Projections in several complex variables

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PROJECTEURS EN PLUSIEURS VARIABLES COMPLEXES

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This Thesis consists of an introduction and the following papers:

Paper I: On the singularities of the Szegö projection for $(0, q)$ forms

Paper II: On the singularities of the Bergman projection for $(0, q)$ forms

Introduction

The Bergman and Szegö projections are classical subjects in several complex variables and complex geometry. By Kohn's regularity theorem for the $\overline{\partial}$-Neumann problem (1963, [11]), the boundary behavior of the Bergman kernel is highly dependent on the Levi curvature of the boundary. The study of the boundary behavior of the Bergman kernel on domains with positive Levi curvature (strictly pseudoconvex domains) became an important topic in the field then. In 1965, L. Hörmander ([9]) determined the boundary behavior of the Bergman kernel. C. Fefferman (1974, [7]) established an asymptotic expansion at the diagonal of the Bergman kernel. More complete asymptotics of the Bergman kernel was obtained by Boutet de Monvel and Sjöstrand (1976, [6]). They also established an asymptotic expansion of the Szegö kernel on strongly pseudoconvex boundaries. All these developments concerned pseudoconvex domains. For the nonpseudoconvex domain, there are few results. R. Beals and P. Greiner (1988, [1]) proved that the Szegö projection is a Heisenberg pseudodifferential operator, under certain Levi curvature assumptions. Hörmander (2004, [10]) determined the boundary behavior of the Bergman kernel when the Levi form is negative definite by computing the leading term of the Bergman kernel on a spherical shell in $\mathbb{C}^n$.

Other developments recently concerned the Bergman kernel for a high power of a holomorphic line bundle. D. Catlin (1997, [4]) and S. Zelditch (1998, [16]) adapted a result of Boutet de Monvel-Sjöstrand for the asymptotics of the Szegö kernel on a strictly pseudoconvex boundary to establish the complete asymptotic expansion of the Bergman kernel for a high power of a holomorphic line bundle with positive curvature. Recently, a new proof of the existence of the complete asymptotic expansion was obtained by B. Berndtsson, R. Berman and J. Sjöstrand (2004, [3]). Without the positive curvature assumption, R. Berman and J. Sjöstrand (2005, [2]) obtained a full asymptotic expansion of the Bergman kernel for a high power of a line bundle when the curvature is non-degenerate. The approach of Berman and Sjöstrand builds on the heat equation method of Menikoff-Sjöstrand (1978, [15]). The expansion was obtained independently by X. Ma and G. Marinescu (2006, [14]) (without a phase function) by using a spectral gap estimate for the Hodge Laplacian.

Recently, Hörmander (2004, [10]) studied the Bergman projection for $(0,q)$ forms. In that paper (page 1306), Hörmander suggested: "A carefull microlocal analysis along the lines of Boutet de Monvel-Sjöstrand should give the asymptotic expansion of the Bergman projection for $(0,q)$ forms when the Levi form is non-degenerate."

The main goal for this thesis is to achieve Hörmander's wish-more precisely, to obtain an asymptotic expansion of the Bergman projection for $(0,q)$ forms. The first step of my research is to establish an asymptotic expansion of the Szegö projection for $(0,q)$ forms. Then, find a suitable operator defined on the boundary of domain which plays the same role as the Kohn Laplacian in the approach of Boutet de Monvel-Sjöstrand.

This thesis consists two parts. In the first paper, we completely study the heat equation method of Menikoff-Sjöstrand and apply it to the Kohn Laplacian defined on a compact orientable connected CR manifold. We then get the full asymptotic expansion of the Szegö projection for $(0,q)$ forms when the Levi form is non-degenerate. We also compute the leading term of the Szegö projection.

In the second paper, we introduce a new operator analogous to the Kohn Laplacian defined on the boundary of a domain and we apply the method of Menikoff-Sjöstrand to this operator. We obtain a description of a new Szegö projection up to smoothing operators. Finally, by using the Poisson operator, we get the full asymptotic expansion of the Bergman projection for $(0,q)$ forms when the Levi form is non-degenerate.

In order to describe the results more precisely, we introduce some notations. Let Ω be a C^∞ paracompact manifold equipped with a smooth density of integration. We let $T(\Omega)$ and $T^*(\Omega)$ denote the tangent bundle of Ω and the cotangent bundle of Ω respectively. The complexified tangent bundle of Ω and the complexified cotangent bundle of Ω will be denoted by $\mathbb{C}T(\Omega)$ and $\mathbb{C}T^*(\Omega)$ respectively. We write $\langle\ ,\ \rangle$ to denote the pointwise duality between $T(\Omega)$ and $T^*(\Omega)$. We extend $\langle\ ,\ \rangle$ bilinearly to $\mathbb{C}T(\Omega)\times\mathbb{C}T^*(\Omega)$.

Let E be a C^∞ vector bundle over Ω. The spaces of smooth sections of E over Ω and distribution sections of E over Ω will be denoted by $C^\infty(\Omega;\, E)$ and $\mathscr{D}'(\Omega;\, E)$ respectively. Let $\mathscr{E}'(\Omega;\, E)$ be the subspace of $\mathscr{D}'(\Omega;\, E)$ of sections with compact support in Ω and let $C_0^\infty(\Omega;\, E)=C^\infty(\Omega;\, E)\bigcap\mathscr{E}'(\Omega;\, E)$.

Let C, D be C^∞ vector bundles over Ω. Let

$$A: C_0^\infty(\Omega;\, C)\to\mathscr{D}'(\Omega;\, D).$$

From now on, we write $K_A(x,y)$ or $A(x,y)$ to denote the distribution kernel of A. Let

$$B: C_0^\infty(\Omega;\, C)\to\mathscr{D}'(\Omega;\, D).$$

We write

$$A\equiv B$$

if

$$K_A(x,y)=K_B(x,y)+F(x,y),$$

where $F(x,y)\in C^\infty(\Omega\times\Omega;\, \mathscr{L}(C_y,D_x))$.

0.1 The Szegö projection

For the precise definitions of some standard notations in CR geometry, see section 2 of paper I. Let $(X, \Lambda^{1,0}T(X))$ be a compact orientable connected CR manifold of dimension $2n-1$, $n \geq 2$. We take a smooth Hermitian metric $(\,|\,)$ on $\mathbb{C}T(X)$ so that $\Lambda^{1,0}T(X)$ is orthogonal to $\Lambda^{0,1}T(X)$ and $(u\,|\,v)$ is real if u, v are real tangent vectors, where $\Lambda^{0,1}T(X) = \overline{\Lambda^{1,0}T(X)}$. The Hermitian metric $(\,|\,)$ on $\mathbb{C}T(X)$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*(X)$ that we shall also denote by $(\,|\,)$. For $q \in \mathbb{N}$, let $\Lambda^{0,q}T^*(X)$ be the bundle of $(0,q)$ forms of X. The Hermitian metric $(\,|\,)$ on $\mathbb{C}T^*(X)$ induces a Hermitian metric on $\Lambda^{0,q}T^*(X)$ also denoted by $(\,|\,)$.

We take (dm) as the induced volume form on X and let $(\,|\,)$ be the inner product on $C^\infty(X; \Lambda^{0,q}T^*(X))$ defined by

$$(f\,|\,g) = \int_X (f(z)\,|\,g(z))(dm),\ f, g \in C^\infty(X; \Lambda^{0,q}T^*(X)).$$

Since X is orientable, there is a globally defined real 1 form $\omega_0(z)$ of length one which is pointwise orthogonal to $\Lambda^{1,0}T^*(X) \oplus \Lambda^{0,1}T^*(X)$, where

$$\Lambda^{1,0}T^*(X) = \overline{\Lambda^{0,1}T^*(X)}.$$

There is a real non-vanishing vector field Y which is pointwise orthogonal to $\Lambda^{1,0}T(X) \oplus \Lambda^{0,1}T(X)$. We take Y so that

$$\langle Y, \omega_0 \rangle = -1,\ \|Y\| = 1.$$

The Levi form $L_p(Z, \overline{W})$, $p \in X$, Z, $W \in \Lambda^{1,0}T_p(X)$, is the Hermitian quadratic form on $\Lambda^{1,0}T_p(X)$ defined as follows:

$$\begin{aligned}&\text{For any } Z,\ W \in \Lambda^{1,0}T_p(X), \text{ pick } \widetilde{Z},\ \widetilde{W} \in C^\infty(X; \Lambda^{1,0}T(X)) \text{ that satisfy}\\ &\widetilde{Z}(p) = Z,\ \widetilde{W}(p) = W. \text{ Then } L_p(Z, \overline{W}) = \frac{1}{2i}\left\langle [\widetilde{Z}\,, \overline{\widetilde{W}}](p)\,, \omega_0(p)\right\rangle.\end{aligned} \tag{0.1}$$

The eigenvalues of the Levi form at $p \in X$ are the eigenvalues of the Hermitian form L_p with respect to the inner product $(\,|\,)$ on $\Lambda^{1,0}T_p(X)$.

Let $\Box_b$ be the Kohn Laplacian on X and let $\Box_b^{(q)}$ denote the restriction to $(0,q)$ forms. Let

$$\pi^{(q)} : L^2(X; \Lambda^{0,q}T^*(X)) \to \operatorname{Ker}\Box_b^{(q)}$$

be the Szegö projection, i.e. the orthogonal projection onto the kernel of $\Box_b^{(q)}$. Let

$$K_{\pi^{(q)}}(x, y) \in \mathscr{D}'(X \times X; \mathscr{L}(\Lambda^{0,q}T_y^*(X), \Lambda^{0,q}T_x^*(X)))$$

be the distribution kernel of $\pi^{(q)}$ with respect to (dm). Formally,

$$(\pi^{(q)}u)(x) = \int K_{\pi^{(q)}}(x, y)u(y)dm(y),\quad u(y) \in C^\infty(X; \Lambda^{0,q}T^*(X)).$$

We recall

Definition 0.1. Given q, $0 \leq q \leq n-1$, the Levi form is said to satisfy condition $Y(q)$ at $p \in X$ if for any $|J| = q$, $J = (j_1, j_2, \ldots, j_q)$, $1 \leq j_1 < j_2 < \cdots < j_q \leq n-1$, we have

$$\left|\sum_{j \notin J} \lambda_j - \sum_{j \in J} \lambda_j\right| < \sum_{j=1}^{n-1} \left|\lambda_j\right|,$$

where λ_j, $j = 1, \ldots, (n-1)$, are the eigenvalues of L_p. If the Levi form is non-degenerate at p, then the condition is equivalent to $q \neq n_+$, n_-, where (n_-, n_+), $n_- + n_+ = n-1$, is the signature of L_p.

When $Y(q)$ holds at each point of X, Kohn (1972, [8]) proved that

$$K_{\pi^{(q)}}(x, y) \in C^\infty(X \times X; \mathscr{L}(\Lambda^{0,q} T^*(X), \Lambda^{0,q} T^*(X))).$$

When condition $Y(q)$ fails, one is interested in the Szegö projection on the level of $(0, q)$ forms. If the Levi form is positive definite at each point of X, Boutet de Monvel and Sjöstrand (1976, [6]) obtained the full asymptotic expansion for $K_{\pi^{(0)}}(x, y)$. If $Y(q)$ fails, $Y(q-1)$, $Y(q+1)$ hold and the Levi form is non-degenerate, Beals and Greiner (1988, [1]) proved that $\pi^{(q)}$ is a Heisenberg pseudodifferential operator. In particular, $\pi^{(q)}$ is a pseudodifferential operator of order 0 type $(\frac{1}{2}, \frac{1}{2})$.

The statement of the main results of paper I

Let Σ be the characteristic manifold of $\Box_b^{(q)}$. We have

$$\Sigma = \left\{(x, \xi) \in T^*(X) \setminus 0;\ \xi = \lambda\omega_0(x), \lambda \neq 0\right\}.$$

Put

$$\Sigma^+ = \left\{(x, \xi) \in T^*(X) \setminus 0;\ \xi = \lambda\omega_0(x), \lambda > 0\right\},$$
$$\Sigma^- = \left\{(x, \xi) \in T^*(X) \setminus 0;\ \xi = \lambda\omega_0(x), \lambda < 0\right\}.$$

We assume that the Levi form is non-degenerate at each point of X. Then the Levi form has constant signature (n_-, n_+), $n_- + n_+ = n-1$. We define

$$\hat{\Sigma} = \Sigma^+ \text{ if } n_+ = q \neq n_-,$$
$$\hat{\Sigma} = \Sigma^- \text{ if } n_- = q \neq n_+,$$
$$\hat{\Sigma} = \Sigma^+ \bigcup \Sigma^- \text{ if } n_+ = q = n_-.$$

The main result of the first paper is the following

Theorem 0.2. *Let $(X, \Lambda^{1,0} T(X))$ be a compact orientable connected CR manifold of dimension $2n-1$, $n \geq 2$, with a Hermitian metric $(\,|\,)$. We assume that the Levi form L is non-degenerate at each point of X. Then, the Levi form has constant* *signature* (n_-, n_+), $n_- + n_+ = n-1$. *Let* $q = n_-$ *or* n_+. *Suppose* $\Box_b^{(q)}$ *has closed range. Then* $\pi^{(q)}$ *is a well defined continuous operator*

$$\pi^{(q)} : H^s(X; \Lambda^{0,q} T^*(X)) \to H^s(X; \Lambda^{0,q} T^*(X)),$$

for all $s \in \mathbb{R}$, *and*

$$\mathrm{WF}'(K_{\pi^{(q)}}) = \mathrm{diag}(\hat{\Sigma} \times \hat{\Sigma}),$$

where H^s, $s \in \mathbb{R}$, *is the standard Sobolev space of order* s *and*

$$\mathrm{WF}'(K_{\pi^{(q)}}) = \left\{(x, \xi, y, \eta) \in T^*(X) \times T^*(X); (x, \xi, y, -\eta) \in \mathrm{WF}(K_{\pi^{(q)}})\right\}.$$

Here $\mathrm{WF}(K_{\pi^{(q)}})$ *is the wave front set of* $K_{\pi^{(q)}}$ *in the sense of Hörmander (see Appendix A of the second paper for a review). Moreover, we have*

$$\begin{aligned} &K_{\pi^{(q)}} = K_{\pi^+} \ \text{ if } \ n_+ = q \neq n_-, \\ &K_{\pi^{(q)}} = K_{\pi^-} \ \text{ if } \ n_- = q \neq n_+, \\ &K_{\pi^{(q)}} = K_{\pi^+} + K_{\pi^-} \ \text{ if } \ n_+ = q = n_-, \end{aligned}$$

where $K_{\pi^+}(x, y)$ *satisfies*

$$K_{\pi^+}(x, y) \equiv \int_0^\infty e^{i\phi_+(x,y)t} s_+(x, y, t) dt$$

with

$$s_+(x, y, t) \in S^{n-1}_{1,0}(X \times X \times]0, \infty[; \mathscr{L}(\Lambda^{0,q} T^*_y(X), \Lambda^{0,q} T^*_x(X))),$$

$$s_+(x, y, t) \sim \sum_{j=0}^{\infty} s^j_+(x, y) t^{n-1-j}$$

in the symbol space $S^{n-1}_{1,0}(X \times X \times]0, \infty[; \mathscr{L}(\Lambda^{0,q} T^*_y(X), \Lambda^{0,q} T^*_x(X)))$,

where $S^m_{1,0}$, $m \in \mathbb{R}$, *is the Hörmander symbol space (see Appendix A of the first paper for a review and references),*

$$s^j_+(x, y) \in C^\infty(X \times X; \mathscr{L}(\Lambda^{0,q} T^*_y(X), \Lambda^{0,q} T^*_x(X))), \ j = 0, 1, \ldots,$$

and

$$\phi_+(x, y) \in C^\infty(X \times X), \tag{0.2}$$

$$\phi_+(x, x) = 0, \tag{0.3}$$

$$\phi_+(x, y) \neq 0 \ \textit{ if } x \neq y, \tag{0.4}$$

$$\mathrm{Im}\, \phi_+(x, y) \geq 0, \tag{0.5}$$

$$d_x \phi_+ \neq 0, \ \ d_y \phi_+ \neq 0 \ \textit{ where } \mathrm{Im}\, \phi_+ = 0, \tag{0.6}$$

$$d_x \phi_+(x, y)|_{x=y} = \omega_0(x), \tag{0.7}$$

$$d_y \phi_+(x, y)|_{x=y} = -\omega_0(x), \tag{0.8}$$

$$\phi_+(x, y) = -\overline{\phi}_+(y, x). \tag{0.9}$$

Similarly,

$$K_{\pi^-}(x,y) \equiv \int_0^\infty e^{i\phi_-(x,y)t} s_-(x,y,t)dt$$

with

$$s_-(x,y,t) \in S^{n-1}_{1,0}(X \times X \times]0,\infty[; \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))),$$
$$s_-(x,y,t) \sim \sum_{j=0}^{\infty} s^j_-(x,y)t^{n-1-j}$$
in the symbol space $S^{n-1}_{1,0}(X \times X \times]0,\infty[; \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))),$

where

$$s^j_-(x,y) \in C^\infty(X \times X; \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))),\ j = 0,1,\ldots,$$

and $-\overline{\phi}_-(x,y)$ *satisfies (0.2)-(0.9).*

More properties of the phase $\phi_+(x,y)$ *will be given in Theorem 0.4 and Remark 0.5 below.*

Remark 0.3. We notice that if $Y(q-1)$ and $Y(q+1)$ hold then $\Box^{(q)}_b$ has closed range.

The tangential Hessian of $\phi_+(x,y)$

Until further notice, we assume that the Levi form is non-degenerate at each point of X. The phase $\phi_+(x,y)$ is not unique. we can replace $\phi_+(x,y)$ by

$$\widetilde{\phi}(x,y) = f(x,y)\phi_+(x,y), \tag{0.10}$$

where $f(x,y) \in C^\infty(X \times X)$ is real and $f(x,x) = 1$, $f(x,y) = f(y,x)$. Then $\widetilde{\phi}$ satisfies (0.2)-(0.9). We work with local coordinates $x = (x_1,\ldots,x_{2n-1})$ defined on an open set $\Omega \subset X$. We want to know the Hessian

$$(\phi_+)'' = \begin{bmatrix} (\phi_+)''_{xx} & (\phi_+)''_{xy} \\ (\phi_+)''_{yx} & (\phi_+)''_{yy} \end{bmatrix}$$

of ϕ_+ at $(p,p) \in X \times X$. Let $U,\ V \in \mathbb{C}T_p(X) \times \mathbb{C}T_p(X)$. From (0.10), we can check that

$$\left\langle \widetilde{\phi}''(p,p)U,V \right\rangle = \langle (\phi_+)''(p,p)U,V \rangle + \langle df(p,p),U \rangle \langle d\phi_+(p,p),V \rangle + \langle df(p,p),V \rangle \langle d\phi_+(p,p),U \rangle.$$

Thus, the Hessian $(\phi_+)''$ of ϕ_+ at (p,p) is only well-defined on the space

$$T_{(p,p)}H_+ = \left\{ W \in \mathbb{C}T_p(X) \times \mathbb{C}T_p(X);\ \langle d\phi_+(p,p),W \rangle = 0 \right\}.$$

In view of (0.7) and (0.8), we see that $T_{(p,p)}H_+$ is spanned by

$$(u,v),\ \ (Y(p),Y(p)),\ \ u,v\in\Lambda^{1,0}T_p(X)\oplus\Lambda^{0,1}T_p(X).$$

We define the tangential Hessian of $\phi_+(x,y)$ at (p,p) as the bilinear map:

$$\begin{aligned}&T_{(p,p)}H_+\times T_{(p,p)}H_+\to\mathbb{C},\\ &\qquad\qquad (U,V)\to\left\langle(\phi_+'')(p,p)U,V\right\rangle,\ \ U,V\in T_{(p,p)}H_+.\end{aligned}$$

In the section 9 of the first paper, we completely determined the tangential Hessian of $\phi_+(x,y)$ at (p,p). For the better understanding, we describe it in some special local coordinates. For a given point $p\in X$, let

$$U_1(x),\ldots,U_{n-1}(x)$$

be an orthonormal frame of $\Lambda^{1,0}T_x(X)$ varying smoothly with x in a neighborhood of p, for which the Levi form is diagonalized at p. We take local coordinates

$$x=(x_1,\ldots,x_{2n-1}),\ \ z_j=x_{2j-1}+ix_{2j},\ \ j=1,\ldots,n-1,$$

defined on some neighborhood of p such that

$$\omega_0(p)=\sqrt{2}dx_{2n-1},\ \ x(p)=0,$$

$$(\frac{\partial}{\partial x_j}(p)\,|\,\frac{\partial}{\partial x_k}(p))=2\delta_{j,k},\ \ j,k=1,\ldots,2n-1$$

and

$$U_j=\frac{\partial}{\partial z_j}-\frac{1}{\sqrt{2}}a_j(x)\frac{\partial}{\partial x_{2n-1}}+\sum_{s=1}^{2n-2}c_{j,s}(x)\frac{\partial}{\partial x_s},\ \ j=1,\ldots,n-1,$$

where

$$\frac{\partial}{\partial z_j}=\frac{1}{2}(\frac{\partial}{\partial x_{2j-1}}-i\frac{\partial}{\partial x_{2j}}),\ \ j=1,\ldots,n-1,$$

$a_j\in C^\infty$, $a_j(0)=0$, $j=1,\ldots,n-1$ and

$$c_{j,s}(x)\in C^\infty,\ \ c_{j,s}(0)=0,\ \ j=1,\ldots,n-1,\ \ s=1,\ldots,2n-2.$$

The integrability of $\Lambda^{1,0}T(X)$, i.e. $[U_j,U_k]\in\Lambda^{1,0}T(X)$ implies that

$$\frac{\partial a_j}{\partial z_k}(0)=\frac{\partial a_k}{\partial z_j}(0),\ \ j,k=1,\ldots,n-1. \tag{0.11}$$

Since the Levi form is diagonalized at p with respect to $U_j(p)$, $j=1,\ldots,n-1$, we can check that (see (0.1))

$$\frac{\partial a_j}{\partial\overline{z}_k}(0)-\frac{\partial\overline{a}_k}{\partial z_j}(0)=2i\lambda_j\delta_{j,k},\ \ j,k=1,\ldots,n-1, \tag{0.12}$$

where λ_j, $j=1,\ldots,n-1$, are the eigenvalues of L_p.

If $\hat\phi\in C^\infty(X\times X)$, $\hat\phi(p,p)=0$, $d_{x,y}\hat\phi(p,p)=d_{x,y}\phi_+(p,p)$ and the tangential Hessian of $\hat\phi(x,y)$ at (p,p) is the same as the tangential Hessian of $\phi_+(x,y)$ at (p,p), then

$$\hat\phi(x,y',x_{2n-1})-\phi_+(x,y',x_{2n-1})=O(\left|(x,y')\right|^3)$$

in some neighborhood of (p,p), where $y'=(y_1,\ldots,y_{2n-2})$. Moreover, we have the following

Theorem 0.4. *With the notations used before, in some neighborhood of $(p,p)\in X\times X$, we have*

$$\begin{aligned}
&\phi_+(x,y)=\sqrt{2}(x_{2n-1}-y_{2n-1})+i\sum_{j=1}^{n-1}\left|\lambda_j\right|\left|z_j-w_j\right|^2+\frac{1}{2}\sum_{j,k=1}^{n-1}\Big(\frac{\partial a_j}{\partial z_k}(0)(z_jz_k-w_jw_k)\\
&\quad+\frac{\partial\overline{a}_j}{\partial\overline{z}_k}(0)(\overline{z}_j\overline{z}_k-\overline{w}_j\overline{w}_k)+\frac{\partial a_j}{\partial\overline{z}_k}(0)(z_j\overline{z}_k-w_j\overline{w}_k)+\frac{\partial\overline{a}_j}{\partial z_k}(0)(\overline{z}_jz_k-\overline{w}_jw_k)\Big)\\
&\quad+\sum_{j=1}^{n-1}\Big(i\lambda_j(z_j\overline{w}_j-\overline{z}_jw_j)+\frac{\partial a_j}{\partial x_{2n-1}}(0)(z_jx_{2n-1}-w_jy_{2n-1})\\
&\quad+\frac{\partial\overline{a}_j}{\partial x_{2n-1}}(0)(\overline{z}_jx_{2n-1}-\overline{w}_jy_{2n-1})\Big)+\sqrt{2}(x_{2n-1}-y_{2n-1})f(x,y)+O(\left|(x,y)\right|^3),\\
&\quad f\in C^\infty,\ \ f(0,0)=0,\ \ f(x,y)=\overline{f}(y,x),\\
&\quad y=(y_1,\ldots,y_{2n-1}),\ \ w_j=y_{2j-1}+iy_{2j},\ \ j=1,\ldots,n-1,
\end{aligned}\tag{0.13}$$

where λ_j, $j=1,\ldots,n-1$, are the eigenvalues of L_p and ϕ_+ is as in Theorem 0.2.

Remark 0.5. We use the same notations as in Theorem 0.4. Since

$$\frac{\partial\phi_+}{\partial x_{2n-1}}(0,0)\neq0,$$

from the Malgrange preparation theorem (see Theorem B.6 of the first paper), we have

$$\phi_+(x,y)=g(x,y)(\sqrt{2}x_{2n-1}+h(x',y))$$

in some neighborhood of $(0,0)$, where g, $h\in C^\infty$, $g(0,0)=1$, $h(0,0)=0$ and $x'=(x_1,\ldots,x_{2n-2})$. Put

$$\hat\phi(x,y)=\sqrt{2}x_{2n-1}+h(x',y).$$

From the global theory of Fourier integral operators (see Proposition B.21 of the first paper), we see that $\phi_+(x,y)$ and $\hat\phi(x,y)$ are equivalent at $(p,\omega_0(p))$ in the sense of Melin-Sjöstrand (see Definition B.20 of the first paper). Since $\phi_+(x,y)=-\overline{\phi}_+(y,x)$, we can replace $\phi_+(x,y)$ by

$$\frac{\hat\phi(x,y)-\overline{\hat\phi}(y,x)}{2}.$$

Then $\phi_+(x,y)$ satisfies (0.2)-(0.9). Moreover, we can check that

$$\phi_+(x,y)=\sqrt{2}(x_{2n-1}-y_{2n-1})+i\sum_{j=1}^{n-1}\left|\lambda_j\right|\left|z_j-w_j\right|^2+\frac{1}{2}\sum_{j,k=1}^{n-1}\Big(\frac{\partial a_j}{\partial z_k}(0)(z_jz_k-w_jw_k)$$
$$+\frac{\partial\overline{a}_j}{\partial\overline{z}_k}(0)(\overline{z}_j\overline{z}_k-\overline{w}_j\overline{w}_k)+\frac{\partial a_j}{\partial\overline{z}_k}(0)(z_j\overline{z}_k-w_j\overline{w}_k)+\frac{\partial\overline{a}_j}{\partial z_k}(0)(\overline{z}_jz_k-\overline{w}_jw_k)\Big)$$
$$+\sum_{j=1}^{n-1}\Big(i\lambda_j(z_j\overline{w}_j-\overline{z}_jw_j)+\frac{\partial a_j}{\partial x_{2n-1}}(0)(z_jx_{2n-1}-w_jy_{2n-1})$$
$$+\frac{\partial\overline{a}_j}{\partial x_{2n-1}}(0)(\overline{z}_jx_{2n-1}-\overline{w}_jy_{2n-1})\Big)+O(\left|(x,y)\right|^3),\tag{0.14}$$

where λ_j, $j=1,\ldots,n-1$, are the eigenvalues of L_p. (Compare (0.14) with (0.13).)

The leading term of the Szegö projection

We have the following corollary of Theorem 0.2.

Corollary 0.6. *There exist smooth functions*

$$F_+,G_+,F_-,G_-\in C^\infty(X\times X;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X)))$$

such that

$$K_{\pi^+}=F_+(-i(\phi_+(x,y)+i0))^{-n}+G_+\log(-i(\phi_+(x,y)+i0)),$$

$$K_{\pi^-}=F_-(-i(\phi_-(x,y)+i0))^{-n}+G_-\log(-i(\phi_-(x,y)+i0)).$$

Moreover, we have

$$F_+=\sum_0^{n-1}(n-1-k)!s^k_+(x,y)(-i\phi_+(x,y))^k+f_+(x,y)(\phi_+(x,y))^n,$$
$$F_-=\sum_0^{n-1}(n-1-k)!s^k_-(x,y)(-i\phi_-(x,y))^k+f_-(x,y)(\phi_-(x,y))^n,$$
$$G_+\equiv\sum_0^{\infty}\frac{(-1)^{k+1}}{k!}s^{n+k}_+(x,y)(-i\phi_+(x,y))^k,$$
$$G_-\equiv\sum_0^{\infty}\frac{(-1)^{k+1}}{k!}s^{n+k}_-(x,y)(-i\phi_-(x,y))^k,\tag{0.15}$$

where

$$f_+(x,y),f_-(x,y)\in C^\infty(X\times X;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))).$$

If $w \in \Lambda^{0,1} T^*_z(X)$, let

$$w^{\wedge,*} : \Lambda^{0,q+1} T^*_z(X) \to \Lambda^{0,q} T^*_z(X),\ q \geq 0,$$

be the adjoint of left exterior multiplication

$$w^{\wedge} : \Lambda^{0,q} T^*_z(X) \to \Lambda^{0,q+1} T^*_z(X).$$

That is,

$$(w^{\wedge} u \mid v) = (u \mid w^{\wedge,*} v), \tag{0.16}$$

for all $u \in \Lambda^{0,q} T^*_z(X)$, $v \in \Lambda^{0,q+1} T^*_z(X)$. Notice that $w^{\wedge,*}$ depends anti-linearly on w.

In section 9 of the first paper, we compute $F_+(x,x)$ and $F_-(x,x)$.

Proposition 0.7. *For a given point $x_0 \in X$, let*

$$U_1(x), \ldots, U_{n-1}(x)$$

be an orthonormal frame of $\Lambda^{1,0} T_x(X)$, for which the Levi form is diagonalized at x_0. Let $e_j(x)$, $j = 1, \ldots, n-1$, denote the basis of $\Lambda^{0,1} T^*_x(X)$, which is dual to $\overline{U}_j(x)$, $j = 1, \ldots, n-1$. Let $\lambda_j(x)$, $j = 1, \ldots, n-1$, be the eigenvalues of the Levi form L_x. We assume that $q = n_+$ and that

$$\lambda_j(x_0) > 0 \ \ \text{if} \ \ 1 \leq j \leq n_+.$$

Then

$$F_+(x_0, x_0) = (n-1)! \frac{1}{2} |\lambda_1(x_0)| \cdots |\lambda_{n-1}(x_0)| \pi^{-n} \prod_{j=1}^{j=n_+} e_j(x_0)^{\wedge} e_j(x_0)^{\wedge,*}.$$

Proposition 0.8. *For a given point $x_0 \in X$, let*

$$U_1(x), \ldots, U_{n-1}(x)$$

be an orthonormal frame of $\Lambda^{1,0} T_x(X)$, for which the Levi form is diagonalized at x_0. Let $e_j(x)$, $j = 1, \ldots, n-1$, denote the basis of $\Lambda^{0,1} T^*_x(X)$, which is dual to $\overline{U}_j(x)$, $j = 1, \ldots, n-1$. Let $\lambda_j(x)$, $j = 1, \ldots, n-1$ be the eigenvalues of the Levi form L_x. We assume that $q = n_-$ and that

$$\lambda_j(x_0) < 0 \ \ \text{if} \ \ 1 \leq j \leq n_-.$$

Then

$$F_-(x_0, x_0) = (n-1)! \frac{1}{2} |\lambda_1(x_0)| \cdots |\lambda_{n-1}(x_0)| \pi^{-n} \prod_{j=1}^{j=n_-} e_j(x_0)^{\wedge} e_j(x_0)^{\wedge,*}.$$

0.2 The Bergman projection

For the precise definitions of some standard notations in complex geometry and several complex variables, see section 2 of paper II. In this section, we assume that all manifolds are paracompact. Let M be a relatively compact open subset with C^∞ boundary Γ of a complex manifold M' of dimension n with a smooth Hermitian metric $(\,|\,)$ on its holomorphic tangent bundle.

Let F be a C^∞ vector bundle over M'. Let $C^\infty(\overline{M};F)$, $\mathscr{D}'(\overline{M};F)$ and $H^s(\overline{M};F)$ denote the spaces of restrictions to M of elements in $C^\infty(M';F)$, $\mathscr{D}'(M';F)$ and $H^s(M';F)$ respectively.

Let $\Lambda^{1,0}T(M')$ and $\Lambda^{0,1}T(M')$ be the holomorphic tangent bundle of M' and the anti-holomorphic tangent boundle of M' respectively. We extend the Hermitian metric $(\,|\,)$ to $\mathbb{C}T(M')$ in a natural way by requiring $\Lambda^{1,0}T(M')$ to be orthogonal to $\Lambda^{0,1}T(M')$ and satisfy

$$\overline{(u\,|\,v)}=(\overline{u}\,|\,\overline{v}),\ u,v\in\Lambda^{0,1}T(M').$$

For $p,q\in\mathbb{N}$, let $\Lambda^{p,q}T^*(M')$ be the bundle of (p,q) forms of M'. The Hermitian metric $(\,|\,)$ on $\mathbb{C}T(M')$ induces a Hermitian metric on $\Lambda^{p,q}T^*(M')$ also denoted by $(\,|\,)$. Let (dM') be the induced volume form on M' and let $(\,|\,)_M$ be the inner product on $C^\infty(\overline{M};\Lambda^{p,q}T^*(M'))$ defined by

$$(f\,|\,h)_M=\int_M(f\,|\,h)(dM'),\ \ f,h\in C^\infty(\overline{M};\Lambda^{p,q}T^*(M')). \tag{0.17}$$

Let $r\in C^\infty(M')$ be a defining function of Γ such that r is real, $r=0$ on Γ, $r<0$ on M and $dr\neq0$ near Γ. From now on, we take a defining function r so that

$$\|dr\|=1\ \text{ on }\ \Gamma.$$

Put

$$\omega_0=J^t(dr). \tag{0.18}$$

Here J^t is the complex structure map for the cotangent bundle.

Let $\Lambda^{1,0}T(\Gamma)$ be the holomorphic tangent bundle of Γ. The Levi form $L_p(Z,\overline{W})$, $p\in X$, Z, $W\in\Lambda^{1,0}T_p(\Gamma)$, is the Hermitian quadratic form on $\Lambda^{1,0}T_p(\Gamma)$ defined as in (0.1).

For the convenience of the reader, we review the definition of the Kohn Laplacian on $(0,q)$ forms. Let

$$\overline{\partial}:C^\infty(M';\Lambda^{0,q}T^*(M'))\to C^\infty(M';\Lambda^{0,q+1}T^*(M'))$$

be the part of the exterior differential operator which maps forms of type $(0,q)$ to forms of type $(0,q+1)$ and we denote by

$$\overline{\partial}_f^{\,*}:C^\infty(M';\Lambda^{0,q+1}T^*(M'))\to C^\infty(M';\Lambda^{0,q}T^*(M'))$$

the formal adjoint of $\overline{\partial}$. That is

$$(\overline{\partial} f \mid h)_{M'} = (f \mid \overline{\partial}_f^* h)_{M'},\ f \in C_0^\infty(M'; \Lambda^{0,q} T^*(M')),\ h \in C^\infty(M'; \Lambda^{0,q+1} T^*(M')),$$

where $(\ |\ )_{M'}$ is defined by

$$(g \mid k)_{M'} = \int_{M'} (g \mid k)(dM'),\ \ g, k \in C_0^\infty(M'; \Lambda^{0,q} T^*(M')).$$

We shall also use the notation $\overline{\partial}$ for the closure in L^2 of the $\overline{\partial}$ operator, initially defined on $C^\infty(\overline{M}; \Lambda^{0,q} T^*(M'))$ and $\overline{\partial}^*$ for the Hilbert space adjoint of $\overline{\partial}$. The domain of $\overline{\partial}^*$ consists of all $f \in L^2(M; \Lambda^{0,q+1} T^*(M'))$ such that for some constant $c > 0$,

$$\left|(f \mid \overline{\partial} g)_M\right| \leq c \left\|g\right\|,\ \text{for all } g \in C^\infty(\overline{M}; \Lambda^{0,q} T^*(M')).$$

For such an f,

$$g \to (f \mid \overline{\partial} g)_M$$

extends to a bounded anti-linear functional on $L^2(M; \Lambda^{0,q} T^*(M'))$ so

$$(f \mid \overline{\partial} g)_M = (\widetilde{f} \mid g)_M$$

for some $\widetilde{f} \in L^2(M; \Lambda^{0,q} T^*(M'))$. We have $\overline{\partial}^* f = \widetilde{f}$. The $\overline{\partial}$-Neumann Laplacian on $(0, q)$ forms is then the operator in the space $L^2(M; \Lambda^{0,q} T^*(M'))$

$$\Box^{(q)} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^* \overline{\partial}. \tag{0.19}$$

We have

$$\text{Dom}\,\Box^{(q)} = \{u \in L^2(M; \Lambda^{0,q} T^*(M'));\ u \in \text{Dom}\,\overline{\partial}^* \bigcap \text{Dom}\,\overline{\partial},$$
$$\overline{\partial}^* u \in \text{Dom}\,\overline{\partial}, \overline{\partial} u \in \text{Dom}\,\overline{\partial}^*\}.$$

As before, if $w \in \Lambda^{0,1} T_z^*(M')$, let

$$w^{\wedge,*} : \Lambda^{0,q+1} T_z^*(M') \to \Lambda^{0,q} T_z^*(M') \tag{0.20}$$

be the adjoint of left exterior multiplication $w^\wedge$. (See (0.16).) Let γ denote the operator of restriction to the boundary Γ. Put

$$D^{(q)} = \text{Dom}\,\Box^{(q)} \bigcap C^\infty(\overline{M}\,; \Lambda^{0,q} T^*(M')).$$

We have

$$D^{(q)} = \left\{u \in C^\infty(\overline{M}; \Lambda^{0,q+1} T^*(M'));\ \gamma(\overline{\partial} r)^{\wedge,*} u = 0,\ \gamma(\overline{\partial} r)^{\wedge,*} \overline{\partial} u = 0\right\}. \tag{0.21}$$

The boundary conditions

$$\gamma(\overline{\partial} r)^{\wedge,*}u=0,\ \gamma(\overline{\partial} r)^{\wedge,*}\overline{\partial} u=0,\ u\in C^\infty(\overline{M}\,,\Lambda^{0,q}T^*(M'))$$

are called $\overline{\partial}$-Neumann boundary conditions.

Let

$$\Pi^{(q)}:L^2(M;\Lambda^{0,q}T^*(M'))\to \mathrm{Ker}\,\Box^{(q)}$$

be the Bergman projection, i.e. the orthogonal projection onto the kernel of $\Box^{(q)}$. Let

$$K_{\Pi^{(q)}}(z,w)\in\mathscr{D}'(M\times M;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M')))$$

be the distribution kernel of $\Pi^{(q)}$. Formally,

$$(\Pi^{(q)}u)(z)=\int_M K_{\Pi^{(q)}}(z,w)u(w)dM'(w),\ \ u(w)\in C^\infty_0(M;\Lambda^{0,q}T^*(M')).$$

We recall

Definition 0.9. Given q, $0\le q\le n-1$. The Levi form is said to satisfy condition $Z(q)$ at $p\in\Gamma$ if it has at least $n-q$ positive or at least $q+1$ negative eigenvalues. If the Levi form is non-degenerate at $p\in\Gamma$, let (n_-,n_+), $n_-+n_+=n-1$, be the signature. Then $Z(q)$ holds at p if and only if $q\neq n_-$.

When $Z(q)$ holds at each point of Γ, Kohn (1963, [11]) proved that

$$K_{\Pi^{(q)}}(z.w)\in C^\infty(\overline{M}\times\overline{M};\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))).$$

When condition $Z(q)$ fails, one is interested in the Bergman projection on the level of $(0,q)$ forms. If the Levi form is positive definite at each point of Γ, Kerzman (1971, [13]) proved that

$$K_{\Pi^{(0)}}(z,w)\in C^\infty(\overline{M}\times\overline{M}\setminus\mathrm{diag}(\Gamma\times\Gamma)).$$

A complete asymptotic expansion of $K_{\Pi^{(0)}}(z,z)$ at the boundary was given by Fefferman (1974, [7]): There are functions $a,\,b\in C^\infty(\overline{M})$ such that

$$K_{\Pi^{(0)}}(z,z)=\frac{a(z)}{r(z)^{n+1}}+b(z)\log(-r(z)).$$

Here $a(z)$ is given for $z\in\Gamma$ by Hörmander (1965, [9]). Complete asymptotics of $K_{\Pi^{(0)}}(z,w)$ when z and w approach the same boundary point in an arbitrary way was obtained by Boutet de Monvel and Sjöstrand (1976, [6]).

Boundary reduction

The Hermitian metric $(\,|\,)$ on $\mathbb{C}T(M')$ induces a Hermitian metric $(\,|\,)$ on $\mathbb{C}T(\Gamma)$. For $z\in\Gamma$, we identify $\mathbb{C}T^*_z(\Gamma)$ with the space

$$\left\{u\in\mathbb{C}T^*_z(M');\,(u\mid dr)=0\right\}. \tag{0.22}$$

For $q\in\mathbb{N}$, the bundle of boundary $(0,q)$ forms is the vector bundle $\Lambda^{0,q}T^*(\Gamma)$ with fiber

$$\Lambda^{0,q}T^*_z(\Gamma)=\left\{u\in\Lambda^{0,q}T^*_z(M');\,(u\mid\overline{\partial}r(z)\wedge g)=0,\ \ \forall g\in\Lambda^{0,q-1}T^*_z(M')\right\} \tag{0.23}$$

at $z\in\Gamma$. In view of (0.21), we see that $u\in D^{(q)}$ if and only if

$$\gamma u\in C^\infty(\Gamma;\Lambda^{0,q}T^*(\Gamma)) \tag{0.24}$$

and

$$\gamma\overline{\partial}u\in C^\infty(\Gamma;\Lambda^{0,q+1}T^*(\Gamma)). \tag{0.25}$$

We take $(d\Gamma)$ as the induced volume form on Γ and let $(\,|\,)_\Gamma$ be the inner product on $C^\infty(\Gamma;\Lambda^{0,q}T^*(M'))$ defined by

$$(f\mid g)_\Gamma=\int_\Gamma(f\mid g)d\Gamma,\ \ f,g\in C^\infty(\Gamma;\Lambda^{0,q}T^*(M')). \tag{0.26}$$

We assume that the Levi form is positive definite at each point of Γ and $q=0$. As before, let $\pi^{(0)}$ be the Szegö projection for $(0,0)$ forms on Γ. Let P be the Poisson operator for functions. That is, if $u\in C^\infty(\Gamma)$, then

$$Pu\in C^\infty(\overline{M}),\ \ \overline{\partial_f}^{*}\overline{\partial}Pu=0$$

and

$$\gamma Pu=u.$$

It is well-known (see ([6])) that

$$\gamma\overline{\partial}P\pi^{(0)}\equiv0. \tag{0.27}$$

From this, it is not difficult to see that

$$\Pi^{(0)}=P\pi^{(0)}(P^*P)^{-1}P^*+F, \tag{0.28}$$

where

$$P^*:\mathscr{E}'(\overline{M})\to\mathscr{D}'(\Gamma)$$

is the operator defined by

$$(P^*u\mid v)_\Gamma=(u\mid Pv)_M,\ u\in\mathscr{E}'(\overline{M}),\ v\in C^\infty(\Gamma)$$

and

$$F(z,w) \in C^\infty(\overline{M} \times \overline{M}).$$

From (0.28), we can obtain the full asymptotic expansion of the Bergman projection for functions.

In the case of $(0,q)$ forms, in general, the relation (0.27) doesn't hold. This makes it difficult to obtain a full asymptotic expansion of the Bergman projection directly from the Szegö projection. Instead, we introduce a new operator $\Box_\beta^{(q)}$ and obtain a modified Szegö kernel such that (0.27) holds.

The operator $\Box_\beta^{(q)}$

Let

$$\Box_f^{(q)} = \overline{\partial}\,\overline{\partial_f}^* + \overline{\partial_f}^*\overline{\partial} : C^\infty(M'; \Lambda^{0,q}T^*(M')) \to C^\infty(M'; \Lambda^{0,q}T^*(M')) \tag{0.29}$$

denote the complex Laplace-Beltrami operator on $(0,q)$ forms and denote by $\sigma_{\Box_f^{(q)}}$ the principal symbol of $\Box_f^{(q)}$. Let us consider the map:

$$\begin{aligned} F^{(q)} : H^2(\overline{M}; \Lambda^{0,q}T^*(M')) &\to H^0(\overline{M}; \Lambda^{0,q}T^*(M')) \oplus H^{\frac{3}{2}}(\Gamma; \Lambda^{0,q}T^*(M')), \\ u &\to (\Box_f^{(q)}u, \gamma u). \end{aligned} \tag{0.30}$$

Given q, $0 \le q \le n-1$, we assume that

Assumption 0.10. *$F^{(k)}$ is injective, $q-1 \le k \le q+1$.*

Thus, the Poisson operator for $\Box_f^{(k)}$, $q-1 \le k \le q+1$, is well-defined. (See section 4 of the second paper.) If M' is Kähler, then $F^{(q)}$ is injective for any q, $0 \le q \le n$. (See section 9 of the second paper for the definition and details.)

Let

$$P : C^\infty(\Gamma; \Lambda^{0,q}T^*(M')) \to C^\infty(\overline{M}; \Lambda^{0,q}T^*(M')) \tag{0.31}$$

be the Poisson operator for $\Box_f^{(q)}$. It is well-known (see page 29 of Boutet de Monvel [5]) that P extends continuously

$$P : H^s(\Gamma; \Lambda^{0,q}T^*(M')) \to H^{s+\frac{1}{2}}(\overline{M}; \Lambda^{0,q}T^*(M')),\ \forall\, s \in \mathbb{R}.$$

Let

$$P^* : \mathscr{E}'(\overline{M}; \Lambda^{0,q}T^*(M')) \to \mathscr{D}'(\Gamma; \Lambda^{0,q}T^*(M'))$$

be the operator defined by

$$(P^*u \mid v)_\Gamma = (u \mid Pv)_M,\ u \in \mathscr{E}'(\overline{M}; \Lambda^{0,q}T^*(M')),\ v \in C^\infty(\Gamma; \Lambda^{0,q}T^*(M')).$$

It is well-known (see page 30 of [5]) that P^* is continuous:

$$P^* : L^2(M; \Lambda^{0,q}T^*(M')) \to H^{\frac{1}{2}}(\Gamma; \Lambda^{0,q}T^*(M'))$$

and

$$P^*: C^\infty(\overline{M}; \Lambda^{0,q} T^*(M')) \to C^\infty(\Gamma; \Lambda^{0,q} T^*(M')).$$

We use the inner product $[\,|\,]$ on $H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M'))$ defined as follows:

$$[u \mid v] = (Pu \mid Pv)_M,$$

where $u, v \in H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M'))$. We consider $(\overline{\partial} r)^{\wedge,*}$ as an operator

$$(\overline{\partial} r)^{\wedge,*}: H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M')) \to H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q-1} T^*(M')).$$

Note that $(\overline{\partial} r)^{\wedge,*}$ is the pointwise adjoint of $\overline{\partial} r$ with respect to $(\,|\,)$. Let

$$T: H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M')) \to \operatorname{Ker}(\overline{\partial} r)^{\wedge,*} \tag{0.32}$$

be the orthogonal projection onto $\operatorname{Ker}(\overline{\partial} r)^{\wedge,*}$ with respect to $[\,|\,]$. That is, if $u \in H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M'))$, then

$$(\overline{\partial} r)^{\wedge,*} Tu = 0$$

and

$$[(I - T)u \mid g] = 0, \ \ \forall\, g \in \operatorname{Ker}(\overline{\partial} r)^{\wedge,*}.$$

In section 4 of the second paper, we will show that T is a classical pseudodifferential operator of order 0 with principal symbol

$$2(\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^{\wedge}.$$

Put

$$\overline{\partial_\beta} = T\gamma\overline{\partial} P: C^\infty(\Gamma; \Lambda^{0,q} T^*(\Gamma)) \to C^\infty(\Gamma; \Lambda^{0,q+1} T^*(\Gamma)). \tag{0.33}$$

$\overline{\partial_\beta}$ is a classical pseudodifferential operator of order one from boundary $(0,q)$ forms to boundary $(0,q+1)$ forms,

$$\overline{\partial_\beta} = \overline{\partial_b} + \text{lower order terms}, \tag{0.34}$$

where $\overline{\partial_b}$ is the tangential Cauchy-Riemann operator and

$$(\overline{\partial_\beta})^2 = 0.$$

Let

$$\overline{\partial_\beta}^\dagger: C^\infty(\Gamma; \Lambda^{0,q+1} T^*(\Gamma)) \to C^\infty(\Gamma; \Lambda^{0,q} T^*(\Gamma)),$$

be the formal adjoint of $\overline{\partial_\beta}$ with respect to $[\,|\,]$. $\overline{\partial_\beta}^\dagger$ is a classical pseudodifferential operator of order one from boundary $(0,q+1)$ forms to boundary $(0,q)$ forms and

$$\overline{\partial_\beta}^\dagger = \gamma\overline{\partial_f}^* P.$$

Put

$$\Box_\beta^{(q)} = \overline{\partial_\beta}\,\overline{\partial_\beta}^\dagger + \overline{\partial_\beta}^\dagger\overline{\partial_\beta}: C^\infty(\Gamma; \Lambda^{0,q} T^*(\Gamma)) \to C^\infty(\Gamma; \Lambda^{0,q} T^*(\Gamma)).$$

We assume that the Levi form is non-degenerate. Put

$$\Sigma^-(q) = \{(x, \lambda\omega_0(x)) \in T^*(\Gamma);\ \lambda < 0 \ \text{ and } Z(q) \text{ fails at } x\},$$

$$\Sigma^+(q) = \{(x, \lambda\omega_0(x)) \in T^*(\Gamma);\ \lambda > 0 \ \text{ and } Z(q) \text{ fails at } x\}.$$

We apply the method of Menikoff-Sjöstrand to $\Box_\beta^{(q)}$ and obtain operators

$$A \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\Gamma; \Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma)),\quad B_-, B_+ \in L^{0}_{\frac{1}{2},\frac{1}{2}}(\Gamma; \Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))$$

such that

$$\begin{aligned}
&\Box_\beta^{(q)} A + B_- + B_+ \equiv I,\\
&\mathrm{WF}'(K_{B_-}) = \mathrm{diag}(\Sigma^-(q) \times \Sigma^-(q)),\\
&\mathrm{WF}'(K_{B_+}) = \mathrm{diag}(\Sigma^+(n-1-q) \times \Sigma^+(n-1-q)),\\
&\overline{\partial_\beta} B_- \equiv 0,\ \overline{\partial_\beta}^\dagger B_- \equiv 0,\\
&B_- \equiv B_-^\dagger \equiv B_-^2,
\end{aligned}$$

where $L^m_{\frac{1}{2},\frac{1}{2}}$ is the space of pseudodifferential operators of order m type $(\frac{1}{2}, \frac{1}{2})$, $B_-^\dagger$ is the formal adjoint of B_- with respect to $[\,|\,]$. We prove that

$$\gamma\overline{\partial} P B_- \equiv 0. \tag{0.35}$$

(See section 7 of the second paper.) From this, we deduce the generalization of (0.28)

$$\Pi^{(q)} = P B_- T (P^* P)^{-1} P^* + F, \tag{0.36}$$

where

$$P^* : \mathscr{E}'(\overline{M}; \Lambda^{0,q}T^*(M')) \to \mathscr{D}'(\Gamma; \Lambda^{0,q}T^*(M'))$$

is the operator defined by

$$(P^* u \mid v)_\Gamma = (u \mid P v)_M,\ u \in \mathscr{E}'(\overline{M}; \Lambda^{0,q}T^*(M')),\ v \in C^\infty(\Gamma; \Lambda^{0,q}T^*(M'))$$

and

$$F(z, w) \in C^\infty(\overline{M} \times \overline{M}; \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M'))).$$

The statement of the main results of paper II

We recall the Hörmander symbol spaces

Definition 0.11. Let $m \in \mathbb{R}$. Let U be an open set in $M' \times M'$.

$$S^m_{1,0}(U \times]0, \infty[; \mathscr{L}(\Lambda^{0,q}T^*_y(M'), \Lambda^{0,q}T^*_x(M')))$$

is the space of all $a(x,y,t)\in C^\infty(U\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_x(M')))$ such that for all compact sets $K\subset U$ and all $\alpha\in\mathbb{N}^{2n}$, $\beta\in\mathbb{N}^{2n}$, $\gamma\in\mathbb{N}$, there is a constant $c>0$ such that

$$\left|\partial_x^\alpha\partial_y^\beta\partial_t^\gamma a(x,y,t)\right|\leq c(1+|t|)^{m-|\gamma|},\ (x,y,t)\in K\times]0,\infty[.$$

$S^m_{1,0}$ is called the space of symbols of order m type $(1,0)$. We write $S^{-\infty}_{1,0}=\bigcap S^m_{1,0}$.

Let $S^m_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M')))$ denote the space of restrictions to $U\bigcap(M\times M)\times]0,\infty[$ of elements in

$$S^m_{1,0}(U\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))).$$

Let

$$a_j\in S^{m_j}_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))),\ j=0,1,2,\ldots,$$

with $m_j\searrow-\infty$, $j\to\infty$. Then there exists

$$a\in S^{m_0}_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M')))$$

such that

$$a-\sum_{0\leq j<k}a_j\in S^{m_k}_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))),$$

for every $k\in\mathbb{N}$.

If a and a_j have the properties above, we write

$$a\sim\sum_{j=0}^\infty a_j\ \text{ in the space }\ S^{m_0}_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times[0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))).$$

Let

$$C,D:C^\infty_0(M;\Lambda^{0,q}T^*(M'))\to\mathscr{D}'(M;\Lambda^{0,q}T^*(M'))$$

with distribution kernels

$$K_C(z,w),K_D(z,w)\in\mathscr{D}'(M\times M;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))).$$

We write

$$C\equiv D\ \operatorname{mod}\ C^\infty(U\bigcap(\overline{M}\times\overline{M}))$$

if

$$K_C(z,w)=K_D(z,w)+F(z,w),$$

where

$$F(z,w)\in C^\infty(U\bigcap(\overline{M}\times\overline{M});\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M')))$$

and U is an open set in $M'\times M'$.

The main result of the second paper is the following

Theorem 0.12. *Let M be a relatively compact open subset with C^∞ boundary Γ of a complex analytic manifold M' of dimension n. We assume that the Levi form is non-degenerate at each point of Γ. Let q, $0 \le q \le n-1$. Suppose that $Z(q)$ fails at some point of Γ and that $Z(q-1)$ and $Z(q+1)$ hold at each point of Γ. Let*

$$\Gamma_q = \{z \in \Gamma; Z(q) \text{ fails at } z\} \tag{0.37}$$

so that Γ_q is a union of connected components of Γ. Then

$$K_{\Pi^{(q)}}(z,w) \in C^\infty(\overline{M} \times \overline{M} \setminus \mathrm{diag}(\Gamma_q \times \Gamma_q); \mathscr{L}(\Lambda^{0,q} T^*_w(M'), \Lambda^{0,q} T^*_z(M'))).$$

Moreover, in a neighborhood U of $\mathrm{diag}(\Gamma_q \times \Gamma_q)$, $K_{\Pi^{(q)}}(z,w)$ satisfies

$$K_{\Pi^{(q)}}(z,w) \equiv \int_0^\infty e^{i\phi(z,w)t} b(z,w,t)dt \mod C^\infty(U \bigcap (\overline{M} \times \overline{M})) \tag{0.38}$$

(for the precise meaning of the oscillatory integral $\int_0^\infty e^{i\phi(z,w)t} b(z,w,t)dt$, see Remark 1.4 of the second paper) with

$$b(z,w,t) \in S^n_{1,0}(U \bigcap (\overline{M} \times \overline{M}) \times]0,\infty[; \mathscr{L}(\Lambda^{0,q} T^*_w(M'), \Lambda^{0,q} T^*_z(M'))),$$

$$b(z,w,t) \sim \sum_{j=0}^\infty b_j(z,w) t^{n-j}$$

in the space $S^n_{1,0}(U \bigcap (\overline{M} \times \overline{M}) \times]0,\infty[; \mathscr{L}(\Lambda^{0,q} T^*_w(M'), \Lambda^{0,q} T^*_z(M')))$,

$$b_0(z,z) \neq 0, \ z \in \Gamma_q,$$

where

$$b_j(z,w) \in C^\infty(U \bigcap (\overline{M} \times \overline{M}); \mathscr{L}(\Lambda^{0,q} T^*_w(M'), \Lambda^{0,q} T^*_z(M'))), \ j = 0,1,\ldots,$$

and

$$\phi(z,w) \in C^\infty(U \bigcap (\overline{M} \times \overline{M})), \tag{0.39}$$

$$\phi(z,z) = 0, \ \ z \in \Gamma_q, \tag{0.40}$$

$$\phi(z,w) \neq 0 \ \text{ if } (z,w) \notin \mathrm{diag}(\Gamma_q \times \Gamma_q), \tag{0.41}$$

$$\mathrm{Im}\,\phi \ge 0, \tag{0.42}$$

$$\mathrm{Im}\,\phi(z,w) > 0 \ \text{ if } (z,w) \notin \Gamma \times \Gamma, \tag{0.43}$$

$$\phi(z,w) = -\overline{\phi}(w,z). \tag{0.44}$$

For $p \in \Gamma_q$, we have

$$\begin{gathered}\sigma_{\Box^{(q)}_f}(z, d_z\phi(z,w)) \text{ vanishes to infinite order at } z = p,\\ (z,w) \text{ is in some neighborhood of } (p,p) \text{ in } M'.\end{gathered} \tag{0.45}$$

For $z = w$, $z \in \Gamma_q$, we have

$$d_z\phi = -\omega_0 - i\,dr,$$

$$d_w\phi = \omega_0 - i\,dr.$$

Moreover, we have

$$\phi(z,w) = \phi_-(z,w) \ \ if\ z, w \in \Gamma_q,$$

where $\phi_-(z,w) \in C^\infty(\Gamma_q \times \Gamma_q)$ is the phase appearing in the description of the Szegö projection. See Theorem 0.2, Theorem 0.4 and Remark 0.5.

From (0.45) and Remark 0.5, it follows that

Theorem 0.13. *Under the assumptions of Theorem 0.12, let $p \in \Gamma_q$. We choose local complex analytic coordinates*

$$z = (z_1, \ldots, z_n),\ \ z_j = x_{2j-1} + ix_{2j},\ \ j = 1, \ldots, n,$$

vanishing at p such that the metric on $\Lambda^{1,0}T(M')$ is

$$\sum_{j=1}^{n} dz_j \otimes d\overline{z}_j \ \ at\ \ p$$

and

$$r(z) = \sqrt{2}\,\mathrm{Im}\, z_n + \sum_{j=1}^{n-1} \lambda_j \left|z_j\right|^2 + O(|z|^3),$$

where λ_j, $j = 1, \ldots, n-1$, are the eigenvalues of L_p. (This is always possible.) We also write

$$w = (w_1, \ldots, w_n),\ \ w_j = y_{2j-1} + iy_{2j},\ \ j = 1, \ldots, n.$$

Then, we can take $\phi(z,w)$ so that

$$\begin{aligned}\phi(z,w) = {} & -\sqrt{2}x_{2n-1} + \sqrt{2}y_{2n-1} - ir(z)\Big(1 + \sum_{j=1}^{2n-1} a_j x_j + \frac{1}{2}a_{2n}x_{2n}\Big) \\ & - ir(w)\Big(1 + \sum_{j=1}^{2n-1} \overline{a}_j y_j + \frac{1}{2}\overline{a}_{2n}y_{2n}\Big) + i\sum_{j=1}^{n-1}\left|\lambda_j\right|\left|z_j - w_j\right|^2 \\ & + \sum_{j=1}^{n-1} i\lambda_j(\overline{z}_j w_j - z_j\overline{w}_j) + O\Big(\left|(z,w)\right|^3\Big) \qquad (0.46)\end{aligned}$$

in some neighborhood of (p,p) in $M' \times M'$, where

$$a_j = \frac{1}{2}\frac{\partial \sigma_{\Box_f^{(q)}}}{\partial x_j}(p, -\omega_0(p) - i\,dr(p)),\ \ j = 1, \ldots, 2n.$$

The leading term of the Bergman projection

We have the following corollary of Theorem 0.12

Corollary 0.14. *Under the assumptions of Theorem 0.12 and let U be a small neighborhood of* $\mathrm{diag}(\Gamma_q \times \Gamma_q)$. *Then there exist smooth functions*

$$F, G \in C^\infty(U \bigcap (\overline{M} \times \overline{M})); \mathscr{L}(\Lambda^{0,q} T_w^*(M'), \Lambda^{0,q} T_z^*(M')))$$

such that

$$K_{\Pi^{(q)}} = F(-i(\phi(z,w) + i0))^{-n-1} + G \log(-i(\phi(z,w) + i0)).$$

Moreover, we have

$$\begin{aligned} F &= \sum_{j=0}^{n} (n-j)! b_j(z,w)(-i\phi(z,w))^j + f(z,w)(\phi(z,w))^{n+1}, \\ G &\equiv \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j!} b_{n+j+1}(z,w)(-i\phi(z,w))^j \mod C^\infty(U \bigcap (\overline{M} \times \overline{M})) \end{aligned} \tag{0.47}$$

where

$$f(z,w) \in C^\infty(U \bigcap (\overline{M} \times \overline{M}); \mathscr{L}(\Lambda^{0,q} T_w^*(M'), \Lambda^{0,q} T_z^*(M'))).$$

We have the following

Proposition 0.15. *Under the assumptions of Theorem 0.12, let $p \in \Gamma_q$, $q = n_-$. Let*

$$U_1(z), \ldots, U_{n-1}(z)$$

be an orthonormal frame of $\Lambda^{1,0} T_z(\Gamma)$, $z \in \Gamma$, for which the Levi form is diagonalized at p. Let $e_j(z)$, $j = 1, \ldots, n-1$ denote the basis of $\Lambda^{0,1} T_z^*(\Gamma)$, $z \in \Gamma$, which is dual to $\overline{U}_j(z)$, $j = 1, \ldots, n-1$. Let $\lambda_j(z)$, $j = 1, \ldots, n-1$ be the eigenvalues of the Levi form L_z, $z \in \Gamma$. We assume that

$$\lambda_j(p) < 0 \ \text{ if } \ 1 \le j \le n_-.$$

Then

$$F(p,p) = n! \left|\lambda_1(p)\right| \cdots \left|\lambda_{n-1}(p)\right| \pi^{-n} 2 \Big(\prod_{j=1}^{j=n_-} e_j(p)^\wedge e_j^{\wedge,*}(p) \Big) \circ (\overline{\partial} r(p))^{\wedge,*} (\overline{\partial} r(p))^\wedge, \tag{0.48}$$

where F is as in Corollary 0.14.

For the reader

We recall briefly some microlocal analysis that we used in this thesis in Appendix A and B of paper I. These two papers can be read independently. We hope that this thesis can serve as an introduction to certain microlocal techniques with applications to complex geometry and CR geometry.

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On the singularities of the Szegö projection for $(0, q)$ forms

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Abstract

In this paper we obtain the full asymptotic expansion of the Szegö projection for $(0, q)$ forms. This generalizes a result of Boutet de Monvel and Sjöstrand for $(0, 0)$ forms. Our main tool is Fourier integral operators with complex valued phase functions of Melin and Sjöstrand.

Résumé

Dans ce travail nous obtenons un développement asymptotique complet du projecteur de Szegö pour les $(0, q)$ formes. Cela généralise un resultat de Boutet de Monvel et Sjöstrand pour les $(0, 0)$ formes. Nous utilisons des opérateurs intégraux de Fourier à phases complexes de Melin et Sjöstrand.

Contents

1 Introduction and statement of the main results

Let $(X, \Lambda^{1,0}T(X))$ be a compact orientable connected CR manifold of dimension $2n-1$, $n \geq 2$, (see Definition 2.1) and take a smooth Hermitian metric $(\,|\,)$ on $\mathbb{C}T(X)$ so that $\Lambda^{1,0}T(X)$ is orthogonal to $\Lambda^{0,1}T(X)$ and $(u\,|\,v)$ is real if u, v are real tangent vectors, where $\Lambda^{0,1}T(X) = \overline{\Lambda^{1,0}T(X)}$ and $\mathbb{C}T(X)$ is the complexified tangent bundle. For $p \in X$, let L_p be the Levi form of X at p. (See (1.1) and Definition 2.6.) Given q, $0 \leq q \leq n-1$, the Levi form is said to satisfty condition $Y(q)$ at $p \in X$ if for any $|J| = q$, $J = (j_1, j_2, \ldots, j_q)$, $1 \leq j_1 < j_2 < \cdots < j_q \leq n-1$, we have

$$\left|\sum_{j \notin J} \lambda_j - \sum_{j \in J} \lambda_j\right| < \sum_{j=1}^{n-1} \left|\lambda_j\right|,$$

where λ_j, $j = 1, \ldots, (n-1)$, are the eigenvalues of L_p. (For the precise meaning of the eigenvalues of the Levi form, see Definition 2.8.) If the Levi form is non-degenerate at p, then $Y(q)$ holds at p if and only if $q \neq n_-, n_+$, where (n_-, n_+) is the signature of L_p, i.e. the number of negative eigenvalues of L_p is n_- and $n_+ + n_- = n-1$. Let $\Box_b$ be the Kohn Laplacian on X (see [6] or section 2) and let $\Box_b^{(q)}$ denote the restriction to $(0,q)$-forms. When condition $Y(q)$ holds, Kohn's L^2 estimates give the hypoellipicity with loss of one dervative for the solutions of $\Box_b^{(q)} u = f$. (See [11], [6] and section 3.) The Szegö projection is the orthogonal projection onto the kernel of $\Box_b^{(q)}$ in the L^2 space. When condition $Y(q)$ fails, one is interested in the Szegö projection on the level of $(0,q)$-forms. Beals and Greiner (see [1]) used the Heisenberg group to obtain the principal term of the Szegö projection. Boutet de Monvel and Sjöstrand (see [9]) obtained the full asymptotic expansion for the Szegö projection in the case of functions. We have been influenced by these works. The main inspiration for the present paper comes from Berman and Sjöstrand [3].

We now start to formulate the main results. First, we introduce some notations. Let E be a C^∞ vector bundle over a paracompact C^∞ manifold Ω. The fiber of E at $x \in \Omega$ will be denoted by E_x. Let $Y \subset\subset \Omega$ be an open set. From now on, the spaces of smooth sections of E over Y and distribution sections of E over Y

will be denoted by $C^\infty(Y;E)$ and $\mathscr{D}'(Y;E)$ respectively. Let $\mathscr{E}'(Y;E)$ be the subspace of $\mathscr{D}'(Y;E)$ whose elements have compact support in Y. For $s\in\mathbb{R}$, we let $H^s(Y;E)$ denote the Sobolev space of order s of sections of E over Y.

Let $\mathbb{C}T^*(X)$ be the complexified cotangent bundle. The Hermitian metric $(\,|\,)$ on $\mathbb{C}T(X)$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*(X)$ that we shall also denote by $(\,|\,)$. Let $\Lambda^{0,q}T^*(X)$ be the bundle of $(0,q)$-forms of X. (See (2.5).) The Hermitian metric $(\,|\,)$ on $\mathbb{C}T^*(X)$ induces a Hermitian metric on $\Lambda^{0,q}T^*(X)$ (see (2.3)) also denoted by $(\,|\,)$.

We take (dm) as the induced volume form on X. In local coordinates $x=(x_1,\ldots,x_{2n-1})$, we represent the Hermitian inner product $(\,|\,)$ on $\mathbb{C}T(X)$ by

$$(u\mid v)=\langle Hu,\overline{v}\rangle,\ \ u,v\in\mathbb{C}T(X),$$

where $H(x)\in C^\infty$ and $H(x)$ is positive definite at each point. Let $h(x)$ denote the determinant of H. The induced volume form on X is given by

$$dm=\sqrt{h(x)}dx.$$

Let $(\,|\,)$ be the inner product on $C^\infty(X;\Lambda^{0,q}T^*(X))$ defined by

$$(f\mid g)=\int_X(f(z)\mid g(z))(dm),\ f,g\in C^\infty(X;\Lambda^{0,q}T^*(X)).$$

Let

$$\pi:L^2(X;\Lambda^{0,q}T^*(X))\to\operatorname{Ker}\Box_b^{(q)}$$

be the Szegö projection, i.e. the orthogonal projection onto the kernel of $\Box_b^{(q)}$. Let

$$K_\pi(x,y)\in\mathscr{D}'(X\times X;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X)))$$

be the distribution kernel of π with respect to (dm). Here $\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))$ is the vector bundle with fiber over (x,y) consisting of the linear maps from $\Lambda^{0,q}T^*_y(X)$ to $\Lambda^{0,q}T^*_x(X)$. Formally,

$$(\pi u)(x)=\int K_\pi(x,y)u(y)\sqrt{h(y)}dy,\ \ u(y)\in C^\infty(X;\Lambda^{0,q}T^*(X)).$$

We pause and recall a general fact of distribution theory. (See Hörmander [17].) Let E and F be C^∞ vector bundles over a paracompact C^∞ manifold M equipped with a smooth density of integration. Let

$$A:C_0^\infty(M;E)\to\mathscr{D}'(M;F)$$

with distribution kernel

$$K_A(x,y)\in\mathscr{D}'(M\times M;\mathscr{L}(E_y,F_x)).$$

Then the following two statements are equivalent

(a) A is continuous: $\mathscr{E}'(M;\,E)\to C^\infty(M;\,F)$,

(b) $K_A\in C^\infty(M\times M;\,\mathscr{L}(E_y,F_x))$.

If A satisfies (a) or (b), we say that A is smoothing. Let

$$B: C_0^\infty(M;\,E)\to\mathscr{D}'(M;\,F).$$

From now on, we write $K_B(x,y)$ or $B(x,y)$ to denote the distribution kernel of B and we write $A\equiv B$ or $A\equiv B \bmod C^\infty$ if $A-B$ is a smoothing operator. A is smoothing if and only if A is continuous

$$A: H^s_{\rm comp}(M;\,E)\to H^{s+N}_{\rm loc}(M;\,F) \text{ for all } N\geq 0, s\in\mathbb{R},$$

where

$$H^s_{\rm loc}(M;\,F)=\Big\{u\in\mathscr{D}'(M;\,F);\,\varphi u\in H^s(M;\,F);\,\forall\varphi\in C_0^\infty(M)\Big\}$$

and

$$H^s_{\rm comp}(M;\,E)=H^s_{\rm loc}(M;\,E)\bigcap\mathscr{E}'(M;\,E).$$

For $z\in X$, let $\Lambda^{1,0}T^*_z(X)=\overline{\Lambda^{0,1}T^*_z(X)}$ and let $\Lambda^{1,0}T^*(X)$ denote the vector bundle with fiber $\Lambda^{1,0}T^*_z(X)$ at $z\in X$. Locally we can choose an orthonormal frame

$$\omega_1(z),\ldots,\omega_{n-1}(z)$$

for $\Lambda^{1,0}T^*_z(X)$, then

$$\overline{\omega}_1(z),\ldots,\overline{\omega}_{n-1}(z)$$

is an orthonormal frame for $\Lambda^{0,1}T^*_z(X)$. The $(2n-2)$-form

$$\omega=i^{n-1}\omega_1\wedge\overline{\omega}_1\wedge\cdots\wedge\omega_{n-1}\wedge\overline{\omega}_{n-1}$$

is real and is independent of the choice of the orthonormal frame. Thus ω can be considered as a globally defined $(2n-2)$-form. Locally there is a real 1-form $\omega_0(z)$ of length one which is orthogonal to $\Lambda^{1,0}T^*_z(X)\oplus\Lambda^{0,1}T^*_z(X)$. $\omega_0(z)$ is unique up to the choice of sign. Since X is orientable, there is a nowhere vanishing $(2n-1)$-form Q on X. Thus, ω_0 can be specified uniquely by requiring that

$$\omega\wedge\omega_0=fQ,$$

where f is a positive function. Therefore ω_0, so chosen, is a uniquely determined global 1-form. We call ω_0 the uniquely determined global real 1-form.

Since $(u\mid v)$ is real if u, v are real tangent vectors, there is a real non-vanishing vector field Y which is orthogonal to $\Lambda^{1,0}T(X)\oplus\Lambda^{0,1}T(X)$. We write $\langle\,,\rangle$ to denote the duality between $T_z(X)$ and $T^*_z(X)$. We extend $\langle\,,\rangle$ bilinearly to $\mathbb{C}T_z(X)\times\mathbb{C}T^*_z(X)$. We take Y so that

$$\langle Y,\omega_0\rangle=-1,\ \|Y\|=1.$$

Therefore Y is uniquely determined. We call Y the uniquely determined global real vector field.

We recall that the Levi form L_p, $p\in X$, is the Hermitian quadratic form on $\Lambda^{1,0}T_p(X)$ defined as follows:

$$\begin{array}{l}\text{For any } Z,\ W\in\Lambda^{1,0}T_p(X),\ \text{pick } \widetilde{Z},\ \widetilde{W}\in C^\infty(X;\Lambda^{1,0}T(X)) \text{ that satisfy}\\ \widetilde{Z}(p)=Z,\ \widetilde{W}(p)=W.\ \text{Then } L_p(Z,\overline{W})=\dfrac{1}{2i}\left\langle[\widetilde{Z}\,,\overline{\widetilde{W}}](p)\,,\omega_0(p)\right\rangle.\end{array}\tag{1.1}$$

Let Σ be the characteristic manifold of $\Box^{(q)}_b$. We have

$$\Sigma=\left\{(x,\xi)\in T^*(X)\setminus 0;\ \xi=\lambda\omega_0(x),\lambda\neq 0\right\}.$$

Put

$$\Sigma^+=\left\{(x,\xi)\in T^*(X)\setminus 0;\ \xi=\lambda\omega_0(x),\lambda>0\right\},$$
$$\Sigma^-=\left\{(x,\xi)\in T^*(X)\setminus 0;\ \xi=\lambda\omega_0(x),\lambda<0\right\}.$$

We assume that the Levi form is non-degenerate at each point of X. Then the Levi form has constant signature (n_-,n_+), $n_-+n_+=n-1$. We define

$$\hat{\Sigma}=\Sigma^+\ \text{ if }\ n_+=q\neq n_-,$$
$$\hat{\Sigma}=\Sigma^-\ \text{ if }\ n_-=q\neq n_+,$$
$$\hat{\Sigma}=\Sigma^+\bigcup\Sigma^-\ \text{ if }\ n_+=q=n_-.$$

The main result of this work is the following

Theorem 1.1. *Let $(X,\Lambda^{1,0}T(X))$ be a compact orientable connected CR manifold of dimension $2n-1$, $n\geq 2$, with a Hermitian metric $(\,|\,)$. (See Definition 2.1 and Definition 2.2.) We assume that the Levi form L is non-degenerate at each point of X. Then, the Levi form has constant signature (n_-,n_+), $n_-+n_+=n-1$. Let $q=n_-$ or n_+. Suppose $\Box^{(q)}_b$ has closed range. Then π is a well defined continuous operator*

$$\pi: H^s(X;\Lambda^{0,q}T^*(X))\rightarrow H^s(X;\Lambda^{0,q}T^*(X)),$$

for all $s\in\mathbb{R}$, and

$$\mathrm{WF}'(K_\pi)=\mathrm{diag}(\hat{\Sigma}\times\hat{\Sigma}),$$

where

$$\mathrm{WF}'(K_\pi)=\{(x,\xi,y,\eta)\in T^*(X)\times T^*(X);\,(x,\xi,y,-\eta)\in \mathrm{WF}(K_\pi)\}\,.$$

Here $\mathrm{WF}(K_\pi)$ *is the wave front set of* K_π *in the sense of Hörmander [14].*

Moreover, we have

$$\begin{aligned}&K_\pi=K_{\pi^+}\ \text{ if }\ n_+=q\neq n_-,\\&K_\pi=K_{\pi^-}\ \text{ if }\ n_-=q\neq n_+,\\&K_\pi=K_{\pi^+}+K_{\pi^-}\ \text{ if }\ n_+=q=n_-,\end{aligned}$$

where $K_{\pi^+}(x,y)$ *satisfies*

$$K_{\pi^+}(x,y)\equiv\int_0^\infty e^{i\phi_+(x,y)t}s_+(x,y,t)dt \mod C^\infty$$

with

$$s_+(x,y,t)\in S^{n-1}_{1,0}(X\times X\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))),$$

$$s_+(x,y,t)\sim\sum_{j=0}^\infty s^j_+(x,y)t^{n-1-j}$$

in the symbol space $S^{n-1}_{1,0}(X\times X\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X)))$,

where $S^m_{1,0}$, $m\in\mathbb{R}$, *is the Hörmander symbol space (see Appendix A for a review and references),*

$$s^j_+(x,y)\in C^\infty(X\times X;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))),\ j=0,1,\ldots,$$

and

$$\phi_+(x,y)\in C^\infty(X\times X), \tag{1.2}$$
$$\phi_+(x,x)=0, \tag{1.3}$$
$$\phi_+(x,y)\neq 0\ \text{ if } x\neq y, \tag{1.4}$$
$$\mathrm{Im}\,\phi_+(x,y)\geq 0, \tag{1.5}$$
$$d_x\phi_+\neq 0,\ \ d_y\phi_+\neq 0\ \text{ where } \mathrm{Im}\,\phi_+=0, \tag{1.6}$$
$$d_x\phi_+(x,y)|_{x=y}=\omega_0(x), \tag{1.7}$$
$$d_y\phi_+(x,y)|_{x=y}=-\omega_0(x), \tag{1.8}$$
$$\phi_+(x,y)=-\overline{\phi}_+(y,x). \tag{1.9}$$

Similarly,

$$K_{\pi^-}(x,y) \equiv \int_0^\infty e^{i\phi_-(x,y)t} s_-(x,y,t)dt \mod C^\infty$$

with

$$s_-(x,y,t) \in S^{n-1}_{1,0}(X \times X \times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))),$$
$$s_-(x,y,t) \sim \sum_{j=0}^\infty s^j_-(x,y)t^{n-1-j}$$
in the symbol space $S^{n-1}_{1,0}(X \times X \times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X)))$,

where

$$s^j_-(x,y) \in C^\infty(X \times X;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))),\ j=0,1,\ldots,$$

and when $q = n_- = n_+$,

$$\phi_-(x,y) = -\overline{\phi}_+(x,y).$$

Formulas for $s^0_+(x,x)$ *and* $s^0_-(x,x)$ *will be given in Proposition 1.7 and Proposition 1.8. More properties of the phase* $\phi_+(x,y)$ *will be given in Theorem 1.4 and Remark 1.5 below.*

Remark 1.2. We notice that if $Y(q-1)$ and $Y(q+1)$ hold then $\Box^{(q)}_b$ has closed range. (See section 7.)

Remark 1.3. If $(X,\Lambda^{1,0}T(X))$ is non-orientable, we also have results similar to Theorem 1.1. (See section 10.)

In the rest of this section, we assume that the Levi form is non-degenerate at each point of X. The phase $\phi_+(x,y)$ is not unique. we can replace $\phi_+(x,y)$ by

$$\widetilde{\phi}(x,y) = f(x,y)\phi_+(x,y), \tag{1.10}$$

where $f(x,y) \in C^\infty(X \times X)$ is real and $f(x,x) = 1$, $f(x,y) = f(y,x)$. Then $\widetilde{\phi}$ satisfies (1.2)-(1.9). We work with local coordinates $x = (x_1,\ldots,x_{2n-1})$ defined on an open set $\Omega \subset X$. We want to know the Hessian

$$(\phi_+)'' = \begin{bmatrix} (\phi_+)''_{xx} & (\phi_+)''_{xy} \\ (\phi_+)''_{yx} & (\phi_+)''_{yy} \end{bmatrix}$$

of ϕ_+ at $(p,p) \in X \times X$. Let $U,\ V \in \mathbb{C}T_p(X) \times \mathbb{C}T_p(X)$. From (1.10), we can check that

$$\left\langle \widetilde{\phi}''(p,p)U,V \right\rangle = \langle (\phi_+)''(p,p)U,V\rangle + \langle df(p,p),U\rangle \langle d\phi_+(p,p),V\rangle$$
$$+ \langle df(p,p),V\rangle \langle d\phi_+(p,p),U\rangle .$$

Thus, the Hessian $(\phi_+)''$ of ϕ_+ at (p,p) is only well-defined on the space

$$T_{(p,p)}H_+ = \Big\{W \in \mathbb{C}T_p(X)\times \mathbb{C}T_p(X);\ \langle d\phi_+(p,p), W\rangle = 0\Big\}.$$

In section 8, we will define $T_{(p,p)}H_+$ as the tangent space of the formal hypersurface H_+ (see (8.46)) at $(p,p)\in X\times X$. In view of (1.7) and (1.8), we see that $T_{(p,p)}H_+$ is spanned by

$$(u,v),\ \ (Y(p),Y(p)),\ \ u,v\in \Lambda^{1,0}T_p(X)\oplus\Lambda^{0,1}T_p(X).$$

We define the tangential Hessian of $\phi_+(x,y)$ at (p,p) as the bilinear map:

$$\begin{aligned}&T_{(p,p)}H_+ \times T_{(p,p)}H_+ \to \mathbb{C},\\ &\qquad (U,V)\to \Big\langle (\phi''_+)(p,p)U, V\Big\rangle,\ \ U,V\in T_{(p,p)}H_+.\end{aligned}$$

For $p\in X$, we take local coordinates $x=(x_1,\ldots,x_{2n-1})$ defined on some neighborhood of p such that

$$\omega_0(p)=\sqrt{2}dx_{2n-1},\ \ x(p)=0.$$

If $\hat\phi\in C^\infty(X\times X)$, $\hat\phi(p,p)=0$, $d_{x,y}\hat\phi(p,p)=d_{x,y}\phi_+(p,p)$ and the tangential Hessian of $\hat\phi(x,y)$ at (p,p) is the same as the tangential Hessian of $\phi_+(x,y)$ at (p,p), then

$$\hat\phi(x,y',x_{2n-1})-\phi_+(x,y',x_{2n-1})=O(\left|(x,y')\right|^3)$$

in some neighborhood of (p,p), where $y'=(y_1,\ldots,y_{2n-2})$. Moreover, we have the following (see section 9)

Theorem 1.4. *For $p\in X$, let*

$$U_1(x),\ldots,U_{n-1}(x)$$

be an orthonormal frame of $\Lambda^{1,0}T_x(X)$ varying smoothly with x in a neighborhood of p, for which the Levi form is diagonalized at p. We take local coordinates

$$x=(x_1,\ldots,x_{2n-1}),\ \ z_j=x_{2j-1}+ix_{2j},\ \ j=1,\ldots,n-1,$$

defined on some neighborhood of p such that

$$\omega_0(p)=\sqrt{2}dx_{2n-1},\ \ x(p)=0,$$

$$(\frac{\partial}{\partial x_j}(p)\,|\,\frac{\partial}{\partial x_k}(p))=2\delta_{j,k},\ \ j,k=1,\ldots,2n-1$$

and

$$U_j=\frac{\partial}{\partial z_j}-\frac{1}{\sqrt{2}}a_j(x)\frac{\partial}{\partial x_{2n-1}}+\sum_{s=1}^{2n-2}c_{j,s}(x)\frac{\partial}{\partial x_s},\ \ j=1,\dots,n-1,$$

where

$$\frac{\partial}{\partial z_j}=\frac{1}{2}(\frac{\partial}{\partial x_{2j-1}}-i\frac{\partial}{\partial x_{2j}}),\ \ j=1,\dots,n-1,$$

$a_j\in C^\infty$, $a_j(0)=0$, $\frac{\partial a_j}{\partial z_k}(0)=\frac{\partial a_k}{\partial z_j}(0)$, $j,k=1,\dots,n-1$ *and*

$$c_{j,s}(x)\in C^\infty,\ \ c_{j,s}(0)=0,\ \ j=1,\dots,n-1,\ \ s=1,\dots,2n-2.$$

(This is always possible. see section 9.*) We also write*

$$y=(y_1,\dots,y_{2n-1}),\ \ w_j=y_{2j-1}+iy_{2j},\ \ j=1,\dots,n-1.$$

Then,

$$\begin{aligned}\phi_+(x,y)=&\sqrt{2}(x_{2n-1}-y_{2n-1})+i\sum_{j=1}^{n-1}\left|\lambda_j\right|\left|z_j-w_j\right|^2+\frac{1}{2}\sum_{j,k=1}^{n-1}\Big(\frac{\partial a_j}{\partial z_k}(0)(z_jz_k-w_jw_k)\\&+\frac{\partial\overline{a}_j}{\partial\overline{z}_k}(0)(\overline{z}_j\overline{z}_k-\overline{w}_j\overline{w}_k)+\frac{\partial a_j}{\partial\overline{z}_k}(0)(z_j\overline{z}_k-w_j\overline{w}_k)+\frac{\partial\overline{a}_j}{\partial z_k}(0)(\overline{z}_jz_k-\overline{w}_jw_k)\Big)\\&+\sum_{j=1}^{n-1}\Big(i\lambda_j(z_j\overline{w}_j-\overline{z}_jw_j)+\frac{\partial a_j}{\partial x_{2n-1}}(0)(z_jx_{2n-1}-w_jy_{2n-1})\\&+\frac{\partial\overline{a}_j}{\partial x_{2n-1}}(0)(\overline{z}_jx_{2n-1}-\overline{w}_jy_{2n-1})\Big)+\sqrt{2}(x_{2n-1}-y_{2n-1})f(x,y)+O(\left|(x,y)\right|^3),\\&f\in C^\infty,\ \ f(0,0)=0,\ \ f(x,y)=\overline{f}(y,x),\end{aligned}\tag{1.11}$$

where λ_j, $j=1,\dots,n-1$, *are the eigenvalues of* L_p.

Remark 1.5. We use the same notations as in Theorem 1.4. Since

$$\frac{\partial\phi_+}{\partial x_{2n-1}}(0,0)\neq 0,$$

from the Malgrange preparation theorem (see Theorem B.6), we have

$$\phi_+(x,y)=g(x,y)(\sqrt{2}x_{2n-1}+h(x',y))$$

in some neighborhood of $(0,0)$, where $g,\ h\in C^\infty,\ g(0,0)=1,\ h(0,0)=0$ and $x'=(x_1,\dots,x_{2n-2})$. Put

$$\hat{\phi}(x,y)=\sqrt{2}x_{2n-1}+h(x',y).$$

From the global theory of Fourier integral operators (see Proposition B.21), we see that $\phi_+(x,y)$ and $\hat{\phi}(x,y)$ are equivalent at $(p,\omega_0(p))$ in the sense of Melin-Sjöstrand (see Definition B.20). Since $\phi_+(x,y)=-\overline{\phi}_+(y,x)$, we can replace the phase $\phi_+(x,y)$ by

$$\frac{\hat{\phi}(x,y)-\overline{\hat{\phi}}(y,x)}{2}.$$

Then $\phi_+(x,y)$ satisfies (1.2)-(1.9). Moreover, we can check that

$$\begin{aligned}\phi_+(x,y)&=\sqrt{2}(x_{2n-1}-y_{2n-1})+i\sum_{j=1}^{n-1}\left|\lambda_j\right|\left|z_j-w_j\right|^2+\frac{1}{2}\sum_{j,k=1}^{n-1}\Big(\frac{\partial a_j}{\partial z_k}(0)(z_jz_k-w_jw_k)\\&+\frac{\partial\overline{a}_j}{\partial\overline{z}_k}(0)(\overline{z}_j\overline{z}_k-\overline{w}_j\overline{w}_k)+\frac{\partial a_j}{\partial\overline{z}_k}(0)(z_j\overline{z}_k-w_j\overline{w}_k)+\frac{\partial\overline{a}_j}{\partial z_k}(0)(\overline{z}_jz_k-\overline{w}_jw_k)\Big)\\&+\sum_{j=1}^{n-1}\Big(i\lambda_j(z_j\overline{w}_j-\overline{z}_jw_j)+\frac{\partial a_j}{\partial x_{2n-1}}(0)(z_jx_{2n-1}-w_jy_{2n-1})\\&+\frac{\partial\overline{a}_j}{\partial x_{2n-1}}(0)(\overline{z}_jx_{2n-1}-\overline{w}_jy_{2n-1})\Big)+O(\left|(x,y)\right|^3),\end{aligned}\tag{1.12}$$

where λ_j, $j=1,\ldots,n-1$, are the eigenvalues of L_p. (Compare (1.12) with (1.11).)

We have the following corollary of Theorem 1.1. (See section 9.)

Corollary 1.6. *There exist smooth functions*

$$F_+,G_+,F_-,G_-\in C^\infty(X\times X;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X)))$$

such that

$$K_{\pi^+}=F_+(-i(\phi_+(x,y)+i0))^{-n}+G_+\log(-i(\phi_+(x,y)+i0)),$$
$$K_{\pi^-}=F_-(-i(\phi_-(x,y)+i0))^{-n}+G_-\log(-i(\phi_-(x,y)+i0)).$$

Moreover, we have

$$\begin{aligned}F_+&=\sum_0^{n-1}(n-1-k)!s^k_+(x,y)(-i\phi_+(x,y))^k+f_+(x,y)(\phi_+(x,y))^n,\\F_-&=\sum_0^{n-1}(n-1-k)!s^k_-(x,y)(-i\phi_-(x,y))^k+f_-(x,y)(\phi_-(x,y))^n,\\G_+&\equiv\sum_0^\infty\frac{(-1)^{k+1}}{k!}s^{n+k}_+(x,y)(-i\phi_+(x,y))^k,\\G_-&\equiv\sum_0^\infty\frac{(-1)^{k+1}}{k!}s^{n+k}_-(x,y)(-i\phi_-(x,y))^k,\end{aligned}\tag{1.13}$$

where

$$f_+(x,y),f_-(x,y)\in C^\infty(X\times X;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))).$$

If $w \in \Lambda^{0,1} T^*_z(X)$, let

$$w^{\wedge,*} : \Lambda^{0,q+1} T^*_z(X) \to \Lambda^{0,q} T^*_z(X),\ q \geq 0,$$

be the adjoint of left exterior multiplication

$$w^{\wedge} : \Lambda^{0,q} T^*_z(X) \to \Lambda^{0,q+1} T^*_z(X).$$

That is,

$$(w^{\wedge} u \mid v) = (u \mid w^{\wedge,*} v),$$

for all $u \in \Lambda^{0,q} T^*_z(X)$, $v \in \Lambda^{0,q+1} T^*_z(X)$. Notice that $w^{\wedge,*}$ depends anti-linearly on w.

In section 9, we compute $F_+(x,x)$ and $F_-(x,x)$.

Proposition 1.7. *For a given point $x_0 \in X$, let*

$$U_1(x), \ldots, U_{n-1}(x)$$

be an orthonormal frame of $\Lambda^{1,0} T_x(X)$, for which the Levi form is diagonalized at x_0. Let $e_j(x)$, $j = 1, \ldots, n-1$, denote the basis of $\Lambda^{0,1} T^*_x(X)$, which is dual to $\overline{U}_j(x)$, $j = 1, \ldots, n-1$. Let $\lambda_j(x)$, $j = 1, \ldots, n-1$, be the eigenvalues of the Levi form L_x. We assume that $q = n_+$ and that

$$\lambda_j(x_0) > 0 \ \text{if} \ \ 1 \leq j \leq n_+.$$

Then

$$F_+(x_0, x_0) = (n-1)! \frac{1}{2} |\lambda_1(x_0)| \cdots |\lambda_{n-1}(x_0)| \pi^{-n} \prod_{j=1}^{j=n_+} e_j(x_0)^{\wedge} e_j(x_0)^{\wedge,*}.$$

Proposition 1.8. *For a given point $x_0 \in X$, let*

$$U_1(x), \ldots, U_{n-1}(x)$$

be an orthonormal frame of $\Lambda^{1,0} T_x(X)$, for which the Levi form is diagonalized at x_0. Let $e_j(x)$, $j = 1, \ldots, n-1$, denote the basis of $\Lambda^{0,1} T^*_x(X)$, which is dual to $\overline{U}_j(x)$, $j = 1, \ldots, n-1$. Let $\lambda_j(x)$, $j = 1, \ldots, n-1$ be the eigenvalues of the Levi form L_x. We assume that $q = n_-$ and that

$$\lambda_j(x_0) < 0 \ \text{if} \ \ 1 \leq j \leq n_-.$$

Then

$$F_-(x_0, x_0) = (n-1)! \frac{1}{2} |\lambda_1(x_0)| \cdots |\lambda_{n-1}(x_0)| \pi^{-n} \prod_{j=1}^{j=n_-} e_j(x_0)^{\wedge} e_j(x_0)^{\wedge,*}.$$

In the rest of this section, we will explain how to prove Theorem 1.1. Let M be an open set in $\mathbb{R}^n$ and let $f, g \in C^\infty(M)$. We write

$$f \asymp g$$

if for every compact set $K \subset M$ there is a constant $c_K > 0$ such that

$$f \leq c_K g,\ g \leq c_K f \ \text{ on } K.$$

We will prove the following

Proposition 1.9. *Let $(X, \Lambda^{1,0}T(X))$ be a compact orientable connected CR manifold of dimension $2n-1$, $n \geq 2$, with a Hermitian metric $(\ |\ )$. Let (n_-, n_+), $n_- + n_+ = n-1$, be the signature of the Levi form. Let $q = n_-$ or n_+. Suppose $\Box_b^{(q)}$ has closed range. Then for every local coordinate patch U with local coordinates $x = (x_1, \ldots, x_{2n-1})$, the distribution kernel of π on $U \times U$ is of the form*

$$K_\pi(x,y) \equiv \frac{1}{(2\pi)^{2n-1}} \int e^{i(\psi(\infty,x,\eta) - \langle y,\eta\rangle)} a(\infty,x,\eta) d\eta \mod C^\infty,$$

$$a(\infty,x,\eta) \in S^0_{1,0}(T^*(U)\,; \mathscr{L}(\Lambda^{0,q}T^*(U), \Lambda^{0,q}T^*(U))),$$

$$a(\infty,x,\eta) \sim \sum_0^\infty a_j(\infty,x,\eta)$$
in the symbol space $S^0_{1,0}(T^*(U); \mathscr{L}(\Lambda^{0,q}T^*(U), \Lambda^{0,q}T^*(U)))$,

where $\mathscr{L}(\Lambda^{0,q}T^*(U), \Lambda^{0,q}T^*(U))$ is the vector bundle with fiber (x,η) consisting of linear maps from $\Lambda^{0,q}T^*_x(U)$ to $\Lambda^{0,q}T^*_x(U)$,

$$a_j(\infty,x,\eta) \in C^\infty(T^*(U); \mathscr{L}(\Lambda^{0,q}T^*(U), \Lambda^{0,q}T^*(U))),\ \ j = 0,1,\ldots,$$

$$a_j(\infty,x,\lambda\eta) = \lambda^{-j} a_j(\infty,x,\eta), \lambda \geq 1,\ \left|\eta\right| \geq 1,\ \ j = 0,1,\ldots.$$

Here

$$\psi(\infty,x,\eta) \in C^\infty(T^*(U)),$$
$$\psi(\infty,x,\lambda\eta) = \lambda\psi(\infty,x,\eta), \lambda > 0,$$
$$\operatorname{Im}\psi(\infty,x,\eta) \asymp \left|\eta\right| (\operatorname{dist}((x, \frac{\eta}{\left|\eta\right|}), \Sigma))^2.$$

Moreover, for all $j = 0,1,\ldots$,

$$\begin{cases} a_j(\infty,x,\eta) = 0 \ \textit{ in a conic neighborhood of } \Sigma^+, & \text{if } q = n_-,\ n_- \neq n_+ \\ a_j(\infty,x,\eta) = 0 \ \textit{ in a conic neighborhood of } \Sigma^-, & \text{if } q = n_+,\ n_- \neq n_+. \end{cases} \quad (1.14)$$

From the global theory of Fourier integral operators (see Melin-Sjöstrand [18] and section 8), we get Theorem 1.1.

Now, we sketch the proof of Proposition 1.9. We will use the heat equation method. We work with some real local coordinates $x=(x_1,\ldots,x_{2n-1})$ defined on an open set $\Omega\subset X$. We assume that $q=n_-$ or $q=n_+$. We will say that $a\in C^\infty(\overline{\mathbb{R}}_+\times\Omega\times\mathbb{R}^{2n-1})$ is quasi-homogeneous of degree j if

$$a(t,x,\lambda\eta)=\lambda^j a(\lambda t,x,\eta)$$

for all $\lambda>0$. We consider the problem

$$\begin{cases}(\partial_t+\Box_b^{(q)})u(t,x)=0 \quad \text{in}\ \mathbb{R}_+\times\Omega\\ u(0,x)=v(x)\end{cases}. \tag{1.15}$$

We shall start by making only a formal construction. We look for an approximate solution of (1.15) of the form

$$u(t,x)=A(t)v(x)$$

$$A(t)v(x)=\frac{1}{(2\pi)^{2n-1}}\int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)v(y)dyd\eta \tag{1.16}$$

where formally

$$a(t,x,\eta)\sim\sum_{j=0}^{\infty}a_j(t,x,\eta),$$

$a_j(t,x,\eta)$ is a matrix-valued quasi-homogeneous function of degree $-j$.

We let the full symbol of $\Box_b^{(q)}$ be:

$$\text{full symbol of }\Box_b^{(q)}=\sum_{j=0}^{2}p_j(x,\xi)$$

where $p_j(x,\xi)$ is positively homogeneous of order $2-j$ in the sense that

$$p_j(x,\lambda\eta)=\lambda^{2-j}p_j(x,\eta),\ \left|\eta\right|\geq 1,\ \lambda\geq 1.$$

We apply $\partial_t+\Box_b^{(q)}$ formally inside the integral in (1.16) and then introduce the asymptotic expansion of $\Box_b^{(q)}(ae^{i\psi})$. Set $(\partial_t+\Box_b^{(q)})(ae^{i\psi})\sim 0$ and regroup the terms according to the degree of quasi-homogeneity. The phase $\psi(t,x,\eta)$ should solve

$$\begin{cases}\dfrac{\partial\psi}{\partial t}-ip_0(x,\psi'_x)=O(\left|\operatorname{Im}\psi\right|^N),\ \forall N\geq 0\\ \psi|_{t=0}=\langle x,\eta\rangle\end{cases}. \tag{1.17}$$

This equation can be solved with $\operatorname{Im}\psi(t,x,\eta) \geq 0$ and the phase $\psi(t,x,\eta)$ is quasi-homogeneous of degree 1. Moreover,

$$\psi(t,x,\eta) = \langle x,\eta\rangle \text{ on } \Sigma,\ d_{x,\eta}(\psi - \langle x,\eta\rangle) = 0 \text{ on } \Sigma,$$

$$\operatorname{Im}\psi(t,x,\eta) \asymp (\left|\eta\right| \frac{t\left|\eta\right|}{1+t\left|\eta\right|})\operatorname{dist}((x,\frac{\eta}{\left|\eta\right|}),\Sigma))^2,\ \ \left|\eta\right| \geq 1.$$

Furthermore, there exists a function $\psi(\infty,x,\eta) \in C^\infty(\Omega\times\dot{\mathbb{R}}^{2n-1})$ with a uniquely determined Taylor expansion at each point of Σ such that for every compact set $K \subset \Omega\times\dot{\mathbb{R}}^{2n-1}$ there is a constant $c_K > 0$ such that

$$\operatorname{Im}\psi(\infty,x,\eta) \geq c_K\left|\eta\right|(\operatorname{dist}((x,\frac{\eta}{\left|\eta\right|}),\Sigma))^2,\ \ \left|\eta\right| \geq 1.$$

If $\lambda \in C(T^*\Omega\setminus 0)$, $\lambda > 0$ is positively homogeneous of degree 1 and $\lambda|_\Sigma < \min\lambda_j$, $\lambda_j > 0$, where $\pm i\lambda_j$ are the non-vanishing eigenvalues of the fundamental matrix of $\Box_b^{(q)}$, then the solution $\psi(t,x,\eta)$ of (1.17) can be chosen so that for every compact set $K \subset \Omega\times\dot{\mathbb{R}}^{2n-1}$ and all indices α, β, γ, there is a constant $c_{\alpha,\beta,\gamma,K}$ such that

$$\left|\partial_x^\alpha\partial_\eta^\beta\partial_t^\gamma(\psi(t,x,\eta)-\psi(\infty,x,\eta))\right| \leq c_{\alpha,\beta,\gamma,K}\,e^{-\lambda(x,\eta)t} \text{ on } \overline{\mathbb{R}}_+\times K.$$

(For the detail, see Menikoff-Sjöstrand[20] or section 4.)

We obtain the transport equations

$$\begin{cases} T(t,x,\eta,\partial_t,\partial_x)a_0 = O(\left|\operatorname{Im}\psi\right|^N),\ \forall N \\ T(t,x,\eta,\partial_t,\partial_x)a_j + l_j(t,x,\eta,a_0,\ldots,a_{j-1}) = O(\left|\operatorname{Im}\psi\right|^N),\ \forall N. \end{cases} \tag{1.18}$$

Let p_0^s denote the subprincipal symbol of $\Box_b^{(q)}$ (invariantly defined on Σ). (For the precise meaning of subprincipal symbols, see Definition A.10 and Definition A.26.) Let F_ρ be the fundamental matrix of $\Box_b^{(q)}$ at $\rho \in \Sigma$. (For the precise meaning of the fundamental matrix, see the discussion before Remark A.43.) We write $\widetilde{\operatorname{tr}}\,F_\rho$ to denote $\sum\left|\lambda_j\right|$, where $\pm i\lambda_j$ are the non-vanishing eigenvalues of F_ρ. Let

$$\inf(p_0^s + \frac{1}{2}\widetilde{\operatorname{tr}}\,F) = \inf\left\{\lambda;\ \lambda: \text{eigenvalue of } p_0^s + \frac{1}{2}\widetilde{\operatorname{tr}}\,F\right\}.$$

We have on Σ^+

$$\inf(p_0^s + \frac{1}{2}\widetilde{\operatorname{tr}}\,F)\begin{cases} = 0, & q = n_+ \\ > 0, & q \neq n_+ \end{cases}. \tag{1.19}$$

On Σ^-

$$\inf(p_0^s + \frac{1}{2}\widetilde{\operatorname{tr}}\,F)\begin{cases} = 0, & q = n_- \\ > 0, & q \neq n_- \end{cases}. \tag{1.20}$$

Let

$$c_j(x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))),\ \ j=0,1,\ldots$$

be positively homogeneous functions of degree $-j$. In section 5, we shall show that we can find solutions

$$a(t,x,\eta)\sim\sum_{j=0}^{\infty}a_j(t,x,\eta)$$

of the system (1.18) with

$$a_j(0,x,\eta)=c_j(x,\eta),\ \ j=0,1,\ldots,$$

where $a_j(t,x,\eta)$ is a matrix-valued C^∞ quasi-homogeneous function of degree $-j$. Moreover, $a_j(t,x,\eta)$ has unique Taylor expansions on Σ, for all j. Furthermore, there exists $\varepsilon_0>0$ such that for every compact set $K\subset\Sigma$ and all indices α,β,γ,j there exists a constant $c>0$ such that

$$\begin{aligned}&\left|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta a_j(t,x,\eta)\right|\leq ce^{-\varepsilon_0 t|\eta|}(1+\left|\eta\right|)^{-j-|\beta|+\gamma}\\&\text{on }\overline{\mathbb{R}}_+\times(K\bigcap\Sigma^+)\ \text{ if } q=n_-, n_-\neq n_+\end{aligned}\tag{1.21}$$

and

$$\begin{aligned}&\left|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta a_j(t,x,\eta)\right|\leq ce^{-\varepsilon_0 t|\eta|}(1+\left|\eta\right|)^{-j-|\beta|+\gamma}\\&\text{on }\overline{\mathbb{R}}_+\times(K\bigcap\Sigma^-)\ \text{ if } q=n_+, n_-\neq n_+.\end{aligned}\tag{1.22}$$

Let

$$a(t,x,\eta)\sim\sum_{j=0}^{\infty}a_j(t,x,\eta)$$

be the solutions of the system (1.18) with

$$a(0,x,\eta)=I,$$

where $a_j(t,x,\eta)$ is a C^∞ matrix-valued quasi-homogeneous function of degree $-j$ and $I\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega)\,,\Lambda^{0,q}T^*(\Omega)))$ is the identity map. We write

$$(\partial_t+\Box_b^{(q)})(e^{i\psi}b)\sim 0$$

if b solves the system (1.18), where $b(t,x,\eta)\sim\sum_{j=0}^{\infty}b_j(t,x,\eta)$, $b_j(t,x,\eta)$ is a matrix-valued quasi-homogeneous function of degree $m-j$, $m\in\mathbb{Z}$. We use

$$\overline{\partial}_b\Box_b^{(q)}=\Box_b^{(q+1)}\overline{\partial}_b,\quad \overline{\partial_b}^*\Box_b^{(q)}=\Box_b^{(q-1)}\overline{\partial_b}^*$$

and get

$$\partial_t(\overline{\partial}_b(e^{i\psi}a))+\Box_b^{(q+1)}(\overline{\partial}_b(e^{i\psi}a))\sim 0$$
$$\partial_t(\overline{\partial_b}^*(e^{i\psi}a))+\Box_b^{(q-1)}(\overline{\partial_b}^*(e^{i\psi}a))\sim 0.$$

Put

$$\overline{\partial}_b(e^{i\psi}a)=e^{i\psi}\hat{a},\ \overline{\partial_b}^*(e^{i\psi}a)=e^{i\psi}\widetilde{a}.$$

We have

$$(\partial_t+\Box_b^{(q+1)})(e^{i\psi}\hat{a})\sim 0,$$
$$(\partial_t+\Box_b^{(q-1)})(e^{i\psi}\widetilde{a})\sim 0.$$

In view of (1.21) and (1.22) (see Proposition 5.7), we see that $\hat{a}$ and $\widetilde{a}$ satisfy the same decay estimates as in (1.21) or (1.22). This also applies to

$$\Box_b^{(q)}(ae^{i\psi})=\overline{\partial}_b(\overline{\partial_b}^*ae^{i\psi})+\overline{\partial_b}^*(\overline{\partial}_b ae^{i\psi})$$
$$=\overline{\partial}_b(e^{i\psi}\widetilde{a})+\overline{\partial_b}^*(e^{i\psi}\hat{a}).$$

Thus, $\partial_t(ae^{i\psi})$ satisfies the same decay estimates as in (1.21) or (1.22). Since $\partial_t\psi$ satisfies the same decay estimates as in (1.21) or (1.22), $\partial_t a$ satisfies the same decay estimates as in (1.21) or (1.22). Hence, there exist

$$a_j(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega)\,,\Lambda^{0,q}T^*(\Omega))),\ j=0,1,\ldots,$$

positively homogeneous of degree $-j$, and $\varepsilon_0>0$, such that for every compact set $K\subset\Sigma$ and all indices α,β,j there exists a constant $c>0$ such that

$$\left|\partial_x^\alpha\partial_\eta^\beta(a_j(t,x,\eta)-a_j(\infty,x,\eta))\right|\le ce^{-\varepsilon_0 t|\eta|}(1+\left|\eta\right|)^{-j-|\beta|} \tag{1.23}$$

and for all $j=0,1,\ldots$,

$$\begin{cases}\text{all derivatives of } a_j(\infty,x,\eta) \text{ vanish at } \Sigma^+, & \text{if } q=n_-,\ n_-\neq n_+\\ \text{all derivatives of } a_j(\infty,x,\eta) \text{ vanish at } \Sigma^-, & \text{if } q=n_+,\ n_-\neq n_+\end{cases}. \tag{1.24}$$

Choose $\chi\in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi(\eta)=1$ when $\left|\eta\right|<1$ and $\chi(\eta)=0$ when $\left|\eta\right|>2$. We formally set

$$G=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)$$
$$-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\big)(1-\chi(\eta))dt\Big)d\eta$$

and

$$S = \frac{1}{(2\pi)^{2n-1}} \int (e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)} a(\infty,x,\eta)) d\eta.$$

In section 6, we will show that G is a pseudodifferential operator of order -1 type $(\frac{1}{2}, \frac{1}{2})$. In section 7, we will show that

$$S + \Box_b^{(q)} \circ G \equiv I$$

and

$$\Box_b^{(q)} \circ S \equiv 0.$$

If $\Box_b^{(q)}$ has closed range, then

$$N\Box_b^{(q)} + \pi = I = \Box_b^{(q)} N + \pi,$$

where N is the partial inverse of $\Box_b^{(q)}$. It is not difficult to see that

$$\pi \equiv S$$

and

$$N \equiv (I - S)G.$$

(See section 8.)

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2 Cauchy-Riemann manifolds, $\overline{\partial}_b$-Complex and $\Box_b$, a review

We will give a brief discussion of the basic elements of CR geometry in a setting appropriate for our purpose. General references for this section are the books Boggess [5], Chen-Shaw [6].

Let X be a real compact C^∞ manifold of dimension $2n-1$, $n \geq 2$. Let $T_p(X)$ and $T_p^*(X)$ be the tangent space of X at p and the cotangent space of X at p respectively. We write $T(X)$ and $T^*(X)$ to denote the bundles with fibers $T_z(X)$ and $T_z^*(X)$ at $z \in X$ respectively. Let $\mathbb{C}T_p(X)$ and $\mathbb{C}T_p^*(X)$ be the complexified tangent space of X at p and the complexified cotangent space of X at p respectively. That is,

$$\mathbb{C}T_p(X) = \left\{u + iv;\, u, v \in T_p(X)\right\},\ \ \mathbb{C}T_p^*(X) = \left\{u + iv;\, u, v \in T_p^*(X)\right\}.$$

We write $\mathbb{C}T(X)$ and $\mathbb{C}T^*(X)$ to denote the bundles with fibers $\mathbb{C}T_z(X)$ and $\mathbb{C}T_z^*(X)$ at $z \in X$ respectively.

Definition 2.1. Let X be a real C^∞ manifold of dimension $2n-1$, $n \geq 2$, and let $\Lambda^{1,0}T(X)$ be a subbundle of $\mathbb{C}T(X)$. The pair $(X, \Lambda^{1,0}T(X))$ is called a CR manifold or a CR structure if

(a) $\dim_{\mathbb{C}} \Lambda^{1,0}T_p(X) = n-1$, $p \in X$,

(b) $\Lambda^{1,0}T(X) \bigcap \Lambda^{0,1}T(X) = 0$, where $\Lambda^{0,1}T(X) = \overline{\Lambda^{1,0}T(X)}$,

(c) For any V_1, $V_2 \in C^\infty(U; \Lambda^{1,0}T(X))$, the Lie bracket $[V_1, V_2] \in C^\infty(U; \Lambda^{1,0}T(X))$, where U is any open subset of X.

Definition 2.2. Let $(X, \Lambda^{1,0}T(X))$ be a CR manifold. A Hermitian metric $(\,|\,)$ on $\mathbb{C}T(X)$ is a complex inner product $(\,|\,)$ on each $\mathbb{C}T_p(X)$ depending smoothly on p with the properties that $\Lambda^{1,0}T_p(X)$ is orthogonal to $\Lambda^{0,1}T_p(X)$ and $(u \mid v)$ is real if u, v are real tangent vectors.

Until further notice, we assume that $(X, \Lambda^{1,0}T(X))$ is a compact orientable connected CR manifold of dimension $2n-1$, $n \geq 2$, and we fix a Hermitian metric $(\,|\,)$ on $\mathbb{C}T(X)$. Then there is a real non-vanishing vector field Y on X which is pointwise orthogonal to $\Lambda^{1,0}T(X) \oplus \Lambda^{0,1}T(X)$.

We write $\langle\,,\rangle$ to denote the duality between $T_z(X)$ and $T^*_z(X)$. We extend $\langle\,,\rangle$ bilinearly to $\mathbb{C}T_z(X) \times \mathbb{C}T^*_z(X)$.

The Hermitian metric $(\,|\,)$ on $\mathbb{C}T(X)$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*(X)$ that we shall also denote by $(\,|\,)$ in the following way. For a given point $z \in X$, let Γ be the anti-linear map

$$\Gamma : \mathbb{C}T_z(X) \to \mathbb{C}T^*_z(X)$$

defined by

$$(u \mid v) = \langle u, \Gamma v \rangle, \quad u, v \in \mathbb{C}T_z(X). \tag{2.1}$$

For ω, $\mu \in \mathbb{C}T^*_z(X)$, we put

$$(\omega \mid \mu) = (\Gamma^{-1}\mu \mid \Gamma^{-1}\omega). \tag{2.2}$$

Let $\Lambda^r(\mathbb{C}T^*(X))$, $r \in \mathbb{N}$, be the vector bundle of r forms of X. That is, the fiber of $\Lambda^r(\mathbb{C}T^*(X))$ at $z \in X$ is the vector space $\Lambda^r(\mathbb{C}T^*_z(X))$ of all finite sums of

$$V_1 \wedge \cdots \wedge V_r, \; V_j \in \mathbb{C}T^*_z(X), \; j = 1, \ldots, r.$$

Here $\wedge$ denotes the wedge product. The Hermitian metric $(\,|\,)$ on $\Lambda^r(\mathbb{C}T^*(X))$ is defined by

$$\begin{aligned} (u_1 \wedge \cdots \wedge u_r \mid v_1 \wedge \cdots \wedge v_r) &= \det\big((u_j \mid v_k)\big)_{1 \leq j,k \leq r}, \\ u_j, v_k \in \mathbb{C}T^*(X), \; j,k &= 1, \ldots, r, \end{aligned} \tag{2.3}$$

and we extend the definition to arbitrary forms by sesqui-linearity.

Similarly, let $\Lambda^r(\mathbb{C}T(X))$, $r\in\mathbb{N}$, be the vector bundle with fiber $\Lambda^r(\mathbb{C}T_z(X))$ at $z\in X$, the set of all finite sums of

$$V_1\wedge\cdots\wedge V_r,\ V_j\in\mathbb{C}T_z(X),\ j=1,\ldots,r.$$

The duality $\langle,\rangle$ between $\Lambda^r(\mathbb{C}T(X))$ and $\Lambda^r(\mathbb{C}T^*(X))$ is defined by

$$\langle v_1\wedge\cdots\wedge v_r,u_1\wedge\cdots\wedge u_r\rangle=\det\Big(\Big\langle v_j,u_k\Big\rangle\Big)_{1\le j,k\le r},$$
$$u_j\in\mathbb{C}T^*(X),v_j\in\mathbb{C}T(X),j=1,\ldots,r.$$

and we extend the definition by bilinearity.

For $z\in X$, let $v\in\mathbb{C}T_z(X)$. For $0\le r\le(2n-2)$, the contraction

$$v^{\lrcorner}:\Lambda^{r+1}(\mathbb{C}T^*_z(X))\to\Lambda^r(\mathbb{C}T^*_z(X))$$

is defined by

$$\langle v_1\wedge\cdots\wedge v_r,v^{\lrcorner}u\rangle=\langle v\wedge v_1\wedge\cdots\wedge v_r,u\rangle$$

for all $u\in\Lambda^{r+1}(\mathbb{C}T^*_z(X))$, $v_j\in\mathbb{C}T_z(X)$, $j=1,\ldots,r$.

We have the pointwise orthogonal decomposition

$$\mathbb{C}T(X)=\Lambda^{1,0}T(X)\oplus\Lambda^{0,1}T(X)\oplus\mathbb{C}Y. \tag{2.4}$$

Define the bundle $\Lambda^{1,0}T^*(X)$ of type $(1,0)$ by

$$\Lambda^{1,0}T^*(X)=(\Lambda^{0,1}T(X)\oplus\mathbb{C}Y)^{\perp}\subset\mathbb{C}T^*(X).$$

Similarly, we set

$$\Lambda^{0,1}T^*(X)=(\Lambda^{1,0}T(X)\oplus\mathbb{C}Y)^{\perp}\subset\mathbb{C}T^*(X).$$

For $z\in X$, $u\in\Lambda^{1,0}T_z(X)$, $v\in\Lambda^{0,1}T_z(X)\oplus\mathbb{C}Y(z)$, we have

$$\langle v,\Gamma u\rangle=(v\mid u)=0,$$

where Γ is as in (2.1). Thus, $\Gamma\Lambda^{1,0}T_z(X)\subset\Lambda^{1,0}T^*_z(X)$. Since

$$\dim\Gamma\Lambda^{1,0}T_z(X)=\dim\Lambda^{1,0}T^*_z(X)=n-1,$$

we have

$$\Lambda^{1,0}T^*_z(X)=\Gamma\Lambda^{1,0}T_z(X).$$

Similarly,

$$\Lambda^{0,1}T^*_z(X)=\Gamma\Lambda^{0,1}T_z(X).$$

For $z\in X$, $\omega\in\Lambda^{1,0}T^*_z(X)$, $\mu\in\Lambda^{0,1}T^*_z(X)$, we have

$$(\omega\,|\,\mu)=(\Gamma^{-1}\mu\,|\,\Gamma^{-1}\omega).$$

Since $\Gamma^{-1}\omega\in\Lambda^{1,0}T_z(X)$, $\Gamma^{-1}\mu\in\Lambda^{0,1}T_z(X)$, we have

$$(\omega\,|\,\mu)=0.$$

Thus, $\Lambda^{1,0}T^*(X)$ is pointwise orthogonal to $\Lambda^{0,1}T^*(X)$. For $q\in\mathbb{N}$, define

$$\Lambda^{0,q}T^*(X)=\Lambda^q(\Lambda^{0,1}T^*(X)). \tag{2.5}$$

That is, the fiber of $\Lambda^{0,q}T^*(X)$ at $z\in X$ is the vector space $\Lambda^q(\Lambda^{0,1}T^*_z(X))$ of all finite sums of

$$V_1\wedge\cdots\wedge V_q,\ V_j\in\Lambda^{0,1}T^*_z(X),\ j=1,\ldots,q.$$

Note that $\Lambda^{0,q}T^*(X)=0$ if $q\geq n$. We use the Hermitian metric $(\,|\,)$ on $\Lambda^{0,q}T^*_z(X)$, that is naturally obtained from $\Lambda^q(\mathbb{C}T^*(X))$. Similarly, for $q\in\mathbb{N}$, let $\Lambda^{0,q}T(X)$ be the vector bundle with fiber $\Lambda^q(\Lambda^{0,1}T_z(X))$ at $z\in X$, the set of all finite sums of

$$V_1\wedge\cdots\wedge V_q,\ V_j\in\Lambda^{0,1}T_z(X),\ j=1,\ldots,q.$$

Let

$$d:C^\infty(X;\Lambda^r(\mathbb{C}T^*(X)))\to C^\infty(X;\Lambda^{r+1}(\mathbb{C}T^*(X)))$$

be the usual exterior derivative. We recall that the exterior derivative d has the following properties, where (b), (c) are special case of Cartan's formula:

$$\mathscr{L}_v\omega=v\lrcorner d\omega+d(v\lrcorner\omega).$$

Here v is a smooth vector field, ω is a q-form and $\mathscr{L}_v\omega$ is the Lie derivative of ω along v.

(a) If $f\in C^\infty(X)$ then $\langle V,df\rangle=V(f)$, $V\in C^\infty(X;\mathbb{C}T(X))$.

(b) If $\phi\in C^\infty(X;\mathbb{C}T^*(X))$ then

$$\langle V_1\wedge V_2,d\phi\rangle=V_1(\langle V_2,\phi\rangle)-V_2(\langle V_1,\phi\rangle)-\langle[V_1,V_2],\phi\rangle, \tag{2.6}$$

where V_1, $V_2\in C^\infty(X;\mathbb{C}T(X))$.

(c) If $\phi\in C^\infty(X;\Lambda^{q-1}(\mathbb{C}T^*(X)))$, $q\geq 2$, then

$$\begin{aligned}\left\langle V_1\wedge\cdots\wedge V_q,d\phi\right\rangle=&-\left\langle V_2\wedge\cdots\wedge V_q,d(V_1\lrcorner\phi)\right\rangle\\&+V_1(\left\langle V_2\wedge\cdots\wedge V_q,\phi\right\rangle)\\&-\left\langle[V_1,V_2]\wedge V_3\wedge\cdots\wedge V_q,\phi\right\rangle\\&-\cdots-\left\langle V_2\wedge V_3\wedge\cdots\wedge[V_1,V_q],\phi\right\rangle,\end{aligned} \tag{2.7}$$

where $V_j\in C^\infty(X;\mathbb{C}T(X))$, $j=1,\ldots,q$.

(d) For $\phi_1 \in C^\infty(X; \Lambda^r(\mathbb{C}T^*(X)))$, $\phi_2 \in C^\infty(X; \Lambda^s(\mathbb{C}T^*(X)))$, we have

$$d(\phi_1 \wedge \phi_2) = d\phi_1 \wedge \phi_2 + (-1)^r \phi_1 \wedge d\phi_2.$$

(e) $d^2 = 0$.

Let
$$\pi^{0,q} : \Lambda^q(\mathbb{C}T^*(X)) \to \Lambda^{0,q} T^*(X)$$
be the orthogonal projection map.

Definition 2.3. The tangential Cauchy-Riemann operator:

$$\overline{\partial}_b : C^\infty(X; \Lambda^{0,q} T^*(X)) \to C^\infty(X; \Lambda^{0,q+1} T^*(X))$$

is defined by
$$\overline{\partial}_b = \pi^{0,q+1} \circ d.$$

We will show that

$$\overline{\partial}_b : C^\infty(X; \Lambda^{0,q} T^*(X)) \to C^\infty(X; \Lambda^{0,q+1} T^*(X))$$

is a complex, i.e. $\overline{\partial}_b^2 = 0$. This will follow from the equation $d^2 = 0$ and some computations. We need the following

Lemma 2.4. *Let $q \geq 1$. Let ω be a smooth q-form. We assume that $\omega(x)$ annihilates $\Lambda^{0,q} T_x(X)$, for all $x \in X$. Then $(d\omega)(x)$ annihilates $\Lambda^{0,q+1} T_x(X)$, for all $x \in X$.*

Proof. We proceed by induction over q. For $q = 1$, we let

$$V_1, V_2 \in C^\infty(X; \Lambda^{0,1} T(X)),$$

then
$$\langle V_1 \wedge V_2, d\omega \rangle = V_1(\langle V_2, \omega \rangle) - V_2(\langle V_1, \omega \rangle) - \langle [V_1, V_2], \omega \rangle .$$
Since $[V_1, V_2] \in C^\infty(X; \Lambda^{0,1} T(X))$, we have $\langle [V_1, V_2], \omega \rangle = 0$. Thus,

$$\langle V_1 \wedge V_2, d\omega \rangle = 0.$$

Let $q \geq 2$. Let ω be a smooth q-form. We assume that $\omega(x)$ annihilates $\Lambda^{0,q} T_x(X)$, for all $x \in X$. Let

$$V_1, \dots, V_{q+1} \in C^\infty(X; \Lambda^{0,1} T(X)).$$

From (2.7), we have

$$\left\langle V_1\wedge\cdots\wedge V_{q+1}, d\omega\right\rangle = -\left\langle V_2\wedge\cdots\wedge V_{q+1}, d(V_1^{\lrcorner}\omega)\right\rangle + V_1(\left\langle V_2\wedge\cdots\wedge V_{q+1},\omega\right\rangle)$$
$$-\left\langle [V_1,V_2]\wedge V_3\wedge\cdots\wedge V_{q+1},\omega\right\rangle - \cdots - \left\langle V_2\wedge V_3\wedge\cdots\wedge [V_1,V_{q+1}],\omega\right\rangle.$$

Since

$$[V_1, V_j]\in C^\infty(X;\Lambda^{0,1}T(X)), j=2,\ldots,q+1,$$

and

$$V_1(\left\langle V_2\wedge\cdots\wedge V_{q+1},\omega\right\rangle)=0,$$

we have

$$\left\langle V_1\wedge\cdots\wedge V_{q+1}, d\omega\right\rangle = -\left\langle V_2\wedge\cdots\wedge V_{q+1}, d(V_1^{\lrcorner}\omega)\right\rangle.$$

By the induction assumption, we have

$$\left\langle V_2\wedge\cdots\wedge V_{q+1}, d(V_1^{\lrcorner}\omega)\right\rangle = 0.$$

Thus,

$$\left\langle V_1\wedge\cdots\wedge V_{q+1}, d\omega\right\rangle = 0.$$

The lemma follows. □

Proposition 2.5. *We have*

$$\overline{\partial}_b^2 = 0.$$

Proof. If $f\in C^\infty(X;\Lambda^{0,q}T^*(X))$, then

$$0 = d^2 f = d(\pi^{0,q+1}df + (I-\pi^{0,q+1})df).$$

Now $(I-\pi^{0,q+1})df$ annihilates $\Lambda^{0,q+1}T(X)$ pointwise. In view of Lemma 2.4, we see that

$$\pi^{0,q+2}d((I-\pi^{0,q+1})df) = 0.$$

Thus,

$$\begin{aligned}0 &= \pi^{0,q+2}d^2 f\\ &= \pi^{0,q+2}d\pi^{0,q+1}df + \pi^{0,q+2}d((I-\pi^{0,q+1})df)\\ &= \pi^{0,q+2}d\pi^{0,q+1}df\\ &= \overline{\partial}_b^2 f.\end{aligned}$$

The proposition follows. □

We take (dm) as the induced volume form on X introduced in the begining of the introduction. Let $(\,|\,)$ be the inner product on $C^\infty(X;\Lambda^{0,q}T^*(X))$ defined by

$$(f\mid g)=\int_X (f(z)\mid g(z))(dm),\ f,g\in C^\infty(X;\Lambda^{0,q}T^*(X)). \tag{2.8}$$

Let $\overline{\partial_b}^*$ be the formal adjoint of $\overline{\partial}_b$, that is

$$(\overline{\partial}_b f\mid h)=(f\mid \overline{\partial_b}^* h),\ f\in C^\infty(X;\Lambda^{0,q}T^*(X)), h\in C^\infty(X;\Lambda^{0,q+1}T^*(X)).$$

$\overline{\partial_b}^*$ is a first order differential operator and

$$\overline{\partial_b}^*:\cdots\leftarrow C^\infty(X;\Lambda^{0,q}T^*(X))\leftarrow C^\infty(X;\Lambda^{0,q+1}T^*(X))\leftarrow\cdots.$$

is a complex.

If $w\in\Lambda^{0,1}T^*_z(X)$, $z\in X$, let

$$w^{\wedge,*}:\Lambda^{0,q+1}T^*_z(X)\to\Lambda^{0,q}T^*_z(X)$$

be the adjoint of left exterior multiplication

$$w^\wedge:\Lambda^{0,q}T^*_z(X)\to\Lambda^{0,q+1}T^*_z(X).$$

That is,

$$(w^\wedge u\mid v)=(u\mid w^{\wedge,*}v),$$

for all $u\in\Lambda^{0,q}T^*_z(X)$, $v\in\Lambda^{0,q+1}T^*_z(X)$. Notice that $w^{\wedge,*}$ depends anti-linearly on w. Note that

$$\begin{aligned}(u\mid w\wedge v_1\wedge\cdots\wedge v_q)&=\left\langle\Gamma^{-1}w\wedge\Gamma^{-1}v_1\wedge\cdots\wedge\Gamma^{-1}v_q,u\right\rangle\\&=\left\langle\Gamma^{-1}v_1\wedge\cdots\wedge\Gamma^{-1}v_q,(\Gamma^{-1}w)^\lrcorner u\right\rangle\\&=((\Gamma^{-1}w)^\lrcorner u\mid v_1\wedge\cdots\wedge v_q)\end{aligned}$$

where $u\in\Lambda^{0,q+1}T^*_z(X)$ and $v_j\in\Lambda^{0,1}T^*_z(X)$, $j=1,\ldots,q$. Thus,

$$(\Gamma^{-1}w)^\lrcorner u=w^{\wedge,*}u,\ u\in\Lambda^{0,q+1}T^*_z(X).$$

Locally we can choose an orthonormal frame

$$\omega_1(z),\ldots,\omega_{n-1}(z)$$

for $\Lambda^{1,0}T^*_z(X)$, then

$$\overline{\omega}_1(z),\ldots,\overline{\omega}_{n-1}(z)$$

is an orthonormal frame for $\Lambda^{0,1}T_z^*(X)$. The $(2n-2)$-form

$$\omega = i^{n-1}\omega_1 \wedge \overline{\omega}_1 \wedge \cdots \wedge \omega_{n-1} \wedge \overline{\omega}_{n-1}$$

is real and is independent of the choice of the orthonormal frame. Thus ω can be considered as a globally defined $(2n-2)$-form. Locally there is a real 1-form $\omega_0(z)$ of length one which is orthogonal to

$$\Lambda^{1,0}T_z^*(X) \oplus \Lambda^{0,1}T_z^*(X).$$

$\omega_0(z)$ is unique up to the choice of sign. Since X is orientable, there is a nowhere vanishing $(2n-1)$-form Q on X. Thus, ω_0 can be specified uniquely by requiring that

$$\omega \wedge \omega_0 = fQ,$$

where f is a positive function. Therefore ω_0, so chosen, is a uniquely determined global 1-form. We call ω_0 the uniquely determined global real 1-form.

We have the pointwise orthogonal decomposition:

$$\mathbb{C}T^*(X) = \Lambda^{1,0}T^*(X) \oplus \Lambda^{0,1}T^*(X) \oplus \{\lambda\omega_0;\, \lambda \in \mathbb{C}\}. \tag{2.9}$$

We take Y (already introduced after Definition 2.2 and (2.4)) so that

$$\langle Y, \omega_0 \rangle = -1, \;\; \|Y\| = 1.$$

Therefore Y is uniquely determined. We call Y the uniquely determined global real vector field. Note that

$$\omega_0 = -\Gamma Y.$$

Definition 2.6. For $p \in X$, the Levi form L_p is the Hermitian quadratic form on $\Lambda^{1,0}T_p(X)$ defined as follows:

$$\begin{aligned}&\text{For any } Z,\, W \in \Lambda^{1,0}T_p(X), \text{ pick } \widetilde{Z},\, \widetilde{W} \in C^\infty(X;\Lambda^{1,0}T(X)) \text{ that satisfy}\\ &\widetilde{Z}(p) = Z,\, \widetilde{W}(p) = W. \text{ Then } L_p(Z, \overline{W}) = \frac{1}{2i}\left\langle [\widetilde{Z}\,, \overline{\widetilde{W}}](p)\,, \omega_0(p)\right\rangle.\end{aligned} \tag{2.10}$$

Here

$$[\widetilde{Z}\,, \overline{\widetilde{W}}] = \widetilde{Z}\overline{\widetilde{W}} - \overline{\widetilde{W}}\widetilde{Z}$$

denotes the commutator of $\widetilde{Z}$ and $\overline{\widetilde{W}}$.

The following lemma shows that the definition of the Levi form L_p is independent of the choices of $\widetilde{Z}$ and $\widetilde{W}$.

Lemma 2.7. *Let* $\widetilde{Z}, \widetilde{W} \in C^\infty(X; \Lambda^{1,0}T(X))$. *We have*

$$\frac{1}{2i}\left\langle [\widetilde{Z}\,, \overline{\widetilde{W}}](p)\,, \omega_0(p)\right\rangle = -\frac{1}{2i}\left\langle \widetilde{Z}(p) \wedge \overline{\widetilde{W}}(p), d\omega_0(p)\right\rangle. \tag{2.11}$$

Proof. In view of (2.6), we see that

$$\left\langle \widetilde{Z} \wedge \overline{\widetilde{W}}, d\omega_0\right\rangle = \widetilde{Z}(\left\langle \overline{\widetilde{W}}, \omega_0\right\rangle) - \overline{\widetilde{W}}(\left\langle \widetilde{Z}, \omega_0\right\rangle) - \left\langle [\widetilde{Z}, \overline{\widetilde{W}}], \omega_0\right\rangle.$$

Since ω_0 is pointwise orthogonal to

$$\Lambda^{1,0}T^*(X) \oplus \Lambda^{0,1}T^*(X),$$

it pointwise annihilates

$$\Lambda^{1,0}T(X) \oplus \Lambda^{0,1}T(X).$$

We have

$$\left\langle \overline{\widetilde{W}}, \omega_0\right\rangle = 0,\ \left\langle \widetilde{Z}, \omega_0\right\rangle = 0.$$

Thus,

$$\frac{1}{2i}\left\langle [\widetilde{Z}\,, \overline{\widetilde{W}}](p)\,, \omega_0(p)\right\rangle = -\frac{1}{2i}\left\langle \widetilde{Z}(p) \wedge \overline{\widetilde{W}}(p), d\omega_0(p)\right\rangle.$$

The lemma follows. □

Definition 2.8. The eigenvalues of the Levi form at $p \in X$ are the eigenvalues of the Hermitian form L_p with respect to the inner product $(\,|\,)$ on $\Lambda^{1,0}T_p(X)$.

For $U, V \in C^\infty(X; \Lambda^{1,0}T(X))$, we have $[U\,, \overline{V}](p) \in \mathbb{C}T_p(X)$, $p \in X$. In view of (2.4), we see that

$$[U\,, \overline{V}](p) = \lambda Y(p) + h,\ \ h \in \Lambda^{0,1}T_p(X) \oplus \Lambda^{1,0}T_p(X).$$

Note that

$$\langle h\,, \omega_0(p)\rangle = 0$$

and

$$\langle Y(p)\,, \omega_0(p)\rangle = -1.$$

In view of (2.10), we have

$$[U\,, \overline{V}](p) = -(2i)L_p(U(p)\,, \overline{V(p)})Y(p) + h. \tag{2.12}$$

Next we compute $\overline{\partial}_b$ and $\overline{\partial_b}^*$.

For each point $z_0 \in X$, we can choose an orthonormal frame

$$e_1(z), \ldots, e_{n-1}(z)$$

for $\Lambda^{0,1} T^*_z(X)$ varying smoothly with z in a neighborhood of z_0. Let $Z_j(z)$, $j = 1, \ldots, n-1$, denote the basis of $\Lambda^{0,1} T_z(X)$, which is dual to $e_j(z)$, $j = 1, \ldots, n-1$. We have

$$df = (\sum e_j^\wedge Z_j + \sum \overline{e}_j^\wedge \overline{Z}_j - \omega_0^\wedge Y) f,\ f \in C^\infty(X).$$

If $f(z) e_{j_1}(z) \wedge \cdots \wedge e_{j_q}(z) \in C^\infty(X; \Lambda^{0,q} T^*(X))$ is a typical term in a general $(0,q)$-form, we have

$$\begin{aligned} d(f(z) e_{j_1}(z) \wedge \cdots \wedge e_{j_q}(z)) = & \sum \big((Z_j f) e_j^\wedge + (\overline{Z}_j f) \overline{e}_j^\wedge - (Y f) \omega_0^\wedge \big) e_{j_1} \wedge \cdots \wedge e_{j_q} \\ & + \sum_{k=1}^{q} (-1)^{k-1} f(z) e_{j_1} \wedge \cdots \wedge (d e_{j_k}) \wedge \cdots \wedge e_{j_q}. \end{aligned}$$

Thus,

$$\begin{aligned} \overline{\partial}_b (f(z) e_{j_1}(z) \wedge \cdots \wedge e_{j_q}(z)) = & \sum_{j=1}^{n-1} Z_j(f) e_j(z)^\wedge e_{j_1} \wedge \cdots \wedge e_{j_q} \\ & + \sum_{k=1}^{q} (-1)^{k-1} f(z) e_{j_1} \wedge \cdots \wedge (\overline{\partial}_b e_{j_k}) \wedge \cdots \wedge e_{j_q} \\ = & (\sum_{j=1}^{n-1} Z_j(f) e_j^\wedge) e_{j_1} \wedge \cdots \wedge e_{j_q}) \\ & + (\sum_{j=1}^{n-1} (\overline{\partial}_b e_j)^\wedge e_j^{\wedge,*})(f(z) e_{j_1} \wedge \cdots \wedge e_{j_q}). \end{aligned}$$

For the given orthonormal frame, the map

$$e_j^\wedge \circ Z_j : C^\infty(X; \Lambda^{0,q} T^*(X)) \to C^\infty(X; \Lambda^{0,q+1} T^*(X))$$

is defined by

$$(e_j^\wedge \circ Z_j)(f(z) e_{j_1}(z) \wedge \cdots \wedge e_{j_q}(z)) = Z_j(f) e_j(z)^\wedge e_{j_1} \wedge \cdots \wedge e_{j_q}$$

and we extend the definition by linearity.

So for the given orthonormal frame we have the identification

$$\overline{\partial}_b \equiv \sum_{j=1}^{n-1} (e_j^\wedge \circ Z_j + (\overline{\partial}_b e_j)^\wedge e_j^{\wedge,*})$$

and correspondingly

$$\overline{\partial_b}^* \equiv \sum_{j=1}^{n-1} (e_j^{\wedge,*} \circ Z_j^* + e_j^{\wedge} (\overline{\partial}_b e_j)^{\wedge,*}),$$

where the map

$$e_j^{\wedge,*} \circ Z_j^* : C^\infty(X; \Lambda^{0,q+1} T^*(X)) \to C^\infty(X; \Lambda^{0,q} T^*(X))$$

is defined by

$$(e_j^{\wedge,*} \circ Z_j^*)(f(z) e_{j_1}(z) \wedge \cdots \wedge e_{j_{q+1}}(z)) = Z_j^*(f) e_j(z)^{\wedge,*} e_{j_1} \wedge \cdots \wedge e_{j_{q+1}}$$

and we extend the definition by linearity.

The Kohn Laplacian $\Box_b$ is given by

$$\Box_b = \overline{\partial}_b \overline{\partial_b}^* + \overline{\partial_b}^* \overline{\partial}_b.$$

From now on, we write $\Box_b^{(q)}$ to denote the restriction to $(0,q)$-forms. We have

$$\begin{aligned}
\Box_b^{(q)} &= \sum_{j,k=1}^{n-1} \Big[\big(e_j^{\wedge} \circ Z_j + (\overline{\partial}_b e_j)^{\wedge} e_j^{\wedge,*} \big) \big(e_k^{\wedge,*} \circ Z_k^* + e_k^{\wedge} (\overline{\partial}_b e_k)^{\wedge,*} \big) \\
&\quad + \big(e_k^{\wedge,*} \circ Z_k^* + e_k^{\wedge} (\overline{\partial}_b e_k)^{\wedge,*} \big) \big(e_j^{\wedge} \circ Z_j + (\overline{\partial}_b e_j)^{\wedge} e_j^{\wedge,*} \big) \Big] \\
&= \sum_{j,k=1}^{n-1} \Big[(e_j^{\wedge} \circ Z_j)(e_k^{\wedge,*} \circ Z_k^*) + (e_k^{\wedge,*} \circ Z_k^*)(e_j^{\wedge} \circ Z_j) \Big] \\
&\quad + \varepsilon(Z) + \varepsilon(Z^*) + \text{zero order terms} \\
&= \sum_{j,k=1}^{n-1} (e_j^{\wedge} e_k^{\wedge,*} \circ Z_j Z_k^* + e_k^{\wedge,*} e_j^{\wedge} \circ Z_k^* Z_j) \\
&\quad + \varepsilon(Z) + \varepsilon(Z^*) + \text{zero order terms} \\
&= \sum_{j,k=1}^{n-1} (e_j^{\wedge} e_k^{\wedge,*} + e_k^{\wedge,*} e_j^{\wedge}) \circ Z_k^* Z_j + \sum_{j,k=1}^{n-1} e_j^{\wedge} e_k^{\wedge,*} \circ [Z_j \,, Z_k^*] \\
&\quad + \varepsilon(Z) + \varepsilon(Z^*) + \text{zero order terms},
\end{aligned} \tag{2.13}$$

where $\varepsilon(Z)$ denotes remainder terms of the form $\sum_{k=1}^{n-1} a_k(z) Z_k$ with $a_k(z) \in C^\infty$, matrix-valued and similarly for $\varepsilon(Z^*)$.

Note that

$$e_j^{\wedge} e_k^{\wedge,*} + e_k^{\wedge,*} e_j^{\wedge} = \delta_{j,k}. \tag{2.14}$$

We obtain the following

Proposition 2.9. *The Kohn Laplacian* $\Box_b^{(q)}$ *is given by*

$$\begin{aligned}\Box_b^{(q)} &= \overline{\partial}_b\overline{\partial_b}^* + \overline{\partial_b}^*\overline{\partial}_b \\ &= \sum_{j=1}^{n-1} Z_j^* Z_j + \sum_{j,k=1}^{n-1} e_j^\wedge e_k^{\wedge,*} \circ [Z_j, Z_k^*] \\ &\quad + \varepsilon(Z) + \varepsilon(Z^*) + \text{zero order terms},\end{aligned}$$

where $\varepsilon(Z)$ *denotes remainder terms of the form* $\sum a_k(z)Z_k$ *with* $a_k(z)$ *smooth, matrix-valued and similarly for* $\varepsilon(Z^*)$.

3 The hypoellipicity of $\Box_b$

We work with some real local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on an open set $\Omega \subset X$. In view of Proposition 2.9, we have

$$\begin{aligned}\Box_b^{(q)} &= \overline{\partial}_b\overline{\partial_b}^* + \overline{\partial_b}^*\overline{\partial}_b \\ &= \sum_{j=1}^{n-1} Z_j^* Z_j + \sum_{j,k=1}^{n-1} e_j^\wedge e_k^{\wedge,*} \circ [Z_j, Z_k^*] \\ &\quad + \varepsilon(Z) + \varepsilon(Z^*) + \text{zero order terms},\end{aligned} \tag{3.1}$$

where $\varepsilon(Z)$ and $\varepsilon(Z^*)$ are as in Proposition 2.9. Let q_j, $j = 1, \ldots, n-1$, be the principal symbols of Z_j, $j = 1, \ldots, n-1$. The principal symbol of $\Box_b^{(q)}$ is

$$p_0 = \sum_{j=1}^{n-1} \overline{q}_j q_j. \tag{3.2}$$

The characteristic manifold Σ of $\Box_b^{(q)}$ is

$$\Sigma = \left\{(x, \xi) \in T^*(X) \setminus 0;\ \xi = \lambda\omega_0(x), \lambda \neq 0\right\}, \tag{3.3}$$

where ω_0 is the uniquely determined global real 1-form.

From (3.2), we see that p_0 vanishes to second order at Σ. Thus, Σ is a doubly characteristic manifold of $\Box_b^{(q)}$ and the subprincipal symbol of $\Box_b^{(q)}$ is well-defined on Σ. (For the precise meanings of doubly characteristic manifold and subprincipal symbol, see Definition A.10, Definition A.11, Definition A.25 and Definition A.26.) For an operator of the form $Z_j^* Z_j$ this subprincipal symbol is given by $\frac{1}{2i}\left\{\overline{q}_j, q_j\right\}$ and the contribution from the double sum in (3.1) to the subprincipal symbol of $\Box_b^{(q)}$ is

$$\frac{1}{i}\sum_{j,k=1}^{n-1} e_j^\wedge e_k^{\wedge,*} \circ \left\{q_j, \overline{q}_k\right\},$$

where $\{q_j, \overline{q}_k\}$ denotes the Poisson bracket of q_j and $\overline{q}_k$. (See Definition A.35.) We get the subprincipal symbol of $\Box_b^{(q)}$ on Σ (see Lemma A.12),

$$p_0^s = (\sum_{j=1}^{n-1} -\frac{1}{2i}\{q_j, \overline{q}_j\}) + \sum_{j,k=1}^{n-1} e_j^{\wedge} e_k^{\wedge,*} \frac{1}{i}\{q_j, \overline{q}_k\}. \tag{3.4}$$

From (2.12), we see that

$$[\overline{Z}_k, Z_j] = -(2i)L(\overline{Z}_k, Z_j)Y \bmod (\Lambda^{1,0}T(X) \oplus \Lambda^{0,1}T(X)),$$

where Y is the uniquely determined global real vector field. Note that the principal symbol of $\overline{Z}_k$ is $-\overline{q}_k$. Hence,

$$\{\overline{q}_k, q_j\} = (2i)L(\overline{Z}_k, Z_j)\sigma_{iY} \text{ on } \Sigma, \tag{3.5}$$

where σ_{iY} is the principal symbol of iY. Thus,

$$p_0^s = (\sum_{j=1}^{n-1} L(\overline{Z}_j, Z_j) - \sum_{j,k=1}^{n-1} 2e_j^{\wedge} e_k^{\wedge,*} L(\overline{Z}_k, Z_j))\sigma_{iY} \text{ on } \Sigma. \tag{3.6}$$

In the rest of this section, we need some basic notions of symplectic geometry. See appendix A, after Definition A.26, for a reivew.

From now on, for any $f \in C^\infty(T^*(X))$, we write H_f to denote the Hamilton field of f. (See Definition A.34.) We need the following

Lemma 3.1. *Σ is a symplectic submanifold of $T^*(X)$ if and only if the Levi form is non-degenerate at each point of X.*

Proof. Note that

$$\Sigma = \{(x,\xi) \in T^*(X) \setminus 0;\, q_1(x,\xi) = \cdots = q_{n-1}(x,\xi) = \overline{q}_1(x,\xi) \cdots = \overline{q}_{n-1}(x,\xi) = 0\}.$$

Let $\mathbb{C}T_\rho(\Sigma)$ and $\mathbb{C}T_\rho(T^*(X))$ be the complexifications of $T_\rho(\Sigma)$ and $T_\rho(T^*(X))$ respectively, where $\rho \in \Sigma$. For $\rho \in \Sigma$, we can choose the basis

$$H_{q_1}, \ldots, H_{q_{n-1}}, H_{\overline{q}_1}, \ldots, H_{\overline{q}_{n-1}}$$

for $T_\rho(\Sigma)^\perp$, where $T_\rho(\Sigma)^\perp$ is the orthogonal to $\mathbb{C}T_\rho(\Sigma)$ in $\mathbb{C}T_\rho(T^*(X))$ with respect to canonical two form,

$$\sigma = d\xi \wedge dx.$$

In view of (3.5), we have

$$\sigma(H_{q_j}, H_{\overline{q}_k}) = \{q_j, \overline{q}_k\} = \frac{2}{i}L(\overline{Z}_k, Z_j)\sigma_{iY} \text{ on } \Sigma. \tag{3.7}$$

We notice that $\left\{q_j, q_k\right\} = 0$ on Σ. Thus, if the Levi form is non-degenerate at each point of X, then σ is non-degenerate on $T_\rho(\Sigma)^\perp$, hence also on $\mathbb{C}T_\rho(\Sigma)$ and Σ is therefore symplectic. Notice also that in that case

$$\mathbb{C}T_\rho(\Sigma)\bigcap T_\rho(\Sigma)^\perp = 0.$$

If Σ is a symplectic submanifold of $T^*(X)$, from (3.7), it follows that the Levi form is non-degenerate at each point of X. The lemma follows. □

Let F_ρ be the fundamental matrix of p_0 at $\rho = (p, \xi_0) \in \Sigma$. (see the discussion before Remark A.43.) We can choose the basis

$$H_{q_1}, \ldots, H_{q_{n-1}}, H_{\overline{q}_1}, \ldots, H_{\overline{q}_{n-1}}$$

for $\mathbb{C}T_\rho(T^*(X)) \setminus \mathbb{C}T_\rho(\Sigma)$. We notice that

$$H_{p_0} = \sum_j (\overline{q}_j H_{q_j} + q_j H_{\overline{q}_j}).$$

We compute the linearization of H_{p_0} at ρ

$$\begin{aligned} H_{p_0}(\rho + \sum(t_k H_{q_k} + s_k H_{\overline{q}_k})) = O(|t, s|^2) &+ \sum_{j,k} t_k \left\{q_k, \overline{q}_j\right\} H_{q_j} \\ &+ \sum_{j,k} s_k \left\{\overline{q}_k, q_j\right\} H_{\overline{q}_j}. \end{aligned}$$

So the matrix F_ρ of the linearization is expressed in the basis

$$H_{q_1}, \ldots, H_{q_{n-1}}, H_{\overline{q}_1}, \ldots, H_{\overline{q}_{n-1}}$$

by

$$F_\rho = \begin{pmatrix} \left\{q_k, \overline{q}_j\right\} & 0 \\ 0 & \left\{\overline{q}_k, q_j\right\} \end{pmatrix}. \tag{3.8}$$

Again, from (3.5), we see that the non-vanishing eigenvalues of F_ρ are

$$\pm 2i\lambda_j \sigma_{iY}(\rho), \tag{3.9}$$

where λ_j, $j = 1, \ldots, n-1$, are the eigenvalues of L_p.

To compute further, we assume that the Levi form is diagonalized at the given point $p \in X$. Then

$$\sum_{j,k} 2e_j^\wedge e_k^{\wedge,*} L_p(\overline{Z}_k, Z_j)\sigma_{iY} = \sum_j 2e_j^\wedge e_j^{\wedge,*} L_p(\overline{Z}_j, Z_j)\sigma_{iY}. \tag{3.10}$$

From above, we see that on Σ and on the space of $(0,q)$ forms, $p_0^s+\frac{1}{2}\widetilde{\operatorname{tr}}\,F$ has the eigenvalues

$$\begin{gathered}\sum_{j=1}^{n-1}\left|\lambda_j\right||\sigma_{iY}|+\sum_{j\notin J}\lambda_j\sigma_{iY}-\sum_{j\in J}\lambda_j\sigma_{iY},\ |J|=q,\\ J=(j_1,j_2,\ldots,j_q),\ 1\le j_1<j_2<\cdots<j_q\le n-1,\end{gathered}\tag{3.11}$$

where $\widetilde{\operatorname{tr}}\,F_p$ denotes $\sum\left|\mu_j\right|$, $\pm\mu_j$ are the non-vanishing eigenvalues of F_p. Put

$$\Sigma^-=\left\{(x,\xi)\in T^*(X)\smallsetminus 0;\ \xi=\lambda\omega_0(x),\lambda<0\right\}$$

and

$$\Sigma^+=\left\{(x,\xi)\in T^*(X)\smallsetminus 0;\ \xi=\lambda\omega_0(x),\lambda>0\right\}.$$

We assume that the Levi form is non-degenerate at each point of X. Then the Levi form has constant signature (n_-,n_+), $n_-+n_+=n-1$. Since $\langle Y,\omega_0\rangle=-1$, we have $\sigma_{iY}>0$ on Σ^+, $\sigma_{iY}<0$ on Σ^-.

Let

$$\inf(p_0^s+\frac{1}{2}\widetilde{\operatorname{tr}}\,F)=\inf\left\{\lambda;\ \lambda:\text{eigenvalue of } p_0^s+\frac{1}{2}\widetilde{\operatorname{tr}}\,F\right\}.$$

From (3.11), we see that on Σ^+

$$\inf(p_0^s+\frac{1}{2}\widetilde{\operatorname{tr}}\,F)\begin{cases}=0, & q=n_+\\ >0, & q\neq n_+\end{cases}.\tag{3.12}$$

On Σ^-

$$\inf(p_0^s+\frac{1}{2}\widetilde{\operatorname{tr}}\,F)\begin{cases}=0, & q=n_-\\ >0, & q\neq n_-\end{cases}.\tag{3.13}$$

Definition 3.2. Given q, $0\le q\le n-1$, the Levi form is said to satisfy condition $Y(q)$ at $p\in X$ if for any $|J|=q$, $J=(j_1,j_2,\ldots,j_q)$, $1\le j_1<j_2<\cdots<j_q\le n-1$, we have

$$\left|\sum_{j\notin J}\lambda_j-\sum_{j\in J}\lambda_j\right|<\sum_{j=1}^{n-1}\left|\lambda_j\right|,$$

where λ_j, $j=1,\ldots,(n-1)$, are the eigenvalues of L_p. If the Levi form is non-degenerate at p, then the condition is equivalent to $q\neq n_+$, n_-, where (n_-,n_+), $n_-+n_+=n-1$, is the signature of L_p.

$$\begin{gathered}\text{From now on, we assume that the Levi form}\\ \text{is non-degenerate at each point of } X.\end{gathered}\tag{3.14}$$

From (3.11), (3.12), (3.13) and Definition 3.2, we have the following

Proposition 3.3. *Let (n_-, n_+), $n_- + n_+ = n-1$, be the signature of the Levi form L. $p^s_0 + \frac{1}{2}\widetilde{\operatorname{tr}}\, F$ is positive semi-definite and we have that (3.12), (3.13) hold and $p^s_0 + \frac{1}{2}\widetilde{\operatorname{tr}}\, F$ is positive definite when $Y(q)$ holds, precisely when $q \notin \{n_+, n_-\}$.*

Let Ω be an open set in $\mathbb{R}^N$. Let P be a classical pseudodifferential operator on Ω of order $m > 1$. P is said to be hypoelliptic with loss of one derivative if $u \in \mathscr{E}'(\Omega)$ and $Pu \in H^s_{\rm loc}(\Omega)$ implies $u \in H^{s+m-1}_{\rm comp}(\Omega)$.

We recall classical works by Boutet de Monvel [7] and Sjöstrand [21].

Proposition 3.4. *Let Ω be an open set in $\mathbb{R}^N$. Let P be a classical pseudodifferential operator on Ω of order $m > 1$. The symbol of P takes the form*

$$\sigma_P(x,\xi) \sim p_m(x,\xi) + p_{m-1}(x,\xi) + p_{m-2}(x,\xi) + \cdots,$$

where p_j is positively homogeneous of degree j. We assume that $\Sigma = p^{-1}_m(0)$ is a symplectic submanifold of codimension $2d$, $p_m \geq 0$ and p_m vanishes to precisely second order on Σ. Let F be the fundamental matrix of p_m. Let p^s_m be the subprincipal symbol of P. Then P is hypoelliptic with loss of one derivative if and only if

$$p^s_m(\rho) + \sum^d_{j=1} (\frac{1}{2} + \alpha_j) \left|\mu_j\right| \neq 0$$

at every point $\rho \in \Sigma$ for all $(\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$, where $\pm i\mu_j$ are the eigenvalues of F at ρ.

Proposition 3.4 also holds if P is a matrix-valued classical pseudodifferential operator on Ω of order $m > 1$ with scalar principal symbol.

From Proposition 3.4, we have the following

Proposition 3.5. *We recall that we work with the assumption that the Levi form L is non-degenerate at each point of X. $\Box^{(q)}_b$ is hypoelliptic with loss of one derivative if and only if $Y(q)$ holds at each point of X.*

Remark 3.6. Kohn's L^2 estimates give the hypoellipicity with loss of one dervative for the solutions $\Box^{(q)}_b u = f$ under condition $Y(q)$. (See Folland-Kohn [11].) Kohn's method works as well when the Levi form L is degenerate.

4 The characteristic equation

In this section, we consider the characteristic equation for $\partial_t + \Box^{(q)}_b$.

Let $p_0(x,\xi)$ be the principal symbol of $\Box^{(q)}_b$. We work with some real local coordinates $x = (x_1, x_2, \ldots, x_{2n-1})$ defined on an open set $\Omega \subset X$. We identify Ω with an open set in $\mathbb{R}^{2n-1}$. Let $\Omega^{\mathbb{C}}$ be an almost complexification of Ω. That is, $\Omega^{\mathbb{C}}$ is an open set in $\mathbb{C}^{2n-1}$ with $\Omega^{\mathbb{C}}\bigcap\mathbb{R}^{2n-1}=\Omega$. We identify $T^*(\Omega)$ with $\Omega\times\mathbb{R}^{2n-1}$. Similarly, let $T^*(\Omega)_{\mathbb{C}}$ be an open set in $\mathbb{C}^{2n-1}\times\mathbb{C}^{2n-1}$ with $T^*(\Omega)_{\mathbb{C}}\bigcap(\mathbb{R}^{2n-1}\times\mathbb{R}^{2n-1})=T^*(\Omega)$. In this section, for any function f, we also write f to denote an almost analytic extension. (See Definition B.1.) We look for solutions $\psi(t,x,\eta)\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega)\setminus 0)$ of the problem

$$\begin{cases}\dfrac{\partial\psi}{\partial t}-ip_0(x,\psi'_x)=O(\left|\operatorname{Im}\psi\right|^N),\ \forall N\geq 0,\\ \psi|_{t=0}=\langle x\,,\eta\rangle\end{cases}\tag{4.1}$$

with $\operatorname{Im}\psi(t,x,\eta)\geq 0$. More precisely, we look for solutions $\psi(t,x,\eta)\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega)\setminus 0)$ with $\operatorname{Im}\psi(t,x,\eta)\geq 0$ such that $\psi|_{t=0}=\langle x\,,\eta\rangle$ and for every compact set $K\subset T^*(\Omega)\setminus 0$, $N\geq 0$, there exists $c_{K,N}\geq 0$, such that

$$\left|\frac{\partial\psi}{\partial t}-ip_0(x,\psi'_x)\right|\leq c_{K,N}\left|\operatorname{Im}\psi\right|^N\ \text{ on } \overline{\mathbb{R}}_+\times K.$$

Let $f(x,\xi)$, $g(x,\xi)\in C^\infty(T^*(\Omega)_{\mathbb{C}})$. We write

$$f=g \mod |\operatorname{Im}(x,\xi)|^\infty$$

if, given any compact subset K of $T^*(\Omega)_{\mathbb{C}}$ and any integer $N>0$, there is a constant $c>0$ such that

$$\left|(f-g)(x,\xi)\right|\leq c\,|\operatorname{Im}(x,\xi)|^N,\ \forall(x,\xi)\in K.$$

Let U and V be C^∞ complex vector fields on $T^*(\Omega)_{\mathbb{C}}$. We write

$$U=V \mod |\operatorname{Im}(x,\xi)|^\infty$$

if

$$U(f)=V(f) \mod |\operatorname{Im}(x,\xi)|^\infty$$

and

$$U(\overline{f})=V(\overline{f}) \mod |\operatorname{Im}(x,\xi)|^\infty,$$

for all almost analytic functions f on $T^*(\Omega)_{\mathbb{C}}$. In Appendix B, we discuss the notions of almost analytic vector fields and equivalence of almost analytic vector fields.

In the complex domain, the Hamiltonian field H_{p_0} is given by

$$H_{p_0}=\frac{\partial p_0}{\partial\xi}\frac{\partial}{\partial x}-\frac{\partial p_0}{\partial x}\frac{\partial}{\partial\xi}.$$

We notice that H_{p_0} depends on the choice of almost analytic extension of p_0 but we can give an invariant meaning of the exponential map $\exp(-itH_{p_0})$, $t \geq 0$. Note that H_{p_0} vanishes on Σ. We consider the real vector field

$$-iH_{p_0} + \overline{-iH_{p_0}}.$$

Let $\Phi(t,\rho)$ be the $-iH_{p_0} + \overline{-iH_{p_0}}$ flow. We notice that for every $T > 0$ there is an open neighborhood U of Σ in $T^*(\Omega)_{\mathbb{C}}$ such that for all $0 \leq t \leq T$, $\Phi(t,\rho)$ is well-defined if $\rho \in U$. Since we only need to consider Taylor expansions at Σ, for the convenience, we assume that $\Phi(t,\rho)$ is well-defined for all $t \geq 0$ and $\rho \in T^*(\Omega)_{\mathbb{C}}$. We have the following

Proposition 4.1. *Let $\Phi(t,\rho)$ be as above. Let U be a real vector field on $T^*(\Omega)_{\mathbb{C}}$ such that*

$$U = -iH_{p_0} + \overline{-iH_{p_0}} \mod |\mathrm{Im}(x,\xi)|^\infty.$$

Let $\hat{\Phi}(t,\rho)$ be the U flow. Then, for every compact set $K \subset T^*(\Omega)_{\mathbb{C}}$, $N \geq 0$, there is $c_{N,K}(t) > 0$, such that

$$\left|\Phi(t,\rho) - \hat{\Phi}(t,\rho)\right| \leq c_{N,K}(t)\mathrm{dist}(\rho,\Sigma)^N,\ \rho \in K.$$

Proof. This follows from Proposition B.13. □

For $t \geq 0$, let

$$G_t = \left\{(\rho, \Phi(t,\rho));\ \rho \in T^*(\Omega)_{\mathbb{C}}\right\}, \tag{4.2}$$

where $\Phi(t,\rho)$ is as in Proposistion 4.1. We call G_t the graph of $\exp(-itH_{p_0})$. Since H_{p_0} vanishes on Σ, we have

$$\Phi(t,\rho) = \rho \quad \text{if } \rho \in \Sigma.$$

G_t depends on the choice of almost analytic extension of p_0. Let $\hat{p}_0$ be another almost analytic extension of p_0. Let $\hat{G}_t$ be the graph of $\exp(-itH_{\hat{p}_0})$. From Proposition 4.1, it follows that G_t coincides to infinite order with $\hat{G}_t$ on $\mathrm{diag}(\Sigma \times \Sigma)$, for all $t \geq 0$.

In Menikoff-Sjöstrand [20], it was shown that there exist $g(t,x,\eta)$, $h(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega)_{\mathbb{C}})$ such that

$$G_t = \left\{(x, g(t,x,\eta), h(t,x,\eta), \eta);\ (x,\eta) \in T^*(\Omega)_{\mathbb{C}}\right\}.$$

Moreover, there exists $\psi(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega)_{\mathbb{C}})$ such that

$$g(t,x,\eta) - \psi'_x(t,x,\eta)$$

and

$$h(t,x,\eta)-\psi'_{\eta}(t,x,\eta)$$

vanish to infinite order on Σ, for all $t\geq 0$. Furthermore, when (t,x,η) is real, $\psi(t,x,\eta)$ solves (4.1) and we have,

$$\operatorname{Im}\psi(t,x,\eta)\asymp\frac{t}{1+t}(\operatorname{dist}((x,\eta),\Sigma))^2,\ \ t\geq 0,\ \left|\eta\right|=1. \tag{4.3}$$

For the precise meaning of $\asymp$, see the discussion after Proposition 1.8. Moreover, we have the following

Proposition 4.2. *There exists* $\psi(t,x,\eta)\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega)\setminus 0)$ *such that* $\operatorname{Im}\psi\geq 0$ *with equality precisely on* $(\{0\}\times T^*(\Omega)\setminus 0)\bigcup(\mathbb{R}_+\times\Sigma)$ *and such that* (4.1) *holds where the error term is uniform on every set of the form* $[0,T]\times K$ *with* $T>0$ *and* $K\subset T^*(\Omega)\setminus 0$ *compact. Furthermore,* ψ *is unique up to a term which is* $O(\left|\operatorname{Im}\psi\right|^N)$ *locally uniformly for every* N *and*

$$\psi(t,x,\eta)=\langle x,\eta\rangle\ \text{on}\ \Sigma,\ d_{x,\eta}(\psi-\langle x,\eta\rangle)=0\ \text{on}\ \Sigma.$$

Moreover, we have

$$\operatorname{Im}\psi(t,x,\eta)\asymp\left|\eta\right|\frac{t\left|\eta\right|}{1+t\left|\eta\right|}\operatorname{dist}((x,\frac{\eta}{\left|\eta\right|}),\Sigma))^2,\ \ t\geq 0,\ \left|\eta\right|\geq 1. \tag{4.4}$$

Proposition 4.3. *There exists a function* $\psi(\infty,x,\eta)\in C^\infty(T^*(\Omega)\setminus 0)$ *with a uniquely determined Taylor expansion at each point of* Σ *such that*

> *For every compact set* $K\subset T^*(\Omega)\setminus 0$ *there is a constant* $c_K>0$ *such that*
>
> $\operatorname{Im}\psi(\infty,x,\eta)\geq c_K\left|\eta\right|(\operatorname{dist}((x,\frac{\eta}{\left|\eta\right|}),\Sigma))^2,$
>
> $d_{x,\eta}(\psi(\infty,x,\eta)-\langle x,\eta\rangle)=0$ on Σ.

If $\lambda\in C(T^*(\Omega)\setminus 0)$*,* $\lambda>0$ *and* $\lambda|_\Sigma<\min\left|\lambda_j\right|$*, where* $\pm i\left|\lambda_j\right|$ *are the non-vanishing eigenvalues of the fundamental matrix of* $\Box_b^{(q)}$*, then the solution* $\psi(t,x,\eta)$ *of* (4.1) *can be chosen so that for every compact set* $K\subset T^*(\Omega)\setminus 0$ *and all indices* α*,* β*,* γ*, there is a constant* $c_{\alpha,\beta,\gamma,K}$ *such that*

$$\left|\partial_x^\alpha\partial_\eta^\beta\partial_t^\gamma(\psi(t,x,\eta)-\psi(\infty,x,\eta))\right|\leq c_{\alpha,\beta,\gamma,K}e^{-\lambda(x,\eta)t}\ \text{on}\ \overline{\mathbb{R}}_+\times K. \tag{4.5}$$

For the proofs of Proposition 4.2 and Proposition 4.3, we refer the reader to Menikoff-Sjöstrand [20]. From the positively homogeneity of p_0, it follows that we can choose $\psi(t,x,\eta)$ in Proposition 4.2 to be quasi-homogeneous of degree 1 in the sense that

$$\psi(t,x,\lambda\eta)=\lambda\psi(\lambda t,x,\eta),\ \ \lambda>0.$$

(See Definition 5.1.) This makes $\psi(\infty,x,\eta)$ positively homogeneous of degree 1.

We recall that

$$p_0=q_1\overline{q}_1+\cdots+q_{n-1}\overline{q}_{n-1}.$$

We can take an almost analytic extension of p_0 so that

$$p_0(x,\xi)=\overline{p}_0(\overline{x},\overline{\xi}). \tag{4.6}$$

From (4.6), we have

$$-\frac{\partial\overline{\psi}}{\partial t}(t,x,-\eta)-ip_0(x,\overline{\psi}'_x(t,x,-\eta))=O(\left|\operatorname{Im}\psi\right|^N),\ t\geq 0,$$

for all $N\geq 0$, (x,η) real. Since $p_0(x,-\xi)=p_0(x,\xi)$, we have

$$-\frac{\partial\overline{\psi}}{\partial t}(t,x,-\eta)-ip_0(x,-\overline{\psi}'_x(t,x,-\eta))=O(\left|\operatorname{Im}\psi\right|^N),\ t\geq 0, \tag{4.7}$$

for all $N\geq 0$, (x,η) real. From Proposition 4.2, we can take $\psi(t,x,\eta)$ so that

$$\psi(t,x,\eta)=-\overline{\psi}(t,x,-\eta),\ \ (x,\eta)\text{ is real}. \tag{4.8}$$

Hence,

$$\psi(\infty,x,\eta)=-\overline{\psi}(\infty,x,-\eta),\ \ (x,\eta)\text{ is real}. \tag{4.9}$$

Put

$$\widetilde{G}_t=\left\{(\overline{y},\overline{\eta},\overline{x},\overline{\xi});(x,\xi,y,\eta)\in G_t\right\},$$

where G_t is defined by (4.2). From (4.6), it follows that

$$\Phi(t,\overline{\rho})=\overline{\Phi}(-t,\rho),$$

where $\Phi(t,\rho)$ is as in Proposition 4.1. Thus, for all $t\geq 0$,

$$G_t=\widetilde{G}_t. \tag{4.10}$$

Put

$$C_t=\left\{(x,\psi'_x(t,x,\eta),\psi'_\eta(t,x,\eta),\eta);(x,\eta)\in T^*(\Omega^{\mathbb{C}})\right\} \tag{4.11}$$

and

$$\widetilde{C}_t=\left\{(\overline{y},\overline{\eta},\overline{x},\overline{\xi});(x,\xi,y,\eta)\in C_t\right\}. \tag{4.12}$$

Since C_t coincides to infinite order with G_t on $\operatorname{diag}(\Sigma\times\Sigma)$, for all $t\geq 0$, from (4.10), it follows that C_t coincides to infinite order with $\widetilde{C}_t$ on $\operatorname{diag}(\Sigma\times\Sigma)$, for all $t\geq 0$. Letting $t\to\infty$, we get the following

Proposition 4.4. *Let*

$$C_\infty = \left\{(x, \psi'_x(\infty,x,\eta), \psi'_\eta(\infty,x,\eta), \eta); (x,\eta) \in T^*(\Omega^{\mathbb{C}})\right\} \tag{4.13}$$

and

$$\widetilde{C}_\infty = \left\{(\overline{y}, \overline{\eta}, \overline{x}, \overline{\xi}); (x,\xi,y,\eta) \in C_\infty\right\}. \tag{4.14}$$

Then $\widetilde{C}_\infty$ *coincides to infinite order with* C_∞ *on* $\operatorname{diag}(\Sigma \times \Sigma)$.

From Proposition 4.4 and the global theory of Fourier integral operators (see Proposition B.21), we have the following

Proposition 4.5. *The two phases*

$$\psi(\infty,x,\eta) - \langle y,\eta\rangle \in C^\infty(\Omega\times\Omega\times\dot{\mathbb{R}}^{2n-1}),\ -\overline{\psi}(\infty,y,\eta) + \langle x,\eta\rangle \in C^\infty(\Omega\times\Omega\times\dot{\mathbb{R}}^{2n-1})$$

are equivalent in the sense of Definition B.20.

We recall that

$$\Sigma = \left\{(x,\xi) \in T^*(\Omega) \setminus 0;\ q_1(x,\xi) = \cdots = q_{n-1}(x,\xi) = \overline{q}_1(x,\xi) \cdots = \overline{q}_{n-1}(x,\xi) = 0\right\}.$$

For any function $f \in C^\infty(T^*(\Omega))$, we use $\widetilde{f}$ to denote an almost analytic extension with respect to the weight function $\operatorname{dist}((x,\xi),\Sigma)$. (See Definition B.1.) Set

$$\widetilde{\Sigma} = \left\{(x,\xi) \in T^*(\Omega)_{\mathbb{C}} \setminus 0;\ \widetilde{q}_1(x,\xi) = \cdots = \widetilde{q}_{n-1}(x,\xi) = \widetilde{\overline{q}}_1(x,\xi) \cdots = \widetilde{\overline{q}}_{n-1}(x,\xi) = 0\right\}. \tag{4.15}$$

We say that $\widetilde{\Sigma}$ is an almost analytic extension with respect to the weight function $\operatorname{dist}((x,\xi),\Sigma)$ of Σ. Let $f(x,\xi)$, $g(x,\xi) \in C^\infty(W)$, where W is an open set in $T^*(\Omega)_{\mathbb{C}}$. We write

$$f = g \mod d^\infty_\Sigma$$

if, given any compact subset K of W and any integer $N > 0$, there is a constant $c > 0$ such that

$$\left|(f-g)(x,\xi)\right| \leq c\operatorname{dist}((x,\xi),\Sigma)^N,\ \forall (x,\xi) \in K.$$

From the global theory of Fourier integral operators (see Proposition B.21), we only need to consider Taylor expansions at Σ. We may work with the following coordinates

Proposition 4.6. *Let* $\rho \in \Sigma$. *Then in some open neighborhood* Γ *of* ρ *in* $T^*(\Omega)_{\mathbb{C}}$, *there are* C^∞ *functions*

$$\widetilde{x}_j \in C^\infty(\Gamma),\ \widetilde{\xi}_j \in C^\infty(\Gamma),\ j = 1,\ldots,2n-1,$$

such that

(a) $\tilde{x}_j$, $\tilde{\xi}_j$, $j = 1,\dots,2n-1$, *are almost analytic functions with respect to the weight function* $\mathrm{dist}((x,\xi),\Sigma)$.

(b)

$$\det\Big(\frac{\partial(x,\xi)}{\partial(\tilde{x},\tilde{\xi})}\Big)\neq 0 \text{ on } (\Gamma)_{\mathbb{R}},$$

where $(\Gamma)_{\mathbb{R}} = \Gamma\bigcap T^*(\Omega)$ *and* $\tilde{x} = (\tilde{x}_1,\dots,\tilde{x}_{2n-1})$, $\tilde{\xi} = (\tilde{\xi}_1,\dots,\tilde{\xi}_{2n-1})$.

(c) $\tilde{x}_j$, $\tilde{\xi}_j$, $j = 1,\dots,2n-1$, *form local coordinates of* Γ.

(d) $(\tilde{x},\tilde{\xi})$ *is symplectic to infinite order on* Σ. *That is,*

$$\{\tilde{x}_j,\tilde{x}_k\} = 0 \bmod d^\infty_\Sigma,\ \{\tilde{\xi}_j,\tilde{\xi}_k\} = 0 \bmod d^\infty_\Sigma,$$
$$\{\tilde{\xi}_j,\tilde{x}_k\} = \delta_{j,k} \bmod d^\infty_\Sigma,$$

where j, $k = 1,\dots,2n-1$. *Here* $\{f,g\} = \frac{\partial f}{\partial\xi}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial\xi}$, f, $g\in C^\infty(\Gamma)$.

(e) *We write* $\tilde{x}'$, $\tilde{x}''$, $\tilde{\xi}'$ *and* $\tilde{\xi}''$ *to denote* $(\tilde{x}_1,\dots,\tilde{x}_n)$, $(\tilde{x}_{n+1},\dots,\tilde{x}_{2n-1})$, $(\tilde{\xi}_1,\dots,\tilde{\xi}_n)$ *and* $(\tilde{\xi}_{n+1},\dots,\tilde{\xi}_{2n-1})$ *respectively. Then,* $\tilde{\Sigma}\bigcap\Gamma$ *coincides to infinite order with*

$$\Big\{(\tilde{x},\tilde{\xi});\ \tilde{x}''=0,\ \tilde{\xi}''=0\Big\}$$

on $\Sigma\bigcap(\Gamma)_{\mathbb{R}}$ *and*

$$\Sigma\bigcap(\Gamma)_{\mathbb{R}} = \Big\{(\tilde{x},\tilde{\xi});\ \tilde{x}''=0,\ \tilde{\xi}''=0,\ \tilde{x}' \text{ and } \tilde{\xi}' \text{ are real}\Big\}.$$

Furthermore, there is a $(n-1)\times(n-1)$ *matrix of almost analytic functions* $A(\tilde{x},\tilde{\xi})$ *such that for every compact set* $K\subset\Gamma$ *and* $N\geq 0$, *there is a* $c_{K,N}>0$, *such that*

$$\Big|p_0(\tilde{x},\tilde{\xi}) - i\left\langle A(\tilde{x},\tilde{\xi})\tilde{x}'',\tilde{\xi}''\right\rangle\Big| \leq c_{K,N}\Big|(\tilde{x}'',\tilde{\xi}'')\Big|^N \text{ on } K,$$

and when $\tilde{x}'$ *and* $\tilde{\xi}'$ *are real,* $A(\tilde{x}',0,\tilde{\xi}',0)$ *has only positive eigenvalues*

$$|\lambda_1|,\dots,|\lambda_{n-1}|,$$

where $\pm i\lambda_1,\dots,\pm i\lambda_{n-1}$ *are the non-vanishing eigenvalues of* $F(\tilde{x}',0,\tilde{\xi}',0)$, *the fundamental matrix of* $\Box^{(q)}_b$. *In particular,*

$$\frac{1}{2}\mathrm{tr}\,A(\tilde{x}',0,\tilde{\xi}',0) = \frac{1}{2}\tilde{\mathrm{tr}}\,F(\tilde{x}',0,\tilde{\xi}',0).$$

Formally, we write

$$p_0(\tilde{x},\tilde{\xi}) = i\left\langle A(\tilde{x},\tilde{\xi})\tilde{x}'',\tilde{\xi}''\right\rangle + O(\Big|(\tilde{x}'',\tilde{\xi}'')\Big|^N). \tag{4.16}$$

Proof. See Menikoff-Sjöstrand [20]. $\square$

Remark 4.7. Set

$$E=\left\{(t,x,\xi,y,\eta)\in\overline{\mathbb{R}}_+\times T^*(\Omega)_{\mathbb{C}}\times T^*(\Omega)_{\mathbb{C}};\,(x,\xi,y,\eta)\in C_t\right\},$$

where C_t is defined by (4.11). Let $(\widetilde{x},\widetilde{\xi})$ be the coordinates of Proposition 4.6. In the work of Menikoff-Sjöstrand [20], it was shown that there exists $\widetilde{\psi}(t,\widetilde{x},\widetilde{\eta})\in C^\infty(\overline{\mathbb{R}}_+\times\Gamma)$, where Γ is as in Proposition 4.6, such that

$$\begin{cases}\dfrac{\partial\widetilde{\psi}}{\partial t}-ip_0(\widetilde{x},\widetilde{\psi}'_{\widetilde{x}})=O(\left|(\widetilde{x}'',\widetilde{\eta}'')\right|^N),\text{for all } N>0,\\ \widetilde{\psi}|_{t=0}=\langle\widetilde{x}\,,\widetilde{\eta}\rangle\end{cases}$$

and $\widetilde{\psi}(t,\widetilde{x},\widetilde{\eta})$ is of the form

$$\widetilde{\psi}(t,\widetilde{x},\widetilde{\eta})=\langle\widetilde{x}',\widetilde{\eta}'\rangle+\left\langle e^{-tA(\widetilde{x}',0,\widetilde{\eta}',0)}\widetilde{x}'',\widetilde{\eta}''\right\rangle+\widetilde{\psi}_2(t,\widetilde{x},\widetilde{\eta})+\widetilde{\psi}_3(t,\widetilde{x},\widetilde{\eta})+\cdots,\quad(4.17)$$

where A is as in Proposition 4.6 and $\widetilde{\psi}_j(t,\widetilde{x},\widetilde{\eta})$ is a C^∞ homogeneous polynomial of degree j in $(\widetilde{x}'',\widetilde{\eta}'')$. If $\lambda\in C(T^*(\Omega)\setminus 0)$, $\lambda>0$ and $\lambda|_\Sigma<\min\lambda_j$ with $\lambda_j>0$, where $\pm i\lambda_j$ are the non-vanishing eigenvalues of the fundamental matrix of $\square_b^{(q)}$, then for every compact set $K\subset\Sigma\bigcap(\Gamma)_{\mathbb{R}}$ and all indices α, β, γ, j, there is a constant $c_{\alpha,\beta,\gamma,j,K}$ such that

$$\left|\partial^\alpha_{\widetilde{x}}\partial^\beta_{\widetilde{\eta}}\partial^\gamma_t(\widetilde{\psi}_j(t,\widetilde{x},\widetilde{\eta}))\right|\le c_{\alpha,\beta,\gamma,K}\,e^{-\lambda(\widetilde{x},\widetilde{\eta})t}\text{ on }\overline{\mathbb{R}}_+\times K.\quad(4.18)$$

Put

$$\widetilde{E}=\left\{(t,\widetilde{x},\frac{\partial\widetilde{\psi}}{\partial\widetilde{x}}(t,\widetilde{x},\widetilde{\eta}),\frac{\partial\widetilde{\psi}}{\partial\widetilde{\eta}}(t,\widetilde{x},\widetilde{\eta}),\widetilde{\eta});\,t\in\overline{\mathbb{R}}_+,\,\widetilde{x},\widetilde{\eta}\in C^\infty(\Gamma)\right\}.$$

We notice that $\widetilde{E}$ coincides to infinite order with E on $\overline{\mathbb{R}}_+\times\operatorname{diag}((\Sigma\bigcap(\Gamma)_{\mathbb{R}})\times(\Sigma\bigcap(\Gamma)_{\mathbb{R}}))$. (See [20].)

5 The heat equation, formal construction

We work with some real local coordinates $x=(x_1,\ldots,x_{2n-1})$ defined on an open set $\Omega\subset X$. We identify $T^*(\Omega)$ with $\Omega\times\mathbb{R}^{2n-1}$.

Definition 5.1. We will say that $a\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega))$ is quasi-homogeneous of degree j if $a(t,x,\lambda\eta)=\lambda^j a(\lambda t,x,\eta)$ for all $\lambda>0$.

It is easy to see that if a is quasi-homogeneous of degree j, then $\partial_x^\alpha \partial_\eta^\beta \partial_t^\gamma a$ is quasi-homogeneous of degree $j - \left|\beta\right| + \gamma$.

In this section, we consider the problem

$$\begin{cases} (\partial_t + \Box_b^{(q)})u(t,x) = 0 \quad \text{in } \mathbb{R}_+ \times \Omega \\ u(0,x) = v(x) \end{cases}. \tag{5.1}$$

We shall start by making only a formal construction. We look for an approximate solution of (5.1) of the form

$$u(t,x) = A(t)v(x)$$

$$A(t)v(x) = \frac{1}{(2\pi)^{2n-1}} \int\int e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} a(t,x,\eta)v(y)\,dy\,d\eta \tag{5.2}$$

where formally

$$a(t,x,\eta) \sim \sum_{j=0}^{\infty} a_j(t,x,\eta),\ a_j(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))),$$

$a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$.

We let the full symbol of $\Box_b^{(q)}$ be:

$$\text{full symbol of } \Box_b^{(q)} = \sum_{j=0}^{2} p_j(x,\xi),$$

where $p_j(x,\xi)$ is positively homogeneous of order $2-j$. We apply $\partial_t + \Box_b^{(q)}$ formally under the integral in (5.2) and then introduce the asymptotic expansion of $\Box_b^{(q)}(ae^{i\psi})$. (See Proposition B.16.) Setting $(\partial_t + \Box_b^{(q)})(ae^{i\psi}) \sim 0$ and regrouping the terms according to the degree of quasi-homogeneity. We obtain the transport equations

$$\begin{cases} T(t,x,\eta,\partial_t,\partial_x)a_0 = O(\left|\mathrm{Im}\,\psi\right|^N),\ \forall N \\ T(t,x,\eta,\partial_t,\partial_x)a_j + l_j(t,x,\eta,a_0,\ldots,a_{j-1}) = O(\left|\mathrm{Im}\,\psi\right|^N),\ \forall N. \end{cases} \tag{5.3}$$

Here

$$T(t,x,\eta,\partial_t,\partial_x) = \partial_t - i\sum_{j=1}^{2n-1} \frac{\partial p_0}{\partial \xi_j}(x,\psi'_x)\frac{\partial}{\partial x_j} + q(t,x,\eta)$$

where

$$q(t,x,\eta) = p_1(x,\psi'_x) + \frac{1}{2i}\sum_{j,k=1}^{2n-1} \frac{\partial^2 p_0(x,\psi'_x)}{\partial \xi_j \partial \xi_k} \frac{\partial^2 \psi(t,x,\eta)}{\partial x_j \partial x_k}$$

and l_j is a linear differential operator acting on $a_0, a_1, \ldots, a_{j-1}$. We note that $q(t,x,\eta) \to q(\infty, x, \eta)$ exponentially fast in the sense of (4.5) and that the same is true for the coefficients of l_j.

Let C_t, E be as in (4.11) and Remark 4.7. We recall that for $t \geq 0$,

$$C_t = \left\{(x,\xi,y,\eta) \in T^*(\Omega)_{\mathbb{C}} \times T^*(\Omega)_{\mathbb{C}};\ \xi = \frac{\partial \psi}{\partial x}(t,x,\eta), y = \frac{\partial \psi}{\partial \eta}(t,x,\eta)\right\},$$

$$E = \left\{(t,x,\xi,y,\eta) \in \overline{\mathbb{R}}_+ \times T^*(\Omega)_{\mathbb{C}} \times T^*(\Omega)_{\mathbb{C}};\ (x,\xi,y,\eta) \in C_t\right\}$$

and for $t > 0$,

$$(C_t)_{\mathbb{R}} = \operatorname{diag}(\Sigma \times \Sigma) = \{(x,\xi,x,\xi) \in T^*(\Omega) \times T^*(\Omega);\ (x,\xi) \in \Sigma\}.$$

If we consider $a_0, a_1, \ldots$ as functions on E, then the equations (5.3) involve differentiations along the vector field

$$v = \frac{\partial}{\partial t} - i H_{p_0}.$$

We can consider only Taylor expansions at Σ. Until further notice, our computations will only be valid to infinite order on Σ.

Consider v as a vector field on E. In the coordinates (t,x,η) we can express v:

$$v = \frac{\partial}{\partial t} - i \sum_{j=1}^{2n-1} \frac{\partial p_0}{\partial \xi_j}(x, \psi'_x) \frac{\partial}{\partial x_j}.$$

We can compute

$$\operatorname{div}(v) = \frac{1}{i}\left(\sum_{j=1}^{2n-1} \frac{\partial^2 p_0(x,\psi'_x)}{\partial x_j \partial \xi_j} + \sum_{j,k=1}^{2n-1} \frac{\partial^2 p_0}{\partial \xi_j \partial \xi_k}(x,\psi'_x) \frac{\partial^2 \psi}{\partial x_j \partial x_k}(t,x,\eta)\right). \tag{5.4}$$

For a smooth function $a(t,x,\eta)$ we introduce the $\frac{1}{2}$ density on E

$$\alpha = a(t,x,\eta)\sqrt{dt\,dx\,d\eta}$$

which is well-defined up to some factor i^{μ}. (See Hörmander [14].) The Lie derivative of α along v is

$$L_v(\alpha) = (v(a) + \frac{1}{2}\operatorname{div}(v)a)\sqrt{dt\,dx\,d\eta}.$$

We see from the expression for T that

$$(Ta)\sqrt{dt\,dx\,d\eta} = (L_v + p_0^s(x, \psi'_x(t,x,\eta)))(a\sqrt{dt\,dx\,d\eta}), \tag{5.5}$$

where

$$p_0^s(x,\xi) = p_1(x,\xi) + \frac{i}{2}\sum_{j=1}^{2n-1}\frac{\partial^2 p_0(x,\xi)}{\partial x_j \partial \xi_j}$$

is the subprincipal symbol (invariantly defined on Σ). Now let $(\widetilde{x}, \widetilde{\xi})$ be the coordinates of Proposition 4.6, in which p_0 takes the form (4.16). In these coordinates we have

$$\begin{aligned} H_{p_0}(\widetilde{x}, \widetilde{\xi}) &= i\left\langle A(\widetilde{x}, \widetilde{\xi})\widetilde{x}'', \frac{\partial}{\partial \widetilde{x}''}\right\rangle - i\left\langle {}^tA(\widetilde{x}, \widetilde{\xi})\widetilde{\xi}'', \frac{\partial}{\partial \widetilde{\xi}''}\right\rangle \\ &\quad + \sum_{|\alpha|=1, |\beta|=1} (\widetilde{x}'')^\alpha (\widetilde{\xi}'')^\beta B_{\alpha\beta}(\widetilde{x}, \widetilde{\xi}, \frac{\partial}{\partial \widetilde{x}}, \frac{\partial}{\partial \widetilde{\xi}}) \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} \nu &= \frac{\partial}{\partial t} + \left\langle A(\widetilde{x}, \widetilde{\psi}'_x)\widetilde{x}'', \frac{\partial}{\partial \widetilde{x}''}\right\rangle \\ &\quad + \sum_{|\alpha|=1, |\beta|=1} (\widetilde{x}'')^\alpha (\widetilde{\psi}'_{x''})^\beta C_{\alpha\beta}(\widetilde{x}', \widetilde{\psi}'_{\widetilde{x}}, \frac{\partial}{\partial \widetilde{x}}). \end{aligned} \tag{5.7}$$

Here $\widetilde{\psi}(t, \widetilde{x}, \widetilde{\eta})$ is as in Remark 4.7.

Let $f(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega)_{\mathbb{C}})$, $f(\infty, x, \eta) \in C^\infty(T^*(\Omega)_{\mathbb{C}})$. We say that $f(t,x,\eta)$ converges exponentially fast to $f(\infty, x, \eta)$ if

$$f(t,x,\eta) - f(\infty, x, \eta)$$

satisfies the same kind of estimates as (4.5). Recalling the form of $\widetilde{\psi}$ we obtain

$$\begin{aligned} \nu = \tilde{\nu} &= \frac{\partial}{\partial t} + \left\langle A(\widetilde{x}', 0, \widetilde{\eta}', 0)\widetilde{x}'', \frac{\partial}{\partial \widetilde{x}''}\right\rangle \\ &\quad + \sum_{|\alpha+\beta|=2, \alpha\neq 0} (\widetilde{x}'')^\alpha (\widetilde{\eta}'')^\beta D_{\alpha\beta}(t, \widetilde{x}, \widetilde{\eta}, \frac{\partial}{\partial \widetilde{x}}) \end{aligned} \tag{5.8}$$

where $D_{\alpha\beta}$ converges exponentially fast to some limit as $t \to +\infty$. We have on Σ,

$$\frac{1}{2}\operatorname{div}(\tilde{\nu}) = \frac{1}{2}\operatorname{tr} A(\widetilde{x}', 0, \widetilde{\eta}', 0) = \frac{1}{2}\widetilde{\operatorname{tr}}\, F(\widetilde{x}', 0, \widetilde{\eta}', 0) \tag{5.9}$$

where $F(\widetilde{x}', 0, \widetilde{\eta}', 0)$ is the fundamental matrix of $\Box_b^{(q)}$. We define $\widetilde{a}(t, \widetilde{x}, \widetilde{\eta})$ by

$$\widetilde{a}(t, \widetilde{x}, \widetilde{\eta})\sqrt{dt\,d\widetilde{x}\,d\widetilde{\eta}} = a(t,x,\eta)\sqrt{dt\,dx\,d\eta}. \tag{5.10}$$

Note that the last equation only defines $\widetilde{a}$ up to i^μ. We have

$$(Ta)\sqrt{dt\,dx\,d\eta} = (\widetilde{T}\widetilde{a})\sqrt{dt\,d\widetilde{x}\,d\widetilde{\eta}}$$

where

$$\widetilde{T} = \frac{\partial}{\partial t} + \left\langle A(\widetilde{x}', 0, \widetilde{\eta}', 0)\widetilde{x}'', \frac{\partial}{\partial \widetilde{x}''} \right\rangle$$
$$+ \frac{1}{2}\widetilde{\mathrm{tr}}\, F(\widetilde{x}', 0, \widetilde{\eta}', 0) + p_0^s(\widetilde{x}', 0, \widetilde{\eta}', 0) + Q(t, \widetilde{x}, \widetilde{\eta}, \frac{\partial}{\partial \widetilde{x}}). \tag{5.11}$$

Here

$$Q(t, \widetilde{x}, \widetilde{\eta}, \frac{\partial}{\partial \widetilde{x}}) = \sum_{|\alpha+\beta|=2, \alpha\neq 0} (\widetilde{x}'')^\alpha (\widetilde{\eta}'')^\beta D_{\alpha\beta}(t, \widetilde{x}, \widetilde{\eta}, \frac{\partial}{\partial \widetilde{x}})$$
$$+ \sum_{|\alpha+\beta|=1} (\widetilde{x}'')^\alpha (\widetilde{\eta}'')^\beta E_{\alpha\beta}(t, \widetilde{x}, \widetilde{\eta}).$$

It is easy to see that $E_{\alpha\beta}$ and $D_{\alpha\beta}$ converge exponentially fast to some limits $E_{\alpha\beta}(\infty, \widetilde{x}, \widetilde{\eta})$ and $D_{\alpha\beta}(\infty, \widetilde{x}, \widetilde{\eta})$ respectively. We need the following

Lemma 5.2. *Let A be a $d \times d$ matrix having only positive eigenvalues and consider the map*

$$\mathscr{A} : u \mapsto \left\langle A \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \end{pmatrix} \right\rangle$$

on the space $P^m(\mathbb{R}^d)$ of homogeneous polynomials of degree m. Then

$$\exp(t\mathscr{A})(u) = u \circ (\exp(tA))$$

and the map $\mathscr{A}$ is a bijection except for $m = 0$.

Proof. We notice that $U(t) : u \mapsto u \circ \exp(tA)$ form a group of operators and that

$$(\frac{\partial U(t)}{\partial t})\Big|_{t=0} = \mathscr{A}.$$

This shows that $U(t) = \exp(t\mathscr{A})$. To prove the second statement, suppose that $u \in P^m$, $m \geq 1$ and $\mathscr{A}(u) = 0$. Then $\exp(t\mathscr{A})(u) = u$ for all t, in other words $u(\exp(tA)(x)) = u(x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$. Since $\exp(tA)(x) \to 0$ when $t \to -\infty$, we obtain $u(x) = u(0) = 0$, which proves the lemma. □

Proposition 5.3. *Let*

$$c_j(x, \eta) \in C^\infty(T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Omega), \Lambda^{0,q} T^*(\Omega))), \quad j = 0, 1, \ldots,$$

be positively homogeneous functions of degree $-j$. Then, we can find solutions

$$a_j(t, x, \eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Omega), \Lambda^{0,q} T^*(\Omega))), \quad j = 0, 1, \ldots,$$

of the system (5.3) *with*

$$a_j(0,x,\eta) = c_j(x,\eta), \;\; j = 0,1,\ldots,$$

where $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$ such that $a_j(t,x,\eta)$ has unique Taylor expansions on Σ, for all j. Furthermore, let $\lambda(x,\eta) \in C(T^*(\Omega))$ and $\lambda|_\Sigma < \min \tau_j$, where τ_j are the eigenvalues of $\frac{1}{2}\widetilde{\operatorname{tr}} F + p_0^s$ on Σ. Then for all indices α, β, γ, j and every compact set $K \subset \Sigma$ there exists a constant $c > 0$ such that

$$\left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a_j(t,x,\eta)\right| \le c e^{-t\lambda(x,\eta)} \text{ on } \overline{\mathbb{R}}_+ \times K. \tag{5.12}$$

Proof. We only need to study Taylor expansions on Σ. Let $(\tilde{x}, \tilde{\xi})$ be the coordinates of Proposition 4.6. We define $\tilde{a}_j(t,\tilde{x},\tilde{\eta})$ from $a_j(t,x,\eta)$ as in (5.10). In order to prove (5.12), it is sufficient to prove the corresponding statement for $\tilde{a}_j$. (See section 1 of Menikoff-Sjöstrand[20].) We introduce the Taylor expansion of $\tilde{a}_0$ with respect to $(\tilde{x}'', \tilde{\eta}'')$.

$$\tilde{a}_0(t,\tilde{x},\tilde{\eta}) = \sum_0^\infty \tilde{a}_0^j(t,\tilde{x},\tilde{\eta}),$$

where $\tilde{a}_0^j$ is a homogeneous polynomial of degree j in $(\tilde{x}'', \tilde{\eta}'')$. Let

$$c_0(\tilde{x},\tilde{\eta}) = \sum_{j=0} \tilde{c}_0^j(\tilde{x},\tilde{\eta}),$$

where $\tilde{c}_0^j$ is a homogeneous polynomial of degree j in $(\tilde{x}'', \tilde{\eta}'')$. From $\tilde{T}\tilde{a}_0 = 0$, we get

$$\tilde{a}_0^0(t,\tilde{x}',\tilde{\eta}') = e^{-t(\frac{1}{2}\widetilde{\operatorname{tr}} F + p_0^s)} \tilde{c}_0^0(\tilde{x},\tilde{\eta}).$$

It is easy to see that for all indices α, β, γ and every compact set $K \subset \Sigma$ there exists a constant $c > 0$ such that

$$\left|\partial_t^\gamma \partial_{\tilde{x}}^\alpha \partial_{\tilde{\eta}}^\beta \tilde{a}_0^0\right| \le c e^{-t\lambda(\tilde{x},\tilde{\eta})} \text{ on } \overline{\mathbb{R}}_+ \times K,$$

where $\lambda(\tilde{x},\tilde{\eta}) \in C(T^*(\Omega))$, $\lambda|_\Sigma < \min \tau_j$. Here τ_j are the eigenvalues of $\frac{1}{2}\widetilde{\operatorname{tr}} F + p_0^s$ on Σ.

Again, from $\tilde{T}\tilde{a}_0 = 0$, we get

$$\left(\frac{\partial}{\partial t} + \mathscr{A} + \frac{1}{2}\widetilde{\operatorname{tr}} F + p_0^s\right)\tilde{a}_0^{j+1}(t,\tilde{x},\tilde{\eta}) = \tilde{b}_0^{j+1}(t,\tilde{x},\tilde{\eta})$$

where $\tilde{b}_0^{j+1}(t,\tilde{x},\tilde{\eta})$ satisfies the same kind of estimate as $\tilde{a}_0^0$. By Lemma 5.2, we see that $\exp(-t\mathscr{A})$ is bounded for $t \ge 0$. We deduce a similar estimate for the function $\widetilde{a}_0^{j+1}(t,\widetilde{x},\widetilde{\eta})$. Continuing in this way we get all the desired estimates for $\widetilde{a}_0$. The next transport equation takes the form $\widetilde{T}\widetilde{a}_1 = \widetilde{b}$ where $\widetilde{b}$ satisfies the estimates (5.12). We can repeat the procedure above and conclude that $\widetilde{a}_1$ satisfies the estimates (5.12). From above, we see that $\widetilde{a}_0$, $\widetilde{a}_1$ have the unique Taylor expansions on Σ. Continuing in this way we get the proposition. □

From Proposition 5.3, we have the following

Proposition 5.4. *Suppose condition $Y(q)$ holds. Let*

$$c_j(x,\eta) \in C^\infty(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))), \ \ j = 0,1,\ldots,$$

be positively homogeneous functions of degree $-j$. Then, we can find solutions

$$a_j(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))), \ \ j = 0,1,\ldots,$$

of the system (5.3) *with*

$$a_j(0,x,\eta) = c_j(x,\eta), \ \ j = 0,1,\ldots,$$

where $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$ and $\varepsilon_0 > 0$ such that for all indices α,β,γ,j and every compact set $K \subset \Omega$ there exists a constant $c > 0$ such that

$$\left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a_j(t,x,\eta)\right| \leq c e^{-\varepsilon_0 t|\eta|}(1+|\eta|)^{-j-|\beta|+\gamma} \text{ on } \overline{\mathbb{R}}_+ \times K. \tag{5.13}$$

Proposition 5.5. *Suppose condition $Y(q)$ fails. Let (n_-,n_+), $n_- + n_+ = n-1$, be the signature of the Levi form. Let*

$$c_j(x,\eta) \in C^\infty(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))), \ \ j = 0,1,\ldots,$$

be positively homogeneous functions of degree $-j$. Then, we can find solutions

$$a_j(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))), \ \ j = 0,1,\ldots,$$

of the system (5.3) *with*

$$a_j(0,x,\eta) = c_j(x,\eta), \ \ j = 0,1,\ldots,$$

where $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$ and such that for all indices α,β,γ,j, every $\varepsilon > 0$ and compact set $K \subset \Sigma$ there exists a constant $c > 0$ such that

$$\left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a_j(t,x,\eta)\right| \leq c e^{\varepsilon t|\eta|}(1+|\eta|)^{-j-|\beta|+\gamma} \text{ on } \overline{\mathbb{R}}_+ \times K. \tag{5.14}$$

Furthermore, there exists $\varepsilon_0 > 0$ such that for all indices α, β, γ, j and every compact set $K \subset \Sigma$, there exists a constant $c > 0$ such that

$$\left|\partial_t^{\gamma}\partial_x^{\alpha}\partial_{\eta}^{\beta} a_j(t,x,\eta)\right| \leq c e^{-\varepsilon_0 t\left|\eta\right|}(1+\left|\eta\right|)^{-j-\left|\beta\right|+\gamma}$$
$$\text{on } \overline{\mathbb{R}}_+ \times (K \bigcap \Sigma^+) \ \ \textit{if } q = n_-, n_- \neq n_+ \tag{5.15}$$

and

$$\left|\partial_t^{\gamma}\partial_x^{\alpha}\partial_{\eta}^{\beta} a_j(t,x,\eta)\right| \leq c e^{-\varepsilon_0 t\left|\eta\right|}(1+\left|\eta\right|)^{-j-\left|\beta\right|+\gamma}$$
$$\text{on } \overline{\mathbb{R}}_+ \times (K \bigcap \Sigma^-) \ \ \textit{if } q = n_+, n_- \neq n_+. \tag{5.16}$$

We need the following formula

Proposition 5.6. *Let Q be a C^∞ differential operator on Ω of order $k > 0$ with full symbol $q(x,\xi) \in C^\infty(T^*(\Omega))$. For $0 \leq q, q_1 \leq n-1$, q, $q_1 \in \mathbb{N}$, let*

$$a(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))).$$

Then,

$$Q(x,D_x)(e^{i\psi(t,x,\eta)}a(t,x,\eta)) \;=\; e^{i\psi(t,x,\eta)}\sum_{|\alpha|\leq k}\frac{1}{\alpha!}q^{(\alpha)}(x,\psi'_x(t,x,\eta))(R_\alpha(\psi,D_x)a),$$

where

$$D_x = -i\partial_x,$$
$$R_\alpha(\psi,D_x)a = D_y^{\alpha}\left\{e^{i\phi_2(t,x,y,\eta)}a(t,y,\eta)\right\}\Big|_{y=x},$$
$$\phi_2(t,x,y,\eta) = (x-y)\psi'_x(t,x,\eta) - (\psi(t,x,\eta)-\psi(t,y,\eta)).$$

For $0 \leq q, q_1 \leq n-1$, q, $q_1 \in \mathbb{N}$, let

$$a_j(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))), \ \ j = 0,1,\ldots, \tag{5.17}$$

be quasi-homogeneous functions of degree $m-j$, $m \in \mathbb{Z}$. We assume that

$$a_j(t,x,\eta), \ j = 0,1,\ldots,$$

are the solutions of the system (5.3). From the proof of Proposition 5.3, it follows that for all indices α, β, γ, j, every $\varepsilon > 0$ and compact set $K \subset \Sigma$ there exists a constant $c > 0$ such that

$$\left|\partial_t^{\gamma}\partial_x^{\alpha}\partial_{\eta}^{\beta} a_j(t,x,\eta)\right| \leq c e^{\varepsilon t\left|\eta\right|}(1+\left|\eta\right|)^{m-j-\left|\beta\right|+\gamma} \text{ on } \overline{\mathbb{R}}_+ \times K. \tag{5.18}$$

Let

$$a(t,x,\eta)\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))) \tag{5.19}$$

be the asymptotic sum of $a_j(t,x,\eta)$. (See Definition 6.1 and Remark 6.2 for a precise meaning.) We formally write

$$a(t,x,\eta)\sim\sum_{j=0}^\infty a_j(t,x,\eta).$$

Let

$$(\partial_t+\Box_b^{(q)})(e^{i\psi(t,x,\eta)}a(t,x,\eta))=e^{i\psi(t,x,\eta)}b(t,x,\eta),$$

where

$$b(t,x,\eta)\sim\sum_{j=0}^\infty b_j(t,x,\eta),$$

$$b_j(t,x,\eta)\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))),\ \ j=0,1,\ldots,$$

$b_j(t,x,\eta)$ is a quasi-homogeneous function of degree $m+2-j$.

From Proposition 5.6, we see that for all N, every compact set $K\subset\Sigma$, $\varepsilon>0$, there exists $c>0$ such that

$$\left|b(t,x,\eta)\right|\leq ce^{\varepsilon t\left|\eta\right|}(\left|\eta\right|^{-N}+\left|\eta\right|^{2-N}(\operatorname{Im}\psi(t,x,\eta)^N) \tag{5.20}$$

on $\overline{\mathbb{R}}_+\times K$, $\left|\eta\right|\geq 1$.

Conversely, if

$$(\partial_t+\Box_b^{(q)})(e^{i\psi(t,x,\eta)}a(t,x,\eta))=e^{i\psi(t,x,\eta)}b(t,x,\eta)$$

and b satisfies the same kind of estimates as (5.20), then $a_j(t,x,\eta)$, $j=0,1,\ldots$, solve the system (5.3) to infinite order at Σ. From this and the particular structure of the problem, we will next show

Proposition 5.7. *Let (n_-,n_+), $n_-+n_+=n-1$, be the signature of the Levi form. Suppose condition $Y(q)$ fails. That is, $q=n_-$ or n_+. Let*

$$a_j(t,x,\eta)\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))),\ \ j=0,1,\ldots,$$

be the solutions of the system (5.3) with

$$a_0(0,x,\eta)=I,\ a_j(0,x,\eta)=0\ \text{ when } j>0,$$

where $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$. Then we can find

$$a_j(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))),\ \ j=0,1,\ldots,$$

where $a_j(\infty,x,\eta)$ is a positively homogeneous function of degree $-j$, $\varepsilon_0>0$ such that for all indices α, β, γ, j, every compact set $K\subset\Sigma$, there exists $c>0$, such that

$$\left|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta(a_j(t,x,\eta)-a_j(\infty,x,\eta))\right|\leq ce^{-\varepsilon_0t|\eta|}(1+|\eta|)^{-j-|\beta|+\gamma} \tag{5.21}$$

on $\overline{\mathbb{R}}_+\times K$, $|\eta|\geq 1$.

Furthermore, for all $j=0,1,\ldots$,

$$\begin{cases}\text{all derivatives of } a_j(\infty,x,\eta) \text{ vanish at } \Sigma^+, & \text{if } q=n_-,\ n_-\neq n_+\\ \text{all derivatives of } a_j(\infty,x,\eta) \text{ vanish at } \Sigma^-, & \text{if } q=n_+,\ n_-\neq n_+.\end{cases} \tag{5.22}$$

Proof. We assume that $q=n_-$. Put

$$a(t,x,\eta)\sim\sum_j a_j(t,x,\eta).$$

Since $a_j(t,x,\eta)$, $j=0,1,\ldots$, solve the system (5.3), we have

$$(\partial_t+\Box_b^{(q)})(e^{i\psi(t,x,\eta)}a(t,x,\eta))=e^{i\psi(t,x,\eta)}b(t,x,\eta),$$

where $b(t,x,\eta)$ satisfies (5.20). Note that we have the interwing properties

$$\begin{cases}\overline{\partial}_b\Box_b^{(q)}=\Box_b^{(q+1)}\overline{\partial}_b\\ \overline{\partial}_b^*\Box_b^{(q)}=\Box_b^{(q-1)}\overline{\partial}_b^*\end{cases}. \tag{5.23}$$

Now,

$$\begin{cases}\overline{\partial}_b^*(e^{i\psi}a)=e^{i\psi}\tilde{a}\\ \overline{\partial}_b(e^{i\psi}a)=e^{i\psi}\hat{a},\end{cases}$$

$\tilde{a}\sim\sum_{j=-1}^\infty\tilde{a}_j(t,x,\eta)$, $\hat{a}\sim\sum_{j=-1}^\infty\hat{a}_j(t,x,\eta)$, where

$$\hat{a}_j\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q+1}T^*(\Omega))),\ \ j=0,1,\ldots,$$

$$\tilde{a}_j\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q-1}T^*(\Omega))),\ \ j=0,1,\ldots,$$

and $\hat{a}_j$, $\tilde{a}_j$ are quasi-homogeneous of degree $1-j$. From (5.23), we have

$$(\partial_t+\Box_b^{(q-1)})(e^{i\psi}\tilde{a})=e^{i\psi}b_1,$$
$$(\partial_t+\Box_b^{(q+1)})(e^{i\psi}\hat{a})=e^{i\psi}b_2,$$

where b_1, b_2 satisfy (5.20). Since b_1, b_2 satisfy (5.20), $\tilde{a}_j$, $\hat{a}_j$, $j=0,1,\ldots$, solve the system (5.3) to infinite order at Σ. We notice that

$$q-1\neq n_-,q+1\neq n_-.$$

In view of the proof of Proposition 5.3, we can find $\varepsilon_0 > 0$, such that for all indices α, β, γ, j, every compact set $K \subset \Sigma^-$, there exists $c > 0$ such that

$$\begin{cases} \left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta \widetilde{a}_j(t,x,\eta)\right| \le c e^{-\varepsilon_0 t|\eta|}(1+\left|\eta\right|)^{1-j-|\beta|+\gamma} \\ \left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta \hat{a}_j(t,x,\eta)\right| \le c e^{-\varepsilon_0 t|\eta|}(1+\left|\eta\right|)^{1-j-|\beta|+\gamma} \end{cases} \tag{5.24}$$

on $\overline{\mathbb{R}}_+ \times K$, $\left|\eta\right| \ge 1$.

Now $\Box_b^{(q)} = \overline{\partial}_b \overline{\partial_b}^* + \overline{\partial_b}^* \overline{\partial}_b$, so $\Box_b^{(q)}(e^{i\psi}a) = e^{i\psi}c$, where c satisfies the same kind of estimates as (5.24). From this we see that $\partial_t(e^{i\psi}a) = e^{i\psi}d$, where d has the same properties as c. Since $d = i(\partial_t\psi)a + \partial_t a$ and $\partial_t\psi$ satisfy the same kind of estimates as (5.24), $\partial_t a$ satisfies the same kind of estimates as (5.24). From this we conclude that we can find $a(\infty,x,\eta) \sim \sum_{j=0}^\infty a_j(\infty,x,\eta)$, where $a_j(\infty,x,\eta)$ is a matrix-valued C^∞ positively homogeneous function of degree $-j$, $\varepsilon_0 > 0$, such that for all indices α, β, γ, j and every compact set $K \subset \Sigma^-$, there exists $c > 0$ such that

$$\left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta (a_j(t,x,\eta) - a_j(\infty,x,\eta))\right| \le c e^{-\varepsilon_0 t|\eta|}(1+\left|\eta\right|)^{-j-|\beta|+\gamma}$$

on $\overline{\mathbb{R}}_+ \times K$, $\left|\eta\right| \ge 1$.

If $n_- = n_+$, then

$$q-1 \ne n_+, q+1 \ne n_+.$$

We can repeat the method above to conclude that we can find

$$a(\infty,x,\eta) \sim \sum_{j=0}^\infty a_j(\infty,x,\eta),$$

where $a_j(\infty,x,\eta)$ is a matrix-valued C^∞ positively homogeneous function of degree $-j$, $\varepsilon_0 > 0$, such that for all indices α, β, γ, j and every compact set $K \subset \Sigma^+$, there exists $c > 0$ such that

$$\left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta (a_j(t,x,\eta) - a_j(\infty,x,\eta))\right| \le c e^{-\varepsilon_0 t|\eta|}(1+\left|\eta\right|)^{-j-|\beta|+\gamma}$$

on $\overline{\mathbb{R}}_+ \times K$, $\left|\eta\right| \ge 1$.

Now, we assume that $n_- \ne n_+$. From (5.15), we can find $\varepsilon_0 > 0$, such that for all indices α, β, γ, j and every compact set $K \subset \Sigma^+$, there exists $c > 0$ such that

$$\left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a_j(t,x,\eta)\right| \le c e^{-\varepsilon_0 t|\eta|}(1+\left|\eta\right|)^{-j-|\beta|+\gamma}$$

on $\overline{\mathbb{R}}_+ \times K$, $\left|\eta\right| \ge 1$.

The proposition follows. □

6 Some symbol classes

In this section we continue to work with some real local coordinates

$$x = (x_1, \dots, x_{2n-1})$$

defined on an open set $\Omega \subset X$. We identify $T^*(\Omega)$ with $\Omega \times \mathbb{R}^{2n-1}$.

Definition 6.1. Let $r(x,\eta)$ be a non-negative real continuous function on $T^*(\Omega)$. We assume that $r(x,\eta)$ is positively homogeneous of degree 1, that is, $r(x,\lambda\eta) = \lambda r(x,\eta)$, for $\lambda \geq 1$, $\left|\eta\right| \geq 1$. For $0 \leq q_1, q_2 \leq n-1$, $q_1, q_2 \in \mathbb{N}$ and $k \in \mathbb{R}$, we say that

$$a \in \hat{S}^k_r(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q_1} T^*(\Omega), \Lambda^{0,q_2} T^*(\Omega)))$$

if

$$a \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q_1} T^*(\Omega), \Lambda^{0,q_2} T^*(\Omega)))$$

and for all indices α, β, γ, every compact set $K \subset \Omega$ and every $\varepsilon > 0$, there exists a constant $c > 0$ such that

$$\left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a(t,x,\eta)\right| \leq c e^{t(-r(x,\eta)+\varepsilon\left|\eta\right|)}(1+\left|\eta\right|)^{k+\gamma-\left|\beta\right|},\ x \in K,\ \left|\eta\right| \geq 1.$$

Remark 6.2. It is easy to see that we have the following properties:

(a) If $a \in \hat{S}^k_{r_1}$, $b \in \hat{S}^l_{r_2}$ then $ab \in \hat{S}^{k+l}_{r_1+r_2}$, $a+b \in \hat{S}^{\max(k,l)}_{\min(r_1,r_2)}$.

(b) If $a \in \hat{S}^k_r$ then $\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a \in \hat{S}^{k-\left|\beta\right|+\gamma}_r$.

(c) If $a_j \in \hat{S}^{k_j}_r$, $j = 0,1,2,\dots$ and $k_j \searrow -\infty$ as $j \to \infty$, then there exists $a \in \hat{S}^{k_0}_r$ such that $a - \sum_0^{v-1} a_j \in \hat{S}^{k_v}_r$, for all $v = 1,2,\dots$. Moreover, if $\hat{S}^{-\infty}_r$ denotes $\bigcap_{k\in\mathbb{R}} \hat{S}^k_r$ then a is unique modulo $\hat{S}^{-\infty}_r$.

If a and a_j have the properties of (c), we write

$$a \sim \sum_0^\infty a_j \text{ in the symbol space } \hat{S}^{k_0}_r.$$

From Proposition 5.4, Proposition 5.5 and the standard Borel construction, we get the following

Proposition 6.3. *Let*

$$c_j(x,\eta) \in C^\infty(T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Omega), \Lambda^{0,q} T^*(\Omega))),\ \ j = 0,1,\dots$$

be positively homogeneous functions of degree $-j$. *Then, we can find solutions*

$$a_j(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))), \quad j = 0,1,\ldots$$

of the system (5.3) *with*

$$a_j(0,x,\eta) = c_j(x,\eta), \quad j = 0,1,\ldots,$$

where $a_j(t,x,\eta)$ *is a quasi-homogeneous function of degree* $-j$ *such that*

$$a_j \in \hat{S}_r^{-j}(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega)), \quad j = 0,1,\ldots,$$

for some r *with* $r > 0$ *if* $Y(q)$ *holds and* $r = 0$ *if* $Y(q)$ *fails.*

If the Levi form has signature (n_-, n_+), $n_- + n_+ = n-1$, *then we can take* $r > 0$,

$$\begin{cases} \text{near} \quad \Sigma^+, & \text{if } q = n_-, n_- \neq n_+, \\ \text{near} \quad \Sigma^-, & \text{if } q = n_+, n_- \neq n_+. \end{cases}$$

Again, from Proposition 5.7 and the standard Borel construction, we get the following

Proposition 6.4. *Suppose condition* $Y(q)$ *fails. We assume that the Levi form has signature* (n_-, n_+), $n_- + n_+ = n-1$. *We can find solutions*

$$a_j(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))), \quad j = 0,1,\ldots$$

of the system (5.3) *with*

$$a_0(0,x,\eta) = I, \; a_j(0,x,\eta) = 0 \text{ when } j > 0,$$

where $a_j(t,x,\eta)$ *is a quasi-homogeneous function of degree* $-j$, *such that for some* $r > 0$ *as in Definition 6.1,*

$$a_j(t,x,\eta) - a_j(\infty,x,\eta) \in \hat{S}_r^{-j}(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))), \quad j = 0,1,\ldots,$$

where

$$a_j(\infty,x,\eta) \in C^\infty(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))), \quad j = 0,1,\ldots,$$

and $a_j(\infty,x,\eta)$ *is a positively homogeneous function of degree* $-j$.

Furthermore, for all $j = 0,1,\ldots$,

$$\begin{cases} a_j(\infty,x,\eta) = 0 & \textit{in a conic neighborhood of } \Sigma^+, \quad \text{if } q = n_-, n_- \neq n_+, \\ a_j(\infty,x,\eta) = 0 & \textit{in a conic neighborhood of } \Sigma^-, \quad \text{if } q = n_+, n_- \neq n_+. \end{cases}$$

Let $b(t,x,\eta)\in\hat{S}^k_r$, $r>0$. Our next goal is to define the operator

$$B(t,x,y)=\int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)d\eta$$

as an oscillatory integral. We have the following

Proposition 6.5. *Let*

$$b(t,x,\eta)\in\hat{S}^k_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega)))$$

with $r>0$. *Then we can define*

$$B(t):C^\infty_0(\Omega;\Lambda^{0,q_1}T^*(\Omega))\to C^\infty(\overline{\mathbb{R}}_+;C^\infty(\Omega;\Lambda^{0,q_2}T^*(\Omega)))$$

with distribution kernel

$$B(t,x,y)=\int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)d\eta$$

and $B(t)$ *has a unique continuous extension*

$$B(t):\mathscr{E}'(\Omega;\Lambda^{0,q_1}T^*(\Omega))\to C^\infty(\overline{\mathbb{R}}_+;\mathscr{D}'(\Omega;\Lambda^{0,q_2}T^*(\Omega))).$$

We have

$$B(t,x,y)\in C^\infty(\overline{\mathbb{R}}_+;C^\infty(\Omega\times\Omega\setminus\operatorname{diag}(\Omega\times\Omega);\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega)))),$$

and

$$B(t,x,y)|_{t>0}\in C^\infty(\mathbb{R}_+\times\Omega\times\Omega;\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega))).$$

Proof. Let

$$S^*\Omega=\left\{(x,\eta)\in\Omega\times\dot{\mathbb{R}}^{2n-1};\left|\eta\right|=1\right\},$$

and let

$$V\subset\overline{\mathbb{R}}_+\times S^*\Omega$$

be a neighborhood of

$$(\mathbb{R}_+\times(\Sigma\bigcap S^*\Omega))\bigcup(\{0\}\times S^*\Omega)$$

such that

$$V_t=\{(x,\eta);(t,x,\eta)\in V\}$$

is independent of t for large t. Set

$$W=\left\{(t,x,\eta)\in\overline{\mathbb{R}}_+\times\Omega\times\dot{\mathbb{R}}^{2n-1};(\left|\eta\right|t,x,\frac{\eta}{\left|\eta\right|})\in V\right\}.$$

Let $\chi_V \in C^\infty(\overline{\mathbb{R}}_+ \times S^*(\Omega))$ have its support in V, be equal to 1 in a neighborhood of $(\mathbb{R}_+ \times S^*\Sigma)\bigcup(\{0\} \times S^*\Omega)$, and be independent of t, for large t. Set

$$\chi_W(t,x,\eta) = \chi_V(\left|\eta\right| t, x, \frac{\eta}{\left|\eta\right|}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega \times \dot{\mathbb{R}}^{2n-1}).$$

We have $\chi_W(t,x,\lambda\eta) = \chi_W(\lambda t, x, \eta)$, $\lambda > 0$. We can choose V sufficiently small so that

$$\left|\psi'_x(t,x,\eta) - \eta\right| \le \frac{\left|\eta\right|}{2} \text{ in } W. \tag{6.1}$$

We formally set

$$\begin{aligned} B(t,x,y) &= \int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}(1-\chi_W(t,x,\eta))b(t,x,\eta)d\eta \\ &\quad + \int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}\chi_W(t,x,\eta)b(t,x,\eta)d\eta \\ &= B_1(t,x,y) + B_2(t,x,y) \end{aligned}$$

where in $B_1(t,x,y)$ and $B_2(t,x,y)$ we have introduced the cut-off functions $(1-\chi_W)$ and χ_W respectively. Choose $\chi \in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi(\eta) = 1$ when $\left|\eta\right| < 1$ and $\chi(\eta) = 0$ when $\left|\eta\right| > 2$. Since $\operatorname{Im}\psi > 0$ outside $(\mathbb{R}_+ \times \Sigma)\bigcup(\{0\} \times \dot{\mathbb{R}}^{2n-1})$, we have

$$\operatorname{Im}\psi(t,x,\eta) \ge c\left|\eta\right| \text{ outside } W,$$

where $c > 0$. The kernel

$$B_{1,\varepsilon}(t,x,y) = \int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}(1-\chi_W(t,x,\eta))b(t,x,\eta)\chi(\varepsilon\eta)d\eta$$

converges in $C^\infty(\overline{\mathbb{R}}_+ \times \Omega \times \Omega; \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega)))$ as $\varepsilon \to 0$. This means that

$$B_1(t,x,y) = \lim_{\varepsilon\to 0} B_{1,\varepsilon}(t,x,y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega \times \Omega; \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))).$$

To study $B_2(t,x,y)$ take a $u(y) \in C_0^\infty(K; \Lambda^{0,q_1}T^*(\Omega))$, $K \subset\subset \Omega$ and set

$$\chi_\nu(\eta) = \chi(2^{-\nu}\eta) - \chi(2^{1-\nu}\eta),\ \nu > 0,\ \chi_0(\eta) = \chi(\eta).$$

Then we have

$$\sum_{\nu=0}^{\infty}\chi_\nu = 1 \text{ and } 2^{\nu-1} \le \left|\eta\right| \le 2^{\nu+1} \text{ when } \eta \in \operatorname{supp}\chi_\nu,\ \nu \ne 0.$$

We assume that $b(t,x,\eta) = 0$ if $\left|\eta\right| \le 1$. If $x \in K$, we obtain for all indices α, β and every $\varepsilon > 0$, there exists $c_{\varepsilon,\alpha,\beta,K} > 0$, such that

$$\left|D_x^\alpha D_\eta^\beta(\chi_\nu(\eta)\chi_W(t,x,\eta)b(t,x,\eta))\right| \le c_{\varepsilon,\alpha,\beta,K}\, e^{t(-r(x,\eta)+\varepsilon\left|\eta\right|)}(1+\left|\eta\right|)^{k-\left|\beta\right|}. \tag{6.2}$$

Note that $\left|D^{\alpha}\chi_{\nu}(\eta)\right| \le c_{\alpha}(1+\left|\eta\right|)^{-|\alpha|}$ with a constant independent of ν. We have

$$\begin{aligned}B_{2,\nu+1} &= \int\int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}\chi_{\nu+1}(\eta)\chi_W(t,x,\eta)b(t,x,\eta)u(y)dyd\eta\\ &= 2^{(2n-1)\nu}\int e^{i\lambda(\psi(\lambda t,x,\eta)-\langle y,\eta\rangle)}\chi_1(\eta)\chi_W(t,x,\lambda\eta)b(t,x,\lambda\eta)u(y)dyd\eta,\end{aligned}$$

where $\lambda = 2^{\nu}$. Since (6.2) holds, we have

$$\left|D^{\alpha}_{\eta}(\chi_W(t,x,2^{\nu}\eta)b(t,x,2^{\nu}\eta))\right| \le c2^{k\nu},$$

if $x \in K$, $1 < \left|\eta\right| < 4$, where $c > 0$. Since $d_y(\psi(\lambda t,x,\eta) - \langle y\ ,\eta\rangle) \ne 0$, if $\eta \ne 0$, we can integrate by parts and obtain

$$\left|B_{2,\nu+1}\right| \le c2^{\nu(2n-1+k-m)}\sum_{|\alpha|\le m}\sup|D^{\alpha}u|.$$

Since m can be chosen arbitrary large, we conclude that $\sum_{\nu}\left|B_{2,\nu}\right|$: converges and that $B(t)$ defines an operator

$$B(t) : C^{\infty}_0(\Omega_y; \Lambda^{0,q_1}T^*(\Omega)) \to C^{\infty}(\overline{\mathbb{R}}_+\times\Omega_x; \Lambda^{0,q_2}T^*(\Omega)).$$

Let $B^*(t)$ be the formal adjoint of $B(t)$ with respect to $(\,|\,)$. From (6.1), we see that $\psi'_x(t,x,\eta) \ne 0$ on W. We can repeat the procedure above and conclude that $B^*(t)$ defines an operator

$$C^{\infty}_0(\Omega_x; \Lambda^{0,q_2}T^*(\Omega)) \to C^{\infty}(\overline{\mathbb{R}}_+\times\Omega_y; \Lambda^{0,q_1}T^*(\Omega)).$$

Hence, we can extend $B(t)$ to

$$\mathscr{E}'(\Omega; \Lambda^{0,q_1}T^*(\Omega)) \to C^{\infty}(\overline{\mathbb{R}}_+; \mathscr{D}'(\Omega; \Lambda^{0,q_2}T^*(\Omega)))$$

by the following formula

$$(B(t)u(y)\mid v(x)) = (u(y)\mid B^*(t)v(x)),\ u\in\mathscr{E}'(\Omega; \Lambda^{0,q_1}T^*(\Omega)), v \in C^{\infty}_0(\Omega; \Lambda^{0,q_2}T^*(\Omega)).$$

When $x \ne y$ and $(x,y) \in \Sigma\times\Sigma$, we have $d_{\eta}(\psi(t,x,\eta) - \langle y\ ,\eta\rangle) \ne 0$, we can repeat the procedure above and conclude that

$$B(t,x,y) \in C^{\infty}(\overline{\mathbb{R}}_+; C^{\infty}(\Omega\times\Omega\smallsetminus\operatorname{diag}(\Omega\times\Omega); \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega)))).$$

Finally, in view of the exponential decrease as $t\to\infty$ of the symbol $b(t,x,\eta)$, we see that the kernel $B(t)|_{t>0}$ is smoothing. □

Let $b(t,x,\eta)\in\hat{S}^k_r$ with $r>0$. Our next step is to show that we can also define the operator

$$B(x,y)=\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)dt\Big)d\eta$$

as an oscillatory integral. We have the following

Proposition 6.6. *Let*

$$b(t,x,\eta)\in\hat{S}^k_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega)))$$

with $r>0$. We assume that $b(t,x,\eta)=0$ when $\left|\eta\right|\leq 1$. Then we can define

$$B:C^\infty_0(\Omega;\Lambda^{0,q_1}T^*(\Omega))\to C^\infty(\Omega;\Lambda^{0,q_2}T^*(\Omega))$$

with distribution kernel

$$B(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)dt\Big)d\eta$$

and B has a unique continuous extension

$$B:\mathscr{E}'(\Omega;\Lambda^{0,q_1}T^*(\Omega))\to\mathscr{D}'(\Omega;\Lambda^{0,q_2}T^*(\Omega)).$$

Moreover, we have

$$B(x,y)\in C^\infty(\Omega\times\Omega\setminus\operatorname{diag}(\Omega\times\Omega);\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega))).$$

Proof. Let W and $\chi_W(t,x,\eta)$ be as in Proposition 6.5. We formally set

$$\begin{aligned}B(x,y)=&\frac{1}{(2\pi)^{2n-1}}\int\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}(1-\chi_W(t,x,\eta))b(t,x,\eta)dt\,d\eta\\&+\frac{1}{(2\pi)^{2n-1}}\int\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}\chi_W(t,x,\eta)b(t,x,\eta)dt\,d\eta\\=&\,B_1(x,y)+B_2(x,y)\end{aligned}$$

where in $B_1(x,y)$ and $B_2(x,y)$ we have introduced the cut-off functions $(1-\chi_W)$ and χ_W respectively. Since

$$\operatorname{Im}\psi(t,x,\eta)\geq c'\left|\eta\right|\text{ outside }W,$$

where $c'>0$, we have

$$\left|e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}(1-\chi_W(t,x,\eta))b(t,x,\eta)\right|\leq ce^{-c'\left|\eta\right|}e^{-\varepsilon_0 t\left|\eta\right|}(1+\left|\eta\right|)^k,\ \varepsilon_0>0$$

and similar estimates for the derivatives. From this, we see that $B_1(x,y)\in C^\infty(\Omega\times\Omega;\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega)))$.

Choose $\chi\in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi(\eta)=1$ when $\left|\eta\right|<1$ and $\chi(\eta)=0$ when $\left|\eta\right|>2$. To study $B_2(x,y)$ take a $u(y)\in C_0^\infty(K;\Lambda^{0,q_1}T^*(\Omega))$, $K\subset\subset\Omega$ and set

$$B_{2,\lambda}(x)=\frac{1}{(2\pi)^{2n-1}}\int_0^\infty\Big(\int\int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)\chi_W(t,x,\eta)\chi(\frac{\eta}{\lambda})u(y)dyd\eta\Big)dt.$$

We have

$$\begin{aligned}B_{2,2\lambda}(x)-B_{2,\lambda}(x)=\frac{\lambda^{2n-1}}{(2\pi)^{2n-1}}\int_0^\infty\Big(\int\int &e^{i\lambda(\psi(\lambda t,x,\eta)-\langle y,\eta\rangle)}\chi_W(t,x,\lambda\eta)b(t,x,\lambda\eta)\\&(\chi(\frac{\eta}{2})-\chi(\eta))u(y)dyd\eta\Big)dt.\end{aligned}$$

Since $d_y(\psi(\lambda t,x,\eta)-\langle y,\eta\rangle)\neq0$, $\eta\neq0$, we obtain

$$\begin{aligned}&\left|\int\int e^{i\lambda(\psi(\lambda t,x,\eta)-\langle y,\eta\rangle)}\chi_W(t,x,\lambda\eta)b(t,x,\lambda\eta)(\chi(\frac{\eta}{2})-\chi(\eta))u(y)dyd\eta\right|\\&\quad\le c\lambda^{-N}\sum_{|\alpha|\le N}\sup\left|D^\alpha_{y,\eta}\chi_W(t,x,\lambda\eta)b(t,x,\lambda\eta)(\chi(\frac{\eta}{2})-\chi(\eta))u(y)\right|\\&\quad\le c'\lambda^{-N}e^{-\varepsilon_0t\left|\eta\right|}(1+|\lambda|)^k,\end{aligned}$$

where c, c', $\varepsilon_0>0$. Hence $B_2(x)=\lim_{\lambda\to\infty}B_{2,\lambda}(x)$ exists. Thus, $B(x,y)$ defines an operator

$$C_0^\infty(\Omega_y;\Lambda^{0,q_1}T^*(\Omega))\to C^\infty(\Omega_x;\Lambda^{0,q_2}T^*(\Omega)).$$

Let B^* be the formal adjoint of B with respect to $(\,|\,)$. Since $\psi'_x(t,x,\eta)\neq0$ on W, we can repeat the procedure above and conclude that B^* defines an operator

$$C_0^\infty(\Omega_x;\Lambda^{0,q_2}T^*(\Omega))\to C^\infty(\Omega_y;\Lambda^{0,q_1}T^*(\Omega)).$$

We can extend B to

$$\mathscr{E}'(\Omega;\Lambda^{0,q_1}T^*(\Omega))\to\mathscr{D}'(\Omega;\Lambda^{0,q_2}T^*(\Omega))$$

by the following formula

$$(Bu(y)\,|\,v(x))=(u(y)\,|\,B^*v(x)),\ u\in\mathscr{E}'(\Omega;\Lambda^{0,q_1}T^*(\Omega)),v\in C_0^\infty(\Omega;\Lambda^{0,q_2}T^*(\Omega)).$$

Finally, when $x\neq y$ and $(x,y)\in\Sigma\times\Sigma$, we have

$$d_\eta(\psi(t,x,\eta)-\langle y\ ,\eta\rangle)\neq0,$$

we can repeat the procedure above and conclude that

$$B(x,y)\in C^\infty(\Omega\times\Omega\setminus\operatorname{diag}(\Omega\times\Omega);\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega))).$$

□

Remark 6.7. Let

$$a(t,x,\eta)\in\hat{S}^k_0(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega))).$$

We assume $a(t,x,\eta)=0$, if $\left|\eta\right|\leq 1$ and

$$a(t,x,\eta)-a(\infty,x,\eta)\in\hat{S}^k_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega)))$$

with $r>0$, where

$$a(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega))).$$

Then we can also define

$$A(x,y)=\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)dt\Big)d\eta$$

as an oscillatory integral by the following formula:

$$A(x,y)=\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}(-t)(i\psi'_t(t,x,\eta)a(t,x,\eta)+a'_t(t,x,\eta))dt\Big)d\eta.$$

We notice that

$$(-t)(i\psi'_t(t,x,\eta)a(t,x,\eta)+a'_t(t,x,\eta))\in\hat{S}^{k+1}_r,\ r>0.$$

Let B be as in the proposition 6.6. We can show that B is a matrix of pseudodifferential operators of order k type $(\frac{1}{2},\frac{1}{2})$. We review some facts about pseudodifferential operators of type $(\frac{1}{2},\frac{1}{2})$.

Definition 6.8. Let $k\in\mathbb{R}$ and let $0\leq q\leq n-1$, $q\in\mathbb{N}$.

$$S^k_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

is the space of all

$$a\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

such that for every compact sets $K\subset\Omega$ and all $\alpha\in\mathbb{N}^{2n-1}$, $\beta\in\mathbb{N}^{2n-1}$, there is a constant $c_{\alpha,\beta,K}>0$ such that

$$\left|\partial^\alpha_x\partial^\beta_\xi a(x,\xi)\right|\leq c_{\alpha,\beta,K}(1+|\xi|)^{k-\frac{|\beta|}{2}+\frac{|\alpha|}{2}},\ (x,\xi)\in T^*(\Omega),x\in K.$$

$S^k_{\frac{1}{2},\frac{1}{2}}$ is called the space of symbols of order k type $(\frac{1}{2},\frac{1}{2})$. We write

$$S^{-\infty}_{\frac{1}{2},\frac{1}{2}}=\bigcap_{m\in\mathbb{R}}S^m_{\frac{1}{2},\frac{1}{2}},\quad S^{\infty}_{\frac{1}{2},\frac{1}{2}}=\bigcup_{m\in\mathbb{R}}S^m_{\frac{1}{2},\frac{1}{2}}.$$

Let $a(x,\xi)\in S^k_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$. We can also define

$$A(x,y)=\frac{1}{(2\pi)^{2n-1}}\int e^{i\langle x-y,\xi\rangle}a(x,\xi)d\xi$$

as an oscillatory integral and we can show that A is continuous

$$A: C^\infty_0(\Omega;\Lambda^{0,q}T^*(\Omega))\to C^\infty(\Omega;\Lambda^{0,q}T^*(\Omega))$$

and has unique continuous extension

$$A:\mathscr{E}'(\Omega;\Lambda^{0,q}T^*(\Omega))\to\mathscr{D}'(\Omega;\Lambda^{0,q}T^*(\Omega)).$$

Definition 6.9. Let $k\in\mathbb{R}$ and let $0\le q\le n-1$, $q\in\mathbb{N}$. A pseudodifferential operator of order k type $(\frac{1}{2},\frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Omega)$ to sections of $\Lambda^{0,q}T^*(\Omega)$ is a continuous linear map

$$A: C^\infty_0(\Omega;\Lambda^{0,q}T^*(\Omega))\to\mathscr{D}'(\Omega;\Lambda^{0,q}T^*(\Omega))$$

such that the distribution kernel of A is

$$K_A=A(x,y)=\frac{1}{(2\pi)^{2n-1}}\int e^{i\langle x-y,\xi\rangle}a(x,\xi)d\xi$$

with $a\in S^k_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$. We call $a(x,\xi)$ the symbol of A. We shall write

$$L^k_{\frac{1}{2},\frac{1}{2}}(\Omega;\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))$$

to denote the space of pseudodifferential operators of order k type $(\frac{1}{2},\frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Omega)$ to sections of $\Lambda^{0,q}T^*(\Omega)$. We write

$$L^{-\infty}_{\frac{1}{2},\frac{1}{2}}=\bigcap\nolimits_{m\in\mathbb{R}}L^m_{\frac{1}{2},\frac{1}{2}},\quad L^{\infty}_{\frac{1}{2},\frac{1}{2}}=\bigcup\nolimits_{m\in\mathbb{R}}L^m_{\frac{1}{2},\frac{1}{2}}.$$

We recall the following classical proposition of Calderon-Vaillancourt.

Proposition 6.10. *If $A\in L^k_{\frac{1}{2},\frac{1}{2}}(\Omega;\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))$. Then, for every $s\in\mathbb{R}$, A is continuous*

$$A: H^s_{\rm comp}(\Omega;\Lambda^{0,q}T^*(\Omega))\to H^{s-k}_{\rm loc}(\Omega;\Lambda^{0,q}T^*(\Omega))$$

and

$$A: H^s_{\rm loc}(\Omega;\Lambda^{0,q}T^*(\Omega))\to H^{s-k}_{\rm loc}(\Omega;\Lambda^{0,q}T^*(\Omega))$$

if A is properly supported. (For the precise meaning of properly supported operators, see the discussion before Definition A.6.)

Proof. See Hörmander [15]. $\square$

We need the following properties of the phase $\psi(t,x,\eta)$.

Lemma 6.11. *For every compact set $K\subset\Omega$ and all $\alpha\in\mathbb{N}^{2n-1}$, $\beta\in\mathbb{N}^{2n-1}$, $|\alpha|+\left|\beta\right|\geq 1$, there exists a constant $c_{\alpha,\beta,K}>0$, such that*

$$\left|\partial_x^\alpha\partial_\eta^\beta(\psi(t,x,\eta)-\langle x,\eta\rangle)\right|\leq c_{\alpha,\beta,K}(1+\left|\eta\right|)^{\frac{|\alpha|-|\beta|}{2}}(1+\operatorname{Im}\psi(t,x,\eta))^{\frac{|\alpha|+|\beta|}{2}},\ \text{if } |\alpha|+\left|\beta\right|=1$$

and

$$\left|\partial_x^\alpha\partial_\eta^\beta(\psi(t,x,\eta)-\langle x,\eta\rangle)\right|\leq c_{\alpha,\beta,K}(1+\left|\eta\right|)^{1-|\beta|},\ \text{if } |\alpha|+\left|\beta\right|\geq 2,$$

where $x\in K$, $t\in\overline{\mathbb{R}}_+$, $\left|\eta\right|\geq 1$.

Proof. For $\left|\eta\right|=1$, we consider Taylor expansions of $\partial_{x_j}(\psi(t,x,\eta)-\langle x,\eta\rangle)$, $j=1,\ldots,2n-1$, at $(x_0,\eta_0)\in\Sigma$,

$$\begin{aligned}\partial_{x_j}(\psi(t,x,\eta)-\langle x,\eta\rangle)=&\sum_k\frac{\partial^2\psi}{\partial x_k\partial x_j}(t,x_0,\eta_0)(x_k-x_0^{(k)})\\&+\sum_k\frac{\partial^2\psi}{\partial\eta_k\partial x_j}(t,x_0,\eta_0)(\eta_k-\eta_0^{(k)})\\&+O(|(x-x_0)|^2+\left|(\eta-\eta_0)\right|^2),\end{aligned}$$

where $x_0=(x_0^{(1)},\ldots,x_0^{(2n-1)})$, $\eta_0=(\eta_0^{(1)},\ldots,\eta_0^{(2n-1)})$. Thus, for every compact set $K\subset\Omega$ there exists a constant $c>0$, such that

$$\left|\partial_x(\psi(t,x,\eta)-\langle x,\eta\rangle)\right|\leq c\frac{t}{1+t}\operatorname{dist}((x,\eta),\Sigma)),$$

where $x\in K$, $t\in\overline{\mathbb{R}}_+$, $\left|\eta\right|=1$. From (4.4), we have

$$\operatorname{Im}\psi(t,x,\eta)\asymp(\frac{t}{1+t})\operatorname{dist}((x,\eta),\Sigma))^2,\ \left|\eta\right|=1.$$

Hence,

$$(\frac{t}{1+t})^{\frac{1}{2}}\operatorname{dist}((x,\eta),\Sigma))\asymp(\operatorname{Im}\psi(t,x,\eta))^{\frac{1}{2}},\ \left|\eta\right|=1.$$

Thus, for every compact set $K\subset\Omega$ there exists a constant $c>0$, such that

$$\left|\partial_x(\psi(t,x,\eta)-\langle x,\eta\rangle)\right|\leq c(\frac{t}{1+t})^{\frac{1}{2}}(\operatorname{Im}\psi(t,x,\eta))^{\frac{1}{2}},\ \left|\eta\right|=1,x\in K.$$

From above, we get for $\left|\eta\right|\geq 1$,

$$\begin{aligned}\left|\partial_x(\psi(t,x,\eta)-\langle x,\eta\rangle)\right|&=\left|\eta\right|\left|\partial_x(\psi(t\left|\eta\right|,x,\frac{\eta}{\left|\eta\right|})-\left\langle x,\frac{\eta}{\left|\eta\right|}\right\rangle)\right|\\&\leq c\left|\eta\right|^{\frac{1}{2}}(\frac{t\left|\eta\right|}{1+t\left|\eta\right|})^{\frac{1}{2}}(\operatorname{Im}\psi(t,x,\eta))^{\frac{1}{2}}\\&\leq c'(1+\left|\eta\right|)^{\frac{1}{2}}(1+\operatorname{Im}\psi(t,x,\eta))^{\frac{1}{2}},\end{aligned}$$

where $c,\ c' > 0$, $x \in K$, $t \in \overline{\mathbb{R}}_+$. Here K is as above. Similarly, for every compact set $K \subset \Omega$ there exists a constant $c > 0$, such that

$$\left|\partial_\eta(\psi(t,x,\eta) - \langle x,\eta\rangle)\right| \leq c(1+\left|\eta\right|)^{-\frac{1}{2}}(\operatorname{Im}\psi(t,x,\eta))^{\frac{1}{2}},$$

where $x \in K$, $t \in \overline{\mathbb{R}}_+$ and $\left|\eta\right| \geq 1$.

For $|\alpha| + \left|\beta\right| \geq 2$, we have

$$\left|\partial_x^\alpha \partial_\eta^\beta(\psi(t,x,\eta) - \langle x,\eta\rangle)\right| \leq c(1+\left|\eta\right|)^{1-|\beta|},$$

where c, $x \in K$, $t \in \overline{\mathbb{R}}_+$ and $\left|\eta\right| \geq 1$. Here K is as above.

The lemma follows. □

We also need the following

Lemma 6.12. *For every compact set $K \subset \Omega$ and all $\alpha \in \mathbb{N}^{2n-1}$, $\beta \in \mathbb{N}^{2n-1}$, there exist a constant $c_{\alpha,\beta,K} > 0$ and $\varepsilon > 0$, such that*

$$\left|\partial_x^\alpha \partial_\eta^\beta(t\psi_t'(t,x,\eta))\right| \leq c_{\alpha,\beta,K}(1+\left|\eta\right|)^{\frac{|\alpha|-|\beta|}{2}} e^{-t\varepsilon|\eta|}(1+\operatorname{Im}\psi(t,x,\eta))^{1+\frac{|\alpha|+|\beta|}{2}},$$
$$\text{if } |\alpha| + \left|\beta\right| \leq 1$$

and

$$\left|\partial_x^\alpha \partial_\eta^\beta(t\psi_t'(t,x,\eta))\right| \leq c_{\alpha,\beta,K}(1+\left|\eta\right|)^{1-|\beta|} e^{-t\varepsilon|\eta|},\ \text{ if } |\alpha| + \left|\beta\right| \geq 2,$$

where $x \in K$, $t \in \overline{\mathbb{R}}_+$, $\left|\eta\right| \geq 1$.

Proof. The proof of this lemma is essentially the same as the proof of Lemma 6.11. □

We need the following

Lemma 6.13. *For every compact set $K \subset \Omega$ and all $\alpha \in \mathbb{N}^{2n-1}$, $\beta \in \mathbb{N}^{2n-1}$, there exist a constant $c_{\alpha,\beta,K} > 0$ and $\varepsilon > 0$, such that*

$$\left|\partial_x^\alpha \partial_\eta^\beta(e^{i(\psi(t,x,\eta) - \langle x,\eta\rangle})\right| \leq c_{\alpha,\beta,K}(1+\left|\eta\right|)^{\frac{|\alpha|-|\beta|}{2}} e^{-\operatorname{Im}\psi(t,x,\eta)}(1+\operatorname{Im}\psi(t,x,\eta))^{\frac{|\alpha|+|\beta|}{2}} \quad (6.3)$$

and

$$\left|\partial_x^\alpha \partial_\eta^\beta(e^{i(\psi(t,x,\eta) - \langle x,\eta\rangle} t\psi_t'(t,x,\eta))\right|$$
$$\leq c_{\alpha,\beta,K}(1+\left|\eta\right|)^{\frac{|\alpha|-|\beta|}{2}} e^{-t\varepsilon|\eta|} e^{-\operatorname{Im}\psi(t,x,\eta)}(1+\operatorname{Im}\psi(t,x,\eta))^{1+\frac{|\alpha|+|\beta|}{2}}, \quad (6.4)$$

where $x \in K$, $t \in \overline{\mathbb{R}}_+$, $\left|\eta\right| \geq 1$.

Proof. First, we prove (6.3). We proceed by induction over $|\alpha|+|\beta|$. For $|\alpha|+|\beta|\leq 1$, from Lemma 6.11, we get (6.3). Let $|\alpha|+|\beta|\geq 2$. Then

$$\begin{aligned}&\left|\partial_x^{\alpha}\partial_\eta^{\beta}(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle})\right|\\&=\left|\sum_{\alpha'+\alpha''=\alpha,\beta'+\beta''=\beta,(\alpha'',\beta'')\neq 0}\partial_x^{\alpha'}\partial_\eta^{\beta'}(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle})\partial_x^{\alpha''}\partial_\eta^{\beta''}(i\psi(t,x,\eta)-i\langle x,\eta\rangle)\right|.\end{aligned}$$

By the induction assumption, we have for every compact set $K\subset\Omega$, there exists a constant $c>0$, such that

$$\left|\partial_x^{\alpha'}\partial_\eta^{\beta'}(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle})\right|\leq c(1+|\eta|)^{\frac{|\alpha'|-|\beta'|}{2}}e^{-\operatorname{Im}\psi(t,x,\eta)}(1+\operatorname{Im}\psi(t,x,\eta))^{\frac{|\alpha'|+|\beta'|}{2}},\quad (6.5)$$

where $x\in K$, $t\in\overline{\mathbb{R}}_+$, $|\eta|\geq 1$. From Lemma 6.11, we have

$$\left|\partial_x^{\alpha''}\partial_\eta^{\beta''}(i\psi(t,x,\eta)-i\langle x,\eta\rangle)\right|\leq c(1+|\eta|)^{\frac{|\alpha''|-|\beta''|}{2}}(1+\operatorname{Im}\psi(t,x,\eta))^{\frac{|\alpha''|+|\beta''|}{2}},\quad (6.6)$$

where $x\in K$, $t\in\overline{\mathbb{R}}_+$, $|\eta|\geq 1$. Combining (6.5) with (6.6), we get (6.3).

From Leibniz's formula, Lemma 6.12 and (6.3), we get (6.4). □

Lemma 6.14. *Let*

$$b(t,x,\eta)\in\hat{S}^k_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))$$

with $r>0$. *We assume that* $b(t,x,\eta)=0$ *when* $|\eta|\leq 1$. *Then*

$$q(x,\eta)=\int_0^\infty e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}b(t,x,\eta)dt\in S^{k-1}_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))).$$

Proof. From Leibniz's formula, we have

$$\begin{aligned}&\partial_x^{\alpha}\partial_\eta^{\beta}(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}b(t,x,\eta))\\&=\sum_{\alpha'+\alpha''=\alpha,\beta'+\beta''=\beta}(\partial_x^{\alpha'}\partial_\eta^{\beta'}e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle})(\partial_x^{\alpha''}\partial_\eta^{\eta''}b(t,x,\eta)).\end{aligned}$$

From (6.3) and the definition of $\hat{S}^k_r$, we have for every compact set $K\subset\Omega$, there exist a constant $c>0$ and $\varepsilon>0$, such that

$$\begin{aligned}&\left|\partial_x^{\alpha}\partial_\eta^{\beta}q(x,\eta)\right|\\&\leq c\int_0^\infty e^{-\operatorname{Im}\psi(t,x,\eta)}(1+|\eta|)^{k+\frac{|\alpha|-|\beta|}{2}}(1+\operatorname{Im}\psi(t,x,\eta))^{\frac{|\alpha|+|\beta|}{2}}e^{-\varepsilon t|\eta|}dt\\&\leq c'(1+|\eta|)^{k-1+\frac{|\alpha|-|\beta|}{2}},\end{aligned}$$

where $c'>0$, $x\in K$. The lemma follows. □

We will next show

Proposition 6.15. *Let*

$$b(t,x,\eta)\in\hat{S}^k_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

with $r>0$. *We assume that* $b(t,x,\eta)=0$ *when* $\left|\eta\right|\le 1$. *Let*

$$\begin{aligned}B: C^\infty_0(\Omega;\Lambda^{0,q}T^*(\Omega))&\to C^\infty(\Omega;\Lambda^{0,q}T^*(\Omega)),\\ \mathscr{E}'(\Omega;\Lambda^{0,q}T^*(\Omega))&\to\mathscr{D}'(\Omega;\Lambda^{0,q}T^*(\Omega)).\end{aligned}$$

be as in Proposition 6.6. *Then*

$$B\in L^{k-1}_{\frac{1}{2},\frac{1}{2}}(\Omega;\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))$$

with symbol

$$\begin{aligned}q(x,\eta)&=\int_0^\infty e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}b(t,x,\eta)dt\\ &\in S^{k-1}_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))).\end{aligned}$$

Proof. Choose $\chi\in C^\infty_0(\mathbb{R}^{2n-1})$ so that $\chi(\eta)=1$ when $\left|\eta\right|<1$ and $\chi(\eta)=0$ when $\left|\eta\right|>2$. Take a $u(y)\in C^\infty_0(\Omega;\Lambda^{0,q}T^*(\Omega))$, then

$$\begin{aligned}Bu&=\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{2n-1}}\int_0^\infty\Big(\int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)u(y)\chi(\varepsilon\eta)d\eta\Big)dt\\ &=\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{2n-1}}\int_0^\infty e^{i\langle x-y,\eta\rangle}\Big(\int e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}b(t,x,\eta)u(y)\chi(\varepsilon\eta)\Big)d\eta dt\\ &=\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{2n-1}}\int e^{i\langle x-y,\eta\rangle}q(x,\eta)u(y)\chi(\varepsilon\eta)d\eta.\end{aligned}$$

From Lemma 6.14, we know that $q(x,\eta)\in S^{k-1}_{\frac{1}{2},\frac{1}{2}}$. Thus

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{2n-1}}\int e^{i\langle x-y,\eta\rangle}q(x,\eta)u(y)\chi(\varepsilon\eta)d\eta\in L^{k-1}_{\frac{1}{2},\frac{1}{2}}(\Omega;\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)).$$

□

We need the following

Lemma 6.16. *Let*

$$a(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

be a classical symbol of order k, that is

$$a(\infty,x,\eta)\sim\sum_{j=0}^{\infty}a_j(\infty,x,\eta)$$
$$\text{in the symbol space } S^k_{1,0}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))),$$

where

$$a_j(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))),\ \ j=0,1,\ldots,$$
$$a_j(\infty,x,\lambda\eta)=\lambda^{k-j}a_j(\infty,x,\eta),\ \lambda\ge 1,\ \left|\eta\right|\ge 1,\ \ j=0,1,\ldots,$$

and $S^k_{1,0}$ *is the Hörmander symbol space. We assume that* $a(\infty,x,\eta)=0$ *when* $\left|\eta\right|\le 1$*. Then*

$$p(x,\eta)=\int_0^\infty(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}-e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)})a(\infty,x,\eta)dt$$
$$\in S^{k-1}_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))).$$

Proof. We have

$$p(x,\eta)=\int_0^\infty e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}(-t)i\psi'_t(t,x,\eta)a(\infty,x,\eta)dt. \tag{6.7}$$

From (6.4), we can repeat the procedure in the proof of Lemma 6.14 to get the lemma. □

Remark 6.17. Let

$$a(t,x,\eta)\in\hat{S}^k_0(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))).$$

We assume $a(t,x,\eta)=0$, if $\left|\eta\right|\le 1$ and

$$a(t,x,\eta)-a(\infty,x,\eta)\in\hat{S}^k_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

with $r>0$, where $a(\infty,x,\eta)$ is as in Lemma 6.16. By Lemma 6.14 and Lemma 6.16, we have

$$\begin{aligned}&\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}a(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)}a(\infty,x,\eta)\Big)dt\\&=\int_0^\infty e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}(a(t,x,\eta)-a(\infty,x,\eta))dt\\&\quad+\int_0^\infty(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}-e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)})a(\infty,x,\eta))dt\\&\in S^{k-1}_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))).\end{aligned}$$

Let

$$A(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)-$$
$$e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)dt\Big)d\eta$$

be as in the Remark 6.7. Then as in Proposition 6.15, we can show that

$$A\in L^{k-1}_{\frac{1}{2},\frac{1}{2}}(\Omega;\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))$$

with symbol

$$q(x,\eta)=\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}a(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)}a(\infty,x,\eta)\Big)dt$$
$$\in S^{k-1}_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))).$$

We have the following

Proposition 6.18. *Let*

$$a(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

be a classical symbol of order k. Then

$$a(x,\eta)=e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)}a(\infty,x,\eta)$$
$$\in S^k_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))).$$

Proof. In view of Lemma 6.13, we have for every compact set $K\subset\Omega$ and all $\alpha\in\mathbb{N}^{2n-1}$, $\beta\in\mathbb{N}^{2n-1}$, there exists a constant $c_{\alpha,\beta,K}>0$, such that

$$\left|\partial_x^\alpha\partial_\eta^\beta(e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)})\right|\leq c_{\alpha,\beta,K}(1+\left|\eta\right|)^{\frac{|\alpha|-|\beta|}{2}}e^{-\operatorname{Im}\psi(\infty,x,\eta)}(1+\operatorname{Im}\psi(\infty,x,\eta))^{\frac{|\alpha|+|\beta|}{2}},$$

where $x\in K$, $\left|\eta\right|\geq 1$. From this and Leibniz's formula, we get the proposition. □

7 The heat equation

Until further notice, we work with some real local coordinates

$$x=(x_1,\ldots,x_{2n-1})$$

defined on an open set $\Omega\subset X$. Let

$$b(t,x,\eta)\in\hat{S}^k_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

with $r>0$. We assume that $b(t,x,\eta)=0$ when $\left|\eta\right|\leq 1$. From now on, we write

$$\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)dt\Big)d\eta$$

to denote the kernel of pseudodifferential operator of order $k-1$ type $(\frac{1}{2},\frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Omega)$ to sections of $\Lambda^{0,q}T^*(\Omega)$ with symbol

$$\int_0^\infty e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}b(t,x,\eta)dt\in S^{k-1}_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))).$$

(See Proposition 6.15.)

Let

$$a(t,x,\eta)\in\hat{S}^k_0(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))).$$

We assume that $a(t,x,\eta)=0$ when $\left|\eta\right|\leq 1$ and that

$$a(t,x,\eta)-a(\infty,x,\eta)\in\hat{S}^k_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))\ \text{ with }\ r>0,$$

where

$$a(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

is a classical symbol of order k. From now on, we write

$$\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)dt\Big)d\eta$$

to denote the kernel of pseudodifferential operator of order $k-1$ type $(\frac{1}{2},\frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Omega)$ to sections of $\Lambda^{0,q}T^*(\Omega)$ with symbol

$$\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}a(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)}a(\infty,x,\eta)\Big)dt$$

in $S^{k-1}_{\frac{1}{2},\frac{1}{2}}((T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$. (See Lemma 6.14 and Lemma 6.16.) From Proposition 5.6, we have the following

Proposition 7.1. *Let Q be a C^∞ differential operator on Ω of order m. Let*

$$b(t,x,\eta)\in\hat{S}^k_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

with $r>0$. We assume that $b(t,x,\eta)=0$ when $\left|\eta\right|\leq 1$. Set

$$Q(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta))=e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}c(t,x,\eta),$$
$$c(t,x,\eta)\in\hat{S}^{k+m}_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))),\ r>0.$$

Put

$$B(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)dt\Big)d\eta,$$
$$C(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}c(t,x,\eta)dt\Big)d\eta.$$

We have

$$Q\circ B\equiv C.$$

Proposition 7.2. *Let Q be a C^∞ differential operator on Ω of order m. Let*

$$b(t,x,\eta)\in\hat{S}^k_0(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))).$$

We assume that $b(t,x,\eta)=0$ when $\left|\eta\right|\leq 1$ and that

$$b(t,x,\eta)-b(\infty,x,\eta)\in\hat{S}^k_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

with $r>0$, where

$$b(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

is a classical symbol of order k. Set

$$\begin{aligned}&Q\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\Big)\\&=e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}c(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}c(\infty,x,\eta),\end{aligned}$$

where

$$c(t,x,\eta)\in\hat{S}^{k+m}_0(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))),$$
$$c(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

is a classical symbol of order $k+m$. Then

$$c(t,x,\eta)-c(\infty,x,\eta)\in\hat{S}^{k+m}_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$$

with $r>0$. Put

$$\begin{aligned}B(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(&e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)-\\&e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\Big)dt\Big)d\eta,\end{aligned}$$
$$\begin{aligned}C(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(&e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}c(t,x,\eta)-\\&e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}c(\infty,x,\eta)\Big)dt\Big)d\eta.\end{aligned}$$

We have

$$Q\circ B\equiv C.$$

We return to our problem. From now on, we assume that our operators are properly supported. We assume that $Y(q)$ holds. Let

$$a_j(t,x,\eta)\in\hat{S}^{-j}_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))),\ \ j=0,1,\ldots,\ \ r>0,$$

be as in Proposition 6.3 with $a_0(0,x,\eta)=I$, $a_j(0,x,\eta)=0$ when $j>0$. Let

$$a(t,x,\eta)\sim\sum_{j=0}^{\infty}a_j(t,x,\eta)$$

in the symbol space $\hat{S}^0_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)))$,

where $r>0$. Let

$$(\partial_t+\Box^{(q)}_b)\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)=e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta). \tag{7.1}$$

From Proposition 5.6, we see that for every compact set $K\subset\Omega$, $\varepsilon>0$ and all indices α, β and $N\in\mathbb{N}$, there exists $c_{\alpha,\beta,N,\varepsilon,K}>0$ such that

$$\left|\partial^\alpha_x\partial^\beta_\eta b(t,x,\eta)\right|\leq c_{\alpha,\beta,N,\varepsilon,K}e^{t(-r(x,\eta)+\varepsilon|\eta|)}(\left|\eta\right|^{-N}+\left|\eta\right|^{2-N}(\operatorname{Im}\psi(t,x,\eta))^N), \tag{7.2}$$

where $t\in\overline{\mathbb{R}}_+$, $x\in K$, $\left|\eta\right|\geq1$. Choose $\chi\in C^\infty_0(\mathbb{R}^{2n-1})$ so that $\chi(\eta)=1$ when $\left|\eta\right|<1$ and $\chi(\eta)=0$ when $\left|\eta\right|>2$. Set

$$A(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)(1-\chi(\eta))dt\Big)d\eta. \tag{7.3}$$

We have the following proposition

Proposition 7.3. *Suppose $Y(q)$ holds. Let $A=A(x,y)$ be as in (7.3). We have*

$$\Box^{(q)}_bA\equiv I.$$

Proof. We have

$$\begin{aligned}&\Box^{(q)}_b\Big(\frac{1}{(2\pi)^{2n-1}}e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)(1-\chi(\eta))\Big)\\&\quad=\frac{1}{(2\pi)^{2n-1}}e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)(1-\chi(\eta))\\&\qquad-\frac{1}{(2\pi)^{2n-1}}\frac{\partial}{\partial t}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)(1-\chi(\eta)),\end{aligned}$$

where $b(t,x,\eta)$ is as in (7.1), (7.2). From Proposition 7.1, we have

$$\begin{aligned}\Box^{(q)}_b\circ A\equiv&\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)(1-\chi(\eta))dt\Big)d\eta\\&-\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\frac{\partial}{\partial t}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta.\end{aligned}$$

From (7.2), it follows that

$$\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)(1-\chi(\eta))dt\Big)d\eta$$

is smoothing. Choose a cut-off function $\chi_1(\eta)\in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi_1(\eta)=1$ when $\left|\eta\right|<1$ and $\chi_1(\eta)=0$ when $\left|\eta\right|>2$. Take a

$$u(y)\in C_0^\infty(\Omega;\Lambda^{0,q}T^*(\Omega)),$$

then

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{2n-1}}\int\int\Big(\int_0^\infty\frac{\partial}{\partial t}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle}a(t,x,\eta)\Big)(1-\chi(\eta))\chi_1(\varepsilon\eta)u(y)dt\Big)d\eta dy\\&=\lim_{\varepsilon\to 0}\frac{-1}{(2\pi)^{2n-1}}\int\int e^{i\langle x-y,\eta\rangle}(1-\chi(\eta))\chi_1(\varepsilon\eta)u(y)d\eta dy.\end{aligned}$$

Hence

$$\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\frac{\partial}{\partial t}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta\equiv -I.$$

Thus

$$\Box_b^{(q)}\circ A\equiv I.$$

□

Remark 7.4. We assume that $Y(q)$ holds. From Proposition 7.3, we know that, for every local coordinate patch X_j, there exists a properly supported operator

$$A_j:\mathscr{D}'(X_j;\Lambda^{0,q}T^*(X_j))\to\mathscr{D}'(X_j;\Lambda^{0,q}T^*(X_j))$$

such that

$$A_j:H^s_{\rm loc}(X_j;\Lambda^{0,q}T^*(X_j))\to H^{s+1}_{\rm loc}(X_j;\Lambda^{0,q}T^*(X_j))$$

and

$$\Box_b^{(q)}\circ A_j-I:H^s_{\rm loc}(X_j;\Lambda^{0,q}T^*(X_j))\to H^{s+m}_{\rm loc}(X_j;\Lambda^{0,q}T^*(X_j))$$

for all $s\in\mathbb{R}$ and $m\geq 0$. We assume that $X=\bigcup_{j=1}^k X_j$. Let $\left\{\chi_j\right\}$ be a C^∞ partition of unity subordinate to $\left\{X_j\right\}$ and set

$$Au=\sum_j A_j(\chi_j u),\ u\in\mathscr{D}'(X;\Lambda^{0,q}T^*(X)).$$

A is well-defined as a continuous operator

$$A:H^s(X;\Lambda^{0,q}T^*(X))\to H^{s+1}(X;\Lambda^{0,q}T^*(X))$$

for all $s \in \mathbb{R}$. We notice that A is properly supported. We have

$$\Box_b^{(q)} \circ A - I : H^s(X; \Lambda^{0,q} T^*(X)) \to H^{s+m}(X; \Lambda^{0,q} T^*(X))$$

for all $s \in \mathbb{R}$ and $m \geq 0$.

Now we assume that $Y(q)$ fails. Let

$$a_j(t,x,\eta) \in \hat{S}_0^{-j}(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Omega), \Lambda^{0,q} T^*(\Omega))), \ \ j = 0, 1, \ldots,$$

and

$$a_j(\infty,x,\eta) \in C^\infty(T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Omega), \Lambda^{0,q} T^*(\Omega))), \ \ j = 0, 1, \ldots,$$

be as in Proposition 6.4. We recall that for some $r > 0$,

$$a_j(t,x,\eta) - a_j(\infty,x,\eta) \in \hat{S}_r^{-j}(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Omega), \Lambda^{0,q} T^*(\Omega))), \ \ j = 0, 1, \ldots.$$

Let

$$a(\infty,x,\eta) \sim \sum_{j=0}^{\infty} a_j(\infty,x,\eta)$$
$$\text{in the symbol space } S^0_{1,0}(T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Omega), \Lambda^{0,q} T^*(\Omega))). \tag{7.4}$$

Here $S^0_{1,0}$ is the Hörmander symbol space. Let

$$a(t,x,\eta) \sim \sum_{j=0}^{\infty} a_j(t,x,\eta)$$
$$\text{in the symbol space } \hat{S}^0_0(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Omega), \Lambda^{0,q} T^*(\Omega))). \tag{7.5}$$

We take $a(t,x,\eta)$ so that for every compact set $K \subset \Omega$ and all indices α, β, γ, k, there exists $c > 0$, c is independent of t, such that

$$\left| \partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta \Big(a(t,x,\eta) - \sum_{j=0}^{k} a_j(t,x,\eta)\Big) \right| \leq c\big(1 + \left|\eta\right|\big)^{-k-1+\gamma-|\beta|}, \tag{7.6}$$

where $t \in \overline{\mathbb{R}}_+$, $x \in K$, $\left|\eta\right| \geq 1$, and

$$a(t,x,\eta) - a(\infty,x,\eta) \in \hat{S}^0_r(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Omega), \Lambda^{0,q} T^*(\Omega)))$$

with $r > 0$. Let

$$(\partial_t + \Box_b^{(q)})\Big(e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} a(t,x,\eta)\Big) = e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} b(t,x,\eta). \tag{7.7}$$

Then

$$b(t,x,\eta) \in \hat{S}^2_0(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Omega), \Lambda^{0,q} T^*(\Omega)))$$

and

$$b(t,x,\eta)-b(\infty,x,\eta)\in\hat{S}^2_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))) \tag{7.8}$$

with $r>0$, where $b(\infty,x,\eta)$ is a classical symbol of order 2. Moreover, we have

$$\begin{aligned}&(\partial_t+\Box^{(q)}_b)\Big(\frac{1}{(2\pi)^{2n-1}}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)\Big)\\&=\frac{1}{(2\pi)^{2n-1}}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\Big).\end{aligned} \tag{7.9}$$

From Proposition 5.6, we see that for every compact set $K\subset\Omega$ and all indices α, β and $N\in\mathbb{N}$, there exists $c_{\alpha,\beta,N,K}>0$ such that

$$\left|\partial^\alpha_x\partial^\beta_\eta b(t,x,\eta)\right|\leq c_{\alpha,\beta,N,K}(\left|\eta\right|^{-N}+\left|\eta\right|^{2-N}(\operatorname{Im}\psi(t,x,\eta))^N), \tag{7.10}$$

where $t\in\overline{\mathbb{R}}_+$, $x\in K$, $\left|\eta\right|\geq1$. Thus,

$$\left|\partial^\alpha_x\partial^\beta_\eta b(\infty,x,\eta)\right|\leq c_{\alpha,\beta,N,K}(\left|\eta\right|^{-N}+\left|\eta\right|^{2-N}(\operatorname{Im}\psi(\infty,x,\eta))^N). \tag{7.11}$$

From (7.8), (7.10) and (7.11), it follows that for every compact set $K\subset\Omega$, $\varepsilon>0$ and all indices α, β and $N\in\mathbb{N}$, there exists $c_{\alpha,\beta,N,\varepsilon,K}>0$ such that

$$\begin{aligned}&\left|\partial^\alpha_x\partial^\beta_\eta\Big(b(t,x,\eta)-b(\infty,x,\eta)\Big)\right|\\&\quad\leq c_{\alpha,\beta,N,\varepsilon,K}\Big(e^{t(-r(x,\eta)+\varepsilon\left|\eta\right|)}(\left|\eta\right|^{-N}+\left|\eta\right|^{2-N}(\operatorname{Im}\psi(t,x,\eta))^N)\Big)^{\frac{1}{2}},\end{aligned} \tag{7.12}$$

where $t\in\overline{\mathbb{R}}_+$, $x\in K$, $\left|\eta\right|\geq1$, $r>0$.

Choose $\chi\in C^\infty_0(\mathbb{R}^{2n-1})$ so that $\chi(\eta)=1$ when $\left|\eta\right|<1$ and $\chi(\eta)=0$ when $\left|\eta\right|>2$. Set

$$\begin{aligned}G(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int^\infty_0\Big(&e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)-\\&e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta.\end{aligned} \tag{7.13}$$

Put

$$S(x,y)=\frac{1}{(2\pi)^{2n-1}}\int e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)d\eta. \tag{7.14}$$

We have the following

Proposition 7.5. *We assume that $Y(q)$ fails. Let G and S be as in (7.13) and (7.14) respectively. Then*

$$S+\Box^{(q)}_b\circ G\equiv I$$

and

$$\Box^{(q)}_b\circ S\equiv0.$$

Proof. We have

$$\Box_b^{(q)}\Big(\frac{1}{(2\pi)^{2n-1}}e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)$$
$$=\frac{1}{(2\pi)^{2n-1}}e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}\Big(b(t,x,\eta)-i\frac{\partial\psi}{\partial t}a-\frac{\partial a}{\partial t}\Big),$$

where $b(t,x,\eta)$ is as in (7.7). Letting $t\to\infty$, we get

$$\Box_b^{(q)}\Big(\frac{1}{(2\pi)^{2n-1}}e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)$$
$$=\frac{1}{(2\pi)^{2n-1}}e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta),$$

where $b(\infty,x,\eta)$ is as in (7.8) and (7.11). From (7.11), we have

$$\frac{1}{(2\pi)^{2n-1}}\int e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)d\eta$$

is smoothing. Thus

$$\Box_b^{(q)}\circ S\equiv 0.$$

In view of (7.9), we have

$$\Box_b^{(q)}\Big(\frac{1}{(2\pi)^{2n-1}}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)(1-\chi(\eta))$$
$$-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)(1-\chi(\eta))\Big)\Big)$$
$$=\frac{1}{(2\pi)^{2n-1}}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)$$
$$-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\Big)(1-\chi(\eta))$$
$$-\frac{1}{(2\pi)^{2n-1}}\frac{\partial}{\partial t}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)(1-\chi(\eta)).$$

From Proposition 7.2, we have

$$\Box_b^{(q)}\circ G=\Box_b^{(q)}\Big(\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)$$
$$-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta\Big)$$
$$\equiv\frac{1}{(2\pi)^{2n-1}}\Big(\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)$$
$$-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta$$
$$-\int\Big(\int_0^\infty\frac{\partial}{\partial t}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta\Big).$$

In view of (7.11) and (7.12), we see that

$$\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)$$
$$-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta$$
$$=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}\Big)b(\infty,x,\eta)(1-\chi(\eta))dt\Big)d\eta$$
$$+\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}\Big(b(t,x,\eta)-b(\infty,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta$$

is smoothing.

Choose a cut-off function $\chi_1(\eta)\in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi_1(\eta)=1$ when $\left|\eta\right|<1$ and $\chi_1(\eta)=0$ when $\left|\eta\right|>2$. Take a $u\in C_0^\infty(\Omega;\Lambda^{0,q}T^*(\Omega))$, then

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\frac{\partial}{\partial t}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)(1-\chi(\eta))\chi_1(\varepsilon\eta)u(y)dt\Big)d\eta dy$$
$$=\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{2n-1}}\int e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)(1-\chi(\eta))\chi_1(\varepsilon\eta)u(y)d\eta dy$$
$$-\lim_{\varepsilon\to 0}\int e^{i\langle x-y,\eta\rangle}(1-\chi(\eta))\chi_1(\varepsilon\eta)u(y)d\eta dy.$$

Hence

$$\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\frac{\partial}{\partial t}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta\equiv S-I.$$

Thus

$$S+\Box_{b,q}\circ G\equiv I.$$

□

In the remainder of this section, we recall some facts about Hilbert space theory that will be useful later. We recall that $C^\infty(X;\Lambda^{0,q}T^*(X))$ carries an inner product

$$(u\mid v)=\int_X(u(z)\mid v(z))(dm),\ u\in C^\infty(X;\Lambda^{0,q}T^*(X)),\ v\in C^\infty(X;\Lambda^{0,q}T^*(X)).$$

(See (2.8).) The completion will be denoted $L^2(X;\Lambda^{0,q}T^*(X))$. Let A be as in Remark 7.4. A has a formal adjoint

$$A^*:\mathscr{D}'(X;\Lambda^{0,q}T^*(X))\to\mathscr{D}'(X;\Lambda^{0,q}T^*(X))$$
$$(A^*u\mid v)=(u\mid Av),\ u\in C^\infty(X;\Lambda^{0,q}T^*(X)),\ v\in C^\infty(X;\Lambda^{0,q}T^*(X)).$$

Lemma 7.6. *A^* is well-defined as a continuous operator*

$$A^* : H^s(X; \Lambda^{0,q} T^*(X)) \to H^{s+1}(X; \Lambda^{0,q} T^*(X))$$

for all $s \in \mathbb{R}$. Moreover, we have

$$A^* \equiv A$$

Proof. The first statement is a consequence of the theorem of Calderon and Vaillancourt. (See Proposition 6.10.) In view of Remark 7.4, we see that $\Box_b^{(q)} \circ A \equiv I$. Thus

$$A^* \circ \Box_b^{(q)} \equiv I.$$

We have

$$\begin{aligned} A^* - A &\equiv A^* \circ (\Box_b^{(q)} \circ A) - A \\ &\equiv (A^* \circ \Box_b^{(q)}) \circ A - A \\ &\equiv A - A \\ &\equiv 0. \end{aligned}$$

The lemma follows. □

From this, we get a two-sided parametrix for $\Box_b^{(q)}$.

Proposition 7.7. *We assume that $Y(q)$ holds. Let A be as in Remark* 7.4. *Then*

$$\Box_b^{(q)} \circ A \equiv A \circ \Box_b^{(q)} \equiv I.$$

Proof. In view of Lemma 7.6, we have $A^* \equiv A$. Thus

$$I \equiv \Box_b^{(q)} \circ A \equiv A^* \circ \Box_b^{(q)} \equiv A \circ \Box_b^{(q)}.$$

□

Remark 7.8. The existence of a two-sided parametrix for $\Box_b^{(q)}$ under condition $Y(q)$ is a classical result. See Beals-Greiner [1].

Definition 7.9. Suppose Q is a closed densely defined operator

$$Q : H \supset \operatorname{Dom} Q \to \operatorname{Ran} Q \subset H,$$

where H is a Hilbert space. Suppose that Q has closed range. By the partial inverse of Q, we mean the bounded operator $N : H \to H$ such that

$$Q \circ N = \pi_2, \; N \circ Q = \pi_1 \text{ on } \operatorname{Dom} Q,$$

where π_1, π_2 are the orthogonal projections in H such that

$$\operatorname{Ran}\pi_1 = (\operatorname{Ker} Q)^{\perp},\ \operatorname{Ran}\pi_2 = \operatorname{Ran} Q.$$

In other words, for $u \in H$, let

$$\pi_2 u = Qv,\ v \in (\operatorname{Ker} Q)^{\perp} \bigcap \operatorname{Dom} Q.$$

Then,

$$Nu = v.$$

Set

$$\operatorname{Dom}\Box_b^{(q)} = \left\{ u \in L^2(X; \Lambda^{0,q} T^*(X));\ \Box_b^{(q)} u \in L^2(X; \Lambda^{0,q} T^*(X)) \right\}.$$

Lemma 7.10. *We consider* $\Box_b^{(q)}$ *as an operator*

$$\Box_b^{(q)} : L^2(X; \Lambda^{0,q} T^*(X)) \supset \operatorname{Dom}\Box_b^{(q)} \to L^2(X; \Lambda^{0,q} T^*(X)).$$

If $Y(q)$ *holds, then* $\Box_b^{(q)}$ *has closed range.*

Proof. Suppose $u_j \in \operatorname{Dom}\Box_b^{(q)}$ and

$$\Box_b^{(q)} u_j = v_j \to v \text{ in } L^2(X; \Lambda^{0,q} T^*(X)).$$

We have to show that there exists $u \in \operatorname{Dom}\Box_b^{(q)}$ such that

$$\Box_b^{(q)} u = v.$$

From Proposition 7.7, we have

$$\Box_b^{(q)} A = I - F_1,\ A\Box_b^{(q)} = I - F_2,$$

where F_j, $j = 1,2$, are smoothing operators. Now,

$$A\Box_b^{(q)} u_j = (I - F_2) u_j \to Av \text{ in } L^2(X; \Lambda^{0,q} T^*(X)).$$

Since F_2 is compact, there exists a subsequence

$$u_{j_k} \to u \text{ in } L^2(X; \Lambda^{0,q} T^*(X)),\ k \to \infty.$$

We have $(I - F_2)u = Av$ and

$$\Box_b^{(q)} u_{j_k} \to \Box_b^{(q)} u \text{ in } H^{-2}(X; \Lambda^{0,q} T^*(X)),\ k \to \infty.$$

Thus $\Box_b^{(q)} u = v$. Now $v \in L^2(X; \Lambda^{0,q} T^*(X))$, so $u \in \operatorname{Dom}\Box_b^{(q)}$. We have proved the lemma. □

It follows that $\operatorname{Ran}\Box_b^{(q)} = (\operatorname{Ker}\Box_b^{(q)})^\perp$. Notice also that $\Box_b^{(q)}$ is self-adjoint. Now, we can prove the following classical proposition of Beals and Greiner. (See [1].)

Proposition 7.11. *Suppose $Y(q)$ holds. Then* $\dim\operatorname{Ker}\Box_b^{(q)} < \infty$. *Let π be the orthogonal projection onto* $\operatorname{Ker}\Box_b^{(q)}$ *and N be the partial inverse. Then π is a smoothing operator and $N = A + F$ where A is as in Proposition* 7.7 *and F is a smoothing operator. Let N^* be the formal adjoint of N,*

$$(N^*u \mid v) = (u \mid Nv),\ u \in C^\infty(X;\Lambda^{0,q}T^*(X)),\ v \in C^\infty(X;\Lambda^{0,q}T^*(X)).$$

Then,

$$N^* = N \text{ on } L^2(X;\Lambda^{0,q}T^*(X)).$$

Proof. From Proposition 7.7, we have

$$A\Box_b^{(q)} = I - F_1,\ \Box_b^{(q)}A = I - F_2,$$

where F_1, F_2 are smoothing operators. Thus $\operatorname{Ker}\Box_b^{(q)} \subset \operatorname{Ker}(I-F_1)$. Since F_1 is compact, $\operatorname{Ker}(I-F_1)$ is finite dimensional and contained in $C^\infty(X;\Lambda^{0,q}T^*(X))$. Thus $\dim\operatorname{Ker}\Box_b^{(q)} < \infty$ and $\operatorname{Ker}\Box_b^{(q)} \subset C^\infty(X;\Lambda^{0,q}T^*(X))$.

Let $\{\phi_1, \phi_2, \dots, \phi_m\}$ be an orthonormal basis for $\operatorname{Ker}\Box_b^{(q)}$. The projection π is given by

$$\pi u = \sum_{j=1}^{m} (u \mid \phi_j)\phi_j.$$

Thus π is a smoothing operator. Notice that $I - \pi$ is the orthogonal projection onto $\operatorname{Ran}\Box_b^{(q)}$ since $\Box_b^{(q)}$ is formally self-adjoint with closed range.

For $u \in C^\infty(X;\Lambda^{0,q}T^*(X))$, we have

$$\begin{aligned}
(N-A)u &= (A\Box_b^{(q)} + F_1)Nu - Au \\
&= A(I-\pi)u + F_1Nu - Au \\
&= -A\pi u + F_1N(\Box_b^{(q)}A + F_2)u \\
&= -A\pi u + F_1(I-\pi)Au + F_1NF_2u.
\end{aligned}$$

Here

$$-A\pi, F_1(1-\pi)A : H^s(X;\Lambda^{0,q}T^*(X)) \to H^{s+m}(X;\Lambda^{0,q}T^*(X))$$

for all $s \in \mathbb{R}$ and $m \geq 0$, so $-A\pi$, $F_1(1-\pi)A$ are smoothing operators. Since

$$\begin{aligned}
F_1NF_2 : \mathscr{E}'(X;\Lambda^{0,q}T^*(X)) &\to C^\infty(X;\Lambda^{0,q}T^*(X)) \\
&\to L^2(X;\Lambda^{0,q}T^*(X)) \to C^\infty(X;\Lambda^{0,q}T^*(X)),
\end{aligned}$$

$F_1 N F_2$ is a smoothing operator. Thus

$$N = A + F, \ F \text{ is a smoothing operator.}$$

Since

$$N\pi = \pi N = 0 = N^* \pi = \pi N^* = 0,$$

we have

$$N^* = (N \Box_b^{(q)} + \pi) N^* = N \Box_b^{(q)} N^* = N.$$

The proposition follows. □

Now, we assume that $Y(q)$ fails but that $Y(q-1)$, $Y(q+1)$ hold. In view of Lemma 7.10, we see that $\Box_b^{(q-1)}$ and $\Box_b^{(q+1)}$ have closed range. We write $\overline{\partial}_b^{(q)}$ to denote the map

$$\overline{\partial}_b : C^\infty(X; \Lambda^{0,q} T^*(X)) \to C^\infty(X; \Lambda^{0,q+1} T^*(X)).$$

Let $\overline{\partial}_b^{(q),*}$ denote the formal adjoint of $\overline{\partial}_b$. We have

$$\overline{\partial}_b^{(q),*} : C^\infty(X; \Lambda^{0,q+1} T^*(X)) \to C^\infty(X; \Lambda^{0,q} T^*(X)).$$

Let $N_b^{(q+1)}$ and $N_b^{(q-1)}$ be the partial inverses of $\Box_b^{(q+1)}$ and $\Box_b^{(q-1)}$ respectively. From Proposition 7.11, we have

$$(N_b^{(q+1)})^* = N_b^{(q+1)}, \ (N_b^{(q-1)})^* = N_b^{(q-1)},$$

where $(N_b^{(q+1)})^*$ and $(N_b^{(q-1)})^*$ are the formal adjoints of $N_b^{(q+1)}$ and $N_b^{(q-1)}$ respectively. Let $\pi_b^{(q+1)}$ and $\pi_b^{(q-1)}$ be the orthogonal projections onto the kernels of $\Box_b^{(q+1)}$ and $\Box_b^{(q-1)}$ respectively. Put

$$N = \overline{\partial}_b^{(q),*} (N_b^{(q+1)})^2 \overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)} (N_b^{(q-1)})^2 \overline{\partial}_b^{(q-1),*} \tag{7.15}$$

and

$$\pi = I - (\overline{\partial}_b^{(q),*} N_b^{(q+1)} \overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)} N_b^{(q-1)} \overline{\partial}_b^{(q-1),*}). \tag{7.16}$$

In view of Proposition 7.11, we see that N is well-defined as a continuous operator

$$N : H^s(X; \Lambda^{0,q} T^*(X)) \to H^s(X; \Lambda^{0,q} T^*(X)) \tag{7.17}$$

and π is well-defined as a continuous operator

$$\pi : H^s(X; \Lambda^{0,q} T^*(X)) \to H^{s-1}(X; \Lambda^{0,q} T^*(X)), \tag{7.18}$$

for all $s \in \mathbb{R}$.

Let π^* and N^* be the formal adjoints of π and N respectively. We have the following

Lemma 7.12. *If we consider π and N as operators*

$$\pi, N : \mathscr{D}'(X;\Lambda^{0,q}T^*(X)) \to \mathscr{D}'(X;\Lambda^{0,q}T^*(X)),$$

then

$$\pi^* = \pi,\ N^* = N, \tag{7.19}$$

$$\Box_b^{(q)}\pi = 0 = \pi\Box_b^{(q)}, \tag{7.20}$$

$$\pi + \Box_b^{(q)}N = I = \pi + N\Box_b^{(q)}, \tag{7.21}$$

$$\pi N = 0 = N\pi, \tag{7.22}$$

$$\pi^2 = \pi. \tag{7.23}$$

Proof. From (7.15) and (7.16), we get (7.19).

For $u \in C^\infty(X;\Lambda^{0,q+1}T^*(X))$, we have

$$\begin{aligned}0 &= (\Box_b^{(q+1)}\pi_b^{(q+1)}u \mid \pi_b^{(q+1)}u)\\ &= (\overline{\partial}_b^{(q+1)}\pi_b^{(q+1)}u \mid \overline{\partial}_b^{(q+1)}\pi_b^{(q+1)}u) + (\overline{\partial}_b^{(q),*}\pi_b^{(q+1)}u \mid \overline{\partial}_b^{(q),*}\pi_b^{(q+1)}u).\end{aligned}$$

Thus,

$$\overline{\partial}_b^{(q+1)}\pi_b^{(q+1)} = 0,\ \overline{\partial}_b^{(q),*}\pi_b^{(q+1)} = 0. \tag{7.24}$$

Hence, by taking the formal adjoints

$$\pi_b^{(q+1)}\overline{\partial}_b^{(q+1),*} = 0,\ \pi_b^{(q+1)}\overline{\partial}_b^{(q)} = 0. \tag{7.25}$$

Similarly,

$$\overline{\partial}_b^{(q-1)}\pi_b^{(q-1)} = 0,\ \pi_b^{(q-1)}\overline{\partial}_b^{(q-1),*} = 0. \tag{7.26}$$

Note that

$$\overline{\partial}_b^{(q),*}\Box_b^{(q+1)} = \Box_b^{(q)}\overline{\partial}_b^{(q),*},\ \overline{\partial}_b^{(q-1)}\Box_b^{(q-1)} = \Box_b^{(q)}\overline{\partial}_b^{(q-1)}. \tag{7.27}$$

Now,

$$\begin{aligned}&\Box_b^{(q)}(\overline{\partial}_b^{(q),*}N_b^{(q+1)}\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}N_b^{(q-1)}\overline{\partial}_b^{(q-1),*})\\ &\quad= \overline{\partial}_b^{(q),*}\Box_b^{(q+1)}N_b^{(q+1)}\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}\Box_b^{(q-1)}N_b^{(q-1)}\overline{\partial}_b^{(q-1),*}\\ &\quad= \overline{\partial}_b^{(q),*}(I-\pi_b^{(q+1)})\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}(I-\pi_b^{(q-1)})\overline{\partial}_b^{(q-1),*}\\ &\quad= \Box_b^{(q)}.\end{aligned}$$

Here we used (7.24), (7.26) and (7.27). Hence,

$$\Box_b^{(q)}\pi = 0.$$

We have

$$\pi\Box_b^{(q)} = (\Box_b^{(q)}\pi)^* = 0,$$

where $(\Box_b^{(q)}\pi)^*$ is the formal adjoint of $\Box_b^{(q)}\pi$. We get (7.20).

Now,

$$\begin{aligned}
\Box_b^{(q)} N &= \Big(\overline{\partial}_b^{(q),*}\Box_b^{(q+1)}(N_b^{(q+1)})^2\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}\Box_b^{(q-1)}(N_b^{(q-1)})^2\overline{\partial}_b^{(q-1),*}\Big) \\
&= \overline{\partial}_b^{(q),*}(I-\pi_b^{(q+1)})N_b^{(q+1)}\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}(I-\pi_b^{(q-1)})N_b^{(q-1)}\overline{\partial}_b^{(q-1),*} \\
&= \overline{\partial}_b^{(q),*}N_b^{(q+1)}\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}N_b^{(q-1)}\overline{\partial}_b^{(q-1),*} \\
&= I - \pi.
\end{aligned} \tag{7.28}$$

Here we used (7.24), (7.26) and (7.27). Thus,

$$\Box_b^{(q)} N + \pi = I.$$

We have

$$\pi + N\Box_b^{(q)} = (\Box_b^{(q)} N + \pi)^* = I,$$

where $(\Box_b^{(q)} N + \pi)^*$ is the formal adjoint of $\Box_b^{(q)} N + \pi$. We get (7.21).

Now,

$$\begin{aligned}
N(I-\pi) &= N(\overline{\partial}_b^{(q),*}N_b^{(q+1)}\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}N_b^{(q-1)}\overline{\partial}_b^{(q-1),*}) \\
&= \overline{\partial}_b^{(q),*}(N_b^{(q+1)})^2\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}N_b^{(q+1)}\overline{\partial}_b^{(q)} \\
&\quad + \overline{\partial}_b^{(q-1)}(N_b^{(q-1)})^2\overline{\partial}_b^{(q-1),*}\overline{\partial}_b^{(q-1)}N_b^{(q-1)}\overline{\partial}_b^{(q-1),*}.
\end{aligned}$$

From (7.24), (7.25) and (7.27), we have

$$\begin{aligned}
\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}N_b^{(q+1)} &= (I-\pi_b^{(q+1)})\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}N_b^{(q+1)} \\
&= N_b^{(q+1)}\Box_b^{(q+1)}\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}N_b^{(q+1)} \\
&= N_b^{(q+1)}\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}\Box_b^{(q+1)}N_b^{(q+1)} \\
&= N_b^{(q+1)}\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}(I-\pi_b^{(q+1)}) \\
&= N_b^{(q+1)}\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}.
\end{aligned}$$

Similarly, we have

$$\overline{\partial}_b^{(q-1),*}\overline{\partial}_b^{(q-1)}N_b^{(q-1)} = N_b^{(q-1)}\overline{\partial}_b^{(q-1),*}\overline{\partial}_b^{(q-1)}.$$

Hence,

$$\begin{aligned}
N(I-\pi) &= \overline{\partial}_b^{(q),*}(N_b^{(q+1)})^2 N_b^{(q+1)}\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}\overline{\partial}_b^{(q)} \\
&\quad + \overline{\partial}_b^{(q-1)}(N_b^{(q-1)})^2 N_b^{(q-1)}\overline{\partial}_b^{(q-1),*}\overline{\partial}_b^{(q-1)}\overline{\partial}_b^{(q-1),*} \\
&= \overline{\partial}_b^{(q),*}(N_b^{(q+1)})^2 N_b^{(q+1)}\Box_b^{(q+1)}\overline{\partial}_b^{(q)} \\
&\quad + \overline{\partial}_b^{(q-1)}(N_b^{(q-1)})^2 N_b^{(q-1)}\Box_b^{(q-1)}\overline{\partial}_b^{(q-1),*} \\
&= \overline{\partial}_b^{(q),*}(N_b^{(q+1)})^2(I-\pi_b^{(q+1)})\overline{\partial}_b^{(q)} \\
&\quad + \overline{\partial}_b^{(q-1)}(N_b^{(q-1)})^2(I-\pi_b^{(q-1)})\overline{\partial}_b^{(q-1),*} \\
&= \overline{\partial}_b^{(q),*}(N_b^{(q+1)})^2\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}(N_b^{(q-1)})^2\overline{\partial}_b^{(q-1),*} \\
&= N.
\end{aligned}$$

Here we used (7.25) and (7.26). Thus,

$$N\pi = 0.$$

We have

$$\pi N = (N\pi)^* = 0,$$

where $(N\pi)^*$ is the formal adjoint of $N\pi$. We get (7.22).

Finally,

$$\pi = (\Box_b^{(q)}N + \pi)\pi = \pi^2.$$

We get (7.23).

The lemma follows. □

Lemma 7.13. *If we restrict π to $L^2(X;\Lambda^{0,q}T^*(X))$, then π is the orthogonal projection onto* $\mathrm{Ker}\,\Box_b^{(q)}$. *Thus, π is well-defined as a continuous operator*

$$\pi : L^2(X;\Lambda^{0,q}T^*(X)) \to L^2(X;\Lambda^{0,q}T^*(X)).$$

Proof. From (7.20), we get $\mathrm{Ran}(\pi) \subset \mathrm{Ker}\,\Box_b^{(q)}$ in the space of distributions. From (7.21), we get $\pi u = u$, when $u \in \mathrm{Ker}\,\Box_b^{(q)}$, so $\mathrm{Ran}(\pi) = \mathrm{Ker}\,\Box_b^{(q)}$ and

$$\pi^2 = \pi = \pi^*\pi = \pi^*.$$

For $\varphi, \phi \in C^\infty(X;\Lambda^{0,q}T^*(X))$, we get

$$((1-\pi)\varphi \mid \pi\phi) = 0$$

so $\mathrm{Ran}(I-\pi) \perp \mathrm{Ran}(\pi)$ and $\varphi = (I-\pi)\varphi + \pi\varphi$ is the orthogonal decomposition. It follows that π restricted to $L^2(X;\Lambda^{0,q}T^*(X))$ is the orthogonal projection onto $\mathrm{Ker}\,\Box_b^{(q)}$. □

Lemma 7.14. *If we consider* $\Box_b^{(q)}$ *as an unbounded operator*

$$\Box_b^{(q)} : L^2(X; \Lambda^{0,q} T^*(X)) \supset \mathrm{Dom}\,\Box_b^{(q)} \to L^2(X; \Lambda^{0,q} T^*(X)),$$

then $\Box_b^{(q)}$ *has closed range and*

$$N : L^2(X; \Lambda^{0,q} T^*(X)) \to \mathrm{Dom}\,\Box_b^{(q)} = \Big\{ u \in L^2(X; \Lambda^{0,q} T^*(X))\,; \\ \Box_b^{(q)} u \in L^2(X; \Lambda^{0,q} T^*(X)) \Big\}$$

is the partial inverse.

Proof. From (7.21) and Lemma 7.13, we see that

$$N : L^2(X; \Lambda^{0,q} T^*(X)) \to \mathrm{Dom}\,\Box_b^{(q)}$$

and $\mathrm{Ran}\,\Box_b^{(q)} \supset \mathrm{Ran}\,(I - \pi)$. If

$$\Box_b^{(q)} u = v, \quad u, v \in L^2(X; \Lambda^{0,q} T^*(X)),$$

then $(I-\pi)v = (I-\pi)\Box_b^{(q)} u = v$ since $\pi\Box_b^{(q)} = \Box_b^{(q)}\pi = 0$. Hence

$$\mathrm{Ran}\,\Box_b^{(q)} \subset \mathrm{Ran}\,(I - \pi)$$

so $\Box_b^{(q)}$ has closed range.

From (7.22), we know that $N\pi = \pi N = 0$. Thus, N is the partial inverse. □

From Lemma 7.13 and Lemma 7.14 we get the following classical result. (See also Beals-Greiner [1].)

Proposition 7.15. *We assume that* $Y(q)$ *fails but that* $Y(q-1)$ *and* $Y(q+1)$ *hold. Then* $\Box_b^{(q)}$ *has closed range. Let* N *and* π *be as in (7.15) and (7.16) respectively. Then* N *is the partial inverse of* $\Box_b^{(q)}$ *and* π *is the orthogonal projection onto* $\mathrm{Ker}\,\Box_b^{(q)}$.

8 The Szegö Projection

In this section, we assume that $Y(q)$ fails. From Proposition 7.5, we know that, for every local coordinate patch X_j, there exist

$$G_j \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(X_j; \Lambda^{0,q} T^*(X_j), \Lambda^{0,q} T^*(X_j))$$

and

$$S_j \in L^{0}_{\frac{1}{2},\frac{1}{2}}(X_j; \Lambda^{0,q} T^*(X_j), \Lambda^{0,q} T^*(X_j))$$

such that

$$\begin{cases} S_j + \Box_b^{(q)} G_j \equiv I \\ \Box_b^{(q)} S_j \equiv 0 \end{cases} \tag{8.1}$$

in the space $\mathscr{D}'(X_j \times X_j; \mathscr{L}(\Lambda^{0,q}T^*(X_j), \Lambda^{0,q}T^*(X_j)))$. Furthermore, the distribution kernel K_{S_j} of S_j is of the form

$$K_{S_j}(x,y) = \frac{1}{(2\pi)^{2n-1}} \int e^{i(\psi(\infty,x,\eta) - \langle y,\eta \rangle)} a(\infty,x,\eta) d\eta, \tag{8.2}$$

where $\psi(\infty,x,\eta) \in C^\infty(T^*(X_j))$ and

$$a(\infty,x,\eta) \in C^\infty(T^*(X_j); \mathscr{L}(\Lambda^{0,q}T^*(X_j), \Lambda^{0,q}T^*(X_j)))$$

are as in Proposition 4.3 and (7.4). From now on, we assume that S_j and G_j are properly supported operators.

We assume that $X = \bigcup_{j=1}^k X_j$. Let χ_j be a C^∞ partition of unity subordinate to $\{X_j\}$. From (8.1), we have

$$\begin{cases} S_j \chi_j + \Box_b^{(q)} G_j \chi_j \equiv \chi_j \\ \Box_b^{(q)} S_j \chi_j \equiv 0 \end{cases} \tag{8.3}$$

in the space $\mathscr{D}'(X_j \times X_j; \mathscr{L}(\Lambda^{0,q}T^*(X_j), \Lambda^{0,q}T^*(X_j)))$. Thus,

$$\begin{cases} S + \Box_b^{(q)} G \equiv I \\ \Box_b^{(q)} S \equiv 0 \end{cases} \tag{8.4}$$

in the space $\mathscr{D}'(X \times X; \mathscr{L}(\Lambda^{0,q}T^*(X), \Lambda^{0,q}T^*(X)))$, where

$$S,\ G : \mathscr{D}'(X; \Lambda^{0,q}T^*(X)) \to \mathscr{D}'(X; \Lambda^{0,q}T^*(X)),$$

$$\begin{cases} Su = \sum_{j=1}^k S_j(\chi_j u),\ u \in \mathscr{D}'(X; \Lambda^{0,q}T^*(X)) \\ Gu = \sum_{j=1}^k G_j(\chi_j u),\ u \in \mathscr{D}'(X; \Lambda^{0,q}T^*(X)) \end{cases}. \tag{8.5}$$

Let

$$S^*, G^* : \mathscr{D}'(X; \Lambda^{0,q}T^*(X)) \to \mathscr{D}'(X; \Lambda^{0,q}T^*(X))$$

be the formal adjoints of S and G respectively. As in Lemma 7.6, we see that S^* and G^* are well-defined as continuous operators

$$\begin{cases} S^* : H^s(X; \Lambda^{0,q}T^*(X)) \to H^s(X; \Lambda^{0,q}T^*(X)) \\ G^* : H^s(X; \Lambda^{0,q}T^*(X)) \to H^{s+1}(X; \Lambda^{0,q}T^*(X)) \end{cases}, \tag{8.6}$$

for all $s \in \mathbb{R}$. We have the following

Lemma 8.1. *Let S be as in (8.4), (8.5). We have*

$$S \equiv S^* S.$$

It follows that

$$S \equiv S^*$$

and

$$S^2 \equiv S.$$

Proof. From (8.4), it follows that

$$S^* + G^* \Box_b^{(q)} \equiv I.$$

We have

$$\begin{aligned} S &\equiv (S^* + G^* \Box_b^{(q)}) \circ S \\ &\equiv S^* S + G^* \Box_b^{(q)} S \\ &\equiv S^* S. \end{aligned}$$

The lemma follows. □

Let

$$H = (I - S) \circ G. \tag{8.7}$$

H is well-defined as a continuous operator

$$H : H^s(X; \Lambda^{0,q} T^*(X)) \to H^{s+1}(X; \Lambda^{0,q} T^*(X))$$

for all $s \in \mathbb{R}$. The formal adjoint H^* is well-defined as a continuous operator

$$H^* : H^s(X; \Lambda^{0,q} T^*(X)) \to H^{s+1}(X; \Lambda^{0,q} T^*(X)),$$

for all $s \in \mathbb{R}$.

Lemma 8.2. *Let S and H be as in (8.4), (8.5) and (8.7). Then*

$$SH \equiv 0, \tag{8.8}$$

$$S + \Box_b^{(q)} H \equiv I. \tag{8.9}$$

Proof. We have

$$SH \equiv S(I - S)G \equiv (S - S^2)G \equiv 0$$

since $S^2 \equiv S$, where G is as in (8.4). From (8.4), it follows that

$$\begin{aligned} S + \Box_b^{(q)} H &= S + \Box_b^{(q)} (I - S)G \\ &\equiv I - \Box_b^{(q)} SG \\ &\equiv I. \end{aligned}$$

The lemma follows. □

Lemma 8.3. *Let H be as in (8.7). Then*

$$H \equiv H^*.$$

Proof. Taking the adjoint in (8.9), we get

$$S^* + H^* \Box_b^{(q)} \equiv I.$$

Hence

$$H \equiv (S^* + H^* \Box_b^{(q)})H \equiv S^* H + H^* \Box_b^{(q)} H.$$

From Lemma 8.1 and Lemma 8.2, we have

$$S^* H \equiv SH \equiv 0.$$

Hence

$$H \equiv H^* \Box_b^{(q)} H \equiv H^*.$$

□

Summing up, we get the following

Proposition 8.4. *We assume that $Y(q)$ fails. Let*

$$\begin{cases} S : \mathscr{D}'(X; \Lambda^{0,q} T^*(X)) \to \mathscr{D}'(X; \Lambda^{0,q} T^*(X)) \\ H : \mathscr{D}'(X; \Lambda^{0,q} T^*(X)) \to \mathscr{D}'(X; \Lambda^{0,q} T^*(X)) \end{cases}$$

be as in (8.5) and (8.7). Then, S and H are well-defined as continuous operators

$$S : H^s(X; \Lambda^{0,q} T^*(X)) \to H^s(X; \Lambda^{0,q} T^*(X)), \tag{8.10}$$
$$H : H^s(X; \Lambda^{0,q} T^*(X)) \to H^{s+1}(X; \Lambda^{0,q} T^*(X)), \tag{8.11}$$

for all $s \in \mathbb{R}$. Moreover, we have

$$H \Box_b^{(q)} + S \equiv S + \Box_b^{(q)} H \equiv I, \tag{8.12}$$
$$\Box_b^{(q)} S \equiv S \Box_b^{(q)} \equiv 0, \tag{8.13}$$
$$S \equiv S^* \equiv S^2, \tag{8.14}$$
$$SH \equiv HS \equiv 0, \tag{8.15}$$
$$H \equiv H^*. \tag{8.16}$$

Remark 8.5. If

$$S', H' : \mathscr{D}'(X; \Lambda^{0,q} T^*(X)) \to \mathscr{D}'(X; \Lambda^{0,q} T^*(X))$$

satisfy (8.10)-(8.16), then

$$S' \equiv (H\Box_b^{(q)} + S)S' \equiv SS' \equiv S(\Box_b^{(q)}H' + S') \equiv S$$

and

$$H' \equiv (H\Box_b^{(q)} + S)H' \equiv (H\Box_b^{(q)} + S')H' \equiv H\Box_b^{(q)}H' \equiv H(\Box_b^{(q)}H' + S') \equiv H.$$

Thus,(8.10)-(8.16) determine S and H uniquely up to smoothing operators.

Remark 8.6. Proposition 8.4 is motivated by the work of Boutet de Monvel and Sjöstrand [9]. See also Beals-Greiner [1].

Now we can prove the following

Proposition 8.7. *We assume that $Y(q)$ fails. Suppose $\Box_b^{(q)}$ has closed range. Let N be the partial inverse of $\Box_b^{(q)}$ and let π be the orthogonal projection onto* $\mathrm{Ker}\,\Box_b^{(q)}$. *Then*

$$N = H + F,$$

$$\pi = S + K,$$

where H, S are as in Proposition 8.4*, F, K are smoothing operators.*

Proof. We may replace S by $I - \Box_b^{(q)}H$ and we have

$$\Box_b^{(q)}H + S = I = H^*\Box_b^{(q)} + S^*.$$

Now,

$$\pi = \pi(\Box_b^{(q)}H + S) = \pi S, \tag{8.17}$$

hence

$$\pi^* = S^*\pi^* = \pi = S^*\pi. \tag{8.18}$$

Similarly,

$$S = (N\Box_b^{(q)} + \pi)S = \pi S + NF_1, \tag{8.19}$$

where F_1 is a smoothing operator. From (8.17) and (8.19), we have

$$S - \pi = S - \pi S = NF_1. \tag{8.20}$$

Hence

$$(S^* - \pi)(S - \pi) = F_1^* N^2 F_1.$$

On the other hand,

$$\begin{aligned}(S^* - \pi)(S - \pi) &= S^*S - S^*\pi - \pi S + \pi^2 \\ &= S^*S - \pi \\ &= S - \pi + F_2,\end{aligned}$$

where F_2 is a smoothing operator. Here we used (8.17) and (8.18). Now,

$$\begin{aligned}F_1^*N^2F_1 : \mathscr{D}'(X;\Lambda^{0,q}T^*(X)) &\to C^\infty(X;\Lambda^{0,q}T^*(X))\\ &\to L^2(X;\Lambda^{0,q}T^*(X)) \to C^\infty(X;\Lambda^{0,q}T^*(X)).\end{aligned}$$

Hence $F_1^*N^2F_1$ is smoothing. Thus $S-\pi$ is smoothing.

We have,

$$\begin{aligned}N-H &= N(\Box_b^{(q)}H+S)-H\\ &= (I-\pi)H+NS-H\\ &= NS-\pi H\\ &= N(S-\pi)+F_3\\ &= NF_4+F_3\end{aligned}$$

where F_4 and F_3 are smoothing operators. Now,

$$\begin{aligned}N-H^* &= N^*-H^*\\ &= F_4^*N+F_3^*\\ &= F_4^*(NF_4+F_3+H)+F_3^*.\end{aligned}$$

Note that

$$\begin{aligned}F_4^*NF_4 : \mathscr{D}'(X;\Lambda^{0,q}T^*(X)) &\to C^\infty(X;\Lambda^{0,q}T^*(X))\\ &\to L^2(X;\Lambda^{0,q}T^*(X)) \to C^\infty(X;\Lambda^{0,q}T^*(X)).\end{aligned}$$

and

$$F_4^*H : H^s(X;\Lambda^{0,q}T^*(X)) \to H^{s+m}(X;\Lambda^{0,q}T^*(X))$$

for all $s\in\mathbb{R}$ and $m\geq 0$. Hence $N-H^*$ is smoothing and so is $(N-H^*)^* = N-H$. □

From Proposition 8.4 and Proposition 8.7, we obtain the following

Theorem 8.8. *We assume that $Y(q)$ fails. Let (n_-,n_+), $n_-+n_+=n-1$, be the signature of the Levi form L. Suppose $\Box_b^{(q)}$ has closed range and recall this is the case when $Y(q-1)$ and $Y(q+1)$ hold. Let π be the Szegö projection, that is, π is the orthogonal projection onto* $\operatorname{Ker}\Box_b^{(q)}$. *Then for every local coordinate patch $U\subset X$, the distribution kernel of π on $U\times U$ is of the form*

$$K_\pi(x,y) \equiv \frac{1}{(2\pi)^{2n-1}}\int e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)d\eta \mod C^\infty, \tag{8.21}$$

$$a(\infty,x,\eta)\in S^0_{1,0}(T^*(U)\,;\mathscr{L}(\Lambda^{0,q}T^*(U),\Lambda^{0,q}T^*(U))),$$

$$a(\infty,x,\eta)\sim\sum_0^\infty a_j(\infty,x,\eta)$$
$$\text{in the symbol space } S^0_{1,0}(T^*(U);\mathscr{L}(\Lambda^{0,q}T^*(U),\Lambda^{0,q}T^*(U))),$$

where

$$a_j(\infty,x,\eta)\in C^\infty(T^*(U);\mathscr{L}(\Lambda^{0,q}T^*(U),\Lambda^{0,q}T^*(U))),\ \ j=0,1,\ldots,$$

$$a_j(\infty,x,\lambda\eta)=\lambda^{-j}a_j(\infty,x,\eta),\lambda\geq 1,\ \left|\eta\right|\geq 1,\ \ j=0,1,\ldots,$$

and $S^m_{1,0}$, $m\in\mathbb{R}$, *is the Hörmander symbol space. Here* $\psi(\infty,x,\eta)$ *is as in Proposition 4.3 and (4.9). We recall that*

$$\psi(\infty,x,\eta)\in C^\infty(T^*(U)),$$

$$\begin{aligned}\psi(\infty,x,\lambda\eta)&=\lambda\psi(\infty,x,\eta),\lambda>0,\\ \operatorname{Im}\psi(\infty,x,\eta)&\asymp\left|\eta\right|(\operatorname{dist}((x,\frac{\eta}{\left|\eta\right|}),\Sigma))^2,\\ \psi(\infty,x,\eta)&=-\overline{\psi}(\infty,x,-\eta).\end{aligned}\tag{8.22}$$

Moreover, for all $j=0,1,\ldots$,

$$\begin{cases}a_j(\infty,x,\eta)=0 \ \textit{ in a conic neighborhood of } \Sigma^+, & \text{if } q=n_-,\ n_-\neq n_+,\\ a_j(\infty,x,\eta)=0 \ \textit{ in a conic neighborhood of } \Sigma^-, & \text{if } q=n_+,\ n_-\neq n_+.\end{cases}\tag{8.23}$$

In the rest of this section, we will study the singularities of the distribution kernel of the Szegö projection. We need

Definition 8.9. Let M be a real paracompact C^∞ manifold and let Λ be a C^∞ closed submanifold of M. Let U be an open set in M. We let $C^\infty_\Lambda(U)$ denote the set of equivalence classes of $f\in C^\infty(U)$ under the equivalence relation

$$f\equiv g \ \text{ in the space } C^\infty_\Lambda(U)$$

if for every $z_0\in\Lambda\bigcap U$, there exists a neighborhood $W\subset U$ of z_0 such that

$$f=g+h \ \text{ on } W,$$

where $h\in C^\infty(W)$ and h vanishes to infinite order on $\Lambda\bigcap W$.

In view of Proposition 4.3, we see that $\psi(\infty, x, \eta)$ has a uniquely determined Taylor expansion at each point of Σ. Thus, we can define $\psi(\infty, x, \eta)$ as an element in $C^\infty_\Sigma(T^*(X))$. We also write $\psi(\infty, x, \eta)$ for the equivalence class of $\psi(\infty, x, \eta)$ in the space $C^\infty_\Sigma(T^*(X))$.

Let M be a real paracompact C^∞ manifold and let Λ be a C^∞ closed submanifold of M. If $x_0 \in \Lambda$, we let $A(\Lambda, n, x_0)$ be the set

$$\begin{aligned} A(\Lambda, n, x_0) = \{&(U, f_1, \ldots, f_n);\ U \text{ is an open neighborhood of } x_0, f_1, \ldots, f_n \in C^\infty_\Lambda(U), \\ & f_j|_{\Lambda \bigcap U} = 0,\ j = 1, \ldots, n, \text{ and } df_1, \ldots, df_n \text{ are linearly independent} \\ & \text{over } \mathbb{C} \text{ at each point of } U\}. \end{aligned} \tag{8.24}$$

Definition 8.10. If $x_0 \in \Lambda$, we let $A_{x_0}(\Lambda, n, x_0)$ denote the set of equivalence classes of $A(\Lambda, n, x_0)$ under the equivalence relation

$$\Gamma_1 = (U, f_1, \ldots, f_n) \sim \Gamma_2 = (V, g_1, \ldots, g_n),\ \ \Gamma_1, \Gamma_2 \in A(\Lambda, n, x_0),$$

if there exists an open set $W \subset U \bigcap V$ of x_0 such that

$$g_j \equiv \sum_{k=1}^n a_{j,k} f_k \text{ in the space } C^\infty_\Lambda(W),\ j = 1, \ldots, n,$$

where $a_{j,k} \in C^\infty_\Lambda(W)$, j, $k = 1, \ldots, n$, and $\left(a_{j,k}\right)_{j,k=1}^n$ is invertible.

If $(U, f_1, \ldots, f_n) \in A(\Lambda, n, x_0)$, we write $(U, f_1, \ldots, f_n)_{x_0}$ for the equivalence class of $(U, f_1, \ldots, f_n)$ in $A_{x_0}(\Lambda, n, x_0)$, which is called the germ of $(U, f_1, \ldots, f_n)$ at x_0.

Definition 8.11. Let M be a real paracompact C^∞ manifold and let Λ be a C^∞ closed submanifold of M. A formal manifold Ω of codimensin k at Λ associated to M is given by:

> For each point of $x \in \Lambda$, we assign a germ $\Gamma_x \in A_x(\Lambda, k, x)$ in such a way that for every point $x_0 \in \Lambda$ has an open neighborhood U such that there exist $f_1, \ldots, f_k \in C^\infty_\Lambda(U)$, $f_j|_{\Lambda \bigcap U} = 0$, $j = 1, \ldots, k$, $df_1, \ldots, df_k$ are linearly independent over $\mathbb{C}$ at each point of U, having the following property: whatever $x \in U$, the germ $(U, f_1, \ldots, f_k)_x$ is equal to Γ_x.

Formally, we write

$$\Omega = \{\Gamma_x;\ x \in \Lambda\}.$$

If the codimension of Ω is 1, we call Ω a formal hypersurface at Λ.

Let $\Omega = \{\Gamma_x;\ x \in \Lambda\}$ and $\Omega_1 = \left\{\widetilde{\Gamma}_x;\ x \in \Lambda\right\}$ be two formal manifolds at Λ. If $\Gamma_x = \widetilde{\Gamma}_x$, for all $x \in \Lambda$, we write

$$\Omega = \Omega_1 \text{ at } \Lambda.$$

Definition 8.12. Let $\Omega = \{\Gamma_x;\, x \in \Lambda\}$ be a formal manifold of codimensin k at Λ associated to M, where Λ and M are as above. The tangent space of Ω at $x_0 \in \Lambda$ is given by:

$$\text{the tangent space of } \Omega \text{ at } x_0 = \left\{u \in \mathbb{C}T_{x_0}(M);\, \left\langle d f_j(x_0), u\right\rangle = 0,\ j = 1, \ldots, k\right\},$$

where $\mathbb{C}T_{x_0}(M)$ is the complexified tangent space of M at x_0 and $(U, f_1, \ldots, f_k)$ is a representative of Γ_{x_0}. We write $T_{x_0}(\Omega)$ to denote the tangent space of Ω at x_0.

Let (x, y) be some coordinates of $X \times X$. From now on, we use the notations ξ and η for the dual variables of x and y respectively.

Remark 8.13. For each point $(x_0, \eta_0, x_0, \eta_0) \in \mathrm{diag}(\Sigma \times \Sigma)$, we assign a germ

$$\Gamma_{(x_0,\eta_0,x_0,\eta_0)} = (T^*(X) \times T^*(X), \xi - \psi'_x(\infty, x, \eta), y - \psi'_\eta(\infty, x, \eta))_{(x_0,\eta_0,x_0,\eta_0)}. \tag{8.25}$$

Let C_∞ be the formal manifold at $\mathrm{diag}(\Sigma \times \Sigma)$:

$$C_\infty = \left\{\Gamma_{(x,\eta,x,\eta)};\, (x, \eta, x, \eta) \in \mathrm{diag}(\Sigma \times \Sigma)\right\}.$$

C_∞ is strictly positive in the sense that

$$\frac{1}{i}\sigma(v, \overline{v}) > 0$$

for all $v \in T_\rho(C_\infty) \setminus \mathbb{C}T_\rho(\mathrm{diag}(\Sigma \times \Sigma))$, where $\rho \in \mathrm{diag}(\Sigma \times \Sigma)$. Here σ is the symplectic two form on $\mathbb{C}T^*_\rho(X) \times \mathbb{C}T^*_\rho(X)$.

We have the following

Proposition 8.14. *There exists a formal manifold* $J_+ = \left\{J_{(x,\eta)};\, (x, \eta) \in \Sigma\right\}$ *at* Σ *associated to* $T^*(X)$ *such that*

$$\mathrm{codim}\, J_+ = n - 1 \tag{8.26}$$

and for all $(x_0, \eta_0) \in \Sigma$, *if* $(U, f_1, \ldots, f_{n-1})$ *is a representative of* $J_{(x_0,\eta_0)}$, *then*

$$\left\{f_j, f_k\right\} \equiv 0 \ \text{ in the space } C^\infty_\Sigma(U),\ j, k = 1, \ldots, n-1, \tag{8.27}$$

$$p_0 \equiv \sum_{j=1}^{n-1} g_j f_j \ \text{ in the space } C^\infty_\Sigma(U), \tag{8.28}$$

where $g_j \in C^\infty_\Sigma(U)$, $j = 1, \ldots, n-1$, *and*

$$\frac{1}{i}\sigma(H_{f_j}, H_{\overline{f}_j}) > 0 \ \text{ at } (x_0, \eta_0) \in \Sigma,\ j = 1, \ldots, n-1. \tag{8.29}$$

We also write f_j *to denote an almost analytic extension of* f_j. *Then,*

$$f_j(x, \psi'_x(\infty, x, \eta)) \text{ vanishes to infinite order on } \Sigma,\ j = 1, \ldots, n-1. \tag{8.30}$$

Moreover, we have

$$T_\rho(C_\infty) = \{(v + \sum_{j=1}^{n-1} t_j H_{f_j}(x_0,\eta_0), v + \sum_{j=1}^{n-1} s_j H_{\overline{f}_j}(x_0,\eta_0));$$
$$v \in T_{(x_0,\eta_0)}(\Sigma),\ t_j, s_j \in \mathbb{C}, j = 1,\dots,n-1\}, \tag{8.31}$$

where $\rho = (x_0,\eta_0,x_0,\eta_0) \in \mathrm{diag}(\Sigma\times\Sigma)$ *and* C_∞ *is as in Remark 8.13.*

Proof. See [20]. □

We return to our problem. We need the following

Lemma 8.15. *We have*

$$\psi''_{\eta\eta}(\infty, p, \omega_0(p))\omega_0(p) = 0 \tag{8.32}$$

and

$$\mathrm{Rank}\left(\psi''_{\eta\eta}(\infty, p, \omega_0(p))\right) = 2n-2, \tag{8.33}$$

for all $p \in X$.

Proof. Since $\psi'_\eta(\infty, x, \eta)$ is positively homogeneous of degree 0, it follows that

$$\psi''_{\eta\eta}(\infty, p, \omega_0(p))\omega_0(p) = 0.$$

Thus,

$$\mathrm{Rank}\left(\psi''_{\eta\eta}(\infty, p, \omega_0(p))\right) \leq 2n-2.$$

From

$$\mathrm{Im}\,\psi(\infty, x, \eta) \asymp \left|\eta\right| \mathrm{dist}((x, \frac{\eta}{\left|\eta\right|});\ \Sigma)^2$$

it follows that

$$\mathrm{Im}\,\psi''_{\eta\eta}(\infty, p, \omega_0(p))V \neq 0, \ \text{ if } V \notin \{\lambda\omega_0(p);\, \lambda \in \mathbb{C}\}.$$

Thus, for all $V \notin \{\lambda\omega_0(p);\, \lambda\in\mathbb{C}\}$, we have

$$\left\langle \psi''_{\eta\eta}(\infty, p, \omega_0(p))V, \overline{V}\right\rangle = \left\langle \mathrm{Re}\,\psi''_{\eta\eta}(\infty, p, \omega_0(p))V, \overline{V}\right\rangle$$
$$+\, i\left\langle \mathrm{Im}\,\psi''_{\eta\eta}(\infty, p, \omega_0(p))V, \overline{V}\right\rangle \neq 0.$$

We get (8.33). □

Until further notice, we assume that $q = n_+$. For $p \in X$, we take local coordinates

$$x = (x_1, x_2, \ldots, x_{2n-1})$$

defined on some neighborhood Ω of $p \in X$ such that

$$\omega_0(p) = dx_{2n-1},\ x(p) = 0 \tag{8.34}$$

and

$$\Lambda^{0,1}T_p(X) \oplus \Lambda^{1,0}T_p(X) = \left\{ \sum_{j=1}^{2n-2} a_j \frac{\partial}{\partial x_j};\ a_j \in \mathbb{C},\ \ j = 1, \ldots, 2n-2 \right\}.$$

We take Ω so that if $x_0 \in \Omega$ then $\eta_{0,2n-1} > 0$ where $\omega_0(x_0) = (\eta_{0,1}, \ldots, \eta_{0,2n-1})$.

Until further notice, we work in Ω and we work with the local coordinates x. Choose $\chi(x,\eta) \in C^\infty(T^*(X))$ so that $\chi(x,\eta) = 1$ in a conic neighborhood of $(p, \omega_0(p))$, $\chi(x,\eta) = 0$ outside $T^*(\Omega)$, $\chi(x,\eta) = 0$ in a conic neighborhood of Σ^- and $\chi(x, \lambda\eta) = \chi(x,\eta)$ when $\lambda > 0$. We introduce the cut-off functions $\chi(x,\eta)$ and $(1-\chi(x,\eta))$ in the integral (8.21):

$$\begin{aligned}
&K_\pi(x,y) \equiv K_{\pi^+}(x,y) + K_{\pi^-}(x,y),\\
&K_{\pi^+}(x,y) \equiv \frac{1}{(2\pi)^{2n-1}} \int e^{i(\psi(\infty,x,\eta) - \langle y,\eta\rangle)} \chi(x,\eta) a(\infty,x,\eta) d\eta,\\
&K_{\pi^-}(x,y) \equiv \frac{1}{(2\pi)^{2n-1}} \int e^{i(\psi(\infty,x,\eta) - \langle y,\eta\rangle)} (1-\chi(x,\eta)) a(\infty,x,\eta) d\eta.
\end{aligned} \tag{8.35}$$

Now, we study K_{π^+}. We write t to denote η_{2n-1}. Put $\eta' = (\eta_1, \ldots, \eta_{2n-2})$. We have

$$\begin{aligned}
&K_{\pi^+}(x,y) \equiv\\
&\quad \frac{1}{(2\pi)^{2n-1}} \int e^{i(\psi(\infty,x,(\eta',t)) - \langle y,(\eta',t)\rangle)} \chi(x,(\eta',t)) a(\infty,x,(\eta',t)) d\eta' dt\\
&\quad = \frac{1}{(2\pi)^{2n-1}} \int_0^\infty \Big(\int e^{it(\psi(\infty,x,(w,1)) - \langle y,(w,1)\rangle)} t^{2n-2} \chi(x,(tw,t)) a(\infty,x,(tw,t)) dw \Big) dt
\end{aligned} \tag{8.36}$$

where $\eta' = tw$, $w \in \mathbb{R}^{2n-2}$. The stationary phase method of Melin and Sjöstrand (see Proposition B.15) then permits us to carry out the w integration in (8.36), to get

$$K_{\pi^+}(x,y) \equiv \int_0^\infty e^{it\phi_+(x,y)} s_+(x,y,t) dt \tag{8.37}$$

with

$$s_+(x,y,t) \sim \sum_{j=0}^{\infty} s_+^j(x,y)t^{n-1-j}$$

in the symbol space $S^{n-1}_{1,0}(\Omega\times\Omega\times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X)))$, (8.38)

where $s_+^j(x,y)\in C^\infty(\Omega\times\Omega;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X)))$, $j=0,1,\ldots$, and $\phi_+(x,y)\in C^\infty(\Omega\times\Omega)$ is the corresponding critical value. (See Proposition B.14 for a review.) For $x\in\Omega$, let $\sigma(x)\in\mathbb{R}^{2n-2}$ be the vector:

$$(x,(\sigma(x),1))\in\Sigma^+. \tag{8.39}$$

Since

$$d_w(\psi(\infty,x,(w,1))-\langle y,(w,1)\rangle)=0 \ \text{ at } x=y,\ w=\sigma(x),$$

it follows that when $x=y$, the corresponding critical point is $w=\sigma(x)$ and consequently

$$\phi_+(x,x)=0, \tag{8.40}$$

$$(\phi_+)'_x(x,x)=\psi'_x(\infty,x,(\sigma(x),1))=(\sigma(x),1),\ (\phi_+)'_y(x,x)=-(\sigma(x),1). \tag{8.41}$$

The following is well-known (see Proposition B.14)

Proposition 8.16. *In some open neighborhood Q of p in Ω, we have*

$$\operatorname{Im}\phi_+(x,y)\geq c\inf_{w\in W}\Big(\operatorname{Im}\psi(\infty,x,(w,1))+\Big|d_w(\psi(\infty,x,(w,1))-\langle y,(w,1)\rangle)\Big|^2\Big),$$
$$(x,y)\in Q\times Q, \tag{8.42}$$

where c is a positive constant and W is some open set of the origin in $\mathbb{R}^{2n-2}$.

We have the following

Proposition 8.17. *In some open neighborhood Q of p in Ω, there is a constant $c>0$ such that*

$$\operatorname{Im}\phi_+(x,y)\geq c\left|x'-y'\right|^2,\ (x,y)\in Q\times Q, \tag{8.43}$$

where $x'=(x_1,\ldots,x_{2n-2})$, $y'=(y_1,\ldots,y_{2n-2})$ and

$$\left|x'-y'\right|^2=(x_1-y_1)^2+\cdots+(x_{2n-2}-y_{2n-2})^2.$$

Proof. From

$$\psi(\infty,x,(w,1))-\langle y,(w,1)\rangle=\langle x-y,(w,1)\rangle+O(|w-\sigma(x)|^2)$$

we can check that

$$d_w(\psi(\infty,x,(w,1))-\langle y,(w,1)\rangle)=\langle x'-y',dw\rangle+O(|w-\sigma(x)|),$$

where $\sigma(x)$ is as in (8.39) and $x'=(x_1,\dots,x_{2n-2})$, $y'=(y_1,\dots,y_{2n-2})$. Thus, there are constants c_1, $c_2>0$ such that

$$\left|d_w(\psi(\infty,x,(w,1))-\langle y,(w,1)\rangle)\right|^2\geq c_1\left|x'-y'\right|^2-c_2\left|w-\sigma(x)\right|^2$$

for (x,w) in some compact set of $\Omega\times\dot{\mathbb{R}}^{2n-2}$. If $\frac{c_1}{2}\left|(x'-y')\right|^2\geq c_2\left|w-\sigma(x)\right|^2$, then

$$\left|d_\omega(\psi(\infty,x,\omega)-\langle y,\omega\rangle)\right|^2\geq\frac{c_1}{2}\left|(x'-y')\right|^2. \tag{8.44}$$

Now, we assume that $\left|(x'-y')\right|^2\leq\frac{2c_2}{c_1}\left|w-\sigma(x)\right|^2$. We have

$$\operatorname{Im}\psi(\infty,x,(w,1))\geq c_3\left|w-\sigma(x)\right|^2\geq\frac{c_1c_3}{2c_2}\left|(x'-y')\right|^2, \tag{8.45}$$

for (x,w) in some compact set of $\Omega\times\dot{\mathbb{R}}^{2n-2}$, where c_3 is a positive constant. From (8.44), (8.45) and Proposition 8.16, we have

$$\operatorname{Im}\phi_+(x,y)\geq c\left|(x'-y')\right|^2$$

for x, y in some neighborhood of p, where c is a positive constant. We get the proposition. □

Remark 8.18. For each point $(x_0,x_0)\in\operatorname{diag}(\Omega\times\Omega)$, we assign a germ

$$H_{+,(x_0,x_0)}=(\Omega\times\Omega,\phi_+(x,y))_{(x_0,x_0)}.$$

Let H_+ be the formal hypersurface at $\operatorname{diag}(\Omega\times\Omega)$:

$$H_+=\left\{H_{+,(x,x)};\,(x,x)\in\operatorname{diag}(\Omega\times\Omega)\right\}. \tag{8.46}$$

The formal conic conormal bundle Λ_{ϕ_+t} of H_+ is given by: For each point

$$(x_0,\eta_0,x_0,\eta_0)\in\operatorname{diag}((\Sigma^+\bigcap T^*(\Omega))\times(\Sigma^+\bigcap T^*(\Omega))),$$

we assign a germ

$$\begin{aligned}\Lambda_{(x_0,\eta_0,x_0,\eta_0)}=(T^*(U)\times T^*(U),\xi_j-(\phi_+)'_{x_j}t,\ j=1,\dots,2n-1,\\ \eta_j-(\phi_+)'_{y_j}t,\ j=1,\dots,2n-2,\ \phi_+(x,y))_{(x_0,\eta_0,x_0,\eta_0)},\end{aligned}$$

where $t = \frac{\eta_{2n-1}}{(\phi_+)'_{y_{2n-1}}}$ and $U \subset \Omega$ is an open set of x_0 such that

$$(\phi_+)'_{y_{2n-1}} \neq 0 \ \text{ on } U \times U.$$

Then,

$$\Lambda_{\phi_+ t} = \left\{\Lambda_{(x,\eta,x,\eta)};\, (x,\eta,x,\eta) \in \operatorname{diag}((\Sigma^+ \bigcap T^*(\Omega)) \times (\Sigma^+ \bigcap T^*(\Omega)))\right\}. \tag{8.47}$$

$\Lambda_{\phi_+ t}$ is a formal manifold at $\operatorname{diag}((\Sigma^+ \bigcap T^*(\Omega)) \times (\Sigma^+ \bigcap T^*(\Omega)))$. In fact, $\Lambda_{\phi_+ t}$ is the positive Lagrangean manifold associated to $\phi_+ t$ in the sense of Melin and Sjöstrand. (See [18] and Appendix B.)

Let $(W, f_1(x,\xi,y,\eta), \ldots, f_{4n-2}(x,\xi,y,\eta))$ be a representative of $\Gamma_{(x_0,\eta_0,x_0,\eta_0)}$, where $\Gamma_{(x_0,\eta_0,x_0,\eta_0)}$ is as in (8.25). Put

$$\Gamma'_{(x_0,\eta_0,x_0,\eta_0)} = (W, f_1(x,\xi,y,-\eta), \ldots, f_{4n-2}(x,\xi,y,-\eta))_{(x_0,\eta_0,x_0,\eta_0)}.$$

Let C'_∞ be the formal manifold at $\operatorname{diag}(\Sigma^+ \times \Sigma^+)$:

$$C'_\infty = \left\{\Gamma'_{(x,\eta,x,\eta)};\, (x,\eta,x,\eta) \in \operatorname{diag}(\Sigma^+ \times \Sigma^+)\right\}. \tag{8.48}$$

We notice that $\psi(\infty, x, \eta) - \langle y, \eta\rangle$ and $\phi_+(x,y)t$ are equivalent at each point of $\operatorname{diag}((\Sigma^+ \bigcap T^*(\Omega)) \times (\Sigma^+ \bigcap T^*(\Omega)))$ in the sense of Definition B.20. From the global theory of Fourier integral operators (see Proposition B.21), we get

$$\Lambda_{\phi_+ t} = C'_\infty \ \text{ at } \operatorname{diag}((\Sigma^+ \bigcap T^*(\Omega)) \times (\Sigma^+ \bigcap T^*(\Omega))). \tag{8.49}$$

See Proposition B.7 and Proposition B.21, for the details. Formally,

$$C_\infty = \left\{(x,\xi,y,\eta);\, (x,\xi,y,-\eta) \in \Lambda_{\phi_+ t}\right\}.$$

Put

$$\hat{\phi}_+(x,y) = -\overline{\phi}_+(y,x).$$

We claim that

$$\Lambda_{\phi_+ t} = \Lambda_{\hat{\phi}_+ t} \ \text{ at } \operatorname{diag}((\Sigma^+ \bigcap T^*(\Omega)) \times (\Sigma^+ \bigcap T^*(\Omega))), \tag{8.50}$$

where $\Lambda_{\hat{\phi}_+ t}$ is defined as in (8.47). From Proposition 4.5, it follows that $\phi_+(x,y)t$ and $-\overline{\phi}_+(y,x)t$ are equivalent at each point of $\operatorname{diag}((\Sigma^+ \bigcap T^*(\Omega)) \times (\Sigma^+ \bigcap T^*(\Omega)))$ in the sense of Definition B.20. Again from the global theory of Fourier integral operators we get (8.50).

From (8.50), we get the following

Proposition 8.19. *There is a function $f\in C^\infty(\Omega\times\Omega)$, $f(x,x)\neq 0$, such that*

$$\phi_+(x,y)+f(x,y)\overline{\phi}_+(y,x) \tag{8.51}$$

vanishes to infinite order on $x=y$.

From (8.51), we can replace $\phi_+(x,y)$ by

$$\frac{1}{2}(\phi_+(x,y)-\overline{\phi}_+(y,x)).$$

Thus, we have

$$\phi_+(x,y)=-\overline{\phi}_+(y,x). \tag{8.52}$$

From (8.41), we see that

$$(x,d_x\phi_+(x,x))\in\Sigma^+,\ d_y\phi_+(x,x)=-d_x\phi_+(x,x).$$

We can replace $\phi_+(x,y)$ by

$$\frac{2\phi_+(x,y)}{\left\|d_x\phi_+(x,x)\right\|+\left\|d_x\phi_+(y,y)\right\|}.$$

Thus,

$$d_x\phi_+(x,x)=\omega_0(x),\ d_y\phi_+(x,x)=-\omega_0(x). \tag{8.53}$$

Similarly,

$$K_{\pi^-}(x,y)\equiv\int_0^\infty e^{i\phi_-(x,y)t}s_-(x,y,t)dt,$$

where $K_{\pi^-}(x,y)$ is as in (8.35). From (8.22), it follows that when $q=n_-=n_+$, we can take $\phi_-(x,y)$ so that

$$\phi_+(x,y)=-\overline{\phi}_-(x,y).$$

Our method above only works locally. From above, we know that there exist open sets X_j, $j=1,2,\ldots,k$, $X=\bigcup_{j=1}^k X_j$, such that

$$K_{\pi^+}(x,y)\equiv\int_0^\infty e^{i\phi_{+,j}(x,y)t}s_{+,j}(x,y,t)dt$$

on $X_j\times X_j$, where $\phi_{+,j}$ satisfies (8.40), (8.42), (8.43), (8.49), (8.52), (8.53) and $s_{+,j}(x,y,t)$, $j=0,1,\ldots$, are as in (8.38). From the global theory of Fourier integral operators, we have

$$\Lambda_{\phi_{+,j}t}=C'_\infty=\Lambda_{\phi_{+,k}t}\ \text{ at }\operatorname{diag}((\Sigma^+\bigcap T^*(X_j\bigcap X_k))\times(\Sigma^+\bigcap T^*(X_j\bigcap X_k))),$$

for all j, k, where $\Lambda_{\phi_{+,j}t}$, $\Lambda_{\phi_{+,k}t}$ are defined as in (8.47) and C'_∞ is as in (8.48). Thus, there is a function $f_{j,k} \in C^\infty((X_j \bigcap X_k) \times (X_j \bigcap X_k))$, such that

$$\phi_{+,j}(x,y) - f_{j,k}(x,y)\phi_{+,k}(x,y) \tag{8.54}$$

vanishes to infinite order on $x = y$, for all j, k. Let $\chi_j(x,y)$ be a C^∞ partition of unity subordinate to $\{X_j \times X_j\}$ with

$$\chi_j(x,y) = \chi_j(y,x)$$

and set

$$\phi_+(x,y) = \sum \chi_j(x,y)\phi_{+,j}(x,y).$$

From (8.54) and the global theory of Fourier integral operators, it follows that $\phi_{+,j}(x,y)t$ and $\phi_+(x,y)t$ are equivalent at each point of

$$\operatorname{diag}((\Sigma^+ \bigcap T^*(X_j)) \times (\Sigma^+ \bigcap T^*(X_j)))$$

in the sense of Definition B.20, for all j. Again, from the global theory of Fourier integral operators, we get the main result of this work

Theorem 8.20. *Let $(X, \Lambda^{1,0}T(X))$ be a compact orientable connected CR manifold of dimension $2n-1$, $n \geq 2$. We assume that the Levi form has signature (n_-, n_+), $n_- + n_+ = n-1$. Let $q = n_-$ or n_+. Suppose $\Box_b^{(q)}$ has closed range. Then, we have*

$$K_\pi = K_{\pi^+} \text{ if } n_+ = q \neq n_-,$$
$$K_\pi = K_{\pi^-} \text{ if } n_- = q \neq n_+,$$
$$K_\pi = K_{\pi^+} + K_{\pi^-} \text{ if } n_+ = q = n_-,$$

where $K_{\pi^+}(x,y)$ satisfies

$$K_{\pi^+}(x,y) \equiv \int_0^\infty e^{i\phi_+(x,y)t} s_+(x,y,t)dt \mod C^\infty$$

with

$$s_+(x,y,t) \in S^{n-1}_{1,0}(X \times X \times]0,\infty[; \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))),$$

$$s_+(x,y,t) \sim \sum_{j=0}^\infty s^j_+(x,y)t^{n-1-j}$$
in the symbol space $$S^{n-1}_{1,0}(X \times X \times]0,\infty[; \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))), \tag{8.55}$$

where

$$s^j_+(x,y) \in C^\infty(X \times X; \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))),\ j = 0,1,\ldots,$$

$$
\begin{aligned}
&\phi_+(x,y)\in C^\infty(X\times X),\\
&\phi_+(x,x)=0,\\
&\phi_+(x,y)\neq 0 \ \ \textit{if } x\neq y,\\
&\operatorname{Im}\phi_+(x,y)\geq 0,\\
&d_x\phi_+\neq 0,\ \ d_y\phi_+\neq 0 \ \ \textit{where } \operatorname{Im}\phi_+=0,\\
&d_x\phi_+(x,y)|_{x=y}=\omega_0(x), \qquad (8.56)\\
&d_y\phi_+(x,y)|_{x=y}=-\omega_0(x), \qquad (8.57)\\
&\phi_+(x,y)=-\overline{\phi}_+(y,x)
\end{aligned}
$$

Moreover, $\phi_+(x,y)$ satisfies (8.43) and (8.49). Similarly,

$$K_{\pi^-}(x,y)\equiv\int_0^\infty e^{i\phi_-(x,y)t}s_-(x,y,t)dt \bmod C^\infty$$

with

$$s_-(x,y,t)\in S^{n-1}_{1,0}(X\times X\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))),$$

$$s_-(x,y,t)\sim\sum_{j=0}^\infty s^j_-(x,y)t^{n-1-j}$$

in the symbol space $S^{n-1}_{1,0}(X\times X\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X)))$,

where

$$s^j_-(x,y)\in C^\infty(X\times X;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))),\ j=0,1,\ldots,$$

and when $q=n_-=n_+$,

$$\phi_-(x,y)=-\overline{\phi}_+(x,y).$$

9 The leading term of the Szegö Projection

To compute the leading term of the Szegö projection, we have to know the tangential Hessian of $\phi_+(x,y)$ at each point of $\operatorname{diag}(X\times X)$ (see (9.1)), where $\phi_+(x,y)$ is as in Theorem 8.20. We work with local coordinates $x=(x_1,\ldots,x_{2n-1})$ defined on an open set $\Omega\subset X$. The tangential Hessian of $\phi_+(x,y)$ at $(p,p)\in\operatorname{diag}(X\times X)$ is the bilinear map:

$$
\begin{aligned}
&T_{(p,p)}H_+\times T_{(p,p)}H_+\to\mathbb{C},\\
&\qquad (u,v)\to\left\langle(\phi''_+)(p,p)u,v\right\rangle,\ \ u,v\in T_{(p,p)}H_+, \qquad (9.1)
\end{aligned}
$$

where H_+ is as in (8.46) and

$$(\phi_+)'' = \begin{bmatrix} (\phi_+)''_{xx} & (\phi_+)''_{xy} \\ (\phi_+)''_{yx} & (\phi_+)''_{yy} \end{bmatrix}.$$

From (8.56) and (8.57), we have

$$\Big\langle (d_x\phi_+(p,p), d_y\phi_+(p,p)), (u,v)\Big\rangle = 0,\ \ u,v \in \Lambda^{1,0}T_p(X) \oplus \Lambda^{0,1}T_p(X)$$

and

$$\Big\langle (d_x\phi_+(p,p), d_y\phi_+(p,p)), (Y(p), Y(p))\Big\rangle = 0.$$

Thus, $T_{(p,p)}H_+$ at (p,p) is spanned by

$$(u,v),\ \ (Y(p), Y(p)),\ \ u,v \in \Lambda^{1,0}T_p(X) \oplus \Lambda^{0,1}T_p(X). \tag{9.2}$$

Now, we compute the the tangential Hessian of $\phi_+(x,y)$ at $(p,p) \in \mathrm{diag}(X \times X)$. We need to understand the tangent space of the formal manifold

$$J_+ = \Big\{ J_{(x,\eta)};\, (x,\eta) \in \Sigma \Big\}$$

at $\rho = (p, \lambda\omega_0(p)) \in \Sigma^+$, $\lambda > 0$, where J_+ is as in Proposition 8.14.

Let λ_j, $j = 1,\ldots,n-1$, be the eigenvalues of the Levi form L_p. We recall that $2i\left|\lambda_j\right|\left|\sigma_{iY}(\rho)\right|$, $j = 1,\ldots,n-1$ and $-2i\left|\lambda_j\right|\left|\sigma_{iY}(\rho)\right|$, $j = 1,\ldots,n-1$, are the non-vanishing eigenvalues of the fundamental matrix F_ρ. (See (3.9).) Let $\Lambda^+_\rho \subset \mathbb{C}T_\rho(T^*(X))$ be the span of the eigenspaces of F_ρ corresponding to $2i\left|\lambda_j\right|\left|\sigma_{iY}(\rho)\right|$, $j = 1,\ldots,n-1$. It is well known (see [21], [20] and Boutet de Monvel-Guillemin [8]) that

$$T_\rho(J_+) = \mathbb{C}T_\rho(\Sigma) \oplus \Lambda^+_\rho,\ \ \Lambda^+_\rho = T_\rho(J_+)^\perp,$$

where $T_\rho(J_+)^\perp$ is the orthogonal to $T_\rho(J_+)$ in $\mathbb{C}T_\rho(T^*(X))$ with respect to the sympletic two form σ. We need the following

Lemma 9.1. *Let* $\rho = (p, \lambda\omega_0(p)) \in \Sigma^+$, $\lambda > 0$. *Let*

$$\overline{Z}_1(x), \ldots, \overline{Z}_{n-1}(x)$$

be an orthonormal frame of $\Lambda^{1,0}T_x(X)$ *varying smoothly with* x *in a neighborhood of* p*, for which the Levi form is diagonalized at* p*. Let* $q_j(x,\xi)$*,* $j = 1,\ldots,n-1$*, be the principal symbols of* $Z_j(x)$*,* $j = 1,\ldots,n-1$*. Then,* Λ^+_ρ *is spanned by*

$$\begin{cases} H_{q_j}(\rho), \text{ if } \frac{1}{i}\Big\{q_j, \overline{q}_j\Big\}(\rho) > 0 \\ H_{\overline{q}_j}(\rho), \text{ if } \frac{1}{i}\Big\{q_j, \overline{q}_j\Big\}(\rho) < 0 \end{cases}. \tag{9.3}$$

We recall that (see (3.5))

$$\frac{1}{i}\Big\{q_j, \overline{q}_j\Big\}(\rho) = -2\lambda L_p(\overline{Z}_j, Z_j).$$

Proof. In view of (3.8), we see that $H_{q_j}(\rho)$ and $H_{\overline{q}_j}(\rho)$ are the eigenvectors of the fundamental matrix F_ρ corresponding to $\left\{q_j,\overline{q}_j\right\}(\rho)$ and $\left\{\overline{q}_j,q_j\right\}(\rho)$, for all j. Since Λ^+_ρ is the span of the eigenspaces of the fundamental matrix F_ρ corresponding to $2i\lambda\left|\lambda_j\right|$, $j=1,\ldots,n-1$, where λ_j, $j=1,\ldots,n-1$, are the eigenvalues of the Levi form L_p. Thus, Λ^+_ρ is spanned by

$$\begin{cases} H_{q_j}(\rho), \text{ if } \frac{1}{i}\left\{q_j,\overline{q}_j\right\}(\rho)>0 \\ H_{\overline{q}_j}(\rho), \text{ if } \frac{1}{i}\left\{q_j,\overline{q}_j\right\}(\rho)<0 \end{cases}.$$

□

We assume that $(U,f_1,\ldots,f_{n-1})$ is a representative of J_ρ. We also write f_j to denote an almost analytic extension of f_j, for all j. It is well known that (see [20] and (8.30)) there exist $h_j(x,y)\in C^\infty(X\times X)$, $j=1,\ldots,n-1$, such that

$$\begin{aligned} &f_j(x,(\phi_+)'_x)-h_j(x,y)\phi_+(x,y) \\ &\quad\text{vanishes to infinite order on } x=y, j=1,\ldots,n-1. \end{aligned} \tag{9.4}$$

From Lemma 9.1, we may assume that

$$\begin{cases} H_{f_j}(\rho)=H_{q_j}(\rho), \text{ if } \frac{1}{i}\left\{q_j,\overline{q}_j\right\}(\rho)>0 \\ H_{f_j}(\rho)=H_{\overline{q}_j}(\rho), \text{ if } \frac{1}{i}\left\{q_j,\overline{q}_j\right\}(\rho)<0 \end{cases}. \tag{9.5}$$

Here q_j, $j=1,\ldots,n-1$, are as in Lemma 9.1.

We take local coordinates

$$x=(x_1,\ldots,x_{2n-1}),\ \ z_j=x_{2j-1}+ix_{2j},\ \ j=1,\ldots,n-1,$$

defined on some neighborhood of p such that

$$\omega_0(p)=\sqrt{2}dx_{2n-1},\ \ x(p)=0,$$

$$(\frac{\partial}{\partial x_j}(p)\,|\,\frac{\partial}{\partial x_k}(p))=2\delta_{j,k},\ \ j,k=1,\ldots,2n-1$$

and

$$\overline{Z}_j=\frac{\partial}{\partial z_j}-\frac{1}{\sqrt{2}}a_j(x)\frac{\partial}{\partial x_{2n-1}}+\sum_{s=1}^{2n-2}c_{j,s}(x)\frac{\partial}{\partial x_s},\ \ j=1,\ldots,n-1,$$

where $\overline{Z}_j$, $j=1,\ldots,n-1$, are as in Lemma 9.1,

$$\frac{\partial}{\partial z_j}=\frac{1}{2}(\frac{\partial}{\partial x_{2j-1}}-i\frac{\partial}{\partial x_{2j}}),\ \ j=1,\ldots,n-1,$$

$a_j \in C^\infty$, $a_j(0)=0$, $j=1,\ldots,n-1$ and

$$c_{j,s}(x)\in C^\infty,\ \ c_{j,s}(0)=0,\ \ j=1,\ldots,n-1,\ \ s=1,\ldots,2n-2. \tag{9.6}$$

Since $\left\langle [\overline{Z}_j,\overline{Z}_k](p),\omega_0(p)\right\rangle=0$ and $\left\langle [\overline{Z}_j,Z_k](p),\omega_0(p)\right\rangle=2i\lambda_j\delta_{j,k}$, we can check that

$$\begin{aligned}
&\frac{\partial a_j}{\partial z_k}(0)=\frac{\partial a_k}{\partial z_j}(0),\ \ j,k=1,\ldots,n-1,\\
&\frac{\partial a_j}{\partial \overline{z}_k}(0)-\frac{\partial \overline{a}_k}{\partial z_j}(0)=2i\lambda_j\delta_{j,k},\ \ j,k=1,\ldots,n-1,
\end{aligned}\tag{9.7}$$

where λ_j, $j=1,\ldots,n-1$, are the eigenvalues of L_p. Let $\xi=(\xi_1,\ldots,\xi_{2n-1})$ denote the dual variables of x. We have

$$q_j(x,\xi)=\frac{i}{2}(\xi_{2j-1}+i\xi_{2j})-\frac{i}{\sqrt{2}}\overline{a}_j(x)\xi_{2n-1}+i\sum_{s=1}^{2n-2}\overline{c}_{j,s}(x)\xi_s,\ \ j=1,\ldots,n-1,$$

where q_j, $j=1,\ldots,n-1$, are as in Lemma 9.1. We may assume that

$$\lambda_j>0,\ \ j=1,\ldots,q,\ \ \lambda_j<0,\ \ j=q+1,\ldots,n-1.$$

From (9.5), we can check that

$$\begin{aligned}
&f_j(x,\xi)=-\frac{i}{2}(\xi_{2j-1}-i\xi_{2j})+\frac{i}{\sqrt{2}}a_j(x)\xi_{2n-1}-i\sum_{s=1}^{2n-2}c_{j,s}(x)\xi_s+O(\left|(x,\xi')\right|^2),\\
&\quad j=1,\ldots,q,\ \ \xi'=(\xi_1,\ldots,\xi_{2n-2})\\
&f_j(x,\xi)=\frac{i}{2}(\xi_{2j-1}+i\xi_{2j})-\frac{i}{\sqrt{2}}\overline{a}_j(x)\xi_{2n-1}+i\sum_{s=1}^{2n-2}\overline{c}_{j,s}(x)\xi_s+O(\left|(x,\xi')\right|^2),\\
&\quad j=q+1,\ldots,n-1,\ \ \xi'=(\xi_1,\ldots,\xi_{2n-2}).
\end{aligned}\tag{9.8}$$

We write

$$y=(y_1,\ldots,y_{2n-1}),\ \ w_j=y_{2j-1}+iy_{2j},\ \ j=1,\ldots,n-1,$$

$$\frac{\partial}{\partial w_j}=\frac{1}{2}(\frac{\partial}{\partial y_{2j-1}}-i\frac{\partial}{\partial y_{2j}}),\ \ \frac{\partial}{\partial \overline{w}_j}=\frac{1}{2}(\frac{\partial}{\partial y_{2j-1}}+i\frac{\partial}{\partial y_{2j}}),\ \ j=1,\ldots,n-1$$

and

$$\frac{\partial}{\partial \overline{z}_j}=\frac{1}{2}(\frac{\partial}{\partial x_{2j-1}}+i\frac{\partial}{\partial x_{2j}}),\ \ j=1,\ldots,n-1.$$

From (9.4) and (9.8), we have

$$\begin{aligned}
&-i\frac{\partial\phi_+}{\partial z_j}+\frac{i}{\sqrt{2}}a_j\frac{\partial\phi_+}{\partial x_{2n-1}}=h_j(x,y)\phi_+(x,y)+O(\left|(x,y)\right|^2),\ \ j=1,\ldots,q,\\
&i\frac{\partial\phi_+}{\partial \overline{z}_j}-\frac{i}{\sqrt{2}}\overline{a}_j\frac{\partial\phi_+}{\partial x_{2n-1}}=h_j(x,y)\phi_+(x,y)+O(\left|(x,y)\right|^2),\ \ j=q+1,\ldots,n-1.
\end{aligned}\tag{9.9}$$

From (9.9), it is straight forward to see that

$$\begin{aligned}
&\frac{\partial^2\phi_+}{\partial z_j\partial z_k}(0,0)=\frac{\partial a_j}{\partial z_k}(0),\ \ \frac{\partial^2\phi_+}{\partial z_j\partial \overline{z}_k}(0,0)=\frac{\partial a_j}{\partial \overline{z}_k}(0),\ \ 1\le j\le q,\ \ 1\le k\le n-1,\\
&\frac{\partial^2\phi_+}{\partial \overline{z}_j\partial z_k}(0,0)=\frac{\partial \overline{a}_j}{\partial z_k}(0),\ \ \frac{\partial^2\phi_+}{\partial \overline{z}_j\partial \overline{z}_k}(0,0)=\frac{\partial \overline{a}_j}{\partial \overline{z}_k}(0),\ \ q+1\le j\le n-1,\ \ 1\le k\le n-1,\\
&\frac{\partial^2\phi_+}{\partial z_j\partial w_k}(0,0)=\frac{\partial^2\phi_+}{\partial z_j\partial \overline{w}_k}(0,0)=0,\ \ 1\le j\le q,\ \ 1\le k\le n-1,\\
&\frac{\partial^2\phi_+}{\partial \overline{z}_j\partial w_k}(0,0)=\frac{\partial^2\phi_+}{\partial \overline{z}_j\partial \overline{w}_k}(0,0)=0,\ \ q+1\le j\le n-1,\ \ 1\le k\le n-1,\\
&\frac{\partial^2\phi_+}{\partial z_j\partial x_{2n-1}}(0,0)+\frac{\partial^2\phi_+}{\partial z_j\partial y_{2n-1}}(0,0)=\frac{\partial a_j}{\partial x_{2n-1}}(0),\ \ 1\le j\le q,\\
&\frac{\partial^2\phi_+}{\partial \overline{z}_j\partial x_{2n-1}}(0,0)+\frac{\partial^2\phi_+}{\partial \overline{z}_j\partial y_{2n-1}}(0,0)=\frac{\partial \overline{a}_j}{\partial x_{2n-1}}(0),\ \ q+1\le j\le n-1.
\end{aligned}\tag{9.10}$$

Since $d_x\phi_+|_{x=y}=\omega_0(x)$, we have

$$\overline{f}_j(x,(\phi_+)'_x(x,x))=0,\ \ j=1,\dots,n-1.$$

Thus,

$$\begin{aligned}
&i\frac{\partial\phi_+}{\partial\overline{z}_j}(x,x)-\frac{i}{\sqrt{2}}\overline{a}_j(x)\frac{\partial\phi_+}{\partial x_{2n-1}}(x,x)=O(|x|^2),\ \ j=1,\dots,q,\\
&-i\frac{\partial\phi_+}{\partial z_j}(x,x)+\frac{i}{\sqrt{2}}a_j(x)\frac{\partial\phi_+}{\partial x_{2n-1}}(x,x)=O(|x|^2),\ \ j=q+1,\dots,n-1.
\end{aligned}\tag{9.11}$$

From (9.11), it is straight forward to see that

$$\begin{aligned}
&\frac{\partial^2\phi_+}{\partial \overline{z}_j\partial z_k}(0,0)+\frac{\partial^2\phi_+}{\partial \overline{z}_j\partial w_k}(0,0)=\frac{\partial \overline{a}_j}{\partial z_k}(0), 1\le j\le q,\ \ 1\le k\le n-1,\\
&\frac{\partial^2\phi_+}{\partial \overline{z}_j\partial \overline{z}_k}(0,0)+\frac{\partial^2\phi_+}{\partial \overline{z}_j\partial \overline{w}_k}(0,0)=\frac{\partial \overline{a}_j}{\partial \overline{z}_k}(0),\ \ 1\le j\le q,\ \ 1\le k\le n-1,\\
&\frac{\partial^2\phi_+}{\partial z_j\partial z_k}(0,0)+\frac{\partial^2\phi_+}{\partial z_j\partial w_k}(0,0)=\frac{\partial a_j}{\partial z_k}(0),\ \ q+1\le j\le n-1,\ \ 1\le k\le n-1,\\
&\frac{\partial^2\phi_+}{\partial z_j\partial \overline{z}_k}(0,0)+\frac{\partial^2\phi_+}{\partial z_j\partial \overline{w}_k}(0,0)=\frac{\partial a_j}{\partial \overline{z}_k}(0),\ \ q+1\le j\le n-1,\ \ 1\le k\le n-1,\\
&\frac{\partial^2\phi_+}{\partial \overline{z}_j\partial x_{2n-1}}(0,0)+\frac{\partial^2\phi_+}{\partial \overline{z}_j\partial y_{2n-1}}(0,0)=\frac{\partial \overline{a}_j}{\partial x_{2n-1}}(0),\ \ 1\le j\le q,\\
&\frac{\partial^2\phi_+}{\partial z_j\partial x_{2n-1}}(0,0)+\frac{\partial^2\phi_+}{\partial z_j\partial y_{2n-1}}(0,0)=\frac{\partial a_j}{\partial x_{2n-1}}(0),\ \ q+1\le j\le n-1.
\end{aligned}\tag{9.12}$$

Since $\phi_+(x,y) = -\overline{\phi}_+(y,x)$, from (9.10), we have

$$\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial w_k}(0,0) = -\frac{\partial^2\overline{\phi}_+}{\partial\overline{w}_j\partial z_k}(0,0)$$
$$= -\overline{\frac{\partial^2\phi_+}{\partial w_j\partial\overline{z}_k}(0,0)} = 0, \;\; q+1 \le k \le n-1.$$

Combining this with (9.12), we get

$$\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial z_k}(0,0) = \frac{\partial\overline{a}_j}{\partial z_k}(0), \;\; 1 \le j \le q, \;\; q+1 \le k \le n-1. \tag{9.13}$$

Similarly,

$$\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial\overline{z}_k}(0,0) = \frac{\partial\overline{a}_j}{\partial\overline{z}_k}(0), \;\; 1 \le j \le q, \;\; 1 \le k \le q,$$
$$\frac{\partial^2\phi_+}{\partial z_j\partial z_k}(0,0) = \frac{\partial a_j}{\partial z_k}(0), \;\; q+1 \le j \le n-1, \;\; q+1 \le k \le n-1. \tag{9.14}$$

From (9.10) and (9.12), we have

$$\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial w_k}(0,0) = \frac{\partial\overline{a}_j}{\partial z_k}(0) - \frac{\partial^2\phi_+}{\partial\overline{z}_j\partial z_k}(0,0)$$
$$= \frac{\partial\overline{a}_j}{\partial z_k}(0) - \frac{\partial a_k}{\partial\overline{z}_j}(0) = -2i\lambda_j\delta_{j,k}, \;\; 1 \le j,k \le q. \tag{9.15}$$

Similarly,

$$\frac{\partial^2\phi_+}{\partial z_j\partial\overline{w}_k}(0,0) = 2i\lambda_j\delta_{j,k}, \;\; q+1 \le j,k \le n-1. \tag{9.16}$$

Since $\phi_+(x,x) = 0$, we have

$$\frac{\partial^2\phi_+}{\partial x_{2n-1}\partial x_{2n-1}}(0,0) + 2\frac{\partial^2\phi_+}{\partial x_{2n-1}\partial y_{2n-1}}(0,0) + \frac{\partial^2\phi_+}{\partial y_{2n-1}\partial y_{2n-1}}(0,0) = 0. \tag{9.17}$$

Combining (9.10), (9.12), (9.13), (9.14), (9.15), (9.16) and (9.17), we completely determine the tangential Hessian of $\phi_+(x,y)$ at (p,p).

Theorem 9.2. *For $p \in X$, let*

$$\overline{Z}_1(x), \dots, \overline{Z}_{n-1}(x)$$

be an orthonormal frame of $\Lambda^{1,0}T_x(X)$ varying smoothly with x in a neighborhood of p, for which the Levi form is diagonalized at p. We take local coordinates

$$x = (x_1, \dots, x_{2n-1}), \;\; z_j = x_{2j-1} + ix_{2j}, \;\; j = 1, \dots, n-1,$$

defined on some neighborhood of p such that

$$\omega_0(p)=\sqrt{2}dx_{2n-1},\;\; x(p)=0,$$

$$(\frac{\partial}{\partial x_j}(p)\,|\,\frac{\partial}{\partial x_k}(p))=2\delta_{j,k},\;\; j,k=1,\ldots,2n-1$$

and

$$\overline{Z}_j=\frac{\partial}{\partial z_j}-\frac{1}{\sqrt{2}}a_j(x)\frac{\partial}{\partial x_{2n-1}}+\sum_{s=1}^{2n-2}c_{j,s}(x)\frac{\partial}{\partial x_s},\;\; j=1,\ldots,n-1,$$

where

$$\frac{\partial}{\partial z_j}=\frac{1}{2}(\frac{\partial}{\partial x_{2j-1}}-i\frac{\partial}{\partial x_{2j}}),\;\; j=1,\ldots,n-1,$$

$a_j\in C^\infty$, $a_j(0)=0$, $\frac{\partial a_j}{\partial z_k}(0)=\frac{\partial a_k}{\partial z_j}(0)$, $j,k=1,\ldots,n-1$ *and*

$$c_{j,s}(x)\in C^\infty,\;\; c_{j,s}(0)=0,\;\; j=1,\ldots,n-1,\;\; s=1,\ldots,2n-2.$$

We also write

$$y=(y_1,\ldots,y_{2n-1}),\;\; w_j=y_{2j-1}+iy_{2j},\;\; j=1,\ldots,n-1.$$

Then,

$$\begin{aligned}\phi_+(x,y)&=\sqrt{2}(x_{2n-1}-y_{2n-1})+i\sum_{j=1}^{n-1}\left|\lambda_j\right|\left|z_j-w_j\right|^2+\frac{1}{2}\sum_{j,k=1}^{n-1}\Big(\frac{\partial a_j}{\partial z_k}(0)(z_jz_k-w_jw_k)\\&+\frac{\partial\overline{a}_j}{\partial\overline{z}_k}(0)(\overline{z}_j\overline{z}_k-\overline{w}_j\overline{w}_k)+\frac{\partial a_j}{\partial\overline{z}_k}(0)(z_j\overline{z}_k-w_j\overline{w}_k)+\frac{\partial\overline{a}_j}{\partial z_k}(0)(\overline{z}_jz_k-\overline{w}_jw_k)\Big)\\&+\sum_{j=1}^{n-1}\Big(i\lambda_j(z_j\overline{w}_j-\overline{z}_jw_j)+\frac{\partial a_j}{\partial x_{2n-1}}(0)(z_jx_{2n-1}-w_jy_{2n-1})\\&+\frac{\partial\overline{a}_j}{\partial x_{2n-1}}(0)(\overline{z}_jx_{2n-1}-\overline{w}_jy_{2n-1})\Big)+\sqrt{2}(x_{2n-1}-y_{2n-1})f(x,y)+O(\left|(x,y)\right|^3),\\&f\in C^\infty,\;\; f(0,0)=0,\;\; f(x,y)=\overline{f}(y,x),\end{aligned}\tag{9.18}$$

where λ_j, $j=1,\ldots,n-1$, *are the eigenvalues of* L_p.

We have the classical formulas

$$\int_0^\infty e^{-tx}t^m\,dt=\begin{cases}m!x^{-m-1}, & \text{if } m\in\mathbb{Z}, m\geq 0\\ \frac{(-1)^m}{(-m-1)!}x^{-m-1}(\log x+c-\sum_1^{-m-1}\frac{1}{j}), & \text{if } m\in\mathbb{Z}, m<0\end{cases}.\tag{9.19}$$

Here $x\neq 0$, $\operatorname{Re}x\geq 0$ and c is the Euler constant, i.e.

$$c=\lim_{m\to\infty}(\sum_1^m\frac{1}{j}-\log m).$$

Note that

$$\int_0^\infty e^{i\phi_+(x,y)t}\sum_{j=0}^\infty s_+^j(x,y)t^{n-1-j}dt$$
$$=\lim_{\varepsilon\to 0+}\int_0^\infty e^{-(-i\phi_+(x,y)t+\varepsilon)}\sum_{j=0}^\infty s_+^j(x,y)t^{n-1-j}dt.$$

We have the following corollary of Theorem 8.20

Corollary 9.3. *There exist smooth functions*

$$F_+, G_+, F_-, G_- \in C^\infty(X\times X;\ \mathscr{L}(\Lambda^{0,q}T_y^*(X), \Lambda^{0,q}T_x^*(X)))$$

such that

$$K_{\pi^+} = F_+(-i(\phi_+(x,y)+i0))^{-n} + G_+\log(-i(\phi_+(x,y)+i0)),$$

$$K_{\pi^-} = F_-(-i(\phi_-(x,y)+i0))^{-n} + G_-\log(-i(\phi_-(x,y)+i0)).$$

Moreover, we have

$$F_+ = \sum_0^{n-1}(n-1-k)!s_+^k(x,y)(-i\phi_+(x,y))^k + f_+(x,y)(\phi_+(x,y))^n,$$
$$F_- = \sum_0^{n-1}(n-1-k)!s_-^k(x,y)(-i\phi_-(x,y))^k + f_-(x,y)(\phi_-(x,y))^n,$$
$$G_+ \equiv \sum_0^\infty \frac{(-1)^{k+1}}{k!}s_+^{n+k}(x,y)(-i\phi_+(x,y))^k,$$
$$G_- \equiv \sum_0^\infty \frac{(-1)^{k+1}}{k!}s_-^{n+k}(x,y)(-i\phi_-(x,y))^k, \tag{9.20}$$

where s_+^k, $k=0,1,\ldots$, *are as in (8.55) and*

$$f_+(x,y), f_-(x,y) \in C^\infty(X\times X;\ \mathscr{L}(\Lambda^{0,q}T_y^*(X), \Lambda^{0,q}T_x^*(X))).$$

In the rest of this section, we assume that $q=n_+$. We will compute the leading term of K_{π^+}. For a given point $p\in X$, we can take local coordinates

$$x=(x_1, x_2, \ldots, x_{2n-1})$$

defined on some neighborhood Ω of $p\in X$ such that

$$w_0(p) = dx_{2n-1},\ x(p)=0,$$

$$\Lambda^{1,0}T_p(X)\oplus\Lambda^{0,1}T_p(X)=\left\{\sum_j^{2n-2} a_j\frac{\partial}{\partial x_j};\ a_j\in\mathbb{C}\right\}$$

and

$$(\frac{\partial}{\partial x_j}(p)\mid\frac{\partial}{\partial x_k}(p))=\delta_{j,k}.\ j,k=1,\ldots,2n-1.$$

In the local coordinates x, we represent the Hermitian inner product $(\ |\ )$ on $\mathbb{C}T(X)$ by

$$(u\mid v)=\langle Hu,\overline{v}\rangle,$$

where $u,v\in\mathbb{C}T(X)$, H is a positive definite Hermitian matrix. Let $h(x)$ denote the determinant of H. The induced volume form on X is given by $\sqrt{h(x)}dx$. We have

$$h(p)=1.$$

Now,

$$(K_{\pi^+}\circ K_{\pi^+})(x,y)\equiv\int_0^\infty\int_0^\infty\Big(\int e^{it\phi_+(x,w)+is\phi_+(w,y)}s_+(x,w,t)s_+(w,y,s)\sqrt{h(w)}dw\Big)dt\,ds.$$

Let $s=t\sigma$, we get

$$(K_{\pi^+}\circ K_{\pi^+})(x,y)\equiv\int_0^\infty\int_0^\infty\Big(\int e^{it\phi(x,y,w,\sigma)}s_+(x,w,t)s_+(w,y,t\sigma)t\sqrt{h(w)}dw\Big)d\sigma\,dt,$$

where

$$\phi(x,y,w,\sigma)=\phi_+(x,w)+\sigma\phi_+(w,y).$$

It is easy to see that $\operatorname{Im}\phi(x,y,w,\sigma)\geq 0$,

$$d_w\phi(x,y,w,\sigma)|_{x=y=w}=(\sigma-1)\omega_0(x).$$

Thus,

$$x=y=w,\ \sigma=1,\ x\text{ is real},$$

are real critical points.

Now, we will compute the Hessian of ϕ at $x=y=w=p$, p is real, $\sigma=1$. We write $H_\phi(p)$ to denote the Hessian of ϕ at $x=y=w=p$, p is real, $\sigma=1$. $H_\phi(p)$ has the following form

$$H_\phi(p)=\begin{bmatrix}0 & {}^t(\phi_+)'_x\\ (\phi_+)'_x & (\phi_+)''_{xx}+(\phi_+)''_{yy}\end{bmatrix}.$$

Since

$$(\phi_+)'_x(p) = \omega_0(p) = dx_{2n-1},$$

we have

$$H_\phi(p) = \begin{bmatrix} 0, & 0,\dots,0, & 1 \\ \vdots & A, & * \\ 1 & * & * \end{bmatrix},$$

where A is the linear map

$$A : \Lambda^{1,0}T_p(X) \oplus \Lambda^{0,1}T_p(X) \to \Lambda^{1,0}T_p(X) \oplus \Lambda^{0,1}T_p(X),$$
$$\langle Au, v\rangle = \left\langle ((\phi_+)''_{xx} + (\phi_+)''_{yy})u, v \right\rangle,\ \forall\ u, v \in \Lambda^{1,0}T_p(X) \oplus \Lambda^{0,1}T_p(X).$$

From (9.18), it follows that A has the eigenvalues:

$$2i\left|\lambda_1(p)\right|, 2i\left|\lambda_1(p)\right|, \dots, 2i\left|\lambda_{n-1}(p)\right|, 2i\left|\lambda_{n-1}(p)\right|, \tag{9.21}$$

on $\Lambda^{1,0}T_p(X) \oplus \Lambda^{0,1}T_p(X)$ with respect to $(\,|\,)$, where

$$\lambda_j(p), j = 1, \cdots, (n-1)$$

are the eigenvalues of the Levi form L_p. Since

$$\left(\frac{\partial}{\partial x_j}(p) \mid \frac{\partial}{\partial x_k}(p)\right) = \delta_{j,k}.\ j,k = 1,\dots,2n-1,$$

we have,

$$\det\left(\frac{H_\phi(p)}{i}\right) = 2^{2n-2}\left|\lambda_1(p)\right|^2 \cdots \left|\lambda_{n-1}(p)\right|^2. \tag{9.22}$$

From the stationary phase formula (see Proposition B.15), we get

$$(K_{\pi^+} \circ K_{\pi^+})(x,y) \equiv \int_0^\infty e^{it\phi_1(x,y)} a(x,y,t)\,dt,$$

where

$$a(x,y,t) \sim \sum_{j=0}^{\infty} a_j(x,y) t^{n-1-j}$$

in the symbol space $S^{n-1}_{1,0}(\Omega \times \Omega \times [0,\infty[; \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))),$

$$a_j(x,y) \in C^\infty(\Omega \times \Omega; \mathscr{L}(\Lambda^{0,q}T^*(X), \Lambda^{0,q}T^*(X))),\ j = 0, 1, \dots,$$

and $\phi_1(x,y)$ is the corresponding critical value. Moreover, we have

$$\begin{aligned}a_0(p,p)&=\left(\det\frac{H_\phi(p)}{2\pi i}\right)^{-\frac{1}{2}}s^0_+(p,p)\circ s^0_+(p,p)\sqrt{h(p)}\\&=2\left|\lambda_1(p)\right|^{-1}\cdots\left|\lambda_{n-1}(p)\right|^{-1}\pi^n s^0_+(p,p)\circ s^0_+(p,p),\end{aligned}\tag{9.23}$$

where s^0_+ is as in (8.55). We notice that

$$\phi_1(x,x)=0,\ \ (\phi_1)'_x(x,x)=(\phi_+)'_x(x,x),\ \ (\phi_1)'_y(x,x)=(\phi_+)'_y(x,x).\tag{9.24}$$

From (9.19), it follows that

$$\begin{aligned}(K_{\pi^+}\circ K_{\pi^+})(x,y)&\equiv F_1(-i(\phi_1(x,y)+i0))^{-n}+G_1\log(-i(\phi_1(x,y)+i0))\\&\equiv F_+(-i(\phi_+(x,y)+i0))^{-n}+G_+\log(-i(\phi_+(x,y)+i0)),\end{aligned}\tag{9.25}$$

where

$$F_1=\sum_0^{n-1}(n-1-k)!a_j(-i\phi_1)^k+f_1\phi_1^n,\ f_1\in C^\infty(\Omega\times\Omega;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))),$$

$G_1\in C^\infty(\Omega\times\Omega;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X)))$, F_+ and G_+ are as in Corollary 9.3. From (9.24) and (9.25), we see that

$$s^0_+(x,x)=a_0(x,x).$$

From this and (9.23), we get

$$2\left|\lambda_1(p)\right|^{-1}\cdots\left|\lambda_{n-1}(p)\right|^{-1}\pi^n s^0_+(p,p)\circ s^0_+(p,p)=s^0_+(p,p).\tag{9.26}$$

Let

$$\mathscr{N}_x(p^s_0+\frac{1}{2}\widetilde{\operatorname{tr}}F)=\left\{u\in\Lambda^{0,q}T^*_x(X);\,(p^s_0+\frac{1}{2}\widetilde{\operatorname{tr}}F)(x,\omega_0(x))u=0\right\},$$

where p^s_0 is the subprincipal symbol of $\Box^{(q)}_b$ and F is the fundamental matrix of $\Box^{(q)}_b$. From the asymptotic expansion of $\Box^{(q)}_b(e^{i\phi_+}s_+)$, we see that

$$s^0_+(p,p)u\in\mathscr{N}_p(p^s_0+\frac{1}{2}\widetilde{\operatorname{tr}}F)$$

for all $u\in\Lambda^{0,q}T^*_p(X)$. (See section 5.) Let

$$I_1=(\frac{1}{2}|\lambda_1|\cdots|\lambda_{n-1}|)^{-1}\pi^n s^0_+(p,p).$$

From (9.26), we see that

$$I_1^2 = I_1. \tag{9.27}$$

Since

$$K_{(\pi^+)^*} \equiv K_{\pi^+}$$

and

$$\phi_+(x, y) = -\overline{\phi}_+(y, x),$$

we have

$$(s_+^0)^*(p, p) = s_+^0(p, p)$$

and hence

$$I_1^* = I_1, \tag{9.28}$$

where $(\pi^+)^*$ is the adjoint of π^+, $s_0^*(p, p)$ and I_1^* are the adjoints of $s_0(p, p)$ and I_1 in the space

$$\mathscr{L}(\Lambda^{0,q} T_p^*(X), \Lambda^{0,q} T_p^*(X)))$$

with respect to $(\,|\,)$ respectively. Note that

$$\dim \mathscr{N}_p(p_0^s + \frac{1}{2}\widetilde{\mathrm{tr}}\, F) = 1.$$

(See section 3.) Combining this with (9.27), (9.28) and $s_+^0(p, p) \neq 0$, it follows that

$$I_1 : \Lambda^{0,q} T_p^*(X) \to \Lambda^{0,q} T_p^*(X)$$

is the orthogonal projection onto $\mathscr{N}_p(p_0^s + \frac{1}{2}\widetilde{\mathrm{tr}}\, F)$.

For a given point $p \in X$, let

$$\overline{Z}_1(x), \ldots, \overline{Z}_{n-1}(x)$$

be an orthonormal frame of $\Lambda^{1,0} T_x(X)$, for which the Levi form is diagonalized at p. Let $e_j(x)$, $j = 1, \ldots, n-1$ denote the basis of $\Lambda^{0,1} T_x^*(X)$, which is dual to $Z_j(x)$, $j = 1, \ldots, n-1$. Let $\lambda_j(x)$, $j = 1, \ldots, n-1$ be the eigenvalues of the Levi form L_x. We assume that

$$\lambda_j(p) > 0 \ \text{ if } \ 1 \leq j \leq n_+.$$

Then

$$I_1 = \prod_{j=1}^{j=n_+} e_j(p)^\wedge e_j^{\wedge,*}(p) \ \text{ at } p.$$

(See section 3.) Summing up, we have proved

Proposition 9.4. *For a given point $x_0 \in X$, Let*

$$\overline{Z}_1(x), \ldots, \overline{Z}_{n-1}(x)$$

be an orthonormal frame of $\Lambda^{1,0} T_x(X)$, for which the Levi form is diagonalized at x_0. Let $e_j(x)$, $j = 1, \ldots, n-1$ denote the basis of $\Lambda^{0,1} T_x^*(X)$, which is dual to $Z_j(x)$, $j = 1, \ldots, n-1$. Let $\lambda_j(x)$, $j = 1, \ldots, n-1$ be the eigenvalues of the Levi form L_x. We assume that $q = n_+$ and that

$$\lambda_j(x_0) > 0 \ \text{if} \ 1 \le j \le n_+.$$

Then

$$F_+(x_0, x_0) = (n-1)! \frac{1}{2} |\lambda_1(x_0)| \cdots |\lambda_{n-1}(x_0)| \pi^{-n} \prod_{j=1}^{j=n_+} e_j(x_0)^\wedge e_j(x_0)^{\wedge,*}.$$

10 The Szegö projection on non-orientable CR manifolds

In this section, $(X, \Lambda^{1,0} T(X))$ is a compact connected not necessarily orientable CR manifold of dimension $2n-1$, $n \ge 2$. We will use the same notations as before. The definition of the Levi form (see 2.6) depends on the choices of ω_0. However, the number of non-zero eigenvalues is independent of the choices of ω_0. Thus, it makes sense to say that the Levi-form is non-degenerate. As before, we assume that the Levi form L is non-degenerate at each point of X. We have the following

Lemma 10.1. *Let (n_-, n_+), $n_- + n_+ = n-1$, be the signature of the Levi-form L. (The signature of the Levi-form L depends on the choices of ω_0.) If $n_- \neq n_+$ at a point of X, then X is orientable.*

Proof. Since X is connected, $n_- \neq n_+$ at a point of X implies $n_- \neq n_+$ at each point of X. Let $X = \bigcup U_j$, where U_j is a local coordinate patch of X. On U_j, we can choose an orthonormal frame

$$\omega_{1,j}(x), \ldots, \omega_{n-1,j}(x)$$

for $\Lambda^{1,0} T_x^*(U_j)$, then

$$\overline{\omega}_{1,j}(x), \ldots, \overline{\omega}_{n-1,j}(x)$$

is an orthonormal frame for $\Lambda^{0,1} T_x^*(U_j)$. The $(2n-2)$-form

$$\omega_j = i^{n-1} \omega_{1,j} \wedge \overline{\omega}_{1,j} \wedge \cdots \wedge \omega_{n-1,j} \wedge \overline{\omega}_{n-1,j}$$

is real and is independent of the choice of the orthonormal frame. There is a real 1-form $\omega_{0,j}(x)$ of length one which is orthogonal to $\Lambda^{1,0}T^*_x(U_j)\oplus\Lambda^{0,1}T^*_x(U_j)$. We take $\omega_{0,j}$ so that $n_- < n_+$ on U_j. Since $\omega_{0,j}$ is unique up to sign and $n_- < n_+$ on U_j, for all j, we have

$$\omega_{0,j}(x)=\omega_{0,k}(x) \text{ on } U_j\bigcap U_k,$$

so ω_0 is globally defined. The lemma follows. □

We only need to consider the case $n_- = n_+$. We recall that if $n_- = n_+$ then $\Box_b^{(q)}$ has closed range. In view of the proof of Theorem 8.20, we have the following

Theorem 10.2. *Let (n_-, n_+) be the signature of the Levi form. We assume that $q = n_- = n_+$. Put*

$$\hat{\Sigma}=\left\{(x,\xi)\in T^*(X)\setminus 0;\, \xi=\lambda\omega_0(x), \lambda\neq 0\right\},$$

where ω_0 is the locally unique real 1 form determined up to sign by

$$\|\omega_0\|=1,\ \omega_0\perp(\Lambda^{0,1}T^*(X)\oplus\Lambda^{1,0}T^*(X)).$$

Then π is a well defined operator

$$\pi: H^s_{\rm loc}(X;\Lambda^{0,q}T^*(X))\rightarrow H^s_{\rm loc}(X;\Lambda^{0,q}T^*(X)),$$

for all $s\in\mathbb{R}$, and

$$\mathrm{WF}'(K_\pi)=\mathrm{diag}(\hat{\Sigma}\times\hat{\Sigma}),$$

where

$$\mathrm{WF}'(K_\pi)=\left\{(x,\xi,y,\eta)\in T^*(X)\times T^*(X);\, (x,\xi,y,-\eta)\in \mathrm{WF}(K_\pi)\right\}.$$

Here $\mathrm{WF}(K_\pi)$ is the wave front set of K_π in the sense of Hörmander [14].

For every local coordinate patch U, we fix a ω_0 on U. We define

$$\Sigma^+=\left\{(x,\xi)\in T^*(U)\setminus 0;\, \xi=\lambda\omega_0(x), \lambda>0\right\},$$
$$\Sigma^-=\left\{(x,\xi)\in T^*(U)\setminus 0;\, \xi=\lambda\omega_0(x), \lambda<0\right\}.$$

We have

$$K_\pi=K_{\pi^+}+K_{\pi^-}\ \ on\ \ U\times U,$$

where $K_{\pi^+}(x,y)$ satisfies

$$K_{\pi^+}(x,y)\equiv\int_0^\infty e^{i\phi_+(x,y)t}s_+(x,y,t)dt\ \ on\ U\times U$$

with

$$s_+(x,y,t)\in S^{n-1}_{1,0}(U\times U\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))),$$

$$s_+(x,y,t)\sim\sum_{j=0}^{\infty}s^j_+(x,y)t^{n-1-j}$$
in the symbol space $S^{n-1}_{1,0}(U\times U\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X)))$,

where

$$s^j_+(x,y)\in C^\infty(U\times U;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))),\ j=0,1,\ldots,$$

$$\phi_+(x,y)\in C^\infty(U\times U),$$

$$\phi_+(x,x)=0,$$

$$\phi_+(x,y)\neq 0\ \ \textit{if}\ x\neq y,$$

$$\operatorname{Im}\phi_+(x,y)\geq 0,$$

$$d_x\phi_+\neq 0,\ \ d_y\phi_+\neq 0\ \ \textit{where}\ \operatorname{Im}\phi_+=0,$$

$$d_x\phi_+(x,y)|_{x=y}=\omega_0(x),$$

$$d_y\phi_+(x,y)|_{x=y}=-\omega_0(x),$$

$$\phi_+(x,y)=-\overline{\phi}_+(y,x).$$

Moreover, $\phi_+(x,y)$ *satisfies (9.18). Similarly,*

$$K_{\pi^-}(x,y)\equiv\int_0^\infty e^{i\phi_-(x,y)t}s_-(x,y,t)dt \mod C^\infty$$

with

$$s_-(x,y,t)\in S^{n-1}_{1,0}(U\times U\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))),$$

$$s_-(x,y,t)\sim\sum_{j=0}^{\infty}s^j_-(x,y)t^{n-1-j}$$
in the symbol space $S^{n-1}_{1,0}(U\times U\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X)))$,

where

$$s^j_-(x,y)\in C^\infty(U\times U;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))),\ j=0,1,\ldots,$$

$$\phi_-(x,y)=-\overline{\phi}_+(x,y).$$

A Appendix: Microlocal analysis, a review

We will give a brief discussion of microlocal analysis in a setting appropriate for our purpose. For more details on the subject, see Hörmander [15], Grigis-Sjöstrand [12] and Melin-Sjöstrand [18]. Our presentation is essentially taken from Grigis-Sjöstrand [12] and Hörmander [15].

Let $\Omega \subset \mathbb{R}^n$ be an open set. From now on, we write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, where $x = (x_1, \ldots, x_n)$, $D_{x_j} = -i\partial_{x_j}$. We have the following

Definition A.1. Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $m \in \mathbb{R}$. $S^m(\Omega \times \mathbb{R}^N)$ is the space of all $a \in C^\infty(\Omega \times \mathbb{R}^N)$ such that for all compact sets $K \subset \Omega$ and all $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^N$, there is a constant $c > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq c(1+|\xi|)^{m-|\beta|},\ (x,\xi) \in K \times \mathbb{R}^N.$$

S^m is called the space of symbols of order m. We write $S^{-\infty} = \bigcap S^m$, $S^\infty = \bigcup S^m$.

We next study asymptotic sums of symbols.

Proposition A.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $a_j \in S^{m_j}(\Omega \times \mathbb{R}^N)$, $j = 0,1,2,\ldots$ with $m_j \searrow -\infty$, $j \to \infty$. Then there exists $a \in S^{m_0}(\Omega \times \mathbb{R}^N)$ unique modulo (i.e. up to some element in) $S^{-\infty}(\Omega \times \mathbb{R}^N)$, such that $a - \sum_{0 \leq j < k} a_j \in S^{m_k}(\Omega \times \mathbb{R}^N)$, for every $k \in \mathbb{N}$.*

Proof. See Grigis-Sjöstrand [12] or Hörmander [14]. □

If a and a_j have the properties of Proposition A.2, we write

$$a \sim \sum_0^\infty a_j$$

and we call a the asymptotic sum of a_j.

Definition A.3. Let $\Omega \subset \mathbb{R}^n$ be an open set. The set $S^m_{\mathrm{cl}}(\Omega \times \mathbb{R}^N)$ of all $a \in S^m(\Omega \times \mathbb{R}^N)$ such that

$$a(x,\xi) \sim \sum_0^\infty a_j(x,\xi),$$

where $a_j \in C^\infty(\Omega \times \mathbb{R}^N)$ is positively homogeneous of degree $m - j$ when $|\xi| \geq 1$, will be called the space of classical symbols of order m.

The positively homogeneity in the definition means that

$$a_j(x,\lambda\xi) = \lambda^{m-j} a_j(x,\xi),\ |\xi| \geq 1,\ \lambda \geq 1.$$

Definition A.4. Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $\varphi(x,\xi) \in C^\infty(\Omega \times \dot{\mathbb{R}}^N)$ is called a phase function if for all $(x,\xi) \in \Omega \times \dot{\mathbb{R}}^N$:

(a) $\operatorname{Im}\varphi(x,\xi) \geq 0$,

(b) $\varphi(x,\lambda\xi) = \lambda\varphi(x,\xi)$ for all $\lambda > 0$,

(c) $d\varphi \neq 0$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $\mathscr{D}'(\Omega)$ be the space of Schwartz distributions on Ω. Let

$$\mathscr{E}'(\Omega) = \{u \in \mathscr{D}'(\Omega); \operatorname{supp} u \text{ is compact}\}.$$

Let $\varphi(x,\xi)$ be a phase function on $\Omega \times \dot{\mathbb{R}}^N$ and let $a(x,\xi) \in S^m(\Omega \times \mathbb{R}^N)$. Choose a cut-off function $\chi(\xi) \in C^\infty(\mathbb{R}^N)$ so that $\chi(\xi) = 1$ when $|\xi| < 1$ and $\chi(\xi) = 0$ when $|\xi| > 2$. For all $u \in C_0^\infty(\Omega)$, set

$$\mathrm{I}(a,\varphi)u = \lim_{\varepsilon \to 0} \int e^{i\varphi(x,\xi)} a(x,\xi) u(x) \chi(\varepsilon\xi)\, dx\, d\xi. \tag{A.1}$$

Then $\mathrm{I}(a,\varphi) \in \mathscr{D}'(\Omega)$. More precisely, we have the following

Proposition A.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $\varphi(x,\xi)$ be a phase function on $\Omega \times \dot{\mathbb{R}}^N$. Then there is a unique way of defining $\mathrm{I}(a,\varphi) \in \mathscr{D}'(\Omega)$ for $a \in S^\infty$ such that $\mathrm{I}(a,\varphi)$ is defined by*

$$\mathrm{I}(a,\varphi) = \int e^{i\varphi(x,\xi)} a(x,\xi)\, d\xi$$

when $a \in S^m(\Omega \times \mathbb{R}^N)$, $m < -N$ and such that for every $m \in \mathbb{R}$, the map

$$S^m(\Omega \times \mathbb{R}^N) \ni a \to \mathrm{I}(a,\varphi)$$

is continuous.

Proof. See Grigis-Sjöstrand [12] or Hörmander [14]. □

Let $Y \subset \mathbb{R}^{m_1}$, $Z \subset \mathbb{R}^{m_2}$ be open sets. We recall that the Schwartz kernel theorem(see Hörmander [17]) states that there is a bijection between the set of distributions $K \in \mathscr{D}'(Y \times Z)$ and the set of continuous linear operators

$$A : C_0^\infty(Z) \to \mathscr{D}'(Y).$$

The correspondence is given by

$$\langle Au\,,v\rangle_Y = \langle K\,, v \otimes u\rangle_{Y \times Z}\,,\ u \in C_0^\infty(Z),\ v \in C_0^\infty(Y),$$

where $\langle\,,\rangle_Y$ and $\langle\,,\rangle_{Y\times Z}$ denote the duality brackes for $\mathscr{D}'(Y)\times C_0^\infty(Y)$ and $\mathscr{D}'(Y\times Z)\times C_0^\infty(Y\times Z)$ respectively and $(v\otimes u)(y,z)=v(y)u(z)$. We call K the distribution kernel of A, and write $K=K_A$. Moreover, the following two conditions are equivalent:

(i) $K_A\in C^\infty(Y\times Z)$,

(ii) A is continuous $\mathscr{E}'(Z)\to C^\infty(Y)$.

If A satisfies (i) or (ii), we say that A is smoothing. Let B be a continuous linear operator

$$B: C_0^\infty(Z)\to\mathscr{D}'(Y).$$

We write $A\equiv B$ if $A-B$ is a smoothing operator.

In order to simplify the discussion of composition of some operators, it is convenient to introduce the notion of properly supported operators. Let C be a closed subset of $Y\times Z$. We say that C is proper if the two projections

$$\Pi_y:(y,z)\in C\to y\in Y$$
$$\Pi_z:(y,z)\in C\to z\in Z$$

are proper, that is the inverse image of every compact subset of Y and Z respectively is compact.

A continuous linear operator

$$A: C_0^\infty(Z)\to\mathscr{D}'(Y)$$

is said to be properly supported if $\operatorname{supp}K_A\subset Y\times Z$ is proper. If A is properly supported, then A is continuous

$$C_0^\infty(Z)\to\mathscr{E}'(Y)$$

and A has a unique continuous extension

$$C^\infty(Z)\to\mathscr{D}'(Y).$$

Definition A.6. Let $Y\subset\mathbb{R}^{m_1}, Z\subset\mathbb{R}^{m_2}$ be open sets. Let φ be a phase function on $Y\times Z\times\dot{\mathbb{R}}^N$. Let $a\in S^m(Y\times Z\times\mathbb{R}^N)$. Then a Fourier integral operator of order m is a continuous linear map

$$A: C_0^\infty(Z)\to\mathscr{D}'(Y)$$

such that $K_A=\mathrm{I}(a,\varphi)$. Formally we write

$$Au(y)=\int\!\!\int e^{i\varphi(y,z,\xi)}a(y,z,\xi)u(z)\,dz\,d\xi,\ u\in C_0^\infty(Z).$$

Definition A.7. Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $m \in \mathbb{R}$. A pseudodifferential operator of order m is a continuous linear map:

$$A : C_0^\infty(\Omega) \to \mathscr{D}'(\Omega)$$

such that

$$K_A = \frac{1}{(2\pi)^N} \mathrm{I}(a, \varphi)$$

with $a \in S^m(\Omega \times \Omega \times \mathbb{R}^N)$, $\varphi(x, y, \xi) = (x - y)\xi$. Formally,

$$Au(x) = \frac{1}{(2\pi)^N} \int\int e^{i(x-y)\xi} a(x, y, \xi) u(y) dy d\xi, \; u \in C_0^\infty(\Omega).$$

We shall write $L^m(\Omega)$ to denote the space of pseudodifferential operators of order m.

We collect some facts about pseudodifferential operators. For the proofs, see Grigis-Sjöstrand [12] or Hörmander [15].

Let $\Omega \subset \mathbb{R}^N$ be an open set. Let

$$Au(x) = \frac{1}{(2\pi)^N} \int\int e^{i(x-y)\xi} a(x, y, \xi) u(y) dy d\xi \tag{A.2}$$

be a pseudodifferential operator of order m, where $a \in S^m(\Omega \times \Omega \times \mathbb{R}^N)$. Then we have the following properties:

(a) A is continuous

$$C_0^\infty(\Omega) \to C^\infty(\Omega)$$

and has unique continuous extension

$$\mathscr{E}'(\Omega) \to \mathscr{D}'(\Omega).$$

(b) If $a \in S^{-\infty}(\Omega \times \Omega \times \mathbb{R}^N)$, then $K_A \in C^\infty(\Omega \times \Omega)$ and A is continuous $\mathscr{E}'(\Omega) \to C^\infty(\Omega)$, conversely if $K_A \in C^\infty$, then there is $a \in S^{-\infty}(\Omega \times \Omega \times \mathbb{R}^N)$ such that (A.2) holds.

We write $L^{-\infty}(\Omega)$ to denote the space of operators with $a \in S^{-\infty}(\Omega \times \Omega \times \mathbb{R}^N)$.

(c) We recall that the singular support of K_A is the smallest closed subset L of $\Omega \times \Omega$ such that $K_A \in C^\infty((\Omega \times \Omega) \setminus L)$. We write $\operatorname{sing\,supp} K_A$ to denote the singular support of K_A. Then

$$\operatorname{sing\,supp} K_A \subset \operatorname{diag}(\Omega \times \Omega) = \{(x, x) \in \Omega \times \Omega\}.$$

(d) A has decomposition $A = A' + A''$, where $A' \in L^m(\Omega)$ is properly supported and $A'' \in L^{-\infty}(\Omega)$.

(e) Let $A = A' + A''$ be the decomposition in (d). Set

$$A'u(x) = \frac{1}{(2\pi)^N} \int\int e^{i(x-y)\xi} \hat{a}(x,y,\xi)u(y)dyd\xi,\ u \in C_0^\infty(\Omega).$$

Then

$$a(x,\xi) := e^{-ix\xi} A'(e^{ix\xi}) \in S^m(\Omega \times \mathbb{R}^N)$$

and has the asymptotic expansion

$$a(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^N} \frac{1}{\alpha!} (\partial_\xi^\alpha D_y^\alpha \hat{a}(x,y,\xi))\Big|_{x=y}. \tag{A.3}$$

We call $a(x,\xi)$ the symbol of A. $a(x,\xi)$ is up to some element in $S^{-\infty}(\Omega \times \mathbb{R}^N)$. We write $a(x,\xi) = \sigma_A(x,\xi)$. Moreover,

$$\begin{aligned} Au(x) &\equiv \frac{1}{(2\pi)^N} \int e^{ix\xi} a(x,\xi)\hat{u}(\xi)d\xi \\ &= \frac{1}{(2\pi)^N} \int e^{i\langle x-y,\xi\rangle} a(x,\xi)u(y)dyd\xi, \end{aligned}$$

where $\hat{u}(\xi)$ is the Fourier transform of $u \in C_0^\infty(\Omega)$.

(f) Let

$$(u \mid v) = \int u(y)\overline{v(y)}dy$$

be the inner product on $L^2(\Omega)$. We define the adjoint

$$A^* : C_0^\infty(\Omega) \to \mathscr{D}'(\Omega)$$

by $(Au \mid v) = (u \mid A^*v)$, $u, v \in C_0^\infty(\Omega)$. The distribution kernel of A^* is

$$K_{A^*}(x,y) = \overline{K_A(y,x)}$$

and $A^* \in L^m(\Omega)$. Moreover, $\sigma_{A^*} \in S^m$ and σ_{A^*} has the following asymptotic formula

$$\sigma_{A^*}(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^N} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{\sigma_A(x,\xi)}. \tag{A.4}$$

(g) Let $A \in L^m(\Omega)$, $B \in L^{m'}(\Omega)$, with at least one of A, B properly supported. Then $A \circ B \in L^{m+m'}(\Omega)$ and

$$\sigma_{A\circ B}(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^N} \frac{\partial_\xi^\alpha \sigma_A(x,\xi) D_x^\alpha \sigma_B(x,\xi)}{\alpha!}. \tag{A.5}$$

(h) We write $\mathscr{S}(\mathbb{R}^N)$ to denote the set of all $\phi \in C^\infty(\mathbb{R}^N)$ such that

$$\sup_x \left| x^\beta \partial^\alpha \phi(x) \right| < \infty$$

for all multi-indices α and β. Let $\mathscr{S}'(\mathbb{R}^N)$ be the dual space of $\mathscr{S}(\mathbb{R}^N)$, i.e. $\mathscr{S}'(\mathbb{R}^N)$ is the space of all continuous linear forms on $\mathscr{S}(\mathbb{R}^N)$. We recall that the Sobolev space $H^s(\mathbb{R}^N)$, $s \in \mathbb{R}$, is the space of all $u \in \mathscr{S}'(\mathbb{R}^N)$ such that $\hat{u}(\xi)$ is locally square integrable and

$$\|u\|_s^2 = \frac{1}{(2\pi)^N} \int \left| \hat{u}(\xi)^2 \right| (1 + |\xi|^2)^s d\xi < \infty,$$

where $\hat{u}(\xi)$ is the Fourier transform of u. Define the Fréchet space

$$H^s_{\text{loc}}(\Omega) = \left\{ u \in \mathscr{D}'(\Omega);\ \varphi a \in H^s(\mathbb{R}^N),\ \forall \varphi \in C_0^\infty(\Omega) \right\}$$

and

$$H^s_{\text{comp}}(\Omega) = H^s_{\text{loc}}(\Omega) \bigcap \mathscr{E}'(\Omega).$$

If $A \in L^m(\Omega)$, then A is continuous

$$H^s_{\text{comp}}(\Omega) \to H^{s-m}_{\text{loc}}(\Omega).$$

If $A \in L^m(\Omega)$ is properly supported, then A is continuous

$$H^s_{\text{loc}}(\Omega) \to H^{s-m}_{\text{loc}}(\Omega).$$

Our next aim is to define pseudodifferential operators on a manifold. First we must discuss changes of variables and the notions of principal symbol and subprincipal sympol. Let $\kappa : \Omega \to \Omega_\kappa$ be a diffeomorphism map, where Ω, Ω_κ are open sets in $\mathbb{R}^N$. If $A \in L^m(\Omega)$, we want to study

$$\begin{aligned} \widetilde{A} = (\kappa^{-1})^* \circ A \circ \kappa^* : C_0^\infty(\Omega_\kappa) \to C^\infty(\Omega_\kappa), \\ u \in C_0^\infty(\Omega_\kappa) \to A(u \circ \kappa) \circ \kappa^{-1} \in C^\infty(\Omega_\kappa). \end{aligned}$$

We have the following

Proposition A.8. *Let Ω, $\Omega_\kappa \subset \mathbb{R}^N$ be open sets. Let*

$$\kappa : \Omega \to \Omega_\kappa$$

be a diffeomorphism. If $A \in L^m(\Omega)$. Then

$$\widetilde{A} = (\kappa^{-1})^* \circ A \circ \kappa^* \in L^m(\Omega_\kappa).$$

Moreover, we have the asymptotic expansion

$$\sigma_{\widetilde{A}}(\kappa(x),\xi) \sim \sum_{\alpha\in\mathbb{N}^N} \frac{\partial_\xi^\alpha a(x,{}^t\kappa'(x)\xi) D_y^\alpha e^{i\langle\rho_x(y),\xi\rangle}}{\alpha!}\bigg|_{y=x}, \tag{A.6}$$

where $\rho_x(y) = \kappa(y) - \kappa(x) - \kappa'(x)(y-x)$ *vanishes of second order at* x*. The terms in the series are in* $S^{m-|\frac{\alpha}{2}|}(\Omega\times\mathbb{R}^N)$.

Proof. See Hörmander [14]. □

Definition A.9. Let $\Omega\subset\mathbb{R}^N$ be an open set. Let $A\in L^m(\Omega)$. We define the principal symbol of A as the image of σ_A in

$$(S^m/S^{m-1})(\Omega\times\mathbb{R}^N).$$

We then have a surjective map

$$L^m \to S^m/S^{m-1}(\Omega\times\mathbb{R}^N)$$

which gives rise to a bijection

$$L^m/L^{m-1} \to S^m/S^{m-1}(\Omega\times\mathbb{R}^N).$$

Let $\Omega\subset\mathbb{R}^N$ be an open set. Let $L^m_{\mathrm{cl}}(\Omega)\subset L^m(\Omega)$ be the space of pseudodifferential operators A with $\sigma_A\in S^m_{\mathrm{cl}}$. For such an operator we can identify the principal symbol in $S^m_{\mathrm{cl}}/S^{m-1}_{\mathrm{cl}}$ with the positively homogenous function $a_m(x,\xi)$ in the asymptotic expansion

$$\sigma_A \sim \sum_0^\infty a_{m-j}(x,\xi), \tag{A.7}$$

where $a_{m-j}(x,\xi)$ is positively homogenous function of degree $m-j$.

Returning to the changes of variables for $\widetilde{A}\in L^m(\Omega_\kappa)$, we see from (A.6) that

$$\sigma_{\widetilde{A}}(\kappa(x),\xi) - \sigma_A(x,{}^t\kappa'(x)\xi) \in S^{m-1}(\Omega\times\mathbb{R}^N). \tag{A.8}$$

If a, $\widetilde{a}$ denote the principal symbols of A, $\widetilde{A}$, we get the relation

$$\widetilde{a}(\kappa(x),\xi) = a(x,{}^t\kappa'(x)\xi). \tag{A.9}$$

Definition A.10. Let $\Omega\subset\mathbb{R}^N$ be an open set. Let $A\in L^m_{\mathrm{cl}}(\Omega)$ with symbol $\sigma_A(x,\xi)$ as in (A.7). The subprincipal symbol of A is defined by

$$a^s(x,\xi) = a_{m-1}(x,\xi) + \frac{i}{2}\sum_{j=1}^N \frac{\partial^2 a_m(x,\xi)}{\partial x_j \partial\xi_j}. \tag{A.10}$$

Definition A.11. Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $A \in L^m_{\mathrm{cl}}(\Omega)$ with symbol $\sigma_A(x,\xi)$ as in (A.7). We say that (x_0,ξ_0) is a doubly characteristic point of A if

(a) $a_m(x_0,\xi_0)=0$,

(b) $(\frac{\partial a_m}{\partial \xi_j})(x_0,\xi_0)=0$, $j=1,\dots,N$,

(c) $(\frac{\partial a_m}{\partial x_j})(x_0,\xi_0)=0$, $j=1,\dots,N$.

We need the following (see Sjöstrand [21])

Lemma A.12. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Suppose that $P = AQ_1Q_2 + B$, where $A \in L^m_{\mathrm{cl}}(\Omega)$, $Q_1, Q_2 \in L^0_{\mathrm{cl}}(\Omega)$, $B \in L^{m-1}_{\mathrm{cl}}(\Omega)$ are properly supported classical pseudodifferential operators with principal symbols a, q_1, q_2, b respectively. Let $\rho \in T^*(\Omega)\smallsetminus 0$ be a point where $q_1 = q_2 = 0$. If we write the symbol of P as*

$$p(x,\xi) = p_m(x,\xi) + p_{m-1}(x,\xi) \mod S^{m-2}$$

where p_m and p_{m-1} are positively homogeneous of degree m and $m-1$, then the subprincipal symbol of P at ρ is given by the formula

$$p^s(\rho) = b(\rho) + a(\rho)(2i)^{-1}\{q_1,q_2\}(\rho).$$

In particular, if $E \in L^k_{\mathrm{cl}}(\Omega)$ is an elliptic operator with principal symbol $e(x,\xi)$, that is $e(x,\xi) \neq 0$, for all $(x,\xi) \in \Omega$, then the subprincipal symbols of $E \circ P$ and $P \circ E$ at ρ are $e(\rho)p^s(\rho)$.

Let Ω, Ω_κ be open sets in $\mathbb{R}^N$. Let

$$\kappa : \Omega \to \Omega_\kappa$$

be a diffeomorphism. Let $A \in L^m_{\mathrm{cl}}(\Omega)$ with symbol $\sigma_A(x,\xi)$ as in (A.7). Let

$$\widetilde{A} = (\kappa^{-1})^* \circ A \circ \kappa^* \in L^m_{\mathrm{cl}}(\Omega_\kappa)$$

with symbol

$$\sigma_{\widetilde{A}} \sim \sum_{j=0}^{\infty} \widetilde{a}_{m-j},$$

where $\widetilde{a}_{m-j}$ is positively homogeneous of degree $m-j$. Let $\rho = (x_0, {}^t\kappa'(x_0)\xi_0)$ be a doubly characteristic point of A. From Taylor's formula, we have

$$a_m(x,\xi) = \sum_{j,k=1}^{2N} a_{j,k}(x,\xi) q_j(x,\xi) q_k(x,\xi) \text{ near } \rho,$$

where $a_{j,k}$, $j,k=1,\dots,2N$, are positively homogeneous C^∞ functions of degree m and q_j, $j=1,\dots,2N$, are positively homogeneous C^∞ functions of degree 0 with $q_j(\rho)=0$, $j=1,\dots,2N$. If $Q_j\in L^0_{\rm cl}(\Omega)$ and $A_{j,k}\in L^m_{\rm cl}(\Omega)$ are properly supported classical pseudodifferential operators with principal symbols q_j and $a_{j,k}$ respectively, then

$$A=\sum_{j,k}A_{j,k}Q_jQ_k+B$$

near ρ, where $B\in L^{m-1}_{\rm cl}(\Omega)$. We denote the principal symbol of B by b. We have

$$\widetilde{A}=\sum_{j,k}\widetilde{A}_{j,k}\widetilde{Q}_j\widetilde{Q}_k+\widetilde{B}$$

near $\widetilde{\rho}=(\kappa(x_0),\xi_0)$, where $\widetilde{A}_{j,k}=(\kappa^{-1})^*\circ A_{j,k}\circ\kappa^*\in L^m_{\rm cl}(\Omega_\kappa)$, $\widetilde{Q}_j=(\kappa^{-1})^*\circ Q_j\circ\kappa^*\in L^m_{\rm cl}(\Omega_\kappa)$, $\widetilde{B}=(\kappa^{-1})^*\circ B\circ\kappa^*\in L^m_{\rm cl}(\Omega_\kappa)$. We denote the principal symbols by $\widetilde{a}_{j,k}$, $\widetilde{q}_j$ and $\widetilde{b}$. From Lemma A.12, we have

$$a^s(\rho)=b(\rho)+(2i)^{-1}\sum_{j,k}a_{j,k}(\rho)\big\{q_j,q_k\big\}(\rho)$$

and

$$\widetilde{a}^s(\widetilde{\rho})=\widetilde{b}(\widetilde{\rho})+(2i)^{-1}\sum_{j,k}\widetilde{a}_{j,k}(\widetilde{\rho})\big\{\widetilde{q}_j,\widetilde{q}_k\big\}(\rho),$$

where a^s and $\widetilde{a}^s$ are the subprincipal symbols of A and $\widetilde{A}$ respectively. In view of (A.9), we have $a^s(\rho)=\widetilde{a}^s(\widetilde{\rho})$. Summing up, we have proved

Proposition A.13. *Let Ω, Ω_κ be open sets in $\mathbb{R}^N$. Let*

$$\kappa:\Omega\to\Omega_\kappa$$

be a diffeomorphism. Let $A\in L^m_{\rm cl}(\Omega)$ with symbol $\sigma_A(x,\xi)$ as in (A.7). Then

$$\widetilde{A}=(\kappa^{-1})^*\circ A\circ\kappa^*\in L^m_{\rm cl}(\Omega_\kappa)$$

with symbol

$$\sigma_{\widetilde{A}}\sim\sum_{j=0}^{\infty}\widetilde{a}_{m-j},$$

where $\widetilde{a}_{m-j}$ is positively homogeneous of degree $m-j$. We have

$$\widetilde{a}_m(\kappa(x),\xi)=a_m(x,{}^t\kappa'(x)\xi).$$

Moreover, if $(x_0,{}^t\kappa'(x_0)\xi_0)$ is a doubly characteristic point of A, then $(\kappa(x_0),\xi_0)$ is also a doubly characteristic point of $\widetilde{A}$ and

$$\widetilde{a}^s(\kappa(x_0),\xi_0)=a^s(x_0,{}^t\kappa'(x_0)\xi_0),$$

where a^s, $\widetilde{a}^s$ denote the subprincipal symbols of A and $\widetilde{A}$ respectively.

Definition A.14. Let $m \in \mathbb{R}$. A pseudodifferential operator of order m on a paracompact C^∞ manifold Ω is a continuous linear map

$$A : C_0^\infty(\Omega) \to C^\infty(\Omega)$$

such that for every local coordinate patch $\Omega_\kappa \subset \Omega$ with coordinates

$$\Omega_\kappa \ni x \to \kappa(x) = (x_1, \ldots, x_N) \in \widetilde{\Omega}_\kappa \subset \mathbb{R}^N,$$

we have

$$(\kappa^{-1})^* \circ A \circ \kappa^* \in L^m(\widetilde{\Omega}_\kappa).$$

We shall write $A \in L^m(\Omega)$ and extend A to a map

$$\mathscr{E}'(\Omega) \to \mathscr{D}'(\Omega).$$

Definition A.15. Let Ω be a paracompact C^∞ manifold of dimension N. Let $m \in \mathbb{R}$. $S^m(T^*(\Omega))$ is the set of all $a \in C^\infty(T^*(\Omega))$ such that pullback to $T^*(\widetilde{\Omega}_\kappa) = \widetilde{\Omega}_\kappa \times \mathbb{R}^N$ is in $S^m(\widetilde{\Omega}_\kappa \times \mathbb{R}^N)$ for every coordinate patch Ω_κ with coordinates $\widetilde{\Omega}_\kappa$.

Definition A.16. Let Ω be a paracompact C^∞ manifold of dimension N. $S^m_{\text{cl}}(T^*(\Omega))$ is the set of all $a \in S^m(T^*(\Omega))$ such that pullback to $T^*(\widetilde{\Omega}_\kappa) = \widetilde{\Omega}_\kappa \times \mathbb{R}^N$ is in $S^m_{\text{cl}}(\widetilde{\Omega}_\kappa \times \mathbb{R}^N)$ for every coordinate patch Ω_κ with coordinates $\widetilde{\Omega}_\kappa$. We call $S^m_{\text{cl}}(T^*(\Omega))$ the space of classical symbols of order m.

Lemma A.17. *Let Ω_1 and Ω_2 be open sets in $\mathbb{R}^N$ and let*

$$\phi : \Omega_1 \to \Omega_2$$

and

$$\Phi : \Omega_1 \to \mathrm{GL}(N, \mathbb{R}) \text{(the group of invertible } N \times N \text{ matrices)}$$

be C^∞ maps. Then

$$a_1(x, \xi) = a_2(\phi(x), \Phi(x)\xi)$$

is in $S^m(\Omega_1 \times \mathbb{R}^N)$ for every $a_2 \in S^m(\Omega_2 \times \mathbb{R}^N)$.

Proof. See Hörmander [14]. □

From Lemma A.17, we see that to check that $a \in S^m(T^*(\Omega))$ it is enough to check the requirement of Definition A.15 for an atlas and the definition agrees with our earlier one if $\Omega \subset \mathbb{R}^N$.

Definition A.18. Let Ω be a paracompact C^∞ manifold of dimension N. If $A \in L^m(\Omega)$ then the restriction of A to Ω_κ identified with $\widetilde{\Omega}_\kappa$ defines a symbol in

$$S^m/S^{-\infty}(\widetilde{\Omega}_\kappa \times \mathbb{R}^N),$$

where Ω_κ is a local coordinate patch with coordinates $\widetilde{\Omega}_\kappa$. If

$$a_\kappa \in S^m(T^*(\Omega_\kappa))$$

is the pullback of a representative then

$$a_\kappa - a_{\kappa'} \in S^{m-1}(T^*(\Omega_\kappa \bigcap \Omega_{\kappa'}))$$

by Proposition A.8 for every pair of coordinate patches. With a locally finite partition of unity $\left\{\psi_j\right\}$ subordinate to a covering by coordinate patches Ω_{κ_j} we set

$$a = \sum \psi_j a_{\kappa_j} \in S^m(T^*(\Omega))$$

and obtain

$$a - a_\kappa \in S^{m-1}(T^*(\Omega_\kappa))$$

for every κ. This determines a modulo $S^{m-1}(T^*(\Omega))$ so we define the principal symbol of A as the image of a in

$$S^m/S^{m-1}(T^*(\Omega)).$$

Let Ω be a paracompact C^∞ manifold. Let $A \in L^m_{\mathrm{cl}}(\Omega)$ with symbol $\sigma_A(x,\xi)$ as in (A.7). We identify the principal symbol of A with a_m.

Definition A.19. Let Ω be a paracompact C^∞ manifold. Let $A \in L^m(\Omega)$. If $a \in S^m(T^*(\Omega))$ is the principal symbol of A then A is said to be non-characteristic at $(x_0,\xi_0) \in T^*(\Omega) \setminus 0$ if

$$ab - 1 \in S^{-1}(T^*(\Omega))$$

in a conic neighborhood of (x_0,ξ_0) for some $b \in S^{-m}(T^*(\Omega))$. We say that $(x,\xi) \in T^*(\Omega) \setminus 0$ is a characteristic point of A if (x,ξ) is not a non-characteristic point of A. Let Σ be the set of characteristic points of A. We call Σ the characteristic set of A. Σ is a closed conic subset of $T^*(\Omega) \setminus 0$.

Let Ω be a paracompact C^∞ manifold. Let $A \in L^m_{\mathrm{cl}}(\Omega)$ with symbol $\sigma_A(x,\xi)$ as in (A.7). The condition of the definition A.19 is equivalent to $a_m(x_0,\xi_0) \neq 0$ and

$$\Sigma = \left\{(x,\xi) \in T^*(\Omega);\ a_m(x,\xi) = 0\right\}.$$

Definition A.20. Let Ω be a paracompact C^∞ manifold. Let $A \in L^m_{\mathrm{cl}}(\Omega)$ and let Σ be the characteristic set of A. We say that $(x,\xi) \in \Sigma$ is a doubly characteristic point of A if for every local coordinate patch Ω_κ, $(x,\xi) \in \Omega_\kappa$, with local coordinates $\widetilde{\Omega}_\kappa$, (x,ξ) is a doubly characteristic point of $\widetilde{A}$, where $\widetilde{A}$ is the pull back of A to $\widetilde{\Omega}_\kappa$. If every $(x,\xi) \in \Sigma$ is a doubly characteristic point, we call Σ doubly characteristic set.

Definition A.21. Let Ω be a paracompact C^∞ manifold and let $A \in L^m_{\mathrm{cl}}(\Omega)$. Let $\rho = (x_0, \xi_0)$ be a doubly characteristic point of A. Let Ω_κ, $\rho \in \Omega_\kappa$, be a local coordinate patch with local coordinates $\widetilde{\Omega}_\kappa$ and let $\widetilde{A}$ be the pull back of A to $\widetilde{\Omega}_\kappa$. The subprincipal symbol of A at ρ is the value $\widetilde{a}^s(\rho)$, where $\widetilde{a}^s$ is the subprincipal symbol of $\widetilde{A}$.

In view of Proposition A.13, we see that the definition above makes sense. We must review some facts about pseudodifferential operators between sections of vector bundles. This will be important in this work.

Definition A.22. Let E and F be complex C^∞ vector bundles over a C^∞ manifold Ω. Let $m \in \mathbb{R}$. Then a pseudodifferential operator of order m from sections of E to sections of F is a continuous linear map

$$A : C_0^\infty(\Omega;\, E) \to C^\infty(\Omega;\, F)$$

such that for every open set $Y \subset \Omega$ where E and F are trivialized by

$$\phi_E : E_Y \to Y \times \mathbb{C}^e,\ \phi_F : F_Y \to Y \times \mathbb{C}^f,$$

there is a $f \times e$ matrix of pseudodifferential operators $A_{j,k} \in L^m(\Omega)$ such that

$$(\phi_F(Au)|_Y)_j = \sum_k A_{j,k}(\phi_E u)_k,\ u \in C_0^\infty(Y;\, E).$$

We shall then write $A \in L^m(\Omega;\, E, F)$.

For every coordinate patch $\Omega_\kappa \subset \Omega$, let

$$v^1_\kappa, v^2_\kappa, \ldots, v^e_\kappa$$

and

$$w^1_\kappa, w^2_\kappa, \ldots, w^f_\kappa$$

be local frames of E and F respectively. Then, for every $\alpha \in C^\infty(\Omega_\kappa;\, E)$ and $a \in C^\infty(\Omega_\kappa;\, F)$, we have

$$\alpha = \sum_j \alpha^j_\kappa(x) v^j_\kappa(x), \quad a = \sum_j a^j_\kappa(x) w^j_\kappa(x),$$

where $\alpha^j_\kappa(x), a^j_\kappa(x) \in C^\infty(\Omega_\kappa)$. If $\Omega_{\kappa'} \bigcap \Omega_\kappa \neq \phi$, we write

$$\alpha = \sum_j \alpha^j_\kappa v^j_\kappa = \sum_j \alpha^j_{\kappa'} v^j_{\kappa'},$$

$$a = \sum_j a^j_\kappa w^j_\kappa = \sum_j a^j_{\kappa'} w^j_{\kappa'}.$$

We have

$$\begin{pmatrix} \alpha^1_{\kappa'}(x) \\ \vdots \\ \alpha^e_{\kappa'}(x) \end{pmatrix} = e_{\kappa',\kappa}(x) \begin{pmatrix} \alpha^1_\kappa(x) \\ \vdots \\ \alpha^e_\kappa(x) \end{pmatrix},$$

$$\begin{pmatrix} a^1_{\kappa'}(x) \\ \vdots \\ a^f_{\kappa'}(x) \end{pmatrix} = f_{\kappa',\kappa}(x) \begin{pmatrix} a^1_\kappa(x) \\ \vdots \\ a^f_\kappa(x) \end{pmatrix},$$

where $e_{\kappa',\kappa}(x)$, $f_{\kappa',\kappa}(x)$ are the transition matrices of E and F respectively.

Let $\mathscr{L}(E,F)$ be the vector bundle over $T^*(\Omega)\setminus 0$ with fiber at (x,ξ) consisting of the linear maps from E_x to F_x. Every $\alpha \in C^\infty(T^*(\Omega); \mathscr{L}(E,F))$ is represented by

$$\begin{aligned} &\alpha(x,\xi) = \left(a^{j,k}_{v_\kappa,w_\kappa}(x,\xi)\right), \\ &\alpha(x,\xi) : E_x \to F_x, \\ &\sum_j s_j v^j_\kappa(x) \to \sum_j t_j w^j_\kappa(x),\ t_j = \sum_k a^{j,k}_{v_\kappa,w_\kappa}(x,\xi) s_k,\ x \in \Omega_\kappa. \end{aligned} \tag{A.11}$$

If $\Omega_\kappa \bigcap \Omega_{\kappa'} \neq \phi$, we have

$$\left(a^{j,k}_{v_{\kappa'},w_{\kappa'}}(x,\xi)\right) = f_{\kappa',\kappa}(x) \circ \left(a^{j,k}_{v_\kappa,w_\kappa}(x,\xi)\right) \circ e_{\kappa,\kappa'}(x). \tag{A.12}$$

We have the following

Definition A.23. Let E and F be complex C^∞ vector bundles over a C^∞ manifold Ω. Let $m \in \mathbb{R}$. $S^m(T^*(\Omega); \mathscr{L}(E,F))$ is the set of all

$$\alpha \in C^\infty(T^*(\Omega); \mathscr{L}(E,F))$$

such that if we write $\alpha = (a^{j,k}_{v_\kappa w_\kappa}(x,\xi))$ as in (A.11), then

$$a^{j,k}_{v_\kappa,w_\kappa}(x,\xi) \in S^m(T^*(\Omega_\kappa)) \text{ for every } j,k.$$

Definition A.24. Let E and F be complex C^∞ vector bundles over a C^∞ manifold Ω and let $a(x,\xi) \in S^m(T^*(\Omega); \mathscr{L}(E,F))$. We say that $a(x,\xi)$ is a classical symbol, if we write $a(x,\xi) = \left(a^{j,k}_{v_\kappa,w_\kappa}(x,\xi)\right)$ as in (A.11), then $a^{j,k}_{v_\kappa,w_\kappa}$ is a classical symbol, for every j,k. We shall write $S^m_{\rm cl}(T^*(\Omega); \mathscr{L}(E,F))$ to denote the space of all classical symbols.

Let E and F be complex C^∞ vector bundles over a C^∞ manifold Ω and let $A \in L^m(\Omega; E, F)$. For every coordinate patch Ω_κ, we write

$$\begin{aligned}
&A = (A^{j,k}_{v_\kappa, w_\kappa}), \quad A^{j,k}_{v_\kappa, w_\kappa} \in L^m(\Omega),\\
&A : C_0^\infty(\Omega_\kappa; E) \to C^\infty(\Omega_\kappa; F),\\
&\sum_j s_j(x) v^j_\kappa(x) \mapsto \sum_j t_j(x) w^j_\kappa(x), \ t_j = \sum_k A^{j,k}_{v_\kappa, w_\kappa} s_k.
\end{aligned} \tag{A.13}$$

We define the symbol of A as

$$\sigma_A(x,\xi) = \left(\sigma_{A^{j,k}_{v_\kappa, w_\kappa}}(x,\xi)\right).$$

If $\Omega_{\kappa'} \bigcap \Omega_\kappa \neq \phi$, we have

$$\left(A^{j,k}_{v_{\kappa'}, w_{\kappa'}}\right) = f_{\kappa',\kappa}(x) \circ \left(A^{j,k}_{v_\kappa, w_\kappa}\right) \circ e_{\kappa,\kappa'}(x). \tag{A.14}$$

From (A.14), we have

$$\left(\sigma_{A^{j,k}_{v_{\kappa'}, w_{\kappa'}}}(x,\xi)\right) - f_{\kappa',\kappa}(x)\left(\sigma_{A^{j,k}_{v_\kappa, w_\kappa}}(x,\xi)\right) e_{\kappa,\kappa'}(x) \in S^{m-1}(T^*(\Omega_{\kappa'} \bigcap \Omega_\kappa); \mathscr{L}(E,F)). \tag{A.15}$$

We define the principal symbol of A at $(x,\xi) \in T^*(\Omega_\kappa)$ as the image of $\sigma_A(x,\xi)$ in

$$(S^m/S^{m-1})(T^*(\Omega_\kappa); \mathscr{L}(E,F)).$$

From (A.15), we see that the principal symbol of A is well-defined as an element in

$$(S^m/S^{m-1})(T^*(\Omega); \mathscr{L}(E,F)).$$

We write $L^m_{\rm cl}(\Omega; E, F)$ to denote the space of pseudodifferential operators of order m from sections of E to sections of F with $\sigma_A \in S^m_{\rm cl}(T^*(\Omega); \mathscr{L}(E,F))$.

Definition A.25. Let E and F be complex C^∞ vector bundles over a C^∞ manifold Ω and let $A \in L^m_{\rm cl}(\Omega; E, F)$. For every coordinate Ω_κ, we write $A = (A^{j,k}_{v_\kappa, w_\kappa})$, $A^{j,k}_{v_\kappa, w_\kappa} \in L^m(\Omega_\kappa)$, as in (A.13). We say that $(x_0, \xi_0) \in T^*(\Omega_\kappa) \setminus 0$ is a doubly characteristic point of A if (x_0,ξ_0) is a doubly characteristic point of $A^{j,k}_{v_\kappa, w_\kappa}$ for every j, k.

Definition A.26. Let E and F be complex C^∞ vector bundles over a C^∞ manifold Ω and let $A \in L^m_{\rm cl}(\Omega; E, F)$. Let (x_0, ξ_0) be a doubly characteristic point of A. For every coordinate Ω_κ, $(x_0,\xi_0) \in T^*(\Omega_\kappa) \setminus 0$, we write $A = (A^{j,k}_{v_\kappa, w_\kappa})$, $A^{j,k}_{v_\kappa, w_\kappa} \in L^m(\Omega_\kappa)$, as in (A.13). Let $a^{s,j,k}_{v_\kappa, w_\kappa}(x_0,\xi_0)$ be the subprincipal of $A_{j,k}$ at (x_0,ξ_0), for every j, k. From (A.14) and Lemma A.12 , we have

$$\left(a^{s,j,k}_{v_{\kappa'}, w_{\kappa'}}(x_0,\xi_0)\right) = f_{\kappa',\kappa}(x_0)\left(a^{s,j,k}_{v_\kappa, w_\kappa}(x_0,\xi_0)\right) e_{\kappa,\kappa'}(x_0) \tag{A.16}$$

on $T^*(\Omega_\kappa\bigcap\Omega_{\kappa'})$, where $e_{\kappa',\kappa}(x)$, $f_{\kappa',\kappa}(x)$ are the transition matrices of E and F respectively. We define the subprincipal symbol of A at (x_0,ξ_0) as

$$\left(a^{s,j,k}_{v_\kappa,w_\kappa}(x_0,\xi_0)\right)\in\mathscr{L}(E_{x_0},F_{x_0}).$$

From (A.16), we see that the subprincipal of A is invariantly defined at every doubly characteristic point of A.

We must make some comments on symplectic geometry. This will also be important in this work. Our presentation is essentially taken from Duistermaat [10], Hörmander [15] and Sjöstrand [21].

First, we review some facts about symplectic vector spaces.

Definition A.27. An antisymmetric nondegenerate bilinear form on a finite dimensional vector space E is called a symplectic form on E. A symplectic vector space is a pair (E,σ) consisting of a finite dimensional vector space E and a symplectic form σ on E.

σ is nondegenerate means that

$$\sigma(v,v')=0,\ \forall v'\in E\Rightarrow v=0.$$

Definition A.28. If (E_1,σ_1), (E_2,σ_2) are symplectic vector spaces and

$$T:E_1\to E_2$$

is a linear bijection with $T^*\sigma_2=\sigma_1$, that is

$$\sigma_1(v,w)=\sigma_2(Tv,Tw),\ v,w\in E_1,$$

then T is called a symplectic isomorphism.

In the vector space $T^*(\mathbb{R}^n)=\{(x,\xi);\,x,\xi\in\mathbb{R}^n\}$ the symplectic form

$$\sigma=\sum d\xi_j\wedge dx_j$$

is the bilinear form

$$\sigma((x,\xi),(x',\xi'))=\langle x',\xi\rangle-\langle x,\xi'\rangle.$$

If e_j and ε_j are the unit vectors along the x_j and ξ_j axes respectively, then we have for $j,k=1,\ldots,n$

$$\sigma(e_j,e_k)=\sigma(\varepsilon_j,\varepsilon_k)=0,$$
$$\sigma(\varepsilon_j,e_k)=-\sigma(e_k,\varepsilon_j)=\delta_{jk}$$

where $\delta_{jk} = 1$ when $j = k$ and $\delta_{jk} = 0$ when $j \neq k$.

Let (E, σ) be a symplectic vector space. Note that the linear bijection mapping

$$A : E \to T^*(\mathbb{R}^n)$$

sending the vectors $(e_1, \dots, e_n, f_1, \dots, f_n)$ into the standard basis of $T^*(\mathbb{R}^n)$ is a symplectic isomorphism if and only if for all $j, k = 1, \dots, n$

$$\begin{cases} \sigma(e_j, e_k) = 0, \\ \sigma(f_j, f_k) = 0, \\ \sigma(f_j, e_k) = \delta_{j,k}. \end{cases} \tag{A.17}$$

$(e_1, \dots, e_n, f_1, \dots, f_n)$ is called symplectic coordinates of (E, σ)

Lemma A.29. *Every symplectic vector space (E, σ) admits a linear symplectic isomorphism*

$$T : E \to T^*(\mathbb{R}^n).$$

Proof. See Duistermaat [10] or Hörmander [15]. □

If L is a linear subspace of E then we define its orthocomplement L^σ in E with respect to σ by

$$L^\sigma = \{e \in E; \sigma(e, l) = 0 \text{ for all } l \in L\}.$$

We have the following rules

$$\begin{aligned} & L \subset M \Rightarrow M^\sigma \subset L^\sigma, \\ & (L^\sigma)^\sigma = L, \\ & (L \bigcap M)^\sigma = L^\sigma + M^\sigma, \ (L + M)^\sigma = L^\sigma \bigcap M^\sigma, \\ & \dim L^\sigma = \dim E - \dim L. \end{aligned}$$

Definition A.30. A linear subspace L of E is called isotropic, Lagrangian resp. involutive, if $L \subset L^\sigma$, $L = L^\sigma$ and $L \supset L^\sigma$ respectively.

Definition A.31. A symplectic form on a manifold Ω is a 2-form σ on Ω such that

$$d\sigma = 0$$

and σ_ρ is a symplectic form on $T_\rho(\Omega)$, for each $\rho \in \Omega$. A symplectic manifold is a pair (Ω, σ) consisting of a manifold Ω and a symplectic form σ on Ω.

Definition A.32. A submanifold V of a symplectic manifold is called symplectic, isotropic, Lagrangian, involutive respectively, if at every point of V the tangent space of V has this property.

Suppose (M,σ), (N,τ) are symplectic manifolds. Let

$$\Phi : M \to N$$

be a C^∞-map. Let $D\Phi$ be the differential map, that is,

$$\begin{aligned} D\Phi_\rho : T_\rho(M) &\to T_\rho(N) \\ v &\to \frac{d}{dt}(\Phi\circ\alpha(t))\Big|_{t=0}, \end{aligned}$$

where $\alpha(t)$ is a C^1-curve with $\alpha(0)=\rho$ and $\alpha'(0)=v$.

Definition A.33. If (M,σ), (N,τ) are symplectic manifolds and

$$\Phi : M \to N$$

is a diffeomorphism with $\Phi^*\tau = \sigma$, that is, $D\Phi_\rho$ is a symplectic isomorphism from $(T_\rho(M),\sigma_\rho)$ to $(T_{\Phi(\rho)}(N),\tau_{\Phi(\rho)})$ for all $\rho\in M$, then Φ is called a canonical transformation.

Definition A.34. Let (M,σ) be a symplectic manifold. For any $f\in C^\infty(M)$ the Hamilton field H_f is the unique C^∞ vector field on M such that $H_f \lrcorner \sigma = -df$. We notice that $H_f \lrcorner \sigma$ is a 1-form defined by

$$(H_f \lrcorner \sigma)_\rho(e_\rho) = \sigma_\rho(H_f(\rho), e_\rho),$$

where $e_\rho \in T_\rho(M)$.

Definition A.35. Let (M,σ) be a symplectic manifold. If $f,g\in C^\infty(M)$ then the Poisson brackets $\{f,g\}\in C^\infty(M)$ are defined by

$$\{f,g\} = H_f g = \sigma(H_f, H_g).$$

Remark A.36. In symplectic coordinates (x,ξ), we have

$$H_f = \sum_j \Big(\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_j}\Big).$$

Let Ω be a C^∞ manifold. Let

$$\pi : T^*(\Omega) \to \Omega$$

be the natural projection map.

Definition A.37. The canonical 1-form on $T^*(\Omega)$ is the 1-form α given by

$$\alpha(x,\xi) = \xi \circ D\pi_{(x,\xi)} \text{ for all } (x,\xi) \in T^*(\Omega).$$

Definition A.38. $\sigma = d\sigma$ is called the canonical 2-form on $T^*(\Omega)$.

Remark A.39. (a) In canonical coordinates (x,ξ) we have

$$\alpha = \sum \xi_j dx_j$$

and hence

$$\sigma = \sum d\xi_j \wedge dx_j.$$

(b) $T^*(\Omega)$ with canonical 2-form σ is a symplectic manifold.

Let (M,σ) be a symplectic manifold. If

$$\Phi : M \to T^*(\mathbb{R}^n)$$

is a canonical transformation, then Φ is called a canonical coordinatization. It is well-known that the functions $x_1,\ldots,x_n,\xi_1,\ldots,\xi_n$ form local canonical coordinates of (M,σ) if and only if for all $j,k = 1,\ldots,n$:

$$\left\{x_j, x_k\right\} = 0,\ \left\{\xi_j,\xi_k\right\} = 0,\ \left\{\xi_j, x_k\right\} = \delta_{jk}.$$

We call $\{x_1,\ldots,x_n,\xi_1,\ldots,\xi_n\}$ the symplectic coordinates of (M,σ).

Proposition A.40. *Suppose* (M,σ) *is a symplectic manifold. Then* $\dim M$ *is even, say* $2n$*, and for each* $m_0 \in M$ *there is a canonical coordinates of a neighborhood* U *of* m_0.

Proof. See Duistermaat [10] or Hörmander [15]. □

Let (S,σ) be a symplectic vector space. Let $Q(x,\xi)$ be a real positive semi-definite quadratic form in S. Let

$$Q(X,Y) = \langle X, Q''Y \rangle$$

be the corresponding symmetric form. Here Q'' is the Hessian of $Q(x,\xi)$. We have the following definition

Definition A.41. The linear map F in S is defined by

$$\sigma(Y,FX) = Q(Y,X),\ X,Y \in S$$

will be called the fundamental matrix of Q.

The symmetry of Q means that

$$\sigma(FX,Y) = -\sigma(X,FY),$$

that is, F is skew symmetric with respect to σ. We notice that the preceding definition is still applicable if Q is complex valued provied we replace S by its complexification

$$S_{\mathbb{C}} = \{X + iY;\, X,Y \in S\}$$

with the obvious complex symplectic structure.

Proposition A.42. *Let $Q(x,\xi)$ be a real positive semi-definite quadratic form in S. Let*

$$Q(X,Y) = \langle X, Q''Y\rangle$$

be the corresponding symmetric form. Here Q'' is the Hessian of $Q(x,\xi)$. Then one can choose symplectic coordinates x,ξ such that

$$Q(x,\xi) = \sum_{j=1}^{k} \frac{1}{2}\mu_j(x_j^2 + \xi_j^2) + \sum_{k+1}^{k+l} x_j^2$$

where $\mu_j > 0$.

Proof. See Hörmander [15]. □

From Proposition A.42, we see that $\pm i\mu_j$ are the non-vanishing eigenvalues of the fundamental matrix F of Q. We write $\widetilde{\operatorname{tr}}\, F = \sum_{j=1}^{k} \left|\mu_j\right|$.

Let Ω be a C^∞ manifold of dimension n. Let $p \in \Omega$ and let $f \in C^2(\Omega)$ with $df(p) = 0$. In local coordinates $(x_1, \ldots, x_n)$, we define

$$f''_p = \left(\frac{\partial^2 f}{\partial x_j \partial x_k}(p)\right)_{j,k=1}^{n}.$$

Since $df(p) = 0$, f''_p is well-defined as a linear map

$$f''_p : T_p(\Omega) \to \mathbb{C},$$
$$t, s \to \left\langle t, f''_p s\right\rangle,$$

where $t,\, s \in T_p(\Omega)$.

If (S,σ) is a symplectic manifold and $f \in C^2(S)$ is real valued, $df = 0$ at $\rho \in S$. Let F_ρ be the fundamental matrix of f''_ρ. We also call F_ρ the fundamental matrix of f. In symplectic coordinates (x,ξ) the matrix F becomes

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x \partial \xi} & \frac{\partial^2 f}{\partial \xi \partial \xi} \\ -\frac{\partial^2 f}{\partial x \partial x} & -\frac{\partial^2 f}{\partial \xi \partial x} \end{pmatrix}.$$

From this we see that the fundamental matrix F is the linearization of H_f.

Remark A.43. The symplectic reduction of quadratic forms is an old topic in mechanics. It is also important for the study of hypoelliptic operators. See Boutet de Monvel [7] and Sjöstrand [21].

B Appendix: Almost analytic manifolds, functions and vector fields

We will give a brief discussion of almost analytic manifolds, functions and vector fields in a setting appropriate for our purpose. For more details on the subject, see Melin-Sjöstrand [18].

Let $W \subset \mathbb{C}^n$ be an open set and let $f \in C^\infty(W)$. In this section, we will use the following notations:

$$\overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j$$

and

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j.$$

Definition B.1. Let $W \subset \mathbb{C}^n$ be an open set and let $\phi(z)$ be a positive continuous function on W. If $f \in C^\infty(W)$, we say that f is almost analytic with respect to the weight function ϕ if, given any compact subset K of W and any integer $N \geq 0$, there is a constant $c > 0$ such that

$$\left|\overline{\partial} f(z)\right| \leq c\phi(z)^N, \ \forall z \in K.$$

When $\phi(z) = |\mathrm{Im}\, z|$ we simply say that f is almost analytic.

Definition B.2. Let $f_1, f_2 \in C^\infty(W)$ with W, ϕ as above. We say that f_1 and f_2 are equivalent with respect to the weight function ϕ if, given any compact subset K of W and any integer $N > 0$, there is a constant $c > 0$ such that

$$\left|(f_1 - f_2)(z)\right| \leq c\phi(z)^N, \ \forall z \in K.$$

When $\phi(z) = |\mathrm{Im}\, z|$ we simply say that they are equivalent and we write

$$f_1 \sim f_2.$$

The following proposition is due to Hörmander. For a proof, see [18].

Proposition B.3. *Let $W \subset \mathbb{C}^n$ be an open set and let $W_{\mathbb{R}} = W \bigcap \mathbb{R}^n$. If $f \in C^\infty(W_{\mathbb{R}})$ then f has an almost analytic extension, uniquely determined up to equivalence.*

Definition B.4. Let U be an open subset of $\mathbb{C}^n$ and let Λ be a C^∞ submanifold of codimension $2k$ of U. We say that Λ is an almost analytic manifold if for every point z_0 of $\Lambda\bigcap\mathbb{R}^n$ there exist an open neighborhood V of z_0 in U and k complex C^∞ almost analytic functions $f_1,\dots,f_k$ defined on V such that

$\Lambda\bigcap V$ is defined by the equations $f_1(z)=\cdots=f_k(z)=0$,
$\partial f_1,\dots,\partial f_k$ are linearly independent over $\mathbb{C}$ at every point of V.

Definition B.5. Let Λ_1 and Λ_2 be two C^∞ closed submanifolds of an open set $U\subset\mathbb{C}^n$. We say that Λ_1 and Λ_2 are equivalent (and we write $\Lambda_1\sim\Lambda_2$) if they have the same intersection with $\mathbb{R}^n$ and the same dimension and if for every open set $V\subset\subset U$ and $N\in\mathbb{N}$ we have

$$\operatorname{dist}(z,\Lambda_2)\le c_{N,V}\,|\operatorname{Im}z|^N,\ z\in V\bigcap\Lambda_1,\ c_{N,V}>0.$$

It is trivial that $\sim$ is an equivalence relation and that Λ_1 and Λ_2 are tangential to infinite order in the real points when $\Lambda_1\sim\Lambda_2$. We recall the Malgrange preparation theorem (see Theorem 7.57 in Hörmander [17])

Theorem B.6. *Let $f_j(t,x)$, $j=1,\dots,n$, be complex valued C^∞ functions in a neighborhood of $(0,0)$ in $\mathbb{R}^{n+m}$ with $f_j(0,0)=0$, $j=1,\dots,n$, and*

$$\det\left(\frac{\partial f_j(0,0)}{\partial t_k}\right)_{j,k=1}^n\neq 0.$$

If $g\in C^\infty$ in a neighborhood of $(0,0)$ we can then find $q_j(t,x)\in C^\infty$ in a neighborhood of $(0,0)$, $j=1,\dots,n$, and $r(x)\in C^\infty$ in a neighborhood of 0 so that

$$g(t,x)=\sum_{j=1}^n q_j(t,x)f_j(t,x)+r(x)$$

in a neighborhood of $(0,0)$.

In section 8, we need the following

Proposition B.7. *Let W be an open neighborhood of the origin in $\mathbb{C}^{n+m}$. Let $f_j(z,w)$, $g_j(z,w)$, $j=1,\dots,n$, be almost analytic functions on W with $f_j(0,0)=0$, $g_j(0,0)=0$, $j=1,\dots,n$,*

$$\det\left(\frac{\partial f_j(0,0)}{\partial z_k}\right)_{j,k=1}^n\neq 0$$

and $\partial g_1,\dots,\partial g_n$ are linearly independent over $\mathbb{C}$ at the origin. Let

$$\Lambda_1=\{(z,w)\in W;\ f_1(z,w)=\ldots=f_n(z,w)=0\}$$

and

$$\Lambda_2=\{(z,w)\in W;\ g_1(z,w)=\ldots=g_n(z,w)=0\}.$$

If Λ_1 *coincides to infinite order with* Λ_2 *at* $(0,0)$ *we can then find* $a_{j,k}(z,w)\in C^\infty$ *in a neighborhood of* $(0,0)$, $j,k=1,\ldots,n$, *with* $\det\left(a_{j,k}(0,0)\right)_{j,k=1}^n\neq 0$ *so that*

$$g_j(z,w)-\sum_{k=1}^n a_{j,k}(z,w)f_k(z,w)$$

vanishes to infinite order at $(0,0)$, *for all* j.

Proof. We write $z=x+iy$, $w=u+iv$, where $x,\ y\in\mathbb{R}^n,\ u,\ v\in\mathbb{R}^m$. From Theorem B.6, it follows that

$$g_j(x,u)=\sum_{k=1}^n a_{j,k}(x,u)f_k(x,u)+r(u),\ \ j=1,\ldots,n,$$

in a real neighborhood of $(0,0)$, where $a_{j,k}(x,u)\in C^\infty$ in a real neighborhood of $(0,0)$, $j,k=1,\ldots,n$. Since Λ_1 coincides to infinite order with Λ_2 at $(0,0)$, it follows that $r(u)$ vanishes to infinite order at 0. Since $f_k(0,0)=0$, $k=1,\ldots,n$, we have

$$dg_j(0,0)=\sum_{k=1}^n a_{j,k}(0,0)df_k(0,0),\ \ j=1,\ldots,n.$$

Hence

$$\det\left(a_{j,k}(0,0)\right)_{j,k=1}^n\neq 0.$$

Let $a_{j,k}(z,w)$ be an almost analytic extension of $a_{j,k}(x,u)$ to a complex neighborhood of $(0,0)$, where $j,\ k=1,\ldots,n$. Then

$$g_j(z,w)-\sum_{k=1}^n a_{j,k}(z,w)f_k(z,w)$$

also vanishes to infinite order at $(0,0)$, for all j. □

We need the following

Lemma B.8. *Let* Ω, Ω_κ *be open sets in* $\mathbb{R}^n$. *Let*

$$\kappa:\Omega\to\Omega_\kappa$$

be a diffeomorphism. Let $\Omega^{\mathbb{C}}$ *and* $\Omega_\kappa^{\mathbb{C}}$ *be open sets in* $\mathbb{C}^n$ *with*

$$\Omega^{\mathbb{C}}\bigcap\mathbb{R}^n=\Omega,\ \Omega_\kappa^{\mathbb{C}}\bigcap\mathbb{R}^n=\Omega_\kappa.$$

Let

$$\widetilde{\kappa}:\Omega^{\mathbb{C}}\to\Omega_{\kappa}^{\mathbb{C}}$$

be an almost analytic extension of κ. *We take* $\Omega^{\mathbb{C}}$ *and* $\Omega_{\kappa}^{\mathbb{C}}$ *so that* $\widetilde{\kappa}$ *is a diffeomorphism. If* Λ *is an almost analytic manifold of* $\Omega^{\mathbb{C}}$, *then* $\widetilde{\kappa}(\Lambda)$ *is an almost analytic manifold of* $\Omega_{\kappa}^{\mathbb{C}}$. *Moreover, if*

$$\hat{\kappa}:\Omega^{\mathbb{C}}\to\Omega_{\kappa}^{\mathbb{C}}$$

is another almost analytic extension of κ *and* $\hat{\kappa}$ *is a diffeomorphism, then*

$$\hat{\kappa}(\Lambda)\sim\widetilde{\kappa}(\Lambda).$$

Furthermore, if $\Lambda_1\sim\Lambda_2$, *then*

$$\widetilde{\kappa}(\Lambda_1)\sim\widetilde{\kappa}(\Lambda_2),$$

where Λ_1 *and* Λ_2 *are almost analytic manifolds of* $\Omega^{\mathbb{C}}$.

We shall now generalize the notion of almost analytic manifolds. We have the following

Definition B.9. Let X be a n dimensional real paracompact C^{∞} manifold. An almost analytic manifold Λ associated to X is given by

(a) A locally closed set $\Lambda_{\mathbb{R}}$. (Locally closed means that every point of $\Lambda_{\mathbb{R}}$ has an neighborhood ω in X such that $\Lambda_{\mathbb{R}}\bigcap\omega$ is closed in ω.)

(b) A covering of $\Lambda_{\mathbb{R}}$ by open coordinate patches

$$\kappa_\alpha:X\supset X_\alpha\to\Omega_\alpha\subset\mathbb{R}^n,\ \alpha\in J$$

and almost analytic manifolds $\Lambda_\alpha\subset\Omega_\alpha^{\mathbb{C}}$ with

$$\Lambda_\alpha\bigcap\mathbb{R}^n:=\Lambda_{\alpha\mathbb{R}}=\kappa_\alpha(X_\alpha\bigcap\Lambda_{\mathbb{R}}).$$

Here $\Omega_\alpha^{\mathbb{C}}\subset\mathbb{C}^n$ is some open set with $\Omega_\alpha^{\mathbb{C}}\bigcap\mathbb{R}^n=\Omega_\alpha$ and the Λ_α shall satisfy the following compatibility conditions: If

$$\kappa_{\beta\alpha}=\kappa_\beta\circ\kappa_\alpha^{-1}:\kappa_\alpha(X_\alpha\bigcap X_\beta)\to\kappa_\beta(X_\alpha\bigcap X_\beta)$$

and if $\widetilde{\kappa}_{\beta\alpha}$ is an almost analytic extension of $\kappa_{\beta\alpha}$, then $\widetilde{\kappa}_{\beta\alpha}(\Lambda_\alpha)$ and Λ_β are equivalent near all points of $\kappa_\beta(X_\alpha\bigcap X_\alpha\bigcap\Lambda_{\mathbb{R}})$.

The Λ_α are called local representatives of Λ and we shall say that two almost analytic manifolds Λ, Λ' associated to X are equivalent (and we write $\Lambda\sim\Lambda'$) if $\Lambda_{\mathbb{R}}=\Lambda'_{\mathbb{R}}$ and if the corresponding local representatives are equivalent as in (b).

Similarly we extend the notion of almost analytic functions and equivalence of almost analytic functions.

Definition B.10. Let W be an open subset of $\mathbb{C}^n$ and let V be a complex C^∞ vector field on W. We say that V is almost analytic if $V(f)$ is almost analytic and $V(\overline{f}) \sim 0$ for all almost analytic functions f on W.

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. We shall denote the real coordinates by x_j, y_j, $j = 1,\dots,n$, and the complex coordinates by $z_j = x_j + iy_j$, $j = 1,\dots,n$.

Definition B.11. Let W be an open subset of $\mathbb{C}^n$ and let

$$U = \sum_{j=1}^{n} a_j(z)\frac{\partial}{\partial z_j} + \sum_{j=1}^{n} b_j(z)\frac{\partial}{\partial \overline{z}_j},$$

$$V = \sum_{j=1}^{n} c_j(z)\frac{\partial}{\partial z_j} + \sum_{j=1}^{n} d_j(z)\frac{\partial}{\partial \overline{z}_j},$$

be complex C^∞ vector fields on W, where

$$a_j(z), b_j(z), c_j(z), d_j(z) \in C^\infty(W),\ j = 1,\dots,n.$$

We say that U and V are equivalent if

$$a_j(z) - c_j(z) \sim 0,\ b_j(z) - d_j(z) \sim 0$$

for all j. If U and V are equivalent, we write

$$U \sim V.$$

Clearly U is almost analytic if and only if

$$U \sim \sum_{j=1}^{n} a_j(z)\frac{\partial}{\partial z_j},$$

where a_j, $j = 1,\dots,n$, are almost analytic. We have the following easy lemma

Lemma B.12. *Let W be an open subset of $\mathbb{C}^n$ and let V be an almost analytic vector field on W. Then*

$$V(f) \sim (V + \overline{V})(f) \ \text{for all almost analytic functions } f. \tag{B.1}$$

If U is a real vector field on W and

$$U(f) \sim V(f) \ \text{for all almost analytic functions } f,$$

then

$$U \sim V + \overline{V}.$$

We have the following

Proposition B.13. *Let W be an open subset of $\mathbb{C}^n$ and let Σ be a C^∞ closed submanifold of $\mathbb{R}^n$. Let V be an almost analytic vector field on W. We assume that*

$$V = 0 \ \text{on}\, \Sigma.$$

Let $\Phi(t,\rho)$ be the $V+\overline{V}$ flow. Let U be a real vector field on W such that

$$U \sim V + \overline{V}.$$

Let $\hat{\Phi}(t,\rho)$ be the U flow. Then, for every compact set $K \subset W$, $N \geq 0$, there is a $c_{N,K}(t) > 0$, such that

$$\left|\Phi(t,\rho) - \hat{\Phi}(t,\rho)\right| \leq c_{N,K}(t)\text{dist}(\rho,\Sigma)^N,\ \rho \in K.$$

Proof. We have the following well-known fact: If Z is a smooth vector field on an open set $\Omega \subset \mathbb{R}^n$ with $Z(x_0) = 0$ and $\Psi(t,x) = \exp(tZ)(x)$, then $\Psi(t,x_0) = x_0$ and $\partial_x^\alpha \Psi(t,x)|_{x=x_0}$, $\alpha \in \mathbb{N}^n$, only depend on $(\partial_x^\beta Z)(x_0)$, $\beta \in \mathbb{N}^n$. In our situation, we therefore have that $\Phi(t,\rho)$, $\hat{\Phi}(t,\rho)$ have the same Taylor expansion at every point of Σ. □

The following proposition is useful (see section 2 of Melin-Sjöstrand [18])

Proposition B.14. *Assume that $f(x,w)$ is a C^∞ complex function in a neighborhood of $(0,0)$ in $\mathbb{R}^{n+m}$ and that*

$$\text{Im}\, f \geq 0,\ \ \text{Im}\, f(0,0) = 0,\ \ f'_x(0,0) = 0,\ \ \det f''_{xx}(0,0) \neq 0. \tag{B.2}$$

Let $\widetilde{f}(z,w)$, $z = x+iy$, $w \in \mathbb{C}^m$, denote an almost analytic extension of f to a complex neighborhood of $(0,0)$ and let $z(w)$ denote the solution of

$$\frac{\partial \widetilde{f}}{\partial z}(z(w),w) = 0$$

in a neighborhood of 0 in $\mathbb{C}^m$. Then,

$$\frac{\partial}{\partial w}(\widetilde{f}(z(w),w)) - \frac{\partial}{\partial w}\widetilde{f}(z,w)|_{z=z(w)},\ \ w \text{ is real}$$
$$\text{vanishes to infinite order at } 0 \in \mathbb{R}^m. \tag{B.3}$$

Moreover, there is a constant $c > 0$ such that near the origin we have

$$\text{Im}\,\widetilde{f}(z(w),w) \geq c\,|\text{Im}\, z(w)|^2,\ w \in \mathbb{R}^m \tag{B.4}$$

and

$$\text{Im}\,\widetilde{f}(z(w),w) \geq c \inf_{x\in\Omega}\Big(\text{Im}\, f(x,w) + \left|d_x f(x,w)\right|^2\Big),\ w \in \mathbb{R}^m, \tag{B.5}$$

where Ω is some open set of the origin in $\mathbb{R}^n$.

We call $\widetilde{f}(z(w),w)$ the corresponding critical value.

Proof. For a proof of (B.4), see [18]. We only prove (B.5). In view of the proof of (B.4) (see p.147 of [18]), we see that

$$\operatorname{Im}\widetilde{f}(z(w),w) \geq c\Big(\inf_{t\in\mathbb{R}^n,|t|\leq 1}\operatorname{Im} f(\operatorname{Re} z(w) - t\,|\operatorname{Im} z(w)|, w) + |\operatorname{Im} z(w)|^2\Big) \tag{B.6}$$

for w small, where w is real and c is a positive constant. Using the almost analyticity, we get by Taylor's formula:

$$f'_x(x,w) = f''_{zz}(z(w),w)(x - z(w)) + O(|x - z(w)|^2 + |\operatorname{Im} z(w)|^2) \tag{B.7}$$

for x, w small, where w is real. Since f''_{zz} is invertible near the origin, we have that when $w \in \mathbb{R}^m$ is close to the origin,

$$|\operatorname{Im} z(w)|^2 \geq c\left|f'_x(\operatorname{Re} z(w) - t\,|\operatorname{Im} z(w)|, w)\right|^2$$

for all $t \in \mathbb{R}^n$, $|t| \leq 1$, where c is a positive constant. From this and (B.6), we get (B.5). □

In the following, we let $z = z(w)$ be the point defined as above. We recall the stationary phase formula of Melin and Sjöstrand

Proposition B.15. *Let $f(x,w)$ be as in Proposition B.14. Then there are neighborhoods U and V of the origin in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and differential operators $C_{f,j}$ in x of order $\leq 2j$ which are C^∞ functions of $w \in V$ such that*

$$\left|\int e^{itf(x,w)}u(x,w)dx - \Big(\det\Big(\frac{t\widetilde{f}''_{zz}(z(w),w)}{2\pi i}\Big)\Big)^{-\frac{1}{2}} e^{it\widetilde{f}(z(w),w)}\sum_0^{N-1}(C_{f,j}\widetilde{u})(z(w),w)t^{-j}\right| \leq c_N t^{-N-\frac{n}{2}}, \quad t \geq 1, \tag{B.8}$$

where $u \in C_0^\infty(U \times V)$. Here $\widetilde{f}$ and $\widetilde{u}$ are almost analytic extensions of f and u respectively. The function

$$\Big(\det\Big(\frac{t\widetilde{f}''_{zz}(z(w),w)}{2\pi i}\Big)\Big)^{-\frac{1}{2}}$$

is the branch of the square root of

$$\Big(\det\Big(\frac{t\widetilde{f}''_{zz}(z(w),w)}{2\pi i}\Big)\Big)^{-1}$$

which is continuously deformed into 1 *under the homotopy*

$$s \in [0,1] \to i^{-1}(1-s)\widetilde{f}''_{zz}(z(w),w) + sI \in \mathrm{GL}(n,\mathbb{C}).$$

We need the following asymptotic formula (see Melin-Sjöstrand [18])

Proposition B.16. *Let* $P \in L^m(\mathbb{R}^n)$ *and let* $\varphi(x) \in C^\infty(\mathbb{R}^n)$ *satisfy* $\operatorname{Im}\varphi \geq 0$ *and* $d\varphi \neq 0$ *where* $\operatorname{Im}\varphi = 0$. *Let* $u(x) \in C_0^\infty(\mathbb{R}^n)$. *If* $p(x,\xi)$ *is the full symbol of* P, *then*

$$P(e^{it\varphi(x)}u(x)) \sim e^{it\varphi(x)} \sum_\alpha \frac{1}{\alpha!} \widetilde{p}^{(\alpha)}(x, t\varphi_x'(x)) \frac{1}{\alpha!} D_y^\alpha(u(y)e^{it\rho(x,y)})|_{y=x} \tag{B.9}$$

with asymptotic convergence in $S_{0,1}^m(\mathbb{R}^n \times \mathbb{R}_+)$, *where* $\widetilde{p}$ *is an almost analytic extension of* p *and*

$$\rho(x,y) = \varphi(y) - \varphi(x) - \left\langle y-x, \varphi_x'(x)\right\rangle.$$

Definition B.17. The C^∞ function $\varphi(x,\theta)$ defined in an open conic set $V \subset \mathbb{R}^n \times \mathbb{R}^N \setminus 0$ is called a non-degenerate complex phase function if

(a) $d\varphi \neq 0$.

(b) $\varphi(x,\theta)$ is positively homogeneous of degree 1.

(c) Put
$$C = \left\{(x,\theta) \in V;\ \varphi_\theta'(x,\theta) = 0\right\}.$$
The differentials $d(\frac{\partial\varphi}{\partial\theta_j})$, $j = 1, \ldots, N$, are linearly independent over the complex numbers on C.

(d) $\operatorname{Im}\varphi \geq 0$.

Let $\phi(x,\theta)$ be a non-degenerate complex phase function in a conic open subset Γ of $\mathbb{R}^n \times \dot{\mathbb{R}}^N$. Let
$$C_\phi = \left\{(x,\theta) \in \Gamma;\ \phi_\theta'(x,\theta) = 0\right\}.$$
By Euler's homogeneity relation, we have $\phi(x,\theta) = \theta.\phi_\theta'(x,\theta) = 0$ on C_ϕ and therefore $\operatorname{Im}\phi$ vanishes on C_ϕ. So does $d(\operatorname{Im}\phi)$, for otherwise there would be a change of sign of $\operatorname{Im}\phi$.

Let $\widetilde{\phi}$ be an almost analytic extension of ϕ in a conic open set $\widetilde{\Gamma} \subset \mathbb{C}^n \times \dot{\mathbb{C}}^N$, $\widetilde{\Gamma} \bigcap (\mathbb{R}^n \times \dot{\mathbb{R}}^N) = \Gamma$. We can choose $\widetilde{\phi}$ such that $\widetilde{\phi}$ is homogeneous of degree 1. Set

$$\widetilde{\phi}_{\widetilde{\theta}}' = (\partial_{\widetilde{\theta}_1}\widetilde{\phi}, \ldots, \partial_{\widetilde{\theta}_N}\widetilde{\phi})$$

and

$$\widetilde{\phi}_{\widetilde{x}}' = (\partial_{\widetilde{x}_1}\widetilde{\phi}, \ldots, \partial_{\widetilde{x}_n}\widetilde{\phi}).$$

Let

$$C_{\widetilde{\phi}} = \left\{(\widetilde{x},\widetilde{\theta}) \in \widetilde{\Gamma};\ \widetilde{\phi}_{\widetilde{\theta}}'(\widetilde{x},\widetilde{\theta}) = 0\right\}, \tag{B.10}$$

$$\Lambda_{\widetilde{\phi}} = \left\{(\widetilde{x}, \widetilde{\phi}_{\widetilde{x}}'(\widetilde{x},\widetilde{\theta}));\ (\widetilde{x},\widetilde{\theta}) \in C_{\widetilde{\phi}}\right\}. \tag{B.11}$$

$\Lambda_{\widetilde{\varphi}}$ is an almost analytic manifold and

$$(\Lambda_{\widetilde{\varphi}})_{\mathbb{R}} = \Lambda_{\widetilde{\varphi}} \bigcap (\mathbb{R}^n \times \dot{\mathbb{R}}^n) = \left\{ (x, \phi'_x(x,\theta)); (x,\theta) \in C_\phi \right\}.$$

Let $\widetilde{\phi}_1$ be another almost analytic extension of ϕ in $\widetilde{\Gamma}$. We have

$$\Lambda_{\widetilde{\varphi}} \sim \Lambda_{\widetilde{\varphi}_1}.$$

(See [18].) Moreover, we have the following

Proposition B.18. *For every point* $(x_0, \xi_0) \in (\Lambda_{\widetilde{\varphi}})_{\mathbb{R}}$ *and after suitable change of local* x *coordinates,* $\Lambda_{\widetilde{\varphi}}$ *is equivalent to a manifold*

$$-\widetilde{x} = \frac{\partial h(\widetilde{\xi})}{\partial \widetilde{\xi}}, \ \widetilde{\xi} \in \mathbb{C}^N$$

in some open neighborhood of (x_0, ξ_0)*, where* h *is an almost analytic function and* $\operatorname{Im} h \geq 0$ *on* $\mathbb{R}^N$ *with equality at* ξ_0.

Definition B.19. An almost analytic manifold satisfying the conditions of Proposition B.18 at every real point for some real symplectic coordinate (x, ξ) is called a positive Lagrangean manifold.

Let φ and ψ be non-degenerate phase functions defined in small conic neighborhoods of $(x_0, \theta_0) \in \mathbb{R}^n \times \dot{\mathbb{R}}^N$ and $(x_0, w_0) \in \mathbb{R}^n \times \dot{\mathbb{R}}^M$ respectively. We assume that $\varphi'_\theta(x_0, \theta_0) = 0$, $\psi'_w(x_0, w_0) = 0$ and that

$$\varphi'_x(x_0, \theta_0) = \psi'_x(x_0, w_0) = \xi_0,$$

where the last equation is a definition. Put $\lambda_0 = (x_0, \xi_0)$. We have the following definition

Definition B.20. We say that φ and ψ are equivalent at λ_0 for classical symbols if there is a conic neighborhood Λ of (x_0, θ_0) such that for every distribution

$$A = \int e^{i\varphi(x,\theta)} a(x,\theta) d\theta,$$

where $a(x,\theta) \in S^m_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^N)$ with support in Λ, there exists

$$b(x, w) \in S^{m + \frac{N-M}{2}}_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^M)$$

with support in a conic neighborhood of (x_0, ω_0) such that

$$A - B \in C^\infty,$$

where $B = \int e^{i\psi(x,w)} b(x,w) dw$ and vise versa.

The global theory of Fourier integral operators with complex phase is the following

Proposition B.21. *Let* φ *and* ψ *be non-degenerate phase functions defined in small conic neighborhoods of* $(x_0, \theta_0) \in \mathbb{R}^n \times \dot{\mathbb{R}}^N$ *and* $(x_0, w_0) \in \mathbb{R}^n \times \dot{\mathbb{R}}^M$ *respectively. We assume that* $\varphi'_\theta(x_0, \theta_0) = 0$*,* ${\psi'}_w(x_0, w_0) = 0$ *and that*

$$\varphi'_x(x_0, \theta_0) = {\psi'}_x(x_0, w_0) = \xi_0.$$

Then φ *and* ψ *are equivalent at* (x_0, ξ_0) *for classical symbols if and only if* $\Lambda_{\widetilde{\varphi}}$ *and* $\Lambda_{\widetilde{\psi}}$ *are equivalent in some neighborhood of* (x_0, ξ_0)*, where* $\widetilde{\varphi}$ *and* $\widetilde{\psi}$ *are almost analytic extensions of* φ *and* ψ *respectively and* $\Lambda_{\widetilde{\varphi}}, \Lambda_{\widetilde{\psi}}$ *are as in (B.11).*

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On the singularities of the Bergman projection for $(0, q)$ forms

Chin-Yu Hsiao

Abstract

We obtain the full asymptotic expansion of the Bergman projection for $(0, q)$ forms when the Levi form is non-degenerate. This generalizes a result of Boutet de Monvel and Sjöstrand for $(0, 0)$ forms. We introduce a new operator analogous to the Kohn Laplacian defined on the boundary of a domain and we apply the heat equation method of Menikoff and Sjöstrand to this operator. We obtain a description of a new Szegö projection up to smoothing operators. Finally, by using the Poisson operator, we get our main result.

Résumé

Nous obtenons un développement asymptotique de la singularité du noyau de Bergman pour les $(0, q)$ formes quand la forme de Levi est non-dégénérée. Cela généralise un résultat de Boutet de Monvel et Sjöstrand pour les $(0, 0)$ formes. Nous introduisons un nouvel opérateur analogue au laplacien de Kohn, défini sur le bord du domaine et nous y appliquons la méthode de Menikoff-Sjöstrand. Cela donne une description modulo des opéreateurs régularisants d'un nouvel projecteur de Szegö. Finalement, en utilisant l'opérateur de Poisson, nous obtenons notre résultat principal.

Contents

1 Introduction and statement of the main results

In this paper, we assume that all manifolds are paracompact. (For the precise definition, see page 156 of Kelley [19].) Let M be a relatively compact open subset with C^∞ boundary Γ of a complex manifold M' of dimension n with a smooth Hermitian metric $(\,|\,)$ on its holomorphic tangent bundle. (See (1.1).) The Hermitian metric induces a Hermitian metric on the bundle of $(0,q)$ forms of M' (see the discussion after (1.1) and section 2) and a positive density (dM')(see (1.3)). Let $\Box$ be the $\overline{\partial}$-Neumann Laplacian on M (see Folland and Kohn [9] or (1.6)) and let $\Box^{(q)}$ denote the restriction to $(0,q)$ forms. For $p\in\Gamma$, let L_p be the Levi form of Γ at p (see (1.12) or Definition 2.2). Given q, $0\le q\le n-1$, the Levi form is said to satisfy condition $Z(q)$ at $p\in\Gamma$ if it has at least $n-q$ positive or at least $q+1$ negative eigenvalues. When condition $Z(q)$ holds at each point of Γ, Kohn's L^2 estimates give the hypoellipicity with loss of one dervative for the solutions of $\Box^{(q)}u=f$. (See [9] or Theorem 3.6.) The Bergman projection is the orthogonal projection onto the kernel of $\Box^{(q)}$ in the L^2 space. When condition $Z(q)$ fails at some point of Γ, one is interested in the Bergman projection on the level of $(0,q)$ forms. When $q=0$ and the Levi form is positive definite, the existence of the complete asymptotic expansion of the singularities of the Bergman projection was obtained by Fefferman [8] on the diagonal and subsequently by Boutet de Monvel and Sjöstrand (see [7]) in complete generality. If $q=n-1$ and the Levi form is negative definite, Hörmander [17] obtained the corresponding asymptotics for the Bergman projection in the distribution sense. We have been influenced by these works.

We now start to formulate the main results. First, we introduce some notations. Let Ω be a C^∞ manifold. We let $T(\Omega)$ and $T^*(\Omega)$ denote the tangent bundle of Ω and the cotangent bundle of Ω respectively. The complexified tangent bundle of Ω and the complexified cotangent bundle of Ω will be denoted by $\mathbb{C}T(\Omega)$ and $\mathbb{C}T^*(\Omega)$ respectively. Let E be a C^∞ vector bundle over Ω. The fiber of E at $x \in \Omega$ will be denoted by E_x. Let $Y \subset\subset \Omega$ be an open set. The spaces of smooth sections of E over Y and distribution sections of E over Y will be denoted by $C^\infty(Y; E)$ and $\mathscr{D}'(Y; E)$ respectively. Let $\mathscr{E}'(Y; E)$ be the subspace of $\mathscr{D}'(Y; E)$ of sections with compact support in Y. For $s \in \mathbb{R}$, we let $H^s(Y; E)$ denote the Sobolev space of order s of sections of E over Y. Put

$$H^s_{\rm loc}(Y; E) = \left\{u \in \mathscr{D}'(Y; E);\, \varphi u \in H^s(Y; E),\, \forall \varphi \in C^\infty_0(Y)\right\}$$

and

$$H^s_{\rm comp}(Y; E) = H^s_{\rm loc}(Y; E) \bigcap \mathscr{E}'(Y; E).$$

Let F be a C^∞ vector bundle over M'. Let $C^\infty(\overline{M}; F)$, $\mathscr{D}'(\overline{M}; F)$, $H^s(\overline{M}; F)$ denote the spaces of restrictions to M of elements in $C^\infty(M'; F)$, $\mathscr{D}'(M'; F)$ and $H^s(M'; F)$ respectively. Let $C^\infty_0(M; F)$ be the subspace of $C^\infty(\overline{M}; F)$ of sections with compact support in M.

Let $\Lambda^{1,0}T(M')$ and $\Lambda^{0,1}T(M')$ be the holomorphic tangent bundle of M' and the anti-holomorphic tangent boundle of M' respectively. (See (2.4).) In local coordinates $z = (z_1, \ldots, z_n)$, we represent the Hermitian metric on $\Lambda^{1,0}T(M')$ by

$$\begin{aligned}(u \mid v) &= g(u, \overline{v}),\ \ u, v \in \Lambda^{1,0}T(M'),\\ g &= \sum_{j,k=1}^{n} g_{j,k}(z) dz_j \otimes d\overline{z}_k,\end{aligned} \tag{1.1}$$

where $g_{j,k}(z) = \overline{g}_{k,j}(z) \in C^\infty$, $j, k = 1, \ldots, n$, and $\left(g_{j,k}(z)\right)^n_{j,k=1}$ is positive definite at each point. We extend the Hermitian metric $(\ |\ )$ to $\mathbb{C}T(M')$ in a natural way by requiring $\Lambda^{1,0}T(M')$ to be orthogonal to $\Lambda^{0,1}T(M')$ and satisfy

$$\overline{(u \mid v)} = (\overline{u} \mid \overline{v}),\ u, v \in \Lambda^{0,1}T(M').$$

The Hermitian metric $(\ |\ )$ on $\mathbb{C}T(M)$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*(M)$ that we shall also denote by $(\ |\ )$. (See (2.9).) For $q \in \mathbb{N}$, let $\Lambda^{0,q}T^*(M')$ be the bundle of $(0,q)$ forms of M'. (See (2.6).) The Hermitian metric $(\ |\ )$ on $\mathbb{C}T^*(M')$ induces a Hermitian metric on $\Lambda^{0,q}T^*(M')$ also denoted by $(\ |\ )$. (See (2.11).)

Let $r \in C^\infty(M')$ be a defining function of Γ such that r is real, $r = 0$ on Γ, $r < 0$ on M and $dr \neq 0$ near Γ. From now on, we take a defining function r so that

$$\|dr\| = 1 \ \text{ on } \ \Gamma.$$

The Hermitian metric $(\,|\,)$ on $\mathbb{C}T(M')$ induces a Hermitian metric $(\,|\,)$ on $\mathbb{C}T(\Gamma)$. For $z\in\Gamma$, we identify $\mathbb{C}T^*_z(\Gamma)$ with the space

$$\left\{u\in\mathbb{C}T^*_z(M');\,(u\mid dr)=0\right\}. \tag{1.2}$$

We associate to the Hermitian metric $\sum_{j,k=1}^n g_{j,k}(z)dz_j\otimes d\overline{z}_k$ a real $(1,1)$ form (see page 144 of Kodaira [20])

$$\omega=i\sum_{j,k=1}^n g_{j,k}dz_j\wedge d\overline{z}_k.$$

Let

$$dM'=\frac{\omega^n}{n!} \tag{1.3}$$

be the volume element (see also (2.12)) and let $(\,|\,)_M$ be the inner product on $C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$ defined by

$$(f\mid h)_M=\int_M(f\mid h)(dM')=\int_M(f\mid h)\frac{\omega^n}{n!},\ \ f,h\in C^\infty(\overline{M};\Lambda^{0,q}T^*(M')). \tag{1.4}$$

Similarly, we take $(d\Gamma)$ as the induced volume form on Γ (see (2.18)) and let $(\,|\,)_\Gamma$ be the inner product on $C^\infty(\Gamma;\Lambda^{0,q}T^*(M'))$ defined by

$$(f\mid g)_\Gamma=\int_\Gamma(f\mid g)d\Gamma,\ \ f,g\in C^\infty(\Gamma;\Lambda^{0,q}T^*(M')). \tag{1.5}$$

Let $\triangle_\Gamma$ be the real Laplacian on Γ and denote by $\sigma_{\triangle_\Gamma}$ the principal symbol of $\triangle_\Gamma$.

Let

$$\overline{\partial}:C^\infty(M';\Lambda^{0,q}T^*(M'))\rightarrow C^\infty(M';\Lambda^{0,q+1}T^*(M'))$$

be the part of the exterior differential operator which maps forms of type $(0,q)$ to forms of type $(0,q+1)$ and we denote by

$$\overline{\partial}_f^*:C^\infty(M';\Lambda^{0,q+1}T^*(M'))\rightarrow C^\infty(M';\Lambda^{0,q}T^*(M'))$$

the formal adjoint of $\overline{\partial}$. That is

$$(\overline{\partial}f\mid h)_{M'}=(f\mid\overline{\partial}_f^*h)_{M'},\ f\in C_0^\infty(M';\Lambda^{0,q}T^*(M')),\ h\in C^\infty(M';\Lambda^{0,q+1}T^*(M')),$$

where $(\,|\,)_{M'}$ is defined by

$$(g\mid k)_{M'}=\int_{M'}(g\mid k)(dM'),\ \ g,k\in C_0^\infty(M';\Lambda^{0,q}T^*(M')).$$

We shall also use the notation $\overline{\partial}$ for the closure in L^2 of the $\overline{\partial}$ operator, initially defined on $C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$ and $\overline{\partial}^*$ for the Hilbert space adjoint of $\overline{\partial}$. The domain of $\overline{\partial}^*$ consists of all $f\in L^2(M;\Lambda^{0,q+1}T^*(M'))$ such that for some constant $c>0$,

$$\left|(f\mid\overline{\partial}g)_M\right|\leq c\left\|g\right\|,\ \text{ for all } g\in C^\infty(\overline{M};\Lambda^{0,q}T^*(M')).$$

For such an f,

$$g\to(f\mid\overline{\partial}g)_M$$

extends to a bounded anti-linear functional on $L^2(M;\Lambda^{0,q}T^*(M'))$ so

$$(f\mid\overline{\partial}g)_M=(\tilde{f}\mid g)_M$$

for some $\tilde{f}\in L^2(M;\Lambda^{0,q}T^*(M'))$. We have $\overline{\partial}^*f=\tilde{f}$. The $\overline{\partial}$-Neumann Laplacian on $(0,q)$ forms is then the self-adjoint operator in the space $L^2(M;\Lambda^{0,q}T^*(M'))$ (see chapter I of [9])

$$\Box^{(q)}=\overline{\partial}\,\overline{\partial}^*+\overline{\partial}^*\overline{\partial}. \tag{1.6}$$

We notice that

$$\begin{aligned}\mathrm{Dom}\,\Box^{(q)}=\{&u\in L^2(M;\Lambda^{0,q}T^*(M'));\ u\in\mathrm{Dom}\,\overline{\partial}^*\bigcap\mathrm{Dom}\,\overline{\partial},\\&\overline{\partial}^*u\in\mathrm{Dom}\,\overline{\partial},\ \overline{\partial}u\in\mathrm{Dom}\,\overline{\partial}^*\}\end{aligned} \tag{1.7}$$

and $C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))\bigcap\mathrm{Dom}\,\Box^{(q)}$ is dense in $\mathrm{Dom}\,\Box^{(q)}$. (See also page 14 of [9].)

Let

$$\Box^{(q)}_f=\overline{\partial}\,\overline{\partial}_f^*+\overline{\partial}_f^*\overline{\partial}:C^\infty(M';\Lambda^{0,q}T^*(M'))\to C^\infty(M';\Lambda^{0,q}T^*(M')) \tag{1.8}$$

denote the complex Laplace-Beltrami operator on $(0,q)$ forms and denote by $\sigma_{\Box^{(q)}_f}$ the principal symbol of $\Box^{(q)}_f$. Let γ denote the operator of restriction to the boundary Γ. Let us consider the map:

$$\begin{aligned}F^{(q)}:H^2(\overline{M};\Lambda^{0,q}T^*(M'))&\to H^0(\overline{M};\Lambda^{0,q}T^*(M'))\oplus H^{\frac{3}{2}}(\Gamma;\Lambda^{0,q}T^*(M')),\\u&\to(\Box^{(q)}_f u,\gamma u).\end{aligned} \tag{1.9}$$

Given q, $0\leq q\leq n-1$, we assume that

Assumption 1.1. *$F^{(k)}$ is injective, $q-1\leq k\leq q+1$.*

Thus, the Poisson operator for $\Box^{(k)}_f$, $q-1\leq k\leq q+1$, is well-defined. (See section 4.) If M' is Kähler, then $F^{(q)}$ is injective for any q, $0\leq q\leq n$. (See section 9 for the definition and details.)

We write $\langle\,,\rangle$ to denote the duality between $T_z(M')$ and $T^*_z(M')$. We extend $\langle\,,\rangle$ bilinearly to $\mathbb{C}T_z(M')\times\mathbb{C}T^*_z(M')$. Let $\frac{\partial}{\partial r}$ be the dual vector of dr. That is

$$(u\mid\frac{\partial}{\partial r})=\langle u,dr\rangle, \tag{1.10}$$

for all $u\in\mathbb{C}T(M')$. Put

$$\omega_0=J^t(dr), \tag{1.11}$$

where J^t is the complex structure map for the cotangent bundle. (See (2.2).)

Let $\Lambda^{1,0}T(\Gamma)$ and $\Lambda^{0,1}T(\Gamma)$ be the holomorphic tangent bundle of Γ and the anti-holomorphic tangent bundle of Γ respectively. (See (2.22).) The Levi form $L_p(Z,\overline{W})$, $p\in\Gamma$, Z, $W\in\Lambda^{1,0}T_p(\Gamma)$, is the Hermitian quadratic form on $\Lambda^{1,0}T_p(\Gamma)$ defined as follows:

$$\begin{aligned}&\text{For any } Z,\ W\in\Lambda^{1,0}T_p(\Gamma)\text{, pick } \widetilde{Z},\ \widetilde{W}\in C^\infty(\Gamma;\Lambda^{1,0}T(\Gamma)) \text{ that satisfy}\\ &\widetilde{Z}(p)=Z,\ \widetilde{W}(p)=W.\ \text{Then } L_p(Z,\overline{W})=\frac{1}{2i}\left\langle[\widetilde{Z}\,,\overline{\widetilde{W}}](p)\,,\omega_0(p)\right\rangle.\end{aligned} \tag{1.12}$$

The eigenvalues of the Levi form at $p\in\Gamma$ are the eigenvalues of the Hermitian form L_p with respect to the inner product $(\mid)$ on $\Lambda^{1,0}T_p(\Gamma)$. If the Levi form is non-degenerate at $p\in\Gamma$, let (n_-,n_+), $n_-+n_+=n-1$, be the signature and notice that $Z(q)$ holds at p if and only if $q\neq n_-$.

We recall the Hörmander symbol spaces

Definition 1.2. Let $m\in\mathbb{R}$. Let U be an open set in $M'\times M'$.

$$S^m_{1,0}(U\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_x(M')))$$

is the space of all $a(x,y,t)\in C^\infty(U\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_x(M')))$ such that for all compact sets $K\subset U$ and all $\alpha\in\mathbb{N}^{2n}$, $\beta\in\mathbb{N}^{2n}$, $\gamma\in\mathbb{N}$, there is a constant $c>0$ such that

$$\left|\partial^\alpha_x\partial^\beta_y\partial^\gamma_t a(x,y,t)\right|\leq c(1+|t|)^{m-|\gamma|},\ (x,y,t)\in K\times]0,\infty[.$$

$S^m_{1,0}$ is called the space of symbols of order m type $(1,0)$. We write $S^{-\infty}_{1,0}=\bigcap S^m_{1,0}$.

Let $S^m_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M')))$ denote the space of restrictions to $U\bigcap(M\times M)\times]0,\infty[$ of elements in

$$S^m_{1,0}(U\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))).$$

Let

$$a_j\in S^{m_j}_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))),\ j=0,1,2,\ldots,$$

with $m_j \searrow -\infty$, $j \to \infty$. Then there exists

$$a \in S^{m_0}_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M')))$$

such that

$$a - \sum_{0\le j<k} a_j \in S^{m_k}_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))),$$

for every $k\in\mathbb{N}$. (See Proposition 1.8 of Grigis-Sjöstrand [11] or Hörmander [13].)

If a and a_j have the properties above, we write

$$a \sim \sum_{j=0}^{\infty} a_j \text{ in the space } S^{m_0}_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times[0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))).$$

Let

$$\Pi^{(q)}: L^2(M;\Lambda^{0,q}T^*(M'))\to \mathrm{Ker}\,\Box^{(q)}$$

be the Bergman projection, i.e. the orthogonal projection onto the kernel of $\Box^{(q)}$. Let

$$K_{\Pi^{(q)}}(z,w)\in\mathscr{D}'(M\times M;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M')))$$

be the distribution kernel of $\Pi^{(q)}$. Formally,

$$(\Pi^{(q)}u)(z)=\int_M K_{\Pi^{(q)}}(z,w)u(w)dM'(w),\quad u(w)\in C^\infty_0(M;\Lambda^{0,q}T^*(M')).$$

Let X and Y be C^∞ vector bundles over M'. Let

$$C,D: C^\infty_0(M;X)\to\mathscr{D}'(M;Y)$$

with distribution kernels

$$K_C(z,w), K_D(z,w)\in\mathscr{D}'(M\times M;\mathscr{L}(X_w,Y_z)).$$

We write

$$C\equiv D \mod C^\infty(U\bigcap(\overline{M}\times\overline{M}))$$

if

$$K_C(z,w)=K_D(z,w)+F(z,w),$$

where $F(z,w)\in C^\infty(U\bigcap(\overline{M}\times\overline{M});\mathscr{L}(X_w,Y_z))$ and U is an open set in $M'\times M'$.

Given q, $0\le q\le n-1$. Put

$$\Gamma_q=\{z\in\Gamma; Z(q)\text{ fails at } z\}. \tag{1.13}$$

If the Levi form is non-degenerate at each point of Γ, then Γ_q is a union of connected components of Γ.

The main result of this work is the following

Theorem 1.3. *Let M be a relatively compact open subset with C^∞ boundary Γ of a complex analytic manifold M' of dimension n. We assume that the Levi form is non-degenerate at each point of Γ. Let q, $0\leq q\leq n-1$. Suppose that $Z(q)$ fails at some point of Γ and that $Z(q-1)$ and $Z(q+1)$ hold at each point of Γ. Then*

$$K_{\Pi^{(q)}}(z,w)\in C^\infty(\overline{M}\times\overline{M}\setminus\operatorname{diag}(\Gamma_q\times\Gamma_q);\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))).$$

Moreover, in a neighborhood U of $\operatorname{diag}(\Gamma_q\times\Gamma_q)$, *$K_{\Pi^{(q)}}(z,w)$ satisfies*

$$K_{\Pi^{(q)}}(z,w)\equiv\int_0^\infty e^{i\phi(z,w)t}b(z,w,t)dt \mod C^\infty(U\bigcap(\overline{M}\times\overline{M})) \tag{1.14}$$

(for the precise meaning of the oscillatory integral $\int_0^\infty e^{i\phi(z,w)t}b(z,w,t)dt$, see Remark 1.4 below) with

$$b(z,w,t)\in S^n_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))),$$

$$b(z,w,t)\sim\sum_{j=0}^\infty b_j(z,w)t^{n-j}$$

in the space $S^n_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))),$

$$b_0(z,z)\neq 0,\ z\in\Gamma_q,$$

where

$$b_j(z,w)\in C^\infty(U\bigcap(\overline{M}\times\overline{M});\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))),\ j=0,1,\ldots,$$

and

$$\phi(z,w)\in C^\infty(U\bigcap(\overline{M}\times\overline{M})), \tag{1.15}$$
$$\phi(z,z)=0,\ \ z\in\Gamma_q, \tag{1.16}$$
$$\phi(z,w)\neq 0\ \text{ if } (z,w)\notin\operatorname{diag}(\Gamma_q\times\Gamma_q), \tag{1.17}$$
$$\operatorname{Im}\phi\geq 0, \tag{1.18}$$
$$\operatorname{Im}\phi(z,w)>0\ \text{ if } (z,w)\notin\Gamma\times\Gamma, \tag{1.19}$$
$$\phi(z,w)=-\overline{\phi}(w,z). \tag{1.20}$$

For $p\in\Gamma_q$, we have

$$\sigma_{\Box^{(q)}_f}(z,d_z\phi(z,w))\ \text{ vanishes to infinite order at }\ z=p,$$
$$(z,w)\ \text{is in some neighborhood of}\ (p,p)\ \text{in}\ M'. \tag{1.21}$$

For $z = w$, $z \in \Gamma_q$, we have

$$d_z\phi = -\omega_0 - i\,dr,$$

$$d_w\phi = \omega_0 - i\,dr.$$

Moreover, we have

$$\phi(z,w) = \phi_-(z,w) \ \ \text{if } z, w \in \Gamma_q,$$

where $\phi_-(z,w) \in C^\infty(\Gamma_q \times \Gamma_q)$ is the phase appearing in the description of the Szegö projection in [18]. See Theorem 1.5 and Theorem 7.15 below.

Remark 1.4. Let $\phi(z,w)$ and $b(z,w,t)$ be as in Theorem 1.3. Let $y = (y_1,\ldots,y_{2n-1})$ be local coordinates on Γ and extend $y_1,\ldots,y_{2n-1}$ to real smooth functions in some neighborhood of Γ. We work with local local coordinates

$$w = (y_1,\ldots,y_{2n-1},r)$$

defined on some small neighborhood U of $p \in \Gamma_q$. Let $u \in C_0^\infty(U;\Lambda^{0,q}T^*(M'))$. Choose a cut-off function $\chi(t) \in C^\infty(\mathbb{R})$ so that $\chi(t) = 1$ when $|t| < 1$ and $\chi(t) = 0$ when $|t| > 2$. Set

$$(B_\varepsilon u)(z) = \int\int_0^\infty e^{i\phi(z,w)t} b(z,w,t)\chi(\varepsilon t)u(w)\,dt\,dw.$$

Since $d_y\phi \neq 0$ where $\operatorname{Im}\phi = 0$ (see (7.31)), we can integrate by parts in y and t and obtain

$$\lim_{\varepsilon\to 0}(B_\varepsilon u)(z) \in C^\infty(\overline{M};\Lambda^{0,q}T^*(M')).$$

This means that

$$B = \lim_{\varepsilon\to 0} B_\varepsilon : C^\infty(\overline{M};\Lambda^{0,q}T^*(M')) \to C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$$

is continuous. We write $B(z,w)$ to denote the distribution kernel of B. Formally,

$$B(z,w) = \int_0^\infty e^{i\phi(z,w)t} b(z,w,t)\,dt.$$

From (1.21) and Remark 1.5 of [18] it follows that

Theorem 1.5. *Under the assumptions of Theorem 1.3, let $p \in \Gamma_q$. We choose local complex analytic coordinates*

$$z = (z_1,\ldots,z_n), \ \ z_j = x_{2j-1} + ix_{2j}, \ \ j = 1,\ldots,n,$$

vanishing at p such that the metric on $\Lambda^{1,0}T(M')$ *is*

$$\sum_{j=1}^{n} dz_j \otimes d\overline{z}_j \ \ at \ \ p$$

and

$$r(z) = \sqrt{2}\operatorname{Im} z_n + \sum_{j=1}^{n-1} \lambda_j \left|z_j\right|^2 + O(|z|^3),$$

where λ_j, $j = 1,\dots,n-1$, *are the eigenvalues of* L_p. *(This is always possible. See Lemma* 3.2 *of [17].) We also write*

$$w = (w_1,\dots,w_n),\ \ w_j = y_{2j-1} + iy_{2j},\ \ j = 1,\dots,n.$$

Then, we can take $\phi(z,w)$ *so that*

$$\begin{aligned}\phi(z,w) = &-\sqrt{2}x_{2n-1} + \sqrt{2}y_{2n-1} - ir(z)\Big(1 + \sum_{j=1}^{2n-1} a_j x_j + \frac{1}{2}a_{2n}x_{2n}\Big)\\ &- ir(w)\Big(1 + \sum_{j=1}^{2n-1} \overline{a}_j y_j + \frac{1}{2}\overline{a}_{2n}y_{2n}\Big) + i\sum_{j=1}^{n-1}\left|\lambda_j\right|\left|z_j - w_j\right|^2\\ &+ \sum_{j=1}^{n-1} i\lambda_j(\overline{z}_j w_j - z_j\overline{w}_j) + O\Big(|(z,w)|^3\Big) \qquad (1.22)\end{aligned}$$

in some neighborhood of (p,p) *in* $M' \times M'$, *where*

$$a_j = \frac{1}{2}\frac{\partial \sigma_{\Box_f^{(q)}}}{\partial x_j}(p, -\omega_0(p) - idr(p)),\ \ j = 1,\dots,2n.$$

We have the following corollary of Theorem 1.3

Corollary 1.6. *Under the assumptions of Theorem 1.3 and let U be a small neighborhood of* $\operatorname{diag}(\Gamma_q \times \Gamma_q)$. *Then there exist smooth functions*

$$F, G \in C^\infty(U \textstyle\bigcap (\overline{M} \times \overline{M}); \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M')))$$

such that

$$K_{\Pi^{(q)}} = F(-i(\phi(z,w) + i0))^{-n-1} + G\log(-i(\phi(z,w) + i0)).$$

Moreover, we have

$$\begin{aligned}F &= \sum_{j=0}^{n}(n-j)!b_j(z,w)(-i\phi(z,w))^j + f(z,w)(\phi(z,w))^{n+1},\\ G &\equiv \sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{j!}b_{n+j+1}(z,w)(-i\phi(z,w))^j \mod C^\infty(U \textstyle\bigcap (\overline{M} \times \overline{M})) \qquad (1.23)\end{aligned}$$

where

$$f(z,w)\in C^\infty(U\bigcap(\overline{M}\times\overline{M});\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))).$$

If $w\in\Lambda^{0,1}T^*_z(M')$, let

$$w^{\wedge,*}:\Lambda^{0,q+1}T^*_z(M')\to\Lambda^{0,q}T^*_z(M') \tag{1.24}$$

be the adjoint of left exterior multiplication

$$w^{\wedge}:\Lambda^{0,q}T^*_z(M')\to\Lambda^{0,q+1}T^*_z(M').$$

That is,

$$(w^\wedge u\mid v)=(u\mid w^{\wedge,*}v), \tag{1.25}$$

for all $u\in\Lambda^{0,q}T^*_z(M')$, $v\in\Lambda^{0,q+1}T^*_z(M')$. Notice that $w^{\wedge,*}$ depends anti-linearly on w.

Let $\Lambda^{0,1}T^*(\Gamma)$ be the bundle of boundary $(0,1)$ forms. (See (2.23) and (2.29).)

Proposition 1.7. *Under the assumptions of Theorem 1.3, let $p\in\Gamma_q$, $q=n_-$. Let*

$$\overline{Z}_1(z),\ldots,\overline{Z}_{n-1}(z)$$

be an orthonormal frame of $\Lambda^{1,0}T_z(\Gamma)$, $z\in\Gamma$, for which the Levi form is diagonalized at p. Let $e_j(z)$, $j=1,\ldots,n-1$ denote the basis of $\Lambda^{0,1}T^*_z(\Gamma)$, $z\in\Gamma$, which is dual to $Z_j(z)$, $j=1,\ldots,n-1$. Let $\lambda_j(z)$, $j=1,\ldots,n-1$ be the eigenvalues of the Levi form L_z, $z\in\Gamma$. We assume that

$$\lambda_j(p)<0 \ \text{ if } \ 1\leq j\leq n_-.$$

Then

$$F(p,p)=n!\left|\lambda_1(p)\right|\cdots\left|\lambda_{n-1}(p)\right|\pi^{-n}2\Big(\prod_{j=1}^{j=n_-}e_j(p)^\wedge e_j^{\wedge,*}(p)\Big)\circ(\overline{\partial}r(p))^{\wedge,*}(\overline{\partial}r(p))^\wedge, \tag{1.26}$$

where F is as in Corollary 1.6.

In the rest of this section, we outline the proof of Theorem 1.3. We assume that the Levi form is non-degenerate at each point of Γ. We pause and recall a general fact of distribution theory. (See Hörmander [16].) Let E, F be C^∞ vector bundles over C^∞ manifolds G and H respectively. We take smooth densities of integration on G and H respectively. Let

$$A:C^\infty_0(G;E)\to\mathscr{D}'(H;F)$$

with distribution kernel

$$K_A(x,y)\in\mathscr{D}'(H\times G;\mathscr{L}(E_y,F_x)).$$

Then the following two statements are equivalent

(a) A is continuous: $\mathscr{E}'(G;\, E) \to C^\infty(H;\, F)$,

(b) $K_A \in C^\infty(H \times G;\, \mathscr{L}(E_y, F_x))$.

If A satisfies (a) or (b), we say that A is smoothing. Let

$$B : C_0^\infty(G;\, E) \to \mathscr{D}'(H;\, F).$$

From now on, we write $K_B(x, y)$ or $B(x, y)$ to denote the distribution kernel of B and we write

$$A \equiv B$$

if $A - B$ is a smoothing operator.

Let

$$P : C^\infty(\Gamma;\, \Lambda^{0,q} T^*(M')) \to C^\infty(\overline{M};\, \Lambda^{0,q} T^*(M')) \tag{1.27}$$

be the Poisson operator for $\Box_f^{(q)}$ which is well-defined since we assumed (1.9) to be injective. It is well-known that P extends continuously

$$P : H^s(\Gamma;\, \Lambda^{0,q} T^*(M')) \to H^{s+\frac{1}{2}}(\overline{M};\, \Lambda^{0,q} T^*(M')),\ \ \forall\ s \in \mathbb{R}.$$

(See page 29 of Boutet de Monvel [5].) Let

$$P^* : \mathscr{E}'(\overline{M};\, \Lambda^{0,q} T^*(M')) \to \mathscr{D}'(\Gamma;\, \Lambda^{0,q} T^*(M'))$$

be the operator defined by

$$(P^* u \mid v)_\Gamma = (u \mid Pv)_M,\ \ u \in \mathscr{E}'(\overline{M};\, \Lambda^{0,q} T^*(M')),\ \ v \in C^\infty(\Gamma;\, \Lambda^{0,q} T^*(M')).$$

It is well-known (see page 30 of [5]) that P^* is continuous:

$$P^* : L^2(M;\, \Lambda^{0,q} T^*(M')) \to H^{\frac{1}{2}}(\Gamma;\, \Lambda^{0,q} T^*(M'))$$

and

$$P^* : C^\infty(\overline{M};\, \Lambda^{0,q} T^*(M')) \to C^\infty(\Gamma;\, \Lambda^{0,q} T^*(M')).$$

We use the inner product $[\,|\,]$ on $H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q} T^*(M'))$ defined as follows:

$$[u \mid v] = (Pu \mid Pv)_M,$$

where $u,\, v \in H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q} T^*(M'))$. We consider $(\overline{\partial} r)^{\wedge,*}$ as an operator

$$(\overline{\partial} r)^{\wedge,*} : H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q} T^*(M')) \to H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q-1} T^*(M')).$$

Note that $(\overline{\partial} r)^{\wedge,*}$ is the pointwise adjoint of $\overline{\partial} r$ with respect to $(\,|\,)$. Let

$$T : H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q} T^*(M')) \to \operatorname{Ker}(\overline{\partial} r)^{\wedge,*} \tag{1.28}$$

be the orthogonal projection onto $\mathrm{Ker}(\overline{\partial} r)^{\wedge,*}$ with respect to $[\,|\,]$. That is, if $u \in H^{-\frac{1}{2}}(\Gamma;\Lambda^{0,q}T^*(M'))$, then

$$(\overline{\partial} r)^{\wedge,*} T u = 0$$

and

$$[(I-T)u \mid g] = 0,\ \ \forall\, g \in \mathrm{Ker}(\overline{\partial} r)^{\wedge,*}.$$

In section 4, we will show that T is a classical pseudodifferential operator of order 0 with principal symbol

$$2(\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^{\wedge}.$$

For $q \in \mathbb{N}$, let $\Lambda^{0,q}T^*(\Gamma)$ be the bundle of boundary $(0,q)$ forms. (See (2.29).) If $u \in C^\infty(\Gamma;\Lambda^{0,q}T^*(M'))$, then $u \in \mathrm{Ker}(\overline{\partial} r)^{\wedge,*}$ if and only if $u \in C^\infty(\Gamma;\Lambda^{0,q}T^*(\Gamma))$. Put

$$\overline{\partial_\beta} = T\gamma\overline{\partial} P : C^\infty(\Gamma;\Lambda^{0,q}T^*(\Gamma)) \to C^\infty(\Gamma;\Lambda^{0,q+1}T^*(\Gamma)). \tag{1.29}$$

$\overline{\partial_\beta}$ is a classical pseudodifferential operator of order one from boundary $(0,q)$ forms to boundary $(0,q+1)$ forms. It is easy to see that

$$\overline{\partial_\beta} = \overline{\partial_b} + \text{lower order terms}, \tag{1.30}$$

where $\overline{\partial_b}$ is the tangential Cauchy-Riemann operator. (See [9] or section 6.) In section 6, we will show that

$$(\overline{\partial_\beta})^2 = 0.$$

Let

$$\overline{\partial_\beta}^\dagger : C^\infty(\Gamma;\Lambda^{0,q+1}T^*(\Gamma)) \to C^\infty(\Gamma;\Lambda^{0,q}T^*(\Gamma)),$$

be the formal adjoint of $\overline{\partial_\beta}$ with respect to $[\,|\,]$. $\overline{\partial_\beta}^\dagger$ is a classical pseudodifferential operator of order one from boundary $(0,q+1)$ forms to boundary $(0,q)$ forms. In section 6, we will show that

$$\overline{\partial_\beta}^\dagger = \gamma\overline{\partial_f}^* P.$$

Put

$$\Box_\beta^{(q)} = \overline{\partial_\beta}\,\overline{\partial_\beta}^\dagger + \overline{\partial_\beta}^\dagger\overline{\partial_\beta} : C^\infty(\Gamma;\Lambda^{0,q}T^*(\Gamma)) \to C^\infty(\Gamma;\Lambda^{0,q}T^*(\Gamma)).$$

For simplicity, we assume that $\Gamma = \Gamma_q$, $\Gamma_q \neq \Gamma_{n-1-q}$. We can repeat the method of [18] (see section 7) to construct

$$A \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\Gamma;\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)), B \in L^{0}_{\frac{1}{2},\frac{1}{2}}(\Gamma;\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))$$

such that

$$\begin{aligned}
&A\Box_\beta^{(q)} + B \equiv B + \Box_\beta^{(q)} A \equiv I,\\
&\overline{\partial_\beta} B \equiv 0,\ \overline{\partial_\beta}^\dagger B \equiv 0,\\
&B \equiv B^\dagger \equiv B^2,
\end{aligned}$$

where $L^m_{\frac{1}{2},\frac{1}{2}}$ is the space of pseudodifferential operators of order m type $(\frac{1}{2},\frac{1}{2})$ (see Definition 7.11) and $B^\dagger$ is the formal adjoint of B with respect to $[\,|\,]$. Moreover, $K_B(x,y)$ satisfies

$$K_B(x,y) \equiv \int_0^\infty e^{i\phi_-(x,y)t} b(x,y,t)dt,$$

where $\phi_-(x,y)$ and $b(x,y,t)$ are as in Theorem 7.15. In section 8, we will show that

$$\Pi^{(q)} \equiv PBT(P^*P)^{-1}P^* \mod C^\infty(\overline{M}\times\overline{M})$$

and

$$PBT(P^*P)^{-1}P^*(z,w) \equiv \int_0^\infty e^{i\phi(z,w)t} b(z,w,t)dt \mod C^\infty(\overline{M}\times\overline{M}),$$

where $\phi(z,w)$ and $b(z,w,t)$ are as in Theorem 1.3.

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2 Terminology and notations, a review

In this section, we will review some standard terminology in complex geometry. For more details on the subject, see Kodaira [20].

Let E be a finite dimensional vector space with a complex structure J. By definition, a complex structure J is a $\mathbb{R}$-linear map

$$J : E \to E$$

that satisfies

$$J^2 = -I.$$

Let $\mathbb{C}E$ be the complexification of E. That is,

$$\mathbb{C}E = \{u + iv;\, u, v \in E\}.$$

Any vector in $\mathbb{C}E$ can be written

$$f = u + iv$$

where u, $v \in E$ and any $\mathbb{R}$-linear map between real vector spaces can be extended to a $\mathbb{C}$-linear map between the complexifications, simply by putting

$$Tf = Tu + iTv.$$

In particular, we can extend J to a $\mathbb{C}$-linear map

$$J : \mathbb{C}E \to \mathbb{C}E.$$

Clearly, it still holds that $J^2 = -I$. This implies that we have a decomposition as a direct sum

$$\mathbb{C}E = \Lambda^{1,0}E \oplus \Lambda^{0,1}E$$

where

$$Ju = iu \ \text{ if } \ u \in \Lambda^{1,0}E$$

and

$$Ju = -iu \ \text{ if } \ u \in \Lambda^{0,1}E.$$

Let us now return to our original situation where $E = T_p(M')$, $p \in M'$. Given holomorphic coordinates

$$z_j = x_j + iy_j, \ \ j = 1, \dots, n$$

we get a basis for $T_p(M')$

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}.$$

The complex structure J on $T_p(M')$ is defined by

$$\begin{cases} J(\dfrac{\partial}{\partial x_j}) = \dfrac{\partial}{\partial y_j}, \ j = 1, \dots, n, \\ J(\dfrac{\partial}{\partial y_j}) = -\dfrac{\partial}{\partial x_j}, \ j = 1, \dots, n. \end{cases} \tag{2.1}$$

J does not depend on the choice of holomorphic coordinates.

The complex structure map

$$J^t : T^*_p(M') \to T^*_p(M'),$$

for the cotangent space is defined as the adjoint of J, that is $\langle Ju \, , v \rangle = \langle u \, , J^t v \rangle$, $u \in T_p(M')$, $v \in T^*_p(M')$. We have

$$\begin{cases} J^t(dx_j) = -dy_j, \ j = 1, \dots, n, \\ J^t(dy_j) = dx_j, \ j = 1, \dots, n. \end{cases} \tag{2.2}$$

We can now apply our previous discussion of complex structures on real vector spaces to $T_p(M')$ and $T^*_p(M')$. We then get decompositions

$$\begin{aligned} \mathbb{C}T_p(M') &= \Lambda^{1,0}T_p(M') \oplus \Lambda^{0,1}T_p(M'), \\ \mathbb{C}T^*_p(M') &= \Lambda^{1,0}T^*_p(M') \oplus \Lambda^{0,1}T^*_p(M'). \end{aligned} \tag{2.3}$$

For $u \in \Lambda^{1,0} T_p(M')$, $v \in \Lambda^{0,1} T_p^*(M')$,

$$-i\langle u,v\rangle = \langle u, J^t v\rangle = \langle Ju, v\rangle = i\langle u,v\rangle.$$

Thus,

$$\langle u,v\rangle = 0.$$

In terms of local coordinates $z=(z_1,\dots,z_n)$, $z_j = x_j + iy_j$, $j=1,\dots,n$, we have

$$\Lambda^{1,0} T_p(M') = \left\{ \sum_{j=1}^n a_j \frac{\partial}{\partial z_j};\ a_j \in \mathbb{C},\ j=1,\dots,n \right\},$$

$$\Lambda^{0,1} T_p(M') = \left\{ \sum_{j=1}^n a_j \frac{\partial}{\partial \overline{z}_j};\ a_j \in \mathbb{C},\ j=1,\dots,n \right\} \tag{2.4}$$

and

$$\Lambda^{1,0} T_p^*(M') = \left\{ \sum_{j=1}^n a_j dz_j;\ a_j \in \mathbb{C},\ j=1,\dots,n \right\},$$

$$\Lambda^{0,1} T_p^*(M') = \left\{ \sum_{j=1}^n a_j d\overline{z}_j;\ a_j \in \mathbb{C},\ j=1,\dots,n \right\}. \tag{2.5}$$

Here

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),\ \ \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),\ j=1,\dots,n,$$

and

$$dz_j = dx_j + i\,dy_j,\ \ d\overline{z}_j = dx_j - i\,dy_j,\ j=1,\dots,n.$$

For $p, q \in \mathbb{N}$, the bundle of (p,q) forms of M' is given by

$$\Lambda^{p,q} T^*(M') = \Lambda^p(\Lambda^{1,0} T^*(M')) \wedge \Lambda^q(\Lambda^{0,1} T^*(M')). \tag{2.6}$$

That is, the fiber of $\Lambda^{p,q} T^*(M')$ at $z \in M'$ is the vector space $\Lambda^p(\Lambda^{1,0} T_z^*(M')) \wedge \Lambda^q(\Lambda^{0,1} T_z^*(M'))$ of all finite sums of

$$W_1 \wedge \cdots \wedge W_p \wedge V_1 \wedge \cdots \wedge V_q,$$

where

$$W_k \in \Lambda^{1,0} T_z^*(M'),\ k=1,\dots,p,\ V_j \in \Lambda^{0,1} T_z^*(M'),\ j=1,\dots,q.$$

Here $\wedge$ denotes the wedge product.

We recall that if $\left(g_{j,k}\right)_{1\leq j,k\leq n}$ is a positive definite Hermitian matrix then the $(1,1)$ tensor form $g=\sum_{j,k=1}^{n} g_{j,k}dz_j\otimes d\overline{z}_k$ can be viewed as a Hermitian metric $(\,|\,)$ on $\mathbb{C}T(M')$ in the following way:

$$\begin{aligned}&\frac{1}{2}(g(u,\overline{v})+\overline{g(v,\overline{u})})=(u\mid v)=g(u,\overline{v}),\ \ u,v\in\Lambda^{1,0}T(M'),\\&(u\mid w)=0,\ \ u\in\Lambda^{1,0}T(M'),\ w\in\Lambda^{0,1}T(M'),\\&\overline{(u\mid v)}=(\overline{u}\mid\overline{v}),\ u,v\in\Lambda^{0,1}T(M').\end{aligned}$$

We can check that

$$(Ju\mid Jv)=(u\mid v),\ \ u,v\in\mathbb{C}T(M').$$

For $t,\,s\in T(M')$, we write

$$t=\frac{1}{2}(u+\overline{u}),\ \ s=\frac{1}{2}(v+\overline{v}),\ \ u,v\in\Lambda^{1,0}T(M').$$

Then,

$$(t\mid s)=\frac{1}{4}(u\mid v)+\frac{1}{4}\overline{(u\mid v)}=\frac{1}{2}\mathrm{Re}(u\mid v)$$

is real. Thus, the Hermitian metric g induces a J-invariant Riemannian metric $(\,|\,)$ on $T(M')$.

The Hermitian metric $(\,|\,)$ on $\mathbb{C}T(M)$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*(M)$ that we shall also denote by $(\,|\,)$ in the following way. For a given point $z\in M'$, let A be the anti-linear map

$$A:\mathbb{C}T_z(M')\to\mathbb{C}T^*_z(M')$$

defined by

$$(u\mid v)=\langle u,Av\rangle,\ \ u,v\in\mathbb{C}T_z(M'). \tag{2.7}$$

Since $(\,|\,)$ and $\langle,\rangle$ are real, A maps $T_z(M')$ to $T^*_z(M')$. A simple computation shows that

$$J^tAJ=A,\ \ JA^{-1}J^t=A^{-1}.$$

In particular (since A is anti-linear),

$$A\Lambda^{1,0}T_z(M')=\Lambda^{1,0}T^*_z(M'),\ A\Lambda^{0,1}T_z(M')=\Lambda^{0,1}T^*_z(M'). \tag{2.8}$$

For $\omega,\,\mu\in\mathbb{C}T^*_z(M')$, we put

$$(\omega\mid\mu)=(A^{-1}\mu\mid A^{-1}\omega). \tag{2.9}$$

We have

$$(\omega\mid\mu)=0\ \text{ if }\omega\in\Lambda^{1,0}T^*_z(M'),\ \mu\in\Lambda^{0,1}T^*_z(M'). \tag{2.10}$$

The Hermitian metric $(\ |\ )$ on $\Lambda^{p,q}T^*(M')$ is defined by

$$\begin{aligned}
&(w_1 \wedge \cdots \wedge w_p \wedge u_1 \wedge \cdots \wedge u_q \mid t_1 \wedge \cdots \wedge t_p \wedge v_1 \wedge \cdots \wedge v_q) \\
&\quad = \det\Big((w_j \mid t_k)\Big)_{1\le j,k\le p} \det\Big((u_j \mid v_k)\Big)_{1\le j,k\le q}, \\
&\quad u_j, v_k \in \Lambda^{0,1}T^*(M'),\ j,k=1,\dots,q,\ w_j, t_k \in \Lambda^{1,0}T^*(M'),\ j,k=1,\dots,p, \quad (2.11)
\end{aligned}$$

and we extend the definition to arbitrary (p,q) forms by sesqui-linearity.

We associate to the Hermitian metric $\sum_{j,k=1}^n g_{j,k}(z) dz_j \otimes d\overline{z}_k$ a $(1,1)$ form (see page 144 of Kodaira [20])

$$\omega = i \sum_{j,k=1}^n g_{j,k} dz_j \wedge d\overline{z}_k.$$

ω is a real $(1,1)$ form. Put $G(z) = \det\Big(g_{j,k}(z)\Big)_{j,k=1}^n$. Then by an elementary calculation (see page 146 of Kodaira [20]),

$$\omega^n = i^n n! G(z) dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n. \tag{2.12}$$

Since $\Big(g_{j,k}(z)\Big)_{j,k=1}^n$ is positive definite, $G(z) > 0$. Therefore using

$$dM' = \frac{\omega^n}{n!} \tag{2.13}$$

as the volume element, we define the integral of a continuous function $f(z)$ on $\overline{M}$ by

$$\int_M f(z)(dM') = \int_M f(z)\frac{\omega^n}{n!} = \int_M f(z) i^n G(z) dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n. \tag{2.14}$$

We work with local coordinates

$$z_j = x_{2j-1} + i x_{2j},\ \ j = 1,\dots,n.$$

Put

$$F_{j,k}(z) = \Big(\frac{\partial}{\partial x_j} \mid \frac{\partial}{\partial x_k}\Big),\ \ j,k = 1,\dots,2n$$

and

$$F(z) = \det\Big(F_{j,k}(z)\Big)_{1\le j,k\le 2n}.$$

Lemma 2.1. *We have*

$$\sqrt{F(z)}\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n} = \frac{\omega^n}{n!}. \tag{2.15}$$

Proof. For a given point $p \in M'$, we may assume that

$$\omega(p) = i\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j.$$

Thus,

$$(\frac{\partial}{\partial x_j}(p) \mid \frac{\partial}{\partial x_k}(p)) = 2\delta_{j,k},\ \ j,k = 1,\ldots,2n,$$

and

$$\sqrt{F(p)}dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n} = 2^n dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n}.$$

By an elementary calculation,

$$\begin{aligned}\frac{\omega^n}{n!}(p) &= i^n dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n \\ &= 2^n dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n}.\end{aligned}$$

The lemma follows. □

Let $(\,|\,)_M$ be the inner product on $C^\infty(\overline{M};\Lambda^{p,q}T^*(M'))$ defined by

$$(f \mid h)_M = \int_M (f \mid h)(dM'),\ \ f, h \in C^\infty(\overline{M};\Lambda^{p,q}T^*(M')). \tag{2.16}$$

The Hermitian metric $(\,|\,)$ on $\mathbb{C}T(M')$ induces a Hermitian metric $(\,|\,)$ on $\mathbb{C}T(\Gamma)$. For $p \in \Gamma$, we have

$$T_p(\Gamma) = \left\{u \in T_p(M');\ \langle u, dr\rangle = (u \mid \frac{\partial}{\partial r}) = 0\right\},$$

where

$$\frac{\partial}{\partial r} = A^{-1}dr.$$

Here A is as in (2.7). We take $(d\Gamma)$ as the induced volume form on Γ. Let $x = (x_1,\ldots,x_{2n-1})$ be a system of local coordinates on Γ. Put

$$h_{j,k}(x) = (\frac{\partial}{\partial x_j} \mid \frac{\partial}{\partial x_k}),\ \ j,k = 1,\ldots,2n-1.$$

Put

$$H(x) = \det\left(h_{j,k}(x)\right)_{j,k=1}^{2n-1}. \tag{2.17}$$

Then,

$$d\Gamma = \sqrt{H(x)}dx_1 \wedge \cdots \wedge dx_{2n-1}. \tag{2.18}$$

We identify $\mathbb{C}T_p^*(\Gamma)$ with the space

$$\left\{u\in\mathbb{C}T_p^*(M');\ \left\langle u,\frac{\partial}{\partial r}\right\rangle=(u\mid dr)=0\right\}. \tag{2.19}$$

Put

$$\mathscr{C}_p=T_p(\Gamma)\bigcap JT_p(\Gamma),$$
$$\mathscr{C}_p^*=T_p^*(\Gamma)\bigcap J^tT_p^*(\Gamma)$$

and

$$\omega_0=J^t(dr), \tag{2.20}$$
$$Y=J(\frac{\partial}{\partial r}). \tag{2.21}$$

We have

$$\mathscr{C}_p=\left\{u\in T_p(\Gamma);\ \langle u,\omega_0(p)\rangle=0\right\},\ \ \mathscr{C}_p^*=\left\{u\in T_p^*(\Gamma);\ \langle u,Y(p)\rangle=0\right\}.$$

Note that

$$\dim_{\mathbb{R}}\mathscr{C}_p=\dim_{\mathbb{R}}\mathscr{C}_p^*=2n-2.$$

As before, we have

$$\mathbb{C}\mathscr{C}_p=\Lambda^{1,0}T_p(\Gamma)\oplus\Lambda^{0,1}T_p(\Gamma)$$

and

$$\mathbb{C}\mathscr{C}_p^*=\Lambda^{1,0}T_p^*(\Gamma)\oplus\Lambda^{0,1}T_p^*(\Gamma),$$

where

$$\begin{aligned}&Ju=iu\ \text{ if }\ u\in\Lambda^{1,0}T_p(\Gamma),\\&Ju=-iu\ \text{ if }\ u\in\Lambda^{0,1}T_p(\Gamma)\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}&J^t\mu=i\mu\ \text{ if }\ \mu\in\Lambda^{1,0}T_p^*(\Gamma),\\&J^t\mu=-i\mu\ \text{ if }\ \mu\in\Lambda^{0,1}T_p^*(\Gamma).\end{aligned} \tag{2.23}$$

We have the orthogonal decompositions with respect to $(\,|\,)$

$$\mathbb{C}T_p(\Gamma)=\Lambda^{1,0}T_p(\Gamma)\oplus\Lambda^{0,1}T_p(\Gamma)\oplus\{\lambda Y(p);\ \lambda\in\mathbb{C}\},$$

$$\mathbb{C}T_p^*(\Gamma)=\Lambda^{1,0}T_p^*(\Gamma)\oplus\Lambda^{0,1}T_p^*(\Gamma)\oplus\{\lambda\omega_0(p);\ \lambda\in\mathbb{C}\}.$$

We notice that

$$J(iY+\frac{\partial}{\partial r})=J\Big(iJ(\frac{\partial}{\partial r})+\frac{\partial}{\partial r}\Big)=-i(iY+\frac{\partial}{\partial r}).$$

Thus, $iY+\frac{\partial}{\partial r}\in\Lambda^{0,1}T(M')$. Near Γ, put

$$T^{*,0,1}_z=\Big\{u\in\Lambda^{0,1}T^*_z(M');\,(u\,|\,\overline{\partial}r(z))=0\Big\} \tag{2.24}$$

and

$$T^{0,1}_z=\Big\{u\in\Lambda^{0,1}T_z(M');\,(u\,|\,(iY+\frac{\partial}{\partial r})(z))=0\Big\}. \tag{2.25}$$

We have the orthogonal decompositions with respect to $(\,|\,)$

$$\Lambda^{0,1}T^*_z(M')=T^{*,0,1}_z\oplus\Big\{\lambda(\overline{\partial}r)(z);\,\lambda\in\mathbb{C}\Big\}, \tag{2.26}$$

$$\Lambda^{0,1}T_z(M')=T^{0,1}_z\oplus\Big\{\lambda(iY+\frac{\partial}{\partial r})(z);\,\lambda\in\mathbb{C}\Big\}. \tag{2.27}$$

Note that

$$T^{*,0,1}_z=\Lambda^{0,1}T^*_z(\Gamma),\ \ T^{0,1}_z=\Lambda^{0,1}T_z(\Gamma),\ \ z\in\Gamma.$$

For $q\in\mathbb{N}$, the bundle of boundary $(0,q)$ forms is given by

$$\Lambda^{0,q}T^*(\Gamma)=\Lambda^q(\Lambda^{0,1}T^*(\Gamma)). \tag{2.28}$$

Note that

$$\Lambda^{0,q}T^*_z(\Gamma)=\Big\{u\in\Lambda^{0,q}T^*_z(M');\,(u\,|\,\overline{\partial}r(z)\wedge g)=0,\ \ \forall g\in\Lambda^{0,q-1}T^*_z(M')\Big\}. \tag{2.29}$$

Let $(\,|\,)_\Gamma$ be the inner product on $C^\infty(\Gamma;\Lambda^{0,q}T^*(M'))$ defined by

$$(f\,|\,g)_\Gamma=\int_\Gamma(f\,|\,g)d\Gamma,\ \ f,g\in C^\infty(\Gamma;\Lambda^{0,q}T^*(M')), \tag{2.30}$$

where $d\Gamma$ is as in (2.18).

We recall the following

Definition 2.2. For $p\in\Gamma$, the Levi form $L_p(Z,\overline{W})$, $Z,\,W\in\Lambda^{1,0}T_p(\Gamma)$, is the Hermitian quadratic form on $\Lambda^{1,0}T_p(\Gamma)$ defined as follows:

$$\begin{array}{l}\text{For any } Z,\,W\in\Lambda^{1,0}T_p(\Gamma)\text{, pick } \widetilde{Z},\,\widetilde{W}\in C^\infty(\Gamma;\Lambda^{1,0}T(\Gamma))\text{ that satisfy}\\ \widetilde{Z}(p)=Z,\,\widetilde{W}(p)=W.\text{ Then } L_p(Z,\overline{W})=\dfrac{1}{2i}\Big\langle[\widetilde{Z}\,,\overline{\widetilde{W}}](p)\,,\omega_0(p)\Big\rangle.\end{array} \tag{2.31}$$

Here

$$[\widetilde{Z}\,,\overline{\widetilde{W}}]=\widetilde{Z}\overline{\widetilde{W}}-\overline{\widetilde{W}}\widetilde{Z}$$

denotes the commutator of $\widetilde{Z}$ and $\overline{\widetilde{W}}$.

It is easy to see that the definition of the Levi form L_p is independent of the choices of $\widetilde{Z}$ and $\widetilde{W}$. We give it in detail for the convenience of the reader

Lemma 2.3. *Let* $\widetilde{Z}, \widetilde{W} \in C^\infty(\Gamma; \Lambda^{1,0}T(\Gamma))$. *We have*

$$\frac{1}{2i}\left\langle [\widetilde{Z}, \overline{\widetilde{W}}](p), \omega_0(p)\right\rangle = -\frac{1}{2i}\left\langle \widetilde{Z}(p)\wedge\overline{\widetilde{W}}(p), d\omega_0(p)\right\rangle. \tag{2.32}$$

Proof. See Lemma 2.7 of [18]. □

Definition 2.4. The eigenvalues of the Levi form at $p \in \Gamma$ are the eigenvalues of the Hermitian form L_p with respect to the inner product $(\,|\,)$ on $\Lambda^{1,0}T_p(\Gamma)$.

Now, we work with local coordinates $z = (z_1, \ldots, z_n)$ defined on some neighborhood $U \subset M'$ of $p \in \Gamma$. We have

$$dr = \sum_{j=1}^{n} \frac{\partial r}{\partial z_j} dz_j + \sum_{j=1}^{n} \frac{\partial r}{\partial \overline{z}_j} d\overline{z}_j$$

and

$$\omega_0 = J^t(dr) = i\sum_{j=1}^{n} \frac{\partial r}{\partial z_j} dz_j - i\sum_{j=1}^{n} \frac{\partial r}{\partial \overline{z}_j} d\overline{z}_j.$$

Then by an elementary calculation,

$$d\omega_0 = -2i\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_k \partial \overline{z}_j} dz_k \wedge d\overline{z}_j.$$

From this and Lemma 2.3, we get the following

Proposition 2.5. *Let* $U = \sum_{k=1}^{n} u_k \frac{\partial}{\partial z_k}$, $V = \sum_{j=1}^{n} v_j \frac{\partial}{\partial z_j} \in \Lambda^{1,0}T_p(\Gamma)$. *Then,*

$$L_p(U, \overline{V}) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_k \partial \overline{z}_j}(p) u_k \overline{v}_j. \tag{2.33}$$

3 The $\overline{\partial}$-Neumann problem, a review

The $\overline{\partial}$-Neumann problem is a generalization to several complex variables of the Laplace operator of one complex variable and the Cauchy-Riemann operator $\partial/\partial\overline{z}$. In this section, we will give a brief discussion of the $\overline{\partial}$-Neumann problem in a setting appropriate for our purpose. General references for this section are the books by Hörmander [15], Folland-Kohn [9] and Chen-Shaw [3].

As in section 1, let M be a relatively compact open subset with C^∞ boundary Γ of a complex analytic manifold M' of dimension n with a Hermitian metric

$g=\sum_{j,k=1}^{n} g_{j,k}dz_j\otimes d\overline{z}_k$ on its holomorphic tangent bundle. We will use the same notations as before. Let $x=(x_1,\ldots,x_{2n-1})$ be a system of local coordinates on Γ and extend $x_1,\ldots,x_{2n-1}$ to real smooth functions in some neighborhood of Γ. We recall that (see (2.18)) the induced volume form on Γ is given by

$$d\Gamma=\sqrt{H(x)}dx_1\wedge\cdots\wedge dx_{2n-1},$$

where

$$H(x)=\det\left(h_{j,k}(x)\right)_{j,k=1}^{2n-1},\ h_{j,k}(x)=(\frac{\partial}{\partial x_j}\mid\frac{\partial}{\partial x_k}),\ \ j,k=1,\ldots,2n-1.$$

We assume that

$$dM'=|F(x,r)|\,dx_1\wedge\cdots\wedge dx_{2n-1}\wedge dr,$$

where dM' is as in (2.13) and $F(x,r)\in C^\infty$. From (2.15), we see that

$$d\Gamma=|F(x,0)|\,dx_1\wedge\cdots\wedge dx_{2n-1}. \tag{3.1}$$

We have the following

Lemma 3.1. *For all $f\in C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$, $g\in C^\infty(\overline{M};\Lambda^{0,q+1}T^*(M'))$,*

$$(\overline{\partial}f\mid g)_M=(f\mid\overline{\partial}_f^{\,*}g)_M+(\gamma f\mid\gamma(\overline{\partial}r)^{\wedge,*}g)_\Gamma, \tag{3.2}$$

where $(\overline{\partial}r)^{\wedge,*}$ is defined by (1.24). We recall that $\overline{\partial}_f^{\,*}$ is the formal adjoint of $\overline{\partial}$ and γ is the operator of restriction to the boundary Γ.

Proof. By using a partition of unity we may assume that f and g are supported in a coordinate patch $U\subset M'$. Let $x=(x_1,\ldots,x_{2n-1})$ be a system of local coordinates on Γ. We work with local coordinates $z=(x_1,\ldots,x_{2n-1},r)$. Put

$$dM'=|F(x,r)|\,dx_1\cdots dx_{2n-1}dr.$$

Let H be the Heaviside function. Then

$$\begin{aligned}(\overline{\partial}f\mid g)_M&=\int_{\mathbb{C}^n}H(-r)(\overline{\partial}f\mid g)|F(x,r)|\,dxdr\\&=\int_{\mathbb{C}^n}(\overline{\partial}(H(-r)f)\mid g)|F(x,r)|\,dxdr\\&\quad+\int_{\mathbb{C}^n}\delta(-r)((\overline{\partial}r)^\wedge f\mid g)|F(x,r)|\,dxdr\\&=(f\mid\overline{\partial}_f^{\,*}g)_M+(\gamma f\mid\gamma(\overline{\partial}r)^{\wedge,*}g)_\Gamma.\end{aligned}$$

The lemma follows. □

As in section 1, We also use the notation $\overline{\partial}$ for the closure in L^2 of the $\overline{\partial}$ operator, initially defined on $C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$. We notice that

$$\begin{aligned}\operatorname{Dom}\overline{\partial}=\{&u\in L^2(M;\Lambda^{0,q}T^*(M'));\ \text{there exist } u_j\in C^\infty(\overline{M};\Lambda^{0,q}T^*(M')),\ j=1,2,\ldots,\\ &\text{and } v\in L^2(M;\Lambda^{0,q+1}T^*(M')),\ \text{such that } u_j\to u \text{ in } L^2(M;\Lambda^{0,q}T^*(M')),\\ &j\to\infty,\ \text{and } \overline{\partial}u_j\to v \text{ in } L^2(M;\Lambda^{0,q+1}T^*(M')),\ j\to\infty\}.\end{aligned}\tag{3.3}$$

We write $\overline{\partial}u=v$.

The Hilbert space adjoint $\overline{\partial}^*$ of $\overline{\partial}$ is defined on the domain of $\overline{\partial}^*$ consisting of all $f\in L^2(M;\Lambda^{0,q+1}T^*(M'))$ such that for some constant $c>0$,

$$\left|(f\,|\,\overline{\partial}g)_M\right|\leq c\left\|g\right\|,\ \text{ for all } g\in C^\infty(\overline{M};\Lambda^{0,q}T^*(M')).$$

For such a f,

$$g\to(f\,|\,\overline{\partial}g)_M$$

extends to a bounded anti-linear functional on $L^2(M;\Lambda^{0,q}T^*(M'))$ so

$$(f\,|\,\overline{\partial}g)_M=(\widetilde{f}\,|\,g)_M$$

for some $\widetilde{f}\in L^2(M;\Lambda^{0,q}T^*(M'))$. We have $\overline{\partial}^*f=\widetilde{f}$.

From Lemma 3.1, it follows that

$$\operatorname{Dom}\overline{\partial}^*\bigcap C^\infty(\overline{M};\Lambda^{0,q+1}T^*(M'))=\left\{u\in C^\infty(\overline{M};\Lambda^{0,q+1}T^*(M'));\gamma(\overline{\partial}r)^{\wedge,*}u=0\right\}\tag{3.4}$$

and

$$\overline{\partial}^*=\overline{\partial}_f^*\ \text{ on } \operatorname{Dom}\overline{\partial}^*\bigcap C^\infty(\overline{M};\Lambda^{0,q+1}T^*(M')).\tag{3.5}$$

The $\overline{\partial}$-Neumann Laplacian on $(0,q)$ forms is then the operator in the space $L^2(M;\Lambda^{0,q}T^*(M'))$

$$\Box^{(q)}=\overline{\partial}\,\overline{\partial}^*+\overline{\partial}^*\overline{\partial}.$$

We notice that $\Box^{(q)}$ is self-adjoint. (See chapter I of Folland-Kohn [9].) We have

$$\begin{aligned}\operatorname{Dom}\Box^{(q)}=\{&u\in L^2(M;\Lambda^{0,q}T^*(M'));\ u\in\operatorname{Dom}\overline{\partial}^*\bigcap\operatorname{Dom}\overline{\partial},\\ &\overline{\partial}^*u\in\operatorname{Dom}\overline{\partial},\overline{\partial}u\in\operatorname{Dom}\overline{\partial}^*\}.\end{aligned}$$

Put

$$D^{(q)}=\operatorname{Dom}\Box^{(q)}\bigcap C^\infty(\overline{M}\,;\Lambda^{0,q}T^*(M')).$$

From (3.4), we have

$$D^{(q)}=\left\{u\in C^\infty(\overline{M};\Lambda^{0,q+1}T^*(M'));\gamma(\overline{\partial}r)^{\wedge,*}u=0,\ \gamma(\overline{\partial}r)^{\wedge,*}\overline{\partial}u=0\right\}.\tag{3.6}$$

In view of (2.29), we see that $u \in D^{(q)}$ if and only if

$$\gamma u \in C^\infty(\Gamma; \Lambda^{0,q} T^*(\Gamma)) \tag{3.7}$$

and

$$\gamma \overline{\partial} u \in C^\infty(\Gamma; \Lambda^{0,q+1} T^*(\Gamma)). \tag{3.8}$$

We have the following

Lemma 3.2. *Let $q \geq 1$. For every $u \in \operatorname{Dom} \overline{\partial}^* \bigcap C^\infty(\overline{M}; \Lambda^{0,q+1} T^*(M'))$, we have*

$$\overline{\partial}^* u \in \operatorname{Dom} \overline{\partial}^* \bigcap C^\infty(\overline{M}; \Lambda^{0,q} T^*(M')).$$

Proof. Let $u \in \operatorname{Dom} \overline{\partial}^* \bigcap C^\infty(\overline{M}; \Lambda^{0,q+1} T^*(M'))$. For $g \in C^\infty(\overline{M}; \Lambda^{0,q-1} T^*(M'))$, we have

$$\begin{aligned} 0 = (\overline{\partial}_f^* \overline{\partial}^* u \mid g)_M &= (\overline{\partial}^* u \mid \overline{\partial} g)_M - (\gamma(\overline{\partial} r)^{\wedge,*} \overline{\partial}^* u \mid \gamma g)_\Gamma \\ &= (u \mid \overline{\partial}\,\overline{\partial} g)_M - (\gamma(\overline{\partial} r)^{\wedge,*} \overline{\partial}^* u \mid \gamma g)_\Gamma \\ &= -(\gamma(\overline{\partial} r)^{\wedge,*} \overline{\partial}^* u \mid \gamma g)_\Gamma. \end{aligned}$$

Here we used (3.2). Thus,

$$\gamma(\overline{\partial} r)^{\wedge,*} \overline{\partial}^* u = 0.$$

The lemma follows. □

Definition 3.3. The boundary conditions

$$\gamma(\overline{\partial} r)^{\wedge,*} u = 0,\ \gamma(\overline{\partial} r)^{\wedge,*} \overline{\partial} u = 0,\ u \in C^\infty(\overline{M}, \Lambda^{0,q} T^*(M'))$$

are called $\overline{\partial}$-Neumann boundary conditions.

Definition 3.4. The $\overline{\partial}$-Neumann problem in M is the problem of finding, given a form $f \in C^\infty(\overline{M}; \Lambda^{0,q} T^*(M'))$, another form $u \in D^{(q)}$ verifying

$$\Box^{(q)} u = f.$$

Definition 3.5. Given q, $0 \leq q \leq n-1$. The Levi form is said to satisfy condition $Z(q)$ at $p \in \Gamma$ if it has at least $n-q$ positive or at least $q+1$ negative eigenvalues. If the Levi form is non-degenerate at $p \in \Gamma$, let (n_-, n_+), $n_- + n_+ = n-1$, be the signature. Then $Z(q)$ holds at p if and only if $q \neq n_-$.

We have the following classical results (see Folland-Kohn [9])

Theorem 3.6. *(*Kohn*) We assume that $Z(q)$ is satisfied at each point of Γ. Then* $\operatorname{Ker}\Box^{(q)}$ *is a finite dimensional subspace of $C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$, $\Box^{(q)}$ has closed range and $\Pi^{(q)}$ is a smoothing operator. That is, the distribution kernel*

$$K_{\Pi^{(q)}}(z,w)\in C^\infty(\overline{M}\times\overline{M};\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))).$$

Moreover, there exists an operator

$$N^{(q)}:L^2(M;\Lambda^{0,q}T^*(M'))\to\operatorname{Dom}\Box^{(q)}$$

such that

$$\begin{aligned}&N^{(q)}\Box^{(q)}+\Pi^{(q)}=I\ \ \textit{on}\ \operatorname{Dom}\Box^{(q)},\\&\Box^{(q)}N^{(q)}+\Pi^{(q)}=I\ \ \textit{on}\ L^2(M;\Lambda^{0,q}T^*(M')).\end{aligned}$$

Furthermore, $N^{(q)}\Big(C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))\Big)\subset C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$ and for each $s\in\mathbb{R}$ and all $f\in C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$, there is a constant $c>0$, such that

$$\left\|N^{(q)}f\right\|_{s+1}\le c\left\|f\right\|_s$$

where $\|\ \|_s$ denotes any of the equivalent norms defining $H^s(\overline{M};\Lambda^{0,q}T^*(M'))$.

Theorem 3.7. *(*Kohn*) Suppose that $Z(q)$ fails at some point of Γ and that $Z(q-1)$ and $Z(q+1)$ hold at each point of Γ. Then,*

$$\Pi^{(q)}u=(I-\overline{\partial}N^{(q-1)}\overline{\partial}^*-\overline{\partial}^*N^{(q+1)}\overline{\partial})u,\ u\in\operatorname{Dom}\overline{\partial}^*\bigcap C^\infty(\overline{M};\Lambda^{0,q}T^*(M')),$$

where $N^{(q+1)}$ and $N^{(q-1)}$ are as in Theorem 3.6. In particular,

$$\Pi^{(q)}:\operatorname{Dom}\overline{\partial}^*\bigcap C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))\to D^{(q)}.$$

4 The operator T

By Assumption 1.1, the map

$$\begin{aligned}F^{(q)}:H^2(\overline{M};\Lambda^{0,q}T^*(M'))&\to H^0(\overline{M};\Lambda^{0,q}T^*(M'))\oplus H^{\frac{3}{2}}(\Gamma;\Lambda^{0,q}T^*(M')),\\u&\to(\Box^{(q)}_f u,\gamma u),\end{aligned}$$

is injective. The Poisson operator

$$P:C^\infty(\Gamma;\Lambda^{0,q}T^*(M'))\to C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$$

of $\Box_f^{(q)}$ is well-defined. That is, if

$$u \in C^\infty(\Gamma; \Lambda^{0,q} T^*(M')),$$

then

$$Pu \in C^\infty(\overline{M}; \Lambda^{0,q} T^*(M')), \ \ \Box_f^{(q)} Pu = 0$$

and

$$\gamma Pu = u.$$

Moreover, if $v \in C^\infty(\overline{M}; \Lambda^{0,q} T^*(M'))$ and $\Box_f^{(q)} v = 0$, then

$$v = P\gamma v.$$

It is well-known (see page 29 of [5]) that P extends continuously

$$P : H^s(\Gamma; \Lambda^{0,q} T^*(M')) \to H^{s+\frac{1}{2}}(\overline{M}; \Lambda^{0,q} T^*(M')), \ \ \forall\, s \in \mathbb{R}.$$

Let

$$P^* : \mathscr{E}'(\overline{M}; \Lambda^{0,q} T^*(M')) \to \mathscr{D}'(\Gamma; \Lambda^{0,q} T^*(M'))$$

be the operator defined by

$$(P^* u \mid v)_\Gamma = (u \mid Pv)_M, \ u \in \mathscr{E}'(\overline{M}; \Lambda^{0,q} T^*(M')), \ v \in C^\infty(\Gamma; \Lambda^{0,q} T^*(M')).$$

It is well-known (see page 30 of [5]) that P^* is continuous:

$$P^* : L^2(M; \Lambda^{0,q} T^*(M')) \to H^{\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M'))$$

and

$$P^* : C^\infty(\overline{M}; \Lambda^{0,q} T^*(M')) \to C^\infty(\Gamma; \Lambda^{0,q} T^*(M')).$$

Let L be a classical pseudodifferential operator on a C^∞ manifold. From now on, we let σ_L denote the principal symbol of L. The operator

$$P^* P : C^\infty(\Gamma; \Lambda^{0,q} T^*(M')) \to C^\infty(\Gamma; \Lambda^{0,q} T^*(M'))$$

is a classical elliptic pseudodifferential operator of order -1 and invertible (since P is injective). (See Boutet de Monvel [4].) Let $\sqrt{-\triangle_\Gamma}$ be the square root of $-\triangle_\Gamma$. It is well-known (see [4]) that

$$\sigma_{P^* P} = \sigma_{(2\sqrt{-\triangle_\Gamma})^{-1}}. \tag{4.1}$$

Let

$$(P^* P)^{-1} : C^\infty(\Gamma; \Lambda^{0,q} T^*(M')) \to C^\infty(\Gamma; \Lambda^{0,q} T^*(M'))$$

be the inverse of $P^* P$. $(P^* P)^{-1}$ is a classical elliptic pseudodifferential operator of order 1 with scalar principal symbol. We have

$$\sigma_{(P^* P)^{-1}} = \sigma_{2\sqrt{-\triangle_\Gamma}}. \tag{4.2}$$

Definition 4.1. The Neumann operator $\mathcal{N}^{(q)}$ is the operator on $C^\infty(\Gamma;\Lambda^{0,q}T^*(M'))$ defined as follows:

$$\mathcal{N}^{(q)}f=\gamma\frac{\partial}{\partial r}Pf,\ f\in C^\infty(\Gamma;\Lambda^{0,q}T^*(M')).$$

The following is well-known (see page 95 of [10])

Lemma 4.2.

$$\mathcal{N}^{(q)}:C^\infty(\Gamma;\Lambda^{0,q}T^*(M'))\to C^\infty(\Gamma;\Lambda^{0,q}T^*(M'))$$

is a classical elliptic pseudodifferential operator of order 1 *with scalar principal symbol and we have*

$$\sigma_{\mathcal{N}^{(q)}}=\sigma_{\sqrt{-\triangle_\Gamma}}. \tag{4.3}$$

We use the inner product $[\,|\,]$ on $H^{-\frac{1}{2}}(\Gamma;\Lambda^{0,q}T^*(M'))$ defined as follows:

$$[u\mid v]=(Pu\mid Pv)_M=(P^*Pu\mid v)_\Gamma, \tag{4.4}$$

where $u,v\in H^{-\frac{1}{2}}(\Gamma;\Lambda^{0,q}T^*(M'))$. We consider $(\overline{\partial}r)^{\wedge,*}$ as an operator

$$(\overline{\partial}r)^{\wedge,*}:H^{-\frac{1}{2}}(\Gamma;\Lambda^{0,q}T^*(M'))\to H^{-\frac{1}{2}}(\Gamma;\Lambda^{0,q-1}T^*(M')).$$

Let

$$T:H^{-\frac{1}{2}}(\Gamma;\Lambda^{0,q}T^*(M'))\to \operatorname{Ker}(\overline{\partial}r)^{\wedge,*}=H^{-\frac{1}{2}}(\Gamma;\Lambda^{0,q}T^*(\Gamma)) \tag{4.5}$$

be the orthogonal projection onto $\operatorname{Ker}(\overline{\partial}r)^{\wedge,*}$ with respect to $[\,|\,]$. That is, if $u\in H^{-\frac{1}{2}}(\Gamma;\Lambda^{0,q}T^*(M'))$, then

$$(\overline{\partial}r)^{\wedge,*}Tu=0$$

and

$$[(I-T)u\mid g]=0,\ \forall\ g\in\operatorname{Ker}(\overline{\partial}r)^{\wedge,*}.$$

Lemma 4.3. *T is a classical pseudodifferential operator of order* 0 *with principal symbol*

$$2(\overline{\partial}r)^{\wedge,*}(\overline{\partial}r)^{\wedge}.$$

Moreover,

$$I-T=(P^*P)^{-1}(\overline{\partial}r)^{\wedge}R, \tag{4.6}$$

where

$$R:C^\infty(\Gamma;\Lambda^{0,q}T^*(M'))\to C^\infty(\Gamma;\Lambda^{0,q-1}T^*(M'))$$

is a classical pseudodifferential operator of order -1.

Proof. Let

$$E = 2(\overline{\partial} r)^{\wedge,*}\big((\overline{\partial} r)^{\wedge,*}\big)^{\dagger} + 2\big((\overline{\partial} r)^{\wedge,*}\big)^{\dagger}(\overline{\partial} r)^{\wedge,*},$$
$$E : H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M')) \to H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M')),$$

where $\big((\overline{\partial} r)^{\wedge,*}\big)^{\dagger}$ is the formal adjoint of $(\overline{\partial} r)^{\wedge,*}$ with respect to $[\,|\,]$. That is,

$$[(\overline{\partial} r)^{\wedge,*} u \mid v] = [u \mid \big((\overline{\partial} r)^{\wedge,*}\big)^{\dagger} v],$$
$$u \in H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M')), v \in H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q-1} T^*(M')).$$

We can check that

$$\big((\overline{\partial} r)^{\wedge,*}\big)^{\dagger} = (P^* P)^{-1}(\overline{\partial} r)^{\wedge}(P^* P). \tag{4.7}$$

Thus, the principal symbol of E is

$$2(\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^{\wedge} + 2(\overline{\partial} r)^{\wedge}(\overline{\partial} r)^{\wedge,*}.$$

Since $\|dr\| = 1 = (\left\|\bar{\partial} r\right\|^2 + \|\partial r\|^2)^{\frac{1}{2}}$ on Γ, we have

$$\left\|\bar{\partial} r\right\|^2 = \|\partial r\|^2 = \frac{1}{2} \text{ on } \Gamma. \tag{4.8}$$

From this, we can check that

$$2(\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^{\wedge} + 2(\overline{\partial} r)^{\wedge}(\overline{\partial} r)^{\wedge,*} = I : H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M')) \to H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M')),$$

where I is the identity map. E is a classical elliptic pseudodifferential operator with principal symbol I. Then $\dim \operatorname{Ker} E < \infty$. Let G be the orthogonal projection onto $\operatorname{Ker} E$ and N be the partial inverse. Then G is a smoothing operator and N is a classical elliptic pseudodifferential operator of order 0 with principal symbol I (up to some smoothing operator). We have

$$EN + G = 2\Big((\overline{\partial} r)^{\wedge,*}\big((\overline{\partial} r)^{\wedge,*}\big)^{\dagger} + 2\big((\overline{\partial} r)^{\wedge,*}\big)^{\dagger}(\overline{\partial} r)^{\wedge,*}\Big)N + G = I \tag{4.9}$$

on $H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M'))$. Put

$$\widetilde{T} = 2(\overline{\partial} r)^{\wedge,*}\big((\overline{\partial} r)^{\wedge,*}\big)^{\dagger} N + G.$$

Note that

$$\operatorname{Ker} E = \Big\{u \in H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M')); (\overline{\partial} r)^{\wedge,*} u = 0,\ \big((\overline{\partial} r)^{\wedge,*}\big)^{\dagger} u = 0\Big\}.$$

From this and $(\overline{\partial} r)^{\wedge,*} \circ (\overline{\partial} r)^{\wedge,*} = 0$, we see that

$$\widetilde{T} g \in \operatorname{Ker}(\overline{\partial} r)^{\wedge,*}, \quad g \in H^{-\frac{1}{2}}(\Gamma; \Lambda^{0,q} T^*(M')).$$

From (4.9), we have

$$I-\widetilde{T}=2\big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger}(\overline{\partial}r)^{\wedge,*}N$$

and

$$\begin{aligned}[(I-\widetilde{T})g\mid u]&=[2\big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger}(\overline{\partial}r)^{\wedge,*}Ng\mid u]\\&=[2(\overline{\partial}r)^{\wedge,*}Ng\mid(\overline{\partial}r)^{\wedge,*}u]\\&=0,\ \ u\in\mathrm{Ker}\,(\overline{\partial}r)^{\wedge,*},\,g\in H^{-\frac{1}{2}}(\Gamma;\Lambda^{0,q}T^*(M')).\end{aligned}$$

Thus,

$$g=\widetilde{T}g+(I-\widetilde{T})g$$

is the orthogonal decomposition with respect to $[\,|\,]$. Hence,

$$\widetilde{T}=T.$$

The lemma follows. □

Now, we assume that $Z(q)$ fails at some point of Γ and that $Z(q-1)$ and $Z(q+1)$ hold at each point of Γ. We recall that

$$\Pi^{(q)}u=(I-\overline{\partial}N^{(q-1)}\overline{\partial}^*-\overline{\partial}^*N^{(q+1)}\overline{\partial})u,\ u\in\mathrm{Dom}\,\overline{\partial}^*\bigcap C^\infty(\overline{M};\Lambda^{0,q}T^*(M')).$$

(See Theorem 3.7.)

Proposition 4.4. *We assume that $Z(q)$ fails at some point of Γ and that $Z(q-1)$ and $Z(q+1)$ hold at each point of Γ. Then,*

$$\begin{aligned}\Pi^{(q)}u&=\Pi^{(q)}PT(P^*P)^{-1}P^*u\\&=(I-\overline{\partial}N^{(q-1)}\overline{\partial}^*-\overline{\partial}^*N^{(q+1)}\overline{\partial})PT(P^*P)^{-1}P^*u,\\&\quad u\in C^\infty(\overline{M};\Lambda^{0,q}T^*(M')),\end{aligned}\tag{4.10}$$

where $N^{(q+1)}$, $N^{(q-1)}$ are as in Theorem 3.6 and T is as in (4.5). In particular,

$$\Pi^{(q)}:C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))\to D^{(q)}.$$

Proof. Let $u\in C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$. We claim that

$$u-\Pi^{(q)}PT(P^*P)^{-1}P^*u\in\big(\mathrm{Ker}\,\Box^{(q)}\big)^{\perp}.\tag{4.11}$$

Let $v\in\mathrm{Dom}\,\overline{\partial}^*\bigcap C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$. From Theorem 3.7, we know that

$$\Pi^{(q)}v\in D^{(q)}.$$

Since $\Pi^{(q)}v$ is harmonic,

$$\Pi^{(q)}v = P\gamma\Pi^{(q)}v.$$

Note that

$$P(P^*P)^{-1}P^* : C^\infty(\overline{M}; \Lambda^{0,q}T^*(M')) \to \operatorname{Ker}\Box_f^{(q)}$$

is the orthogonal projection with respect to $(\ |\ )_M$. That is,

$$(f - P(P^*P)^{-1}P^*f \mid P\gamma g)_M = 0, \quad f, g \in C^\infty(\overline{M}; \Lambda^{0,q}T^*(M')).$$

We have

$$\begin{aligned}
&(\Pi^{(q)}PT(P^*P)^{-1}P^*u \mid \Pi^{(q)}v)_M \\
&\quad = (PT(P^*P)^{-1}P^*u \mid \Pi^{(q)}v)_M \\
&\quad = (PT(P^*P)^{-1}P^*u \mid P\gamma\Pi^{(q)}v)_M \\
&\quad = (P(P^*P)^{-1}P^*u \mid P\gamma\Pi^{(q)}v)_M \\
&\quad = (u \mid \Pi^{(q)}v)_M. \qquad (4.12)
\end{aligned}$$

Thus,

$$(u - \Pi^{(q)}PT(P^*P)^{-1}P^*u \mid \Pi^{(q)}v)_M = 0. \qquad (4.13)$$

Since $\operatorname{Dom}\overline{\partial}^* \bigcap C^\infty(\overline{M}; \Lambda^{0,q}T^*(M'))$ is dense in $L^2(M; \Lambda^{0,q}T^*(M'))$, we get (4.11). Thus,

$$\Pi^{(q)}u = \Pi^{(q)}PT(P^*P)^{-1}P^*u.$$

Since $PT(P^*P)^{-1}P^*u \in \operatorname{Dom}\overline{\partial}^* \bigcap C^\infty(\overline{M}; \Lambda^{0,q}T^*(M'))$, we get the last identity in (4.10). The proposition follows. □

5 The principal symbols of $\gamma\overline{\partial}P$ and $\gamma\overline{\partial}_f^*P$

First, we compute the principal symbols of $\overline{\partial}$ and $\overline{\partial}_f^*$. For each point $z_0 \in \Gamma$, we can choose an orthonormal frame

$$t_1(z), \ldots, t_{n-1}(z) \qquad (5.1)$$

for $T_z^{*,0,1}$ varying smoothly with z in a neighborhood of z_0, where $T_z^{*0,1}$ is defined by (2.24). Then (see (2.26))

$$t_1(z), \ldots, t_{n-1}(z), t_n(z) := \frac{\overline{\partial}r(z)}{\left\|\overline{\partial}r(z)\right\|}$$

is an orthonormal frame for $\Lambda^{0,1}T_z^*(M')$. Let

$$T_1(z), \ldots, T_{n-1}(z), T_n(z) \qquad (5.2)$$

denote the basis of $\Lambda^{0,1}T_z(M')$ which is dual to

$$t_1(z),\dots,t_n(z).$$

We have (see (2.27))

$$T_n = \frac{iY + \frac{\partial}{\partial r}}{\left\| iY + \frac{\partial}{\partial r}\right\|}. \tag{5.3}$$

Note that

$$T_1(z),\dots,T_{n-1}(z) \text{ is an orthonormal frame for } \Lambda^{0,1}T_z(\Gamma),\ z\in\Gamma, \tag{5.4}$$

and

$$t_1(z),\dots,t_{n-1}(z) \text{ is an orthonormal frame for } \Lambda^{0,1}T^*_z(\Gamma),\ z\in\Gamma. \tag{5.5}$$

We have

$$\overline{\partial} f = \Big(\sum_{j=1}^{n} t_j^{\wedge} T_j\Big) f,\ f\in C^\infty(M').$$

If $f(z)t_{j_1}(z)\wedge\cdots\wedge t_{j_q}(z)\in C^\infty(M';\Lambda^{0,q}T^*(M'))$ is a typical term in a general $(0,q)$ form, we have

$$\begin{aligned}\overline{\partial} f &= \sum_{j=1}^{n}(T_j f)t_j^{\wedge} t_{j_1}\wedge\cdots\wedge t_{j_q}\\ &\quad+\sum_{k=1}^{q}(-1)^{k-1} f(z)t_{j_1}\wedge\cdots\wedge(\overline{\partial} t_{j_k})\wedge\cdots\wedge t_{j_q}.\end{aligned}$$

So for the given orthonormal frame we have

$$\begin{aligned}\overline{\partial} &= \sum_{j=1}^{n} t_j^{\wedge}\circ T_j + \text{lower order terms}\\ &= \sum_{j=1}^{n-1} t_j^{\wedge}\circ T_j + \frac{(\overline{\partial} r)^{\wedge}}{\left\|\overline{\partial} r\right\|}\circ\frac{iY+\frac{\partial}{\partial r}}{\left\|iY+\frac{\partial}{\partial r}\right\|} + \text{lower order terms}\end{aligned} \tag{5.6}$$

and correspondingly

$$\overline{\partial}_f^{*} = \sum_{j=1}^{n-1} t_j^{\wedge,*}\circ T_j^* + \frac{(\overline{\partial} r)^{\wedge,*}}{\left\|\overline{\partial} r\right\|}\circ\frac{iY-\frac{\partial}{\partial r}}{\left\|iY+\frac{\partial}{\partial r}\right\|} + \text{lower order terms.} \tag{5.7}$$

We consider

$$\gamma\overline{\partial} P: C^\infty(\Gamma;\Lambda^{0,q}T^*(M'))\to C^\infty(\Gamma;\Lambda^{0,q+1}T^*(M'))$$

and

$$\gamma\overline{\partial_f}^* P : C^\infty(\Gamma;\Lambda^{0,q+1}T^*(M')) \to C^\infty(\Gamma;\Lambda^{0,q}T^*(M')).$$

$\gamma\overline{\partial}P$ and $\gamma\overline{\partial_f}^* P$ are classical pseudodifferential operators of order 1. From (4.8), we know that

$$\left\|\overline{\partial}r\right\| = \frac{1}{\sqrt{2}} \ \text{ on } \Gamma.$$

We can check that

$$\left\|iY + \frac{\partial}{\partial r}\right\| = \sqrt{2} \ \text{ on } \Gamma.$$

Combining this with (5.6), (5.7) and (4.3), we get

$$\gamma\overline{\partial}P = \sum_{j=1}^{n-1} t_j^\wedge \circ T_j + (\overline{\partial}r)^\wedge \circ (iY + \sqrt{-\triangle_\Gamma}) + \text{lower order terms} \tag{5.8}$$

and

$$\gamma\overline{\partial_f}^* P = \sum_{j=1}^{n-1} t_j^{\wedge,*} \circ T_j^* + (\overline{\partial}r)^{\wedge,*} \circ (iY - \sqrt{-\triangle_\Gamma}) + \text{lower order terms.} \tag{5.9}$$

From Lemma 3.2, it follows that

$$\gamma\overline{\partial_f}^* P : C^\infty(\Gamma;\Lambda^{0,q+1}T^*(\Gamma)) \to C^\infty(\Gamma;\Lambda^{0,q}T^*(\Gamma)). \tag{5.10}$$

Put

$$\begin{aligned}\Sigma^+ &= \left\{(x,\lambda\omega_0(x)) \in T^*(\Gamma)\setminus 0;\ \lambda > 0\right\},\\ \Sigma^- &= \left\{(x,\lambda\omega_0(x)) \in T^*(\Gamma)\setminus 0;\ \lambda < 0\right\}.\end{aligned} \tag{5.11}$$

We recall that $\omega_0 = J^t(dr)$. In section 8, we need the following

Proposition 5.1. *The map*

$$\gamma(\overline{\partial}r)^{\wedge,*}\overline{\partial}P : C^\infty(\Gamma;\Lambda^{0,q}T^*(\Gamma)) \to C^\infty(\Gamma;\Lambda^{0,q}T^*(\Gamma)).$$

is a classical pseudodifferential operator of order one from boundary $(0,q)$ forms to boundary $(0,q)$ forms and we have

$$\gamma(\overline{\partial}r)^{\wedge,*}\overline{\partial}P = \frac{1}{2}(iY + \sqrt{-\triangle_\Gamma}) + \textit{lower order terms.} \tag{5.12}$$

In particular, it is elliptic outside Σ^-.

Proof. Note that

$$\gamma(\overline{\partial} r)^{\wedge,*}\overline{\partial} P = \gamma(\overline{\partial} r)^{\wedge,*}\overline{\partial} P + \gamma\overline{\partial} P(\overline{\partial} r)^{\wedge,*} \tag{5.13}$$

on the space $C^\infty(\Gamma; \Lambda^{0,q} T^*(\Gamma))$. From (5.8), we have

$$\gamma(\overline{\partial} r)^{\wedge,*}\overline{\partial} P = \sum_{j=1}^{n-1}\Big((\overline{\partial} r)^{\wedge,*} t_j^{\wedge}\Big)\circ T_j + \Big((\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^{\wedge}\Big)\circ(iY+\sqrt{-\triangle_\Gamma})$$
$$+\text{lower order terms}$$

and

$$\gamma\overline{\partial} P(\overline{\partial} r)^{\wedge,*} = \sum_{j=1}^{n-1}\Big(t_j^{\wedge}(\overline{\partial} r)^{\wedge,*}\Big)\circ T_j + \Big((\overline{\partial} r)^{\wedge}(\overline{\partial} r)^{\wedge,*}\Big)\circ(iY+\sqrt{-\triangle_\Gamma})$$
$$+\text{lower order terms.}$$

Thus,

$$\begin{aligned}\gamma(\overline{\partial} r)^{\wedge,*}\overline{\partial} P + \gamma\overline{\partial} P(\overline{\partial} r)^{\wedge,*} &= \sum_{j=1}^{n-1}\Big(t_j^{\wedge}(\overline{\partial} r)^{\wedge,*} + (\overline{\partial} r)^{\wedge,*} t_j^{\wedge}\Big)\circ T_j \\ &\quad+\Big((\overline{\partial} r)^{\wedge}(\overline{\partial} r)^{\wedge,*} + (\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^{\wedge}\Big)\circ(iY+\sqrt{-\triangle_\Gamma}) \\ &\quad+\text{lower order terms.}\end{aligned} \tag{5.14}$$

Note that

$$t_j^{\wedge}(\overline{\partial} r)^{\wedge,*} + (\overline{\partial} r)^{\wedge,*} t_j^{\wedge} = 0,\ \ j=1,\ldots,n-1, \tag{5.15}$$

and

$$(\overline{\partial} r)^{\wedge}(\overline{\partial} r)^{\wedge,*} + (\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^{\wedge} = \frac{1}{2}. \tag{5.16}$$

Combining this with (5.14) and (5.13), we get (5.12).

Note that

$$\sigma_{iY+\sqrt{-\triangle_\Gamma}}(x,\xi) = -\langle Y,\xi\rangle + \|\xi\| = |\xi| + (\omega_0 \mid \xi) \geq 0$$

with equality precisely when $\xi = -\lambda\omega_0$, $\lambda \geq 0$. The proposition follows. □

For $z \in \Gamma$, put

$$I^{0,q} T_z^*(M') = \Big\{u \in \Lambda^{0,q} T_z^*(M');\ u = (\overline{\partial} r)^{\wedge} g,\ g \in \Lambda^{0,q-1} T_z^*(M')\Big\}. \tag{5.17}$$

$I^{0,q} T_z^*(M)$ is orthogonal to $\Lambda^{0,q} T_z^*(\Gamma)$. In section 7, we need the following

Proposition 5.2. *The operator*

$$\gamma(\overline{\partial}r)^{\wedge}\overline{\partial_f}^{*}P(P^{*}P)^{-1}: C^{\infty}(\Gamma; I^{0,q}T^{*}(M')) \to C^{\infty}(\Gamma; I^{0,q}T^{*}(M')).$$

is a classical pseudodifferential operator of order one,

$$\gamma(\overline{\partial}r)^{\wedge}\overline{\partial_f}^{*}P(P^{*}P)^{-1} = (iY - \sqrt{-\triangle_\Gamma})\sqrt{-\triangle_\Gamma} + \textit{lower order terms.} \tag{5.18}$$

It is elliptic outside Σ^{+}.

Proof. Note that

$$\gamma(\overline{\partial}r)^{\wedge}\overline{\partial_f}^{*}P(P^{*}P)^{-1} = \gamma(\overline{\partial}r)^{\wedge}\overline{\partial_f}^{*}P(P^{*}P)^{-1} + \gamma\overline{\partial_f}^{*}P(P^{*}P)^{-1}(\overline{\partial}r)^{\wedge} \tag{5.19}$$

on the space $C^{\infty}(\Gamma; I^{0,q}T^{*}(\Gamma))$. From (5.9) and (4.2), we have

$$\begin{aligned}\gamma(\overline{\partial}r)^{\wedge}\overline{\partial_f}^{*}P(P^{*}P)^{-1} &= \sum_{j=1}^{n-1}\Big((\overline{\partial}r)^{\wedge}t_j^{\wedge,*}\Big)\circ\Big(T_j^{*}\circ 2\sqrt{-\triangle_\Gamma}\Big)\\ &+\Big((\overline{\partial}r)^{\wedge}(\overline{\partial}r)^{\wedge,*}\Big)\circ\Big((iY-\sqrt{-\triangle_\Gamma})\circ 2\sqrt{-\triangle_\Gamma}\Big) + \text{lower order terms}\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}\gamma\overline{\partial_f}^{*}P(P^{*}P)^{-1}(\overline{\partial}r)^{\wedge} &= \sum_{j=1}^{n-1}\Big(t_j^{\wedge,*}(\overline{\partial}r)^{\wedge}\Big)\circ\Big(T_j^{*}\circ 2\sqrt{-\triangle_\Gamma}\Big)\\ &+\Big((\overline{\partial}r)^{\wedge,*}(\overline{\partial}r)^{\wedge}\Big)\circ\Big((iY-\sqrt{-\triangle_\Gamma})\circ 2\sqrt{-\triangle_\Gamma}\Big) + \text{lower order terms.}\end{aligned} \tag{5.21}$$

Thus,

$$\begin{aligned}&\gamma(\overline{\partial}r)^{\wedge}\overline{\partial_f}^{*}P(P^{*}P)^{-1} + \gamma\overline{\partial_f}^{*}P(P^{*}P)^{-1}(\overline{\partial}r)^{\wedge}\\ &= \sum_{j=1}^{n-1}\Big(t_j^{\wedge,*}(\overline{\partial}r)^{\wedge} + (\overline{\partial}r)^{\wedge}t_j^{\wedge,*}\Big)\circ\Big(T_j^{*}\circ 2\sqrt{-\triangle_\Gamma}\Big)\\ &+\Big((\overline{\partial}r)^{\wedge,*}(\overline{\partial}r)^{\wedge} + (\overline{\partial}r)^{\wedge}(\overline{\partial}r)^{\wedge,*}\Big)\circ\Big((iY-\sqrt{-\triangle_\Gamma})\circ 2\sqrt{-\triangle_\Gamma}\Big) + \text{lower order terms.}\end{aligned} \tag{5.22}$$

Combining this with (5.19), (5.15) and (5.16), we get (5.18). The proposition follows. □

6 The operator $\Box_\beta^{(q)}$

Put

$$\overline{\partial_\beta} = T\gamma\overline{\partial}P : C^{\infty}(\Gamma; \Lambda^{0,q}T^{*}(\Gamma)) \to C^{\infty}(\Gamma; \Lambda^{0,q+1}T^{*}(\Gamma)). \tag{6.1}$$

We recall that (see (4.5)) the orthogonal projection T onto $\mathrm{Ker}(\overline{\partial} r)^{\wedge,*}$ with respect to $[\,|\,]$ is a classical pseudodifferential operator of order 0 with principal symbol

$$2(\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^{\wedge}.$$

(See Lemma 4.3.) $\overline{\partial_\beta}$ is a classical pseudodifferential operator of order one from boundary $(0,q)$ forms to boundary $(0,q+1)$ forms.

Lemma 6.1. *We have*

$$(\overline{\partial_\beta})^2 = 0.$$

Proof. Let $u, v \in C^\infty(\Gamma; \Lambda^{0,q}T^*(\Gamma))$. We claim that

$$[T\gamma\overline{\partial} P(I-T)\gamma\overline{\partial} Pu \mid v] = 0. \tag{6.2}$$

We have

$$\begin{aligned}
[T\gamma\overline{\partial} P(I-T)\gamma\overline{\partial} Pu \mid v] &= [\gamma\overline{\partial} P(I-T)\gamma\overline{\partial} Pu \mid v] \qquad (\text{since } v \in \mathrm{Ker}(\overline{\partial} r)^{\wedge,*}) \\
&= (P\gamma\overline{\partial} P(I-T)\gamma\overline{\partial} Pu \mid Pv)_M \\
&= (\overline{\partial} P(I-T)\gamma\overline{\partial} Pu \mid Pv)_M \quad (\overline{\partial} Pf \in \mathrm{Ker}\,\Box_f^{(q)},\ \forall\ f \in C^\infty) \\
&= (P(I-T)\gamma\overline{\partial} Pu \mid \overline{\partial_f}^* Pv)_M \quad (\text{since } Pv \in \mathrm{Dom}\,\overline{\partial}^*) \\
&= [(I-T)\gamma\overline{\partial} Pu \mid \gamma\overline{\partial_f}^* Pv].
\end{aligned}$$

From Lemma 3.2, we have

$$\gamma\overline{\partial_f}^* Pv \in \mathrm{Ker}(\overline{\partial} r)^{\wedge,*}.$$

Thus,

$$[(I-T)\gamma\overline{\partial} Pu \mid \gamma\overline{\partial_f}^* Pv] = 0.$$

We get (6.2), and hence

$$T\gamma\overline{\partial} P\gamma\overline{\partial} Pu = T\gamma\overline{\partial} PT\gamma\overline{\partial} Pu,\ \ u \in C^\infty(\Gamma; \Lambda^{0,q}T^*(\Gamma)).$$

Now,

$$(\overline{\partial_\beta})^2 = T\gamma\overline{\partial} PT\gamma\overline{\partial} P = T\gamma\overline{\partial} P\gamma\overline{\partial} P = T\gamma\overline{\partial}^2 P = 0.$$

The lemma follows. □

We pause and recall the tangential Cauchy-Riemann operator. For $z \in \Gamma$, let

$$\pi_z^{0,q} : \Lambda^{0,q}T_z^*(M') \to \Lambda^{0,q}T_z^*(\Gamma)$$

be the orthogonal projection map (with respect to $(\,|\,)$). We can check that

$$\pi_z^{0,q} = 2(\overline{\partial} r(z))^{\wedge,*}(\overline{\partial} r(z))^{\wedge}.$$

For an open set $U \subset \Gamma$, the tangential Cauchy-Riemann operator:

$$\overline{\partial_b} : C^\infty(U; \Lambda^{0,q} T^*(\Gamma)) \to C^\infty(U; \Lambda^{0,q+1} T^*(\Gamma))$$

is now defined as follows: for any $\phi \in C^\infty(U; \Lambda^{0,q} T^*(\Gamma))$, let $\widetilde{U}$ be an open set in M' with $\widetilde{U} \cap \Gamma = U$ and pick $\phi_1 \in C^\infty(\widetilde{U}; \Lambda^{0,q} T^*(M'))$ that satisfies $\pi_z^{0,q}(\phi_1(z)) = \phi(z)$, for all $z \in U$. Then $\overline{\partial_b}\phi$ is defined to be a smooth section in $C^\infty(U; \Lambda^{0,q+1} T^*(\Gamma))$:

$$z \to \pi_z^{0,q}(\gamma \overline{\partial} \phi_1(z)).$$

It is not difficult to check that the definition of $\overline{\partial_b}$ is independent of the choice of ϕ_1. Since $\overline{\partial}^2 = 0$, we have $\overline{\partial_b}^2 = 0$ and we have the following boundary complex

$$\overline{\partial_b} : \cdots \to C^\infty(U\,; \Lambda^{0,q} T^*(\Gamma)) \to C^\infty(U\,; \Lambda^{0,q+1} T^*(\Gamma)) \to \cdots .$$

Let $\overline{\partial_b}^*$ be the formal adjoint of $\overline{\partial_b}$ with respect to $(\,|\,)_\Gamma$, that is

$$(\overline{\partial}_b f \mid h)_\Gamma = (f \mid \overline{\partial_b}^* h)_\Gamma, \ f \in C_0^\infty(U\,; \Lambda^{0,q} T^*(\Gamma)), h \in C^\infty(U\,; \Lambda^{0,q+1} T^*(\Gamma)).$$

$\overline{\partial_b}^*$ is a differential operator of order one from boundary $(0, q+1)$ forms to boundary $(0, q)$ forms and

$$\overline{\partial_b}^* : \cdots \leftarrow C^\infty(U\,; \Lambda^{0,q} T^*(\Gamma)) \leftarrow C^\infty(U\,; \Lambda^{0,q+1} T^*(\Gamma)) \leftarrow \cdots .$$

is a complex.

From the definition of $\overline{\partial_b}$, we know that

$$\overline{\partial_b} = 2(\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^{\wedge} \gamma \overline{\partial} P.$$

Since the principal symbol of T is $2(\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^{\wedge}$, it follows that

$$\overline{\partial_\beta} = \overline{\partial_b} + \text{lower order terms.} \tag{6.3}$$

Let

$$\overline{\partial_\beta}^\dagger : C^\infty(\Gamma; \Lambda^{0,q+1} T^*(\Gamma)) \to C^\infty(\Gamma; \Lambda^{0,q} T^*(\Gamma)), \tag{6.4}$$

be the formal adjoint of $\overline{\partial_\beta}$ with respect to $[\,|\,]$, that is

$$[\overline{\partial_\beta} f \mid h] = [f \mid \overline{\partial_\beta}^\dagger h], \ f \in C^\infty(\Gamma; \Lambda^{0,q} T^*(\Gamma)), h \in C^\infty(\Gamma; \Lambda^{0,q+1} T^*(\Gamma)).$$

$\overline{\partial_\beta}^\dagger$ is a classical pseudodifferential operator of order one from boundary $(0, q+1)$ forms to boundary $(0, q)$ forms.

Lemma 6.2. *We have*

$$\overline{\partial_\beta}^{\dagger} = \gamma \overline{\partial_f}^{*} P.$$

Proof. Let $u \in C^\infty(\Gamma; \Lambda^{0,q} T^*(\Gamma))$, $v \in C^\infty(\Gamma; \Lambda^{0,q+1} T^*(\Gamma))$. We have

$$\begin{aligned}[\overline{\partial_\beta} u \mid v] &= [T\gamma \overline{\partial} P u \mid v] = [\gamma \overline{\partial} P u \mid v] \\ &= (P\gamma \overline{\partial} P u \mid P v)_M = (\overline{\partial} P u \mid P v)_M \\ &= (P u \mid \overline{\partial_f}^{*} P v)_M = [u \mid \gamma \overline{\partial_f}^{*} P v],\end{aligned}$$

and the lemma follows. □

Remark 6.3. We can check that on boundary $(0,q)$ forms, we have

$$\overline{\partial_\beta}^{\dagger} = \gamma \overline{\partial_f}^{*} P = \overline{\partial_b}^{*} + \text{lower order terms}. \tag{6.5}$$

Set

$$\Box_\beta^{(q)} = \overline{\partial_\beta}^{\dagger} \overline{\partial_\beta} + \overline{\partial_\beta}\,\overline{\partial_\beta}^{\dagger} : \mathscr{D}'(\Gamma; \Lambda^{0,q} T^*(\Gamma)) \to \mathscr{D}'(\Gamma; \Lambda^{0,q} T^*(\Gamma)). \tag{6.6}$$

$\Box_\beta^{(q)}$ is a classical pseudodifferential operator of order two from boundary $(0,q)$ forms to boundary $(0,q)$ forms. We recall that the Kohn Laplacian on Γ is given by

$$\Box_b^{(q)} = \overline{\partial_b}\,\overline{\partial_b}^{*} + \overline{\partial_b}^{*} \overline{\partial_b} : \mathscr{D}'(\Gamma; \Lambda^{0,q} T^*(\Gamma)) \to \mathscr{D}'(\Gamma; \Lambda^{0,q} T^*(\Gamma)).$$

From (6.3) and (6.5), we see that

$$\sigma_{\Box_b^{(q)}} = \sigma_{\Box_\beta^{(q)}}$$

and the characteristic manifold of $\Box_\beta^{(q)}$ is

$$\Sigma = \Sigma^+ \bigcup \Sigma^-,$$

where Σ^+, Σ^- are given in (5.11). (See section 3 of [18].) Moreover, $\sigma_{\Box_\beta^{(q)}}$ vanishes to second order on Σ and we have

$$\Box_\beta^{(q)} = \Box_b^{(q)} + L_1, \tag{6.7}$$

where L_1 is a classical pseudodifferential operator of order one with

$$\sigma_{L_1} = 0 \ \text{ at each point of } \Sigma. \tag{6.8}$$

The following is well-known (see Lemma 3.1 of [18])

Lemma 6.4. *Σ is a symplectic submanifold of $T^*(\Gamma)$ if and only if the Levi form is non-degenerate at each point of Γ.*

Let p^s_β denote the subprincipal symbol of $\Box^{(q)}_\beta$ (invariantly defined on Σ) and let $F_\beta(\rho)$ denote the fundamental matrix of $\sigma_{\Box^{(q)}_\beta}$ at $\rho\in\Sigma$. We write $\widetilde{\operatorname{tr}}\,F_\beta(\rho)$ to denote $\sum\left|\lambda_j\right|$, where $\pm i\lambda_j$ are the non-vanishing eigenvalues of $F_\beta(\rho)$. From (6.7) and (6.8), we see that

$$p^s_\beta+\frac{1}{2}\widetilde{\operatorname{tr}}\,F_\beta=p^s_b+\frac{1}{2}\widetilde{\operatorname{tr}}\,F_b\ \text{ on }\ \Sigma,$$

where p^s_b is the subprincipal symbol of $\Box^{(q)}_b$ and F_b is the fundamental matrix of $\sigma_{\Box^{(q)}_b}$. We have the following

Lemma 6.5. *Let $\rho=(p,\xi)\in\Sigma$. Then*

$$\frac{1}{2}\widetilde{\operatorname{tr}}\,F_\beta+p^s_\beta=\sum_{j=1}^{n-1}\left|\lambda_j\right||\sigma_{iY}|+\Big(\sum_{j=1}^{n-1}L_p(\overline{T}_j,T_j)-\sum_{j,k=1}^{n-1}2t^\wedge_j t^{\wedge,*}_k L_p(\overline{T}_k,T_j)\Big)\sigma_{iY}\ \text{ at }\ \rho, \tag{6.9}$$

where λ_j, $j=1,\ldots,n-1$, are the eigenvalues of L_p and T_j, t_j, $j=1,\ldots,n-1$, are as in (5.4) and (5.5).

Proof. See section 3 of [18]. □

It is not difficult to see that on Σ the action of $\frac{1}{2}\widetilde{\operatorname{tr}}\,F_\beta+p^s_\beta$ on boundary $(0,q)$ forms has the eigenvalues

$$\begin{gathered}\sum_{j=1}^{n-1}\left|\lambda_j\right||\sigma_{iY}|+\sum_{j\notin J}\lambda_j\sigma_{iY}-\sum_{j\in J}\lambda_j\sigma_{iY},\ |J|=q,\\ J=(j_1,j_2,\ldots,j_q),\ 1\le j_1<j_2<\cdots<j_q\le n-1.\end{gathered} \tag{6.10}$$

(See section 3 of [18].) We assume that the Levi form is non-degenerate at $p\in\Gamma$. Let (n_-,n_+), $n_-+n_+=n-1$, be the signature of L_p. Since $\langle Y,\omega_0\rangle=-1$, we have $\sigma_{iY}>0$ on Σ^+, $\sigma_{iY}<0$ on Σ^-.

Let

$$\inf(p^s_\beta+\frac{1}{2}\widetilde{\operatorname{tr}}\,F_\beta)(\rho)=\inf\left\{\lambda;\lambda:\text{eigenvalue of }(p^s_\beta+\frac{1}{2}\widetilde{\operatorname{tr}}\,F_\beta)(\rho)\right\},\ \rho\in\Sigma.$$

From (6.10), we see that at $(p,\omega_0(p))\in\Sigma^+$,

$$\inf(p^s_\beta+\frac{1}{2}\widetilde{\operatorname{tr}}\,F_\beta)\begin{cases}=0, & q=n_+\\ >0, & q\neq n_+\end{cases}. \tag{6.11}$$

At $(p,-\omega_0(p))\in\Sigma^-$,

$$\inf(p^s_\beta+\frac{1}{2}\widetilde{\operatorname{tr}}\,F_\beta)\begin{cases}=0, & q=n_-\\ >0, & q\neq n_-\end{cases}. \tag{6.12}$$

Definition 6.6. Given q, $0 \le q \le n-1$, the Levi form is said to satisfy condition $Y(q)$ at $p \in \Gamma$ if for any $|J| = q$, $J = (j_1, j_2, \ldots, j_q)$, $1 \le j_1 < j_2 < \cdots < j_q \le n-1$, we have

$$\left|\sum_{j \notin J} \lambda_j - \sum_{j \in J} \lambda_j\right| < \sum_{j=1}^{n-1} |\lambda_j|,$$

where λ_j, $j = 1, \ldots, (n-1)$, are the eigenvalues of L_p. If the Levi form is non-degenerate at p, then the condition is equivalent to $q \ne n_+, n_-$, where (n_-, n_+), $n_- + n_+ = n - 1$, is the signature of L_p.

$$\text{From now on, we assume that the Levi form is non-degenerate at each point of } \Gamma. \tag{6.13}$$

By classical works of Boutet de Monvel [6] and Sjöstrand [26], we get the following

Proposition 6.7. *$\Box_\beta^{(q)}$ is hypoelliptic with loss of one derivative if and only if $Y(q)$ holds at each point of Γ.*

7 The heat equation for $\Box_\beta^{(q)}$

In this section, we will apply some results of Menikoff and Sjöstrand [24] to construct approximate orthogonal projection for $\Box_\beta^{(q)}$. Our presentation is essentially taken from [18]. The reader who is familiar with [18] may go directly to Theorem 7.15.

Until further notice, we work with real local coordinates $x = (x_1, x_2, \ldots, x_{2n-1})$ defined on a connected open set $\Omega \subset \Gamma$. Thus, the Levi form has constant signature on Ω. For any C^∞ function f, we also write f to denote an almost analytic extension. (For the precise meaning of almost analytic functions, we refer the reader to Definition 1.1of [23].) We let the full symbol of $\Box_\beta^{(q)}$ be:

$$\text{full symbol of } \Box_\beta^{(q)} \sim \sum_{j=0}^{\infty} q_j(x, \xi),$$

where $q_j(x, \xi)$ is positively homogeneous of order $2 - j$.

First, we consider the characteristic equation for $\partial_t + \Box_\beta^{(q)}$. We look for solutions $\psi(t, x, \eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega) \setminus 0)$ of the problem

$$\begin{cases} \dfrac{\partial \psi}{\partial t} - i q_0(x, \psi'_x) = O(\left|\operatorname{Im} \psi\right|^N), \ \forall N \ge 0, \\ \psi|_{t=0} = \langle x\ , \eta \rangle \end{cases} \tag{7.1}$$

with $\operatorname{Im}\psi(t,x,\eta)\geq 0$.

Let U be an open set in $\mathbb{R}^n$ and let $f,\ g\in C^\infty(U)$. We write

$$f\asymp g$$

if for every compact set $K\subset U$ there is a constant $c_K>0$ such that

$$f\leq c_K g,\ g\leq c_K f\ \text{ on } K.$$

We have the following

Proposition 7.1. *There exists* $\psi(t,x,\eta)\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega)\setminus 0)$ *such that* $\operatorname{Im}\psi\geq 0$ *with equality precisely on* $(\{0\}\times T^*(\Omega)\setminus 0)\bigcup(\mathbb{R}_+\times\Sigma)$ *and such that* (7.1) *holds where the error term is uniform on every set of the form* $[0,T]\times K$ *with* $T>0$ *and* $K\subset T^*(\Omega)\setminus 0$ *compact. Furthermore,*

$$\psi(t,x,\eta)=\langle x,\eta\rangle\ \text{on}\,\Sigma,\ d_{x,\eta}(\psi-\langle x,\eta\rangle)=0\ \text{on}\,\Sigma,$$

$$\psi(t,x,\lambda\eta)=\lambda\psi(\lambda t,x,\eta),\ \ \lambda>0,$$

$$\operatorname{Im}\psi(t,x,\eta)\asymp\left|\eta\right|\frac{t\left|\eta\right|}{1+t\left|\eta\right|}\operatorname{dist}((x,\frac{\eta}{\left|\eta\right|}),\Sigma))^2,\ \ t\geq 0,\ \left|\eta\right|\geq 1. \tag{7.2}$$

Proposition 7.2. *There exists a function* $\psi(\infty,x,\eta)\in C^\infty(T^*(\Omega)\setminus 0)$ *with a uniquely determined Taylor expansion at each point of* Σ *such that*

For every compact set $K\subset T^*(\Omega)\setminus 0$ *there is a constant* $c_K>0$ *such that*

$\operatorname{Im}\psi(\infty,x,\eta)\geq c_K\left|\eta\right|(\operatorname{dist}((x,\frac{\eta}{\left|\eta\right|}),\Sigma))^2,$

$d_{x,\eta}(\psi(\infty,x,\eta)-\langle x,\eta\rangle)=0$ on Σ.

If $\lambda\in C(T^*(\Omega)\setminus 0)$, $\lambda>0$ *and* $\lambda|_\Sigma<\min\left|\lambda_j\right|$, *where* $\pm i\left|\lambda_j\right|$ *are the non-vanishing eigenvalues of the fundamental matrix of* $\Box^{(q)}_\beta$, *then the solution* $\psi(t,x,\eta)$ *of* (7.1) *can be chosen so that for every compact set* $K\subset T^*(\Omega)\setminus 0$ *and all indices* $\alpha,\ \beta,\ \gamma$, *there is a constant* $c_{\alpha,\beta,\gamma,K}$ *such that*

$$\left|\partial^\alpha_x\partial^\beta_\eta\partial^\gamma_t(\psi(t,x,\eta)-\psi(\infty,x,\eta))\right|\leq c_{\alpha,\beta,\gamma,K}e^{-\lambda(x,\eta)t}\ \text{on}\ \overline{\mathbb{R}}_+\times K. \tag{7.3}$$

For the proof of Proposition 7.1 and Proposition 7.2, we refer the reader to Menikoff-Sjöstrand [24].

Definition 7.3. We will say that $a\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega))$ is quasi-homogeneous of degree j if $a(t,x,\lambda\eta)=\lambda^j a(\lambda t,x,\eta)$ for all $\lambda>0$.

We consider the problem

$$\begin{cases} (\partial_t + \Box_\beta^{(q)})u(t,x) = 0 \quad \text{in } \mathbb{R}_+ \times \Omega \\ u(0,x) = v(x) \end{cases}. \tag{7.4}$$

We shall start by making only a formal construction. We look for an approximate solution of (7.4) of the form

$$u(t,x) = A(t)v(x)$$

$$A(t)v(x) = \frac{1}{(2\pi)^{2n-1}} \int\int e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} a(t,x,\eta) v(y) dy d\eta \tag{7.5}$$

where formally

$$a(t,x,\eta) \sim \sum_{j=0}^{\infty} a_j(t,x,\eta),\ a_j(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))),$$

$a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$.

We apply $\partial_t + \Box_\beta^{(q)}$ formally under the integral in (7.5) and then introduce the asymptotic expansion of $\Box_\beta^{(q)}(ae^{i\psi})$. Setting $(\partial_t + \Box_\beta^{(q)})(ae^{i\psi}) \sim 0$ and regrouping the terms according to the degree of quasi-homogeneity. We obtain the transport equations

$$\begin{cases} T(t,x,\eta,\partial_t,\partial_x)a_0 = O(\left|\operatorname{Im}\psi\right|^N),\ \forall N \\ T(t,x,\eta,\partial_t,\partial_x)a_j + l_j(t,x,\eta,a_0,\ldots,a_{j-1}) = O(\left|\operatorname{Im}\psi\right|^N),\ \forall N. \end{cases} \tag{7.6}$$

Here

$$T(t,x,\eta,\partial_t,\partial_x) = \partial_t - i\sum_{j=1}^{2n-1} \frac{\partial q_0}{\partial \xi_j}(x,\psi'_x)\frac{\partial}{\partial x_j} + q(t,x,\eta)$$

where

$$q(t,x,\eta) = q_1(x,\psi'_x) + \frac{1}{2i}\sum_{j,k=1}^{2n-1} \frac{\partial^2 q_0(x,\psi'_x)}{\partial \xi_j \partial \xi_k}\frac{\partial^2 \psi(t,x,\eta)}{\partial x_j \partial x_k}$$

and $l_j(t,x,\eta)$ is a linear differential operator acting on $a_0, a_1, \ldots, a_{j-1}$. We can repeat the method of [18] (see Proposition 5.7 of [18]) to get the following

Proposition 7.4. *Let* (n_-, n_+), $n_- + n_+ = n-1$, *be the signature of the Levi form on* Ω. *We can find solutions*

$$a_j(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))),\ \ j = 0,1,\ldots, \tag{7.7}$$

of the system (7.6) *with*

$$a_0(0,x,\eta) = I,\ a_j(0,x,\eta) = 0 \ \text{ when } j > 0, \tag{7.8}$$

where $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$, such that a_j has unique Taylor expansions on Σ. Moreover, we can find

$$a_j(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))),\ \ j=0,1,\ldots,$$

where $a_j(\infty,x,\eta)$ is a positively homogeneous function of degree $-j$, $\varepsilon_0>0$ such that for all indices α, β, γ, j, every compact set $K\subset\Sigma$, there exists $c>0$, such that

$$\left|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta(a_j(t,x,\eta)-a_j(\infty,x,\eta))\right|\le ce^{-\varepsilon_0 t\left|\eta\right|}(1+\left|\eta\right|)^{-j-\left|\beta\right|+\gamma} \tag{7.9}$$

on $\overline{\mathbb{R}}_+\times K$, $\left|\eta\right|\ge 1$.

Furthermore, for all $j=0,1,\ldots$,

$$\begin{cases}\text{all derivatives of } a_j(\infty,x,\eta) \text{ vanish at } \Sigma^+ & \text{if } q\neq n_+\\ \text{all derivatives of } a_j(\infty,x,\eta) \text{ vanish at } \Sigma^- & \text{if } q\neq n_-\end{cases} \tag{7.10}$$

and

$$\begin{cases}a_0(\infty,x,\eta)\neq 0 \text{ at each point of } \Sigma^+ & \text{if } q=n_+\\ a_0(\infty,x,\eta)\neq 0 \text{ at each point of } \Sigma^- & \text{if } q=n_-\end{cases}. \tag{7.11}$$

Definition 7.5. Let $r(x,\eta)$ be a non-negative real continuous function on $T^*(\Omega)$. We assume that $r(x,\eta)$ is positively homogeneous of degree 1, that is, $r(x,\lambda\eta)=\lambda r(x,\eta)$, for $\lambda\ge 1$, $\left|\eta\right|\ge 1$. For $0\le q\le n-1$ and $k\in\mathbb{R}$, we say that

$$a\in\hat{S}^k_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)))$$

if

$$a\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)))$$

and for all indices α, β, γ, every compact set $K\subset\Omega$ and every $\varepsilon>0$, there exists a constant $c>0$ such that

$$\left|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta a(t,x,\eta)\right|\le ce^{t(-r(x,\eta)+\varepsilon\left|\eta\right|)}(1+\left|\eta\right|)^{k+\gamma-\left|\beta\right|},\ x\in K,\ \left|\eta\right|\ge 1.$$

Remark 7.6. It is easy to see that we have the following properties:

(a) If $a\in\hat{S}^k_{r_1}$, $b\in\hat{S}^l_{r_2}$ then $ab\in\hat{S}^{k+l}_{r_1+r_2}$, $a+b\in\hat{S}^{\max(k,l)}_{\min(r_1,r_2)}$.

(b) If $a\in\hat{S}^k_r$ then $\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta a\in\hat{S}^{k-\left|\beta\right|+\gamma}_r$.

(c) If $a_j\in\hat{S}^{k_j}_r$, $j=0,1,2,\ldots$ and $k_j\searrow-\infty$ as $j\to\infty$, then there exists $a\in\hat{S}^{k_0}_r$ such that $a-\sum_0^{v-1}a_j\in\hat{S}^{k_v}_r$, for all $v=1,2,\ldots$. Moreover, if $\hat{S}^{-\infty}_r$ denotes $\bigcap_{k\in\mathbb{R}}\hat{S}^k_r$ then a is unique modulo $\hat{S}^{-\infty}_r$.

If a and a_j have the properties of (c), we write

$$a \sim \sum_0^\infty a_j \text{ in the symbol space } \hat{S}_r^{k_0}.$$

From Proposition 7.4 and the standard Borel construction, we get the following

Proposition 7.7. *Let (n_-, n_+), $n_- + n_+ = n-1$, be the signature of the Levi form on Ω. We can find solutions*

$$a_j(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))), \quad j=0,1,\ldots$$

of the system (7.6) *with*

$$a_0(0,x,\eta) = I, \ a_j(0,x,\eta) = 0 \text{ when } j>0,$$

where $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$, such that for some $r>0$ as in Definition 7.5,

$$a_j(t,x,\eta) - a_j(\infty,x,\eta) \in \hat{S}_r^{-j}(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))), \quad j=0,1,\ldots,$$

where

$$a_j(\infty,x,\eta) \in C^\infty(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))), \quad j=0,1,\ldots,$$

and $a_j(\infty,x,\eta)$ is a positively homogeneous function of degree $-j$.

Furthermore, for all $j=0,1,\ldots,$

$$\begin{cases} a_j(\infty,x,\eta) = 0 \ \textit{in a conic neighborhood of } \Sigma^+, & \text{if } q \neq n_+, \\ a_j(\infty,x,\eta) = 0 \ \textit{in a conic neighborhood of } \Sigma^-, & \text{if } q \neq n_-. \end{cases}$$

Remark 7.8. Let

$$b(t,x,\eta) \in \hat{S}_r^k(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma)))$$

with $r>0$. We assume that $b(t,x,\eta)=0$ when $\left|\eta\right| \leq 1$. Let $\chi \in C_0^\infty(\mathbb{R}^{2n-1})$ be equal to 1 near the origin. Put

$$B_\varepsilon(x,y) = \int \Big(\int_0^\infty e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} b(t,x,\eta) dt \Big) \chi(\varepsilon\eta) d\eta.$$

For $u \in C_0^\infty(\Omega; \Lambda^{0,q}T^*(\Gamma))$, we can show that

$$\lim_{\varepsilon \to 0} (\int B_\varepsilon(x,y) u(y) dy) \in C^\infty(\Omega; \Lambda^{0,q}T^*(\Gamma))$$

and

$$B : C_0^\infty(\Omega; \Lambda^{0,q} T^*(\Gamma)) \to C^\infty(\Omega; \Lambda^{0,q} T^*(\Gamma))$$
$$u \to \lim_{\varepsilon\to 0}(\int B_\varepsilon(x,y)u(y)dy),$$

is continuous. Formally,

$$B(x,y) = \int \Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)} b(t,x,\eta)dt\Big) d\eta.$$

Moreover, B has a unique continuous extension

$$B : \mathscr{E}'(\Omega; \Lambda^{0,q} T^*(\Gamma)) \to \mathscr{D}'(\Omega; \Lambda^{0,q} T^*(\Gamma))$$

and

$$B(x,y) \in C^\infty(\Omega\times\Omega \smallsetminus \mathrm{diag}(\Omega\times\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Gamma), \Lambda^{0,q} T^*(\Gamma))).$$

For the details, we refer the reader to Proposition 6.6 of [18].

Remark 7.9. Let

$$a(t,x,\eta) \in \hat{S}_0^k(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Gamma), \Lambda^{0,q} T^*(\Gamma))).$$

We assume $a(t,x,\eta) = 0$, if $\left|\eta\right| \leq 1$ and

$$a(t,x,\eta) - a(\infty,x,\eta) \in \hat{S}_r^k(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Gamma), \Lambda^{0,q} T^*(\Gamma)))$$

with $r > 0$, where

$$a(\infty,x,\eta) \in C^\infty(T^*(\Omega); \mathscr{L}(\Lambda^{0,q} T^*(\Gamma), \Lambda^{0,q} T^*(\Gamma))).$$

Then we can also define

$$A(x,y) = \int \Big(\int_0^\infty \Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)} a(t,x,\eta) - e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)} a(\infty,x,\eta)\Big) dt\Big) d\eta \tag{7.12}$$

as an oscillatory integral by the following formula:

$$A(x,y) = \int \Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)} (-t)(i\psi'_t(t,x,\eta)a(t,x,\eta) + a'_t(t,x,\eta))dt\Big) d\eta.$$

We notice that

$$(-t)(i\psi'_t(t,x,\eta)a(t,x,\eta) + a'_t(t,x,\eta)) \in \hat{S}_r^{k+1},\ r > 0.$$

We recall the following

Definition 7.10. Let $k\in\mathbb{R}$. $S^k_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)))$ is the space of all

$$a\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)))$$

such that for every compact sets $K\subset\Omega$ and all $\alpha\in\mathbb{N}^{2n-1}$, $\beta\in\mathbb{N}^{2n-1}$, there is a constant $c_{\alpha,\beta,K}>0$ such that

$$\left|\partial^\alpha_x\partial^\beta_\xi a(x,\xi)\right|\le c_{\alpha,\beta,K}(1+|\xi|)^{k-\frac{|\beta|}{2}+\frac{|\alpha|}{2}},\ (x,\xi)\in T^*(\Omega), x\in K.$$

$S^k_{\frac{1}{2},\frac{1}{2}}$ is called the space of symbols of order k type $(\frac{1}{2},\frac{1}{2})$.

Definition 7.11. Let $k\in\mathbb{R}$. A pseudodifferential operator of order k type $(\frac{1}{2},\frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Gamma)$ to sections of $\Lambda^{0,q}T^*(\Gamma)$ is a continuous linear map

$$A: C^\infty_0(\Omega;\Lambda^{0,q}T^*(\Gamma))\to\mathscr{D}'(\Omega;\Lambda^{0,q}T^*(\Gamma))$$

such that the distribution kernel of A is

$$K_A=A(x,y)=\frac{1}{(2\pi)^{2n-1}}\int e^{i\langle x-y,\xi\rangle}a(x,\xi)d\xi$$

with $a\in S^k_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)))$. We call $a(x,\xi)$ the symbol of A. We shall write

$$L^k_{\frac{1}{2},\frac{1}{2}}(\Omega;\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))$$

to denote the space of pseudodifferential operators of order k type $(\frac{1}{2},\frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Gamma)$ to sections of $\Lambda^{0,q}T^*(\Gamma)$.

We recall the following classical proposition of Calderon-Vaillancourt. (See Hörmander [14].)

Proposition 7.12. *If* $A\in L^k_{\frac{1}{2},\frac{1}{2}}(\Omega;\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))$. *Then, for every* $s\in\mathbb{R}$, A *is continuous*

$$A: H^s_{\rm comp}(\Omega;\Lambda^{0,q}T^*(\Gamma))\to H^{s-k}_{\rm loc}(\Omega;\Lambda^{0,q}T^*(\Gamma)).$$

We have the following

Proposition 7.13. *Let*

$$a(t,x,\eta)\in\hat{S}^k_0(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))).$$

We assume $a(t,x,\eta)=0$, *if* $\left|\eta\right|\le 1$ *and*

$$a(t,x,\eta)-a(\infty,x,\eta)\in\hat{S}^k_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)))$$

with $r>0$, *where*

$$a(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))).$$

Let

$$A(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)-\\ e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)dt\Big)d\eta$$

be as in (7.12). Then

$$A\in L^{k-1}_{\frac{1}{2},\frac{1}{2}}(\Omega;\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))$$

with symbol

$$q(x,\eta)=\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}a(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)}a(\infty,x,\eta)\Big)dt\\ \in S^{k-1}_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))).$$

Proof. See Lemma 6.16 and Remark 6.17 of [18]. □

From now on, we write

$$\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)dt\Big)d\eta$$

to denote the kernel of pseudodifferential operator of order $k-1$ type $(\frac{1}{2},\frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Gamma)$ to sections of $\Lambda^{0,q}T^*(\Gamma)$ with symbol

$$\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}a(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)}a(\infty,x,\eta)\Big)dt,$$

where $a(t,x,\eta)$ and $a(\infty,x,\eta)$ are as in Proposition 7.13.

The following is essentially well-known (See page 72 of [24].)

Proposition 7.14. *Let* Q *be a properly supported pseudodifferential operator on* Ω *of order* $k>0$ *with classical symbol* $q(x,\xi)\in C^\infty(T^*(\Omega))$. *Let*

$$b(t,x,\eta)\in\hat{S}^m_0(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))).$$

We assume that $b(t,x,\eta)=0$ *when* $|\eta|\leq 1$ *and that*

$$b(t,x,\eta)-b(\infty,x,\eta)\in\hat{S}^m_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)))$$

with $r>0$, where

$$b(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)))$$

is a classical symbol of order m. Then,

$$Q(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta))=e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}c(t,x,\eta)+d(t,x,\eta), \tag{7.13}$$

where

$$c(t,x,\eta)\in\hat{S}^{k+m}_0(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))), \tag{7.14}$$

$$c(t,x,\eta)\sim\sum_\alpha\frac{1}{\alpha!}q^{(\alpha)}(x,\psi'_x(t,x,\eta))(R_\alpha(\psi,D_x)b) \tag{7.15}$$

in the symbol space $\hat{S}^{k+m}_0(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)))$,

$$c(t,x,\eta)-c(\infty,x,\eta)\in\hat{S}^{k+m}_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))),\ r>0, \tag{7.16}$$

$$d(t,x,\eta)\in\hat{S}^{-\infty}_0(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))), \tag{7.17}$$

$$d(t,x,\eta)-d(\infty,x,\eta)\in\hat{S}^{-\infty}_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))),\ r>0. \tag{7.18}$$

Here

$$c(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))) \tag{7.19}$$

is a classical symbol of order $k+m$,

$$d(\infty,x,\eta)\in S^{-\infty}_{1,0}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(M'),\Lambda^{0,q}T^*(M')))$$

(For the precise meaning of $S^{-\infty}_{1,0}$, see Definition 1.2.) and

$$R_\alpha(\psi,D_x)b=D^\alpha_y\left\{e^{i\phi_2(t,x,y,\eta)}b(t,y,\eta)\right\}\Big|_{y=x},$$

$$\phi_2(t,x,y,\eta)=(x-y)\psi'_x(t,x,\eta)-(\psi(t,x,\eta)-\psi(t,y,\eta)).$$

Moreover, put

$$B(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)- e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\Big)dt\Big)d\eta,$$

$$C(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}c(t,x,\eta)- e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}c(\infty,x,\eta)\Big)dt\Big)d\eta.$$

We have

$$Q\circ B\equiv C.$$

As in section 1, we put

$$\Gamma_q = \{z \in \Gamma; Z(q) \text{ fails at } z\}$$

and set

$$\Sigma^-(q) = \{(x,\xi) \in \Sigma^-; Z(q) \text{ fails at } x\}, \;\; \Sigma^+(q) = \{(x,\xi) \in \Sigma^+; Z(q) \text{ fails at } x\}.$$

From Proposition 7.7 and Proposition 7.14, we can repeat the method of [18] to get the following

Theorem 7.15. *We recall that we work with the assumption that the Levi form is non-degenerate at each point of* Γ. *Given* q, $0 \leq q \leq n-1$. *Suppose that* $Z(q)$ *fails at some point of* Γ. *Then there exist*

$$A \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\Gamma; \Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma)), B_-, B_+ \in L^0_{\frac{1}{2},\frac{1}{2}}(\Gamma; \Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))$$

such that

$$\begin{aligned} &\mathrm{WF}'(K_{B_-}) = \mathrm{diag}(\Sigma^-(q) \times \Sigma^-(q)), \\ &\mathrm{WF}'(K_{B_+}) = \mathrm{diag}(\Sigma^+(n-1-q) \times \Sigma^+(n-1-q)) \end{aligned} \tag{7.20}$$

and

$$A\Box^{(q)}_\beta + B_- + B_+ \equiv B_- + B_+ + \Box^{(q)}_\beta A \equiv I, \tag{7.21}$$

$$\overline{\partial_\beta} B_- \equiv 0, \; \overline{\partial_\beta}^\dagger B_- \equiv 0, \tag{7.22}$$

$$\overline{\partial_\beta} B_+ \equiv 0, \; \overline{\partial_\beta}^\dagger B_+ \equiv 0, \tag{7.23}$$

$$B_- \equiv B_-^\dagger \equiv B_-^2, \tag{7.24}$$

$$B_+ \equiv B_+^\dagger \equiv B_+^2, \tag{7.25}$$

where $B_-^\dagger$ *and* $B_+^\dagger$ *are the formal adjoints of* B_- *and* B_+ *with respect to* $[\,|\,]$ *respectively and*

$$\mathrm{WF}'(K_{B_-}) = \{(x,\xi,y,\eta) \in T^*(\Gamma) \times T^*(\Gamma); (x,\xi,y,-\eta) \in \mathrm{WF}(K_{B_-})\}.$$

Here $\mathrm{WF}(K_{B_-})$ *is the wave front set of* K_{B_-} *in the sense of Hörmander [13]. See Definition A.4 for a review.*

Moreover near $\mathrm{diag}(\Gamma_q \times \Gamma_q)$, $K_{B_-}(x,y)$ *satisfies*

$$K_{B_-}(x,y) \equiv \int_0^\infty e^{i\phi_-(x,y)t} b(x,y,t)dt$$

with

$$b(x,y,t)\in S^{n-1}_{1,0}(\Gamma\times\Gamma\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_x(\Gamma))),$$
$$b(x,y,t)\sim\sum_{j=0}^{\infty}b_j(x,y)t^{n-1-j}\ \ in\ \ S^{n-1}_{1,0}(\Gamma\times\Gamma\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_x(\Gamma))),$$
$$b_0(x,x)\neq 0\ \ if\ x\in\Gamma_q, \tag{7.26}$$

(A formula for $b_0(x,x)$ will be given in Proposition 7.17.) where $S^m_{1,0}$, $m\in\mathbb{R}$, is the Hörmander symbol space (see Definition 1.2),

$$b_j(x,y)\in C^\infty(\Gamma\times\Gamma;\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_x(\Gamma))),\ j=0,1,\ldots,$$

and

$$\phi_-(x,y)\in C^\infty(\Gamma\times\Gamma), \tag{7.27}$$
$$\phi_-(x,x)=0, \tag{7.28}$$
$$\phi_-(x,y)\neq 0\ \ if\ x\neq y, \tag{7.29}$$
$$\operatorname{Im}\phi_-(x,y)\geq 0, \tag{7.30}$$
$$d_x\phi_-\neq 0,\ \ d_y\phi_-\neq 0\ \ where\ \operatorname{Im}\phi_-=0, \tag{7.31}$$
$$d_x\phi_-(x,y)|_{x=y}=-\omega_0(x), \tag{7.32}$$
$$d_y\phi_-(x,y)|_{x=y}=\omega_0(x), \tag{7.33}$$
$$\phi_-(x,y)=-\overline{\phi}_-(y,x). \tag{7.34}$$

Similarly, near $\operatorname{diag}(\Gamma_{n-1-q}\times\Gamma_{n-1-q})$*,*

$$K_{B_+}(x,y)\equiv\int_0^\infty e^{i\phi_+(x,y)t}c(x,y,t)dt$$

with

$$c(x,y,t)\in S^{n-1}_{1,0}(\Gamma\times\Gamma\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_x(\Gamma))),$$
$$c(x,y,t)\sim\sum_{j=0}^{\infty}c_j(x,y)t^{n-1-j}\ \ in\ \ S^{n-1}_{1,0}(\Gamma\times\Gamma\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_x(\Gamma))),$$

where

$$c(x,y)\in C^\infty(\Gamma\times\Gamma;\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_x(\Gamma))),\ j=0,1,\ldots,$$

and $-\overline{\phi}_+(x,y)$ satifies (7.27)-(7.34).

We only give the outline of the proof of Theorem 7.15. For all the details, we refer the reader to section 7 and section 8 of [18]. Let

$$a_j(t,x,\eta)\in\hat{S}_0^{-j}(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))),\ \ j=0,1,\ldots,$$

and

$$a_j(\infty,x,\eta)\in C^\infty(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))),\ \ j=0,1,\ldots,$$

be as in Proposition 7.7. We recall that for some $r>0$

$$a_j(t,x,\eta)-a_j(\infty,x,\eta)\in\hat{S}_r^{-j}(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))),\ \ j=0,1,\ldots.$$

Let

$$a(\infty,x,\eta)\sim\sum_{j=0}^{\infty}a_j(\infty,x,\eta)\ \text{ in }\ S^0_{1,0}(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))).\tag{7.35}$$

Let

$$a(t,x,\eta)\sim\sum_{j=0}^{\infty}a_j(t,x,\eta)\ \text{ in }\ \hat{S}^0_0(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))).\tag{7.36}$$

We take $a(t,x,\eta)$ so that for every compact set $K\subset\Omega$ and all indices α, β, γ, k, there exists $c>0$, c is independent of t, such that

$$\left|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta\Big(a(t,x,\eta)-\sum_{j=0}^{k}a_j(t,x,\eta)\Big)\right|\le c(1+\left|\eta\right|)^{-k-1+\gamma-|\beta|},\tag{7.37}$$

where $t\in\overline{\mathbb{R}}_+$, $x\in K$, $\left|\eta\right|\ge1$, and

$$a(t,x,\eta)-a(\infty,x,\eta)\in\hat{S}^0_r(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)))$$

with $r>0$.

Choose $\chi\in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi(\eta)=1$ when $\left|\eta\right|<1$ and $\chi(\eta)=0$ when $\left|\eta\right|>2$. Set

$$A(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta.\tag{7.38}$$

Put

$$B(x,y)=\frac{1}{(2\pi)^{2n-1}}\int e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)d\eta.\tag{7.39}$$

Since $a_j(t,x,\eta)$, $j=0,1,\ldots$, solve the transport equations (7.6), we can check that

$$B+\Box_\beta^{(q)}A\equiv I,$$

$$\Box_\beta^{(q)}B\equiv 0.$$

From the global theory of Fourier integral operators (see Melin-Sjöstrand [23]), we get

$$K_B\equiv K_{B_-}+K_{B_+},$$

wher K_{B_-} and K_{B_+} are as in Theorem 7.15. By using a partition of unity we get the global result.

Remark 7.16. For more properties of the phase $\phi_-(x,y)$, see section 8 and section 9 of [18].

We can repeat the computation of the leading term of the Szegö projection (see section 9 of [18]), to get the following

Proposition 7.17. *Let $p\in\Gamma_q$, $q=n_-$. Let*

$$\overline{Z}_1(x),\ldots,\overline{Z}_{n-1}(x)$$

be an orthonormal frame of $\Lambda^{1,0}T_x(\Gamma)$, for which the Levi form is diagonalized at p. Let $e_j(x)$, $j=1,\ldots,n-1$, denote the basis of $\Lambda^{0,1}T_x^*(\Gamma)$, which is dual to $Z_j(x)$, $j=1,\ldots,n-1$. Let $\lambda_j(x)$, $j=1,\ldots,n-1$, be the eigenvalues of the Levi form L_x. We assume that

$$\lambda_j(p)<0 \ \ if \ \ 1\leq j\leq n_-.$$

Then

$$b_0(p,p)=\frac{1}{2}\left|\lambda_1(p)\right|\cdots\left|\lambda_{n-1}(p)\right|\pi^{-n}\prod_{j=1}^{j=n_-}e_j(p)^\wedge e_j(p)^{\wedge,*},$$

where b_0 is as in (7.26).

In section 8, we need the following

Proposition 7.18. *Suppose that $Z(q)$ fails at some point of Γ. Let B_- be as in Theorem 7.15. Then,*

$$\gamma\overline{\partial}PB_-\equiv 0. \tag{7.40}$$

Proof. In view of Theorem 7.15, we know that

$$T\gamma\overline{\partial}PB_-=\overline{\partial_\beta}B_-\equiv 0,\ \ \gamma\overline{\partial_f}^*PB_-=\overline{\partial_\beta}^\dagger B_-\equiv 0.$$

Combining this with $(\overline{\partial}\,\overline{\partial}_f^{\,*}+\overline{\partial}_f^{\,*}\overline{\partial})P=0$, we have

$$\gamma\overline{\partial}_f^{\,*}P\gamma\overline{\partial}PB_-=-\gamma\overline{\partial}P\gamma\overline{\partial}_f^{\,*}PB_-\equiv 0$$

and

$$\gamma\overline{\partial}_f^{\,*}P(I-T)\gamma\overline{\partial}PB_-=\gamma\overline{\partial}_f^{\,*}P\gamma\overline{\partial}PB_--\gamma\overline{\partial}_f^{\,*}PT\gamma\overline{\partial}PB_-\equiv 0. \tag{7.41}$$

Combining this with (4.6), we get

$$\gamma\overline{\partial}_f^{\,*}P(P^*P)^{-1}(\overline{\partial}r)^\wedge R\gamma\overline{\partial}PB_-\equiv 0.$$

Thus,

$$\gamma(\overline{\partial}r)^\wedge\overline{\partial}_f^{\,*}P(P^*P)^{-1}(\overline{\partial}r)^\wedge R\gamma\overline{\partial}PB_-\equiv 0. \tag{7.42}$$

In view of Proposition 5.2, we know that

$$\gamma(\overline{\partial}r)^\wedge\overline{\partial}_f^{\,*}P(P^*P)^{-1}:C^\infty(\Gamma;\,I^{0,q}T^*(M'))\to C^\infty(\Gamma;\,I^{0,q}T^*(M'))$$

is elliptic near Σ^-, where $I^{0,q}T_z^*(M')$ is as in (5.17). Since

$$\mathrm{WF}'(K_{B_-})\subset\mathrm{diag}(\Sigma^-\times\Sigma^-),$$

we get

$$(\overline{\partial}r)^\wedge R\gamma\overline{\partial}PB_-\equiv 0.$$

(See Proposition A.6 and Proposition A.7.) Thus, by (4.6),

$$(I-T)\gamma\overline{\partial}PB_-\equiv 0.$$

The proposition follows. □

8 The Bergman projection

Given q, $0\le q\le n-1$. In this section, we assume that $Z(q)$ fails at some point of Γ and that $Z(q-1)$ and $Z(q+1)$ hold at each point of Γ. In view of Proposition 4.4, we know that

$$\Pi^{(q)}:C^\infty(\overline{M};\,\Lambda^{0,q}T^*(M'))\to D^{(q)}.$$

Put

$$K=\gamma\Pi^{(q)}P:C^\infty(\Gamma;\,\Lambda^{0,q}T^*(\Gamma))\to C^\infty(\Gamma;\,\Lambda^{0,q}T^*(\Gamma)). \tag{8.1}$$

Let $K^\dagger$ be the formal adjoint of K with respect to $[\,|\,]$. That is,

$$K^\dagger:\mathscr{D}'(\Gamma;\,\Lambda^{0,q}T^*(\Gamma))\to\mathscr{D}'(\Gamma;\,\Lambda^{0,q}T^*(\Gamma))$$
$$[K^\dagger u\mid v]=[u\mid Kv],\ u\in\mathscr{D}'(\Gamma;\,\Lambda^{0,q}T^*(\Gamma)),\ v\in C^\infty(\Gamma;\,\Lambda^{0,q}T^*(\Gamma)).$$

Lemma 8.1. *We have*

$$K^\dagger v = Kv,$$

$v \in C^\infty(\Gamma; \Lambda^{0,q} T^*(\Gamma))$.

Proof. For $u, v \in C^\infty(\Gamma; \Lambda^{0,q} T^*(\Gamma))$, we have

$$\begin{aligned}[Ku \mid v] &= [\gamma \Pi^{(q)} Pu \mid v] \\ &= (\Pi^{(q)} Pu \mid Pv)_M \\ &= (Pu \mid \Pi^{(q)} Pv)_M \\ &= [u \mid Kv].\end{aligned}$$

Thus,

$$K^\dagger v = Kv.$$

The lemma follows. □

We can extend K to

$$\mathscr{D}'(\Gamma; \Lambda^{0,q} T^*(\Gamma)) \to \mathscr{D}'(\Gamma; \Lambda^{0,q} T^*(\Gamma))$$

by the following formula

$$[Ku \mid v] = [u \mid K^\dagger v], \ u \in \mathscr{D}'(\Gamma; \Lambda^{0,q} T^*(\Gamma)), v \in C^\infty(\Gamma; \Lambda^{0,q} T^*(\Gamma)).$$

Lemma 8.2. *Let* $u \in \mathscr{D}'(\Gamma; \Lambda^{0,q} T^*(\Gamma))$. *We have*

$$\mathrm{WF}(Ku) \subset \Sigma^-.$$

Proof. Let $u \in \mathscr{D}'(\Gamma; \Lambda^{0,q} T^*(\Gamma))$. We have

$$(\gamma(\overline{\partial} r)^{\wedge,*} \overline{\partial} P)(Ku) = 0.$$

In view of Proposition 5.1, we know that $\gamma(\overline{\partial} r)^{\wedge,*} \overline{\partial} P$ is elliptic outside Σ^-. The lemma follows. □

Lemma 8.3. *Let* B_- *be as in Theorem 7.15. We have*

$$B_- K \equiv K B_- \equiv K.$$

Proof. Let A, B_- and B_+ be as in Theorem 7.15. In view of Theorem 7.15, we have

$$B_- + B_+ + A \Box_\beta^{(q)} \equiv I.$$

We may replace B_+ by $I - A\Box_\beta^{(q)} - B_-$ and get

$$B_- + B_+ + A\Box_\beta^{(q)} = I.$$

It is easy to see that

$$\Box_\beta^{(q)} K = 0.$$

Thus,

$$K = (B_- + B_+ + A\Box_\beta^{(q)})K = (B_- + B_+)K. \tag{8.2}$$

Let $u \in \mathscr{D}'(\Gamma; \Lambda^{0,q} T^*(\Gamma))$. From Lemma 8.2, we know that

$$\mathrm{WF}(Ku) \subset \Sigma^-.$$

Note that

$$\mathrm{WF}'(K_{B_+}) \subset \mathrm{diag}(\Sigma^+ \times \Sigma^+).$$

Thus,

$$B_+ K u \in C^\infty,$$

so $B_+ K$ is smoothing and

$$(B_- + B_+)K \equiv B_- K.$$

From this and (8.2), we get

$$K \equiv B_- K$$

and

$$K = K^\dagger \equiv K^\dagger B_-^\dagger \equiv K B_-.$$

The lemma follows. $\Box$

We pause and introduce some notations. Let X and Y be C^∞ vector bundles over M' and Γ respectively. Let

$$C, D : C^\infty(\Gamma; Y) \to \mathscr{D}'(M; X)$$

with distribution kernels

$$K_C(z, y), K_D(z, y) \in \mathscr{D}'(M \times \Gamma; \mathscr{L}(Y_y, X_z)).$$

We write

$$C \equiv D \mod C^\infty(\overline{M} \times \Gamma)$$

if

$$K_C(z, y) = K_D(z, y) + F(z, y),$$

where $F(z, y) \in C^\infty(\overline{M} \times \Gamma; \mathscr{L}(Y_y, X_z))$.

Lemma 8.4. *We have*

$$\Pi^{(q)} P B_- \equiv \Pi^{(q)} P \mod C^\infty(\overline{M} \times \Gamma). \tag{8.3}$$

Proof. From Lemma 8.3, we have

$$K = \gamma \Pi^{(q)} P \equiv K B_- = \gamma \Pi^{(q)} P B_-.$$

Thus,

$$\Pi^{(q)} P = P\gamma \Pi^{(q)} P \equiv P\gamma \Pi^{(q)} P B_- \mod C^\infty(\overline{M} \times \Gamma).$$

We get (8.3). □

Put

$$Q = P B_- T (P^* P)^{-1} P^* : C^\infty(\overline{M}; \Lambda^{0,q} T^*(M')) \to C^\infty(\overline{M}; \Lambda^{0,q} T^*(M')), \tag{8.4}$$

where T is as in (4.5).

Proposition 8.5. *We have*

$$Q \equiv \Pi^{(q)} \mod C^\infty(\overline{M} \times \overline{M}).$$

Proof. We have

$$\Pi^{(q)} Q = \Pi^{(q)} P B_- T (P^* P)^{-1} P^* \equiv \Pi^{(q)} P T (P^* P)^{-1} P^* \mod C^\infty(\overline{M} \times \overline{M}). \tag{8.5}$$

Here we used (8.3). From (8.5) and the first part of (4.10), we get

$$\Pi^{(q)} Q \equiv \Pi^{(q)} \mod C^\infty(\overline{M} \times \overline{M}). \tag{8.6}$$

From Theorem 3.7, we have

$$\Pi^{(q)} Q = (I - \overline{\partial}^* N^{(q+1)} \overline{\partial} - \overline{\partial} N^{(q-1)} \overline{\partial}^*) Q,$$

where $N^{(q+1)}$ and $N^{(q-1)}$ are as in Theorem 3.6. From (7.40), (7.22) and Lemma 6.2, we see that

$$\overline{\partial} Q \equiv 0, \ \ \overline{\partial}^* Q \equiv 0 \mod C^\infty(\overline{M} \times \overline{M}).$$

Thus,

$$\Pi^{(q)} Q = (I - \overline{\partial}^* N^{(q+1)} \overline{\partial} - \overline{\partial} N^{(q-1)} \overline{\partial}^*) Q \equiv Q \mod C^\infty(\overline{M} \times \overline{M}).$$

From this and (8.6), the proposition follows. □

Let $x=(x_1,\ldots,x_{2n-1})$ be a system of local coordinates on Γ and extend

$$x_1,\ldots,x_{2n-1}$$

to real smooth functions in some neighborhood of Γ. We write

$$(\xi_1,\ldots,\xi_{2n-1},\theta)$$

to denote the dual variables of $(x_1,\ldots,x_{2n-1},r)$. We write

$$z=(x_1,\ldots,x_{2n-1},r),\ \ x=(x_1,\ldots,x_{2n-1},0)$$

and

$$\xi=(\xi_1,\ldots,\xi_{2n-1}),\ \ \zeta=(\xi_1,\ldots,\xi_{2n-1},\theta).$$

Until further notice, we work with the local coordinates $z=(x,r)$ defined on some neighborhood of $p\in\Gamma$.

We represent the Riemannian metric on $T(M')$ by

$$h=\sum_{j,k=1}^{2n}h_{j,k}(z)dx_j\otimes dx_k,\ \ dx_{2n}=dr,$$

where $h_{j,k}(z)=h_{k,j}(z)$, j, $k=1,\ldots,n$, and $\left(h_{j,k}(z)\right)_{1\le j,k\le 2n}$ is positive definite at each point of M'. Put

$$\left(h_{j,k}(z)\right)^{-1}_{1\le j,k\le 2n}=\left(h^{j,k}(z)\right)_{1\le j,k\le 2n}.$$

It is well-known (see page 99 of [25]) that

$$\Box^{(q)}_f=-\frac{1}{2}\Big(h^{2n,2n}(z)\frac{\partial^2}{\partial r^2}+2\sum_{j=1}^{2n-1}h^{2n,j}(z)\frac{\partial^2}{\partial r\partial x_j}+T(r)\Big)+\text{lower order terms},\quad (8.7)$$

where

$$T(r)=\sum_{j,k=1}^{2n-1}h^{j,k}(z)\frac{\partial^2}{\partial x_j\partial x_k}.\quad (8.8)$$

Note that

$$T(0)=\triangle_\Gamma+\text{lower order terms}\quad (8.9)$$

and

$$h^{2n,2n}(x)=1,\ \ h^{2n,j}(x)=0,\ \ j=1,\ldots,2n-1.\quad (8.10)$$

We let the full symbol of $\Box^{(q)}_f$ be:

$$\text{full symbol of }\ \Box^{(q)}_f=\sum_{j=0}^{2}q_j(z,\zeta)$$

where $q_j(z,\zeta)$ is a homogeneous polynomial of order $2-j$ in ζ. We have the following

Proposition 8.6. *Let $\phi_- \in C^\infty(\Gamma\times\Gamma)$ be as in Theorem 7.15. Then, in some neighborhood U of $\operatorname{diag}(\Gamma_q\times\Gamma_q)$ in $M'\times M'$, there exists a smooth function*

$$\widetilde{\phi}(z,y)\in C^\infty((\overline{M}\times\Gamma)\bigcap U)$$

such that

$$\begin{aligned}&\widetilde{\phi}(x,y)=\phi_-(x,y),\\&\operatorname{Im}\widetilde{\phi}\geq 0,\\&d_z\widetilde{\phi}\neq 0,\ d_y\widetilde{\phi}\neq 0\ \ \textit{where}\operatorname{Im}\widetilde{\phi}=0,\\&\operatorname{Im}\widetilde{\phi}>0\ \ \textit{if}\ r\neq 0,\end{aligned}\tag{8.11}$$

and

$$q_0(z,\widetilde{\phi}'_z)$$

vanishes to infinite order on $r=0$. We write $\frac{\partial}{\partial r(z)}$ to denote $\frac{\partial}{\partial r}$ acting in the z variables. We have

$$\frac{\partial}{\partial r(z)}\widetilde{\phi}(z,y)|_{r=0}=-i\sqrt{-\sigma_{\triangle_\Gamma}(x,(\phi_-)'_x)}\tag{8.12}$$

in some neighborhood of $x=y$, where

$$\operatorname{Re}\sqrt{-\sigma_{\triangle_\Gamma}(x,(\phi_-)'_x)}>0.$$

Proof. From (8.7) and (8.8), we have

$$\begin{aligned}&q_0(z,\zeta)=\frac{1}{2}h^{2n,2n}(z)\theta^2+\sum_{j=1}^{2n-1}h^{2n,j}(z)\theta\xi_j+g(z,\xi),\\&g(x,\xi)=-\frac{1}{2}\sigma_{\triangle_\Gamma},\end{aligned}\tag{8.13}$$

where $g(z,\xi)$ is the principal symbol of $-\frac{1}{2}T(r)$.

We consider the Taylor expansion of $q_0(z,\zeta)$ with respect to r,

$$q_0(z,\zeta)=\frac{1}{2}\theta^2-\frac{1}{2}\sigma_{\triangle_\Gamma}+\sum_{j=1}^\infty g_j(x,\xi)r^j+\sum_{j=1}^\infty s_j(x,\zeta)\theta r^j.\tag{8.14}$$

We introduce the Taylor expansion of $\widetilde{\phi}(z,y)$ with respect to r,

$$\widetilde{\phi}(z,y)=\phi_-(x,y)+\sum_1^\infty\phi_j(x,y)r^j.$$

Let

$$\phi_1(x,y)=-i\sqrt{-\sigma_{\triangle_\Gamma}(x,(\phi_-)'_x)}.$$

Since $(\phi_-)'_x|_{x=y} = -\omega_0(x)$ is real, we choose the branch of $\sqrt{-\sigma_{\triangle_\Gamma}(x,(\phi_-)'_x)}$ so that

$$\mathrm{Re}\sqrt{-\sigma_{\triangle_\Gamma}(x,(\phi_-)'_x)}>0$$

in some neighborhood of $x=y$, $r=0$. Put

$$\widetilde{\phi}_1(z,y)=\phi_-(x,y)+r\phi_1(x,y).$$

We have

$$q_0(z,(\widetilde{\phi}_1)'_z)=O(r).$$

Similarly, we can find $\phi_2(x,y)$ so that

$$q_0(z,(\widetilde{\phi}_2)'_z)=O(r^2),$$

where $\widetilde{\phi}_2(z,y)=\phi_-(x,y)+r\phi_1(x,y)+r^2\phi_2(x,y)$. Continuing in this way we get the phase $\widetilde{\phi}(z,y)$ such that

$$\widetilde{\phi}(x,y)=\phi_-(x,y)$$

and

$$q_0(z,\widetilde{\phi}'_z)$$

vanishes to infinite order on $r=0$. The proposition follows. □

Remark 8.7. Let $\widetilde{\phi}(z,y)$ be as in Proposition 8.6 and let

$$d(z,y,t)\in S^m_{1,0}(\overline{M}\times\Gamma\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_z(M')))$$

with support in some neighborhood of $\mathrm{diag}(\Gamma_q\times\Gamma_q)$. (For the meaning of the space $S^m_{1,0}(\overline{M}\times\Gamma\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_z(M')))$, see Definition 1.2.) Choose a cut-off function $\chi(t)\in C^\infty(\mathbb{R})$ so that $\chi(t)=1$ when $|t|<1$ and $\chi(t)=0$ when $|t|>2$. For all $u\in C^\infty(\Gamma;\Lambda^{0,q}T^*(M'))$, set

$$(D_\varepsilon u)(z)=\int\int_0^\infty e^{i\widetilde{\phi}(z,y)t}d(z,y,t)\chi(\varepsilon t)u(y)dtdy.$$

Since $\mathrm{Im}\,\widetilde{\phi}\geq 0$ and $d_y\widetilde{\phi}\neq 0$ where $\mathrm{Im}\,\widetilde{\phi}=0$, we can integrate by parts in y, t and obtain

$$\lim_{\varepsilon\to 0}(D_\varepsilon u)(z)\in C^\infty(\overline{M};\Lambda^{0,q}T^*(M')).$$

This means that

$$D=\lim_{\varepsilon\to 0}D_\varepsilon: C^\infty(\Gamma;\Lambda^{0,q}T^*(M'))\to C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$$

is continuous. Formally,

$$D(z,y)=\int_0^\infty e^{i\widetilde{\phi}(z,y)t}d(z,y,t)dt.$$

Proposition 8.8. *Let*

$$B_-(x,y)=\int_0^\infty e^{i\phi_-(x,y)t}b(x,y,t)dt$$

be as in Theorem 7.15. We have

$$PB_-(z,y)\equiv\int_0^\infty e^{i\widetilde{\phi}(z,y)t}\widetilde{b}(z,y,t)dt \mod C^\infty(\overline{M}\times\Gamma)$$

with

$$\widetilde{b}(z,y,t)\in S^{n-1}_{1,0}(\overline{M}\times\Gamma\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_z(M'))),$$

$$\widetilde{b}(z,y,t)\sim\sum_{j=0}^\infty\widetilde{b}_j(z,y)t^{n-1-j}$$

in the space $S^{n-1}_{1,0}(\overline{M}\times\Gamma\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_z(M')))$, *where*

$$\widetilde{b}_j(z,y)\in C^\infty(\overline{M}\times\Gamma;\mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_z(M'))),\ j=0,1,\ldots.$$

Proof. Put

$$b(x,y,t)\sim\sum_{j=0}^\infty b_j(x,y)t^{n-1-j}$$

and formally set

$$\widetilde{b}(z,y,t)\sim\sum_{j=0}^\infty\widetilde{b}_j(z,y)t^{n-1-j}.$$

We notice that

$$B_-(x,y)\in C^\infty(\Gamma\times\Gamma\setminus\operatorname{diag}(\Gamma_q\times\Gamma_q);\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_z(\Gamma))).$$

For simplicity, we may assume that $b(x,y,t)=0$ outside some small neighborhood of $\operatorname{diag}(\Gamma_q\times\Gamma_q)$. Put

$$\Box^{(q)}_f(\widetilde{b}(z,y,t)e^{i\widetilde{\phi}t})=\widetilde{c}(z,y,t)e^{i\widetilde{\phi}t}.$$

From (7.29) and (8.11), we know that near $\operatorname{diag}(\Gamma_q\times\Gamma_q)$, $\widetilde{\phi}(z,y)=0$ if and only if $x=y$, $r=0$. From this observation, we see that if $\widetilde{c}(z,y,t)$ vanishes to infinite order on $\operatorname{diag}(\Gamma_q\times\Gamma_q)\times\overline{\mathbb{R}}_+$, we can integrate by parts and obtain

$$\lim_{\varepsilon\to0}\int_0^\infty e^{i\widetilde{\phi}t}\widetilde{c}(z,y,t)\chi(\varepsilon t)dt\equiv0 \mod C^\infty(\overline{M}\times\Gamma),$$

where $\chi(t)$ is as in Remark 8.7. Thus, we only need to consider the Taylor expansion of $\widetilde{b}(z,y,t)$ on $x=y$, $r=0$. We introduce the asymptotic expansion of $\Box_f^{(q)}(\widetilde{b}e^{i\widetilde{\phi}t})$. Setting

$$\Box_f^{(q)}(\widetilde{b}e^{i\widetilde{\phi}t})\sim 0$$

and regrouping the terms according to the degree of homogeneity. We obtain the transport equations

$$\begin{cases} T(z,y,\partial_z)\widetilde{b}_0(z,y)=0 \\ T(z,y,\partial_z)\widetilde{b}_j(z,y)+l_j(z,y,\widetilde{b}_0(z,y),\ldots,\widetilde{b}_{j-1}(z,y))=0,\ j=1,2,\ldots. \end{cases} \tag{8.15}$$

Here

$$T(z,y,\partial_z)=-i\sum_{j=1}^{2n-1}\frac{\partial q_0}{\partial \xi_j}(z,\widetilde{\phi}'_z)\frac{\partial}{\partial x_j}-i\frac{\partial q_0}{\partial\theta}(z,\widetilde{\phi}'_z)\frac{\partial}{\partial r}$$
$$+R(z,y),$$

where

$$R(z,y)=q_1(z,\widetilde{\phi}'_z)+\frac{1}{2i}\sum_{j,k=1}^{2n}\frac{\partial^2 q_0(z,\widetilde{\phi}'_z)}{\partial\xi_j\partial\xi_k}\frac{\partial^2\widetilde{\phi}}{\partial x_j\partial x_k},\ \ x_{2n}=r,\ \ \xi_{2n}=\theta,$$

and l_j is a linear differential operator acting on $\widetilde{b}_0(z,y),\ldots,\widetilde{b}_{j-1}(z,y)$.

We introduce the Taylor expansion of $\widetilde{b}_0(z,y)$ with respect to r,

$$\widetilde{b}_0(z,y)=b_0(x,y)+\sum_1^\infty b_0^j(x,y)r^j.$$

Since

$$\frac{\partial q_0}{\partial\theta}|_{r=0}=\theta$$

and

$$\widetilde{\phi}'_r|_{r=0}=-i\sqrt{-\sigma_{\triangle_\Gamma}(x,(\phi_-)'_x)},$$

we have

$$\frac{\partial q_0}{\partial\theta}(z,\widetilde{\phi}'_z)|_{r=0}\neq 0$$

in some neighborhood of $x=y$. Thus, we can find $b_0^1(x,y)r$ such that

$$T(z,y,\partial_z)(b_0(x,y)+b_0^1(x,y)r)=O(|r|)$$

in some neighborhood of $r=0$, $x=y$. We can repeat the procedure above to find $b_0^2(x,y)$ such that

$$T(z,y,\partial_z)(b_0(x,y)+\sum_{k=1}^2 b_0^k(x,y)r^k)=O(|r|^2)$$

in some neighborhood of $r=0$, $x=y$. Continuing in this way we solve the first transport equation to infinite order at $r=0$, $x=y$.

For the second transport equation, we can repeat the method above to solve the second transport equation to infinite order at $r=0$, $x=y$. Continuing in this way we solve (8.15) to infinite order at $r=0$, $x=y$.

Put

$$\widetilde{B}(z,y)=\int_0^\infty e^{i\widetilde{\phi}(z,y)t}\widetilde{b}(z,y,t)dt.$$

From the construction above, we see that

$$\Box_f^{(q)}\widetilde{B}\equiv 0 \mod C^\infty(\overline{M}\times\Gamma),\ \ \gamma\widetilde{B}\equiv B_-. \tag{8.16}$$

It is well-known (see chapter XX of [14]) that there exists

$$G: C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))\to C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$$

such that

$$G\Box_f^{(q)}+P\gamma=I \ \text{ on } \ C^\infty(\overline{M};\Lambda^{0,q}T^*(M')). \tag{8.17}$$

From this and (8.16), we have

$$\widetilde{B}=(G\Box_f^{(q)}+P\gamma)\widetilde{B}\equiv PB_- \mod C^\infty(\overline{M}\times\Gamma).$$

The proposition follows. $\square$

From Proposition 8.8, we have

$$C(z,y):=PB_-T(P^*P)^{-1}(z,y)\equiv\int_0^\infty e^{i\widetilde{\phi}(z,y)t}c(z,y,t)dt \mod C^\infty(\overline{M}\times\Gamma)$$

with

$$c(z,y,t)\in S^n_{1,0}(\overline{M}\times\Gamma\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_z(M'))),$$
$$c(z,y,t)\sim\sum_{j=0}^\infty c_j(z,y)t^{n-j}$$

in the space $S^n_{1,0}(\overline{M}\times\Gamma\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_z(M')))$. Let

$$C^*: C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))\to\mathscr{D}'(\Gamma;\Lambda^{0,q}T^*(M'))$$

be the operator defined by

$$(C^*u\mid v)_\Gamma=(u\mid Cv)_M,\ \ u\in C^\infty(\overline{M};\Lambda^{0,q}T^*(M')), v\in C^\infty(\Gamma;\Lambda^{0,q}T^*(M')).$$

The distribution kernel of C^* is

$$C^*(y,z) \equiv \int_0^\infty e^{-i\overline{\widetilde{\phi}}(z,y)t} c^*(y,z,t)dt \mod C^\infty(\Gamma \times \overline{M}) \tag{8.18}$$

where

$$\begin{aligned}
&c^*(y,z,t) \in S^n_{1,0}(\Gamma \times \overline{M} \times]0,\infty[; \mathscr{L}(\Lambda^{0,q} T^*_z(M'), \Lambda^{0,q} T^*_y(M'))),\\
&(c^*(y,z,t)\mu \mid v) = (\mu \mid c(y,z,t)v),\ \ \mu \in \Lambda^{0,q} T^*_z(M'),\ v \in \Lambda^{0,q} T^*_y(M'),\\
&c^*(y,z,t) \sim \sum_{j=0}^\infty c^*_j(y,z)t^{n-j}
\end{aligned}$$

in $S^n_{1,0}(\Gamma \times \overline{M} \times]0,\infty[; \mathscr{L}(\Lambda^{0,q} T^*_z(M'), \Lambda^{0,q} T^*_y(M')))$. The oscillatory integral (8.18) is defined as follows: Let $u \in C^\infty(\overline{M}; \Lambda^{0,q} T^*(M'))$. Set

$$(C^*_\varepsilon u)(y) = \int \int_0^\infty e^{-i\widetilde{\phi}(z,y)t} c^*(y,z,t)\chi(\varepsilon t)u(z)dt\,dz,$$

where χ is as in Remark 8.7. Since $d_x\widetilde{\phi} \neq 0$ where $\operatorname{Im}\widetilde{\phi} = 0$, we can integrate by parts in x and t and obtain

$$\lim_{\varepsilon\to 0}(C^*_\varepsilon u)(y) \in C^\infty(\Gamma; \Lambda^{0,q} T^*(M')).$$

This means that

$$C^* = \lim_{\varepsilon\to 0} C^*_\varepsilon : C^\infty(\overline{M}; \Lambda^{0,q} T^*(M')) \to C^\infty(\Gamma; \Lambda^{0,q} T^*(M'))$$

is continuous.

We can repeat the proof of Proposition 8.6 to find

$$\phi(z,w) \in C^\infty(\overline{M} \times \overline{M})$$

such that

$$\begin{aligned}
&\phi(z,y) = \widetilde{\phi}(z,y),\\
&\operatorname{Im}\phi \geq 0,\\
&\operatorname{Im}\phi > 0 \ \text{ if } (z,w) \notin \Gamma \times \Gamma
\end{aligned}$$

and

$$q_0(w, -\overline{\phi}'_w)$$

vanishes to infinite order on $r = 0$. Since $\phi_-(x,y) = -\overline{\phi}_-(y,x)$, we can take $\phi(z,w)$ so that

$$\phi(z,w) = -\overline{\phi}(w,z).$$

As in the proof of Proposition 8.8, we can find

$$a^*(w,z,t)\in S^n_{1,0}(\overline{M}\times\overline{M}\times[0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_z(M'),\Lambda^{0,q}T^*_w(M'))),$$
$$a^*(w,z,t)\sim\sum_{j=0}^{\infty}a^*_j(w,z)t^{n-j}$$

in $S^n_{1,0}(\overline{M}\times\overline{M}\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_z(M'),\Lambda^{0,q}T^*_w(M')))$, such that

$$a^*(y,z,t)=c^*(y,z,t)$$

and

$$\Box^{(q)}_f(a^*(w,z,t)e^{-i\overline{\phi}(z,w)t})$$

vanishes to infinite order on $\mathrm{diag}(\Gamma_q\times\Gamma_q)\times\overline{\mathbb{R}}_+$. From (8.17), we have

$$PC^*(w,z)\equiv\int_0^\infty e^{-i\overline{\phi}(z,w)t}a^*(w,z,t)dt \mod C^\infty(\overline{M}\times\overline{M}).$$

Thus,

$$CP^*(z,w)=PB_-T(P^*P)^{-1}P^*(z,w)\equiv\int_0^\infty e^{i\phi(z,w)t}a(z,w,t)dt \mod C^\infty(\overline{M}\times\overline{M}),$$
$$a(z,w,t)\in S^n_{1,0}(\overline{M}\times\overline{M}\times[0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))),$$
$$a(z,w,t)\sim\sum_{j=0}^{\infty}a_j(z,w)t^{n-j}$$

in the space $S^n_{1,0}(\overline{M}\times\overline{M}\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M')))$. From this and Proposition 8.5, we get the main result of this work

Theorem 8.9. *Given q, $0\leq q\leq n-1$. Suppose that $Z(q)$ fails at some point of Γ and that $Z(q-1)$ and $Z(q+1)$ hold at each point of Γ. Then*

$$K_{\Pi^{(q)}}(z,w)\in C^\infty(\overline{M}\times\overline{M}\setminus\mathrm{diag}(\Gamma_q\times\Gamma_q);\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))).$$

Moreover, in a neighborhood U of $\mathrm{diag}(\Gamma_q\times\Gamma_q)$, $K_{\Pi^{(q)}}(z,w)$ satisfies

$$K_{\Pi^{(q)}}(z,w)\equiv\int_0^\infty e^{i\phi(z,w)t}a(z,w,t)dt \mod C^\infty(U\bigcap(\overline{M}\times\overline{M})) \tag{8.19}$$

with

$$a(z,w,t)\in S^n_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))),$$

$$a(z,w,t) \sim \sum_{j=0}^{\infty} a_j(z,w)t^{n-j}$$

in the space $S^n_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))),$

$$a_0(z,z)\neq 0,\ z\in\Gamma_q,$$

where

$$a_j(z,w)\in C^\infty(U\bigcap(\overline{M}\times\overline{M});\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))),\ j=0,1,\ldots,$$

and

$$\phi(z,w)\in C^\infty(U\bigcap(\overline{M}\times\overline{M})), \tag{8.20}$$

$$\phi(z,z)=0,\ \ z\in\Gamma_q, \tag{8.21}$$

$$\phi(z,w)\neq 0\ \ \textit{if}\ (z,w)\notin \operatorname{diag}(\Gamma_q\times\Gamma_q), \tag{8.22}$$

$$\operatorname{Im}\phi\geq 0, \tag{8.23}$$

$$\operatorname{Im}\phi(z,w)>0\ \ \textit{if}\ (z,w)\notin\Gamma\times\Gamma, \tag{8.24}$$

$$\phi(z,w)=-\overline{\phi}(w,z). \tag{8.25}$$

For $p\in\Gamma_q$*, we have*

$$\sigma_{\Box^{(q)}_f}(z,d_z\phi(z,w))\ \textit{vanishes to infinite order at}\ z=p,$$
$$(z,w)\ \textit{is in some neighborhood of}\ (p,p)\ \textit{in}\ M'. \tag{8.26}$$

For $z=w$*,* $z\in\Gamma_q$*, we have*

$$d_z\phi=-\omega_0-idr,$$

$$d_w\phi=\omega_0-idr.$$

As before, we put

$$B_-(x,y)\equiv\int_0^\infty e^{i\phi_-(x,y)t}b(x,y,t)dt,$$

$$b(x,y,t)\sim\sum_{j=0}^{\infty}b_j(x,y)t^{n-1-j},$$

and

$$K_{\Pi^{(q)}}(z,w)\equiv\int_0^\infty e^{i\phi(z,w)t}a(z,w,t)dt$$

$$a(z,w,t)\sim\sum_{j=0}^{\infty}a_j(z,w)t^{n-j}.$$

Since $\Pi^{(q)} \equiv PB_- T(P^*P)^{-1}P^*$,

$$(P^*P)^{-1} = 2\sqrt{-\triangle_\Gamma} + \text{lower order terms}$$

and

$$T = 2(\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^{\wedge} + \text{lower order terms},$$

we have

$$a_0(x,x) = 2\sigma_{\sqrt{-\triangle_\Gamma}}(x, (\phi_-)'_y(x,x))b_0(x,x)2(\overline{\partial} r(x))^{\wedge,*}(\overline{\partial} r(x))^{\wedge}, \quad x \in \Gamma.$$

Since $(\phi_-)'_y(x,x) = \omega_0(x)$ and $\|\omega_0\| = 1$ on Γ, it follows that

$$a_0(x,x) = 4b_0(x,x)(\overline{\partial} r(x))^{\wedge,*}(\overline{\partial} r(x))^{\wedge}. \tag{8.27}$$

From this and Proposition 7.17, we get the following

Proposition 8.10. *Under the assumptions of Theorem 8.9, let $p \in \Gamma_q$, $q = n_-$. Let*

$$\overline{Z}_1(z), \ldots, \overline{Z}_{n-1}(z)$$

be an orthonormal frame of $\Lambda^{1,0}T_z(\Gamma)$, $z \in \Gamma$, for which the Levi form is diagonalized at p. Let $e_j(z)$, $j = 1, \ldots, n-1$ denote the basis of $\Lambda^{0,1}T^*_z(\Gamma)$, $z \in \Gamma$, which is dual to $Z_j(z)$, $j = 1, \ldots, n-1$. Let $\lambda_j(z)$, $j = 1, \ldots, n-1$ be the eigenvalues of the Levi form L_z, $z \in \Gamma$. We assume that

$$\lambda_j(p) < 0 \;\; \text{if} \;\; 1 \le j \le n_-.$$

Then

$$a_0(p,p) = \left|\lambda_1(p)\right| \cdots \left|\lambda_{n-1}(p)\right| \pi^{-n} 2\Big(\prod_{j=1}^{j=n_-} e_j(p)^{\wedge} e_j^{\wedge,*}(p)\Big) \circ (\overline{\partial} r(p))^{\wedge,*}(\overline{\partial} r(p))^{\wedge}. \tag{8.28}$$

9 Examples

The aim of this section is to illustrate the results in some simple examples. First, we will show that when M' is Kähler (see below), then $F^{(q)}$ is injective for any q, $0 \le q \le n$. We recall that $F^{(q)}$ is defined by (1.9).

As before, let $g = \sum_{j,k=1}^{n} g_{j,k} dz_j \otimes d\overline{z}_k$ be the Hermitian metric on $\Lambda^{1,0}T(M')$ and let

$$\omega = i \sum_{j,k=1}^{n} g_{j,k} dz_j \wedge d\overline{z}_k$$

be the associated real $(1,1)$-form. We call g a Kähler metric if $d\omega = 0$. ω is then called its Kähler form. A complex manifold endowed with a Kähler metric is called a Kähler manifold.

Let

$$\partial : C^\infty(M'; \Lambda^{p,q} T^*(M')) \to C^\infty(M'; \Lambda^{p+1,q} T^*(M'))$$

be the part of the exterior differential operator which maps forms of type (p,q) to forms of type $(p+1,q)$ and we denote by

$$\partial_f^* : C^\infty(M'; \Lambda^{p+1,q} T^*(M')) \to C^\infty(M'; \Lambda^{p,q} T^*(M'))$$

the formal adjoint of ∂. The following is well-known (see page 113 of Morrow and Kodaira [25])

Proposition 9.1. *If M' is Kähler, then*

$$\partial \partial_f^* + \partial_f^* \partial = \overline{\partial}\, \overline{\partial_f}^* + \overline{\partial_f}^* \overline{\partial}$$

on the space $C^\infty(M'; \Lambda^{p,q} T^*(M'))$, $0 \le p, q \le n$.

Note that

$$\partial_f^* u = 0, \;\; u \in C^\infty(M'; \Lambda^{0,q} T^*(M')), \;\; \overline{\partial} u = 0, \;\; u \in C^\infty(M'; \Lambda^{p,n} T^*(M'))$$

and

$$\overline{\partial_f}^* u = 0, \;\; u \in C^\infty(M'; \Lambda^{p,0} T^*(M')), \;\; \partial u = 0, \;\; u \in C^\infty(M'; \Lambda^{n,q} T^*(M')).$$

Now, we assume that M' is Kähler. We claim that $F^{(q)}$ is injective, for any q, $0 \le q \le n$. Given q, $0 \le q \le n$. Let $u \in \operatorname{Ker} F^{(q)}$. Then, $u \in C^\infty(\overline{M}; \Lambda^{0,q} T^*(M'))$, $\overline{\partial} u = 0$, $\overline{\partial_f}^* u = 0$ and $\gamma u = 0$. From Proposition 9.1, we can check that

$$\partial u = 0.$$

We work with local coordinates $z = (z_1, \ldots, z_n)$ defined on some neighborhood of $p \in \Gamma$. We write

$$u = \sum_J u_J d\overline{z}^J, \;\; J = (j_1, \ldots, j_q), \;\; 1 \le j_1 < j_2 < \cdots < j_q \le n.$$

Since $\partial u = 0$, for any J,

$$\frac{\partial u_J}{\partial z_j} = 0, \;\; j = 1, \ldots, n.$$

$\overline{u}_J$ is holomorphic and $\gamma \overline{u}_J = 0$. Thus, $u_J = 0$ and consequently $u = 0$. We have proved the claim.

9.1 Complex projective space

For a point $(\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{C}^{n+1} \setminus (0, 0, \dots, 0)$,

$$\xi = \{(\lambda\xi_0, \lambda\xi_1, \dots, \lambda\xi_n); \lambda \in \mathbb{C}\}$$

is a complex line through $0 = (0, 0, \dots, 0)$. The collection of all complex lines through 0 is called the n-dimensional complex projective space and denoted by $\mathbb{CP}^n$. A point ξ of $\mathbb{CP}^n$ represents a complex line

$$\xi = \{(\lambda\xi_0, \dots, \lambda\xi_n); \lambda \in \mathbb{C}\}.$$

$\xi_0, \dots, \xi_n$ are called the homogeneous coordinates of $\xi \in \mathbb{CP}^n$ and denoted by $\xi = (\xi_0, \dots, \xi_n)$. The equality

$$(\xi_0', \dots, \xi_n') = (\xi_0, \dots, \xi_n)$$

means that $\xi_0' = \lambda\xi_0, \dots, \xi_n' = \lambda\xi_n$ for some $\lambda \neq 0$, $\lambda \in \mathbb{C}$. Put

$$U_j = \left\{\xi = (\xi_0, \dots, \xi_n) \in \mathbb{CP}^n; \xi_j \neq 0\right\}. \tag{9.1}$$

$\xi \in U_j$ is represented as

$$\xi = (\frac{\xi_0}{\xi_j}, \dots, \frac{\xi_{j-1}}{\xi_j}, 1, \frac{\xi_{j+1}}{\xi_j}, \dots, \frac{\xi_n}{\xi_j}).$$

The map

$$\begin{aligned} z^j : \xi \to z^j(\xi) &= (\frac{\xi_0}{\xi_j}, \dots, \frac{\xi_{j-1}}{\xi_j}, \frac{\xi_{j+1}}{\xi_j}, \dots, \frac{\xi_n}{\xi_j}) \\ &= (z_1, \dots, z_n) \end{aligned} \tag{9.2}$$

gives local coordinates on U_j where $\mathscr{U}_j = z^j(U_j) = \mathbb{C}^n$. If

$$z = (z_1, \dots, z_n) \in z^j(U_j \bigcap U_k), \quad j > k,$$

then $z_k \neq 0$. The coordinates transformations

$$\begin{aligned} &\tau_{k,j} : z^j(U_j \bigcap U_k) \to z^k(U_j \bigcap U_k), \\ &\quad (z_1, \dots, z_n) \to (\frac{z_1}{z_k}, \dots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \dots, \frac{z_{j-1}}{z_k}, \frac{1}{z_k}, \frac{z_{j+1}}{z_k}, \dots, \frac{z_n}{z_k}) \end{aligned} \tag{9.3}$$

are biholomorphic. $\mathbb{CP}^n$ is a complex manifold obtained by glueing the $(n+1)$-copies of $\mathbb{C}^n$ via the isomorphisms (9.3).

On U_ν, put

$$a_\nu(\xi) = \frac{1}{|\xi_\nu|^2}(|\xi_0|^2 + \cdots + |\xi_n|^2), \ \ \xi = (\xi_0, \ldots, \xi_n). \tag{9.4}$$

Now, we work with local coordinates

$$z = (z_1, \ldots, z_n) = (\frac{\xi_0}{\xi_\nu}, \ldots, \frac{\xi_{\nu-1}}{\xi_\nu}, \frac{\xi_{\nu+1}}{\xi_\nu}, \ldots, \frac{\xi_n}{\xi_\nu}) \tag{9.5}$$

defined on U_ν. Then,

$$a_\nu(z) = 1 + \sum_{j=1}^n \left| z_j \right|^2.$$

We can check that

$$\partial\overline{\partial} \log a_\nu(z) = \sum_{j,k=1}^n g_{j,k} dz_j \wedge d\overline{z}_k,$$

where

$$g_{j,k} = \frac{1}{a_\nu(z)^2}(a_\nu(z)\delta_{j,k} - \overline{z}_j z_k). \tag{9.6}$$

Here $\delta_{j,k} = 0$ if $j \neq k$, $\delta_{j,k} = 1$ if $j = k$. The Hermitian metric $\sum_{j,k=1}^n g_{j,k} dz_j \otimes d\overline{z}_k$ is easily seen to be positive definite. Note that

$$a_\nu(z) = \left| e_{\nu,\mu}(z) \right|^2 a_\mu(z), \ \ e_{\nu,\mu} = \frac{\xi_\mu}{\xi_\nu} \ \text{ on } \ U_\nu \bigcap U_\mu.$$

$e_{\nu,\mu}$ is a non-vanishing holomorphic function. Hence

$$\partial\overline{\partial} \log a_\nu(z) = \partial\overline{\partial} \log a_\mu(z) \ \text{ on } \ U_\nu \bigcap U_\mu.$$

Thus, $\sum_{j,k=1}^n g_{j,k} dz_j \otimes d\overline{z}_k$ gives a Hermitian metric on $\mathbb{CP}^n$. Its associated real $(1,1)$-form is given by

$$\begin{aligned} \omega &= i\partial\overline{\partial} \log a_\nu(z) \\ &= i d\overline{\partial} \log a_\nu(z) \ \text{ on } \ U_\nu, \end{aligned}$$

since $d = \partial + \overline{\partial}$. Therefore $d\omega = 0$ and $\sum_{j,k=1}^n g_{j,k} dz_j \otimes d\overline{z}_k$ is a Kähler metric on $\mathbb{CP}^n$.

Now, we fix the Hermitian metric above on $\mathbb{CP}^n$. Let

$$r(\xi) = \frac{1}{|\xi|^2}(-\lambda_0 |\xi_0|^2 - \lambda_1 |\xi_1|^2 - \cdots - \lambda_{k_0} \left|\xi_{k_0}\right|^2 + \lambda_{k_0+1} \left|\xi_{k_0+1}\right|^2 + \cdots + \lambda_n |\xi_n|^2),$$

where $|\xi|^2 = |\xi_0|^2 + \cdots + |\xi_n|^2$, $\lambda_j > 0$, $j = 0, 1, \ldots, n$. Put

$$\Gamma = \{\xi \in \mathbb{CP}^n; \, r(\xi) = 0\}, \ \ M = \{\xi \in \mathbb{CP}^n; \, r(\xi) < 0\}.$$

It is easy to see that $dr \neq 0$ at each point of Γ.

Lemma 9.2. Γ *is connected.*

Proof. Let $\xi=(\xi_0,\ldots,\xi_n)$, $\xi'=(\xi'_0,\ldots,\xi'_n)\in\Gamma$. We have

$$\lambda_0|\xi_0|^2+\lambda_1|\xi_1|^2+\cdots+\lambda_{k_0}\left|\xi_{k_0}\right|^2=\lambda_{k_0+1}\left|\xi_{k_0+1}\right|^2+\cdots+\lambda_n|\xi_n|^2$$

and

$$\lambda_0\left|\xi'_0\right|^2+\lambda_1\left|\xi'_1\right|^2+\cdots+\lambda_{k_0}\left|\xi'_{k_0}\right|^2=\lambda_{k_0+1}\left|\xi'_{k_0+1}\right|^2+\cdots+\lambda_n\left|\xi'_n\right|^2.$$

Since $\mathbb{C}$ is connected, we can find continuous curves

$$c_j(t):[0,1]\to\mathbb{C},\ \ j=0,1,\ldots,n,$$

such that

$$c_j(0)=\xi_j,\ \ c_j(1)=\xi'_j,\ \ j=0,1,\ldots,n.$$

Put

$$d(t)^2=\frac{\lambda_0|c_0(t)|^2+\cdots+\lambda_{k_0}\left|c_{k_0}(t)\right|^2}{\lambda_{k_0+1}\left|c_{k_0+1}(t)\right|^2+\cdots+\lambda_n|c_n(t)|^2}.$$

Then,

$$d(0)=d(1)=1.$$

Put

$$\begin{aligned}c(t):[0,1]&\to\Gamma\\ t&\to(c_0(t),\ldots,c_{k_0}(t),d(t)c_{k_0+1}(t),\ldots,d(t)c_n(t)).\end{aligned}$$

Then,

$$c(0)=\xi,\ \ c(1)=\xi'.$$

The lemma follows. □

Now, we work with local coordinates

$$(z_1,\ldots,z_n)=(\frac{\xi_1}{\xi_0},\frac{\xi_2}{\xi_0},\ldots,\frac{\xi_n}{\xi_0})$$

defined on U_0. Then,

$$r(z)=\frac{1}{a_0(z)}(-\lambda_0-\lambda_1|z_1|^2-\cdots-\lambda_{k_0}\left|z_{k_0}\right|^2+\lambda_{k_0+1}\left|z_{k_0+1}\right|^2+\cdots+\lambda_n|z_n|^2),$$

where $a_0(z)=1+\sum_{j=1}^n\left|z_j\right|^2$. Let $U=\sum_{k=1}^n u_k\frac{\partial}{\partial z_k}$, $V=\sum_{j=1}^n v_j\frac{\partial}{\partial z_j}\in\Lambda^{1,0}T_p(\Gamma)$, $p\in\Gamma$. From (2.33), we have

$$L_p(U,\overline{V})=\frac{1}{a_0(p)\left\|dr(p)\right\|}(-\sum_{j=1}^{k_0}\lambda_j u_j\overline{w}_j+\sum_{j=k_0+1}^{n}\lambda_j u_j\overline{w}_j).\tag{9.7}$$

We notice that $U=\sum_{j=1}^{n} u_j \frac{\partial}{\partial z_j} \in \Lambda^{1,0} T_p(\Gamma)$ if and only if

$$\sum_{j=1}^{n} \frac{\partial r}{\partial z_j} u_j = -\sum_{j=1}^{k_0} \lambda_j \overline{z}_j u_j + \sum_{k_0+1}^{n} \lambda_j \overline{z}_j u_j = 0, \ \ p=(z_1,\dots,z_n). \tag{9.8}$$

Proposition 9.3. *The Levi form is non-degenerate at each point of* Γ *and we have* $\Gamma = \Gamma_{k_0}$. *That is, the number of negative eigenvalues of the Levi form is* k_0.

Proof. Let $p=(z_1,\dots,z_n)\in\Gamma$ and let $U=\sum_{j=1}^{n} u_j \frac{\partial}{\partial z_j} \in \Lambda^{1,0} T_p(\Gamma)$. If $L_p(U,\overline{W})=0$ for all $W\in\Lambda^{1,0}T_p(\Gamma)$, from (9.7), (9.8), we see that

$$(u_1,\dots,u_n)=c(z_1,\dots,z_n), \ \ c\in\mathbb{C}.$$

From (9.8), we have

$$-\sum_{j=1}^{k_0} \lambda_j \overline{z}_j u_j + \sum_{k_0+1}^{n} \lambda_j \overline{z}_j u_j = c(-\sum_{j=1}^{k_0} \lambda_j \left|z_j\right|^2 + \sum_{k_0+1}^{n} \lambda_j \left|z_j\right|^2) = \lambda_0 c = 0.$$

Thus, $c=0$ and consequently the Levi form is non-degenerate at p.

We compute the signature of the Levi form at

$$z_0 = (0,\dots,0,\sqrt{\frac{\lambda_0}{\lambda_n}})\in\Gamma.$$

From (9.8), we have

$$\Lambda^{1,0} T_{z_0}(\Gamma) = \left\{ \sum_{j=1}^{n-1} u_j \frac{\partial}{\partial z_j};\ u_j\in\mathbb{C},\ j=1,\dots,n-1 \right\}.$$

For $U=\sum_{k=1}^{n-1} u_k \frac{\partial}{\partial z_k}$, $V=\sum_{j=1}^{n-1} v_j \frac{\partial}{\partial z_j} \in \Lambda^{1,0} T_{z_0}(\Gamma)$, we have

$$L_{z_0}(U,\overline{V}) = \frac{1}{a_0(z_0)\|dr(z_0)\|}(-\sum_{j=1}^{k_0} \lambda_j u_j \overline{w}_j + \sum_{j=k_0+1}^{n-1} \lambda_j u_j \overline{w}_j).$$

Thus, the number of negative eigenvalues of L_{z_0} is k_0. The proposition follows. □

From Theorem 1.3, we know that if $q\neq k_0$, then

$$K_{\Pi^{(q)}} \in C^\infty(\overline{M}\times\overline{M};\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))).$$

If $q = k_0$, then $Z(q-1)$ and $Z(q+1)$ hold at each point of Γ. From Theorem 1.3, we have

$$K_{\Pi^{(q)}} \equiv \int_0^\infty e^{i\phi(z,w)t} b(z,w,t) dt \mod C^\infty(\overline{M} \times \overline{M})$$

with

$$b(z,w,t) \in S^n_{1,0}(\overline{M} \times \overline{M} \times]0,\infty[; \mathscr{L}(\Lambda^{0,q} T^*_w(M'), \Lambda^{0,q} T^*_z(M'))),$$

$$b(z,w,t) \sim \sum_{j=0}^\infty b_j(z,w) t^{n-j}$$

in the space $S^n_{1,0}(\overline{M} \times \overline{M} \times]0,\infty[; \mathscr{L}(\Lambda^{0,q} T^*_w(M'), \Lambda^{0,q} T^*_z(M')))$,

$$b_0(z,z) \neq 0, \ z \in \Gamma.$$

We continue to work with local coordinates

$$(z_1, \ldots, z_n) = (\frac{\xi_1}{\xi_0}, \frac{\xi_2}{\xi_0}, \ldots, \frac{\xi_n}{\xi_0})$$

defined on U_0. We study the leading term of $K_{\Pi^{(q)}}$ at

$$z_0 = (0, \ldots, 0, \sqrt{\frac{\lambda_0}{\lambda_n}}).$$

It is straight forward to see that

$$g(z_0) = \sum_{j=1}^{n-1} \frac{1}{a_0(z_0)} dz_j \otimes d\overline{z}_j + \frac{1}{a_0(z_0)^2} dz_n \otimes d\overline{z}_n,$$

$$\|dr(z_0)\|^2 = 2\lambda_n \lambda_0$$

and

$$\overline{\partial} r(z_0) = \frac{1}{a_0(z_0)} \sqrt{\lambda_0 \lambda_n} d\overline{z}_n,$$

where

$$a_0(z_0) = \frac{\lambda_n + \lambda_0}{\lambda_n}.$$

We can check that

$$\sqrt{a_0(z_0)} \frac{\partial}{\partial \overline{z}_1}, \ldots, \sqrt{a_0(z_0)} \frac{\partial}{\partial \overline{z}_{n-1}}$$

is an orthonormal frame of $\Lambda^{0,1} T_{z_0}(\Gamma)$ and the eigenvalues of L_{z_0} are

$$-\frac{1}{\|dr(z_0)\|} \lambda_1, \ldots, -\frac{1}{\|dr(z_0)\|} \lambda_{k_0}, \frac{1}{\|dr(z_0)\|} \lambda_{k_0+1}, \ldots, \frac{1}{\|dr(z_0)\|} \lambda_n.$$

Put

$$e_1 = \frac{1}{\sqrt{a_0(z_0)}} d\overline{z}_1, \dots, e_{n-1} = \frac{1}{\sqrt{a_0(z_0)}} d\overline{z}_{n-1}.$$

From Proposition 1.7, we have

$$b_0(z_0, z_0) = 2|\lambda_1| \cdots |\lambda_{n-1}| |dr(z_0)|^{-(n+1)} \pi^{-n} \Big(\prod_{j=1}^{k_0} e_j^{\wedge} e_j^{\wedge,*} \Big) \circ (\overline{\partial} r(z_0))^{\wedge,*} (\overline{\partial} r(z_0))^{\wedge}.$$

9.2 Spherical shell in $\mathbb{C}^n$

Consider the spherical shell

$$M = \{z \in \mathbb{C}^n;\, R_0 < |z| < R_1\},$$

where $0 < R_0 < R_1$, $n \geq 3$ and

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

We take the Hermitian metric

$$g = \sum_{j,k=1}^{n} dz_j \otimes d\overline{z}_k$$

on $\Lambda^{1,0} T(\mathbb{C}^n)$. The Levi form of

$$\Gamma = \{z \in \mathbb{C}^n;\, |z| = R_0\} \bigcup \{z \in \mathbb{C}^n;\, |z| = R_1\}$$

has $n-1$ positive eigenvalues at the outer boundary but $n-1$ negative eigenvalues at the inner one. We consider $(0, n-1)$ forms. Since $n \geq 3$, $Z(n-2)$ and $Z(n)$ hold at each point of Γ. From Theorem 1.3, we know that

$$K_{\Pi^{(n-1)}} \in C^\infty(\overline{M} \times \overline{M} \setminus \mathrm{diag}(\Gamma_{n-1} \times \Gamma_{n-1});\, \mathscr{L}(\Lambda^{0,n-1} T^*_w(\mathbb{C}^n), \Lambda^{0,n-1} T^*_z(\mathbb{C}^n))).$$

In a neighborhood U of $\mathrm{diag}(\Gamma_{n-1} \times \Gamma_{n-1})$, $K_{\Pi^{(n-1)}}(z, w)$ satisfies

$$K_{\Pi^{(n-1)}}(z, w) \equiv \int_0^\infty e^{i\phi(z,w)t} b(z, w, t) dt \mod C^\infty(U \bigcap (\overline{M} \times \overline{M})) \tag{9.9}$$

with

$$b(z, w, t) \in S^n_{1,0}(U \bigcap (\overline{M} \times \overline{M}) \times]0, \infty[;\, \mathscr{L}(\Lambda^{0,n-1} T^*_w(\mathbb{C}^n), \Lambda^{0,n-1} T^*_z(\mathbb{C}^n))),$$

$$b(z, w, t) \sim \sum_{j=0}^{\infty} b_j(z, w) t^{n-j}$$

in the space $S^n_{1,0}(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\mathscr{L}(\Lambda^{0,n-1}T^*_w(\mathbb{C}^n),\Lambda^{0,n-1}T^*_z(\mathbb{C}^n)))$,

$$b_0(z,z)\neq 0,\ z\in\Gamma_{n-1}.$$

Put

$$r(z)=\sqrt{2}(R_0-|z|)\ \text{ near }\ \Gamma_{n-1}.$$

We have $dr=1$ and

$$\overline{\partial}r=-\frac{1}{\sqrt{2}|z|}(z_1d\overline{z}_1+\cdots+z_nd\overline{z}_n).$$

We consider

$$p=(0,\ldots,0,R_0)\in\Gamma_{n-1}.$$

We can check that

$$\frac{\partial}{\partial\overline{z}_1},\ldots,\frac{\partial}{\partial\overline{z}_{n-1}}$$

is an orthonormal frame of $\Lambda^{0,1}T_p(\Gamma)$ and the eigenvalues of L_p are

$$-\frac{1}{\sqrt{2}R_0},\ldots,-\frac{1}{\sqrt{2}R_0}.$$

From Proposition 1.7, we have

$$b_0(p,p)=(\sqrt{2}R_0)^{-(n-1)}\pi^{-n}\Big(\prod_{j=1}^{n-1}d\overline{z}_j^{\wedge}d\overline{z}_j^{\wedge,*}\Big)\circ(d\overline{z}_n^{\wedge,*}d\overline{z}_n^{\wedge}).$$

A Appendix: The wave front set of a distribution, a review

We will give a brief discussion of wave front set in a setting appropriate for our purpose. For more details on the subject, see Hörmander [14], Hörmander [16] and Grigis-Sjöstrand [11]. Our presentation is essentially taken from [11]. For all the proofs of this section, we refer the reader to chapter 7 of [11], chapter $VIII$ of [14] and chapter $XVIII$ of [16].

We will asume the reader is familiar with some basic notions and facts of microlocal analysis such as: Hörmander symbol spaces, pseudodifferential operators. Nevertheless we recall briefly some of this notions.

Let $\Omega\subset\mathbb{R}^n$ be an open set. From now on, we write $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$, $D_x^\alpha=D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$ and $|\alpha|=\alpha_1+\cdots+\alpha_n$, where $x=(x_1,\ldots,x_n)$, $D_{x_j}=-i\partial_{x_j}$. We have the following

Definition A.1. Let $m \in \mathbb{R}$. $S^m_{1,0}(\Omega \times \mathbb{R}^N)$ is the space of all $a \in C^\infty(\Omega \times \mathbb{R}^N)$ such that for all compact sets $K \subset \Omega$ and all $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^N$, there is a constant $c > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq c(1+|\xi|)^{m-|\beta|},\ (x,\xi) \in K \times \mathbb{R}^N.$$

$S^m_{1,0}$ is called the space of symbols of order m. We write $S^{-\infty}_{1,0} = \bigcap S^m_{1,0}$, $S^\infty_{1,0} = \bigcup S^m_{1,0}$.

Let $Y \subset \mathbb{R}^{m_1}$, $Z \subset \mathbb{R}^{m_2}$ be open sets. We recall that the Schwartz kernel theorem(see Hörmander [16]) states that there is a bijection between the set of distributions $K \in \mathscr{D}'(Y \times Z)$ and the set of continuous linear operators

$$A : C_0^\infty(Z) \to \mathscr{D}'(Y).$$

The correspondence is given by

$$\langle Au\ , v\rangle_Y = \langle K\ , v \otimes u\rangle_{Y\times Z}\,,\ u \in C_0^\infty(Z),\ v \in C_0^\infty(Y),$$

where $\langle\ ,\rangle_Y$ and $\langle\ ,\rangle_{Y\times Z}$ denote the duality brackes for $\mathscr{D}'(Y) \times C_0^\infty(Y)$ and $\mathscr{D}'(Y \times Z) \times C_0^\infty(Y \times Z)$ respectively and $(v \otimes u)(y,z) = v(y)u(z)$. We call K the distribution kernel of A, and write $K = K_A$. Moreover, the following two conditions are equivalent:

(i) $K_A \in C^\infty(Y \times Z)$,

(ii) A is continuous $\mathscr{E}'(Z) \to C^\infty(Y)$.

If A satisfies (i) or (ii), we say that A is smoothing. Let B be a continuous linear operator

$$B : C_0^\infty(Z) \to \mathscr{D}'(Y).$$

We write $A \equiv B$ if $A - B$ is a smoothing operator.

In order to simplify the discussion of composition of some operators, it is convenient to introduce the notion of properly supported operators. Let C be a closed subset of $Y \times Z$. We say that C is proper if the two projections

$$\Pi_y : (y,z) \in C \to y \in Y$$
$$\Pi_z : (y,z) \in C \to z \in Z$$

are proper, that is the inverse image of every compact subset of Y and Z respectively is compact.

A continuous linear operator

$$A : C_0^\infty(Z) \to \mathscr{D}'(Y)$$

is said to be properly supported if $\operatorname{supp} K_A \subset Y \times Z$ is proper. If A is properly supported, then A is continuous

$$C_0^\infty(Z) \to \mathscr{E}'(Y)$$

and A has a unique continuous extension

$$C^\infty(Z) \to \mathscr{D}'(Y).$$

Definition A.2. Let $k \in \mathbb{R}$. A pseudodifferential operator of order k type $(1,0)$ is a continuous linear map

$$A : C_0^\infty(\Omega) \to \mathscr{D}'(\Omega)$$

such that the distribution kernel of A is

$$K_A = A(x,y) = \frac{1}{(2\pi)^n} \int e^{i\langle x-y,\xi\rangle} a(x,\xi) d\xi$$

with $a \in S^k_{1,0}(T^*(\Omega))$. We shall write $L^k_{1,0}(\Omega)$ to denote the space of pseudodifferential operators of order k type $(1,0)$.

Definition A.3. Let

$$A = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,\xi) d\xi \in L^m_{1,0}(\Omega),\ a \in S^m_{1,0}(T^*(\Omega)).$$

Then A is said to be elliptic at $(x_0,\xi_0) \in T^*(\Omega) \smallsetminus 0$ if

$$ab - 1 \in S^{-1}_{1,0}(T^*(\Omega))$$

in a conic neighborhood of (x_0,ξ_0) for some $b \in S^{-m}_{1,0}(T^*(\Omega))$.

From now on, all pseudofifferential operators in this section will be assumed properly supported.

Definition A.4. Let $u \in \mathscr{D}'(\Omega)$, $(x_0,\xi_0) \in T^*(\Omega) \setminus 0$. We say that u is C^∞ near (x_0,ξ_0) if there exists $A \in L^0_{1,0}(\Omega)$ elliptic at (x_0,ξ_0), such that $Au \in C^\infty(\Omega)$. We let $\mathrm{WF}(u)$ be the set of points in $T^*(\Omega) \setminus 0$, where u is not C^∞.

Lemma A.5. *Let $u \in \mathscr{D}'(\Omega)$. Then $u \in C^\infty(\Omega)$ if and only if* $\mathrm{WF}(u) = \emptyset$.

Let $K \in \mathscr{D}'(\Omega \times \Omega)$. Put

$$\mathrm{WF}'(K) = \left\{(x,\xi,y,\eta);\,(x,\xi,y,-\eta) \in \mathrm{WF}(K)\right\}.$$

Proposition A.6. *Let*

$$A = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x,\xi) d\xi \in L^m_{1,0}(\Omega),\ a \in S^m_{1,0}(T^*(\Omega)).$$

Let Λ *be the smallest closed cone in* $T^*(\Omega) \setminus 0$ *such that for any* $\chi \in C^\infty(T^*(\Omega))$, $\chi(x,\lambda\xi) = \chi(x,\xi)$ *and* $\chi = 0$ *in some conic neighborhood of* Λ, *we have*

$$\chi a \in S^{-\infty}_{1,0}(T^*(\Omega)).$$

Then

$$\mathrm{WF}'(K_A) = \mathrm{diag}(\Lambda \times \Lambda).$$

Moreover, let $u \in \mathscr{D}'(\Omega)$. *Then,*

$$\mathrm{WF}(Au) \subset \Lambda \bigcap \mathrm{WF}(u).$$

Proposition A.7. *Let*

$$\mathscr{K} : C^\infty_0(\Omega) \to \mathscr{D}'(\Omega)$$

with distribution kernel $K \in \mathscr{D}'(\Omega \times \Omega)$. *We assume that*

$$\mathrm{WF}'(K) \subset \left\{(x,\xi,x,\xi); (x,\xi) \in T^*(\Omega) \setminus 0\right\}.$$

Then, there is a unique way of defining $\mathscr{K} u$ *for every* $u \in \mathscr{E}'(\Omega)$ *so that the map*

$$u \in \mathscr{E}'(\Omega) \to \mathscr{K} u \in \mathscr{D}'(\Omega)$$

is continuous. Moreover, we have

$$\begin{aligned}\mathrm{WF}(\mathscr{K} u) \subset &\{(x,\xi); (x,\xi,x,\xi) \in \mathrm{WF}'(K)\\ &\textit{for some } (x,\xi) \in \mathrm{WF}(u)\}.\end{aligned}$$

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