ETAT TOPOLOGIQUE DE L'ESPACE TEMPS A ECHELLE 0

Igor Bogdanoff

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THESE

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Igor BOGDANOFF

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ETAT TOPOLOGIQUE DE L'ESPACE-TEMPS A L'ECHELLE ZERO

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Gabriel SIMONOFF	Président
Roman JACKIW	Rapporteur
Jack MORAVA	Rapporteur
Hans JAUSLIN	Examinateur
Daniel STERNHEIMER	Examinateur
Jac VERBAARSCHOT	Examinateur

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- " KMS State of the Spacetime at the Planck Scale "
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- " Topological Origin of Inertia "
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INTRODUCTION

L'objectif de la présente recherche, dans le contexte de la supergravité N=2, consiste à proposer une solution au problème posé par l'existence et la nature de la Singularité Initiale de l'espace-temps propre au modèle cosmologique standard. L'une des insuffisances (sans doute la plus préoccupante) du modèle type "Big Bang" reste en effet son inaptitude à fournir une approche de l'origine singulière de l'univers. Associée à l'échelle zéro de l'espace-temps, la Singularité Initiale ne peut être décrite par la théorie physique (perturbative) en raison des divergences non renormalisables qui la caractérisent. En revanche, nous proposons ici, notamment dans l'article publié en réf[1] et ci-joint en annexe [A1], l'existence d'une solution dans le cadre d'une théorie duale, non perturbative, relevant de la *théorie topologique des champs* [2].

L'originalité de cette solution réside en ce qu'elle implique, en sus de l'état physique, l'existence possible, en deçà de l'échelle de Planck, d'un "état topologique" de la métrique du (pré)espace-temps. Un tel état s'inscrit logiquement dans le cadre de la théorie Euclidienne des champs proposée par J. Schwinger et appliquée il y a longtemps déjà par S.Hawking en cosmologie quantique (voir la réf[3] pour les travaux typiques de ces deux auteurs). Toutefois, l'interprétation purement "topologique" des contraintes propres à la gravité quantique résulte d'une série de résultats récents obtenus par G.Bogdanoff [3] et indiquant l'existence probable de "fluctuations quantiques" (ou q-superposition) de la signature de la métrique à l'échelle de Planck. En effet, il a été montré qu'à cette échelle (i.e. échelle de supergravité N = ...) la signature Lorentzienne de la métrique (+++–) ne devrait plus être considérée comme *fixe* mais est très probablement soumise à des *fluctuations quantiques* (+++±) jusqu'à la limite d'échelle zéro.En outre, il a été établi, toujours en réf [3], qu'en raison de contraintes algébriques tout autant que physiques, il est également probable qu'une telle fluctuation de signature ne peut intervenir qu'entre la forme Lorentzienne (3, 1) et la forme Euclidienne (4,0), à l'exclusion de la forme ultra-hyperbolique (2,2) (sur ce point, voir les chaps. 1, 3, 4 de la réf[3]). La théorie de fluctuation de la signature de la métrique a été développée ou appliquée de différentes façons depuis, notamment en réf.[4]. Ces développements récents conduisent dans tous les cas à des résultats sensiblement analogues à ceux de la réf.[3].

Du point de vue physique, cette notion de fluctuation quantique de la signature peut être vue comme une conséquence directe de la condition de KUBO-MARTIN-SCHWINGER (KMS) [5] à laquelle est très probablement soumis le système thermodynamique formé par l'espace-temps à l'échelle de Planck. Cette approche a été établie pour la première fois en réf[3] et développée par nous dans la réf[1, A1] déjà citée ainsi que, plus spécifiquement, dans les réf[6,7] ci-jointes en annexes [A2-A3]. Nos propositions formalisées dans les travaux cités indiquent en effet que, compte tenu des importants résultats de Dolan et Jackiw [8] puis de Weinberg [9] concernant le comportement thermique de l'univers primordial à haute température, il est raisonnable de considérer que le (pré)espace-temps se trouve en état d'équilibre thermique à l'échelle de Planck. Il est alors naturel d'en déduire qu'en tant que système thermodynamique, ce même (pré)espace-temps est soumis à la condition KMS à la même échelle. Sur la base de cette approche, nous établissons (notamment en termes d'algèbres d'opérateurs) qu'à l'échelle zéro, la signature de la métrique doit donc être considérée comme *Euclidienne*. Nous présentons nos principales propositions dans ce domaine en section I. Celles-ci sont exposées pour l'essentiel dans les articles [1-A1] et [6-A2, 7-A3] annexés à la présente thèse.

Par ailleurs, de manière évidente, la notion de fluctuation de signature de la métrique entre l'échelle de Planck (limite infrarouge de la théorie de fluctuation) et l'échelle zéro (limite ultraviolette) apparaît comme une conséquence naturelle de la *non-commutativité* de la géométrie de l'espace-temps à l'échelle quantique [10]. Dans un tel contexte a d'ailleurs été construit, en termes d'algèbres de Hopf et toujours en réf [3], le "produit bicroisé cocyclique"

$$U_q(so(4))^{op} \overset{\psi}{\triangleright\!\blacktriangleleft} U_q(so(3,1)) \qquad (1)$$

où $U_q(so(4))^{op}$ représente l'algèbre de Hopf (ou "groupe quantique") Euclidien et $U_q(so(3,1))$ le groupe quantique Lorentzien, le symbole $\triangleright\!\blacktriangleleft$ désignant un produit bicroisé et ψ un 2-cocycle de déformation (voir les réfs [11,12]) pour des développements plus précis). Le produit bicroisé (1) suggère alors un genre inattendu d'"unification" entre les algèbres de Hopf Lorentzienne et Euclidienne à l'échelle de Planck et induit la possibilité d'une "q-déformation" de la signature de la forme Lorentzienne (physique) à la forme Euclidienne (topologique) [3-13]. En outre, l'équ.(1) définit implicitement une transformation de (semi)dualité (au sens de Majid [12]) entre les groupes quantiques Lorentzien et Euclidien (cf. equ.(27)). Nous revenons sur cet important résultat au paragraphe (2.2).

Du point de vue des groupes classiques, la fluctuation de la signature de la forme Lorentzienne vers la forme Euclidienne peut être décrite par l'espace homogène symétrique construit en réf.[3]:

$$\sum_h = \frac{SO(3,1) \otimes SO(4)}{SO(3)} \qquad (2)$$

SO(3) étant plongé diagonalement dans le produit SO(3, 1) $\otimes$ SO(4). A partir de $\sum_h$, l'on peut construire l'espace topologique quotient $\sum_{top} = \dfrac{\mathbb{R}^{3,\,1} \oplus \mathbb{R}^4}{SO(3)}$, espace topologique séparé susceptible de décrire la possible *superposition* des deux métriques Lorentzienne et Riemanniennes. Il a été montré dans [3] que $\sum_{top}$ comporte un point singulier unique S pouvant correspondre à l'origine de l'espace de superposition .

Revenons à présent aux aspects physiques de la théorie de superposition. Comme nous l'indiquons au §(5.1) de la réf [1-A1], il devrait exister, à l'échelle de Planck, une limite à la température - et à la courbure - du (pré)espace-temps, limite postulée par Hagedorn, et précisée par Atick et Witten [14], au delà de laquelle l'on devrait considérer un secteur purement *topologique* de l'espace-temps, décrit par la théorie topologique des champs de Witten ou Donaldson. Le premier invariant de Donaldson est une forme algébrique "Riemannienne" dont nous suggérons au §(2.3) l'isomorphisme avec l'invariant topologique caractérisant, selon notre approche, la limite d'échelle 0 (cf. [1-A1]). A une telle échelle, la théorie ne devrait donc plus être considérée comme singulière mais devrait plutôt être redéfinie sous une nouvelle forme, Euclidienne et topologique. Cette approche repose sur deux idées essentielles :

((**i**) Conformément à certains résultats en théorie des (super)cordes, notamment ceux de E. Kiritsis et C. Kounnas dans [15], nous considérons l'hypothèse selon laquelle, à très haute courbure (i.e. à l'échelle de Planck $T \sim M_{Planck}$) la gravitation classique, décrite par l'approximation $O(1/\mathbf{M}_{Planck})$ n'est plus valable. Nous proposons donc d'introduire, dans le Lagrangien "quantique" de la théorie, des termes de dérivées supérieures en R^2 (tout en considérant, en dimension 4, la possibilité d'un "cut off" des termes de dérivées plus hautes sur la limite R^2, ce qui élimine les termes en $R^3 + \ldots + R^n$ de la théorie des cordes). Les détails de cette construction peuvent être examinés dans la réf[3].

(**ii**) Suite à nos résultats publiés en [1-A1] et [6-A2], nous conjecturons que ces termes peuvent autoriser la superposition (3, 1) $\leftrightarrow$ (4, 0) de la signature de la métrique dans le cadre d'une théorie élargissant la gravitation classique de type Einstein. A partir des indications du §(5.1) de [1-A1] selon lesquelles l'espace-temps à l'échelle de Planck devrait être vu comme soumis à la condition KMS, nous reprenons l'approche établie en réf[3] concernant l'existence de *deux* potentiels gravitationnels distincts. Nous conjecturons alors qu'en supergravité $R + R^2$ (et en N = 2), l'approximation linéarisée de la métrique de Schwartzschild peut être considérée comme une solution locale *exacte* de la théorie étendue. Nous rapprochons cette conjecture des résultats physiques obtenus en [3], selon lesquels la présence de termes non-linéaires R^2 dans le Lagrangien effectif de supergravité peut autoriser la superposition (3, 1) / (4, 0) de la signature de la métrique à partir de l'échelle de Planck

Au §(1) de la réf [1-A1], nous précisons le contenu du Lagrangien quadratique qui nous paraît le plus naturellement adapté aux conditions de très hautes courbures de la variété, lorsque l'échelle $ß << l_{Planck}$ (i.e. pour des échelles de longueur "inférieures" à la longueur de Planck). Notons qu'au sens strict, la notion "inférieur à la longueur de Planck" n'a plus de signification en termes de distance, en raison même de la perturbation portant sur la métrique Lorentzienne. Notre Lagrangien "étendu" est alors :

$$\mathbf{L}_{supergravit\acute{e}} = \hat{\beta} R + \frac{1}{g^2} R^2 + \alpha RR^* \qquad (3)$$

avec une composante physique Lorentzienne (le terme d'Einstein $\hat{\beta} R$) et une composante *topologique* Euclidienne (le terme topologique αRR^*). L'interpolation entre ces deux composantes, selon un mécanisme que nous suggérons ci-dessous, nous incite donc à considérer que $\mathbf{L}_{supergravit\acute{e}}$ décrit correctement les deux pôles (physique et topologique) d'une *même* théorie (la superposition) ainsi que les deux métriques associées.

Nous indiquons ainsi qu'à la limite d'échelle $ß = 0$, la théorie, de dimension D = 4, réduite à αRR^*, dominée par des instantons gravitationnels de dimension 0, peut être vue comme purement topologique. Dans ce secteur, la métrique est statique, définie positive Euclidienne (+ + + +). Le domaine de validité de l'évolution Euclidienne s'étend jusqu'à l'échelle de Planck $ß \sim l_{Planck}$. Au delà de l'échelle de Planck ($ß > l_{Planck}$), la théorie est de type Lorentzien et également de dimension D = 4. Enfin, dans le secteur de gravité quantique ($0 < ß < l_{Planck}$), la théorie, définie par la quantification du groupe de Lorentz, possède une dimension supplémentaire (D = 5), laquelle autorise la superposition des deux classes Lorentzienne et Euclidienne (ce qui induit une phase de "fluctuation" des signatures $(3, 1) \leftrightarrow (4, 0)$. La dynamique du (pré)espace-temps pourrait alors correspondre à l'expansion d'un monopôle gravitationnel de dimension 5 tandis que l'état de superposition quantique de la métrique peut être associée (après compactification de la quatrième coordonnée spatiale du monopôle D = 5) à une dualité monopôle-Instanton d'un genre nouveau en dimension 4 (sur ce point, voir encore la réf.[3]).

Enfin, lorsque $ß > l_{Planck}$, l'espace-temps entre dans la phase Lorentzienne conventionnelle de l'expansion cosmologique.

En fonction de ce qui précède, l'un des résultats les plus importants présentés en réf.[1-A1] est donc qu'à l'échelle zéro, la signature du (pré)espace-temps peut à nouveau être considérée comme fixe, mais sous une forme *Euclidienne* (++++). Ceci est important dans la mesure où la théorie Euclidienne peut être interprétée comme la plus simple des théories topologiques des champs. Nous présentons en section 2 nos résultats propres obtenus dans le cadre de la théorie topologique des champs. Ces résultats sont détaillés dans l'article [1-A1] joint en annexe. Une autre application possible de la théorie topologique en cosmologie est également proposée dans l'article [16-A4].

Nous suggérons alors ci-après que la Singularité Initiale de l'espace-temps correspond à un instanton gravitationnel singulier de taille zéro, caractérisé par une configuration Riemannienne de la métrique en dimension D=4. Dans cette perspective, le problème posé par la Singularité Initiale peut trouver une solution dans le cadre de la théorie topologique des champs. Plus précisément, nous suggérons que l'échelle singulière zéro peut être décrite en termes d'invariants topologiques (en particulier le premier invariant de Donaldson $\sum_i (-1)^{n_i}$). Nous introduisons ainsi un nouvel indice topologique, reliée à l'échelle zéro de l'espace-temps, de la forme

$$Z_{ß=0} = \mathrm{Tr}(-1)^{S} \qquad (4)$$

que nous appelons "Invariant de Singularité". Cette approche topologique de l'échelle zéro, fondée sur la théorie des instantons gravitationnels, comporte plusieurs conséquences intéressantes. Parmi celles-ci, il nous a paru pertinent de mettre en évidence l'existence possible, à l'échelle zéro de l'espace-temps, d'une "amplitude topologique" (reliée à la charge topologique de l'instanton gravitationnel singulier de taille zéro). Nous en tirons une conjecture inattendue, selon laquelle l’interaction inertielle, hors de portée de la théorie des champs, pourrait en revanche être correctement décrite dans le cadre de la théorie topologique des champs. Nous développons cette conjecture publiée en réf. [16-A4].

La présentation de notre recherche est organisée comme suit. En section I, nous rappelons nos résultats publiés en [5-A2, 6-A3] suggérant que le (pré)espace-temps, en équilibre thermique à l'échelle de Planck, est soumis à la condition KMS. En section 2, nous résumons nos principales démonstrations et exemples publiés principalement en [1-A1] et indiquant que la limite d'échelle zéro du (pré)espace-temps (dans le contexte de la supergravité N=2) peut être décrite par la théorie topologique des champs. Nous proposons alors une solution nouvelle au problème posé par la Singularité initiale de l'espace-temps dans le cadre de la théorie topologique des champs. Enfin, en section 3, nous présentons nos résultats publiés en réf[16], en particulier la conjecture (4.2) suggérant l'existence d'une amplitude topologique au voisinage de la Singularité Initiale de l'espace-temps. Ces résultats sont joints en annexe [A4].

CHAPITRE 1

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ETAT KMS DE l'ESPACE-TEMPS
A L'ECHELLE DE PLANCK

Nous fondons notre approche du (pré)espace-temps à l'échelle quantique sur l'une des conditions physiques les plus naturelles prédites par le Modèle Standard à l'échelle de Planck. En accord avec [7,8], et en particulier avec les récents résultats de Kounnas *et al* [17,18], nous prétendons dans la présente thèse qu'à l'échelle de Planck, le (pré)espace-temps, en tant que système thermodynamique, est en état d'*équilibre thermodynamique* [3]. Nous introduisons ce point de vue au §(5.1) de notre article publié en réf[1-A1]. Or selon des résultats plus spécifiques présentés en [6-A2, 7-A3], l'importante conséquence de cette approche est que le (pré)espace-temps à l'équilibre à l'échelle de Planck devrait donc être considéré comme soumis à la condition de Kubo-Martin-Schwinger (KMS) [5]. De manière inattendue, la théorie KMS et la théorie modulaire pourraient comporter des conséquences spectaculaires sur la physique à l'échelle de Planck. Ceci en raison des " effets KMS ". En effet, nous montrons dans les articles publiés en réf. [1-A1, 6-A2, 7-A3] qu'appliquées à l'espace-temps quantique, les propriétés KMS sont telles qu'à l'intérieur des limites de la " bande KMS (i.e. entre l'échelle zéro ß = 0 et l'échelle de Planck ß = ℓ_{Planck}), la direction genre temps du système devrait être considérée comme *complexe* : $t \mapsto \tau = t_r + it_i$. Dans ce cas, lorsque *ß* → 0, la théorie est projetée sur la limite purement imaginaire $t \mapsto \tau = it_i$ de la bande KMS.Inversement, sur la limite infrarouge ß ≥ ℓ_{Planck}, la direction genre temps devient purement réelle $t \mapsto \tau = t_r$ Ceci signifie qu'à l'intérieur des limites de la bande KMS, les métriques Lorentzienne et Euclidienne devraient être considérées en état de " superposition quantique " (ou couplées), ceci induisant une unification (ou couplage) entre l'état physique (Lorentzien) et l'état topologique (Euclidien) du (pré)espace-temps à l'échelle de Planck.

Commençons par rappeler les principaux résultats concernant le possible équilibre thermique du (pré)espace-temps à l'échelle de Planck.

1.1 Equilibre thermique du (pré)espace-temps à l'échelle de Planck

Il est bien connu qu'à l'échelle de Planck, l'on doit s'attendre à une phase de transition thermodynamique, étroitement reliée (i) à l'existence d'une limite supérieure à la croissance de la température (la température de Hagedorn) [14] et (ii) l'état d'*équilibre thermodynamique* caractérisant globalement le (pré)espace-temps à cette échelle [3].
Dans ce contexte, les investigations déjà citées de Dolan-Jackiw [8] et S. Weinberg [9] puis plus tard de plusieurs autres (voir [3]) ont renouvelé l'idée de Hagedorn concernant l'existence, à très haute température, d'une limite restreignant la croissance de l'excitation des états du système.Plus récemment, J.J. Atik et E.Witten ont montré l'existence d'une limite de Hagedorn en théorie des cordes [15]. La raison est que, comme rappelé par C. Kounnas en théorie des supercordes N = 4 , à température finie, la fonction de partition *Z(ß)* et l'énergie moyenne *U(ß)* présentent des pôles singuliers en $ß \equiv T^{-1}$, dans la mesure où la densité des états du système croît exponentiellement avec l'énergie E (cf. réf[18]) :

$$Z(ß) = \int dE \rho(E) e^{-ßE} \sim \frac{1}{(ß - b)^{(k-1)}}$$

$$U(ß) = \frac{\partial}{\partial ß} \ln Z \sim (k-1) \frac{1}{\beta - b}$$

Manifestement, il existe donc au voisinage de l'échelle de Planck une température critique $T_H = b^{-1}$, température limite à laquelle le système (pré)espace-temps peut être considéré en état d'équilibre thermodynamique, comme rappelé en réf. [3]. En fait, *a(t)* représentant le facteur d'échelle cosmologique, la température globale T du système obéit à la loi bien connue :

$$T(t) \approx T_p \frac{a(t_p)}{a(t)} \tag{5}$$

et selon le Modèle Cosmologique Standard, au voisinage de l'échelle de Planck, T atteint la température limite $T_p \approx \frac{E_p}{k_B} \approx \left(\frac{\hbar C^5}{G}\right)^{\frac{1}{2}} k_B^{-1} \approx 1{,}4.10^{32} K$. En fait, il est couramment admis en théorie des cordes que, avant la phase d'inflation, le rapport entre le taux d'interactions (Γ) des champs initiaux et l'expansion (H) du (pré)espace-temps est $\frac{\Gamma}{H}$ << 1, de sorte que le système peut raisonnablement être considéré à l'équilibre [3]. Ceci a été établi d'abord dans le cadre des travaux précurseurs déjà cités de Weinberg [9], Atick et Witten [14] et plusieurs autres. En théorie des cordes, l'on se référera aux travaux récents de C. Kounnas *et al* [17].Nous développons certains des arguments cités en réfs [1-A1, 6-A2].

Or, cette notion naturelle d'équilibre thermodynamique, lorsqu'elle est considérée comme une condition de jauge globale, comporte des conséquences importantes et inattendues concernant la physique à l'échelle de Planck. Parmi ces conséquences, la plus décisive, mise en évidence en réf. [3, 6-A2], est très probablement que le (pré)espace-temps à l'échelle de Planck devrait être considéré comme soumis à la " condition KMS ". Rappelons dans le paragraphe ci-après ce que signifie la condition KMS.

1.2 Condition KMS à l'échelle de Planck

Nous fournissons une description détaillée de l'application de la condition KMS à l'espace-temps dans l'article publié en réf [6]. Rappelons brièvement la définition mathématique d'un état d'équilibre.

Définition 1.2.1 *Soit A un élément de la C*-algèbre de von Neumann* $\boldsymbol{A}$. *H étant un opérateur autoadjoint, l'équilibre φ de ce système est décrit par la condition de Gibbs*

$\varphi(A) = \mathrm{Tr}\left(e^{\,ßH} A\right) / \,\mathrm{Tr}\left(e^{\,ßH}\right)$

et satisfait la condition KMS.

Nous fournissons une définition plus technique dans l'article publié en réf[6]. La définition (1.2.1) a été rappelée notamment dans la réf [10]. A présent, il est habituel (et naturel) d'opposer la notion d'équilibre à celle d'évolution d'un système. En fait, la célèbre théorie modulaire de Tomita-Takesaki a établi que la dynamique intrinsèque d'un système quantique correspond, d'une manière unique, au groupe d'automorphismes à un paramètre fortement continu α_t d'une C^* - algèbre de von Neumann $\boldsymbol{A}$ [19] :

$$\alpha_t(A) = e^{iHt} A \ e^{-iHt} \tag{6}$$

Ce groupe à un paramètre décrit l'évolution temporelle des observables du système et correspond à l'algèbre de Heisenberg. Cependant, nous devons aussi tenir compte de la remarquable découverte de Takesaki et Winnink, reliant le groupe d'évolution $\alpha_t\,(A)$ du système (plus précisément le groupe modulaire $M = \Delta^{it} M \,\Delta^{-it}$) avec l'état d'équilibre $\varphi(A) = \dfrac{\mathrm{Tr}(Ae^{-ßH})}{\mathrm{Tr}(e^{-ßH})}$ de ce même système [10]. La condition KMS n'est autre que cette relation entre évolution $\alpha_t\,(A)$ et équilibre $\varphi(A)$ d'un système. Plus précisément, dans le cadre de la mécanique statistique quantique, la condition KMS fournit une formulation mathématique rigoureuse de la coexistence de *différents* états d'équilibre possibles à la *même* température donnée T.

En effet, il a été établi [5] qu'un état φ sur une C^*-algèbre $\boldsymbol{A}$ et le groupe d'automorphismes à un paramètre $\alpha_t\,(A)$ à la température ß = 1 / k T vérifient la condition KMS si, pour tout couple *A, B* de la $*$ - sous-algèbre de $\boldsymbol{A}$, il existe une fonction $f(t_C)$ holomorphe dans la bande $\{t_C = t + i\,ß \in \mathbb{C}$, Im $t_C \in [\,0\,,\,ß\,]\,\}$ telle que :

24. $f\,(t) = \varphi(A\,(\,\alpha_t\,B))$,

(ii) $\quad f\,(t + i\,ß\,) = \varphi(\,\alpha_t\,(B)A)\,, \qquad \forall\; t \in \mathbb{R}.$ (7)

En outre, un état φ sur la C^*-algèbre $\boldsymbol{A}$ est *séparateur* si la représentation algébrique donnée est une algèbre de von Neumann W^* munie d'un vecteur cyclique et séparateur. Les ensembles

$I_l = \{A \in \boldsymbol{A} \quad \varphi(\boldsymbol{A}^* \boldsymbol{A}) = 0\}$
et
$I_r = \{A \in \boldsymbol{A} : \varphi(\boldsymbol{A}\,\boldsymbol{A}^*) = 0\}$

forment un idéal à gauche et à droite dans $\boldsymbol{A}$. Pour tout état KMS, l'on a $I_l = I_r$.

La définition ci-dessus exprime la relation bijective entre état l'équilibre, état holomorphe des paramètres de mesure et condition KMS.

A présent, comme nous le rappelons en [6-A2], si l'on admet qu'au voisinage de l'échelle de Planck un état ω du système (pré)espace-temps satisfait la condition d'équilibre d'équilibre $\int_{-\infty}^{+\infty} \omega([h, \alpha_t(A)])dt = 0, \quad \forall A \in U$, alors, d'après [10], l'on est amené à admettre que ce système est soumis à la condition KMS.

A partir de ce résultat établi pour la première fois en [3] et développé depuis (quoique dans une contexte différent) par Derredinger et Lucchesi dans [20, 21], nous considérons à présent la possible transformation holomorphe de la coordonnée genre temps du système (pré)espace-temps en état KMS.

1.3 Complexification de la coordonnée genre temps à l'échelle de Planck

L'une des conséquences de la condition KMS est qu'elle induit de manière naturelle l'existence d'un nouveau degré de liberté sur le signe de la direction genre temps g_{00} de la métrique. En effet, à l'intérieur de la bande KMS, l'on doit considérer la transformation [10-21] :

$$t \to \tau = t_r + i\, t_i \qquad (8)$$

De même, la température (réelle) devrait également être considérée comme complexe à l'échelle de Planck :

$$T_C = T_r + iT_i \tag{9}$$

Une telle transformation est liée au fait qu'étant donné un système quelconque soumis à la condition KMS et considérant une algèbre de von Neumann $\boldsymbol{M}$ sur laquelle peut être déterminé un état φ (ainsi que deux éléments A, B de $\boldsymbol{M}$), alors il existe une fonction f(z) holomorphe dans la bande $\{z \in \mathbb{C},\ \text{im}\ z \in [0,\ \hbar ß]\}$ telle que :

$$f(t) = \varphi(A\ (\alpha_t\ B)) \tag{10}$$

$$\text{et } f(t_r + i\hbar ß) = \varphi(\alpha_t\ (B)A),\ \forall\ t \in \mathbb{R}. \tag{11}$$

Ici, t est le paramètre temporel du système, de même que le paramètre d'échelle $\hbar ß = \hbar / kT$. Ainsi dans notre cas, le système thermodynamique à l'équilibre étant l'espace-temps lui-même, à l'intérieur des limites de la bande KMS, i.e. de l'échelle zéro (ß = 0) à l'échelle de Planck (ß = ℓ_{Planck}), la direction genre temps du système doit être étendue à la variable complexe $t_C = t_r + i\ t_i \in \mathbb{C}$, Im $t_C \in [i\ t_i\ ,\ t_r]$ Naturellement, selon la théorie modulaire de Tomita [10], la condition KMS, appliquée au système (pré)espace-temps, autorise, à l'intérieur de la bande KMS, l' existence d'un groupe d'automorphismes étendu (holomorphe) représentant le groupe d' "évolution quantique " du système à l'échelle de Planck. Celui-ci dépend de l'algèbre de von Neumann $\boldsymbol{M}_q$ dont la forme générale, construite en réf [3] est :

$$\boldsymbol{M}_q \mapsto \sigma_{ß_C}(\boldsymbol{M}_q) = e^{\ Hß_C} \boldsymbol{M}_q\ \ e^{\ \ Hß_C} \tag{12}$$

le paramètre *ß* étant formellement *complexe,* interprétable comme un temps *t* et / ou une température *T* complexes, la " signature KMS " de la métrique étant (+++±). Ainsi, la condition KMS suggère l'existence à l'échelle de Planck, d'un potentiel effectif à une boucle couplé, en supergravité N=2, au dilaton complexe $\varphi = \frac{1}{g^2} + i\alpha$ induisant la forme dynamique de la métrique $\eta_{\mu\nu}$ = diag(1 , 1 , 1 , $e^{i\theta}$).

La signature de $\eta_{\mu\nu}$ est alors Lorentzienne (i.e. physique) pour $\theta = \pm\ \pi$ et peut devenir Euclidienne (topologique) pour $\theta = 0$. Par conséquent, la signature KMS de la métrique est bien superposée, de la forme (+++±). Il est intéressant de noter que dans le contexte différent des supercordes, J.J. Atick et E. Witten ont été les premiers à proposer dès 1988 une telle extension de la température réelle vers le domaine complexe [14]. Récemment, en théorie des cordes supersymmétriques N=4, I. Antoniadis, J.P. Deredinger et C. Kounnas [17] ont également suggéré d'étendre la température réelle T vers les valeurs imaginaires pures, cette extension résultant de l'identification de T avec l'inverse du rayon R du cercle S^1 représentant le temps Euclidien compactifié du système ($R = 1 / 2\pi T$). En conséquence, l'on peut introduire une température complexe dans l'espace des modules thermique, la partie imaginaire étant reliée au champ antisymétrique $B_{\mu\nu}$ sous dualités de cordes :

$$\boldsymbol{IIA} \xleftrightarrow{S/T/U} \text{type } \boldsymbol{IIB} \xleftrightarrow{S/T/U} \boldsymbol{Hétérotique} \;.$$

Plus précisément, dans l'approche d'Antoniadis *et al*, le champ contrôlant la température provient du produit des parties réelles de trois champs complexes : s= Re **S**, t= Re **T** and u= Re **U**. Dans notre approche KMS, les parties imaginaires pures des modules **S**, **T**, **U** peuvent être interprétées en termes de température Euclidienne.

A présent, il apparaît clairement que le groupe d'automorphismes de Tomita-Takesaki $\sigma_{ß_c}(\boldsymbol{M}_q)$ induit, dans le champ KMS, l'existence de deux flots duaux l'un de l'autre. Tiré de l'équ. (12), le premier, de la forme :

$$\sigma_t(\boldsymbol{M}_q) = e^{iHß_i} M_q \; e^{-iHß_i} \qquad (13)$$

correspond à l'algèbre des observables du système et au flot Lorentzien en temps réel. Dans cette perspective, ce courant représente un “ flot physique ”, que nous appelons $\mathbf{P}^f_{ß>0}$. Quant au deuxième flot possible, en partant à nouveau de l'equ. (12), il prend nécessairement la forme :

$$\sigma_{iß}(\boldsymbol{M}_q) = e^{\beta_i H} \boldsymbol{M}_q \; e^{-\beta_i H} \qquad (14)$$

donnant sur $\boldsymbol{M}$q le semi groupe d'opérateur non stellaire. Considérant la continuation analytique entre les équations (13) et (14 représente un “ courant en temps imaginaire ” ou, de manière équivalente, un courant topologique que nous appelons $\mathbf{T}^f$. Nous considérons maintenant la condition KMS en termes d'algèbres de von Neumann.

1.4 Etat KMS et facteurs. Dans les états KMS, les algèbres de von Neumann impliquées sont ce que l'on appelle des " facteurs ", i.e. un type particulier d'algèbre de von Neumann, dont le centre est réduit aux scalaires $a \in \mathbb{C}$. Il existe trois types de facteurs : le type I et le type II (en particulier ici le II_∞) – naturellement non commutatifs mais munis d'une *trace* et que l'on peut donc interpréter comme " commutatifs sous la trace "- et le type III, sans trace. Une trace τ sur un facteur M est une forme linéaire telle que $\tau(AB) = \tau(BA), \ \forall A,B \in M$. Lorsque la mesure sur M n'est pas définie (ce qui est le cas du type III), la notion de trace disparaît et est remplacée par celle de " poids ", qui représente une application linéaire de M_+ sur $\mathbb{R}_+ = [0, +\infty]$. Les facteurs de type III sont importants ici dans la mesure où ce sont les seuls facteurs intéressants dans le contexte des états KMS (les états KMS liés aux types II et III étant triviaux, cette trivialité résultant de celle du groupe modulaire sous-jacent dans ces deux cas). Nous utilisons ici les facteurs de type " III_λ ", $\lambda \in]0, 1[$, caractérisés par l'invariant $S(M) = \lambda^{\mathbb{Z}} \cup \{0\}$. Pour des définitions plus précises, voir les réf [3] et [1-A1].A présent, nous suggérons dans l'article publié en réf [6-A2] que la condition KMS appliquée au (pré)espace-temps à l'échelle de Planck définit trois différentes échelles sur le cône de lumière, depuis l'échelle zéro jusqu'à l'échelle de Planck. Ces trois domaines peuvent être décrits par trois différents types de facteurs.

L'échelle topologique zéro (ß = 0, signature {++++}) : cette échelle initiale, que nous proposons d'appeler en réf.[1-A1] " échelle topologique " correspond au sommet imaginaire du cône de lumière, i.e. un instanton gravitationnel de taille zéro. Toutes les mesures réalisées sur la métrique Euclidienne étant ρ-équivalentes jusqu'à l'infini, le système est ergodique. Comme montré par A. Connes, tout flot ergodique pour une mesure invariante dans la classe de mesure de Lebesgue donne un unique facteur hyperfini de type type II_∞ [11]. Ceci suggère fortement que l'échelle singulière zéro devrait être décrit par un facteur de type II_∞, muni d'une trace hyperfinie notée Tr_∞. Par hyperfinie, nous entendons simplement que la trace du facteur II_∞ n'est pas finie. Nous appelons $\boldsymbol{M}_{Top}^{0,1}$ un tel facteur " topologique ", qui est un produit tensoriel infini $\otimes^\infty$ d'algèbres de matrices (ITPFI) du type d'Araki-Woods $R_{O,1}$ [10]. D'après [3], l'état initial sur $\boldsymbol{M}_{Top}^{0,1}$, correspondant, dans l'ex. (2.1) de [1-A1] aux valeurs divergentes du champ dilaton $\frac{1}{g^2}$, peut être donné par :

$$\varphi(\boldsymbol{M}_{Top}^{0,1}) = \frac{\mathrm{Tr}_{\infty}(e^{-ßH}\boldsymbol{M}_{Top}^{0,1})}{\mathrm{Tr}_{\infty}(e^{-ßH})} \tag{15}$$

et considérant le caractère hyperfini de la trace Tr_∞, l'on a de manière équivalente :

$$\varphi(\boldsymbol{M}_{Top}^{0,1}) = \mathrm{Tr}_{\infty}(e^{\;ßH}\boldsymbol{M}_{Top}^{0,1}\;\; e^{ßH}) \tag{16}$$

où $\varphi(\boldsymbol{M}_{Top}^{0,1})$ représente un type de " courant " particulier, que nous proposons d'appeler " courant tracial " $\mathbf{T}^f$. Clairement, l'invariant hyperfini $\mathbf{T}^f$ est une pure amplitude topologique [2, 22] et, en tant que telle, se " propage " en temps imaginaire de l'échelle zéro à l'infini. En ce sens, $\varphi(\boldsymbol{M}_{Top}^{0,1})$ peut être vu comme un " cycle topologique zéro " représentant une " pseudo dynamique Euclidienne " intrinsèque contrôlant l'éclatement de la Singularité Initiale de l'espace-temps [3]. C'est pourquoi nous suggérons de décrire par $\varphi(\boldsymbol{M}_{Top}^{0,1})$ l'évolution " topologique " du (pré)espace-temps au voisinage de l'échelle zéro

(ii) ***l'échelle quantique*** ($0 < ß < \ell_{Planck}$, signature {+++±}) : nous abordons le domaine KMS [5]. Considérant les fluctuations quantiques de $g_{\mu\nu}$, il n'existe plus de mesure invariante sur la métrique non commutative. Par conséquent, selon la théorie des algèbres de von Neumann, le " bon facteur " adressant de telles contraintes est uniquement une algèbre non commutative sans trace, i.e. un facteur de type III [10] (le seul type de facteur impliqué dans des états KMS non triviaux). Plus précisément, il s'agît d'un type III_λ que nous appelons $\boldsymbol{M}_q$, muni de la période $\lambda \in]\,0, 1\,[$. Dans ce cas, la notion de trace doit obligatoirement être remplacée par celle de poids de l'algèbre sous-jacente (ce qui nous introduit de manière naturelle à la notion de flot des poids de l'algèbre A). Nous considérons alors [1] que le seul objet pertinent pour décrire une possible " évolution " à l'échelle quantique est le flot des poids de l'algèbre de type III_λ caractérisant le système (pré)espace-temps à cette échelle. Or, il a été démontré par A. Connes que tout facteur de type III_λ peut être décomposé de façon *canonique* selon le produit croisé suivant [10] : $\mathrm{III}_\lambda = \mathrm{II}_\infty \rtimes_\theta \mathbb{R}^*_+$

C'est donc ainsi qu'apparaît obligatoirement dans notre construction le $\mathbb{R}^{*}{}_{+}$ (dual of $\mathbb{R}$) agissant de manière périodique sur le facteur de type II_{∞}. Alors, la périodicité (*ß*-dépendante) de l'action de $\mathbb{R}^{*}{}_{+}$ sur $\boldsymbol{M}_{Top}^{0,1}$ prend la forme : $\boldsymbol{M}$q = $\boldsymbol{M}_{Top}^{0,1} \rtimes_{\theta} \mathbb{R}_{+}^{*}/ß\mathbb{Z} \equiv \boldsymbol{M}_{Top}^{0,1} \rtimes_{\theta} ß\, S_1$

La relation entre λ et ß est telle que $\lambda = \frac{2\pi}{ß}$, de sorte que lorsque $ß \to \infty$, l'on obtient $\lambda \to 0$ (la périodicité est supprimée). A présent, comme la théorie est définie sur l'espace de Hilbert $\mathscr{L}(\mathfrak{h}) = \mathscr{L}\left[L^2\left(\mathbb{R}_{+}^{*}/ß\mathbb{Z}\right)\right]$, $\boldsymbol{M}$q devient : $\boldsymbol{M}$q = $\boldsymbol{M}_{Top}^{0,1} \rtimes_{\theta} \mathscr{L}\left[L^2\left(\mathbb{R}_{+}^{*}/ß\mathbb{Z}\right)\right]$

Le facteur $\mathscr{L}\left[L^2\left(\mathbb{R}_{+}^{*}/ß\mathbb{Z}\right)\right]$, de type I_{∞}, induit le flot modulaire (périodique) d'*évolution* du système.

Le facteur KMS $\boldsymbol{M}$q de type III_{λ} connecte le facteur " topologique " de type II_{∞} $\boldsymbol{M}_{Top}^{0,1}$ avec le facteur " physique " de type I_{∞} que nous appelons

$$\boldsymbol{M}_{Phys} : \boldsymbol{M}\mathrm{q} = \boldsymbol{M}_{Top}^{0,1} \rtimes_{\theta} \boldsymbol{M}_{Phys} \tag{17}$$

En termes de flots, l' équ. (17) connecte le flot topologique des poids $\boldsymbol{M}_{Top}^{0,1}$ et le flot modulaire physique induit par $\mathscr{L}\left[L^2\left(\mathbb{R}_{+}^{*}/ß\mathbb{Z}\right)\right]$. Ceci fournit une bonne image de l'unification entre états physiques et topologiques, à rapprocher du produit bicroisé (41) $U_q(so(4)^{op} \overset{\psi}{\bowtie} U_q(so(3,1))$ unifiant les groupes quantiques Euclidien et Lorentzien. Le flot quantique $\sigma_{ß_c}(\boldsymbol{M}_{q\ flow}) = e^{ß_c H}\boldsymbol{M}_{q\ flow}e^{ß_c H}$ est construit à la prop. (1.4.1).

(iii) ***l'échelle physique*** ($ß > \ell_{Planck}$, signature $\{+++-\}$) : cette dernière échelle représente la partie physique (relativiste) du cône de lumière. Par conséquent, la notion de mesure (de Lebesgue) est pleinement définie. L'algèbre de von Neumann impliquée est donc munie d'une trace hyperfinie et est donnée sur l'espace de Hilbert infini $\mathscr{L}(\mathfrak{h})$, avec $\mathfrak{h} = L^2$®. Alors, $\mathscr{L}(L^2$®) est un facteur de *type* I_{∞} indexé par le groupe réel $\mathbb{R}$, que nous appelons $\boldsymbol{M}_{Phys}$. Ainsi, $\mathscr{L}(L^2$®) = $\boldsymbol{M}_{Phys}$ et le flot induit par $\boldsymbol{M}_{Phys}$ est simplement le flot d'évolution en temps réel, donné par le groupe modulaire :

$$\sigma_t(\boldsymbol{M}_{Phys}) = e^{iHt}\boldsymbol{M}_{Phys}\ e^{-iHt}$$

Dans ce cas, tous les automorphismes sont intérieurs. Nous appelons “ flot physique ” $\mathbf{P}^{f}_{ß>0}$ ce flot d’évolution en temps réel. Naturellement, $\sigma_t(\boldsymbol{M}_{Phys})$ donne simplement l’algèbre des observables du système [19].A partir de cette construction, détaillée dans les réfs [1-A1] et [6-A2], nous proposons d’établir que l’état KMS “ unifie ” le flot physique et le courant topologique.

Proposition 1.4.1 *A l’échelle KMS 0 < ß < l_{Planck} , les deux groupes d’automorphismes* $\sigma_t(M_{Phys})$ *et* $\sigma_ß(M^{0,1}_{Top})$ *sont couplés au sein de l’unique facteur de type III*λ *de la forme* $\boldsymbol{M}$q = $\boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathscr{L}\left[L^2\left(\mathbb{R}^*_+ / ß\mathbb{Z}\right)\right]$. *Le groupe d’automorphismes étendu (complexe) associé décrivant l’évolution à l’échelle quantique est de la forme* $M_q \mapsto \sigma_ß(M_q) = e^{Hß_C} M_q \; e^{-Hß_C}$ *Mq correspond au couplage entre le groupe d’automorphismes à un paramètre donnant le flot physique et le semi-groupe d’automorphismes donnant le flot topologique du système.* **Preuve** Comme établi en [1] et rappelé ci-dessus, l’état KMS du (pré)espace-temps peut être donné de manière canonique par l’unique facteur IIIλ de la forme :

$$\mathbf{M}\text{q} = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathscr{L}\left[L^2\left(\mathbb{R}^*_+ / ß\mathbb{Z}\right)\right] = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \boldsymbol{M}_{Phys} \tag{18}$$

qui représente ce que nous proposons d’appeler l’ ”unification KMS ” de l’état topologique et de l’état physique de la métrique du (pré)espace-temps à l’échelle de Planck. Or, les résultats généraux obtenus dans la réf [3] permettent de considérer l’existence d’un poids opératoriel de $\boldsymbol{M}$q sur son sous-groupe $\boldsymbol{M}^{0,1}_{Top}$, l’état d’équilibre φ sur $\boldsymbol{M}$q étant donné par l’état sur $\boldsymbol{M}^{0,1}_{Top}$. Nous exprimons alors l’état φ sous la nouvelle forme proposée en [6] :

$$\varphi(\boldsymbol{M}_{q\text{-état}}) = \mathrm{Tr}_\infty\,(e^{\;ßH}\boldsymbol{M}^{0,1}_{Top}\; e^{ßH}) \tag{19}$$

Ceci représente ce que nous proposons d’appeler le “ courant tracial ” engendré par le facteur topologique $\boldsymbol{M}^{0,1}_{Top}$. Cependant, Connes-Takesaki ont montré [10] que le flot des poids sur un facteur de type III est donné par le flot des poids sur le facteur de type II$_\infty$ associé. Car il existe un homomorphisme OUT III$_\lambda \rightarrow$ OUT II$_\infty$ tel que la séquence (20) est exacte :

$$\{\,1\,\} \rightarrow H^1(F) \xrightarrow{\overline{\partial}M} \mathrm{OUT}M \xrightarrow{\overline{\gamma}} OUT_{\theta,\tau}(N) \rightarrow \{\,1\,\} \tag{20}$$

L'action multiplicative de $\mathbb{R}$: $\tau \circ \theta_S = e^{-S}\tau,\ \ s \in \mathbb{R}$ on $\boldsymbol{M}^{0,1}_{Top}$ *translate* la trace τ de $\boldsymbol{M}^{0,1}_{Top}$, ce qui engendre le flot des poids sur $\boldsymbol{M}^{0,1}_{Top}$ et $\boldsymbol{M}$q (cf.[10]). Ainsi, $\varphi(\boldsymbol{M}_{q\text{-état}})$ devient un automorphisme de semi-groupe *ß*-dépendant :$\sigma_ß\ (\boldsymbol{M}_{q\text{-état}}) = e^{\ ßH}\boldsymbol{M}^{0,1}_{Top}\ \ e^{ßH}$

L'équ. (19) décrit donc le flot des poids de $\boldsymbol{M}$q. Mais comme souligné en [6], l'on peut également interpréter l'équ.(19) comme un " flot modulaire en temps imaginaire " *it,* dual du flot modulaire usuel en temps réel donné par : $\sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-evolution}}})$ `$= e^{\ iHt}\boldsymbol{M}_{\boldsymbol{Phys}}e^{\ -\ iHt}$, t$\in \mathbb{R}$.

Une interprétation de ce type a également été proposée (quoique dans un contexte différent) par Derrendinger et Lucchesi en [13]. Finalement, le flot KMS connecte le flot des poids $\sigma_ß(\boldsymbol{M}_{\boldsymbol{q\text{- state}}})$ au groupe modulaire $\sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-evolution}}})$:

$$\begin{aligned}\sigma_{ß_C}(\boldsymbol{M}_{q\quad flow}) &= \sigma_ß(\boldsymbol{M}^{0,1}_{Top}) \otimes \sigma_t(\boldsymbol{M}_{Phys})\\ &= e^{\ (ß+it)H}\boldsymbol{M}_{q\quad flow}\ \ e^{(ß+it)H}\\ &= e^{\,ß_C H}\boldsymbol{M}_{q\quad flow}e^{\ \ ß_C H}\end{aligned}$$

ce qui est indexé par la variable en temps complexe $ß_C$. Encore une fois, un tel flot exprime l'unification entre le flot *physique* $\sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-evolution}}}) = \sigma_t(\boldsymbol{M}_{Phys})$ et le flot *topologique* $\sigma_ß(\boldsymbol{M}_{\boldsymbol{q\text{- state}}}) = \sigma_ß(\boldsymbol{M}^{0,1}_{Top})$ au sein d'un flot KMS (ou flot quantique) *unique* $\underset{0<\mathbf{ß}<\ell P}{\mathbf{Q}^f}$ donné par le groupe d'automorphismes $\boldsymbol{M}_q$:

$$\sigma_{ß_C}(\boldsymbol{M}_{q\quad flow}) = \sigma_ß(\boldsymbol{M}_{\boldsymbol{q\text{- state}}}) \bigoplus \sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-evolution}}})$$

La bande KMS étalée de l'échelle zéro à l'échelle de Planck du (pré)espace-temps admet donc zéro comme borne inférieure et l'échelle de Planck comme borne supérieure. Entre ces deux limites, le flot topologique Euclidien et le flot physique Lorentzien sont donc unifiés de manière naturelle au sein du flot " quantique " holomorphe

$$\mathbf{Q}^{f}_{0<\text{ß}<\ell P} \rightarrow \ \sigma_{ß_C}(\boldsymbol{M}_{q-flot}) = e^{ß_C H} \boldsymbol{M}_{q-flot} e^{-ß_C H} .$$

Une autre façon de vérifier le couplage de M_{Phys} et $M^{0,1}_{Top}$ au sein de l'unique facteur de type III$_\lambda$ réside dans l'invariant de Connes :

$$\delta \in : \mathbb{R} \rightarrow \text{OUT M} = \frac{\text{AUT M}}{\text{INT M}} \tag{21}$$

(automorphismes de M quotientés par les automorphismes intérieurs, nécessairement présents dans le cas non commutatif). Cet invariant de M représente un flot ergodique $\{W(M)\ ,\ W_\lambda\}$ où W_λ est un groupe de transformations à un paramètre – i.e. un flot – qui admet une description en termes de classes de poids et dont le paramètre naturel est $\mathbb{R}_+{}^*$. Nous considérons à présent le facteur de type III$_\lambda$ Mq de l'équ.(18). Partant de l'équ.(20), l'on peut construire l'extension *Ext* (notée $\top$) de OUT Mq par INT Mq dans AUT Mq :

$$\text{AUT Mq} \equiv \ \text{OUT Mq} \top \text{INT Mq} \tag{22}$$

avec $\{x,y\} \in$ OUT Mq et $\{x',y'\} \in$ INTMq. Le groupe des automorphismes intérieurs INTMq est un sous-groupe normal de AUT Mq . Considérant alors deux poids φ et ψ de Mq, et appliquant le théorème de Radon-Nikodym [10], il existe un unitaire de Mq tel que

$$\sigma_t^{\psi}(x) = u_t \sigma_t^{\varphi}(x) u_t^*$$

avec $u_t = (D\psi\ ;\ D\varphi)_t$ et $\sigma_t^{\psi}(x) \in$ INT Mq pour une certaine classe d'automorphismes modulaires. Considérons alors que sous la trace du facteur II$_\infty$ impliqué dans le produit croisé $\boldsymbol{M}q = M^{0,1}_{Top} \rtimes_\theta \mathbb{R}^*{}_+$ tous les automorphismes modulaires sont (nécessairement) *intérieurs*. Il en résulte la restriction de INTMq à un sous-groupe du groupe des automorphismes modulaires, sous-groupe que nous appelons INT$_{mod}$ Mq. Puis, nous cherchons l'image du groupe modulaire intérieur dans OUTMq. Dans une certaine classe de cohomologie $\{K\}$, le groupe $\sigma_t^{\psi}(x)$ est donné par INT$_{mod}$Mq, tandis que les transformations non unitaires $\sigma_ß(x)$ sont données par OUT Mq. L'on obtient alors pour le flot " physique " :

$\sigma_t^{\psi}(x) = e^{iHt} M_q \, e^{\ iHt} \in \mathrm{INT}_{mod}\,\mathrm{Mq}$

tandis que le flot “ topologique ” des poids de Mq est donné par :

$\sigma_{ß}(x) = e^{\ ßH} M_q \, e^{ßH} \in \mathrm{OUT\ Mq}$

et de l’extension AUT Mq ≡ OUT Mq $\top$ INT_{mod} Mq l’on peut tirer [1] :

$$\sigma_{(t \,\top\, ß)} = \sigma_t^{\psi}(x) \top \sigma_{ß}(x) \tag{23}$$

A l’intérieur du groupe général des extensions $\{Ext\}$, l’on obtient alors le sous-groupe holomorphe trivial :

$\sigma_{ß+it}(M_q) = e^{\ (ß+it)H} M_q \ \ e^{(ß+it)H} = \sigma_{ß_C}(M_q) = e^{\ Hß_C} M_q \ \ e^{\ -Hß_C}$

qui correspond à l’état KMS et “ unifie ” au sein de la forme étendue unique $\sigma_{ß_C}(Mq)$ le flot physique $\sigma_t^{\psi}(x)$ et le courant topologique $\sigma_{ß}(x)$. Clairement, l’on obtient $\sigma_{ß_C}(Mq) \subset$ OUT Mq $\top$ INT_{mod} Mq. Une nouvelle fois l’on trouve le résultat ci-dessus :

$$\sigma_{ß_C}(\boldsymbol{M}_{q-flot}) = \sigma_{ß}(\boldsymbol{M}_{q\text{-état}}) \oplus \sigma_t(\boldsymbol{M}_{q\text{-évolution}}) \quad \textbf{qed}$$

Il résulte de (1.4.1) ainsi que de plusieurs autres propositions publiées en référence (notamment le §(5) de (1-A1)) l’un des principaux résultats de la présente recherche : l’échelle zéro de l’espace-temps (ou du (pré)espace-temps) est de nature topologique. Nous apportons au§(2) de la réf[1] plusieurs exemples de nature à illustrer ce résultat. En particulier, considérant toujours le (pré)espace-temps en tant que système thermodynamique, nous montrons :

Exemple 1.4.2 *La limite d’échelle zéro du noyau de la chaleur du système thermique (pré)espace-temps est topologique*

La démonstration de l’ex. (1.4.2) est donnée au §(2.1) de la réf [1]. De même, nous établissons la limite ultraviolette d’une autre approche standard :

Exemple 1.4.3 *La limite d'échelle zéro de l'espace-temps de Minkowski, donnée en termes d'intégrales de chemins de Feynmann, est topologique.*

La démonstration est fournie à l'exemple (2.2) de la réf[1-A1]. Ensuite, nous proposons un nouvel exemple montrant que la limite d'échelle zéro $ß \to 0$ de la théorie supersymmétrique N=2 est topologique. En effet, considérant la variété espace-temps $\boldsymbol{M}$ l'on peut montrer [23] que l'espace des états d'énergie zéro décrits par l'Hamiltonien du système est donné par l'ensemble des formes harmoniques sur $\boldsymbol{M}$ et est égal au nombre de Betti de $\boldsymbol{M}$. L'on a ainsi pour $ß \to 0$:

$$\mathrm{Tr}(-1)^{F} = \sum_{k=0}^{4} (-1)^{k} b_i = \chi(\boldsymbol{M})$$

où b_i est le i^{th} nombre de Betti et $\chi(\boldsymbol{M})$ la caractéristique d'Euler-Poincaré de $\boldsymbol{M}$. Finalement, sur la limite d'échelle zéro, nous retrouvons l'indice topologique [2] correspondant à toute théorie topologique des champs standard.

Pour finir, nous indiquons dans l'ex.(2.4) de [1] qu'il est possible d'obtenir un résultat analogue dans le contexte de la supergravité N = 2.

Exemple 1.4.4 *La limite d'échelle zéro du (pré)espace-temps en supergravité N=2 est topologique.*

En fait, pour une variété spinorielle, l'on peut exprimer H en termes de l'opérateur de Dirac $\not{D}$. Alors, en dimension D=4, nous avons montré que la limite $ß \to 0$ du système décrit par l'opérateur de Dirac est donné par l'invariant topologique suivant :

$$\mathrm{Ind}(\not{D}_{+}) = \frac{\dim M}{8\pi^2}\int \mathrm{Tr}(R \wedge R) - \frac{1}{8\pi^2}\int \mathrm{Tr}(F \wedge F)$$

Nous retrouvons une nouvelle fois la limite topologique de la théorie pour $ß \to 0$.

L'ensemble de ces résultats nous amène à présent à formaliser le deuxième résultat principal de notre recherche : la limite d'échelle zéro de l'espace-temps est purement topologique.

CHAPITRE 2

LA LIMITE TOPOLOGIQUE DE L'ESPACE-TEMPS A L'ECHELLE ZERO

2.1 Théorie topologique des champs

L'on définit habituellement, à partir de Witten [2], la théorie topologique des champs comme la quantification de zéro, le Lagrangien de la théorie étant (**i**) soit un mode 0, soit (**ii**) une classe caractéristique $c_n(V)$ d'un fibré vectoriel $V \xrightarrow{\pi} M$ construit sur l'espace-temps. Nous proposons alors au §(2.2) une nouvelle limite topologique de la théorie, non triviale, fondée non plus sur H = 0 mais sur ß = 0 et donc *indépendante* de H. La limite topologique ordinaire de la théorie quantique des champs, décrite par l'invariant de Witten Z = Tr$(-1)^n$ [24] est donnée par la limite de la fonction de partition

$$Z = \mathrm{Tr}(-1)^n \, e^{-ßH} \qquad (24)$$

calculée sur le (3+1)- espace-temps Minkoskien pour les valeurs nulles (ou invariantes) de H. *n* représente le nombre d'états d'énergie zéro de la théorie, par exemple le nombre fermionique dans les théories supersymmétriques [23]. Alors, **Z** décrit tous les états d'énergie zéro pour les valeurs nulles de l'Hamiltoniern H.

2.2 Une nouvelle limite topologique

A présent, nous proposons ici une nouvelle limite topologique de la théorie quantique des champs, limite non triviale (i.e. correspondant au minimum non trivial de l'action). Construite à partir du mode zéro de l'échelle du système $ß \to 0$ et *indépendante* de H, cette limite inattendue (en dimension D=4) est alors donnée par la limite de température (température de Hagedorn) du système physique D=(3+1). En un certain sens, cette suggestion peut être rapprochée de la "conjecture holographique"[24] selon laquelle les états de la gravité quantique en *d* dimensions ont une description naturelle dans le cadre d'une théorie *(d– 1)*- dimensionnelle.

Si, conformément aux résultats de [3] repris en réf[1-A1], nous admettons que l'espace-temps est soumis à la condition KMS à l'échelle de Planck, alors dans ce cas, sur la limite d'échelle zéro associée, dans la fonction de partition de l'équ.(24) à $ß = 0$, la théorie est projetée sur la limite imaginaire pure $t \mapsto \tau = it_i$ de la bande KMS. Ainsi, la fonction de partition (2) donne l'état topologique connecté au mode zéro de l'échelle :

$$Z_{ß \to 0} = \mathrm{Tr}\,(-1)^s \qquad (25)$$

où **s** représente le nombre instantonique de la théorie. Ce nouvel invariant topologique construit pour la première fois dans [3], isomorphe à l'invariant de Witten Z = Tr $(-1)^F$, peut explicitement être associé à la singularité initiale du (pré)espace-temps, atteinte pour la valeur ß = 0 de la fonction de partition des états **Z** = Tr $(-1)^s\, e^{-ßH}$. Nous proposons d'appeler "invariant de singularité" ce nouvel invariant, associé à l'instanton gravitationnel singulier de taille 0. Cette intéressante relation avec la théorie instanton suggère naturellement un lien, proposé pour la première fois dans la réf. [3], entre le premier invariant de Donaldson [25] et l'échelle zéro de l'espace-temps.

2.3 Echelle zéro et premier invariant de Donaldson

Rappelons au préalable une définition générale des invariants de Donaldson :

Définition 2.3.1 *Soit $\boldsymbol{M}$ une variété quadridimensionnelle. L'invariant de Donaldson $q_d(\boldsymbol{M})$ est un polynôme entier symétrique de degré d dans la 2-homologie $H_2(\boldsymbol{M};\boldsymbol{Z})$ de $\boldsymbol{M}$*

$$q_d(\boldsymbol{M}) : H_2(\boldsymbol{M}) \times \ldots \times H_2(\boldsymbol{M}) \to \boldsymbol{Z}$$

$\mathcal{M}_{\mathrm{mod}}^{(k)}$ étant l'espace des modules des instantons de degré k, l'invariant de Donaldson est défini par l'application

$m\ :\ H_2(\boldsymbol{M}) \rightarrow H^2(\boldsymbol{M})\mathcal{M}^{(k)}_{\text{mod}}$

Il a été précisé en réf. [3] en quel sens l'image de la symétrie 0, décrite par le groupe de jauge non brisé du type SU(2) $\otimes$ SU(2), peut être donnée par le premier invariant de Donaldson, associé ici à l'existence d'une "amplitude topologique" caractérisant la théorie. Lorsque la dimension dim$\boldsymbol{\mathcal{M}}$ de l'espace des modules des instantons est non nul, les invariants de Donaldson sont donnés par la fonction de corrélation de la théorie :

$$Z(\gamma_1\ \dots\ \gamma_r) = \int DX\, e^{-S} \prod_{i=1}^{r} \int_{\gamma i} W_{k_1} = \left\langle \prod_{i=1}^{r} \int_{\gamma_i} W_{k_i} \right\rangle \quad (\text{Dim } \mathcal{M}_{\mathrm{k}} \neq 0) \tag{26}$$

Or, notre résultat formel le plus surprenant est qu'à l'échelle ß = 0 associée à la limite des hautes températures, l'espace des modules des instantons étant nul sur cette limite, la fonction de partition, donnée par

$$\underset{ß=0}{Z} = \mathrm{Tr}(-1)^{\mathrm{S}}\, e^{-\beta H} \tag{27}$$

doit redonner le premier invariant de Donaldson

$$\mathrm{I} = \sum_i (-1)^{n_i}\ , \tag{28}$$

invariant topologique non polynomial, réduit à un entier pour dim $\boldsymbol{\mathcal{M}}_{\mathrm{k}} = 0$. Nous formalisons ceci dans la prop. qui suit :

Proposition 2.3.2 *La limite de haute température de la théorie quantique des champs, correspondant à ß $\rightarrow$ 0 dans la fonction de partition Z = Tr (-1) ^{S} e $^{-ßH}$ donne le premier invariant de Donaldson. La métrique sous-jacente de la variété de dimension 4 représentant l'espace-temps à l'échelle zéro est Euclidienne* (+ + + +).

Remarque Soit la fonction de partition des états de la métrique Z = Tr(-1)$^{\mathrm{S}}$ $e^{-ßH}$. La limite ß $\rightarrow$ 0 de la fonction de partition Z, de la forme

$$Z_{ß=0} = \mathrm{Tr}(-1)^{S}$$

correspond à une symétrie généralisée de tous les états possibles de la métrique, tous les états instantoniques de $g_{\mu\nu}$, donnés par la charge topologique de l'instanton gravitationnel singulier, étant équivalents à l'échelle 0. Nous appelons "symétrie 0" la symétrie généralisée caractérisant l'échelle singulière 0.

Preuve Selon des arguments standard, l'on peut écrire :

$$\mathrm{Tr}\,(-1)^{S}\, e^{-ßH} = \int_{\mathrm{CPB}} d\phi(t)\, d\psi(t) \exp - S_E(\phi,\ \psi)$$

A la suite des travaux de Witten [26], il a été montré par L. Alvarez-Gaumé [27] qu'étant donnée une théorie quantique des champs supersymétrique, l'on peut définir l'invariant topologique I = Tr $(-1)^{f}$, f étant le nombre fermionique. Nous suggérons d'étendre en supergravité ce résultat, afin qu'il soit possible à partir de l'indice de l'opérateur de Dirac de la variété spinorielle (pré)espace-temps, de définir l'invariant topologique

$$\dot{\imath} = \mathrm{Tr}\,(-1)^{S}$$

$\dot{\imath}$ donne la différence entre le nombre d'instantons et de monopoles gravitationnels dans l'espace de Hilbert de la théorie à l'échelle 0 et s désigne le nombre d'instantons. Les propriétés de la supergravité sont telles que l'indice supergravitationnel dépend seulement des modes 0 des états d'énergie - les valeurs propres de l'Hamiltonien $D\!\!\!/^2$ paramétrant l'énergie - , les états d'énergie non nuls induisant l'existence de paires monopôles - instantons. Tr $(-1)^{S}$ est invariant sous les déformations continues de $D\!\!\!/^2$ et constitue donc un indice topologique de la théorie de déformation quantique de la signature d'espace-temps. Le calcul de l'indice de l'opérateur de Dirac à partir de la régularisation de la trace donnée par l'équ.(30) donne l'indice $\dot{\imath}$ de l'opérateur de Dirac :

$$\dot{\imath} = \mathrm{Tr}\,\Gamma\, e^{-ß_c D\!\!\!/^2} = \mathrm{Tr}\,(-1)^{S}\, e^{-ß_c D\!\!\!/^2} = \int_{cpl} [Dx]\,[D\psi]\, e^{-\int_0^{ß} dt\, L} \qquad (29)$$

avec $ß_c \in \mathbb{C}$. Lorsque $ß_c = 0$, la limite de la fonction de partition $Z = \mathrm{Tr}\,(-1)^S\, e^{-ß_c H}$ est réduite à :

$$Z_0 = \mathrm{Tr}\,(-1)^S \tag{30}$$

Witten a montré [26] que Tr $(-1)^n$ peut être compris comme l'indice d'un opérateur agissant sur l'espace de Hilbert $\mathcal{H}$ du système. Partageons $\mathcal{H}$ en un sous-espace monopôle et un sous-espace instanton $\mathcal{H} = \mathcal{H}_m + \mathcal{H}_i$. $\mathcal{H}_m$ et $\mathcal{H}_i$ définissent les états monopôles et instantons à l'échelle 0. Q étant un générateur de supersymétrie, il résulte de (7.19) $Q|\psi\rangle = 0$, $Q^*|\psi\rangle = 0$. Comme Q est adjoint de Q^* en regard de la norme de l'espace de Hilbert, l'on a $\mathrm{Tr}(-1)^S = \mathrm{Ker}\,Q - \mathrm{Ker}\,Q^*$, de sorte qu'en tant qu'indice topologique, Tr $(-1)^S$ est invariant sous les déformations continues des paramètres de la théorie qui ne modifient pas le comportement asymptotique de l'Hamiltonien à haute énergie. Le Hamiltonien correspond au Laplacien sur les formes $H = dd^* + d^*d$ et l'espace des états d'énergie 0 est donné par l'ensemble des formes harmoniques paires sur M_n:

$$\mathrm{Tr}\,(-1)^S\, e^{-ßH} = \chi\,(M) = \sum_{k=0}^{n} (-1)^k b_k \tag{31}$$

où $\chi(M)$ est la caractéristique d'Euler de M et b_i le i ème nombre de Betti. $\Delta =$ Tr $(-1)^S$ est indépendant de ß, les seules contributions à Δ provenant du secteur topologique d'énergie 0 : $\Delta = n_i\,(E=0) - n_m\,(E=0)$· Δ est donc un invariant topologique. Montrons que cet invariant est isomorphe au premier invariant de Donaldson. La constante de couplage g de la théorie est dimensionnelle : $g \rightarrow g'(\rho)$, ρ étant le rayon de l'instanton. La limite ß = 0 implique donc $\rho = 0$ et correspond au secteur des instantons de taille 0. Or, sur la limite ß = 0, $\mathrm{Dim}\, M_k = 0$. Lorsque la dimension de l'espace des modules des instantons est non nul, les invariants de Donaldson sont donnés par :

$$Z(\gamma_1 \ldots \gamma_r) = \int DX\, e^{-S} \prod_{i=1}^{r} \int_{\gamma i} W_{k_1} = \left\langle \prod_{i=1}^{r} \int_{\gamma_i} W_{k_i} \right\rangle \quad (\mathrm{Dim}\, M_k \neq 0) \tag{32}$$

A présent, qu'en est-il de ces mêmes invariants lorsque l'espace des modules est de dimension 0? La solution est dans la correspondance entre les invariants de Donaldson sur les variétés de dimension 4 et les groupes d'homologie de A. Floer [28] sur les variétés de dimension 3. Coupons la 4 - variété M en deux parties non fermées M^+ et M^- :

$M = M^+ \cup_h \; M^-$ (33)

où les bords de M^+ et M^- sont des 3-sphères d'homologie. Soient S^+ et S^- les sphères d'homologie formant les bords de M^+ et M^-. Considérons leur homologie de Floer $HF_*(M^+)$ et $HF_*(M^-)$. Pour une charge topologique donnée k, nous considérons les instantons gravitationnels sur les 4-variétés M^+ et M^-. Les solutions des conditions aux bords permettant de définir la connexion sur les bords $S^{\mp}$ sont notées $C^{\mp}$. Dans ce cas, C.Nash a montré [29] que l'espace des modules des instantons sur la variété fermée M devient $C^+ \cap C^- = M_k$. Les conditions au bord permettent de construire deux classes d'homologie de Floer $[C^{\mp}] = HF_*(S^{\mp})$. Donaldson établit que le couplage de ces classes fondées sur la dualité de Poincaré donne $[C^+] \bigstar [C^-] = q_d(M)$, où $\bigstar$ représente le couplage des cycles d'homologie. Or, $d = 0$ correspond à la dimension 0 de l'espace des modules. Dans ce cas, comme montré dans [29], les invariants de Donaldson deviennent des entiers. En effet, l'évaluation de l'invariant $q_d(M)$ implique $d = \dim M_k/2$. La charge topologique k doit donc satisfaire l'égalité de Witten, soit $\dim M_k = 8\, p_1(E) - \frac{3}{2}\,(\chi(M) + \sigma(M))$, où $p_1(E)$ est le premier nombre de Pontryagin du fibré donnant la charge topologique de la configuration, $\chi(M)$ la caractéristique d'Euler et $\sigma(M)$ la signature de M. Nous avons alors $8p_1(E) - \frac{3}{2}(\chi(M) + \sigma(M)) = 0$ et l'espace modulaire M_k est réduit à un ensemble discret de points. Pour $\dim M_k = 0$, les invariants de Donaldson se réduisent à l'évaluation de la fonction de partition Z, exprimée comme une somme algébrique alternée sur les instantons :

$$Z = \sum_i (-1)^{n_i} \tag{34}$$

i désignant le i ème instanton et $n_i = 0$ ou 1 déterminant le signe de sa contribution à Z. Donaldson a montré sur des bases topologiques [25] que lorsque $\dim M_k = 0$, alors $\sum_i (-1)^{n_i}$ est un invariant topologique non polynomial, réduit à un entier. Nous retrouvons le même résultat à partir de $T_{\alpha\beta} = \left\{Q\,,\ \lambda_{\alpha\beta}\right\}$. La fonction de partition Z à la température $ß^{-1}$ a la forme générale $Z_q = \mathrm{Tr}\,(-1)^S\, e^{-ßH}$. Pour ß = 0, Z_q devient $\underset{ß=0}{Z} = \mathrm{Tr}\,(-1)^S$. Or, $\mathrm{Tr}\,(-1)^S$ est isomorphe à $\sum_i (-1)^{n_i}$, s et n_i donnant dans les deux cas le nombre d'instantons de la théorie. $\underset{ß=0}{Z} = \mathrm{Tr}\,(1)^S$ redonne donc le premier invariant de Donaldson, et projette la théorie physique Lorentzienne sur la limite topologique Euclidienne pour dim $M_k = 0$. Une autre manière de parvenir à ce résultat consiste à poser :

$$< P > = \frac{1}{Z} \sum_n DF \exp[-S] \; P(F)$$

Pour S = 0, l'on obtient, d' après Donaldson [25] $< P > = \frac{1}{Z} \sum_{\mathcal{M}_k} P_n$. Or, lorsque dim $M_k = 0$, $< P >$ se réécrit :

$$< P > = \frac{1}{Z} \sum_i (-1)^{n_i} = k,$$ de sorte que $$Z = \sum_i (-1)^{n_i}$$, comme requis.

L'essentiel de la démonstration ci-dessus a été repris dans la prop. (3.2) de l'article [1-A1]. A la limite des hautes températures $ß^{-1} = 0$ paramétrant l' échelle 0 de la théorie, la fonction de partition Z donne donc le premier invariant de Donaldson décrit par l'équ.(28), projetant la théorie physique Lorentzienne sur la limite topologique euclidienne.

Comme établi en réf.[3], ce qui précède suggère ainsi l'existence d'une profonde correspondance, - *une symétrie de dualité*- (voir la réf. [30]), entre théorie physique et théorie topologique.En effet, la théorie des champs considérée ici est une théorie thermique supersymmétrique [20,21] en dimension D=4 [31]. Le contenu du supermultiplet thermique a été détaillé dans un autre travail [3]. Comme précisé plus haut, la théorie appartient à la classe de supergravité N=2 [22], le Hamiltonien étant donné par le carré de l'opérateur de Dirac $\not{D}^2$ [11-29]. En tant que tel, le plus simple multiplet bosonique se réduit à un champ vectoriel et à deux scalaires exhibant une géométrie Kählérienne spéciale. En fait, la théorie N=2 est particulièrement intéressante dans notre contexte, pour deux raisons :

(**i**) Les champs scalaires complexes de la théorie (par exemple le dilaton S [30] ou le champ T [27]) peuvent être vus comme des "signatures" de la condition KMS [3] à laquelle pourrait être soumise l'espace-temps à l'échelle de Planck.De tels champs pourraient également être considérés comme l'une des meilleures clés pour comprendre la possible dualité entre observables physiques (infrarouge) et états topologiques (ultraviolet) :

$$\textit{Vide topologique } (\beta = 0,\ \textit{instanton}) \xleftrightarrow{i-\textit{dualité}} \textit{Vide physique } (\beta = \ell_{Planck},\ \textit{monopole})$$

Ceci est basé sur la dualité instantons / monopoles initialement suggérée en réf.[3] et récemment prouvée dans le contexte des supercordes C.P. Bacchas, P. Bain et M.B. Green [32]. En outre, toujours dans le contexte des supercordes a été conjecturée une symétrie du type U=S⊗T [3] à partir de laquelle l'on peut inférer la dualité ci-dessus entre observables (physique) et cycles (topologique) sur une 4-variété M :

Si à présent l'on associe, de manière naturelle, l'état "physique" de l'espace-temps à la forme Lorentzienne de la métrique (échelle de Planck) et l'état topologique à la forme Euclidienne (échelle zéro), alors il est également naturel d'en déduire qu'entre l'échelle zéro et l'échelle de Planck, il devrait exister une *superposition* (+++±) entre les structures métriques (et algébriques) Lorentzienne (physique) and Euclidienne (topologique).

2.4 Dualité entre mode Physique et mode topologique

Afin d'établir d'un point de vue algébrique les hypothèses de superposition et de dualité évoquées ci-dessus, nous suggérons à présent d'adopter la démarche proposée en réf[3], consistant à considérer la métrique d'espace-temps comme soumise à une q-déformation (i.e. déformation quantique au sens de la réf[13]) à l'échelle de Planck. Cet état quantique de la métrique donne lieu à une nouvelle description algébrique non plus en termes de groupes classiques mais de *groupes quantiques*. Dans ce sens, nous utilisons ici une application du résultat général obtenu en [3] sous la forme du produit bicroisé cocyclique :

$$\boldsymbol{M}_{\chi}(H) = H^{op} \overset{\psi}{\triangleright\!\blacktriangleleft} H_{\chi} \qquad (35)$$

où H est une algèbre de Hopf, $\triangleright\!\blacktriangleleft$ un produit bicroisé (i.e. un type spécial de produit croisé, défini en réf[12]) et χ un 2-cocycle ou"twist" au sens de Drinfeld [33]. Cette construction est inspirée par l'idée d'*unifier* deux groupes quantiques différents au sein d'une structure algébrique unique. Le point intéressant est que les deux groupes quantiques impliqués sont reliés dans l'équ.(35) par une relation de dualité, plus exactement de "semidualité", en un sens expliqué dans la réf[3]. Pour mettre en évidence cette propriété qui fournit un cadre algébrique à la dualité annoncée entre mode physique et mode topologique, nous proposons à présent de construire la semidualisation des données correspondant à

$$\mathrm{H} \overset{\psi}{\triangleright\!\blacktriangleleft} \mathrm{A}$$

La forme exacte de l'objet résultant $\mathrm{A}^{*} \underset{\chi}{\triangleright\!\triangleleft} \mathrm{H}$ a été conjecturée dans [3] mais demeure non explicite à certains égards, malgré les progrès effectués dans [34]. Alors, en rapportant les conditions ci-dessus aux éléments de A^{*} , nous avons :

Proposition 2.4.1 *Le produit bicroisé* $H \overset{\psi}{\triangleright\!\blacktriangleleft} A$ *admet la semidualisation suivante :*

(i) Considérant deux bigèbres X et H, X est un H-module cogébrique à gauche, i.e.
$\Delta(h \triangleright x) = h_{(1)} \triangleright x_{(1)} \otimes h_{(2)} \triangleright x_{(2)}$ *et* $\varepsilon(h \triangleright x) = \varepsilon(h)\varepsilon(x)$

(ii) H est un X-module cogébrique cocyclique à droite dans le sens nouveau :

$$\Delta(h \triangleleft x) = h_{(1)} \triangleleft x_{(1)} \otimes h_{(2)} \triangleleft x_{(2)} \quad et \quad \varepsilon(h \triangleleft x) = \varepsilon(h)\varepsilon(x)$$

$$(h_{(1)} \triangleleft x_{(1)}) \triangleleft y_{(1)}\chi(h_{(2)}, x_{(2)}, y_{(2)}) = \chi(h_{(1)}, x_{(1)}, y_{(1)})h_{(2)} \triangleleft (x_{(2)}y_{(2)})$$

où

$$\chi(h_{(1)} \triangleleft x_{(1)}, y_{(1)}, z_{(1)})\chi(h_{(2)}, x_{(2)}, y_{(2)}z_{(2)}) = \chi(h_{(1)}, x_{(1)}, y_{(1)})\chi(h_{(2)}, x_{(2)}y_{(2)}, z_{(2)})$$

$$\chi(h,1,x) = \chi(h,x,1) = \varepsilon(h)\varepsilon(x)$$

(iii) les deux algèbres de Hopf sont compatibles dans le sens

$h \triangleright 1 = \varepsilon(h),1 \quad 1 \triangleleft x = \varepsilon(x), \quad \chi(1,x,y) = \varepsilon(x)\varepsilon(y)$

et

(A) $\chi(h_{(1)},x_{(1)},y_{(1)})h_{(2)} \triangleright (x_{(2)}y_{(2)}) = h_{(1)} \triangleright x_{(1)} \quad (h_{(2)} \triangleleft x_{(2)}) \triangleright y$

(B) $(hg) \triangleleft x = h \triangleleft (g_{(1)} \triangleright x_{(1)}) \quad g_{(2)} \triangleleft x_{(2)}$

(C) $h_{(2)} \triangleright x_{(2)} \otimes h_{(1)} \triangleleft x_{(1)} = h_{(1)} \triangleright x_{(1)} \otimes h_{(2)} \triangleleft x_{(2)}$

(D) $\chi(hg,x,y) = \chi(h_{(1)},\ g_{(1)} \triangleright x_{(1)},(g_{(2)} \triangleleft x_{(2)}) \triangleright y_{(1)})\chi(g_{(3)},x_{(3)},y_{(2)})$

Remarque Il résulte de ces données l'existence d'un certain type de double produit croisé cocyclique de la forme

$X \underset{\chi}{\triangleright\!\triangleleft} H$, dont la structure, d'abord conjecturée dans [3], a été précisée dans [34]. A partir de (ii), il est clair qu'il s'agît d'une forme de quasi-algèbre de Hopf duale, où le produit serait associatif sous conjugaison par une fonctionnelle Φ construite à partir de χ.

Démonstration Pour réaliser la semidualisation, l'on suppose que A est de dimension finie. Soit X = A*. Les conditions résultantes conservent leur signification pour tout X. Le fait que A soit un H-module algébrique à droite implique que X est un H-module cogébrique à gauche, en accord avec

$\langle a \triangleleft h, x\rangle = \langle a, h \triangleright x\rangle \quad \forall x \in A^{*}$

Ensuite, nous définissons χ sur H $\otimes$ X $\otimes$ X

$\chi(h,x,y) = \langle x \otimes y, \psi(h)\rangle$

et l'on peut vérifier que H devient un X-module cogébrique à droite, comme annoncé. L'action de X est donnée ici par la coaction de A selon :

$h \triangleleft x = \left\langle x, h^{(\bar{1})}\right\rangle h^{(\bar{2})} \quad \forall h \in H$

Finalement, l'on parvient à semidualiser les conditions de compatibilité (A) et (D). Concernant (B) et (C), ils sont dualisés selon les formules de la réf [12] pour les produits bicroisés usuels (le cocycle n'intervient pas). Pour (A), nous considérons x $\otimes$ y pour obtenir

$$\chi(h_{(1)},x_{(1)},y_{(1)})\langle x_{(2)}y_{(2)}, a \triangleleft h_{(2)}\rangle = \langle x_{(1)},\ a_{(1)} \triangleleft h_{(1)}\rangle\langle y,\ a_{(2)} \triangleleft (h_{(2)} \triangleleft x_{(2)})\rangle$$

ou encore, en utilisant les définitions ci-dessus :

$$\chi(h_{(1)},x_{(1)},y_{(1)})\langle h_{(2)} \triangleright (x_{(2)}y_{(2)}),\ a\rangle = \langle h_{(1)} \triangleright x_{(1)}, a_{(1)}\rangle\langle (h_{(2)} \triangleleft x_{(2)}) \triangleright y,\ a_{(2)}\rangle$$

$$= \langle (h_{(1)} \triangleright x_{(1)})(h_{(2)} \triangleleft x_{(2)}) \triangleright y,\ a\rangle$$

pour tout a $\in$ A, qui est la condition (A)- énoncée. De même pour (D). □

A présent, appliquons la construction générale obtenue ci-dessus et considérons les structures algébriques Lorentzienne et Euclidienne.Alors, nous suggérons la proposition suivante (voir la prop(4.1) de la réf[1]) :

Proposition 2.4.2 *Les algèbres de Hopf Euclidienne et Lorentzienne sont reliées par le produit bicroisé cocyclique de la forme* $U_q(so(4))^{op} \overset{\psi}{\triangleright\!\blacktriangleleft} U_q(so(3, 1))$

Preuve Considérant l'approche en termes d'algèbres enveloppantes, à partir de l'algèbre de Hopf Euclidienne H = $U_q(so(4))$, nous avons la décomposition bien connue H = $U_q(su(2)) \otimes U_q(su(2))$ ainsi que l'algèbre "opposée" $H^{op} = U_q(su(2))^{op} \otimes U_q(su(2))^{op}$, tandis que la forme Lorentzienne est A = $H_\chi = U_q(su(2)) \blacktriangleright\!\blacktriangleleft U_q(su(2)) \cong U_q(so(3, 1))$. Comme expliqué en [3], le cocycle de déformation est $\chi = \mathfrak{R}_{23}$. Alors, l'action et la coaction sont :

$$(a \otimes b) \triangleleft (h \otimes g) = h_{(1)}aSh_{(2)} \otimes g_{(1)}bSg_{(2)}$$

$$ß(h \otimes g) = (h_{(1)} \otimes g_{(1)}).(Sh_{(3)} \otimes Sg_{(3)}) \otimes h_{(2)} \otimes g_{(2)}$$

$$= h_{(1)}Sh_{(3)} \otimes g_{(1)}Sg_{(3)} \otimes h_{(2)} \otimes g_{(2)} \qquad (36)$$

où l'on retrouve la structure de produit tensoriel de l'action et de la coaction pour chaque copie Uq(su(2)). D'autre part, la cocycle $h,g \in$ Uq(su(2)) est :

$$\psi(h\otimes g) = (h_{(1)}\otimes g_{(1)})(1\otimes \mathfrak{R}^{(1)})(Sh_{(4)}\otimes Sg_{(4)})(1\otimes \mathfrak{R}^{-(1)})\otimes$$
$$(h_{(2)}\otimes g_{(2)})(\mathfrak{R}^{(2)}\otimes 1)(Sh_{(3)}\otimes Sg_{(3)})(\mathfrak{R}^{-(2)}\otimes 1)$$

où le produit est en H = Uq(su(2))⊗ Uq(su(2)). Ceci donne :

$$\psi(h\otimes g) = h_{(1)}Sh_{(4)}\otimes g_{(1)}\mathfrak{R}^{(1)}Sg_{(4)}\mathfrak{R}^{-(1)}\otimes h_{(2)}\mathfrak{R}^{(2)}Sh_{(3)}\mathfrak{R}^{-(2)}\otimes g_{(2)}Sg_{(3)}$$

$$= h_{(1)}Sh_{(4)}\otimes g_{(1)}\mathfrak{R}^{(1)}Sg_{(2)}\mathfrak{R}^{-(1)}\otimes\ h_{(2)}\mathfrak{R}^{(2)}Sh_{(3)}\mathfrak{R}^{-(2)}\otimes\ 1 \qquad (37)$$

pour les structures explicites de produit bicroisé. **qed**

Clairement, la prop.(2.4.1) prouve la possible "unification" (à l'échelle de Planck dans notre modèle) entre les algèbres de Hopf q-Lorentzienne et q-Euclidienne.

Par ailleurs, le résultat ci-dessus suggère un certain type de dualité entre les groupes quantiques Lorentzien et Euclidien. Pour mettre en évidence cette dualité, l'étape suivante consiste à montrer l'existence d'une intéressante relation de "semidualité" (proposée dans le cas général par S. Majid [12]) entre algèbres de Hopf Lorentzienne et Euclidienne. Mieux, une telle dualité fournit une description de la *transition* entre le groupe quantique q-Euclidien et le groupe quantique q-Lorentzien [3]. D'où la proposition :

Proposition 2.4.3 *Le groupe quantique Euclidien* $U_{q^{-1}}(su(2)) \otimes U_q(su(2)) \cong U_q(su(2))^{op} \triangleright\!\blacktriangleleft U_q(su(2))$ *est connecté par semidualité au groupe quantique Lorentzien* $U_q(su(2)) \triangleright\!\triangleleft U_q(su(2))^{op*} \cong \mathfrak{D}(U_q(su(2)))$. *Alors, la semidualité relie une version de* Uq(so(4)) (*Euclidien) à une version de* Uq(so(3, 1)) (*Lorentzien*).

Il existe dans la réf. [3] une démonstration complète de la prop. (2.4.2), basée sur les propriétés du "double de Drinfeld" $\mathfrak{D}$(Uq(su(2))). Alors, utilisant la construction en termes de cocycle $M_\chi(H)$, nous obtenons la relation :

$$\mathrm{U_q(su(2))} \overset{\psi}{\triangleright\!\blacktriangleleft} \mathrm{U_q(su(2))} \cong \mathrm{U_q(so(4))} \xleftrightarrow{\textit{semidualisation}} \mathrm{U_q(su(2))^*} \underset{\chi}{\triangleright\!\triangleleft} \mathrm{U_q(su(2))} \sim \mathrm{U_q(so(3,1))} \tag{38}$$

La "q-déformation" de l'algèbre de Hopf q-Euclidienne vers l'algèbre de Hopf q-Lorentzienne correspond à une transformation de dualité et induit l'existence d'un 2-cocycle de déformation. De même, le produit bicroisé cocyclique

$$\mathrm{Uq(so(4)^{op}} \overset{\psi}{\triangleright\!\blacktriangleleft} \mathrm{Uq(so(3,\ 1))} \tag{39}$$

définit implicitement la nouvelle transformation de (semi)dualité :

$$\mathrm{U_q(so(4))^{op}} \overset{\psi}{\triangleright\!\blacktriangleleft} \mathrm{U_q(so(3,\ 1))} \cong \mathrm{U_q(so(4))} \xleftrightarrow{\textit{semidualisation}} \mathrm{SO_q(3,1)} \underset{\chi}{\triangleright\!\triangleleft} \mathrm{U_q(so(4))^{op}}$$

où χ est construit à partir de ψ, celui-ci étant dérivé de la structure quasitriangulaire $\mathfrak{R}$ de $\mathrm{U_q(su(2))}$.

Naturellement, l'on peut également semidualiser à partir des autres facteurs pour construire certains types de quasi-algèbres de Hopf $A \overset{\chi}{\blacktriangleright\!\blacktriangleleft} H^*$, associé à $H \overset{\psi}{\triangleright\!\blacktriangleleft} A$. Cette fois, la coaction cocyclique de A sur H est dualisée en une coaction cocyclique de A sur H^* tandis que l'action de H sur A est remplacée par une coaction de H^* sur A. La construction devient alors générale, de la forme $A \overset{\chi}{\blacktriangleright\!\blacktriangleleft} Y$ (où Y joue le rôle de H^*). L'on obtient ainsi des exemples du type $\mathrm{Uq(su(2))} \overset{\chi}{\blacktriangleright\!\blacktriangleleft} \mathrm{Uq(su(2))^*}$, $\mathrm{Uq(so(3,\ 1))} \overset{\chi}{\blacktriangleright\!\blacktriangleleft} \mathrm{SOq(4)^{cop}}$ etc., par semidualisation de cette forme. Ceux-ci sont duaux des constructions précédentes.

A présent, une conséquence intéressante de ces résultats concerne les propriétés de dualité au niveau de la q-déformation de l'espace-temps lui-même. En effet, à la lumière des constructions précédentes, l'on parvient à l'importante observation qui suit concernant la transition de la métrique q-Euclidienne à la métrique q-Lorentzienne:

Corollaire 3.4.4 *Pour $q \neq 1$, la transition de la métrique q-Euclidienne à la métrique q-Lorentzienne au rayon unité est une dualité de $*$ -algèbre de Hopf Uq(su(2)) $\leftrightarrow$ SUq(2).*

Démonstration Selon une construction introduite par Majid [12,13] et appliquée dans la réf[3], l'on décrit le q-espace-temps $\mathbb{R}^{3,1}$ par l'algèbre tressée BMq(2), Mq(2) ayant une $*$ - structure unitaire correspondant à SUq(2) (par $*$ - structure, nous entendons "structures réelles", au sens précisé dans la réf[12]). Ici, BMq(2) admet une description en tant que matrices hermitiennes tressées. L'on écrit alors $\mathbb{R}_q^{3,\,1} \equiv$ BMq(2). Par ailleurs, l'on note $\overline{\mathrm{Mq}(2)}$ la structure algébrique résultat du twist (au sens de la réf[13]) de Mq(2). Enfin, il a été montré [3] que la $\bigstar$ - structure de $\mathbb{R}_q^4 = \overline{\mathrm{Mq}(2)}$ est donnée par la $*$ - structure unitaire de Mq(2) qui, au rayon $\rho = 1$, donne le $*$ - groupe quantique SUq(2), construction duale de celle associée à la $*$ - algèbre de Hopf Uq(su(2)). Explicitement, la $\bigstar$ - structure de $\mathbb{R}_q^4 \equiv \overline{\mathrm{Mq}(2)}$ est $\begin{pmatrix} a^{\bigstar} & b^{\bigstar} \\ c^{\bigstar} & d^{\bigstar} \end{pmatrix} = \begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix}$ et coïncide avec celle de la $*$ - structure unitaire de Mq(2) sur l'identification des deux espaces vectoriels. Or, Uq(su(2)) $\cong$ B(Uq(su(2))) en tant que $*$ - algèbre sous transmutation [12]. Cette transformation, combinée avec l'auto-dualité de ces groupes de tresse, donne l'isomorphisme de $*$ - algèbre Uq(su(2)) $\cong$ BSUq(2) comme expliqué ci-dessus. Il en résulte (cf. [3]) que les structures naturelles de l'espace q-Euclidien $\mathbb{R}_q^4$ et celle du q-Lorentzien $\mathbb{R}_q^{3,\,1}$, covariantes sous Uq(so(4)) et Uq(so(3, 1)) [32] sont reliées comme suit :

$$\begin{tikzcd} U_q(su(2)) \arrow[rr, leftrightarrow, "\text{dualité de } *\text{-algèbres de Hopf}"] \arrow[d, leftrightarrow, "\textbf{Transmutation}"', "\approx"] \arrow[drr, phantom, "\approx"] & & SU_q(2) \sim \mathbb{R}_q^4 / \rho = 1 \arrow[d, leftrightarrow, "\textbf{q- changement de signature}"] \\ BU_q(su(2)) \arrow[rr, leftrightarrow, "\text{autodualité de groupes } \bigstar\text{- tressés}"'] & & BSU_q(2) = \mathbb{R}_q^{3,\,1} / \rho = 1 \end{tikzcd} \tag{40}$$

et cette construction rend explicite le changement de signature comme équivalent à une dualité de $*$ - algèbre de Hopf. □

L'on remarquera que nous obtenons une relation de dualité entre les q-espaces $\mathbb{R}_q^4$ et $\mathbb{R}_q^{3,\,1}$ comme une sorte de T-dualité [27]. Cette interprétation est possible seulement lorsque $q \neq 1$ - i.e. à l'échelle de

Planck -. L'on peut étendre ces résultats, obtenus à partir des groupes quantiques Lorentzien et Euclidien, au groupe de q-Poincaré

$$\mathbb{R}_q^{3,\,1} > \!\!\overline{\lhd}\; U_q(\widetilde{so}\,(3\;,\,1)) \tag{41}$$

vu, bien sûr, comme dual du groupe de q-Poincaré Euclidien

$$\mathbb{R}_q^{4} > \!\!\overline{\lhd}\; U_q(\widetilde{so}\,(4)) \tag{42}$$

En revenant sur la remarque précédente, la dualité d'algèbre de Hopf a été récemment reliée à la T-dualité en théorie des supercordes par C.Klimcik et P.Sevara [36]. De telles dualités en termes de groupes quantiques ont également été proposées par S. Majid [13].

De manière intéressante, les résultats ci-dessus fournissent ainsi certaines indications sur l'origine algébrique de la fluctuation de signature à l'échelle de Planck, considérée comme transformation de dualité. Une remarque importante est que certains des isomorphismes ci-dessus sont valides seulement lorsque $q \neq 1$, i.e. en théorie non classique. Notons également que la dualité d'algèbres de Hopf au niveau semi-classique est une dualité de bigèbres de Lie et a été comprise physiquement comme une T-dualité non abélienne pour des modèles σ sur G, G^* [36], de sorte que la dualité mise en évidence ici est reliée à d'autres types de dualités en physique.

A présent, appliquons les résultats algébriques obtenus précédemment dans un contexte plus physique. En effet, la dualité entre les groupes quantiques Lorentzien et Euclidien peut être étendue à une dualité entre secteur "physique" (Lorentzien) et secteur "topologique" (Euclidien) de la théorie. D'où la proposition formulée au § (4.3) de l'article [1-A1] :

Proposition 2.4.4 *Il existe, à l'échelle de Planck, une symétrie de dualité entre l'anneau de cohomologie BRST (secteur physique de la théorie) et l'anneau de cohomologie de l'espace des modules des instantons (secteur topologique)*

Ainsi, partant de la forme générique des groupes de cohomologie BRST(cf. réf. [22]), soit

$$H^{(g)}_{BRST} = \frac{\ker Q^{(g)}_{BRST}}{\mathrm{im} Q^{(g-1)}_{BRST}} \tag{43}$$

nous montrons à la prop. (4.3) de (1-A1) que la théorie topologique réalise l'injection d'anneaux :

$$H^{\bigstar}_{BRST} = \otimes^{\Delta U_k}_{g=0} \; H^{g}_{BRST} \xrightarrow{\iota} H^{\bigstar}\left(\mathcal{M}^{(k)}_{\mathrm{mod}}\right) = \otimes^{d_k}_{i=0} H^{(i)}\left(\mathcal{M}^{(k)}_{\mathrm{mod}}\right) \tag{44}$$

l'équ. (44) fournissant un chemin injectif du mode physique dans le mode topologique. En termes d'observables 0_i et de cycles d'homologie $H_i \subset \mathrm{M}_{\mathrm{mod}}$ dans l'espace des modules $\mathrm{M}_{\mathrm{mod}}$ des configurations du type instantons gravitationnels $\Im[\phi(x)]$ sur les champs gravitationnels ϕ de la théorie, nous relevons l'équivalence :

$$\langle 0_1 0_2 \; ... \; 0_n \rangle = \#(H_1 \cap H_2 \cap \; ... \; \cap H_n)$$

où le secteur physique de la théorie est décrit par les observables 0_i et le secteur dual, de type topologique, par les cycles d'homologie $H_i \subset \mathrm{M}_{\mathrm{mod}}$. L'oscillation de signature entre secteur physique et secteur topologique est alors induite par la divergence $\Delta U_k = \int \partial^{\mu} j_{\mu} \, d^4x$ du courant-fantôme j_{μ} [22]. Lorsque $\Delta \mathrm{U} = 0$, comme il n'existe pas d'espace de plongement pour l'espace des modules, nous suggérons que la théorie est alors projetée dans la branche de Coulomb, à l'origine de $\mathrm{M}_{\mathrm{mod}}$, sur un instanton singulier de taille 0 [37] que nous identifions à l'espace-temps à l'échelle 0. La théorie est ramifiée sur le secteur purement topologique H_i, la signature correspondant à ce secteur étant Euclidienne (+ + + +).

2.5 Transition entre état topologique et état physique

Nous concluons cette section par une question importante : comment peut-on expliquer, d'un point de vue cosmologique, la transition de l'état topologique à l'état physique de l'espace-temps? Nous suggérons une approche dans le cadre de la théorie KMS dans le § (5.2) de l'article [1-A1]. Cette approche est formalisée dans la conjecture suivante :

Conjecture 2.5.1 *A l'échelle infrarouge* $ß \geq \ell_{Planck}$*, la brisure de l'état KMS du (pré)espace-temps induit le découplage entre le flot topologique et le flot physique de la théorie.*

Rappelons que nous avons montré en [1-A1] que l'état KMS du (pré)espace-temps à l'échelle de Planck induit de manière naturelle le couplage entre l'état physique et l'état topologique de la métrique. Par conséquent, l'hypothèse nouvelle développée en [1-A1] est que la transition entre état topologique (à l'échelle zéro) et état physique de l'espace-temps (au delà de l'échelle de Planck) peut être décrite en termes de brisure de l'état KMS. Une telle approche est d'ailleurs renforcée par le fait que la brisure d'état KMS au delà de l'échelle de Planck pourrait elle-même être liée à la brisure de supersymétrie attendue à la même échelle. Cette importante relation est d'ailleurs annoncée par les intéressants résultats de Deredinger et Luchiesi dans la réf [20], résultats que nous détaillons au §(5.2.1) de [1-A1].

En fait, Derrendinger et Lucchesi ont clairement confirmé l'existence d'une étroite relation entre la supersymétrie thermique et la condition KMS. Cette relation est réalisée au niveau des coordonnées de Grassman thermiques, en raison d'une condition d'(anti)périodicité décrite par les équations (45) et (46) :

$$\hat{\theta}^{\alpha}(t+iß) = -\hat{\theta}^{\alpha}(t) \tag{45}$$

$$\hat{\bar{\theta}}^{\dot{\alpha}}(t+iß) = -\hat{\bar{\theta}}^{\dot{\alpha}}(t) \tag{46}$$

Les auteurs ont prouvé de manière convaincante qu'en l'absence de la corrélation supersymétrie/état KMS au niveau de la métrique d'espace-temps, les bosons (périodiques) et les fermions (antipériodiques) ne *peuvent pas* appartenir au même multiplet de supersymétrie. A présent, que se passe-t-il au delà de l'échelle de Planck, lorsque l'état KMS est brisé? Dans ce cas, un point X du superespace est à nouveau muni des coordonnées de Grassmann usuelles

$$X = (x^{\mu}, \theta^{\alpha}, \bar{\theta}^{\alpha})$$

Cette condition est équivalente à la supersymétrie à température nulle, pour laquelle les paramètres de transformation (i.e. les Grassmanniennes θ and $\overline{\theta}$) sont des *constantes*. Or précisément, le résultat principal des réfs [20,21] établit qu'à température finie, il est impossible d'utiliser des paramètres constants dans les règles de transformations de supersymétrie. Les paramètres de supersymétrie doivent être des variables dépendant du temps, (anti)périodiques en temps imaginaire. Ainsi, d'une manière naturelle, les coordonnées thermiques Grassmanniennes $\hat{X} = (x^{\mu}, \hat{\theta}^{\alpha}(t), \hat{\overline{\theta}}^{\dot{\alpha}}(t))$ doivent être "translatées" en temps imaginaire et sont par conséquent soumises aux conditions d'antipériodicité $\hat{\theta}^{\alpha}(t+iß) = -\hat{\theta}^{\alpha}(t)$ et $\hat{\overline{\theta}}^{\dot{\alpha}}(t+iß) = -\hat{\overline{\theta}}^{\dot{\alpha}}(t)$ des équ.(45) et (46). Comme montré en [1-A1], l'application de ces conditions implique que globalement, le système espace-temps doit être soumis à la condition KMS à l'échelle de Planck. Cet important résultat peut être considéré comme une confirmation de la conjecture (5.5) proposée en réf. [1-A1].

Nous concluons notre étude de l'échelle zéro de l'espace-temps par la recherche d'une possible amplitude topologique caractérisant l'échelle zéro.

CHAPITRE 3

AMPLITUDE TOPOLOGIQUE
A L'ECHELLE ZERO

Nous avons suggéré en [1-A1] que la Singularité Initiale de l'espace-temps, non réductible dans le cadre habituel de la théorie quantique des champs, peut être décrite dans le cadre de la théorie topologique des champs (voir section 2). De ce point de vue, l'échelle zéro de l'espace-temps peut être identifiée à un instanton gravitationnel singulier de taille zéro [3, 37]. Nous poussons ici l'une des conséquences de cette identification, relevant de l'existence d'une "amplitude topologique" à l'échelle zéro de l'espace-temps.

3.1 Charge topologique de l'instanton singulier de taille zéro

Nos principaux résultats publiés en [16-A4] suggèrent que la géométrie de l'instanton peut être identifiée à celle de la boule B^4 dont le bord est la sphère S^3. L'on peut alors montrer (voir encore [16-A4]) que le bord de l'espace-temps peut être identifié au bord S^3 de l'instanton gravitationnel singulier B^4. Dans un tel contexte, l'échelle zéro de l'espace-temps (i.e. la Singularité Initiale) peut être entièrement caractérisée par la charge topologique Q_S de l'instanton gravitationnel singulier B^4, soit

$$Q_S = \int d^4x \, R_{\mu\nu}\tilde{R}^{\mu\nu} \tag{47}$$

Par construction, Q_S est un invariant topologique, indépendant de la taille de l'instanton et qui reste donc défini à l'échelle ß = 0. De ce point de vue, la "propagation" (i.e. pseudo-dynamique

Euclidienne au sens fixé en ([16]) de la singularité initiale est induite par l'existence de l'amplitude topologique associée à Q_S, détectable sur le bord S^3 de l'instanton gravitationnel muni de la topologie B^4. Les pseudo-observables sont ici interprétées comme cocycles sur l'espace des modules des instantons et sont associées aux cycles γ_i de la 4-variété B^4 (application de Donaldson). Considérant un point X de B^4, l'amplitude topologique assurant la propagation de la charge instantonique Q_S prend alors la forme:

$$\left\langle 0_{S^3} .\ 0_X \right\rangle = \#(S^3, X) \tag{48}$$

L'amplitude topologique de la théorie est donnée par les pseudo-observables du membre de gauche, tandis que le membre de droite désigne le nombre d'intersections des $\gamma_i \subset B^4$. La fonction $\#(S^3, X)$ est nulle si le point X est situé hors de la sphère S^3 et vaut 1 si X est à l'intérieur de S^3 (i.e. si $X \in B^4$), cas où il existe une amplitude topologique.

3.2 Conjecture : origine topologique de l'interaction inertielle

A titre d'application nous conjecturons une approche nouvelle, selon laquelle l'interaction inertielle pourrait être correctement décrite dans le cadre de la théorie topologique des champs, proposée par Edward Witten en 1988 [2]. Plus précisément, nous suggérons en réf. [16-A4] que le caractère non local de la charge topologique Q_S peut être relié à la nature non locale de l'interaction inertielle. Nous conjecturons alors que cette propriété observable ne peut être expliquée en théorie quantique des champs mais pourrait trouver une solution dans le cadre de la théorie topologique des champs. En effet, l'évaluation de la contribution inertielle (ou potentiel inertiel) totale résultant de la somme des masses de l'univers, de la forme :

$$\mathbf{U}_{\text{inertiel total}} = \sum_{univers} \frac{GM}{c^2 r} \approx 1 \tag{49}$$

s'avère être un invariant pour chaque masse locale. Or, la charge topologique de l'instanton gravitationnel singulier, de la forme

$$Q = \frac{1}{32\pi^2}\int d^4x \, R_{\mu\nu}\tilde{R}^{\mu\nu} = 1 \qquad (50)$$

représente également un invariant de type topologique. L'égalité entre la masse inertielle et la masse gravitationnelle est ici expliquée en termes de quantification de la charge topologique de l'instanton gravitationnel singulier. Nous en tirons un modèle de "pseudo-propagation" de l'amplitude topologique $\mathbf{Int}_{top}$, susceptible d'être décrite par les transformations conformes Conf (S^3) de la sphère S^3(voir, par exemple la réf. [38] pour rappel). Conf (S^3) peut être décrit par le groupe de Möbius Möb(3) [38], défini à partir de l' inversion de S^3. D' où :

Proposition 3.2.1 *Pour toute similitude* $h \in \mathrm{Sim}(\mathbb{R}^3)$, *l'application générale définissant la charge topologique de l'instanton, i.e.* $f : S^3 \to S^3$ *, définie par* $f(n) = n$ *et* $f = g^{-1} \circ h \circ g$ *sur* $S^3 \setminus n$ *appartient au groupe de Möbius Möb(3), groupe conforme de* S^3.

La prop. (3.2.1) a été établie dans les réf.[3] et [16-A4].

L'on poursuit en suggérant dans la prop. (3.2.2) que Möb(3) est le groupe conforme Conf (S^3) de S^3. Posons que Conf (S^3) décrit l'invariance d'échelle (i.e. invariance conforme) de la sphère identifiée ici, suivant l'inclusion $S^3 \subset SL(2, C)$, à l' espace physique, compactifié de $\mathbb{R}^3$.

Proposition 3.2.2 *Soit* $\text{Möb}^{\pm}(3) = \text{Conf}^{\pm}(S^3)$. $\forall$ *le rayon* $r \to 0$ *de* S^3 *engendrant* $S^3_{r\to 0}$, *et* $\forall f \in$ *Möb(3), alors* $S^3_{r\to 0}$ *appartient au faisceau* $f(S^3)$*de sphères* S^3. *Réciproquement, une bijection de* S^3 *vérifiant cette propriété appartient à Möb (3) . Le groupe Möb (3) présente un isomorphisme naturel avec* $PO(\alpha)$ *de la quadrique d' équation* $q = -\sum_{i=1}^{4} x^2 + x_5^2$.

La prop.(3.2.2) a été établie en réf[3]. Dans un tel cadre, la principale conjecture de l'article ci-joint en annexe A4 est alors que le fondement sur lequel repose le "principe de Mach" [39] (tout comme l'interaction inertielle) ne doit pas être considérée comme classiquement "physique" mais relève de la théorie topologique des champs. Nous tirons de notre approche cf. [16-A4], l'existence d'un "principe de Mach topologique", explicité dans la conj. (3.2.3).

Conjecture 3.2.3 *Les amplitudes topologiques associées à la propagation de la charge topologique de l'instanton gravitationnel singulier de taille zéro correspondant à la Singularité Initiale de l'espace-temps déterminent le comportement inertiel des masses locales.*

A titre d'illustration de la conjecture 3.2.3, nous considérons l'expérience du pendule de Foucault $\mathcal{F}$, qui ne peut trouver d'explication satisfaisante en mécanique classique ou relativiste [40]. Rappelons que le problème essentiel consiste en l'invariance angulaire du plan d'oscillation de $\mathcal{F}$. Alors, le "principe de Mach topologique" énonce que l'interaction entre $\mathcal{F}$ et l'espace-temps global **E** au sens de Mach est de type topologique - ce qui pourrait expliquer les propriétés d'invariance globale du système formé par la plan d'oscillation de $\mathcal{F}$ et le reste de l'univers.

Finalement, les perspectives conjecturales introduites dans la réf [16-A4], confèrent une certaine pertinence à la proposition de "principe de Mach topologique" et, plus généralement, à l'approche topologique du problème de la singularité initiale. L'on peut espérer que cette approche préliminaire permettra d'ouvrir des perspectives nouvelles sur l'origine de l'espace temps ainsi que sur plusieurs autres questions non résolues (notamment celles concernant la nature de l'interaction inertielle).

CONCLUSION

Nos principaux résultats, à travers notamment la théorie des groupes quantiques [12], les algèbres d'opérateurs et la géométrie non commutative [10] et, enfin, la théorie topologique des champs [2] suggèrent l'existence d'une possible transition de phase concernant la métrique de l'espace-temps depuis l'échelle topologique zéro jusqu'à l'échelle physique (au delà de l'échelle dent Planck). Ces résultats font suite à ceux obtenus par G.Bogdanoff dans [3] concernant la fluctuation attendue de la signature de la métrique à l'échelle de Planck.

A ce stade, nous résumons nos résultats les plus significatifs :

(**i**) la métrique de l'espace-temps à l'échelle zéro peut être considérée comme Euclidienne (++++) i.e. *topologique* ;

(**ii**) la singularité initiale de l'espace-temps pourrait être décrite par identification à un instanton singulier de taille zéro ;

(**iii**) à partir de (**i**) et (**ii**), nous suggérons l'existence d'une symétrie de dualité (que nous appelons *i* -dualité), entre état physique (échelle de Planck) et état topologique (échelle zéro) de l'espace-temps. Une telle symétrie découle directement de la condition KMS à laquelle l'espace-temps devrait être soumis à l'échelle de ¨Planck.

Alors, la résolution possible de la Singularité Initiale dans le cadre de la théorie topologique nous amène à envisager l'existence, avant l'échelle de Planck, d'une première phase d'expansion purement topologique du (pré)espace-temps, paramétrée par la croissance de la dimension de l'espace des modules dim***M*** et décrite par la "pseudo-dynamique" Euclidienne

$$\sigma_{ß}(M^{0,1}_{Top}) = e^{-ßH}\; M^{0,1}_{Top}\; e^{ßH}$$

Ainsi, la chaîne d'évènements susceptibles d'expliquer la transition depuis la phase topologique zéro jusqu'à la phase physique de l'espace-temps (au delà de l'échelle de Planck) pourrait prendre la forme suivante :

$$\{brisure\ \ de\ \ Supersymmetrie\}$$
$$\rightarrow \{brisure\ \ de\ \ l'\acute{e}quilibre\ \ thermodynamique\}$$
$$\rightarrow \{brisure\ \ de\ \ l'\acute{e}tat\ \ KMS\}$$
$$\rightarrow \{d\acute{e}couplage\ \ temps\ \ imaginaire\ \ /\ temps\ r\acute{e}el\}$$
$$\rightarrow \{d\acute{e}couplage\ \ \acute{e}tat\ \ topologique\ \ /\ \acute{e}tat\ \ physique\}$$

En termes de C*-algèbres, les transformations ci-dessus sont données par :

$$II_\infty \otimes \mathbb{R}_+^* \xrightarrow[0<ß<\ell_P]{\text{Flot KMS}} \mathbf{Q}^f_ß \rightarrow \alpha_{ß_c}(\boldsymbol{M_q}) = e^{-ß_c H}\boldsymbol{M_q}\, e^{ß_c H} \quad \begin{cases} \xrightarrow{\text{Flot topologique}} \mathbf{T}^f_{ß=0} \rightarrow \alpha_ß(\boldsymbol{M}^{0,1}_{Top}) = e^{-ßH}\boldsymbol{M}^{0,1}_{Top}\, e^{ßH} \\ \xrightarrow{\text{Flot physique}} \mathbf{P}^f_{ß>0} \rightarrow \alpha_t(\boldsymbol{M}_{Phys}) = e^{iHt}\boldsymbol{M}_{Phys}\, e^{iHt} \end{cases}$$

Enfin, nous tirons des résultats de la présente recherche l'existence de trois phases au cours de l'expansion du (pré)espace-temps :

(i) - *Echelle zéro (Singularité Initiale) : état topologique pur*

(ii)- *Echelle de Planck (état KMS) : état topologique + état physique*

(iii)- *Echelle classique (brisure de l'état KMS) : état physique pur*

L'existence de la phase (ii) de fluctuation de la signature de la métrique de l'espace-temps à l'échelle de Planck a été établie dans la réf.[3]. Les conclusions de [3] sont d'ailleurs renforcées par les approches récentes de compactification 4D → 3D de l'espace-temps à l'échelle quantique (sur ce point, voir par exemple la réf[41]. En effet, il est généralement admis qu'à haute température (température de Planck) une théorie dynamique (i.e. la théorie des champs) perd un degré de liberté par réduction dimensionnelle de la direction genre temps. Dans ce nouveau contexte, il convient par exemple de relever que la théorie physique de Yang et Mills est transformée en une théorie de Jauge à trois dimensions. Le point intéressant est alors que le terme de Chern-Simon, introduit par Deser, Templeton et Jackiw [42], devient naturellement relevant à l'échelle correspondant au couplage entre théorie topologique à quatre dimensions et théorie physique. Or, le terme de Chern-Simon étant, à l'échelle de Planck, associé à une 3-surface genre espace, sa présence dans la théorie apparaît ici comme la *condition de fluctuation* de la signature de la métrique entre la direction genre temps et la direction genre espace. Ceci simplement parce que le système devenant indépendant de la quatrième coordonnée, celle-ci peut être indifféremment considérée comme genre temps et/ou genre espace sans que le terme de Chern-Simon ne soit perturbé.

Dans un travail ultérieur, nous proposons de développer l'idée selon laquelle à l'échelle zéro, la dynamique Lorentzienne devrait être remplacée par une "dynamique Euclidienne" (ou pseudo-dynamique) intrinsèque. Cette pseudo-dynamique Euclidienne, engendrée par les automorphismes non-stellaires du facteur "topologique" $M_{Top}^{0,1}$ implique, suivant les résultats de la réf [7], une "croissance spectrale" du diamètre de l'espace des états $d(\varphi, \psi)$ en temps Euclidien (dual de l'espace des observables en temps Lorentzien). Cette pseudo-dynamique, liée aux automorphismes de semi-groupe $\sigma_ß(M_{Top}^{0,1})$ peut être décrite de manière naturelle par le flot des poids (dans le sens de Connes-Takesaki [43]) de l'algèbre $\boldsymbol{M}_q$. Ceci achève de relier, à l'échelle zéro de l'espace-temps, le contenu topologique de la Singularité Initiale à la première phase d'expansion du pré-univers, dont la source, de manière inattendue, pourrait être également topologique.

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ANNEXES

Note : Les tirés à part des articles originaux ont été annexés au manuscrit de la thèse. Pour des raisons techniques, il n'a pas été possible de les faire figurer dans ce dossier et ils ont été remplacés par leurs préprints en PDF.

PUBLICATIONS ANNEXEES

- " Topological Field Theory of the Initial Singularity of Spacetime "
  Class. and Quantum Gravity vol **18** n° 21 (2001)

- " Spacetime Metric and the KMS Condition at the Planck Scale "
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- " KMS State of the Spacetime at the Planck Scale "
  Ch. J. of Phys. (2002)

- " Topological Origin of Inertia "
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Topological field theory of the initial singularity of spacetime

Grichka Bogdanoff and Igor Bogdanoff
Mathematical physics Laboratory CNRS UPRES A 5029 BOURGOGNE University, France

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Abstract

We suggest here a new solution of the initial space-time singularity. In this approach the initial singularity of space-time corresponds to a zero size singular gravitational instanton characterized by a Riemannian metric configuration (++++) in dimension D = 4. Connected with some unexpected topological datas corresponding to the zero scale of space-time, the initial singularity is thus not considered in terms of divergences of physical fields but can be resolved in the frame of topological field theory. Then it is suggested that the "zero scale singularity" can be understood in terms of topological invariants (in particular the first Donaldson invariant $\sum_i (-1)^{n_i}$). In this perspective we here introduce a new topological index, connected with 0 scale, of the form $Z_{ß=0} = \mathrm{Tr}\ (-1)^s$, which we call "singularity invariant". Interestingly, this invariant corresponds also to the invariant topological current yield by the hyperfinite II_∞ von Neumann algebra describing the zero scale of space-time. Then we suggest that the (pre)space-time is in thermodynamical equilibrium at the Planck scale state and is threrefore subject to the KMS condition. This might correspond to a unification phase between "physical state" (Planck scale) and "topological state" (zero scale). Then we conjecture that the transition from the topological phase of the space-time (around the scale zero) to the physical phase observed beyond the Planck scale should be deeply connected to the supersymmetry breaking of the N=2 supergravity.

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INTRODUCTION

One of the limits of the standard space-time model remains its inability to provide a description of the singular origin of space-time. Here we suggest, in the context of N=2 supergravity, that the initial singularity, associated with zero scale of space-time, cannot be described by (perturbative) physical theory but might be resolved by a (non-perturbative) dual theory of topological type. Such an approach is based on our recent results [6-7] concerning the quantum fluctuations (or q-superposition) of the signature of the metric at the Planck scale. We have suggested that the signature of the space-time metric (+++–) is not anymore frozen at the Planck scale ℓ_p and presents quantum fluctuations (+++±) until zero scale where it becomes Euclidean (++++). Such a suggestion appears as a natural consequence of the non-commutativity of the space-time geometry at the Planck scale [11]. In this non-commutative setting, we have constructed (cf. 4.1) the "cocyle bicrossproduct" [6] :

$$U_q(so(4))^{op} \overset{\psi}{\triangleright\!\blacktriangleleft} U_q(so(3,1)) \tag{1}$$

where $U_q(so(4))^{op}$ and $U_q(so(3,1))$ are Hopf algebras (or "quantum groups"[16]), the symbol $\triangleright\!\blacktriangleleft$ a (bi)crossproduct and ψ a 2-cocycle of deformation (for more specific definitions, see ref[29]). The bicrossproduct (1) suggests an unexpected kind of "unification" between the Lorentzian and the Euclidean Hopf algebras at the Planck scale and yields the possibility of a "q-deformation" of the signature from the Lorentzian (physical) mode to the Euclidean (topological) mode [6-30]. Moreover equ.(1) defines implicitly a (semi)duality transformation between Lorentzian and Euclidean quantum groups (see equ.(42)). This is important insofar we consider that the Euclidean theory is the simplest topological field theory.

In other respect, it has been stated in string theory [25] that the behavior of string amplitudes at very high temperature (Hagedorn limit) reveals the existence of a possible phase transition and the restoration of large-scale symmetries of the system. In the context of this "unbroken phase", generally expected at the Planck scale, the theory is characterized by a general covariance preserving the exact symmetry of the system. The metric $g_{\mu\nu}$ is developed around zero and there exists at this level neither light cone, wave propagation, nor movement. The exploration of this unbroken (and non-physical) phase of the system is accessible only in the framework of a new kind of field theory proposed by E. Witten under the name "topological field theory" [37].

Topological field theory is usually defined as the quantization of zero, the Lagrangian of the theory being either (**i**) a zero mode or (**ii**) a characteristic class $c_n(V)$ of a vectorial bundle $V \xrightarrow{\pi} M$ built on space-time [31]. Starting from the Bianchi identity Tr$R \wedge R^* = \frac{1}{30}$ Tr$F \wedge F^*$, our approach of 4D supergravity leads us to describe the energy content of the pre-space-time system by the curvature ***R***. We therefore put $\mathcal{L} \sim R \wedge R^*$. The value of the topological action $S_{class} = \int_M \mathcal{L}_{class} = \int_M c_n(V) = k \in \mathbb{Z}$ is then either zero or corresponds to an integer. The topological limit of quantum field theory, described in particular by the Witten invariant Z = Tr $(-1)^n$ [36] is then given by the usual quantum statistical partition function taken over the (3+1) Minkowskian space-time

$$\mathbf{Z} = \mathrm{Tr}\,(-1)^n \; e^{-ßH} \tag{2}$$

with $ß = \frac{1}{kT}$ and n being the zero energy states number of the theory, for example the fermion number in supersymmetric theories [1]. Then **Z** describes all zero energy states for null values of the Hamiltonian H.

Now, we propose here (§(1.2)) a *new topological limit* of quantum field theory, non-trivial (i.e. corresponding to the non-trivial minimum of the action). Built from scale $ß \rightarrow 0$ and *independent* of H, this unexpected topological limit (in 4D dimensions) is then given by the temperature limit (Hagedorn temperature) of the physical system (3+1)D. In a way this can be derived from the "holographic conjecture" [42] following which the states of quantum gravity in d dimensions have a natural description in terms of a $(d-1)$-dimensional theory. In agreement with [4-34-39] and, in particular, the recent results of C. Kounnas *and al* [3-27], we argue in §(5.1.1) that on the hereabove limit (i.e. at the Planck scale), the "space-time system" is in a *thermodynamical equilibrium state* [34] and, therefore, is subject to the Kubo-Martin-Schwinger (KMS) condition [24]. A similar point of view has also been successfully developed in the context of thermal supersymmetry by Derendinger and Lucchesi in [13-28]. Surprisingly, the KMS and modular theories [11] might have dramatic consequences onto Planck scale physics. Indeed, when applied to quantum space-time, the KMS properties are such that the time-like direction of the system, within the limits of the "KMS strip" (i.e.

between the zero scale and the Planck scale) should be considered as *complex* : $t \mapsto \tau = t_r + it_i$. In this case, on the $ß \to 0$ limit, the theory is projected onto the pure imaginary boundary $t \mapsto \tau = it_i$ of the KMS strip. Then the partition function (2) gives the pure topological state connected with the zero mode of the scale :

$$\underset{ß\to 0}{Z} = \mathrm{Tr}\,(-1)^{\mathrm{S}} \tag{3}$$

where **s** represents the instantonic number. This new "singularity invariant" [6-7]), isomorphic to the Witten index Z = Tr $(-1)^{\mathrm{F}}$, can be connected with the initial singularity of space-time, reached for $ß$ = 0 in the partition function **Z** = Tr $(-1)^{\mathrm{S}}\, e^{-ßH}$. According to sec. 3, when $ß \to 0$, the partition function **Z** gives the first Donaldson invariant

$$\mathrm{I} = \sum_i (-1)^{n\,i} \tag{4}$$

a non-polynomial topological invariant, reduced to an integer for $\dim \mathcal{M}^{(k)}_{\mathrm{mod}} = 0$ ($\dim \mathcal{M}^{(k)}_{\mathrm{mod}}$ being the dimension of the instanton moduli space). This suggests that the (topological) origin of space-time might be successfully represented by a singular zero size gravitational instanton [41]. A good image of this euclidean point-like object is the "transitive point", whose orbits under the action of $\mathbb{R}$ are dense everywhere from zero to infinity. Then at zero scale, the observables 0_i should be replaced by the homology cycles $H_i \subset \mathcal{M}^{(k)}_{\mathrm{mod}}$ in the moduli space of gravitational instantons. We get then a deep correspondence -*a symmetry of duality*- [2-19-32], between physical theory and topological theory. More precisely, it may exist, at the Planck scale, a duality transformation (which we call "*i*-duality"[6]) between the BRST cohomology ring (physical mode) and the cohomology ring of instanton moduli space (topological mode) [19]. In the context of quantum groups [16-17], we have shown that transition from q-Euclidean to q-Lorentzian spaces [30-35] can also be viewed as a Hopf algebra duality [29]. Interestingly, the Hopf algebra duality has been recently connected to superstrings T-duality by C. Klimcik and P. Sevara [26].

The present article is organized as follows. In section 1 we define the topological field theory and suggest that there exists at the scale limit $ß \to 0$ a non-trivial topological limit of quantum field theory, *dual* to the topological limit associated with $ß \to \infty$. In section 2 we evidence that the $ß \to 0$ limit of some standard theories is topological. We give several examples of such a topological limit. In section 3, we show that the high temperature limit of quantum field theory corresponding to $ß \to 0$ should give the first Donaldson invariant. The signature of the metric of the underlying 4-dimensional manifold is therefore expected to be Euclidean (++++) at the scale zero. In section 4, we emphasize, in the quantum groups context, the existence of a symmetry of duality between the Planck scale (physical sector of the theory) and zero scale (topological sector). In section 5, we discuss in the framework of KMS state and von Neumann C^*- algebras a way to understand the transition from the topological (ultraviolet) phase of space-time to the standard physical (infrared) phase.

1. TOPOLOGICAL THEORY AT SCALE 0

1.1 Preliminaries

The field theory considered here is thermal supersymmetric [13-28] and in the context of D4 manifolds [40]. We have detailed the content of the (thermal) supermultiplet in a previous work [6]. The theory belongs to the class of N=2 supergravities [19], the Hamiltonian being given by the squared Dirac operator $\not{D}^2$ [11-31]. As such, the simplest bosonic multiplet reduces to a vector field plus two scalars exhibiting a special Kähler geometry. Rightly, N=2 is here of a particular interest, for two main reasons :

(**i**) the complex scalar fields of the theory (for example the dilaton S-field [32] or the T-field [2]) can be seen as "signatures" of the KMS condition [11-25] to which the space-time might be subject at the Planck scale. They might also be one of the best keys to understand the possible *duality* between physical observables (infrared) and topological states (ultraviolet) :

$$\textit{Topological vacuum } (ß = 0,\ \textit{instanton}) \xleftrightarrow{i-\textit{dualité}} \textit{Physical vacuum } (ß = \ell_{Planck},\ \textit{monopole})$$

This is based on the instantons / monopoles duality initially suggested by us in [6] and recently proved in the superstrings context by C.P. Bacchas, P. Bain and M.B. Green [5]. Moreover, in string theory again, has been conjectured a U=S⊗T-symmetry [25] from which we can infer the hereabove duality between (physical) observables and (topological) cycles on a four-manifold M :

$$\langle 0_1 0_2 \ \dots \ 0_n \rangle \xleftrightarrow{U-duality} \chi(\gamma_1, \gamma_2, \dots, \gamma_n)$$

Then the main contribution of the present article would be to emphasize that, as for conifolds cycles, a zero topological cycle might control the blow up of the space-time Initial Singularity.

(**ii**) From another point of view, the **S/T** fields are closely related to the existence, in the Lagrangian, of *non-linear terms.* As recalled by A. Gregori, C. Kounnas and P. M. Petropoulos [23], in the frame of N=2 supergravity, the theory is generally inducing some non perturbative corrections and a BPS-saturated coupling with higher derivative terms $R^2 + \dots$ As our model is proposed in 4D dimensions, the development of higher derivative terms can be limited in a natural way to the R^2 term. Then the Lagrangian usually considered in supergravity is :

$$L = \int d^4x \ \sqrt{g} \left\{ l^2 (\alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2) + R + \kappa L_M \right\} \tag{5}$$

from which we pull the simplified Lagrangian density that we use here :

$$\mathbf{L}_{supergravit\acute{e}} = \hat{\beta} R + \frac{1}{g^2} R^2 + \alpha RR^* \tag{6}$$

This type of Lagrangian density is coupling the physical component (the Einstein term $\hat{\beta} R$) with the topological term RR^*. This is of crucial interest since, as observed in ex.(2.1), when ß → 0, we are only left with the topological term RR^*(decoupled, on this limit, of the axion field α).

Now, let's begin with a brief reminder of topological field theory as originally introduced by E. Witten in 1988 [37] :

Definition 1.1 *Topological field theory is defined by a cohomological field such that a correlation function of n physical observables* $\langle O_1 O_2 \dots\dots O_n \rangle$ *can be interpreted as the number of intersections*

$$\langle O_1 O_2 \dots O_n \rangle = \#(H_1 \cap H_2 \cap \dots \cap H_n)$$

of n cycles of homology $H_i \subset \mathcal{M}^{(k)}_{\mathrm{mod}}$, *in moduli space* $\mathcal{M}^{(k)}_{\mathrm{mod}}$ *of configurations of the instanton type* [$\phi(x)$]*, on the fields* ϕ *of the theory.*

The content of "cohomological fields" (for which the general covariance is exact) is given by the field variations (which induce a Fadeev-Popov ghost contribution and gauge fixing part). The point, however, is that the total gauge fixed action is a BRST commutator and the energy-momentum tensor is BRST invariant [19-37]. In other words, the correlation functions of cohomological fields are independent of the metric. Now, the topological field theory (for D = 4) is established when the Hamiltonian (or the Lagrangian) of the system is H=0, such as the theory is independent of the underlying metric. We propose to extend this definition, stating that a theory can also be topological *if it does not depend on the Hamiltonian* H (or the Lagrangian $\boldsymbol{L}$) of the system.

Definition 1.2 *A theory is topological if (the Lagrangian* $\boldsymbol{L}$ *being non-trivial) it does not depend on* $\boldsymbol{L}$*.*

Def (1.2) means that $\boldsymbol{L}$ is a topological invariant of the form $\boldsymbol{L} = R \wedge R^*$. Based on this definition, we suggest that there exists a *second* topological limit of the theory, dual to that given by H = 0. In this case, we can have H $\neq$ 0, but the theory is taken at the limit of *scale zero* associated with $ß \rightarrow 0$. Then the minimum of the action is not zero (as it is in the trivial case) but has a non-trivial (invariant) value.

We consider the possible existence of such a "topological field " at the high temperature limit of the system.

1.2 A new topological limit

Proposition 1.3 *There exists at the scale* $ß \rightarrow 0$ *a non-trivial topological limit of the theory, dual to the topological limit corresponding to* $ß \rightarrow \infty$.

Proof The (thermo)dynamical content of the quantum field theory can be described by the partition function :

Z = Tr $(-1)^{n}\, e^{-ßH}$ (7)

where n is the "metric number" of the theory. When $ß \rightarrow 0$, the theory is no longer dependent on H. On this limit, such that the temperature T $\rightarrow$ T_{Hag}(Hagedorn limit), equ.(7) becomes Z_0 = Tr$(-1)^{n}$, H vanishing from the metric states partition function. $ß$ plays the role of a coupling constant, such that it exists an infinite number of states not interacting with each other and independent of H. The point is that for $ß = 0$, the action S is projected onto a non trivial minimum, corresponding to the self-duality condition $R = \pm R^{*}$. But in this case, the field configuration is necessarily *Euclidean* and defines a gravitational instanton, i.e. a topological configuration [6]. We are therefore confronted to a 4D pure *topological* theory , as described by the first Donaldson invariant [14] :

$$\mathrm{I} = \sum_i (-1)^{n_i}$$

n_i being the instanton number. The limit $ß = 0$ is here dual (in a sense precised in §(4)) to the usual topological limit $ß \rightarrow \infty$ given by H = 0. The density operator of the (pre)space-time system is written as :

$$\rho = e^{-\beta H + \lambda_0}$$

λ_0 being (classically) a factor of re-normalization of the system. When $ß = 0$, the density operator is thus reduced to $\rho = e^{\lambda_0}$, which is independent of H $\neq$ 0, characteristic of a second topological limit of the theory.

Now we propose to show, through some very simple examples, that interesting contacts with topological field theory can be made in taking the $ß \rightarrow 0$ limit of some established standard results. To be as demonstrative as possible, we shall most often proceed in a heuristic way.

2. THE $ß \rightarrow 0$ LIMIT OF SOME STANDARD THEORIES

To warm up, we first consider the $ß \rightarrow 0$ topological limit of the standard (quantum) thermal field theory.

Example 2.1 *The topological 0-scale limit of the heat kernel*

(i) One famous mathematical proof of the Atiyah-Singer theorem (given, for example, by E.Getzler [20]) lies in the heat equation [21-22]. Considering the heat operator $e^{-ßH}$ acting on the differential forms on a closed, oriented manifold X, the $ß \rightarrow 0$ limit of this operator corresponds to the local curvature invariants of the manifold [31].

Let's consider a (quantum) thermal field theory on a system defined by the first order elliptic differential operator P and it's adjoint P*. We put the laplacian Δ = PP* et Δ' = P*P. For any $ß > 0$, we can evaluate the partition function $K = \mathrm{Tr}\left(e^{-ß\Delta}\right)$ giving the states of the metric of the system. Now, to get the asymptotic $ß \rightarrow 0$ limit, we take the symbol of $\mathrm{Tr}\left(e^{-ß\Delta}\right)$ (which can be expressed in terms of $\sigma(\Delta)$ and its derivatives) and we get :

$$\mathrm{Tr}\left(e^{-ß\Delta}\right) \cong \mathrm{Tr}\int_M \sigma(e^{-ß\Delta})\,dxdk \tag{8}$$

For $ß \rightarrow 0$, K degenerates on the Dirac mass and the right-hand side of (8) has an asymptotic expansion such that

$$\mathrm{Tr}\left(e^{-ß\Delta}\right) \cong \sum_0^\infty t^{i-n/2} B_i$$

and as a result, we get the well known $ß$-independent topological index (in the Atiyah-Singer sense [22]) :

$$Ind(P) = B_n[\Delta] - B_n[\Delta']$$

With this index we see in a simple way that the $ß \rightarrow 0$ limit of thermal field theory is topological.

Another important argument lies in the fact that at ℓ_{Planck}, the (pre)space-time might enter a phase of thermodynamical equilibrium (§(5.1.1)). Consequently (§(5.1.2)) it should be subject to the KMS condition [24]. As evidenced in §(5.1.3) and in the ex.(5.2.1), this implies the holomorphicity of the time-like direction, the real time-like and the real space-like directions given by g_{44}^c being compactified on the two circles S^1_{t-like} and S^1_{s-like} [6]. But one can easily see that this configuration is equivalent to the dimensional reduction of the 4D Lorentzian theory onto a 3D theory. This type of reduction has been described by Seiberg and Witten [33]. We then are left with three-manifold invariants, in particular the Floer invariant of a supersymmetric non linear σ-model [18]. In this case, the three dimensional pseudo-gravity $\Gamma_{(3)}$ is coupled to the $\mathbf{S},\mathbf{T}$ complex scalar fields :

(1) $\mathbf{S} = \dfrac{1}{g^2} \pm i.\alpha_i$ (axion) with $\mathbf{S}$ and $\bar{\mathbf{S}}$

(2) $\mathbf{T} = g_{44} \pm i g^{*}{}_{i4}$ with $\mathbf{T}$ and $\bar{\mathbf{T}}$

Those scalar fields are propagating. Then the coupling of the **S/T**-fields with the 3D pseudo-gravity is given by the extended σ-model :

$$\sum = SO(3) \times \frac{SL(2, \mathbb{R})}{SO(2)} \times \frac{SL(2, \mathbb{R})}{SO(1,1)} \tag{9}$$

As the theory is *independent* of g_{44}, the 2D field $\dfrac{SL(2, \mathbb{R})}{SO(1,1)}$ in the Lorentzian case and $\dfrac{SL(2, \mathbb{R})}{SO(2)}$ in the Euclidean case can be viewed as equivalent. Thus the corresponding "superposition state" of the signature $(+++\pm)$ is able to be described by the symmetric homogeneous space

$$\sum_h = \frac{SO(3,1) \otimes SO(4)}{SO(3)}$$

SO(3) being diagonally embedded in SO(3, 1) ⊗ SO(4). Next step, as suggested in [6], a "monopoles+instantons" configuration can be associated to this 5D metric configuration at the Planck scale. Instantons and monopoles are here connected by a S-field. The form of the 5D metric induced by the σ-model (9) and constructed in [6] is :

$$ds^2 = a(w)^2 d\Omega^2_{(3)} + \frac{dw^2}{g^2} - dt^2 \tag{10}$$

where the axion term is $a = f(w\ ,\ t)$, the 3-geometry being $d\Omega^2_{(3)} = f(x,y,z)$. Clearly the expected values of the running coupling constant (dilaton) $\varphi = \dfrac{1}{g^2}$ are giving the two 4D limits of the 5D metric of equ.(10). Thus we get: **- *Infrared*** : *ß* → ∞. In this strong coupling sector we have $\frac{dw^2}{g^2} \to 0$ and the *w* direction of $\mathbf{\Gamma}^5$ is cancelled. So after dimensional reduction (D=5 → D= 4) the metric on $\mathbf{\Gamma}^5$ becomes 4D Lorentzian :

$$ds^2 = a(w)^2 d\Omega^2_{(3)} - dt^2 \tag{11}$$

The σ-model (9) is reduced to the usual Lorentzian symmetry :

$$SO(3) \times \frac{SL(2, \mathbb{R})}{SO(1,1)} \xrightarrow{ß(g) \xrightarrow{infrared} \infty} SO(3, 1) \tag{12}$$

Likewise, when g → ∞, the R^2 term cancels in the 5D Lagrangian density $\mathbf{L}_{supergravit\acute{e}} = \hat{\beta} R + \frac{1}{g^2} R^2 + \alpha RR^*$, and, as $R = R^*$, the topological term αRR^* is also suppressed. So, we get $\mathbf{L} = \hat{\beta} R$.

Let's see now what happens on the (dual) ultraviolet limit, when *ß* → 0.

- *Ultraviolet* : *ß* → 0. We can construct a boundary of equ.(10), corresponding to the small coupling constant sector of the coupled theory and we get divergent values for the real dilaton field $\varphi = \frac{1}{g^2}$. Then naïvely, we can apply one of the results of [23] saying that the axion field is decoupled of the theory on this limit and we are left with the divergent dilaton field only. So, we have for the metric on $\mathbf{\Gamma}^5$ the new *Euclidean* form :

$$ds^2 = a(w)^2 d\Omega^2_{(3)} + \frac{dw^2}{g^2} \tag{13}$$

Therefore, in the ultraviolet, the σ-model (9) is reduced to the four-dimensional target space :

$$SO(3) \times \frac{SL(2, \mathbb{R})}{SO(2)} \xrightarrow{ß(g) \xrightarrow{ultraviolet} 0} SO(3) \times SO(3) = SO(4) \tag{14}$$

and on this small coupling limit, the reduced theory becomes *Euclidean*, i.e. topological.. Again, it appears reasonable to conclude that the ß → 0 limit has a pure topological content. Incidentally, this result could as well be understood in the frame of the isodimensional instanton-monopole duality proposed by us in [6] and proved in the string context by Bacchas *and al* [5]. Indeed, we have shown that the q-deformed 5D theory is dominated by the (3+1)D monopoles in the infrared ($ß \rightarrow \ell_{Planck}$) and by the 4D instantons in the ultraviolet ($ß \rightarrow 0$) [6]. In this sense, the Euclidean signature (++++) can be seen as *i*-dual to the Lorentzian one (+++–). Likewise, the topological limit $ß \rightarrow 0$ should be viewed as *i*-dual to the physical limit $ß \rightarrow \ell_{Planck}$. This might be an unexpected application of the Seiberg-Witten **S/T**-duality [32].

At present, let's explore the ultraviolet limit of another standard result, i.e. the Feynmann Path integral [39].

Example 2.2 *The topological 0-scale limit of the Feynmann (3+1) path-integral approach*

(i) It's well known that in quantized Minkowski space-time, the amplitude $(g_2, \phi_2, \sigma_2 \mid g_1, \phi_1, \sigma_1)$ is given by :

$$(g_2, \phi_2, \sigma_2 \mid g_1, \phi_1, \sigma_1) = \int D[\phi] \exp [i\, S(\phi)]$$

To include the point-like (0-modes) configurations of $g_{\mu\nu}$, we put $\mathrm{Tr}(-1)^n$ in the integral and we get :

$$(g_2, \phi_2, \sigma_2 \mid g_1, \phi_1, \sigma_1) = \int \mathrm{Tr}(-1)^n D[\phi] \exp [i\, S(\phi)] \qquad (15)$$

So, the trivial {t=ß → 0, S=0} Lorentzian vacuum is distinct of the "topological vacuum" connected to the minimum of the *Euclidean* action $S_E = \frac{8\pi^2}{g^2}$. But it has been shown [15-37] that the zero modes in the expansion about the minima of S are tangent to the instanton moduli space $\boldsymbol{M}_k$, so the topological vacuum should be viewed as the "true vacuum" of the theory. Then equ.(15) becomes for $ß \rightarrow 0$:

$$(g_0, \phi_0, \sigma_0 \mid g_0, \phi_0, \sigma_0) = I_0 = \int \mathrm{Tr}(-1)^n D[\phi_0] \qquad (16)$$

To define I_0, one can assume that at zero scale, the measure $D[\phi_0]$ is concentrated on one *unique* point and becomes a pure state, i.e. a positive trace class operator with unit trace. Concerning **ϕ**, the field content can be given by the non linear term R^2, so that the ß-dependant typical form of the Lagrangian density is, as seen in [6] :

$$\mathbf{L}_{supergravité} = \hat{\beta} R + \frac{1}{g^2} R^2 + \alpha R R^* \tag{17}$$

Now, for g= ß → 0, the Einstein term R is cancelled and as $R = R^*$, the only remaining term in equ.(17) is the topological invariant RR^*(itself decoupled from the axion field α). So, equ.(15) takes the new form :

$$(g_0, \phi_0, \sigma_0 \mid g_0, \phi_0, \sigma_0) \rightarrow \int \mathrm{Tr}\, R^2 = \int \mathrm{Tr} R R^* = \mathrm{I}_0 \tag{18}$$

and I_0 becomes a topological invariant. As

$$-\int_X \mathrm{Tr}(R(A)^2) = 8\pi^2 k(E)$$

and we apply the Gauss-Bonnet theorem to find :

$$\chi(M) = \frac{1}{32\pi^2} \int_X \varepsilon_{abcd} R_{ab} R_{cd} \tag{19}$$

Therefore, the ß → 0 limit of the Feynmann path integral is giving the Euler Characteristic, i.e. the "true vaccum" mentioned hereabove and corresponding to the topological pole of the theory.

Next, we provide a new example showing that the ß → 0 limit of the N=2 supersymmetric theory is topological.

Example 2.3 *The topological 0-scale limit of the (supersymmetric) quantum field theory*

We apply here a well known quantum mechanical account of Morse theory due to Witten [40]. First, we start from the standard supersymmetry algebra $\{Q_i, Q_j\} = Q_i Q_j + Q_j Q_i = 0$. Next, we express this superalgebra in terms of data provided only by the space-time manifold $\boldsymbol{M}$. To do so, let's define a set of coboundary operators, the conjugation of d by $e^{ßH}$ being parametrised by $ß = \frac{1}{kT}$:

$$d_{ß} = e^{ßH} d\, e^{-ßH}$$
$$d_{ß}^{*} = e^{ßH} d^{*}\, e^{-ßH} \tag{20}$$

for a Morse function $H(\mathrm{x})$. Then the spectrum of the *ß*-dependant Hamiltonian is:

$$H_{ß} = d_{ß} d_{ß}^{*} + d_{ß}^{*} d_{ß} \tag{21}$$

Now, let's send *ß* onto zero. We get for the Hamiltonian the invariant value :

$$H_0 = dd^{*} + d^{*}d = \bigoplus_{p\geq 0} \Delta_p \tag{22}$$

But this invariant is nothing else than the Betti numbers of ***M***, given by $b_p = \dim ker \Delta_p$, which is a discrete function, independent of *ß*. Consequently, the space of zero energy states of H is given by the set of even (odd) harmonic forms on ***M*** and equals the Betti number of ***M***. So we have, for $ß \rightarrow 0$:

$$\mathrm{Tr}(-1)^{\mathrm{F}} = \sum_{k=0}^{4} (-1)^k b_i \quad = \chi(\boldsymbol{M}) \tag{24}$$

where b_i is the i^{th} Betti number and $\chi(\boldsymbol{M})$ the Euler-Poincaré characteristic of ***M***. Finally, on the zero scale limit, we recover the topological index [37] corresponding to any standard topological field theory.

To finish, we obtain in the last following example some analog results in the frame of full (N=2) supergravity.

Example 2.4 *The topological ß → 0 limit of (N=2) supergravity*

As a matter of fact, for a spin manifold, we can express H in terms of the Dirac operator $\not{D}$. Then in dimension D=4, we can calculate on the $ß \rightarrow 0$ limit the index of the squared Dirac operator :

$$\mathrm{Ind}(\not{D}_+) = \lim_{ß \rightarrow 0} \mathrm{Str}\left[\left(e^{-ß\mathcal{D}_+^2}\right)\right] =$$
$$\frac{1}{(2\pi)^{\mathrm{n}}} \int_{\mathrm{T}^*\mathrm{M}} \mathrm{Str}\left(e^{-|\xi|^2 + \frac{1}{2}\mathrm{R}\left(\xi, \frac{\partial}{\partial\xi}\right) + \frac{1}{16}(\mathrm{R} \wedge \mathrm{R})\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\xi}\right) + \mathrm{B}}\right) \mathrm{dx}\, \mathrm{d}\xi$$

By the Mehler formula, we find the Dirac index in function of the Dirac genus $\hat{\mathrm{A}}(M)$:

$$\text{ind}\,(\not{D}_+) = \int_M ch(B)\hat{A}(M) \tag{25}$$

ch being the Chern character, B the curvature and $\hat{A}(M)$ the Dirac genus of the auxillary fiber bundle. Since the spinors are interacting with Yang-Mills fields, the $\hat{A}(M)$ term is coming from the gravitational part whereas the rest of equ.(25) comes from the gauge part. As $\text{Ch(B)} = \text{Tr}\left(e^{-B/2 i \pi}\right)$, we get :

$$\hat{A}(M) = \prod_{j=1}^{k} \frac{x_j / 2}{\sinh (x_j / 2)} \tag{26}$$

and we can express the complete Yang-Mills + gravity index through the following invariant :

$$\text{Ind}(\not{D}_+) = \frac{\dim M}{8\pi^2}\int \text{Tr}(R \wedge R) - \frac{1}{8\pi^2}\int \text{Tr}(F \wedge F) \tag{27}$$

Finally, in Yang-Mills + gravity context, we obtain again a topological invariant on the $ß \to 0$ limit.

Now, to go further, the next step consists to detect, on the $ß \to 0$ limit, the nature of the topological invariant involved. We shall discover that Donaldson invariants are playing a very important role on this boundary.

3. $ß \to 0$ SCALE AND DONALDSON INVARIANTS

From a topological point of view, Donaldson invariants are obtained from characteristic classes of an infinite dimensional bundle on the manifold equally infinite and canonically associated with a 4-dimensional manifold :

Definition 3.1 *Let* $\boldsymbol{M}$ *be a 4-dimensional manifold . The Donaldson invariant* $q_d(\boldsymbol{M})$ *is a symmetric integer polynomial of degree d in the 2-homology* $H_2(\boldsymbol{M};\boldsymbol{Z})$ *of* $\boldsymbol{M}$

$$q_d(\boldsymbol{M}) : H_2(\boldsymbol{M}) \times \ldots \times H_2(\boldsymbol{M}) \to \boldsymbol{Z}$$

$\mathcal{M}^{(k)}_{\text{mod}}$ *being the instanton moduli space of degree* k, *the Donaldson invariant is defined by the map*

$$m : H_2(\boldsymbol{M}) \to H^2(\boldsymbol{M})\mathcal{M}^{(k)}_{\text{mod}}$$

Now, we suggest that on the $ß \to 0$ limit, the 4D field theory is projected onto the first Donaldson invariant.

Proposition 3.2 *The high temperature limit of quantum field theory corresponding to ß → 0 in the partition function* $Z = \mathrm{Tr}\,(-1)^{S}\, e^{-ßH}$ *gives the first Donaldson invariant. The signature of the metric of the underlying 4-dimensional zero scale manifold is therefore Euclidean* (+ + + +).

Proof Let the partition function $Z = \mathrm{Tr}(-1)^{S}\, e^{-ßH}$ connected with a set described by the density matrix :

$$Q = (-1)^{S}\, e^{-ßH} \qquad (28)$$

According to standard arguments, we can write :

$$\mathrm{Tr}\,(-1)^{S}\, e^{-ßH} = \int_{CPB} d\phi(t)\, d\psi(t) \exp - S_E(\phi,\ \psi) \qquad (29)$$

It has been shown [1-36] that given a supersymmetric QFT, one can define the invariant $I = \mathrm{Tr}\,(-1)^{f}$, f being the fermionic number. We propose to extend equ.(29) to supergravity and to define the topological invariant

$$\dot{\imath} = \mathrm{Tr}\,(-1)^{S} \qquad (30)$$

where S is the instanton number. So, the regularization of the trace (30) gives the index $\dot{\imath}$ of the Dirac operator :

$$\dot{\imath} = \mathrm{Tr}\,\Gamma\, e^{-ß_c \not{D}^2} = \mathrm{Tr}\,(-1)^{S}\, e^{-ß_c \not{D}^2} = \int_{cpl} [Dx]\,[D\psi]\, e^{-\int_0^{ß} dt\, L} \qquad (31)$$

with $ß_c \in \mathbb{C}$. Then for $ß_c = 0$, the value of the partition function $Z = \mathrm{Tr}\,(-1)^{S}\, e^{-ß_c H}$ is :

$$Z_0 = \mathrm{Tr}\,(-1)^{S} \qquad (32)$$

and $\mathrm{Tr}\,(-1)^{S}$ can be seen as the index of an operator acting on the Hilbert space $\mathscr{H}$. Dividing $\mathscr{H}$ in monopole and instanton sub-spaces $\mathscr{H} = \mathscr{H}_{\mathfrak{M}} + \mathscr{H}_{\mathrm{i}}$ and Q being a generator of supersymmetry, we get :

$$Q|\psi\rangle = 0\ ,\quad Q^{*}|\psi\rangle = 0 \qquad (33)$$

So $\mathrm{Tr}\,(-1)^{S} = \mathrm{Ker}\,Q - \mathrm{Ker}\,Q^{*}$ such that as topological index, $\mathrm{Tr}\,(-1)^{S}$ is invariant under continuous deformations of parameters which do not modify the asymptotic behavior of the Hamiltonian H at high energy. H is given by $H = dd^* + d^*d$, the space of zero energy states corresponding to the set of even harmonic forms on M_n:

$$\mathrm{Tr}\,(-1)^S\, e^{-ßH} = \chi\,(M) = \sum_{k=0}^{n} (-1)^k b_k \tag{34}$$

$\Delta = \mathrm{Tr}\,(-1)^S$ is independent of *ß*, the sole contributions to Δ coming from the topological sector of zero energy : $\Delta = n_i^{E=0} - n_m^{E=0}$. On formal basis, $n_i^{E=0} - n_m^{E=0}$ can be seen as the trace of the operator $(-1)^S$. Then Δ is a topological invariant, i.e. the first Donaldson invariant. The coupling constant g being dimensional, the limit $ß = 0$ implies $\rho = 0$ and corresponds to the sector of zero size instantons [41]. So, $\mathrm{Dim}\,\mathcal{M}^{(k)}_{\mathrm{mod}} = 0$. When $\mathrm{Dim}\,\mathcal{M}^{(k)}_{\mathrm{mod}} \neq 0$, the Donaldson invariants are given by :

$$Z(\gamma_1 \;\ldots\; \gamma_r) = \int DX\, e^{-S} \prod_{i=1}^{r} \int_{\gamma i} W_{k_1} = \left\langle \prod_{i=1}^{r} \int_{\gamma_i} W_{k_i} \right\rangle \quad (\mathrm{Dim}\ \mathcal{M}^{(k)}_{\mathrm{mod}} \neq 0) \tag{35}$$

What happens then ? The solution is in the correspondence between the Donaldson invariants on 4D manifolds and the Floer homology groups [18] on 3D manifolds. Indeed, Donaldson invariants amount to the calculus of the partition function Z, expressed as an algebraic sum over the instantons [15]:

$$Z \mapsto \underset{\mathcal{M}^{(k)}_{\mathrm{mod}}=0}{Z} = \sum_i (-1)^{n_i} \tag{36}$$

i indicating the *i* th instanton and $n_i = 0$ or 1 determining the sign of its contribution to Z. Donaldson has shown on topological grounds [14] that when $\dim \mathcal{M}^{(k)}_{\mathrm{mod}} = 0$, then $\sum_i (-1)^{n_i}$ is a non-polynomial topological invariant, reduced to an integer. We find the same result starting from $T_{\alpha\beta} = \left\{ Q\ ,\ \lambda_{\alpha\beta} \right\}$. In fact, the partition function of the system at temperature $ß^{-1}$ has the general form $Z_q = \mathrm{Tr}\,(-1)^S\, e^{-ßH}$. For $ß = 0$, Z_q becomes $\underset{ß=0}{Z} = \mathrm{Tr}(-1)^S$, which is isomorphic to $\sum_i (-1)^{n_i}$, s and n_i giving in both cases the instanton number of the theory.

This result strongly suggests that on the high temperature limit $ß \to 0$ parameterizing the 0 scale of the theory, the partition function $\underset{\mathcal{M}^{(k)}_{\mathrm{mod}}=0}{Z}$ projects the Lorentzian physical theory onto the Euclidean topological limit.

Now, starting from hereabove, we suggest the existence of a deep correspondence, of the duality symmetry type, between physical sector ($\lambda \geq$ Planck scale) and topological sector (0 scale) of the (pre)space-time.

4. DUALITY SYMMETRY BETWEEN PHYSICAL AND TOPOLOGICAL STATES

Ideally, the duality we are looking for (which we call "*i*-duality" $t \to \frac{1}{it}$ [5], of the type i = S⊗T) should exchange real time in strong coupling / large radius with imaginary time in weak coupling / small radius. In this sense, Planck (physical) scale should be *i*-dual to zero (topological) scale.

Let's first outline a few formal aspects of Lorentzian/Euclidean duality in terms of Hopf algebras.

4.1 Duality between q-Lorentzian and q-Euclidean Hopf algebras

Considering the non commutative constraints at the Planck scale, it appears interesting to adopt an approach in terms of "quantum groups" at this scale. So we have shown that in D=4, it should exist a *superposition* (+++±) between Lorentzian (physical) and Euclidean (topological) algebraic structures. Then we have constructed, in the enveloping algebras setting, the q-deformation of the cocyle bicrossproduct [6]:

$$\boldsymbol{M}_{\chi}(H) = H^{op} \ {}^{\psi}\!\!\triangleright\!\!\blacktriangleleft H_{\chi} \tag{37}$$

where H is a Hopf algebra, $\triangleright\!\!\blacktriangleleft$ a bicrossproduct (i.e. a special type of crossproduct, defined in [29]) and χ a 2-cocycle or"twist" in the Drinfeld sense [16-17]. This is inspired by the idea to unify two different quantum groups within a *unique* algebraic structure. So, we propose the following :

Proposition 4.1 *The Euclidean and the Lorentzian Hopf algebras are related by the cocycle bicrossproduct*

$$U_q(so(4))^{op} \ {}^{\psi}\!\!\triangleright\!\!\blacktriangleleft U_q(so(3, 1))$$

Proof Starting, in the setting of enveloping algebras, from the Euclidean Hopf algebra H = $U_q(so(4))$, we have the well known decomposition H = $U_q(su(2)) \otimes U_q(su(2))$ and the "opposite" $H^{op} = U_q(su(2))^{op} \otimes U_q(su(2))^{op}$, whereas the Lorentzian form is A = $H_{\chi} = U_q(su(2)) \ \blacktriangleright\!\!\blacktriangleleft$

$U_q(su(2)) \cong U_q(so(3,1))$. As explained in [6], the cocycle of deformation is $\chi = \mathcal{R}_{23}$. Then the action and the coaction are :

$$(a \otimes b) \triangleleft (h \otimes g) = h_{(1)} a S h_{(2)} \otimes g_{(1)} b S g_{(2)}$$

$$\begin{aligned} ß(h \otimes g) &= (h_{(1)} \otimes g_{(1)}).(Sh_{(3)} \otimes Sg_{(3)}) \otimes h_{(2)} \otimes g_{(2)} \\ &= h_{(1)} Sh_{(3)} \otimes g_{(1)} Sg_{(3)} \otimes h_{(2)} \otimes g_{(2)} \end{aligned} \tag{38}$$

where we find the structure of tensor product of the action and the coaction for each $U_q(su(2))$ copy. On the other hand, the cocycle for $h,g \in U_q(su(2))$ is :

$$\psi(h \otimes g) = (h_{(1)} \otimes g_{(1)})(1 \otimes \mathcal{R}^{(1)})(Sh_{(4)} \otimes Sg_{(4)})(1 \otimes \mathcal{R}^{-(1)}) \otimes \\ (h_{(2)} \otimes g_{(2)})(\mathcal{R}^{(2)} \otimes 1)(Sh_{(3)} \otimes Sg_{(3)})(\mathcal{R}^{-(2)} \otimes 1)$$

where the product is in $H = U_q(su(2)) \otimes U_q(su(2))$. This gives:

$$\begin{aligned} \psi(h \otimes g) &= h_{(1)} Sh_{(4)} \otimes g_{(1)} \mathcal{R}^{(1)} Sg_{(4)} \mathcal{R}^{-(1)} \otimes h_{(2)} \mathcal{R}^{(2)} Sh_{(3)} \mathcal{R}^{-(2)} \otimes g_{(2)} Sg_{(3)} \\ &= h_{(1)} Sh_{(4)} \otimes g_{(1)} \mathcal{R}^{(1)} Sg_{(2)} \mathcal{R}^{-(1)} \otimes h_{(2)} \mathcal{R}^{(2)} Sh_{(3)} \mathcal{R}^{-(2)} \otimes 1 \end{aligned} \tag{39}$$

for the explicit bicrossproduct structures. **qed**

Clearly, prop.(4.1) proves the possible "unification" between the q-Lorentzian and the q-Euclidean Hopf algebras at the Planck scale.We give a detailed demonstration of this proposition in [6]. But also, the hereabove result suggests a certain type of "duality" between Lorentzian (physical) and Euclidean (topological) quantum groups. To see this, the next step consists in showing the existence of a very interesting "semidualisation" (proposed in the general case by S. Majid [29]) between Lorentzian and Euclidean Hopf algebras. Better still, such a duality allows a description of the *transition* from the q-Euclidean group to the q-Lorentzian group [30] :

Proposition 4.2 $U_{q^{-1}}(su(2)) \otimes U_q(su(2)) \cong U_q(su(2))^{op} \triangleright\!\blacktriangleleft U_q(su(2))$ *is connected by semidualisation to* $U_q(su(2)) \triangleright\!\triangleleft U_q(su(2))^{op*} \cong \mathfrak{D}(U_q(su(2)))$. *Then the semidualisation connects a version of* $U_q(so(4))$ *to a version of* $U_q(so(3,1))$.

We have given in [6] a complete demonstration of prop. (4.2), based on the properties of the Drinfeld double $\mathfrak{D}(U_q(su(2)))$. Then, using our general cocycle construction $M_\chi(H)$, we get the interesting relation :

$$U_q(su(2)) \,{}^{\psi}\!\triangleright\!\!\blacktriangleleft U_q(su(2)) \cong U_q(so(4)) \xleftrightarrow{\textit{semidualisation}} U_q(su(2))^* \triangleright\!\!\triangleleft_{\chi} U_q(su(2)) \sim U_q(so(3,1)) \qquad (40)$$

The "q-deformation" from q-Euclidean to q-Lorentzian Hopf algebras corresponds to a duality transformation and induces the existence of a 2-cocycle of deformation. Likewise, the cocycle bicrossproduct

$$U_q(so(4)^{op} \,{}^{\psi}\!\triangleright\!\!\blacktriangleleft U_q(so(3,1)) \qquad (41)$$

defines implicitly the new (semi)duality transformation

$$U_q(so(4))^{op} \,{}^{\psi}\!\triangleright\!\!\blacktriangleleft U_q(so(3,1)) \cong U_q(so(4)) \xleftrightarrow{\textit{semidualisation}} SO_q(3,1) \triangleright\!\!\triangleleft_{\chi} U_q(so(4))^{op}$$

where χ is constructed from ψ, this one being derived from the quasitriangular structure $\mathfrak{R}$ of $U_q(su(2))$ [5].

Now, an interesting consequence of those results concerns some duality characteristics at the level of q-deformation of space-time itself. We have shown [6] that the natural structures of the q-Euclidean space $\mathbb{R}_q^4$ and of the q-Lorentzian space $\mathbb{R}_q^{3,1}$, covariant under $U_q(so(4))$ and $U_q(so(3,1))$ [8] are connected as follows :

$$\begin{array}{lcl} U_q(su(2)) & \xleftrightarrow{*\text{-Hopf algebras duality}} & SU_q(2) \sim \mathbb{R}_q^4 / \rho = 1 \\ \textbf{Transmutation} \updownarrow \approx & & \updownarrow \textbf{q - signature change} \\ & \approx & \\ BU_q(su(2)) & \xleftrightarrow[\bigstar\text{- braided groups autoduality}]{} & BSU_q(2) = \mathbb{R}_q^{3,1} / \rho = 1 \end{array} \qquad (42)$$

where we get a duality relation between $\mathbb{R}_q^4$ and $\mathbb{R}_q^{3,1}$ as a kind of T-duality [2]. This interpretation is possible only when $q \neq 1$ - i.e. at the Planck scale -. We can extend those results to q-Poincaré groups

$$\mathbb{R}_q^{3,\,1} \mathrel{>\!\overline{\lhd}}\ \widetilde{U_q(so(3\,,\,1))} \quad (43)$$

seen as dual to the Euclidean q-Poincaré group

$$\mathbb{R}_q^{4} \mathrel{>\!\overline{\lhd}}\ \widetilde{U_q(so(4))} \quad (44)$$

Interestingly, the Hopf algebra duality has been recently related to superstrings T-duality by C.Klimcik and P.Sevara [26]. Such dualities in terms of quantum groups have also been proposed by S. Majid [29].

Now, we apply the hereabove results into a more physical context. So, we propose the following :

Proposition 4.3 *There exists, at the Planck scale, a symmetry of duality between the BRST cohomology ring (physical sector of the theory) and the cohomology ring of instanton moduli space (topological sector).*

Proof Let be, at the Planck scale, BRST cohomology groups, of which the generic form, reviewed in [37], is :

$$H_{BRST}^{(g)} = \frac{\ker Q_{BRST}^{(g)}}{\mathrm{im} Q_{BRST}^{(g-1)}} \quad (45)$$

where $Q_{BRST}^{(g)}$ is the BRST charge acting on operators of the ghost number g. From the theory of Donaldson [14-15], we conclude the existence, at 0 scale of space-time, of cohomology groups constructed by de Rham :

$$H^{(i)}(M_{\mathrm{mod}}^{(k)}) = \frac{\ker d^{(i)}}{\mathrm{im}\, d^{(i-1)}} \quad (46)$$

where $d^{(i)}$ represents the external derivative acting on the differential forms of degree i on $\mathcal{M}_{\mathrm{mod}}^{(k)}$.

Topological theory then brings about ring injection which follows:

$$H_{BRST}^{\bigstar} = \otimes_{g\,=\,0}^{\Delta U_k}\ H_{BRST}^{g} \xrightarrow{\ \iota\ } H^{\bigstar}\left(\mathcal{M}_{\mathrm{mod}}^{(k)}\right) = \otimes_{i\,=\,0}^{d_k} H^{(i)}\left(\mathcal{M}_{\mathrm{mod}}^{(k)}\right) \quad (47)$$

and which, according to conditions given in [19], becomes a ring isomorphism. There exists therefore an injective path from the physical mode to the topological mode. Now let 0_i be the physical observables considered, such that a correlation function of *n* observables is the number given by the matrix of intersections H_i :

$$\langle 0_1\, 0_2\, \dots\, 0_n \rangle = \#(H_1 \cap H_2 \cap \dots \cap H_n) \tag{48}$$

number associated with *n* cycles of homology $H_i \subset \mathrm{M}_{\mathrm{mod}}$, in moduli space $\mathcal{M}^{(k)}_{\mathrm{mod}}$ of configurations of the gravitational instanton type $\Im[\phi(x)]$, on the gravitational fields ϕ of the theory. The physical sector of the theory is described by the left hand side of equation (48) and the topological sector by the right hand side. One observes that $\langle 0_1 0_2\, \dots\, 0_n \rangle \neq 0$, i.e. the theory has a physical content if $\Delta U_k = \int \partial^{\mu} j_{\mu}\, d^4 x$, with j_{μ} being the "ghost flow" of degree *k*, ΔU its integral anomaly and $d_i = gh\,[0_i]$ the ghost number of 0_i. Moreover

$$d_k = \dim_{\mathbb{R}} \mathcal{M}^{(k)}_{\mathrm{mod}} \tag{49}$$

is the dimension of moduli space of degree *k.* Following the theorem of Atiyah-Singer [21], one can show that $\Delta U_k = d_k$. From this point of view, the correlation functions of a set of local observables

$$G(x_1\, \dots\, x_n) = \langle 0(x_1) \dots 0(x_n) \rangle \tag{50}$$

amounts to the integral over moduli space of the number of cohomology classes of space. The associated BRST charge Q is of the form $Q = \sum (-1)^n$. When the divergence of the ghost flow is non-zero, i.e. $\partial^{\mu}\, \mathrm{j}\, \boldsymbol{M} \neq 0$, then the theory oscillates between (0_i) and (H_i) - i.e. between the Coulomb branch and the Higgs branch in metric superposition space - . For the 0 mode of the scale, $\partial^{\mu}\, \mathrm{j}\, \boldsymbol{M} = 0$, then

$$\langle 0_1 0_2\, \dots\, 0_n \rangle = 0 \tag{51}$$

which suggests that on this limit, $\dim \mathcal{M}^{(k)}_{\mathrm{mod}} = 0$. In fact, after functional integration over the empty degrees of liberty of the theory, the physical observables are reduced to closed forms Ω_i of degree d_i, which signifies :

$$\Delta U = \dim \mathcal{M}_{\text{mod}}^{(k)}$$

and when $\Delta U = 0$, there exists no embedding space for moduli space and the theory is projected into the Coulomb branch, at the origin of $\mathcal{M}_{\text{mod}}^{(k)}$, on a singular instanton of zero size, identified to space-time at zero scale. The corresponding signature in this sector of the theory is therefore Euclidean (+ + + +). **qed**

This result suggests once more that at zero scale, the theory is no longer physical but purely topological.

Now, here is a critical question raised by this paper : how do we go from the topological state of the (pre)space-time around the origin to the usual physical state ? In the last section, we shall try to answer this question.

5. TRANSITION FROM INITIAL TOPOLOGICAL PHASE TO STANDARD PHYSICAL PHASE

Considering all the preceding developments, it's of crucial interest to worry about how the initial (generally covariant) topological phase possibly characterizing the (pre)space-time at the vicinity of the Initial Singularity does break down to the universe we observe to day. We then propose some (hopefully) stimulant tracks able to be worked out within some further researches.

On general basis, we claim hereafter that the transition *Topological phase*→ *Physical phase* might be deeply related to the breaking of the N = 2 supergravity at the Planck scale. In other words, supersymmetry breaking, as showed by C. Kounnas *and al* in superstrings context [3-27], is characterized by the loss of the thermodynamical equilibrium of the system. To sum up, the D-dimensional space-time supersymmetry is spontaneously broken in (D-1) dimensions by thermal effects. For this reason, supersymmetry breaking might bring about the *decoupling* of the topological and the physical states of the (pre)space-time system. How is it so ? To see it, according to [4-27], let's recall that at the Planck scale, the (pre)space-time is generally characterized by two fundamental properties : (i) the thermodynamical equilibrium state [34] and (ii) the non-commutativity of the underlying geometry [11]. Those two properties are very often considered, together or separately. However, it is critical to realize that for any system, properties (i) and (ii) are inducing the famous "Kubo-Martin-Schwinger"(KMS) condition [24]. Therefore, we propose now to consider that, most likely, space-time, as a thermodynamical system, is subject to the KMS condition at the Planck scale [6]. Consequently, in the interior of the "KMS strip", i.e. from $ß = 0$ to $ß = \ell_{\text{Planck}}$, the fourth coordinate g_{44} should be considered as *complex*, the two real poles being $ß = 0$ (topological pole) and $ß = \ell_{\text{Planck}}$ (physical pole). This is a direct (and standard) consequence of the KMS condition. So, we suggest [6] that within the KMS strip, the Lorentzian and the Euclidean metric are in a "quantum superposition state" (or coupled), this entailing a "unification" (or coupling) between the topological (Euclidean) and the physical (Lorentzian) states of space-time. Conversely, the transition from the topological state to the physical state of the space-time can be seen in terms of "KMS breaking" (cf. conj. (5.2.5)).

Now, let's begin with the hypothesis of global thermodynamical equilibrium at the Planck scale.

5.1 Thermodynamical equilibrium and KMS state of the space-time at the Planck scale

5.1.1 *Thermodynamical equilibrium of space-time*

From a thermodynamical point of view, it appears that the Planck temperature

$$ß^{-1}_{\text{planck}} \approx T_P \approx \frac{E_P}{k_B} \approx \left(\frac{\hbar c^5}{G}\right)^{1/2} k_B^{-1} \approx 1,4 \times 10^{32}\,\text{K}$$

represents the upper limit of the *physical* temperature of the system. Indeed, it is currently admitted that, before the inflationary phase, the ratio between the interaction rate (Γ) of the initial fields and the (pre)space-time expansion (H) is $\frac{\Gamma}{H} >> 1$, so that the system can reasonably be considered in equilibrium state. This has been established a long time ago within some precursor works of S.

Weinberg [34], E. Witten [4] and several others. It has recently been shown by C. Kounnas *and al* in the superstrings context [27]. However, this natural notion of equilibrium, when viewed as a global gauge condition, has dramatic consequences regarding physics at the Planck scale. Which kind of consequences? To answer, let's see on formal basis what an equilibrium state is.

Definition 5.1 *H being an autoadjoint operator and* $\mathfrak{H}$ *the Hilbert space of a finite system, the equilibrium state* ω *of this system is described by the Gibbs condition* $\varphi(A) = \dfrac{\mathrm{Tr}_{\mathfrak{h}}(e^{\ ßH} A)}{\mathrm{Tr}_{\mathfrak{h}}(e^{\ ßH})}$ *and satisfies the KMS condition.*

Here, Tr is the usual trace, $ß = \dfrac{1}{kT}$ is the inverse of the temperature, H the Hamiltonian, i.e. the generator of the one parameter group of the system. Of course, **A** is a von Neumann C^*- algebra (see §(5.1.4) for definitions). The equilibrium state implies that $ß$ must be seen as a periodic (imaginary) time interval $[0, ß = \ell_{Planck}]$. Now, the famous Tomita-Takesaki modular theory [10-11] has established that to each state $\varphi(A)$ of the system corresponds, in a unique manner, the strongly continuous one parameter $*$ - automorphisms group α_t :

$$\alpha_t(\boldsymbol{A}) = e^{iHt}\, \boldsymbol{A}\ e^{-iHt} \tag{52}$$

with $t \in \mathbb{R}$. This one parameter group describes the time evolution of the observables and corresponds to the well known Heisenberg algebra. At this stage, we are brought to find the remarkable discovery of Takesaki and Winnink, connecting (**i**) the evolution group $\alpha_t(\boldsymbol{A})$ of a system (i.e. the modular group $\boldsymbol{M} = \Delta^{it} A\ \Delta^{-it}$) with (**ii**) its equilibrium state $\varphi(A) = \dfrac{\mathrm{Tr}(Ae^{-ßH})}{\mathrm{Tr}(e^{-ßH})}$ [11]. The famous "KMS condition" [24] is nothing else than this relation between $\alpha_t(\boldsymbol{A})$ and $\varphi(\boldsymbol{A})$, the content of this relation being precised in (i) and (ii) of §(5.1).

Then we claim in a natural way that the space-time, in equilibrium state at the Planck scale, is therefore subject to the KMS condition at this scale.

5.1.2 *The (pre)space-time in KMS state at the Planck scale*

When viewed as a hyperfinite system at the Planck scale, the (pre)space-time may be described by a von Neumann C^*-algebra **A** (a von Neumann algebra is *hyperfinite* if it is generated by an increasing sequence of finite dimensional sub-agebras). Now, let's see the essence of the KMS condition, given by the Haag-Hugenholtz-Winnink theorem [23] : a state ω on the C^*-algebra **A** and the continuous one parameter automorphisms group of **A** at the temperature $ß = 1 / k\,T$ verify the KMS condition if, for any pair A, B of the $*$ - sub-algebras of **A,** it exists a $f(t_C)$ function *holomorphic* in the strip $\{t_C = t + i\,ß \in \mathbb{C}\ ,\ \mathrm{Im}\, t_C \in [\,0\ ,\ ß\,]\}$ such that :

$$\text{(i)}\quad f\,(t) = \varphi(A\,(\,\alpha_t\,B))\ ,$$
$$\text{(ii)}\quad f\,(t + i\,ß\,) = \varphi(\,\alpha_t\,(B)A)\ ,\qquad \forall\, t \in \mathbb{R}. \tag{53}$$

Then we observe with (i) and (ii), the two crucial properties of the KMS condition : the *holomorphicity* of the KMS strip and of course, due to the cyclicity of the trace, the *non commutativity* $\varphi((\alpha_t A)B) = \varphi(B(\alpha_{t+i\ ß}A))$ characterizing any "KMS space" (in fact, the two boundaries of the strip do not commute with each other).

Now, if we admit that around ℓ_{Planck}, the hyperfinite (pre)space-time system is in a thermal equilibrium state, then according to [24], we are also bound to admit that this system is in a KMS state. Incidentally, another good reason to apply the KMS condition to the space-time at ℓ_{Planck} is that at such a scale, the notion of commutative geometry vanishes and should be replaced by *non commutative* geometry [11]. In this new framework, the notion of "point" in the usual space collapses and is replaced by the "algebra of functions" defined on a non commutative manifold. Non commutative geometry and quantum groups theory [16-29] are addressing such non-commutative constraints. But the non-commutativity induced by the KMS state is in natural correspondence with the expected non commutativity of the space-time geometry at the Planck scale.

Next, let's push forwards the consequences raised by the holomorphicity of the KMS strip.

5.1.3 *Holomorphic time flow at the Planck scale*

As a consequence of the application of the KMS condition to space-time itself, we are induced to consider that the time-like coordinate g_{00} becomes *holomorphic* within the limits of the KMS strip. So we should have [11-24] :

$$t \rightarrow \tau = t_r + i\, t_i \qquad (54)$$

as showed in [6]. In the same way, the physical (real) temperature becomes also complex at the Planck scale :

$$T \rightarrow T_C = T_r + i T_i \qquad (55)$$

as proposed by Atick and Witten in another context [4]. So, the KMS condition suggests the existence at the Planck scale, of an effective one loop potential coupled, in N = 2 supergravity, to the complex dilaton + axion field $\varphi = \dfrac{1}{g^2} + i\alpha$ and yielding the following dynamical form of the metric

$$\eta_{\mu\nu} = \mathrm{diag}(1\ ,\ 1\ ,\ 1\ ,\ e^{i\theta}) \qquad (56)$$

The signature of (56) is Lorentzian (physical) for $\theta = \pm\, \pi$ and can become Euclidean (topological) for $\theta = 0$. This unexpected effect is simply due to the fact that, within the boundaries of the analytic KMS field -i.e. from the scale zero up to the Planck scale- the "time-like" direction is extended to the complex variable $t_C = t_r + i\, t_i \in \mathbb{C}$, Im $t_C \in [i\, t_i\ ,\ t_r]$, the function *f(t)* being analytic within the limits of the KMS field and continuous on the boundaries. What happens on the $ß = 0$ limit ? Applying the KMS properties, we find that the time like direction t becomes pure imaginary so that the signature is Euclidean (++++). Conversely, t is pure real for $ß \geq \ell_{Planck}$ (+++–). So, according to Tomita's modular theory [11], the KMS condition, when applied to the space-time, induces, within the KMS strip, the existence of the "extended" (holomorphic) automorphisms group :

$$\boldsymbol{M}_q \mapsto \sigma_{ß_C}(\boldsymbol{M}_q) = e^{\;Hß_C}\boldsymbol{M}_q\ e^{\;Hß_C} \qquad (57)$$

with the *ß* parameter being formally *complex* and able to be interpreted as a complex time *t* and / or temperature *T*. It is interesting to remark that in the totally different context of superstrings, J.J. Atick and E. Witten were the first to propose such an extension of the real temperature towards a complex domain [4]. Recently, in N=4 supersymmetric string theory, I. Antoniadis, J.P. Deredinger and C. Kounnas [3] have also suggested to shift the real temperature to imaginary one by identification with the inverse radius of a compactified Euclidean time on S^1, with $R = 1 / 2\pi T$. Consequently, one can introduce a complex temperature in the thermal moduli space, the imaginary part coming from the $B_{\mu\nu}$ antisymmetric field under type ***IIA*** $\xleftrightarrow{S/T/U}$ type ***IIB*** $\xleftrightarrow{S/T/U}$ ***Heterotic*** string-string dualities. More precisely, in Antoniadis *and al* approach, the field controlling the temperature comes from the product of the real parts of three complex fields : s= Re **S**, t= Re **T** and u= Re **U**. Within our KMS approach, the imaginary parts of the moduli **S**, **T**, **U** can be interpreted in term of Euclidean temperature. Indeed, from our point of view, a good reason to consider the temperature as complex at the Planck scale is that a system in thermodynamical equilibrium state must be considered as subject to the KMS condition [24].

Now, let's step forward a more algebraic comprehension of KMS state, in terms of von Neumann algebras.

5.1.4 *KMS state in terms of von Neumann algebras*

The von Neumann algebras are, naïvely speaking, the *non commutative* analogs of measure theory. They have a critical importance in our understanding of non commutativity of space-time around the Planck scale.

In the KMS state, the only von Neumann algebras involved are what is called "factors", i.e. a special type of von Neumann algebra, whose the center is reduced to the scalars $a \in \mathbb{C}$. There exists three types of factors : the type I and type II (in particular here II_∞) -which are commutative and endowed with a *trace*- and the type III, *non commutative* and traceless. A trace τ on a factor M is a linear form such that $\tau(AB) = \tau(BA)$, $\forall A, B \in M$. In this case, any measure on M is invariant. When the measure on M is ill defined (which is the case of type III), the notion of trace vanishes and is replaced by the one of "weight", which is a linear map from M_+ to $\mathbb{R}_+ = [0, +\infty]$. The type III factors have no definite trace. They are very important hereafter as far as they are the only one involved in KMS states. We work here with "III_λ" factors, $\lambda \in]0, 1[$, characterized by the invariant $S(M) = \lambda^{\mathbb{Z}} \cup \{0\}$.

Rightly, the KMS condition, when applied to the (pre)space-time at the Planck scale, cuts up three different scales on the (pre)light cone, which can be described by three different types of von Neumann algebras (or "factors").

5.1.5 *From the topological scale to the physical scale of the space-time*

(i) ***the topological scale*** (ß = 0, signature {++++}) : this initial "topological" scale correspond to the imaginary vertex of the light cone, i.e. a zero-size gravitational instanton. All the measures performed on the Euclidean metric being ρ-equivalent up to infinity, the system is ergodic. As shown by A. Connes, any ergodic flow for an invariant measure in the Lebesgue measure class gives a unique type II_∞ hyperfinite factor [11]. This strongly suggests that the singular 0-scale should be described by a type II_∞ factor, endowed with a hyperfinite trace noted Tr_∞. By hyperfinite, we simply mean that the trace of the II_∞ factor is not finite. We call $\boldsymbol{M}_{Top}^{0,1}$ such a "topological" factor, which is an infinite tensor product $\otimes^\infty$ of matrices algebra (ITPFI) of the $R_{O,1}$ Araki-Woods type [11]. Now, the initial state on $\boldsymbol{M}_{Top}^{0,1}$, corresponding in ex. (2.1) to the divergent values of the dilaton field $\frac{1}{g^2}$, is given by :

$$\varphi(\boldsymbol{M}_{Top}^{0,1}) = \frac{\mathrm{Tr}_\infty(e^{-ßH}\boldsymbol{M}_{Top}^{0,1})}{\mathrm{Tr}_\infty(e^{-ßH})} \tag{58}$$

and, considering the hyperfinite characteristic of the trace Tr_∞, we have equivalently :

$$\varphi(\boldsymbol{M}_{Top}^{0,1}) = \mathrm{Tr}_\infty(e^{\;ßH}\boldsymbol{M}_{Top}^{0,1}\; e^{ßH}) \tag{59}$$

where $\varphi(\boldsymbol{M}_{Top}^{0,1})$ represents a very special type of "current", that we propose to call 'trace current" $\mathbf{T}^f$. Clearly, the invariant hyperfinite II_∞ trace current $\mathbf{T}^f$ is a pure topological amplitude [19-37] and, as such, "propagates" in imaginary time from zero to infinity. In this sense, $\varphi(\boldsymbol{M}_{Top}^{0,1})$ can be seen as a "zero topological cycle" which represents an intrinsic "Euclidean dynamic" controlling the blow up of the space-time Initial Singularity [6].

(ii) ***the quantum scale*** (0 < ß < ℓ_{Planck} , signature {+++±}) : we reach the KMS domain [24]. Considering the quantum fluctuations of $g_{\mu\nu}$, there is no more invariant measure on the non commutative metric. Therefore, according to von Neumann algebra theory, the "good factor" addressing those constraints is uniquely a non commutative *traceless* algebra, i.e. a type III factor [9] (the only one able to be involved in KMS state). More precisely, it is a type III$_\lambda$ that we call $\boldsymbol{M}_q$, with the period $\lambda \in]\,0, 1\,[$. Important, it has been demonstrated that any type III$_\lambda$ factor can be canonically decomposed into the following way [9] :

$$\mathrm{III}_\lambda = \mathrm{II}_\infty \rtimes_\theta \mathbb{R}^*_+ \tag{60}$$

$\mathbb{R}^*_+$ (dual of $\mathbb{R}$) acting periodically on the "II$_\infty$ factor". Then the *ß*-dependant *periodicity* of the action of $\mathbb{R}^*_+$ on $\boldsymbol{M}^{0,1}_{Top}$ takes the form :

$$\boldsymbol{M}\mathrm{q} = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathbb{R}^*_+ / ß\mathbb{Z} \equiv \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta ß\, \mathrm{S}_1 \tag{61}$$

The relation between λ and ß is such that $\lambda = \frac{2\pi}{ß}$, so that when $ß \to \infty$, we get $\lambda \to 0$ (the periodicity is suppressed). Now, the theory being given on the infinite Hilbert space $\mathscr{L}(\mathfrak{h}) = \mathscr{L}\left[L^2\left(\mathbb{R}^*_+ / ß\mathbb{Z}\right)\right]$, $\boldsymbol{M}$q becomes :

$$\boldsymbol{M}\mathrm{q} = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathscr{L}\left[L^2\left(\mathbb{R}^*_+ / ß\mathbb{Z}\right)\right] \tag{62}$$

The type I$_\infty$ factor $\mathscr{L}\left[L^2\left(\mathbb{R}^*_+ / ß\mathbb{Z}\right)\right]$ yields the modular flow of (periodic) *evolution* of the system. So, the KMS type III$_\lambda$ factor $\boldsymbol{M}$q connects the "topological" type II$_\infty$ factor $\boldsymbol{M}^{0,1}_{Top}$ with the "physical" type I$_\infty$ factor $\boldsymbol{M}_{Phys}$:

$$\boldsymbol{M}\mathrm{q} = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \boldsymbol{M}_{Phys} \tag{63}$$

In terms of "flows", Equ. (63) connects the topological flow of weights of $\boldsymbol{M}^{0,1}_{Top}$ with the physical modular flow raised by $\mathscr{L}\left[L^2\left(\mathbb{R}^*_+ / ß\mathbb{Z}\right)\right]$. This furnishes a good image of the unification between topological and physical states, to be compared to the bicrossproduct (41) Uq(so(4)$^{\mathrm{op}}$ $\overset{\psi}{\triangleright\!\blacktriangleleft}$ Uq(so(3,1)) unifying Euclidean and Lorentzian q-groups. The quantum flow $\sigma_{ß_c}(\boldsymbol{M}_{q\ \ flow}) = e^{ß_c H}\boldsymbol{M}_{q\ \ flow} e^{\ ß_c H}$ is constructed in prop. (5.2).

(iii) ***the physical scale*** ($ß > \ell_{Planck}$, signature {+++–}) : this last scale represents the physical part of the light cone and, consequently, the notion of (Lebesgue) measure is fully defined. Therefore, the (commutative) algebra involved is endowed with a hyperfinite trace and is given on the infinite Hilbert space $\mathscr{L}(\mathfrak{h})$, with $\mathfrak{h} = L^2(\mathbb{R})$. Then $\mathscr{L}(L^2(\mathbb{R}))$ is a *type* I_∞ *factor*, indexed by the real group $\mathbb{R}$, which we call $\boldsymbol{M}_{Phys}$. So, $\mathscr{L}(L^2(\mathbb{R})) = \boldsymbol{M}_{Phys}$ and the flow raised by $\boldsymbol{M}_{Phys}$ is simply the (real) time evolution, given by the modular group :

$$\sigma_t(\boldsymbol{M}_{Phys}) = e^{iHt}\boldsymbol{M}_{Phys}\ e^{-iHt} \tag{64}$$

In this case (type I_∞ factor) all the automorphisms are inner automorphisms. We call "physical flow" $\mathbf{P}^f_{ß>0}$ this evolution flow in real time. Of course, $\sigma_t(\boldsymbol{M}_{Phys})$ is simply giving the usual algebra of observables [12].

At present, we shall evidence that the KMS state "unifies" the physical flow and the topological current.

Proposition 5.2 *At the KMS scale $0 < ß < l_{Planck}$, the two automorphisms groups $\sigma_t(M_{Phys})$ and $\sigma_ß(M^{0,1}_{Top})$ are coupled up to Planck scale within a unique IIIλ factor of the form* $\boldsymbol{M}$q = $\boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathscr{L}\left[L^2\left(\mathbb{R}^*_+ / ß\mathbb{Z}\right)\right]$. *The corresponding extended one (complex) automorphisms group describing the quantum evolution is*

$$M_q \mapsto \sigma_ß(M_q) = e^{Hß_c} M_q\ e^{-Hß_c}$$

Mq corresponds to the coupling between the one parameter automorphisms group giving the physical flow and the automorphisms semi-group giving the topological flow of the system.

Proof The KMS state of the (pre)space-time is yield by the unique IIIλ factor given by equ. (60) :

$$\mathbf{M}\mathrm{q} = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathscr{L}\left[L^2\left(\mathbb{R}^*_+ / ß\mathbb{Z}\right)\right] = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \boldsymbol{M}_{Phys} \tag{65}$$

which represents the KMS "unification" of the topological state and the physical state of the (pre)space-time at the Planck scale. Now, since it exists an operatorial weight of $\boldsymbol{M}$q on its sub-group $\boldsymbol{M}^{0,1}_{Top}$, the equilibrium state φ on $\boldsymbol{M}$q is given by the state on $\boldsymbol{M}^{0,1}_{Top}$. We express the state φ under the new form constructed in [6] :

$\varphi(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) = \mathrm{Tr}_\infty(e^{\;ßH}\boldsymbol{M}^{0,1}_{Top}\; e^{ßH})$

This represents what we have called in (5.1.5) the "trace current" of the "topological factor" $\boldsymbol{M}^{0,1}_{Top}$. However, Connes-Takesaki have shown [10] that the flow of weights on a factor is given by the flow of weights on the associated II_∞ factor. For it exists an homomorphism OUT $\mathrm{III}_\lambda \rightarrow$ OUT II_∞ such that the sequence (66) is exact :

$$\{\,1\,\} \rightarrow H^1(F) \xrightarrow{\overline{\partial} M} \mathrm{OUT}M \xrightarrow{\overline{\gamma}} OUT_{\theta,\tau}(N) \rightarrow \{\,1\,\} \tag{66}$$

The multiplicative action of $\mathbb{R}$: $\tau \circ \theta_S = e^{-S}\tau,\;\; s \in \mathbb{R}$ on $\boldsymbol{M}^{0,1}_{Top}$ *translates* the trace τ of $\boldsymbol{M}^{0,1}_{Top}$, which generates the flow of weights on $\boldsymbol{M}^{0,1}_{Top}$ and $\boldsymbol{M}$q (cf.[10]). So, $\varphi(\boldsymbol{M}_{\boldsymbol{q\text{-}state}})$ becomes a *ß*-dependant automorphism (semi)group :

$$\sigma_ß\,(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) = e^{\;ßH}\boldsymbol{M}^{0,1}_{Top}\; e^{ßH} \tag{67}$$

Equ. (67) describes the *flow of weights* [10] of the type III_λ factor $\boldsymbol{M}$q. But as pointed in [6], we can also interpret equ.(67) as a " modular flow in imaginary time" *it*, dual to the modular flow in real time given by :

$\sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}})` = e^{\;iHt}\boldsymbol{M}_{\boldsymbol{Phys}}e^{\,-\,iHt}$, t $\in \mathbb{R}$.

An interpretation of this type has also been proposed (in a different context, however) by Derendinger and Lucchesi in [13]. Finally, the KMS flow connects the flow of weights $\sigma_ß(\boldsymbol{M}_{\boldsymbol{q\text{-}state}})$ to the modular group $\sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}})$:

$$\begin{aligned}\sigma_{ß_C}(\boldsymbol{M}_{q\;\;flow}) &= \sigma_ß(\boldsymbol{M}^{0,1}_{Top}) \otimes \sigma_t(\boldsymbol{M}_{Phys})\\ &= e^{\;(ß+it)H}\boldsymbol{M}_{q\;\;flow}\;\; e^{(ß+it)H}\\ &= e^{\,ß_C H}\boldsymbol{M}_{q\;\;flow}e^{\;\;ß_C H}\end{aligned}$$

which is indexed by the complex time variable $ß_C$. Again, this flow is expressing the unification between the *physical* flow $\sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}}) = \sigma_t(\boldsymbol{M}_{Phys})$ and the *topological* flow $\sigma_ß(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) = \sigma_ß(\boldsymbol{M}^{0,1}_{Top})$ within the *unique* KMS (or quantum) flow $\underset{0<\mathbf{ß}<\ell P}{\mathbf{Q}^f}$ given by the automorphisms group of $\boldsymbol{M}_q$:

$$\sigma_{ß_C}(\boldsymbol{M}_{q\;\;flow}) = \sigma_ß(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) \oplus \sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}})$$

The (pre)space-time KMS strip has zero as infimum and the Planck scale as supremum. So between those bounds, the Euclidean topological flow and the Lorentzian physical flow are unified in a natural way within the holomorphic "quantum flow" $\underset{0<ß<\ell_P}{\mathbf{Q}^f} \rightarrow \ \sigma_{ß_C}(\boldsymbol{M}_{q \ flow}) = e^{ß_C H}\boldsymbol{M}_{q \ flow} e^{\ ß_C H}$.

Another way to verify the coupling of M_{Phys} and $M_{Top}^{0,1}$ in the unique type III$_\lambda$ factor lies in the Conne's invariant

$$\delta : \ \mathbb{R} \rightarrow \text{OUT M} = \frac{\text{AUT M}}{\text{INT M}} \tag{68}$$

(automorphisms of M quotiented by the inner automorphisms, necessarily present in the non commutative case). This M invariant represents an ergodic flow $\{W(M) \ , \ W_\lambda\}$ where W_λ is a one parameter group of transformations - i.e. a flow - which admits a description in terms of class of weights and whose the natural parameter is $\mathbb{R}_+^{\ *}$. We consider now the type III$_\lambda$ factor Mq of equ.(61). Starting from equ. (68), we can construct the extension *Ext* (noted $\top$) of OUT Mq by INT Mq in AUT Mq :

$$\text{AUT Mq} \equiv \ \text{OUT Mq} \top \text{INT Mq} \tag{69}$$

with $\{x,y\} \in$ OUT Mq and $\{x',y'\} \in$ INTMq. The inner automorphisms group INTMq is a normal sub-group of AUT Mq . Considering two weights φ et ψ of Mq, and applying the Radon-Nikodym theorem [10], it exists a unitary of Mq such that $\sigma_t^{\psi}(x) = u_t \sigma_t^{\varphi}(x) u_t^*$, with $u_t = (D\psi \ ; \ D\varphi)_t$ and $\sigma_t^{\psi}(x) \in$ INT Mq for a certain class of modular automorphisms. Considering the fact that under the trace of the factor II$_\infty$ involved in the crossproduct $\boldsymbol{M}q \ = M_{Top}^{0,1} \ \rtimes_\theta \ \mathbb{R}^*{}_+$ all the modular automorphisms are *inner* automorphisms, we restrict INTMq to the sub-group of the modular automorphisms, which we call INT$_{mod}$ Mq. Then we look for the image of the inner modular group in OUTMq. Within a certain cohomology class $\{K\}$, the group $\sigma_t^{\psi}(x)$ is given by INT$_{mod}$Mq, whereas the non-unitary transformations $\sigma_ß(x)$ are given by OUT Mq. We get then for the "physical" flow :

$$\sigma_t^{\psi}(x) \ = e^{iHt} M_q \ e^{\ iHt} \in \text{INT}_{mod} \text{ Mq} \tag{70}$$

whereas the "topological" flow of weights of *Mq* is given by :

$$\sigma_ß(x) \ = e^{\ ßH} M_q \ e^{ßH} \in \text{OUT Mq} \tag{71}$$

and the extension AUT Mq ≡ OUT Mq $\top$ INT$_{mod}$ Mq yields :

$$\sigma_{(t \top ß)} = \sigma_t^{\psi}(x) \top \sigma_ß(x) \tag{72}$$

Within the general group of extensions $\{Ext\}$, we get the trivial holomorphic sub-group :

$\sigma_{ß+it}(M_q) = e^{(ß+it)H} M_q \; e^{(ß+it)H} = \sigma_{ß_C}(M_q) = e^{Hß_C} M_q \; e^{-Hß_C}$ or

which corresponds to the KMS state and"unifies" within the unique extended form $\sigma_{ß_C}(Mq)$ the physical flow $\sigma_t^{\psi}(x)$ and the topological current $\sigma_ß(x)$. Clearly, we get $\sigma_{ß_C}(Mq) \subset$ OUT Mq $\top$ INT$_{mod}$ Mq. Again we find : $\sigma_{ß_C}(\boldsymbol{M}_{q\ \ flow}) = \sigma_ß(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) \oplus \sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}})$ **qed**

Now, let's get over the last step. Our aim is to explain the transition from the topological state to the physical state (**TP** transition) of the space-time. We shall cope with this problem following two different ways :

(i) we conjecture that such a transition could be related to the N=2 supergravity breaking beyond ℓ_{Planck} ;

(ii) likewise, **TP** transition could be explained in terms of "decoupling", beyond the Planck scale, between the (Euclidean) "topological current" (raised by $\boldsymbol{M}_{Top}^{0,1}$) and the (Lorentzian) physical flow (yield by $\boldsymbol{M}_{Phys}$).

5.2 TP transition, supersymmetry breaking and flows decoupling

First of all, let's put in evidence the link between KMS state and supersymmetry. To do this, we propose hereafter a relevant example able to be seen as a good toy model expliciting the deep correspondence between thermal states, supersymmetry and extended space-time (i.e. extension of the time-like direction in the complex plane).

Example 5.2.1 *thermal states, supersymmetry and KMS condition*

In the following, we shall focus on some important results recently obtained by J.P. Derendinger and C. Lucchesi [13]. Interestingly, it has been demonstrated that thermal supersymmetry (as opposed to T=0 supersymmetry) must be considered in the context of thermal (i.e. KMS) superspace. We remark here that the authors apply the KMS condition to the thermal superspace (i.e. the thermal supersymmetric space) in a general setting. In our own approach, as suggested in ref. 6 and in the present paper as well, we apply the KMS condition to the *thermal (pre) space-time at the Planck scale*. Considering that in the standard “hot big-bang” theory the (pre) space-time is generally viewed as supersymmetric, such an identification is natural. Namely, the authors have established that the thermal supersymmetry parameters must be both *time dependant and (anti)periodic in imaginary time* on the interval [0, ß], where *ß = 1/T*. In other words, focusing on field representations of the thermal super-Poincaré algebra and on chiral supermultiplet, one can straightfully see that thermal superfields are characterized by their time/ temperature *periodicity* properties. To explicit this, let's simply recall that at zero temperature, supersymmetry can heuristically be represented as a set of "generalized translations", including Grassmann variables that are translated by the supersymmetry generators. Therefore, a "point" X in superspace has coordinates

$$X = (x^{\mu}, \theta^{\alpha}, \overline{\theta}^{\alpha}) \tag{73}$$

where θ and $\overline{\theta}$ are the usual Grassmannian objets. Since at zero temperature the parameters of supersymmetry transformations are constant, the zero-temperature superspace coordinates are also space-time constants. In fact, at T=0, the (anticommuting) Grassmann coordinates simply turn bosonic commutation relations into fermionic anticommutators and conversely. Now, what happens at finite temperature (i.e. the case of primordial universe investigated here)? As a matter of fact, the situation is not so simple, because fermion and boson statistics involve, in addition, the appropriate statistical weight in field theory Green's functions. In such a context, as pointed in refs [13] and [28], it is natural to require that the variables which are translated by the effect of thermal supersymmetry transformation express the same properties as the thermal supersymmetry parameters. Therefore, the construction of thermal supersymmetry requires that the Grassmann variables get promoted to be time-dependant and (anti)periodic in imaginary time. To see this, let's precise that the thermal average $\langle \ldots \rangle_{ß}$ of a field operator O is, as usual, given by

$$\langle 0 \rangle_{ß} \equiv \frac{1}{Z(ß)} \mathrm{Tr}\left(e^{-ßH} 0\right) \tag{74}$$

with the lowest energy state being $E_0 = 0$, so that we have on the zero temperature limit :

$$\langle 0 \rangle_{ß} \xrightarrow{\beta \to \infty} \langle 0 | 0 | 0 \rangle$$

Now, at finite temperature, the Green's functions are necessarily subject to periodicity constraints in imaginary time. However, as showed in [6], those constraints are *exactly defining the KMS condition*. To verify this important point, we now review those conditions for bosonic and fermionic fields. Let's first begin with a free scalar (i.e. bosonic) field ϕ at $x = (t, \mathbf{x})$ whose evolution is such that :

$$\phi(x) = e^{iHt} \phi(0, x) e^{-iHt} \tag{75}$$

where the time coordinate t is allowed to be *complex*. Then the n-point thermal Green's function G_{nC} of the system is :

$$G_{nC}(x_1, \ldots, x_n) = \left\langle \mathrm{T}_C \phi(x_1) \ldots \phi(x_n) \right\rangle_{ß} \tag{76}$$

T_C being the path-ordering operator and $\langle \ldots \rangle_{\beta}$ the canonical thermal average. Then the thermal path-ordered propagator takes the form (D_c being the thermal propagator of the theory) :

$$D_C(x_1, x_2) = \left\langle \theta_C(t_1 - t_2) D_C^{>}(x_1, x_2) + \theta_C(t_2 - t_1) D_C^{<}(x_1, x_2) \ldots \phi(x_n) \right\rangle \tag{77}$$

where θ_c is the path Heaviside function. Then the thermal bosonic two-point functions $D_C^{>}$, $D_C^{<}$ are defined as :

$$\begin{aligned} D_C^{>}(x_1, x_2) &= \left\langle \phi(x_1) \phi(x_2) \right\rangle_{ß} \\ D_C^{<}(x_1, x_2) &= \left\langle \phi(x_2) \phi(x_1) \right\rangle_{ß} \end{aligned} \tag{78}$$

At this stage, as proposed in [6], the Boltzman weight $e^{-ßH}$ can be seen as an evolution operator in Euclidean time, so that after a translation in imaginary time we get the formula (79) :

$$e^{-ßH}\phi(t,\mathrm{x})e^{ßH} = \phi(t+iß,\mathrm{x}) \tag{79}$$

which is exactly the KMS condition formulated in equ. (53). Then $D_C^{>}(x_1,x_2)$ in equ.(78) becomes :

$$D_C^{>}(x_1,x_2) = \frac{1}{Z(ß)}\mathrm{Tr}\left[e^{-ßH}\phi(x_1)\phi(x_2)\right] \tag{80}$$

Likewise for $D_C^{<}(x_1,x_2)$. So using the cyclicity of the thermal trace and the notion of evolution in Euclidean time *it*, one can construct the "bosonic KMS condition" [13-28]. Interestingly, such a condition relates $D_C^{>}$ and $D_C^{<}$ by a translation in Euclidean (imaginary) time :

$$D_C^{>}(t_1;\mathrm{x}_1,t_2;\mathrm{x}_2) = D_C^{<}(t_1+iß;\mathrm{x}_1,t_2;\mathrm{x}_2) \tag{81}$$

Of course the same construction holds for fermions. Indeed, defining the fermionic two-point function $S_{C_{ab}}^{>}$ and $S_{C_{ab}}^{<}$, (with a,b = 1 ...4 for Dirac four-components spinors) as

$$S_{C_{ab}}^{>}(x_1,x_2) = \left\langle \psi_a(x_1)\overline{\psi}_b(x_2)\right\rangle_ß$$

$$S_{C_{ab}}^{<}(x_1,x_2) = -\left\langle \overline{\psi}_b(x_2)\psi_a(x_1)\right\rangle_ß \tag{82}$$

and as in the bosonic case, the fermionic KMS condition takes the form :

$$S_{C_{ab}}^{>}(t_1;\mathrm{x}_1,t_2;\mathrm{x}_2) = -S_{C_{ab}}^{<}(t_1+iß;\mathrm{x}_1,t_2;\mathrm{x}_2) \tag{83}$$

which differs from the bosonic condition only by a relative sign. From the structure of equ.(81) and equ.(83), we deduce that when the temperature of the supersymmetric system (here the (pre)space-time) is *not* zero, then bosonic fields are *periodic* in imaginary time whereas fermionic fields are *antiperiodic*. Let's remark that supersymmetry algebra is not sensible to this periodicity-antiperiodicity distinction. If (as pointed in [13-28]) it is true that the supersymmetry breaking is "encoded" in this difference, the breaking becomes effective only when the KMS state is cancelled. For this reason, as demonstrated in the hereabove refs., the KMS condition must be applied to the *superfields* of the theory. In [13-28], the superfields are superspace expansions which contain as components the bosonic and fermionic degrees of freedom of supermultiplets. Such superfields are usually formulated using two-component Weyl spinors ψ_α and $\overline{\psi}^{\dot{\alpha}}$, related to Dirac spinors through $\psi_a = \dfrac{\psi_\alpha}{\overline{\psi}^{\dot{\alpha}}}$. Then the KMS condition for Dirac spinors can be extended to Weyl spinors and, in the same way, to Majorana spinors. The fermionic KMS condition for majorana spinors takes the form :

$$\begin{aligned} S^{>\ \dot{\beta}}_{C_\alpha}(x_1, x_2) &= \left\langle \psi_\alpha(x_1) \overline{\psi}^{\dot{\beta}}(x_2) \right\rangle_ß \\ S^{<\ \dot{\beta}}_{C_\alpha}(x_1, x_2) &= -\left\langle \overline{\psi}^{\dot{\beta}}(x_2) \psi_\alpha(x_1) \right\rangle_ß \end{aligned} \qquad (84)$$

Now, one can realize that imposing the KMS condition to superfields components implies that one must also allow Grassmann parameters to depend on imaginary time. In fact, in the context of supersymmetry, the main question is the following : under thermal constraints, how do we successfully achieve the transformation of periodic bosons into antiperiodic fermions and vice-versa? The answer, developed in [13-28], consists in constructing the *thermal superspace*, i.e. in introducing time dependant and antiperiodic space-time coordinates. Henceforth, a point in thermal superspace has "KMS coordinates", given by a new set of Grassmannian variables:

$$\hat{X} = (x^\mu, \hat{\theta}^\alpha(t), \hat{\overline{\theta}}^{\dot{\alpha}}(t)) \qquad (85)$$

where the symbol "^" denotes the thermal quantities and $\hat{\theta}^\alpha(t), \hat{\overline{\theta}}^{\dot{\alpha}}(t)$ are subject to the antiperiodicity conditions

$$\hat{\theta}^\alpha(t + iß) = -\hat{\theta}^\alpha(t), \quad \hat{\overline{\theta}}^{\dot{\alpha}}(t + iß) = -\hat{\overline{\theta}}^{\dot{\alpha}}(t) \qquad (86)$$

Consequently, the condition (86) induces a temperature-dependant constraint on the time-dependant superspace Grassmann coordinates $\hat{\theta}^\alpha(t)$ and $\hat{\overline{\theta}}^{\dot{\alpha}}(t)$. From equ. (85), we finally observe that the KMS condition must be applied to the space-time metric itself, as formulated in §(5.1.2). Among the consequences, we are therefore induced to consider that the time-like direction must be extended in the complex plan (see §(5.1.3)).

Now, what does these results mean in the context of our research? As a matter of fact, Derendinger and Lucchesi have clearly confirmed that there exists a deep relation between (thermal) supersymmetry and KMS condition. This relation is implemented at the level of thermal Grasmann coordinates, because of the (anti)periodicity conditions given by equ.(86). Indeed, it has been proved by the authors that the only way to preserve supersymmetry in the thermal context is to consider that the space-time metric itself must be subject to the KMS condition. Otherwise, the periodic bosons and antiperiodic fermions *could not* be related by supersymmetry. Now, let's put this simple question : what happens when the KMS state collapses? The analysis of the "KMS Grassmann coordinates", in particular the equ.(86), clearly show that supersymmetry cannot be implemented without applying the KMS condition to space-time coordinates. The reason of this is that when the space-time system is *not* subject to the KMS state (e.g. non-equilibrium state), a point X of superspace is endowed again with the usual Grassmann coordinates

$$X = (x^{\mu}, \theta^{\alpha}, \overline{\theta}^{\alpha})$$

This is equivalent to T=0 supersymmetry, for which the parameters of transformation (i.e. the Grassmannians θ and $\overline{\theta}$) are space-time *constants*. But rightly, the main result of refs [13-28] establishes without ambiguity that at finite temperature, one *cannot* make use of constant parameters in supersymmetry transformations rules. The supersymmetry parameters must be time dependant *variables*, (anti)periodic in imaginary time. So, in a natural way, the thermal Grassmann coordinates $\hat{X} = (x^{\mu}, \hat{\theta}^{\alpha}(t), \hat{\overline{\theta}}^{\dot{\alpha}}(t))$ must be "translated" in imaginary time and are consequently subject to the antiperiodicity conditions $\hat{\theta}^{\alpha}(t + i\beta) = -\hat{\theta}^{\alpha}(t)$ and $\hat{\overline{\theta}}^{\dot{\alpha}}(t + i\beta) = -\hat{\overline{\theta}}^{\dot{\alpha}}(t)$ of equ.(86). Obviously, the only way to implement such a condition is to consider that globally, the space-time system is in KMS state at a given scale (i.e. in our case between the scale zero and the Planck scale). Incidentally, the hereabove approach can be seen as a confirmation that the ß → 0 limit is topological. As a matter of fact, the ß → 0 limit of equ.(79) is given by the scalar field $\phi(x)$, which, by construction, is a topological configuration marking the origin of the imaginary time direction of the theory.

From hereabove we can now conclude that (thermal) supersymmetry and KMS states are linked in such a manner that *the breaking of the KMS state beyond the Planck scale should induce the breaking of supersymmetry at the same scale*. Let's go further in the exploration of such a breaking. In a very stimulating way, Derendinger and Lucchesi have emphasized the fact that the thermal field boundary conditions characterizing KMS state carry information that is of global nature in space-time. By construction, the supersymmetry algebra being a *local* structure is insensitive to this global information. What is the nature of this "global information"? Indeed, the translation of Grassmannians variables into imaginary (topological) time clearly indicates that the natural state of such a global information is a topological state, correctly described by topological field theory (which is precisely a *non local* theory). More exactly, the boundary conditions characterizing the euclidean time dependence of the supersymmetry parameters can be seen as topological invariants. In this perspective, supersymmetry breaking can then be investigated in terms of cancellation of such topological invariants. Let's now explore this occurrence.

5.2.2 *Supersymmetry and topological invariants*

In a famous precursor paper [36], and in some others, E. Witten has clearly put in evidence that if we want supersymmetry breaking to occur, the various four-manifolds invariants (such that the Donaldson invariant, the Euler number, the Witten index etc..) must necessarily vanish. The outline of the argument is that the canceling of the supersymmetry index $Tr(-1)^F$ is canceling the zero energy modes, which consequently breaks the Bose-Fermi pairs [1]. At this stage, if we agree with supersymmetry theory, a reasonable conclusion is that (N=2) supergravity breaking could be viewed as related to the canceling of topological configurations. Let's now go further : can supersymmetry breaking explain the Topological → Physical transition? In a certain sense, the answer might be yes. In fact, since the context of the theory is supergravity N= 2, we may precise the conditions of topological modes canceling within supersymmetry breaking. So :

Conjecture 5.2.3 *On a D = 4 Riemannian (pre)space-time manifold, the N = 2 supergravity breaking at the Planck scale is related to the canceling of the Euler characteristic and of the topological mode of the manifold.*

Let $\boldsymbol{M}$ be the four dimensional Riemannian N=2 supersymmetric (pre)space-time. The Euler characteristic of $\boldsymbol{M}$ is

$$\chi(\boldsymbol{M}) = \frac{1}{32\pi}\int_M \varepsilon_{\mu\nu\rho\sigma} R_{\mu\nu} \wedge R_{\rho\sigma}$$

We have shown in prop. (3.2) that this invariant is given by $\mathrm{Tr}(-1)^S$. Now, according toWitten's results [36], a discontinuous change of $\mathrm{Tr}(-1)^S$ is possible, due to the asymptotic behavior of the manifold, allowing, for large field strengths, some energy states to "move in from infinity". For instance, let's consider the potential

$$V(\phi) = (m\phi - g\phi^2)$$

One can easily observe that arbitrarily small $g \neq 0$ induces the existence of extra low-energy states at $\phi \sim m/g$ which have no counterpart for the pure g = 0 value. Therefore, $\mathrm{Tr}(-1)^F$ will change discontinuously from g = 0 to $g \neq 0$. The same result can be extended to$\mathrm{Tr}(-1)^S$, when coupling the instanton radius to g. In this case, we meet again the conclusions of (ii) in example (2.1) (i.e. the instanton configuration is cancelled for large values of g).

Next, we have seen (5.1.2) that the (pre)space-time should be in KMS state at ℓ_{Planck}, so that the time like direction t becomes holomorphic within the KMS strip. The metric configuration is described by the symmetric homogeneous space

$$\sum\nolimits_h = \frac{\mathrm{SO}(3,1) \otimes \mathrm{SO}(4)}{\mathrm{SO}(3)} \qquad (87)$$

SO(3) being diagonally embedded in SO(3, 1) ⊗ SO(4) [6]. To $\sum_h$ corresponds, at the level of the underlying spaces involved, the topological quotient space $\sum_{top} = \dfrac{\mathbb{R}^{3,1} \oplus \mathbb{R}^4}{\mathrm{SO}(3)}$ from which, assuming that the compact part of the 3-geometry is a sphere S^3, the topology of the five dimensional (pre)space-time can be viewed as isomorphic to $S^3 \otimes \mathbb{R}^{\pm}$ ($\mathbb{R}^+$ being the space-like direction and

$\mathbb{R}^-$ the time-like direction, out of the orbit of the action of SO(3) on $\mathbb{R}^{3,1} \oplus \mathbb{R}^4$). We then meet again the equivalent form $S^3 \otimes \mathbb{R}^+ \otimes \mathbb{R}^-$ of the five dimensional manifold described in (2.1). The point is that $\mathbb{R}^-$ allows us to define the boundary conditions of the (pre)space-time 5-geometry $\mathbf{\Gamma}^5$. Therefore, the form of the 5D metric is [6] :

$$ds^2 = a(\omega)^2\, d\Omega^2_{(3)} + \frac{d\omega^2}{g^2} - dt^2 \tag{88}$$

where the axion term is $a = f(\omega\,,\ t)$, the 3-geometry $d\Omega^2_{(3)} = f(x, y, z)$. Then, as showed in (2.1), on the (infrared) strong coupling bound (i.e. the Planck scale, in respect of the (ultraviolet) zero scale), condition (**i**) imply $\frac{1}{g^2} \rightarrow 0$ and the ω direction of $\mathbf{\Gamma}^5$ is cancelled. So, we get a dimensional reduction (D=4 → D=3) of the compact Riemannian 4-geometry embedded in the five dimensional (pre)space-time manifold $\mathbf{\Gamma}^5$. We have for the metric :

$$(++++-) \xrightarrow{w \text{ compactification on } S^1 \rightarrow 0} (+++(0)-) \xrightarrow{\text{Dimensional reduction}} (+++-).$$

Obviously, the boundary condition $ß \rightarrow \infty$ gives rise to the asymptotic cancellation of the $\mathbf{\Gamma}^4$ Euler characteristic:

$$\chi(M) = \frac{1}{32\pi}\int_M \varepsilon_{uv\rho\sigma} R_{uv} \wedge R_{\rho\sigma} = \mathrm{Tr}(-1)^F = 0 \tag{89}$$

Likewise, the asymptotic flatness condition [6] for $ß \rightarrow \infty$ gives $R_{uv} \wedge R_{\rho\sigma} \rightarrow 0$, which implies that the dimension D of the asymptotic manifold must be *odd*, so that, again, we get $\chi = 0$ for the (3+1) usual space-time. Therefore, according to ref. [36], the supersymmetry is broken. Simultaneously, the topological state, given by *even* values of the Euler number χ vanishes, implying the "TP transition" :

$$\textit{Topological mode} \xrightarrow{TP \text{ transition}} \textit{Physical mode}\ .$$

To finish, we meet a novel problem : could TP transition be, in some way, related to the breaking of the KMS state described in (5.1)? This question is discussed in the last paragraph.

5.2.4 *TP transition and decoupling between topological flow and physical flow*

In answer to the hereabove question, we now conjecture that for $ß \geq \ell_{Planck}$, i.e. at the (semi-classical) scale where supersymmetry is being broken, the topological flow (evolution in imaginary time) corresponding to the zero topological pole of the theory is *decoupled* from the physical flow (evolution in real time).

According to most of the models, supergravity is considered as broken for scales greater than the Planck scale [25]. But thermal supersymmetry breaking is also closely connected to the cancellation of the thermodynamical equilibrium state [27-28]. Indeed, as already pointed in this paper, I. Antoniadis *and al* have recently demonstrated that a five-dimensional (N = 4) supersymmetry can effectively be described by a four-dimensional theory in which supersymmetry is spontaneously broken by finite thermal effects [3]. In a similar way, Derendinger and Lucchiesi have outlined the fact that thermal supersymmetry is a global (i.e. topological) property of the space-time in KMS state [13-28]. In this context, the cancellation of the thermodynamical equilibrium state necessarily cancels the KMS state and, consequently, breaks the supersymmetry [6]. This scenario is typically the one characterizing our setting. As a matter of fact, the five dimensional supersymmetric theory evoked hereabove correponds to the five dimensional supersymmetric (pre)space-time in KMS state. Then the (thermal) supersymmetry breaking is characterized in Kounnas approach, by a D=5 $\rightarrow$ D=4 dimensional reduction, which corresponds exactly, in our case, to the decoupling between imaginary time and real time. Indeed, we could have :

$$\mathbb{R}^3 \otimes \mathbb{C} \; (\textit{five dimensional KMS space-time}) \xrightarrow{\textit{ss breaking}} \begin{cases} \mathbb{R}^3 \otimes \mathbb{R}^+ \; (\textit{four dimensional topological space - tim} \\ \mathbb{R}^3 \otimes \mathbb{R} \;\; (\textit{four dimensional physical space - time} \end{cases}$$

So, supersymmetry breaking, KMS breaking and topological $\rightarrow$ physical transition appear as deeply connected. To see this, let's come back to the KMS state. We call "KMS breaking" the end of the KMS state beyond the Planck's scale. The observed cancellation of the thermodynamical equilibrium beyond the Planck scale (which gives the inflationary phase and the beginning of the cosmological expansion) is inducing KMS breaking (see ex. (5.2.1)). Such a breaking must be seen as the inverse of the KMS coupling between equilibrium state and physical evolution of the system. And logically, such a breaking should bring about the transition from the pure (non perturbative) topological phase around the Initial Singularity to the physical phase of the universe we can observe to day.

Now, here is our conjecture :

Conjecture 5.2.5 *In the infrared ß ≥ ℓ_{Planck} scale, KMS breaking is inducing the decoupling between the topological flow and the physical flow of the theory.*

Considering the KMS state of space-time at the Planck scale, the KMS flow, as shown in prop. (5.2), is :

$$\sigma_{ß_c}(\boldsymbol{M}_q) = \sigma_ß(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) \oplus \sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}}) = e^{ß_c H}\boldsymbol{M}_q e^{\ ß_c H} \tag{90}$$

or

$$\sigma_{ß_c}(\boldsymbol{M}_q) = e^{\ (ß+it)H}\boldsymbol{M}\mathrm{q}\ e^{(ß+it)H} \tag{91}$$

Now, starting from $\boldsymbol{M}\mathrm{q} = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathscr{L}\left[L^2\left(\mathbb{R}^*_+ \big/ ß\mathbb{Z}\right)\right] = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathscr{L}[L^2(ß\mathrm{S}^1)]$, we can say that $ß >$ ℓ_{Planck} is equivalent to $ß \to \infty$, with respect to the scale zero. So, when $ß > \ell_{Planck}$, the period of the system is so large that we can consider it as supressed from equ.(62), whereas the circle S^1 is decompactified on the straight line R. Moreover, this limit corresponds to $\lambda = \dfrac{2\pi}{ß} \to 0$. So, on this limit $\mathbb{R}^*_+ \big/ ß\mathbb{Z} \to \mathbb{R}^*{}_+$. But the suppression of the period $\mathbb{R}^*_+ \big/ ß\mathbb{Z}$ is equivalent to the cancellation of the equilibrium state and therefore induces the breaking of the KMS state. To see this, we can write the "extended" automorphisms group corresponding to the KMS state :

$$\sigma_{ß_c}(\boldsymbol{M}_q) = e^{ß_c H}[\boldsymbol{M}^{0,1}_{Top} \rtimes_\alpha \mathbb{R}^*{}_+]e^{\ ß_c H} = e^{\ (ß+it)H}[\boldsymbol{M}^{0,1}_{Top} \rtimes_\alpha \mathbb{R}^*{}_+]\ e^{(ß+it)H} \tag{92}$$

Then for $ß >> \ell_{Planck}$, we get $\mathbb{R}^*{}_+ \to \infty$ so the corresponding weight φ on $\boldsymbol{M}$q is such that $\varphi \to \infty$. But, according to Connes-Takesaki [10], the infinite dominant weight on $\boldsymbol{M}$q is dual to the hyperfinite *trace* on $\boldsymbol{M}^{0,1}_{Top}$. Therefore, the image of the "flow of infinite weights" on $\boldsymbol{M}$q becomes, under the ergodic action of $\mathbb{R}^*{}_+$:

$$\sigma_{\beta\to\infty}(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) = \mathrm{Tr}_\infty\,(e^{\ ßH}\boldsymbol{M}^{0,1}_{Top}\ \ e^{ßH}) \tag{93}$$

where we meet again the topological "trace current" $\mathbf{T}^f$ of $\boldsymbol{M}_{Top}^{0,1}$, independent of *ß*. But the independence of $\mathbf{T}^f$ with respect to *ß* implies in the same way that $\mathbf{T}^f$ is also independent of $\mathbb{R}^*_+$ on this limit. So, $\mathbb{R}^*_+$ must be decoupled of $\boldsymbol{M}_{Top}^{0,1}$, which means that the modular evolution group $\sigma_t(\boldsymbol{M}_{Phys}) = \Delta^{it}\boldsymbol{M}_{Phys}\Delta^{-it}$ is itself decoupled from the crossed product (65). Moreover, since the hyperfinite trace (93) is independent of *ß*, we are left with the "topological" state :

$$\sigma_{\beta\to\infty}(\boldsymbol{M}_{\boldsymbol{q}\text{-}\boldsymbol{state}}) \equiv \mathrm{Tr}_\infty(\boldsymbol{M}_{Top}^{0,1})$$

which is equivalent to say that the only value of *ß* contributing to equ. (79) is *ß* = 0. So, on this boundary, (see equ.(64)), $\sigma_{ß_c}(\boldsymbol{M}_q)$ is reduced to the real pole, so that :

$$\sigma_{ß_c}(\boldsymbol{M}_q) \to \sigma_t(\boldsymbol{M}_{\boldsymbol{q}\text{-}\boldsymbol{evolution}}) = e^{iHt}\boldsymbol{M}_{\boldsymbol{q}\text{-}\boldsymbol{evolution}}e^{-iHt},$$

But of course, in this case **M**q, as type III algebra, is also suppressed. This is simply because, on the infinite limit of the action of $\mathbb{R}$ on $\boldsymbol{M}_{Top}^{0,1}$, the infinite trace Tr_∞ on $\boldsymbol{M}_{Top}^{0,1}$, dual to the dominant weight on ***M***q, is left invariant. Applying a result of [11] on infinite weights, one can find that the infinite weight φ_∞ on ***M***q is invariant under the inner automorphisms of ***M***q. Therefore, φ_∞ is a trace, which is a sufficient condition to cancel ***M***q as a III_λ factor. But this is equivalent to say that on this limit, the action of $\mathbb{R}$ is *decoupled* of $\boldsymbol{M}_{Top}^{0,1}$. Therefore, the crossed product (65) is broken into its two subgroups $\boldsymbol{M}_{Top}^{0,1}$ and $\mathscr{L}(L^2(\mathbb{R}^*_+))$. This is as it should be, since beyond the Planck scale, i.e. at the classical scale, the KMS state is broken and the measure space on the metric is again well defined, so that the underlying algebra must be endowed with a trace. Consequently, it cannot be **M**q anymore. So, the new algebra involved should be a type I_∞ sub-algebra of **M**q. Considering the decomposition $\boldsymbol{M}_{Top}^{0,1} \rtimes_\alpha \mathscr{L}(L^2(\mathbb{R}^*_+))$, this sub-algebra is necessarily $\mathscr{L}(L^2(\mathbb{R}^*_+)) = \boldsymbol{M}_{Phys}$. Then $\sigma_t(\boldsymbol{M}_{\boldsymbol{q}\text{-}\boldsymbol{evolution}})$ becomes simply :

$$\sigma_t(\boldsymbol{M}_{Phys}) = e^{iHt}\boldsymbol{M}_{Phys}e^{-iHt} \tag{94}$$

This corresponds to the usual modular group giving the physical evolution of the space-time. So the product $\boldsymbol{M}_{Top}^{0,1} \rtimes_\alpha \boldsymbol{M}_{Phys}$ shrinks onto $\boldsymbol{M}_{Phys}$ so that we finally get $\underset{ß>\ell_{Planck}}{\boldsymbol{M}_q} = \boldsymbol{M}_{Phys}$ in the infrared.

In the same way, applying the result of prop. 5.2, we see that the breaking of KMS state implies

AUT Mq $\equiv$ OUT Mq $\top$ INT$_{mod}$ Mq

reduces to the well known case of a factor I, where all the automorphisms of the algebra are inner automorphisms:

AUT Mq $\equiv$ INT Mq

So obviously, this transition causes the decoupling between OUTMq and INT$_{mod}$ Mq , i.e. between the topological current $\sigma_{ß}(x)$ and the physical flow $\sigma_t^{\psi}(x)$.

As a result of prop.(5.2.5), we finally can conclude that the breaking of the KMS state beyond the Planck scale induces the decoupling between the physical flow $\sigma_t(\boldsymbol{M}_{Phys}) = e^{\,iHt}\boldsymbol{M}_{Phys}e^{\,-\,iHt}$ and the zero topological current

$\sigma_ß\,(\boldsymbol{M}^{0,1}_{Top}) = \ \mathrm{Tr}_\infty\,(e^{\,-\,ß\,H}\ \ \boldsymbol{M}^{0,1}_{Top}\ \ e^{\,ß\,H}\ )$:

$$\sigma_{ß_c}(\boldsymbol{M}_q) = e^{ß_c H}\,\boldsymbol{M}^{0,1}_{Top}\ e^{\ ß_c H}\ \xrightarrow{KMS\ \ breaking} \begin{cases} \rightarrow \mathrm{Tr}_\infty(e^{-\,ßH}\boldsymbol{M}^{0,1}_{Top}\ e^{\,ßH}) \\ \rightarrow e^{\,iHt}\boldsymbol{M}_{Phys}e^{\,-\,iHt} \end{cases} \tag{95}$$

At the level of the von Neumann algebras, starting from the KMS algebra $\boldsymbol{M}$q = $\boldsymbol{M}^{0,1}_{Top}\ \rtimes_\theta\ \boldsymbol{M}_{Phys}$, the KMS breaking can be seen as the decoupling between $\boldsymbol{M}^{0,1}_{Top}$ and $\boldsymbol{M}_{Phys}$. This decoupling describes the transition from the topological phase (zero scale) to the physical phase (beyond the Planck scale).

6. CONCLUSION

Even though certain of the hereabove results might seem mysterious, their interest is to outline, through quantum groups theory and non commutative geometry, a possible phase transition from the topological zero scale to the physical Planck scale. We describe with more details in a forthcoming paper the unexpected "algebraic blow up" of the topological initial singularity. At this stage, we propose to draw the following main ideas :

(**i**) the metric, onto the zero scale, might be considered as Euclidean (++++) i.e. *topological* ;

(**ii**) the Initial Singularity of space-time could be understood as a 0-size singular gravitational instanton;

(**iii**) From (**i**) and (**ii**), we suggest the existence of a deep symmetry, of the duality type (*i* - duality), between physical state (Planck scale) and topological state (zero scale).

Then the possible resolution of the initial singularity in the framework of topological theory allows us to envisage the existence, before the Planck scale, of a purely topological first phase of expansion of space-time, parameterized by the growth of the dimension of moduli space dim***M*** and described by the Euclidean "pseudo-dynamic" :

$$\sigma_{ß}\,(M_{Top}^{0,1}) = e^{-ß H}\; M_{Top}^{0,1}\; e^{ß H}$$

So, the chain of events able to explain the transition from the zero topological phase to the physical phase of the space-time might be the following :

$$\{Supersymmetry\ breaking\} \to \{thermodynamical\ \ equilibrium\ \ breaking\} \to \{KMS\, state\ \ breaking\} \to$$
$$\to \{imaginary\ \ time\,/\,real\ \ time\ decoupling\} \to \{topological\ \ state\ \ /\ physical\ state\ \ decoupling\}$$

In terms of C*-algebras, the hereabove transformations are given by :

$$\mathrm{II}_{\infty} \otimes \mathbb{R}_{+}^{*} \xrightarrow[0<ß<\ell P]{\text{KMS flow}\quad \mathbf{Q}^{f}} \alpha_{ß_c}(\boldsymbol{M_q}) = e^{-ß_c H}\boldsymbol{M_q}\, e^{ß_c H} \begin{cases} \xrightarrow{\text{Topological flow}} \underset{ß=0}{\mathbf{T}^{f}} \to \alpha_{ß}(\boldsymbol{M}_{Top}^{0,1}) = e^{-ßH}\boldsymbol{M}_{Top}^{0,1} e^{ßH} \\ \xrightarrow{\text{Physical flow}} \underset{ß>0}{\mathbf{P}^{f}} \to \alpha_t(\boldsymbol{M}_{Phys}) = e^{iHt}\,\boldsymbol{M}_{Phys}\; e^{-iHt} \end{cases}$$

In a forthcoming article, we push forward the idea following which, that at 0 scale, the Lorentzian dynamic is replaced by an intrinsic "Euclidean dynamic". A first path to follow would be to investigate the zero limit of the The Euclidean dynamic engendered by the non-stellar automorphisms of the algebra $M_{Top}^{0,1}$ implies, following the results of [6], a "spectral increase" in the diameters of the space of states d(φ , ψ) in Euclidean time (dual to the space of observables in Lorentzian time). This Euclidean pseudo-dynamic, linked with semi-group automorphisms $\sigma_{ß}(M_{Top}^{0,1})$ is described in a natural way by the flow of weights (in the Connes-Takesaki [9] sense) of algebra M_q ; we suggest equally (**ii**) that the Euclidean modular flow representing the evolution of a system in imaginary time can be associated with an increase in the spectral distance separating the states of the system. Finally, it has been proposed by one of us [7] that the Euclidean dynamic raised above results from the existence of the topological amplitude yield by the topological charge Q = $\theta\int d^4x \operatorname{Tr} R_{\mu\nu} \tilde{R}^{\mu\nu}$ of the zero size singular gravitational instanton connected to the (topological) origin of space-time.

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Spacetime Metric and the KMS Condition at the Planck Scale

Grichka Bogdanoff and Igor Bogdanoff

Mathématical Physics Laboratory CNRS UMR 5029 Bourgogne University
U.F.R. Sciences et Techniques
B.P. 47870
F-21078 Dijon Cedex FRANCE

Abstract

Considering the expected thermal equilibrium chcaracterising the physices at the Planck scale, it is here stated, for the first time, that, as a global system, the space-time at the Planck scale must be considered as subject to the Kubo-Martin-Schwinger (KMS) condition. Consequently, in the interior of the KMS strip, i.e. from the scale ß = 0 to the scale ß = ℓ_{Planck}, the fourth coordinate g_{44} must be considered as complex, the two real poles being ß = 0 and ß = ℓ_{Planck}. This means that within the limits of the KMS strip, the Lorentzian and the Euclidean metric are in a "quantum superposition state" (or coupled), this entailing a "unification" (or coupling) between the topological (Euclidean) and the physical (Lorentzian) states of space-time.

1. INTRODUCTION

Many descriptions have been recently proposed in the literature regarding the physical state of the universe at the vicinity of the Planck scale. Non commutative geometry, string theory, supergravity or quantum gravity have contributed, independently of each others, to establish on solid basis the data of a "transition phase" in the physical content and the geometric structures of the (pre)spacetime at such a scale. But what is the nature of this dramatic change? In the present paper, we propose a novel approach, based on one of the most natural and realistic physical condition predicted by the Standard Model for the (pre)universe. In agreement with some well-established results [1-2-3] (recalled in [4]) and, more recently, the approach of C. Kounnas *and al* [5-6], we argue that at the Planck scale, the "spacetime system" is in a *thermodynamical equilibrium state* [2]. This notion of equilibrium state at the Planck scale has been recently stated with some new interesting arguments in [7-8-9]. As a consequence, according to [10], we suggest hereafter that the (pre)universe should be considered as subject to the Kubo-Martin-Schwinger (KMS) condition [11] at such a scale. Surprisingly, the well known KMS and modular theories [12] have never been applied to the study of the metric properties in the context of quantum cosmology ; however, the KMS condition might have dramatic consequences onto Planck scale physics. For which reasons? Because, when applied to quantum spacetime, the KMS statistics are such that , within the limits of the "KMS strip" (i.e. between the scale zero and the Planck scale), the time like direction of the system should be considered as *complex* : $t \mapsto t_c = t_r + it_i$. G.Bogdanoff has showed in [10] that at the scale zero, the theory is projected onto the pure imaginary boundary $t \mapsto t_c = it_i$ of the KMS strip. Namely, there exists, around $ß \to 0$, a non-trivial *topological* limit of quantum field theory, *dual* to the usual topological limit associated with $ß \to \infty$ in the partition function (2). Such a topological state of the (pre)spacetime can be described by the topological invariant

$$ {}_{ß \to 0}Z = \mathrm{Tr}\,(-1)^{S} \tag{1}$$

given by the $ß \to 0$ limit of the partition function (2). S is here the instanton number of the theory. It has been demonstrated [10] that this topological index is isomorphic to the first Donaldson Invariant [13]. This suggests that at zero scale, the observables 0_i must be replaced by the homology cycles $H_i \subset \mathcal{M}^{(k)}_{\mathrm{mod}}$ in the moduli space of gravitational instantons [14]. We get then a deep correspondence -a symmetry of duality- [15-16], between physical theory and topological field theory. Conversely, on the (classical) infrared limit $ß \geq \ell_{Planck}$, the imaginary component cancels and the time like direction becomes pure real $t \mapsto t_c = t_r$ So, within the limits of the KMS strip (i.e. for $0 < ß < \ell_{Planck}$), the Lorentzian and the Euclidean metric are in a "quantum superposition state" (or coupled). Such a superposition state, described in detail in [10], is entailing a "unification" (or coupling) between the topological (Euclidean) and the physical (Lorentzian) states of spacetime. The (Lorentzian / Euclidean) states of the metric $g_{\mu\nu}$ are given by the partition function

$$\mathbf{Z} = \mathrm{Tr}\,(-1)^S \; e^{-ßH} \qquad (2)$$

with $ß = \dfrac{1}{kT}$ as usual and *n* being the "metric number" of the theory. To avoid any difficulty of interpretation, let's remark that the scale parameter *ß* admits two possible intepretations : (i) either $ß = \hbar/kT$ can be seen as a real time parameter of a Lorentzian (3+1)-dimensional theory [17] or (ii) $ß = it$ can be interpreted as the fourth space-like direction of a Riemannian 4-dimensional Euclidean theory (e.g. $ß = \ell_{Planck}$) [18]. In the second case, *ß* is a periodic imaginary time interval [5-6]. Considering the hypothesis of holomorphicity formulated in section 4, we use here the two interpretations. In such a context, the KMS state of the (pre)spacetime may be considered as a transition phase from the *Euclidean topological phase* (ß = 0) to the *Lorentzian physical phase*, beyond the Planck scale [10-19].

The present article is organized as follows. In section 2 we recall that at the Planck scale, the "spacetime system" should most likely be considered as being in a thermodynamical equilibrium state. In section 3, we show that, as a natural consequence of this equilibrium state, the spacetime must be considered as subject to the KMS condition. In section 4, we suggest that, considering the KMS properties, the time-like direction g_{44} of the metric should be seen as complex $t_C = t_r + i\, t_i$ within the limits of the KMS strip. In section 5, we discuss briefly the transition from imaginary time t_i to real time t_r in terms of KMS breaking beyond the Planck scale.

2. THERMODYNAMICAL EQUILIBRIUM OF THE SPACETIME AT THE PLANCK SCALE

It is well known that at the Planck scale, one must expect a thermodynamical "phase transition", closely related to (**i**) the existence of an upper limit in the temperature growth -the Hagedorn temperature [1]- and (**ii**) the *equilibrium state* characterizing, most likely, the (pre)spacetime at such a scale. In such a context the seminal investigations of K. Huang and S. Weinberg [2], Dolan and R. Jackiw [3] , then later of several others, have renewed the initial idea of Hagedorn concerning the existence, at very high temperature, of a limit restricting the growth of states excitation. Already some time ago, J.J. Atik and E.Witten have shown the existence of a Hagedorn limit around the Planck scale in string theory [1]. The reason is that, as recently recalled by C. Kounnas [6] in the context of N = 4 superstrings, at finite temperature, the partition function *Z(ß)* and the mean energy *U(ß)* develop some power pole singularities in $ß \equiv T^{-1}$ since the density of states of a system grows exponentially with the energy E :

$$Z(ß) = \int dE \rho(E) e^{-ßE} \sim \frac{1}{(ß - b)^{(k-1)}}$$

$$U(ß) = \frac{\partial}{\partial ß} \ln Z \sim (k-1) \frac{1}{\beta - b} \qquad (3)$$

Clearly, equ.(3) exhibits the existence, around the Planck scale, of a critical temperature $T_H = b^{-1}$, where the (pre)spacetime system must be viewed as in a thermodynamical equilibrium state. Indeed, *a(t)* being the cosmological scale factor, the global temperature T follows the well-known law :

$$T(t) \approx T_p \frac{a(t_p)}{a(t)}$$

and around the Planck time, *T* is reaching the critical limit $T_p \approx \frac{E_p}{k_B} \approx \left(\frac{\hbar C^5}{G} \right)^{\frac{1}{2}} k_B^{-1} \approx 1{,}4.10^{32} K$. In fact, it is currently admitted in string theory that, before the inflationary phase, the ratio between the interaction rate (Γ) of the initial fields and the (pre)spacetime expansion (H) is $\frac{\Gamma}{H} << 1$, so that the system can reasonably be considered in *equilibrium state*. This has been established a long time ago within some precursor works already quoted in ref. [1,2,3]. Recently the same approach has been successfully considered by C. Kounnas *et al* in the superstrings context [5]. However, this natural notion of equilibrium, when viewed as a global gauge condition, has dramatic consequences regarding physics at the Planck scale. Among those consequences, unquestionably the most important is that the (pre)spacetime at the Planck scale must be considered as subject to the famous "KMS condition", a very special and interesting physical state that we are now going to describe.

3. THE (PRE)SPACETIME IN KMS STATE AT THE PLANCK SCALE

Let's first recall on mathematical basis what an equilibrium state is.

Definition *H being an autoadjoint operator and $\mathfrak{H}$ the Hilbert space of a finite system, the equilibrium state ω of this system is described by the Gibbs condition* $\omega(A) = \frac{Tr_{\mathfrak{H}}(e^{-ßH} A)}{Tr_{\mathfrak{H}}(e^{-ßH})}$ *and satisfy the KMS condition.*

This well-known definition has been proposed for the first time in [11]. Now, it is usual (and natural) to oppose the notion of equilibrium to the one of *evolution* of a system. In fact, the famous Tomita-Takesaki modular theory has established that the "intrinsic" dynamic of a quantum system corresponds, in a unique manner, to the strongly continuous one parameter $*$ - automorphism group α_t of some von Neumann C^* - algebra **A** [12] :

$$\alpha_t(\mathbf{A}) = e^{iHt}\, \mathbf{A}\, e^{-iHt} \tag{4}$$

This one parameter group describes the time evolution of the observables of the system and corresponds to the well-known Heisenberg algebra. Nothing mysterious at this stage. However, we are here brought to find the remarkable discovery of Takesaki and Winnink, connecting the evolution group $\alpha_t(\mathbf{A})$ of a system (more precisely the modular group $M = \Delta^{it} M \Delta^{-it}$) with the equilibrium state $\varphi(A) = \frac{\mathrm{Tr}(Ae^{-ßH})}{\mathrm{Tr}(e^{-ßH})}$ of this system [11-20]. With this more or less unexpected relation between evolution $\alpha_t(\mathbf{A})$ and equilibrium $\varphi(\mathbf{A})$, we now meet the famous "KMS condition". More exactly, in the frame of quantum statistical mechanic, the KMS condition provides a rigorous mathematical formulation about the coexistence of *different* possible equilibrium states at the *same* given temperature T.

Let's recall now how such a relation between equilibrium state and evolution of a system is realized by the KMS condition. It has been clearly established [11] that a state ω on the C^*-algebra **A** and the continuous one parameter automorphism group of **A** at the temperature $ß = 1 / k\,T$ verify the KMS condition if, for any pair A, B of the $*$ - sub-algebra of **A,** α_t - invariant and of dense norm, it exists a $f(t_C)$ function holomorphic in the strip $\{t_C = t + i\,ß \in \mathbb{C}$, $\operatorname{Im} t_C \in [\,0\,,\,ß\,]\,\}$ such that :

(i) $f\,(t) = \varphi(A\,(\,\alpha_t\,B))$,
(ii) $f\,(t + i\,ß\,) = \varphi(\,\alpha_t\,(B)A)$, $\quad \forall\, t \in \mathbb{R}$. (5)

Moreover, a state φ on the C^*-algebra **A** is *separator* if the given algebraic representation is a von Neumann algebra W^* endowed with a separator and cyclic vector. The sets

$I_l = \{A \in \mathbf{A}\ \ \varphi\,(\mathbf{A}^*\,\mathbf{A}\,) = 0\}$

and

$I_r = \{A \in \mathbf{A}\ :\varphi\,(\,\mathbf{A}\,\mathbf{A}^*\,) = 0\,\}$

are forming a left and right ideal in **A**. For any KMS state, we have $I_l = I_r$.

The above definition expresses the bijective relation between equilibrium state, holomorphic state of the measure parameters and KMS state.

Now, considering the general properties raised by the KMS condition, if we admit that around the Planck scale, the (pre)spacetime system is in a thermal equilibrium state, then we are also bound to admit that this system is in a KMS state. Indeed, it has been shown a long time ago [11] that if a state of a system ω satisfies the equilibrium condition $\int_{-\infty}^{+\infty} \omega([h, \alpha_t(A)])dt = 0, \ \ \forall A \in U$, then, ω satisfies the KMS condition. So, there is a biunivoque relation between equilibrium state and KMS state. So, if we admit that around ℓ_{Planck}, the (pre)spacetime system is in a thermal equilibrium state, then according to [11], we are also bound to admit that this system is in a KMS state.

Next, let's push forwards the consequences raised by the holomorphicity of the KMS strip.

4. HOLOMORPHIC TIME FLOW AT THE PLANCK SCALE

As a critical consequence of the KMS condition, we are induced to consider that the time-like coordinate g_{00} becomes holomorphic within the limits of the KMS strip. Indeed, as demonstrated in details in [10-19], within the KMS strip, we necessarily should have :

$$t \to t_C = t_r + i\, t_i \tag{6}$$

In the same way, the physical (real) temperature should also be considered as complex at the Planck scale :

$$T \to T_C = T_r + iT_i \tag{7}$$

as proposed by Atick and Witten in another context [1]. This unexpected effect is simply due to the fact that, given a von Neumann algebra W^* and two elements A, B of W^*, then there exists a function $f(t_C)$ holomorphic in the strip $\{t_c \in \mathbb{C},\ \mathrm{im}\ t_c \in [0,\ \hbar ß]\}$ such that :

$$f(t) = \varphi(A\ (\alpha_t\ B)) \text{ and } f(t_r + i\hbar ß) = \varphi(\alpha_t\ (B)A),\ \forall\ t \in \mathbb{R} \tag{8}$$

Here, t is the usual time parameter of the 3D theory, like $\hbar ß = \hbar / kT$. So in our case, within the limits of the KMS strip, i.e. from the scale zero ($ß = 0$) to the Planck scale ($ß = \ell_{Planck}$), the "time-like" direction of the system must be extended to the complex variable

$$t_C = t_r + i\, t_i \in \mathbb{C},\ \mathrm{Im}\ t_C \in [i\, t_i\ ,\ t_r] \tag{9}$$

Of course, the holomorphicity of the time like direction of the spacetime is induced in a natural manner by the fact that in our approach, the thermodynamical system is the spacetime itself. Such a situation has been investigated in details in [10] in the context of "quantum groups" and non-commutative geometry.

Indeed, according to Tomita's modular theory [20], the KMS condition, when applied to the spacetime as a system, allows, within the KMS strip, the existence of an "extended" (holomorphic) automorphism "group of evolution", which depends, in the classification of factors [12], on a "type IIIλ factor" $\boldsymbol{M}_q$ (a factor is a special type of von Neumann algebra, whose the center is reduced to the scalars $a \in \mathbb{C}$). The extended automorphism group has the following form :

$$\boldsymbol{M}_q \mapsto \sigma_{ß_C}(\boldsymbol{M}_q) = e^{Hß_C}\boldsymbol{M}_q\ e^{Hß_C} \tag{10}$$

with the $ß_c = ß_r + i\hbar ß_i$ parameter being formally *complex* and able to be interpreted as a complex time t and / or temperature $T \to T_C = T_r + iT_i$. So, the KMS condition suggests the existence at the Planck scale, of an effective one loop potential coupled, in N = 2 supergravity, to the complex dilaton + axion field $\varphi = \frac{1}{g^2} + i\alpha$ and yielding the dynamical form $\eta_{\mu\nu} = \mathrm{diag}(1, 1, 1, e^{i\theta})$ for the metric. The signature of $\eta_{\mu\nu}$ is Lorentziann (i.e. physical) for $\theta = \pm \pi$ and can become Euclidean (topological) for $\theta = 0$. Consequently, the "KMS signature" of the metric is (+++±). This is as it should be since, considering the quantum fluctuations of $g_{\mu\nu}$, there is no more invariant measure on the non commutative metric. Therefore, according to von Neumann algebra theory, the "good factor" addressing those constraints is uniquely a non commutative *traceless* algebra, i.e. the type IIIλ factor **M**q, of the general form constructed by Connes [12] :

$$\boldsymbol{M}\mathrm{q} = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathbb{R}^*_+ / ß\mathbb{Z} \equiv \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta ß\, S_1 \tag{11}$$

$\boldsymbol{M}^{0,1}_{Top}$ being a type II_∞ factor and $\mathbb{R}^*_+ / ß\mathbb{Z}$ the group acting periodically on $\boldsymbol{M}^{0,1}_{Top}$. The relation between the periods λ and $ß$ is such that $\lambda = \frac{2\pi}{ß}$, so that when $ß \to \infty$, we get $\lambda \to 0$ (the periodicity is suppressed).

At the Lie group level, this "superposition state" can simply be given by the symmetric homogeneous space constructed in [10] :

$$\Sigma h = \frac{SO(3\,,\,1) \otimes SO\,(4)}{SO\,(3)} \tag{12}$$

to which corresponds, at the level of the underlying metric spaces involved, the topological quotient space :

$$\Sigma top = \frac{\mathbb{R}^{3,\,1} \oplus \mathbb{R}^4}{SO(3)} \tag{13}$$

In the non commutative context, G. Bogdanoff has constructed, again in [10], the "cocyle bicrossproduct" :

$$U_q(so(4)^{op} \overset{\psi}{\triangleright\!\!\blacktriangleleft} U_q(so(3,\,1)) \tag{14}$$

where $U_q(so(4)^{op}$ and $U_q(so(3, 1))$ are Hopf algebras (or "quantum groups"[21]) and ψ a 2-cocycle of q-deformation. The bicrossproduct (7) suggests an unexpected kind of "unification" between the Lorentzian and the Euclidean Hopf algebras at the Planck scale and yields the possibility of a "q-deformation" of the signature from the Lorentzian (physical) mode to the Euclidean (topological) mode [10-19].

Now, starting from equ. (10), it appears clearly that the Tomita-Takesaki modular automorphisms group $\sigma_{ß_c}(\boldsymbol{M}_q)$ corresponds to the "unification" given by equ.(14) and induces, within the KMS field, the existence of *two* dual flows.

(**i**) On the boundary $ß_i \geq \ell_{Planck}$, the first possible flow, of the form

$$\sigma_t(\boldsymbol{M}_q) = e^{iH\beta} M_q \ e^{-iH\beta} \tag{15}$$

corresponds to the algebra of observables of the system and to the Lorentzian flow in real time. In this perspective, this flow is a "physical flow", that we call $\mathbf{P}^f_{ß>0}$. This scale represents the physical part of the light cone and, consequently, the notion of (Lebesgue) measure is fully defined. Therefore, the (commutative) algebra involved at such a scale is endowed with a hyperfinite trace and is given on the infinite Hilbert space $\mathscr{L}(\mathfrak{h})$, with $\mathfrak{h} = L^2(\mathbb{R})$. Then, $\mathscr{L}(L^2(\mathbb{R}))$ is a *type* I_∞ factor, indexed by the real group $\mathbb{R}$, which we call $\boldsymbol{M}_{Phys}$. Of course, at this scale, the theory is Lorentzian, controlled by SO(3, 1).

(**ii**) On the "zero scale" $ß_i = 0$ limit, the second flow takes necessarily the non unitary form :

$$\sigma_{iß}(\boldsymbol{M}_q) = e^{\beta_r H} \boldsymbol{M}_q \ e^{-\beta_r H} \tag{16}$$

giving on $\boldsymbol{M}_q$ the semi-group of unbounded and non-stellar operators. This initial «topological» scale corresponds to the imaginary vertex of the light cone, i.e. a zero-size gravitational instanton [19]. All the measures performed on the Euclidean metric being ρ-equivalent up to infinity, the system is ergodic. As shown by A. Connes, any ergodic flow for an invariant measure in the Lebesgue measure class gives a unique type II_∞ hyperfinite factor [12]. This strongly suggests that the singular 0-scale should be described by a type II_∞ factor, endowed with a hyperfinite trace noted Tr_∞. We have called $\boldsymbol{M}^{0,1}_{Top}$ such a "topological" factor, which is an infinite tensor product $\otimes^\infty$ of matrices algebra (ITPFI) of the $R_{O,1}$ Araki-Woods type [22]. Since $\boldsymbol{M}q = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta ß\ S_1$, on the $ß\ i = 0$ limit, we get $\boldsymbol{M}q \equiv \boldsymbol{M}^{0,1}_{Top}$ .With respect to the analytic continuation between () and (), $\sigma_{iß}(\boldsymbol{M}_q)$ represents a "current in imaginary time". We have stated in [19] that this current is another way to interpret the "flow of weights" of the algebra $\boldsymbol{M}q$ [23]. Clearly, according to [23], the flow of weights of $\boldsymbol{M}q$ is an ergodic flow, which represents an invariant of $\boldsymbol{M}q$. Then, $\sigma_{iß}(\boldsymbol{M}_q)$ yields a pure topological amplitude [24] and, as such, "propagates" in imaginary time from zero to infinity. $\sigma_{iß}(\boldsymbol{M}_q)$ is not defined on the whole algebra $\boldsymbol{M}q$ but on an ideal $\{\mathfrak{I}\}$ of $\boldsymbol{M}q$. One can demonstrate that in this case, the theory is Riemannian, the isometries of the metric being given by SO(4). As showed in [10, 19] this zero scale corresponds to the first Donaldson invariant $I = \sum_i (-1)^{n\,i}$ and can be described by the topological quantum field theory proposed by E. Witten in [24].

To finish, let's observe that the topological flow does not commute with the physical flow. Again, this is a direct and natural consequence of the KMS condition.

5. DISCUSSION

It is interesting to remark that in the totally different context of superstrings, J.J. Atick and E. Witten were the first to propose such an extension of the real temperature towards a complex domain [1]. Recently, in N=4 supersymmetric string theory, I. Antoniadis, J.P. Deredinger and C. Kounnas [5] have also suggested to shift the real temperature to imaginary one by identification with the inverse radius of a compactified Euclidean time on S^1, with $R = 1 / 2\pi T$. Consequently, one can introduce a complex temperature in the thermal moduli space, the imaginary part coming from the $B_{\mu\nu}$ antisymmetric field under type ***IIA*** $\xleftrightarrow{S/T/U}$ type ***IIB*** $\xleftrightarrow{S/T/U}$ ***Heterotic*** string-string dualities. More precisely, in Antoniadis *and al* approach, the field controlling the temperature comes from the product of the real parts of three complex fields : s= Re **S**, t= Re **T** and u= Re **U**. Within our KMS approach, the imaginary parts of the moduli **S**, **T**, **U** can be interpreted in term of Euclidean temperature. Indeed, from our point of view, a good reason to consider the temperature as complex at the Planck scale is that a system in thermodynamical equilibrium state must be considered as subject to the KMS condition [11].

(ii) On the other hand, according to most of the models, supergravity is considered as broken for scales greater than the Planck scale. But supersymmetry breaking is also closely connected to the cancellation of the thermodynamic equilibrium state. C. Kounnas has recently demonstrated that a five-dimensional (N=4) supersymmetry can be described by a four-dimensional theory in which supersymmetry is spontaneously broken by finite thermal effects [6]. This scenario may be applied to our setting. As a matter of fact, the end of the thermal equilibrium phase at the Planck scale might bring about the breaking of KMS state and of supersymmetry N = 4. This corresponds exactly, in our case, to the decoupling between imaginary time and real time.

To sum up, the chain of events able to explain the transition from the topological phase to the physical phase of the spacetime might be the following :

$$\{\textit{thermodynamical equilibrium breaking}\} \to \{\textit{KMS state breaking}\} \to \{\textit{imaginary time / real time decoupling}\}$$
$$\to \{\textit{topological state / physical state decoupling}\} \to \{\textit{Supersymmetry breaking}\}$$

We have given a detailed description of such a transition in [19]. Likewise, the supersymmetry is broken in [5-6] by the finite temperature, which corresponds in our view to the decoupling between real and imaginary (topological) temperature (the topological temperature being identified, in Kounnas model, with the inverse radius of a compactified Euclidean time on S^1 : $2\pi T = 1/R$). Applying this representation, the partition function in our case is given by the (super)trace over the thermal spectrum of the theory in 4 dimensions. According to this, supersymmetry breaking and transition from topological state to physical state might be deeply connected.

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The KMS State of Spacetime at the Planck Scale

Igor Bogdanoff

Mathématical Physics Laboratory CNRS UMR 5029 Bourgogne University
U.F.R. Sciences et Techniques
B.P. 47870
F-21078 Dijon Cedex FRANCE

Considering the expected thermal equilibrium characterizing the physics at the Planck scale it is here stated, for the first time, that as a system the spacetime at the Planck scale must be considered as subject to the Kubo-Martin-Schwinger (KMS) condition. Consequently in the interior of the KMS strip, i.e. from the scale ß = 0 to the scale ß = ℓ_{Planck}, the fourth coordinate g_{44} must be considered as complex, the two real poles being ß = 0 and ß = ℓ_{Planck}. This means that within the limits of the KMS strip the Lorentzian and the Euclidean metric are in a "quantum superposition state" (or coupled), this entailing a "unification" (or coupling) between the topological (Euclidean) and the physical (Lorentzian) states of spacetime.

1. INTRODUCTION

Much have been recently said regarding the physical state of the universe at the vicinity of the Planck scale. String theory, non commutative geometry, supergravity or quantum gravity have contributed, independently of each others, to establish on solid basis the data of a "transition phase" in the physical content and the geometric structures of the (pre)spacetime at such a scale. But what is the nature of this dramatic change? In the present paper, we propose a novel approach, based on one of the most natural and realistic physical condition predicted by the Standard Model for the (pre)universe. In agreement with some well-established results [1-2-3] (recalled in [4]) and, more recently, the approach of C. Kounnas *and al* [5-6], we argue that at the Planck scale, the "spacetime system" is in a *thermodynamical equilibrium state* [2]. This notion of equilibrium state at the Planck scale has been recently stated with some new interesting arguments in [7-8-9]. As a consequence, according to [10], we suggest hereafter that the (pre)universe should be considered as subject to the Kubo-Martin-Schwinger (KMS) condition [11] at such a scale. Surprisingly, the well known KMS and modular theories [12] have never been applied to the study of the metric properties in the context of quantum cosmology ; however, the KMS condition might have dramatic consequences onto Planck scale physics. For which reasons? Because, when applied to quantum spacetime, the KMS statistics are such that , within the limits of the "KMS strip" (i.e. between the scale zero and the Planck scale), the time like direction of the system should be considered as *complex* : $t \mapsto t_c = t_r + it_i$. G.Bogdanoff has showed in [10] that at the scale zero, the theory is projected onto the pure imaginary boundary $t \mapsto t_c = it_i$ of the KMS strip. Namely, there exists, around $ß \to 0$, a non-trivial *topological* limit of quantum field theory, *dual* to the usual topological limit associated with $ß \to \infty$ in the partition function (2). Such a topological state of the (pre)spacetime can be described by the topological invariant

$$\underset{ß \to 0}{Z} = \mathrm{Tr}\,(-1)^{S} \qquad (1)$$

given by the $ß \to 0$ limit of the partition function (2). S is here the instanton number of the theory. It has been demonstrated [10] that this topological index is isomorphic to the first Donaldson Invariant [13]. This suggests that at zero scale, the observables 0_i must be replaced by the homology cycles $H_i \subset \mathcal{M}^{(k)}_{\mathrm{mod}}$ in the moduli space of gravitational instantons [14]. We get then a deep correspondence -a symmetry of duality- [15-16], between physical theory and topological field theory. Conversely, on the (classical) infrared limit $ß \geq \ell_{Planck}$, the imaginary component cancels and the time like direction becomes pure real $t \mapsto t_c = t_r$ So, within the limits of the KMS strip (i.e. for $0 < ß < \ell_{Planck}$), the Lorentzian and the Euclidean metric are in a "quantum superposition state" (or coupled). Such a superposition state, described in detail in [10], is entailing a "unification" (or coupling) between the topological (Euclidean) and the physical (Lorentzian) states of spacetime. The (Lorentzian / Euclidean) states of the metric $g_{\mu\nu}$ are given by the partition function

$$\mathbf{Z} = \mathrm{Tr}\,(-1)^{s}\ e^{-ßH} \qquad (2)$$

with $ß = \frac{1}{kT}$ as usual and *n* being the "metric number" of the theory. To avoid any difficulty of interpretation, let's remark that the scale parameter *ß* admits two possible intepretations : (i) either $ß = \hbar / kT$ can be seen as a real time parameter of a Lorentzian (3+1)-dimensional theory [17] or (ii) $ß = it$ can be interpreted as the fourth space-like direction of a Riemannian 4-dimensional Euclidean theory (e.g. $ß = \ell_{Planck}$) [18]. In the second case, *ß* is a periodic imaginary time interval [5-6]. Considering the hypothesis of holomorphicity formulated in section 4, we use here the two interpretations. In such a context, the KMS state of the (pre)spacetime may be considered as a transition phase from the *Euclidean topological phase* (ß = 0) to the *Lorentzian physical phase*, beyond the Planck scale [10-19].

The present article is organized as follows. In section 2 we recall that at the Planck scale, the "spacetime system" should most likely be considered as being in a thermodynamical equilibrium state. In section 3, we show that, as a natural consequence of this equilibrium state, the spacetime must be considered as subject to the KMS condition. In section 4, we suggest that, considering the KMS properties, the time-like direction g_{44} of the metric should be seen as complex $t_C = t_r + i\, t_i$ within the limits of the KMS strip. In section 5, we discuss briefly the transition from imaginary time t_i to real time t_r in terms of KMS breaking beyond the Planck scale.

2. THERMODYNAMICAL EQUILIBRIUM OF THE SPACETIME AT THE PLANCK SCALE

It is well known that at the Planck scale, one must expect a thermodynamical "phase transition", closely related to (**i**) the existence of an upper limit in the temperature growth -the Hagedorn temperature [1]- and (**ii**) the *equilibrium state* characterizing, most likely, the (pre)spacetime at such a scale. In such a context the seminal investigations of K. Huang and S. Weinberg [2], Dolan and R. Jackiw [3] , then later of several others, have renewed the initial idea of Hagedorn concerning the existence, at very high temperature, of a limit restricting the growth of states excitation. Already some time ago, J.J. Atik and E.Witten have shown the existence of a Hagedorn limit around the Planck scale in string theory [1]. The reason is that, as recently recalled by C. Kounnas [6] in the context of N = 4 superstrings, at finite temperature, the partition function *Z(ß)* and the mean energy *U(ß)* develop some power pole singularities in $ß \equiv T^{-1}$ since the density of states of a system grows exponentially with the energy E :

$$Z(ß) = \int dE\rho(E)e^{-ßE} \sim \frac{1}{(ß-b)^{(k-1)}}$$

$$U(ß) = \frac{\partial}{\partial ß}\ln Z \sim (k-1)\frac{1}{\beta - b} \qquad (3)$$

Clearly, equ.(3) exhibits the existence, around the Planck scale, of a critical temperature $T_H = b^{-1}$, where the (pre)spacetime system must be viewed as in a thermodynamical equilibrium state. Indeed, *a(t)* being the cosmological scale factor, the global temperature T follows the well-known law :

$$\{thermodynamical\ equilibrium\ breaking\} \rightarrow \{KMS\ state\ breaking\} \rightarrow \{imaginary\ time /\ real\ time\ decoupling\}$$
$$\rightarrow \{topological\ state /\ physical\ state\ decoupling\} \rightarrow \{Supersymmetry\ breaking\}$$

and around the Planck time, T is reaching the critical limit $T_p \approx \frac{E_p}{k_B} \approx \left(\frac{\hbar C^5}{G}\right)^{\frac{1}{2}} k_B^{-1} \approx 1{,}4.10^{32}\,K$. In fact, it is currently admitted in string theory that, before the inflationary phase, the ratio between the interaction rate (Γ) of the initial fields and the (pre)spacetime expansion (H) is $\frac{\Gamma}{H} << 1$, so that the system can reasonably be considered in *equilibrium state*. This has been established a long time ago within some precursor works already quoted in ref. [1,2,3]. Recently the same approach has been successfully considered by C. Kounnas *et al* in the superstrings context [5]. However, this natural notion of equilibrium, when viewed as a global gauge condition, has dramatic consequences regarding physics at the Planck scale. Among those consequences, unquestionably the most important is that the (pre)spacetime at the Planck scale must be considered as subject to the famous "KMS condition", a very special and interesting physical state that we are now going to describe.

3. THE (PRE)SPACETIME IN KMS STATE AT THE PLANCK SCALE

Let's first recall on mathematical basis what an equilibrium state is.

Definition *H being an autoadjoint operator and $\mathfrak{H}$ the Hilbert space of a finite system, the equilibrium state ω of this system is described by the Gibbs condition* $\omega(A) = \dfrac{Tr_{\mathfrak{H}}(e^{-\beta H} A)}{Tr_{\mathfrak{H}}(e^{-\beta H})}$ *and satisfy the KMS condition.*

This well-known definition has been proposed for the first time in [11]. Now, it is usual (and natural) to oppose the notion of equilibrium to the one of *evolution* of a system. In fact, the famous Tomita-Takesaki modular theory has established that the "intrinsic" dynamic of a quantum system corresponds, in a unique manner, to the strongly continuous one parameter $*$ - automorphism group α_t of some von Neumann C^* - algebra **A** [12] :

$$\alpha_t(\boldsymbol{A}) = e^{iHt}\,\boldsymbol{A}\,e^{-iHt} \tag{4}$$

This one parameter group describes the time evolution of the observables of the system and corresponds to the well-known Heisenberg algebra. Nothing mysterious at this stage. However, we are here brought to find the remarkable discovery of Takesaki and Winnink, connecting the evolution group $\alpha_t(\boldsymbol{A})$ of a system (more precisely the modular group $M = \Delta^{it} M \Delta^{-it}$) with the equilibrium state $\varphi(A) = \dfrac{\mathrm{Tr}(Ae^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$ of this system [11-20]. With this more or less unexpected relation between evolution $\alpha_t(\boldsymbol{A})$ and equilibrium $\varphi(\boldsymbol{A})$, we now meet the famous "KMS condition". More exactly, in the frame of quantum statistical mechanic, the KMS condition provides a rigorous mathematical formulation about the coexistence of *different* possible equilibrium states at the *same* given temperature T.

Let's recall now how such a relation between equilibrium state and evolution of a system is realized by the KMS condition. It has been clearly established [11] that a state ω on the C^*-algebra **A** and the continuous one parameter automorphism group of **A** at the temperature $ß = 1 / k\,T$ verify the KMS condition if, for any pair A, B of the $*$ - sub-algebra of **A**, α_t - invariant and of dense norm, it exists a $f(t_C)$ function holomorphic in the strip $\{t_C = t + i\,ß \in \mathbb{C}\ ,\ \mathrm{Im}\, t_C \in [\,0\ ,\ ß\,]\,\}$ such that :

(i) $f\,(t) = \varphi(A\,(\,\alpha_t\,B))$,

(ii) $f\,(t + i\,ß\,) = \varphi(\,\alpha_t\,(B)A)$, $\quad \forall\, t \in \mathbb{R}$. (5)

Moreover, a state φ on the C^*-algebra **A** is *separator* if the given algebraic representation is a von Neumann algebra W^* endowed with a separator and cyclic vector. The sets

$I_l = \{A \in \mathbf{A}\ \ \varphi\,(\mathbf{A}^*\,\mathbf{A}\,) = 0\}$

and

$I_r = \{A \in \mathbf{A}\ :\varphi\,(\,\mathbf{A}\,\mathbf{A}^*\,) = 0\,\}$

are forming a left and right ideal in **A**. For any KMS state, we have $I_l = I_r$.

The above definition expresses the bijective relation between equilibrium state, holomorphic state of the measure parameters and KMS state.

Now, considering the general properties raised by the KMS condition, if we admit that around the Planck scale, the (pre)spacetime system is in a thermal equilibrium state, then we are also bound to admit that this system is in a KMS state. Indeed, it has been shown a long time ago [11] that if a state of a system ω satisfies the equilibrium condition $\int_{-\infty}^{+\infty} \omega([h, \alpha_t(A)])dt = 0, \quad \forall A \in U$, then, ω satisfies the KMS condition. So, there is a biunivoque relation between equilibrium state and KMS state. So, if we admit that around ℓ_{Planck}, the (pre)spacetime system is in a thermal equilibrium state, then according to [11], we are also bound to admit that this system is in a KMS state.

Next, let's push forwards the consequences raised by the holomorphicity of the KMS strip.

4. HOLOMORPHIC TIME FLOW AT THE PLANCK SCALE

As a critical consequence of the KMS condition, we are induced to consider that the time-like coordinate g_{00} becomes holomorphic within the limits of the KMS strip. Indeed, as demonstrated in details in [10-19], within the KMS strip, we necessarily should have :

$$t \rightarrow t_C = t_r + i\, t_i \qquad (6)$$

In the same way, the physical (real) temperature should also be considered as complex at the Planck scale :

$$T \rightarrow T_C = T_r + iT_i \qquad (7)$$

as proposed by Atick and Witten in another context [1]. This unexpected effect is simply due to the fact that, given a von Neumann algebra W^* and two elements A, B of W^*, then there exists a function $f(t_C)$ holomorphic in the strip $\{t_c \in \mathbb{C},\ \text{im}\ t_c \in [0,\ \hbar ß]\}$ such that :

$$f(t) = \varphi(A\ (\alpha_t\ B)) \text{ and } f(t_r + i\hbar ß) = \varphi(\alpha_t\ (B)A),\ \forall\ t \in \mathbb{R} \qquad (8)$$

Here, t is the usual time parameter of the 3D theory, like $\hbar ß = \hbar / kT$. So in our case, within the limits of the KMS strip, i.e. from the scale zero ($ß$ = 0) to the Planck scale ($ß = \ell_{Planck}$), the "time-like" direction of the system must be extended to the complex variable

$$t_C = t_r + i\, t_i \in \mathbb{C},\ \text{Im}\ t_C \in [i\, t_i\ ,\ t_r] \qquad (9)$$

Of course, the holomorphicity of the time like direction of the spacetime is induced in a natural manner by the fact that in our approach, the thermodynamical system is the spacetime itself. Such a situation has been investigated in details in [10] in the context of "quantum groups" and non-commutative geometry.

Indeed, according to Tomita's modular theory [20], the KMS condition, when applied to the spacetime as a system, allows, within the KMS strip, the existence of an "extended" (holomorphic) automorphism "group of evolution", which depends, in the classification of factors [12], on a "type IIIλ factor" $\boldsymbol{M}_q$ (a factor is a special type of von Neumann algebra, whose the center is reduced to the scalars $a \in \mathbb{C}$). The extended automorphism group has the following form :

$$\boldsymbol{M}_q \mapsto \sigma_{ß_c}(\boldsymbol{M}_q) = e^{Hß_c} \boldsymbol{M}_q \; e^{Hß_c} \qquad (10)$$

with the $ß_c = ß_r + i\hbar ß_i$ parameter being formally *complex* and able to be interpreted as a complex time t and / or temperature $T \rightarrow T_c = T_r + iT_i$. So, the KMS condition suggests the existence at the Planck scale, of an effective one loop potential coupled, in N = 2 supergravity, to the complex dilaton + axion field $\varphi = \frac{1}{g^2} + i\alpha$ and yielding the dynamical form $\eta_{\mu\nu} = \mathrm{diag}(1\,,\,1\,,\,1\,,\,e^{i\theta})$ for the metric. The signature of $\eta_{\mu\nu}$ is Lorentziann (i.e. physical) for $\theta = \pm\,\pi$ and can become Euclidean (topological) for $\theta = 0$. Consequently, the "KMS signature" of the metric is (+++±). This is as it should be since, considering the quantum fluctuations of $g_{\mu\nu}$, there is no more invariant measure on the non commutative metric. Therefore, according to von Neumann algebra theory, the "good factor" addressing those constraints is uniquely a non commutative *traceless* algebra, i.e. the type IIIλ factor **M**q, of the general form constructed by Connes [12] :

$$\boldsymbol{M}_q = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathbb{R}^*_+ / ß\mathbb{Z} \;\equiv\; \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta ß\, S_1 \qquad (11)$$

$\boldsymbol{M}^{0,1}_{Top}$ being a type II_∞ factor and $\mathbb{R}^*_+/ß\mathbb{Z}$ the group acting periodically on $\boldsymbol{M}^{0,1}_{Top}$. The relation between the periods λ and ß is such that $\lambda = \frac{2\pi}{ß}$, so that when $ß \rightarrow \infty$, we get $\lambda \rightarrow 0$ (the periodicity is suppressed).

At the Lie group level, this "superposition state" can simply be given by the symmetric homogeneous space constructed in [10] :

$$\sum h = \frac{SO(3\,,\,1) \otimes SO\,(4)}{SO\,(3)} \qquad (12)$$

to which corresponds, at the level of the underlying metric spaces involved, the topological quotient space :

$$\Sigma\text{top} = \frac{\mathbb{R}^{3,\,1} \oplus \mathbb{R}^4}{SO(3)} \tag{13}$$

In the non commutative context, G. Bogdanoff has constructed, again in [10], the "cocyle bicrossproduct" :

$$U_q(so(4)^{op} \overset{\psi}{\rhd\!\blacktriangleleft} U_q(so(3,\,1)) \tag{14}$$

where $U_q(so(4)^{op}$ and $U_q(so(3, 1))$ are Hopf algebras (or "quantum groups"[21]) and ψ a 2-cocycle of q-deformation. The bicrossproduct (7) suggests an unexpected kind of "unification" between the Lorentzian and the Euclidean Hopf algebras at the Planck scale and yields the possibility of a "q-deformation" of the signature from the Lorentzian (physical) mode to the Euclidean (topological) mode [10-19].

Now, starting from equ. (10), it appears clearly that the Tomita-Takesaki modular automorphisms group $\sigma_{ß_c}(\boldsymbol{M}_q)$ corresponds to the "unification" given by equ.(14) and induces, within the KMS field, the existence of *two* dual flows.

(**i**) On the boundary $ß_i \geq \ell_{Planck}$, the first possible flow, of the form

$$\sigma_t(\boldsymbol{M}_q) = e^{iH\beta} M_q \;\; e^{-iH\beta} \tag{15}$$

corresponds to the algebra of observables of the system and to the Lorentzian flow in real time. In this perspective, this flow is a "physical flow", that we call $\mathbf{P}^f_{\mathbf{ß}>0}$. This scale represents the physical part of the light cone and, consequently, the notion of (Lebesgue) measure is fully defined. Therefore, the (commutative) algebra involved at such a scale is endowed with a hyperfinite trace and is given on the infinite Hilbert space $\mathscr{L}(\mathfrak{h})$, with $\mathfrak{h} = L^2(\mathbb{R})$. Then, $\mathscr{L}(L^2(\mathbb{R}))$ is a *type* I_∞ factor, indexed by the real group $\mathbb{R}$, which we call $\boldsymbol{M}_{Phys}$. Of course, at this scale, the theory is Lorentzian, controlled by SO(3, 1).

(**ii**) On the "zero scale" $ß_i = 0$ limit, the second flow takes necessarily the non unitary form :

$$\sigma_{iß}(\boldsymbol{M}_q) = e^{\beta_r H} \boldsymbol{M}_q \; e^{-\beta_r H} \tag{16}$$

giving on $\boldsymbol{M}$q the semi-group of unbounded and non-stellar operators. This initial «topological» scale corresponds to the imaginary vertex of the light cone, i.e. a zero-size gravitational instanton [19]. All the measures performed on the Euclidean metric being ρ-equivalent up to infinity, the system is ergodic. As shown by A. Connes, any ergodic flow for an invariant measure in the Lebesgue measure class gives a unique type II_∞ hyperfinite factor [12]. This strongly suggests that the singular 0-scale should be described by a type II_∞ factor, endowed with a hyperfinite trace noted Tr_∞. We have called $\boldsymbol{M}^{0,1}_{Top}$ such a "topological" factor, which is an infinite tensor product $\otimes^\infty$ of matrices algebra (ITPFI) of the $R_{O,1}$ Araki-Woods type [22]. Since $\boldsymbol{M}$q = $\boldsymbol{M}^{0,1}_{Top} \rtimes_\theta ß\, S_1$, on the $ß_i = 0$ limit, we get $\boldsymbol{M}$q $\equiv \boldsymbol{M}^{0,1}_{Top}$ .With respect to the analytic continuation between () and (), $\sigma_{iß}(\boldsymbol{M}_q)$ represents a "current in imaginary time". We have stated in [19] that this current is another way to interpret the "flow of weights" of the algebra $\boldsymbol{M}$q [23]. Clearly, according to [23], the flow of weights of $\boldsymbol{M}$q is an ergodic flow, which represents an invariant of $\boldsymbol{M}$q . Then, $\sigma_{iß}(\boldsymbol{M}_q)$ yields a pure topological amplitude [24] and, as such, "propagates" in imaginary time from zero to infinity. $\sigma_{iß}(\boldsymbol{M}_q)$ is not defined on the whole algebra $\boldsymbol{M}$q but on an ideal $\{\mathfrak{I}\}$ of $\boldsymbol{M}$q. One can demonstrate that in this case, the theory is Riemannian, the isometries of the metric being given by SO(4). As showed in [10, 19] this zero scale corresponds to the first Donaldson invariant $\mathrm{I} = \sum_i (-1)^{n_i}$ and can be described by the topological quantum field theory proposed by E. Witten in [24].

To finish, let's observe that the topological flow does not commute with the physical flow. Again, this is a direct and natural consequence of the KMS condition.

5. DISCUSSION

It is interesting to remark that in the totally different context of superstrings, J.J. Atick and E. Witten were the first to propose such an extension of the real temperature towards a complex domain [1].

Recently, in N=4 supersymmetric string theory, I. Antoniadis, J.P. Deredinger and C. Kounnas [5] have also suggested to shift the real temperature to imaginary one by identification with the inverse radius of a compactified Euclidean time on S^1, with $R = 1 / 2\pi T$. Consequently, one can introduce a complex temperature in the thermal moduli space, the imaginary part coming from the $B_{\mu\nu}$ antisymmetric field under type $\boldsymbol{IIA} \xleftrightarrow{S/T/U}$ type $\boldsymbol{IIB} \xleftrightarrow{S/T/U} \boldsymbol{Heterotic}$ string-string dualities. More precisely, in Antoniadis *and al* approach, the field controlling the temperature comes from the product of the real parts of three complex fields : s= Re **S**, t= Re **T** and u= Re **U**. Within our KMS approach, the imaginary parts of the moduli **S**, **T**, **U** can be interpreted in term of Euclidean temperature. Indeed, from our point of view, a good reason to consider the temperature as complex at the Planck scale is that a system in thermodynamical equilibrium state must be considered as subject to the KMS condition [11].

(ii) On the other hand, according to most of the models, supergravity is considered as broken for scales greater than the Planck scale. But supersymmetry breaking is also closely connected to the cancellation of the thermodynamic equilibrium state. C. Kounnas has recently demonstrated that a five-dimensional (N=4) supersymmetry can be described by a four-dimensional theory in which supersymmetry is spontaneously broken by finite thermal effects [6]. This scenario may be applied to our setting. As a matter of fact, the end of the thermal equilibrium phase at the Planck scale might bring about the breaking of KMS state and of supersymmetry N = 4. This corresponds exactly, in our case, to the decoupling between imaginary time and real time.

To sum up, the chain of events able to explain the transition from the topological phase to the physical phase of the spacetime might be the following :

$$\{\textit{thermodynamical equilibrium breaking}\} \rightarrow \{\textit{KMS state breaking}\} \rightarrow \{\textit{imaginary time / real time decoupling}\}$$
$$\rightarrow \{\textit{topological state / physical state decoupling}\} \rightarrow \{\textit{Supersymmetry breaking}\}$$

We have given a detailed description of such a transition in [19]. Likewise, the supersymmetry is broken in [5-6] by the finite temperature, which corresponds in our view to the decoupling between real and imaginary (topological) temperature (the topological temperature being identified, in Kounnas model, with the inverse radius of a compactified Euclidean time on S^1 : $2\pi T = 1/R$). Applying this representation, the partition function in our case is given by the (super)trace over the thermal spectrum of the theory in 4 dimensions. According to this, supersymmetry breaking and transition from topological state to physical state might be deeply connected.

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Topological Origin of Inertia

Igor BOGDANOV

Mathematical physics Laboratory CNRS UPRES A 5029
BOURGOGNE University, FRANCE

The purpose of the present article, following "Mach's principle" (the main elements of which have contributed to the foundations of general relativity) is to propose a new (non-local) interpretation of the inertial interaction. We then suggest that the inertial interaction can be correctly described by the topological field theory proposed by Witten in 1988. In such a context, the instantaneous propagation and the infinite range of the inertial interaction might be explained in terms of the topological amplitude connected with the singular zero size gravitational instanton corresponding to the Initial Singularity of space-time.

1. INTRODUCTION

The phenomenon of inertia - or "pseudo-force" according to E. Mach [1] - has recently been presented by J.P. Vigier [2] as one of the "unsolved mysteries of modern physics". Indeed, our point of view is that this important question, which is well formulated in the context of Mach's principle, cannot be resolved or even understood in the framework of conventional field theory.

Here we suggest a novel approach, a direct outcome of the topological field theory proposed by Edward Witten in 1988 [3]. According to this approach, beyond the interpretation proposed by Mach, we consider inertia as a *topological field,* linked to the topological charge Q = 1 of the "singular zero size gravitational instanton" [4] which, according to [5], can be identified with the initial singularity of space-time in the standard model.

Evaluation of the total inertial (or inertial potential) contribution, resulting from the sum of the masses in the universe may be given by:

$$\mathbf{U}_{\text{inertial total}} = \sum_{univers} \frac{GM}{c^2 r} \approx 1 \tag{1}$$

which turns out to be an invariant for each local mass. On the other hand, the topological charge of the singular gravitational instanton, of the form

$$Q = \frac{1}{32\pi^2}\int d^4x\, R_{\mu\nu}\tilde{R}^{\mu\nu} = 1 \qquad (2)$$

is also an invariant equal to unity, just as the total inertial contribution. Unexpectedly, the equivalence between the inertial mass and the gravitational mass might be explained here in terms of the quantization of the topological charge of the singular gravitational instanton [6].

We therefore claim that the topological charge of the singular gravitational instanton of zero size represents the (non-local) source of the inertial interaction. Like the topological charge, the inertial interaction is also invariant and propagates itself "instantaneously" from one point to another in space-time. Such a property is not explicable within the framework of field theory but may find a solution in topological field theory. In this new context, the initial singularity can be viewed as the source of a "topological amplitude" [3] corresponding to the charge of the singular zero size gravitational instanton, i.e. $Q = \int d^4x\, R_{\mu\nu}\tilde{R}^{\mu\nu}$. This amplitude can be detected at the boundary S^3 of the singular gravitational instanton, which, according to [5], might have the topology of the 4-dimensional Euclidean ball B^4 bounded by the sphere S^3. So, the pseudo-observables of Riemannian space-time in the vicinity of the origin are here interpreted as co-cycles on instanton moduli space and are associated with γ_i cycles on the B^4 four-dimensional manifold (Donaldson mapping [7]). Considering a point X in B^4, the topological amplitude inducing the propagation of the instantonic charge can take the general form :

$$\left\langle 0_{S^3} .\ 0_X \right\rangle = \#(S^3, X)$$

Starting from this, the pseudo-observables 0_i of the left member are giving the topological amplitudes of the theory whereas the right member designates the number of intersections of $\gamma_i \subset B^4$. The function $\#(S^3, X)$ is zero if the point $X \notin B^4$ and is equal to one if $X \in B^4$ (the case for which there exists topological amplitude).

The present article is organized as follows. In paragraph 2, we briefly recall the context in which the problem of inertia is formulated (in classical mechanics) [2]. Likewise, we recall the canonical formulation of Mach's principle, which suggests a non-local approach of the inertial interaction [1]. In paragraph 3, we consider the " topological Mach's principle", a new formulation of Mach's principle in the context of topological field theory [3]. In paragraph 4, we suggest that the propagation of the "causal signal" connected with the Initial Singularity of space-time [5] can conveniently be described by the topological amplitude related to a singular gravitational instanton of size zero [4]. We complete this approach by showing, from another point of view, that if the "topological signal" at the origin has a Dirac mass (a distribution with a support reduced to zero), then this signal is sent to infinity - i.e. onto the sphere S^3 representing the co-boundary of the space-time E^4 and of the singular gravitational instanton B^4 [5]. Last, we discuss these diverse results and conjectures in paragraph 6.

2. REMINDER : INERTIA AND MACH'S PRINCIPLE

It is well known in classical mechanics that the inertial interaction is interpreted as an instantaneous reaction to acceleration of any material object :

$$f_i = - m a$$

a being the acceleration of the system itself. According the Newtonian point of view, - and, to some extent, in special relativity - , this reflects an inherent property of matter, which does not conflict with the conception of absolute space and time.

On the contrary, according to general relativity, although independent of the observer, the intrinsic properties of space-time and its geometry are described by a metric configuration inseparable from the distribution of the matter. This relationship has been defined on axiomatic basis in the "equivalence principle" between inertia and gravitational mass. However, how can one explain the *instantaneous* propagation (i.e. non-physical in the strict sense) of the inertial interaction ? How does this interaction propagate itself at "infinite speed" from one point to another in space-time ?

The first attempt to provide a global response was qualitatively formulated by E. Mach [1] for whom, on the basis of the relativity of all motion, the origin of inertia is the "interconnection" of all matter in the universe. We take from this the generic expression of Mach's principle as initially formulated :

Definition 2.1. (E. Mach) : *the inertial reference point defined by local physics coincides with the reference point in which distant objects are at rest, where it results that the most distant masses distributed in the universe determine the inertial behavior of local masses.*

Numerous other variants of Mach's principle still exist, developed notably by Brans-Dicke [8], Bondi-Samuel [9] or Brill [10] and some others. However, none of these approaches appears able to explain, in a framework compatible with relativistic principles, the nature of the inertial interaction as well as its mode of propagation.

We now propose hereafter to renew the approach of Mach's principle in the context of topological field theory.

3. TOPOLOGICAL MACH'S PRINCIPLE

Our main suggestion within this paper is that the foundation on which Mach's principle lies (as the notion of inertia) is not "physical" but falls within topological field theory, defined by Witten in 1988 [3]. As an example, we consider Foucault's pendulum experiment $\mathcal{G}$, which cannot be explained satisfactorily in either classical or relativistic mechanics. We recall that the problem is the angular invariance of the plane of oscillation of the pendulum $\mathcal{G}$. Then, the "topological Mach's principle" assumes that the interaction between $\mathcal{G}$ and "global space-time " **E** is itself of the topological type - which by its nature explains precisely the invariant and global properties of the system formed by the oscillation plane of the pendulum and the rest of the universe.

In [5] the existence of a zero scale limit in pre- space-time was suggested, such a limit concentrating the energy density of the "whole universe", in the Mach sense. We begin by showing that within such an approach, the zero scale configuration has no *physical* content (there exists no stable physical length lower than the Planck length) but is a *topological* configuration, corresponding to a singular gravitational instanton of zero size.

3.1 Zero scale of space-time and topological field theory

As Weinberg first stated [11], one can reasonably consider that space-time at the Planck scale forms a system in global thermodynamical equilibrium. From an algebraic point of view, an equilibrium state is a state on a C^{*}- quasi-local algebra A, generated by a sub-algebra corresponding to the kinematic observables of the sub-system. Such a state φ on A can be given, at the temperature β^{-1}, by the Gibb's state :

$$\varphi(A) = \frac{\mathrm{Tr}_{\mathfrak{h}}(e^{\ \beta H} A)}{\mathrm{Tr}_{\mathfrak{h}}(e^{\ \beta H})}$$

On the other hand, in the context of the Tomita-Takesaki modular theory, the famous Kubo-Martin-Schwinger "(KMS) condition" [12] has established that each state $\varphi(A)$ of the system can be connected, in a unique manner, to the strongly continuous one parameter $*$ - automorphisms group α_t giving the time evolution of the observables and corresponding to the well known Heisenberg algebra :

$$\alpha_t(\boldsymbol{A}) = e^{iHt}\, \boldsymbol{A}\ e^{-iHt}$$

Now, starting from the expected equilibrium state of the universe around the Planck scale, it has been shown in [5] that (pre)space-time at the Planck scale can be seen as subject to the KMS condition [12]. Therefore, within the holomorphic KMS strip (i.e. between the scale zero ($\beta = 0$) and the Planck scale ($\beta = \ell_{Planck}$), it is therefore natural to consider the timelike direction of the metric as *holomorphic*, so that we should have :

$$t \rightarrow \tau = t_r + i\, t_i$$

as showed in [5]. In the same way, the physical (real) temperature should also be viewed as complex at the Planck scale :

$$T \rightarrow T_C = T_r + i T_i$$

as proposed by Atick and Witten in another context [13]. Recently, in N=4 supersymmetric string theory, I. Antoniadis, J.P. Deredinger and C. Kounnas [14] have also introduced at the Planck scale a complex temperature in the thermal moduli space, the imaginary part coming from the $B_{\mu\nu}$ antisymmetric field under type ***IIA*** $\xleftrightarrow{S/T/U}$ type ***IIB*** $\xleftrightarrow{S/T/U}$ ***Heterotic*** string-string dualities. So, the KMS condition suggests the existence at the Planck scale of an effective one loop potential coupled, in N = 2 supergravity, to the complex dilaton + axion field $\varphi = \frac{1}{g^2} + i\alpha$ and yielding the following dynamical form of the metric

$$\eta_{\mu\nu} = \text{diag}(1\ ,\ 1\ ,\ 1\ ,\ e^{i\theta}) \qquad \eta_{\mu\nu} = \text{diag}(1\ ,\ 1\ ,\ 1\ ,\ e^{i\theta}) \tag{3}$$

The signature of the metric (3), endowed with a supplementary degree of freedom in the g_{44} component, is Lorentzian for $\theta = \pm\pi$ and can become Euclidean for $\theta = 0$. The modular theory of Tomita [15] suggests then the "dualisation" of the signature given by the generalized automorphisms of the "quantum von Neumann algebra" $\boldsymbol{M}_q$, which can be written :

$$M_q \mapsto \sigma_{ß_c}(M_q) = e^{Hß_c} M_q \ e^{-Hß_c} \tag{4}$$

The temporal flow associated with (4) is formally holomorphic in the variable $ß_c = ß_r + iß_i \in \mathbb{C}$. The group of modular automorphisms $\sigma_{ß_c}$ generates two *dual* flows. The first one, of the form :

$$\sigma_t(\boldsymbol{M}_q)` = e^{iHt} \boldsymbol{M}_q \ e^{-iHt} \quad , \ t \in \mathbb{R}. \tag{5}$$

corresponds to the algebra of observables and to the Lorentzian flow in *real* time. On the other hand, the dual current, is :

$$\sigma_\beta(\boldsymbol{M}_q) = e^{-ßH} \boldsymbol{M}_q \ e^{ßH} \tag{6}$$

giving on the topological von Neumann algebra $\boldsymbol{M}_q$ a semi-group of non-bounded and non-stellar operators. The flow σ_β of $\boldsymbol{M}_q$ is defined not throughout the whole of $\boldsymbol{M}_q$ but on an ideal $\{\mathfrak{I}\}$ of $\boldsymbol{M}_q$ and coupled to the topological flow in pure $ß = i\ t$ imaginary time. In the model constructed in [5], the algebra of observables described by (5) is replaced at the scale zero of space-time by an "algebra of states", dual to the algebra of Heisenberg in the form (6). Of course, at the singular ß = 0 scale, it is no longer possible to conserve the notion of physical observables; instead, we consider homology cycles in moduli space of (zero size) gravitational instantons. This latter conclusion remains true, in pure imaginary time, for all real ß > 0. Such an approach allows us to distinguish three different domains in the "cosmological light cone", each of these domains being described by a specific von Neumann algebra called a "factor" [16]. If we call $\boldsymbol{M}_{Top}^{0,1} = R \otimes F$ the factor $R_{0,1}$ of type II_∞ corresponding to the singular zero scale, as all ergodic transformations starting from $\boldsymbol{M}_{Top}^{0,1}$ (flows associated with zero scale) are weakly equivalent, therefore $\boldsymbol{M}_{Top}^{0,1}$ is a hyperfinite factor of the Araki-Woods type [17]. The factor $\boldsymbol{M}_{Top}^{0,1}$ is then canonical. More generally, there exist thus three scales (corresponding to the three regions of the cosmological light cone) and described by three different "factors" :

(**i**) the *classical* scale (ß >, Planck), described by the tracial factor $\boldsymbol{M}_{Phys}$ of type I_∞;

(**ii**) the *quantum* superposition scale ($0 < ß < \ell_{Planck}$) described by the traceless type III_λ factor such that $III_\lambda = II_\infty \rtimes_\theta \mathbb{R}^*_+$. Then, we write $\boldsymbol{M}_q = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathbb{R}^*_+$;

(iii) the *topological* scale (zero scale associated with ß = 0) described by the tracial type II_∞ factor $\boldsymbol{M}^{0,1}_{Top}$.

To finish, we note that the algebraic "flow of weights" [18] associated with the factor $\boldsymbol{M}^{0,1}_{Top}$ of type II_∞ at zero scale of space-time is an invariant of $\boldsymbol{M}^{0,1}_{Top}$. Thus, it has also been shown, again in [5], that the initial space-time singularity, can be approached not in terms of divergences of physical fields but in terms of topological field symmetries and associated invariants (such as the first Donaldson invariant [7-19]) :

$$I = \sum_i (-1)^{n_i}$$

n_i being defined by Donaldson as the instanton number of the theory [7]. i is indicating the i th instanton and $n_i = 0$ or 1 determines the sign of its contribution to Z. The contribution of the ith instanton to Z is $(-1)^{n_i}$. The index $\sum_i (-1)^{n_i}$ is a non-polynomial topological invariant, reduced to an integer when the dimension of the instanton moduli space $\dim \mathcal{M}^{(k)}_{\text{mod}} = 0$ [19]. Then, a possible resolution of the Initial Singularity consists in considering that the singular zero scale, which cannot be described by the (perturbative) physical theory, should be described by the (non-perturbative) dual theory, of the topological type.

3.2 Topological invariant of singularity

Starting from Witten [3], one normally defines topological field theory as the quantization of zero, the Lagrangian of the theory being either (**i**) a zero mode, or (**ii**) a characteristic class $c_n(V)$ of a vectorial bundle $V \xrightarrow{\pi} M$ built on space-time. A new topological limit of the theory has therefore been defined [5], which is both non-trivial and no longer based on H = 0 but on *ß* = 0 and hence *independent* of H. *ß* plays the role of a coupling constant, such that it exists an infinite number of states not interacting with each other and independent of H. The point is that for *ß* = 0, the action S is projected onto a non trivial minimum, corresponding to the self-duality condition $R = \pm R^*$. Then, the minimum of the action is not zero (as it is in the trivial case) but has a non-trivial (invariant) value. But in this case, the field configuration is necessarily *Euclidean* and defines a gravitational instanton, i.e. a topological configuration.

Let's recall that the ordinary topological limit of quantum field theory is described by the Witten invariant

$$Z_n = \mathrm{Tr}(-1)^n \qquad (7)$$

n being the zero energy states of the theory, for example the fermion number in supersymmetric theories [20]. Equ. 7 is given by the limit of the partition function $Z = \mathrm{Tr}(-1)^n e^{-ßH}$ for zero (or invariant) values of H. Here, $Z_n = \mathrm{Tr}(-1)^n e^{-ßH}$ is simply the partition function at temperature β^{-1} connected with a set described by the density matrix Q = $(-1)^n e^{-ßH}$.

Now, we propose to extend Witten's results into the context of N=2 supergravity [21]. In such a context, we analyze the zero mode of the space-time scale (i.e. the $ß \rightarrow 0$ limit). Hence, as observed hereabove and according to the results of [5], at the Planck scale, the space-time, in thermodynamical equilibrium, can be considered as submitted to the "KMS condition" [12]. Therefore, within the limits of the KMS strip - i.e. between the scale zero ($ß = 0$) and the Planck scale ($ß = \ell_{Planck}$)- the time like direction of the metric might be viewed as *holomorphic*. The corresponding quantum fluctuation of the signature of the metric from the Lorentzian form to the Euclidean one and conversely has been exhibited in [5] and can be described by the symmetric homogeneous space

$$\Sigma_h = \frac{SO(3,1) \otimes SO(4)}{SO(3)}$$

SO(3) being diagonally embedded in SO(3, 1) $\otimes$ SO(4). In terms of metric field, to such a superposition state might correspond a "monopoles+instantons" configuration, the action of the superposed theory being of the general form :

$$\mathscr{L}_{m+i} = \frac{1}{g^2(\rho)} \int d^4x\, R_{\mu\nu} R^{\mu\nu} + \vartheta \int d^4x\, \mathrm{Tr}\; R_{\mu\nu} \tilde{R}^{\mu\nu} + \frac{g^2(\bar{\rho})}{2} \left\{ \|R\|^2 + \|D_\mu \phi\|^2 \right\}$$

According to the results of E. Kiritsis et C. Kounnas on a class of heterotic ground states in 4D [5], equ.(11) can be true in certain classes of N = 2 supergravities (resulting of spontaneous breakdown of N = 4). Such Lagrangians contain a scalar potential representing the kinetic terms and a Kähler potential [21]. This type of action results typically from the Instanton-Monopole symmetry of duality found in [5] and recently verified by C.P. Bacchas, P. Bain and M.B. Green in the context of string theory [22]. Namely, 4-instantons (Euclidean metric) and (3+1)-monopôles (Lorentzian metric) are connected by a complex dilaton field (or S-field) [21] of the form $\left(\frac{1}{g^2}+i\theta\right)$ playing the role of coupling constant of the superposed theory. So, on the infrared limit (ß ≥ ℓ_{Planck}) corresponding to g → ∞, the instanton contribution to the functional integral, given by $e^{-\frac{1}{g^2(\rho)}\int R^2}$ is suppressed. This is a well-known result, demonstrated a long time ago [7]. On the contrary, it has been pointed in [5] that the monopole configuration is well defined on this limit. So, the "superposed action" is reduced to :

$$\mathscr{L}_{i+m} \rightarrow \mathscr{L}_m = \frac{g^2(\overline{\rho})}{2}\left\{\|R\|^2 + \|D_\mu \phi\|^2\right\}$$

corresponding to a 4D Lorentzian configuration of the metric on the infrared limit. Conversely, on the ultraviolet limit ß→0, it has been showed, again in [5], that monopole contribution is suppressed from the functional integral, just like the perturbative part of the instantonic action (tunneling effect). However, one can easily observe that the topological charge Q = $\vartheta\int d^4x \,\mathrm{Tr}\; R_{\mu\nu}\tilde{R}^{\mu\nu}$ is conserved in $\mathscr{L}_{m+i}$, so that on the $ß \rightarrow 0$ limit, the theory contains only a Riemannian topological contribution coming from zero size instantons [5]. Of course, for $0 < ß < \ell_{Planck}$, the action takes the general "superposed" form $\mathscr{L}_{m+i}$ mixing Lorentzian and Euclidean states of the metric field.

Now, we shall consider the partition function Z_s describing this generalized symmetry of all possible states (Lorentzian or Euclidean) of the metric beyond ℓ_{Planck} :

$$Z_s = \mathrm{Tr}(-1)^S\, e^{-ß_c H} \tag{8}$$

with $ß_c \in \mathbb{C}$ and s representing the number of instantons of the theory. According to N=2 supergravity properties, H can be adequately described by the square of the Dirac operator $\not{D}^2$. Now, let's consider the scale ß = 0. Equ.(8) becomes :

$$Z_s = \mathrm{Tr}(-1)^S$$

This new index, isomorphic to the Witten invariant Z = $\mathrm{Tr}(-1)^n$, can be explicitly associated with the initial singularity of the pre-space-time, reached for the value ß = 0 (zero mode of the scale) of the metric states partition function. One can therefore extend the iso-dimensional monopole / instanton duality by suggesting that such a duality symmetry links the BRST cohomology ring (physical sector of the theory) and the cohomology ring of the instanton moduli space (topological sector). The BRST cohomology groups [21] having the generic form

$$H^{(g)}_{BRST} = \frac{\ker Q^{(g)}_{BRST}}{\mathrm{im} Q^{(g-1)}_{BRST}} \tag{9}$$

we then consider that the topological field theory allows the injection of rings :

$$H^{\bigstar}_{BRST} = \bigotimes^{\Delta U_k}_{g=0} H^{g}_{BRST} \xrightarrow{\iota} H^{\bigstar}\left(\mathcal{M}^{(k)}_{\mathrm{mod}}\right) = \bigotimes^{d_k}_{i=0} H^{(i)}\left(\mathcal{M}^{(k)}_{\mathrm{mod}}\right) \tag{10}$$

which provides an injective path of the physical mode into the topological mode. In terms of observables $_{0_i}$ and homology cycles $H_i \subset \mathcal{M}^{(k)}_{\mathrm{mod}}$ in moduli space $\mathcal{M}^{(k)}_{\mathrm{mod}}$ of configurations of the gravitational instanton type $\Im[\phi(x)]$ on the gravitational fields ϕ of the theory, we bring out the equivalence :

$$\langle 0_1 0_2 \ \ldots \ 0_n \rangle = \#(H_1 \cap H_2 \cap \ \ldots \ \cap H_n) \tag{11}$$

where the physical sector of the theory is described by the observables $_{0_i}$ and the dual sector, of the topological type, by the homology cycles $H_i \subset \mathcal{M}^{(k)}_{\mathrm{mod}}$. The oscillation of the signature of the metric between physical and topological sectors is then induced by the divergence $\Delta U_k = \int \partial^{\mu} j_{\mu} \, d^4x$ of the ghost flow [16][15] $_{j_\mu}$. When ΔU = 0, as there exists no embedding space for moduli space, we suggest that the theory is then projected in the Coulomb branch, at the origin of $\mathcal{M}^{(k)}_{\mathrm{mod}}$, on a singular instanton of zero size [4] which we identify to space-time at zero scale. The theory is ramified on the purely topological sector $_{H_i}$, the corresponding signature at this sector being Euclidean (++++).

From this viewpoint, the image of Euclidean zero symmetry, described by the non-broken gauge group of type SU(2) ⊗ SU(2), is given by the first Donaldson invariant [7] associated with the existence of a "topological amplitude" characterizing the theory. When $\dim \mathcal{M}_{\mathrm{mod}}^{(k)} \neq 0$, the Donaldson invariants are given by the correlation function of the theory :

$$Z(\gamma_1 \ldots \gamma_{\mathrm{r}}) = \int DXe^{-S} \prod_{i=1}^{r} \int W_{k_i} = \left\langle \prod_{i=1}^{r} \int W_{k_i} \right\rangle \qquad (\mathrm{Dim}\ \mathcal{M}_{\mathrm{mod}}^{(k)} \neq 0)$$

(12)

where W_{k_i} is a k form, for $0 \leq k \leq 4$. In particular, as observed by E. Witten [3], if γ is a k dimensional homology cycle on the manifold M, then the integral $I(\gamma) = \int_{\gamma} W_k$ is BRST invariant, since $\{Q,I\} = \int_{\gamma} W_k = 0$.

Now, what happens when $\dim \mathcal{M}_{\mathrm{mod}}^{(k)} = 0$? The solution is in the correspondence between the Donaldson invariants on 4D manifolds and the Floer homology groups on 3D manifolds [23]. In fact, Donaldson invariants amount to the calculus of the partition function Z, expressed as an algebraic sum over the instantons :

$$\mathrm{Z} \mapsto \underset{\mathcal{M}_{\mathrm{mod}}^{(k)}=0}{\mathrm{Z}} = \sum_i (-1)^{n_i}$$

Donaldson has shown on topological grounds [7] that when $\dim \mathcal{M}_{\mathrm{mod}}^{(k)} = 0$, then $\sum_i (-1)^{n_i}$ is a non-polynomial topological invariant, reduced to an integer. We find the same result starting from $T_{\alpha\beta} = \{Q\ ,\ \lambda_{\alpha\beta}\}$.

Now, let's go further into the understanding of the zero scale limit of space-time. We claim that, even if this limit is singular within the context of (perturbative) quantum field theory, it can be resolved in the (non-perturbative) dual theory, i.e., the topological field theory. As noticed previously, when ß → 0, the partition function

$$Z_{\mathrm{s}} = \mathrm{Tr}(-1)^{\mathrm{S}}\, e^{-ß_c H} \qquad (13)$$

describing the symmetry of all the (Lorentzian + Euclidean) states of the metric is reduced to

$$\underset{ß=0}{Z} = \mathrm{Tr}(-1)^S \tag{14}$$

What is here the meaning of Tr $(-1)^S$? In fact, at least in the context of supersymmetric theories [20], it can be seen as the index of an operator acting on the Hilbert space $\mathcal{H}$. Dividing $\mathcal{H}$ in monopole and instanton sub-spaces $\mathcal{H} = \mathcal{H}_{\mathfrak{M}} + \mathcal{H}_{\mathrm{i}}$ and Q being a generator of supersymmetry, it follows from (21) that $Q|\psi\rangle = 0$ and $Q^*|\psi\rangle = 0$. Then Tr $(-1)^S$ = Ker Q - Ker Q^* so that as topological index, Tr $(-1)^S$ is invariant under continuous deformations of parameters which do not modify the asymptotic behavior of the Hamiltonian H at high energy. At the scale $ß = 0$, H is given by $H = dd^* + d^*d$, the space of zero energy states corresponding to the set of even harmonic forms on M_n:

$$\mathrm{Tr}\,(-1)^S = \chi\,(\boldsymbol{M}) = \sum_{k=0}^{n}(-1)^k b_k \tag{15}$$

where b_i is the i^{th} Betti number and $\chi\,(\boldsymbol{M})$ the Euler-Poincaré characteristic of $\boldsymbol{M}$. Obviously, Δ = Tr $(-1)^S$ is independent of $ß$, the sole contributions to Δ coming from the topological sector of zero energy. Consequently, Δ is a topological invariant, i.e. the first Donaldson invariant. The coupling constant g being dimensional, the limit $ß = 0$ implies $\rho = 0$ and corresponds to the sector of zero size instantons. Therefore, (15) is a topological index, corresponding to the accumulation point of the zero size instantons of the theory. Let's note that all instantonic (i.e. Euclidean) states of $g_{\mu\nu}$, given by the topological charge of the singular gravitational instanton are equivalent at zero scale. We call "symmetry zero" the generalized symmetry characterizing the singular zero scale.

To conclude this part, we suggest as a geometric model of the instanton the ball B^4 bounded by the sphere S^3. The so-called "propagation" of the solution depends then on the support of the gravitational instanton : in the region of zero limit, there exists an accumulation of topological charge above the singular point S_0 such that the topological charge density $RR^* \to \infty$; in the dual situation, corresponding to the fundamental state, the support of the instanton is extended to infinity and $RR^* \to 0$. The "transition" from zero to infinity is then described by the conformal transformations of the sphere (see in § 4 the Möbius group action (4.5)).

From this point of view, the Machian interaction evidenced between the pendulum $\mathcal{G}$ and the global universe could be exactly described by the interaction between $\mathcal{G}$ and the zero size instanton - almost like a renormalisation transformation. We therefore propose a new formulation of the Mach's principle in the context of topological field theory :

3.3 Topological Mach's principle. *The topological amplitudes associated with the propagation of the topological charge of the singular zero size gravitational instanton corresponding to the Initial Singularity of space-time determines the inertial behavior of local masses.*

In the following paragraph, we put forward a number of natural arguments to establish topological Mach's principle. In particular, we consider that the propagation of information characterizing the Initial Singularity of space-time can conveniently be described by the topological amplitude associated with the S_0 singular zero size gravitational instanton.

4. SINGULAR ZERO SIZE INSTANTON AND TOPOLOGICAL INTERACTION

The principal argument demonstrating the existence of an interaction between $\mathcal{G}$ and the charge Q of S_0 lies in the fact that the topological charge of the instanton is entirely determined by the asymptotic behavior of the gauge field (A_μ in the Yang et Mills case and $g_{\mu\nu}$ in N = 2 supergravity). The field F, non-zero in $E^4 = \mathbb{R}^4$ instanton space, cancels at the boundary $\partial E^4 = S^3$, the gauge potential becoming a pure gauge on the boundary. Hence, we suggest that $\partial E^4 = S^3$ represents the curved and compact three-dimensional physical space $\mathbb{R}^3$ within which the rotation plane $\mathcal{G}$ is inscribed. We draw from this a relation of topological type between the topological charge defined by ∂E^4 and the fixed orientation of the oscillation plane $\boldsymbol{\Pi^2}$ of $\mathcal{G}$, also defined by $\partial E^4 = S^3$.

Proposition 4.1 *The topological charge **Q** of the Yang-Mills instanton is completely determined by the asymptotic behavior of the gauge field A_μ, at the boundary ∂E^4 represented by the three-dimensional sphere S^3 where the gauge potential becomes a pure gauge. $\partial E^4 = S^3$ represents the curved and compact three-dimensional physical space $\mathbb{R}^3$ in which the plane of rotation Π^2 of $\mathcal{G}$* ***is embedded.***

Demonstration The Lagrangian of the Yang and Mills theory in Euclidean space-time has the form :

$$\mathbf{L} = \frac{1}{4g^2}\int d^4 x F^{\alpha}_{uv} F^{\alpha uv} \tag{16}$$

with f^{abc} being the structure constants of the gauge group SU(2) :

$$F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + f^{abc} A^b_\mu A^c_\nu \tag{17}$$

The classical Yang and Mills action can then be written :

$$S = \frac{1}{8g^2}\int d^4 x \left\{ (F^a_{\mu\nu} \pm \tilde{F}^{a\mu\nu})^2 \right\} \mp \frac{8\pi^2}{g^2} Q \tag{18}$$

with $\tilde{F}^{a\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} F^a_{\mu\nu}$, Q being the topological charge (or Pontryagin index) of the instanton:

$$\mathrm{Q} = \frac{1}{32\pi^2}\int d^4 x\, F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \tag{19}$$

As the action of the configuration must be finite, on spheres at infinity of radius $r \to \infty$, the field $F^a_{\mu\nu}$ must fall to 0 more rapidly than r^{-3} :

$$F^a_{\mu\nu}(|x| = \mathrm{r}) \to 0 \tag{20}$$

such that A^a_μ must be a pure gauge at infinity ($r \to \infty$):

$$A^a_\mu T^a \to iU(x)\partial_\mu U^{-1}(x) \tag{21}$$

T^a being the generators of the gauge group, i.e. in the case of the fundamental representation with Pauli matrices σ^a , $T^a = \frac{1}{2}\sigma^a$ for SU(2). The element $U \in SU(2)$. Considering the charge Q, it is possible to represent it by an integral over the total derivative :

$$\mathrm{Q} = \frac{1}{32\pi^2}\int d^4 x\, \partial_\mu K_\mu \tag{22}$$

that is to say, according to the Gauss theorem :

$$Q = \frac{1}{32\pi^2} \int_{|x|=r} K_\mu \, dS_\mu \tag{23}$$

where the vector K_μ represents the Chern-Simons current :

$$K_\mu = 2\varepsilon_{\mu\nu\alpha\beta}(A_\nu^a \partial_\alpha A_\beta^a + \frac{1}{3} f^{abc} A_\nu^a A_\alpha^b A_\beta^c) \tag{24}$$

From equ. (24) we conclude that the topological charge Q is thus entirely determined by the asymptotic behavior of the field A_μ^a. As established in [21], Q depends solely on the global properties of the function $A_\mu^a(|x| = r)$. Indeed, at infinity we have :

$$F \xrightarrow{|x| \to \infty} 0$$

but this is not (necessarily) the case for the gauge potential A_μ^a which becomes a pure gauge :

$$A(x) \xrightarrow{|x| \to \infty} U(x)\partial_\mu U^{-1}(x) \tag{25}$$

The gauge elements $U(x) \in \mathrm{SU}(2)$, $x \in S^3$ are such that :

$$U = A + i\vec{\sigma}\,\vec{B},\ A^2 + \vec{B}^2 = 1 \tag{26}$$

and $U(x)$ represents :

$$U\colon\ S^3 \to \mathrm{SU}(2) \cong S^3 \tag{27}$$

where we find the applications of the sphere S^3 representing the compact physical space E^3, boundary of the space E^4, on the isotopic space of SU(2), equally isomorphic to S^3. We draw the identification of S^3, boundary of the 4-dimensional instanton solution to physical space, of the double embedding of SU(2) in SL(2, C) - the universal convening of the Lorentz group - and in SU(2) ⊗ SU(2), the convening of SO(4). As SU(2) → S^3, we therefore propose to interpret S^3 both (**i**) as the 3-dimensional boundary of the 4-dimensional instanton Euclidean solution B^4 and (**ii**) as the 3-dimensional boundary of space-time . From this identification, the application $S^3 \to S^3$ is designated by $\pi_3(S^3)$ and is such that :

$$\pi_3(S^3) = \mathbb{Z} \tag{28}$$

such that the applications $S^3 \to S^3$ are classified according to the homotopy classes characterized by integers, in our case n = 1. Thus, the 2-dimensional plane of oscillation $\prod^2$ of Foucault's pendulum is embedded in the 3-dimensional physical space corresponding precisely to S^3. As the topological charge **Q** of the instanton is determined by the behavior of the gauge field on S^3, it follows from this that $\prod^2$ is determined by **Q**, as required. **qed**

We have established the relation between **Q** and $\prod^2$ in the case of Yang and Mills instantons. One can extend this result to supergravity, insofar as, in the context of non-linear curvature theories, the field R is, like F, asymptotically free. The action of the gravitational instanton becomes :

$$S = = \frac{1}{g^2(\rho)} \int d^4x \, R_{\mu\nu} R^{\mu\nu} + \vartheta \int d^4x \, \mathrm{Tr} \; R_{\mu\nu} \tilde{R}^{\mu\nu} \tag{29}$$

the topological term being :

$$Q = \vartheta \int d^4x \, \mathrm{Tr} \; R_{\mu\nu} \tilde{R}^{\mu\nu} \tag{30}$$

In this case, the surface term K, associated with the Chern-Simons topological flow, becomes :

$$\begin{aligned} Q_3^0(\Gamma, R) &\sim \mathrm{tr}\left(\Gamma \mathrm{d}\Gamma + \frac{2}{3}\Gamma^3 \right) \\ Q_3^0(\omega, R) &\sim \mathrm{tr}\left(\omega \mathrm{d}\omega + \frac{2}{3}\omega^3 \right) \end{aligned} \tag{31}$$

Γ representing the Christophel symbols, ω the spinorial connections and R the curvature of the 4-manifold. In putting R = $d\Gamma + \Gamma^2$ or R = $d\omega + \omega^2$, we therefore find again the result of the first part of (4.1.) in supergravity. Concerning the Chern-Simons form, we recall that Witten has shown [24] that general relativity in dimension (2+1) is equivalent to Chern-Simons topological theory in dimension 3. The Chern-Simons Lagrangian is :

$$h(A) = \int Tr(A \wedge dA) + \frac{2}{3} A \wedge A \wedge A) \tag{32}$$

Hence, the action of the theory of gravity in dimension (2+1) has the form :

$$L = \varepsilon^{abc}\varepsilon^{ijk}e_i^a R_{jk}^{bc}(\omega)$$
$$R_{jk}^{bc} = \partial_j \omega_k^{bc} + \omega_j^{be}\omega_k^{ec} - (j \leftrightarrow k) \quad (33)$$

Analysis of the two Lagrangians (32) and (33) shows the equivalence between gravity (2+1) and the Chern-Simon theory at the limit of zero scale associated with zero energy modes. In dimension 4, we find then that at the limit of zero scale, Lorentzian (3+1) field theory must be replaced by Donaldson topological field theory (4,0) [7].

One can therefore consider that the propagation of the initial singularity is induced by the existence of a topological amplitude – the charge of the zero size gravitational instanton, i.e. $Q = \int d^4x\, R_{\mu\nu}\tilde{R}^{\mu\nu}$, - detectable at the boundary S^3 of the singular gravitational instanton provided with the topology of the B^4 Euclidean ball of dimension D=4. As noticed in (1), the pseudo-observables in 4D Euclidean space are connected with cycles γ_i of the B^4 4-manifold (Donaldson application).

Now, we return to the inertial interaction. In the following conjecture, we suggest that the source of inertia is not, strictly speaking, physical but has a *topological* content, linked to the topological charge Q=1 of the singular gravitational instanton.

Conjecture 4.2. *The inertial interaction may be interpreted as a topological interaction, of which the source is the topological charge of the zero size singular gravitational instanton.*

Elements of demonstration. We have shown that the total inertial force likely to contribute in the case where the entire Universe would be submitted to an acceleration ***a*** in respect to a given object, could be obtained by summing the forces $F_{12}^{'}$ on masses other than m itself :

$$F_{inertielle} = ma\sum \frac{GM}{c^2 r} \quad (34)$$

If M represents the totality of the mass of the Universe, a good approximation of $\sum \frac{GM}{c^2 r}$ gives us :

$$F_{totale} = \sum_{univers} \frac{GM}{c^2 r} \approx 1 \quad (35)$$

The total inertial interaction F_{totale} is therefore of the order of unity and represents a topological invariant -i.e. invariant of gravitational gauge and scale invariant -. In all points of space, the contribution of F_{totale} of is identical. Hence, we suggest that the total inertia F_{totale} has its origin in the topological charge Q :

$$Q = \frac{1}{32\pi^2}\int d^4x\, R_{\mu\nu}\tilde{R}^{\mu\nu} = 1 \quad (36)$$

Indeed, we have seen that the topological charge Q of the instanton is entirely determined by the asymptotic behavior of the gauge field at the boundary ∂E^4 represented by the three-dimensional sphere S^3 where the gauge potential becomes a pure gauge. We show then that at the zero scale limit, F_{totale} is isomorphic to Q :

$$F_{totale} \equiv Q$$

At the zero scale limit, $\sum_{univers} \frac{GM}{c^2 r}$ is identifiable to the Schwartzshild radius l_S of space-time quotiented by the radius r → ε of the configuration in region of zero scale, i.e. :

$$\sum_{univers} \frac{GM}{c^2 r} = \sum_{univers} \frac{l_S}{r} \qquad (37)$$

We therefore draw from (37) that $\frac{GM}{c^2 r}$ represents the Gaussian curvature of the manifold, which gives :

$$\frac{GM}{c^2 r} = K_{r\varphi} \quad (38)$$

$K_{r\varphi}$ being the Gaussian curvature of the geodesic surface (r, φ). Hence, at the Planck scale, we must extend Einstein's gravitational term R towards an asymptotically free non-linear theory. Our approach is that the presence of terms in R^2 in the Lagrangian is the result of a superposition of two scales of quantum gravity : (**i**) the gravitational scale - i.e. macroscopic scale - characterized by the presence of an interaction in R and (**ii**) the quantum scale - i.e. microscopic scale - characterized by the presence of dual gravity terms, in R* = iR. The curvature term $K_{r\varphi}$ takes then the new form in supergravity :

$$K_{r\varphi}\,\textbf{\textit{(Planck)}} \rightarrow R_{\mu\nu}R^{\mu\nu}$$

such that $\sum_{univers} \frac{l_S}{r}$ is identifiable to

$$\Theta = \int d^4x\, R_{\mu\nu}R^{\mu\nu} \qquad (39)$$

As the theory is auto-dual, one has therefore R = R*****, and Θ becomes :

$$\Theta = \int d^4x\, R_{\mu\nu}\tilde{R}^{\mu\nu} \qquad (40)$$

which, by construction, is isomorphic to the topological charge Q of the gravitational instanton, as required. **qed**

From the hereabove results, we can conclude that the inertial interaction might admit the topological charge of the zero size gravitational instanton as source. The topological nature of the inertial interaction could explains (i) its global invariance properties and (ii) its instantaneous propagation between two points in space-time .

We complete the conjecture (4.2) in specifying why the plane of oscillation Π^2 should be static.

Corollary 4.3. *The static behavior of the plane of oscillation Π^2 of the pendulum $\mathcal{G}$ should be induced by the structure of topological invariant of* ***Q*** *:*

Elements of demonstration. Instanton theory has established that Q is a topological invariant [6]. In our case, we have Q = 1. The resulting vacuum topology θ - and, as a consequence, that of physical space S^3, boundary of E^4, in which Π^2 is embedded - is also invariant. Indeed, the invariance of the topological charge implies the invariance of the topological structure of S^3, i.e. of the three-dimensional physical space in the course of temporal evolution. It results from this that $\Pi^2 \subset S^3$ is itself invariant - i.e. static -. Furthermore, the static character of Π^2 implies that the underlying symmetry is Euclidean and can be described by the action of the group :

$$G_S = SU(2) \otimes SU(2) \tag{41}$$

G_S being, precisely, the symmetry group describing the instanton configuration characterized by the topological charge Q, as required. **qed**

We now show that a good model of the propagation of the topological interaction $\mathbf{Int}_{top}$ can be given by the conformal transformations Conf (S^3) of the sphere S^3.

4.4 Conformal group of S^3

We propose to describe the transformations Conf (S^3) by the Möbius group [25], defined from the inversion of S^3. Recalling that the pole inversion c, of strength a $\in \mathbb{R}^*$ is the application $i = i_{c,\alpha} : X \setminus c \to X \setminus$ where c is defined, in X vectorialised in c, by $i(x) = \frac{\alpha}{\|x\|^2} . x$. From this :

Definition 4.5 *An inversion of S^3 is meaning all restriction to S^3 of an inversion of R^4 conserving S^3. The Möbius group Möb(3) of S^3 is the sub-group of the S^3 bijection group induced by inversions of S^3.*

At this point, we can establish a link between the application defining the topological charge of the instanton and the similitudes of $\mathbb{R}^4$. We recall first that the group GL(n k) operates on M(n, k) by similitude :

$$(P, M) \mapsto P.M = P.M.P^{-1} \tag{42}$$

The equivariant canonical bijection associated with the matrix M is then a diffeomorphism, representing the *similitude class* of M :

$$\varphi(M) : GL(n, \mathbb{R}) / Z(M) \rightarrow 0\,(M) \tag{43}$$

which leads us to the important definition, taken from the reduction of endomorphisms [22] :

Definition 4.6 *The application $\psi : M\,(n, \mathbb{R}) \rightarrow 0\,(M)$ describing the similitude class of M is given by :*

$$X \mapsto \exp(X).M.\exp(-X) \tag{44}$$

From this definition, we establish in the two following propositions that the application defining the topological charge of the instanton belongs to the conformal group of S^3.

Proposition 4.7 *For all similitude $h \in \mathrm{Sim}\,(\mathbb{R}^3)$, the application defining the topological charge of the instanton, i.e. $f : S^3 \rightarrow S^3$, defined by $f(n) = n$ and $f = g^{-1} \circ h \circ g$ on $S^3 \setminus n$ belongs to Möb (3)).*

Demonstration Let n be the north pole and s the south pole of the sphere S^3. We can then show that $g^{-1} \circ h \circ g$, conveniently extended to the whole of S^3, is in Möb(3) if h is an inversion or a hyperplane symmetry. Yet, the h induce the existence of the group of similitudes Sim ($\mathbb{R}^3$). Indeed, considering $\vec{f}$, it has been shown [25] that if the kernel

$$\mathrm{Ker}\,(\vec{f} - \mathrm{Id}_{\vec{X}}) = \{0\} \tag{45}$$

then f admits a unique fixed point corresponding to its center. Moreover, $f \in$ Sim (X)\Is (X), there exists a unique $\omega \in$ X such that $f(\omega) = \omega$. The point ω is the center of the similitude f and one can write :

$$f = h \circ g = g \circ h, \qquad h \in H_{\omega,\mu} \text{ et } g \in \mathrm{Is}_{\omega}(X) \tag{46}$$

The above assumes convenient extensions, the most immediate consisting of attaching a point at infinity on $\mathbb{R}^3$ and extending g to $S^3 \to \mathbb{R}^3$ by g(n) = ∞. Thus, following [25], it exists in Möb (3) the applications

$$f_{\lambda} = g^{-1} \circ H_{0,\lambda} \circ g \mapsto e^{-\lambda} \circ H_{0,\lambda} \circ e^{\lambda} \tag{47}$$

associated with vectorial homothetias of $\mathbb{R}^3$. If $\lambda > 1$, the application f_λ admits the north pole as an attractor and the south pole as a repeller, i.e. the iterates $f_{\lambda}^{\,n}$ (n $\in \mathbb{N}$) making all points of $S^3 \setminus$ s converge towards n. The only point of S^3 out of the attraction of n is the south pole s. **qed**

We show now that Möb(3) is the conformal group Conf(S^3) of S^3. Letting Conf (S^3) describe the scale invariance (i.e. the conformal invariance) of the sphere identified here (following the inclusion $S^3 \subset$ SL(2, C)) to physical space $\mathbb{R}^3$ compactification.

Proposition 4.8 *Let Möb$^{\pm}$ (3) = Conf$^{\pm}$ (S^3). $\forall$ the radius r $\to$ 0 of S^3 engendering* $S^3_{r\to 0}$, *and* $\forall$ *$f \in$ Möb(3), then* $S^3_{r\to 0}$ *belongs to the bundle f(S^3) of spheres S^3. Reciprocally, a bijection of S^3 verifying this property necessarily belongs to Möb(3) . The group Möb(3) presents a natural isomorphism with PO(α) of the quadric of equation*

$$q = -\sum_{i=1}^{4} x^2 + x_5^2 .$$

Demonstration Let $i_{c,\ \alpha}$ be an inversion of X of dimension n. Let its derivative be $\boldsymbol{i}'(x)$ composed of the vectorial hyperplane symmetry $x^{\perp}$ and the homothetia of ratio $\alpha / \|x\|^2$. We show then that $\boldsymbol{i}'(x)$ in all $x \in X \setminus$ c is a direct similitude for $\alpha^n < 0$ and an indirect similitude for $\alpha^n > 0$. In fact, $\boldsymbol{i}'(x)$ conserves the right and right-oriented angles. Since the product of two conformal applications is conformal, then Möb(3)$\subset$Conf(S^3). Reciprocally, as Möb(3) $\subset$ Conf (S^3) is transitive on S^3, then $f \in$ Conf (S^3) leaves the north pole n fixed. According to the stereographic projection g of n, for f(n) = n, we get :

$$g \circ f \circ g^{-1} \in \text{Conf}\,(\mathbb{R}^3) \qquad (48)$$

g and f being conformal. Applying Liouville's similitudes theorem, we have

$$g \circ f \circ g^{-1} \in \text{Sim}\,(\mathbb{R}^3) \Longrightarrow f \in \text{Möb}(3) \qquad (49)$$

It follows from the inversion properties [25] that $f(\sigma)$ conserves the structure of the sphere S^3 when the radius $r \rightarrow 0$. Reciprocally, on putting $f(n) = n$, $g \circ f \circ g^{-1}$ transforms the $\mathbb{R}^3$ (half) lines into (half) lines, such that $S^3_{r \rightarrow 0}$ belongs to the bundle $f(S^3)$ of spheres S^3.

Finally, it has been established in [25] that Möb(3) = INCORPORER "Equation.3" * mergeformat i.e. the Möbius group of S3 corresponds to the restriction of the group PO(_) on im (_). qed

We now conjecture that the plane of oscillation of F conserves the initial singularity S for inertial reference point, whatever the orientation of this plane in physical space R3

Conjecture 4.9 Whatever the orientation in physical space R3 of the plane of oscillation ∏2 of the pendulum F, this 2-dimensional plane necessarily intersects the initial singularity S, i.e. ∏2 is always aligned on S .

Elements of demonstration We have established the identification between physical space R3 compactification and S3, boundary of space-time and equally boundary of the singular gravitational instanton solution. Each orientation of the plane of oscillation ∏2 corresponds therefore to an orientation in S3. We have also established that S3 can be identified to physical space :

$$S^3 \leftrightarrow \mathbb{R}^3$$

such that the three-dimensional information coming from physical 3-geometry is concentrated on the 3-surface S^3 but is not detectable in the interior of the sphere. Thus, the conformal invariance of S^3 implies that the temporal direction x^4 is necessarily orthogonal to the tangent space in a point of S^3. Putting this point as the south pole **s** of S^3, we have shown above that there exists in Möb (3) the applications

$$f_\lambda = g^{-1} \circ H_{0,\lambda} \circ g \;\mapsto e^{-\lambda} \circ H_{0,\lambda} \circ e^{\lambda} \qquad (50)$$

associated with the vectorial homothetias of $\mathbb{R}^3$. If $\lambda > 1$, the application f_λ admits the north pole as attractor and the south pole as repeller, i.e. the iterations $f_\lambda^{\,n}$ ($\mathbf{n} \in \mathbb{N}$) make all points in $S^3 \setminus \mathbf{s}$ converge towards $\mathbf{n}$. The only point in S^3 escaping the attraction of $\mathbf{n}$ is the south pole $\mathbf{s}$. The north pole $\mathbf{n}$ of the 3-sphere is therefore the fixed point of the conformal transformation Conf(S^3) and the temporal direction x^4, orthogonal to the tangential plane of the sphere in $\mathbf{s}$, intersects necessarily the center O of S^3 as well as the north pole $\mathbf{n}$. Since, by construction, the plane of oscillation $\boldsymbol{\Pi}^2$ contains x^4, then :

$$x^4 \subset \boldsymbol{\Pi}^2$$

it follows that the plane of oscillation $\boldsymbol{\Pi}^2$ is orthogonal to the tangential plane of the sphere at the south pole $\mathbf{s}$ and meets therefore necessarily the center O of S^3 as well as the north pole $\mathbf{n}$, the singular attractor point of S^3. qed

The above conjecture suggests therefore that the symmetry of rotation in $\mathbb{R}^3$ (exhibited by the plane of oscillation of Foucault's pendulum) is explicitly linked to the symmetry of the zero instanton configuration, $SU(2) \approx S^3$ being a sub-group of both $SU(2) \otimes SU(2)$ and $SL(2, C)$. Once the identification Initial Singularity / instanton zero is admitted, the above approach allows us reasonably to consider that, whatever the orientation of the plane of the pendulum in physical space, this plane remains necessarily aligned with the singular origin of space-time, identified here to the singular origin $\mathbf{n}$ of the sphere S^3, $\mathbf{n}$ being the north pole of the 3-sphère, the unique fixed point of the system. Indeed all possible orientations in physical space of the plane of oscillation of the pendulum are given by all possible orthogonal directions to the tangent plane at S^3. We obtain the different orientations of $\boldsymbol{\Pi}^2$ in $\mathbb{R}^3$ by making the south pole $\mathbf{s}$ "turn" on the 3-surface S^3, this rotation conserving the alignment between $\mathbf{s}$, O and $\mathbf{n}$ in the same plane $\boldsymbol{\Pi}^2$.

We draw from the above that whatever the orientation, the plane of oscillation of Foucault's pendulum is necessarily aligned with the initial singularity marking the origin of physical space S^3, that of Euclidean space E^4 (described by the family of instantons I_β of whatever radius ß) and, finally, that of Lorentzian space-time M^4.

The angular invariance of $\boldsymbol{\Pi}^2$ comes therefore *in fine* from the fact that the direction x^4 represents equally the fourth direction (in ß = $\boldsymbol{i}$ t imaginary time) of the Euclidean configuration of the type instanton E^4 bounded by S^3, such that at each point **s** of S^3, Π^2 cuts O and the origin of E^4 represented by the north pole **n**. We suggest then that this interpretation of x^4 as imaginary time explains in a non-trivial way the nature of the inertial force as well as its instantaneous propagation from one point to another in space-time .

5. DIRAC SIGNAL AND INITIAL SINGULARITY

We complete this paper by suggesting a subsidiary argument concerning the propagation of topological type of a "causal information" from the singular point S, the origin of the system, to the boundary of space-time. In the following, we consider that the topological signal at the origin represents a Dirac shock and is sent to infinity - i.e. to S^3, boundary of space-time .

In effect, the initial singularity can be interpreted as a causal signal giving rise to, at instant zero, a shock at the origin corresponding to a Dirac signal [26]. The shock at the origin, or Imp(t), distributed at the zero scale of space-time, must satisfy :

(i) $\forall t \in R, \; f(t) \geq 0$

$$\text{(ii)} \quad \mathrm{Imp}(t) = \begin{cases} 0 & \text{si} \quad t \neq 0 \\ \infty & \text{si} \quad t = 0 \end{cases} \tag{51}$$

(iii) $\int_R \mathrm{Imp}(t)dt = 1$

The unity signal at the singular origin $\mathbb{S}$ can then be considered as an ideal signal, of causal type.

Proposition 5.1 *The initial singularity, distribution of zero support, can be interpreted as a Dirac signal. It follows from this that the Fourier transform is a function that can be extended in the complex plane under the form of a holomorphic entire function, or bilateral Laplace transform.*

Proof It has been shown in [5] that the initial singularity can be interpreted as a singular zero size gravitational instanton, configuration built by E. Witten in [3]. The support of all associated distributions δ is therefore reduced to the singular point $\mathbb{S}$. The function δ associated with the curvature is therefore a Dirac distribution, such that, as established in [26], its Fourier transform is holomorphic and given by :

$$f[\delta] = \left\langle \delta(x), \, e^{-2i\pi\nu\beta} \right\rangle = [e^{-2i\pi\nu\beta}]_{\beta=0} \tag{52}$$

or, when the scale *ß* (or the time t) of the theory is zero :

$$f[\delta] = 1 \tag{53}$$

becomes a real and even distribution which, insofar as $f[1] = \delta$, must satisfy :

$$f[\delta] = \bar{f}[\delta] = 1 \tag{54}$$

The holomorphic function resulting from the Fourier transform of the δ function of zero support can equally be written in the form of a bilateral Laplace transform $f(ß)$:

$$f(ß_C) = = \int f(H) e^{-ß_c H} dH \tag{55}$$

where ß is a complex variable. By decomposing *ß* into real and imaginary parts, i.e. $ß_C = ß_r + iß_i$, we observe that, for $ß_r = 0$:

$$f(iß_i) = \int f(H) e^{-ißH} dH \tag{56}$$

which, up to the change of variable, is the Fourier transform of f(H). For a fixed $ß_r$, we have:

$$f(ß_r + iß_i) = \int e^{-ßH} f(H)\ e^{-ißH} dH \tag{57}$$

which is the FT of $f(H)\ e^{-ßH}$. Deriving under the sum the expression for $f(ß_C)$:

$$d/dß_C\ f(ß_C) = \int - ß_c f(H)\ e^{-ß_c H} dH \tag{58}$$

or, in general :

$$d^m/d(ß_C)^m\ f(ß_C) = = \int -(H)^m f(H) e^{-ß_c H} dH \tag{59}$$

and the summability abscissae of $-(H)^m f(H)$ are the same as those of f(H), such that $f(ß_C)$ is indefinitely derivable for all values of $ß_C$ where $f(ß_C)$ exists. $f(ß_r + iß_i)$ is therefore holomorphic in all the band of summability, i.e. for all values situated to the right of 0 in the complex plane formed by $ß_r > 0$ and $ß_i > 0$. The function $ß_C$ is therefore analytic in this domain of the complex plane. As the Dirac signal at the origin has for support the point $\mathbb{S}$, its FT describes therefore an impulsional response non-decreasing at infinity. **qed**

To understand the necessarily non-compact character of the impulsional response giving the evolution of the system, we complete the above proposition by the following corollary :

Corollary 5.2 *The Fourier transform of the singular distribution δ_S of punctual support describing the impulsional response of the space-time system cannot be of compact support.*

Demonstration Let's consider the Dirac signal $\delta_S \in E(\mathbb{R})$ and $f = \hat{\delta}_s$. The function f is analytic on $\mathbb{R}$, insofar as it is the trace of a holomorphic function on $\mathbb{C}$. We suppose that f is of compact support, hence f cancels itself out on an non-empty open set of $\mathbb{R}$. Thus, if a general analytical function on $\mathbb{R}$ is zero on an non-empty open set, then it is identically zero. It follows from this that f cannot be of compact support. **qed**

The above results suggest in fact that the (topological) interaction here considered is ergodic. As the Dirac signal at the origin has for support the point $\mathbb{S}$, its FT describes therefore an impulsion response non-decreasing at infinity. Indeed, (i) the behavior of $\prod^2$ is scale invariant and (ii) the zero size singular gravitational instanton characterizing, according to [4], the initial singularity, represents a critical point S_0 in the system formed by the pre- space-time manifold and $\mathcal{G}$, such that the correlation length of the system $\xi \to \infty$. From this viewpoint, the interaction $\mathbf{Int}_{top}$ is subject to the action of a renormalisation group G_n assuring the scale invariance of the system.

6. DISCUSSION

In this paper, taking the example of Foucault's experiment, we have suggested a new approach concerning two open problems :

(**i**) the problem of the invariance of the inertial interaction onto all points in space-time ;

(**ii**) the problem of the "instantaneous propagation" of inertia from one point to another in space-time - i.e. the Machian principle according to which the inertial reference frame defined by local physics coincides with the reference frame in which distant objects are at rest. It follows that the masses distributed most distantly in the universe determine the inertial behavior of local masses.

As a result of the hereabove research, our point of view is that the problem of inertia, well formulated in the context of Mach's principle, cannot be resolved by ordinary field theory. Indeed, as suggested in [5], beyond the Planck scale, quantum field theory must be analytically extended towards topological field theory. In such a context, the initial singularity can be viewed as a singular zero size gravitational instanton. This point like solution, endowed with an

Euclidean metric, corresponds to the origin of the topological quotient space $\sum_{top} = \dfrac{\mathbb{R}^{3,\,1} \oplus \mathbb{R}^4}{SO(3)}$ describing the q-deformation (quantum {Lorentzian $\oplus$ Euclidean} superposition) of the metric of the (pre)space-time between the scale zero and the Planck scale [5].

Our conclusion is then that onto zero scale, the topological charge Q = 1 of the singular zero size instanton might represent the source of the global topological inertial interaction. This could be tested by the angular invariance of the plane of oscillation of Foucault's pendulum or by the Thirring- Lense effect.

In a subsequent paper, we will consider that this result can be reinforced by the hypothesis of a correlation between the singular scale (t_0, x_0) and the macroscopic scale (t, x). We begin from the observation of the cosmological radiation at 2.7 ° K and draw from this the existence of a thermal Green function - or Euclidean Green function G_E - describing the correlation between the zero scale (ß = 0) and the macroscopic scale of space-time . Such an approach suggests the topological nature of the interaction between zero and macroscopic scales. Indeed, the correlation described by

$$G_E(t_0, x_0 \,;\, t, x) = \int \langle \phi(t_0, x_0)\phi(t, x) \rangle e^{-\xi} d\phi \qquad (60)$$

is such that all points P of space-time are correlated - by an Euclidean path - to the singular point $\mathbf{S_0}$. Insofar as the path between $\mathbf{S_0}$ and P is Euclidean - which is the case since the space-time system, considered at the non-zero temperature T = 2.7° K is the concern of statistical mechanics - the interaction between $\mathbf{S_0}$ and P depends only on the boundary conditions and is thus purely topological. Following this, we specify in this paper to come the notion of "Euclidean propagation" of inertia, according to the flow of weights of the algebra of states describing space-time in the region of the initial singularity. Starting from the von Neumann algebra $\sum$ describing the zero size instantonic state, we conjecture, according to [16] that the sole data from the algebra $\sum$ implies the existence of a "pseudo-dynamic" associated with $\sum$ and characterized by the flow of weights of $\sum$. Such a flow assures the propagation of the topological charge Q of the zero instanton. In agreement with the results of Connes [16], the homomorphism defining the canonical dynamic δ is such that δ : $\mathbb{R} \rightarrow \text{Out} \sum = \dfrac{\text{Aut}\,\Sigma}{\text{Int}\,\Sigma}$, this invariant having an intrinsic description in terms of flow of weights of $\sum$. We suggest then that this "intrinsic dynamic" is based on the semi-group of automorphisms :

$$\alpha_\tau(M) = e^{-ß \mathbb{D}^2} \, M \, e^{ß \mathbb{D}^2} \qquad (61)$$

corresponding to the evolution in imaginary time $i\,t$ of the state M - i.e. to the expansion of the space of states $\mathbb{E}$. This expansion of $\mathbb{E}$ is indexed by increasing values of ß, the radius of $\mathbb{E}$. As stated in [5], equ. (61) describes the flow of weights of the system and this flow being ergodic, *ß* is necessarily increasing in the interval $[0, \infty]$.

Then, we claim that a satisfactory test of the topological nature of the interaction existing between the zero size singular gravitational instanton and local systems should be provided by the angular invariance of the plane of oscillation of Foucault's pendulum. We have shown that Möb(3) is the conformal group Conf (S^3) of S^3. Conf(S^3) describes the scale invariance (i.e. conformal invariance) of the sphere identified here, following the inclusion $S^3 \subset SL(2, C)$, to physical space $\mathbb{R}^3$ compactification. We have then suggested that the flow of weights of the algebra M giving the modular flow $\alpha_t(M)$ on S^3 belongs to the class of similitude S^3.

Finally, an interesting consequence of the above approach is that it allows us to establish an explicit relation between the automorphism semi-group of the algebra of states A and the renormalisation semi-group of the theory [27]. Introduced by Wilson then by Kadanoff [28], the renormalisation program - in particular the renormalization group - allows us to encompass in a unique formalism the different scales of the theory. Observation shows that the behavior of Foucault's pendulum, notably the angular invariance of the plane of oscillation, is scale invariant. Everything occurs therefore as if the dynamic of $\mathcal{G}$ were subject to the action of a renormalization group $G_{\mathbf{R}}$, the group whose structure we define below. The calculation of the correlation length ξ between two localized variables at different points takes then the form, considering the variables ψ_n :

$$\langle \psi_n, \psi_m \rangle - \langle \psi_n \rangle \langle \psi_m \rangle = e^{-ß / ß_0} \qquad (62)$$

where the distance ß depends on the number of points on the lattice between n et m. At zero scale, the correlation length becomes, considering the coupling g_0 :

$$ß_0(g_0) \to \infty \qquad (63)$$

The correlation length is infinite, so that at zero scale, there exists an instantaneous interaction between the point $\mathbf{S_0}$ representing the initial singularity of space-time and the boundary at infinity of the 4-manifold, representing the 3-dimensional physical space.

Finally, all the hereabove results seem to confer a certain relevance (i) to the formulation of the so-called "topological Mach's Principle" and, more generally (ii) to the "Singularity Principle". Hopefully, this preliminary approach might open some new and interesting perspectives on the origin of space-time and on some other open questions.

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