BRIGITTE BIDÉGARAY
Étude qualitative d'équations d'ondes dispersives, 1994

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MM	F. BETHUEL	
	T. CAZENAVE	Rapporteur
	J.M. GHIDAGLIA	
	J. GINIBRE	
	J.C. GUILLOT	Rapporteur
	J.C. SAUT	

Век живи, век учись, дураком умрёшь !

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Introduction

Cette thèse est composée de trois parties traitant de différentes équations aux dérivées partielles non linéaires.
Dans la première et la troisième parties, on traite le problème de Cauchy associé à des équations de type Schrödinger provenant de la physique des plasmas et de l'optique non linéaire.
Dans la deuxième partie, on construit des mesures invariantes par le flot de divers systèmes hamiltoniens.

0.1 Etude d'une équation de Zakharov non locale.

L'équation de Zakharov non locale a été dérivée par V.E. Zakharov en 1972 [15] pour la description de la turbulence de Langmuir qui est un phénomène non linéaire apparaîssant dans des plasmas non magnétisés.
Cette équation qui décrit plus particulièrement le collapse des ondes de Langmuir (qui sont piégées dans des zones où le plasma est de faible densité, s'auto-focalisent et s'effondrent en temps fini) fait intervenir l'enveloppe du champ électrique de l'onde plasma $\vec{E}$ et la variation de la densité ionique δn. Après adimensionalisation des équations, on obtient :

$$\begin{cases} i\dfrac{\partial E}{\partial t} + \Delta E = -\nabla\Delta^{-1}\text{div } (\delta n E), \\ \\ \dfrac{1}{c^2}\dfrac{\partial^2 \delta n}{\partial t^2} - \Delta \delta n = \Delta |E|^2, \end{cases} \tag{1.1}$$

où c est la vitesse ion-acoustique des ondes sonores libres.
On peut relier naturellement l'étude de cette équation à celle de l'équation de Zakharov (locale), dans laquelle $\nabla\Delta^{-1}\text{div}$ est remplacé par l'opérateur identité. Le problème de Cauchy pour l'équation de Zakharov sur $\mathbf{R}^n$ a été étudié par C. Sulem et P.L. Sulem (1979) [13], S.H. Schochet et M.I. Weinstein (1986) [12], H. Added

et S. Added (1988) [1], et T. Ozawa et Y. Tsutsumi (1990) [9] et pour des données périodiques aux bords sur $\mathbb{R}$ par J. Bourgain (1993) [2]. Par ailleurs, L. Glangetas et F. Merle (1992) [5, 6] ont étudié les solutions auto-similaires explosant en temps fini de cette équation en dimension 2.
Je démontre ici pour l'équation non locale des résultats similaires à ceux de Schochet et Weinstein et de Ozawa et Tsutsumi.
Le premier théorème concerne l'étude du problème de Cauchy pour des données initiales dans un espace de Sobolev assez gros. On montre que ce problème est localement bien posé en dimension 1, 2 et 3. Ce résultat est obtenu en effectuant une méthode de point fixe mise en œuvre sur une formulation intégrale du problème.

Théorème 1.1 *Pour le système (1.1) sur* $\mathbb{R}^N, N = 1, 2, 3$, *avec les données initiales*

$$\begin{cases} \delta n(x,0) = n_0(x), \\ \partial_t \delta n(x,0) = n_1(x), \\ E(x,0) = E_0(x), \end{cases}$$

où $n_0 \in H^1, n_1 \in L^2$ *et* $E_0 \in H^2$, *il existe un temps* $T > 0$ *dépendant uniquement de* $\|n_0\|_{H^1}$, $\|n_1\|_{L^2}$, $\|E_0\|_{H^2}$ *et* N *et une unique solution* $(E(t), \delta n(t))$ *vérifiant*

$$\begin{cases} E(t) & \in & C^0([0,T]; H^2) \cap C^1([0,T]; L^2), \\ E(t) & \in & W^{1,8/N}(0,T; L^4), \\ \delta n(t) & \in & C^0([0,T]; H^1) \cap C^1([0,T]; L^2) \cap C^2([0,T]; H^{-1}). \end{cases}$$

Ce résultat devient global dans le cas de la dimension 1 ou pour de petites données initiales dans le cas des dimensions supérieures.
On s'intéresse ensuite à la limite des solutions de cette équation quand c tend vers l'infini. Ceci demande de faire à nouveau une étude d'existence pour le problème de Cauchy mais dans des espaces de Sobolev plus petits. Pour avoir des estimations uniformes en c, qui permettront le passage à la limite, on écrit les équations de Zakharov non locales comme une perturbation dispersive d'un système hyperbolique.
Ce nouveau résultat pour le problème de Cauchy s'écrit

Théorème 1.2 *Soit* $s \geq \left[\frac{N}{2}\right] + 3$. *Il existe unique solution du système (1.1) sur* $\mathbb{R}^N, N = 1, 2, 3$, *avec les données initiales*

$$\begin{cases} \delta n^c(x,0) = n_0(x), \\ \partial_t \delta n^c(x,0) = n_1(x), \\ E^c(x,0) = E_0(x), \end{cases}$$

sur un intervalle de temps $[0, T]$, T *ne dépendant pas de c mais uniquement de* $\|n_0\|_{H^s}$, $\|n_1\|_{H^{s-1}}$ *and* $\|E_0\|_{H^{s+1}}$, *que nous supposons finis.*
De plus pour tout $t \in [0, T]$ *nous avons l'estimation*

$$\|E^c\|_{H^{s+1}} + \|E^c_t\|_{H^{s-1}} + \|n^c\|_{H^s} + \frac{1}{c}\|n^c_t\|_{H^{s-1}} + \frac{1}{c^2}\|n^c_{tt}\|_{H^{s-2}} \leq const.$$

La limite formelle du système quand c tend vers $+\infty$ est

$$i\frac{\partial E}{\partial t} + \Delta E = \nabla \Delta^{-1}\mathrm{div}\ (|E|^2 E). \tag{1.2}$$

Cette équation de Schrödinger non linéaire non locale a été étudiée par T. Colin [4] et correspond physiquement au cas où les ions atteignent un état d'équilibre en un temps négligeable. On montre que la limite formelle est en fait rigoureuse au sens suivant

Théorème 1.3 *Quand c tend vers ∞,*
$\delta n^c + |E^c|^2 \to 0$ *dans* $C^0([0,T] \times \mathbf{R}^k)$,
$\nabla(\delta n^c + |E^c|^2) \to 0$ *dans* $C^0([0,T]; H^{s-2})$,
$E^c \to \tilde{E}$ *dans* $C^1([0,T] \times \mathbf{R}^k) \cap \mathcal{C}([0,T]; \mathcal{C}^2)$,
où $\tilde{E}$ est l'unique solution de (1.2).

0.2 Mesures invariantes pour certaines équations aux dérivées partielles.

La motivation pour la construction de mesures invariantes a été la recherche d'une explication au phénomène de Fermi-Pasta-Ulam qui a été observé aussi bien numériquement qu'expérimentalement dans le cadre d'ondes hydrodynamiques en grande profondeur régies par l'équation cubique de Schrödinger et dans beaucoup d'autres équations dispersives non linéaires, et ceci pour des systèmes non intégrables. Ce phénomène est le retour périodique en temps dans un voisinage de la donnée initiale.

La construction de la mesure est effectuée en suivant une idée de P.E. Zhidkov [16]. On considère uniquement des équations hamiltoniennes de la forme :

$$\dot{u}(t) = JH'(u(t)), \tag{2.3}$$

où J est un opérateur anti-adjoint.
On dispose donc d'au moins une quantité conservée, l'hamiltonien H. La construction de la mesure se fait d'abord pour un problème de dimension finie approchant le problème initial (méthode de Galerkin) puis on passe à la limite. Ainsi, on construit une mesure cylindrique μ sur l'espace des phases. Celle-ci est invariante par le flot de l'équation, c'est-à-dire que si on note $u(t) = f(u(0), t)$, où u est solution de (2.3), si Ω est un ouvert de l'espace des phases et $\Omega_t = f(\Omega, t)$ alors $\mu(\Omega_t) = \mu(\Omega)$ pour tout t.
Un tel résultat suppose évidemment des conditions supplémentaires sur l'équation considérée. Ces limitations sont dues au fait qu'il faut que l'objet construit soit bien une mesure (et pas seulement une pré-mesure) et que le problème soit bien posé globalement en temps.
Si, de plus, on dispose d'une autre quantité invariante, on peut montrer par faible compacité que

Théorème 2.4 *Presque toutes les trajectoires sont récurrentes au sens de Poisson.*

Pour la justification de ces constructions, nous utilisons les résultats dus à J. Bourgain d'existence de solutions pour des données L^2 avec des conditions aux bords périodiques.

Outre à des problèmes "continus" tels l'équation de Schrödinger non linéaire ou l'équation de la chaleur, on peut aussi appliquer ces résultats à l'étude de schémas numériques. Le cas des méthodes de Galerkin a été traité lors de la construction générale. D'autres discrétisations en espace sont possibles dans la mesure où elles conservent deux quantités, c'est le cas de certains schémas aux différences finies, de schémas spectraux et pseudo-spectraux discrétisant les problèmes ci-dessus mais aussi le système de Zakharov.
Une telle démarche ne permet pas de justifier pleinement le phénomène de récurrence effectivement observé numériquement. Une autre approche de ce problème est la théorie KAM développée dans les années 1960 par Kolmogorov, Arnold et Moser [10]. On peut feuilleter l'espace des phases de systèmes intégrables (comme KdV périodique) par des N-tores. Certaines discrétisations de ces problèmes héritent de cette structure feuilletée et en dimension finie, sous de petites perturbations, ce feuilletage par des N-tores persiste. Le flot de l'équation est alors "piégé" dans ces tores, d'où la récurrence.
Ceci n'explique cependant pas le phénomène de récurrence observé dans des équations aux dérivées partielles en dimension d'espace plus grande que 2 (voir par exemple Yuen et Ferguson [14] pour des équations non linéaires de Schrödinger, ou Rachid [17] pour des systèmes de Davey-Stewartson).

0.3 Etude d'équations intervenant en optique non linéaire.

Dans cette partie, on s'intéresse plus particulièrement à deux équations qui décrivent l'interaction laser-matière. Les modélisations choisies sont celles développées par A.C. Newell et J.V. Moloney [8].

0.3.1 L'équation de Maxwell-Debye.

La première de ces équations est le système de Maxwell-Debye

$$\begin{cases} \left(\dfrac{\partial}{\partial z} + \dfrac{n_0}{c}\dfrac{\partial}{\partial t}\right) A - \dfrac{i}{2k}\nabla_1^2 A + i\dfrac{\omega_0}{c}\delta n A = 0, \\ \\ \tau\dfrac{\partial \delta n}{\partial t} + \delta n = n_2 |A|^2, \end{cases} \tag{3.4}$$

qui décrit l'interaction d'une onde électromagnétique (d'enveloppe A) avec un milieu résonnant (d'indice $n_0 + \delta n$) qui admet un temps de réponse non négligeable τ.

Il est possible de transformer ce système en une unique équation intégro-différentielle

$$\frac{\partial \tilde{A}}{\partial t}(t;x,y) - \frac{ic}{2kn_0}\nabla_1^2\tilde{A}(t;x,y) +$$
$$+ i\frac{\omega_0}{n_0}\left\{\delta\tilde{n}(t_0;x,y) + \int_{t_0}^{t}\frac{n_2}{\tau}|\tilde{A}(\zeta;x,y)|^2 e^{\frac{\zeta}{\tau}}d\zeta\right\} e^{-\frac{t}{\tau}}\tilde{A}(t;x,y) = 0, \quad (3.5)$$

où on a éliminé la variable z par une méthode de caractéristiques.
On s'intéresse alors au problème de Cauchy pour des données initiales en $t = 0$ dans différents espaces de Sobolev.
On obtient un premier résultat d'existence et d'unicité locales en temps pour des données régulières.

Théorème 3.5 *i) Pour tout (φ, ν) appartenant à $H^s \times H^s$ avec $s > 1$, l'équation (3.5) pour les conditions initiales*

$$\tilde{A}(0) = \varphi, \qquad \delta\tilde{n}(0) = \nu,$$

admet une unique solution dans $X = L^\infty(0,T;H^s)$ pour un T suffisamment petit.
ii) Les solutions dépendent continuement des données initiales, à savoir :
si $\tilde{A} \in L^\infty(0,T;H^s)$ est solution de l'équation de Maxwell-Debye pour les données initiales (φ,ν), φ_p et ν_p tendent respectivement vers φ et ν dans H^s alors pour un p suffisamment grand la solution $\tilde{A}_p$ de l'équation de Maxwell-Debye associée aux données initiales φ_p et ν_p tend vers $\tilde{A}$ dans $L^\infty(0,T;H^s)$.

Ensuite nous démontrons un résultat analogue pour des données plus faibles

Théorème 3.6 *i) Pour tout (φ,ν) appartenant à $H^1 \times H^1$, l'équation (3.5) admet une unique solution dans $X' = L^\infty(0,T;H^1)$ pour un T suffisamment petit.*
ii) Pour tout (φ,ν) appartenant à $L^2 \times L^\infty$, l'équation (3.5) admet une unique solution appartenant à $X'' = L^4(0,T;L^4) \cap \mathcal{C}([0,T];L^2)$ pour un T suffisamment petit. De plus $\tilde{A}$ appartient à $L^q(0,T;L^r)$ pour toute paire admissible (q,r).
iii) Les solutions dépendent continuement des données initiales dans un sens analogue à celui donné dans le théorème 3.5

A chaque fois, on trouve la solution comme point fixe d'une fonctionnelle issue d'une formulation intégrale du problème.
On étudie ensuite la limite des solutions quand le retard τ tend vers 0. La limite formelle est solution de l'équation de Schrödinger cubique

$$\frac{\partial \tilde{A}}{\partial t}(t;x,y) - \frac{ic}{2kn_0}\nabla_1^2\tilde{A}(t;x,y) + i\frac{\omega_0 n_2}{n_0}|\tilde{A}(t;x,y)|^2\tilde{A}(t;x,y) = 0. \quad (3.6)$$

On montre à la fois pour des données initiales régulières ou faibles que cette limite est rigoureuse. Ainsi, on obtient les théorèmes

Théorème 3.7 *On suppose que les données initiales (pour $\tilde{A}$ et $\tilde{n}$) sont bornées uniformément dans $X = L^\infty(0,T;H^s)$, $s > 3$, et que quand τ tend vers 0, la donnée initiale φ tend fortement vers ψ dans H^s. Soit A, la solution de l'équation de Schrödinger cubique associée à cette donnée ψ. Alors la suite des $\tilde{A}$ quand τ tend vers 0 tend fortement vers A dans X.*

Théorème 3.8 *On suppose que les données initiales (pour $\tilde{A}$ et $\tilde{n}$) sont bornées uniformément dans $X = L^\infty(0,T;H^1)$, et que quand τ tend vers 0, la donnée initiale φ tend fortement vers ψ dans H^1. Soit A, la solution de l'équation de Schrödinger cubique associée à cette donnée ψ. Alors la suite des $\tilde{A}$ quand τ tend vers 0 tend fortement vers A dans X.*

0.3.2 L'équation de Maxwell-Bloch.

Le second exemple d'équation de ce type est le système de Maxwell-Bloch

$$\begin{cases} \dfrac{\partial A}{\partial z} + \dfrac{1}{c}\dfrac{\partial A}{\partial t} - i\dfrac{c}{2\omega}\nabla_1^2 A + \dfrac{\kappa}{c}A = \dfrac{i\omega}{2\varepsilon_0 c}L, \\[2ex] \dfrac{\partial L}{\partial t} + (\gamma_{12} + i(\omega_{12} - \omega))L = \dfrac{ip^2}{\hbar}AN, \\[2ex] \dfrac{\partial N}{\partial t} + \gamma_{11}(N - N_0) = \dfrac{2i}{\hbar}(A^*L - AL^*), \end{cases} \tag{3.7}$$

qui décrit l'interaction d'une onde électromagnétique d'enveloppe A avec un milieu résonnant constitué de gaz à deux niveaux d'énergie. L représente l'enveloppe de la polarisation et N est le nombre d'inversion qui décrit le passage d'un niveau d'énergie à l'autre. La constante N_0 est due à un apport d'énergie au milieu. Cette fois-ci, on ne peut pas se ramener à une unique équation comme dans le cas précédent.
Ce problème a déjà été étudié mathématiquement mais sous des formes quelque peu simplifiées, c'est-à-dire, par exemple, dans le cas où on ne tient pas compte du Laplacien par rapport aux variables transversales avec des données périodiques aux bords en z. Le système développe alors une dynamique très complexe.
On commence ici par étudier le comportement des solutions de l'approximation adiabatique, c'est-à-dire le cas où on néglige $\dfrac{\partial L}{\partial t}$ et $\dfrac{\partial N}{\partial t}$. Le système de Maxwell-Bloch devient alors

$$\frac{\partial A}{\partial z} + \frac{1}{c}\frac{\partial A}{\partial t} - i\frac{c}{2\omega}\nabla_1^2 A + \frac{\kappa}{c}A = \frac{i\omega}{2\varepsilon_0 c}L \tag{3.8}$$

où

$$L = \frac{ip^2}{\hbar}\frac{(\gamma_{12} - i(\omega_{12} - \omega))N_0 A}{\gamma_{12}^2 + (\omega_{12} - \omega)^2 + \frac{4p^2\gamma_{12}}{\hbar^2\gamma_{11}}|A|^2}, \tag{3.9}$$

et on obtient alors le résultat

Théorème 3.9 *Le problème de Cauchy est globalement bien posé dans L^2 et dans H^1 pour l'approximation adiabatique de l'équation de Maxwell-Bloch. De plus pour certaines valeurs des paramètres* $\left(\kappa > \frac{\omega p^2}{2\varepsilon_0 \hbar} \cdot \frac{-\gamma_{12} N_0}{\gamma_{12}^2 + (\omega_{12} - \omega)^2}\right)$, *les normes L^2 de A et de L tendent vers 0 quand t tend vers $+\infty$.*

Pour le système complet, on étudie ensuite le problème de Cauchy pour des données régulières

Théorème 3.10 *i) Pour tout $(\varphi, \lambda, \mu) \in L^\infty(\xi; H^s) \times L^\infty(\xi; H^s) \times L^\infty(\xi; H^s)$, où $\xi = ct - z$, l'équation (3.7) pour les données initiales*

$$A(0) = \varphi, \qquad L(0) = \lambda, \qquad N(0) = \nu,$$

admet une unique solution dans $X^3 = (L^\infty(\xi, 0, T; H^s))^3$ pour un T suffisamment petit.
ii) Les solutions dépendent continuement des données initiales dans un sens analogue à celui donné dans le théorème 3.5.

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Partie I

Etude d'une équation de Zakharov non locale.

Chapitre 1

On a nonlocal Zakharov equation.

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ON A NONLOCAL ZAKHAROV EQUATION

B. BIDEGARAY
*CMLA, Ecole Normale Supérieure de Cachan, 61 av. du Pdt Wilson
94235 Cachan CEDEX, France.*

ABSTRACT

This article deals with the Cauchy problem for a nonlocal Zakharov equation. We will first recall the physical motivation (due to V.E. Zakharov) of this system. Then we will study the local Cauchy problem for certain initial data, and will identify the limit of the solutions when the ion velocity tends to infinity.

1 Origin of the nonlocal Zakharov system

The Physical theory which follows has been developped by V.E. Zakharov[5,6] to describe the Langmuir Oscillations in Plasma Physics.
We assume that:

1. the plasma is sufficiently uniform,
2. the magnetic field is sufficiently weak,
3. the nonlinearity level is not too high,
4. there are no transverse high frequency electromagnetic waves.

We consider the following equations that modelise the phenomena:
Linearised hydrodynamical equations

$$\frac{\partial}{\partial t}\delta n_e + \operatorname{div}(n_0+\delta n)\vec{V}_e = 0, \tag{1}$$

$$\frac{\partial}{\partial t}\delta \vec{V}_e + \frac{3V_{T_e}^2}{n}\vec{\nabla}\delta n = \frac{e}{m_e}\vec{E}. \tag{2}$$

Maxwell's equation

$$\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \operatorname{curl}\operatorname{curl}\vec{E} + \frac{4\pi e}{c^2}(n_0+\delta n)\frac{\partial}{\partial t}\delta\vec{V}_e = 0. \tag{3}$$

Vlasov's equation

$$\frac{\partial f_i}{\partial t} + (V.\vec{\nabla})f_i - n\frac{e}{m_i}\vec{\nabla}\varphi_{el}\frac{\partial f_i}{\partial V} = 0, \tag{4}$$

$$\delta n_i = \frac{en_0}{T_e}(\varphi_{el} - \varphi). \tag{5}$$

We moreover set

$$\vec{E} = \frac{1}{2}\left(\vec{\bar{E}}\exp(-i\omega_{pl}t) + \vec{\bar{E}}^{*}\exp(i\omega_{pl}t)\right), \tag{6}$$

and

$$\vec{\bar{E}} = \vec{\nabla}\psi. \tag{7}$$

Eq. 1-7 imply that

$$\Delta(2i\omega_{pl}\psi_t + 3V_{T_e}^2\Delta\psi) = \omega_{pl}^2\mathrm{div}(\frac{\delta n}{n_0}\nabla\psi). \tag{8}$$

Two different hypothesis can be made:

First hypothesis

The nonlinear phenomena have such a long period that the ions have enough time to reach the Boltzmann distribution law in a low frequency field:

$$\frac{\delta n}{n_0} = -e\frac{\varphi_{el}}{T_i}. \tag{9}$$

After some computations and a change of scale, Eq. 8 and 9 lead to

$$\Delta(i\psi_t + \Delta\psi) = \mathrm{div}(|\nabla\psi|^2\nabla\psi). \tag{10}$$

This equation has been widely studied by T. Colin[1,2].

Second hypothesis

The ions do not have the time to reach this distribution:

$$\left(\frac{\partial^2}{\partial t^2} + 2\gamma_s\frac{\partial}{\partial t} - c_s^2\Delta\right)\delta n = \frac{1}{16\pi m_i}\Delta|E|^2. \tag{11}$$

We will suppose that the damping rate is zero, i.e. we neglect $2\gamma_s\frac{\partial}{\partial t}$.
After a change of scale, Eq. 11 and 8 become

$$\left\{\begin{array}{l} \Delta(i\psi_t + \Delta\psi) = \mathrm{div}(n\nabla\psi), \\ \frac{1}{c^2}n_{tt} - \Delta n = \Delta(|\nabla\psi|^2). \end{array}\right. \tag{12}$$

In what follows, we will not give any complete proof. They may be found in an article which is to appear[7].

2 Existence and uniqueness for the Cauchy Problem

We first consider the system 12 with $c = 1$:

$$\begin{cases} \Delta(i\psi_t + \Delta\psi) = \mathrm{div}(n\nabla\psi), \\ n_{tt} - \Delta n = \Delta(|\nabla\psi|^2). \end{cases} \tag{13}$$

We set $B = \nabla(-\Delta)^{-1}\nabla.$ and $\varphi = \nabla\psi$. Eq. 13 is equivalent to

$$\begin{cases} i\varphi_t + \Delta\varphi = -B(n\varphi), \\ n_{tt} - \Delta n = \Delta(|\varphi|^2). \end{cases} \tag{14}$$

B is homogeneous of order 0 in the Fourier variables. Thanks to Calderón-Zygmund's theorem we have the following result:
For all $1 < p < \infty$, there exists $C_{s,p}$ such that

$$\|Bf\|_{W^{s,p}} \leq C_{s,p}\|f\|_{W^{s,p}}. \tag{15}$$

2.1 *Theorem for the Cauchy Problem*

Theorem 1 *Let us consider the problem on R^N, $N = 1, 2, 3$.*

$$\begin{cases} i\dot{\varphi} + \Delta\varphi = \nabla\Delta^{-1}\nabla.(n\varphi), \\ \ddot{n} - \Delta n = \Delta|\varphi|^2, \\ n(x,0) = n_0(x), \\ \partial_t n(x,0) = n_1(x), \\ \varphi(x,0) = \varphi_0(x), \end{cases} \tag{16}$$

with $n_0 \in H^1, n_1 \in L^2$ and $\varphi_0 \in H^2$.
Then there exists a time $T > 0$ depending only on $\|n_0\|_{H^1}$, $\|n_1\|_{L^2}$, $\|\varphi_0\|_{H^2}$ and N and a unique solution $(\varphi(t), n(t))$ to Eq. 16 which satisfies

$$\begin{cases} \varphi(t) \in C^0([0,T]; H^2) \cap C^1([0,T]; L^2), \\ \varphi(t) \in W^{1,8/N}(0,T; L^4), \\ n(t) \in C^0([0,T]; H^1) \cap C^1([0,T]; L^2) \cap C^2([0,T]; H^{-1}). \end{cases} \tag{17}$$

Thanks to Eq. 15, we have been inspired for the idea of the proof by the work of Ozawa and Tsutsumi[3]. We first make a change of variables (in order not to loose regularity at each step). We then use a fixed point method to find existence and uniqueness in a subset of the original functional space. To conclude, we prove uniqueness in the whole space and return to the initial variables.

2.2 *Setting of the fixed point method*

Setting $F = \partial_t\varphi$, we formally get

$$\begin{cases} iF_t + \Delta F = -B(\partial_t n(\varphi_0 + \int_0^t F ds) + nF), \\ n_{tt} - \Delta n = \Delta(|\varphi|^2), \\ \varphi = (-\Delta + 1)^{-1}\{iF + B(n(\varphi_0 + \int_0^t F ds)) + (\varphi_0 + \int_0^t F ds)\}, \end{cases} \tag{18}$$

with $F(0) = F_0$, $n(0) = n_0$, $\partial_t n(0) = n_1$.
We work in R^N with $1 \leq N \leq 3$. We use the following functional space:

$$X = [L^\infty(I; L^2) \cap L^{8/N}(I; L^4)] \oplus [L^\infty(I; H^1) \cap W^{1,\infty}(I; L^2)], \tag{19}$$

where $I = [0, T]$.
We set $N = (N_1, N_2)$ with

$$\begin{cases} N_1[F,n](t) = U(t)F_0 + i\int_0^t U(t-s)\{B(\partial_s n(\varphi_0 + \int_0^s F d\tau) + nF)\}ds, \\ N_2[F,n](t) = \cos(\omega t)n_0 + \omega^{-1}\sin(\omega t)n_1 + \int_0^t \omega^{-1}\sin(\omega(t-s))\Delta|\varphi(s)|^2 ds, \\ (-\Delta + 1)\varphi = \{iF + B(n(\varphi_0 + \int_0^t F ds)) + (\varphi_0 + \int_0^t F ds)\}, \end{cases} \tag{20}$$

where $U(t)$ is the group generated by the Schrödinger operator and ω denotes $\sqrt{-\Delta}$ (multiplication by $|\xi|$ in Fourier variables).
We set

$$a = \max\{\|\varphi_0\|_{L^2}, \|\varphi_0\|_{L^4}, \|\Delta\varphi_0 + B(n_0\varphi_0)\|_{L^2}, \|n_0\|_{H^1} + \|n_1\|_{L^2}\}, \tag{21}$$

$$Y = \{(F(t), n(t)) \in X / \ \|F\|_{L^\infty(I;L^2)} \leq 2a, \|F\|_{L^{8/N}(I;L^4)} \leq 2\delta a, \\ \|n\|_{L^\infty(I;H^1)} \leq 2, \left\|\frac{dn}{dt}\right\|_{L^\infty(I;L^2)} \leq 2a\}. \tag{22}$$

The fixed point method will consist in solving

$$\begin{cases} N1[F,n] = F, \\ N2[F,n] = n. \end{cases} \tag{23}$$

$N : Y \rightarrow Y$ is a contraction if T is sufficiently small, thus we may find a fixed point.

2.3 *Return to the initial problem*

The fixed point method yields:

$$\left\{\begin{array}{lcl} F(t) & \in & \left[\bigcap_{j=0}^{1} C^j([0,T];H^{-2j})\right] \bigcap L^{8/N}(0,T;L^4), \\ n(t) & \in & \bigcap_{j=0}^{2} C^j([0,T];H^{1-j}), \\ \varphi(t) & \in & C([0,T];H^2). \end{array}\right. \tag{24}$$

This regularity enables us to return to the initial variables; we get the desired regularity and two conservation laws:

$$\int |\varphi(t)|^2 = \int |\varphi_0|^2, \tag{25}$$

$$\int (|\nabla\varphi(t)|^2 + n(t)|\varphi(t)|^2 + \frac{1}{2}(\nabla\Phi(t))^2 + \frac{1}{2}n^2(t)) = \int (|\nabla\varphi_0|^2 + n_0|\varphi_0|^2 + \frac{1}{2}(\nabla\Phi_0)^2 + \frac{1}{2}n_0^2), \tag{26}$$

where $-\Delta\Phi = \dfrac{\partial n}{\partial t}$.

3 Limit as c tends to ∞

We now consider the problem

$$\left\{\begin{array}{l} \frac{1}{c^2}n_{tt} - \Delta(n + |E|^2) = 0, \\ iE_t + \Delta E + B(nE) = 0. \end{array}\right. \tag{27}$$

The result obtained in Section 2 still holds but T depends on c.

We shall work with variables in H^s with $s > \left[\frac{k}{2}\right] + 3$.

There is no need to prove uniqueness since this has been already done in some larger space.

The formal limit of the solution is the couple $(-|\bar{E}|^2, \bar{E})$ solution of

$$i\bar{E}_t + \Delta\bar{E} - B(|\bar{E}|^2\bar{E}) = 0. \tag{28}$$

3.1 *The Theorem for the Limit*

Theorem 2 *When c tends to ∞,*

$n^c + |E^c|^2 \to 0$ *in* $C^0([0,T] \times R^k)$,

$\nabla(n^c + |E^c|^2) \to 0$ *in* $C^0([0,T];H^{s-2})$,

$E^c \to \bar{E}$ *in* $C^1([0,T] \times R^k) \cap C([0,T];C^2)$,

where $\bar{E}$ is the unique solution of

$$i\bar{E}_t + \Delta\bar{E} - B(|\bar{E}|^2\bar{E}) = 0. \tag{29}$$

This time we adapt the method of Schochet and Weinstein[4]. We first carry out a transformation of the system into a dispersive perturbation of a symmetric hyperbolic one. Then we prove the existence of a regular solution for a time independent of c and pass to the limit when c tends to ∞.

3.2 *Transformation of the system*

We set

$$\begin{cases} V = -\frac{1}{c}\Delta^{-1}\nabla n_t, \\ Q = n + |E|^2, \end{cases} \tag{30}$$

$$\sqrt{2}E = F + iG \text{ et } \sqrt{2}\nabla E = H + iL, \tag{31}$$

and

$$U = {}^t(Q, V, F, G, H, L), \tag{32}$$

then the system may be rewritten as

$$U_t + \sum_{j=1}^{k}\{R(A^j(U)U_{x_j}) + cC^jU_{x_j}\} + S(\bar{B}(U)U) = K\Delta U. \tag{33}$$

R and S are Calderón-Zygmund's operators.
$\bar{B}$ is a nonlocal operator.
K is an antisymmetric matrix.
C^j and $A^j(U)$ are symmetric matrices.
These properties will be very useful to make estimation in Sobolev spaces and to use the classical theory of hyperbolic equations.

3.3 *Existence of a regular solution for a time independent of c*

To this end, we use the following iteration scheme:

$$U^0(x,t) = U_0(x), \tag{34}$$

$$\frac{\partial U^{p+1}}{\partial t} + \sum_{j=1}^{k}\{R(A^j(U^p)U^{p+1}_{x_j}) + cC^jU^{p+1}_{x_j}\} + S(\bar{B}(U^p)U^{p+1}) = K\Delta U^{p+1}, \tag{35}$$

$$U^{p+1}(x,0) = U_0(x). \tag{36}$$

One can show the following estimates.

- $\forall p \geq 0 \qquad |||U^p|||_{s,T} \leq \delta,$

- $\forall p \geq 0 \qquad \|U^{p+1} - U^p\|_{0,T} \leq C\|U^p - U^{p-1}\|_{0,T}$, with $C < 1$.

- Then
$$U^p \to U \text{ in } L^\infty(0,T;L^2),$$
$$U^p \text{ is bounded in } L^\infty(0,T;H^s). \tag{37}$$

Then $U \in C([0,T];C^1)$ and the solution is a classical one.
Moreover we have
$$U \in Lip([0,T];H^{s-2}),$$
$$U \in C([0,T];H^s) \cap C^1([0,T];H^{s-2}). \tag{38}$$

The time T is independent of c (because C^j is symmetric).
To gain some regularity we use the theory of commutators.
There is no problem in returning to the initial variables and we get

Theorem 3 *Let $s \geq \left[\frac{k}{2}\right] + 3$. For $n_0 \in H^s$, $n_1 \in H^{s-1}$ and $E_0 \in H^{s+1}$, there exists a unique solution to system 27 endowed with the following initial data:*

$$\begin{cases} n(0,x) = n_0(x), \\ \partial_t n(0,x) = n_1(x), \\ E(0,x) = E_0(x), \end{cases} \tag{39}$$

on a time interval $[0,T]$, T not depending on c but only on $\|n_0\|_{H^s}$, $\|n_1\|_{H^{s-1}}$ and $\|E_0\|_{H^{s+1}}$.
Moreover, for all $t \in [0,T]$ we have the estimate

$$\|E^c\|_{H^{s+1}} + \|E^c_t\|_{H^{s-1}} + \|n^c\|_{H^s} + \frac{1}{c}\|n^c_t\|_{H^{s-1}} + \frac{1}{c^2}\|n^c_{tt}\|_{H^{s-2}} \leq \text{Cst} \tag{40}$$

Thanks to the two conservation laws Eq. 25 and 26 we can, as for the classical Zakharov equation ($B = -I$), show that the solutions are global in time in the 1-dimensional case and also in the 2-dimensional case when the initial data are sufficiently small.

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Chapitre 2

On a nonlocal Zakharov equation

Abstract

We study the Cauchy problem for a nonlocal Zakharov equation, namely

$$\begin{cases} i\varphi_t + \Delta\varphi = \nabla(-\Delta)^{-1}\nabla.(n\varphi), \\ \lambda^{-2}n_{tt} - \Delta n = \Delta|\varphi|^2. \end{cases}$$

We first study the Cauchy problem for a fixed λ and then, in a smaller functional space, the limit of the solutions when λ tends to ∞.

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Theory, Methods and Applications.

2.1 Introduction.

Our aim is to prove some results about a nonlocal Zakharov equation introduced by Zakharov (see [14],[15]). The derivation of this system is carried out for $x \in \mathbf{R}^3$, however we will suppose that $x \in \mathbf{R}^N$, $N = 1, 2, 3$.
We study the following system

$$\left\{ \begin{array}{l} i\dot{\phi} + \Delta\phi = -B(n\phi), \\ \lambda^{-2}\ddot{n} - \Delta n = \Delta|\phi|^2, \end{array} \right.$$

where $B = \nabla\Delta^{-1}\nabla.$.
We consider the initial value problem, that is

$$\left\{ \begin{array}{l} \phi(x,0) = \phi_0(x), \\ n(x,0) = n_0(x), \\ n_t(x,0) = n_1(x). \end{array} \right.$$

This article is divided into three sections. In section 2, we derive the equation from the physical equations according to Zakharov (see [14]). In sedction 3, we study the local Cauchy problem for $(\phi_0, n_0, n_1) \in H^2 \times H^1 \times L^2$. In section 4, we study the limit of the solution to the equation when λ tends to zero.

The nonlocal Zakharov equation may be connected with two other equations : the classical Zakharov equation and a nonlinear nonlocal Schrödinger equation.

The classical Zakharov equation is the same one with $B = -I$. There is a link between the two problems. For example, they are the same in a one-dimensional space and also if we only consider radial solutions (because $-B$ is the L^2-projection on the gradients (see [4]). The point is to try to adapt the results about the classical equation. We may cite different people who worked on this equation : H. Added - S. Added ([1], [2]), C. Sulem - P.L. Sulem ([12]), S.H. Schochet - M.I. Weinstein ([10]), T. Ozawa - Y. Tsutsumi ([9]).

For the existence and the uniqueness for the Cauchy problem, we reason as in [9]. It is possible to adapt the method because of the continuity of the nonlocal operator B in a large number of spaces. The same results on the Cauchy problem would be obtained if we have an other operator instead of B, provided we always have these continuity results.

To pass to the limit when λ tends to ∞, we adapt the proof in [10]. Lots of complications are due to the nonlocal term. First of all we have to write the initial system as the dispersive perturbation of a symmetric hyperbolic system, the nonlocal term yields some additional terms which make the derivation far more complicated. Then we encounter some difficulties in estimating the different nonlocal operators of this new system and need the use of the theory of commutators to solve these new problems. As for the previous problem, we may want to extend our results to some other operator B. We need the same properties as before, and some others. For example, we need to be able to define an operator A such that $\nabla B\Phi = A\nabla\Phi$.

A nonlinear, nonlocal Schrödinger equation is studied by T. Colin ([4]-[5]). It consists in taking $\lambda = \infty$. He obtains some results about the local and global Cauchy problem, some finite time blow-up results and also standing waves and their stability. The limit we obtain when λ tends to ∞ turns out to be the solutions to this equation.

2.2 Origin of the nonlocal Zakharov system.

This article deals with some equations introduced by V.E. Zakharov (cf. e.g. [14], [15]), to describe Langmuir plasma turbulence. The physical description follows exactly these two articles.

We consider the hydrodynamical equations, the system of the Vlasov equations for the particle distribution functions and the Maxwell equations for the fields. This system is quite complicated, so we first simplify it. The idea is based on the fact that we can distinguish slow and fast processes in a plasma. We assume

- the plasma is sufficiently uniform,
- the magnetic field is sufficiently weak,
- the nonlinearity level is not too high,
- there are no tranverse high-frequency electromagnetic waves,

then the fastest process is the Langmuir oscillation, whose period is $\tau_L \sim 1/\omega_{pl}$, ω_{pl} being the Langmuir frequency.
The other time scale (when there is no magnetic field) is the period of ion-acoustical oscillations, their minimal value being $\sqrt{m_i/m_e}$ times higher than τ_L, where m_i and m_e denote the ion and electron masses.
We average the dynamical equations on a period τ_L. We only consider long wave oscillations with phase velocities far larger than thermal ones. We neglect quasi-linear effects.
We also neglect interactions between the different high-frequency oscillations, and we obtain the linearized hydrodynamical equations :

$$\frac{\partial}{\partial t}\delta n_e + \text{div } (n_0 + \delta n)\vec{V}_e = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}\delta \vec{V}_e + \frac{3V_{T_e}^2}{n}\vec{\nabla}\delta n = \frac{e}{m_e}\vec{E}. \tag{2.2}$$

Maxwell's equation reads :

$$\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \text{curl curl } \vec{E} + \frac{4\pi e}{c^2}(n_0 + \delta n)\frac{\partial}{\partial t}\delta\vec{V}_e = 0. \tag{2.3}$$

Moreover, we suppose that :

$$n = n_0 + \delta n + \delta n_e,$$

$$\delta n, \delta n_e \ll n_0.$$

n is the electron density, δn is the given lowfrequency plasma nonuniformity and δn_e is the density variation due to the Langmuir oscillations. $\vec{E}$ is the electric field and δV_e is the variation of the electron velocity. c is the ion sound velocity. Thus, we have

$$\frac{1}{c^2}\left(\frac{\partial^2\vec{E}}{\partial t^2}+\omega_{pl}^2\vec{E}\right)+\text{curl curl }\vec{E}-\frac{3V_{T_e}^2}{c^2}\vec{\nabla}\text{div }\vec{E}+\omega_{pl}^2\frac{\delta n}{c^2 n_0}\vec{E}=0. \tag{2.4}$$

Next, we suppose that $\vec{E}=\frac{1}{2}\left(\vec{\tilde{E}}\exp(-i\omega_{pl}t)+\vec{\tilde{E}}^*\exp(i\omega_{pl}t)\right)$, where $\vec{\tilde{E}}$ slowly varies ($\partial\tilde{E}/\partial t \ll \omega_{pl}\tilde{E}$). Neglecting $\partial^2\tilde{E}/\partial t^2$, we have :

$$\frac{\partial^2\vec{E}}{\partial t^2}=-\omega_{pl}^2\vec{E}+i\omega_{pl}\left(\frac{\partial\vec{\tilde{E}}}{\partial t}\exp(-i\omega_{pl}t)+\frac{\partial\vec{\tilde{E}}^*}{\partial t}\exp(i\omega_{pl}t)\right).$$

Substituting this in (2.4), we obtain :

$$-2i\frac{\omega_{pl}}{c^2}\frac{\partial\vec{\tilde{E}}}{\partial t}+\text{curl curl }\vec{\tilde{E}}-\frac{3V_{T_e}^2}{c^2}\vec{\nabla}\text{div }\vec{\tilde{E}}=-\omega_{pl}^2\frac{\delta n}{c^2 n_0}\vec{\tilde{E}}. \tag{2.5}$$

We need a second equation in order to relate δn and $|\tilde{E}|^2$.
We suppose that the electrons are distributed according to the Boltzmann law :

$$n=n_0\exp\{-\frac{(e\varphi_{el}-\phi)}{T_e}\},\qquad \frac{\delta n}{n_0}=\frac{e\varphi_{el}-\phi}{T_e}\ll 1. \tag{2.6}$$

The ion distribution function obeys Vlasov's equation :

$$\frac{\partial f_i}{\partial t}+(V.\vec{\nabla})f_i-\frac{e}{m_i}\vec{\nabla}\varphi_{el}\frac{\partial f_i}{\partial V}=0,\ \text{and } \delta n_i=\frac{en_0}{T_e}(\varphi_{el}-\phi).$$

We set $\vec{\tilde{E}}=\vec{\nabla}\psi$, so div $\vec{\tilde{E}}=\Delta\psi$.

$$-2i\frac{\omega_{pl}}{c^2}\frac{\partial}{\partial t}(\text{div }\vec{\tilde{E}})-\frac{3V_{T_e}^2}{c^2}\text{div }\vec{\nabla}\text{div }\vec{\tilde{E}}=-\frac{\omega_{pl}^2}{c^2}\text{div }(\frac{\delta n}{n_0}\vec{\tilde{E}}),$$

$$-2i\frac{\omega_{pl}}{c^2}\frac{\partial}{\partial t}\Delta\psi-\frac{3V_{T_e}^2}{c^2}\Delta^2\psi=-\frac{\omega_{pl}^2}{c^2}\text{div }(\frac{\delta n}{n_0}\nabla\psi),$$

thus

$$\Delta(2i\omega_{pl}\psi_t+3V_{T_e}^2\Delta\psi)=\omega_{pl}^2\text{div }(\frac{\delta n}{n_0}\nabla\psi). \tag{2.7}$$

We multiply (2.7) by ψ^* and integrate by parts. The imaginary part yields that

$$N_0=\int|\nabla\psi|^2dx$$

is conserved.

first hypothesis : The nonlinear phenomena have such a long period ($\tau^{-1} \ll kV_{T_i}$) that the ions have enough time to reach the Boltzmann distribution law in a low-frequency electric field :

$$\frac{\delta n}{n_0} = -e\frac{\varphi_{el}}{T_i}.$$

Now, we also know that $\dfrac{\delta n}{n_0} = \dfrac{e\varphi_{el} - \phi}{T_e}$, therefore :

$$\frac{\delta n}{n_0} = \frac{-\phi}{T_i + T_e}. \tag{2.8}$$

ϕ equals $\dfrac{e^2}{4m_e\omega_{pl}^2}|\vec{\vec{E}}|^2$, hence

$$\frac{\delta n}{n_0} = \frac{-e^2}{4m_e\omega_{pl}^2}\frac{|\nabla\psi|^2}{(T_i + T_e)} = \frac{-\varepsilon_0}{4n_0}\frac{|\nabla\psi|^2}{(T_i + T_e)}.$$

This result together with (2.7) yields :

$$\Delta(2i\omega_{pl}\psi_t + 3V_{T_e}^2\Delta\psi) = \frac{-e^2}{4m_e(T_i + T_e)}\text{div }(|\nabla\psi|^2\nabla\psi),$$

$$\Delta(i\psi_t + \frac{3}{2}\frac{V_{T_e}^2}{\omega_{pl}}\Delta\psi) = \frac{-e^2}{4m_e\omega_{pl}(T_i + T_e)}\text{div }(|\nabla\psi|^2\nabla\psi),$$

$$\Delta(i\psi_t + \frac{3}{2}\lambda_D\omega_{pl}\Delta\psi) = \frac{-\varepsilon_0\omega_{pl}}{8n_0(T_i + T_e)}\text{div }(|\nabla\psi|^2\nabla\psi). \tag{2.9}$$

second hypothesis : $\tau^{-1} \gg kV_{T_i}$.
For low-frequency motions the hydrodynamical approximation is valid :

$$\left(\frac{\partial^2}{\partial t^2} + 2\gamma_s\frac{\partial}{\partial t} - c_s^2\Delta\right)\delta n = \frac{1}{16\pi m_i}\Delta|E|^2. \tag{2.10}$$

We assume that the damping rate γ_s is zero.

After some changes of scale, we obtain the equations, we study.
For the first hypothesis, we obtain :

$$\Delta(i\psi_t + \Delta\psi) = \text{div }(|\nabla\psi|^2\nabla\psi).$$

We can find some results conserning this equation in [4] and [5].
For the second hypothesis, we have the following system :

$$\begin{cases} \Delta(i\psi_t + \Delta\psi) = \text{div }(n\nabla\psi), \\ \dfrac{1}{c^2}n_{tt} - \Delta n = \Delta(|\nabla\psi|^2). \end{cases}$$

2.3 Existence and Uniqueness for the Cauchy Problem.

In the study of the Cauchy problem, we will omit the constant c^2. It is obvious to see that the result will be valid for every value of c. Thus we study :

$$\left\{\begin{array}{l} \Delta(i\dot{\psi}+\Delta\psi)=\nabla.(n\nabla\psi), \\ \ddot{n}-\Delta n=\Delta|\nabla\psi|^2, \end{array}\right. \tag{3.11}$$

that is

$$\left\{\begin{array}{l} i\dot{\psi}+\Delta\psi=\Delta^{-1}\nabla.(n\nabla\psi), \\ \ddot{n}-\Delta n=\Delta|\nabla\psi|^2, \end{array}\right.$$

$$\left\{\begin{array}{l} i\nabla\dot{\psi}+\Delta\nabla\psi=\nabla\Delta^{-1}\nabla.(n\nabla\psi), \\ \ddot{n}-\Delta n=\Delta|\nabla\psi|^2. \end{array}\right.$$

We set $\phi=\nabla\psi$. This leads to the system that we are going to study :

$$\left\{\begin{array}{l} i\dot{\phi}+\Delta\phi=\nabla\Delta^{-1}\nabla.(n\phi), \\ \ddot{n}-\Delta n=\Delta|\phi|^2. \end{array}\right. \tag{3.12}$$

In the study, we omit the fact that ϕ is a gradient. We nevertheless have $i\dot{\phi}+\Delta\phi=\nabla f$ with $f=\Delta^{-1}\nabla.(n\phi)$. Taking the Fourier transform, we get

$$i\widehat{\phi_t}-|\xi|^2\hat{\phi}=i\xi\hat{f},,$$

$$\widehat{\phi_t}+i|\xi|^2\hat{\phi}=\xi\hat{f},$$

$$\frac{d}{dt}\left(\hat{\phi}e^{i|\xi|^2t}\right)=\xi\hat{f}e^{i|\xi|^2t},$$

$$\hat{\phi}e^{i|\xi|^2t}-\hat{\phi}(0)=\xi\int_0^t\hat{f}(\xi,s)e^{i|\xi|^2s}ds.$$

As we have $\hat{\phi}(0)=\xi\hat{\psi}(0)$,

$$\hat{\phi}(\xi,t)=\xi\left(\hat{\phi}(\xi,0)e^{-i|\xi|^2t}+\int_0^t\hat{f}(\xi,s)e^{-i|\xi|^2(t-s)}ds\right),$$

and ϕ is a gradient.

These computations are quite justified. We set $B=\nabla(-\Delta)^{-1}\nabla$.

This operator is homogeneous of order 0 in the Fourier variable. Calderón-Zygmund's theorem tells us that B is continuous in $L^p(\mathbf{R}^m)$, for all $1<p<\infty$, i.e.

$$\exists C_p\ /\forall f\in L^p\quad \|Bf\|_{L^p}\leq C_p\|f\|_{L^p}. \tag{3.13}$$

A simple argument tells us that B is also continuous in every $W^{s,p}$, for all $1 < p < \infty$:

$$\exists C_{s,p} \,/\forall f \in W^{s,p} \quad \|Bf\|_{W^{s,p}} \le C_{s,p}\|f\|_{W^{s,p}}. \tag{3.14}$$

We now follow the work of Ozawa and Tsutsumi [9].
The system (3.12) reads

$$\begin{cases} i\dot{\phi} + \Delta\phi = -B(n\phi), \\ \ddot{n} - \Delta n = \Delta|\phi|^2. \end{cases} \tag{3.15}$$

Setting $F = \partial_t\phi$, we formally get

$$\begin{cases} i\dot{F} + \Delta F = -B(\partial_t n(\phi_0 + \int_0^t F ds) + nF), \\ \ddot{n} - \Delta n = \Delta|\phi|^2, \\ \phi = (-\Delta+1)^{-1}\left\{iF + B(n(\phi_0 + \int_0^t F ds)) + (\phi_0 + \int_0^t F ds)\right\}. \end{cases} \tag{3.16}$$

We consider the following initial conditions

$$\begin{cases} F(0) = i(\Delta\phi_0 + B(n_0\phi_0)) = F_0, \\ n(0) = n_0, \\ \partial_t n(0) = n_1. \end{cases} \tag{3.17}$$

We work in $\mathbf{R}^N$ with $1 \le N \le 3$.
We set

$$X = [L^\infty(I;L^2) \cap L^{8/N}(I;L^4)] \oplus [L^\infty(I;H^1) \cap W^{1,\infty}(I;L^2)],$$

where $I = [0,T]$. We suppose that $(\phi_0, n_0, n_1) \in H^2 \times H^1 \times L^2$, and we set

$$a = \max\{\|\phi_0\|_{L^2}, \|\phi_0\|_{L^4}, \|\Delta\phi_0 + B(n_0\phi_0)\|_{L^2}, \|n_0\|_{H^1} + \|n_1\|_{L^2}\}$$

(since we have $(\phi_0, n_0, n_1) \in H^2 \times H^1 \times L^2$, a is finite).

To begin with, we want to obtain the existence and the uniqueness of a solution of (3.16-3.17) in X using a fixed point method.
We set $N = (N_1, N_2)$ with

$$\begin{cases} N_1[F,n](t) &= U(t)F_0 \\ &+ \; i\int_0^t U(t-s)\{B(\partial_s n(\phi_0 + \int_0^s F d\tau) + nF)\}ds, \\ N_2[F,n](t) &= \cos(\omega t)n_0 + \omega^{-1}\sin(\omega t)n_1 \\ &+ \; \int_0^t \omega^{-1}\sin(\omega(t-s))\Delta|\phi(s)|^2 ds, \\ (-\Delta+1)\phi &= \{iF + B(n(\phi_0 + \int_0^t F ds)) + (\phi_0 + \int_0^t F ds)\}, \end{cases} \tag{3.18}$$

where $U(t)$ is the semi-group generated by the Schrödinger operator and ω denotes $\sqrt{-\Delta}$ ($|\xi|$ in Fourier variable).

We set

$$Y = \{(F(t), n(t)) \in X / \quad \|F\|_{L^\infty(I;L^2)} \leq 2a, \|F\|_{L^{8/N}(I;L^4)} \leq 2\delta a,$$
$$\|n\|_{L^\infty(I;H^1)} \leq 2a, \left\|\frac{dn}{dt}\right\|_{L^\infty(I;L^2)} \leq 2a\}.$$

We first show that N is a contraction from Y into Y.

To this end we use the following lemmas :

1. $L^p - L^q$ estimate (see [6] or [7]).
   If $2 \leq p \leq \infty$ and $1/p + 1/q = 1$,
   then $\|U(t)v\|_{L^p} \leq (4\pi|t|)^{-N/2+N/p}\|v\|_{L^q}, \quad t \neq 0$

2. Strichartz estimate (see [9]).
   We set $\alpha(N) = \infty$ if $N = 1, 2$; $\alpha(N) = \dfrac{2N}{N-2}$ if $N \geq 3$.
   If $2 \leq q < \alpha(N)$ and $(N/2 - N/q)r = 2$,
   then there exists $K_1(N, q)$ such that
   $\|U(\cdot)v\|_{L^r(\mathbf{R};L^q)} \leq K_1\|v\|_{L^2}$ for all $v \in L^2$.

3. (see [13]).
   If $1 \leq q', r' \leq 2, 2 \leq q < \alpha(N)$ and $(N/2 - N/q)r = 2$,
   then there exists $K_2(N, q')$ such that

$$\left\|\int_0^t U(t-s)f(s)ds\right\|_{L^\infty(I;L^2)} \leq K_2\|f\|_{L^{r'}(I;L^{q'})} \qquad \forall f \in L^{r'}(I; L^{q'}).$$

4. For all $t \in \mathbf{R}^+$, for all $v \in L^2$, we have
   $\|\cos(\omega t)v\|_{L^2} \leq \|v\|_{L^2}$,
   $\|(1+\omega^2)^{1/2}\omega^{-1}\sin(\omega t)v\|_{L^2} \leq (1+t)\|v\|_{L^2}$.

first step : N maps Y into Y.
The second lemma yields that

$$\|U(\cdot)i(\Delta\phi_0 + B(n_0\phi_0))\|_{L^{8/N}(\mathbf{R};L^4)} \leq K_1\|\Delta\phi_0 + B(n_0\phi_0)\|_{L^2} \leq \delta a. \tag{3.19}$$

This fixes δ. We also know that

$$\|U(\cdot)i(\Delta\phi_0 + B(n_0\phi_0))\|_{L^\infty(\mathbf{R};L^2)} = \|\Delta\phi_0 + B(n_0\phi_0)\|_{L^2} \leq a, \tag{3.20}$$

because $U(t)$ is unitary on L^2. Using the first lemma and the continuity property (3.13) of B, we have

$$\begin{aligned}
\|N_1[F,n](t)\|_{L^4} \leq\ & \|U(t)i(\Delta\phi_0 + B(n_0\phi_0))\|_{L^4} \\
+\ & \int_0^t \left\| U(t-s)\left\{B(\partial_s n(\phi_0 + \int_0^s F d\tau) + nF)\right\}\right\|_{L^4} ds, \\
\leq\ & \|U(t)i(\Delta\phi_0 + B(n_0\phi_0))\|_{L^4} \\
+\ & \int_0^t C|t-s|^{-N/4} \left\| B(\partial_s n(\phi_0 + \int_0^s F d\tau) + nF)\right\|_{L^{4/3}} ds, \\
\leq\ & \|U(t)i(\Delta\phi_0 + B(n_0\phi_0))\|_{L^4} \\
+\ & \int_0^t C|t-s|^{-N/4}\Big\{\|\partial_s n\|_{L^2}(\|\phi_0\|_{L^4} + \int_0^s \|F\|_{L^4} d\tau) \\
& \qquad + \|n\|_{L^2}\|F\|_{L^4}\Big\} ds.
\end{aligned}$$

$$\begin{aligned}
\|N_1[F,n]\|_{L^{8/N}(I;L^4)} \leq\ & \delta a \\
+\ & C\Bigg\{ \left\|\frac{dn}{dt}\right\|_{L^\infty(I;L^2)} \|\phi_0\|_{L^4} \left\| \int_0^t |t-s|^{-N/4} ds \right\|_{L^{8/N}(I)} \\
+\ & \left\|\frac{dn}{dt}\right\|_{L^\infty(I;L^2)} \|\phi_0\|_{L^4} \\
& \left\| \int_0^t |t-s|^{-N/4} (\int_0^s \|F\|_{L^4} d\tau) ds \right\|_{L^{8/N}(I)} \\
+\ & \|n\|_{L^\infty(I;L^2)} \left\| \int_0^t |t-s|^{-N/4} \|F\|_{L^4} ds \right\|_{L^{8/N}(I)} \Bigg\}.
\end{aligned}$$

Now

$$\left\| \int_0^T T^{-N/4} ds \right\|_{L^{8/N}(I)} = \|T^{1-N/4}\|_{L^{8/N}(I)} = T^{1-N/8},$$

$$\begin{aligned}
\left\| \int_0^T T^{-N/4} (\int_0^T \|F\|_{L^4} d\tau) ds \right\|_{L^{8/N}(I)} \leq\ & \left\| T^{2-N/4} \|F\|_{L^4} ds \right\|_{L^{8/N}(I)}, \\
=\ & T^{2-N/4} \|F\|_{L^{8/N}(I;L^4)},
\end{aligned}$$

and

$$\left\| \int_0^T T^{-N/4} \|F\|_{L^4} ds \right\|_{L^{8/N}(I)} \leq T^{1-N/4} \|F\|_{L^{8/N}(I;L^4)}.$$

Hence, we have

$$\begin{aligned}
\|N_1[F,n]\|_{L^{8/N}(I;L^4)} &\leq \delta a + CT^{1-N/4}\|n\|_{L^\infty(I;L^2)}\|F\|_{L^{8/N}(I;L^4)} \\
&\quad + CT^{1-N/8}\left\|\frac{dn}{dt}\right\|_{L^\infty(I;L^2)}\|\phi_0\|_{L^4} \\
&\quad + CT^{2-N/8}\left\|\frac{dn}{dt}\right\|_{L^\infty(I;L^2)}\|F\|_{L^{8/N}(I;L^4)}, \\
&\leq \delta a + C(T^{1-N/4}\delta a^2 + T^{1-N/8}a^2 + T^{2-N/4}\delta a^2), \\
&\leq \delta a + C(T^{1-N/4}a + T^{1-N/8}\delta^{-1}a + T^{2-N/4}a)\delta a.
\end{aligned}$$

We choose T sufficiently small so that $C(T^{1-N/4}a + T^{1-N/8}\delta^{-1}a + T^{2-N/4}a) \leq 1$. Hence

$$\|N_1[F,n]\|_{L^{8/N}(I;L^4)} \leq 2\delta a. \tag{3.21}$$

Using the third lemma and (3.13), we obtain that

$$\begin{aligned}
\|N_1[F,n]\|_{L^\infty(I;L^2)} &\leq \|U(t)i(\Delta\phi_0 + B(n_0\phi_0))\|_{L^\infty(I;L^2)} \\
&\quad + C\|B(\partial_t n(\phi_0 + \int_0^t F ds) + nF)\|_{L^{8/(8-N)}(I;L^{4/3})}, \\
&\leq a + C\|\partial_t n(\phi_0 + \int_0^t F ds) + nF\|_{L^{8/(8-N)}(I;L^{4/3})}, \\
&\leq a + C\|\|\partial_t n\|_{L^2}\left(\|\phi_0\|_{L^4} + \int_0^t \|F\|_{L^4} ds\right) \\
&\quad + \|n\|_{L^2}\|F\|_{L^4}\|_{L^{8/(8-N)}(I)}, \\
&\leq a \\
&\quad + C\left\|\frac{dn}{dt}\right\|_{L^\infty(I;L^2)}\|\phi_0\|_{L^4}\|1\|_{L^{8/(8-N)}(I)} \\
&\quad + C\left\|\frac{dn}{dt}\right\|_{L^\infty(I;L^2)}\left\|\int_0^T \|F\|_{L^4} ds\right\|_{L^{8/(8-N)}(I)} \\
&\quad + C\|n\|_{L^\infty(I;L^2)}\|1\|_{L^{4/(4-N)}(I)}\|F\|_{L^{8/N}(I;L^4)}, \\
&\leq a \\
&\quad + CT^{1-N/8}\left\|\frac{dn}{dt}\right\|_{L^\infty(I;L^2)}\|\phi_0\|_{L^4} \\
&\quad + CT^{2-N/4}\left\|\frac{dn}{dt}\right\|_{L^\infty(I;L^2)}\|F\|_{L^{8/N}(I;L^4)} \\
&\quad + CT^{1-N/4}\|n\|_{L^\infty(I;L^2)}\|F\|_{L^{8/N}(I;L^4)}, \\
&\leq a + C(T^{1-N/8}a^2 + T^{2-N/4}\delta a^2 + T^{1-N/4}\delta a^2),
\end{aligned}$$

$$\leq \ a + C(T^{1-N/8}a + T^{2-N/4}\delta a + T^{1-N/4}\delta a)a.$$

We choose T sufficiently small so that $C(T^{1-N/8}a+T^{2-N/4}\delta a+T^{1-N/4}\delta a) \leq 1$. Hence

$$\|N_1[F,n]\|_{L^\infty(I;L^2)} \leq 2a. \tag{3.22}$$

In order to treat the second coordinate of N, we first estimate ϕ in $L^\infty(I;H^2)$.

$$\phi(t) = (-\Delta+1)^{-1}\left\{iF + B(n(\phi_0 + \int_0^t F ds)) + (\phi_0 + \int_0^t F ds)\right\},$$

$$(-\Delta+1)\phi(t) = iF + B(n(\phi_0 + \int_0^t F ds)) + \phi_0 + \int_0^t F ds,$$

$$\|\phi\|_{H^2} \leq \|F\|_{L^2} + C\|n(\phi_0 + \int_0^t F ds)\|_{L^2} + \|\phi_0\|_{L^2} + \int_0^t \|F\|_{L^2} ds,$$

$$\begin{aligned}\|\phi\|_{L^\infty(I;H^2)} \ \leq \ & \|F\|_{L^\infty(I;L^2)} + C\|n\|_{L^\infty(I;L^4)}\left(\|\phi_0\|_{L^4} + \int_0^T \|F\|_{L^4} ds\right) \\ & + \|\phi_0\|_{L^2} + \int_0^T \|F\|_{L^2} ds.\end{aligned}$$

Now

$$\int_0^T \|F\|_{L^4} dt \leq CT^{1-N/8}\|F\|_{L^{8/N}(I;L^4)},$$

then

$$\|\phi\|_{L^\infty(I;H^2)} \leq C(a + a^2 + T^{1-N/8}\delta a^2 + Ta). \tag{3.23}$$

$$\begin{aligned}N_2[F,n](t) \ = \ & \cos(\omega t)n_0 + \omega^{-1}\sin(\omega t)n_1 \\ + \ & \int_0^t \omega^{-1}\sin(\omega(t-s))\Delta|\phi(s)|^2 ds.\end{aligned}$$

Using the fourth lemma and (3.23), we have

$$\begin{aligned}\|N_2[F,n](t)\|_{H^1} \ = \ & \|(1-\Delta)^{1/2}N_2[F,n](t)\|_{L^2}, \\ \leq \ & \|(1+\omega^2)^{1/2}\cos(\omega t)n_0\|_{L^2} \\ & + \|(1+\omega^2)^{1/2}\omega^{-1}\sin(\omega t)n_1\|_{L^2} \\ & + \int_0^t \|(1+\omega^2)^{1/2}\omega^{-1}\sin(\omega(t-s))\Delta|\phi(s)|^2\|_{L^2} ds,\end{aligned}$$

$$\begin{aligned}
&\leq \|n_0\|_{H^1} + (1+t)\|n_1\|_{L^2} \\
&\quad + \int_0^t (1+(t-s))\|\Delta|\phi(s)|^2\|_{L^2} ds, \\
&\leq (1+T)a + \int_0^T (1+T)\|\Delta|\phi(s)|^2\|_{L^2} ds, \\
&\leq (1+T)a + T(1+T)\|\Delta|\phi|^2\|_{L^\infty(I;L^2)}, \\
&\leq (1+T)a + CT(1+T)\|\phi\|^2_{L^\infty(I;H^2)},
\end{aligned}$$

because $\|\Delta|\phi|^2\|_{L^\infty(I;L^2)} \leq C\|\phi\|^2_{L^\infty(I;H^2)}$.

$$\|N_2[F,n]\|_{L^\infty(I;H^1)} \leq a + [T + CT(1+T)(1+a+T^{1-N/8}\delta a + T)^2 a]a.$$

We choose T sufficiently small so that $T + CT(1+T)(1+a+T^{1-N/8}\delta a+T)^2 a \leq 1$. Hence

$$\|N_2[F,n]\|_{L^\infty(I;H^1)} \leq 2a. \tag{3.24}$$

$$\begin{aligned}
\frac{d}{dt}N_2[F,n](t) &= -\omega\sin(\omega t)n_0 + \cos(\omega t)n_1 \\
&+ \int_0^t \cos(\omega(t-s))\Delta|\phi(s)|^2 ds.
\end{aligned}$$

Using once more the fourth lemma and (3.23) :

$$\begin{aligned}
\|\frac{d}{dt}N_2[F,n](t)\|_{L^\infty(I;L^2)} &\leq \|n_0\|_{H^1} + \|n_1\|_{L^2} + \int_0^T \|\Delta|\phi(s)|^2\|_{L^2} ds, \\
&\leq \|n_0\|_{H^1} + \|n_1\|_{L^2} + CT\|\phi\|^2_{L^\infty(I;H^2)}, \\
&\leq a + CT(1+a+T^{1-N/8}\delta a + T)^2 a.a.
\end{aligned}$$

We choose T sufficiently small so that $CT(1+a+T^{1-N/8}\delta a+T)^2 a \leq 1$. Hence

$$\|\frac{d}{dt}N_2[F,n]\|_{L^\infty(I;L^2)} \leq 2a. \tag{3.25}$$

Collecting the four results (3.21 - 3.22 - 3.24 - 3.25), we conclude that N maps Y into itself (for T small).

<u>second step</u> : N is a contraction in Y, i.e.
$\forall (F,n),(F',n') \in Y$ and T sufficiently small

$$|||N[F,n] - N[F',n']||| \leq \frac{1}{2}|||(F,n)-(F',n')|||$$

where $|||\cdot|||$ denotes the natural the norm of X.
The computation which follow are essentially the same as in the first step.

$$\begin{aligned} N_1[F,n](t)-N_1[F',n'](t) &= i\int_0^t U(t-s)\Big\{B(\partial_s n(\phi_0+\int_0^s F d\tau)+nF) \\ &\quad -B(\partial_s n'(\phi_0+\int_0^s F' d\tau)+n'F')\Big\}ds, \end{aligned}$$

$$\begin{aligned} N_1[F,n]-N_1[F',n'] &= i\int_0^t U(t-s)\Big\{B((\partial_s n-\partial_s n')\phi_0) \\ &\quad +B\left((\partial_s n-\partial_s n')\int_0^s F d\tau+\partial_s n'(\int_0^s (F-F')d\tau)\right) \\ &\quad +B\left((n-n')F+n'(F-F')\right)\Big\}ds, \end{aligned}$$

$$\begin{aligned} \|N_1[F,n](t) &- N_1[F',n'](t)\|_{L^4} \\ &\le \int_0^t (4\pi|t-s|)^{-N/4}\Big\{\|\partial_s n-\partial_s n'\|_{L^2}\|\phi_0\|_{L^4} \\ &\quad +\|\partial_s n-\partial_s n'\|_{L^2}\int_0^T \|F\|_{L^4}d\tau+\|\partial_s n'\|_{L^2}\int_0^T \|F-F'\|_{L^4}d\tau \\ &\quad +\|n-n'\|_{L^2}\|F\|_{L^4}+\|n'\|_{L^2}\|F-F'\|_{L^4}\Big\}ds, \end{aligned}$$

$$\begin{aligned} \|N_1[F,n] &- N_1[F',n']\|_{L^{8/N}(I;L^4)} \\ &\le C\Big\{\left\|\int_0^t T^{-N/4}\|\partial_s n-\partial_s n'\|_{L^2}\|\phi_0\|_{L^4}ds\right\|_{L^{8/N}(I)} \\ &\quad +\left\|\int_0^t T^{-N/4}\|\partial_s n'\|_{L^2}\int_0^T \|F-F'\|_{L^4}d\tau\right\|_{L^{8/N}(I)} \\ &\quad +\left\|\int_0^t T^{-N/4}\|\partial_s n-\partial_s n'\|_{L^2}\int_0^T \|F\|_{L^4}d\tau\right\|_{L^{8/N}(I)} \\ &\quad +\left\|\int_0^t T^{-N/4}\|n-n'\|_{L^2}\int_0^T \|F\|_{L^4}d\tau\right\|_{L^{8/N}(I)} \\ &\quad +\left\|\int_0^t T^{-N/4}\|n'\|_{L^2}\int_0^T \|F-F'\|_{L^4}d\tau\right\|_{L^{8/N}(I)}\Big\}, \\ &\le C\Big\{T^{1-N/8}\|\frac{d}{dt}(n-n')\|_{L^\infty(I;L^2)}\|\phi_0\|_{L^4} \end{aligned}$$

$$
\begin{aligned}
&+T^{2-N/4}\|\frac{d}{dt}(n-n')\|_{L^\infty(I;L^2)}\|F\|_{L^{8/N}(I;L^4)}\\
&+T^{2-N/4}\|\frac{d}{dt}n'\|_{L^\infty(I;L^2)}\|F-F'\|_{L^{8/N}(I;L^4)}\\
&+T^{1-N/4}\|n-n'\|_{L^\infty(I;L^2)}\|F\|_{L^{8/N}(I;L^4)}\\
&+T^{1-N/4}\|n'\|_{L^\infty(I;L^2)}\|F-F'\|_{L^{8/N}(I;L^4)}\Big\},\\
\leq\ & C\{T^{1-N/8}a+T^{2-N/4}(\delta a+a)+T^{1-N/4}(\delta a+a)\}\\
&\qquad\qquad\times|||(F,n)-(F',n')|||.
\end{aligned}
$$

We choose T sufficiently small so that

$$
C\{T^{1-N/8}a+T^{2-N/4}(\delta a+a)+T^{1-N/4}(\delta a+a)\}\leq\frac{1}{8}.
$$

Hence

$$
\|N_1[F,n]-N_1[F',n']\|_{L^{8/N}(I;L^4)}\leq\frac{1}{8}|||(F,n)-(F',n')|||. \tag{3.26}
$$

$$
\begin{aligned}
\|N_1[F,n]\ -\ & N_1[F',n']\|_{L^\infty(I;L^2)}\\
\leq\ & C\|B((\partial_t n-\partial_t n')\phi_0)\\
&+B((\partial_t n-\partial_t n')\int_0^t F ds+(\partial_t n'(\int_0^t(F-F')ds))\\
&+B((n-n')F+n'(F-F'))\|_{L^{8/(8-N)}(I;L^{4/3})},\\
\leq\ & C\Big\{\|(\partial_t n-\partial_t n')\phi_0\|_{L^{8/(8-N)}(I;L^{4/3})}\\
&+\|(\partial_t n-\partial_t n')\int_0^t F ds\|_{L^{8/(8-N)}(I;L^{4/3})}\\
&+\|\partial_t n'\int_0^t(F-F')ds\|_{L^{8/(8-N)}(I;L^{4/3})}\\
&+\|(n-n')F\|_{L^{8/(8-N)}(I;L^{4/3})}\\
&+\|n'(F-F')\|_{L^{8/(8-N)}(I;L^{4/3})}\Big\},\\
\leq\ & C\Big\{\ \|\|\partial_t n-\partial_t n'\|_{L^2}\|\phi_0\|_{L^4}\|_{L^{8/(8-N)}(I)}\\
&+\left\|\|\partial_t n-\partial_t n'\|_{L^2}\int_0^T\|F\|_{L^4}ds\right\|_{L^{8/(8-N)}(I)}\\
&+\left\|\|\partial_t n'\|_{L^2}\int_0^T\|F-F'\|_{L^4}ds\right\|_{L^{8/(8-N)}(I)}\\
&+\|\|n-n'\|_{L^2}\|F\|_{L^4}\|_{L^{8/(8-N)}(I)}\\
&+\|\|n'\|_{L^2}\|F-F'\|_{L^4}\|_{L^{8/(8-N)}(I)}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\Big\{T^{1-N/8}\left\|\frac{d}{dt}(n-n')\right\|_{L^\infty(I;L^2)}\|\phi_0\|_{L^4} \\
&\quad +T^{2-N/4}\left\|\frac{d}{dt}(n-n')\right\|_{L^\infty(I;L^2)}\|F\|_{L^{8/N}(I;L^4)} \\
&\quad +T^{2-N/4}\left\|\frac{d}{dt}n'\right\|_{L^\infty(I;L^2)}\|F-F'\|_{L^{8/N}(I;L^4)} \\
&\quad +T^{1-N/4}\|n-n'\|_{L^\infty(I;L^2)}\|F\|_{L^{8/N}(I;L^4)} \\
&\quad +T^{1-N/4}\|n'\|_{L^\infty(I;L^2)}\|F-F'\|_{L^{8/N}(I;L^4)}\Big\}, \\
&\leq C\{T^{1-N/8}a+T^{2-N/4}(\delta a+a)+T^{1-N/4}(\delta a+a)\} \\
&\qquad\qquad \times|||(F,n)-(F',n')|||.
\end{aligned}
$$

We choose T sufficiently small so that $C(T^{1-N/8}a+T^{2-N/4}\delta a+T^{1-N/4}\delta a)\leq\frac{1}{8}$. Hence

$$
\|N_1[F,n]-N_1[F',n']\|_{L^\infty(I;L^2)}\leq\frac{1}{8}|||(F,n)-(F',n')|||. \tag{3.27}
$$

$$
\begin{aligned}
\phi(t)-\phi'(t) &= (-\Delta+1)^{-1}\Big\{i(F-F')+B((n-n')\phi_0) \\
&\quad +B((n-n')\int_0^t F ds)+B(n'\int_0^t(F-F')ds) \\
&\quad +\int_0^t(F-F')ds\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\|\phi(t)-\phi'(t)\|_{H^2} &\leq C\Big\{\|F-F'\|_{L^2}+\|(n-n')\phi_0\|_{L^2} \\
&\quad +\left\|(n-n')\int_0^T F ds\right\|_{L^2}+\left\|n'\int_0^T(F-F')ds\right\|_{L^2} \\
&\quad +\left\|\int_0^T(F-F')ds\right\|_{L^2}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\|\phi-\phi'\|_{L^\infty(I;H^2)} &\leq C\Big\{\|F-F'\|_{L^\infty(I;L^2)}+\|n-n'\|_{L^\infty(I;L^4)}\|\phi_0\|_{L^4} \\
&\quad +\|n-n'\|_{L^\infty(I;L^4)}\int_0^T\|F\|_{L^4}ds \\
&\quad +\|n'\|_{L^\infty(I;L^4)}\int_0^T\|F-F'\|_{L^4}ds \\
&\quad +\int_0^T\|F-F'\|_{L^2}ds\Big\}.
\end{aligned}
$$

So

$$\|\phi-\phi'\|_{L^\infty(I;H^2)} \le C\{1+a+T^{1-N/8}(\delta a+a)+T\}|||(F,n)-(F',n')|||. \qquad (3.28)$$

$$\begin{aligned} N_2[F,n](t) &- N_2[F',n'](t) \\ &= \int_0^t \omega^{-1}\sin(\omega(t-s))(\Delta|\phi(s)|^2-\Delta|\phi'(s)|^2)ds. \end{aligned}$$

$$\begin{aligned} \|N_2[F,n](t) &- N_2[F',n'](t)\|_{H^1} \\ &\le \int_0^t \|(1+\omega^2)^{1/2}\omega^{-1}\sin(\omega(t-s))(\Delta|\phi|^2-\Delta|\phi'|^2)\|_{L^2}ds, \\ &\le \int_0^t (1+(t-s))\|\Delta(|\phi(s)|^2-|\phi'(s)|^2)\|_{L^2}ds. \end{aligned}$$

We have $\|\Delta(|\phi|^2-|\phi'|^2)\|_{L^2} \le C(\|\phi\|_{H^2}+\|\phi'\|_{H^2})\|\phi-\phi'\|_{H^2}$ therefore

$$\begin{aligned} \|N_2[F,n](t) &- N_2[F',n'](t)\|_{H^1} \\ &\le CT(1+T)(\|\phi(t)\|_{H^2}+\|\phi'(t)\|_{H^2})\|(\phi-\phi')(t)\|_{H^2}, \end{aligned}$$

$$\begin{aligned} \|N_2[F,n] &- N_2[F',n']\|_{L^\infty(I;H^1)} \\ &\le CT(1+T)(\|\phi\|_{L^\infty(I;H^2)}+\|\phi'\|_{L^\infty(I;H^2)})\|\phi-\phi'\|_{L^\infty(I;H^2)}, \\ &\le CT(1+T)(a+a^2+T^{1-N/8}\delta a^2+Ta) \\ &\quad \times(1+a+T^{1-N/8}(\delta a+a)+T)|||(F,n)-(F',n')|||. \end{aligned}$$

We choose T sufficiently small so that

$$CT(1+T)(a+a^2+T^{1-N/8}\delta a^2+Ta)(1+a+T^{1-N/8}(\delta a+a)+T) \le \frac{1}{8}.$$

Hence

$$\|N_2[F,n]-N_2[F',n']\|_{L^\infty(I;H^1)} \le \frac{1}{8}|||(F,n)-(F',n')|||. \qquad (3.29)$$

$$\begin{aligned} \frac{d}{dt}N_2[F,n](t) &- \frac{d}{dt}N_2[F',n'](t) \\ &= \int_0^t \cos(\omega(t-s))\Delta(|\phi(s)|^2-(|\phi'(s)|^2)ds. \end{aligned}$$

$$\begin{aligned}
\|\frac{d}{dt}N_2[F,n](t) &- \frac{d}{dt}N_2[F',n'](t)\|_{L^2} \\
&\leq \int_0^t \|\Delta(|\phi(s)|^2-(|\phi'(s)|^2)\|_{L^2}ds, \\
\|\frac{d}{dt}(N_2[F,n] &- N_2[F',n'])\|_{L^\infty(I;L^2)} \\
&\leq CT(\|\phi\|_{L^\infty(I;H^2)}+\|\phi'\|_{L^\infty(I;H^2)}) \\
&\quad \|\phi-\phi'\|_{L^\infty(I;H^2)}, \\
&\leq CT(a+a^2+T^{1-N/8}\delta a^2+Ta) \\
&\quad \times(1+a+T^{1-N/8}(\delta a+a)+T)|||(F,n)-(F',n')|||.
\end{aligned}$$

We choose T sufficiently small so that

$$CT(a+a^2+T^{1-N/8}\delta a^2+Ta)(1+a+T^{1-N/8}(\delta a+a)+T)\leq\frac{1}{8}.$$

Hence

$$\left\|\frac{d}{dt}(N_2[F,n]-N_2[F',n'])\right\|_{L^\infty(I;L^2)}\leq\frac{1}{8}|||(F,n)-(F',n')|||. \tag{3.30}$$

Summing up all the above estimates (3.26 - 3.27 - 3.29 - 3.30), we obtain

$$|||N(F,n)-N(F',n')|||\leq\frac{1}{2}|||(F,n)-(F',n')|||. \tag{3.31}$$

N is also a contraction from Y into Y for T sufficiently small.

N has also a unique fixed point in Y such that

$$\left\{\begin{aligned}
F(t) &= U(t)F_0 \\
&+ i\int_0^t U(t-s)\left\{B(\partial_s n(\phi_0+\int_0^s Fd\tau)+nF)\right\}ds, \\
n(t) &= \cos(\omega t)n_0+\omega^{-1}\sin(\omega t)n_1 \\
&+ \int_0^t \omega^{-1}\sin(\omega(t-s))\Delta|\phi(s)|^2ds, \\
\phi(t) &= (-\Delta+1)^{-1}\left\{iF+B(n(\phi_0+\int_0^t Fds))+(\phi_0+\int_0^t Fds)\right\}.
\end{aligned}\right. \tag{3.32}$$

We obtain immediately that

$$\left\{\begin{array}{l}
F(0)=i(\Delta\phi_0+B(n_0\phi_0))=F_0, \\
n(0)=n_0, \qquad \partial_t(0)=n_1, \\
\phi(0)=\phi_0.
\end{array}\right.$$

(F, n, ϕ) satisfies (3.16-3.17) in the integral form.
$F_0 \in L^2$ so $U(t)F_0 \in \mathcal{C}(I; L^2) \cap \mathcal{C}^1(I; H^{-2})$, and $\partial_t U(t)F_0 = iU(t)\Delta F_0 = i\Delta(U(t)F_0)$.
Since $L^{4/3} \hookrightarrow H^{-1}$, $\quad B(\partial_s n(\phi_0 + \int_0^s F d\tau) + nF) \in L^1(I; H^{-1})$,
hence $\int_0^t U(t-s)\left\{B(\partial_s n(\phi_0 + \int_0^s F d\tau) + nF)\right\} ds \in \mathcal{C}(I; H^{-1}) \cap W^{1,1}(I; H^{-3})$,
and

$$
\begin{aligned}
\partial_t \int_0^t & U(t-s)\{B(\partial_s n(\phi_0 + \int_0^s F d\tau) + nF)\} ds \\
&= i\Delta \int_0^t U(t-s)\{B(\partial_s n(\phi_0 + \int_0^s F d\tau) + nF)\} ds \\
&\quad + B(\partial_t n(\phi_0 + \int_0^t F d\tau) + nF), \\
&= i \int_0^t U(t-s)\Delta\{B(\partial_s n(\phi_0 + \int_0^s F d\tau) + nF)\} ds \\
&\quad + B(\partial_t n(\phi_0 + \int_0^t F d\tau) + nF),
\end{aligned}
$$

so that

$$
\begin{aligned}
\partial_t F(t) &= i\Delta \left(U(t)F_0 + i \int_0^t U(t-s)\{B(\partial_s n(\phi_0 + \int_0^s F d\tau) + nF)\} ds\right) \\
&\quad + iB(\partial_t n(\phi_0 + \int_0^t F d\tau) + nF).
\end{aligned}
$$

that is $\partial_t F(t) = i\Delta F(t) + iB(\partial_t n(\phi_0 + \int_0^t F d\tau) + nF)$,
i.e. $i\dot{F} + \Delta F = -B(\partial_t n(\phi_0 + \int_0^t F d\tau) + nF)$.

Furthermore $n_0 \in H^1$, $n_1 \in L^2$, and $\Delta|\phi|^2 \in L^2(0, T; L^2)$.

$$n(t) \in \mathcal{C}([0,T]; H^1), \partial_t n(t) \in \mathcal{C}([0,T]; L^2) \text{ and } \partial_{t^2} n(t) \in \mathcal{C}([0,T]; H^{-1})$$

and in H^{-1} we have

$$\partial_{t^2} n - \Delta n = \Delta|\phi|^2.$$

(F, n, ϕ) also verifies the system (3.16-3.17) with the equalities in H^{-1}.

We shall now show that

$$
\left\{
\begin{array}{lll}
F(t) & \in & \left[\bigcap_{j=0}^{1} \mathcal{C}^j([0,T]; H^{-2j})\right] \bigcap L^{8/N}(0, T; L^4), \\
n(t) & \in & \bigcap_{j=0}^{2} \mathcal{C}^j([0,T]; H^{1-j}), \\
\phi(t) & \in & \mathcal{C}([0,T]; H^2).
\end{array}
\right. \tag{3.33}
$$

We know, in fact, that if $1 \le q', r' \le 2, 2 \le q < \alpha(N)$ and $(N/2 - N/q)r = 2$, then

$$\int_0^t U(t-s)f(s)ds \in C^1(0,T;L^2) \qquad \text{if } f \in L^{r'}(I;L^{q'}).$$

here we have $F_0 \in L^2$ and $B(\partial_t n(\phi_0 + \int_0^t F d\tau) + nF) \in L^{8/(8-N)}(I;L^{4/3})$,
so that $U(t)F_0 \in C(0,T;L^2) \cap C^1(0,T;H^{-2})$ (cf Kato [8]).

$$\int_0^t U(t-s)\left\{B(\partial_s n(\phi_0 + \int_0^s F d\tau) + nF)\right\} ds \in C(0,T;L^2),$$

so that $F(t) \in C(0,T;L^2)$.
We have already seen that $n(t) \in \bigcap_{j=0}^{2} C^j(0,T;H^{1-j})$.
On the other hand $B(\partial_t n(\phi_0 + \int_0^t F d\tau) + nF) \in C(0,T;L^{4/3}) \subset C(0,T;H^{-1})$,
and $\Delta F(t) \in C(0,T;H^{-2})$, so that $\partial_t F(t) \in C(0,T;H^{-2})$,
and $(-\Delta + 1)\phi(t) = iF + B(n(\phi_0 + \int_0^t F ds)) + (\phi_0 + \int_0^t F ds) \in C(0,T;L^2)$,
so that $\phi(t) \in C([0,T];H^2)$.
Then we have the system (3.33).

This enables us to differentiate once more in time the equation :

$$(-\Delta + 1)\phi(t) = iF + B(n(\phi_0 + \int_0^t F ds)) + (\phi_0 + \int_0^t F ds).$$

We obtain

$$(-\Delta + 1)\frac{\partial \phi}{\partial t}(t) = i\frac{\partial F}{\partial t}(t) + B\left(\frac{\partial n}{\partial t}(\phi_0 + \int_0^t F ds) + nF(t)\right) + F(t)$$

in H^{-2}.
On the other hand $\partial_t F(t) = i\Delta F(t) + iB(\partial_t n(\phi_0 + \int_0^t F ds) + nF)$,
thus $(-\Delta + 1)F = i\dot{F} + B(\partial_t n(\phi_0 + \int_0^t F ds) + nF) + F$.
This yields $\frac{\partial}{\partial t}\phi(t) = F(t)$ in H^{-2}.
Furthermore

$$\frac{\partial}{\partial t}\phi(t) = (-\Delta + 1)^{-1}\left(i\frac{\partial F}{\partial t} + B(\frac{\partial n}{\partial t}(\phi_0 + \int_0^t F ds) + nF(t)) + F(t)\right).$$

The right hand term belongs to $C(0,T;L^2)$ so that $\phi(t) \in C^1(0,T;L^2)$.

$$(-\Delta + 1)\phi(t) = iF + B(n(\phi_0 + \int_0^t F ds)) + (\phi_0 + \int_0^t F ds).$$

As we know that $\int_0^t F ds = \phi(t) - \phi_0$, therefore

$$(-\Delta + 1)\phi(t) = i\frac{\partial \phi}{\partial t} + B(n\phi) + \phi,$$

which yields

$$i\frac{\partial \phi}{\partial t} + \Delta\phi = -B(n\phi).$$

This gives us the existence of a solution of (3.15).
This solution verifies :

$$\left\| \frac{\partial \phi}{\partial t} \right\|_{L^\infty(I;L^2)} \leq 2a,$$

$$\left\| \frac{\partial \phi}{\partial t} \right\|_{L^{8/N}(I;L^4)} \leq 2\delta a,$$

$$\|n\|_{L^\infty(I;H^1)} \leq 2a,$$

$$\left\| \frac{\partial n}{\partial t} \right\|_{L^\infty(I;L^2)} \leq 2a.$$

On the other hand $i\dot{\phi} + \Delta\phi = -B(n\phi)$, this gives us the following conservation law :

$$\|\phi(t)\|_{L^2} = \|\phi_0\|_{L^2} \leq a. \tag{3.34}$$

A second conservation law may be obtained by setting

$$-\Delta\Phi = \frac{\partial n}{\partial t}.$$

Then we have

$$\int (|\nabla\phi(t)|^2 + n(t)|\phi(t)|^2 + \frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}n^2(t)) = \text{ Cst.} \tag{3.35}$$

There only remains to prove the uniqueness of (ϕ, n) in a convenient space, which is equivalent to show the uniqueness of (F, n) in X. For the time being, we only know that (F, n) is unique in Y.
If we have two solutions to the problem, (F, n) and (F', n'), we may associate a maximal time of existence in X (T and T') and a value a and a' such that

$$\begin{array}{ll}
\|F\|_{L^\infty(I;L^2)} \leq 2a, & \|F'\|_{L^\infty(0,T';L^2)} \leq 2a', \\
\|F\|_{L^{8/N}(I;L^4)} \leq 2\delta a, & \|F'\|_{L^{8/N}(0,T';L^4)} \leq 2\delta a', \\
\|n\|_{L^\infty(0,T;H^1)} \leq 2a, & \|n'\|_{L^\infty(0,T';H^1)} \leq 2a', \\
\|n_t\|_{L^\infty(I;L^2)} \leq 2a, & \|n'_t\|_{L^\infty(0,T';L^2)} \leq 2a',
\end{array}$$

with

$$\max\{\|\phi_0\|_{L^2}, \|\phi_0\|_{L^4}, \|\Delta\phi_0 + B(n_0\phi_0)\|_{L^2}, \|n_0\|_{H^1} + \|n_1\|_{L^2}\} \leq a \text{ and } a'.$$

We set $\tau = \min(T, T')$ and $\alpha = \max(a, a')$.
Then we have the above estimations with τ and α.
Following exactly the same lines as above with Y defined using τ and α, we obtain the uniqueness of the solution as a fixed point of N.

The result is also as follows :

Theorem 3.1 *Let us consider the problem on* $\mathbf{R}^N, N = 1, 2, 3.$

$$\begin{cases} i\dot{\phi} + \Delta\phi = \nabla\Delta^{-1}\nabla.(n\phi), \\ \ddot{n} - \Delta n = \Delta|\phi|^2, \\ n(x,0) = n_0(x), \\ \partial_t n(x,0) = n_1(x), \\ \phi(x,0) = \phi_0(x), \end{cases}$$

with $n_0 \in H^1, n_1 \in L^2$ *and* $\phi_0 \in H^2$.
Then there exists a time $T > 0$ *depending only on* $\|n_0\|_{H^1}$, $\|n_1\|_{L^2}$, $\|\phi_0\|_{H^2}$ *and* N *and a unique solution* $(\phi(t), n(t))$ *with*

$$\begin{cases} \phi(t) & \in & C^0([0,T]; H^2) \cap C^1([0,T]; L^2), \\ \phi(t) & \in & W^{1,8/N}(0,T; L^4), \\ n(t) & \in & C^0([0,T]; H^1) \cap C^1([0,T]; L^2) \cap C^2([0,T]; H^{-1}). \end{cases}$$

Remark The former result is a result about the local Cauchy problem. Thanks to the two conservation laws (3.34) and (3.35) we may show, as for the classical Zakharov equation (see [12]), that the solutions are global in time in the 1-dimensional case as well as in the 2-dimensional case for sufficiently small initial data.

2.4 Limit when $\lambda \longrightarrow \infty$.

We consider the problem

$$\begin{cases} \frac{1}{\lambda^2} n_{tt} - \Delta(n + |E|^2) = 0, \\ iE_t + \Delta E + B(nE) = 0, \end{cases} \tag{4.36}$$

where $B = \nabla(\Delta^{-1})\nabla.$ and

$$\begin{cases} E \ : \mathbf{R}_x^k \times \mathbf{R}_t^+ \to \mathbf{C}^k, \\ n \ : \mathbf{R}_x^k \times \mathbf{R}_t^+ \to \mathbf{R}. \end{cases}$$

Our goal is to show that the solutions (n, E) to (4.36) tend to $(-|\tilde{E}|^2, \tilde{E})$ when λ goes to infinity, where $\tilde{E}$ is the solution to $i\tilde{E}_t + \nabla \tilde{E} = B(|\tilde{E}|^2 \tilde{E})$. We follow the work of Schochet and Weinstein [10]. We have encountered some new difficulties due to the nonlocal term. This complicates the transformation of the initial system and makes useful the use of commutators.
We first have to prove the existence of solutions in H^s with $s > \left[\frac{k}{2}\right] + 3$ for an interval of time which may be very small but independent of λ. For that aim we proceed in two steps : the first one consists in writing the system as the perturbation of a symmetric hyperbolic system and the second one in computing the solution to this equivalent system as the limit of a sequence.

2.4.1 Transformation of the system.

We want to describe the system (4.36) as a dispersive perturbation of a symmetric hyperbolic system.
We set

$$\begin{cases} V = -\frac{1}{\lambda} \Delta^{-1} \nabla n_t, \\ Q = n + |E|^2, \end{cases}$$

where

$$\begin{cases} V : \mathbf{R}_x^k \times \mathbf{R}_t^+ \to \mathbf{R}^k, \\ Q : \mathbf{R}_x^k \times \mathbf{R}_t^+ \to \mathbf{R}. \end{cases}$$

Hence, (4.36) formally reads

$$\begin{cases} Q_t + \lambda\nabla.V - (|E|^2)_t = 0, \\ V_t + \lambda\nabla Q = 0, \\ iE_t + \Delta E - B(|E|^2E) + B(QE) = 0. \end{cases} \tag{4.37}$$

$$iE_t + \Delta E - B(|E|^2E) + B(QE) = 0,$$

$$i^tE_t\bar{E} +^t \Delta E\bar{E} -^t B(|E|^2E)\bar{E} +^t B(QE)\bar{E} = 0,$$

$$i(|E|^2)_t +^t \Delta E\bar{E} -^t \Delta\bar{E}E -^t B(|E|^2E)\bar{E} +^t B(|E|^2\bar{E})E,$$

$$+^tB(QE)\bar{E} -^t B(Q\bar{E})E = 0.$$

Moreover

$$i\partial_jE_t + \Delta\partial_jE - \partial_jB(|E|^2E) + \partial_jB(QE) = 0,$$

where ∂_j is the differentiation with respect to x_j.
This may be written in a condensed form

$$i\nabla E_t + \Delta\nabla E - \nabla B(|E|^2E) + \nabla B(QE) = 0,$$

where, for a vector ${}^t(\Phi^1, \ldots, \Phi^k)$, we have

$$\nabla\Phi =^t (\partial_1\Phi^1, \partial_1\Phi^2, \ldots, \partial_1\Phi^k, \partial_2\Phi^1, \ldots, \partial_k\Phi^k).$$

We will call these k^2-components vectors, 2-vectors.
We also want to write the 2-vector $\nabla B\Phi$ in the form $A\nabla\Phi$ where A is always an operator of order 0.
As $\nabla.\Phi = \sum_{j=1}^{k} \partial_j\Phi^j$, we have $\nabla.\Phi = \Gamma.\nabla\Phi$,
where Γ is the 2-vector ${}^t(1, 0, \ldots, 0; 0, 1, \ldots, 0; \ldots; 0, \ldots, 0, 1)$,
and hence $A = \nabla\nabla(\Delta^{-1})\Gamma$. (transforms a 2-vector into a 2-vector). We also have

$$i\nabla E_t + \Delta\nabla E - A\nabla(|E|^2E) + A\nabla(QE) = 0.$$

Now

$$\begin{aligned} \nabla(QE) &= {}^t(\partial_1(QE^1), \partial_1(QE^2), \ldots, \partial_k(QE^k)), \\ &= Q\nabla E +^t (\partial_1 QE^1, \partial_1 QE^2, \ldots, \partial_k QE^k), \\ &= Q\nabla E + \alpha(E)\nabla Q, \end{aligned}$$

where $\alpha(\Phi)$ is the matrix $(k^2 - k)$ which has Φ on its "diagonal" and zeros everywhere else.

Similarly $\nabla(|E|^2E) = |E|^2\nabla E + \alpha(E)\nabla(|E|^2)$,

$$\begin{aligned}\alpha(E)\nabla(|E|^2) &= \sum_{j=1}^{k} {}^t(\partial_1 E^j\bar{E}^jE^1, \partial_1 E^j\bar{E}^jE^2, \dots, \partial_k E^j\bar{E}^jE^k)\\ &\quad+\sum_{j=1}^{k} {}^t(\partial_1 \bar{E}^jE^jE^1, \partial_1 \bar{E}^jE^jE^2, \dots, \partial_k \bar{E}^jE^jE^k).\end{aligned}$$

Then we have $\alpha(E)\nabla(|E|^2) = \mathcal{A}_1(E)\nabla E + \mathcal{A}_2(E)\nabla\bar{E}$, where $\mathcal{A}_1(E)$ (resp. $\mathcal{A}_2(E)$) is the block-diagonal matrix $(k^2 - k^2)$ such that each block is equal to $E^t\bar{E}$ (resp. E^tE). Hence

$$i\nabla E_t + \Delta\nabla E - A(|E|^2\nabla E + \mathcal{A}_1(E)\nabla E + \mathcal{A}_2(E)\nabla\bar{E}) + A(Q\nabla E + \alpha(E)\nabla Q) = 0.$$

We now set

$$\sqrt{2}E = F + iG \text{ et } \sqrt{2}\nabla E = H + iL.$$

Hence, we have

$$\begin{cases} F, G \ : \mathbf{R}_x^k \times \mathbf{R}_t^+ \to \mathbf{R}^k,\\ H, L \ : \mathbf{R}_x^k \times \mathbf{R}_t^+ \to \mathbf{R}^{k^2}.\end{cases}$$

Moreover, we notice that B and A preserve the real and imaginary parts.

$$\begin{gathered}(|E|^2)_t = i({}^t\Delta E\bar{E} - {}^t\Delta\bar{E}E - {}^tB(|E|^2E)\bar{E} + {}^tB(|E|^2\bar{E})E\\ + {}^tB(QE)\bar{E} - {}^tB(Q\bar{E})E),\\ {}^t\Delta E.\bar{E} = \sum_{j=1}^{k}(\bar{E}^j\sum_{l=1}^{k}\partial_l(\partial_l E^j)) = {}^t\bar{E}\nabla.(\nabla E),\end{gathered}$$

where $\nabla.$ is the multiplication by the following $(k - k^2)$ matrix

$$\begin{pmatrix} \partial_1 & & 0 & \partial_2 & & 0 & & \partial_k & & 0\\ & \ddots & & & \ddots & & \cdots & & \ddots & \\ 0 & & \partial_1 & 0 & & \partial_2 & & 0 & & \partial_k\end{pmatrix}.$$

$$\begin{aligned}(|E|^2)_t &= i({}^t\Delta E\bar{E} - {}^t\Delta\bar{E}E - {}^tB(|E|^2E)\bar{E} + {}^tB(|E|^2\bar{E})E\\ &\quad + {}^tB(QE)\bar{E} - {}^tB(Q\bar{E})E),\\ &= i(-i{}^tG\nabla.H + i{}^tF\nabla.L - i\frac{{}^tB}{4}(((|F|^2 + |G|^2)(F + iG))(F - iG)\\ &\quad + i\frac{{}^tB}{4}(((|F|^2 + |G|^2)(F - iG))(F + iG)\\ &\quad + i{}^tB(Q(F + iG))(F - iG) - i{}^tB(Q(F - iG))(F + iG)),\\ &= ({}^tG\nabla.H - {}^tF\nabla.L) + (-\frac{{}^tB}{2}(((|F|^2 + |G|^2)F)G\\ &\quad + \frac{{}^tB}{2}(((|F|^2 + |G|^2)G)F) + ({}^tB(QF)G - {}^tB(QG)F).\end{aligned}$$

$$\begin{aligned} Q_t &+ \lambda\nabla.V +^t F\nabla.L -^t G\nabla.H + \frac{^tB}{2}((|F|^2+|G|^2)F)G \\ &- \frac{^tB}{2}((|F|^2+|G|^2)G)F -^t B(QF)G +^t B(QG)F = 0. \end{aligned}$$

$$\begin{gathered} V_t + \lambda\nabla Q = 0, \\ i(F+iG)_t + \Delta(F+iG) - \frac{1}{2}B((|F|^2+|G|^2)(F+iG)) + B(Q(F+iG)) = 0, \\ F_t + \Delta G - \frac{1}{2}B((|F|^2+|G|^2)G) + B(QG) = 0, \\ G_t - \Delta F + \frac{1}{2}B((|F|^2+|G|^2)F) - B(QF) = 0. \end{gathered}$$

$$\begin{aligned} &i(H+iL)_t + \Delta(H+iL) \\ &-\frac{A}{2}((|F|^2+|G|^2)(H+iL) + \mathcal{A}_1(F+iG)(H+iL) + \mathcal{A}_2(F+iG)(H-iL)) \\ &+A(Q(H+iL) + \alpha(F+iG)\nabla Q) = 0. \end{aligned}$$

In order to treat $\mathcal{A}_1$ and $\mathcal{A}_2$, we have :

$$\begin{aligned} E^t\bar{E} &= (F+iG)^t(F-iG) = F^tF + G^tG + i(G^tF - F^tG), \\ E^tE &= (F+iG)^t(F+iG) = F^tF - G^tG + i(G^tF + F^tG). \end{aligned}$$

Let $\mathcal{M}$ be the operator which creates a (k^2-k^2)-dimensional block-diagonal matrix with identical blocks. Then

$$\begin{aligned} &\mathcal{A}_1(F+iG)(H+iL) + \mathcal{A}_2(F+iG)(H-iL) \\ &= \mathcal{M}(F^tF+G^tG)H - \mathcal{M}(G^tF-F^tG)L \\ &\quad +\mathcal{M}(F^tF-G^tG)H + \mathcal{M}(G^tF+F^tG)L \\ &+i[\mathcal{M}(F^tF+G^tG)L + \mathcal{M}(G^tF-F^tG)H \\ &\quad -\mathcal{M}(F^tF-G^tG)L + \mathcal{M}(G^tF+F^tG)H], \\ &= 2\mathcal{M}(F^tF)H + 2\mathcal{M}(F^tG)L + 2i[\mathcal{M}(G^tG)L + \mathcal{M}(G^tF)H]. \end{aligned}$$

$$\begin{aligned} H_t + \Delta L - A(\frac{1}{2}(|F|^2+|G|^2)L + \mathcal{M}(G^tG)L + \mathcal{M}(G^tF)H) \\ +A(QL + \alpha(G)\nabla Q) = 0, \\ L_t - \Delta H + A(\frac{1}{2}(|F|^2+|G|^2)H + \mathcal{M}(F^tF)H + \mathcal{M}(F^tG)L) \\ -A(QH + \alpha(F)\nabla Q) = 0. \end{aligned}$$

Summing up, we have the following system :

$$\begin{aligned} Q_t &+ \lambda\nabla.V +^t F\nabla.L -^t G\nabla.H + \frac{^tB}{2}((|F|^2+|G|^2)F)G \\ &- \frac{^tB}{2}((|F|^2+|G|^2)G)F -^t B(QF)G +^t B(QG)F = 0, \end{aligned}$$

$$
\begin{aligned}
V_t + \lambda\nabla Q &= 0,\\
F_t + \Delta G - \frac{1}{2}B((|F|^2+|G|^2)G) + B(QG) &= 0,\\
G_t - \Delta F + \frac{1}{2}B((|F|^2+|G|^2)F) - B(QF) &= 0,\\
H_t + \Delta L - A(\frac{1}{2}(|F|^2+|G|^2)L + \mathcal{M}(G^tG)L + \mathcal{M}(G^tF)H)&\\
+A(QL + \alpha(G)\nabla Q) &= 0,\\
L_t - \Delta H + A(\frac{1}{2}(|F|^2+|G|^2)H + \mathcal{M}(F^tF)H + \mathcal{M}(F^tG)L)&\\
-A(QH + \alpha(F)\nabla Q) &= 0.
\end{aligned}
$$

which also reads

$$
\begin{aligned}
Q_t \;&+\; \lambda\nabla.V +^t F\nabla.L -^t G\nabla.H + \frac{^tB}{2}((|F|^2+|G|^2)F)G\\
&-\; \frac{^tB}{2}((|F|^2+|G|^2)G)F -^t B(QF)G +^t B(QG)F = 0,
\end{aligned} \tag{4.38}
$$

$$
V_t + \lambda\nabla Q = 0, \tag{4.39}
$$

$$
F_t - \frac{1}{2}B((|F|^2+|G|^2)G) + B(QG) = -\Delta G, \tag{4.40}
$$

$$
G_t + \frac{1}{2}B((|F|^2+|G|^2)F) - B(QF) = \Delta F, \tag{4.41}
$$

$$
H_t - A(\frac{1}{2}(|F|^2+|G|^2)L + \mathcal{M}(G^tG)L + \mathcal{M}(G^tF)H) + A(QL + \alpha(G)\nabla Q) = -\Delta L, \tag{4.42}
$$

$$
L_t + A(\frac{1}{2}(|F|^2+|G|^2)H + \mathcal{M}(F^tF)H + \mathcal{M}(F^tG)L) - A(QH + \alpha(F)\nabla Q) = \Delta H. \tag{4.43}
$$

We introduce the vector with $1+3k+2k^2$ components : $U =^t (Q, V, F, G, H, L)$, and we want to write the above system in the form :

$$
U_t + \sum_{j=1}^{k}\{R(A^j(U)U_{x_j}) + \lambda C^j U_{x_j}\} + S(\tilde{B}(U)U) = K\Delta U, \tag{4.44}
$$

where R and S are nonlocal operators.

Let us describe all the operators :

$$
K = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -I & 0 & 0\\
0 & 0 & I & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -I\\
0 & 0 & 0 & 0 & I & 0
\end{pmatrix}
$$

$$C^j = \begin{pmatrix} 0 & {}^t\epsilon^j & 0 & 0 & 0 & 0 \\ \epsilon^j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where ϵ^j is the jth vector of the canonical basis of $\mathbf{R}^k$.

$$A^j(U) = \begin{pmatrix} 0 & 0 & 0 & 0 & -{}^t(\alpha(G)\epsilon^j) & {}^t(\alpha(F)\epsilon^j) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha(G)\epsilon^j & 0 & 0 & 0 & 0 & 0 \\ \alpha(F)\epsilon^j & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & -A & 0 \\ 0 & 0 & 0 & 0 & 0 & -A \end{pmatrix}$$

We denote by $\tilde{B}_j(U)$ the jth column of $\tilde{B}_j$, and we have

$$\tilde{B}_1(U) = \tilde{B}_2(U) \equiv 0,$$

$$\tilde{B}_3(U) = \begin{pmatrix} -\frac{1}{2}{}^tB((|F|^2 + |G|^2)G) + {}^tB(QF) \\ 0 \\ 0 \\ -\frac{1}{2}(|F|^2 + |G|^2) + Q \\ 0 \\ 0 \end{pmatrix}$$

$$\tilde{B}_4(U) = \begin{pmatrix} \frac{1}{2}{}^tB((|F|^2 + |G|^2)F) - {}^tB(QG) \\ 0 \\ \frac{1}{2}(|F|^2 + |G|^2) - Q \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\tilde{B}_5(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{M}(G^tF) \\ -\frac{1}{2}(|F|^2 + |G|^2) - \mathcal{M}(F^tF) + Q \end{pmatrix}$$

$$\tilde{B}_6(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2}(|F|^2 + |G|^2) + \mathcal{M}(G^tG) - Q \\ -\mathcal{M}(F^tG) \end{pmatrix}$$

$\tilde{B}(U)$ has a nonlocal term in its first component.

$$S = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & -B & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 & -A & 0 \\ 0 & 0 & 0 & 0 & 0 & -A \end{pmatrix}$$

2.4.2 Existence of a regular solution for an independent time of λ.

We set $N = 2k^2 + 3k + 1$.
We consider the following iteration scheme :

$$U^0(x,t) = U_0(x),$$

$$\frac{\partial U^{p+1}}{\partial t} + \sum_{j=1}^{k}\{R(A^j(U^p)U^{p+1}_{x_j}) + \lambda C^j U^{p+1}_{x_j}\} + S(\tilde{B}(U^p)U^{p+1}) = K\Delta U^{p+1},$$

$$U^{p+1}(x,0) = U_0(x).$$

We set

$$|||u|||_s = |||u|||_{s,T} = \sup_{t\in[0,T]} \|u(\cdot,t)\|_s.$$

In what follows, we assume that $s > \left[\frac{k}{2}\right] + 3$.
Moreover we set $|||U_0|||_s = \varepsilon$ and let $\delta > \varepsilon$.
We argue as follows :

- $\forall p \geq 0 \qquad |||U^p|||_{s,T} \leq \delta.$
- $\forall p \geq 0 \qquad |||U^{p+1} - U^p|||_{0,T} \leq C|||U^p - U^{p-1}|||_{0,T}$
  with $C < 1$.
- Then
  $U^p \to U$ in $L^\infty(0,T;L^2)$,
  U^p is bounded in $L^\infty(0,T;H^s)$.

Thanks to the convexity of the $\|\ \|_s$ norms, we have

$$\|(U^p - U)(\cdot,t)\|_{s'} \to 0 \ \ \forall s' < s.$$

Since $s > k/2+1$, we may choose $s' > k/2+1$.
Then $U \in \mathcal{C}([0,T];\mathcal{C}^1)$ and the solution is a classical one.
Moreover we have

$$U \in Lip([0,T];H^{s-2}),$$
$$U \in \mathcal{C}([0,T];H^s) \cap \mathcal{C}^1([0,T];H^{s-2}).$$

There remains to specify the first two points.

A lemma.

In what follows, we use the following result :
Let us consider the problem

$$\partial_t u + a(u) = f,$$

with $a(u) = \sum_{j=1}^{k}\{R(A^j(v)\partial_j u) + \lambda C^j \partial_j u\} + S(\tilde{B}(v)u) - K\Delta u$,
where $|||v|||_{s,T} \le \delta$.

$$((\partial_t u, u)) + ((a(u),u)) \le \|f\|_0 \|u\|_0.$$

C^j is symmetric with constant coefficients, so we have $((\lambda C^j\partial_j u, u)) = 0$. This leads to a result which does not depend on λ.
K being antisymmetric, $((K\Delta u, u)) = 0$.
Hence $((a(u),u)) = \sum_{j=1}^{k}((R(A^j(v)\partial_j u), u)) + ((S(\tilde{B}(v)u), u))$.
$|((S(\tilde{B}(v)u),u))| \le \|S(\tilde{B}(v)u)\|_0 \|u\|_0 \le C\|u\|_0^2$, where C only depends on δ.

There also remains to estimate $((R(A^j(v)\partial_j u), u))$ for all $0 \le j \le k$.

$$\begin{aligned}
((R(A^j(v)\partial_j u),u)) &= ((R(\partial_j(A^j(v)u)),u)) - \underbrace{((R([A^j(v),\partial_j]u),u))}_{\alpha},\\
&= ((\partial_j R(A^j(v)u),u)) - \alpha,\\
&= -((R(A^j(v)u),\partial_j u)) - \alpha,\\
&= -((A^j(v)u, R(\partial_j u))) - \alpha,\\
&= -((u, A^j(v)R(\partial_j u))) - \alpha,\\
&= -((u, R(A^j(v)\partial_j u))) - \alpha - ((([A^j(v),R](\partial_j u),u)),\\
2((R(A^j(v)\partial_j u),u)) &= -((R([A^j(v),\partial_j]u),u)) - ((([A^j(v),R](\partial_j u),u)).
\end{aligned}$$

Hence $2((R(A^j(v)\partial_j u),u)) = ((R([\partial_j, A^j(v)]u),u)) + ((([R,A^j(v)](\partial_j u),u))$. $[\partial_j, A^j(v)]$ is a commutator of order 0, hence

$$|((R([\partial_j, A^j(v)]u),u))| \le C\|u\|_0^2,$$

and the constant only depends on δ.
$[R, A^j(v)]$ is a commutator of order -1, therefore

$$|(([R, A^j(v)]\partial_j u), u))| \leq C\|u\|_0^2,$$

and the constant only depends on δ.
Also, we have $\frac{d}{dt}\|u(t)\|_0^2 \leq C\|u(t)\|_0^2 + 2\|f\|_0\|u\|_0$, $(*)$
where C only depends on δ.

Estimation for the large norm.

We are going to show the result by induction : we consider the following assumption
(H_p) $\forall 0 \leq l \leq p$ $|||U^l|||_{s,T} \leq \delta$.
This is valid for $p = 0$ (because $\varepsilon \leq \delta$).
We want to show that $(H_p) \Rightarrow (H_{p+1})$ (for a good choice of T).
We choose α such that $|\alpha| \leq s$, and we set $U_\alpha^{p+1} = D^\alpha U^{p+1}$.

$$\begin{aligned}
\partial_t U_\alpha^{p+1} &+ \sum_{j=1}^{k}\{R(A^j(U^p)\partial_j U_\alpha^{k+1}) + \lambda C^j \partial_j U_\alpha^{p+1}\} \\
&+ S(\tilde{B}(U^p)U_\alpha^{p+1}) - K\Delta U_\alpha^{p+1} \\
&= (\sum_{j=1}^{k}[RA^j(U^p)\partial_j, D^\alpha] + [S\tilde{B}(U^p), D^\alpha])U^{p+1}.
\end{aligned}$$

The commutator $\sum_{j=1}^{k}[RA^j(U^p)\partial_j, D^\alpha] + [S\tilde{B}(U^p), D^\alpha]$ is an operator of order $|\alpha|$.
This equation is in the form

$$\partial_t U_\alpha^{p+1} + a(U_\alpha^{p+1}) = f_1,$$

with $v = U^p$ (and hence $|||v|||_{s,T} \leq \delta$),
and $f_1 = [\sum_{j=1}^{k} RA^j(U^p)\partial_j + S\tilde{B}(U^p), D^\alpha]U^{p+1}$.

$$\|f_1\|_0 \leq C\|U^{p+1}\|_{|\alpha|} \leq C\|U^{p+1}\|_s,$$

and C only depends δ.
Using $(*)$, we obtain

$$\frac{d}{dt}\|U_\alpha^{p+1}(t)\|_0^2 \leq C(\|U_\alpha^{p+1}(t)\|_0^2 + \|U^{p+1}\|_s\|U_\alpha^{p+1}\|_0).$$

Hence

$$\frac{d}{dt}\|U^{p+1}(t)\|_s^2 \leq C\|U^{p+1}(t)\|_s^2,$$

$$\|U^{p+1}(t)\|_s^2 \leq e^{Ct}\|U^{p+1}(0)\|_s^2 \leq e^{Ct}\varepsilon^2.$$

and C only depends on δ.
We choose T sufficiently small such that

$$\|U^{p+1}(t)\|_s \leq \delta \qquad\qquad \forall 0 \leq t \leq T.$$

T does not depend on p (nor on λ) because C does not depend on them. Therefore, we have the estimation for the large norm.

Convergence in low norm.

$$\partial_t U^{p+1} + \sum_{j=1}^{k}\{R(A^j(U^p)U^{p+1}_{x_j}) + \lambda C^j U^{p+1}_{x_j}\} + S(\tilde{B}(U^p)U^{p+1}) - K\Delta U^{p+1} = 0,$$

$$\partial_t U^{p} + \sum_{j=1}^{k}\{R(A^j(U^{p-1})U^{p}_{x_j}) + \lambda C^j U^{p}_{x_j}\} + S(\tilde{B}(U^{p-1})U^{p}) - K\Delta U^{p} = 0.$$

Setting $V^p = U^{p+1} - U^p$, we obtain

$$\begin{cases} \partial_t V^p & + & \sum_{j=1}^{k}\{R(A^j(U^p)V^p_{x_j}) + \lambda C^j V^p_{x_j}\} + S(\tilde{B}(U^p)V^p) - K\Delta V^p \\ & = & -\sum_{j=1}^{k} R((A^j(U^p) - A^j(U^{p-1}))U^p_{x_j}) \\ & & -S((\tilde{B}(U^p) - \tilde{B}(U^{p-1}))U^p), \\ V^p(0,x) & = & 0. \end{cases}$$

This equation is in the form

$$\partial_j V^p + a(V^p) = f_2,$$

with $v = U^p$,
and $f_2 = -\sum_{j=1}^{k} R((A^j(U^p) - A^j(U^{p-1}))U^p_{x_j}) - S((\tilde{B}(U^p) - \tilde{B}(U^{p-1}))U^p)$.
A^j is linear and $\tilde{B}$ only consists in terms of the first, second and third order, we have also :

$$\begin{aligned} \|R((A^j(U^p) - A^j(U^{p-1}))U^p_{x_j})\|_0 &\leq C\|U^p - U^{p-1}\|_0\|U^p_{x_j}\|_\infty, \\ &\leq C\|U^p - U^{p-1}\|_0\|U^p\|_s, \\ &\leq C\|U^p - U^{p-1}\|_0, \\ \|S((\tilde{B}(U^p) - \tilde{B}(U^{p-1}))U^p)\|_0 &\leq C\|U^p - U^{p-1}\|_0\|U^p\|_\infty, \\ &\leq C\|U^p - U^{p-1}\|_0\|U^p\|_s, \\ &\leq C\|U^p - U^{p-1}\|_0. \end{aligned}$$

where the constants only depends on δ.
Using $(*)$, we have

$$\frac{d}{dt}\|V^p(t)\|_0^2 \leq C\|V^p\|_0^2 + D\|V^{p-1}\|_0\|V^p\|_0.$$

Hence $\frac{d}{dt}\|V^p(t)\|_0^2 \leq C\|V^p\|_0^2 + \beta\|V^{p-1}\|_0^2 + \frac{D^2}{4\beta}\|V^p\|_0^2$.
We consider β such that $C + \frac{D^2}{4\beta} = \beta$ ($4\beta^2 - C\beta - D^2 = 0$ has a positive solution β)

$$\frac{d}{dt}\|V^p(t)\|_0^2 \leq \beta\|V^p\|_0^2 + \beta\|V^{p-1}\|_0^2.$$

then

$$\|V^p(t)\|_0^2 \leq e^{\beta t}\underbrace{\|V^p(0)\|_0^2}_{0} + (e^{\beta t} - 1)\|V^{p-1}(t)\|_0^2$$

We choose T sufficiently small such that $e^{\beta t} - 1 < 1 \;\; \forall 0 \leq t \leq T$.
Hence we obtain the convergence in the low norm.

Return to the initial problem.

We suppose that the Cauchy problem is posed for the following initial data :

$$\begin{cases} n(0,x) = n_0(x), \\ \partial_t n(0,x) = n_1(x), \\ E(0,x) = E_0(x). \end{cases} \tag{4.45}$$

$V_t + \lambda\nabla Q = 0$ is one of the given equations.
$F_t - \frac{1}{2}B((|F|^2 + |G|^2)G) + B(QG) = -\Delta G$,
and $G_t + \frac{1}{2}B((|F|^2 + |G|^2)F) - B(QF) = \Delta F$
immediately yields $E_t + \Delta E - B(|E|^2 E) + B(QE) = 0$,
where we have set $E = \frac{1}{\sqrt{2}}(F + iG)$.
Let us set $W = (\nabla F - H, \nabla G - L)$, this is zero at time $t = 0$ and therefore for all t, hence $\nabla E = \frac{1}{\sqrt{2}}(H + iL)$ and using the last equations of the perturbated hyperbolic system we have $Q_t + \lambda\nabla.V - (|E|^2)_t = 0$.
Therefore we find again the initial equations.
We can also get some regularity :

$$E \in C([0,T]; H^{s+1}) \cap C^1([0,T]; H^{s-1}),$$

$$n \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1}) \cap C^2([0,T]; H^{s-2}).$$

Hence

$$\begin{cases} \frac{1}{\lambda^2} n_{tt} - \Delta(n + |E|^2) = 0, \\ iE_t + \Delta E + B(nE) = 0, \end{cases}$$

and the solutions are classical ones.
In what follows, we call (n^λ, E^λ), the solutions to the initial system (4.36) which is associated to the parameter λ. Then we have the following estimation :

$$\|E^\lambda\|_{H^{s+1}} + \|E_t^\lambda\|_{H^{s-1}} + \|n^\lambda\|_{H^s} + \frac{1}{\lambda}\|n_t^\lambda\|_{H^{s-1}} + \frac{1}{\lambda^2}\|n_{tt}^\lambda\|_{H^{s-2}} \leq \text{const} \qquad (4.46)$$

We may state the theorem :

Theorem 4.2 *Let $s \geq \left[\frac{k}{2}\right] + 3$. There exists a unique solution to (4.36, 4.45) on a time interval $[0,T]$, T not depending on λ but only on $\|n_0\|_{H^s}$, $\|n_1\|_{H^{s-1}}$ and $\|E_0\|_{H^{s+1}}$, which we suppose are finite.*
Moreover, for all $t \in [0,T]$ we have the estimation (4.46).

2.4.3 The limit when λ tends to ∞.

Theorem 4.3 *When λ tends to ∞,*
$n^\lambda + |E^\lambda|^2 \to 0$ *in* $C^0([0,T] \times \mathbf{R}^k)$,
$\nabla(n^\lambda + |E^\lambda|^2) \to 0$ *in* $C^0([0,T]; H^{s-2})$,
$E^\lambda \to \tilde{E}$ *in* $C^1([0,T] \times \mathbf{R}^k) \cap C([0,T]; C^2)$,
where $\tilde{E}$ is the unique solution to

$$i\tilde{E}_t + \Delta\tilde{E} - B(|\tilde{E}|^2\tilde{E}) = 0.$$

The demonstration is carried out as follows.
We differentiate with respect to the time the equation (4.44), and obtain :

$$\begin{aligned} U_{tt} &+ \sum_{j=1}^{k}\{R(A^j(U)U_{x_jt} + A^j(U_t)U_{x_j}) + \lambda C^j U_{x_jt}\} \\ &+ S(\tilde{B}_t(U)U + \tilde{B}(U)U_t) = K\Delta U_t. \end{aligned}$$

As in the former proof, we may then show that :

$$\|U_t^\lambda(0)\|_{H^{s-2}} \leq C \Rightarrow \|U_t^\lambda(t)\|_{H^{s-2}} \leq C' \ \ \forall t \in [0,T].$$

Indeed :
Let α be such that $|\alpha| \leq s-2$,

$$\begin{aligned} \partial_t(U_{t\alpha}) &+ \sum_{j=1}^{k}\{R(A^j(U)(U_{t\alpha})_{x_j}) + \lambda C^j(U_{t\alpha})_{x_j}\} \\ &+ \tilde{B}(U)U_{t\alpha} - K\Delta U_{t\alpha} \\ &= D^\alpha(-\sum_{j=1}^{k} R(A^j(U_t)U_{x_j}) + S(\tilde{B}_t(U)U)) \\ &+ \left(\sum_{j=1}^{k}[RA_j(U)\partial_j, D^\alpha] + [S\tilde{B}(U), D^\alpha]\right) U_t. \end{aligned}$$

We set

$$f_3 = D^\alpha(-\sum_{j=1}^{k} R(A^j(U_t)U_{x_j}) - S(\tilde{B}_t(U)U)),$$

$$f_4 = \left(\sum_{j=1}^{k}[RA_j(U)\partial_j, D^\alpha] + [S\tilde{B}(U), D^\alpha]\right) U_t.$$

$$\|f_4\|_0 \leq C\|U_t\|_{s-2},$$

$$\begin{aligned} \|f_3\|_0 &\leq \|\sum_{j=1}^{k}(R(A^j(U_t)U_{x_j}) + S(\tilde{B}_t(U)U)\|_{s-2}, \\ &\leq C\sum_{j=1}^{k}\|A^j(U_t)U_{x_j}\|_{s-2} + C\|\tilde{B}_t(U)U)\|_{s-2}, \\ &\leq C\|U_t\|_{s-2}. \end{aligned}$$

We have an equation in the form

$$\partial(U_{t\alpha}) + a(U_{t\alpha}) = f_3 + f_4,$$

with $v = U$ and $\|f_3 + f_4\|_0 \leq C\|U_t\|_{s-2}$.
(*) yields

$$\frac{d}{dt}\|U_{t\alpha}(t)\|_0^2 \leq \|U_{t\alpha}(t)\|_0^2,$$

$$\|U_t(t)\|_{s-2}^2 \leq e^{Ct}\|U_t(0)\|_{s-2}^2.$$

and $\|U_t(0)\|_{s-2}^2$ is finite using the initial regularity and equations (4.38-4.43). $V_t + \lambda\nabla Q = 0$,
then $V_t^\lambda \in L^\infty([0,T]; H^{s-2}) \Rightarrow \lambda\nabla Q^\lambda \in L^\infty([0,T]; H^{s-2})$,

$$\begin{aligned} \Rightarrow \|Q^\lambda\|_{C^0([0,T]\times\mathbf{R}^k)} &\leq C\|D^{s-1}Q^\lambda\|_{L^2}^{k/2(s-1)}\|Q^\lambda\|_{L^2}^{1-k/2(s-1)}, \\ &\leq C\lambda^{-k/2(s-1)}, \\ \text{and } \|\nabla Q^\lambda\|_{C^0([0,T];H^{s-2})} &\leq C\lambda^{-1}. \end{aligned}$$

When λ tends to $+\infty$, $\|Q^\lambda\|_{C^0([0,T]\times\mathbf{R}^k)} \to 0$ and $\|\nabla Q^\lambda\|_{C^0([0,T];H^{s-2})} \to 0$,
hence $n^\lambda + |E^\lambda|^2 \to 0$ in $C^0([0,T]\times\mathbf{R}^k)$.
This allows us to obtain the first two results of convergence. To obtain the last one, we set $v^\lambda = (F^\lambda, G^\lambda, H^\lambda, L^\lambda)$.
$\{v^\lambda\}$ is bounded in $C^0([0,T]; H^s)$,
$\{v_t^\lambda\}$ is bounded in $C^0([0,T]; H^{s-2})$ (which yields the equicontinuity of v^λ).
Therefore using Ascoli-Arzela's theorem, we have the convergence of a subsequence in $C^0([0,T]; H_{loc}^{s-2})$.
Using interpolation and the boundedness in $C^0([0,T]; H^s)$, we have the convergence

of a subsequence in $\mathcal{C}^0([0,T];H^{s-\varepsilon}_{loc})$ $\forall \varepsilon > 0$.
Let $(\tilde{F},\tilde{G},\tilde{H},\tilde{L})$ be the limit of this subsequence.
Thanks to the equations governing $(F^\lambda_t, G^\lambda_t, H^\lambda_t, L^\lambda_t)$, we obtain the convergence in the sense of $\mathcal{C}^1([0,T];H^{s-2-\varepsilon}_{loc}), \forall \varepsilon > 0$.
The equations pass to the limit :

$$\tilde{F}_t - \frac{1}{2}B((|\tilde{F}|^2 + |\tilde{G}|^2)\tilde{G}) = -\Delta\tilde{G}, \tag{4.47}$$

$$\tilde{G}_t + \frac{1}{2}B((|\tilde{F}|^2 + |\tilde{G}|^2)\tilde{F}) = \Delta\tilde{F}, \tag{4.48}$$

$$\tilde{H}_t - A(\frac{1}{2}(|\tilde{F}|^2 + |\tilde{G}|^2)\tilde{L} + \mathcal{M}(\tilde{G}^t\tilde{G})\tilde{L} + \mathcal{M}(\tilde{G}^t\tilde{F})\tilde{H}) = -\Delta\tilde{L}, \tag{4.49}$$

$$\tilde{L}_t + A(\frac{1}{2}(|\tilde{F}|^2 + |\tilde{G}|^2)\tilde{H} + \mathcal{M}(\tilde{F}^t\tilde{F})\tilde{H} + \mathcal{M}(\tilde{F}^t\tilde{G})\tilde{L}) = \Delta\tilde{H}. \tag{4.50}$$

The regularity of the solutions enables us to return as before to the initial equations and we have :

$$i\tilde{E}_t + \Delta\tilde{E} = B(|\tilde{E}|^2\tilde{E}).$$

Remark The different constants which are used in this section are said to depend only on δ (they, of course, also depend on the space dimension). We may be more precise and give additional information about how these constants depend on δ in order to estimate T.
In the proof of the lemma, we obtain in fact the following estimate

$$\frac{d}{dt}\|u(t)\|_0^2 \le K_I(\delta + \delta^3)\|u(t)\|_0^2 + 2\|f\|_0\|u\|_0.$$

While computing in the large norm we have found the following estimates :

$$\|f_1\|_0 \le K_1(\delta + \delta^3)|U^{p+1}\|_s,$$

$$\frac{d}{dt}\|U^{p+1}(t)\|_s^2 \le K_{II}(\delta + \delta^3)\|U^{p+1}(t)\|_s^2,$$

$$\|U^{p+1}(t)\|_s^2 \le e^{K_{II}(\delta+\delta^3)t}\varepsilon^2.$$

We also have to have the following estimate on T :

$$T \le \frac{2}{K_{II}(\delta + \delta^3)}\log\frac{\delta}{\varepsilon}.$$

Studying the convergence in low norm, we have the following results :

$$\|f_2\|_0 \le K_2(\delta + \delta^3)|V^{p-1}\|_0,$$

$$\beta = K_{III}(\delta + \delta^3),$$

$$\|V^p(t)\|_0^2 \le (e^{\beta t} - 1)\|V^{p-1}(t)\|_0^2.$$

As we want $\beta T < \log 2$, this yields

$$T < \frac{\log 2}{K_{III}(\delta + \delta^3)}.$$

Taking everything into account, we obtain that

$$T \leq \frac{2}{K_0(\delta + \delta^3)} \log \frac{\delta}{\varepsilon}.$$

In the rest of the section, no other restriction on T has been made.

We may think of some extensions of this study. For example, the same arguments as in [9] enable us to study the Cauchy problem for $(\phi_0, n_0, n_1) \in H^m \times H^{m-1} \times H^{m-2}$. There are opened problems for the Zakharov equation which are opened for this equation too. We may for example think of the global Cauchy problem in 3 dimensions.

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Partie II

Mesures invariantes pour certaines équations aux dérivées partielles.

Chapitre 3

Invariant measures for some partial differential equations

Abstract

We construct invariant measures for Hamiltonian systems such as the nonlinear Schrödinger equation or the wave equation in order to prove Poisson's recurrence. The particular case of schemes (finite dimensional spaces) is also treated in order to explain the recurrence phenomenon which is observed during numerical simulations.

Submitted to Physica D

3.1 Introduction.

Fermi-Pasta-Ulam recurrence phenomenon has first been noticed in the context of the numerical study of a chain of balls with nonlinear interactions. This phenomenon has also been observed experimentally by Yuen, Lake and Ferguson [22] in the frame of deep water waves governed by the cubic nonlinear Schrödinger equation. This phenomenon may be described as follows. The energy of an initial data with a finite number of modes spreads to higher modes and after a certain lapse of time we observe a return to the initial modes. Such a behavior is "almost" periodic in time.
The propagation of energy to higher modes is connected with an other phenomenon which also occurs when dealing with certain equations with periodic boundary conditions : Benjamin-Feir instability. This new phenomenon is the instability of spatially uniform solutions for perturbations with a certain frequency.
The link between both effects is made clear in [20] by Yuen and Ferguson. Their numerical results tend to prove that a simple recurrence seems to appear only in the case when the higher modes of the perturbation are stable according to a Benjamin-Feir analysis. These observations are carried out in the 1-dimensional case and generalized by the same authors to the 2-dimensional case in [21]. However, this time, they do not produce the link with Benjamin-Feir instability. This link is given in the article [16] by Martin and Yuen. Thanks to a multiple scale method for the time variable, Janssen (cf. [11]) proves the recurrence for certain perturbations of the uniform solution and shows that in such a case the recurrence time is connected with the amplitude in a straightforward way. Using an Ansatz on the form of the solutions (3 Fourier modes) Infeld in [10] shows the recurrence in time of these modes. In [19] Weideman and Herbst use such an approach for this equation but they consider it as schemes and not as Ansatz. Recurrence for the Davey-Stewartson equations has also been studied by K. Rachid using different methods.
In [3], Bourgain shows that the solution to the Korteweg-de Vries equation is almost periodic in time for initial data in L^2 using the theory of Hill's operator with a periodic L^2 potential. The same sort of result has been previously obtained by McKean and Trubowitz [15] for initial data in C^∞.
Lax in [12] constructs some particular solutions to the Korteweg-de Vries equation verifying a minimization problem with constraints. He proves that these solutions are quasi-periodic, i.e. they return to their initial shape up to a translation. Some numerical results due to Hyman correspond to this theoretical result (cf. [12] or [9]).
Our point of view is completely different. The matter is to find invariant measures on Hilbert spaces which are the phase spaces of Hamiltonian systems with at least two conservation laws. Next we use this construction to prove a Poisson's recurrence like theorem. Such a construction has first been carried out by Friedlander [6] for the wave equation with a cubic nonlinearity but some details in the proof seem obscure. It is also Zhidkov's point of view in different articles. In [24], he carries out this study for the equation $iu_t + u_{xx} + f(x, |u|^2)u = 0$ where the nonlinearity is very weak. This study is generalized to the equation $iu_t + u_{xx} + |u|^2u = 0$ in [26], except that the existence result of solutions to this equation in $L^2(0, A)$ can not be deduced

from Tsutsumi's results in [18], who makes use of $L^p(\mathbb{R}) - L^q(\mathbb{R})$ estimates which are not valid in the periodic case. On the other hand, we may now base the proof on a result of Bourgain [2]. In [27], Zhidkov studies in the same way the wave equation $u_{tt} - u_{xx} + f(x,u) = 0$, once more for weak nonlinearities. The article [28] is the insertion of all former results in a wider frame of certain Hamiltonian systems.
The present paper is the enlightenment of this last article as well as the application to new classes of examples, in particular to numerical schemes. In connection with that we give a brief survey of hamiltonian schemes for approximating nonlinear partial differential equations of hamiltonian type. The outline is the following. In section 2, we construct invariant measures for Hamiltonian systems and we prove the Poisson's recurrence. The process we use is exactly the same as Zhidkov's but it's more explicit. Section 3 is devoted to the study of a few fields of application of this general theory. Finally section 4 deals particulary with recurrence in the case of schemes which is the phenomenon one actually observes during numerical simulations.
We may regret the fact that this kind of study does not fulfill the original aim. Indeed we do not really prove recurrence but the fact that the solutions come infinitely often near the initial data (with time intervals which may be not constant). Moreover the notion of proximity to the initial data is not the one we may commonly observe on numerical computations ; we test whether two neighborhoods (one for the initial data and one for the solution at time t) which may have a very complex structure have a nonempty intersection. In return, the results which are proved here may be applied to a far wider class of initial data than perturbations of spatially uniform solutions.
While finishing the drafting of this article we have been informed of some very similar work done by Bourgain (cf. [4]) for the nonlinear Schrödinger equation inspired by the work done by Lebowitz, Rose and Speer [14].

3.2 Construction of invariant measures.

The construction of invariant measures is made up of many steps. The first one consists in associating to the initial equation and the initial functional spaces projected equations on nested finite dimensional functional spaces. In these spaces we construct invariant Gaussian measures for the projected problem. The last step consists in passing to the limit as the dimension of the spaces tend to $+\infty$. This makes up the construction of so-called cylindrical measures on the whole of the phase space X.

3.2.1 Setting the problem.

We study the following Cauchy problem :

$$\begin{cases} \dot{u}(t) = JH'(u(t)), \\ u(t_0) = \phi \in X, \end{cases} \tag{2.1}$$

where X is a Hilbert space such that $\mathcal{D}$ $(= \mathcal{C}_0^\infty)$ is dense in X^*.
Let Y be a Hilbert space which is dense in X, and let us assume that
the functional H is $\mathcal{C}^1$ from Y into $\mathbf{R}$,
the functional J is linear from X^* into X,
for all $g, h \in \mathcal{D}$, $g(Jh) = -h(Jg)$.
These properties imply that $H(u(t))$ does not depend on t. This yields a conservation law in X for the system (2.1). Besides, we assume that X may be endowed with a norm which is as well invariant. This is crucial in order to prove theorem 2.8.
We split H in two parts

$$g(u) = H(u) - \frac{1}{2}(Su, u)_X.$$

The part $g(u)$ has to contain all the nonlinearity of the initial equation and has to be defined for functions belonging to X.
The operator S is assumed to be positive and self-adjoint on X and g defined on X, real valued and continuous.
We assume (H2.0) that we know how to solve the problem (2.1) in X and that the solution is continuous with respect to the initial data, that is : for every $t_0 \in \mathbf{R}$, $\varepsilon > 0$, $T > 0$, there exists $\delta > 0$ such that

$$\|u_1(t_0) - u_2(t_0)\|_X < \delta \Rightarrow \|u_1(t) - u_2(t)\|_X < \varepsilon$$

for all $t \in I = [t_0 - T, t_0 + T]$.
We associate to this problem a sequence of finite dimensional problems. With that aim we construct a sequence of Hilbert subspaces of Y :

$$X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots \subset Y \subset X$$

where $d_n < \infty$ is the dimension of the space X_n and we assume that $\bigcup_n X_n$ is dense in Y.
We denote by P_n the orthogonal projector from X onto X_n, and we obtain a new problem set on this space :

$$\begin{cases} \dot{u}^n(t) = P_n[JH'(P_n u^n(t))], \\ u^n(t_0) = P_n\phi \in X_n. \end{cases} \tag{2.2}$$

As in the case of (2.1), this system admits an invariant which is $H(P_n u^n(t))$.
We assume (H2.1) that $u^n(t)$ exists globally in time for every initial data $\phi \in X$ and that for all $t_0 \in \mathbb{R}$, $\varepsilon > 0$, $T > 0$, there exists $\delta > 0$ such that for all n

$$\|u_1^n(t_0) - u_2^n(t_0)\|_X < \delta \Rightarrow \|u_1^n(t) - u_2^n(t)\|_X < \varepsilon$$

for all $t \in I$. The fact that δ does not depend on n is fundamental in the proof of theorem 2.7.
We impose compatibility conditions over the different problems :
(H2.2) The solution u^n to (2.2) converges to the solution u to (2.1) in $\mathcal{C}(I;X)$ (uniformly with respect to n (cf. theorem 2.7)).
Remark : The two former uniformity properties H2.2 and H2.3 imply the aforementionned property H2.1 of continuity with respect to the initial data for the problem (2.1).
(H2.3) The operator S^{-1} is nuclear (which means for example that the sum of its eigenvalues is convergent) and maps X_n into X_n.
(H2.4) The operator J is defined on X_n^* and $P_n J = J P_n^*$.

3.2.2 Invariant measures in finite dimension.

To begin with, we will construct an invariant measure for each finite dimensional system. The construction is based on the classical Liouville theorem (see for example Arnold [1]) :

Theorem 2.1 (Liouville) *Let us consider the equation $\dot{z} = f(z)$ and $\rho(C) = \int_C \lambda(z)dz$ where λ is a positive function and C is a Borel set of $\mathbb{R}^m$. Then ρ is invariant if and only if $\sum_{i=1}^{m} \frac{\partial}{\partial z_i}(\lambda f_i) = 0$.*

We will now carry out the construction of the invariant measure μ_n on the phase space of the finite dimensional system on X_n.
Let $(e_1, \ldots, e_{d_n})$ be the eigenvectors of S which generate X_n (cf. $S^{-1} : X_n \to X_n$) and let F be a Borel set of $\mathbb{R}^{d_n}$. Then we define the cylindrical set M by :

$$M = \{x \in X / [(x, e_1)_X, \ldots, (x, e_{d_n})_X] \in F\}.$$

Let $\mathcal{A}_n$ be the algebra whose elements are these cylinders and let w_n be the function defined on $\mathcal{A}_n$ by

$$w_n(M) = (2\pi)^{-d_n/2} \prod_{j=1}^{d_n} \lambda_j^{1/2} \int_F e^{-\frac{1}{2}\sum_{j=1}^{d_n} \lambda_j y_j^2} dy,$$

where λ_j is the eigenvalue of S corresponding to the eigenvector e_j.
In that way we do have constructed a measure w_n on $\mathcal{A}_n$.
We set $u^n(t) = \sum_{j=1}^{d_n} a_j(t)e_j$, $a = (a_1, \ldots, a_{d_n})$ and $h(a) = H\left(\sum_{j=1}^{d_n} a_j(t)e_j\right)$.
For every Borel set A of $\mathbf{R}^{d_n}$, we define the measure

$$\mu_n'(A) = (2\pi)^{-d_n/2} \prod_{j=1}^{d_n} \lambda_j^{1/2} \int_A e^{-h(a)} da,$$

and notice that

$$\dot{a}(t) = J\nabla_a h(a).$$

We apply Liouville's theorem with $\lambda(a) = e^{-h(a)}$, which yields the invariance of μ_n' on the Borel sets of $\mathbf{R}^{d_n}$. The inverse change of coordinates implies that μ_n is invariant on $\mathcal{A}_n$ where

$$\mu_n(M) = (2\pi)^{-d_n/2} \prod_{j=1}^{d_n} \lambda_j^{1/2} \int_M e^{-H(u)} du.$$

This ends the construction of invariant measures in finite dimensional spaces.

3.2.3 Invariant measures in infinite dimension.

Let $\mathcal{A} = \bigcup_n \mathcal{A}_n$. We associate to this new algebra $\mathcal{A}$ the minimal Borel σ-algebra $\mathcal{M}$ containing $\mathcal{A}$.
For each M belonging to $\mathcal{M}$, we set $w_n(M) = w_n(M \cap X_n)$. This allows us to extend the measures w_n over the whole of $\mathcal{M}$. Such a construction is licit since $M \cap X_n$ belongs to $\mathcal{M}$. The σ-additivity of the measure is conserved thanks to the following lemma (cf. Dalecky and Fomin [5]) and the fact that w_n is a Gaussian measure with S^{-1} as correlation operator.

Lemma 2.2 *The measure w_n is σ-additive on the algebra $\mathcal{M}$ if and only if S^{-1} is a nuclear operator.*

The proof of this lemma may be found in the appendix B.
We notice that for a fixed element M in $\mathcal{A}$, and from a certain range n, the sequence $w_n(M)$ is constant. We take this value as the value of $w(M)$ and we extend w over the σ-algebra $\mathcal{M}$.

Lemma 2.3 (Zhidkov) *The sequence $\{w_n\}$ is weakly convergent to w in X.*

Proof:
The only possible limit for w_n is w since $w_n(M)$ tends to $w(M)$ for each element M of $\mathcal{A}$ and the extension to $\mathcal{M}$ is unique. There is also only left to prove that the sequence $\{w_n\}$ is weakly compact. For that aim we use Prohorov's theorem.

Theorem 2.4 (Prohorov) *A subset $\mathcal{N}$ of the set of finite positive Borel measures on a complete separable metric space (X, ρ) is precompact if and only if*
(i) there exists $M < \infty$, such that $\nu(X) \leq M$ for all $\nu \in \mathcal{N}$,
(ii) for all $\varepsilon > 0$, there exists a compact set K_ε in X, such that $\nu(X \backslash K_\varepsilon) < \varepsilon$ for all $\nu \in \mathcal{N}$.

We take $\mathcal{N} = \{w_n\}$. It is obvious that for all n, $w_n(X) = 1$, therefore (i) is true.
Concerning (ii), the construction of the compact set K_ε is carried out as follows.
Since S^{-1} is a nuclear operator, $\mathrm{Tr} S^{-1} = \sum \lambda_k^{-1} < \infty$.
There exists a function p defined on $[0, \infty[$ such that $\lim_{x \to +\infty} p(x) = +\infty$ and $\sum_k \lambda_k^{-1} p(\lambda_k) < \infty$.
We set $T = p(S)$ and $Q = S^{-1}T$.
Hence $\mathrm{Tr} Q = \sum_k \lambda_k^{-1} p(\lambda_k)$ ($< +\infty$ according to the assumption).
Let $B_R = \left\{ u \in X / \|T^{1/2}u\|_X \leq R \right\}$ and $\mathcal{B} = \overline{B_R}^X$.
Let $\psi_m = \sum_k a_k^m \varphi_k$, be a sequence of elements of B_R, φ_k denoting the eigenvectors of S.
Now $T^{1/2}\psi_m = \sum_k p(\lambda_k)^{1/2} a_k^m \varphi_k$ and $|T^{1/2}\psi_m|_X = |\sum_k p(\lambda_k)(a_k^m)^2|^{1/2} \leq R$.
In order to prove the compactness of $\mathcal{B}$, we only need to show that it is possible to make an estimate of the remainders of the series ψ_m which is uniform in m.
Indeed $\sum_{k=K}^{+\infty} |a_k^m|^2 = \sum_{k=K}^{+\infty} \frac{p(\lambda_k)(a_k^m)^2}{p(\lambda_k)}$.
We choose K such that for all $k \geq K$, $\frac{1}{p(\lambda_k)} \leq \frac{R}{\varepsilon}$.
This shows that for all ε, there exists an integer K such that for all m

$$\left| \sum_{k=K}^{+\infty} a_k^m \varphi_k \right|_X^2 \leq \varepsilon.$$

Then $\mathcal{B}$ is compact for all R.
Now, (cf. Lemma B.2 in appendix B) for a Gaussian measure μ_B with B as correlation operator we have $\mu_B\{x : (Ax, x)_X \geq 1\} \leq \mathrm{Tr} AB$.
We take $B = P_n S^{-1}$ and $A = T$, hence

$$\begin{aligned} w_n\{x/(Tu, u)_X \geq 1\} &\leq \mathrm{Tr} P_n S^{-1} T \leq \mathrm{Tr} Q, \\ w_n(X \backslash B) \leq w_n(X \backslash B_R) = w_n\{x/(Tu, u)_X \geq R\} &\leq \frac{\mathrm{Tr} Q}{R^2}: \end{aligned}$$

Therefore $w_n(X\backslash B) \leq \dfrac{\mathrm{Tr}Q}{R^2}$.

For each ε, we set $R = \sqrt{\dfrac{\mathrm{Tr}Q}{\varepsilon}}$ and $K_\varepsilon = B$.

Consequently and thanks to Prohorov's theorem, w_n is weakly compact and w_n tends weakly to the only possible limit, w. ∎

As in the case of w_n, we set $\mu_n(M) = \mu_n(M \cap X_n)$, for each element M of $\mathcal{M}$. For each Borel set Ω of X, let us set

$$\mu(\Omega) = \int_\Omega e^{-g(u)} w(du).$$

Then we get the following result

Lemma 2.5 *Let Ω be an opened set of X such that $\mu(\Omega) < \infty$. Then*

$$\liminf_{n\to\infty} \mu_n(\Omega) \geq \mu(\Omega).$$

Proof:

Let Ω be an open Borel set of X, then for every ε, there exists a function θ defined on X, with $0 \leq \theta(u) \leq 1$, such that

$$\int_\Omega \theta(u) e^{-g(u)} w(du) \geq \mu(\Omega) - \varepsilon.$$

Let us set $M = \Omega \cap X_n$ and let F be the Borel set of $\mathbf{R}^{d_n}$ which is associated to M.

$$\begin{aligned}
\mu_n(\Omega) &= (2\pi)^{-d_n/2} \prod_{j=1}^{d_n} \lambda_j^{1/2} \int_F e^{-h(y)} dy, \\
&= \int_F e^{-g(\sum_{j=1}^{d_n} y_j e_j)} (2\pi)^{-d_n/2} \prod_{j=1}^{d_n} \lambda_j^{1/2} e^{-\frac{1}{2}\lambda_j y_j^2} dy, \\
&= \int_\Omega e^{-g(u)} w_n(du).
\end{aligned}$$

$$\begin{aligned}
\liminf_{n\to\infty} \mu_n(\Omega) &= \liminf_{n\to\infty} \int_\Omega e^{-g(u)} w_n(du), \\
&\geq \liminf_{n\to\infty} \int_\Omega \theta(u) e^{-g(u)} w_n(du), \\
&= \int_\Omega \theta(u) e^{-g(u)} w(du), \\
&\geq \mu(\Omega) - \varepsilon.
\end{aligned}$$

Letting ε tend to 0, we obtain $\liminf\limits_{n\to\infty} \mu_n(\Omega) \geq \mu(\Omega)$. ∎

Corollary 2.6 *Let Φ be a closed set of X. Then*

$$\limsup_{n\to\infty} \mu_n(\Phi) \leq \mu(\Phi).$$

We denote

$f(\phi, t) = u(t + t_0)$ where u is the solution to (2.1),
$f_n(\phi, t) = u^n(t + t_0)$ where u^n is the solution to (2.2).

Theorem 2.7 *Let Ω be an opened set of X and $\Omega_t = f(\Omega, t)$. Then $\mu(\Omega) = \mu(\Omega_t)$.*

Proof:
The continuity assumption with respect to the initial data yields that Ω_t is as well an open set of X. We only make the proof in the case when $\mu(\Omega)$ and $\mu(\Omega_t)$ are assumed to be finite. Then for all $\varepsilon > 0$, there exists a compact set K such that $\mu(\Omega \backslash K) \leq \varepsilon$. It is obvious that $K_t \subset \Omega_t$ is compact.
Let $\alpha = \min\{\mathrm{dist}(K, \partial\Omega); \mathrm{dist}(K_t, \partial\Omega_t)\}$. For each element u in K, there exists an open ball $B(u)$ with center $u \in \Omega$ such that $\mathrm{dist}(f_n(u,t); f_n(v,t)) < \frac{\alpha}{3}$, for all $v \in B(u)$ and for all n according to the continuity assumption with respect to the initial data for the problem (2.2).
We set $\Omega_\beta = \{v \in \Omega_t / \mathrm{dist}(v, \partial\Omega_t) \geq \beta\}$, and we choose a finite covering $B(u_1)$, $\ldots$,$B(u_l)$ of K. We set $D = \bigcup_{i=1}^{l} B(u_i)$. Since $u^n(t)$ converges uniformly with respect to n to $u(t)$, $f_n(D, t) \subset \Omega_{\frac{\alpha}{4}}$ for a sufficiently large n.

$$\begin{aligned}
\mu(\Omega) &\leq \mu(D) + \varepsilon, \\
&\leq \liminf_{n\to\infty} \mu_n(D) + \varepsilon, \\
&\leq \liminf_{n\to\infty} \mu_n(D \cap X_n) + \varepsilon, \\
&\leq \liminf_{n\to\infty} \mu_n(f_n(D \cap X_n, t)) + \varepsilon, \\
&\leq \limsup_{n\to\infty} \mu_n(\Omega_{\frac{\alpha}{4}}) + \varepsilon, \\
&\leq \mu(\Omega_t) + \varepsilon.
\end{aligned}$$

Hence $\mu(\Omega) \leq \mu(\Omega_t)$, and since time has no privileged direction, $\mu(\Omega) = \mu(\Omega_t)$. ∎

3.2.4 Poisson's Recurrence.

Theorem 2.8 *For almost every initial data ϕ, the trajectory $f(\phi, t)$ is Poisson recurrent.*

Proof:
Thanks to theorem 2.7, for each neighborhood $\mathcal{V}$ of the initial data, $\mu(\mathcal{V}) = \mu(\mathcal{V}_t)$.
Let $B(x_i, \varepsilon)$ be a finite covering of the phase space which is weakly compact thanks to the first conserved quantity. There exists an increasing sequence t_n tending to infinity and x_i such that $\mu(\mathcal{V}_{t_n} \cap B(x_i, \varepsilon)) > 0$.
Let K be a compact subset of the union of the $\mathcal{V}_{t_n}$. There exists a finite sequence t_{n_i} such that $\{\mathcal{V}_{t_{n_i}}\}$ is a finite covering of K. Hence there exists an index j and a subsequence t'_n of t_n such that $\mu(\mathcal{V}_{t_{n_j}} \cap \mathcal{V}_{t'_n}) > 0$. Let us set $\tilde{t}_n = t'_n - t_{n_j}$, this defines a sequence which tends to infinity and such that $\mu(\mathcal{V} \cap \mathcal{V}_{\tilde{t}_n}) > 0$. So the solution comes for almost every initial data infinitely often near its initial value. ∎

Remark. This last argument is still valid in the case of schemes with a time discretisation (cf. section 3.4).
This last theorem is the only one which uses an invariant quantity in X. The construction of measures holds even if this condition is not fulfilled. Theorem 2.8 is also valid in the case where the total measure is finite ($\mu(X) < \infty$).

3.3 Applications.

Different classical equations may be written in the form (2.1). This is for example the case of the Schrödinger or the wave equations.
These applications are nevertheless limited by the following facts :
* The nonlinearity of the equation has to be defined for functions belonging to X.
* The space X endowed by the norm which is the invariant defined for the less regular functions has to be a Banach space.
* The operator S^{-1} has to be nuclear.
Each of the following examples will be explained in two steps :
* The setting up of the equation into an Hamiltonian form. We determine explicitly the invariants of the equation, the spaces X and Y, the operators J and S as well as the functional g.
In this section we will have to check the following hypothesis :
(H3.5) X and Y are Hilbert spaces, and we have appropriate density results,
(H3.6) $H : Y \to \mathbf{R}$ is C^1,
(H3.7) $J : X^* \to X$ is linear and skew-adjoint,
(H3.8) there is at least one invariant norm,
(H3.9) $g(u)$ is continuous and defined on X,
(H3.10) $S > 0$ is self-adjoint,
(H3.11) S^{-1} is nuclear,
(H3.12) $J : X_n^* \to X_n$ and $P_n J = J P_n$;
* the testing of the remaining hypotheses, that is :
(H3.13) there exist solutions in X and X_n,
(H3.14) these solutions are continuous with respect to the initial data.

3.3.1 Setting up.

The nonlinear Schrödinger equation.
We consider the problem

$$\begin{cases} iu_t + \Delta u + f(x, |u|^2)u = 0, & x \in (0, A), t \in \mathbf{R}, \\ u(0,t) = u(A,t), \\ u(x, t_0) = u_0(x). \end{cases} \tag{3.1}$$

We transform this problem setting $u = (u_1, u_2)$.
The partial differential equation becomes

$$\begin{cases} u_{1t} + \Delta u_2 + f(x, (u_1)^2 + (u_2)^2)u_2 &= 0, \\ u_{2t} - \Delta u_1 - f(x, (u_1)^2 + (u_2)^2)u_1 &= 0. \end{cases}$$

We set $F(x, s) = \dfrac{1}{2} \displaystyle\int_0^s f(x, \sigma) d\sigma$. We know two invariants for this equation

$$E(u_1, u_2) = \frac{1}{2} \int_0^A \left\{ (u_1)^2 + (u_2)^2 \right\} dx,$$

$$H(u_1,u_2)=\int_0^A\left\{\frac{1}{2}((\nabla u_1)^2+(\nabla u_2)^2)-F(x,(u_1)^2+(u_2)^2)\right\}dx.$$

The gradient of this last invariant is :

$$H'(u_1,u_2)=\begin{pmatrix}-\Delta u_1+f(x,(u_1)^2+(u_2)^2)u_1\\ -\Delta u_2+f(x,(u_1)^2+(u_2)^2)u_2\end{pmatrix}.$$

The functional spaces we will consider are $X=L^2\times L^2$ and $Y=H^1\times H^1$, the operators J et S being respectively equal to

$$J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\qquad\text{and}\qquad S=\begin{pmatrix}-\Delta & 0\\ 0 & -\Delta\end{pmatrix}.$$

This allows us to compute

$$g(u_1,u_2)=-\int_0^A F(x,(u_1)^2+(u_2)^2)dx.$$

The wave equation.
We consider the problem

$$\begin{cases}u_{tt}-u_{xx}+f(x,u)=0, & x\in(0,A),t\in\mathbf{R},\\ u(0,t)=u(A,t), & \\ u(x,t_0)=u_0(x), & u_t(x,t_0)=v_0(x)\end{cases}\tag{3.2}$$

We transform this problem setting $v=u_t$.
The partial differential equation becomes

$$\begin{cases}u_t-v &= 0,\\ v_t-u_{xx}+f(x,u) &= 0.\end{cases}$$

We set $F(x,u)=\frac{1}{2}\int_0^u f(x,s)ds$, and we may find an invariant in the form :

$$H(u,v)=\int_0^A\left\{\frac{1}{2}((v)^2+(u_x)^2)+F(x,u)\right\}dx.$$

The computation of its gradient yields :

$$H'(u,v)=\begin{pmatrix}-\Delta u+f(x,u)\\ v\end{pmatrix}.$$

We consider the functional spaces $X=L^2\times H^{-1}$ and $Y=H^1\times L^2$. The operators J and S are respectively equal to

$$J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\qquad\text{and}\qquad S=\begin{pmatrix}-\Delta & 0\\ 0 & -\Delta\end{pmatrix}.$$

The computation of g gives

$$g(u,v)=\int_0^A F(x,u)dx.$$

3.3.2 Testing the hypotheses.

The nonlinear Schrödinger equation.
The global existence of solutions to (2.1) for an initial data in L^2 is given in an article by Bourgain [2] and the conditions that should be imposed on the nonlinearity f are studied in the appendix A. Concerning the finite dimensional problem (2.2), it may be written in the form

$$\dot{u}^n = P_n \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} -\Delta u_1^n + f(x,(u_1^n)^2 + (u_2^n)^2)u_1^n \\ -\Delta u_2^n + f(x,(u_1^n)^2 + (u_2^n)^2)u_2^n \end{pmatrix}.$$

Since X_n is supposed to be stable with respect to S, the operator representing the linear part is the same one and the estimates obtained for the continuous case are still valid. The assumption on the nonlinear part still holds. Hence there exists a unique global solution.
The continuity with respect to the initial data may be studied in a classical way thanks to estimates which are analogous to those of the proof for the local existence. The nonlinearities we choose here verify for example

$$\|f(x,|u|^2)\|_2 \leq C \sum_\gamma \|u\|_4^\gamma$$

$$\text{and } \|\nabla_s f(x,|u|^2)\|_4 \leq C \sum_\eta \sup_{u \in B(0,M)} \|u\|_4^\eta \text{ when } u \in B(0,M),$$

the sum over γ and η dealing with a finite number of terms with $0 \leq \gamma$ and $-1 \leq \eta$. These conditions are weaker than those initialy chosen by Zhidkov (cf. [28]), that is

$$|f(x,s)| + |(1+s)\nabla_s f(x,s)| < C \text{ for all } x, s.$$

He also carries out the application of the method to the cubic nonlinear Schrödinger equation. In the present paper our aim is to find the weakest assumptions under which the construction is possible. The method used by Zhidkov for the cubic Schrödinger equation is slighlty different. He defines a sequence of Schrödinger equations with very weak nonlinearities which tend in a certain sense to the cubic nonlinearity. He constructs an invariant measure for each of these equations and then passes to the limit over the measures. This yields an invariant measure for the cubic Schrödinger equation.

The wave equation.
Here we will keep Zhidkov's hypotheses on the nonlinearity, that is

$$f(x,u)| \leq C(1+u^2)^{1/2}$$

and

$$|\partial_u f(x,u)| \leq C.$$

We may remark that these conditions are non fulfiled in the case of the cubic wave equation which is the case studied by Friedlander [6].
We refer to Zhidkov's article [27] for the proof of the wellposedness for this this equation. The wellposedness for the discretized equation is made with the same type of arguments than in the case of the nonlinear Schrödinger equation.

3.3.3 Why this approach fails in the Korteweg-de Vries equation case.

In [28] Zhidkov treats the case of a Korteweg-de Vries equation, but it is a linear one :

$$u_t + (a(x)u)_x + u_{xxx} = 0.$$

This induces us to study the usual Korteweg-de Vries equation. We will show why it's is impossible to apply the general theory to this problem. We will also consider the equation

$$\begin{cases} u_t + u_{xxx} + u^k u_x = 0, & x \in (0, A), t \in \mathbf{R}, \\ u(0,t) = u(A,t), & \\ u(x,t_0) = u_0(x). & \end{cases} \tag{3.3}$$

This equation has the following invariants :

$$H_0(u) = \int_0^A u dx,$$

$$H_1(u) = \frac{1}{2}\int_0^A u^2 dx,$$

$$H_2(u) = \int_0^A \left\{\frac{1}{2}u_x^2 - \frac{1}{(k+1)(k+2)}u^{k+2}\right\} dx,$$

and in the case when $k = 1$, there are some additional invariants, the first one being

$$H_3(u) = \int_0^A \left\{\frac{1}{2}u_{xx}^2 - \frac{5}{6}uu_x^2 + \frac{5}{72}u^4\right\} dx.$$

Their gradients are respectively equal to

$$H_1'(u) = u,$$

$$H_2'(u) = -u_{xx} - \frac{1}{(k+1)}u^{k+1},$$

$$H_3'(u) = u_{xxxx} + \frac{5}{6}u_x^2 + \frac{5}{2}uu_{xx} + \frac{5}{18}u^3.$$

The following problem faces us. The nonlinear part imposes X to be included in H^1. Since we dispose of various invariants we could take H_3 as Hamiltonian (in the case $k = 1$), unfortunately it is not possible to set the Korteweg-de Vries equation in a Hamiltonian form with this invariant (but it is possible for H_2 taking $J_2 = -\partial_x$, $S_2 = -\Delta$. and $g_2(u) = -\int_0^A \frac{1}{(k+1)(k+2)}|u|^{k+2}dx$). It seems also impossible to apply this method to this equation.

3.4 Particular case of numerical schemes.

In the case of schemes, only the finite dimensional construction is useful. The choice of spaces becomes now indifferent since all norms are equivalent. This solves also the limitations on possible nonlinearities and the operator S^{-1} becomes necessarily nuclear. In the case when S is singular, we have to consider its reduction to the orthogonal space of its kernel and choose a nonlinearity such that the dynamical systems preserves this subspace. In the examples presented here S is a discretization of the Laplacian which happens to be singular in the periodic case (the kernel is the constants) and nonsingular in the zero boundary case.
We will now consider equations with two invariants and try to find a discretization which conserves analogous invariants. That way we may hope that the observed phenomena which are connected to the existence of invariants for the discretized equation may extend to the continuous case. The case of semi-discretizations in space by a Galerkin method in bases of eigenvectors of S has already been treated in the theoretical part (cf. section 3.2.2). We will produce here other types of discretizations.

3.4.1 The nonlinear Schrödinger equation.

We remind that the cubic Schrödinger equation

$$u_t = iu_{xx} + iq|u|^2u, \tag{4.1}$$

considered on the interval $[-\frac{L}{2}, \frac{L}{2}]$ with periodic boundary conditions admits two classical invariants :

$$\frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} |u|^2 dx = c_1, \tag{4.2}$$

$$\frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} (|u_x|^2 - \frac{1}{2}q|u|^4) dx = c_2. \tag{4.3}$$

Space discretizations.

In what follows we will denote by δ the forward-space derivative

$$\delta U_j = h^{-1}(U_{j+1} - U_j), \tag{4.4}$$

and δ^2 the second central-space derivative

$$\delta^2 U_j = h^{-2}(U_{j+1} - 2U_j + U_{j-1}). \tag{4.5}$$

The indexes for the space variable will be j, for the Fourier variable k and the superscript for the time variable will be n.

We discretise $[-\frac{L}{2}, \frac{L}{2}]$ in K equal intervals. Let $h = \dfrac{L}{K}$.
We consider here three types of space discretizations : finite difference schemes, spectral schemes and pseudo-spectral schemes.
Finite difference scheme.
Let $U_j(t)$ be an approximation of $u(-\frac{L}{2} + (j-1)h, t)$. The most classical finite difference scheme is given by

$$\dot{U}_j = i\delta^2 U_j + iq|U_j|^2 U_j. \tag{4.6}$$

Spectral scheme.
We make the following Ansatz : u is reduced to a finite number of frequences, i.e. it may be written in the form

$$u(x,t) = \sum_{-K/2}^{K/2} A_k(t)\exp(i\mu_k x).$$

Hence we obtain

$$\dot{A}_k = -i\mu_k^2 A_k + iq \sum_{l_1+l_2-l_3=k} A_{l_1} A_{l_2} A_{l_3}^*. \tag{4.7}$$

Pseudo-spectral scheme.
It is based on the fact that we may use a FFT to solve the problem numerically. Therefore we define an analogue of the Fourier transform

$$F_k U_j = A_k = \frac{1}{K} \sum_{-K/2}^{(K/2)-1} U_j \exp(-i\mu_k x_j),$$

and of the inverse Fourier transform

$$F_j^{-1} A_k = U_j = \sum_{-K/2}^{(K/2)-1} A_k \exp(i\mu_k x_j).$$

The equation may also be discretised using

$$\dot{A}_k = -i\mu_k^2 A_k + iqF_k(U_j|U_j|^2) \tag{4.8}$$

or equivalently

$$\dot{U}_j = -iF_k^{-1}(\mu_k^2 F_k(U_j)) + iq|U_j|^2 U_j. \tag{4.9}$$

A variant of the finite difference scheme is the following (cf. Herbst and Ablowitz [8]).
Integrable scheme.

$$\dot{U}_j = i\delta^2 U_j + i\frac{1}{2}q|U_j|^2(U_{j+1} + U_{j-1}). \tag{4.10}$$

Analysis of space discretizations.

Finding two conservation laws allows the use of the general theory for these schemes. Each of the three first schemes have two invariant quantities which are similar to those of equation (4.1). The setting in an hamiltonian form according to the notations of section 3.2 is very similar to the continuous case.

Finite difference scheme.

$$\frac{h}{2}\sum_{-K/2}^{(K/2)-1}|U_j|^2 = C_{1,1},$$

$$\frac{h}{2}\sum_{-K/2}^{(K/2)-1}\left\{|\delta U_j|^2 - \frac{1}{2}q|U_j|^4\right\} = C_{1,2}.$$

We set $U = (V_{-K/2},\dots,V_{(K/2)-1},W_{-K/2},\dots,W_{(K/2)-1})^T$ where $U_j = V_j + iW_j$. The scheme (4.6) becomes

$$\begin{cases} \dot V_j = -\delta^2 W_j - q(V_j^2+W_j^2)W_j, \\ \dot W_j = \delta^2 V_j + q(V_j^2+W_j^2)V_j. \end{cases}$$

Then $J = \begin{pmatrix} 0 & hI_K \\ -hI_K & 0 \end{pmatrix}$ and $S = \begin{pmatrix} -D_K^2 & 0 \\ 0 & -D_K^2 \end{pmatrix}$

where $D_K^2 = \dfrac{1}{h^2}\begin{pmatrix} -2 & 1 & & 1 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & -2 \end{pmatrix}$

and $H'(U) = (\dots,\delta^2 V_j + q(V_j^2+W_j^2)V_j,\dots,\delta^2 W_j + q(V_j^2+W_j^2)W_j,\dots)^T$.

Spectral scheme.

$$\frac{L}{2}\sum_{-K/2}^{K/2}|A_k|^2 = C_{2,1},$$

$$\frac{L}{2}\left\{\sum_{-K/2}^{K/2}\mu_k^2|A_k|^2 - \frac{1}{2}q\sum_{-K/2\le l_1,l_2,l_3\le K/2} A_{l_1}A_{l_2}A_{l_3}^*A_{l_1+l_2-l_3}^*\right\} = C_{2,2}.$$

Pseudo-spectral scheme.

$$\frac{h}{2}\sum_{-K/2}^{(K/2)-1}|U_j|^2 = C_{3,1},$$

$$\frac{L}{2}\sum_{k=-K/2}^{(K/2)-1}\mu_k^2|A_k|^2 - \frac{1}{4}qh\sum_{j=-K/2}^{(K/2)-1}|U_j|^4 = C_{3,2}.$$

Integrable scheme.

In the case of the integrable scheme we may also find two invariants but the second one does not seem to have a counterpart at the level of the continuous equation :

$$\frac{h}{2}\sum_{-K/2}^{(K/2)-1}|U_j|^2 = C_{4,1},$$

$$\frac{h}{2}\sum_{-K/2}^{(K/2)-1}\left\{U_j^*(U_{j+1}+U_{j-1}) - 4q^{-1}\ln(1+\frac{1}{2}q|U_j|^2)\right\} = C_{4,2}.$$

Remark : An other proof of recurrence.
A direct and straightforward proof of the periodicity of these three schemes for a small number of modes as well as the expression of the recurrence time thanks to an elliptic integral may be found in the article of Weideman and Herbst [19].

A full discretization.

For the effective computation of the former schemes we have to choose a time discretization which has to conserve the different quantities. We may use the mid-point scheme which is a finite difference scheme in both the time and the space variables :

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} = i\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2h^2} + i\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{2h^2}$$
$$+ i\frac{q}{4}\left(|u_j^n|^2 + |u_j^{n+1}|^2\right)\left(u_j^n + u_j^{n+1}\right). \tag{4.11}$$

Analysis of the full discretization.

Two quantities are conserved

$$\frac{h}{2}\sum_{-K/2}^{(K/2)-1}|u_j^n|^2 = \frac{h}{2}\sum_{-K/2}^{(K/2)-1}|u_j^{n+1}|^2,$$

$$\frac{h}{2}\sum_{-K/2}^{(K/2)-1}\left(|\delta u_j^n|^2 - \frac{1}{2}q|u_j^n|^4\right) = \frac{h}{2}\sum_{-K/2}^{(K/2)-1}\left(|\delta u_j^{n+1}|^2 - \frac{1}{2}q|u_j^{n+1}|^4\right).$$

Recurrence has been actually observed for this scheme but it seems nevertheless impossible to apply the above theory to such full discretized schemes, indeed setting for example $U^n = (v_{-K/2}^n, \dots, v_{(K/2)-1}^n, w_{-K/2}^n, \dots, w_{(K/2)-1}^n)^T$ where $u_j^n = v_j^n + iw_j^n$ we may use the same operator S as in the space discretization case. On the other hand it is not possible to find an operator J satisfying

$$\frac{U^{n+1}-U^n}{\Delta t} = JH'(U^{n+1})$$

This is the only restriction since we noticed (cf. the end of section 3.2) that Theorem 2.8 also holds for schemes with a time discretization.

3.4.2 The wave equation.

A space discretization.

Let us consider a nonlinear wave equation in the form :

$$u_{tt} - u_{xx} + f(u) = 0. \tag{4.12}$$

A space finite difference discretization is possible :

$$\ddot{u}_j - \delta^2 u_j + f(u_j) = 0. \tag{4.13}$$

Setting $F(s) = \frac{1}{2}\int_0^s f(\sigma)d\sigma$, we obtain the following invariant quantity for (4.13) :

$$\sum_{-K/2}^{(K/2)-1} \left\{ \frac{1}{2}\left(|\dot{u}_j|^2 + |\delta u_j|^2 \right) + F(u_j) \right\} = C.$$

Let $v_j = \dot{u}_j$,
setting $U = (u_{-K/2}, \dots, u_{(K/2)-1})^T$ and $V = (v_{-K/2}, \dots, v_{(K/2)-1})^T$, we set the discretized wave equation in the the hamiltonian form using

$$H(U,V) = -\frac{1}{2}(D_K^2 U, U) + \frac{1}{2}(V,V) + F(U)$$

$$H'(U,V) = \begin{pmatrix} -D_K^2 U + f(U) \\ V \end{pmatrix}, \; J = \begin{pmatrix} 0 & hI_K \\ -hI_K & 0 \end{pmatrix}, \; S = \begin{pmatrix} -D_K^2 & 0 \\ 0 & I_K \end{pmatrix}.$$

A full discretization.

To obtain an invariant for a full finite difference scheme for the wave equation, we have to restrict ourselves once more to a smaller family of nonlinearities that is $f(u) = Au^{2m-1}$. Then we use for example the scheme :

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} - \frac{1}{2}\delta^2 u_j^{n+1} - \frac{1}{2}\delta^2 u_j^{n-1} + \frac{A}{2m}\sum_{l=0}^{2m-1} (u_j^{n+1})^{2m-1-l}(u_j^{n-1})^l = 0. \tag{4.14}$$

For this scheme the invariant quantity is :

$$\sum_{-K/2}^{(K/2)-1} \left\{ \frac{|u_j^{n-1} - u_j^n|^2}{\Delta t^2} + \frac{1}{2}(|\delta u_j^{n+1}|^2 + |\delta u_j^n|^2) + \frac{A}{2m}((u_j^{n+1})^{2m} + (u_j^n)^{2m}) \right\} = C.$$

In this case we are facing the same problem for the setting in an hamiltonian form as in the case of the full discretization of the cubic nonlinear Schrödinger equation.

3.4.3 The Zakharov equations.

The continuous equations.

In the context of plasma physics, Zakharov [23] has introduced the following system

$$\begin{cases} iE_t + E_{xx} = NE, \\ N_{tt} - N_{xx} = \dfrac{\partial^2}{\partial x^2}(|E|^2). \end{cases} \tag{4.15}$$

Setting $M = N_t$ and $\sqrt{2}E = F + iG$, we obtain the following formulation.

$$\begin{cases} F_t = -G_{xx} + NG, \\ G_t = F_{xx} - NF, \\ N_t = M, \\ M_t = N_{xx} + \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}(F^2 + G^2). \end{cases} \tag{4.16}$$

Considering these equations with periodic boundary conditions on the interval $[-\frac{L}{2}, \frac{L}{2}]$ and setting $v = -u_x$ where $u_{xx} = N_t$, we formally get two invariant quantities

$$\frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} (F^2 + G^2)dx = c_1, \tag{4.17}$$

$$\frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} \left((F_x)^2 + (G_x)^2 + v^2 + N^2 + N(F^2 + G^2)\right) dx = c_2. \tag{4.18}$$

Making an attempt at setting (formally) this system in an hamiltonian form with the notations of section 3.2 we set :

$$X = L^2 \times L^2 \times H^{-1} \times H^{-2}$$

and

$$Y = H^1 \times H^1 \times L^2 \times H^{-1},$$

$$S = \begin{pmatrix} -\Delta & 0 & 0 & 0 \\ 0 & -\Delta & 0 & 0 \\ 0 & 0 & -\Delta & 0 \\ 0 & 0 & 0 & -\Delta \end{pmatrix} \quad \text{and} \quad u = (F, G, N, M)$$

and we get

$$\frac{1}{2}(Su, u)_X = \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} \left((F_x)^2 + (G_x)^2 + v^2 + N^2\right) dx$$

and thus

$$g(u) = \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} N(F^2 + G^2)dx.$$

With $J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_{xx} \\ 0 & 0 & \partial_{xx} & 0 \end{pmatrix}$ and $H'(u) = \begin{pmatrix} -F_{xx} + NF \\ -G_{xx} + NG \\ N + \frac{1}{2}(F^2 + G^2) \\ -\int_0^x (\int_0^\xi M(\zeta)d\zeta)d\xi \end{pmatrix}$,

the system (4.16) is written in an hamiltonian form with the second invariant (4.18) as hamiltonian.
The hope of getting a result for this continuous equation in such a large space as X is very small, but we can study numerical schemes.

A space discretization.

Using notations (4.4) and (4.5) for the space derivative as for the previous finite difference schemes, it is possible to discretize (4.15) in the following way.

$$\begin{cases} i\dot{E}_j + \delta^2 E_j &= N_j E_j, \\ \ddot{N}_j - \delta^2 N_j &= \delta^2(|E_j|^2), \end{cases} \tag{4.19}$$

or setting $\sqrt{2}E_j = F_j + iG_j$ and $M_j = \dot{N}_j$,

$$\begin{cases} \dot{F}_j &= -\delta^2 G_j + N_j G_j, \\ \dot{G}_j &= \delta^2 F_j - N_j F_j, \\ \dot{N}_j &= M_j, \\ \dot{M}_j &= \delta^2 N_j + \frac{1}{2}\delta^2(F_j^2 + G_j^2). \end{cases} \tag{4.20}$$

It is possible to find two invariants for this scheme, namely

$$\frac{h}{2}\sum_{-K/2}^{(K/2)-1} \left\{F_j^2 + G_j^2\right\} = C_1, \tag{4.21}$$

$$\frac{h}{2}\sum_{-K/2}^{(K/2)-1} \left\{(\delta F_j)^2 + (\delta G_j)^2 + (\delta u_j)^2 + N_j^2 + N_j(F_j^2 + G_j^2)\right\} = C_1, \tag{4.22}$$

where $\delta^2 u_j = \dot{N}_j$. The setting into an hamiltonian form is made through the following notations : we set $U = (\dots, F_j, \dots, G_j, \dots, N_j, \dots, M_j, \dots)$ and

$$H(U) = \frac{1}{2}\{(-(D_K^2 F, F) - (D_K^2 G, G) + (D_K^2 u, u) - (\mathcal{G}M, M) +$$

$$+(N, N) + h\sum_j N_j(F_j^2 + G_j^2)\},$$

$$S = \begin{pmatrix} -D_K^2 & 0 & 0 & 0 \\ 0 & -D_K^2 & 0 & 0 \\ 0 & 0 & I_K & 0 \\ 0 & 0 & 0 & \mathcal{G} \end{pmatrix}$$

where

$$\mathcal{G}_{ij} = \begin{cases} x_i(1 - \dfrac{x_j}{L}) & \text{si} \quad x_i \leq x_j, \\ \\ x_j(1 - \dfrac{x_i}{L}) & \text{si} \quad x_j \leq x_i. \end{cases}$$

An the hamiltonian form is obtained with :

$$H'(U) = \begin{pmatrix} -D_K^2 F + N.F \\ -D_K^2 G + N.G \\ N + \frac{1}{2}(F.F + G.G) \\ -\mathcal{G}M \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_K & 0 & 0 \\ -I_K & 0 & 0 & 0 \\ 0 & 0 & 0 & -D_K^2 \\ 0 & 0 & D_K^2 & 0 \end{pmatrix},$$

where $u.v$ denotes the vector whose components are the $u_j v_j$.

A full discretization.

The following full discretization of the Zakharov equations is due to Glassey [7].

$$\begin{cases} i\dfrac{E_j^{n+1} - E_j^n}{\Delta t} + \dfrac{1}{2}\delta^2 E_j^n + \dfrac{1}{2}\delta^2 E_j^{n+1} = \dfrac{1}{4}(N_j^n + N_j^{n+1})(E_j^n + E_j^{n+1}), \\ \\ \dfrac{N_j^{n+1} - 2N_j^n + N_j^{n-1}}{\Delta t^2} - \dfrac{1}{2}\delta^2 N_j^{n+1} - \dfrac{1}{2}\delta^2 N_j^{n-1} = \delta^2 \left(|E_j^n|^2\right). \end{cases} \tag{4.23}$$

As in the nonlinear Schrödinger case, we want to write this scheme in an hamiltonian form. With this aim we set as in the continuous case $\sqrt{2}E_j^n = F_j^n + iG_j^n$ and $M_j^n = \dfrac{N_j^n - N_j^{n-1}}{\Delta t}$ which yields the new scheme :

$$\begin{cases} \dfrac{F_j^{n+1} - F_j^n}{\Delta t} = -\frac{1}{2}\delta^2 G_j^n - \frac{1}{2}\delta^2 G_j^{n+1} + \frac{1}{4}(N_j^n + N_j^{n+1})(G_j^n + G_j^{n+1}), \\ \\ \dfrac{G_j^{n+1} - G_j^n}{\Delta t} = \frac{1}{2}\delta^2 F_j^n + \frac{1}{2}\delta^2 F_j^{n+1} - \frac{1}{4}(N_j^n + N_j^{n+1})(F_j^n + F_j^{n+1}), \\ \\ \dfrac{N_j^{n+1} - N_j^n}{\Delta t} = M_j^{n+1}, \\ \\ \dfrac{M_j^{n+1} - M_j^n}{\Delta t} = \frac{1}{2}\delta^2 N_j^{n+1} + \frac{1}{2}\delta^2 N_j^{n-1} + \frac{1}{2}\delta^2 \left((F_j^n)^2 + (G_j^n)^2\right). \end{cases} \tag{4.24}$$

Glassey proves that for initial data M_j^1 such that $\sum_{j=-K/2}^{(K/2)-1} M_j^1 = 0$ (zero mean) the numerical scheme is well posed at each step.
For this scheme we have similar invariants to (4.17) and (4.18) :

$$\frac{h}{2} \sum_{-K/2}^{(K/2)-1} ((F_j^n)^2 + (G_j^n)^2) = C_1, \tag{4.25}$$

$$\frac{h}{2}\sum_{j=-K/2}^{(K/2)-1}\Big\{(\delta F_j^{n+1})^2+(\delta G_j^{n+1})^2+(\delta u_j^n)^2+\frac{1}{2}((N_j^n)^2+(N_j^{n+1})^2)$$
$$+\frac{1}{2}(N_j^n+N_j^{n+1})((F_j^{n+1})^2+(G_j^n)^2)\Big\}=C_2. \qquad (4.26)$$

where $\delta^2 u_j^n = \dfrac{N_j^{n+1}-N_j^n}{\Delta t}$.

3.4.4 The Korteweg-de Vries equation.

We do not know of any finite difference scheme for the Korteweg-de Vries equation with two laws of conservation corresponding to H_1 and H_2. In return, problems connected with the nature of the nonlinearity are solved.
As far as we know, the only numerical method which allows a large number of conservation laws is the one developped by Hyman (cf. [9]). This method is based on the work of Lax [12] who shows that a certain class of solutions to the Korteweg-de Vries equation are solutions to a minimization problem. The main idea is to minimize the N-th invariant quantity under constraints which are the previous invariants. This is done thanks to an augmented Lagrangian method. This method is not based on an hamiltonian form of the equation. So it is impossible to study it as the restriction of a continuous problem on a finite dimensional phase space. The argument which is used for the different numerical schemes for the nonlinear Schrödinger equation does not apply in the present case.

3.5 Conclusion.

It seems very difficult to make some great improvements for applying Zhidkov's theoretical frame to partial differential equations. Two facts are responsible for that. First, there are some strong limitations on the nature of both the linear operator and the nonlinearity. Second, in general we need an existence theory in a large space like L^2 and such results are known for a very limited number of equations especially for periodic boundary conditions. Bourgain has for example also proved some existence results for the Kadomtsev-Petviashvilli II equations in L^2 for periodic boundary conditions. It seems very difficult to write an hamiltonian numerical scheme for these equations since it is some generalization of the Korteweg-de Vries equation.

For the sake of completeness, we cite here the proofs of some results which are only stated in the body of the article. It concerns global existence and uniqueness for the Cauchy problems for the different equations we have studied. We also give useful measure theory results.

A Global existence and uniqueness results.

We retranscript here Part 4 of Bourgain's article [2] with the adaptation to other nonlinearities. We use the estimates on tori which are explicited in the second part of the same article.
We consider the NLS equation

$$\begin{aligned} \Delta u + i\partial_t u + f(x,|u|^2)u &= 0, \\ u(x,0) &= \phi(x), \end{aligned}$$

where u is periodic in the x variable.
Let us set $w = f(x,|u|^2)u$, the associated integral equation is

$$u(\cdot,t) = U(t)\phi + i\int_0^t U(t-\tau)w(\cdot,\tau)d\tau,$$

where $U(t) = e^{it\Delta}$.
We want to use estimates on tori, hence it is useful to make a time localization. For that aim we introduce a cut-off function ψ_1 which is equal to 1 on $[-\delta,\delta]$ with a support in $[-2\delta,2\delta]$. It also possible to write the integral equation in the form

$$u(\cdot,t) = \psi_1(t)U(t)\phi + i\psi_1(t)\int_0^t U(t-\tau)w(\cdot,\tau)d\tau.$$

Using the Fourier transform, we find

$$\begin{aligned} u(x,t) &= \psi_1(t)\sum_{n\in\mathbf{Z}} \hat{\phi}(n)e^{2\pi i(nx+n^2t)} \\ &+ i\psi_1(t)\sum_{n\in\mathbf{Z}} e^{2\pi i(nx+n^2t)}\int_{-\infty}^{+\infty}\frac{e^{2\pi i(\lambda-n^2)t}-1}{2\pi i(\lambda-n^2)}\hat{w}(n,\lambda)d\lambda. \end{aligned}$$

We introduce a new cut-off function ψ_2 which is equal to 1 on $[-B,B]$ and with support in $[-2B,2B]$. Hence we get

$$\begin{aligned} &\psi_1(t)\int_{-\infty}^{+\infty}\frac{e^{2\pi i(\lambda-n^2)t}-1}{\lambda-n^2}\hat{w}(n,\lambda)d\lambda = \\ &+ \sum_{k\geq 1}\frac{(2\pi i)^k}{k!}\psi_1(t)t^k\int_{-\infty}^{+\infty}\psi_2(\lambda-n^2)(\lambda-n^2)^{k-1}\hat{w}(n,\lambda)d\lambda \\ &+ \psi_1(t)\int_{-\infty}^{+\infty}(1-\psi_2)(\lambda-n^2)\frac{e^{2\pi i(\lambda-n^2)t}}{\lambda-n^2}\hat{w}(n,\lambda)d\lambda \\ &- \psi_1(t)\int_{-\infty}^{+\infty}(1-\psi_2)(\lambda-n^2)\frac{\hat{w}(n,\lambda)}{\lambda-n^2}d\lambda. \end{aligned}$$

It is also necessary to estimate the following terms :

$$I = \psi_1(t)\sum_{n\in\mathbb{Z}}\hat{\phi}(n)e^{2\pi i(nx+n^2t)},$$

$$\begin{aligned} II &= \frac{1}{2B}\sum_{k\geq 1}\frac{(2\pi i)^k}{k!}(2Bt)^k\psi_1(t) \\ & \left\{\sum_{n\in\mathbb{Z}}\left[\int_{-\infty}^{+\infty}\psi_2(\lambda-n^2)\left(\frac{\lambda-n^2}{2B}\right)^{k-1}\hat{w}(n,\lambda)d\lambda\right]e^{2\pi i(nx+n^2t)}\right\}, \end{aligned}$$

$$III = \psi_1(t)\sum_{n\in\mathbb{Z}}e^{2\pi inx}\int_{-\infty}^{+\infty}\frac{(1-\psi_2)(\lambda-n^2)}{\lambda-n^2}e^{2\pi i\lambda t}\hat{w}(n,\lambda)d\lambda,$$

$$IV = -\psi_1(t)\sum_{n\in\mathbb{Z}}e^{2\pi i(nx+n^2t)}\int_{-\infty}^{+\infty}\frac{(1-\psi_2)(\lambda-n^2)}{\lambda-n^2}e^{2\pi i\lambda t}\hat{w}(n,\lambda)d\lambda.$$

In the same way as in [2], we get

$$\|I\|_{L^4(dxdt)} \leq c\|\phi\|_2,$$

$$\|II\|_{L^4(dxdt)} \leq c\delta B\|w\|_{L^{4/3}(dxdt)},$$

$$\|III\|_{L^4(dxdt)} \leq CB^{-1/4}\|w\|_{L^{4/3}(dxdt)},$$

$$\|IV\|_{L^4(dxdt)} \leq CB^{-1/4}\|w\|_{L^{4/3}(dxdt)}.$$

With the aim to have the same sort of estimates than Bourgain we impose for example that

$$\|w\|_{4/3} \leq C\sum_{\beta}\|u\|_4^{\beta},$$

$$\|w_1-w_2\|_{4/3} \leq C\sum_{\gamma}(\|u_1\|_4+\|u_2\|_4)^{\gamma}\,\|u_1-u_2\|_4,$$

where $w_i = f(x,\|u_i\|^2)u_i$, $i=1,2$, the sums over β or γ containing a finite number of terms and $1\leq\beta$, $0\leq\gamma$. Let us set

$$\begin{aligned} Tu(x,t) &= \psi_1(t)\sum_{n\in\mathbb{Z}}\hat{\phi}(n)e^{2\pi i(nx+n^2t)} \\ &+ i\psi_1(t)\sum_{n\in\mathbb{Z}}e^{2\pi i(nx+n^2t)}\int_{-\infty}^{+\infty}\frac{e^{2\pi i(\lambda-n^2)t}-1}{2\pi i(\lambda-n^2)}\hat{w}(n,\lambda)d\lambda. \end{aligned}$$

The proof of the global existence in L^2 is carried out as follows :
- there exists a constant M such that T maps $B(0,M)$ in itself ;
- on this ball T is a contraction (for these two points we have to impose conditions on δ and B) ;
- thanks to a fixed point argument the problem is locally well posed in L^2 ;

- according to the conservation of the L^2 norm for this equation, the solution is global in time.
Les us check the first two points :

$$\|Tu\|_4 \le c_1 \left\{ \|\phi\|_2 + \delta B \left(\sum_\beta \|u\|_4^\beta\right) + B^{-1/4}\left(\sum_\beta \|u\|_4^\beta\right)\right\},$$

which may be brought to be lower than M, for a sufficiently large M, since it is possible to fix δB and $B^{-1/4}$ arbitrarily small.
Hence $\|u\|_4 \le M$ implies that $\|Tu\|_4 \le M$.

$$\begin{aligned}\|Tu_1 - Tu_2\|_4 &\le c_1(\delta B + B^{-1/4}) \sum_\gamma (\|u_1\|_4 + \|u_2\|_4)^\gamma \|u_1 - u_2\|_4, \\ &\le 2c_1(\delta B + B^{-1/4}) \sum_\gamma (M)^\gamma \|u_1 - u_2\|_4.\end{aligned}$$

The quantity $2c_1(\delta B + B^{-1/4})\sum_\gamma (M)^\gamma$ may be brought to be lower than $\frac{1}{2}$ up to a new decreasing of δB and $B^{-1/4}$ which is compatible with the former one.
We now give a few fields of application for this result.
The case which is studied by Bourgain is $w = |u|^\alpha u$. He shows that the two hypotheses on w are verified in the case when $0 < \alpha \le 2$.
The two nonlinearities Zhidkov proposes are $w = \dfrac{u|u|^2}{1+|u|^2}$ and $w = 1 - e^{-\alpha u}$.
In the frame we adopt here, we want that

$$\|f(x,|u|^2)u\|_{4/3} \le C \sum_\beta \|u\|_4^\beta,$$

$$\|f(x,|u|^2)u - f(x,|v|^2)v\|_{4/3} \le C \sum_\gamma (\|u\|_4 + \|v\|_4)^\gamma \|u - v\|_4.$$

The first estimate is true if

$$\|f(x,|u|^2)\|_2 \le C \sum_\gamma \|u\|_4^\gamma.$$

The second may be fulfilled under one of the two following conditions :

$$\sup_{u\in B(0,M)} \|\nabla_s f(x,|u|^2)u\|_2 \le C \sum_\gamma \sup_{u\in B(0,M)} \|u\|_4^\gamma \text{ when } u \in B(0,M),$$

$$\text{or } \|\nabla_s f(x,|u|^2)\|_4 \le C \sum_\eta \sup_{u\in B(0,M)} \|u\|_4^\eta \text{ when } u \in B(0,M),$$

the sum over η containing a finite number of terms and $-1 \le \eta$.

B Gaussian measures on a Hilbert space.

The following ingredients are contained in Dalecky and Fomin's book [5]. We give them for the reader's convenience, and in particular in order to make clear the proof of Lemma 2.2. Let us begin with a theorem which gives a condition for a premeasure to be σ-additive.

Let ν be a premeasure on an algebra $\mathcal{U}$ of subsets of a set X. We say that a class $\mathcal{K} \subset \mathcal{U}$ approximates ν from below if for each $A \in \mathcal{U}$ and each $\varepsilon > 0$, there exists a $K \in \mathcal{K}$ such that $K \subset A$ and $|\nu|(A \backslash K) < \varepsilon$.
According to this definition, we may state :

Proposition B.1 *Let ν be a premeasure on an algebra $\mathcal{U}$ of subsets of a topological space X and let $\mathcal{F} \subset \mathcal{U}$ be a class of closed subsets of X, which is closed under intersection and approximates ν from below.*
If for every $\varepsilon > 0$, there exists a compact set K_ε such that for all $F \in \mathcal{F}$,

$$F \cap K_\varepsilon = \emptyset \Rightarrow |\nu|(F) < \varepsilon,$$

then ν is σ-additive, i.e. it is a measure.

On a Hilbert space, we can define particular measures called Gaussian cylindrical measures. The construction is carried out as follows.
Let ν be a measure on $\mathbf{R}^n$, its characteristic functional χ_ν is given by the formula

$$\chi_\nu(\xi) = \int_{\mathbf{R}^n} e^{i\xi \cdot x} \rho(x) dx,$$

where ρ is the density of ν with respect to Lebesgue's measure dx. The measure ν is said to be Gaussian if its characteristic functional may be expressed in the form

$$\chi_\nu(y) = \exp\left\{-\frac{1}{2}(By, y) + i(\alpha, y)\right\},$$

where B is a positive operator. It is said to be centered if α is zero.
In a Hilbert space X we call cylindrical a set in the form

$$\amalg = \{x \in X; Px \in \mathcal{M}\},$$

where P is a projection onto a subset of a finite dimension of X, and $\mathcal{M}$ is a Borel set of this subspace called the base of $\amalg$.
We define a cylindrical measure ν over X by the measure of the cylindrical sets of X. The measure of a cylindrical set $\amalg$ is chosen to be equal to the measure of the base $\mathcal{M}$ of $\amalg$ in PX.
Such a measure is said to be a centered Gaussian measure if each projection onto

a finite dimensional space is a centered Gaussian measure. Hence its characteristic functional is in the form

$$\chi_\nu(y) = e^{-b(y,y)/2}$$

where b is a nonnegative bilinear form which is continuous on every finite dimensional subset and is called the correlation of ν. If $b(y, y)$ may be written in the form (By, y), B is called the correlation operator of ν. We will denote by ν_B the centered Gaussian cylindrical measure with correlation operator B.
Now we have the lemma

Lemma B.2 *Let A be a positive operator then*

$$\nu_B\{x : (Ax, x)_X \geq 1\} \leq TrAB,$$

$$\nu_B\{x : |(Ax, x) - TrAB| \leq c\sqrt{TrAB}\} \geq 1 - \frac{2}{c^2}\|AB\|_X.$$

It is now possible to prove lemma 2.2 which gives a necessary and sufficient condition of σ-additivity for a centered Gaussian cylindrical measure.

Lemma 2.2. *A centered Gaussian cylindrical measure ν on X is σ-additive if and only if its correlation operator is in the form $b(y_1, y_2) = (By_1, y_2)$, where B is a nuclear positive operator X.*

Sufficient condition :
If ν is σ-additive then its characteristic functional and its correlation are continuous on X, hence $b(y, y) = (By, y)$ with $B \in \mathcal{L}(X)$. Let us show by contradiction that B is nuclear. If it is not the case, we may find a projection onto a finite dimensional subspace such that its trace $\mathrm{Tr}PBP = R$ is arbitrarily large. Then we set

$$\text{II} = \{x \in X : |\|Px\|_X^2 - R| < \alpha\sqrt{R}\}.$$

The intersection of this cylinder with the ball of center 0 and radius $R - \alpha\sqrt{R}$ is empty.
According to lemma 1.1, we have

$$\nu(\text{II}) \geq 1 - \frac{2\|b\|}{\alpha^2} = \frac{1}{2} \text{ if we take } \alpha = 2\sqrt{\|B\|}.$$

We may consider balls of X centered in 0, with an arbitrarily large radius and with a measure which is lower than $\frac{1}{2}$. This is impossible since $\nu(X) = 1$.

Necessary condition :
Let II be a cylinder which does not intersect the ball in X with center 0 and radius R. Its base $\mathcal{M}$ also does not intersect the ball of center 0 and radius R in PX. According to lemma B.2, we get

$$\nu(\text{II}) \leq \frac{\mathrm{Tr}PBP}{R^2} \leq \frac{\mathrm{Tr}B}{R^2},$$

which is finite since we assume that B is nuclear. We use proposition B.1 with $\mathcal{U}$ equal to the Borel σ-algebra of X and $\mathcal{F}$ to the set of cylinders in X.
For all $\varepsilon > 0$, there exists $K_\varepsilon = B(0, R)$ with $\varepsilon = \dfrac{\mathrm{Tr}B}{R^2}$ such that, for all $F \in \mathcal{F}$ such that $F \cap B(0, R) = \emptyset$, then $\nu(F) \leq \varepsilon$. ∎

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Partie III

Etude d'équations de l'optique non linéaire.

Chapitre 4

Sur une équation de Maxwell-Bloch.

Résumé

On étudie le problème de Cauchy associé à deux systèmes d'équations (Maxwell-Debye et Maxwell-Bloch) décrivant des phénomènes d'interaction laser-matière. On montre que ces problèmes sont bien posés localement en temps pour des données initiales appartenant à différents espaces de Sobolev. Dans le cas du système de Maxwell-Debye, qui comporte un terme de retard, on étudie la limite des solutions quand ce retard tend vers 0. On considère également une approximation adiabatique du système de Maxwell-Bloch.

Introduction

Notre but est une étude analytique d'équations régissant la propagation de la lumière dans un milieu qui interagit avec le champ électromagnétique qui correspond à cette onde lumineuse. On étudie ainsi deux systèmes d'équations qui ont une structure mathématique relativement semblable : le système de Maxwell-Debye

$$\begin{cases} \left(\dfrac{\partial}{\partial z} + \dfrac{n_0}{c}\dfrac{\partial}{\partial t}\right) A - \dfrac{i}{2k}\nabla_1^2 A + i\dfrac{\omega_0}{c}\delta n A = 0, \\ \tau\dfrac{\partial \delta n}{\partial t} + \delta n = n_2 |A|^2, \end{cases}$$

qui décrit l'interaction d'une onde électromagnétique avec un milieu non résonnant admettant un temps de réponse non négligeable, et celui de Maxwell-Bloch

$$\begin{cases} \dfrac{\partial A}{\partial z} + \dfrac{1}{c}\dfrac{\partial A}{\partial t} - i\dfrac{c}{2\omega}\nabla_1^2 A + \dfrac{\kappa}{c} A = \dfrac{i\omega}{2\varepsilon_0 c} L, \\ \dfrac{\partial L}{\partial t} + (\gamma_{12} + i(\omega_{12} - \omega))L = \dfrac{ip^2}{\hbar} AN, \\ \dfrac{\partial N}{\partial t} + \gamma_{11}(N - N_0) = \dfrac{2i}{\hbar}(A^* L - AL^*), \end{cases}$$

qui décrit l'interaction d'une onde électromagnétique avec un milieu résonnant constitué de gaz à deux niveaux d'énergie.

Il existe déjà divers articles sur une version plus simple de l'équation de Maxwell-Bloch qui consiste à négliger le Laplacien (∇_1^2) en les variables d'espace x et y. On néglige ainsi les variations transversales du champ, c'est-à-dire que l'on considère un problème à (1+1) dimensions.

Dans l'article de Constantin, Foias et Gibbon [4], cette équation à (1+1) dimensions est étudiée pour des données aux bord en z périodiques. Le système est alors hyperbolique non linéaire. Ils étudient l'existence globale de solutions dans L^2 et construisent un attracteur universel de dimension finie dans ce même espace. Cet attracteur est constitué de fonctions C^∞. Ce système admet le système complexe de Lorenz comme sous-système correspondant aux solutions qui ne dépendent pas de la variable d'espace z. Ceci permet d'entrevoir la complexité dynamique de ces équations.

Ikeda, Otsuka et Matsumoto [7] étudient les états turbulents du système de Maxwell-Bloch qui correspondent à l'oscillation laser multi-mode. C'est à nouveau un problème en (1+1) dimensions avec des conditions aux bords périodiques en z qui est étudié.

Feng, Moloney et Newell [5] effectuent une étude de la stabilité linéaire pour un problème en (1+1) dimensions mais, cette fois-ci, c'est la variable d'espace x qui est conservée.

Après avoir précisé dans la première partie la dérivation des deux modèles étudiés, nous consacrons la deuxième partie à l'étude du système de Maxwell-Debye et plus particulièrement, nous montrons que le problème de Cauchy est localement bien posé dans H^s pour $s > 1$ (solutions régulières) puis dans H^1 et dans L^2 (solutions faibles). Dans le cas où $s \geq 1$, nous montrons également que, quand le retard τ tend vers 0, les solutions de l'équation de Maxwell-Debye tendent dans H^s vers celle de l'équation de Schrödinger qui est la limite formelle. La troisième partie regroupe quelques résultats à propos de l'équation de Maxwell-Bloch. Nous commençons par étudier cette équation après approximation adiabatique; et ensuite nous étudions le problème de Cauchy dans H^s, $s > 1$ pour le système complet.

1 Les équations de l'optique non linéaire.

On étudie des modèles permettant de décrire la propagation de la lumière dans un milieu actif résonnant. Le milieu ayant un grand nombre de degrés de liberté, on se limite à considérer un petit nombre d'entre eux en faisant diverses hypothèses. Le champ électrique est supposé être une collection de trains d'ondes presque monochromatiques et de plus on suppose que les degrés de liberté qui résonnent directement ou indirectement avec le champ électrique ont une influence cumulative à long terme. Tous les ingrédients de la modélisation qui suit se trouvent dans le livre de Newell et Moloney [8].

On commence tout d'abord par dériver les équations de Bloch qui décrivent la dynamique des oscillateurs (atomes excités) de la matière.
L'état de la matière est décrit par la fonction d'onde ψ et l'opérateur hamiltonien H dont les valeurs propres sont les niveaux d'énergie (quantifiés) et les fonctions propres sont les états de base.
Dans l'état non perturbé, l'hamiltonien est noté H_0, les niveaux d'énergie $E_j = \hbar\omega_j$ et H_0 admet des fonctions propres ψ_j qui vérifient $H_0\psi_j = \hbar\omega_j\psi_j$.
$\{\psi_j\}$ est une base orthonormée de l'espace des phases et on peut choisir les fonctions propres telles que on ait de plus $\int \vec{R}\psi_j(\vec{R})d\vec{R} = 0$ et $\int \vec{R}\psi_j\psi_j^* d\vec{R} = 0$.
On veut calculer le vecteur de polarisation induit par le champ $\vec{E}$. Il est donné de façon générale par

$$\vec{P} = n_a e \int \vec{R}\psi\psi^* d\vec{R}$$

où n_a est la densité volumique des atomes et e est la charge d'un électron.

La première étape consiste à utiliser l'équation de Schrödinger. On suppose que ψ satisfait

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

où H est la somme de H_0 et du potentiel de perturbation $\delta V = -e\int \vec{E}.d\vec{R}$. Comme $\vec{E}$ varie peu sur des distance atomiques, on considèrera que $\delta V = -e\vec{E}.\vec{R}$.

La deuxième étape consiste à chercher la forme des états non perturbés. On suppose à cette fin que $\psi(\vec{R},t) = \sum_{j=1}^{N} a_j(t)\psi_j(\vec{R})$. On obtient alors immédiatement que si $H = H_0$, alors $a_j(t) = a_j(0)e^{-i\omega_j t}$.

La troisième étape est le calcul du vecteur de polarisation. On a

$$\vec{P}_{atome} = e\int \vec{R}\psi\psi^* d\vec{R} = \sum_{jk} \rho_{jk}\vec{P}_{kj} = \mathrm{Tr}\,\vec{p}\rho$$

où l'élément de la matrice de densité $\rho_{jk} = a_j a_k^*$ dépend du temps et l'élément

de la matrice dipolaire $\vec{p}_{jk} = e\int \vec{R}\psi_j^*\psi_k d\vec{R}$ n'en dépend pas. On a évidemment $\vec{P} = n_a\vec{P}_{atome}$.
Il reste à écrire les équations dynamiques qui régissent les ρ_{jk}.

La quatrième étape est la dérivation des équations de Bloch "brutes" (au sens où elles sont en trop grand nombre). En utilisant la forme particulière de H, on obtient

$$\frac{\partial a_k}{\partial t} = -i\omega_k a_k + \frac{i\vec{E}}{\hbar}.\sum_{l=1}^{N}\vec{p}_{kl}a_l$$

et on obtient donc pour les éléments de la matrice de densité les équations :

$$\frac{\partial \rho_{jk}}{\partial t} = -i(\omega_j - \omega_k)\rho_{jk} + \frac{i\vec{E}}{\hbar}.\sum_{l=1}^{N}\vec{p}_{jl}\rho_{lk} - \frac{i\vec{E}}{\hbar}.\sum_{l=1}^{N}\vec{p}_{lk}\rho_{jl}.$$

La cinquième étape est la simplification de ces équations. On sait résoudre les équations précédentes si on néglige les termes en $\vec{E}$, mais ceci ne peut être effectué que s'ils sont effectivement négligeables. En pratique, on peut négliger tous les termes de $\frac{i\vec{E}}{\hbar}.\sum_{l=1}^{N}\vec{p}_{jl}\rho_{lk} - \frac{i\vec{E}}{\hbar}.\sum_{l=1}^{N}\vec{p}_{lk}\rho_{jl}$ sauf ceux qui ont des fréquences proches de $\omega_{jk} = \omega_j - \omega_k$. Ainsi un grand nombre de termes sont négligés, leur somme constitue un ensemble non négligeable. On modélise la perte graduelle d'énergie du petit nombre de modes auxquels on s'intéresse vers le grand nombre des autres par l'addition d'un terme de la forme $-\gamma_{jk}\rho_{jk}$ ($\gamma_{jk} > 0$) dans les équations régissant les n niveaux d'énergie que l'on considère. On n'étudie plus les autres équations. On a ainsi opéré une simplification des équations ainsi qu'une réduction de leur nombre.

La sixième étape consiste en l'identification des résonances possibles.
Il peut y avoir une résonance directe, c'est-à-dire que la fréquence ω de $\vec{E}$ est proche de ω_{jk} (produits du type $\vec{E}.\sum\vec{p}_{jk}\rho_{kk}$).
Il peut y avoir une rectification D.C. (termes du type $\vec{E}.\sum\vec{p}_{jl}\rho_{lj}$ dans lesquels la fréquence $-\omega$ de $\vec{E}$ élimine la fréquence $\omega_{lj} \sim \omega$ de ρ_{lj}, d'où un effet cumulatif à long terme sur ρ_{jj}).
Il peut y avoir enfin une résonance paramétrique lorsque l'une des combinaisons binaires des fréquences du champ $\pm\omega_r$ avec les différences de fréquences $\pm\omega_{lk}$ du dipôle sont égales à d'autres différences de fréquence $\pm\omega_{jk}$.

Ainsi pour les n niveaux considérés, on a

$$\frac{\partial \rho_{jk}}{\partial t} = -i(\omega_j - \omega_k)\rho_{jk} + \frac{i\vec{E}}{\hbar}.\sum_{l=1}^{n}\vec{p}_{jl}\rho_{lk} - \frac{i\vec{E}}{\hbar}.\sum_{l=1}^{n}\vec{p}_{lk}\rho_{jl} \tag{1.1}$$

et pour le vecteur polarisation

$$\vec{P} = n_a \mathrm{Tr}\,\vec{p}\rho. \tag{1.2}$$

Enfin on dérive l'équation de Maxwell qui régit les variations de l'enveloppe de $\vec{E}$.
Les équations de Maxwell sont

$$\begin{aligned}
\vec{\nabla}.\vec{D} &= \rho, && \text{équation de Gauss électrique,}\\
\vec{\nabla}.\vec{B} &= 0, && \text{équation de Gauss magnétique,}\\
\vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}, && \text{équation de Faraday,}\\
\vec{\nabla}\times\vec{H} &= \frac{\partial\vec{D}}{\partial t}+\vec{\jmath}, && \text{équation d'Ampère.}
\end{aligned}$$

$\vec{E}$ est le champ électrique, $\vec{B}$ est l'induction magnétique, $\vec{D}$ est l'induction électrique et $\vec{H}$ est le champ magnétique. Ces champs sont de plus reliés par les relations $\vec{B} = \mu\vec{H}$ et $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ où ε et μ sont respectivement la permittivité et la perméabilité du milieu. On considère ici que la densité de charge électrique ρ et la densité de courant électrique $\vec{\jmath}$ sont nuls et que μ est constant et égal à $\mu_0 = \frac{1}{\varepsilon_0 c^2}$ où c est la vitesse de la lumière dans le vide et ε_0 la permittivité du vide.
On obtient alors aisément

$$\vec{\nabla}^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial^2 t} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2\vec{P}}{\partial^2 t}. \tag{1.3}$$

2 Les équations de Maxwell-Debye.

2.1 Modélisation.

La présence d'un champ électromagnétique fait varier l'indice de réfraction du milieu que l'on considère ici non résonnant avec un temps de réponse non négligeable τ. Soit le vecteur électrique de déplacement $\vec{D} = \varepsilon_0 n(\omega)\vec{E}$ où $n(\omega)$ est appelé indice du milieu et est de la forme

$$n(\omega) = n_0(\omega) + \delta n(E).$$

On obtient comme équation pour δn, l'équation de Debye, à savoir

$$\tau\frac{\partial \delta n}{\partial t} + \delta n = n_2|E|^2.$$

Cette loi d'évolution est relativement intuitive si on remarque que pour un milieu qui réagit "instantanément", on a $n(\omega) = n_0(\omega) + n_2|E|^2$. On décompose le vecteur polarisation en une polarisation linéaire $\vec{P}_L = \varepsilon_0(n_0^2(\omega) - 1)\vec{E}$ et une polarisation non linéaire $\vec{P}_{NL} = 2\varepsilon_0 n_0 \delta n(E)\vec{E}$. Dans le cas d'une onde unidirectionnelle, on peut poser $\vec{E} = \hat{e}A(\vec{r}, t)e^{i(kz-\omega t)} + c.c.$ et (1.3) devient

$$\left\{\begin{array}{l} \left(\dfrac{\partial}{\partial z} + \dfrac{n_0}{c}\dfrac{\partial}{\partial t}\right)A - \dfrac{i}{2k}\nabla_1^2 A + i\dfrac{\omega_0}{c}\delta n A = 0, \\ \tau\dfrac{\partial \delta n}{\partial t} + \delta n = n_2|A|^2. \end{array}\right. \tag{2.1}$$

Pour des commodités d'écriture, on remplacera dans la suite δn par n.

2.2 Le problème de Cauchy local.

Mise en forme.

On pose $\xi = \frac{c}{n_0}t - z$ et $A(x,y,z,t) = \bar{A}(\xi,t;x,y)$, $n(x,y,z,t) = \bar{n}(\xi,t;x,y)$. Les équations de Maxwell-Debye (2.1) réécrites après ce changement de variable sont de la forme

$$\begin{cases} \frac{n_0}{c}\frac{\partial \bar{A}}{\partial t} - \frac{i}{2k}\nabla_1^2 \bar{A} + i\frac{\omega_0}{c}\bar{n}\bar{A} = 0, \\ \tau\frac{\partial \bar{n}}{\partial t} + \frac{\tau c}{n_0}\frac{\partial \bar{n}}{\partial \xi} + \bar{n} = n_2|\bar{A}|^2. \end{cases} \tag{2.2}$$

On effectue le nouveau changement de variable $\bar{n} = \bar{m}e^{-\frac{n_0}{\tau c}\xi}$, d'où le nouveau système :

$$\begin{cases} \frac{n_0}{c}\frac{\partial \bar{A}}{\partial t} - \frac{i}{2k}\nabla_1^2 \bar{A} + i\frac{\omega_0}{c}\bar{m}e^{-\frac{n_0}{\tau c}\xi}\bar{A} = 0, \\ \tau\frac{\partial \bar{m}}{\partial t} + \frac{\tau c}{n_0}\frac{\partial \bar{m}}{\partial \xi} = n_2|\bar{A}|^2 e^{\frac{n_0}{\tau c}\xi}. \end{cases}$$

On étudie l'équation de transport avec t comme variable d'évolution, et donc en posant $\Xi(t) = \xi_0 + \frac{c}{n_0}t$, où ξ_0 désigne une caractéristique particulière.

$$\begin{aligned} \frac{d}{dt}\left(\bar{m}(\Xi(t),t;x,y)\right) &= \left(\frac{\partial \bar{m}}{\partial \xi}\Xi'(t) + \frac{\partial \bar{m}}{\partial t}\right)(\Xi(t),t;x,y), \\ &= \left(\frac{c}{n_0}\frac{\partial \bar{m}}{\partial \xi} + \frac{\partial \bar{m}}{\partial t}\right)(\Xi(t),t;x,y), \\ &= \frac{n_2}{\tau}|\bar{A}\left(\Xi(t),t;x,y\right)|^2 e^{\frac{n_0}{\tau c}\Xi(t)}. \end{aligned}$$

$$\bar{m}(\Xi(t),t;x,y) = \bar{m}(\Xi(t_0),t_0;x,y) + \int_{t_0}^{t}\frac{n_2}{\tau}|\bar{A}\left(\Xi(\zeta),\zeta;x,y\right)|^2 e^{\frac{n_0}{\tau c}\Xi(\zeta)}d\zeta.$$

A partir de maintenant, on se place sur la caractéristique passant par $(t,\xi) = (0,\xi_0)$, d'où $\Xi(t) = \xi_0 + \frac{c}{n_0}t$ et $\Xi(\zeta) = \xi_0 + \frac{c}{n_0}\zeta$.

$$\begin{aligned} \bar{m}(\xi_0 + \frac{c}{n_0}t,t;x,y) &= \bar{m}(\xi_0 + \frac{c}{n_0}t_0,t_0;x,y) \\ &+ \int_{t_0}^{t}\frac{n_2}{\tau}|\bar{A}(\xi_0 + \frac{c}{n_0}\zeta,\zeta;x,y)|^2 e^{\frac{n_0}{\tau c}(\xi_0+\frac{c}{n_0}\zeta)}d\zeta. \end{aligned}$$

La première variable ne sert à rien puisque l'on reste sur une caractéristique donnée, on pose donc

$$\begin{cases} \bar{A}(\xi_0 + \frac{c}{n_0}t,t;x,y) = \tilde{A}(t;x,y), \\ \bar{m}(\xi_0 + \frac{c}{n_0}t,t;x,y) = \tilde{m}(t;x,y), \\ \bar{n}(\xi_0 + \frac{c}{n_0}t,t;x,y) = \tilde{n}(t;x,y). \end{cases}$$

$$\tilde{m}(t;x,y) = \tilde{m}(t_0;x,y) + \int_{t_0}^{t} \frac{n_2}{\tau} |\tilde{A}(\zeta;x,y)|^2 e^{\frac{n_0}{\tau c}(\xi_0 + \frac{c}{n_0}\zeta)} d\zeta.$$

et donc

$$\tilde{n}(t;x,y) = \tilde{n}(t_0;x,y) + \int_{t_0}^{t} \frac{n_2}{\tau} |\tilde{A}(\zeta;x,y)|^2 e^{\frac{\zeta}{\tau}} d\zeta.$$

On introduit ce résultat dans l'équation de Schrödinger. On ne choisit pas encore t_0 mais on le suppose fixé.

$$\frac{\partial \tilde{A}}{\partial t}(t;x,y) - \frac{ic}{2kn_0} \nabla_1^2 \tilde{A}(t;x,y) +$$
$$+ i\frac{\omega_0}{n_0} \left\{ \tilde{n}(t_0;x,y) + \int_{t_0}^{t} \frac{n_2}{\tau} |\tilde{A}(\zeta;x,y)|^2 e^{\frac{\zeta}{\tau}} d\zeta \right\} e^{-\frac{t}{\tau}} \tilde{A}(t;x,y) = 0.$$

On écrit maintenant la formulation intégrale de Duhamel de cette équation en utilisant l'opérateur $U(t)$ associé à la partie linéaire $\frac{\partial \tilde{A}}{\partial t} - \frac{ic}{2kn_0}\nabla_1^2 \tilde{A} = 0$ de l'équation précédente et en omettant les variables x et y.

$$\tilde{A}(t) = U(t-t_1)\tilde{A}(t_1) - \int_{t_1}^{t} U(t-\theta) i\frac{\omega_0}{n_0} \left\{ \tilde{n}(t_0) + \int_{t_0}^{\theta} \frac{n_2}{\tau} |\tilde{A}(\zeta)|^2 e^{\frac{\zeta}{\tau}} d\zeta \right\} e^{-\frac{\theta}{\tau}} \tilde{A}(\theta) d\theta.$$

Fixons les conditions initiales, c'est-à-dire les valeurs de t_0 et t_1.
Prenons par exemple $t_0 = t_1 = 0$. On appelle les données initiales pour $\tilde{A}$ et $\tilde{n}$, φ et ν respectivement. On a alors

$$\tilde{A}(t) = U(t)\varphi - \int_{0}^{t} U(t-\theta) i\frac{\omega_0}{n_0} \left\{ \nu + \int_{0}^{\theta} \frac{n_2}{\tau} |\tilde{A}(\zeta)|^2 e^{\frac{\zeta}{\tau}} d\zeta \right\} e^{-\frac{\theta}{\tau}} \tilde{A}(\theta) d\theta. \tag{2.3}$$

Proposition 2.1 *Si A et m appartiennent à $L^\infty(0,T;L^2)$ alors les problèmes (2.2) et (2.3) sont équivalents.*

C'est sur la formulation (2.3) que nous allons essayer d'effectuer un raisonnement de point fixe pour montrer l'existence locale pour le problème de Cauchy. A partir d'ici, on traite $t > 0$ comme une variable de temps et on considère deux variables d'espace x et y. Les conditions au bord en x et y pour A sont la nullité en $+\infty$ et $-\infty$.
Nous allons effectuer un raisonnement de point fixe. Pour cela, on pose

$$\Phi\tilde{A}(t) = U(t)\varphi - \int_{0}^{t} U(t-\theta) i\frac{\omega_0}{n_0} \left\{ \nu + \int_{0}^{\theta} \frac{n_2}{\tau} |\tilde{A}(\zeta)|^2 e^{\frac{\zeta}{\tau}} d\zeta \right\} e^{-\frac{\theta}{\tau}} \tilde{A}(\theta) d\theta.$$

On veut donc montrer que pour un certain espace fonctionnel X, $R > 0$ et $0 < \alpha < 1$ et pour tout $A, B \in B_X(0,R)$, on a $\Phi A \in B_X(0,R)$ et $\|\Phi A - \Phi B\|_X \leq \alpha \|A - B\|_X$.

Existence et unicité de solutions régulières.

La première idée consiste à chercher des solutions régulières, c'est-à-dire se placer dans le cas où X est une algèbre, nous allons donc prendre $X = L^\infty(0,T;H^s)$ avec $s > 1$.
Remarque : Si on considère les variables de départ, ceci correspond à une régularité $L^\infty(\xi; L^\infty(0,T;H^s))$.

Théorème 2.2 *i) Pour tout (φ, ν) appartenant à $H^s \times H^s$ avec $s > 1$, l'équation (2.3) admet une unique solution dans $X = L^\infty(0,T;H^s)$ pour un T suffisamment petit.*
ii) Les solutions dépendent continuement des données initiales, à savoir :
si $\tilde{A} \in L^\infty(0,T;H^s)$ est solution de l'équation de Maxwell-Debye pour les données initiales (φ,ν), φ_p et ν_p tendent respectivement vers φ et ν dans H^s alors pour un p suffisamment grand la solution $\tilde{A}_p$ de l'équation de Maxwell-Debye associée aux données initiales φ_p et ν_p tend vers $\tilde{A}$ dans $L^\infty(0,T;H^s)$.

Remarque : Dans la démonstration qui suit, on s'attachera à exprimer la dépendance en τ des constantes de majoration. Ceci n'a pas un intérêt immédiat car dans le théorème ci-dessus τ est fixé, mais nous le ferons tendre plus tard vers 0 en ayant besoin d'estimations uniformes en τ.

On utilise le résultat suivant sur l'opérateur U :

Lemme 2.3 *Il existe un $K > 0$ tel que pour tout $f \in L^1(0,T;H^s)$,*

$$\left\| \int_0^t U(t-\theta) f(\theta) d\theta \right\|_{L^\infty(0,T;H^s)} \leq K \|f\|_{L^1(0,T;H^s)}.$$

Pour la preuve de cette estimation, et de celle du lemme 2.5 utilisée plus loin pour l'étude des solutions faibles, on peut consulter l'article de Ginibre et Velo [6].

Preuve du i) du théorème 2.2
Posons $\Phi\tilde{A}(t) = I + II + III$ avec

$$I = U(t)\varphi,$$

$$II = -i\frac{\omega_0}{n_0}\int_0^t U(t-\theta)\nu e^{-\frac{\theta}{\tau}}\tilde{A}(\theta)d\theta,$$

$$III = -i\frac{\omega_0 n_2}{\tau n_0}\int_0^t U(t-\theta)\left(\int_0^\theta |\tilde{A}(\zeta)|^2 e^{\frac{1}{\tau}\zeta} d\zeta\right) e^{-\frac{1}{\tau}\theta}\tilde{A}(\theta)d\theta.$$

Grâce à l'estimation ci-dessus, on obtient directement :

$$\|I\|_{L^\infty(0,T;H^s)} = \|\varphi\|_{H^s},$$

$$\|II\|_{L^\infty(0,T;H^s)} \leq C\tau\left(1-e^{-\frac{T}{\tau}}\right)\|\nu\|_{H^s}\|\tilde{A}\|_{L^\infty(0,T;H^s)},$$
$$\|III\|_{L^\infty(0,T;H^s)} \leq C\left[T+\tau\left(e^{-\frac{T}{\tau}}-1\right)\right]\|\tilde{A}\|^3_{L^\infty(0,T;H^s)}.$$

Donc

$$\|\Phi\tilde{A}\|_X \leq \|\varphi\|_{H^s} + C\tau\left(1-e^{-\frac{T}{\tau}}\right)\|\nu\|_{H^s}\|\tilde{A}\|_X + C\left[T+\tau\left(e^{-\frac{T}{\tau}}-1\right)\right]\|\tilde{A}\|^3_X.$$

Soit $a=\|\varphi\|_{H^s}$, on pose $R=2a$. Pour un temps T suffisamment petit, on a bien

$$\|\Phi\tilde{A}\|_X \leq a + C\tau\left(1-e^{-\frac{T}{\tau}}\right)\|\nu\|_{H^s}R + C\left[T+\tau\left(e^{-\frac{T}{\tau}}-1\right)\right]R^3 \leq R.$$

Il faut vérifier ensuite que l'on a bien une contraction.

$$(\Phi\tilde{A}-\Phi\tilde{B})(t) = I' + II'$$

où

$$I' = -i\frac{\omega_0}{n_0}\int_0^t U(t-\theta)\nu e^{-\frac{\theta}{\tau}}(\tilde{A}-\tilde{B})(\theta)d\theta,$$
$$II' = -i\frac{\omega_0 n_2}{\tau n_0}\int_0^t U(t-\theta)e^{-\frac{1}{\tau}\theta}\int_0^\theta e^{\frac{1}{\tau}\zeta}\left[|\tilde{A}(\zeta)|^2\tilde{A}(\theta)-|\tilde{B}(\zeta)|^2\tilde{B}(\theta)\right]d\zeta d\theta.$$

Les mêmes estimations donnent :

$$\|I'\|_{L^\infty(0,T;H^s)} \leq C\tau\left(1-e^{-\frac{T}{\tau}}\right)\|\nu\|_{H^s}\|\tilde{A}-\tilde{B}\|_{L^\infty(0,T;H^s)},$$

$$\begin{aligned}\|II'\|_{L^\infty(0,T;H^s)} \ \leq\ & C\left[T+\tau\left(e^{-\frac{T}{\tau}}-1\right)\right]\left(\|\tilde{A}\|^2_{L^\infty(0,T;H^s)}+\|\tilde{B}\|^2_{L^\infty(0,T;H^s)}\right)\\ & \times\|\tilde{A}-\tilde{B}\|_{L^\infty(0,T;H^s)}.\end{aligned}$$

Donc

$$\begin{aligned}\|(\Phi\tilde{A}-\Phi\tilde{B})\|_X \leq C\Big\{&\tau\left(1-e^{-\frac{1}{\tau}T}\right)\|\nu\|_{H^s}\\ &+C\left[T+\tau\left(e^{-\frac{T}{\tau}}-1\right)\right]\left(\|\tilde{A}\|^2_X+\|\tilde{B}\|^2_X\right)\Big\}\|\tilde{A}-\tilde{B}\|_X.\end{aligned}$$

Quitte à diminuer encore T (mais en conservant le R précédent), Φ est contractante dans $B_X(0,R)$. On obtient donc l'existence et l'unicité locale en temps de solutions régulières à l'équation de Maxwell-Debye sous forme intégrale. Ceci achève la démonstration du premier point du théorème 2.2.

Preuve du ii) Le second point se démontre à l'aide du même type d'estimations. Pour un p assez grand on peut construire sur un même intervalle de temps I, une solution A_p qui appartient à la boule $B_X(0,R)$, où R ne dépend pas de p.
$\tilde{A}$ est solution de (2.3) :

$$\tilde{A}(t) = U(t)\varphi - \int_0^t U(t-\theta)i\frac{\omega_0}{n_0}\left\{\nu + \int_0^\theta \frac{n_2}{\tau}|\tilde{A}(\zeta)|^2 e^{\frac{\zeta}{\tau}}d\zeta\right\}e^{-\frac{\theta}{\tau}}\tilde{A}(\theta)d\theta.$$

ainsi que $\tilde{A}_p$:

$$\tilde{A}_p(t) = U(t)\varphi_p - \int_0^t U(t-\theta) i\frac{\omega_0}{n_0}\left\{\nu_p + \int_0^\theta \frac{n_2}{\tau}|\tilde{A}_p(\zeta)|^2 e^{\frac{\zeta}{\tau}} d\zeta\right\} e^{-\frac{\theta}{\tau}}\tilde{A}_p(\theta)d\theta.$$

On obtient alors

$$\begin{aligned} \|\tilde{A}(t) - \tilde{A}_p(t)\|_{H^s} &\leq \|\varphi - \varphi_p\|_{H^s} \\ &+ C\tau\left(1 - e^{-\frac{T}{\tau}}\right)\left[\|\nu - \nu_p\|_{H^s}\|\tilde{A}_p\|_X + \|\nu\|_{H^s}\|\tilde{A} - \tilde{A}_p\|_X\right] \\ &+ C\left[T + \tau\left(e^{-\frac{T}{\tau}} - 1\right)\right]\left(\|\tilde{A}\|_X^2 + \|\tilde{A}_p\|_X^2\right)\|\tilde{A} - \tilde{A}_p\|_X. \end{aligned}$$

Donc

$$\begin{aligned} \|\tilde{A}(t) - \tilde{A}_p(t)\|_{H^s} &\leq \|\varphi - \varphi_p\|_{H^s} \\ &+ C\tau\left(1 - e^{-\frac{T}{\tau}}\right) R\left[\|\nu - \nu_p\|_{H^s} + \|\tilde{A} - \tilde{A}_p\|_X\right] \\ &+ C\left[T + \tau\left(e^{-\frac{T}{\tau}} - 1\right)\right] R^2\|\tilde{A} - \tilde{A}_p\|_X. \end{aligned}$$

On prend T suffisamment petit pour que

$$\|\tilde{A} - \tilde{A}_p\|_X \leq \|\varphi - \varphi_p\|_{H^s} + C\|\nu - \nu_p\|_{H^s} + \frac{1}{2}\|\tilde{A} - \tilde{A}_p\|_X$$

c'est-à-dire

$$\|\tilde{A} - \tilde{A}_p\|_X \leq 2\|\varphi - \varphi_p\|_{H^s} + 2C\|\nu - \nu_p\|_{H^s}.$$

Donc $\tilde{A}_p$ tend vers $\tilde{A}$ dans X. Ceci achève la démonstration. ∎

Existence et unicité de solutions plus faibles.

Etudions maintenant des solutions moins régulières, i.e. dans $X' = L^\infty(0,T;H^1)$ et $X" = L^\infty(0,T;L^2)$.
On définit tout d'abord la notion de paire admissible :
La paire (q,r) est dite admissible si $\left(\frac{N}{2} - \frac{N}{q}\right) r = 2$ où N est la dimension d'espace et $r \in [2, \frac{2N}{N-2})$ ($[2,\infty)$ si $N = 2$, $[2,\infty]$ si $N = 1$).

Théorème 2.4 *i) Pour tout (φ,ν) appartenant à $H^1 \times H^1$, l'équation (2.3) admet une unique solution dans $X' = L^\infty(0,T;H^1)$ pour un T suffisamment petit.*
ii) Pour tout (φ,ν) appartenant à $L^2 \times L^\infty$, l'équation (2.3) admet une unique solution appartenant à $X" = L^4(0,T;L^4) \cap C([0,T];L^2)$ pour un T suffisamment petit. De plus $\tilde{A}$ appartient à $L^q(0,T;L^r)$ pour toute paire admissible (q,r).
iii) Les solutions dépendent continuement des données initiales dans un sens analogue à celui donné dans le théorème 2.2

Remarque : Contrairement à tous les autres résultats, le ii) ne fournit pas d'estimations uniformes en τ. Il n'y a donc pas d'espoir de pouvoir passer à la limite.
Pour démontrer ce théorème, nous allons utiliser différents lemmes :

Lemme 2.5 *Si (q,r) et (γ,ρ) sont des paires admissibles, on a*

$$\left\| \int_0^t U(t-s)f(s)ds \right\|_{L^q(0,T;L^r)} \leq K \|f\|_{L^{\gamma'}(0,T;L^{\rho'})},$$

pour tout $f \in L^{\gamma'}(0,T;L^{\rho'})$.

l'inégalité de Strichartz (cf. [9])

Lemme 2.6 *Si $\left(\frac{N}{2} - \frac{N}{q}\right) r = 2$, il existe une constante C, ne dépendant que de N et r, telle que pour tout $\varphi \in L^2$,*

$$\|U(t)\varphi\|_{L^q(\mathbf{R};L^r(\mathbf{R}^n))} \leq C\|\varphi\|_{L^2}.$$

et le fait que

$$\|f\|_{L^4(0,T;L^4)} \leq CT^{1/4}\|f\|_{L^\infty(0,T;H^1)}.$$

Preuve du i)
Montrons que Φ envoie une boule $B_{X'}(0,R)$ dans elle-même.

On obtient alors les estimations suivantes :

$$\|I\|_{L^\infty(0,T;H^1)} = \|\varphi\|_{H^1},$$

$$\begin{aligned}
\|II\|_{L^\infty(0,T;L^2)} &\leq C\left\|e^{-\frac{\theta}{\tau}}\nu\tilde{A}(\theta)\right\|_{L^2(0,T;L^1)}, \\
&\leq C\left\|e^{-\frac{\theta}{\tau}}\nu\right\|_{L^2(0,T;L^2)} \|\tilde{A}\|_{L^\infty(0,T;L^2)}, \\
&\leq C\tau\left(1-e^{-\frac{2}{\tau}T}\right)^{1/2} \|\nu\|_{L^2}\|\tilde{A}\|_{L^\infty(0,T;L^2)}.
\end{aligned}$$

$$\begin{aligned}
\|III\|_{L^\infty(0,T;L^2)} &\leq \frac{C}{\tau}\left\|\left(\int_0^\theta |\tilde{A}(\zeta)|^2 e^{\frac{\zeta}{\tau}}d\zeta\right)\tilde{A}(\theta)e^{-\frac{\theta}{\tau}}\right\|_{L^{4/3}(0,T;L^{4/3})}, \\
&\leq \frac{C}{\tau}\left\|\int_0^\theta |\tilde{A}(\zeta)|^2 e^{\frac{1}{\tau}(\zeta-\theta)}d\zeta\right\|_{L^2(0,T;L^2)} \|\tilde{A}\|_{L^4(0,T;L^4)}.
\end{aligned}$$

Or d'après l'inégalité de Cauchy-Schwarz et le lemme de Fubini

$$\begin{aligned}
\left\| \int_0^\theta |\tilde{A}(\zeta)|^2 e^{\frac{1}{\tau}(\zeta-\theta)} d\zeta \right\|_{L^2_x} &\leq \left(\int_0^\theta e^{\frac{\zeta-\theta}{\tau}} d\zeta \right)^{1/2} \left(\int_0^\theta \|\tilde{A}(\zeta)\|_{L^4}^4 e^{\frac{\zeta-\theta}{\tau}} d\zeta \right)^{1/2}, \\
&\leq \left(\int_0^\theta e^{\frac{\zeta-\theta}{\tau}} d\zeta \right) \|\tilde{A}\|^2_{L^\infty(0,T;L^4)}, \\
&\leq \tau \left(1 - e^{-\frac{\theta}{\tau}}\right) \|\tilde{A}\|^2_{L^\infty(0,T;L^4)},
\end{aligned}$$

et

$$\|\tau \left(1 - e^{-\frac{\theta}{\tau}}\right)\|_{L^2(0,T)} = \tau \left[T - 2\tau(1 - e^{-\frac{T}{\tau}}) + \frac{\tau}{2}(1 - e^{-\frac{2T}{\tau}}) \right]^{1/2},$$

d'où

$$\left\| \int_0^\theta |\tilde{A}(\zeta)|^2 e^{\frac{1}{\tau}(\zeta-\theta)} d\zeta \right\|^2_{L^2(0,T;L^2)} \leq C\tau^2 \left[T - 2\tau(1 - e^{-\frac{T}{\tau}}) + \frac{\tau}{2}(1 - e^{-\frac{2T}{\tau}}) \right] \|\tilde{A}\|^4_{L^\infty(0,T;L^4)}.$$

Ceci donne

$$\begin{aligned}
\|III\|_{L^\infty(0,T;L^2)} &\leq C \left[T - 2\tau(1 - e^{-\frac{T}{\tau}}) + \frac{\tau}{2}(1 - e^{-\frac{2T}{\tau}}) \right] \|\tilde{A}\|_{L^4(0,T;L^4)} \|\tilde{A}\|^2_{L^\infty(0,T;L^4)}, \\
&\leq C \left[T - 2\tau(1 - e^{-\frac{T}{\tau}}) + \frac{\tau}{2}(1 - e^{-\frac{2T}{\tau}}) \right] T^{1/4} \|\tilde{A}\|^3_{L^\infty(0,T;H^1)}.
\end{aligned}$$

Etudions maintenant les gradients :

Posons $\nabla_1 II = II_1 + II_2$ avec

$$II_1 = -i\frac{\omega_0}{n_0} \int_0^t U(t-\theta) \nu e^{-\frac{\theta}{\tau}} \nabla_1 \tilde{A}(\theta) d\theta,$$

$$II_2 = -i\frac{\omega_0}{n_0} \int_0^t U(t-\theta) \nabla_1 \nu e^{-\frac{\theta}{\tau}} \tilde{A}(\theta) d\theta.$$

$$\begin{aligned}
\|II_1\|_{L^\infty(0,T;L^2)} &\leq C \left\| e^{-\frac{\theta}{\tau}} \nu \nabla_1 \tilde{A}(\theta) \right\|_{L^2(0,T;L^1)}, \\
&\leq C \left\| e^{-\frac{\theta}{\tau}} \nu \right\|_{L^2(0,T;L^2)} \|\nabla_1 \tilde{A}\|_{L^\infty(0,T;L^2)}, \\
&\leq C\sqrt{\tau} \left(1 - e^{-\frac{2}{\tau}T}\right)^{1/2} \|\nu\|_{L^2} \|\nabla_1 \tilde{A}\|_{L^\infty(0,T;L^2)}.
\end{aligned}$$

$$\begin{aligned}
\|II_2\|_{L^\infty(0,T;L^2)} &\leq C \left\| e^{-\frac{\theta}{\tau}} \nabla_1 \nu \tilde{A}(\theta) \right\|_{L^2(0,T;L^1)}, \\
&\leq C \left\| e^{-\frac{\theta}{\tau}} \nabla_1 \nu \right\|_{L^2(0,T;L^2)} \|\tilde{A}\|_{L^\infty(0,T;L^2)}, \\
&\leq C\sqrt{\tau} \left(1 - e^{-\frac{2}{\tau}T}\right)^{1/2} \|\nabla_1 \nu\|_{L^2} \|\tilde{A}\|_{L^\infty(0,T;L^2)}.
\end{aligned}$$

On peut maintenant écrire $\nabla_1 III = III_1 + III_2 + III_3$ avec

$$III_1 = -i\frac{\omega_0 n_2}{\tau n_0}\int_0^t U(t-\theta)\left(\int_0^\theta |\tilde{A}(\zeta)|^2 e^{\frac{1}{\tau}\zeta}d\zeta\right)e^{-\frac{1}{\tau}\theta}\nabla_1\tilde{A}(\theta)d\theta,$$

$$III_2 = -i\frac{\omega_0 n_2}{\tau n_0}\int_0^t U(t-\theta)\left(\int_0^\theta \nabla_1\tilde{A}(\zeta)\tilde{A}^*(\zeta)e^{\frac{1}{\tau}\zeta}d\zeta\right)e^{-\frac{1}{\tau}\theta}\tilde{A}(\theta)d\theta,$$

$$III_3 = -i\frac{\omega_0 n_2}{\tau n_0}\int_0^t U(t-\theta)\left(\int_0^\theta \tilde{A}(\zeta)\nabla_1\tilde{A}^*(\zeta)e^{\frac{1}{\tau}\zeta}d\zeta\right)e^{-\frac{1}{\tau}\theta}\tilde{A}(\theta)d\theta.$$

$$\|III_1\|_{L^\infty(0,T;L^2)} \leq \frac{C}{\tau}\left\|\int_0^\theta |\tilde{A}(\zeta)|^2 e^{\frac{\zeta-\theta}{\tau}}d\zeta\right\|_{L^{4/3}(0,T;L^4)}\|\nabla_1\tilde{A}\|_{L^\infty(0,T;L^2)}.$$

Or

$$\begin{aligned}\left\|\int_0^\theta |\tilde{A}(\zeta)|^2 e^{\frac{\zeta-\theta}{\tau}}d\zeta\right\|_{L^4_x} &\leq \left(\int_0^\theta e^{\frac{4}{3}\frac{\zeta-\theta}{2\tau}}d\zeta\right)^{3/4}\left(\int_0^\theta e^{4\frac{\zeta-\theta}{2\tau}}\|\tilde{A}(\zeta)\|_{L^8}^8 d\zeta\right)^{1/4},\\ &\leq \left(\int_0^\theta e^{\frac{4}{3}\frac{\zeta-\theta}{2\tau}}d\zeta\right)^{3/4}\left(\int_0^\theta e^{4\frac{\zeta-\theta}{2\tau}}d\zeta\right)^{1/4}\|\tilde{A}\|^2_{L^\infty(0,T;L^8)},\\ &= C\tau\left(1-e^{-\frac{4}{3}\frac{\theta}{2\tau}}\right)^{3/4}\left(1-e^{-4\frac{\theta}{2\tau}}\right)^{1/4}\|\tilde{A}\|^2_{L^\infty(0,T;L^8)},\end{aligned}$$

et

$$\left\|\left(1-e^{-\frac{4}{3}\frac{\theta}{2\tau}}\right)^{3/4}\left(1-e^{-4\frac{\theta}{2\tau}}\right)^{1/4}\right\|_{L^{4/3}(0,T)} \leq C\left[T-\frac{3\tau}{2}\left(1-e^{-\frac{2}{3\tau}T}\right)\right]^{3/4}.$$

D'où

$$\|III_1\|_{L^\infty(0,T;L^2)} \leq C\left[T-\frac{3\tau}{2}\left(1-e^{-\frac{2}{3\tau}T}\right)\right]^{3/4}\|\tilde{A}\|^3_{L^\infty(0,T;H^1)}.$$

On a également :

$$\begin{aligned}\|III_2\|_{L^\infty(0,T;L^2)} &\leq \frac{C}{\tau}\left\|\int_0^\theta \nabla_1\tilde{A}(\zeta)\tilde{A}^*(\zeta)e^{\frac{\zeta-\theta}{\tau}}d\zeta\right\|_{L^{4/3}(0,T;L^{8/5})}\|\tilde{A}\|_{L^\infty(0,T;L^8)},\\ &\leq \frac{C}{\tau}\left\|\left(\int_0^\theta e^{\frac{8}{3}\frac{\zeta-\theta}{2\tau}}d\zeta\right)^{3/8}\left(\int_0^\theta e^{\frac{8}{5}\frac{\zeta-\theta}{2\tau}}d\zeta\right)^{5/8}\right\|_{L^{4/3}(0,T)}\|\nabla_1\tilde{A}\|_{L^\infty(0,T;L^2)}\\ &\qquad\times\|\tilde{A}^*\|_{L^\infty(0,T;L^8)}\|\tilde{A}\|_{L^\infty(0,T;L^8)},\\ &\leq C\left[T-\frac{3\tau}{4}\left(1-e^{-\frac{4}{3\tau}T}\right)\right]^{3/4}\|\nabla_1\tilde{A}\|_{L^\infty(0,T;L^2)}\|\tilde{A}\|^2_{L^\infty(0,T;L^8)},\\ &\leq C\left[T-\frac{3\tau}{4}\left(1-e^{-\frac{4}{3\tau}T}\right)\right]^{3/4}\|\tilde{A}\|^3_{L^\infty(0,T;H^1)}.\end{aligned}$$

De même

$$\|III_3\|_{L^\infty(0,T;L^2)} \leq C\left[T - \frac{3\tau}{4}\left(1 - e^{-\frac{4}{3\tau}T}\right)\right]^{3/4} \|\tilde{A}\|^3_{L^\infty(0,T;H^1)}.$$

Donc si on prend par exemple $R = 2\|A_0\|_{X'}$, et T suffisamment petit Φ envoie la boule $B_{X'}(0, R)$ dans elle-même.
Démontrons maintenant que l'on a une contraction. La majoration de I' et de son gradient s'effectue exactement comme celle de II. On obtient donc

$$\|I'\|_{L^\infty(0,T;L^2)} \leq C\tau\left(1 - e^{-\frac{2}{\tau}T}\right)^{1/2} \|\nu\|_{L^2}\|\tilde{A} - \tilde{B}\|_{L^\infty(0,T;L^2)},$$

$$\begin{aligned}\|\nabla_1 I'\|_{L^\infty(0,T;L^2)} &\leq C\sqrt{\tau}\left(1 - e^{-\frac{2}{\tau}T}\right)^{1/2} \|\nu\|_{L^2}\|\nabla_1(\tilde{A} - \tilde{B})\|_{L^\infty(0,T;L^2)} \\ &\quad + C\sqrt{\tau}\left(1 - e^{-\frac{2}{\tau}T}\right)^{1/2} \|\nabla_1\nu\|_{L^2}\|\tilde{A} - \tilde{B}\|_{L^\infty(0,T;L^2)}.\end{aligned}$$

Pour le traitement de II', on utilise le fait que

$$\begin{aligned}\left||A(\zeta)|^2 A(\theta) - |B(\zeta)|^2 B(\theta)\right| &\leq |A(\zeta)|^2|A(\theta) - B(\theta)| \\ &+ |A(\zeta) - B(\zeta)|(|A(\zeta)| + |B(\zeta)|)|B(\theta)|.\end{aligned}$$

De la même façon que pour le calcul pour III, on obtient :

$$\begin{aligned}\|II'\|_{L^\infty(0,T;L^2)} &\leq C\left[T - 2\tau(1 - e^{-\frac{T}{\tau}}) + \frac{\tau}{2}(1 - e^{-\frac{2T}{\tau}})\right] \\ &\quad \times \left(\|\tilde{A}\|^2_{L^\infty(0,T;L^4)} + \|\tilde{B}\|^2_{L^\infty(0,T;L^4)}\right)\|\tilde{A} - \tilde{B}\|_{L^4(0,T;L^4)} \\ &\leq C\left[T - 2\tau(1 - e^{-\frac{T}{\tau}}) + \frac{\tau}{2}(1 - e^{-\frac{2T}{\tau}})\right] T^{1/4} \\ &\quad \times \left(\|\tilde{A}\|^2_{L^\infty(0,T;H^1)} + \|\tilde{B}\|^2_{L^\infty(0,T;H^1)}\right)\|\tilde{A} - \tilde{B}\|_{L^\infty(0,T;H^1)}.\end{aligned}$$

Il reste a voir les estimations des gradients des différences.

$$\begin{aligned}&\nabla_1\left(|\tilde{A}(\zeta)|^2\tilde{A}(\theta) - |\tilde{B}(\zeta)|^2\tilde{B}(\theta)\right) \\ &= |\tilde{A}(\zeta)|^2\nabla_1\tilde{A}(\theta) - |\tilde{B}(\zeta)|^2\nabla_1\tilde{B}(\theta) \\ &\quad + \nabla_1\tilde{A}(\zeta)\tilde{A}^*(\zeta)\tilde{A}(\theta) - \nabla_1\tilde{B}(\zeta)\tilde{B}^*(\zeta)\tilde{B}(\theta) \\ &\quad + \tilde{A}(\zeta)\nabla_1\tilde{A}^*(\zeta)\tilde{A}(\theta) - \tilde{B}(\zeta)\nabla_1\tilde{B}^*(\zeta)\tilde{B}(\theta), \\ &= (|\tilde{A}(\zeta)|^2 - |\tilde{B}(\zeta)|^2)\nabla_1\tilde{A}(\theta) + |\tilde{B}(\zeta)|^2(\nabla_1\tilde{A}(\theta) - \nabla_1\tilde{B}(\theta)) \\ &\quad + (\nabla_1\tilde{A}(\zeta)\tilde{A}^*(\zeta) - \nabla_1\tilde{B}(\zeta)\tilde{B}^*(\zeta))\tilde{A}(\theta) + \nabla_1\tilde{B}(\zeta)\tilde{B}^*(\zeta)(\tilde{A}(\theta) - \tilde{B}(\theta)) \\ &\quad + (\tilde{A}(\zeta)\nabla_1\tilde{A}^*(\zeta) - \tilde{B}(\zeta)\nabla_1\tilde{B}^*(\zeta))\tilde{A}(\theta) + \tilde{B}(\zeta)\nabla_1\tilde{B}^*(\zeta)(\tilde{A}(\theta) - \tilde{B}(\theta)).\end{aligned}$$

De même que pour $\nabla_1 III$, on pose :

$$\nabla_1 II' = II'_1 + II'_2 + II'_3 + II'_4 + II'_5 + II'_6,$$

où $\left(\text{avec } C_1 = -i\dfrac{\omega_0 n_2}{\tau n_0}\right)$

$$
\begin{aligned}
II'_1 &= C_1\int_0^t U(t-\theta)\left(\int_0^\theta (|\tilde{A}(\zeta)|^2 - |\tilde{B}(\zeta)|^2)e^{\frac{1}{\tau}\zeta}d\zeta\right)e^{-\frac{1}{\tau}\theta}\nabla_1\tilde{A}(\theta)d\theta,\\
II'_2 &= C_1\int_0^t U(t-\theta)\left(\int_0^\theta |\tilde{B}(\zeta)|^2 e^{\frac{1}{\tau}\zeta}d\zeta\right)e^{-\frac{1}{\tau}\theta}(\nabla_1\tilde{A}(\theta) - \nabla_1\tilde{B}(\theta))d\theta,\\
II'_3 &= C_1\int_0^t U(t-\theta)\left(\int_0^\theta (\nabla_1\tilde{A}(\zeta)\tilde{A}^*(\zeta) - \nabla_1\tilde{B}(\zeta)\tilde{B}^*(\zeta))e^{\frac{1}{\tau}\zeta}d\zeta\right)e^{-\frac{1}{\tau}\theta}\tilde{A}(\theta)d\theta,\\
II'_4 &= C_1\int_0^t U(t-\theta)\left(\int_0^\theta \nabla_1\tilde{B}(\zeta)\tilde{B}^*(\zeta)e^{\frac{1}{\tau}\zeta}d\zeta\right)e^{-\frac{1}{\tau}\theta}(\tilde{A}(\theta) - \tilde{B}(\theta))d\theta,\\
II'_5 &= C_1\int_0^t U(t-\theta)\left(\int_0^\theta (\tilde{A}(\zeta)\nabla_1\tilde{A}^*(\zeta) - \tilde{B}(\zeta)\nabla_1\tilde{B}^*(\zeta))e^{\frac{1}{\tau}\zeta}d\zeta\right)e^{-\frac{1}{\tau}\theta}\tilde{A}(\theta)d\theta,\\
II'_6 &= C_1\int_0^t U(t-\theta)\left(\int_0^\theta \tilde{B}(\zeta)\nabla_1\tilde{B}^*(\zeta)e^{\frac{1}{\tau}\zeta}d\zeta\right)e^{-\frac{1}{\tau}\theta}(\tilde{A}(\theta) - \tilde{B}(\theta))d\theta.
\end{aligned}
$$

II'_1 et II'_2 se majorent en suivant le modèle de III_1. Par contre la majoration des quatre autres termes se fait selon celle de III_2, d'où les résultats :

$$
\begin{aligned}
\|II'_1\|_{L^\infty(0,T;L^2)} \le C\left[T - \frac{3\tau}{2}\left(1 - e^{-\frac{2}{3\tau}T}\right)\right]^{3/4} \|\tilde{A} - \tilde{B}\|_{L^\infty(0,T;H^1)}\\
\times\left(\|\tilde{A}\|_{L^\infty(0,T;H^1)} + \|\tilde{B}\|_{L^\infty(0,T;H^1)}\right)\|\tilde{A}\|_{L^\infty(0,T;H^1)}.
\end{aligned}
$$

$$
\|II'_2\|_{L^\infty(0,T;L^2)} \le C\left[T - \frac{3\tau}{2}\left(1 - e^{-\frac{2}{3\tau}T}\right)\right]^{3/4} \|\tilde{B}\|^2_{L^\infty(0,T;H^1)}\|\tilde{A} - \tilde{B}\|_{L^\infty(0,T;H^1)}.
$$

$$
\begin{aligned}
\|II'_3\|_{L^\infty(0,T;L^2)} \le C\left[T - \frac{3\tau}{4}\left(1 - e^{-\frac{4}{3\tau}T}\right)\right]^{3/4} \|\tilde{A} - \tilde{B}\|_{L^\infty(0,T;H^1)}\\
\times(\|\tilde{A}\|_{L^\infty(0,T;H^1)} + \|\tilde{B}\|_{L^\infty(0,T;H^1)})\|\tilde{A}\|_{L^\infty(0,T;H^1)}.
\end{aligned}
$$

$$
\|II'_4\|_{L^\infty(0,T;L^2)} \le C\left[T - \frac{3\tau}{4}\left(1 - e^{-\frac{4}{3\tau}T}\right)\right]^{3/4} \|\tilde{B}\|^2_{L^\infty(0,T;H^1)}\|\tilde{A} - \tilde{B}\|_{L^\infty(0,T;H^1)}.
$$

$$
\begin{aligned}
\|II'_5\|_{L^\infty(0,T;L^2)} \le C\left[T - \frac{3\tau}{4}\left(1 - e^{-\frac{4}{3\tau}T}\right)\right]^{3/4} \|\tilde{A} - \tilde{B}\|_{L^\infty(0,T;H^1)}\\
\times(\|\tilde{A}\|_{L^\infty(0,T;H^1)} + \|\tilde{B}\|_{L^\infty(0,T;H^1)})\|\tilde{A}\|_{L^\infty(0,T;H^1)}
\end{aligned}
$$

$$\|II_6'\|_{L^\infty(0,T;L^2)} \leq C\left[T - \frac{3\tau}{4}\left(1 - e^{-\frac{4}{3\tau}T}\right)\right]^{3/4} \|\tilde{B}\|^2_{L^\infty(0,T;H^1)}\|\tilde{A} - \tilde{B}\|_{L^\infty(0,T;H^1)}.$$

Quitte à réduire à nouveau T, Φ est une contraction de $B_{X'}(0,R)$ dans elle-même. Ceci démontre la première partie du théorème 2.4.

Preuve du ii)
La plupart des estimations nécessaires ont été effectuées dans la preuve du i). On suit ici la démonstration de Cazenave et Weissler [3] pour l'équation de Schrödinger dans L^2 avec un exposant critique.
Soit (q,r), une paire admissible.

$$\|I\|_{L^q(0,T;L^r)} \leq C_1\|\varphi\|_{L^2},$$

$$\|II\|_{L^q(0,T;L^r)} \leq C_2\|\nu\|_{L^\infty}\|\tilde{A}\|_{L^4(0,T;L^4)},$$

$$\|III\|_{L^q(0,T;L^r)} \leq \frac{CT}{\tau^{1/2}}\|\tilde{A}\|^3_{L^4(0,T;L^4)}.$$

En choisissant dans un premier temps $r = q = 4$ et en faisant des estimations du même type sur des différences, on obtient le fait que Φ est une contraction de $B_{L^4(0,T;L^4)}(0,R)$ pour un T suffisamment petit et R choisi tel que $C_1\|\varphi\|_{L^2}$ et $C_2\|\nu\|_{L^\infty}$ soient inférieurs à $\dfrac{R}{3}$.
Ceci fournit l'existence et l'unicité d'une solution dans cet espace. En prenant à nouveau (q,r) quelconque, on obtient l'appartenance de $\tilde{A}$ à $\mathcal{C}([0,T];L^2)$ et à $L^q(0,T;L^r)$.

Preuve du iii)
La continuité par rapport aux données initiales se démontre de façon analogue à celle du théorème 2.2 en utilisant le même type d'estimations que ci-dessus. ■

Remarque : On observe que la régularité demandée à ν pour effectuer les calculs ci-dessus n'est a priori pas conservée par le flot. Ceci constituerait un grave problème si on avait conservation de la norme H^1 et donc un espoir de prolonger les solutions sur de plus grands temps. Nous n'avons pas trouvé de telle loi de conservation pour l'équation de Maxwell-Debye. On trouve néanmoins aisément la conservation de la quantité :

$$M(\xi,t) = \int_{\mathbf{R}^2(x,y)} |A(\xi,t)|^2.$$

Dans ce qui suit, on démontre que le temps d'existence de la solution de Maxwell-Debye pour une condition initiale donnée est la même dans tous les espaces H^s, $s > 1$. Ce résultat est donné par

Théorème 2.7 *Soit $(\varphi,\nu)\in H^{1+\varepsilon}\times H^{1+\varepsilon}$, et $\tilde{A}$ la solution maximale de l'équation de Maxwell-Debye dans $H^{1+\varepsilon}$. Soit $T_{1+\varepsilon}$ son temps d'existence. Supposons de plus que $(\varphi,\nu)\in H^s\times H^s$ avec $s>1+\varepsilon$, alors $\tilde{A}$ est solution de l'équation de Maxwell-Debye dans $L^\infty(0,T_{1+\varepsilon};H^s)$.*

Preuve :
Pour démontrer ceci, on considère l'équation de Maxwell-Debye sous la forme intégro-différentielle :

$$\frac{\partial\tilde{A}}{\partial t}-\frac{ic}{2kn_0}\nabla_1^2\tilde{A}+i\frac{\omega_0}{n_0}\left\{\nu+\int_0^t\frac{n_2}{\tau}|\tilde{A}(\zeta)|^2e^{\frac{\zeta}{\tau}}d\zeta\right\}e^{-\frac{t}{\tau}}\tilde{A}(t)=0.$$

Soit J^s l'opérateur $(1-\nabla_1^2)^s$. On note $\langle\cdot,\cdot\rangle$, le produit scalaire de $L^2(dx,dy)$.

$$\mathrm{Re}\,\langle\frac{\partial\tilde{A}}{\partial t},J^s\tilde{A}\rangle=\frac{1}{2}\frac{d}{dt}\|\tilde{A}(t)\|_{H^s}^2,$$

$$\mathrm{Re}\,\langle i\nabla_1^2\tilde{A},J^s\tilde{A}\rangle=0,$$

$$\begin{aligned}\left|\langle\nu e^{-\frac{t}{\tau}}\tilde{A}(t),J^s\tilde{A}\rangle\right| &\leq \langle J^{s/2}\nu e^{-\frac{t}{\tau}}\tilde{A}(t),J^{s/2}\tilde{A}\rangle,\\ &\leq \|\tilde{A}(t)\|_{H^s}\|\nu\tilde{A}(t)\|_{H^s},\\ &\leq C\|\tilde{A}(t)\|_{H^s}\left(\|\nu\|_{L^\infty}\|\tilde{A}(t)\|_{H^s}+\|\nu\|_{H^s}\|\tilde{A}(t)\|_{L^\infty}\right),\end{aligned}$$

$$\begin{aligned}\left|\langle\int_0^t|\tilde{A}(\zeta)|^2\tilde{A}(t)e^{\frac{(\zeta-t)}{\tau}}d\zeta,J^s\tilde{A}\rangle\right| &\leq \langle J^{s/2}\int_0^t|\tilde{A}(\zeta)|^2\tilde{A}(t)e^{\frac{(\zeta-t)}{\tau}}d\zeta,J^{s/2}\tilde{A}\rangle,\\ &\leq \|\tilde{A}(t)\|_{H^s}\int_0^t\||\tilde{A}(\zeta)|^2\tilde{A}(t)\|_{H^s}d\zeta,\\ &\leq C\|\tilde{A}(t)\|_{H^s}\int_0^t(\|\tilde{A}(\zeta)\|_{H^s}\|\tilde{A}(\zeta)\|_{L^\infty}\|\tilde{A}(t)\|_{L^\infty}\\ &\qquad+\|\tilde{A}(\zeta)\|_{L^\infty}^2\|\tilde{A}(t)\|_{H^s})d\zeta,\\ &\leq C\|\tilde{A}(t)\|_{H^s}\|\tilde{A}(t)\|_{L^\infty}\int_0^t\|\tilde{A}(\zeta)\|_{H^s}\|\tilde{A}(\zeta)\|_{L^\infty}d\zeta\\ &\qquad+C\|\tilde{A}(t)\|_{H^s}^2\int_0^t\|\tilde{A}(\zeta)\|_{L^\infty}^2d\zeta.\end{aligned}$$

On a supposé que $\tilde{A}\in L^\infty(0,T;H^{1+\varepsilon})$ (i.e. $T<T_{1+\varepsilon}$,) et que $\|\nu\|_{H^s}$ et $\|\varphi\|_{H^s}$ sont finis pour $s>1+\varepsilon$. En utilisant le fait que $H^{1+\varepsilon}\hookrightarrow L^\infty$, et en posant $y=\int_0^t\|\tilde{A}(\zeta)\|_{H^s}d\zeta$, on obtient l'inégalité différentielle :

$$y'y''\leq C(y'^2+yy').$$

Comme $y'=\|\tilde{A}(t)\|_{H^s}>0$, on peut diviser par cette quantité et poser $z=y+y'$. z vérifie alors l'inégalité $z'\leq(C+1)z$ et $z(0)=\|\phi\|_{H^s}$, d'où

$$\|\tilde{A}(t)\|_{H^s}=y'\leq z'\leq\|\phi\|_{H^s}e^{(C+1)t}.$$

■

2.3 Limite quand le retard tend vers 0.

Existence des solutions sur un intervalle de temps indépendant de τ.

Les estimations effectuées précédemment sont uniformes en τ (quand celui-ci tend vers 0). En effet, pour l'étude dans H^s, on a

$$\begin{aligned}
\|\Phi\tilde{A}\|_X &\leq \|\tilde{A}(0)\|_{H^s} + C\tau\left(1-e^{-\frac{T}{\tau}}\right)\|\tilde{n}(0)\|_{H^s}\|\tilde{A}\|_X \\
&\quad + C\left[T+\tau\left(e^{-\frac{T}{\tau}}-1\right)\right]\|\tilde{A}\|_X^3, \\
&\leq \|\tilde{A}(0)\|_{H^s} + C\tau\|\tilde{n}(0)\|_{H^s}\|\tilde{A}\|_X + CT\|\tilde{A}\|_X^3.
\end{aligned}$$

$$\begin{aligned}
\|(\Phi\tilde{A}-\Phi\tilde{B})\|_X &\leq C\Big\{\tau\left(1-e^{-\frac{1}{\tau}T}\right)\|\tilde{n}(0)\|_{H^s} \\
&\quad + C\left[T+\tau\left(e^{-\frac{T}{\tau}}-1\right)\right]\left(\|\tilde{A}\|_X^2+\|\tilde{B}\|_X^2\right)\Big\}\|\tilde{A}-\tilde{B}\|_X, \\
&\leq C\Big\{\tau\|\tilde{n}(0)\|_{H^s} + CT\left(\|\tilde{A}\|_X^2+\|\tilde{B}\|_X^2\right)\Big\}\|\tilde{A}-\tilde{B}\|_X.
\end{aligned}$$

On remarque qu'il en est de même dans le cas des estimations effectuées dans H^1. On peut donc en déduire que les solutions de l'équation de Maxwell-Debye (dans les deux cadres fonctionnels précédemment étudiés) existent sur un intervalle de temps $[0,T]$ qui ne dépend pas de τ ($\in [0,\tau_0]$). Munis de ce résultat, nous allons pouvoir étudier la limite des solutions de l'équation de Maxwell-Debye quand τ tend vers 0. Comme les équations tendent formellement vers l'équation de Schrödinger cubique

$$\frac{\partial\tilde{A}}{\partial t}(t;x,y) - \frac{ic}{2kn_0}\nabla_1^2\tilde{A}(t;x,y) + i\frac{\omega_0 n_2}{n_0}|\tilde{A}|^2\tilde{A}(t;x,y) = 0, \tag{2.4}$$

on espère que les solutions vont tendre vers la solution de cette équation.

Passage à limite pour des solutions fortes.

Théorème 2.8 *On suppose que les données initiales (pour $\tilde{A}$ et $\tilde{n}$) sont bornées uniformément dans $X = L^\infty(0,T;H^s)$, $s>3$, et que quand τ tend vers 0, la donnée initiale φ tend fortement vers ψ dans H^s. Soit A, la solution de l'équation de Schrödinger cubique associée à cette donnée ψ. Alors la suite des $\tilde{A}$ quand τ tend vers 0 tend fortement vers A dans X.*

Pour démontrer ce théorème, nous allons utiliser comme ingrédient principal le théorème d'Ascoli-Arzela.

Les hypothèses sur les données initiales du théorème 2.8 assurent que la suite des solutions est uniformément bornée dans X. Par ailleurs

$$\begin{aligned}
&\frac{\partial\tilde{A}}{\partial t}(t;x,y) - \frac{ic}{2kn_0}\nabla_1^2\tilde{A}(t;x,y) + \\
&\quad + i\frac{\omega_0}{n_0}\left\{\nu(x,y) + \int_0^t \frac{n_2}{\tau}|\tilde{A}(\zeta;x,y)|^2 e^{\frac{\zeta}{\tau}}d\zeta\right\} e^{-\frac{t}{\tau}}\tilde{A}(t;x,y) = 0.
\end{aligned}$$

On se place dans le cas où H^{s-2} est une algèbre (pour une plus grande simplicité des calculs), c'est-à-dire que $s > 3$. On a alors

$$\begin{aligned}\left\|\frac{\partial\tilde{A}}{\partial t}\right\|_{H^{s-2}} \leq\ & \frac{C}{2kn_0}\left\|\tilde{A}(t)\right\|_{H^s} \\ & +\frac{\omega_0}{n_0}\|\nu\|_{H^{s-2}}e^{-\frac{t}{\tau}}\left\|\tilde{A}(t)\right\|_{H^{s-2}} \\ & +\frac{\omega_0 n_2}{n_0}\left(\int_0^t\frac{e^{\frac{\zeta-t}{\tau}}}{\tau}\left\|\tilde{A}(\zeta)\right\|^2_{H^{s-2}}d\zeta\right)\left\|\tilde{A}(t)\right\|_{H^{s-2}}.\end{aligned}$$

Pour des τ et T suffisamment petits, ceci est borné uniformément en τ dans X. D'après le théorème d'Ascoli-Arzela, on peut affirmer qu'il existe une sous-suite $\tilde{A}$ qui tend dans $\mathcal{C}(0,T;H^{s-\varepsilon})$ pour tout $\varepsilon > 0$ vers une fonction A quand τ tend vers 0.

Il reste donc à vérifier que cette limite est bien solution de l'équation de Schrödinger cubique (2.4).

On remarque aisément que $\nu e^{-\frac{t}{\tau}}\tilde{A}(t)$ tend vers 0 quand τ tend vers 0. Le seul terme qui pose problème est le terme non linéaire. On veut donc montrer que $\int_0^t\frac{1}{\tau}|\tilde{A}(\zeta;x,y)|^2e^{\frac{\zeta-t}{\tau}}d\zeta$ tend en un sens à préciser vers $|A(t)|^2$.

$\int_0^t\frac{1}{\tau}|\tilde{A}(\zeta)|^2e^{\frac{\zeta-t}{\tau}}d\zeta-|A(t)|^2$

$$\begin{aligned}&=\int_0^{t-\eta}\frac{1}{\tau}|\tilde{A}(\zeta)|^2e^{\frac{\zeta-t}{\tau}}d\zeta+\int_{t-\eta}^t\frac{1}{\tau}|\tilde{A}(\zeta)|^2e^{\frac{\zeta-t}{\tau}}d\zeta-|A(t)|^2,\\ &=\int_0^{t-\eta}\frac{1}{\tau}|\tilde{A}(\zeta)|^2e^{\frac{\zeta-t}{\tau}}d\zeta+\int_{t-\eta}^t\frac{1}{\tau}(|\tilde{A}(\zeta)|^2-|A(\zeta)|^2)e^{\frac{\zeta-t}{\tau}}d\zeta\\ &\quad+\int_{t-\eta}^t\frac{1}{\tau}(|A(\zeta)|^2-|A(t)|^2)e^{\frac{\zeta-t}{\tau}}d\zeta-e^{-\frac{\eta}{\tau}}|A(t)|^2.\end{aligned}$$

On majore très facilement chacun des termes de cette somme en norme H^σ où σ est strictement inférieur à s :

D'après le résultat de convergence, pour tout α, il existe un τ_0 tel que, pour tout $\tau < \tau_0$,

$$\left\|\int_{t-\eta}^t\frac{1}{\tau}(|\tilde{A}(\zeta)|^2-|A(\zeta)|^2)e^{\frac{\zeta-t}{\tau}}d\zeta\right\|_{H^\sigma}\leq\alpha.$$

D'après la continuité et donc l'uniforme continuité sur $[0,T]$ de A

$$\left\||A(\zeta)|^2-|A(t)|^2\right\|_{H^\sigma}\leq\alpha,$$

dès que $\zeta - t < \eta$ et donc

$$\left\|\int_{t-\eta}^t\frac{1}{\tau}(|A(\zeta)|^2-|A(t)|^2)e^{\frac{\zeta-t}{\tau}}d\zeta\right\|_{H^\sigma}\leq\varepsilon.$$

Pour un ε donné ceci fixe η. Pour cet η,

$$\left\| \int_0^{t-\eta} \frac{1}{\tau} |\tilde{A}(\zeta)|^2 e^{\frac{\zeta - t}{\tau}} d\zeta \right\|_{H^\sigma} \leq \left\| \tilde{A} \right\|_X^2 \left(e^{-\frac{\eta}{\tau}} - e^{-\frac{t}{\tau}} \right),$$

qui tend vers 0 uniformément sur $[0, T]$. Et enfin $\left\| e^{-\frac{\eta}{\tau}} |A(t)|^2 \right\|_{H^\sigma}$ tend vers 0 quand τ tend vers 0.
On remarque que tous ces résultats sont uniformes en temps, on obtient donc la convergence forte dans $L^\infty(0, T; H^{s-\varepsilon})$.
On a donc bien vérifié qu'une sous-suite (et donc toute la suite) de $\tilde{A}$ converge dans $L^\infty(0, T; H^{s-\varepsilon})$ vers la solution de l'équation de Schrödinger cubique.

Passage à la limite pour des solutions faibles.

On procède exactement de la même façon que dans le cas des solutions régulières. On ne peut plus utiliser la structure d'algèbre, mais les majorations sont encore faciles à obtenir. En effet :

$$\begin{aligned}
\left\| \frac{\partial \tilde{A}}{\partial t} \right\|_{L^\infty(0,T;H^{-1})} &\leq \frac{C}{2kn_0} \left\| \tilde{A}(t) \right\|_{L^\infty(0,T;H^1)} + \frac{\omega_0}{n_0} \left\| \nu e^{-\frac{t}{\tau}} \tilde{A}(t) \right\|_{L^\infty(0,T;H^{-1})} \\
&\quad + \frac{\omega_0 n_2}{n_0} \left\| \left(\int_0^t \frac{e^{\frac{\zeta - t}{\tau}}}{\tau} \tilde{A}(\zeta) d\zeta \right) \tilde{A}(t) \right\|_{L^\infty(0,T;H^{-1})}, \\
&\leq \frac{C}{2kn_0} \left\| \tilde{A}(t) \right\|_{L^\infty(0,T;H^1)} + \frac{\omega_0}{n_0} \left\| \nu e^{-\frac{t}{\tau}} \tilde{A}(t) \right\|_{L^\infty(0,T;L^{3/4})} \\
&\quad + \frac{\omega_0 n_2}{n_0} \left\| \left(\int_0^t \frac{e^{\frac{\zeta - t}{\tau}}}{\tau} \tilde{A}(\zeta) d\zeta \right) \tilde{A}(t) \right\|_{L^\infty(0,T;L^{3/4})}, \\
&\leq \frac{C}{2kn_0} \left\| \tilde{A}(t) \right\|_{L^\infty(0,T;H^1)} + \frac{\omega_0}{n_0} \|\nu\|_{L^{3/2}} \left\| \tilde{A}(t) \right\|_{L^\infty(0,T;L^{3/2})} \\
&\quad + \frac{\omega_0 n_2}{n_0} \left\| \int_0^t \frac{e^{\frac{\zeta - t}{\tau}}}{\tau} \tilde{A}(\zeta) d\zeta \right\|_{L^\infty(0,T;L^2)} \left\| \tilde{A}(t) \right\|_{L^2(0,T;L^4)}, \\
&\leq \frac{C}{2kn_0} \left\| \tilde{A}(t) \right\|_{L^\infty(0,T;H^1)} + \frac{\omega_0}{n_0} \|\nu\|_{H^1} \left\| \tilde{A}(t) \right\|_{L^\infty(0,T;H^1)} \\
&\quad + C \frac{\omega_0 n_2}{n_0} T^{1/2} \left\| \tilde{A}(t) \right\|_{L^\infty(0,T;H^1)}^3 .
\end{aligned}$$

La suite du raisonnement est analogue à celle effectuée précédemment. On peut donc énoncer le théorème suivant :

Théorème 2.9 *On suppose que les données initiales (pour $\tilde{A}$ et $\tilde{n}$) sont bornées uniformément dans $X = L^\infty(0, T; H^1)$, et que quand τ tend vers 0, la donnée initiale φ tend fortement vers ψ dans H^1. Soit A, la solution de l'équation de Schrödinger cubique associée à cette donnée ψ. Alors la suite des $\tilde{A}$ quand τ tend vers 0 tend fortement vers A dans X.*

3 Les équations de Maxwell-Bloch.

3.1 Modélisation.

Nous allons maintenant écrire les équations de Maxwell-Bloch qui décrivent les interactions d'une onde électromagnétique unidirectionnelle avec un milieu constitué par un gaz d'atomes à deux niveaux. Comme on néglige l'effet Doppler, la polarisation est de la forme

$$\vec{P} = n_a(\vec{p}_{12}\rho_{12} + \vec{p}_{21}\rho_{21}).$$

On suppose de plus que le champ est polarisé dans une unique direction perpendiculaire à sa direction de propagation z. Sans perte de généralité, on peut supposer que la direction de l'élément de la matrice dipolaire est parallèle à celui du champ électrique. On a donc

$$\vec{E} = \vec{A}(x,y,z,t)e^{i(\omega/c)z}e^{-i\omega_c t} + c.c.$$

(où $c.c.$ désigne le complexe conjugué) et

$$\vec{P} = \vec{L}(x,y,z,t)e^{i(\omega/c)z}e^{-i\omega_c t} + c.c.$$

On fait de plus l'approximation de la variation lente de l'enveloppe. L'équation (1.3) devient alors

$$\frac{\partial \vec{A}}{\partial z} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} - i\frac{c}{2\omega}\nabla_1^2\vec{A} + \frac{\kappa}{c}\vec{A} = \frac{i\omega}{2\varepsilon_0 c}\vec{L}$$

où κ décrit des déperditions dues par exemple à des miroirs.
Notons $\hat{e}$ la direction de polarisation du champ (c'est-à-dire que $\vec{A}(x,y,z,t) = \hat{e}A(x,y,z,t)$) et p le module de $\vec{p}_{12}$. On a alors

$$n_a p \rho_{12} = L(x,y,z,t)e^{i(\omega/c)z - i\omega t}.$$

En écrivant l'équation de Bloch pour ρ_{12} et $\rho_{22} - \rho_{11} = \dfrac{N}{n_a}$ (on néglige les termes qui contiennent les secondes harmoniques $e^{\pm 2i\omega t}$), en supposant que $\gamma_{11} = \gamma_{22}$ et en posant $\omega_{12} = \omega_1 - \omega_2 > 0$, on obtient

$$\left\{\begin{array}{l} \dfrac{\partial A}{\partial z} + \dfrac{1}{c}\dfrac{\partial A}{\partial t} - i\dfrac{c}{2\omega}\nabla_1^2 A + \dfrac{\kappa}{c}A = \dfrac{i\omega}{2\varepsilon_0 c}L, \\[2ex] \dfrac{\partial L}{\partial t} + (\gamma_{12} + i(\omega_{12} - \omega))L = \dfrac{ip^2}{\hbar}AN, \\[2ex] \dfrac{\partial N}{\partial t} + \gamma_{11}N = \dfrac{2i}{\hbar}(A^*L - AL^*). \end{array}\right.$$

Il faut fournir de l'énergie au milieu pour le maintenir actif, c'est-à-dire que l'on force une partie des atomes à être dans un état excité. La simulation de cet apport d'énergie est donnée par un terme constant de la forme $\gamma_{11}N_0$ dans l'équation qui régit le nombre d'inversion N. Les équations de Maxwell-Bloch régissant une onde unidirectionnelle, polarisée dans une seule direction dans un milieu constitué d'atomes à deux niveaux d'énergie sont donc

$$\begin{cases} \dfrac{\partial A}{\partial z} + \dfrac{1}{c}\dfrac{\partial A}{\partial t} - i\dfrac{c}{2\omega}\nabla_1^2 A + \dfrac{\kappa}{c}A = \dfrac{i\omega}{2\varepsilon_0 c}L, \\ \dfrac{\partial L}{\partial t} + (\gamma_{12} + i(\omega_{12} - \omega))L = \dfrac{ip^2}{\hbar}AN, \\ \dfrac{\partial N}{\partial t} + \gamma_{11}(N - N_0) = \dfrac{2i}{\hbar}(A^*L - AL^*). \end{cases} \tag{3.1}$$

3.2 Etude de l'équation quasi-stationnaire.

Quand on néglige la variation en temps de L et de M (approximation adiabatique), on obtient les nouvelles équations :

$$\begin{cases} \dfrac{\partial A}{\partial z} + \dfrac{1}{c}\dfrac{\partial A}{\partial t} - i\dfrac{c}{2\omega}\nabla_1^2 A + \dfrac{\kappa}{c}A = \dfrac{i\omega}{2\varepsilon_0 c}L, \\ (\gamma_{12} + i(\omega_{12} - \omega))L = \dfrac{ip^2}{\hbar}AN, \\ \gamma_{11}(N - N_0) = \dfrac{2i}{\hbar}(A^* L - AL^*). \end{cases}$$

En effectuant des substitutions, on obtient

$$\frac{\partial A}{\partial z} + \frac{1}{c}\frac{\partial A}{\partial t} - i\frac{c}{2\omega}\nabla_1^2 A + \frac{\kappa}{c}A = \frac{i\omega}{2\varepsilon_0 c}L \tag{3.2}$$

où

$$L = \frac{ip^2}{\hbar}\frac{(\gamma_{12} - i(\omega_{12} - \omega))N_0 A}{\gamma_{12}^2 + (\omega_{12} - \omega)^2 + \frac{4p^2\gamma_{12}}{\hbar^2\gamma_{11}}|A|^2}. \tag{3.3}$$

Si on pose $\xi = ct - z$ et $A(x, y, z, t) = \bar{A}(\xi, t; x, y)$ et des notations analogues pour $\bar{L}$ et $\bar{N}$, on obtient

$$\frac{\partial \bar{A}}{\partial t} - i\frac{c^2}{2\omega}\nabla_1^2 \bar{A} + \kappa \bar{A} = \frac{i\omega}{2\varepsilon_0}\bar{L}.$$

On remarque alors aisément que

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbf{R}^2}|\bar{A}(t)|^2 dxdy \quad &+ \quad \kappa\int_{\mathbf{R}^2}|\bar{A}(t)|^2 dxdy \\ &= \quad -\frac{\omega p^2}{2\varepsilon_0\hbar}\int_{\mathbf{R}^2}\frac{\gamma_{12}N_0|\bar{A}(t)|^2}{\gamma_{12}^2 + (\omega_{12} - \omega)^2 + \frac{4p^2\gamma_{12}}{\hbar^2\gamma_{11}}|\bar{A}(t)|^2}dxdy \end{aligned} \tag{3.4}$$

Il existe donc une constante D telle que

$$\frac{\partial}{\partial t}\int_{\mathbf{R}^2}|\bar{A}(t)|^2 dxdy \leq D\int_{\mathbf{R}^2}|\bar{A}(t)|^2 dxdy.$$

Le second membre est du même signe que $-N_0$. Dans le cas où $N_0 \geq 0$, on peut majorer ce second membre par 0 et poser $D = -2\kappa < 0$. Dans le cas où $N_0 < 0$, on peut majorer le second membre par

$$-\frac{\omega p^2}{2\varepsilon_0\hbar}\frac{\gamma_{12}N_0}{\gamma_{12}^2 + (\omega_{12} - \omega)^2}\int_{\mathbf{R}^2}|\bar{A}(t)|^2 dxdy$$

et on pose $D = -2\kappa - \dfrac{\omega p^2}{\varepsilon_0\hbar}\cdot\dfrac{\gamma_{12}N_0}{\gamma_{12}^2 + (\omega_{12} - \omega)^2}$, qui est négatif si

$$\kappa > \frac{\omega p^2}{2\varepsilon_0\hbar}\cdot\frac{-\gamma_{12}N_0}{\gamma_{12}^2 + (\omega_{12} - \omega)^2}.$$

D'après le lemme de Gronwall

$$\int_{\mathbf{R}^2} |\bar{A}(t)|^2 dxdy \leq \left(\int_{\mathbf{R}^2} |\bar{A}(0)|^2 dxdy\right) e^{Dt}. \tag{3.5}$$

Sur un intervalle de temps fini la norme L^2 reste donc bornée. Cette norme est bornée indépendamment du temps et tend vers 0 quand le temps tend vers $+\infty$ dans tous les cas où on peut choisir $D < 0$.

On a de façon évidente l'existence et l'unicité locale en temps dans L^2 et (3.5) assure également que ce résultat est en fait global.

Pour établir une théorie dans H^1, il suffit de remarquer que

$$\nabla_1 L = \frac{CN_0\nabla_1\bar{A}(\alpha+\beta|\bar{A}|^2) + CN_0A(\alpha+\beta[\bar{A}\nabla_1\bar{A}^* + \bar{A}^*\nabla_1\bar{A}])}{(\alpha+\beta|\bar{A}|^2)^2}$$

où $C = \frac{ip^2}{\hbar}(\gamma_{12} - i(\omega_{12}-\omega))$, $\alpha = \gamma_{12}^2 + (\omega_{12}-\omega)^2$ et $\beta = \frac{4p^2\gamma_{12}}{\hbar^2\gamma_{11}}$. En effectuant des estimations du même genre que de celles effectuées dans le cadre de l'étude de l'équation de Maxwell-Debye dans H^1, on obtient l'existence et l'unicité locale d'une solution de cette équation quasi-stationnaire dans H^1.

Une autre estimation possible est

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(\gamma_{12}^2 + (\omega_{12}-\omega)^2)\int_{\mathbf{R}^2}|\bar{A}(t)|^2 dxdy + \frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}}\int_{\mathbf{R}^2}|\bar{A}(t)|^4 dxdy\right]$$
$$+\left[\kappa(\gamma_{12}^2 + (\omega_{12}-\omega)^2) + \frac{\omega p^2\gamma_{12}}{2\varepsilon_0\hbar}N_0\right]\int_{\mathbf{R}^2}|\bar{A}(t)|^2 dxdy + 4\kappa\frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}}\int_{\mathbf{R}^2}|\bar{A}(t)|^4 dxdy = 0.$$

Dans le cas où on avait $D < 0$ précédemment, la quantité

$$\kappa(\gamma_{12}^2 + (\omega_{12}-\omega)^2) + \frac{\omega p^2\gamma_{12}}{2\varepsilon_0\hbar}N_0$$

est positive. $4\kappa\frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}}$ est également toujours positif. Il existe alors une constante C telle que

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(\gamma_{12}^2 + (\omega_{12}-\omega)^2)\int_{\mathbf{R}^2}|\bar{A}(t)|^2 dxdy + \frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}}\int_{\mathbf{R}^2}|\bar{A}(t)|^4 dxdy\right]$$
$$+C\left[\frac{1}{2}(\gamma_{12}^2 + (\omega_{12}-\omega)^2)\int_{\mathbf{R}^2}|\bar{A}(t)|^2 dxdy + \frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}}\int_{\mathbf{R}^2}|\bar{A}(t)|^4 dxdy\right] \leq 0,$$

et d'après le lemme de Gronwall

$$\left[\frac{1}{2}(\gamma_{12}^2 + (\omega_{12}-\omega)^2)\int_{\mathbf{R}^2}|\bar{A}(t)|^2 dxdy + \frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}}\int_{\mathbf{R}^2}|\bar{A}(t)|^4 dxdy\right]$$
$$\leq \left[\frac{1}{2}(\gamma_{12}^2 + (\omega_{12}-\omega)^2)\int_{\mathbf{R}^2}|\bar{A}(0)|^2 dxdy + \frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}}\int_{\mathbf{R}^2}|\bar{A}(0)|^4 dxdy\right] e^{-Ct}.$$

Les normes L^2 et L^4 de $A(t)$ décroissent donc avec le temps.

On remarque de plus que $\|\bar{L}(t)\|_{L^2} \leq \frac{p^2}{\hbar(\gamma_{12}^2 + (\omega_{12}-\omega)^2)^{1/2}}|N_0|\|\bar{A}(t)\|_{L^2}$.

Théorème 3.10 *Le problème de Cauchy est globalement bien posé dans L^2 et dans H^1 pour l'approximation adiabatique de l'équation de Maxwell-Bloch. De plus pour certaines valeurs des paramètres $\left(\kappa > \dfrac{\omega p^2}{2\varepsilon_0\hbar}\cdot\dfrac{-\gamma_{12}N_0}{\gamma_{12}^2+(\omega_{12}-\omega)^2}\right)$, les normes L^2 de $\bar{A}$ et de $\bar{L}$ tendent vers 0 quand t tend vers $+\infty$.*

Remarque : Cet amortissement est clairement dû aux deux coefficients positifs κ et γ_{12}.

3.3 Le problème de Cauchy local.

Mise en forme.

Considérons à nouveau les équations de Maxwell-Bloch sous la forme :

$$\begin{cases} \dfrac{\partial A}{\partial z} + \dfrac{1}{c}\dfrac{\partial A}{\partial t} - i\dfrac{c}{2\omega}\nabla_1^2 A + \dfrac{\kappa}{c}A = \dfrac{i\omega}{2\varepsilon_0 c}L, \\ \dfrac{\partial L}{\partial t} + (\gamma_{12} + i(\omega_{12} - \omega))L = \dfrac{ip^2}{\hbar}AN, \\ \dfrac{\partial N}{\partial t} + \gamma_{11}(N - N_0) = \dfrac{2i}{\hbar}(A^*L - AL^*). \end{cases}$$

Les équations de Maxwell-Bloch réécrites, après le changement de variable $\xi = ct - z$ et avec les mêmes notations que pour l'équation quasi-stationnaire, sont de la forme :

$$\begin{cases} \dfrac{\partial \bar{A}}{\partial t} - i\dfrac{c^2}{2\omega}\nabla_1^2 \bar{A} + \kappa\bar{A} = \dfrac{i\omega}{2\varepsilon_0}\bar{L}, \\ c\dfrac{\partial \bar{L}}{\partial \xi} + \dfrac{\partial \bar{L}}{\partial t} + (\gamma_{12} + i(\omega_{12} - \omega))\bar{L} = \dfrac{ip^2}{\hbar}\bar{A}\bar{N}, \\ c\dfrac{\partial \bar{N}}{\partial \xi} + \dfrac{\partial \bar{N}}{\partial t} + \gamma_{11}(\bar{N} - N_0) = \dfrac{2i}{\hbar}(\bar{A}^*\bar{L} - \bar{A}\bar{L}^*). \end{cases} \tag{3.6}$$

On pose $\bar{M} = \bar{N} - N_0$ d'où :

$$\begin{cases} \dfrac{\partial \bar{A}}{\partial t} - i\dfrac{c^2}{2\omega}\nabla_1^2 \bar{A} + \kappa\bar{A} = \dfrac{i\omega}{2\varepsilon_0}\bar{L}, \\ \dfrac{\partial \bar{L}}{\partial t} + c\dfrac{\partial \bar{L}}{\partial \xi} + (\gamma_{12} + i(\omega_{12} - \omega))\bar{L} = \dfrac{ip^2}{\hbar}(\bar{A}N_0 + \bar{A}\bar{M}), \\ \dfrac{\partial \bar{M}}{\partial t} + c\dfrac{\partial \bar{M}}{\partial \xi} + \gamma_{11}\bar{M} = \dfrac{2i}{\hbar}(\bar{A}^*\bar{L} - \bar{A}\bar{L}^*). \end{cases}$$

On écrit ceci sous forme intégrale en utilisant l'opérateur U associé à l'équation linéaire $\dfrac{\partial A}{\partial t} - i\dfrac{c^2}{2\omega}\nabla_1^2 A = 0$ et en considérant comme temps initial $t_0 = 0$. Les données initiales pour $\bar{A}$, $\bar{L}$ et $\bar{M}$ sont appelées respectivement φ, λ et μ.

$$\left\{\begin{array}{rcl} \bar{A}(\xi,t;x,y) & = & U(t)\varphi(\xi;x,y) \\ & + & \displaystyle\int_0^t U(t-\theta)\left[-\kappa\bar{A}+\frac{i\omega}{2\varepsilon_0}\bar{L}\right](\xi,\theta;x,y)d\theta, \\ \bar{L}(\xi,t;x,y) & = & \lambda(\xi-ct;x,y) \\ & + & \displaystyle\int_0^t \left[-(\gamma_{12}+i(\omega_{12}-\omega))\bar{L}+\frac{ip^2}{c\hbar}(\bar{A}N_0+\bar{A}\bar{M})\right] \\ & & (\xi-c(t-\theta),\theta;x,y)d\theta, \\ \bar{M}(\xi,t;x,y) & = & \mu(\xi-ct;x,y) \\ & + & \displaystyle\int_0^t \left[-\gamma_{11}\bar{M}+\frac{2i}{\hbar}(\bar{A}^*\bar{L}-\bar{A}\bar{L}^*)\right](\xi-c(t-\theta),\theta;x,y)d\theta. \end{array}\right. \tag{3.7}$$

Proposition 3.11 *Si $\bar{A}$, $\bar{L}$ et $\bar{M}$ appartiennent à $L^\infty(\xi;0,T;L^2)$ alors les problèmes (3.6) et (3.7) sont équivalents.*

On veut effectuer une méthode de point fixe, on pose donc

$$\left\{\begin{array}{rcl} \Phi_{\bar{A}}(\xi,t;x,y) & = & U(t)\varphi(\xi;x,y) \\ & + & \displaystyle\int_0^t U(t-\theta)\left[-\kappa\bar{A}+\frac{i\omega}{2\varepsilon_0}\bar{L}\right](\xi,\theta;x,y)d\theta, \\ \Phi_{\bar{L}}(\xi,t;x,y) & = & \lambda(\xi-ct;x,y) \\ & + & \displaystyle\int_0^t \left[-(\gamma_{12}+i(\omega_{12}-\omega))\bar{L}+\frac{ip^2}{c\hbar}(\bar{A}N_0+\bar{A}\bar{M})\right] \\ & & (\xi-c(t-\theta),\theta;x,y)d\theta, \\ \Phi_{\bar{M}}(\xi,t;x,y) & = & \mu(\xi-ct;x,y) \\ & + & \displaystyle\int_0^t \left[-\gamma_{11}\bar{M}+\frac{2i}{\hbar}(\bar{A}^*\bar{L}-\bar{A}\bar{L}^*)\right](\xi-c(t-\theta),\theta;x,y)d\theta. \end{array}\right.$$

Existence et unicité de solutions régulières.

On va chercher une solution dans $(L^\infty(\xi;L^\infty(0,T;H^s(x,y))))^3 =: X^3$ pour $s>1$.

Théorème 3.12 *i) Pour tout $(\varphi,\lambda,\mu)\in L^\infty(\xi;H^s)\times L^\infty(\xi;H^s)\times L^\infty(\xi;H^s)$, l'équation (3.7) admet une unique solution dans $X^3=(L^\infty(\xi,0,T;H^s))^3$ pour un T suffisamment petit.*
ii) Les solutions dépendent continuement des données initiales dans un sens analogue à celui donné dans le théorème 2.2

Preuve :

$$\begin{aligned}
\|\Phi_{\bar{A}}(\xi)\|_{L^\infty(0,T;H^s(x,y))} &\leq \|U(t)\varphi(\xi)\|_{L^\infty(0,T;H^s(x,y))} \\
&+ \left\|\int_0^t U(t-\theta)\left[-\kappa\bar{A} + \frac{i\omega}{2\varepsilon_0}\bar{L}\right](\xi;\theta)d\theta\right\|_{L^\infty(0,T;H^s(x,y))}, \\
&\leq \|\varphi(\xi)\|_{H^s(x,y)} \\
&+ K\left\|-\kappa\bar{A}(\xi) + \frac{i\omega}{2\varepsilon_0}\bar{L}(\xi)\right\|_{L^1(0,T;H^s(x,y))}, \\
&\leq \|\varphi(\xi)\|_{H^s(x,y)} \\
&+ KT\left(\left\|\bar{A}(\xi)\right\|_{L^\infty(0,T;H^s(x,y))} + \left\|\bar{L}(\xi)\right\|_{L^\infty(0,T;H^s(x,y))}\right).
\end{aligned}$$

Donc

$$\|\Phi_{\bar{A}}\|_X \leq \|\varphi\|_{\bar{L}^\infty(\xi;H^s(x,y))} + KT\left(\left\|\bar{A}\right\|_X + \left\|\bar{L}\right\|_X\right).$$

$$\begin{aligned}
\|\Phi_{\bar{L}}\|_X \leq &\|\lambda\|_{L^\infty(\xi;H^s(x,y))} \\
&+T\left\|-(\gamma_{12} + i(\omega_{12}-\omega))\bar{L} + \frac{ip^2}{\hbar}(\bar{A}N_0 + \bar{A}\bar{M})(\xi - c(t-\theta),\theta)\right\|_X, \\
\leq &\|\lambda\|_{L^\infty(\xi;H^s(x,y))} + KT\left(\left\|\bar{L}\right\|_X + \left\|\bar{A}\right\|_X N_0 + \left\|\bar{A}\right\|_X \left\|\bar{M}\right\|_X\right).
\end{aligned}$$

De même

$$\|\Phi_{\bar{M}}\|_X \leq \|\mu\|_{L^\infty(\xi;H^s(x,y))} + KT\left(\left\|\bar{M}\right\|_X + \left\|\bar{A}\right\|_X \left\|\bar{L}\right\|_X\right).$$

Donc en prenant

$$\frac{R}{2} = \sup\left(\|\varphi\|_{L^\infty(\xi;H^s(x,y))}, \|\lambda\|_{L^\infty(\xi;H^s(x,y))}, \|\mu\|_{L^\infty(\xi;H^s(x,y))}\right),$$

Φ envoie la boule $B_{X^3}(0,R)$ dans elle-même pour un T suffisamment petit.
Pour ce qui est de la contraction, si on considère deux solutions $(\bar{A}_1, \bar{L}_1, \bar{M}_1)$ et

$(\bar{A}_2, \bar{L}_2, \bar{M}_2)$ du système initialisés aux mêmes valeurs en $t = 0$, on obtient

$$\left\{\begin{array}{rcl}(\Phi_{\bar{A}_1} - \Phi_{\bar{A}_2})(\xi;t;x,y) & = & \int_0^t U(t-\theta)\left[-\kappa(\bar{A}_1 - \bar{A}_2) + \frac{i\omega}{2\varepsilon_0}(\bar{L}_1 - \bar{L}_2)\right](\xi,\theta;x,y)d\theta, \\ (\Phi_{\bar{L}_1} - \Phi_{\bar{L}_2})(\xi;t;x,y) & = & \int_0^t \Big[-(\gamma_{12} + i(\omega_{12} - \omega))(\bar{L}_1 - \bar{L}_2) \\ & + & \frac{ip^2}{\hbar}\Big((\bar{A}_1 - \bar{A}_2)N_0 + (\bar{A}_1 - \bar{A}_2)\bar{M}_1 + (\bar{M}_1 - \bar{M}_2)\bar{A}_2\Big)\Big] \\ & & (\xi - c(t-\theta),\theta;x,y)d\theta, \\ (\Phi_{\bar{M}_1} - \Phi_{\bar{M}_2})(\xi;t;x,y) & = & \int_0^t \Big[-\gamma_{11}(\bar{M}_1 - \bar{M}_2) \\ & + & \frac{2i}{\hbar}\Big((\bar{A}_1^* - \bar{A}_2^*)\bar{L}_1 - (\bar{A}_1 - \bar{A}_2)\bar{L}_1^* \\ & & +(\bar{L}_1 - \bar{L}_2)\bar{A}_2^* - (\bar{L}_1 - \bar{L}_2)^*\bar{A}_2\Big)\Big](\xi - c(t-\theta),\theta;x,y)d\theta. \end{array}\right.$$

En faisant le même type d'estimations que précédemment, on obtient (en réduisant éventuellement l'intervalle de temps) le fait que Φ est une contraction dans la boule considérée. Ceci fournit donc l'existence et l'unicité dans X^3, c'est-à-dire la première partie du théorème. On démontre la continuité par rapport aux données initiales comme dans le cas de l'équation de Maxwell-Debye. ∎

De même que pour l'équation de Maxwell-Debye, le temps d'existence est le même dans tous les H^s. Ceci est donné par le théorème

Théorème 3.13 *Soit* $(\varphi, \lambda, \mu) \in L^\infty(\xi; H^{1+\varepsilon}) \times L^\infty(\xi; H^{1+\varepsilon}) \times L^\infty(\xi; H^{1+\varepsilon})$, *et* $(\bar{A}, \bar{L}, \bar{M})$ *la solution maximale de l'équation de Maxwell-Bloch dans* $H^{1+\varepsilon}$. *Soit* $T_{1+\varepsilon}$ *sont temps d'existence. Supposons de plus que* $(\varphi, \lambda, \mu) \in L^\infty(\xi; H^s) \times L^\infty(\xi; H^s) \times L^\infty(\xi; H^s)$ *avec* $s > 1 + \varepsilon$, *alors* $(\bar{A}, \bar{L}, \bar{M})$ *est solution de l'équation de Maxwell-Bloch dans* $(L^\infty(0, T_{1+\varepsilon}; H^s))^3$.

Preuve :
La méthode est la même que dans le cas de l'équation de Maxwell-Debye, mais cette fois-ci nous ne disposons plus d'une unique équation.
On garde la forme initiale pour l'équation donnant A et on considère les équations pour L et M après application de la méthode des caractéristiques, on se place alors sur une seule caractéristique et on définit $\tilde{A}$, $\tilde{L}$ et $\tilde{M}$ comme pour l'équation de

Maxwell-Debye, on obtient alors :

$$\left\{\begin{array}{rcl} \dfrac{\partial \tilde{A}}{\partial t}(t) & = & i\dfrac{c^2}{2\omega}\nabla_1^2\tilde{A}(t) - \kappa\tilde{A}(t) + \dfrac{i\omega}{2\varepsilon_0}\tilde{L}(t), \\ \tilde{L}(t) & = & \lambda + \displaystyle\int_0^t \left[-(\gamma_{12} + i(\omega_{12} - \omega))\tilde{L} + \dfrac{ip^2}{c\hbar}(\tilde{A}N_0 + \tilde{A}\tilde{M})\right](\theta)d\theta, \\ \tilde{M}(t) & = & \mu + \displaystyle\int_0^t \left[-\gamma_{11}\tilde{M} + \dfrac{2i}{\hbar}(\tilde{A}^*\tilde{L} - \tilde{A}\tilde{L}^*)\right](\theta)d\theta. \end{array}\right. \tag{3.8}$$

On multiplie la première équation par $J^s A$ et on prend la partie réelle du produit scalaire dans L^2. Ceci fournit la première majoration

$$\frac{1}{2}\frac{d}{dt}\|\tilde{A}(t)\|_{H^s}^2 + \kappa\|\tilde{A}(t)\|_{H^s}^2 \leq C\|\tilde{L}(t)\|_{H^s}\|\tilde{A}(t)\|_{H^s},$$

ce qui compte tenu du fait que κ est positif implique que

$$\|\tilde{A}(t)\|_{H^s} \leq \|\varphi\|_{H^s} + C\int_0^t \|\tilde{L}(\theta)\|_{H^s}d\theta.$$

On prend ensuite la norme H^s des deux équations pour L et M, ce qui donne en tenant compte du fait que la norme $H^{1+\varepsilon}$ est bornée sur l'intervalle de temps considérée :

$$\begin{array}{rcl} \|\tilde{L}(t)\|_{H^s} & \leq & \|\lambda\|_{H^s} \\ & + & C\displaystyle\int_0^t \{\|\tilde{L}(\theta)\|_{H^s} + \|\tilde{A}(\theta)\|_{H^s} + \|\tilde{M}(\theta)\|_{H^s}\}d\theta, \end{array}$$

$$\begin{array}{rcl} \|\tilde{M}(t)\|_{H^s} & \leq & \|\mu\|_{H^s} \\ & + & C\displaystyle\int_0^t \{\|\tilde{M}(\theta)\|_{H^s} + \|\tilde{A}(\theta)\|_{H^s} + \|\tilde{L}(\theta)\|_{H^s}\}d\theta. \end{array}$$

On a donc

$$\begin{array}{rcl} \|\tilde{A}(t)\|_{H^s} + \|\tilde{L}(t)\|_{H^s} + \|\tilde{M}(t)\|_{H^s} & \leq & (\|\varphi\|_{H^s} + \|\lambda\|_{H^s} + \|\mu\|_{H^s}) \\ & + & C\displaystyle\int_0^t \left(\|\tilde{A}(\theta)\|_{H^s} + \|\tilde{L}(\theta)\|_{H^s} + \|\tilde{M}(\theta)\|_{H^s}\right)d\theta \end{array}$$

ce qui par le lemme de Gronwall assure que le temps d'existence des solutions de l'équation de Maxwell-Bloch pour des données initiales dans H^s est le même que dans H^2. ■

4 Conclusion

Les modèles choisis ici sont parmi les plus simples en optique non linéaire. Il existe entre autres de nombreux systèmes du même type que celui de Maxwell-Bloch mais avec par exemple un plus grand nombre d'états excités possibles. Pour un atome à trois niveaux, on obtient deux équations de Schrödinger couplées entre elles et avec 6 équations de transport :

$$\begin{aligned}
&\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)A_1+\frac{i}{2k}\nabla_1^2A_1+\alpha_1A_1=i\frac{\omega_1}{2\varepsilon_0c}n_ap_{1m}\sigma_{m1},\\
&\left(\pm\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)A_2+\frac{i}{2k}\nabla_1^2A_2+\alpha_2A_2=i\frac{\omega_2}{2\varepsilon_0c}n_ap_{2m}\sigma_{m2},\\
&\frac{\partial}{\partial t}\sigma_{11}+\gamma_{\|}^1(\sigma_{11}-\sigma_{11}^0)=\frac{i}{\hbar}A_1^*p_{1m}\sigma_{m1}-\frac{i}{\hbar}A_1p_{m1}\sigma_{1m},\\
&\frac{\partial}{\partial t}\sigma_{22}+\gamma_{\|}^2(\sigma_{22}-\sigma_{22}^0)=\frac{i}{\hbar}A_2^*p_{2m}\sigma_{m2}-\frac{i}{\hbar}A_2p_{m2}\sigma_{2m},\\
&\frac{\partial}{\partial t}\sigma_{mm}+\gamma_{\|}^3(\sigma_{mm}-\sigma_{mm}^0)=\frac{i}{\hbar}(A_1p_{1m}\sigma_{1m}+A_2p_{m2}\sigma_{m2})-\frac{i}{\hbar}(A_1^*p_{1m}\sigma_{1m}+A_2^*p_{2m}\sigma_{2m}),\\
&\frac{\partial}{\partial t}\sigma_{1m}+(\gamma_{1m}+i(\omega_1+\omega_{1m}))\sigma_{1m}=\frac{i}{\hbar}p_{1m}A_1^*(\sigma_{mm}-\sigma_{11})-\frac{i}{\hbar}p_{2m}A_2^*\sigma_{12},\\
&\frac{\partial}{\partial t}\sigma_{2m}+(\gamma_{2m}+i(\omega_2+\omega_{2m}))\sigma_{2m}=\frac{i}{\hbar}p_{2m}A_2^*(\sigma_{mm}-\sigma_{22})-\frac{i}{\hbar}p_{1m}A_1^*\sigma_{21},\\
&\frac{\partial}{\partial t}\sigma_{12}+(\gamma_{12}+i(\omega_1-\omega_2+\omega_{12}))\sigma_{12}=\frac{i}{\hbar}p_{1m}A_1^*\sigma_{m2}-\frac{i}{\hbar}p_{m2}A_2\sigma_{1m}.
\end{aligned}$$

L'équation de Maxwell-Debye admet également des généralisations. C'est le cas lorsque l'on remplace l'onde unidirectionnelle par deux ondes se propageant en sens contraires : $\vec{E}=\hat{e}\left(A_1e^{i(kz-\omega t)}+A_2e^{-i(kz-\omega t)}+c.c.\right)$. On obtient alors deux équations de Schrödinger couplées avec deux équations de retard :

$$\begin{aligned}
\left(\frac{\partial}{\partial z}+\frac{n_0}{c}\frac{\partial}{\partial t}\right)A_1-\frac{i}{2k_0}\nabla_1^2A_1 &= -i\frac{\omega_0}{c}\left(\delta n_0A_1+\delta n_1A_2\right),\\
\left(-\frac{\partial}{\partial z}+\frac{n_0}{c}\frac{\partial}{\partial t}\right)A_2-\frac{i}{2k_0}\nabla_1^2A_2 &= +i\frac{\omega_0}{c}\left(\delta n_0A_2+\delta n_1^*A_1\right),\\
\tau\frac{\partial}{\partial t}\delta n_0+\delta n_0 &= n_2\left(A_1A_1^*+A_2A_2^*\right),\\
\tau\frac{\partial}{\partial t}\delta n_1+\delta n_1 &= n_2A_1A_2^*.
\end{aligned}$$

Pour ces équations, l'étude du problème de Cauchy se fait sur le modèle de celles effectuées dans cet article. D'autres résultats sont en préparation tels l'explosion en temps fini ou l'existence d'ondes solitaires.

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