free. Now, as $v_1, ..., v_5 \in R$, pairwise they should be at unbalanced distance at most 2. Hence, each pair of vertices v_i and v_j must be connected by an unbalanced 2-path using an internal vertex $v_{ij} \neq v$ (it is not necessary to have these vertices all distinct from each other), for $1 \leq i < j \leq k$.

Suppose $v_{15} \neq v_{24}$. Now, without loss of generality, also suppose that $v_2v_{24} \in \Sigma$ and $v_{24}v_4 \in \Sigma^c$. Now, we must have $v_{24} = v_{13}$ and $v_{24} = v_{35}$ to keep the graph planar. Also we will have $v_{24} = v_{14}$ and $v_{24} = v_{25}$ to keep the graph planar.

Now, $v_{24} = v_{14}$ implies $v_1v_{24} \in \Sigma$. This and $v_{24} = v_{13}$ imples $v_{24}v_3 \in \Sigma^c$.

Similarly, $v_{24} = v_{25}$ implies $v_{24}v_5 \in \Sigma^c$. This and $v_{24} = v_{35}$ imples $v_3v_{24} \in \Sigma$.

But this is a contradiction.

Hence, we must have $v_{15} = v_{24}$. This will force $v_{15} = v_{24} = v_{13} = v_{35}$. Now, without loss of generality, also suppose that $v_2v_{24} \in \Sigma$ and $v_{24}v_4 \in \Sigma^c$.

Then we can argue similar to the previous case and arrive to a contradiction. $\diamondsuit$

Now note that, for any $z \in S$, we have $d(z) \geq 2$ as otherwise the vertex z does not help connecting any two (or more) vertices of R by an unbalanced 2-path with the internal vertex being itself (that is, z) and hence can be deleted to get a signified planar triange-free graph with the signified relative clique number equal to that of (G) but with order less that (G) which contradicts the minimality of (G). Now, a vertex z of S must connect at least two vertices of R by an unbalanced 2-path with the internal vertex being itself (that is, z).

Now for each vertex $z \in S$ with $d(z) \leq 5$, assume that the neighbors of z are $v_1, v_2, ..., v_k$. Fix a planar embedding of (G) and assume that the neighbors of z are arranged in a clockwise order around z. Now delete the vertex z and add the edges v_1v_2, v_2v_3, ..., $v_{k-1}v_k$, v_kv_1 (for $d(z) = 2$ add only one edge v_1v_2) to obtain a new graph.

Note that this new graph is also planar and the degree of each vertex in the new graph, which were in (G) also, is at least as much as the degree of the corresponding vertex in (G). Hence, there is a vertex v in the new graph, which belongs to R and has degree at most 5.

Hence, there will be a neighbor u of v and at least 5 vertices of $R \setminus (N(v) \cup \{v\})$ which are connected to v with an unbalanced 2-path with internal vertex u which is a contradiction to Claim 1. So, we can conclude that $\omega_{rs}((G)) \leq 26$.

(d) (proof of the upper bound)

We will now prove the upper bound by contradiction.

Assume that (G, Σ) is a planar signified graph with girth at least 6 of minimum order with $\omega_{rs}((G)) > 4$. Moreover assume (G, Σ) is such that, if we delete any edge of (G, Σ), the new graph will have signified relative chromatic number greater than 5. Let R be a signified relative clique of maximum order of (G). Let, $S = V(G) \setminus R$. Note that S induces an independent set of (G) as deleting any edge between two vertices of S will not decrease the signified relative clique number of the graph (G).

First note that, for any $z \in S$, we have $d(z) \geq 2$ as otherwise the vertex z does not help connecting any two (or more) vertices of R by an unbalanced 2-path with the internal vertex being itself (that is, z) and hence can be deleted to get a signified planar graph with girth at least 6 and with relative signified chromatic number equal to that of (G) but with order less than (G) which contradicts the minimality of (G).

Also, for any $z \in V(G)$, we must have $|N(z) \cap R| \leq 2$. If not, then we will have $|N^{\alpha}(z) \cap R| \geq 2$ for some $\alpha \in \{+, -\}$. Now to have unbalanced distance at most 2 between two vertices of $N^{\alpha}(z) \cap R$, there should be an unbalanced 2-path joining the two vertices. This will create a cycle of length 4, which is a contradiction. Hence for any $z \in S$ we have, $d(z) = 2$ and that z must be an internal vertex of a 2-dipath with two terminal vertices from R.

Note that every signified forest has signified relative clique number at most 3. Therefore, (G) must have a cycle. Also, (G) must contain a cycle $abcdef$ of length 6 (that is, the edges of the cycle are ab, bc, cd, de, ef and fa) using Theorem 3.18(d) whose proofs are independent from this proof. As S is an independent set in (G), we can have at most three vertices (should be non-adjacent) of the cycle from S.

As the vertices of S are non-adjacent, without loss of generality, we can assume that $a, c, e \in R$. If another vertex of R is in the cycle, then, without loss of generality, we can assume it to be d. If so, then d and a must be connected by an unbalanced 2-path or be adjacent which creates a cycle of length less than 6, hence is not possible. Therefore, exactly three vertices of the cycle are from R.

Now, for any $w \in R$, we have w connected by an unbalanced 2-path to each of a, c, e. If we have two such vertices w and x, which are adjacent or connected by a 2-dipath, it will create a subdivision of the complete graph K_5 in (G) which contradicts the planarity of $\overrightarrow{G}$. □

A.2 Proof of Lemma 5.20

Let (G, Γ) be a planar sclique with $|V(G)| > 15$. Assume that G is triangulated and has domination number 2.

We define the partial order $\prec$ for the set of all dominating sets of order 2 of G as follows: for any two dominating sets $D = \{x, y\}$ and $D' = \{x', y'\}$ of order 2 of (G), $D' \prec D$ if and only if $|N(x') \cap N(y')| < |N(x) \cap N(y)|$.

Let $D = \{x, y\}$ be a maximal dominating set of order 2 of G with respect to $\prec$. Also for the rest of this article, $t, t', \alpha, \overline{\alpha}, \beta, \overline{\beta}$ are variables satisfying $\{t, t'\} = \{x, y\}$ and $\{\alpha, \overline{\alpha}\} = \{\beta, \overline{\beta}\} = \{+, -\}$.

Now, we fix the following notations (Fig: A.1):

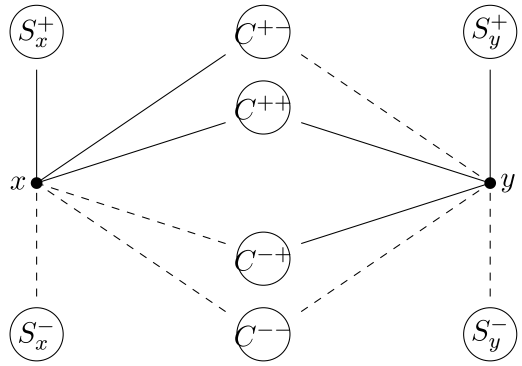


Figure A.1: Structure of G (not a planar embedding)

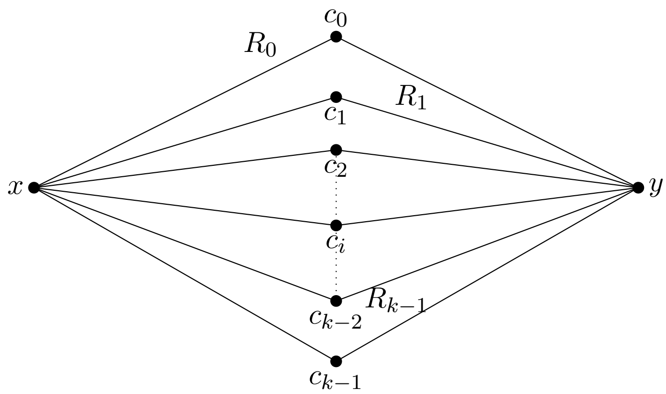


Figure A.2: A planar embedding of $und(H)$

$$C = N(x) \cap N(y),\ C^{\alpha\beta} = N^\alpha(x) \cap N^\beta(y),\ C_t^\alpha = N^\alpha(t) \cap C,$$
$$S_t = N(t) \setminus C,\ S_t^\alpha = S_t \cap N^\alpha(t) \text{ and } S = S_x \cup S_y.$$

Hence we have,

$$16 \leq |(G,\Gamma)| = |D| + |C| + |S|. \tag{A.1}$$

Let (H) be the signified graph obtained from the induced subgraph $(G)[D \cup C]$ of (G) by deleting all the edges between the vertices of D and all the edges between the vertices of C. Note that it is possible to extend the planar embedding of H given in Fig A.2 to a planar embedding of G for some particular ordering of the elements of, say $C = \{c_0, c_1, ..., c_{k-1}\}$.

Notice that H has k faces, namely the unbounded face F_0 and the faces F_i bounded by edges $xc_{i-1}, c_{i-1}y, yc_i, c_ix$ for $i \in \{1, ..., k-1\}$. Geometrically, H divides the plane into k connected components. The *region* R_i of (G) is the i^{th} connected component (corresponding to the face F_i) of the plane. *Boundary points* of a region R_i are c_{i-1} and c_i for $i \in \{1, ..., k-1\}$ and, c_0 and c_{k-1} for $i = 0$. Two regions are *adjacent* if they have at least one common boundary point (hence, a region is adjacent to itself).

Now for the different possible values of $|C|$, we want to show that H cannot be extended to a planar sclique of order at least 16. Note that, for extending H to (G), we can add new vertices only from S. Any vertex $v \in S$ will be inside one of the regions R_i. If there is at least one vertex of S in a region R_i, then R_i is *non-empty* and *empty* otherwise. In fact, when there is no chance of confusion, R_i might represent the set of vertices of S contained in the region R_i.

As any two distinct non-adjacent vertices of (G) must be connected by an unbalanced 2-path, we have the following three lemmas:

Lemma A.1. *(a) If $(u,v) \in S_x \times S_y$ or $(u,v) \in S_t^\alpha \times S_t^\alpha$, then u and v are in adjacent regions.*

(b) If $(u,c) \in S_t^\alpha \times C_t^\alpha$, then c is a boundary point of a region adjacent to the region containing u.

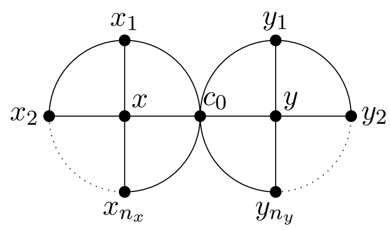


Figure A.3: For $|C| = 1$ while x and y are non-adjacent

Lemma A.2. *Let R, R^1, R^2 be three distinct regions such that R is adjacent to R^i with common boundary point c^i while the other boundary points of R^i is $\overline{c^i}$ for all $i \in \{1, 2\}$. If $v \in S_t^\alpha \cap R$ and $u^i \in ((S_t^\alpha \cup S_{t'}) \cap R^i) \cup (\{\overline{c^i}\} \cap C_t^\alpha)$, then v disagrees with u^i on c^i, where $i \in \{1, 2\}$. If both u^1 and u^2 exist, then $|S_t^\alpha \cap R| \leq 1$.*

Lemma A.3. *For any edge uv in (G), we have $|N^\alpha(u) \cap N^\beta(v)| \leq 3$.*

Now we ask the question "How small $|C|$ can be?" and try to prove possible lower bounds of $|C|$. The first result regarding the lower bound of $|C|$ is proved below.

Lemma A.4. $|C| \geq 2$.

Proof. We know that x and y are either connected by an unbalanced 2-path or by an edge. If x and y are adjacent, then as (G) is triangulated, we have $|C| \geq 2$. If x and y are non-adjacent, then $|C| \geq 1$. Hence it is enough to show that we cannot have $|C| = 1$ while x and y are non-adjacent.

If $|C| = 1$ and x and y are non-adjacent, then the triangulation will force the configuration depicted in Fig A.3 as a subgraph of G, where $C = \{c_o\}$, $S_x = \{x_1, ..., x_{n_x}\}$ and $S_y = \{y_1, ..., y_{n_y}\}$. Without loss of generality we may assume $|S_y| \geq |S_x|$. Then by equation (A.1) we have,

$$n_y = |S_y| \geq \lceil (16 - 2 - 1)/2 \rceil = 7.$$

Clearly $n_x \geq 3$ as otherwise $\{c_0, y\}$ is a dominating set with at least two common neighbors $\{y_1, y_{n_y}\}$ which contradicts the maximality of D.

For $n_x = 3$, we know that c_0 is not adjacent to x_2 as otherwise $\{c_0, y\}$ is a dominating set with at least two common neighbors $\{y_1, y_{n_y}\}$ contradicting the maximality of D. But then x_2 should be adjacent to y_i for some $i \in \{1, ..., n_y\}$ as otherwise $d(x_2, y) > 2$. Now the triangulation will force x_2 and y_i to have at least two common neighbors. Also x_2 cannot be adjacent to y_j for any $j \neq i$, as it will create a dominating set $\{x_2, y\}$ with at least two common neighbors $\{y_i, y_j\}$ contradicting the maximality of D. Hence, x_2 and y_i are adjacent to both x_1 and x_3. Note that t_{ℓ_t} and t_{ℓ_t+k} are adjacent if and only if $k = 1$, as otherwise $d(t_{\ell_t+1}, t') > 2$ for $1 \leq \ell_t < \ell_t + k \leq n_t$. In this case, by equation (A.1) we have,

$$n_y = |S_y| \geq 16 - 2 - 1 - 3 = 10.$$

Assume $i \geq 5$. Hence, c_0 is adjacent to y_j for all $j = 1, 2, 3$, as otherwise $d(y_j, x_3) > 2$. This implies $d(y_2, x_2) > 2$, a contradiction. Similarly $i < 5$ will also force a contradiction. Hence $n_x \geq 4$.

For $n_x = 4$, c_0 cannot be adjacent to both x_3 and $x_{n_x-2} = x_2$ as it creates a dominating set $\{c_0, y\}$ with at least two common neighbors $\{y_1, y_{n_y}\}$ contradicting the maximality of D. For $n_x \geq 5$, c_0 is adjacent to x_3 implies, either for all $i \geq 3$ or for all $i \leq 3$, x_i is adjacent to c_0, as otherwise $d(x_i, y) > 2$. Either of these cases will force c_0 to become adjacent to y_j, as otherwise we will have either $d(x_1, y_j) > 2$ or $d(x_{n_x}, y_j) > 2$ for all $j \in \{1, 2, ..., n_y\}$. But then we will have a dominating set $\{c_0, x\}$ with at least two common vertices contradicting the maximality of D. Hence for $n_x \geq 5$, c_0 is not adjacent to x_3. Similarly we can show, for $n_x \geq 5$, that c_0 is not adjacent to either x_3 or x_{n_x-2}.

So, for $n_x \geq 4$, without loss of generality we can assume that c_0 is not adjacent to x_3. We know that $d(y_1, x_3) \leq 2$. We have already noted that t_{l_t} and t_{l_t+k} are adjacent if and only if $k = 1$ for any $0 \leq l_t < l_t + k \leq n_t$. Hence to have $d(y_1, x_3) \leq 2$, we must have one of the following edges: y_1x_2, y_1x_3, y_1x_4 or y_2x_3. The first edge will imply the edges x_2y_j as otherwise $d(x_1, y_j) > 2$ for all $j = 3, 4, 5$. These three edges will imply $d(x_4, y_3) > 2$. Hence we do not have y_1x_2.

The other three edges, assuming we cannot have y_1x_2, will force the edges x_2c_0 and x_1c_0 for having $d(x_2, y) \leq 2$ and $d(x_1, y) \leq 2$. This will imply $d(x_1, y_4) > 2$, a contradiction. Hence we cannot have the other three edges also.

Hence we are done. □

Now we will prove that, for $2 \leq |C| \leq 5$, at most one region of (G) can be non-empty. Later, using this result, we will improve the lower bound of $|C|$.

Lemma A.5. *If $2 \leq |C| \leq 5$, then at most one region of (G) is non-empty.*

Proof. For pictorial help one can look at Fig A.2. For $|C| = 2$, if x and y are adjacent, then the region that contains the edge xy is empty, as otherwise triangulation will force x and y to have a common neighbor other than c_0 and c_1. So for the rest of the proof we can assume x and y to be non-adjacent for $|C| = 2$.

Step 0: First we shall show that it is not possible to have either $S_x = \emptyset$ or $S_y = \emptyset$ and have at least two non-empty regions. Without loss of generality assume that $S_x = \emptyset$. Then x and y are non-adjacent, as otherwise y will be a dominating vertex which is not possible.

For $|C| = 2$, if both $S_y \cap R_0$ and $S_y \cap R_1$ are non-empty, then triangulation will force, either multiple edges c_0c_1 (one in each region) or a common neighbor of x, y other than c_0, c_1, a contradiction.

For $|C| = 3, 4$ and 5, triangulation implies the edges $c_0c_1, ..., c_{k-2}c_{k-1}, c_{k-1}c_0$. Hence every $v \in S_y$ must be connected to x by an unbalanced 2-path through c_i for some $i \in \{1, 2, ..., k-1\}$. Now assume $|S_y^\alpha| \geq |S_y^{\overline{\alpha}}|$ for some $\alpha \in \{+, -\}$. Then by equation (A.1) we have,

$$|S_y^\alpha| \geq \lceil (16 - 2 - 5)/2 \rceil = 5.$$

Now by Lemma A.1, we know that the vertices of S_y^α will be contained in two adjacent regions for $|C| = 4, 5$. For $|C| = 3$, $S_y^\alpha \cap R_i$ for all $i \in \{0, 1, 3\}$ implies $|S_y^\alpha| \leq 3$ by Lemma A.2. Hence, without loss of generality, we may assume $S_y^\alpha \subseteq R_1 \cup R_2$. If both $S_y^\alpha \cap R_1$ and $S_y^\alpha \cap R_2$ are non-empty, then by Lemma A.2, each vertex of $S_y^\alpha \cap R_1$ disagrees with each vertex of $S_y^\alpha \cap R_2$ on c_1. Then $\{c_1, y\}$ becomes a dominating set with at least six common neighbors (c_0, c_2 and four vertices from S_y^α) which contradicts the maximality of D.

Hence, all the vertices of S_y^α must be contained in one region, say R_1. Then each of them should be connected to x by an unbalanced 2-path with internal vertex either c_0 or c_1. However, the vertices that are connected to x by an unbalanced 2-path with internal vertex c_0 should have unbalanced distance at most 2 with the vertices connected to x by an unbalanced 2-path with internal vertex c_1. It is not possible to connect them unless they are all adjacent to either c_0 or c_1. But then it will contradict the maximality of D.

Hence both S_x and S_y are non-empty.

Step 1: Now we will prove that at most four sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, for all $t \in \{x, y\}$ and $i \in \{0, 1, ..., k-1\}$. It is trivial for $|C| = 2$. For $|C| = 4$ and 5, the statement follows from Lemma A.1. For $|C| = 3$, we consider the following two cases:

(i) Assume $S_t \cap R_i \neq \emptyset$ for all $t \in \{x, y\}$ and for all $i \in \{0, 1, 2\}$. Then by Lemma A.2 we have, $|S_t \cap R_i| \leq 1$ for all $t \in \{x, y\}$ and for all $i \in \{0, 1, 2\}$. Then by equation (A.1) we have,

$$16 \leq |(G)| = 2 + 3 + 4 = 9.$$

This is a contradiction.

(ii) Assume that five out of the six sets $S_t \cap R_i$ are non-empty and the other one is empty, where $t \in \{x, y\}$ and $i \in \{0, 1, 2\}$. Without loss of generality we can assume $S_x \cap R_0 = \emptyset$. By Lemma A.2 we have $|S_t \cap R_i| \leq 1$ for all $(t, i) \in \{(x, 1), (x, 2), (y, 0)\}$. In particular, $|S_x| \leq 2$.

Now, all verticex of $S_t \cap R_i$ is adjacent to c_1, for being at unbalanced distance at most 2 from each other, by Lemma A.2. That means, every vertex of S_x is adjacent to c_1. Hence, there can be at most three vertices in $(S_y \cap R_1) \cup (S_y \cap R_2)$ as otherwise the dominating set $\{c_1, y\}$ will contradict the maximality of D. Hence, $|S_y| \leq 4$.

Therefore by equation (A.1) we have,

$$16 \leq |(G)| = 2 + 3 + (2 + 4) = 11.$$

This is a contradiction.

Hence at most four sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, where $t \in \{x, y\}$ and $i \in \{0, 1, ..., k - 1\}$.

Step 2: Now assume that exactly four sets out of the sets $S_t \cap R_i$ are non-empty, for all $t \in \{x, y\}$ and $i \in \{0, ..., k - 1\}$. Without loss of generality we have the following three cases (by Lemma A.1):

(i) Assume the four non-empty sets are $S_x \cap R_1, S_y \cap R_0, S_y \cap R_1$ and $S_y \cap R_2$ (only possible for $|C| \geq 3$). We have the edges c_0c_{k-1} and c_1c_2 by triangulation. Lemma A.2 implies that $S_x \cap R_1 = \{x_1\}$ and that the vertices of $S_y \cap R_0$ and the vertices of $S_y \cap R_2$ disagree with x_1 on c_0 and c_1 respectively. Hence by Lemma A.3, we have $|S_y \cap R_0|, |S_y \cap R_2| \leq 3$.

For $|C| = 3$, if every vertex from $S_y \cap R_1$ is adjacent to either c_0 or c_1, then $\{c_0, c_1\}$ will be a dominating set with at least four common neighbors $\{x, y, x_1, c_2\}$ contradicting the maximality of D. If not, then triangulation will force x_1 to be adjacent to at least two vertices y_1, y_2 (say) from S_y. But then $\{x_1, y\}$ will become a dominating set with at least four common neighbors $\{y_1, y_2, c_0, c_1\}$ contradicting the maximality of D.

For $|C| = 4$ and 5, Lemma A.1 implies that vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_2$ disagree with each other on y. Now by Lemma A.2, any vertex of $S_y \cap R_1$ is adjacent to either c_0 (if it agrees with the vertices of $S_y \cap R_0$ on y) or c_1 (if it agrees with the vertices of $S_y \cap R_2$ on y). Also vertices of $S_y \cap R_0$ and $S_y \cap R_2$ are connected to x_1 by an unbalanced 2-path through c_0 and c_1 respectively.

Now by equation (A.1) we have,

$$|S_y| \geq (16 - 2 - 5 - 1) = 8.$$

Hence, without loss of generality, at least four vertices y_1, y_2, y_3, y_4 of S_y are adjacent to c_0. Hence $\{c_0, y\}$ is a dominating set with at least five common neighbors $\{y_1, y_2, y_3, y_4, c_{k-1}\}$ contradicting the maximality of D for $|C| = 4$.

For $|C| = 5$, each vertex of $S_y \cap R_1$ disagree with c_3 by Lemma A.1 and hence without loss of generality are all adjacent to c_0. Now $|S_y \cap R_2| \leq 3$ and $|S_y| \geq 8$ implies $|S_y \cap (R_0 \cup R_1)| \geq 5$. But every vertex of $S_y \cap (R_0 \cup R_1)$ and c_4 are adjacent to c_0. Hence $\{c_0, y\}$ is a dominating set with at least six common neighbors, contradicting the maximality of D for $|C| = 5$.

(ii) Assume the four non-empty sets are $S_x \cap R_0, S_x \cap R_1, S_y \cap R_0$ and $S_y \cap R_1$. For $|C| = 2$ every vertex in S is adjacent to either c_0 or c_1 (by Lemma A.2). So, $\{c_0, c_1\}$ is a dominating set. Hence no vertex $w \in S$ can be adjacent to both c_0 and c_1 because otherwise $\{c_0, c_1\}$ will be a dominating set with at least three common neighbors $\{x, y, w\}$ contradicting the maximality of D. By equation (A.1) we have,

$$|S| \geq 16 - 2 - 2 = 12.$$

Hence, without loss of generality, we may assume $|S_x \cap R_0| \geq 3$. Assume $\{x_1, x_2, x_3\} \subseteq S_x \cap R_0$. Now all vertices of $S_x \cap R_0$ must be adjacent to c_0 (or c_1), as otherwise it will force all vertices of $S_y \cap R_1$ to be adjacent to both c_0 and c_1 (by Lemma A.2). Without loss of generality assume all vertices of $S_x \cap R_0$ are adjacent to c_0. Then all $w \in S_y$ will be adjacent to c_0, as otherwise $d(w, x_i) > 2$ for some $i \in \{1, 2, 3\}$. But then $\{c_0, x\}$ will be a dominating set with at least three common vertices $\{x_1, x_2, x_3\}$ contradicting the maximality of D.

For $|C| = 3, 4$, every vertex of S will be adjacent to c_0 (by Lemma A.2). By equation (A.1) we have,

$$|S| \geq (16 - 2 - 4) = 10.$$

Hence, without loss of generality, $|S_x| \geq 5$. Hence $\{c_o, x\}$ is a dominating set with at least five common neighbors $S_x \cup \{y\}$ contradicting the maximality of D for $|C| = 3, 4$.

For $|C| = 5$, every vertex of $S_t \cap R_i$ disagree with c_{i+2} on t and hence $|S_t \cap R_i| \leq 3$ for $i \in \{0, 1\}$ by Lemma A.1. Assume, $|S_x \cap R_0| = 3$ and $S_x \cap R_0 = \{x_1, x_2, x_3\}$. Now assume without loss of generality that $c_2 \in N^{\alpha}(x)$. Hence, we must have $\{x_1, x_2, x_3\} \subseteq N^{\overline{\alpha}}(x)$.

Note that x_1, x_2, x_3 must agree on c_0 in order to be at unbalanced distance at most 2 with the vertices of $S_y \cap R_1$. Further assume that $\{x_1, x_2, x_3\} \subseteq N^{\beta}(c_0)$. But then as all the three vertices $\{x_1, x_2, x_3\}$ are adjacent to both x and c_0, the only way each of them can be at unbalanced distance 2 with c_3 is by an unbalanced 2-path with internal vertex x. Hence we have $c_3 \in N^{\alpha}(x)$. This implies that $x_4 \in N^{\overline{\alpha}}(x)$ for any vertex $x_4 \in S_x \cap R_1$. But then the vertices of $S_x \cap R_1$ must disagree with vertices of $S_x \cap R_0$ on c_0 making it impossible for the vertices of $S_1 \cap R_0$ to be at unbalanced distance at most 2 with x_1, x_2, x_3 and with the vertices of $S_x \cap R_1$. Hence we must have $|S_x \cap R_0| \leq 2$.

Similarly we can prove $|S_t \cap R_i| \leq 2$ for $i \in \{0, 1\}$.

Now we will show that it is not possible to have $|S_t \cap R_i| = 2$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$.

Suppose we have $|S_t \cap R_i| = 2$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$. Then clearly, the vertices of $S_t \cap R_i$ disagree with c_{i+2} and c_{i+3} on t. Hence, the vertices of $S_t \cap R_0$ agree with the vertices of $S_t \cap R_1$ on t. Therefore, the vertices of $S_t \cap R_0$ must disagree with the vertices of $S_t \cap R_1$ on c_0.

Then it will not be possible to have both the vertices of $S_x \cap R_0$ at unbalanced distance at most 2 with all the four vertices of S_y.

Therefore, we have $|S| \leq 7$. Hence by equation (A.1) we have,

$$16 \leq |(G)| \leq 2 + 5 + 7 = 14.$$

This is a contradiction. Hence we are done.

(iii) Assume the four non-empty sets are $S_x \cap R_1, S_x \cap R_2, S_y \cap R_0$ and $S_y \cap R_1$ (only possible for $|C| = 3$). Now Lemma A.2 implies that every vertex of $(S_x \cap R_1) \cup (S_y \cap R_0)$ is adjacent to c_0 and every vertex of $(S_x \cap R_2) \cup (S_y \cap R_1)$ is adjacent to c_1.

Moreover triangulation forces the edges $c_0 c_2$ and $c_1 c_2$. Triangulation also forces some vertex $v_1 \in S_y \cap R_1$ to be adjacent to c_0. This will create the dominating set $\{c_0, c_1\}$ with at least four common neighbors $\{x, y, v_1, c_2\}$ contradicting the maximality of D.

Hence at most three sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, where $t \in \{x, y\}$ and $i \in \{0, 1, ..., k - 1\}$.

Step 3: Now assume that exactly three sets out of the sets $S_t \cap R_i$ are non-empty, where $t \in \{x, y\}$ and $i \in \{0, ..., k - 1\}$. Without loss of generality we have the following two cases (by Lemma A.1):

(i) Assume the three non-empty sets are $S_x \cap R_0, S_y \cap R_0$ and $S_y \cap R_1$. Triangulation implies the edge $c_0 c_1$ inside the region R_1.

For $|C| = 2$, there exists $u \in S_y \cup R_1$ such that u is adjacent to both c_0 and c_1 by triangulation. Now if $|S_y \cup R_1| \geq 2$, then some other vertex $v \in S_y \cup R_1$ must be adjacent to either c_0 or c_1. Without loss of generality we may assume that v is adjacent to c_0. Then every $w \in S_x \cap R_0$ will be adjacent to c_0 to have $d(v, w) \leq 2$. But then $\{c_0, y\}$ will be a dominating set with at least three common neighbors $\{c_1, u, v\}$ contradicting the maximality of D.

So we must have $|S_y \cup R_1| = 1$. Now let us assume that $S_y \cup R_1 = \{u\}$. Then any $w \in S_x \cap R_0$ is adjacent to either c_0 or c_1. If $|S_x| \geq 5$, then without loss of generality we can assume that at least three vertices of S_x are adjacent to c_0. Now to have at most distance 2 with all those three vertices, every vertex of S_y will be adjacent to c_0. This will create the dominating set $\{c_0, x\}$ with at least three common neighbors contradicting the maximality of D.

Also $|S_x| = 1$ clearly creates the dominating set $\{c_0, y\}$ (as x_1 is adjacent to c_0 by triangulation) with at least three common neighbors (a vertex from $S_y \cap R_0$ by triangulation, u and c_1) contradicting the maximality of D.

For $2 \leq |S_x| \leq 4$, c_0 (or c_1) can be adjacent to at most two vertices of $S_y \cap R_0$ because otherwise there will be one vertex $v \in S_y \cap R_0$ which will force c_0 (or c_1) to be adjacent to all vertices of $w \in S_x$ in order to satisfy $d(v, w) \leq 2$ and create a dominating set $\{c_0, y\}$ that contradicts the maximality of D.

Also, not all vertices of S_x can is adjacent to c_0 (or c_1) as otherwise $\{c_o, y\}$ (or $\{c_1, y\}$) will be a dominating set with at least three common neighbors (u, c_1 (or c_0) and a vertex from $S_y \cap R_0$) contradicting the maximality of D.

Note that, by equation (A.1), we have,

$$|S_y \cap R_0| \geq 10 - S_x.$$

Assume $S_x = \{x_1, ..., x_n\}$ with triangulation forcing the edges c_0x_1, x_1x_2, ..., $x_{n-1}x_n$, x_nc_1 for $n \in \{2, 3, 4\}$.

For $|S_x| = 2$, at most four vertices of $S_y \cap R_0$ can be adjacent to c_0 or c_1. Hence there will be at least four vertices of $S_y \cap R_0$ each connected to x by an unbalanced 2-path through x_1 or x_2. Without loss of generality x_1 will be adjacent to at least 2 vertices of S_y and hence $\{x_1, y\}$ will be a dominating set contradicting the maximality of D.

For $|S_x| = 3$, without loss of generality assume that x_2 is adjacent to c_0. To satisfy $d(x_1, v) \leq 2$ for all $v \in S_y \cap R_0$, at least four vertices of S_y will be adjacent connected to x_1 by an unbalanced 2-path through x_2 (as, according to previous discussions, at most two vertices of S_y can be adjacent to c_0). This will create the dominating set $\{x_2, y\}$ contradicting the maximality of D.

For $|S_x| = 4$ we have x_2c_0 and x_3c_1 as otherwise at least three vertices of S_x will be adjacent to either c_0 or c_1 which is not possible (because it forces all vertices of S_y to be adjacent to c_0 or c_1). Now each vertex $v \in S_y \cap R_0$ must be adjacent to either c_0 or x_2 (to satisfy $d(v, x_1) \leq 2$) and also to either c_1 or x_3 (to satisfy $d(v, x_4) \leq 2$) which is not possible to do keeping the graph planar.

For $|C| = 3, 4, 5$ by Lemma A.2, each vertex of S_x disagree with each vertex of $S_y \cap R_1$ on c_0. We also have the edge x_1c_2 for some $x_1 \in S_x$ by triangulation. Now by equation (A.1) we have,

$$|S| \geq (16 - 2 - |C|) = 13 - |C|.$$

Hence $|S_x| \leq 2$ for $|C| = 3, 4$, as otherwise every vertex $u \in S_y$ will be adjacent to c_0 creating a dominating set $\{c_0, t\}$ with at least $(|C| + 1)$ common neighbors $S_t \cup \{c_1\}$ for some $t \in \{x, y\}$ contradicting the maximality of D. For $|C| = 5$, as every vertex in $S_x \cap R_0$ agree with each other on x (as they all must disagree with c_2 on x) and on c_0 (as they all disagree with vertices of $S_y \cap R_1$ on c_0). So, by Lemma A.3, we have $|S_x \cap R_0| \leq 3$. But if $|S_x \cap R_0| = 3$ then every vertex of S_y will be adjacent to c_0 creating a dominating set $\{c_0, y\}$ with at least six common neighbors $S_y \cup \{c_1\}$ contradicting the maximality of D.

Hence $|S_x| \leq 2$ for $|C| = 3, 4$ and 5.

Now for $|C| = 3$, we can assume x and y are non-adjacent as otherwise $\{c_0, y\}$ will be a dominating set with at least four common neighbors (x, c_1 and, two other vertices each from the sets $S_y \cap R_0$, $S_y \cap R_1$ by triangulation) contradicting the maximality of D. Hence triangulation will imply the edge c_1c_2. Now for $|S_x| \leq 2$, either $\{c_0, c_2\}$ is a dominating set with at least four common neighbors $\{x, y, c_1, x_1\}$ contradicting the maximality of D or x_1 is adjacent to at least two vertices $y_1, y_2 \in S_y \cap R_0$ creating a dominating set $\{x_1, y\}$ (the other vertex in S_x must be adjacent to x_1 by triangulation) with at least four common neighbors $\{y_1, y_2, c_0, c_2\}$ contradicting the maximality of D.

For $|C| = 4$ we have $|S_y \cap R_1| \leq 2$ as otherwise we will have the dominating set $\{c_0, y\}$ with at least five common neighbors (c_1, vertices of $S_y \cap R_1$ and one vertex of $S_y \cap R_0$ by triangulation) contradicting the maximality of D. Now by equation (A.1) we have,

$$\begin{aligned} |S_y \cap R_0| &\geq (16 - |D| - |C| - |S_x| - |S_y \cap R_1|) \\ &\geq (16 - 2 - 4 - 2 - 2) = 6. \end{aligned}$$

Now, at most two vertices of $S_y \cap R_0$ can be adjacent to c_0 as otherwise $\{c_0, y\}$ will be a dominating set with at least five common neighbors (c_1, vertices of $S_y \cap R_0$ and one vertex of $S_y \cap R_1$ by triangulation) contradicting the maximality of D.

Also by triangulation in R_3 we either have the edge xy or have the edge c_2c_3. But, if we have the edge xy, then $|S_y \cap R_1| = 1$ as otherwise the dominating set $\{c_0, y\}$ will contradict the maximality of D. Hence, by triangulation, and to have unbalanced distance at most 2 with the vertices of S_x, each vertex of $S_y \cap R_0$ will be adjacent either to c_3 or to x_1. This will create a dominating set $\{x_1, y\}$ or $\{c_3, y\}$ that contradicts the maximality of D. Hence, we do not have the edge xy (not even in other regions) and have the edge c_2c_3.

For $|S_x| \leq 2$, the vertices of $S_y \cap R_0$ will be adjacent to either c_3 or c_0 or x_1 to have unbalanced distance at most 2 with x. But then triangulation will force at least one vertex of $S_y \cap R_0$ to be common neighbor of c_3 and x_1 and another vertex of $S_y \cap R_0$ to be common neighbor of c_3 and x_1 or the edge c_0c_3. It is not difficult to check, casewise, (drawing a picture for individual cases will help in understanding the scenario) that one of the sets $\{c_0, y\}$, $\{c_3, y\}$ or $\{x_1, y\}$ will be a dominating set contradicting the maximality of D.

For $|C| = 5$ by Lemma A.1, each vertex of $S_y \cap R_i$ must disagree with c_{i+2} on y. If vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_1$ agree with each other on y, then they must disagree with each other on c_0 which implies $|S_y \cap R_i| \leq 3$ for all $i \in \{0, 1\}$. If vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_1$ disagree with each other on y, then vertices of $S_y \cap R_i$ must agree with c_{3-i} on y. Then, by Lemma A.2, each vertex of $S_y \cap R_i$ must be connected to c_{3-i} by an unbalanced 2-path through c_{4-3i} which implies $|S_y \cap R_i| \leq 3$ for all $i \in \{0, 1\}$.

Assume, we have $|S_y \cap R_0| = 3$ and $|S_y \cap R_1| = 3$. Then each vertex of $S_y \cap R_i$ must disagree with both c_{i+2} and c_{i+3} on y. This will imply that the vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_1$ disagree with each other on c_0. Now there will be no way to have unbalanced distance at most 2 between a vertex of S_x and all the six vertices of S_y.

Hence we must have $|S_y| \leq 5$. Then by equation (A.1) we have,

$$16 \leq |(G)| \leq 2 + 5 + (2 + 5) = 14.$$

This is a contradiction. This concludes this particular subcase.

(ii) Assume the three non-empty sets are $S_x \cap R_1, S_y \cap R_0$ and $S_y \cap R_2$ (only possible for $|C| \geq 3$). By Lemma A.2, we have $S_x = \{x_1\}$ and the fact that each vertex of $S_y \cap R_i$ disagrees with $c_{i^2/4}$ on x_1 for $i \in \{0, 2\}$. Triangulation implies the edges x_1c_0, x_1c_1, $c_{k-1}c_0$, c_0c_1 and c_1c_2.

For $|C| = 3$, $\{c_0, c_1\}$ is a dominating set with at least four common neighbors $\{x, y, c_2, x_1\}$ contradicting the maximality of D. For $|C| = 4$ and 5 we have, every vertex of $S_y \cap R_0$ disagree with every vertex of $S_y \cap R_2$ on y. Hence, by Lemma A.3, we have $|S_y \cap R_i| \leq 3$ for all $i \in \{0, 2\}$. Hence by equation (A.1) we have

$$\begin{aligned} 16 \leq |(G)| &= |D| + |C| + |S| \\ &\leq [2 + 5 + (1 + 3 + 3)] = 14. \end{aligned}$$

This is a contradiction.

Step 4: Hence at most two sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, where $t \in \{x, y\}$ and $i \in \{0, 1, ..., k-1\}$.

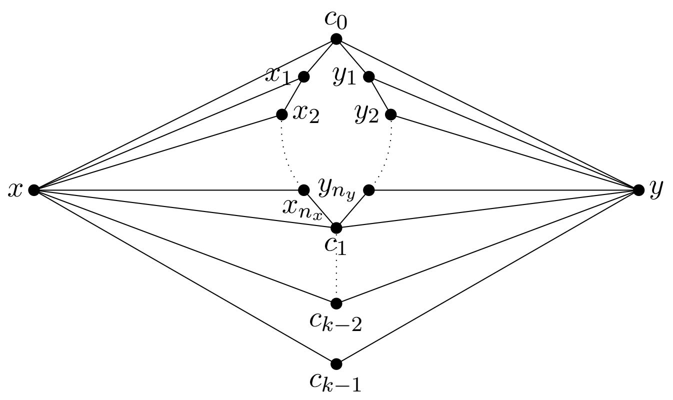


Figure A.4: The only non-empty region is R_1

Now assume that exactly two sets out of the sets $S_t \cap R_i$ are non-empty, where $t \in \{x, y\}$ and $i \in \{0, ..., k-1\}$, yet there are two non-empty regions. Without loss of generality assume that the two non-empty sets are $S_x \cap R_0$ and $S_y \cap R_1$. Triangulation will force x and y to have a common neighbor other than c_0 and c_1 for $|C| = 2$ which is a contradiction.

For $|C| = 3, 4, 5$ triangulation implies the edges $c_{k-1}c_0$ and c_0c_1. By Lemma A.2, we know that each vertex of S is adjacent to c_0. By equation (A.1) we have,

$$|S| \geq (16 - 2 - 5) = 9.$$

Hence, without loss of generality, we may assume $|S_x| \geq 4$. Then $\{c_0, x\}$ will be a dominating set with at least six common neighbors $S_x \cup \{c_{k-1}, c_1\}$ contradicting the maximality of D.

Hence we are done. $\square$

The lemma proved above was one of the key steps to prove the theorem. Now we will improve the lower bound of $|C|$.

Lemma A.6. $|C| \geq 6$.

Proof. For $|C| = 2, 3, 4, 5$ without loss of generality by Lemma A.5, we may assume R_1 to be the only non-empty region. Then triangulation will force the configuration depicted in Fig A.4 as a subgraph of G, where $C = \{c_o, ..., c_{k-1}\}$, $S_x = \{x_1, ..., x_{n_x}\}$ and $S_y = \{y_1, ..., y_{n_y}\}$. Without loss of generality we may assume,

$$|S_y| = n_y \geq n_x = |S_x|.$$

Then by equation (A.1) we have,

$$n_y = |S_y| \geq (16 - 2 - |C| - |S_x|) = 14 - |C| - |S_x|. \tag{A.2}$$

First of all assume $n_x = 0$. Then x is non-adjacent to y as otherwise y will dominate the whole graph. So we have the edges $c_0c_1, c_1c_2, ..., c_{k-1}c_0$ by triangulation. Then by equation A.2 we have,

$$|S_y| \geq 14 - 5 = 9.$$

Now to have $Ud(x, y_i) \leq 2$, every y_i must be connected to x by an unbalanced 2-path with internal vertex either c_0 or c_1. Hence at least four vertices of S_y must be adjacent to either c_0 or c_1. Note that c_0 is also adjacent to c_{k-1}, c_1 and that c_1 is also adjacent to c_0, c_2. So, the dominating set $\{c_0, y\}$ or $\{c_1, y\}$ will contradict the maximality of D. Hence $n_x \geq 1$.

Claim 1: $|C| = 5$ is not possible.

Proof of claim 1: Assume that $|C| = 5$. Then by equation A.2 we have,

$$|S_y| \geq 14 - 5 - n_x = 9 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 4$. Now every vertex of S_y disagree with c_3 on y. They also must disagree with y on c_2 as otherwise all of them will be connected to c_2 by unbalanced 2-paths with internal vertex c_1 and imply $Ud(y_1, y_4) > 2$. For similar reason, the vertices of S_y must disagree with c_4 on y.

Moreover, the edge c_0c_1 does not exist because it will force each vertex of S_y to be connected to vertices of S_x by unbalanced 2-paths with internal vertex either c_0 or c_1. In fact, for $n_x \geq 2$, as not all vertices of S_x can be adjacent to both c_0 and c_1, every vertex of S_y will be connected to the vertices of S_x by unbalanced 2-paths with internal vertex being exactly one of c_0, c_1 implying $Ud(y_1, y_4) > 2$. For $n_x = 1$, as $n_y \geq 7$, at least four vertices of S_y will be connected to the vertices of S_x by unbalanced 2-paths with internal vertex being exactly one of c_0, c_1 implying $Ud(y_i, y_{i+3}) > 2$ for some $i \in \{1, 2, ..., n_y\}$. Hence the edge c_0c_1 does not exist.

Also, if we have the edge y_1y_4 and without loss of generality assume the edge y_1y_3 by triangulation, then every vertex of S_x must be connected to y_2 by unbalanced 2-paths with internal vertex y_1. In this case $\{y_1, y\}$ is a dominating set with at least n_y common neighbors (c_0 and $n_y - 1$ common neighbors from S_y). Hence, to avoid contradicting the maximality of D, we must have $n_y \leq 5$. Then we must also have $n_x \geq 3$. But then, as every vertex of S_x agree on c_0 and on x (as they all disagree with c_3 on x), they must disagree with c_1, c_2 and c_4 to have unbalanced distance at most 2 with them. Also the vertices of S_y must disagree on c_1 to have unbalanced distance at most 2 with it. Hence the vertex c_4 and c_1 agree with each other on x and y. Hence we have $Ud(c_4, c_1) > 2$ as the edge c_0c_1 does not exist. This is a contradiction. Hence we do not have the edge y_1y_4.

Therefore, y_1 and y_4 must be connected by an unbalanced 2-path with an internal vertex x_j from S_x for some $j \in \{1, 2, .., n_x\}$. As we cannot have the edge y_1y_4, this will imply that every vertex of $S \setminus \{x_j\}$ will be adjacent to x_j to be at unbalanced distance at most 2 from each other. Then we can arrive to a contradiction exactly like the case described in the paragraph above.

This proves the claim. $\Diamond$

Claim 2: $|C| = 4$ is not possible.

Proof of claim 2: Assume that $|C| = 4$. Then by equation A.2 we have,

$$|S_y| \geq 14 - 4 - n_x = 10 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 5$.

Now we will show that every vertex of S_y disagree with c_2 and c_3 on y. First note that no vertex can agree with both c_2 and c_3 on y as otherwise it must be adjacent to both c_0 and c_1 which is impossible as $n_y \geq 5$. So, basically, if the claim is not true, then some vertices of S_y will agree with c_2 on y and the other vertices of S_y will agree with c_3 on y.

Also at most three vertices of S_y can agree with c_2 (or c_3) on y. So, $n_y \leq 6$. Hence, $n_x \geq 3$.

Now, three vertices agree on, say, c_2, then they will all disagree with c_2 on c_1 and every vertex (there are at least three such vertices) of S_x will disagree with those three vertices on c_1. Then, to have unbalanced distance at most 2 with the vertices of S_x, the other vertices (there are at least two such vertices) of S_y should be adjacent to c_1 which is not possible as they are already connected to c_3 with unbalanced 2-paths with internal vertex c_0.

The rest of the proof is similar to the proof Claim 1. Using similar arguments it is possible to show that the edge c_0c_1 does not exist, the edge y_1y_4 does not exist and it is not possible to have an unbalanced 2-path with internal vertex from S_x connecting y_1 and y_4. $\Diamond$

Proof of claim 3: Assume that $|C| = 3$. Then by equation A.2 we have,

$$|S_y| \geq 14 - 3 - n_x = 11 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 5$.

First note that it is not possible to have the edge c_0c_1 as this will force some three vertices of S_y to be connected to vertices of S_x by unbalanced 2-paths with internal vertex c_0 (or c_1) making $\{c_0, y\}$ (or $\{c_1, y\}$) a dominating set that contradicts the maximality of D.

For $n_x \geq 7$, there are at least 4 vertices in S_y that agree with each other on y. We need to have unbalanced distance at most 2 between them. Let those four vertices be y_i, y_j, y_k, y_l with $i > j > k > l$.

Now assume we have the edge y_iy_l. Then every vertex of S_x will be adjacent to either y_i or y_l. Without loss of generality assume that every vertex of S_x is adjacent to y_i. But then $\{y_i, y\}$ will be a dominating set with at least 4 common neighbors contradicting the maximality of D. Hence $n_y \leq 6$. Therefore we must have $n_x \geq 4$.

For $n_y = 5, 6$, one can show that these cases are not possible without creating a dominating set that contradicts the maximality of D. If one just tries to have unbalanced distance at most 2 between the vertices of S, the proof will follow. The proof of this part is also similar to the ones done before and, though a bit tedious, is not difficult to check. $\Diamond$

Proof of claim 4: Assume that $|C| = 2$. Then by equation A.2 we have,

$$|S_y| \geq 14 - 2 - n_x = 12 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 6$.

This is actually the easiest of the four claims. The case $n_y \geq 7$ can be argued as in the previous proof. For $n_y = 6$, we must have $n_x \geq 5$. If one just tries to have unbalanced distance at most 2 between the vertices of S, the proof will follow. The proof of this part is also similar to the ones done beforeand, though a bit tedious, is not difficult to check. $\Diamond$

This completes the proof of the lemma. $\square$

So, now we have proved that the value of $|C|$ is at least 6. This is an answer to our question "how small $|C|$ can be?". Now we will ask the question "How big $|C|$ can be?" and try to provide upper bounds for the value of $|C|$. The following lemma will help us to do so.

Lemma A.7. *If $|C| \geq 6$, then the following holds:*

(a) $|C^{\alpha\beta}| \leq 3$, $|C_t^{\alpha}| \leq 6$, $|C| \leq 12$. Moreover, if $|C^{\alpha\beta}| = 3$, then $(G)[C^{\alpha\beta}]$ is an unbalanced 2-path.

(b) $|C_t^{\alpha}| \geq 5$ (respectively $4, 3, 2, 1, 0$) implies $|S_t^{\alpha}| \leq 0$ (respectively $1, 3, 4, 5, 6$).

Proof. **(a)** If $|C^{\alpha\beta}| \geq 4$, then there will be two vertices $u, v \in C^{\alpha\beta}$ with $d(u, v) > 2$ which is a contradiction. Hence we have the first inequality which implies the other two.

Also if $|C^{\alpha\beta}| = 3$, then the only way to connect the two non-adjacent vertices u, v of $C^{\alpha\beta}$ is to connected them with an unbalanced 2-path through the other vertex (other than u, v) of $C^{\alpha\beta}$.

(b) Lemma A.1(b) implies that if all elements of C_t^{α} do not belong to the set of four boundary points of any three consecutive regions (like R, R^1, R^2 in Lemma A.2), then $|S_t^{\alpha}| = 0$. Hence we have $|C_t^{\alpha}| \geq 5$ implies $|S_t^{\alpha}| \leq 0$.

By Lemma A.2, if all the elements of C_t^{α} belong to the set of four boundary points $c^1, c^2, \overline{c^1}, \overline{c^2}$ of three consecutive regions R, R^1, R^2 (like in Lemma A.2) and contains both $\overline{c^1}, \overline{c^2}$, then $|S_t^{\alpha}| \leq 1$. Also $S_t^{\alpha} \subseteq R$ by Lemma A.2. Hence we have,

$$|C_t^{\alpha}| \geq 4 \text{ implies } |S_t^{\alpha}| \leq 1.$$

Now assume that all the elements of C_t^{α} belongs to the set of three boundary points $c^1, c^2, \overline{c^1}$ of two adjacent regions R, R^1 (like in Lemma A.2) and contains both $\overline{c^1}, c^2$. Then by Lemma A.1, $v \in S_t^{\alpha}$ implies v is in R or R^1.

Now if both $S_t^{\alpha} \cap R$ and $S_t^{\alpha} \cap R^1$ are non-empty, then each vertex of $(S_t^{\alpha} \cap R) \cup \{c^2\}$ disagrees with each vertex of $(S_t^{\alpha} \cap R^1) \cup \{\overline{c^1}\}$ on c^1 (by Lemma A.2).

Hence by Lemma A.3 we have,

$$|(S_t^\alpha \cap R) \cup \{\overline{c^1}\}|, |(S_t^\alpha \cap R^1) \cup \{c^2\}| \leq 3.$$

This clearly implies,

$$|S_t^\alpha \cap R|, |S_t^\alpha \cap R^1| \leq 2 \text{ and } |S_t^\alpha| \leq 4.$$

Now suppose we have $|S_t^\alpha| = 4$ and hence also $|S_t^\alpha \cap R|, |S_t^\alpha \cap R^1| = 2$. Then $S_{t'} = \emptyset$ as the only way for a vertex of $S_{t'}$ to have unbalanced distance at most 2 with every vertex of S_t is by being connected by an unbalanced 2-path with internal vertex c_1, which is impossible as the vertices of $S_t^\alpha \cap R$ disagree with the vertices $S_t^\alpha \cap R^1$ on c_1.

In fact, for the same reason, it is impossible to have unbalanced distance at most 2 between all the vertices of S_t and t' unless we have the edge tt' (that is the edge xy). But then the edge tt' makes t a vertex that dominates the whole graph contradicting the domination number of the graph being 2. Therefore, it is not possible to have $|S_t^\alpha| = 4$. Hence we have $|S_t^\alpha| = 3$ in this case.

Also if one of $S_t^\alpha \cap R$ and $S_t^\alpha \cap R^1$ is empty then we must have $|S_t^\alpha| \leq 3$ by Lemma A.2 and A.3.

Hence we have

$$|C_t^\alpha| \geq 3 \text{ implies } |S_t^\alpha| \leq 3.$$

Let $R, R^1, R^2, c^1, c^2, \overline{c^1}, \overline{c^2}$ be like in Lemma A.2 and assume $C_t^\alpha = \{c^1, c^2\}$. By Lemma A.1, $v \in S_t^\alpha$ implies v is in R, R^1 or R^2 and also that both $S_t^\alpha \cap R^1$ and $S_t^\alpha \cap R^2$ can not be non-empty. Hence, without loss of generality, assume $S_t^\alpha \cap R^2 = \emptyset$.

Then by Lemma A.2, vertices of $S_t^\alpha \cap R^1$ disagree with vertices of $(S_t^\alpha \cap R) \cup \{c^2\}$ on c^1. Hence by Lemma A.3 we have,

$$|S_t^\alpha \cap R^1|, |(S_t^\alpha \cap R) \cup \{c^2\}| \leq 3.$$

This implies $|S_t^\alpha| \leq 5$.

Now if $S_t^\alpha \cap R^1 = \emptyset$, then we have $S_t^\alpha = S_t^\alpha \cap R$. Let $|S_t^\alpha \cap R| \geq 6$. Now consider the induced graph $(O) = (G)[(S \cap R) \cup \{c^1, c^2\}]$. In this graph the vertices of $(S_t^\alpha \cap R) \cup \{c^1, c^2\}$ are pairwise at unbalanced distance at most 2. Hence $\omega_{rs}((O)) \geq 8$. But this is a contradiction as (O) is an outerplanar graph and every outerplanar graph has a signified relative clique number at most 7 (see Chapter 5, Section 5.3, Theorem 5.16(a) for details). Hence,

$$|C_t^\alpha| \geq 2 \text{ implies } |S_t^\alpha| \leq 5.$$

Now suppose we have $|S_t^\alpha| = 5$. Then we must have $S_{t'} = \emptyset$ as otherwise it is not possible to have unbalanced distance at most 2 between the vertices of S.

We also do not have the edge xy as it will contradict the domination number of the graph being 2 (t will dominate the graph). So, by triangulation we have the edges c^1c^2 and $c^{\overline{1}}c^1$. So, each vertex of S_t must be connected to t' with an unbalanced 2-path with internal vertices from $\{c^{\overline{1}}, c^1, c^2\}$. But then it will not be possible to have unbalanced distance at most 2 between the five vertices of S_t.

Hence,

$$|C_t^\alpha| \geq 2 \text{ implies } |S_t^\alpha| \leq 4.$$

In general S_t^α is contained in two distinct adjacent regions by Lemma A.1. Without loss of generality assume $S_t^\alpha \subseteq R_1 \cup R_2$. If both $S_t^\alpha \cap R_1$ and $S_t^\alpha \cap R_2$ are non-empty, then by Lemma A.2 we know that vertices of $S_t^\alpha \cap R_1$ disagree with vertices of $S_t^\alpha \cap R_2$ on c_1. Hence $|S_t^\alpha \cap R_1|, |S_t^\alpha \cap R_2| \leq 3$ which implies $|S_t^\alpha| \leq 6$.

Now assume only one of the two sets $S_t^\alpha \cap R_1$ and $S_t^\alpha \cap R_2$ is non-empty. Without loss of generality assume $S_t^\alpha \cap R_1 \neq \emptyset$. If $c_0, c_1 \notin C_t^\alpha$ and $|C_t^\alpha| = 1$, then we have $|S_t^\alpha \cap R_1| \leq 3$ by Lemma A.2 and A.3. In the induced outerplanar graph $(O) = (G)[(S \cap R_1) \cup \{c_1, c_2\}]$ vertices of $S_t^\alpha \cup (c_t^\alpha \cap \{c_1, c_2\})$ are pairwise at unbalanced distance at most 2.

Hence $7 \geq \chi_s((O)) \geq |S_t^\alpha \cup (c_t^\alpha \cap \{c_1, c_2\})|$. Therefore,

$$|C_t^\alpha| \geq 1 \text{ (respectively 0) implies } |S_t^\alpha| \leq 6 \text{ (respectively 7)}.$$

Now, when both the equalities hold, we must have $S_{t'} = \emptyset$ as otherwise $C_t^\alpha \cup S_t \cup S_{t'}$ will contain an signified outerplanar graph with signified chromatic number at least 8, which is not possible, in order to have all the vertices of S at unbalanced distance at most 2.

Now, $S_{t'} = \emptyset$ will imply that the edge xy is not there as otherwise t will dominate the whole graph. Hence, each vertex of S_t must be connected to t' by an unbalanced 2-path with internal vertex c_i for some $i \in \{0, 1, 2\}$. But this will force $|S_t| \leq 5$ as otherwise the vertices of S_t will no longer be at unbalanced distance at most 2 from each other.

Hence,

$$|C_t^\alpha| \geq 1 \text{ (respectively 0) implies } |S_t^\alpha| \leq 5 \text{ (respectively 6)}.$$

Hence we are done. □

Now we will prove that the value of $|C|$ can be at most 5 which contradicts our previously proven lower bound of $|C|$.

Lemma A.8. $|C| \leq 5$.

Proof. Without loss of generality we can suppose $|C_x^\alpha| \geq |C_y^\beta| \geq |C_y^{\overline{\beta}}| \geq |C_x^{\overline{\alpha}}|$ (the last inequality is forced). We know that $|C| \leq 12$ and $|C_x^\alpha| \leq 6$ (Lemma A.7(a)). So it is enough to show that $|S| \leq 13 - |C|$ for all possible values of $(|C|, |C_x^\alpha|, |C_y^\beta|)$ since it contradicts (A.1).

For $(|C|, |C_x^\alpha|, |C_y^\beta|) = (12, 6, 6)$, $(11, 6, 6)$, $(10, 6, 6)$, $(10, 6, 5)$, $(10, 5, 5)$, $(9, 5, 5)$, $(8, 4, 4)$ we have $|S| \leq 13 - |C|$ using Lemma A.7(b).

For $(|C|, |C_x^\alpha|, |C_y^\beta|) = (8, 6, 6)$, $(7, 6, 6)$, $(7, 6, 5)$,$(6, 6, 6)$, $(6, 6, 5)$, $(6, 6, 4)$, $(6, 5, 5)$ we are forced to have,

$$|C^{\alpha\beta}| > 3.$$

This is a contradiction by Lemma A.7(a).

So, $(|C|, |C_x^\alpha|, |C_y^\beta|) \neq (12, 6, 6)$, $(11, 6, 6)$, $(10, 6, 6)$, $(10, 6, 5)$, $(10, 5, 5)$, $(9, 5, 5)$, $(8, 4, 4)$, $(8, 6, 6)$, $(7, 6, 6)$, $(7, 6, 5)$,$(6, 6, 6)$, $(6, 6, 5)$, $(6, 6, 4)$, $(6, 5, 5)$.

We will be done if we prove that $(|C|, |C_x^\alpha|, |C_y^\beta|)$ cannot take the other possible values also. That leaves us checking a lot of cases. We will check just a few cases and observe that the other cases can be checked using similar logic.

Case 1: Assume $(|C|, |C_x^\alpha|, |C_y^\beta|) = (9, 6, 6)$.

Then we are forced to have, $|C^{\alpha\beta}| = |C^{\alpha\overline{\beta}}| = |C^{\overline{\alpha}\beta}| = 3$ in order to satisfy the first inequality of Lemma A.7(a). So $(G)[C^{\alpha\beta}]$, $(G)[C^{\alpha\overline{\beta}}]$ and $(G)[C^{\overline{\alpha}\beta}]$ are unbalanced 2-paths by Lemma A.7(a). Without loss of generality we can assume $C^{\alpha\overline{\beta}} = \{c_0, c_1, c_2\}$ and $C^{\overline{\alpha}\beta} = \{c_3, c_4, c_5\}$. Hence by Lemma A.1 we have $u \in R_1 \cup R_2$ and $v \in R_4 \cup R_5$ for any $(u, v) \in S_y^{\overline{\beta}} \times S_x^{\overline{\alpha}}$. Hence by Lemma A.1, either $S_y^{\overline{\beta}}$ or $S_x^{\overline{\alpha}}$ is empty. Without loss of generality assume $S_y^{\overline{\beta}} = \emptyset$. Therefore we have, $|S| = |S_x| = |S_x^{\overline{\alpha}}| \leq 3$ (by Lemma A.7(b)). So this case is not possible.

Case 2: Assume $(|C|, |C_x^{\alpha}|, |C_y^{\beta}|) = (7, 6, 4)$.

So, without loss of generality, we can assume that $(G)[C^{\alpha\beta}]$ and $(G)[C^{\alpha\overline{\beta}}]$ are unbalanced 2-paths and, $C^{\alpha\beta} = \{c_0, c_1, c_2\}$, $C^{\alpha\overline{\beta}} = \{c_3, c_4, c_5\}$ and $C^{\overline{\alpha}\beta} = \{c_6\}$.

By Lemma A.7 we have $|S_x| \leq 5$ and $|S_y| \leq 3 + 1 = 4$. So we are done if either $S_x = \emptyset$ or $S_y = \emptyset$.

So assume both S_x and S_y are non-empty. First assume that $S_y^{\beta} \neq \emptyset$. Then by Lemma A.1 we have $S_y^{\beta} \subseteq R_5$, $S_x^{\overline{\alpha}} \subseteq R_5 \cup R_6$ and hence $S_y^{\overline{\beta}} = \emptyset$. By Lemma A.2, vertices of S_y^{β} and vertices of $S_x^{\overline{\alpha}} \cap R_5$ must disagree with c_6 on c_5 while disagreeing with each other on c_5, which is not possible. Hence, $S_x^{\overline{\alpha}} \cap R_5 = \emptyset$. Also $|S_x^{\overline{\alpha}} \cap R_6| \leq 3$ as they all disagree on c_5 with the vertex of S_y^{β}. So $|S| \leq 4$ when $S_y^{\beta} \neq \emptyset$.

Now assume $S_y^{\beta} = \emptyset$ hence $S_y^{\overline{\beta}} \neq \emptyset$. Then by Lemma A.1 we have $S_y^{\overline{\beta}} \subseteq R_1 \cup R_2$, $S_x^{\overline{\alpha}} \subseteq R_0 \cup R_1$ and hence $S_y^{\beta} = \emptyset$. Assume $S_y^{\overline{\beta}} \cap R_2 = \emptyset$ as otherwise vertices of $S_x^{\overline{\alpha}}$ will be adjacent to both c_0 and c_1 (to be connected to c_6 and vertices of $S_y^{\overline{\beta}} \cap R_2$ by an unbalanced 2-path) implying $|S_x^{\overline{\alpha}}| \leq 1$ implying $|S| \leq 5$. If $S_x^{\overline{\alpha}} \cap R_0 \neq \emptyset$, then $|S_y^{\overline{\beta}} \cap R_1| = 1$, $|S_y^{\overline{\alpha}} \cap R_1| \leq 1$ and $|S_y^{\overline{\alpha}} \cap R_0| \leq 3$ by Lemma A.2 and hence $|S| \leq 5$. If $S_x^{\overline{\alpha}} \cap R_0 = \emptyset$ then we have $|S_y^{\overline{\beta}} \cap R_1| \leq 2$, $|S_y^{\overline{\alpha}} \cap R_1| \leq 3$ and hence $|S| \leq 5$. So this case is not possible.

In a similar way one can handle the other cases. □

This proves Lemma 5.20. □

A.3 Proof of Theorem 6.31

(a) The lower bound follows from the fact that the unbalanced 4-cycle is an [s]-clique.

Now we prove the upper bound.

Assume that $[G, \Sigma]$ is a signed outerplanar graph of minimum order with $\omega_{[rs]}([G, \Sigma]) > 4$. Moreover, assume $[G, \Sigma]$ is such that if we delete any edge of $[G, \Sigma]$, it will no longer have signed relative clique number greater than 4.

Let R be a signed relative clique of maximum order in $[G, \Sigma]$ and let $S = V(G) \setminus R$. Note that S induces an independent set of $[G, \Sigma]$ as deleting any edge between two vertices of S will not decrease the signed relative clique number of the graph $[G, \Sigma]$.

First note that, for any $z \in S$, we have $d(z) \geq 2$ as otherwise the vertex z does not help connecting any two (or more) vertices of R by an unbalanced 4-cycle and hence can be deleted to get a signed planar graph with relative signed chromatic number equal to that of $[G, \Sigma]$ but with order less than $[G, \Sigma]$, which contradicts the minimality of $[G, \Sigma]$.

In fact, any $z \in S$ with $d(z) = 2$ must be the internal vertex of a 2-path that connects two vertices of R. But, we can replace that 2-path by an edge and obtain another signed outerplanar graph to contradict the minimality of $[G, \Sigma]$. Hence, $d(z) \geq 3$ for all $z \in S$.

As $[G, \Sigma]$ is an outerplanar graph, there exists a vertex $x \in V(G)$ with $d(x) \leq 2$. By the above discussion we know that $x \in R$. Clearly $d(x) = 2$ as otherwise $|R| \leq 4$.

Assume that $N(x) = \{w, z\}$. Now as $|R \setminus \{x, w, z\}| \geq 2$, without loss of generality, we can assume that at least two vertices of R are connected to x by unbalanced 4-cycles. Hence at least three vertices of R, including x, must be connected to both w and z. It is easy to note that it is not possible to obtain this keeping the graph outerplanar. So, this is a contradiction.

(b) As any two non-adjacent vertices of a signed relative clique must be part of an unbalanced 4-cycle, it is not possible to have a outerplanar signed relative clique with girth at least 5 of order more than 2. Hence the upper bound.

An edge has signed relative clique number 2. Hence the lower bound. □

A.4 Proof of Theorem 6.32

(a) The upper bound follows from Theorem 6.23(a).

Consider the graph obtained by deleting the vertices x_4, x_5, y_4, y_5 from the signified graph (B_0) (see Fig. 6.6). Notice that the new graph is a presentation of an [s]-clique of order 8. Hence the lower bound.

(b) Let $[G, \Sigma]$ be a triangle-free planar signed graph of minimum order with $\omega_{[rs]}([G, \Sigma]) > 17$. Let R be a signed relative clique of maximum order of $[G, \Sigma]$ and let $S = V(G) \setminus R$.

Claim 1: For any $v \in V(G)$ we have, $|N(v) \cap R| \leq 4$.

Proof of Claim 1: Resign the neighbors of v in such a way that we obtain a presentation of $[G, \Sigma]$ with $N(v) = N^+(v)$. Now, as $[G, \Sigma]$ is triangle-free, each pair of neighbors of v must be part of an unbalanced 4-cycle. For a pair of neighbors of v to be part of an unbalanced 4-cycle, we must have an unbalanced 2-path connecting them (note that they are already connected by a 2-path, with internal vertex v, which is not unbalanced). Now by the proof of Claim 1 of the proof of Theorem 5.17(b) (see Chapter A, Section A.1), we know this is not possible. Hence, our claim is proved. $\Diamond$

Now note that for any $z \in S$, we have $d(z) \geq 2$ as otherwise z can be deleted to get a signed triange-free planar graph whose signed relative chromatic number is equal to that of $[G]$ but with order less than $[G]$, which contradicts the minimality of $[G]$. Now, a vertex z of S must connect at least two vertices of R by a 2-path with the internal vertex being itself (that is, z).

Now for each vertex $z \in S$ with $d(z) \leq 5$, assume that the neighbors of z are $v_1, v_2, ..., v_k$. Fix a planar embedding of G and assume that the neighbors of z are arranged in a clockwise order around z. Now delete the vertex z and add the edges v_1v_2, v_2v_3, ..., $v_{k-1}v_k$, v_kv1 (for $d(z) = 2$ add only one edge v_1v_2) to obtain a new graph.

Note that this new graph is also planar (may not be triangle-free) and the degree of each vertex in the new graph, which were in G also, is at least as much as the degree of the vertex in G. Hence, there is a vertex v in the new graph, which belongs to R, with degree at most 5.

As each vertex from $R \setminus N(v) \cup \{v\}$ is adjacent to at least two neighbors of v for being part of the same unbalanced 4-cycle with v, there will be a neighbor u of v and at least five neighbors from $R \setminus N(v) \cup \{v\}$ which is a contradiction to Claim 1. So, we can conclude that $\omega_{[rs]}([G, \Sigma]) \leq 17$.

(c) As any two non-adjacent vertices of a signed relative clique must be part of an unbalanced 4-cycle, the vertices of a signed relave clique in a planar graph with girth at least 5 must be all adjacent to each other. So, it is not possible to have a planar signed relative clique with girth at least 5 of order more than 2. Hence the upper bound.

An edge has signed relative clique number 2. Hence the lower bound. $\square$

A.5 Proof of Lemma 6.40

Let a, b, c and d be the original vertices of the K_4 from which $[G, \Sigma]$ is constructed. Let t_{uv} be the length of the path joining u and v in $[G, \Sigma]$ for $u, v \in \{a, b, c, d\}$.

Then we have

$$\begin{aligned} t_{ab} + t_{bc} + t_{ca} &= t_{ab} + t_{bd} + t_{ad} \\ &= t_{ac} + t_{cd} + t_{ad} \\ &= t_{bc} + t_{cd} + t_{bd} \\ &= 2k \end{aligned} \tag{A.3}$$

From equation A.3 we have

$$t_{uv} = t_{wx} \text{ for } \{u, v, w, x\} = \{a, b, c, d\}. \tag{A.4}$$

Let u and v be a pair of vertices of $[G, \Sigma]$. If u and v are connected by an odd path then we are done. Hence assume that all the paths connecting u and v are even for the rest of this proof.

If u and v are both vertices of a particular facial cycle of $[G, \Sigma]$, then there are two paths of even length at most $2k$ connecting u, v such that one of them has an odd number of negative edges and the other one has an even number of negative edges.

If there is no facial cycle of $[G, \Sigma]$ containing both u and v, then we may assume, without loss of generality, that u is a vertex of the path obtained by subdividing ab and v is a vertex of the path obtained by subdividing cd.

Suppose that u has partitioned the path obtained by subdividing ab to paths of length t'_{ab} (length of the path connecting a and u) and t''_{ab} (length of the path connecting u and b) and that v has partitioned the path obtained by subdividing cd into paths of length t'_{cd} (length of the path connecting c and v) and t''_{cd} (length of the path connecting v and d).

Note that by equation 6.3 we have

$$\begin{aligned} t'_{ab} + t''_{ab} &= t'_{cd} + t''_{cd} \\ &= t_{ab} = t_{cd}. \end{aligned} \tag{A.5}$$

If $t_{ab} = t_{cd}$ is even, then t'_{ab} and t''_{ab} have the same parity and t'_{cd} and t''_{cd} have the same parity. Moreover, this will imply that t_{ac} and t_{ad} have the same parity.

On the other hand, if $t_{ab} = t_{cd}$ is odd, then t'_{ab} and t''_{ab} have different parities and t'_{cd} and t''_{cd} have different parities. Moreover, this will imply that t_{ac} and t_{ad} have different parities.

Now the path connecting u, v that contains the vertices a, c is of length $t'_{ab} + t_{ac} + t'_{cd}$ and the path connecting u, v that contains the vertices b, d is of length $t''_{ab} + t_{bd} + t''_{cd}$. Clearly, $t'_{ab} + t_{ac} + t'_{cd}$ and $t''_{ab} + t_{bd} + t''_{cd}$ have the same parity irrespective of the parity of $t_{ab} = t_{cd}$. Now note that

$$\begin{aligned} t'_{ab} + t_{ac} + t'_{cd} + t''_{ab} + t_{bd} + t''_{cd} &= t_{ab} + t_{ac} + t_{cd} + t_{bd} \text{ (by equation A.5)} \\ &= 2(t_{ab} + t_{ac}) \text{ (by equation 6.3)} \\ &= 2(t_{ab} + t_{ac} + t_{bc}) - 2t_{bc} \\ &= 4k - 2t_{bc} \text{ (by equation 6.2)} \\ &\leq 4k - 2 \text{ (as } t_{bc} \geq 1) \end{aligned}$$

Hence we have $min\{(t'_{ab} + t_{ac} + t'_{cd}), (t'_{ab} + t_{bd} + t'_{cd})\} \leq (2k - 2)$. Therefore, there is an even path of length atmost $(2k - 2)$ connecting u and v. Similarly, we can show that $min\{(t'_{ab} + t_{ad} + t''_{cd}), (t''_{ab} + t_{bc} + t'_{cd})\} \leq 2k$.

Let the number of negative edges contained in the path connecting u, v that contains the vertices a, c be neg_{ac}. Similarly define neg_{bc}, neg_{ad} and neg_{bd}. Note that the parity of neg_{ac} and neg_{bd} is different from the parity of neg_{ad} and neg_{bc}.

Therefore, there are two even paths, one with even number of negative edges and the other with odd number of negative edges, of length at most $2k - 2$ connecting u and v. □

A.6 Proof of Theorem 6.41

Consider a K_4 on four vertices a, b, c and d. Let G_1 be a subdivision of this K_4 where edges ab and cd each are subdivided into $2k - 2$ edges. Let (G_1, Σ_1) be the signified graph with the new edge incedent to a (created by the subdivision of ab) and the new edge incedent to c (created by the subdivision of cd) being negative. Thus the signed graph $[G_1, \Sigma_1]$ is a subdivision of K_4 in which all the four faces are cycles of length $2k$. Hence by Lemma 6.40 we have

$$\omega([G_1]^{(2k-2)}) = |V(G_1)| = 4k - 2.$$

In the following we will build a sequence of signified graphs (G_i, Σ_i), for $i \in \{1, 2, \cdots, 2k-2\}$, such that each (G_{i+1}, Σ_{i+1}) contains (G_i, Σ_i) as a subgraph, the signed graph $[G_{i+1}, \Sigma_{i+1}]$ is bipartite planar and of unbalanced-girth $2k$ and such that $\omega([G_{i+1}]^{(2k-2)}) > \omega([G_i]^{(2k-2)})$. At the final step we will have,

$$\omega([G_{2k-2}]^{(2k-2)}) \geq 2^{2k-1}.$$

We start with the following partial construction. Suppose (G_i, Σ_i) is built and let $P = uv_1v_2\cdots v_rw$ be a maximal thread, that is, a path P connecting u and w such that all v_j's are of degree 2 in G_i but u and w are of degree at least 3, of (G_i, Σ_i) for $j \in \{1, 2, ..., r\}$. Furthermore, assume that P is either part of a path of length $2k-2$ connecting a and b or part of a path of length $2k-2$ connecting c and d.

Since P is a thread, if we add the new edge uw in (G_i), the resulting graph will still be planar. So we add such an edge and subdivide it r times to obtain the new thread $P' = uv'_1v'_2\cdots v'_rw$. Also we assign signs of the new edges in a way such that the edges uv'_1 and v_rw has the same sign, the edges v'_rw and uv_1 has the same sign and the edges $v'_iv'_{i+1}$ and $v_{r-i+1}v_{r-i}$ has the same sign.

Consider a planar drawing of the graph in which P and P' form a facial cycle of length $2r$. In the face PP' connect v_1 and v'_r by a new edge. Subdivide this new edge $2k-r-2$ times (that is, into $2k-r-1$ edges), so that each of the facial cycles containing the new thread is of length $2k$. Choose signs of the edges of this new path in such a way that each of the facial cycles containing the new thread is unbalanced.

Let (G'_i, Σ'_i) be the resulting signified graph. We first note that G'_i is also of unbalanced-girth $2k$. Now suppose a maximal clique W of $[G_i]^{(2k-2)}$ contains v_j of the thread P. Then we claim that $W \cup \{v'_j\}$ is also a clique of $[G'_i]^{(2k-2)}$.

To prove this let x be any vertex of W. If x is not in P, then consider a path of length r from v_j to x. Each time this path uses a part of P replace it with the corresponding part from P'. If r is odd, then we are done. If not, then there should be two even-paths P_1 and P_2, each of length at most $(2k-2)$, connecting x and v_j with P_1 having odd number of negative edges and P_2 having even number of negative edges. Each time P_1 or P_2 path uses a part of P, replace it with the corresponding part from P', to obtain even-paths P'_1 and P'_2, each of length at most $(2k-2)$, connecting x and v'_j. Note that according to our construction one of P'_1 and P'_2 has odd number of negative edges while the other has even number of negative edges.

If $x \in P$, then, without loss of generality, assume that P is part of a path of length $2k-2$ connecting a and b. Consider the subgraph induced by this path together with c, P' and the $v_1...v'_r$ thread we added to build G'_i. This induced subgraph is a subdivision of K_4 in which all the faces are unbalanced-cycles of length $2k$. Thus, by Lemma 6.40 we are done. In particular if all vertices of P are in W, then $W \cup \{v'_1, v'_2, \cdots, v'_r\}$ is a clique in $[G'_i]^{(2k-2)}$.

Now we describe our general construction. At first we have (G_1, Σ_1) on $4k-2$ vertices and two maximal threads. By Lemma 6.40 all the vertices of these two threads are parts of the unique clique of order $4k-2$ in $[G_1]^{(2k-1)}$. We apply the previously mentioned construction on both threads to build (G_2, Σ_2) which will have four maximal threads each of length $2k-3$ (we are only considering maximal threads that are part of a path of length $2k-2$ connecting a, b or c, d). There is a clique of order $(4k-2) + 2(2k-3)$ in $[G_2]^{(2k-2)}$, and there are four maximal threads of length $2k-3$, each a part of a path of length $2k-2$ either connecting a and b or c and d.

Continuing this construction, in general, there is a clique W_i of $[G_i]^{(2k-2)}$ $(2 \leq i \leq 2k-2)$ which is of order $(4k-2) + \sum_{j=1}^{i-1} 2^j(2k-j-2)$ and there are 2^i maximal threads of length $2k-i-1$ which are part of a path of length $2k-2$ connecting a and b or c and d.

Note that $[G_i, \Sigma_i]$ at each step is a planar bipartite signed graph of unbalanced-girth $2k$. The clique W_{2k-2} of $[G_{2k-2}, \Sigma_{2k-2}]^{(2k-1)}$ has order equal to

$$
\begin{aligned}
4k - 2 + \sum_{j=1}^{2k-2} 2^j(2k - j - 2) &= 4k - 2 + (2k - 2) \sum_{j=1}^{2k-2} 2^j - 2 \sum_{j=1}^{2k-2} j2^{j-1} \\
&= 4k - 2 + [(2k - 2)(2^{2k-1} - 2)] - \\
&\quad 2[(1 - 2^{2k-1}) - (-1)(2k - 1)2^{2k-2}] \\
&= 4k - 2 + [k2^{2k} - 4k - 2^{2k} + 4] - \\
&\quad [2 - 2^{2k} + k2^{2k} - 2^{2k-1}] \\
&= 2^{2k-1}.
\end{aligned}
$$

This completes the proof. $\square$