On the coefficient problem and multifractality of whole-plane SLE

Thanh Binh Le

UNIVERSITÉ D'ORLÉANS

École doctorale Mathématiques, Informatiques, Physique théorique et Ingéniérie des systèmes

LABORATOIRE : MAPMO

THÈSE présentée par :

Thanh Binh LE

soutenue le : **05 Décembre 2016**

pour obtenir le grade de : **Docteur de l'université d'Orléans**

Discipline : **Mathématiques**

SUR LE PROBLÈME DE COEFFICIENT ET LA MULTIFRACTALITÉ DE WHOLE-PLANE SLE

THÈSE DIRIGÉE PAR :

Michel ZINSMEISTER	Professeur, Université d'Orléans - Directeur de thèse

RAPPORTEURS :

Steffen ROHDE	Professeur, Université de Washington
Fredrik VIKLUND	Professeur, KTH Stockholm

JURY :

Matthieu ASTORG	MCF, Université d'Orléans
Julien BARRAL	Professeur, Université Paris 13
Athanasios BATAKIS	MCF, Université d'Orléans
Bertrand DUPLANTIER	Directeur de Recherche, IPhT, Saclay
Xiong JIN	Lecturer, Université de Manchester
Fredrik VIKLUND	Professeur Associée, KTH Stockholm
Michel ZINSMEISTER	Professeur, Université d'Orléans

Remerciements

Ce manuscrit conclut trois ans de travail, et je tiens en ces quelques lignes à exprimer ma reconnaissance envers tous ceux qui de près ou de loin y ont contribué.

En premier lieu, je tiens à remercier Michel Zinsmeister, mon directeur de thèse pour son encadrement, ses conseils, ainsi que pour ses encouragements, son enthousiasme, sa confiance et sa patience. Je lui suis reconnaissante pour son aide précieuse dans la relecture et la correction de ma thèse.

Je remercie cordialement Messieurs Steffen Rohde et Fredrik Viklund d'avoir accepté d'être rapporteurs de cette thèse. Je souhaite adresser également mes remerciements à Messieurs Matthieu Astorg, Julien Barral, Athanasios Batakis, Bertrand Duplantier, Xiong Jin et Fredrik Viklund pour l'honneur qu'ils me font en acceptant d'être membres de mon jury.

Je tien à remercier particulièrement à Bertrand Duplantier et Hieu Ho pour toutes nos discussions à mon travail. Les discussions que j'ai eu avec eux m'ont beaucoup apporté.

Je souhaite aussi remercier le Gouvernement Vietnamien pour financer mes études. Il m'a donné l'occasion de travailler dans un bon environnement de recherche en France.

Inscrit en thèse de Mathématiques à l'Université d'Orléans, j'ai passé trois ans au sein du laboratoire MAPMO. Je sais gré à chaque membre du laboratoire et du département de son soutien et de ses témoignages d'amitié. Un merci tout particulier aux (ex-)thésards, à Anne Liger, Marie-France Grespier, Marie-Laurence Poncet, secrétaires du laboratoire du MAPMO pour leur convivialité et leurs aides.

Je remercie mes amis, Anh, Lan, Hieu, Nga, Duy, Khoa, Nhat, Tien, Amina, Gregoire, Han-Yong pour leurs conseils et leur soutien pendant ces trois années loin de ma famille. Un sincère remerciement à Nicole pour m'avoir aidée en français.

Enfin, ce travail de thèse aurait été impossible sans le soutien affectif de ma petite famille: ma femme Xuan Quynh et ma petite fille Quynh Trang qui m'ont permis de persévérer toutes ces années. Je souhaite également remercier profondément mes parents pour leur amour et leur soutien sans faille.

Contents

Introduction

The starting point of this thesis was to study the logarithmic coefficients of conformal maps associated with the *interior whole-plane* SLE_κ *process.* The starting motivation was to revisit Bieberbach's conjecture in the framework of SLE_κ theory. In this section, we recall Bieberbach's conjecture as well as the history of the proof. We will also give definition for SLE_κ.

Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a holomorphic function in the unit disk $\mathbb{D}$. Bieberbach proved in 1916 [3] that if f is further assumed to be injective, then

$$|a_2| \leq 2|a_1|,$$

and he conjectured that

$$|a_n| \leq n|a_1|$$

for all $n > 2$. This famous conjecture has been finally proved in 1984 by de Branges [4]. His proof was made possible by the addition of a new idea (an inequality of Askey and Gasper) to a series of methods and results developed in almost a century. The earliest important contribution to the proof of Bieberbach's conjecture is the proof [14] by Charles Loewner in 1923 for $n = 3$. De Branges' one indeed used Loewner's idea in a essential way.

Let $\gamma : [0,\infty) \to \mathbb{C}$ be a simple curve such that $|\gamma(t)| \to +\infty$ as $t \to +\infty$ and such that $\gamma(t) \neq 0, t \geq 0$. Define then for each $t > 0$, the slit domain $\Omega_t = \mathbb{C} \setminus \gamma([t,\infty))$ being a simply connected domain containing 0 and we can thus consider the Riemann mapping $f_t : \mathbb{D} = \{|z| < 1\} \to \Omega_t$ characterized by $f_t(0) = 0, f_t'(0) > 0$. By the Caratheodory convergence theorem, f_t converges as $t \to 0$ to $f := f_0$, the Riemann mapping of Ω_0. Assuming without loss of generality that $f_0'(0) = 1$ and changing the time t if necessary, we may choose the normalization $f_t'(0) = e^t$, $t \geq 0$.

The key idea of Loewner is to observe that the sequence of domains Ω_t is increasing, which translates into the fact that $\Re\big(\frac{\partial f_t}{\partial t}/z\frac{\partial f_t}{\partial z}\big) > 0$ or, equivalently, that this quantity is the Poisson integral of a positive measure on the unit circle, actually a probability measure, due to the above normalization. Since the domains Ω_t are slit domains, this probability measure must be a Dirac mass at $\lambda(t) = f_t^{-1}(\gamma(t)) \in \partial\mathbb{D}$. We notice that λ is a continuous function from $[0,\infty)$ to the unit circle. Loewner has shown that the Loewner chain (f_t) associated with (Ω_t) is driven by the function λ, in the sense that (f_t) satisfies the following PDE:

$$\frac{\partial}{\partial t} f_t(z) = z\frac{\partial}{\partial z} f_t(z)\frac{\lambda(t) + z}{\lambda(t) - z}. \tag{0.0.1}$$

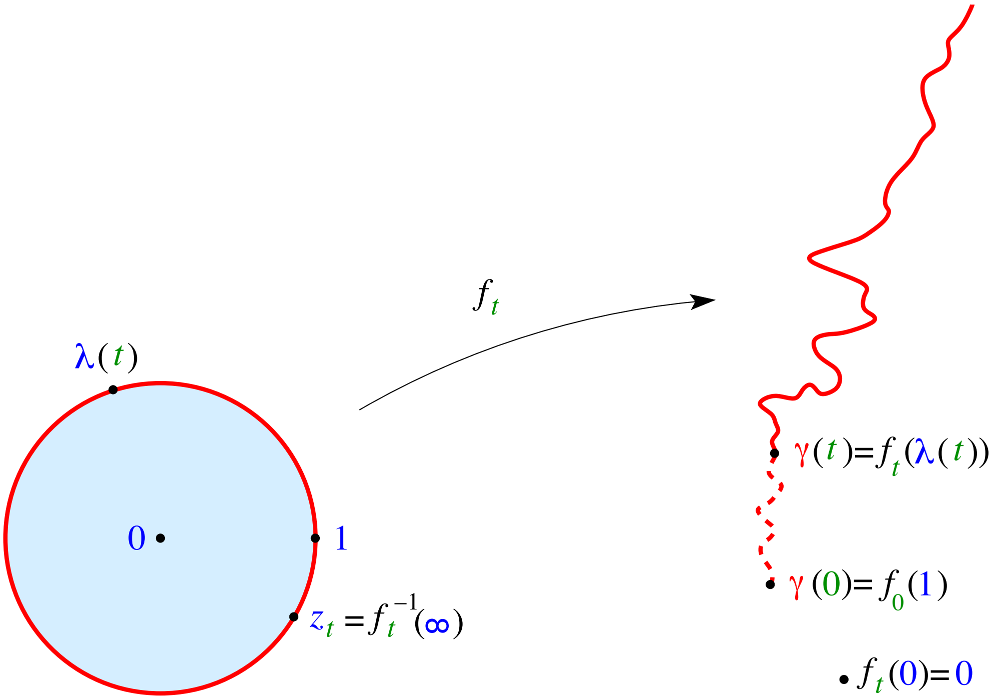

Figure 1: *Loewner map $z \mapsto f_t(z)$ from $\mathbb{D}$ to the slit domain $\Omega_t = \mathbb{C}\backslash\gamma([t,\infty))$ (here slit by a single curve $\gamma([t,\infty))$ for* $\mathrm{SLE}_{\kappa\leq 4}$*). One has $f_t(0) = 0, \forall t \geq 0$. At $t = 0$, the driving function $\lambda(0) = 1$, so that the image of $z = 1$ is at the tip $\gamma(0) = f_0(1)$ of the curve (Fig. 1 in [6]).*

This PDE is named after him. In 1923, Loewner used his equation, with the sole information that $|\lambda(t)| = 1$, to prove that $|a_3| \leq 3|a_1|$. We shortly summarize Loewner's method as follows:

By extending both sides of the Loewner equation as power series, with $f_t(z) = e^t(z + a_2(t)z^2 + a_3(t)z^3 + \cdots)$, and identifying the coefficients, one gets

$$\dot{a}_2 - a_2 = 2\bar{\lambda},$$
$$\dot{a}_3 - 2a_3 = 4a_2\bar{\lambda} + 2\bar{\lambda}^2.$$

The dot means a t-derivative and $\bar{\lambda}$ means the complex conjugate of λ. Because of the uniform bound $(|a_2(t)| \leq C_2 < +\infty, |a_3(t)| \leq C_3 < +\infty, \forall t \geq 0)$, these yield

$$a_2(t) = -2e^t \int_t^{+\infty} e^{-s}\bar{\lambda}(s)ds,$$
$$a_3(t) = 4e^{2t}\left(\int_t^{+\infty} e^{-s}\bar{\lambda}(s)ds\right)^2 - 2e^{2t}\int_t^{+\infty} e^{-2s}\bar{\lambda}^2(s)ds.$$

The first equation implies that $|a_2| \leq 2\int_t^{+\infty} e^{-s}ds \leq 2$, a new proof of Bieberbach's conjecture in the $n = 2$ case. For a_3, one remarks that it suffices to prove that $\Re(a_3) \le 3$. From the remaining equation and applying the Cauchy-Schwarz inequality, one can complete the proof (see Appendix A of Ref. [6] for details).

Besides Loewner's theory of growth processes, de Branges' proof also relied much on the consideration, developed by Grunsky [7] and later Lebedev and Milin [10], of *logarithmic coefficients*. More precisely, if $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and injective with $f(0) = 0$, we may consider the power series,

$$\log \frac{f(z)}{z} = 2 \sum_{n \ge 1} \gamma_n z^n.$$

The purpose of introducing this logarithm was to prove Robertson's conjecture [20], which was known to imply Bieberbach's. Let f be in the class $\mathcal{S}$ of schlicht functions, i.e., holomorphic and injective in the unit disk, and normalized as $f(0) = 0, f'(0) = 1$. There is a branch $f^{[2]}$ of $z \mapsto \sqrt{f(z^2)}$ which is an odd function in $\mathcal{S}$. We then write

$$f^{[2]}(z) := z\sqrt{f(z^2)/z^2} = \sum_{n=0}^{\infty} b_{2n+1} z^{2n+1}, \tag{0.0.2}$$

with $b_1 = 1$. Robertson's conjecture states that:

$$\forall n \ge 0,\ \sum_{k=0}^{n} |b_{2k+1}|^2 \le n + 1. \tag{0.0.3}$$

Since

$$a_n = b_1 b_{2n-1} + b_3 b_{2n-3} + \cdots + b_{2n-1} b_1,$$

it is apparent that Robertson's conjecture implies Bieberbach's. For $n = 2$, Robertson's conjecture is the same as $|a_2| \le 2$. Using Loewner's method, Robertson proved in 1936 that the conjecture is true for $n = 3$.

Lebedev and Milin approached Robertson's conjecture with observing that

$$\log \frac{f^{[2]}(\sqrt{z})}{\sqrt{z}} = \frac{1}{2} \log \frac{f(z)}{z},$$

and consequently that

$$\sum_{n=0}^{\infty} b_{2n+1} z^n = \exp\left(\sum_{n=1}^{\infty} \gamma_n z^n \right).$$

They proved an inequality that is now called the second Lebedev-Milin inequality, a relation between the coefficients of any power series to those of its exponential, namely

$$\forall n \ge 0,\ \sum_{k=0}^{n} |b_{2k+1}|^2 \le (n+1) \exp\left(\frac{1}{n+1} \sum_{m=1}^{n} \sum_{k=1}^{m} \left(k|\gamma_k|^2 - \frac{1}{k} \right) \right). \tag{0.0.4}$$

From the above inequality, Milin [16] naturally conjectured that

$$\forall f \in \mathcal{S}, \forall n \geq 1,\ \sum_{m=1}^{n} \sum_{k=1}^{m} \left(k|\gamma_k|^2 - \frac{1}{k} \right) \leq 0.$$

Milin's conjecture, proved in 1984 by de Branges, implies Robertson's, hence Bieberbach's conjecture.

We now return to Loewner's theory. It is remarkable that Loewner's method has a converse: given any continuous function $\lambda : [0, \infty) \to \mathbb{C}$ with $|\lambda(t)| = 1$ for $t \geq 0$, then the Loewner equation (0.0.1), supplemented by the boundary (initial) condition $\lim_{t \to +\infty} f_t(e^{-t}z) = z$, has a solution $f_t(z)$, such that $(f_t(z))_{t \geq 0}$ is a chain of Riemann maps onto simply connected domains (Ω_t) that are increasing with t.

However, Loewner's ideas go far beyond Bieberbach's conjecture: In 1999, Oded Schramm [21] introduced into the Loewner equation (0.0.1) the *random* driving function,

$$\lambda(t) := e^{i\sqrt{\kappa}B_t}, \tag{0.0.5}$$

where B_t is standard one dimensional Brownian motion and κ a non-negative parameter, thereby making Eq. (0.0.1) a stochastic PDE and creating the celebrated *Schramm-Loewner Evolution* (or *Stochastic Loewner Evolution*) with parameter κ (SLE_κ). There exist several variants of SLE_κ known as *chordal, radial.*

The associated conformal maps f_t from $\mathbb{D}$ to $\mathbb{C} \setminus \gamma([t, \infty))$, obeying (0.0.1) for (0.0.5), define the *interior whole-plane* SLE_κ. Their coefficients $a_n(t)$, which are random variables, are defined by the normalized series expansion:

$$f_t(z) = e^t \Big(z + \sum_{n \geq 2} a_n(t) z^n \Big), \tag{0.0.6}$$

The starting point of this thesis was to study their logarithmic coefficients $\gamma_n(t)$, which are also random variables, defined as

$$\log \frac{e^{-t} f_t(z)}{z} = 2 \sum_{n \geq 1} \gamma_n(t) z^n. \tag{0.0.7}$$

Besides, we also consider some other problems such as *McMullen's asymptotic variance*, or *Grunsky coefficients matrix.* For the reader's convenience, we describe briefly the content of each chapter in this thesis.

In chapter one, we first recall a result obtained in Ref. [6], that is an explicit expression for the expectations of the coefficients $a_n := a_n(0)$ of the expansion (0.0.6). By the same method as used in [6], we will give an explicit expression for the expectations of the logarithmic coefficients $\gamma_n := \gamma_n(0)$ of the expansion (0.0.7). We end the chapter by introducing an algorithm to compute $\mathbb{E}(|\gamma_n(0)|^2)$, together with computations.

Chapter 2 is devoted to introduce the main results obtained in our article [5]. In particular, we give expressions in closed form for the mixed moments,

$$\mathbb{E}\left(\frac{(f'(z))^{p/2}}{(f(z))^{q/2}} \right); \quad \mathbb{E}\left(\frac{|f'(z)|^p}{|f(z)|^q} \right) \text{ (here } f(z) := f_0(z)),$$

along an integrability curve $\mathcal{R}$, which is a *parabola* in the (p, q) plane depending on the SLE parameter κ.

If $p = 2$ and $q = 0$ then, due to Parseval's formula, explicit expression for $\mathbb{E}\left(|f(z)|^2\right)$ yields $\mathbb{E}\left(|a_n|^2\right)$. For example, as given in Ref. [6], $\mathbb{E}\left(|a_n|^2\right) = 1$ for SLE_6 and $\mathbb{E}\left(|a_n|^2\right) = n$ for SLE_2. Similarly, if one knows $\mathbb{E}\left(|f'(z)/f(z)|^2\right)$ then $\mathbb{E}\left(|\gamma_n|^2\right)$ is known. In particular, we prove that $\mathbb{E}\left(|\gamma_n|^2\right) = 1/2n^2$ for SLE_2.

In chapter 3, we first give expressions for the SLE_κ functions,

$$F(z) := \mathbb{E}\left(\frac{(f'(z))^{p/2}}{(f(z)/z)^{q/2}}\right); \quad G(z, \bar{z}) := \mathbb{E}\left(\frac{|f'(z)|^p}{|f(z)/z|^q}\right),$$

for some special parameters (p, q). The rest of this chapter is devoted to the study on the generalized integral means spectrum, $\beta(p, q; \kappa)$, corresponding to the singular behavior of the mixed moments $\mathbb{E}\left(|f'(z)|^p / |f(z)|^q\right)$. This is also an important part in our work [5].

In chapter 4, we study the (expected) McMullen asymptotic variance of SLE_2. More precisely, we determine explicit expression for $\mathbb{E}(|\log f'(z)|^2)$ for SLE_2, whereby we prove that this function satisfies the following formula:

$$\lim_{p \to 0} \frac{2\bar{\beta}(p, f)}{p^2} = \lim_{r \to 1^-} \frac{1}{4\pi |\log(1-r)|} \int_{|z|=r} \mathbb{E}(|\log f'(z)|^2)|dz|,$$

where $\bar{\beta}(p, f)$ is the average integral means spectrum of the (time 0) interior whole-plane SLE_2 map f.

In chapter 5, we study the Grunsky coefficients, in expectation, for interior whole-plane SLE_κ. In particular, with the support of MAPLE, we can compute the coeficients $d_{n,m}$ defined by the power series,

$$G(z, \xi) := \mathbb{E}\left((z - \xi)^q \frac{f'(z)^p f'(\xi)^p}{(f(z) - f(\xi))^q}\right) = \sum_{n,m=0}^{\infty} d_{n,m} z^n \xi^m, \quad (z, \xi) \in \mathbb{D} \times \mathbb{D},$$

for all $p, q \in \mathbb{R}$. We will give explicit expressions for $G(z, \xi)$ for special values of $(p, q; \kappa)$.

Chapter 1

Radial, whole-plane SLE_κ processes

1.1 Mathematical background

Loewner equation involves Riemann mapping theorem for simply connected domains. Therefore, we begin this section with a brief introduction to this subject.

1.1.1 Simply connected domains

An arc in a metric space X is a continuous mapping $\gamma : [a, b] \subset \mathbb{R} \to X$. Such an arc is said to be closed if $\gamma(a) = \gamma(b)$. Two arcs γ_1, γ_2 defined on the same interval $[a, b]$ are said to be *homotopic* if there exists $\Gamma : [a, b] \times [0, 1] \to X$ continuous such that

$$\forall s \in [a, b], \Gamma(s, 0) = \gamma_1(s), \Gamma(s, 1) = \gamma_2(s).$$

Definition 1.1.1. The space X is called simply-connected if it is connected and if every closed arc $\gamma : [a, b] \to X$ is homotopic to a constant arc $\gamma_0 : [a, b] \to \gamma(a)$.

When X is a plane domain we have the following equivalent characterizations of simply connected domains:

Theorem 1.1.2. *For a connected open subset Ω of $\mathbb{C}$ the followings are equivalent:*

i. Ω is simply connected,

ii. $\bar{\mathbb{C}} \setminus \Omega$ is connected,

iii. For any closed arc γ whose image lies in Ω and any $z \notin \Omega$, $\mathrm{Ind}(z, \gamma) = 0$.

We recall that $\mathrm{Ind}(z, \gamma)$ stands for the variation of the argument (measured in number of turns) of $\gamma(t) - z$ along $[a, b]$. When γ is piecewise C^1 this quantity is also equal to

$$\frac{1}{i2\pi} \int_a^b \frac{\gamma'(s)}{\gamma(s) - z} ds = \frac{1}{i2\pi} \int_\gamma \frac{1}{\zeta - z} d\zeta.$$

Theorem 1.1.3. (Riemann Mapping Theorem). *Let Ω be a simply connected proper subdomain of $\mathbb{C}$ and $w \in \mathbb{C}$. Then there exists a unique biholomorphic map $g : \Omega \to \mathbb{D}$ such that $g(\omega) = 0, g'(\omega) > 0$.*

An equivalent statement is that there exists a unique holomorphic bijection $f : \mathbb{D} \to \Omega$ sending 0 to $z_0 \in \Omega$ and $f'(0) > 0$. The specific map f will be called the *Riemann map* for z_0.

1.1.2 Caratheodory convergence theorem

Definition 1.1.4. Let U_n be a sequence of open sets in $\mathbb{C}$ containing 0. Let V_n be the connected component of the interior of $\bigcap_{k\leq n} U_k$ containing 0. The *kernel* of the sequence is defined to be the union of the V_n's, provided that it is non-empty; otherwise it is defined to be $\{0\}$. Thus the kernel is either a connected open set containing 0 or the one point set $\{0\}$.

The sequence is said to converge to a kernel if each subsequence has the same kernel. We now recall the Caratheodory convergence theorem.

Theorem 1.1.5. (Caratheodory convergence theorem). *Let (f_n) be a sequence of holomorphic univalent functions on the unit disk $\mathbb{D}$, normalized so that $f_n(0) = 0$ and $f_n'(0) > 0$. Then f_n converges uniformly on compacta in $\mathbb{D}$ to a function f if and only if $U_n = f_n(\mathbb{D})$ converges to its kernel and this kernel is not $\mathbb{C}$. If this kernel is $\{0\}$, then $f = 0$. Otherwise the kernel is a connected open set U, f is univalent on $\mathbb{D}$ and $f(\mathbb{D}) = U$.*

1.1.3 Brownian motion - Itô formula

Definition 1.1.6. A *standard, one-dimentional Brownian motion* is a continuous-time stochastic process $(B_t)_{t\geq 0}$ characterised by the following properties:

i. $B_0 = 0$ and B_t is continuous in t (a.s.);

ii. Stationarity: if $0 \leq s \leq t$, the $B_t - B_s$ has the same law as B_{t-s};

iii. Markov property: $B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_k} - B_{t_{k-1}}$ are independent for all $0 \leq t_1 < t_2 < \cdots < t_k$. We say that B_t has *independent increments.*

iv. Gaussianity: B_t has a normal distribution with mean 0 and variance t.

Definition 1.1.7. A *standard, n-dimentional Brownian motion* is a (n-dimentional) continuous-time stochastic process $B_t = (B_t^{(1)}, \cdots, B_t^{(n)})$ such that each $B_t^{(i)}$ is a standard Brownian motion and the $B_t^{(i)}$'s are independent of each other.

Theorem 1.1.8. (The one $-$ dimensional Itô formula) *Let X_t be an Itô process given by*

$$dX_t = udt + vdB_t.$$

Let $g(t,x) \in C^2([0,+\infty)\times\mathbb{R})$ *(i.e. g is twice continuously differentiable on* $[0,+\infty)\times\mathbb{R}$*). Then*

$$Y_t = g(t, X_t)$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\cdot(dX_t)^2,$$

where $(dX_t)^2 = (dX_t)\cdot(dX_t)$ *is computed according to the rules*

$$dt\cdot dt = dt\cdot dB_t = dB_t\cdot dt = 0,\ dB_t\cdot dB_t = dt.$$

Theorem 1.1.9. (The general Itô formula) *Let*

$$dX_t = udt + vdB_t$$

be an n-dimensional Itô process. Let $g(t,x) = (g_1(t,x),\cdots,g_p(t,x))$ *be a* C^2 *map from* $[0,+\infty)\times\mathbb{R}^n$ *into* $\mathbb{R}^p$*. Then the process*

$$Y(t,\omega) = g(t, X_t)$$

is again an Itô process, whose component number k, Y_k*, is given by*

$$dY_k = \frac{\partial g_k}{\partial t}(t,X)dt + \sum_i \frac{\partial g_k}{\partial x_i}(t,X)dX_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial_{x_i}\partial_{x_j}}(t,X)dX_i dX_j,$$

where $dB_t^{(i)}\cdot dB_t^{(j)} = \delta_{ij}dt, dB_t^{(i)}\cdot dt = dt\cdot dB_t^{(i)} = 0.$

Readers can see chapter 4 of Ref. [17] to know the definitions of 1-dimensional and multi-dimensional Itô processes.

1.1.4 Radial, whole-plane stochastic Loewner evolution

The stochastic Loewner evolution (or Schramm-Loewner evolution) with parameter κ (SLE_κ), discoverd by Oded Schramm (2000), is a family of random planar curves that have been proven to be a scaling limit of a variety of two-dimentional lattice models in statistical mechanics. Given a parameter κ and a domain in the complex plane U, it gives a family of random curves in U with κ controlling how much the curve turns. SLE_κ is the Loewner process driven by the function

$$\lambda(t) = e^{i\sqrt{\kappa}B_t}$$

in the whole-plane and radial cases, and

$$\lambda(t) = \sqrt{\kappa}B_t$$

in the chordal case. $\kappa \in [0, \infty)$ and B_t is a standard, one-dimensional Brownian motion.
SLE$_\kappa$ is conjectured or proved to describe the scaling limit of various stochastic processes in the plane, the most famous one is critical percolation. The characteristic function of the process $\sqrt{\kappa}B_t$ has the form

$$\mathbb{E}(e^{i\xi\sqrt{\kappa}B_t}) = e^{-t\kappa\xi^2/2} \tag{1.1.1}$$

We end the first section with recalling the definitions of (standard, inner) radial SLE$_\kappa$ and the (interior) whole-plane SLE$_\kappa$ version which we study in this work.

Definition 1.1.10. Let $\lambda(t) = \exp(i\sqrt{\kappa}B_t)$ be a two-sided Brownian motion on the unit circle. The standard inner radial SLE$_\kappa$ process is the the family of conformal maps $(g_t)_{t\geq 0}$ satisfying

$$\partial_t g_t(z) = g_t(z)\frac{\lambda(t) + g_t(z)}{\lambda(t) - g_t(z)}, z \in \mathbb{D} \setminus K_t, \tag{1.1.2}$$

with initial condition $g_0(z) = z$.

Here, $(K_t)_{t\geq 0}$ (so-called SLE *hulls*) is a random increasing family of subsets of the unit disk that grows towards the origin 0. The map g_t is the unique conformal map from $\mathbb{D} \setminus K_t$ onto $\mathbb{D}$, such that $g_t(0) = 0$ and $g_t'(0) = e^t$. It can be continued to negative times (via the two-sided Brownian motion B_t in the driving function $\lambda(t)$).

Definition 1.1.11. The interior whole-plane SLE$_\kappa$ process driven by $\lambda(t) = e^{i\sqrt{\kappa}B_t}$ is the family of conformal maps $(f_t)_{t\geq 0}$, from $\mathbb{D}$ onto the slit domains (Ω_t), satisfying

$$\frac{\partial}{\partial t} f_t(z) = z\frac{\partial}{\partial z} f_t(z)\frac{\lambda(t) + z}{\lambda(t) - z},\ z \in \mathbb{D},$$

with initial condition

$$\lim_{t\to+\infty} f_t(e^{-t}z) = z.$$

1.2 Expectation of $f_0(z)$

In Ref. [6], the authors give an explicit expression for the expectations of the coefficients $a_n(0)$ of the expansion (0.0.6) of the interior whole-plane SLE$_\kappa$ map (at time zero) f_0, thereby obtaining the expectation of the map, $\mathbb{E}[f_0(z)]$. We now recall this result.

Theorem 1.2.1. *For $n \geq 3$, setting $a_n := a_n(0)$ and $a_2 := a_2(0)$,*

$$\begin{aligned} a_n(0) &\overset{\text{(law)}}{=} e^{i(n-1)\sqrt{\kappa}B_t} a_n(t), \\ \mathbb{E}(a_n) &= -2\frac{\prod_{k=1}^{n-2}(\frac{\kappa}{2}k^2 - k - 2)}{\prod_{k=1}^{n-1}(\frac{\kappa}{2}k^2 + k)}, \\ \mathbb{E}(a_2) &= -\frac{4}{\kappa + 2}. \end{aligned}$$

Corollary 1.2.2. *The expected conformal map* $\mathbb{E}[f_0(z)]$ *of the interior whole-plane* SLE_κ *is polynomial of degree* $k+1$ *if* $\frac{\kappa}{2}k^2 = k+2$.

In the same way, we also obtain an explicit expression for the expectations of the logarithmic coefficients $\gamma_n(0)$ of the expansion (0.0.7). This result is introduced in the next section.

1.3 Expectation of $\log \frac{f_0(z)}{z}$

Differentiating both sides of (0.0.7) with respect to t and using Loewner equation (0.0.1) lead to

$$2\sum_{k=1}^{\infty} \dot{\gamma}_k(t) z^k + 1 = z\frac{f_t'(z)}{f_t(z)} \frac{\lambda(t)+z}{\lambda(t)-z}. \tag{1.3.1}$$

By expanding the right hand side of the quation (1.3.1) as power series, and identifying coefficients, one gets the set of equations

$$\dot{\gamma}_1(t) - \gamma_1(t) = \overline{\lambda(t)}, \tag{1.3.2}$$

$$\dot{\gamma}_n(t) - n\gamma_n(t) = \overline{\lambda(t)}^n + 2\sum_{k=1}^{n-1} k\overline{\lambda(t)}^{n-k}\gamma_k(t), \ n \geq 2; \tag{1.3.3}$$

where the dot means a t-derivative and $\overline{\lambda(t)} = 1/\lambda(t) = e^{-i\sqrt{\kappa}B_t}$.

From Eq. (1.3.2), we have

$$\gamma_1(t) = -e^t \int_t^{+\infty} e^{-s}\overline{\lambda(s)}ds,$$

and hence

$$\mathbb{E}(\gamma_1(0)) = -\frac{2}{\kappa+2}. \tag{1.3.4}$$

We now use the auxiliary function

$$\beta_n(t) := e^{-nt}\gamma_n(t), \tag{1.3.5}$$

and the shorthand notation

$$X_t := e^{-t-i\sqrt{\kappa}B_t}. \tag{1.3.6}$$

The differential recursion (1.3.3) then becomes

$$\dot{\beta}_n(t) = 2\sum_{k=1}^{n-1} kX_t^{n-k}\beta_k(t) + X_t^n, \ n \geq 2, \tag{1.3.7}$$

with

$$\beta_1(t) = e^{-t}\gamma_1(t) = -\int_t^{+\infty} X_s ds. \tag{1.3.8}$$

Note that one can rewrite the differential recursion (1.3.7) as

$$\dot{\beta}_n = X_t[\dot{\beta}_{n-1} + 2(n-1)\beta_{n-1}]. \tag{1.3.9}$$

Because Eq. (1.3.9) coincides with Eq. (64) used for proving Theorem 3.1 in [6] (denoted there by $u_n(t)$ with $u_1(t) = 1$), we therefore use the same method and arguments for our problem. The proof of Theorem 1.2.1 (Theorem 3.1 in [6]) will be introduced in Appendix B.

From Eqs. (1.3.8) and (1.3.9), we see that the solution $\beta_n(t)$, for $n \geq 1$, can be expressed in the form

$$\beta_n(t) = -2\int_t^{+\infty} X_s\alpha_n(s)ds, \tag{1.3.10}$$

with $\alpha_1(s) = \frac{1}{2}$. Substituting (1.3.10) into (1.3.9) yields

$$\begin{aligned} \alpha_n(t) &= X_t\alpha_{n-1}(t) - 2(n-1)\int_t^{+\infty} X_s\alpha_{n-1}(s)ds \\ &= \mathcal{X}\alpha_{n-1}(t) + (n-1)\mathcal{J}\alpha_{n-1}(t) \\ &= [\mathcal{X} + (n-1)\mathcal{J}]\alpha_{n-1}(t), \end{aligned} \tag{1.3.11}$$

where the operators $\mathcal{X}$ and $\mathcal{J}$ are defined as

$$\mathcal{X}\alpha(t) := X_t\alpha(t), \tag{1.3.12}$$

$$\mathcal{J}\alpha(t) := -2\int_t^{+\infty} X_s\alpha(s)ds. \tag{1.3.13}$$

Owing to (1.3.10), (1.3.11) and (1.3.13), we obtain for $n \geq 2$,

$$\beta_n = \mathcal{J} \circ [\mathcal{X} + (n-1)\mathcal{J}] \circ \cdots \circ [\mathcal{X} + \mathcal{J}]\alpha_1 = \frac{1}{2}\mathcal{J}\prod_{k=1}^{n-1} \circ [\mathcal{X} + k\mathcal{J}]\mathbb{1}, \tag{1.3.14}$$

where $\mathbb{1}$ is the constant function equal to 1 on $\mathbb{R}^+$.

On the other hand, because of the strong Markov property of Brownian motion ($\forall s \geq t,\ B_s \overset{\text{(law)}}{=} B_t + \tilde{B}_{s-t}$ with $\tilde{B}_{s'}$ being an independent copy of the Brownian motion B_s), we have the identity in law,

$$X_s \overset{\text{(law)}}{=} X_t\tilde{X}_{s-t}, \quad \forall s \geq t, \tag{1.3.15}$$

with $\tilde{X}_{s'} := e^{-s' - i\sqrt{\kappa}\tilde{B}_{s'}}$ ($s' \geq 0$) being an independent copy of the process X_s (1.3.6).

Using (1.3.15) in the operator $\mathcal{J}$ (1.3.13) yields the indentity in law,

$$\begin{aligned} \mathcal{J}\alpha(t) &\overset{\text{(law)}}{=} -2X_t\int_0^{+\infty} \tilde{X}_u\alpha(u+t)du \\ &= \mathcal{X} \circ \tilde{\mathcal{J}}\alpha(t), \end{aligned} \tag{1.3.16}$$

with the operator $\tilde{\mathcal{J}}$ defined as $\tilde{\mathcal{J}}\alpha(t) := -2\int_0^{+\infty} \tilde{X}_u \alpha(u+t)du$.
From (1.3.16), we can rewrite (1.3.14) as

$$\beta_n \overset{(\text{law})}{=} \frac{1}{2}\mathcal{J}\prod_{k=1}^{n-1} \circ [\mathcal{X}(1+k\tilde{\mathcal{J}}^{[k]})]\mathbb{1}, \quad n \geq 2, \tag{1.3.17}$$

where the operators $\tilde{\mathcal{J}}^{[k]}, k = 1, \cdots, n-1$, involve successive *independent* copies, $\tilde{X}^{[k]}_{s_k}, k = 1, \cdots, n-1$, of the original process X_s (i.e. $\tilde{\mathcal{J}}^{[k]}\alpha(t) := -2\int_0^{+\infty} \tilde{X}^{[k]}_{s_k}\alpha(s_k + t)ds_k$). Moreover, owing to $X_{s_k+t} \overset{(\text{law})}{=} X_t \tilde{X}^{[k]}_{s_k}$ with $k = \overline{1, n-1}$, we obtain

$$\tilde{\mathcal{J}}^{[k]} X_t^{k-1} = -2\int_0^{+\infty} \tilde{X}^{[k]}_{s_k}(X_{s_k+t})^{k-1} ds_k \overset{(\text{law})}{=} -2X_t^{k-1}\int_0^{+\infty} \big(\tilde{X}^{[k]}_{s_k}\big)^k ds_k,$$

and thus,

$$[\mathcal{X}(1+k\tilde{\mathcal{J}}^{[k]})]X_t^{k-1} \overset{(\text{law})}{=} X_t^k\Big(1 - 2k\int_0^{+\infty} \big(\tilde{X}^{[k]}_{s_k}\big)^k ds_k\Big), \quad k = 1, \cdots, n-1. \tag{1.3.18}$$

The indentities in law (1.3.17) and (1.3.18) lead us to the following explicit representation of $\beta_n(t)$ (1.3.14):

$$\beta_n(t) \overset{(\text{law})}{=} -\int_t^{+\infty} X_s^n \prod_{k=1}^{n-1}\Big(1 - 2k\int_0^{+\infty} \big(\tilde{X}^{[k]}_{s_k}\big)^k ds_k\Big)ds. \tag{1.3.19}$$

By applying the identity (1.3.15) in (1.3.19), we get

$$\begin{aligned} e^{in\sqrt{\kappa}B_t}\gamma_n(t) = (X_t)^{-n}\beta_n(t) &\overset{(\text{law})}{=} -\int_0^{+\infty} \tilde{X}_s^n \prod_{k=1}^{n-1}\Big(1 - 2k\int_0^{+\infty} \big(\tilde{X}^{[k]}_{s_k}\big)^k ds_k\Big)ds \\ &\overset{(\text{law})}{=} \beta_n 0 = \gamma_n(0), \end{aligned} \tag{1.3.20}$$

which implies that the logarithm of the *conjugate* whole-plane Schramm Loewner evolution $e^{-i\sqrt{\kappa}B_t}f_t(e^{i\sqrt{\kappa}B_t}z)$ has the *same law* as $\log\frac{f_0(z)}{z}$.

Since $\tilde{X}_s$, $\tilde{X}^{[k]}_{s_k}, k = 1, \cdots, n-1$, are successive independent copies of the original process X_s, we can then compute $\mathbb{E}[\gamma_n(0)]$ by the identity (1.3.20) (also recall that $\mathbb{E}[(\tilde{X}_s)^k] = e^{-(\frac{\kappa}{2}k^2+k)s}$). In particular,

$$\begin{aligned} \mathbb{E}[\gamma_n(0)] &= -\int_0^{+\infty} \mathbb{E}[\tilde{X}_s^n]\prod_{k=1}^{n-1}\Big(1 - 2k\int_0^{+\infty} \mathbb{E}[(\tilde{X}^{[k]}_{s_k})^k]ds_k\Big)ds \\ &= -\frac{1}{\frac{\kappa}{2}n^2+n}\prod_{k=1}^{n-1}\Big(1 - \frac{2k}{\frac{\kappa}{2}k^2+k}\Big). \end{aligned} \tag{1.3.21}$$

Finally, from (1.3.4), (1.3.20) and (1.3.21), we state the following theorem:

Theorem 1.3.1. *For $n \geq 2$, setting $\gamma_n := \gamma_n(0)$ and $\gamma_1(0) := \gamma_1$,*

$$\begin{aligned}
\gamma_n(0) &\overset{\text{(law)}}{=} e^{in\sqrt{\kappa}B_t}\gamma_n(t), \\
E(\gamma_n) &= -\frac{\prod_{k=1}^{n-1}\left(\frac{\kappa}{2}k^2 - k\right)}{\prod_{k=1}^{n}\left(\frac{\kappa}{2}k^2 + k\right)}, \qquad (1.3.22)\\
E(\gamma_1) &= -\frac{2}{\kappa+2}.
\end{aligned}$$

Corollary 1.3.2. *The expected conformal map $\mathbb{E}\big[\log\frac{f_0(z)}{z}\big]$ of the logarithm of the interior whole-plane* SLE_κ *is polynomial of degree k if $\kappa = \frac{2}{k}$.*

Proof. From Theorem 1.3.1, one sees that $\mathbb{E}\big[\log\frac{f_0(z)}{z}\big]$ is polynomial if there exists $k \in \mathbb{N}^*$ such that $\frac{\kappa}{2}k^2 = k$, i.e. $\kappa = 2/k$, as all $\mathbb{E}(\gamma_n)$ then vanish for $n \geq k+1$. □

Corollary 1.3.3. *In the* SLE_2 *case,*

$$\mathbb{E}(\gamma_n) = \begin{cases} -1/2, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

The formula (1.3.22) gives for the first terms:

$$\begin{aligned}
\mathbb{E}(\gamma_2) &= -\frac{\kappa-2}{2(\kappa+1)(\kappa+2)}, \\
\mathbb{E}(\gamma_3) &= -\frac{2(\kappa-1)(\kappa-2)}{3(\kappa+1)(\kappa+2)(3\kappa+2)}, \\
\mathbb{E}(\gamma_4) &= -\frac{(\kappa-1)(\kappa-2)(3\kappa-2)}{4(\kappa+1)(\kappa+2)(3\kappa+2)(2\kappa+1)}, \\
\mathbb{E}(\gamma_5) &= -\frac{2(\kappa-1)(\kappa-2)(3\kappa-2)(2\kappa-1)}{5(\kappa+1)(\kappa+2)(3\kappa+2)(2\kappa+1)(5\kappa+2)}, \\
\mathbb{E}(\gamma_6) &= -\frac{(\kappa-1)(\kappa-2)(3\kappa-2)(2\kappa-1)(5\kappa-2)}{6(\kappa+1)(\kappa+2)(3\kappa+2)(2\kappa+1)(5\kappa+2)(3\kappa+1)}.
\end{aligned}$$

1.4 Logarithmic coefficient quadratic expectations

In 2010, we computed $\mathbb{E}(|\gamma_n|^2)$ for small n (see [9]). In particular, we got

$$\begin{aligned}
\mathbb{E}(|\gamma_1|^2) &= \frac{2}{\kappa+2}, \\
\mathbb{E}(|\gamma_2|^2) &= \frac{\kappa^2+16\kappa+12}{4(\kappa+1)(\kappa+2)(\kappa+6)}.
\end{aligned}$$

Therefore, as a continuation of this work, we find an algorithm that we have implemented on MATLAB to compute $\mathbb{E}(|\gamma_n|^2)$. This section is devoted to present the algorithm and the results which we obtain for γ_3 to γ_9.

1.4.1 Computational experiments

This algorithm is the same as the one used for computing $\mathbb{E}(|a_n|^2)$ in [6]. In particular, the algorithm is divided into two parts: the first encodes the computation of γ_n, while the second uses it to compute $\mathbb{E}(|\gamma_n|^2)$.
For the encoding of γ_n, we observe that they are linear combinations of successive integrals of the form

$$\int_t^\infty ds_1 \int_{s_1}^\infty ds_2 \ldots \int_{s_{k-1}}^\infty e^{-(i\alpha_1\sqrt{\kappa}B_{s_1}+\beta_1 s_1)-(i\alpha_2\sqrt{\kappa}B_{s_2}+\beta_2 s_2)-\ldots-(i\alpha_k\sqrt{\kappa}B_{s_k}+\beta_k s_k)} ds_k. \tag{1.4.1}$$

Their expectations are encoded as

$$(\alpha_1, \beta_1)\ldots(\alpha_k, \beta_k) \quad (1 \leq k \leq n), \tag{1.4.2}$$

and are explicitly computed by using as above the strong Markov property and the Gaussian characteristic function (1.1.1):

$$(\alpha_1, \beta_1)\ldots(\alpha_k, \beta_k) = \prod_{j=0}^{k-1}[\beta_k + \beta_{k-1} + \ldots + \beta_{k-j} + \eta(\alpha_k + \alpha_{k-1} + \ldots + \alpha_{k-j})]^{-1}.$$

where $\eta(\xi) := \kappa\xi^2/2$. Next, in order to compute $\mathbb{E}(|\gamma_n|^2)$, we have to evaluate the expectation of products of integrals such as (1.4.1) with complex conjugate of others, that we symbolically denote by

$$[(\alpha_1, \beta_1)\ldots(\alpha_k, \beta_k); (-\alpha'_1, \beta'_1)\ldots(-\alpha'_\ell, \beta'_\ell)] \quad (1 \leq k, \ell \leq n). \tag{1.4.3}$$

The product integrals may be written as a sum of $\binom{k+\ell}{k}$ ordered integrals with $k+\ell$ variables: the k first ones and the ℓ last ones are ordered and the number of ordered integrals corresponds to the number of ways of shuffling k cards in the left hand with ℓ cards in the right hand. This sum is quite large and, in order to systematically compute it, we write its expectation as the sum of expectations of integrals of the form (1.4.2) that begin with a term of type (α_1, β_1) or with a term of type $(-\alpha'_1, \beta'_1)$, thus reducing the work to a computation at lower order.

1.4.2 Computations of coefficients for higher orders

Using dynamic programing, we perform computations (formal up to $n = 9$ and numerical up to $n = 16$) on a usual computer. Here are the results for γ_3 to γ_9:

$$E\big(|\gamma_3|^2\big) = \frac{2}{9}\frac{\kappa^4 + 31\kappa^3 + 302\kappa^2 + 356\kappa + 120}{(2+3\kappa)(1+\kappa)(2+\kappa)(6+\kappa)(10+\kappa)};$$

$$E\big(|\gamma_4|^2\big) = \frac{1}{16}\frac{3\kappa^7 + 154\kappa^6 + 3105\kappa^5 + 28534\kappa^4 + 91464\kappa^3 + 106672\kappa^2 + 54288\kappa + 10080}{(14+\kappa)(1+\kappa)(2+3\kappa)(2+\kappa)(6+\kappa)(10+\kappa)(1+2\kappa)(3+\kappa)};$$

$$
\begin{aligned}
&E\big(|\gamma_5|^2\big)\\
&= \frac{2}{25}\big(6\kappa^9 + 431\kappa^8 + 12818\kappa^7 + 198509\kappa^6 + 1583120\kappa^5 + 5077844\kappa^4\\
&\qquad + 6741152\kappa^3 + 4456176\kappa^2 + 1442304\kappa + 181440\big)\\
&\qquad /\big[(18+\kappa)(14+\kappa)(1+\kappa)(2+3\kappa)(2+\kappa)(6+\kappa)(10+\kappa)\\
&\qquad\quad (1+2\kappa)(3+\kappa)(2+5\kappa)\big];
\end{aligned}
$$

$$
\begin{aligned}
&E\big(|\gamma_6|^2\big)\\
&= \frac{1}{36}\big(30\kappa^{12} + 3025\kappa^{11} + 131609\kappa^{10} + 3216919\kappa^9 + 47640321\kappa^8\\
&\qquad + 422153664\kappa^7 + 2049303168\kappa^6 + 5164417376\kappa^5 + 6709663264\kappa^4\\
&\qquad + 4916208896\kappa^3 + 2042489088\kappa^2 + 447056640\kappa + 39916800\big)\\
&\qquad /\big[(22+\kappa)(5+\kappa)(1+3\kappa)(18+\kappa)(2+5\kappa)(14+\kappa)(2+\kappa)(1+\kappa)\\
&\qquad\quad (6+\kappa)(2+3\kappa)(10+\kappa)(1+2\kappa)(3+\kappa)\big];
\end{aligned}
$$

$$
\begin{aligned}
&E\big(|\gamma_7|^2\big)\\
&= \frac{2}{49}\big(90\kappa^{14} + 11565\kappa^{13} + 650066\kappa^{12} + 20991690\kappa^{11} + 427582066\kappa^{10}\\
&\qquad + 5621167065\kappa^9 + 46208313378\kappa^8 + 217731354480\kappa^7 + 556664001568\kappa^6\\
&\qquad + 772618416480\kappa^5 + 635527192256\kappa^4 + 317710798080\kappa^3\\
&\qquad + 94432843776\kappa^2 + 15299447040\kappa + 1037836800\big)\\
&\qquad /\big[(26+\kappa)(22+\kappa)(18+\kappa)(2+5\kappa)(2+7\kappa)(5+\kappa)(14+\kappa)(3+\kappa)\\
&\qquad\quad (10+\kappa)(6+\kappa)(1+\kappa)(1+3\kappa)(1+2\kappa)(2+3\kappa)(2+\kappa)\big];
\end{aligned}
$$

$$
\begin{aligned}
&E\big(|\gamma_8|^2\big)\\
&= \frac{1}{64}\big(1260\kappa^{18} + 212220\kappa^{17} + 15972721\kappa^{16} + 710467025\kappa^{15}\\
&\qquad + 20772022248\kappa^{14} + 418608472462\kappa^{13} + 5900754980045\kappa^{12}\\
&\qquad + 57555045631737\kappa^{11} + 376658350364054\kappa^{10} + 1602643590508876\kappa^9\\
&\qquad + 4346741124397272\kappa^8 + 7510565809793696\kappa^7 + 8440476804641728\kappa^6\\
&\qquad + 6321345332817792\kappa^5 + 3168229268416512\kappa^4 1045199986728192\kappa^3\\
&\qquad + 216982225128960\kappa^2 + 25613536435200\kappa + 1307674368000\big)\\
&\qquad /\big[(1+\kappa)(1+2\kappa)(2+\kappa)(1+3\kappa)(3+\kappa)(1+4\kappa)(3+2\kappa)(2+3\kappa)(5+\kappa)(2+5\kappa)\\
&\qquad\quad (6+\kappa)(7+\kappa)(2+7\kappa)(10+\kappa)(14+\kappa)(18+\kappa)(22+\kappa)(26+\kappa)(30+\kappa)\big];
\end{aligned}
$$

$$
\begin{aligned}
&E\big(|\gamma_9|^2\big)\\
&= \frac{2}{81}\big(15120\kappa^{21} + 3130020\kappa^{20} + 292381824\kappa^{19} + 16336358935\kappa^{18}\\
&\qquad + 609663097723\kappa^{17} + 16046685312588\kappa^{16} + 305815223149938\kappa^{15}\\
&\qquad + 4249934755510755\kappa^{14} + 42691956825394491\kappa^{13} + 303775371493846966\kappa^{12}\\
&\qquad + 1500161524181640952\kappa^{11} + 5069881966077351360\kappa^{10}\\
&\qquad + 11650539687097032144\kappa^{9} + 18247361263799099424\kappa^{8}\\
&\qquad + 19713761511688071936\kappa^{7} + 14894859141855481600\kappa^{6}\\
&\qquad + 7892452487790806272\kappa^{5} + 2905579088214538752\kappa^{4}\\
&\qquad + 724920375549081600\kappa^{3} + 116517087890841600\kappa^{2}\\
&\qquad + 10856675483136000\kappa + 444609285120000\big)\\
&\qquad /\big[(1+\kappa)(1+2\kappa)(2+\kappa)(1+3\kappa)(3+\kappa)(1+4\kappa)(3+2\kappa)(2+3\kappa)(5+\kappa)\\
&\qquad\quad (2+5\kappa)(6+\kappa)(7+\kappa)(2+7\kappa)(2+9\kappa)(10+\kappa)(10+3\kappa)(14+\kappa)\\
&\qquad\quad (18+\kappa)(22+\kappa)(26+\kappa)(30+\kappa)(34+\kappa)\big].
\end{aligned}
$$

These results lead us to three comments:

i. All the coefficients of the polynomial expansions in κ are positive;

ii. For $\kappa \to \infty$, these expectations vanish as $\frac{1}{\kappa}$.

iii. For all explicitly computed coefficients ($n \leq 9$), and for all numerically computed ones ($n \leq 16$), that the conclusion $\mathbb{E}(|\gamma_n|^2) = \frac{1}{2n^2}$ holds in the SLE_2 case. The rigorous proof of this property is given in the next chapter.

Chapter 2

Logarithmic coefficients of whole-plane SLE_κ processes

2.1 Expectations of logarithmic coefficients

The starting point was to consider Milin's conjecture. It yields the study of the logarithmic coefficients γ_n of the interior whole-plane SLE_κ map $f(z) := f_0(z)$. By Corollary 1.3.3, we have already seen a computation of $\mathbb{E}\left(\log \frac{f(z)}{z}\right)$ for SLE_2. There is an alternative proof of this corollary, which is based on the study of *SLE one-point function*. We first restate Corollary 1.3.3 as

Theorem 2.1.1. *Let $f(z) := f_0(z)$ be the interior whole-plane SLE_κ map at time 0, such that*

$$\log \frac{f(z)}{z} = 2 \sum_{n \geq 1} \gamma_n z^n; \tag{2.1.1}$$

then, for $\kappa = 2$,

$$\mathbb{E}(\gamma_n) = \begin{cases} -1/2, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

Differentiating both sides of (2.1.1), we get

$$z \frac{f'(z)}{f(z)} = 1 + 2 \sum_{n \geq 1} n \gamma_n z^n. \tag{2.1.2}$$

We now consider the SLE one-point function

$$F(z) := \mathbb{E}\left(z \frac{f'(z)}{f(z)} \right), \tag{2.1.3}$$

and, following Ref. [2], aim at finding a partial differential equation satisfied by F. Before we enter the details of the proof, let us, for the benefit of the reader not familiar with Ref. [2], detail the strategy of that paper that we will apply in various contexts here.

The starting point is to consider the standard inner radial SLE$_\kappa$ $(g_t)_{t\geq 0}$ as defined in Definition 1.1.10. We denote by g_t^{-1} the inverse map of g_t, which maps $\mathbb{D}$ onto $\mathbb{D}\setminus K_t$. The whole-plane SLE$_\kappa$ map f is rather related to the map g_t^{-1}, but this last function satisfies, by Loewner's theory, a PDE not well-suited to Itô calculus. To overcome this difficulty, one notices that g_t^{-1} and g_{-t} have the same law up to the conjugation by $e^{i\sqrt{\kappa}B_t}$. This is the purpose of the following lemma (Lemma 1 in Ref. [2]).

Lemma 2.1.2. *Let g_t be a radial* SLE$_\kappa$*, then for all $t \in \mathbb{R}$ the map $z \mapsto g_{-t}(z)$ has the same distribution as the map $z \mapsto g_t^{-1}(z\lambda(t))/\lambda(t)$.*

We therefore redefine a radial SLE$_\kappa$ $\tilde{f}_t$, $t \geq 0$, as the continuation g_{-t} of the standard inner radial SLE process g_t to negative times, as follows.

Definition 2.1.3. The (conjugate, inverse) radial SLE$_\kappa$ process $\tilde{f}_t$ is defined, for $t \in \mathbb{R}$, as

$$\tilde{f}_t(z) := g_{-t}(z) \overset{\text{(law)}}{=} g_t^{-1}(z\lambda(t))/\lambda(t), \tag{2.1.4}$$

thus mapping $\mathbb{D}$ onto $\mathbb{D}\setminus K_t$.

Lemma 2.1.2 implies in particular that $\tilde{f}_t$ is a solution to the ODE:

$$\partial_t \tilde{f}_t(z) = \tilde{f}_t(z)\frac{\tilde{f}_t(z)+\lambda(t)}{\tilde{f}_t(z)-\lambda(t)},\ \tilde{f}_0(z) = z, \tag{2.1.5}$$

which is the right property needed to apply Itô calculus. In order to apply stochastic calculus, we use Lemma 2 in Ref. [2], which is a version of the SLE's Markov property,

$$\tilde{f}_t(z) = \lambda(s)\tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s)),\ s \leq t. \tag{2.1.6}$$

To finish, one has to connect the whole-plane SLE with the (modified) radial one. This is done through Lemma 3 in Ref. [2], which is in our present setting (with a change of an e^{-t} convergence factor there to an e^t factor here, when passing from the exterior to the interior of the unit disk $\mathbb{D}$):

Lemma 2.1.4. *The limit in law, $\lim_{t\to+\infty} e^t \tilde{f}_t(z)$, exists, and has the same law as the (time zero) interior whole-plane random map $f_0(z)$:*

$$\lim_{t\to+\infty} e^t \tilde{f}_t(z) \overset{\text{(law)}}{=} f_0(z).$$

Let us now return to the proof of Theorem 2.1.1.

Proof. Let us introduce the auxiliary, time-dependent, radial variant of the SLE one-point function $F(z)$ (2.1.3) above,

$$\widetilde{F}(z,t) := \mathbb{E}\left(z\frac{\tilde{f}_t'(z)}{\tilde{f}_t(z)}\right), \tag{2.1.7}$$

where $\tilde{f}_t$ is a modified radial SLE map at time t as in Definition 2.1.3. Owing to Lemma (2.1.4), we have

$$\lim_{t\to+\infty} \widetilde{F}(z,t) = F(z). \tag{2.1.8}$$

We then use a martingale technique to obtain an equation satisfied by $\widetilde{F}(z,t)$. For $s \leq t$, define $\mathcal{M}_s := \mathbb{E}\left(\frac{\tilde{f}_t'(z)}{\tilde{f}_t(z)}|\mathcal{F}_s\right)$, where $\mathcal{F}_s$ is the σ-algebra generated by $\{B_u,\, u \leq s\}$. $(\mathcal{M}_s)_{s\geq 0}$ is by construction a martingale. Because of the Markov property of SLE, we have

$$\begin{aligned}
\mathcal{M}_s = \mathbb{E}\left(\frac{\tilde{f}_t'(z)}{\tilde{f}_t(z)}|\mathcal{F}_s\right) &= \mathbb{E}\left(\frac{\tilde{f}_s'(z)}{\lambda(s)}\frac{\tilde{f}_{t-s}'(\tilde{f}_s(z)/\lambda(s))}{\tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s))}|\mathcal{F}_s\right)\\
&= \frac{\tilde{f}_s'(z)}{\lambda(s)}\mathbb{E}\left(\frac{\tilde{f}_{t-s}'(\tilde{f}_s(z)/\lambda(s))}{\tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s))}|\mathcal{F}_s\right)\\
&= \frac{\tilde{f}_s'(z)}{\tilde{f}_s(z)}\widetilde{F}(z_s,\tau),
\end{aligned}$$

where $z_s := \tilde{f}_s(z)/\lambda(s)$, and $\tau := t-s$.

We have from Eq. (2.1.5)

$$\begin{aligned}
\partial_s \log \tilde{f}_s' &= \frac{\partial_z\left[\tilde{f}_s\frac{\tilde{f}_s+\lambda(s)}{\tilde{f}_s-\lambda(s)}\right]}{\tilde{f}_s'} = \frac{\tilde{f}_s+\lambda(s)}{\tilde{f}_s-\lambda(s)} - \frac{2\lambda(s)\tilde{f}_s}{(\tilde{f}_s-\lambda(s))^2} \\
&= 1-\frac{2}{(1-z_s)^2},
\end{aligned} \tag{2.1.9}$$

$$\partial_s \log \tilde{f}_s = \frac{\partial_s\tilde{f}_s}{\tilde{f}_s} = \frac{z_s+1}{z_s-1}, \tag{2.1.10}$$

$$dz_s = z_s\left[\frac{z_s+1}{z_s-1}-\frac{\kappa}{2}\right]ds - iz_s\sqrt{\kappa}dB_s. \tag{2.1.11}$$

The coefficient of the ds-drift term of the Itô derivative of $\mathcal{M}_s$ is obtained from the above as,

$$\frac{\tilde{f}_s'(z)}{\tilde{f}_s(z)}\left[-\frac{2z_s}{(1-z_s)^2}+z_s\left(\frac{z_s+1}{z_s-1}-\frac{\kappa}{2}\right)\partial_z - \partial_\tau - \frac{\kappa}{2}z_s^2\partial_z^2\right]\widetilde{F}(z_s,\tau), \tag{2.1.12}$$

and vanishes by the (local) martingale property. Because $\tilde{f}_s$ is univalent, $\tilde{f}_s'$ does not vanish in $\mathbb{D}$, therefore the bracket above vanishes.

Owing to the existence of the limit (2.1.8), we can now take the $\tau \to +\infty$ limit in the above, and get the ODE satisfied by $F(z)$,

$$\begin{aligned}\mathcal{P}(\partial)[F(z)] &:= -\frac{2z}{(1-z)^2}F(z) + z\left(\frac{z+1}{z-1} - \frac{\kappa}{2}\right)F'(z) - \frac{\kappa}{2}z^2F''(z) \\ &= \left[-\frac{2z}{(1-z)^2} + z\left(\frac{z+1}{z-1}\right)\partial_z - \frac{\kappa}{2}(z\partial_z)^2\right]F(z) = 0.\end{aligned} \tag{2.1.13}$$

Following Ref. [6], we now look for solutions to (2.1.13) of the form $\varphi_\gamma(z) := (1-z)^\gamma$. We have

$$\mathcal{P}(\partial)[\varphi_\gamma] = A(2,2,\gamma)\varphi_\gamma + B(2,\gamma)\varphi_{\gamma-1} + C(2,\gamma)\varphi_{\gamma-2},$$

where, in anticipation of the notation that will be introduced in Section 2.2 below,

$$\begin{aligned}A(2,2,\gamma) &:= \gamma - \frac{\kappa}{2}\gamma^2, \\ B(2,\gamma) &:= 2 - \left(3 + \frac{\kappa}{2}\right)\gamma + \kappa\gamma^2, \\ C(2,\gamma) &:= -2 + \left(2 + \frac{\kappa}{2}\right)\gamma - \frac{\kappa}{2}\gamma^2,\end{aligned}$$

with, identically, $A + B + C = 0$. The linear independence of $\varphi_\gamma, \varphi_{\gamma-1}, \varphi_{\gamma-2}$ thus shows that $\mathcal{P}(\partial)[\varphi_\gamma] = 0$ is equivalent to $A = B = C = 0$, which yields $\kappa = 2, \gamma = 1$, and $F(z) = 1 - z$. From Definition (2.1.3), we thus get

Lemma 2.1.5. *Let $f(z) = f_0(z)$ be the interior whole-plane* SLE_2 *map at time* 0*, we then have*

$$\mathbb{E}\left(z\frac{f'(z)}{f(z)}\right) = 1 - z.$$

Theorem 2.1.1 follows from Lemma 2.1.5 and the series expansion (2.1.2). □

2.2 SLE one-point function

We can generalize the result of Lemma 2.1.5 by considering SLE one-point function with general exponents. In particular, we prove the following theorem.

Theorem 2.2.1. *Let $f(z) = f_0(z)$ be the interior whole-plane SLE_κ map at time zero. Consider the curve $\mathcal{R}$, defined parametrically by*

$$p = -\frac{\kappa}{2}\gamma^2 + \left(2 + \frac{\kappa}{2}\right)\gamma, \quad 2p - q = \left(1 + \frac{\kappa}{2}\right)\gamma, \quad \gamma \in \mathbb{R}. \tag{2.2.1}$$

On $\mathcal{R}$, the whole-plane SLE_κ *one-point function has the integrable form,*

$$\mathbb{E}\left(\frac{(f'(z))^{\frac{p}{2}}}{(f(z)/z)^{\frac{q}{2}}}\right) = (1 - z)^\gamma.$$

Remark 2.2.1. *Eq.* (2.2.1) *describes a parabola in the* (p, q) *plane (see Fig. 2.1), which is given in Cartesian coordinates by*

$$2\kappa\left(\frac{2p-q}{2+\kappa}\right)^2-(4+\kappa)\,\frac{2p-q}{2+\kappa}+p=0, \tag{2.2.2}$$

with two branches,

$$\begin{aligned}&\gamma=\gamma_0^{\pm}(p):=\frac{1}{2\kappa}\left(4+\kappa\pm\sqrt{(4+\kappa)^2-8\kappa p}\right),\ \ p\leq\frac{(4+\kappa)^2}{8\kappa},\\ &q=2p-\left(1+\frac{\kappa}{2}\right)\gamma_0^{\pm}(p).\end{aligned} \tag{2.2.3}$$

or, equivalently,

$$2p=q+\frac{2+\kappa}{8\kappa}\left(6+\kappa\pm\sqrt{(6+\kappa)^2-16\kappa q}\right),\ \ q\leq\frac{(6+\kappa)^2}{16\kappa}. \tag{2.2.4}$$

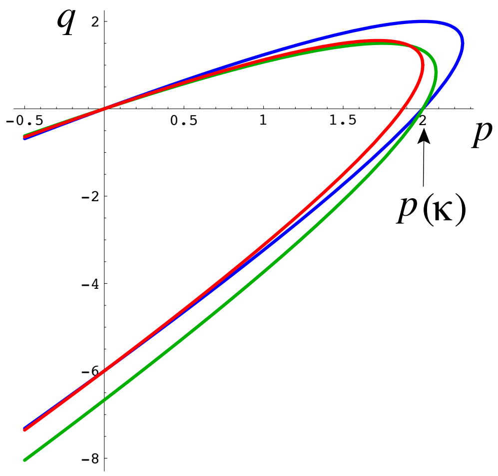

Figure 2.1: *Integral curves* $\mathcal{R}$ *of Theorem 2.2.1, for* $\kappa = 2$ *(blue),* $\kappa = 4$ *(red), and* $\kappa = 6$ *(green). In addition to the origin, the* $q = 0$ *intersection point with the* p*-axis is at* $p(\kappa) := (6 + \kappa)(2 + \kappa)/8\kappa$*, with* $p(2) = p(6) = 2$ *[6, 11].*

Proof. Our aim is to derive an ODE satisfied by the whole-plane SLE$_\kappa$ one-point function,

$$F(z):=\mathbb{E}\left(z^{\frac{q}{2}}\frac{(f'(z))^{\frac{p}{2}}}{(f(z))^{\frac{q}{2}}}\right), \tag{2.2.5}$$

which, by construction, stays finite at the origin and such that $F(0) = 1$.

We introduce the shorthand notation,

$$X_t(z):=\frac{(\tilde{f}_t'(z))^{\frac{p}{2}}}{(\tilde{f}_t(z))^{\frac{q}{2}}}, \tag{2.2.6}$$

where $\tilde{f}_t$ is the conjugate, reversed radial SLE process in $\mathbb{D}$, as introduced in Definition 2.1.3, and such that by Lemma 2.1.4, the limit, $\lim_{t\to+\infty} e^t \tilde{f}_t(z)$, is the same in law as the whole-plane map at time zero. Applying the same method as in the previous section, we consider the auxiliary, time-dependent function

$$\widetilde{F}(z,t) := \mathbb{E}\left(z^{\frac{q}{2}} X_t(z)\right), \tag{2.2.7}$$

such that

$$\lim_{t\to+\infty} \exp\left(\frac{p-q}{2}t\right)\widetilde{F}(z,t) = F(z). \tag{2.2.8}$$

Consider now the martingale $(\mathcal{M}_s)_{t\geq s\geq 0}$, defined by

$$\mathcal{M}_s = \mathbb{E}(X_t(z)|\mathcal{F}_s).$$

By the SLE Markov property we get, setting $z_s := \tilde{f}_s(z)/\lambda(s)$,

$$\mathcal{M}_s = X_s(z)\widetilde{F}(z_s,\tau), \quad \tau := t-s. \tag{2.2.9}$$

As before, the partial differential equation satisfied by $\widetilde{F}(z_s,\tau)$ is obtained by expressing the fact that the ds-drift term of the Itô differential of Eq. (2.2.9),

$$d\mathcal{M}_s = \widetilde{F}\, dX_s + X_s\, d\widetilde{F},$$

vanishes. The differential of X_s is simply computed from Eqs. (2.1.9) and (2.1.10) above as:

$$\begin{aligned} dX_s(z) &= X_s(z)F_1(z_s)ds, \\ F_1(z) &:= \frac{p}{2}\left[1-\frac{2}{(1-z)^2}\right]-\frac{q}{2}\left[1-\frac{2}{1-z}\right]. \end{aligned} \tag{2.2.10}$$

The Itô differential $d\widetilde{F}$ brings in the ds terms proportional to $\partial_{z_s}\widetilde{F}$, $\partial^2_{z_s}\widetilde{F}$, and $\partial_\tau\widetilde{F}$; therefore, in the PDE satisfied by $\widetilde{F}$, the latter terms are exactly the same as in the PDE (2.1.12). We therefore directly arrive at the vanishing condition of the overall drift term coefficient in $d\mathcal{M}_s$,

$$X_s(z)\left[F_1(z_s)+z_s\left(\frac{z_s+1}{z_s-1}-\frac{\kappa}{2}\right)\partial_z-\partial_\tau-\frac{\kappa}{2}z_s^2\partial_z^2\right]\widetilde{F}(z_s,\tau)=0. \tag{2.2.11}$$

Since $X_s(z)$ does not vanish in $\mathbb{D}$, the bracket in (2.2.11) must identically vanish:

$$\left[F_1(z_s)+z_s\frac{z_s+1}{z_s-1}\partial_z-\partial_\tau-\frac{\kappa}{2}(z_s\partial_z)^2\right]\widetilde{F}(z_s,\tau)=0, \tag{2.2.12}$$

where we used $z\partial_z + z^2\partial_z^2 = (z\partial_z)^2$.

To derive the ODE satisfied by $F(z)$ (2.2.5), we first recall its expression as the limit (2.2.8), which further implies

$$\lim_{\tau\to+\infty} \exp\left(\frac{p-q}{2}\tau\right)\partial_\tau \tilde{F}(z,\tau) = -\frac{p-q}{2}F(z).$$

Multiplying the PDE (2.2.11) satisfied by $\widetilde{F}$ by $\exp(\frac{p-q}{2}\tau)$ and letting $\tau \to +\infty$, we obtain

$$\begin{aligned}\mathcal{P}(\partial)[F(z)] &:= \left[-\frac{\kappa}{2}(z\partial_z)^2 - \frac{1+z}{1-z}z\partial_z + F_1(z) + \frac{p-q}{2}\right]F(z)\\ &= \left[-\frac{\kappa}{2}(z\partial_z)^2 - \frac{1+z}{1-z}z\partial_z - \frac{p}{(1-z)^2} + \frac{q}{1-z} + p - q\right]F(z) = 0.\end{aligned} \tag{2.2.13}$$

We now look specifically for solutions to (2.2.13), together with the boundary condition $F(0) = 1$, of the form $\varphi_\gamma(z) = (1-z)^\gamma$. This function satisfies the simple differential operator algebra [6]

$$\mathcal{P}(\partial)[\varphi_\gamma] = A(p,q,\gamma)\varphi_\gamma + B(q,\gamma)\varphi_{\gamma-1} + C(p,\gamma)\varphi_{\gamma-2}, \tag{2.2.14}$$

where

$$A(p,q,\gamma) := p - q + \gamma - \frac{\kappa}{2}\gamma^2, \tag{2.2.15}$$

$$B(q,\gamma) := q - \left(3 + \frac{\kappa}{2}\right)\gamma + \kappa\gamma^2, \tag{2.2.16}$$

$$C(p,\gamma) := -p + \left(2 + \frac{\kappa}{2}\right)\gamma - \frac{\kappa}{2}\gamma^2, \tag{2.2.17}$$

such that, identically, $A + B + C = 0$. Because $\varphi_\gamma, \varphi_{\gamma-1}, \varphi_{\gamma-2}$ are linearly independent, the condition $\mathcal{P}(\partial)[\varphi_\gamma]$ is equivalent to the system $A = C = 0$, hence $C(p,\gamma) = 0$ and $A(p,q,\gamma) - C(p,\gamma) = 2p - q - (1+\kappa/2)\gamma = 0$. It yields precisely the parabola parametrization (2.2.1) given in Theorem 2.2.1, and has for solution (2.2.3). □

2.3 SLE two-point function

In this section, we aim at proving the important property mentioned at the end of chapter 1. That is the following theorem.

Theorem 2.3.1. *Let $f(z) := f_0(z)$ be the time 0 interior whole-plane* SLE$_\kappa$ *map, in the same setting as in Theorem 2.1.1, then, for $\kappa = 2$,*

$$\mathbb{E}(|\gamma_n|^2) = \frac{1}{2n^2}, \quad \forall n \geq 1.$$

The proof of Theorem 2.3.1 is a first motivation of our article [5]. The idea behind the proof, which is a development of Parseval's formula, is to use the series expansion (2.1.2) and to compute $\mathbb{E}\left(\left|z\frac{f'(z)}{f(z)}\right|^2\right)$. We indeed prove:

Theorem 2.3.2. *Let f be the interior whole-plane* SLE_κ *map, in the same setting as in Theorem 2.3.1; then for $\kappa = 2$,*

$$\mathbb{E}\left(\left|z\frac{f'(z)}{f(z)}\right|^2\right) = \frac{(1-z)(1-\bar{z})}{1-z\bar{z}}.$$

We now establish systematically the proof of Theorem 2.3.2.

2.3.1 Beliaev–Smirnov type equations

In order to get Theorem 2.3.2, we will determine the moduli one-point function, $\mathbb{E}\left(|z|^q\frac{|f'(z)|^p}{|f(z)|^q}\right)$, for (p,q) belonging to the same parabola $\mathcal{R}$ as in Theorem 2.2.1, and where $f = f_0$ is the (time zero) interior whole-plane SLE_κ map.

In contradistinction to the method used in Refs. [2, 6] for writing a PDE obeyed by $\mathbb{E}(|f'(z)|^p)$, we shall use here a slightly different approach, building on the results obtained in Section 2.2. We shall study the SLE two-point function for $z_1, z_2 \in \mathbb{D}$,

$$G(z_1,\bar{z}_2) := \mathbb{E}\left(z_1^{\frac{q}{2}}\frac{(f'(z_1))^{\frac{p}{2}}}{(f(z_1))^{\frac{q}{2}}}\overline{\left[z_2^{\frac{q}{2}}\frac{(f'(z_2))^{\frac{p}{2}}}{(f(z_2))^{\frac{q}{2}}}\right]}\right). \tag{2.3.1}$$

As before, we define a time-dependent, auxiliary two-point function,

$$\begin{aligned}\widetilde{G}(z_1,\bar{z}_2,t) &:= \mathbb{E}\left(z_1^{\frac{q}{2}}\frac{(\tilde{f}_t'(z_1))^{\frac{p}{2}}}{(\tilde{f}_t(z_1))^{\frac{q}{2}}}\overline{\left[z_2^{\frac{q}{2}}\frac{(\tilde{f}_t'(z_2))^{\frac{p}{2}}}{(\tilde{f}_t(z_2))^{\frac{q}{2}}}\right]}\right)\\ &= \mathbb{E}\left(z_1^{\frac{q}{2}}X_t(z_1)\overline{z_2^{\frac{q}{2}}X_t(z_2)}\right),\end{aligned} \tag{2.3.2}$$

where as above $\tilde{f}_t$ is the reverse radial SLE_κ process 2.1.3, and where we used the shorthand notation (2.2.6). This time, the two-point function (2.3.1) is the limit

$$\lim_{t\to+\infty} e^{(p-q)t}\widetilde{G}(z_1,\bar{z}_2,t) = G(z_1,\bar{z}_2). \tag{2.3.3}$$

Let us define the two-point martingale $(\mathcal{M}_s)_{t\geq s\geq 0}$, with

$$\mathcal{M}_s := \mathbb{E}(X_t(z_1)\overline{X_t(z_2)}|\mathcal{F}_s).$$

By the Markov property of SLE,

$$\mathbb{E}\big(X_t(z_1)\overline{X_t(z_2)}|\mathcal{F}_s\big) = X_s(z_1)\overline{X_s(z_2)}\,\widetilde{G}(z_{1s},\bar{z}_{2s},\tau),\ \ \tau := t-s, \tag{2.3.4}$$

where

$$z_{1s} := \tilde{f}_s(z_1)/\lambda(s); \;\; \bar{z}_{2s} := \overline{\tilde{f}_s(z_2)/\lambda(s)} = \overline{\tilde{f}_s(z_2)}\lambda(s). \tag{2.3.5}$$

Their Itô differentials, dz_{1s} and $d\bar{z}_{2s}$, are as in (2.1.11),

$$\begin{aligned} dz_{1s} &= z_{1s}\left[\frac{z_{1s}+1}{z_{1s}-1} - \frac{\kappa}{2}\right] ds - i\sqrt{\kappa}\, z_{1s}\, dB_s, \\ d\bar{z}_{2s} &= \bar{z}_{2s}\left[\frac{\bar{z}_{2s}+1}{\bar{z}_{2s}-1} - \frac{\kappa}{2}\right] ds + i\sqrt{\kappa}\, \bar{z}_{2s}\, dB_s. \end{aligned} \tag{2.3.6}$$

As before, the partial differential equation satisfied by $\widetilde{G}(z_{1s}, \bar{z}_{2s}, \tau)$ is obtained by expressing the fact that the ds-drift term of the Itô differential of Eq. (2.3.4),

$$d\mathcal{M}_s = [dX_s(z_1)\overline{X_s(z_2)} + X_s(z_1)d\overline{X_s(z_2)}]\, \widetilde{G} + X_s(z_1)\overline{X_s(z_2)}\, d\widetilde{G}, \tag{2.3.7}$$

vanishes.

The differentials of X_s, $\overline{X_s}$ are as in Eq. (2.2.10) above:

$$\begin{aligned} dX_s(z_1) &= X_s(z_1)F_1(z_{1s})ds, \;\; d\overline{X_s(z_2)} = \overline{X_s(z_2)}F_1(\bar{z}_{2s})ds, \\ F_1(z) &:= \frac{p}{2} - \frac{q}{2} - \frac{p}{(1-z)^2} + \frac{q}{1-z}. \end{aligned} \tag{2.3.8}$$

We thus obtain the simple expression

$$d\mathcal{M}_s = X_s(z_1)\overline{X_s(z_2)}\left[[F_1(z_{1s}) + F_1(\bar{z}_{2s})]\, \widetilde{G}\, ds + d\widetilde{G}\right], \tag{2.3.9}$$

and the vanishing of the ds-drift term in $d\mathcal{M}_s$ requires that of the drift term in the right-hand side bracket in (2.3.9), since $X_s(z)$ does not vanish in $\mathbb{D}$.

The Itô differential of $\widetilde{G}(z_{1s}, \bar{z}_{2s}, \tau)$ can be obtained from Eqs. (2.3.6) and Itô calculus as

$$\begin{aligned} d\widetilde{G}(z_{1s}, \bar{z}_{2s}, \tau) =& \partial_1\widetilde{G}\, dz_{1s} + \bar{\partial}_2\widetilde{G}\, d\bar{z}_{2s} - \partial_\tau\widetilde{G}\, ds \\ &- \frac{\kappa}{2} z_{1s}^2\, \partial_1^2\widetilde{G}\, ds - \frac{\kappa}{2}\bar{z}_{2s}^2\, \bar{\partial}_2^2\widetilde{G}\, ds + \kappa z_{1s}\, \bar{z}_{2s}\, \partial_1\bar{\partial}_2\widetilde{G}\, ds, \end{aligned} \tag{2.3.10}$$

where use was made of the shorthand notations, $\partial_1 := \partial_{z_1}$ and $\bar{\partial}_2 := \partial_{\bar{z}_2}$. We observe that the only coupling between the $z_{1s}, \bar{z}_{2s}$ variables arises in the last term of (2.3.10), the other terms simply resulting from the independent contributions of the z_{1s} and $\bar{z}_{2s}$ parts.

Using again the Itô differentials (2.3.6), we can rewrite (2.3.10) as

$$\begin{aligned} d\widetilde{G} = & -i\sqrt{\kappa}\left(z_{1s}\partial_1 - \bar{z}_{2s}\bar{\partial}_2\right)\widetilde{G}\, dB_s \\ &+ \frac{z_{1s}+1}{z_{1s}-1} z_{1s}\partial_1\widetilde{G}\, ds + \frac{\bar{z}_{2s}+1}{\bar{z}_{2s}-1}\bar{z}_{2s}\bar{\partial}_2\widetilde{G}\, ds - \partial_\tau\widetilde{G}\, ds \\ &- \frac{\kappa}{2}(z_{1s}\partial_1 - \bar{z}_{2s}\,\bar{\partial}_2)^2\widetilde{G}\, ds, \end{aligned} \tag{2.3.11}$$

where we used the obvious formal identity

$$(z_1\partial_1)^2 + (\bar{z}_2\,\bar{\partial}_2)^2 - 2z_1\partial_1\bar{z}_2\,\bar{\partial}_2 = (z_1\partial_1 - \bar{z}_2\,\bar{\partial}_2)^2. \tag{2.3.12}$$

At this stage, comparing the computations (2.3.9) and (2.3.11) above with those in the one-point martingale study in Section 2.2, it is clear that the PDE obeyed by $\widetilde{G} = \widetilde{G}(z_{1s}, \bar{z}_{2s}, \tau)$ is obtained as two duplicates of Eq. (2.2.12), completed as in (2.3.12) by the derivative coupling between variables z_{1s}, $\bar{z}_{2s}$:

$$\left[F_1(z_{1s}) + z_{1s}\frac{z_{1s}+1}{z_{1s}-1}\partial_1 + F_1(\bar{z}_{2s}) + \bar{z}_{2s}\frac{\bar{z}_{2s}+1}{\bar{z}_{2s}-1}\bar{\partial}_2 - \partial_\tau - \frac{\kappa}{2}(z_{1s}\partial_1 - \bar{z}_{2s}\bar{\partial}_2)^2\right]\widetilde{G} = 0. \tag{2.3.13}$$

The existence of the limit (2.3.3) further implies that of

$$\lim_{\tau\to\infty} e^{(p-q)\tau}\partial_\tau\tilde{G}(z_1, \bar{z}_2, \tau) = -(p-q)G(z_1, \bar{z}_2).$$

Multiplying the PDE (2.3.13) satisfied by $\widetilde{G}$ by $\exp((p-q)\tau)$ and letting $\tau \to +\infty$, then gives the expected PDE for $G(z_1, \bar{z}_2)$. It can be most compactly written in terms of the ODE (2.2.13) as

$$\left[\mathcal{P}(\partial_1) + \mathcal{P}(\bar{\partial}_2) + \kappa z_1\partial_1\bar{z}_2\bar{\partial}_2\right] G(z_1, \bar{z}_2) = 0, \tag{2.3.14}$$

and its fully explicit expression is

$$\begin{aligned}\mathcal{P}(D)[G(z_1, \bar{z}_2)] &= -\frac{\kappa}{2}(z_1\partial_1 - \bar{z}_2\bar{\partial}_2)^2 G - \frac{1+z_1}{1-z_1}z_1\partial_1 G - \frac{1+\bar{z}_2}{1-\bar{z}_2}\bar{z}_2\bar{\partial}_2 G \\ &+ \left[-\frac{p}{(1-z_1)^2} - \frac{p}{(1-\bar{z}_2)^2} + \frac{q}{1-z_1} + \frac{q}{1-\bar{z}_2} + 2p - 2q\right]G = 0.\end{aligned} \tag{2.3.15}$$

2.3.2 Moduli one-point function

We note that one can take the $z_1 = z_2 = z$ case in Definition (2.3.1) above, thereby obtaining the whole-plane SLE$_\kappa$ moduli one-point function,

$$G(z, \bar{z}) = \mathbb{E}\left(|z|^q\frac{|f'(z)|^p}{|f(z)|^q}\right). \tag{2.3.16}$$

Because of Eq. (2.3.15), it obeys the corresponding PDE,

$$\begin{aligned}\mathcal{P}(D)[G(z, \bar{z})] &= -\frac{\kappa}{2}(z\partial - \bar{z}\bar{\partial})^2 G - \frac{1+z}{1-z}z\partial G - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}G \\ &+ \left[-\frac{p}{(1-z)^2} - \frac{p}{(1-\bar{z})^2} + \frac{q}{1-z} + \frac{q}{1-\bar{z}} + 2p - 2q\right]G = 0,\end{aligned} \tag{2.3.17}$$

which is the generalization to $q \neq 0$ of the Beliaev–Smirnov equation studied in Refs. [6] and [11].

2.3.3 Integrable case

We begin with a Lemma concerning the uniqueness of the solutions to Eq.(2.3.17).

Lemma 2.3.3. *The space of formal series* $F(z_1, \bar{z}_2) = \sum_{k,\ell \in \mathbb{N}} a_{k,\ell} z_1^k \bar{z}_2^\ell$, *with complex coefficients and that are solutions of the PDE* (2.3.15), *is one-dimensional.*

Proof. We assume that F is a solution to (2.3.15) with $F(0,0) = 0$; it suffices to prove that, necessarily, $F = 0$. We argue by contradiction: If not, consider the minimal (necessarily non constant) term $a_{k,l} z^k \bar{z}^\ell$ in the series of F, with $a_{k,\ell} \neq 0$ and $k + \ell$ minimal (and non vanishing). Then $\mathcal{P}(D)[F]$ (2.3.15) will have a minimal term, equal to $-a_{k,\ell} \left[\frac{\kappa}{2}(k-\ell)^2 + k + \ell\right] z_1^k \bar{z}_2^\ell$, which is non-zero, contradicting the fact that $\mathcal{P}(D)[F]$ vanishes. □

As a second step, following Ref. [6], let us consider the action of the operator $\mathcal{P}(D)$ of (2.3.15) on a function of the factorized form $\varphi(z_1)\varphi(\bar{z}_2)P(z_1, \bar{z}_2)$, which we write, in a shorthand notation, as $\varphi\bar{\varphi}P$. By Leibniz's rule, it is given by

$$\begin{aligned}
\mathcal{P}(D)[\varphi\bar{\varphi}P] = &-\frac{\kappa}{2}\varphi\bar{\varphi}(z_1\partial_1 - \bar{z}_2\bar{\partial}_2)^2 P - \kappa(z_1\partial_1 - \bar{z}_2\bar{\partial}_2)(\varphi\bar{\varphi})(z_1\partial_1 - \bar{z}_2\bar{\partial}_2)P \\
&+ \kappa(z_1\partial_1\varphi)(\bar{z}_2\bar{\partial}_2\bar{\varphi})P - \varphi\bar{\varphi}\frac{1+z_1}{1-z_1}z_1\partial_1 P - \varphi\bar{\varphi}\frac{1+\bar{z}_2}{1-\bar{z}_2}\bar{z}_2\bar{\partial}_2 P \\
&- \left[\frac{\kappa}{2}\bar{\varphi}(z_1\partial_1)^2\varphi + \frac{\kappa}{2}\varphi(\bar{z}_2\bar{\partial}_2)^2\bar{\varphi} + \bar{\varphi}\frac{1+z_1}{1-z_1}z_1\partial_1\varphi + \varphi\frac{1+\bar{z}_2}{1-\bar{z}_2}\bar{z}_2\bar{\partial}_2\bar{\varphi}\right]P \\
&+ \left[-\frac{p}{(1-z_1)^2} - \frac{p}{(1-\bar{z}_2)^2} + \frac{q}{1-z_1} + \frac{q}{1-\bar{z}_2} + 2p - 2q\right]\varphi\bar{\varphi}P.
\end{aligned}$$

Note that the operator $z_1\partial_1 - \bar{z}_2\bar{\partial}_2$ is antisymmetric with respect to $z_1, \bar{z}_2$; therefore, if we choose a symmetric function, $P(z_1, \bar{z}_2) = P(z_1\bar{z}_2)$, the first line of $\mathcal{P}(D)[\varphi\bar{\varphi}P]$ above identically vanishes.

One then looks for solutions to (2.3.15) of the particular form,

$$G(z_1, \bar{z}_2) = \varphi_\gamma(z_1)\varphi_\gamma(\bar{z}_2)P(z_1\bar{z}_2),$$

where, as before, $\varphi_\gamma(z) = (1-z)^\gamma$. The action of the differential operator then takes the simple form,

$$\begin{aligned}
\mathcal{P}(D)[\varphi_\gamma\bar{\varphi}_\gamma P] = &z_1\bar{z}_2\varphi_{\gamma-1}\bar{\varphi}_{\gamma-1}\left(\kappa\gamma^2 P - 2(1 - z_1\bar{z}_2)P'\right) \\
&+ \mathcal{P}(\partial_1)[\varphi_\gamma]\bar{\varphi}_\gamma P + \mathcal{P}(\bar{\partial}_2)[\bar{\varphi}_\gamma]\varphi_\gamma P,
\end{aligned}$$

where P' is the derivative of P with respect to $z_1\bar{z}_2$, and $\mathcal{P}(\partial)$ is the so-called boundary operator (2.2.13) [6].

The ODE, $\kappa\gamma^2 P(x) - 2(1-x)P'(x) = 0$ with $x = z_1\bar{z}_2$ and $P(0) = 1$, has for solution $P(z_1\bar{z}_2) = (1 - z_1\bar{z}_2)^{-\kappa\gamma^2/2}$. It is then sufficient to pick for γ the value $\gamma = \gamma_0^\pm(p)$ (2.2.3) such that $\mathcal{P}(\partial)[\varphi_\gamma] = 0$, as obtained in the proof of Theorem 2.2.1, to get a solution of the PDE, $\mathcal{P}(D)[\varphi_\gamma\bar{\varphi}_\gamma P] = 0$ (2.3.15). By uniqueness of the solution with $G(0,0) = 1$, it gives the explicit form of the SLE two-point function,

$$G(z_1, \bar{z}_2) = \varphi_\gamma(z_1)\varphi_\gamma(\bar{z}_2)(1 - z_1\bar{z}_2)^{-\kappa\gamma^2/2}.$$

We thus get:

Theorem 2.3.4. *Let $f(z) = f_0(z)$ be the interior whole-plane* SLE_κ *map in the setting of Theorem* (2.1.1)*; then, for (p,q) belonging to the parabola $\mathcal{R}$ defined in Theorem 2.2.1 by Eqs.* (2.2.1) *or* (2.2.2) *or* (2.2.3), *and for any pair $(z_1, z_2) \in \mathbb{D}\times\mathbb{D}$,*

$$\mathbb{E}\left(z_1^{\frac{q}{2}} \frac{(f'(z_1))^{\frac{p}{2}}}{(f(z_1))^{\frac{q}{2}}} \overline{\left[z_2^{\frac{q}{2}} \frac{(f'(z_2))^{\frac{p}{2}}}{(f(z_2))^{\frac{q}{2}}} \right]} \right) = \frac{(1-z_1)^\gamma (1-\bar{z}_2)^\gamma}{(1-z_1\bar{z}_2)^\beta}, \quad \beta = \frac{\kappa}{2}\gamma^2.$$

Corollary 2.3.5. *In the same setting as in Theorem 2.3.4, we have for $z \in \mathbb{D}$,*

$$\mathbb{E}\left(|z|^q \frac{|f'(z)|^p}{|f(z)|^q} \right) = \frac{(1-z)^\gamma (1-\bar{z})^\gamma}{(1-z\bar{z})^\beta}, \quad \beta = \frac{\kappa}{2}\gamma^2,$$

for

$$\gamma = \gamma_0^\pm(p) := \frac{1}{2\kappa}\left(4 + \kappa \pm \sqrt{(4+\kappa)^2 - 8\kappa p}\right), \; p \le \frac{(4+\kappa)^2}{8\kappa},$$
$$q = 2p - \left(1 + \frac{\kappa}{2}\right)\gamma_0^\pm(p).$$

Let us stress some particular cases of interest. First, the $p = 0$ case gives the integral means of f.

Corollary 2.3.6. *The interior whole-plane* SLE_κ *map has the integrable moment*

$$\mathbb{E}\left(\left[\frac{f(z_1)}{z_1}\right]^{\frac{(2+\kappa)(4+\kappa)}{4\kappa}} \left[\overline{\frac{f(z_2)}{z_2}}\right]^{\frac{(2+\kappa)(4+\kappa)}{4\kappa}} \right) = \frac{(1-z_1)^{\frac{4+\kappa}{\kappa}}(1-\bar{z}_2)^{\frac{4+\kappa}{\kappa}}}{(1-z_1\bar{z}_2)^{\frac{(4+\kappa)^2}{2\kappa}}},$$

$$\mathbb{E}\left(\left|\frac{f(z)}{z}\right|^{\frac{(2+\kappa)(4+\kappa)}{2\kappa}} \right) = \frac{(1-z)^{\frac{4+\kappa}{\kappa}}(1-\bar{z})^{\frac{4+\kappa}{\kappa}}}{(1-z\bar{z})^{\frac{(4+\kappa)^2}{2\kappa}}}.$$

Second, taking $p = q$ gives the logarithmic integral means we started with:

Corollary 2.3.7. *The interior whole-plane* SLE_κ *map $f(z) = f_0(z)$ has the integrable logarithmic moment*

$$\mathbb{E}\left(\left[z_1 \frac{f'(z_1)}{f(z_1)}\right]^{\frac{2+\kappa}{2\kappa}} \left[\bar{z}_2 \overline{\frac{f'(z_2)}{f(z_2)}}\right]^{\frac{2+\kappa}{2\kappa}} \right) = \frac{(1-z_1)^{\frac{2}{\kappa}}(1-\bar{z}_2)^{\frac{2}{\kappa}}}{(1-z_1\bar{z}_2)^{\frac{2}{\kappa}}},$$

$$\mathbb{E}\left(\left|z\frac{f'(z)}{f(z)}\right|^{\frac{2+\kappa}{\kappa}} \right) = \frac{(1-z)^{\frac{2}{\kappa}}(1-\bar{z})^{\frac{2}{\kappa}}}{(1-z\bar{z})^{\frac{2}{\kappa}}}.$$

Theorem 2.3.2 is the $\kappa = 2$ case of the latter result.

2.3.4 Proof of Theorem 2.3.1

Using (2.1.2), we get

$$\left|z\frac{f'(z)}{f(z)}\right|^2 = 1 + 2\sum_{n\geq 1} n\gamma_n(z^n + \bar{z}^n) + \sum_{n\geq 1}\sum_{m\geq 1} nm\gamma_n\bar{\gamma}_m z^n\bar{z}^m. \tag{2.3.18}$$

On the other hand, by Theorem 2.3.2,

$$\mathbb{E}\left(\left|z\frac{f'(z)}{f(z)}\right|^2\right) = \frac{(1-z)(1-\bar{z})}{(1-z\bar{z})} = 1 - \sum_{n\geq 0} z^{n+1}\bar{z}^n - \sum_{n\geq 0} z^n\bar{z}^{n+1} + 2\sum_{n\geq 1} z^n\bar{z}^n.$$

Identifying the latter with the expectation of (2.3.18), we get the expected coefficients

$$\begin{aligned}
&\mathbb{E}(\gamma_1) = -1/2,\ \ \mathbb{E}(\gamma_n) = 0,\ \ n \geq 2,\\
&\mathbb{E}(|\gamma_n|^2) = \frac{1}{2n^2},\ \ n \geq 1,\\
&\mathbb{E}(\gamma_n\bar{\gamma}_{n+1}) = -\frac{1}{n(n+1)},\ \ \mathbb{E}(\gamma_n\bar{\gamma}_{n+k}) = 0,\ \ n \geq 1, k \geq 2,
\end{aligned}$$

which encompasses Theorems 2.3.1 and 2.1.1.

We end this chapter by briefly returning to the Lebedev-Milin theory. By Theorem 2.3.1, we have for SLE_2,

$$\mathbb{E}\left(\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right)\right) = -\frac{1}{2}\sum_{m=1}^{n}\sum_{k=1}^{m}\frac{1}{k} = -\frac{n+1}{2}\sum_{k=2}^{n+1}\frac{1}{k},$$

which gives an example of the validity "in expectation" of the Milin conjecture. Recalling Definition (0.0.2), we also obtain, in expectation, a check of Robertson's conjecture (0.0.3):

$$\mathbb{E}\left(\log\frac{\sum_{k=0}^{n}|b_{2k+1}|^2}{n+1}\right) \leq \frac{1}{n+1}\mathbb{E}\left(\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right)\right) = -\frac{1}{2}\sum_{k=2}^{n+1}\frac{1}{k},$$

owing to the second Lebedev-Milin inequality (0.0.4),

$$\forall n \geq 0,\ \sum_{k=0}^{n}|b_{2k+1}|^2 \leq (n+1)\exp\left(\frac{1}{n+1}\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right)\right).$$

Chapter 3

Generalized spectrum

The starting point of this chapter was to study a new parabola along which we obtain the general expressions for the whole-plane SLE$_\kappa$ functions (2.2.5) and (2.3.16). Next, we determine all parameters (p, q) such that the SLE$_\kappa$ one-point function (2.2.5) is a polynomial of degree n (for any positive integer n), and consider the $p = q = -2 - \kappa$ case. We end this chapter by studying the generalized integral means spectrum, $\beta(p, q; \kappa)$, corresponding to the singular behavior of the mixed moments $\mathbb{E}\big(|f'(z)|^p/|f(z)|^q\big)$, in the whole parameter space $(p, q) \in \mathbb{R}^2$.

3.1 A new parabola

In Theorem 2.2.1, by looking for solutions to (2.2.13) of the form $\varphi_\gamma(z) = (1-z)^\gamma$, we obtained the parabola $\mathcal{R}$ defined by (2.2.1) or (2.2.2). In the following theorem, we will give a new parabola, denoted by $\mathcal{R}_1$, by looking for solutions to (2.2.13) of the particular form,

$$(1 - \alpha)\varphi_{\gamma-1}(z) + \alpha\varphi_\gamma(z), \; \gamma, \alpha \in \mathbb{R}. \tag{3.1.1}$$

Theorem 3.1.1. *Let $f(z) = f_0(z)$ be the interior whole-plane SLE$_\kappa$ map at time zero. Consider the curve $\mathcal{R}_1$, defined parametrically by*

$$p = -\frac{\kappa}{2}\gamma^2 + \left(2 + \frac{3\kappa}{2}\right)\gamma - \kappa - 2, \quad 2p - q = \left(1 + \frac{3\kappa}{2}\right)\gamma - \kappa - 2, \;\; \gamma \in \mathbb{R}. \tag{3.1.2}$$

On $\mathcal{R}_1$, the whole-plane SLE$_\kappa$ *one-point function has the integrable form,*

$$\mathbb{E}\left(\frac{(f'(z))^{\frac{p}{2}}}{(f(z)/z)^{\frac{q}{2}}}\right) = \left(2 - \frac{2\kappa}{2+\kappa}\gamma\right)(1-z)^{\gamma-1} + \left(\frac{2\kappa}{2+\kappa}\gamma - 1\right)(1-z)^\gamma. \tag{3.1.3}$$

Remark 3.1.1. *Eq.* (3.1.2) *describes a parabola in the* (p, q) *plane (see Fig. 3.1), which is given in Cartesian coordinates by*

$$2\kappa\left(\frac{2p - q + \kappa + 2}{2 + 3\kappa}\right)^2 - (4 + 3\kappa)\frac{2p - q + \kappa + 2}{2 + 3\kappa} + \kappa + 2 + p = 0, \tag{3.1.4}$$

with two branches,

$$\begin{aligned} \gamma = \gamma_1^{\pm}(p) &:= \frac{1}{2\kappa}\left(4+3\kappa \pm \sqrt{(4+\kappa)^2 - 8\kappa p}\right), \quad p \leq \frac{(4+\kappa)^2}{8\kappa}, \\ q &= 2p - \left(1+\frac{3\kappa}{2}\right)\gamma_1^{\pm}(p) + \kappa + 2, \end{aligned} \tag{3.1.5}$$

or, equivalently,

$$2p = q + \frac{2+3\kappa}{8\kappa}\left(6+3\kappa \pm \sqrt{-7\kappa^2 + 4\kappa(1-4q)+36}\right) - \kappa - 2, \quad q \leq \frac{-7\kappa^2+4\kappa+36}{16\kappa}. \tag{3.1.6}$$

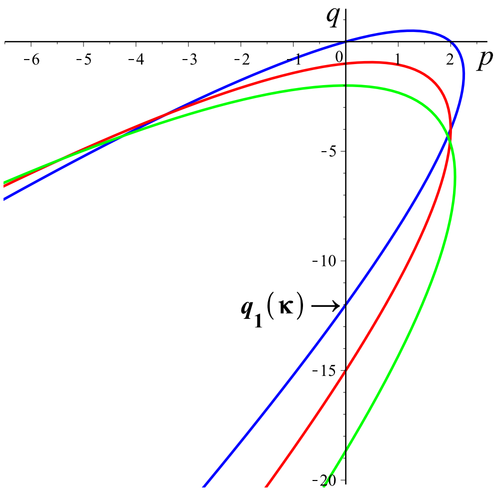

Figure 3.1: *Integral curves $\mathcal{R}_1$ of Theorem 3.1.1, for $\kappa = 2$ (blue), $\kappa = 4$ (red), and $\kappa = 6$ (green). The $p = 0$ intersection points with the q-axis are at $q_1(\kappa) := -2(2+\kappa)(1+\kappa)/\kappa$, with $q_1(2) = -12$, and $q_2(\kappa) := (2-\kappa)/2$ (not marked).*

Proof. As mentioned above, let us look specifically for solutions to the ODE (2.2.13), together with the boundary condition $F(0) = 1$, of the particular form (3.1.1). According to (2.2.14), the function (3.1.1) satisfies the simple differential operator algebra

$$\begin{aligned} \mathcal{P}(\partial)[(1-\alpha)\varphi_{\gamma-1} + \alpha\varphi_\gamma] &= \alpha A(p,q,\gamma)\varphi_\gamma + [\alpha B(q,\gamma) + (1-\alpha)A(p,q,\gamma-1)]\varphi_{\gamma-1} \\ &\quad + [\alpha C(p,\gamma) + (1-\alpha)B(q,\gamma-1)]\varphi_{\gamma-2} + (1-\alpha)C(p,\gamma-1)\varphi_{\gamma-3}, \end{aligned} \tag{3.1.7}$$

where, as defined by (2.2.15), (2.2.16) and (2.2.17),

$$\begin{aligned} A(p,q,\gamma) &:= p - q + \gamma - \frac{\kappa}{2}\gamma^2, \\ B(q,\gamma) &:= q - \left(3+\frac{\kappa}{2}\right)\gamma + \kappa\gamma^2, \\ C(p,\gamma) &:= -p + \left(2+\frac{\kappa}{2}\right)\gamma - \frac{\kappa}{2}\gamma^2. \end{aligned}$$

Owing to $A+B+C=0$, identically, it follows that the sum of the coefficients of $\varphi_\gamma, \varphi_{\gamma-1}, \varphi_{\gamma-2}, \varphi_{\gamma-3}$ in (3.1.7) identically equals 0. Because $\varphi_\gamma, \varphi_{\gamma-1}, \varphi_{\gamma-2}, \varphi_{\gamma-3}$ are linearly independent, the condition $\mathcal{P}(\partial)[(1-\alpha)\varphi_{\gamma-1}+\alpha\varphi_\gamma]$ is equivalent to the system $A(p,q,\gamma)=C(p,\gamma-1)=\alpha B(q,\gamma)+(1-\alpha)A(p,q,\gamma-1)=0$, hence

$$C(p,\gamma-1)=-p-\frac{\kappa}{2}\gamma^2+\left(2+\frac{3\kappa}{2}\right)\gamma-\kappa-2=0,$$

$$A(p,q,\gamma)-C(p,\gamma-1)=2p-q-\left(1+\frac{3\kappa}{2}\right)\gamma+k+2=0,$$

and

$$\alpha=\frac{2\kappa}{2+\kappa}\gamma-1.$$

It yields precisely the parabola parametrization (3.1.2) and the expression (3.1.3) given in Theorem 3.1.1. □

Next, we will determine the SLE$_\kappa$ moduli one-point function $G(z,\bar{z})$ (2.3.16) for (p,q) belonging to the parabola $\mathcal{R}_1$ as in Theorem 3.1.1. Relying on some results which we obtained for some specific values of p,q and κ, we aim at looking for solutions to the PDE (2.3.17) with (p,q) given by Eq. (3.1.2) of the particular form,

$$(1-z)^{\gamma-1}(1-\bar{z})^{\gamma-1}(P_0(z\bar{z})+(z+\bar{z})P_1(z\bar{z})), \tag{3.1.9}$$

where P_0 and P_1 are symmetric functions, i.e. $P_0(z,\bar{z})=P_0(z\bar{z})$, $P_1(z,\bar{z})=P_1(z\bar{z})$. Substituting (3.1.9) into the corresponding PDE yields the following system,

$$\begin{cases}4u(u-1)P_1'(u)+((2\gamma^2\kappa-\kappa-2)u-\kappa-2)P_1(u)+(2+\kappa-2\gamma\kappa)P_0(u)=0,\\ 4u(u-1)P_0'(u)+2(\gamma^2\kappa-2)uP_0(u)+8(\kappa+1-\gamma\kappa)uP_1(u)=0,\end{cases} \tag{3.1.10}$$

where $u:=z\bar{z}$. Solving the above system by MAPLE and using the boundary condition $G(0,0)=1$, we find the particular solution (P_0,P_1) defined as

$$P_0(u)=\frac{1}{(1-u)^{\beta(\gamma,\kappa)}}\operatorname{hypergeom}\left(\left[1-(\gamma-1)\kappa,\frac{3}{2}-\frac{4\gamma-3}{4}\kappa\right],\left[\frac{1}{2}+\frac{\kappa}{4}\right],u\right), \tag{3.1.11}$$

$$\begin{aligned}&P_1(u)\\ &=\frac{(4\gamma-3)\kappa-6}{2(\kappa+2)(1-u)^{\beta(\gamma,\kappa)}}(u-1)\operatorname{hypergeom}\left(\left[2-(\gamma-1)\kappa,\frac{5}{2}-\frac{4\gamma-3}{4}\kappa\right],\left[\frac{3}{2}+\frac{\kappa}{4}\right],u\right)\\ &\quad-\frac{1}{2(1-u)^{\beta(\gamma,\kappa)}}\operatorname{hypergeom}\left(\left[1-(\gamma-1)\kappa,\frac{3}{2}-\frac{4\gamma-3}{4}\kappa\right],\left[\frac{1}{2}+\frac{\kappa}{4}\right],u\right),\end{aligned} \tag{3.1.12}$$

with $\beta(\gamma,\kappa):=\frac{\kappa}{2}\gamma^2+\kappa\gamma-\kappa-2$. The **hypergeom(n, d, z)** calling sequence is the generalized hypergeometric function $F(n,d,z)$, as introduced in Appendix A.

We thus get:

Theorem 3.1.2. *Let $f(z) = f_0(z)$ be the interior whole-plane* SLE_κ *map at time zero; then, for (p, q) belonging to the parabola $\mathcal{R}_1$ defined in Theorem 3.1.1 by Eqs.* (3.1.2) *or* (3.1.4) *or* (3.1.5)*, and for any $z \in \mathbb{D}$,*

$$\mathbb{E}\left(|z|^q \frac{|f'(z)|^p}{|f(z)|^q}\right) = (1-z)^{\gamma-1}(1-\bar{z})^{\gamma-1}\big(P_0(z\bar{z}) + (z+\bar{z})P_1(z\bar{z})\big),$$

where the functions P_0 and P_1 are defined by Eqs. (3.1.11) *and* (3.1.12)*, respectively.*

From Theorem 3.1.2, together with the support of MAPLE, we obtain the following expressions for $G(z, \bar{z})$ for some specific values of (κ, γ).

- In the $(\kappa = 6, \gamma = 1)$ case,
$$\mathbb{E}\left(\left|\frac{f(z)}{z}\right|^2\right) = \frac{2-(z+\bar{z})}{2(1-z\bar{z})}.$$
- In the $(\kappa = 10, \gamma = 1)$ case,
$$\mathbb{E}\left(\left|\frac{f(z)}{z}\right|^4\right) = \frac{3 - z\bar{z} + (z+\bar{z})(z\bar{z}-2)}{3(1-z\bar{z})^3}.$$
- In the $(\kappa = 14, \gamma = 1)$ case,
$$\mathbb{E}\left(\left|\frac{f(z)}{z}\right|^6\right) = \frac{2(z\bar{z})^2 - 10z\bar{z} + 20 + (z+\bar{z})(-3(z\bar{z})^2 + 12z\bar{z} - 15)}{20(1-z\bar{z})^5}.$$
- In the $(\kappa = 3, \gamma = 2)$ case,
$$\mathbb{E}\left(|f'(z)|^2\left|\frac{f(z)}{z}\right|^2\right) = \frac{(1-z)(1-\bar{z})\big(5(z\bar{z})^2 + 18z\bar{z} + 5 - 7(z+\bar{z})(1+z\bar{z})\big)}{5(1-z\bar{z})^7}.$$
- In the $(\kappa = 4, \gamma = 2)$ case,
$$\mathbb{E}\left(|f'(z)|^2\left|\frac{f(z)}{z}\right|^4\right)$$
$$= \frac{(1-z)(1-\bar{z})\big(3(z\bar{z})^3 + 21(z\bar{z})^2 + 21z\bar{z} + 3 - (z+\bar{z})(5(z\bar{z})^2 + 14z\bar{z} + 5)\big)}{3(1-z\bar{z})^{10}}.$$
- In the $(\kappa = 2, \gamma = 3)$ case,
$$\mathbb{E}\left(|f'(z)|^2\left|\frac{f(z)}{z}\right|^4\right) = (1-z)^2(1-\bar{z})^2\big(P_0(z\bar{z}) + (z+\bar{z})P_1(z\bar{z})\big),$$
where $P_0(z\bar{z}), P_1(z\bar{z})$ are defined as
$$P_0(z\bar{z}) = \frac{(z\bar{z})^3 + 9(z\bar{z})^2 + 9(z\bar{z}) + 1}{(1-z\bar{z})^{11}},$$
$$P_1(z\bar{z}) = -\frac{2(z\bar{z})^2 + 6(z\bar{z}) + 2}{(1-z\bar{z})^{11}}.$$

- In the $(\kappa = 1, \gamma = 4)$ case,

$$\mathbb{E}\Big(|f'(z)|^3\Big|\frac{f(z)}{z}\Big|\Big) = (1-z)^3(1-\bar{z})^3\frac{3(z\bar{z})^2 + 14z\bar{z} + 3 - 5(z+\bar{z})(z\bar{z}+1)}{3(1-z\bar{z})^9}.$$

Although Eqs. (3.1.11) and (3.1.12) can't give us the explicit expressions for P_0 and P_1 when $\gamma = 1$ and $\kappa \in \{4, 8, 12, 16, 20\}$, however, by solving the system (3.1.10) corresponding to each specific value of (γ, κ), together with the condition $G(0,0) = 1$, we still obtain the explicit expressions for P_0 and P_1 (given in the polar coordinate system (r, θ)) in these case. For instance, we have:

- In the $(\kappa = 4, \gamma = 1)$ case,

$$\mathbb{E}\Big(\Big|\frac{f(z)}{z}\Big|\Big) = \frac{\ln\left(\frac{1+r}{1-r}\right)}{2r} + \frac{2r - (1+r^2)\ln\left(\frac{1+r}{1-r}\right)}{4r^2}\cos(\theta).$$

- In the $(\kappa = 8, \gamma = 1)$ case,

$$\begin{aligned}\mathbb{E}\Big(\Big|\frac{f(z)}{z}\Big|^3\Big) &= \frac{6r(1+r^2) - 3(1-r^2)^2\ln\left(\frac{1+r}{1-r}\right)}{16r^3(1-r^2)^2}\\ &\quad + \frac{9(r^6 - r^4 - r^2 + 1)\ln\left(\frac{1+r}{1-r}\right) - 18r^5 + 12r^3 - 18r}{32r^4(1-r^2)^2}\cos(\theta).\end{aligned}$$

- In the $(\kappa = 12, \gamma = 1)$ case,

$$\mathbb{E}\Big(\Big|\frac{f(z)}{z}\Big|^5\Big) = \frac{5}{512}\frac{Q_1(r) + Q_2(r)\ln\left(\frac{1+r}{1-r}\right) + \left(Q_3(r) + Q_4(r)\ln\left(\frac{1+r}{1-r}\right)\right)\cos(\theta)}{r^6(1-r^2)^4},$$

where the polynomials $Q_i(i = \overline{1,4})$ are defined as

$$\begin{aligned}Q_1(r) &= -12r^2 + 44r^4 + 44r^6 - 12r^8,\\ Q_2(r) &= 6(r - 4r^3 + 6r^5 - 4r^7 + r^9),\\ Q_3(r) &= 30r - 80r^3 + 36r^5 - 80r^7 + 30r^9,\\ Q_4(r) &= -15 + 45r^2 - 30r^4 - 30r^6 + 45r^8 - 15r^{10}.\end{aligned}$$

Remark 3.1.2. *In the polar coordinate system (r, θ), the PDE (2.3.17) satisfied by G becomes*

$$\begin{aligned}\mathcal{P}(D)[G(re^{i\theta}, re^{i\theta})] &= \frac{\kappa}{2}\partial^2_{\theta,\theta}G + \frac{r(r^2-1)}{r^2 - 2r\cos(\theta) + 1}\partial_r G - \frac{2r\sin(\theta)}{r^2 - 2r\cos(\theta) + 1}\partial_\theta G\\ &+ \left[p\Big(\frac{r^4 + 4r^2(1 - r\cos(\theta)) - 1}{(r^2 - 2r\cos(\theta) + 1)^2} + 1\Big) + q\frac{2r\cos(\theta) - 2r^2}{r^2 - 2r\cos(\theta) + 1}\right]G = 0.\end{aligned}$$

One can thus verifies the accuracy of the solutions above.

3.2 The case $F(z)$ polynomial

In this section, our aim is to answer the following question: Let n be an arbitrary positive integer, what cases will the SLE$_\kappa$ one-point fuction $F(z)$ (2.2.5) be a polynomial n?

We approach to this question consisted in observing that: $F(z)$ is an arbitrary polynomial of degree n if, and only if, $F(z)$ is a linear combination of the functions $\varphi_i(z) = (1-z)^i$ with $i = \overline{0,n}$. As a result, we now look for solutions to (2.2.13), together with the boundary condition $F(0) = 1$, of the particular form,

$$F(z) = \alpha_0\varphi_\gamma(z) + \alpha_1\varphi_{\gamma-1}(z) + \cdots + \alpha_k\varphi_{\gamma-k}(z), \tag{3.2.1}$$

where k is a given nonnegative integer, and $\alpha_0, \cdots, \alpha_k$ are real numbers such that

$$\alpha_0 + \cdots + \alpha_k = 1.$$

Recall that Theorem 2.2.1 and Theorem 3.1.1 are obtained by considering the cases of $k = 0$ and $k = 1$, respectively. According to (2.2.14), the function (3.2.1) satisfies the differential operator algebra

$$\begin{aligned}
&\mathcal{P}(\partial)\left[\alpha_0\varphi_\gamma(z) + \alpha_1\varphi_{\gamma-1}(z) + \cdots + \alpha_k\varphi_{\gamma-k}(z)\right] \qquad (3.2.2)\\
&= \sum_{i=0}^{k} \alpha_i\left[A(p,q,\gamma-i)\varphi_{\gamma-i} + B(q,\gamma-i)\varphi_{\gamma-i-1} + C(p,\gamma-i)\varphi_{\gamma-i-2}\right]\\
&= \alpha_0 A(p,q,\gamma)\varphi_\gamma + [\alpha_0 B(q,\gamma) + \alpha_1 A(p,q,\gamma-1)]\varphi_{\gamma-1}\\
&\quad + [\alpha_0 C(p,\gamma) + \alpha_1 B(q,\gamma-1) + \alpha_2 A(p,q,\gamma-2)]\varphi_{\gamma-2} + \cdots + \alpha_k C(p,\gamma-k)\varphi_{\gamma-k-2},
\end{aligned}$$

where, as before, $A(p,q,\gamma)$, $B(q,\gamma)$, $C(p,\gamma)$ are defined by (2.2.15), (2.2.16) and (2.2.17), such that, identically, $A + B + C = 0$. Because $\varphi_\gamma, \varphi_{\gamma-1}, \cdots, \varphi_{\gamma-k-2}$ are linearly independent, one has $A(p,q,\gamma) = C(p,\gamma-k) = 0$, hence

$$\begin{aligned}
C(p,\gamma-k) &= -p - \frac{\kappa}{2}\gamma^2 + \big(2 + (2k+1)\frac{\kappa}{2}\big)\gamma - k(k+1)\frac{\kappa}{2} - 2k = 0,\\
A(p,q,\gamma) - C(p,\gamma-k) &= 2p - q - \big(1 + (2k+1)\frac{\kappa}{2}\big)\gamma + k(k+1)\frac{\kappa}{2} + 2k = 0,
\end{aligned}$$

which are equivalent to

$$\begin{aligned}
p &= -\frac{\kappa}{2}\gamma^2 + \left(2 + (2k+1)\frac{\kappa}{2}\right)\gamma - k(k+1)\frac{\kappa}{2} - 2k,\\
2p - q &= \left(1 + (2k+1)\frac{\kappa}{2}\right)\gamma - k(k+1)\frac{\kappa}{2} - 2k.
\end{aligned} \tag{3.2.3}$$

Remark 3.2.1. *Eq.* (3.2.3) *describes a parabola, denoted by $\mathcal{R}_k$, in the (p,q) plane, which is given in Cartesian coordinates by*

$$\begin{aligned}
2\kappa\left(\frac{2p - q + k(k+1)\frac{\kappa}{2} + 2k}{2 + (2k+1)\kappa}\right)^2 - (4 + (2k+1)\kappa)\frac{2p - q + k(k+1)\frac{\kappa}{2} + 2k}{2 + (2k+1)\kappa}&\\
+ k(k+1)\frac{\kappa}{2} + 2k + p = 0,&
\end{aligned} \tag{3.2.4}$$

with two branches,

$$\begin{aligned} &\gamma = \gamma_k^{\pm}(p) := \frac{1}{2\kappa}\left(4 + (2k+1)\kappa \pm \sqrt{(4+\kappa)^2 - 8\kappa p}\right), \quad p \le \frac{(4+\kappa)^2}{8\kappa}, \\ &q = 2p - \left(1 + (2k+1)\frac{\kappa}{2}\right)\gamma_k^{\pm}(p) + k(k+1)\frac{\kappa}{2} + 2k, \end{aligned} \tag{3.2.5}$$

or, equivalently,

$$\begin{aligned} &\gamma = \gamma_k^{\pm}(q) := \frac{1}{4\kappa}\left(6 + (2k+1)\kappa \pm \sqrt{(1 - 4k(k+1))\kappa^2 + 4\kappa(3 - 2k - 4q) + 36}\right), \\ &q \le \frac{(1 - 4k(k+1))\kappa^2 + 4\kappa(3-2k) + 36}{16\kappa}, \\ &2p = q + \left(1 + (2k+1)\frac{\kappa}{2}\right)\gamma_k^{\pm}(q) - k(k+1)\frac{\kappa}{2} - 2k. \end{aligned}$$

Obviously, one regets the parabola $\mathcal{R}$ (2.2.2) *for* $k = 0$*, and the parbola* $\mathcal{R}_1$ (3.1.4) *for* $k = 1$.

Eq. (3.2.3) can be rewritten in a simplier form,

$$\begin{aligned} p &= 2(\gamma - k) - \frac{\kappa}{2}(\gamma - k)(\gamma - k - 1), \\ q &= p + \gamma - \frac{\kappa}{2}\gamma^2. \end{aligned} \tag{3.2.6}$$

Solving the ODE (2.2.13) with (p, q) given by Eq. (3.2.6), together with the boundary condition $F(0) = 1$, we obtain

$$\begin{aligned} F(z) &= (1-z)^{\gamma-k}\,\mathrm{hypergeom}\left(\left[-k, \frac{(2\gamma-k)\kappa - 2}{\kappa}\right], \left[\frac{\kappa+2}{\kappa}\right], z\right) \\ &= (1-z)^{\gamma-k}\left(1 + \sum_{m=1}^{\infty}\frac{1}{m!}\cdot\frac{\prod_{j=0}^{m-1}(k-j)\big((k-2\gamma-j)\kappa+2\big)}{\prod_{j=1}^{m}(j\kappa+2)}z^m\right) \\ &= \begin{cases} (1-z)^{\gamma} & \text{if } k = 0, \\ (1-z)^{\gamma-k}\left(1 + \sum_{m=1}^{k}\frac{1}{m!}\cdot\frac{\prod_{j=0}^{m-1}(k-j)((k-2\gamma-j)\kappa+2)}{\prod_{j=1}^{m}(j\kappa+2)}z^m\right) & \text{if } k \ge 1. \end{cases} \end{aligned} \tag{3.2.7}$$

Eq. (3.2.7) shows the determination of real numbers α_i $(i = \overline{0,k})$ in (3.2.1), and the fact that $F(z)$ is a polynomial of degree n when $\gamma = n$ and $k \le n$,

$$F(z) = \begin{cases} (1-z)^{n} & \text{if } k = 0, \\ (1-z)^{n-k}\left(1 + \sum_{m=1}^{k}\frac{1}{m!}\cdot\frac{\prod_{j=0}^{m-1}(k-j)((k-2n-j)\kappa+2)}{\prod_{j=1}^{m}(j\kappa+2)}z^m\right) & \text{if } 1 \le k \le n. \end{cases}$$

From this, we get the following proposition.

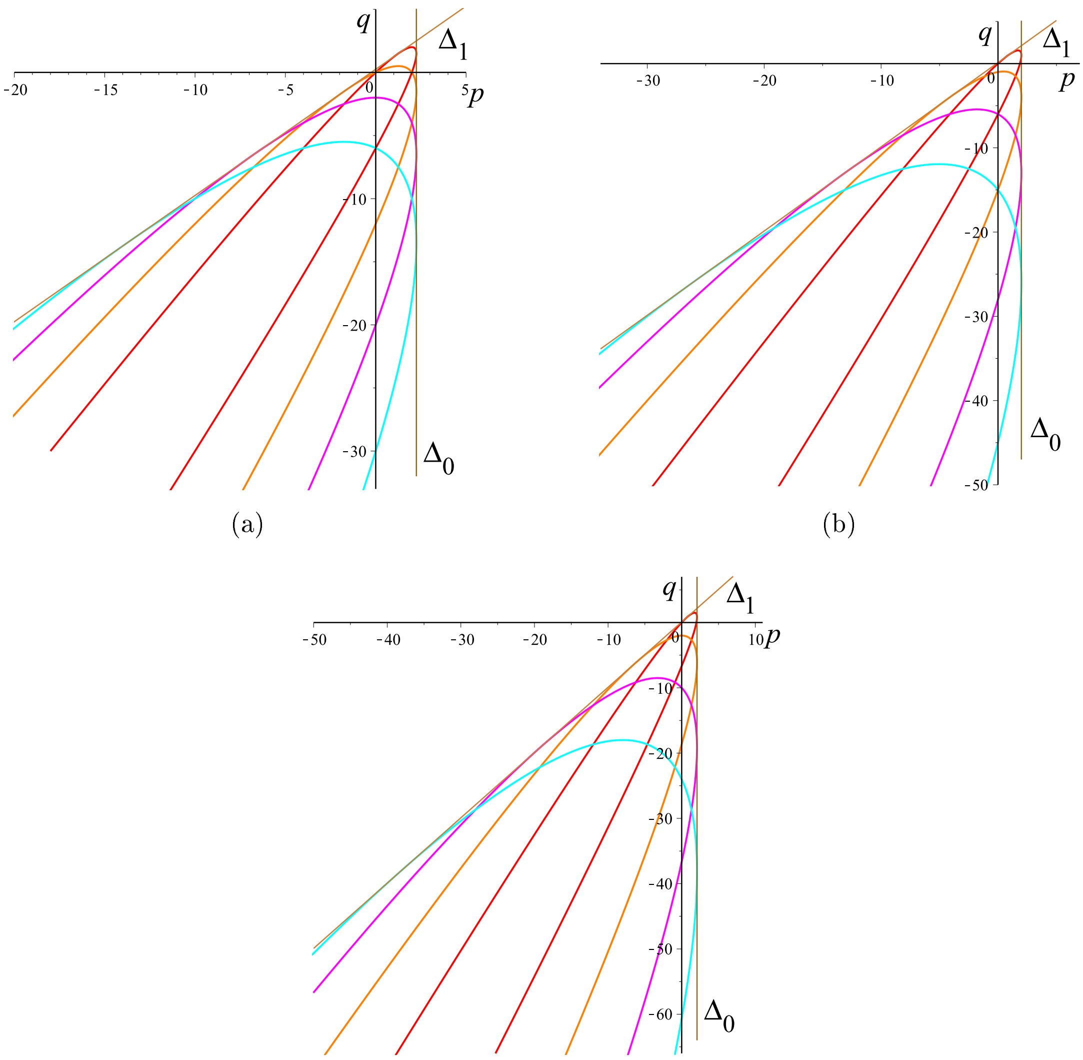

Figure 3.2: Parabolas $\mathcal{R}$ (red), $\mathcal{R}_1$ (coral), $\mathcal{R}_2$ (magenta), $\mathcal{R}_3$ (cyan) of the family $(\mathcal{R}_k)_{k\in\mathbb{N}}$, defined by Eqs. (3.2.3) or (3.2.4), for (**a**) $\kappa = 2$, (**b**) $\kappa = 4$, and (**c**) $\kappa = 6$. The two lines $\Delta_0 := \{p = (4+\kappa)^2/8\kappa\}$ and $\Delta_1 := \{(p,q) \in \mathbb{R}^2, q = p + 1/2\kappa\}$ are tangent lines to each parabola of this family.

Proposition 3.2.1. *The* SLE$_\kappa$ *one-point function* $F(z)$ (2.2.5) *is a polynomial of degree* $n > 0$ *if, and only if,*

$$p = 2(n-k) - \frac{\kappa}{2}(n-k)(n-k-1),$$
$$q = p + n - \frac{\kappa}{2}n^2, \quad k = \overline{0,n}.$$

In other words, if and only if (p,q) *is one of the points defined by* $\gamma = n$ *on* $(n+1)$ *parabolas* $\mathcal{R}, \mathcal{R}_1, \cdots, \mathcal{R}_n$.

Here are the examples for $n = 2, n = 3$ and $n = 4$:

Example 3.2.1. *The function $F(z)$ is a polynomial of degree 2 if, and only if,*

$$\begin{cases} p = 0,\ q = 2 - 2\kappa; \\ or, \\ p = 2,\ q = 4 - 2\kappa; \\ or, \\ p = 4 - \kappa,\ q = 6 - 3\kappa; \end{cases}$$

Example 3.2.2. *The function $F(z)$ is a polynomial of degree 3 if, and only if,*

$$\begin{cases} p = 0,\ q = 3 - \dfrac{9}{2}\kappa; \\ or, \\ p = 2,\ q = 5 - \dfrac{9}{2}\kappa; \\ or, \\ p = 4 - \kappa,\ q = 7 - \dfrac{11}{2}\kappa; \\ or, \\ p = 6 - 3\kappa,\ q = 9 - \dfrac{15}{2}\kappa. \end{cases}$$

Example 3.2.3. *The function $F(z)$ is a polynomial of degree 4 if, and only if,*

$$\begin{cases} p = 0,\ q = 4 - 8\kappa; \\ or, \\ p = 2,\ q = 6 - 8\kappa; \\ or, \\ p = 4 - \kappa,\ q = 8 - 9\kappa; \\ or, \\ p = 6 - 3\kappa,\ q = 10 - 11\kappa; \\ or, \\ p = 8 - 6\kappa,\ q = 12 - 14\kappa. \end{cases}$$

3.3 The $p = q = -2 - \kappa$ case

In this section, we introduce a result in the $p = q = -2 - \kappa$ case. Let us turn to the following theorem.

Theorem 3.3.1. *For $p = q = -2 - \kappa$, the whole-plane* SLE_κ *moduli one-point function* (2.3.16) *is given by*

$$\begin{aligned}
&\mathbb{E}\left(\left|\frac{f(z)}{zf'(z)}\right|^{\kappa+2}\right) \qquad (3.3.1)\\
&= \frac{1}{(1-z\bar{z})^{\frac{\kappa+2}{2}}}\Bigg[\mathit{hypergeom}\Big(\Big[-\frac{\kappa}{2}-1, -\frac{3\kappa}{4}-\frac{3}{2}\Big], \Big[\frac{\kappa}{4}+\frac{1}{2}\Big], z\bar{z}\Big)\\
&\qquad + \Big(\frac{z}{1-z}+\frac{\bar{z}}{1-\bar{z}}\Big)\Big(\mathit{hypergeom}\Big(\Big[-\frac{\kappa}{2}-1, -\frac{3\kappa}{4}-\frac{3}{2}\Big], \Big[\frac{\kappa}{4}+\frac{1}{2}\Big], z\bar{z}\Big)\\
&\qquad\qquad + (1-z\bar{z})\,\mathit{hypergeom}\Big(\Big[-\frac{\kappa}{2}, -\frac{3\kappa}{4}-\frac{1}{2}\Big], \Big[\frac{\kappa}{4}+\frac{3}{2}\Big], z\bar{z}\Big)\Big)\Bigg].
\end{aligned}$$

Proof. Let us give some remarks for the proof of this theorem. We first note that: for any $\kappa > 0$, if $p = q = -2 - \kappa$ then all the extra diagonals, except for the main diagonal, of the coefficient matrix of $G(z, \bar{z})$, defined by its corresponding power series, are identical. This fact implies that for all the extra diagonals, their respective sums have a common factor which is a symmetric function (i.e., $P(z, \bar{z}) = P(z\bar{z})$). Obviously, the sum of the main diagonal is a symmetric function.

Moreover, relying on the specific results which we obtained for $\kappa = 2$ and $\kappa = 4$, we realized the existence of the factor $1/(1 - z\bar{z})^{(\kappa+2)/2}$ given in (3.3.1). Thus we now look specifically for solutions to the PDE (2.3.17), together with the boundary condition $G(0, 0) = 1$, of the form

$$G(z, \bar{z}) = \frac{Q_1(z\bar{z}) + \big(\frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}\big)Q_2(z\bar{z})}{(1-z\bar{z})^{\frac{\kappa+2}{2}}}. \qquad (3.3.2)$$

Substituting (3.3.2) into the PDE (2.3.17) with $p = q = -2-\kappa$, we get the following system

$$\begin{cases}
(u^2-1)Q_1'(u) + 2u(1-u)Q_2'(u) - \big(\frac{\kappa+2}{2}u + \frac{5(\kappa+2)}{2}\big)Q_1(u)\\
\qquad\qquad\qquad\qquad + (\kappa+2)(u+2)Q_2(u) = 0,\\
4(u-u^2)Q_1'(u) + 4u(u-1)Q_2'(u) + (\kappa+2)(4u+2)Q_1(u)\\
\qquad\qquad\qquad\qquad - (\kappa+2)(5u+1)Q_2(u) = 0,
\end{cases}$$

where $u := z\bar{z}$. Solving the above system by MAPLE, we find the particular solution (Q_1, Q_2) such that the function (3.3.2) equals 1 at $(0, 0)$, given by

$$\begin{aligned}
Q_1(u) &= \text{hypergeom}\left(\left[-\frac{\kappa}{2}-1, -\frac{3\kappa}{4}-\frac{3}{2}\right], \left[\frac{\kappa}{4}+\frac{1}{2}\right], u\right),\\
Q_2(u) &= \text{hypergeom}\left(\left[-\frac{\kappa}{2}-1, -\frac{3\kappa}{4}-\frac{3}{2}\right], \left[\frac{\kappa}{4}+\frac{1}{2}\right], u\right)\\
&\qquad + (1-u)\,\text{hypergeom}\left(\left[-\frac{\kappa}{2}, -\frac{3\kappa}{4}-\frac{1}{2}\right], \left[\frac{\kappa}{4}+\frac{3}{2}\right], u\right).
\end{aligned}$$

□

Theorem (3.3.1) gives for some specific values of κ:

- In the $\kappa = 2/3$ case,
$$\mathbb{E}\Big(\Big|\frac{f(z)}{zf'(z)}\Big|^{8/3}\Big) = \frac{1}{(1-z\bar{z})^{4/3}}\Big(Q_1(z\bar{z}) + (\frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}})Q_2(z\bar{z})\Big),$$
where $Q_1(z\bar{z}), Q_2(z\bar{z})$ are defined as
$$Q_1(z\bar{z}) = \frac{2(z\bar{z})^2 + 20z\bar{z} + 5}{5} \quad , \quad Q_2(z\bar{z}) = \frac{(z\bar{z})^2 + 16z\bar{z} + 10}{5}.$$
- In the $\kappa = 2$ case,
$$\mathbb{E}\Big(\Big|\frac{f(z)}{zf'(z)}\Big|^{4}\Big) = \frac{1}{(1-z\bar{z})^{2}}\Big(Q_1(z\bar{z}) + (\frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}})Q_2(z\bar{z})\Big),$$
where $Q_1(z\bar{z}), Q_2(z\bar{z})$ are defined as
$$Q_1(z\bar{z}) = 3(z\bar{z})^2 + 6z\bar{z} + 1,$$
$$Q_2(z\bar{z}) = 2(z\bar{z})^2 + 6z\bar{z} + 2.$$
- In the $\kappa = 4$ case,
$$\mathbb{E}\Big(\Big|\frac{f(z)}{zf'(z)}\Big|^{6}\Big) = \frac{1}{(1-z\bar{z})^{3}}\Big(Q_1(z\bar{z}) + (\frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}})Q_2(z\bar{z})\Big),$$
where $Q_1(z\bar{z}), Q_2(z\bar{z})$ are defined as
$$Q_1(z\bar{z}) = \frac{15(z\bar{z})^3 + 63(z\bar{z})^2 + 45z\bar{z} + 5}{5},$$
$$Q_2(z\bar{z}) = \frac{10(z\bar{z})^3 + 54(z\bar{z})^2 + 54z\bar{z} + 10}{5}.$$
- In the $\kappa = 6$ case,
$$\mathbb{E}\Big(\Big|\frac{f(z)}{zf'(z)}\Big|^{8}\Big) = \frac{1}{(1-z\bar{z})^{4}}\Big(Q_1(z\bar{z}) + (\frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}})Q_2(z\bar{z})\Big),$$
where $Q_1(z\bar{z}), Q_2(z\bar{z})$ are defined as
$$Q_1(z\bar{z}) = 3(z\bar{z})^4 + 20(z\bar{z})^3 + 30(z\bar{z})^2 + 12z\bar{z} + 1,$$
$$Q_2(z\bar{z}) = 2(z\bar{z})^4 + 16(z\bar{z})^3 + 30(z\bar{z})^2 + 16z\bar{z} + 2.$$
- In the $\kappa = 8$ case,
$$\mathbb{E}\Big(\Big|\frac{f(z)}{zf'(z)}\Big|^{10}\Big) = \frac{1}{(1-z\bar{z})^{5}}\Big(Q_1(z\bar{z}) + (\frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}})Q_2(z\bar{z})\Big),$$
where $Q_1(z\bar{z}), Q_2(z\bar{z})$ are defined as
$$Q_1(z\bar{z}) = \frac{63(z\bar{z})^5 + 585(z\bar{z})^4 + 1430(z\bar{z})^3 + 1170(z\bar{z})^2 + 315z\bar{z} + 21}{21},$$
$$Q_2(z\bar{z}) = \frac{42(z\bar{z})^5 + 450(z\bar{z})^4 + 1300(z\bar{z})^3 + 1300(z\bar{z})^2 + 450z\bar{z} + 42}{21}.$$

3.4 Integral means spectrum

3.4.1 Introduction

This section is devoted to summarize the integral means spectrum study of Ref. [5]. In particular, in Section 6 of Ref. [5], we generalized to our work the average integral means spectrum analysis of Refs. [2] and [6] (see also [11, 12, 13]) concerning the whole-plane SLE_κ. The original work by Beliaev–Smirnov [2] dealt with the exterior version, whereas Ref. [6] and our work concern the interior case. We thus look for the singular behavior of the integral,

$$\int_{r\partial\mathbb{D}} \mathbb{E}\left(\frac{|f'(z)|^p}{|f(z)|^q}\right)|dz|, \tag{3.4.1}$$

for $r \to 1^-$, where f stands for the interior whole-plane SLE map at time zero. The generalized average integral means spectrum $\beta(p,q)$ corresponding to this generalized moment integral is the exponent such that

$$\int_{r\partial\mathbb{D}} \mathbb{E}\left(\frac{|f'(z)|^p}{|f(z)|^q}\right)|dz| \overset{(\mathrm{r}\to 1^-)}{\asymp} (1-r)^{-\beta(p,q)}, \tag{3.4.2}$$

in the sense of the equivalence of the logarithms of both terms.

It is interesting to remark that the map $\hat{f}$,

$$\zeta \in \mathbb{C}\setminus\overline{\mathbb{D}} \mapsto \hat{f}(\zeta) := 1/f(1/\zeta),$$

is just the *exterior* whole-plane map from $\mathbb{C}\setminus\overline{\mathbb{D}}$ to the slit plane considered by Beliaev and Smirnov in Ref. [2]. We identically have for $0 < r < 1$:

$$\int_{r^{-1}\partial\mathbb{D}} \mathbb{E}\left(|\hat{f}'(\zeta)|^p\right)|d\zeta| = r^{2p-2}\int_{r\partial\mathbb{D}} \mathbb{E}\left(\frac{|f'(z)|^p}{|f(z)|^{2p}}\right)|dz|. \tag{3.4.3}$$

We thus see that the standard integral mean of order $(p, q = 0)$ for the exterior whole-plane map studied in Ref. [2] *coincides* (up to an irrelevant power of r) with the (p,q) integral mean (3.4.1) for $q = 2p$, for the interior whole-plane map.

Remark 3.4.1. *Exterior-Interior Duality. More generally, we obviously have*

$$\int_{r^{-1}\partial\mathbb{D}} \mathbb{E}\left(\frac{|\hat{f}'(\zeta)|^p}{|\hat{f}(\zeta)|^{q'}}\right)|d\zeta| = r^{2p-2}\int_{r\partial\mathbb{D}} \mathbb{E}\left(\frac{|f'(z)|^p}{|f(z)|^{2p-q'}}\right)|dz|, \tag{3.4.4}$$

so that the (p,q') exterior integral means spectrum coincides with the (p,q) interior integral means spectrum for $q+q' = 2p$. In particular, the $(p,0)$ interior derivative moments studied in Ref. [6] correspond to the $(p,2p)$ mixed moments of the Beliaev–Smirnov exterior map.

In the original Beliaev–Smirnov case, the integral means spectrum successively involves three functions [2]:

$$\beta_{\rm tip}(p,\kappa) := -p-1+\frac{1}{4}\big(4+\kappa-\sqrt{(4+\kappa)^2-8\kappa p}\,\big), \tag{3.4.5}$$

$$\text{for } p \le p_0'(\kappa) := -1-\frac{3\kappa}{8}; \tag{3.4.6}$$

$$\beta_0(p,\kappa) := -p+\frac{4+\kappa}{4\kappa}\big(4+\kappa-\sqrt{(4+\kappa)^2-8\kappa p}\,\big), \tag{3.4.7}$$

$$\text{for } p_0'(\kappa) \le p \le p_0(\kappa);$$

$$\beta_{\rm lin}(p,\kappa) := p-\frac{(4+\kappa)^2}{16\kappa}, \tag{3.4.8}$$

$$\text{for } p \ge p_0(\kappa) := \frac{3(4+\kappa)^2}{32\kappa}. \tag{3.4.9}$$

As shown in Refs. [6, 11, 12, 13] in the interior case, because of the unboundedness of the interior whole-plane SLE_κ map, there exists a phase transition at $p = p^*(\kappa)$, with

$$\begin{aligned} p^*(\kappa) &:= \frac{1}{16\kappa}\left((4+\kappa)^2-4-2\sqrt{2(4+\kappa)^2+4}\right) \\ &= \frac{1}{32\kappa}\left(\sqrt{2(4+\kappa)^2+4}-6\right)\left(\sqrt{2(4+\kappa)^2+4}+2\right). \end{aligned} \tag{3.4.10}$$

The integral means spectrum is afterwards given by

$$\beta(p,\kappa) := 3p-\frac{1}{2}-\frac{1}{2}\sqrt{1+2\kappa p}, \text{ for } p \ge p^*(\kappa). \tag{3.4.11}$$

Since $p^*(\kappa) < p_0(\kappa)$ (3.4.9), this transition precedes and supersedes the transition from the bulk spectrum (3.4.7) towards the linear behavior (3.4.8).

The singularity analysis given in Ref. [6] led us to introduce the σ-dependent function

$$\beta_+^\sigma(p,\kappa) = (1-2\sigma)p-\frac{1}{2}\big(1+\sqrt{1-2\sigma\kappa p}\,\big). \tag{3.4.12}$$

For $\sigma = -1$, it recovers the integral means spectrum (3.4.11) above for the interior whole-plane SLE, while for $\sigma = +1$ it introduces a new spectrum,

$$\beta_+^{(+1)}(p,\kappa) = -p-\frac{1}{2}\big(1+\sqrt{1-2\kappa p}\,\big), \tag{3.4.13}$$

the relevance of which for the exterior whole-plane SLE case is analyzed in a joint work of M. Zinsmeister and B. Duplantier with D. Beliaev [1].

For general real values of σ,

$$\sigma := q/p-1, \tag{3.4.14}$$

we can rewrite (3.4.12) as a function of (p, q, κ),

$$\beta_+^\sigma(p, \kappa) = \beta_1(p, q; \kappa) := 3p - 2q - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa(p - q)}. \tag{3.4.15}$$

The multifractal spectrum (3.4.15) is defined only for $2\kappa(q-p) \leq 1$, hence for points in the (p, q) plane *below* the oblique line (see Fig. 3.3):

$$\Delta_1 := \left\{(p, q) \in \mathbb{R}^2, q = p + 1/2\kappa\right\}. \tag{3.4.16}$$

Note also that both the tip spectrum (3.4.5) and the bulk spectrum (3.4.7) are defined only to the *left* of a vertical line in the (p, q) plane, as given by (see Fig. 3.3)

$$\Delta_0 := \left\{p = \frac{(4+\kappa)^2}{8\kappa}, q \in \mathbb{R}\right\}. \tag{3.4.17}$$

We claim that the generalized spectrum generated by the integral means (3.4.1) in the general (p, q) case will involve the standard multifractal spectra (3.4.5), (3.4.7), (3.4.8), that are independent of q, and also the new (p, q)-dependent multifractal spectrum (3.4.15). The phase transitions between these spectra will occur along lines drawn in the real (p, q) plane.

Here we will simply describe the corresponding partition of the (p, q) plane into the respective domains of validity of the four spectra above. We thus need to determine the boundary curves where pairs (possibly triplets) of these spectra coincide, which are signaling the onset of the respective transitions.

3.4.2 Phase transition lines

3.4.2.1 The Red Parabola

The parabola $\mathcal{R}$ of Theorems 2.2.1 and 2.3.4, which we will hereafter call (and draw in) **red** (see Fig. 3.3), is given by the simultaneous conditions,

$$A(p, q, \gamma) = 0,\ C(p, \gamma) = 0, \tag{3.4.18}$$

which recovers the parametric form (2.2.1)

$$\begin{aligned} p = p_{\mathcal{R}}(\gamma) &:= \left(2 + \frac{\kappa}{2}\right)\gamma - \frac{\kappa}{2}\gamma^2, \\ q = q_{\mathcal{R}}(\gamma) &:= \left(3 + \frac{\kappa}{2}\right)\gamma - \kappa\gamma^2,\ \gamma \in \mathbb{R}. \end{aligned} \tag{3.4.19}$$

We then showed that the B–S bulk spectrum $\beta_0(p)$ and the novel spectrum $\beta_1(p, q; \kappa)$ coincide along the finite sector of parabola $\mathcal{R}$ located between its tangency points T_0 and T_1 with Δ_0 and Δ_1, i.e., corresponding to the domain where $\gamma \in [1/\kappa, 2/\kappa + 1/2]$ (see Fig. 3.3).

In Cartesian coordinates, the red parabola $\mathcal{R}$ (3.4.19) has for equation (2.2.2).

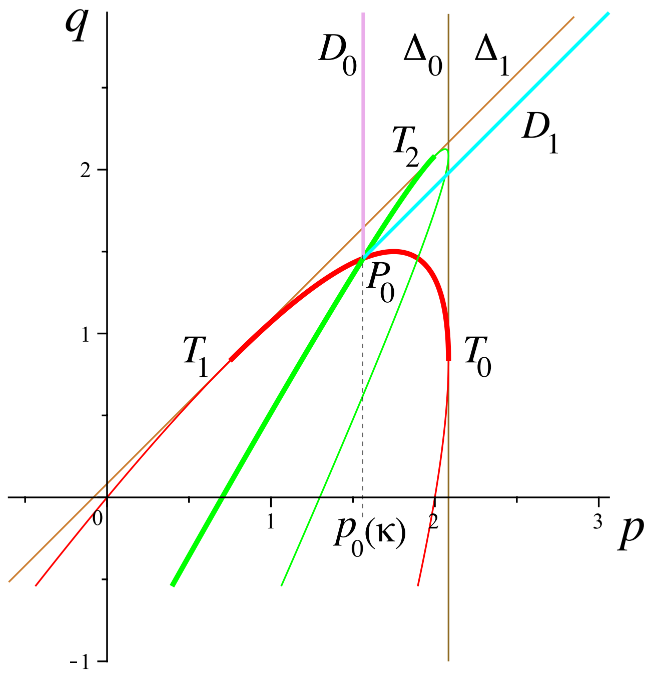

Figure 3.3: *Red parabola* $\mathcal{R}$ (3.4.19) *and green parabola* $\mathcal{G}$ (3.4.21) *(for* $\kappa = 6$*). From the intersection point* P_0 (3.4.22) *originate the two (half)-lines* D_0 (3.4.24) *and* D_1 (3.4.25)*. The bulk spectrum* $\beta_0(p)$ *and the generalized spectrum* $\beta_1(p,q)$ *coincide along the arc of red parabola between its tangency points* T_0 *and* T_1 *with* Δ_0 *and* Δ_1 *(thick red line). They also coincide along the infinite left branch of the green parabola, up to its tangency point* T_2 *to* Δ_1 *(thick green line). The* $\beta_0(p)$ *spectrum and the linear one* $\beta_{\text{lin}}(p)$ *coincide along* D_0*, whereas* $\beta_1(p,q)$ *and* $\beta_{\text{lin}}(p)$ *coincide along* D_1*.*

3.4.2.2 The Green Parabola

A second parabola in the (p,q) plane, hereafter called (and drawn in) green (see Fig. 3.3) and denoted by $\mathcal{G}$, is such that the multifractal spectra $\beta_0(p)$ and $\beta_1(p,q;\kappa)$ coincide on part of it. By using the *duality* property of the spectrum function [5, 6], we set the simultaneous seed conditions,

$$\begin{aligned} &A(p,q,\gamma') = 0,\ C(p,\gamma'') = 0, \\ &\gamma' + \gamma'' = 2/\kappa + 1/2, \end{aligned} \tag{3.4.20}$$

where γ' and γ'' are *dual* of each other. Eqs. (2.2.15) and (2.2.17) immediately give the parametric form for the green parabola,

$$\begin{aligned} p = p_{\mathcal{G}}(\gamma') &:= \frac{(4+\kappa)^2}{8\kappa} - \frac{\kappa}{2}\gamma'^2, \\ q = q_{\mathcal{G}}(\gamma') &:= \frac{(4+\kappa)^2}{8\kappa} + \gamma' - \kappa\gamma'^2,\ \gamma' \in \mathbb{R}. \end{aligned} \tag{3.4.21}$$

We showed [5] that the multifractal spectra $\beta_0(p)$ and $\beta_1(p,q;\kappa)$ coincide along the infinite left branch of parabola $\mathcal{G}$ below its tangency point T_2 with Δ_1, i.e., corresponding to the domain where $\gamma' \in [1/\kappa, +\infty)$ (Fig. 3.3).

3.4.2.3 Quadruple point

The intersection of the **red** and **green** parabolas (3.4.19) and (3.4.21) can be found by combining the seed equations (3.4.18) and (3.4.20). We find either $\gamma = \gamma' = 1/\kappa + 1/4$, or $\gamma = 2/\kappa + 1/4, \gamma' = -1/4$, which lead to the two intersection points,

$$P_0 : p_0 = p_0(\kappa) = \frac{3(4+\kappa)^2}{32\kappa}, \qquad q_0 = \frac{(4+\kappa)(8+\kappa)}{16\kappa}, \tag{3.4.22}$$

$$P_1 : p_1 = \frac{(8+\kappa)(8+3\kappa)}{32\kappa}, \qquad q_0 = \frac{(4+\kappa)(8+\kappa)}{16\kappa}. \tag{3.4.23}$$

Note that these points have same ordinate, while the abscissa of the left-most one, P_0, is $p_0(\kappa)$ (3.4.9), where the integral means spectrum transits from the B–S bulk form (3.4.7) to its linear form (3.4.8).

Through this intersection point P_0 further pass two important straight lines in the (p,q) plane.

Definition 3.4.1. D_0 and D_1 are, respectively, the vertical line and the slope one line passing through point P_0, of equations

$$D_0 := \{(p,q) : p = p_0\}, \tag{3.4.24}$$

$$D_1 := \left\{(p,q) : q - p = q_0 - p_0 = \frac{16-\kappa^2}{32\kappa}\right\}. \tag{3.4.25}$$

A key property of D_1 is the following. The difference,

$$\beta_1(p,q;\kappa) - \beta_{\mathrm{lin}}(p,\kappa) = \frac{1}{\kappa}\left(\frac{\kappa}{4} - \sqrt{1+2\kappa(p-q)}\right)^2, \tag{3.4.26}$$

is always positive, and vanishes only on line D_1, where

$$\forall (p,q) \in D_1, \;\; \beta_1(p,q;\kappa) = \beta_{\mathrm{lin}}(p,\kappa) = p - \frac{(4+\kappa)^2}{16\kappa}. \tag{3.4.27}$$

3.4.2.4 The Blue Quartic

A third locus, the **blue** quartic $\mathcal{Q}$, will also play an important role, that is where the B–S tip-spectrum, $\beta_{\mathrm{tip}}(p;\kappa)$ (3.4.5), coincides with the novel spectrum, $\beta_1(p,q;\kappa)$ (3.4.15). In the (p,q) plane, the tip condition (3.4.6) [2] descibes the domain to the left of the straight line D'_0 (Fig. 3.4), defined by

$$D'_0 := \{(p,q) : p = p'_0(\kappa) = -1 - 3\kappa/8\}. \tag{3.4.28}$$

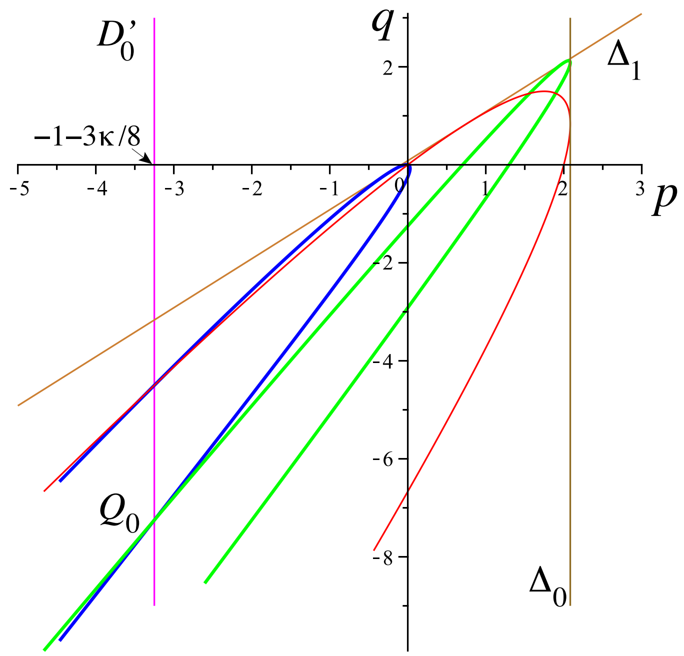

Figure 3.4: *The blue quartic 3.4.29 for* $\kappa = 6$. *It intersects the green parabola at point* Q_0 (3.4.31) *and the red parabola at point* Q_1 (3.4.30) *(not marked), both of abscissa* $p_0'(\kappa) = -1 - 3\kappa/8$.

As described in Ref. [5], the parametric form for the blue quartic $\mathcal{Q}$ is given by

$$\begin{aligned} p &= p_{\mathcal{Q}}(\gamma) := \frac{\kappa}{16} + \left(1 + \frac{\kappa}{4}\right)\gamma - \frac{\kappa}{2}\gamma^2 - \frac{1}{8}\Delta^{\frac{1}{2}}(\gamma), \\ q &= q_{\mathcal{Q}}(\gamma) := p_{\mathcal{Q}}(\gamma) + \gamma - \frac{\kappa}{2}\gamma^2, \quad \gamma \in \mathbb{R}, \end{aligned} \tag{3.4.29}$$

where

$$\Delta(\gamma) := 4\kappa^2\gamma^2 - 2\kappa(4+\kappa)\gamma + \frac{1}{4}(8+\kappa)^2 + 4\kappa.$$

The intersection of the **blue** quartic $\mathcal{Q}$ (3.4.29) with the **red** parabola $\mathcal{R}$ (3.4.19) is located at

$$Q_1 : p_0' = -1 - \frac{3\kappa}{8},\ q = -\frac{1}{2}(3+\kappa);\ \ \gamma = -\frac{1}{2}, \tag{3.4.30}$$

followed by a second intersection at the origin, $p = q = 0$, for $\gamma = 2/\kappa$.

The intersection of the **blue** quartic $\mathcal{Q}$ (3.4.29) with the **green** parabola $\mathcal{G}$ (3.4.21) is located at

$$Q_0 : p_0' = -1 - \frac{3\kappa}{8},\ q_0' := -2 - \frac{7\kappa}{8};\ \ \gamma = \gamma' = 1 + \frac{2}{\kappa}. \tag{3.4.31}$$

The tip spectrum and the generalized one coincide in both γ-intervals $[1/\kappa, 1 + 2/\kappa]$ and $[1 + 2/\kappa, +\infty)$, which together parameterize the branch of the quartic located below its contact with Δ_1 (see Fig. 3.4). Because of the tip relevance condition (3.4.6), only the interval $[1+2/\kappa, +\infty)$ describing the lower infinite branch of the quartic located to the *left* of Q_0 will matter for the integral means spectrum.

3.4.3 Whole-plane SLE_κ generalized spectrum

3.4.3.1 Four-domain structure for the generalized spectrum $\beta(p, q)$

The only possible scenario which thus emerges to construct the average generalized integral means spectrum by a continuous matching of the 4 different spectra along the phase transition lines described above, is the partition of the (p, q) plane in 4 different regions as indicated in Fig. 3.5:

- a part (I) to the left of D'_0 and located above the blue quartic up to point Q_0, where the average integral means spectrum is $\beta_{\mathrm{tip}}(p)$;
- an upper part (II) bounded by lines D'_0, D_0, and located above the section of the green parabola between points Q_0 and P_0, where the spectrum is given by $\beta_0(p)$;
- an infinite wedge (III) of apex P_0 located between the upper half-lines D_0 and D_1, where the spectrum is given by $\beta_{\mathrm{lin}}(p)$;
- a lower part (IV) whose boundary is the blue quartic up to point Q_0, followed by the arc of green parabola between points P_0 and Q_0, followed by the half-line D_1 above P_0 where the spectrum is $\beta_1(p, q)$.

The two wings T_1P_0 and P_0T_0 of the red parabola (Fig. 3.3), where we know from Theorem 2.3.4 that the average spectrum is given by $\beta_0(p) = \beta_1(p, q)$, can thus be seen as the respective extensions of region IV into II and of region II into IV.

This is summarized by the following theorem (Proposition 6.1 in Ref. [5]).

Theorem 3.4.2. *The separatrix curves for the generalized integral means spectrum of whole-plane* SLE_κ *are in the* (p, q) *plane (Fig. 3.5):*

- *(i) the vertical half-line* D_0 *above* $P_0 = (p_0, q_0)$ (3.4.22), *where* $p_0 = 3(4+\kappa)^2/32\kappa, q_0 = (4+\kappa)(8+\kappa)/16\kappa$;
- *(ii) the unit slope half-line* D_1 *originating at* P_0*, whose equation is* $q - p = (16-\kappa^2)/32\kappa$ *with* $p \geq p_0$;
- *(iii) the section of green parabola, with parametric coordinates* $\big(p_{\mathcal{G}}(\gamma'), q_{\mathcal{G}}(\gamma')\big)$ (3.4.21) *for* $\gamma' \in [1/4 + 1/\kappa, 1 + 2/\kappa]$*, between* P_0 *and* $Q_0 = (p'_0, q'_0)$ (3.4.31), *where* $p'_0 = -1 - 3\kappa/8, q'_0 = -2 - 7\kappa/8$;
- *(iv) the vertical half-line* D'_0 *above point* Q_0;
- *(v) the branch of the blue quartic from* Q_0 *to* ∞*, with parametric coordinates* $\big(p_{\mathcal{Q}}(\gamma), q_{\mathcal{Q}}(\gamma)\big)$ (3.4.29) *for* $\gamma \in [1 + 2/\kappa, +\infty)$.

As mentioned above, the whole-plane SLE_κ case studied by Beliaev and Smirnov corresponds to the *B–S line* $q = 2p$. Owing to Eq. (2.2.2), it intersects the red parabola $\mathcal{R}$ only at $p = 0$. The green parabola $\mathcal{G}$ (3.4.21) has for Cartesian equation,

$$\frac{\kappa}{2}(2p-q)^2 - \frac{1}{8}(4+\kappa)^2\,(2p-q) + p + \frac{1}{128}(4+\kappa)^2(8+\kappa) = 0, \tag{3.4.32}$$

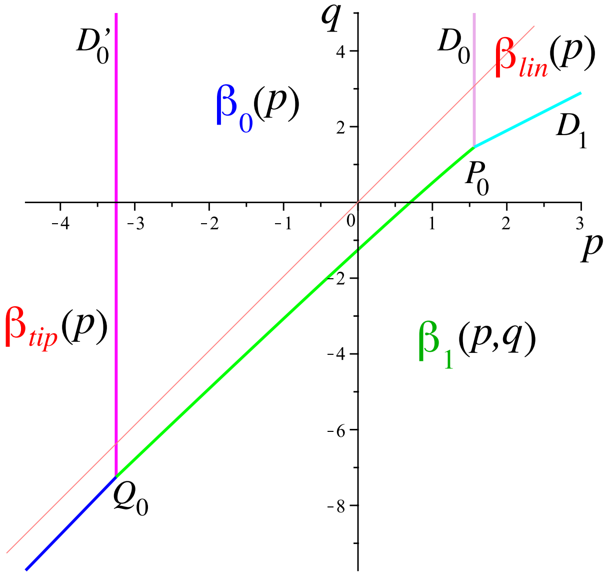

Figure 3.5: *Respective domains of validity of integral means spectra $\beta_{\text{tip}}(p)$, $\beta_0(p)$, $\beta_{\text{lin}}(p)$, and $\beta_1(p,q)$. The thin straight line (coral) $q = 2p$ corresponds to the version of whole-plane SLE studied in Ref. [2]. It does not intersect the lower domain where β_1 holds.*

which shows that it intersects the B–S line at [1]

$$p = p_0''(\kappa) := -\frac{1}{128}(4+\kappa)^2(8+\kappa), \tag{3.4.33}$$

which is to the *left* of the tip transition line at $p_0'(\kappa) = -1 - \frac{3}{8}\kappa$ (3.4.6). The quartic $\mathcal{Q}$ (3.4.29) obeys

$$\left[\left(2p - q - \frac{\kappa}{16}\right)^2 - \frac{c}{4}\right]\left(2p - q - 1 - \frac{\kappa}{8}\right)(2p-q) = \frac{\kappa}{2}(p-q)\left(2p - q - \frac{1}{4} - \frac{\kappa}{8}\right)^2$$
$$c = c(\kappa) := \frac{1}{64}(8+\kappa)^2 + \frac{\kappa}{4}, \tag{3.4.34}$$

which immediately shows that the B–S line $q = 2p$ intersects $\mathcal{Q}$ only at the origin and stays above its lower branch.

The B–S line therefore does not intersect the segment of green parabola $\mathcal{G}$ between P_0 and Q_0, nor the quartic $\mathcal{Q}$ below Q_0 (Fig. 3.5). Thus the novel spectrum β_1 *does not* a priori appear in the version of whole-plane SLE$_\kappa$ considered in Ref. [2]. The B–S line nevertheless intersects $\mathcal{G}$ at p_0'' (3.4.33) to the left of Q_0, in a domain lying above the quartic.

We summarize the proof of Theorem 3.4.2 as follows: we first prove that the generalized integral means spectrum $\beta(p,q)$ has in the upward wedge sector located between the upper half-lines D_0 and D_1 the *linear* form $\beta_{\rm lin}(p)$. Next, we show that the generalized spectrum $\beta(p,q)$ is given in the whole infinite domain of the (p,q) plane located to the left of the infinite lower branch of the green parabola $\mathcal{G}$ (3.4.21) below point P_0 and to the left of the half-line D_0 above P_0 (Fig. 3.6) by $\beta_0(p)$ for $p \geq -1-3\kappa/8$ or by $\beta_{\rm tip}(p)$ in the reverse case. Next, we show that

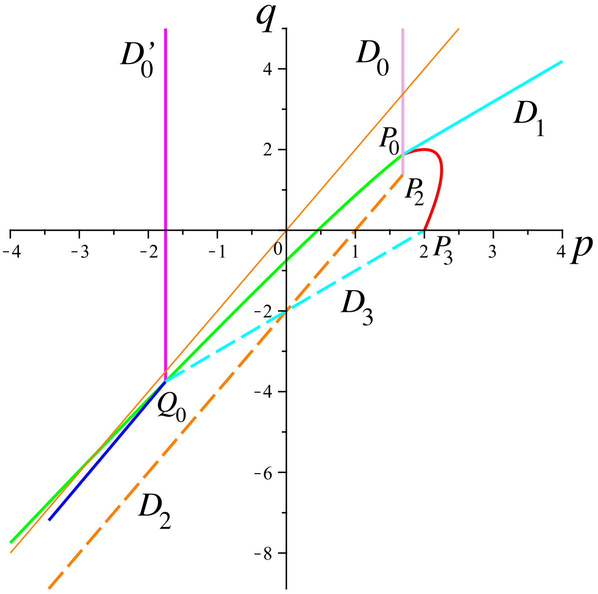

Figure 3.6: *Various geometrical elements appearing in the proof of Theorem 3.4.2, when establishing the respective domains of validity of integral means spectra $\beta_{tip}(p), \beta_0(p), \beta_{lin}(p)$, and $\beta_1(p,q)$. The dashed coral line D_2 corresponds to Eq.* (3.4.35) *and intersects D_0 at point P_2. The dashed cyan line D_3 corresponds to Eq.* (3.4.36), *and intersects D'_0 at Q_0 and the red parabola $\mathcal{R}$ at point P_3. The $q=2p$ continuous straight line in coral , corresponding to the whole-plane* SLE *version studied in Ref. [2], does not intersect the blue quartic, but intersects the green parabola at a point of abscissa* (3.4.33). *(In the particular $\kappa=2$ case shown here, the intersection point P_3 of D_3 with $\mathcal{R}$ coincides with the intersection point* $(p(2)=2, q=0)$ *of $\mathcal{R}$ with the p-axis.)*

the generalized integral means spectrum $\beta(p,q)=\beta_{\rm tip}$ in the infinite wedge sector of apex Q_0, between the green parabola $\mathcal{G}$ end the blue quartic $\mathcal{Q}$, and located to the left of the line D'_0 (Fig. 3.6). We further show that $\beta(p,q)=\beta_1(p,q)$ in the semi-infinite strip (Fig. 3.6) located under the blue quartic $\mathcal{Q}$ up to Q_0, under the branch of the green parabola $\mathcal{G}$ between Q_0 and P_0, and above the straight line D_2 of equation,

$$q = 2p - p(\kappa), \quad p(\kappa) := \frac{(2+\kappa)(6+\kappa)}{8\kappa}. \tag{3.4.35}$$

To the right, this domain is closed by the vertical segment of line D_0 located above the intersection point of D_0 with D_2, $P_2 = \big(p_0(\kappa), q_2 = \frac{(4+\kappa)^2+8}{16\kappa}\big)$, and up to point P_0

(Fig. 3.6). Finally, we show that the generalized integral means spectrum $\beta(p,q) = \beta_1(p,q)$ in a finite domain located above the straight line D_3 of equation (Fig. 3.6),

$$p - q = \hat{p}(\kappa), \quad \hat{p}(\kappa) := 1 + \kappa/2. \tag{3.4.36}$$

This line intersects the green parabola $\mathcal{G}$ at point Q_0 and the red one $\mathcal{R}$ at point $P_3 = \left(1+\frac{2}{\kappa}, \frac{4-\kappa^2}{2\kappa}\right)$. This finite domain is enclosed by the segment of line D_2 between its intersection point with D_3, $\left(\frac{6-3\kappa}{4\kappa}\hat{p}(\kappa), \frac{6-7\kappa}{4\kappa}\hat{p}(\kappa)\right)$, and P_2, followed by the vertical segment of line D_0 located above P_2 and up to point P_0, followed by the section of the red parabola $\mathcal{R}$ between P_0 and P_3, followed by the segment of line D_3 between its intersection point with D_2 and P_3.

Thus, the validity of the various spectra is established in an infinite domain of the (p,q)-plane in Fig. 3.6, located above a frontier line obtained by following by continuity the set of lines: D_2 up to its intersection point with D_3, D_3 up to P_3, the section of the red parabola $\mathcal{R}$ between P_3 and P_0, and D_1 from P_0 up to infinity. This concludes the proof of Theorem 3.4.2. For convenience, we will denote by $\mathcal{J}$ this frontier line.

3.4.3.2 Checks for points below the frontier line $\mathcal{J}$

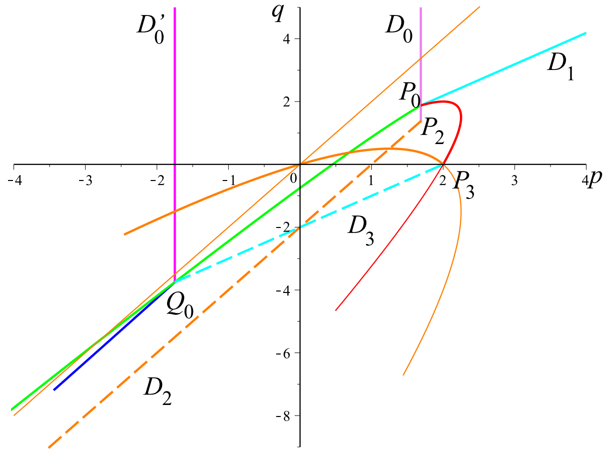

Figure 3.7: *The coral parabola* $\mathcal{R}_1$ (3.4.37) *and the red parabola* $\mathcal{R}$ (3.4.19) *have the same intersection point* P_3 *with the straight line* D_3 (3.4.36). *The infinite branch of* $\mathcal{R}_1$ *below* P_3 *(thin coral line), with parametric coordinates* $\left(p_{\mathcal{C}}(\gamma), q_{\mathcal{C}}(\gamma)\right)$ (3.4.37) *for* $\gamma \in [1+2/\kappa, +\infty)$. *The infinite branch of* $\mathcal{R}$ *below* P_3 *(thin red line), with parametric coordinates* $\left(p_{\mathcal{R}}(\gamma), q_{\mathcal{R}}(\gamma)\right)$ (3.4.19) *for* $\gamma \in [1+2/\kappa, +\infty)$. *(In the particular* $\kappa = 2$ *case shown here, point* P_3 *coincides with the intersection point* $(p(2) = 2, q = 0)$ *of* $\mathcal{R}$ *with the p-axis.)*

The parabola $\mathcal{R}_1$ of Theorems 3.1.1 and 3.1.2, which we will hereafter call (and draw in) coral (Fig. 3.7), is given by the simultaneous conditions,

$$A(p,q,\gamma) = 0, \; C(p,\gamma-1) = 0,$$

which recovers the parametric form (3.1.2)

$$\begin{aligned} p = p_{\mathcal{C}}(\gamma) &:= \left(2+\frac{3}{2}\kappa\right)\gamma - \frac{\kappa}{2}\gamma^2 - \kappa - 2, \\ q = q_{\mathcal{C}}(\gamma) &:= \left(3+\frac{3}{2}\kappa\right)\gamma - \kappa\gamma^2 - \kappa - 2, \ \gamma \in \mathbb{R}. \end{aligned} \tag{3.4.37}$$

In Cartesian coordinates, the coral parabola $\mathcal{R}_1$ (3.4.37) has for equation (3.1.4).

It is interesting to note that the coral parabola $\mathcal{R}_1$ and the red parabola $\mathcal{R}$ have the same intersection point $P_3 = \big(1+\frac{2}{\kappa}, \frac{4-\kappa^2}{2\kappa}\big)$ with the straight line D_3 (3.4.36) (Fig. 3.7). The unique intersection point of the straight line D_2 (3.4.35) with the coral parabola $\mathcal{R}_1$, $\big(\frac{5(2+\kappa)(6-\kappa)}{32\kappa}, \frac{(2+\kappa)(18-7\kappa)}{16\kappa}\big)$, is between that of D_2 with the red parabola $\mathcal{R}$, $\big(\frac{(6+\kappa)(10+3\kappa)}{32\kappa}, \frac{(6+\kappa)^2}{16\kappa}\big)$, and that of D_2 with D_3, $\big(\frac{6-3\kappa}{4\kappa}\hat{p}(\kappa), \frac{6-7\kappa}{4\kappa}\hat{p}(\kappa)\big)$ (Fig. 3.7).

Using Theorem 3.1.2, we now calculate the generalized spectrum $\beta(p,q;\kappa)$ for some points lying on the infinite branch of the coral parabola $\mathcal{R}_1$ below P_3 (Fig. 3.7), with parametric coordinates $\big(p_{\mathcal{C}}(\gamma), q_{\mathcal{C}}(\gamma)\big)$ (3.4.37) for $\gamma \in [1+2/\kappa, +\infty)$, in the $\kappa = 2$ and $\kappa = 3$ cases.

- For $\kappa = 2$ and $\gamma = 3$, we obtain $(p,q) = (p_{\mathcal{C}}(\gamma), q_{\mathcal{C}}(\gamma)) = (2,-4)$ and

$$G\big(re^{i\theta}, re^{-i\theta}\big) = \frac{\big(r^2+1-2r\cos\theta\big)^2\,\big(r^6+9\,r^4+9\,r^2+1-\big(4\,r^5+12\,r^3+4\,r\big)\cos\theta\big)}{(1-r)^{11}(1+r)^{11}}.$$

Then,

$$\frac{1}{2\pi}\int_0^{2\pi} G(re^{i\theta}, re^{-i\theta})d\theta = \frac{\big(r^2+1\big)\big(r^8+20\,r^6+58\,r^4+20\,r^2+1\big)}{(1+r)^{11}(1-r)^{11}} \underset{\asymp}{(\mathrm{r}\to 1^-)} \frac{1}{(1-r)^{11}},$$

which implies $\beta(2,-4;2) = 11 = \beta_1(2,-4;2)$.

- For $\kappa = 2$ and $\gamma = 4$, we obtain $(p,q) = (p_{\mathcal{C}}(\gamma), q_{\mathcal{C}}(\gamma)) = (0,-12)$ and

$$G\big(re^{i\theta}, re^{-i\theta}\big) = \frac{\big(r^2+1-2r\cos\theta\big)^3\big(P_0(r)+P_1(r)\cos\theta\big)}{(1-r)^{20}(1+r)^{20}},$$

where

$$\begin{aligned} P_0(r) &= r^{10}+25\,r^8+100\,r^6+100\,r^4+25\,r^2+1, \\ P_1(r) &= -\big(6\,r^9+60\,r^7+120\,r^5+60\,r^3+6\,r\big). \end{aligned}$$

Then, $\frac{1}{2\pi}\int\limits_0^{2\pi} G(re^{i\theta}, re^{-i\theta})d\theta$

$$= \frac{r^{16}+52\,r^{14}+568\,r^{12}+2144\,r^{10}+3290\,r^8+2144\,r^6+568\,r^4+52\,r^2+1}{(1-r)^{20}\,(1+r)^{20}}$$

$$\underset{\asymp}{(\mathrm{r}\to 1^-)} \frac{1}{(1-r)^{20}},$$

which implies $\beta(0,-12;2) = 20 = \beta_1(0,-12;2)$.

- For $\kappa = 2$ and $\gamma = 5$, we obtain $(p, q) = (p_{\mathcal{C}}(\gamma), q_{\mathcal{C}}(\gamma)) = (-4, -24)$ and

$$G\big(re^{i\theta}, re^{-i\theta}\big) = \frac{\big(r^2 + 1 - 2r\cos\theta\big)^4 \big(P_0(r) + P_1(r)\cos\theta\big)}{(1-r)^{31}(1+r)^{31}},$$

where

$$\begin{aligned} P_0(r) &= r^{14} + 49\,r^{12} + 441\,r^{10} + 1225\,r^8 + 1225\,r^6 + 441\,r^4 + 49\,r^2 + 1, \\ P_1(r) &= -\big(8\,r^{13} + 168\,r^{11} + 840\,r^9 + 1400\,r^7 + 840\,r^5 + 168\,r^3 + 8\,r\big). \end{aligned}$$

Then,

$$\frac{1}{2\pi}\int_0^{2\pi} G(re^{i\theta}, re^{-i\theta})d\theta = \frac{(r^2+1)\,P(r)}{(1-r)^{31}(1+r)^{31}} \overset{(\mathrm{r}\to 1^-)}{\asymp} \frac{1}{(1-r)^{31}},$$

which implies $\beta(-4, -24; 2) = 31 = \beta_1(-4, -24; 2)$, where

$$\begin{aligned} P(r) = r^{20} &+ 96\,r^{18} + 2029\,r^{16} + 15616\,r^{14} + 51694\,r^{12} + 77344\,r^{10} \\ &+ 51694\,r^8 + 15616\,r^6 + 2029\,r^4 + 96\,r^2 + 1. \end{aligned}$$

- For $\kappa = 3$ and $\gamma = 2$, we obtain $(p, q) = (p_{\mathcal{C}}(\gamma), q_{\mathcal{C}}(\gamma)) = (2, -2)$ and

$$G\big(re^{i\theta}, re^{-i\theta}\big) = \frac{\big(r^2 + 1 - 2r\cos\theta\big)\,\big(5\,r^4 + 18\,r^2 + 5 - 14\,r\big(r^2+1\big)\cos\theta\big)}{5(1-r)^7(1+r)^7}.$$

Then,

$$\frac{1}{2\pi}\int_0^{2\pi} G(re^{i\theta}, re^{-i\theta})d\theta = \frac{\big(r^2+1\big)\big(5\,r^4 + 32\,r^2 + 5\big)}{5(1+r)^7(1-r)^7} \overset{(\mathrm{r}\to 1^-)}{\asymp} \frac{1}{(1-r)^7},$$

which implies $\beta(2, -2; 3) = 7 = \beta_1(2, -2; 3)$.

- For $\kappa = 3$ and $\gamma = 3$, we obtain $(p, q) = (p_{\mathcal{C}}(\gamma), q_{\mathcal{C}}(\gamma)) = (1, -19/2)$ and

$$G\big(re^{i\theta}, re^{-i\theta}\big) = \frac{\big(r^2 + 1 - 2r\cos\theta\big)^2 \big(P_0(r) + P_1(r)\cos\theta\big)}{15(1-r)^{35/2}(1+r)^{35/2}},$$

where

$$\begin{aligned} P_0(r) &= 15\,r^{10} + 315\,r^8 + 1190\,r^6 + 1190\,r^4 + 315\,r^2 + 15, \\ P_1(r) &= -\big(78\,r^9 + 728\,r^7 + 1428\,r^5 + 728\,r^3 + 78\,r\big). \end{aligned}$$

Then, $\frac{1}{2\pi}\int\limits_0^{2\pi} G(re^{i\theta}, re^{-i\theta})d\theta$

$$= \frac{\big(r^2+1\big)\big(15\,r^{12} + 516\,r^{10} + 3561\,r^8 + 7016\,r^6 + 3561\,r^4 + 516\,r^2 + 15\big)}{15\,(r-1)^{35/2}\,(r+1)^{35/2}}$$

$$\overset{(\mathrm{r}\to 1^-)}{\asymp} \frac{1}{(1-r)^{35/2}},$$

which implies $\beta(1, -19/2; 3) = 35/2 = \beta_1(1, -19/2; 3)$.

- For $\kappa = 3$ and $\gamma = 4$, we obtain $(p,q) = (p_{\mathcal{C}}(\gamma), q_{\mathcal{C}}(\gamma)) = (-3,-23)$ and

$$G\big(re^{i\theta}, re^{-i\theta}\big) = \frac{\big(r^2+1-2r\cos\theta\big)^3\big(P_0(r)+P_1(r)\cos\theta\big)}{3315(1-r)^{31}(1+r)^{31}},$$

where

$$\begin{aligned} P_0(r) &= 3315\,r^{16} + 175032\,r^{14} + 1973972\,r^{12} + 7592200\,r^{10} + 11723250\,r^8 \\ &\qquad + 7592200\,r^6 + 1973972\,r^4 + 175032\,r^2 + 3315, \\ P_1(r) &= -\big(25194\,r^{15} + 646646\,r^{13} + 4327554\,r^{11} + 10606750\,r^9 + 10606750\,r^7 \\ &\qquad + 4327554\,r^5 + 646646\,r^3 + 25194\,r\big). \end{aligned}$$

Then,

$$\frac{1}{2\pi}\int_0^{2\pi} G(re^{i\theta}, re^{-i\theta})d\theta = \frac{(r^2+1)\,P(r)}{3315(1-r)^{31}(1+r)^{31}} \overset{(\mathrm{r}\to 1^-)}{\asymp} \frac{1}{(1-r)^{31}},$$

which implies $\beta(-3,-23;3) = 31 = \beta_1(-3,-23;3)$, where

$$\begin{aligned} P(r) &= 3315\,r^{20} + 277134\,r^{18} + 5468645\,r^{16} + 40345964\,r^{14} + 130356040\,r^{12} \\ &+ 193312844\,r^{10} + 130356040\,r^8 + 40345964\,r^6 + 5468645\,r^4 + 277134\,r^2 + 3315. \end{aligned}$$

Next, we use Corollary 2.3.5 to calculate the spectrum $\beta(p,q;\kappa)$ for some points lying on the infinite branch of the red parabola $\mathcal{R}$ below P_3 (Fig. 3.7), with parametric coordinates $\big(p_{\mathcal{R}}(\gamma), q_{\mathcal{R}}(\gamma)\big)$ (3.4.19) for $\gamma \in [1+2/\kappa, +\infty)$, in the $\kappa = 2$ and $\kappa = 3$ cases.

- For $\kappa = 2$ and $\gamma = 3$, we obtain $(p,q) = (p_{\mathcal{R}}(\gamma), q_{\mathcal{R}}(\gamma)) = (0,-6)$ and

$$G(re^{i\theta}, re^{-i\theta}) = \frac{\big(r^2+1-2r\cos\theta\big)^3}{(1-r)^9(1+r)^9}.$$

Then,

$$\frac{1}{2\pi}\int_0^{2\pi} G(re^{i\theta}, re^{-i\theta})d\theta = \frac{(r^2+1)\,(r^4+8\,r^2+1)}{(1-r)^9(1+r)^9} \overset{(\mathrm{r}\to 1^-)}{\asymp} \frac{1}{(1-r)^9},$$

which implies $\beta(0,-6;2) = 9 = \beta_1(0,-6;2)$.

- For $\kappa = 2$ and $\gamma = 5$, we obtain $(p,q) = (p_{\mathcal{R}}(\gamma), q_{\mathcal{R}}(\gamma)) = (-10,-30)$ and

$$G(re^{i\theta}, re^{-i\theta}) = \frac{\big(r^2+1-2r\cos\theta\big)^5}{(1-r)^{25}(1+r)^{25}}.$$

Then,

$$\frac{1}{2\pi}\int_0^{2\pi} G(re^{i\theta}, re^{-i\theta})d\theta = \frac{(r^2+1)\,(r^8+24\,r^6+76\,r^4+24\,r^2+1)}{(1-r)^{25}(1+r)^{25}} \overset{(\mathrm{r}\to 1^-)}{\asymp} \frac{1}{(1-r)^{25}},$$

which implies $\beta(-10,-30;2) = 25 = \beta_1(-10,-30;2)$.

- For $\kappa = 3$ and $\gamma = 2$, we obtain $(p,q) = (p_{\mathcal{R}}(\gamma), q_{\mathcal{R}}(\gamma)) = (1, -3)$ and

$$G(re^{i\theta}, re^{-i\theta}) = \frac{(r^2+1-2r\cos\theta)^2}{(1-r)^6(1+r)^6}.$$

  Then,

$$\frac{1}{2\pi}\int_0^{2\pi} G(re^{i\theta}, re^{-i\theta})d\theta = \frac{r^4+4\,r^2+1}{(1-r)^6(1+r)^6} \overset{(r\to 1^-)}{\asymp} \frac{1}{(1-r)^6},$$

  which implies $\beta(1,-3;3) = 6 = \beta_1(1,-3;3)$.
- For $\kappa = 3$ and $\gamma = 5$, we obtain $(p,q) = (p_{\mathcal{R}}(\gamma), q_{\mathcal{R}}(\gamma)) = (-20, -105/2)$ and

$$G(re^{i\theta}, re^{-i\theta}) = \frac{(r^2+1-2r\cos\theta)^5}{(1-r)^{75/2}(1+r)^{75/2}}.$$

  Then,

$$\frac{1}{2\pi}\int_0^{2\pi} G(re^{i\theta}, re^{-i\theta})d\theta = \frac{(r^2+1)\,(r^8+24\,r^6+76\,r^4+24\,r^2+1)}{(1-r)^{75/2}(1+r)^{75/2}} \overset{(r\to 1^-)}{\asymp} \frac{1}{(1-r)^{75/2}},$$

  which implies $\beta(-20,-105/2;3) = 75/2 = \beta_1(-20,-105/2;3)$.

As above, one always gets the spectrum $\beta(p,q) = \beta_1(p,q)$ for the given points (p,q) below the frontier line $\mathcal{J}$. This is consistent with the proposed partition of the (p,q) plane (Fig. 3.5).

Chapter 4

McMullen's asymptotic variance and SLE_2

4.1 Starting motivation

Let us consider a general analytic one-parameter family (ϕ_t), $t \in U$, a neighborhood of $t = 0$, of conformal maps with $\phi_0 = \text{id}$ and $\phi_t(0) = 0, \forall t \in U$. Then

$$\phi_t(z) = \int_0^z e^{\log \phi_t'(u)} du, \quad z \in \mathbb{D},$$

and

$$\frac{\partial}{\partial t}\phi_t(z) = \int_0^z \frac{\partial}{\partial t}(\log \phi_t'(u)) e^{\log \phi_t'(u)} du.$$

From which follows that

$$V(z) = \frac{\partial}{\partial t}\phi_t(z)\Big|_{t=0} = \int_0^z \frac{\partial}{\partial t}\big(\log \phi_t'(u)\big)\Big|_{t=0} du,$$

and $b(z) = V'(z) = \frac{\partial}{\partial t}\big(\log \phi_t'(z)\big)\big|_{t=0}$ belongs to the Bloch space $\mathcal{B}$, which is defined as follows:

$$\mathcal{B} = \Big\{ b \text{ holomorphic in } \mathbb{D};\ \sup_{\mathbb{D}} (1 - |z|^2)|b'(z)| < \infty \Big\}.$$

We recall that the *McMullen's asymptotic variance* of a Bloch function $b \in \mathcal{B}$ is defined by

$$\sigma^2(b) := \limsup_{r \longrightarrow 1^-} \frac{1}{2\pi |\log(1-r)|} \int_0^{2\pi} |b(re^{i\theta})|^2 d\theta.$$

In [15], McMullen asked under which condition on the family of (ϕ_t) it is true that

$$\frac{d^2}{dt^2} \text{H.}\dim(\phi_t(\partial\mathbb{D}))\Big|_{t=0} = \lim_{r \to 1^-} \frac{1}{4\pi |\log(1-r)|} \int_{|z|=r} |b(z)|^2 |dz| = \frac{\sigma^2(b)}{2}, \qquad (4.1.2)$$

where $\text{H.}\dim(\phi_t(\partial\mathbb{D}))$ denotes Hausdorff dimension of $\phi_t(\partial\mathbb{D})$ (see Appendix A).

Conversely, starting from a function $b \in \mathcal{B}$, it is known that if we put

$$\phi_t(z) = \int_0^z e^{tb(u)} du, \quad b \in \mathcal{B}, \tag{4.1.3}$$

then (ϕ_t) is an analytic family. There exists a neighborhood U of 0 such that if $t \in U$ then ϕ_t is a conformal map with quasiconformal extension.

In Ref. [8], using a probability argument, N. Le and M. Zinsmeister described a relatively large family of function $b \in \mathcal{B}$ for which if ϕ_t is defined by (4.1.3) (t being real), then (4.1.2) is true with Hausdorff dimension replaced by Minkowski dimension (see also Appendix A),

$$\frac{d^2}{dt^2}\mathrm{M.dim}(\phi_t(\partial\mathbb{D}))\Big|_{t=0} = \lim_{r\to 1^-} \frac{1}{4\pi|\log(1-r)|} \int_{|z|=r} |b(z)|^2|dz|. \tag{4.1.4}$$

In particular, they proved that

$$\lim_{p\to 0} \frac{2\beta(p,\phi)}{p^2} = \lim_{r\to 1^-} \frac{1}{4\pi|\log(1-r)|} \int_0^{2\pi} |b(re^{i\theta})|^2 d\theta, \tag{4.1.5}$$

which, together with Proposition 3 in Ref. [8], implies (4.1.4), where $\beta(p,\phi)$ is the *integral means spectrum* of ϕ ($\phi'(z) = \exp b(z)$) defined as

$$\beta(p,\phi) = \limsup_{r\longrightarrow 1^-} \frac{\log\left(\int_0^{2\pi} |\phi'(re^{i\theta})|^p d\theta\right)}{|\log(1-r)|}, \ p \in \mathbb{R}.$$

The starting motivation of this chapter was to prove an analog of (4.1.5), in expectation, for the interior whole-plane SLE_2 map at time 0. In particular, we will prove the following theorem.

Theorem 4.1.1. *Let $f := f_0$ be the interior whole-plane SLE_κ map at time zero and $\bar\beta(p,f)$ be the average integral means spectrum of f defined as*

$$\bar\beta(p,f) = \limsup_{r\longrightarrow 1^-} \frac{\log\left(\int_0^{2\pi} \mathbb{E}[|f'(re^{i\theta})|^p] d\theta\right)}{|\log(1-r)|}, \ p \in \mathbb{R};$$

then, for $\kappa = 2$,

$$\lim_{p\to 0} \frac{2\bar\beta(p,f)}{p^2} = \lim_{r\to 1^-} \frac{1}{4\pi|\log(1-r)|} \int_0^{2\pi} \mathbb{E}(|\log f'(re^{i\theta})|^2) d\theta. \tag{4.1.6}$$

On the other hand, as shown in Ref. [6] and mentioned in Section 3.4 of Chapter 3, the average integral means spectrum of the interior whole-plane SLE_κ map f is given by

$$\begin{aligned}
\beta_{\text{tip}}(p,\kappa) &:= -p-1+\frac{1}{4}\left(4+\kappa-\sqrt{(4+\kappa)-8\kappa p}\right), && p\le p_0'(\kappa),\\
\beta_0(p,\kappa) &:= -p+\frac{4+\kappa}{4\kappa}\left(4+\kappa-\sqrt{(4+\kappa)^2-8\kappa p}\right), && p_0'(\kappa)\le p\le p^*(\kappa),\\
\beta(p,\kappa) &:= 3p-\frac{1}{2}-\frac{1}{2}\sqrt{1+2\kappa p}, && p^*(\kappa)\le p,
\end{aligned}$$

where the transition values $p_0'(\kappa)$ and $p^*(\kappa)$ are as in (3.4.6) and (3.4.10),

$$\begin{aligned}
p_0'(\kappa) &:= -1-\frac{3\kappa}{8},\\
p^*(\kappa) &:= \frac{1}{32\kappa}\left(\sqrt{2(4+\kappa)^2+4}-6\right)\left(\sqrt{2(4+\kappa)^2+4}+2\right).
\end{aligned}$$

Therefore, $\bar{\beta}(p,f)$ has the following development at $p=0$:

$$\bar{\beta}(p,f)=\frac{2\kappa}{(4+\kappa)^2}p^2+\circ(p^2).$$

Eq. (4.1.6) is equivalent to

$$\lim_{r\to 1^-}\frac{1}{4\pi|\log(1-r)|}\int_0^{2\pi}\mathbb{E}(|\log f'(re^{i\theta})|^2)d\theta=\frac{4\kappa}{(4+\kappa)^2}, \tag{4.1.7}$$

and equals 2/9 for $\kappa=2$.

From this, the proof of Theorem 4.1.1 is twofold:

- Using a martingale technique, we derive a PDE satisfied by $\mathbb{E}(|\log f'(z)|^2)$ for $\kappa=2$.
- From that PDE together with the support of MAPLE, we determine the coefficient matrix of $\mathbb{E}(|\log f'(z)|^2)$ whereby we can find the explicit expression for $\mathbb{E}(|\log f'(z)|^2)$ and show that Eq. (4.1.7) is true for $\kappa=2$.

4.2 SLE logarithm one-point function

Let us now consider the whole-plane SLE logarithm one-point function,

$$F(z)=\mathbb{E}(\log f'(z)), \tag{4.2.1}$$

which, by construction, stay finite at the origin and such that $F(0)=0$. Our aim is to derive an ODE satisfied by F (4.2.1).

We consider the auxiliary, time-dependent, radial variant of the function $F(z)$ above,

$$\widetilde{F}(z,t):=\mathbb{E}(\log[e^t\tilde{f}_t'(z)]), \tag{4.2.2}$$

where $\tilde{f}_t$ is a modified radial SLE map at time t as in Definition 2.1.3.

Owing to Lemma (2.1.4), we have

$$\lim_{t\to+\infty} \widetilde{F}(z,t) = F(z). \tag{4.2.3}$$

Applying the same method as in chapter 2, we use a martingale technique to obtain an equation satisfied by $\widetilde{F}(z,t)$ (4.2.2). For $s \le t$, define $\mathcal{M}_s := \mathbb{E}(\log \tilde{f}_t'(z)|\mathcal{F}_s)$, where $\mathcal{F}_s$ is the σ-algebra generated by $\{B_u, u \le s\}$. $(\mathcal{M}_s)_{s\ge 0}$ is by construction a martingale. By the SLE Markov property we have

$$\begin{aligned}\mathcal{M}_s := \mathbb{E}\big(\log \tilde{f}_t'(z)|\mathcal{F}_s\big) &= \mathbb{E}\big(\log \tilde{f}_s'(z) + \log \tilde{f}_\tau'(z_s)|\mathcal{F}_s\big)\\ &= \log \tilde{f}_s'(z) + \mathbb{E}\big(\log[e^\tau \tilde{f}_\tau'(z_s)]|\mathcal{F}_s\big) - \tau\\ &= \log \tilde{f}_s'(z) + \widetilde{F}(z_s,\tau) - \tau,\end{aligned}$$

where $z_s := \tilde{f}_s(z)/\lambda(s)$, and $\tau := t - s$.

From Eqs. (2.1.9) and (2.1.11) in section 2.1, we directly arrive at the vanishing condition of the coefficient of the ds-drift term of the Itô derivative of $\mathcal{M}_s$,

$$2\left(1 - \frac{1}{(1-z_s)^2}\right) + \left[z_s\left(\frac{z_s+1}{z_s-1} - \frac{\kappa}{2}\right)\partial_z - \partial_\tau - \frac{\kappa}{2}z_s^2\partial_z^2\right]\widetilde{F}(z_s,\tau) = 0, \tag{4.2.4}$$

by the (local) martingale property.

The existence of the limit (4.2.3) further implies that of

$$\lim_{\tau\to+\infty} \partial_\tau \widetilde{F}(z,\tau) = 0.$$

Letting $\tau \to +\infty$, we obtain the expected ODE for $F(z)$ (4.2.1),

$$\mathcal{P}(\partial)[F(z)] := 2\left(1 - \frac{1}{(1-z)^2}\right) + z\left(\frac{z+1}{z-1} - \frac{\kappa}{2}\right)\partial_z F(z) - \frac{\kappa}{2}z^2\partial_z^2 F(z) = 0. \tag{4.2.5}$$

Solving the ODE (4.2.5) by MAPLE, together with the condition $F(0) = 0$, we get explicit expressions for $F(z)$ for some values of κ. In particular,

- In the $\kappa = 1$ case,

$$F(z) = \frac{7}{30}z^2 - \frac{28}{15}z + \frac{4}{5}\log(1-z). \tag{4.2.6}$$

- In the $\kappa = 2$ case,

$$F(z) = -\frac{4}{3}z + \frac{2}{3}\log(1-z). \tag{4.2.7}$$

Remark 4.2.1. *Obviously, the ODE* (4.2.5) *yields a first-order differential equation satisfied by* $F'(z) = \mathbb{E}\big(\frac{f''(z)}{f'(z)}\big)$. *Solving this ODE, we obtain*

$$F'(z) = \frac{4}{\kappa}\frac{(z-1)^{\frac{4}{\kappa}}}{z^{\frac{2}{\kappa}+1}}\int \frac{z^{\frac{2}{\kappa}}(z-2)}{(z-1)^{\frac{4}{\kappa}+2}}dz,$$

with the condition $F'(0) = 2\mathbb{E}(a_2) = -\frac{8}{\kappa+2}$.

4.3 SLE logarithm two-point function

4.3.1 Nonhomogeneous Beliaev-Smirnov type equations

In this section, we study the whole-plane SLE logarithm two-point function for $z_1, z_2 \in \mathbb{D}$,

$$G(z_1, \bar{z}_2) := \mathbb{E}\big(\log f'(z_1)\overline{\log f'(z_2)}\big). \tag{4.3.1}$$

As before, we define a time-dependent, auxiliary two-point function,

$$\widetilde{G}(z_1, \bar{z}_2, t) := \mathbb{E}\Big(\log[e^t \tilde{f}_t'(z_1)]\overline{\log[e^t \tilde{f}_t'(z_2)]}\Big), \tag{4.3.2}$$

where $\tilde{f}_t$ is the conjugate, reversed radial SLE process in $\mathbb{D}$, as introduced in Definition 2.1.3.

Because of Lemma (2.1.4), the two-point function (4.3.1) is the limit

$$\lim_{t\to+\infty} \widetilde{G}(z_1, \bar{z}_2, t) = G(z_1, \bar{z}_2). \tag{4.3.3}$$

Let us now consider the two-point martingale $(\mathcal{M}_s)_{t\geq s\geq 0}$, defined by

$$\mathcal{M}_s := \mathbb{E}\Big(\log \tilde{f}_t'(z_1)\overline{\log \tilde{f}_t'(z_2)}|\mathcal{F}_s\Big),$$

and introduce the shorthand notation,

$$X_t(z) := \log \tilde{f}_t'(z). \tag{4.3.4}$$

By the Markov property of SLE we get

$$\begin{aligned}\mathcal{M}_s = {} & X_s(z_1)\overline{X_s(z_2)} + \tau^2 - \tau\big(X_s(z_1) + \overline{X_s(z_2)}\big) \\ & + X_s(z_1)\overline{\widetilde{F}}(\bar{z}_{2s}, \tau) + \overline{X_s(z_2)}\widetilde{F}(z_{1s}, \tau) \\ & - \tau\big(\widetilde{F}(z_{1s}, \tau) + \overline{\widetilde{F}}(\bar{z}_{2s}, \tau)\big) + \widetilde{G}(z_{1s}, \bar{z}_{2s}, \tau),\ \tau := t - s,\end{aligned} \tag{4.3.5}$$

where, as defined in (2.3.5),

$$z_{1s} := \tilde{f}_s(z_1)/\lambda(s); \quad \bar{z}_{2s} := \overline{\tilde{f}_s(z_2)/\lambda(s)} = \overline{\tilde{f}_s(z_2)}\lambda(s),$$

and $\overline{\widetilde{F}}(\bar{z}, t) := \overline{\widetilde{F}(z, t)}$ with $\widetilde{F}(z, t)$ defined in (4.2.2).

As before, the partial differential equation satisfied by $\widetilde{G}(z_{1s}, \bar{z}_{2s}, \tau)$ is obtained by expressing the fact that the ds-drift term of the Itô differential of Eq. (4.3.5),

$$\begin{aligned}d\mathcal{M}_s = {} & (X_s(z_1) - \tau)\big[d\overline{X_s(z_2)} + ds + d\overline{\widetilde{F}}\,\big] + (\overline{X_s(z_2)} - \tau)\big[dX_s(z_1) + ds + d\widetilde{F}\,\big] \\ & + (dX_s(z_1) + ds)\overline{\widetilde{F}} + (d\overline{X_s(z_2)} + ds)\widetilde{F} + d\widetilde{G},\end{aligned} \tag{4.3.6}$$

vanishes.

The differentials of $X_s, \overline{X_s}$ are as in Eq. (2.1.9),

$$dX_s(z_1) = \Big(1 - \frac{2}{(1-z_{1s})^2}\Big)ds,\ d\overline{X_s(z_2)} = \Big(1 - \frac{2}{(1-\bar z_{2s})^2}\Big)ds.$$

Note that the coefficient of the ds-drift term in $\big[dX_s(z_1) + ds + d\widetilde{F}\,\big]$ is the left hand side of the ODE (4.2.4), hence vanishes. The coefficient of the ds-drift term in $\big[d\overline{X_s(z_2)} + ds + d\overline{\widetilde{F}}\,\big]$ aslo vanishes, since it is the left hand side of the complex conjugate equation of the ODE (4.2.4).

In addition, the Itô differential of $\widetilde{G}(z_{1s}, \bar z_{2s}, \tau)$ is given as in (2.3.11),

$$\begin{aligned} d\widetilde{G} = {} & -i\sqrt{\kappa}\,\big(z_{1s}\partial_1 - \bar z_{2s}\bar\partial_2\big)\,\widetilde{G}\,dB_s \\ & + \frac{z_{1s}+1}{z_{1s}-1} z_{1s}\partial_1\widetilde{G}\,ds + \frac{\bar z_{2s}+1}{\bar z_{2s}-1}\bar z_{2s}\bar\partial_2\widetilde{G}\,ds - \partial_\tau\widetilde{G}\,ds \\ & - \frac{\kappa}{2}(z_{1s}\partial_1 - \bar z_{2s}\,\bar\partial_2)^2\widetilde{G}\,ds, \end{aligned}$$

where we used the shorthand notations $\partial_1 := \partial_{z_1}$, $\bar\partial_2 := \partial_{\bar z_2}$ and the obvious formal identity

$$(z_1\partial_1)^2 + (\bar z_2\,\bar\partial_2)^2 - 2z_1\partial_1\bar z_2\,\bar\partial_2 = (z_1\partial_1 - \bar z_2\,\bar\partial_2)^2.$$

We therefore directly arrive at the vanishing condition of the overall drift term coefficient in $d\mathcal{M}_s$,

$$\begin{aligned} & \left[z_{1s}\frac{z_{1s}+1}{z_{1s}-1}\partial_1 + \bar z_{2s}\frac{\bar z_{2s}+1}{\bar z_{2s}-1}\bar\partial_2 - \partial_\tau - \frac{\kappa}{2}(z_{1s}\partial_1 - \bar z_{2s}\bar\partial_2)^2\right]\widetilde{G} \\ & + 2\left(1 - \frac{1}{(1-z_{1s})^2}\right)\overline{\widetilde{F}}(\bar z_{2s},\tau) + 2\left(1 - \frac{2}{(1-\bar z_{2s})^2}\right)\widetilde{F}(z_{1s},\tau) = 0. \end{aligned} \tag{4.3.7}$$

The existence of the limit (4.3.3) further implies that of

$$\lim_{\tau\to+\infty}\partial_\tau\widetilde{G}(z_1,\bar z_2,\tau) = 0.$$

Letting $\tau \to +\infty$, we get the expected PDE for $G(z_1, \bar z_2)$ (4.3.1),

$$\begin{aligned} \mathcal{P}(D)[G(z_1,\bar z_2)] = {} & -\frac{\kappa}{2}(z_1\partial_1 - \bar z_2\bar\partial_2)^2 G - \frac{1+z_1}{1-z_1}z_1\partial_1 G - \frac{1+\bar z_2}{1-\bar z_2}\bar z_2\bar\partial_2 G \\ & + 2\left(1 - \frac{1}{(1-z_1)^2}\right)\overline{F}(\bar z_2) + 2\left(1 - \frac{2}{(1-\bar z_2)^2}\right)F(z_1) = 0. \end{aligned} \tag{4.3.8}$$

Recall that

$$F(z_1) = \mathbb{E}(\log f'(z_1)),\ \ \overline{F}(\bar z_2) = \overline{\mathbb{E}(\log f'(z_2))} = \mathbb{E}(\log\overline{f'(z_2)}\,).$$

4.3.2 Moduli logarithm one-poin function

Note that one can take the $z_1 = z_2 = z$ case in Definition (4.3.1) above, thereby obtaining the moduli logarithm one-point function,

$$G(z,\bar z) = \mathbb{E}\big(|\log f'(z)|^2\big). \tag{4.3.9}$$

Because of Eq. (4.3.8), it obeys the corresponding PDE,

$$\begin{aligned}\mathcal{P}(D)[G(z,\bar{z})] = &-\frac{\kappa}{2}(z\partial - \bar{z}\bar{\partial})^2 G - \frac{1+z}{1-z}z\partial G - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}G \\ &+ 2\left(1 - \frac{1}{(1-z)^2}\right)\overline{F}(\bar{z}) + 2\left(1 - \frac{2}{(1-\bar{z})^2}\right)F(z) = 0.\end{aligned} \tag{4.3.10}$$

Let us now look for the expression for the moduli logarithm one-point function $G(z,\bar{z})$ (4.3.9) in the $\kappa = 2$ case.

4.3.3 The $\kappa = 2$ case

For $\kappa = 2$, owing to (4.2.7), the PDE (4.3.10) becomes

$$\begin{aligned}&-(z\partial - \bar{z}\bar{\partial})^2 G - \frac{1+z}{1-z}z\partial G - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}G + 2\left(1 - \frac{1}{(1-z)^2}\right)\left(-\frac{4}{3}\bar{z} + \frac{2}{3}\log(1-\bar{z})\right) \\ &+ 2\left(1 - \frac{2}{(1-\bar{z})^2}\right)\left(-\frac{4}{3}z + \frac{2}{3}\log(1-z)\right) = 0.\end{aligned} \tag{4.3.11}$$

One can write $G(z,\bar{z})$ in the following form,

$$G(z,\bar{z}) = F(z)\overline{F(z)} + R(z,\bar{z}), \tag{4.3.12}$$

where

$$F(z)\overline{F(z)} = \frac{4}{9}\big(2z - \log(1-z)\big)\big(2\bar{z} - \log(1-\bar{z})\big). \tag{4.3.13}$$

As introduced in Section 4.1, the second step of the proof is to accurately compute the coefficients of $G(z,\bar{z})$ defined by its corresponding power series. From the PDE (4.3.11), we construct a system of recursive equations for these coefficients and then get the coefficient matrix of $G(z,\bar{z})$ by using MAPLE. Owing to (4.3.12) and (4.3.13), we also find the coefficient matrix of $R(z,\bar{z})$. In particular, we introduce here a part of the coefficient matrix of $G(z,\bar{z})$,

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 8 & -\frac{2}{3} & \frac{4}{9} & \frac{1}{3} & \frac{4}{15} & \frac{2}{9} & \frac{4}{21} & \frac{1}{6} & \frac{4}{27} & \frac{2}{15} & \frac{4}{33} & \cdots \\
0 & -\frac{2}{3} & \frac{5}{3} & -\frac{10}{27} & \frac{1}{18} & \frac{2}{45} & \frac{1}{27} & \frac{2}{63} & \frac{1}{36} & \frac{2}{81} & \frac{1}{45} & \frac{2}{99} & \cdots \\
0 & \frac{4}{9} & -\frac{10}{27} & \frac{64}{81} & -\frac{5}{27} & \frac{4}{135} & \frac{2}{81} & \frac{4}{189} & \frac{1}{54} & \frac{4}{243} & \frac{2}{135} & \frac{4}{297} & \cdots \\
0 & \frac{1}{3} & \frac{1}{18} & -\frac{5}{27} & \frac{17}{36} & -\frac{1}{9} & \frac{1}{54} & \frac{1}{63} & \frac{1}{72} & \frac{1}{81} & \frac{1}{90} & \frac{1}{99} & \cdots \\
0 & \frac{4}{15} & \frac{2}{45} & \frac{4}{135} & -\frac{1}{9} & \frac{8}{25} & -\frac{2}{27} & \frac{4}{315} & \frac{1}{90} & \frac{4}{405} & \frac{2}{225} & \frac{4}{495} & \cdots \\
0 & \frac{2}{9} & \frac{1}{27} & \frac{2}{81} & \frac{1}{54} & -\frac{2}{27} & \frac{19}{81} & -\frac{10}{189} & \frac{1}{108} & \frac{2}{243} & \frac{1}{135} & \frac{2}{297} & \cdots \\
0 & \frac{4}{21} & \frac{2}{63} & \frac{4}{189} & \frac{1}{63} & \frac{4}{315} & -\frac{10}{189} & \frac{80}{441} & -\frac{5}{126} & \frac{4}{567} & \frac{2}{315} & \frac{4}{693} & \cdots \\
0 & \frac{1}{6} & \frac{1}{36} & \frac{1}{54} & \frac{1}{72} & \frac{1}{90} & \frac{1}{108} & -\frac{5}{126} & \frac{7}{48} & -\frac{5}{162} & \frac{1}{180} & \frac{1}{198} & \cdots \\
0 & \frac{4}{27} & \frac{2}{81} & \frac{4}{243} & \frac{1}{81} & \frac{4}{405} & \frac{2}{243} & \frac{4}{567} & -\frac{5}{162} & \frac{88}{729} & -\frac{2}{81} & \frac{4}{891} & \cdots \\
0 & \frac{2}{15} & \frac{1}{45} & \frac{2}{135} & \frac{1}{90} & \frac{2}{225} & \frac{1}{135} & \frac{2}{315} & \frac{1}{180} & -\frac{2}{81} & \frac{23}{225} & -\frac{2}{99} & \cdots \\
0 & \frac{4}{33} & \frac{2}{99} & \frac{4}{297} & \frac{1}{99} & \frac{4}{495} & \frac{2}{297} & \frac{4}{693} & \frac{1}{198} & \frac{4}{891} & -\frac{2}{99} & \frac{32}{363} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix};$$

a part of the coefficient matrix of $R(z, \bar{z})$,

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 4 & -\frac{4}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & -\frac{4}{3} & \frac{14}{9} & -\frac{4}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -\frac{4}{9} & \frac{20}{27} & -\frac{2}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -\frac{2}{9} & \frac{4}{9} & -\frac{2}{15} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -\frac{2}{15} & \frac{68}{225} & -\frac{4}{45} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & -\frac{4}{45} & \frac{2}{9} & -\frac{4}{63} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{4}{63} & \frac{76}{441} & -\frac{1}{21} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{21} & \frac{5}{36} & -\frac{1}{27} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{27} & \frac{28}{243} & -\frac{4}{135} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{4}{135} & \frac{22}{225} & -\frac{4}{165} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{4}{165} & \frac{92}{1089} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix};$$

together with a part of the coefficient matrix of $(9/4) \cdot R(z, \bar{z})$,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 9 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -3 & \frac{7}{2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -1 & \frac{5}{3} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{3}{10} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -\frac{3}{10} & \frac{17}{25} & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{5} & \frac{1}{2} & -\frac{1}{7} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & \frac{19}{49} & -\frac{3}{28} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{28} & \frac{5}{16} & -\frac{1}{12} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{12} & \frac{7}{27} & -\frac{1}{15} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{15} & \frac{11}{50} & -\frac{3}{55} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{55} & \frac{23}{121} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is remarkable to note that the coefficient matrix of $(9/4) \cdot R(z, \bar{z})$ is a symmetric *tridiagonal* matrix. From the above coefficients, we determine the general rules for its three diagonals,

$$\begin{gathered} a_{1,1} = 9, \ a_{n,n} = \frac{12}{n^2} + \frac{1}{n} \quad \forall n \geq 2, \\ a_{n,n+1} = a_{n+1,n} = -\frac{6}{n(n+1)} \quad \forall n \geq 1, \\ a_{n,m} = 0 \quad \text{otherwise}, \end{gathered}$$

whereby we get the explicit expression for $(9/4) \cdot R(z, \bar{z})$ as follows:

$$\frac{9}{4} \cdot R(z, \bar{z}) = -4z\bar{z} + \frac{6(z+\bar{z})}{z\bar{z}} \int_0^{z\bar{z}} \log(1-x)dx - 12 \int_0^{z\bar{z}} \frac{\log(1-x)}{x} dx - \log(1-z\bar{z}). \tag{4.3.14}$$

From Eqs. (4.3.12), (4.3.13) and (4.3.14), we obtain for $\kappa = 2$,

$$\begin{aligned} G(z, \bar{z}) = \frac{4}{9}(2z - \log(1-z))(2\bar{z} - \log(1-\bar{z})) - \frac{16}{9} z\bar{z} + \frac{8(z+\bar{z})}{3z\bar{z}} \int_0^{z\bar{z}} \log(1-x)dx \\ - \frac{16}{3} \int_0^{z\bar{z}} \frac{\log(1-x)}{x} dx - \frac{4}{9} \log(1-z\bar{z}). \end{aligned} \tag{4.3.15}$$

Using the two identities

$$\int_0^{z\bar{z}} \ln(1-x)dx = (z\bar{z}-1)\ln(1-z\bar{z}) - z\bar{z}$$

and

$$\int_0^{z\bar{z}} \frac{\ln(1-x)}{x}dx = -\operatorname{dilog}(1-z\bar{z}) = -\sum_{n=1}^{\infty} \frac{(z\bar{z})^n}{n^2},$$

where the *dilog* function is defined as

$$\operatorname{dilog}(x) = \int_1^x \frac{\ln(t)}{1-t}dt = \sum_{n=1}^{\infty} \frac{(1-x)^n}{n^2},$$

we can rewrite (4.3.15) as

$$\begin{aligned} G(z,\bar{z}) = &-\frac{8}{9}z\log(1-\bar{z}) - \frac{8}{9}\bar{z}\log(1-z) + \frac{4}{9}\log(1-z)\log(1-\bar{z}) - \frac{8}{3}(z+\bar{z}) \\ &+ \frac{24(z+\bar{z})(z\bar{z}-1)-4z\bar{z}}{9z\bar{z}}\log(1-z\bar{z}) + \frac{16}{3}\sum_{n=1}^{\infty}\frac{(z\bar{z})^n}{n^2}. \end{aligned} \tag{4.3.17}$$

4.4 The proof of Theorem 4.1.1

Because of (4.3.17), we have

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \mathbb{E}(|\log f'(re^{i\theta})|^2)d\theta &= \frac{16}{9}r^2 + \frac{4}{9}\sum_{n=1}^{\infty}\frac{r^{2n}}{n^2} - \frac{4}{9}\log(1-r^2) + \frac{16}{3}\sum_{n=1}^{\infty}\frac{r^{2n}}{n^2} \\ &= \frac{16}{9}r^2 - \frac{4}{9}\big(\log(1-r)+\log(1+r)\big) + \frac{52}{9}\sum_{n=1}^{\infty}\frac{r^{2n}}{n^2}. \end{aligned}$$

It follows that

$$\frac{\int_0^{2\pi}\mathbb{E}(|\log f'(re^{i\theta})|^2)d\theta}{-4\pi\log(1-r)} = \frac{2}{9} + \frac{\frac{16}{9}r^2 - \frac{4}{9}\log(1+r) + \frac{52}{9}\sum_{n=1}^{\infty}\frac{r^{2n}}{n^2}}{-2\log(1-r)} \longrightarrow \frac{2}{9}$$

as $r \to 1^-$, which encompasses Theorem 4.1.1.

Chapter 5

GRUNSKY MATRICES AND SLE$_\kappa$

5.1 Grunsky matrices

Let f be a holomorphic and injective (univalent) in the unit disk $\mathbb{D}$, normalized so that $f(0) = 0$ and $f'(0) = 1$ (i.e. $f \in \mathcal{S}$). The corresponding Grunsky coefficients $b_{n,m} = b_{m,n}$ of f are defined by the power series

$$\log \frac{f(z) - f(\xi)}{z - \xi} = \sum_{n,m=0}^{\infty} b_{n,m} z^n \xi^m, \tag{5.1.1}$$

which is convergent for $|z| < 1$, $|\xi| < 1$. Note that $b_{0,n} = b_{n,0}$ is given by $\log(f(z)/z) = \sum_{n=1}^{\infty} b_{0,n}\, z^n$, $b_{0,0} = 0$.

We recall here an important inequality related to the Grunsky coefficients $(b_{n,m})$, which is called *Grunsky inequality* [18].

Theorem 5.1.1. *(**Grunsky Inequality**) Let f be a holomorphic and injective function in the unit disk and $b_{n,m}$ be the Grunsky coefficients of f. Then for any complex vector $(\lambda_1, \lambda_2, ..., \lambda_k, ...)$, the following inequality holds:*

$$\left| \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{n,m} \lambda_n \lambda_m \right| \leq \sum_{n=1}^{\infty} \frac{|\lambda_n|^2}{n}. \tag{5.1.2}$$

Grunsky [7] showed that a holomorphic function f is injective in the unit disk if and only if the Grunsky coefficients of f satisfy the Grunsky inequality (5.1.2). In other words, the Grunsky inequality give necessary and sufficient conditions that a holomorphic function be injective in the unit disk.

From (5.1.2), it will be convenient to work with the *normalized Grunsky coefficients* [18],

$$c_{n,m} = c_{m,n} = \sqrt{nm}\, b_{n,m}, \text{ for } n, m = 1, 2, ...$$

Then, the *normalized Grunsky matrix* $(c_{n,m})$ satisfies

$$\left| \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{n,m} \lambda_n \lambda_m \right| \leq \sum_{n=1}^{\infty} |\lambda_n|^2, \tag{5.1.3}$$

for any complex vector $(\lambda_1, \lambda_2, ..., \lambda_k, ...)$.

It is remarkable to note that the Grunsky matrix $(c_{n,m})$ of f is *unitary* if and only if the complement of the image of f has Lebesgue measure zero, i.e., $|\mathbb{C} \setminus f(\mathbb{D})| = 0$. So, roughly speaking, the image $f(\mathbb{D})$ is a slit region in the complex plane. For instance, it is unitary for SLE$_{\kappa \leq 4}$.

Besides, the important special case that $\lambda_k = 0$ for $k > m$ can be expressed in terms of linear algebra. In particular, if $f \in \mathcal{S}$ then the Grunsky matrix $\mathcal{C} = (c_{k,l})_{k,l=1,...,m}$ of order m $(m = 1, 2, ...)$ can be written in the form [18]

$$\mathcal{C} = \mathcal{U}' \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{pmatrix} \mathcal{U}, \ \mathcal{U} \text{ unitary} \tag{5.1.4}$$

and $\mathcal{U}'$ its transpose with $|d_k| \leq 1$ $(k = 1, ..., m)$.

Recall that a complex square matrix $\mathcal{U} = (u_{i,j})$ is unitary if its conjugate transpose $\mathcal{U}^* = (u^*_{i,j})$, where $u^*_{i,j} = \overline{u_{j,i}}$, is also its inverse (i.e. $\mathcal{U}\mathcal{U}^* = \mathcal{U}^*\mathcal{U} = I$).

5.2 Grunsky matrices for SLE$_\kappa$ processes

Let us now return to the whole-plane SLE$_\kappa$. We consider (5.1.1) in expectation,

$$\mathbb{E}\left(\log \frac{f(z) - f(\xi)}{z - \xi}\right) = \sum_{n,m=0}^{\infty} b_{n,m}\, z^n \xi^m, \tag{5.2.1}$$

where f is the interior whole-plane SLE$_\kappa$ map at time zero.

By differentiating two sides of (5.2.1) with respect to z and ξ successively, we get

$$\mathbb{E}\left(\frac{f'(z)f'(\xi)}{(f(z) - f(\xi))^2}\right) - \frac{1}{(z - \xi)^2} = \sum_{n,m \geq 1} nm\, b_{n,m}\, z^{n-1}\xi^{m-1} \ (z \neq \xi), \tag{5.2.2}$$

or, equivalently,

$$\mathbb{E}\left((z - \xi)^2 \frac{f'(z)f'(\xi)}{(f(z) - f(\xi))^2}\right) = 1 + (z - \xi)^2 \sum_{n,m \geq 1} nm\, b_{n,m}\, z^{n-1}\xi^{m-1}. \tag{5.2.3}$$

Owing to (5.2.3), in order to study the Grunsky coefficient matrix $(\sqrt{nm}\, b_{n,m})$ of f $(n, m = 1, 2, ...)$, we aim at determining the two-point mixed moment below,

$$\mathbb{E}\left((z - \xi)^2 \frac{f'(z)f'(\xi)}{(f(z) - f(\xi))^2}\right), \tag{5.2.4}$$

More generally, we will study the natural generalization of (5.2.4) defined as

$$G(z, \xi) := \mathbb{E}\left((z - \xi)^q \frac{f'^p(z)f'^p(\xi)}{(f(z) - f(\xi))^q}\right), \ \ (p, q) \in \mathbb{R}^2, \tag{5.2.5}$$

where the $(p = 1, q = 2)$ case is the most important one. Our method is to use a martingale technique to obtain a PDE obeyed by $G(z, \xi)$, and then compute coefficients of $G(z, \xi)$, with the support of MAPLE, for any $p, q \in \mathbb{R}$ and $\kappa > 0$.

5.2.1 Beliaev-Smirnov type PDE for $G(z, \xi)$

As introduced above, our aim is to derive a PDE satisfied by the SLE two-point function $G(z, \xi)$ (5.2.5), which, by construction, stays finite at the origin and such that $G(0, 0) = 1$.

Let us introduce the shorthand notation,

$$X_t(z, \xi) := \frac{\tilde{f}_t'^p(z)\tilde{f}_t'^p(\xi)}{\left(\tilde{f}_t(z) - \tilde{f}_t(\xi)\right)^q}, \tag{5.2.6}$$

where, as before, $\tilde{f}_t$ is the conjugate, reversed radial SLE process in $\mathbb{D}$, as introduced in Definition 2.1.3. Applying the same method as in chapter 2, we consider the auxiliary, time-dependent two-point function

$$\widetilde{G}(z, \xi, t) := \mathbb{E}((z - \xi)^q X_t(z, \xi)), \tag{5.2.7}$$

such that

$$\lim_{t \to +\infty} e^{(2p-q)t}\widetilde{G}(z, \xi, t) = G(z, \xi). \tag{5.2.8}$$

Let us consider the two-point martingale $(\mathcal{M}_s)_{t \geq s \geq 0}$, defined by

$$\mathcal{M}_s = \mathbb{E}(X_t(z, \xi)|\mathcal{F}_s).$$

By the Markov property of SLE we get

$$\mathcal{M}_s = X_s(z, \xi)\widetilde{G}(z_s, \xi_s, \tau), \quad \tau := t - s, \tag{5.2.9}$$

where

$$z_s := \tilde{f}_s(z)/\lambda(s); \;\; \xi_s := \tilde{f}_s(\xi)/\lambda(s).$$

Their Itô differentials, dz_s and $d\xi_s$, are as in (2.1.11),

$$\begin{aligned} dz_s &= z_s\left[\frac{z_s + 1}{z_s - 1} - \frac{\kappa}{2}\right]ds - i\sqrt{\kappa}\, z_s\, dB_s, \\ d\xi_s &= \xi_s\left[\frac{\xi_s + 1}{\xi_s - 1} - \frac{\kappa}{2}\right]ds - i\sqrt{\kappa}\, \xi_s\, dB_s. \end{aligned} \tag{5.2.10}$$

As before, the partial differential equation satisfied by $\widetilde{G}(z_s, \xi_s, \tau)$ is obtained by expressing the fact that the ds-drift term of the Itô differential of Eq. (5.2.9),

$$d\mathcal{M}_s = \widetilde{G}dX_s + X_s d\widetilde{G}, \tag{5.2.11}$$

vanishes. The Itô differential of X_s is simply computed from Eqs. (2.1.9) and (2.1.10) as:

$$\begin{aligned} dX_s(z) &= X_s(z,\xi)F(z_s,\xi_s)ds, \\ F(z,\xi) &:= 2p\left[1 - \frac{1}{(1-z)^2} - \frac{1}{(1-\xi)^2}\right] - q\frac{z\xi - z - \xi - 1}{(1-z)(1-\xi)}. \end{aligned} \tag{5.2.12}$$

We thus get the simple expression

$$d\mathcal{M}_s = X_s(z,\xi)\left[F(z_s,\xi_s)\,\widetilde{G}\,ds + d\widetilde{G}\right], \tag{5.2.13}$$

and the vanishing of the ds-drift term in $d\mathcal{M}_s$ requires that of the drift term in the right-hand side bracket in (5.2.13), since $X_s(z,\xi)$ does not vanish in $\mathbb{D}\times\mathbb{D}$.

The Itô differential of $\widetilde{G}(z_s,\xi_s,\tau)$ can be obtained from Eqs. (5.2.10) and Itô calculus as

$$\begin{aligned} d\widetilde{G}(z_s,\xi_s,\tau) = {}& \partial_z\widetilde{G}\,dz_s + \partial_\xi\widetilde{G}\,d\xi_s - \partial_\tau\widetilde{G}\,ds \\ & - \frac{\kappa}{2}z_s^2\,\partial_z^2\widetilde{G}\,ds - \frac{\kappa}{2}\xi_s^2\,\partial_\xi^2\widetilde{G}\,ds - \kappa z_s\,\xi_s\,\partial_z\partial_\xi\widetilde{G}\,ds, \end{aligned} \tag{5.2.14}$$

We observe that the only coupling between the z_s,ξ_s variables arises in the last term of (5.2.14), the other terms simply resulting from the independent contributions of the z_s and ξ_s parts.

Using again the Itô differentials (5.2.10), we can rewrite (5.2.14) as

$$\begin{aligned} d\widetilde{G} = {}& -i\sqrt{\kappa}\,(z_s\partial_z + \xi_s\partial_\xi)\,\widetilde{G}\,dB_s \\ & + \frac{z_s+1}{z_s-1}z_s\partial_z\widetilde{G}\,ds + \frac{\xi_s+1}{\xi_s-1}\xi_s\partial_\xi\widetilde{G}\,ds - \partial_\tau\widetilde{G}\,ds \\ & - \frac{\kappa}{2}(z_s\partial_z + \xi_s\,\partial_\xi)^2\widetilde{G}\,ds, \end{aligned} \tag{5.2.15}$$

where we used the obvious formal identity

$$(z\partial_z)^2 + (\xi\,\partial_\xi)^2 + 2z\,\partial_z\xi\,\partial_\xi = (z\,\partial_z + \xi\,\partial_\xi)^2. \tag{5.2.16}$$

We therefore directly arrive at the vanishing condition of the overall drift term coefficient in $d\mathcal{M}_s$,

$$\left[F(z_s,\xi_s) + z_s\frac{z_s+1}{z_s-1}\partial_z + \xi_s\frac{\xi_s+1}{\xi_s-1}\partial_\xi - \partial_\tau - \frac{\kappa}{2}(z_s\partial_z + \xi_s\partial_\xi)^2\right]\widetilde{G} = 0. \tag{5.2.17}$$

The existence of the limit (5.2.8) further implies that of

$$\lim_{\tau\to\infty} e^{(2p-q)\tau}\partial_\tau\tilde{G}(z,\xi,\tau) = -(2p-q)G(z,\xi).$$

Multiplying the PDE (5.2.17) satisfied by $\widetilde{G}$ by $\exp((2p-q)\tau)$ and letting $\tau\to+\infty$, we get

$$\left[-\frac{\kappa}{2}(z\partial_z + \xi\partial_\xi)^2 - \frac{1+z}{1-z}z\partial_z - \frac{1+\xi}{1-\xi}\xi\partial_\xi + F(z,\xi) + 2p - q\right]G = 0, \tag{5.2.18}$$

and its fully explicit expression is

$$\begin{aligned}\mathcal{P}(D)[G(z,\xi)] := & -\frac{\kappa}{2}(z\partial_z + \xi\partial_\xi)^2 G - \frac{1+z}{1-z}z\partial_z G - \frac{1+\xi}{1-\xi}\xi\partial_\xi G \\ & + \left[-\frac{2p}{(1-z)^2} - \frac{2p}{(1-\xi)^2} - q\frac{z\xi - z - \xi - 1}{(1-z)(1-\xi)} + 4p - q \right] G = 0.\end{aligned} \tag{5.2.19}$$

5.3 Some observations for coefficient matrix of $G(z, \xi)$

We write $G(z,\xi)$ (5.2.5) in the form of a power series,

$$G(z,\xi) := \sum_{n,m=0}^{\infty} d_{n,m} z^n \xi^m, \quad (z,\xi) \in \mathbb{D} \times \mathbb{D}. \tag{5.3.1}$$

Substituting (5.3.1) into the PDE (5.2.19) and identifying coefficients, we obtain a system of recursive equations for the coefficients $d_{n,m}$. From that system of recursive equations, we can accurately compute the coefficients $d_{n,m}$ for any values of p, q and κ. However, in the $(p = 1, q = 2)$ case, we have not yet found any exact formula for $G(z,\xi)$ for specific values of κ. Thus, we only introduce here the square matrix of order 10, $(d_{n,m})_{n,m=0,1,\ldots,9}$, for $\kappa = 2$.

$$(p=1,\quad q=2,\quad \kappa=2)$$

$$\begin{bmatrix}
1 & 0 & -\frac{1}{3} & -\frac{2}{9} & -\frac{2}{15} & -\frac{56}{675} & -\frac{22}{405} & -\frac{176}{4725} & -\frac{484}{18225} & -\frac{112288}{5740875} & \cdots \\
0 & \frac{2}{3} & \frac{2}{9} & -\frac{2}{45} & -\frac{74}{675} & -\frac{1564}{14175} & -\frac{9448}{99225} & -\frac{2806}{35721} & -\frac{367646}{5740875} & -\frac{115466984}{2210236875} & \cdots \\
-\frac{1}{3} & \frac{2}{9} & \frac{16}{45} & \frac{26}{135} & \frac{1019}{14175} & \frac{884}{99225} & -\frac{18217}{893025} & -\frac{261232}{8037225} & -\frac{79944173}{2210236875} & -\frac{2613504436}{72937816875} & \cdots \\
-\frac{2}{9} & -\frac{2}{45} & \frac{26}{135} & \frac{526}{2835} & \frac{2452}{19845} & \frac{62968}{893025} & \frac{10274}{297675} & \frac{5283482}{442047375} & -\frac{39055234}{24312605625} & -\frac{26876491178}{2844574858125} & \cdots \\
-\frac{2}{15} & -\frac{74}{675} & \frac{1019}{14175} & \frac{2452}{19845} & \frac{19646}{178605} & \frac{3278708}{40186125} & \frac{24416488}{442047375} & \frac{15367078}{442047375} & \frac{2700606214}{135455945625} & \frac{55052880026}{5752362490875} & \cdots \\
-\frac{56}{675} & -\frac{1564}{14175} & \frac{884}{99225} & \frac{62968}{893025} & \frac{3278708}{40186125} & \frac{159258446}{2210236875} & \frac{277167518}{4862521125} & \frac{40182709222}{948191619375} & \frac{2594538280798}{86285437363125} & \frac{552925132190594}{27179912769384375} & \cdots \\
-\frac{22}{405} & -\frac{9448}{99225} & -\frac{18217}{893025} & \frac{10274}{297675} & \frac{24416488}{442047375} & \frac{277167518}{4862521125} & \frac{48036018236}{948191619375} & \frac{3607387633702}{86285437363125} & \frac{298696562738746}{9059970923128125} & \frac{10246714507825942}{407698691540765625} & \cdots \\
-\frac{176}{4725} & -\frac{2806}{35721} & -\frac{261232}{8037225} & \frac{5283482}{442047375} & \frac{15367078}{442047375} & \frac{40182709222}{948191619375} & \frac{3607387633702}{86285437363125} & \frac{339710562151462}{9059970923128125} & \frac{173278838082904}{5435982553876875} & \frac{60459573479696542}{2310292585397671875} & \cdots \\
-\frac{484}{18225} & -\frac{367646}{5740875} & -\frac{79944173}{2210236875} & -\frac{39055234}{24312605625} & \frac{2700606214}{135455945625} & \frac{2594538280798}{86285437363125} & \frac{298696562738746}{9059970923128125} & \frac{173278838082904}{5435982553876875} & \frac{2665604273426513}{92411703415906875} & \frac{1063203926041206766}{42416971867901255625} & \cdots \\
-\frac{112288}{5740875} & -\frac{115466984}{2210236875} & -\frac{2613504436}{72937816875} & -\frac{26876491178}{2844574858125} & \frac{55052880026}{5752362490875} & \frac{552925132190594}{27179912769384375} & \frac{10246714507825942}{407698691540765625} & \frac{60459573479696542}{2310292585397671875} & \frac{1063203926041206766}{42416971867901255625} & \frac{118464398617767647908}{5180930135293653365625} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

On the other hand, we found some special cases that their respective coefficient matrices are "*finite*" in the sense that they have a finite numbers of nonzero entries. We can therefore obtain closed-form expressions for $G(z, \xi)$ in such cases. Let us now turn to the observations (*predictions*) below for more details.

Observation 5.3.1. *For any nonnegative integer l, if letting*

$$\kappa = \frac{2}{2l+1} \quad , \quad p = 0 \quad , \quad q = -\frac{l(3l+5)+2}{2(2l+1)},$$

then, the corresponding coefficient matrix $(d_{n,m})$ is a square matrix of order $(3l+3)$, and we obtain

$$\begin{aligned} G(z, \xi) &= \sum_{m=0}^{3l+2} |d_{3l+2-m,m}|(1-z)^{3l+2-m}(1-\xi)^m, \qquad (5.3.2)\\ G(z) &:= \lim_{\xi \to z} G(z, \xi) = \mathbb{E}\left((f'(z))^{2p-q}\right) = (1-z)^{3l+2}. \end{aligned}$$

In the specific matrices below, the coefficients $d_{3l+2-m,m}$ in (5.3.2) are in bold.

$$(l = 0) \quad \begin{bmatrix} 1 & -1 & \mathbf{\frac{1}{3}} \\ -1 & \mathbf{\frac{1}{3}} & 0 \\ \mathbf{\frac{1}{3}} & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} G(z, \xi) &= \frac{1}{3}(1-z)^2 + \frac{1}{3}(1-z)(1-\xi) + \frac{1}{3}(1-\xi)^2, \\ G(z) &= (1-z)^2. \end{aligned}$$

$$(l = 1) \quad \begin{bmatrix} 1 & -\frac{5}{2} & 3 & -2 & \frac{5}{7} & -\mathbf{\frac{3}{28}} \\ -\frac{5}{2} & 4 & -3 & \frac{8}{7} & -\mathbf{\frac{5}{28}} & 0 \\ 3 & -3 & \frac{9}{7} & -\mathbf{\frac{3}{14}} & 0 & 0 \\ -2 & \frac{8}{7} & -\mathbf{\frac{3}{14}} & 0 & 0 & 0 \\ \frac{5}{7} & -\mathbf{\frac{5}{28}} & 0 & 0 & 0 & 0 \\ -\mathbf{\frac{3}{28}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} G(z, \xi) = &\frac{3}{28}(1-z)^5 + \frac{5}{28}(1-z)^4(1-\xi) + \frac{3}{14}(1-z)^3(1-\xi)^2 \\ &+ \frac{3}{14}(1-z)^2(1-\xi)^3 + \frac{5}{28}(1-z)(1-\xi)^4 + \frac{3}{28}(1-\xi)^5, \\ G(z) = &(1-z)^5. \end{aligned}$$

$$
(l=2)\quad \begin{bmatrix}
1 & -4 & 8 & -10 & \frac{25}{3} & -\frac{14}{3} & \frac{56}{33} & -\frac{4}{11} & \mathbf{\frac{5}{143}} \\
-4 & 12 & -18 & \frac{50}{3} & -10 & \frac{42}{11} & -\frac{28}{33} & \mathbf{\frac{12}{143}} & 0 \\
8 & -18 & 20 & -\frac{40}{3} & \frac{60}{11} & -\frac{14}{11} & \mathbf{\frac{56}{429}} & 0 & 0 \\
-10 & \frac{50}{3} & -\frac{40}{3} & \frac{200}{33} & -\frac{50}{33} & \mathbf{\frac{70}{429}} & 0 & 0 & 0 \\
\frac{25}{3} & -10 & \frac{60}{11} & -\frac{50}{33} & \mathbf{\frac{25}{143}} & 0 & 0 & 0 & 0 \\
-\frac{14}{3} & \frac{42}{11} & -\frac{14}{11} & \mathbf{\frac{70}{429}} & 0 & 0 & 0 & 0 & 0 \\
\frac{56}{33} & -\frac{28}{33} & \mathbf{\frac{56}{429}} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{4}{11} & \mathbf{\frac{12}{143}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{\frac{5}{143}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
\begin{aligned}
G(z,\xi) &= \frac{5}{143}(1-z)^8 + \frac{12}{143}(1-z)^7(1-\xi) + \frac{56}{429}(1-z)^6(1-\xi)^2 \\
&+ \frac{70}{429}(1-z)^5(1-\xi)^3 + \frac{25}{143}(1-z)^4(1-\xi)^4 + \frac{70}{429}(1-z)^3(1-\xi)^5 \\
&+ \frac{56}{429}(1-z)^2(1-\xi)^6 + \frac{12}{143}(1-z)(1-\xi)^7 + \frac{5}{143}(1-\xi)^8, \\
G(z) &= (1-z)^8.
\end{aligned}
$$

$$
(l=3)\quad \begin{bmatrix}
1 & -\frac{11}{2} & \frac{275}{18} & -\frac{55}{2} & 35 & -\frac{98}{3} & \frac{294}{13} & -\frac{150}{13} & \frac{55}{13} & -\frac{55}{52} & \frac{11}{68} & \mathbf{-\frac{7}{612}} \\
-\frac{11}{2} & \frac{220}{9} & -55 & 80 & -\frac{245}{3} & \frac{784}{13} & -\frac{420}{13} & \frac{160}{13} & -\frac{165}{52} & \frac{110}{221} & \mathbf{-\frac{11}{306}} & 0 \\
\frac{275}{18} & -55 & 100 & -\frac{350}{3} & \frac{1225}{13} & -\frac{700}{13} & \frac{280}{13} & -\frac{75}{13} & \frac{825}{884} & \mathbf{-\frac{275}{3978}} & 0 & 0 \\
-\frac{55}{2} & 80 & -\frac{350}{3} & \frac{1400}{13} & -\frac{875}{13} & \frac{1120}{39} & -\frac{105}{13} & \frac{300}{221} & \mathbf{-\frac{275}{2652}} & 0 & 0 & 0 \\
35 & -\frac{245}{3} & \frac{1225}{13} & -\frac{875}{13} & \frac{1225}{39} & -\frac{245}{26} & \frac{735}{442} & \mathbf{-\frac{175}{1326}} & 0 & 0 & 0 & 0 \\
-\frac{98}{3} & \frac{784}{13} & -\frac{700}{13} & \frac{1120}{39} & -\frac{245}{26} & \frac{392}{221} & \mathbf{-\frac{98}{663}} & 0 & 0 & 0 & 0 & 0 \\
\frac{294}{13} & -\frac{420}{13} & \frac{280}{13} & -\frac{105}{13} & \frac{735}{442} & \mathbf{-\frac{98}{663}} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{150}{13} & \frac{160}{13} & -\frac{75}{13} & \frac{300}{221} & \mathbf{-\frac{175}{1326}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{55}{13} & -\frac{165}{52} & \frac{825}{884} & \mathbf{-\frac{275}{2652}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{55}{52} & \frac{110}{221} & \mathbf{-\frac{275}{3978}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{11}{68} & \mathbf{-\frac{11}{306}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{-\frac{7}{612}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
\begin{aligned}
G(z,\xi) &= \frac{7}{612}(1-z)^{11} + \frac{11}{306}(1-z)^{10}(1-\xi) + \frac{275}{3978}(1-z)^9(1-\xi)^2 \\
&+ \frac{275}{2652}(1-z)^8(1-\xi)^3 + \frac{175}{1326}(1-z)^7(1-\xi)^4 + \frac{98}{663}(1-z)^6(1-\xi)^5 \\
&+ \frac{98}{663}(1-z)^5(1-\xi)^6 + \frac{175}{1326}(1-z)^4(1-\xi)^7 \frac{275}{2652}(1-z)^3(1-\xi)^8 \\
&+ \frac{275}{3978}(1-z)^2(1-\xi)^9 \frac{11}{306}(1-z)(1-\xi)^{10} + \frac{7}{612}(1-\xi)^{11}, \qquad G(z) = (1-z)^{11}.
\end{aligned}
$$

$$(l = 4)$$

$$\begin{bmatrix}
1 & -7 & \tfrac{273}{11} & -\tfrac{637}{11} & 98 & -126 & 126 & -99 & \tfrac{2079}{34} & -\tfrac{1001}{34} & \tfrac{7007}{646} & -\tfrac{1911}{646} & \mathbf{\tfrac{182}{323}} & -\tfrac{14}{209} & \mathbf{\tfrac{18}{4807}} \\
-7 & \tfrac{455}{11} & -\tfrac{1365}{11} & 245 & -350 & 378 & -315 & \tfrac{3465}{17} & -\tfrac{3465}{34} & \tfrac{25025}{646} & -\tfrac{7007}{646} & \tfrac{1365}{646} & -\tfrac{910}{3553} & \mathbf{\tfrac{70}{4807}} & 0 \\
\tfrac{273}{11} & -\tfrac{1365}{11} & 315 & -525 & 630 & -567 & \tfrac{6615}{17} & -\tfrac{3465}{17} & \tfrac{51975}{646} & -\tfrac{15015}{646} & \tfrac{3003}{646} & -\tfrac{4095}{7106} & \mathbf{\tfrac{2730}{81719}} & 0 & 0 \\
-\tfrac{637}{11} & 245 & -525 & 735 & -735 & \tfrac{9261}{17} & -\tfrac{5145}{17} & \tfrac{40425}{323} & -\tfrac{24255}{646} & \tfrac{5005}{646} & -\tfrac{637}{646} & \mathbf{\tfrac{9555}{163438}} & 0 & 0 & 0 \\
98 & -350 & 630 & -735 & \tfrac{10290}{17} & -\tfrac{6174}{17} & \tfrac{51450}{323} & -\tfrac{16170}{323} & \tfrac{3465}{323} & -\tfrac{455}{323} & \mathbf{\tfrac{637}{7429}} & 0 & 0 & 0 & 0 \\
-126 & 378 & -567 & \tfrac{9261}{17} & -\tfrac{6174}{17} & \tfrac{55566}{323} & -\tfrac{18522}{323} & \tfrac{4158}{323} & -\tfrac{567}{323} & \mathbf{\tfrac{819}{7429}} & 0 & 0 & 0 & 0 & 0 \\
126 & -315 & \tfrac{6615}{17} & -\tfrac{5145}{17} & \tfrac{51450}{323} & -\tfrac{18522}{323} & \tfrac{4410}{323} & -\tfrac{630}{323} & \mathbf{\tfrac{945}{7429}} & 0 & 0 & 0 & 0 & 0 & 0 \\
-99 & \tfrac{3465}{17} & -\tfrac{3465}{17} & \tfrac{40425}{323} & -\tfrac{16170}{323} & \tfrac{4158}{323} & -\tfrac{630}{323} & \tfrac{\mathbf{990}}{7429} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tfrac{2079}{34} & -\tfrac{3465}{34} & \tfrac{51975}{646} & -\tfrac{24255}{646} & \tfrac{3465}{323} & -\tfrac{567}{323} & \mathbf{\tfrac{945}{7429}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\tfrac{1001}{34} & \tfrac{25025}{646} & -\tfrac{15015}{646} & \tfrac{5005}{646} & -\tfrac{455}{323} & \mathbf{\tfrac{819}{7429}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tfrac{7007}{646} & -\tfrac{7007}{646} & \tfrac{3003}{646} & -\tfrac{637}{646} & \mathbf{\tfrac{637}{7429}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\tfrac{1911}{646} & \tfrac{1365}{646} & -\tfrac{4095}{7106} & \mathbf{\tfrac{9555}{163438}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tfrac{182}{323} & -\tfrac{910}{3553} & \mathbf{\tfrac{2730}{81719}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\tfrac{14}{209} & \mathbf{\tfrac{70}{4807}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{\tfrac{18}{4807}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\begin{aligned}
G(z,\xi) = {} & \frac{18}{4807}(1-z)^{14} + \frac{70}{4807}(1-z)^{13}(1-\xi) + \frac{2730}{81719}(1-z)^{12}(1-\xi)^2 + \frac{9555}{163438}(1-z)^{11}(1-\xi)^3 + \frac{637}{7429}(1-z)^{10}(1-\xi)^4 \\
& + \frac{819}{7429}(1-z)^9(1-\xi)^5 + \frac{945}{7429}(1-z)^8(1-\xi)^6 + \frac{990}{7429}(1-z)^7(1-\xi)^7 + \frac{945}{7429}(1-z)^6(1-\xi)^8 + \frac{819}{7429}(1-z)^5(1-\xi)^9 \\
& + \frac{637}{7429}(1-z)^4(1-\xi)^{10} + \frac{9555}{163438}(1-z)^3(1-\xi)^{11} + \frac{2730}{81719}(1-z)^2(1-\xi)^{12}\frac{70}{4807}(1-z)(1-\xi)^{13} + \frac{18}{4807}(1-\xi)^{14}, \\
G(z) = {} & (1-z)^{14}.
\end{aligned}$$

Besides, in this case, note that $\lambda = 0$ is not a root of the characteristic polynomial $P(\lambda)$ of the coefficient matrix $(d_{n,m})_{n,m=0,1,\dots,3l+2}$. So, $\lambda = 0$ is not an eigenvalue of the matrix. Using MAPLE, we get approximate values of eigenvalues λ. For instance:

- For $l=0$, $P(\lambda) = -\lambda^3 + (4/3)\,\lambda^2 + (7/9)\,\lambda - 1/27$,

  $$\lambda = \big\{ -0.473688871436316;\ 0.0443581584495084;\ 1.76266404632014\big\}.$$

- For $l=1$,

  $$P(\lambda) = \lambda^6 - \frac{44}{7}\,\lambda^5 - \frac{7733}{392}\,\lambda^4 + \frac{76689}{5488}\,\lambda^3 + \frac{162063}{87808}\,\lambda^2 - \frac{15795}{614656}\,\lambda - \frac{2025}{120472576},$$
  $$\lambda = \big\{ -2.72880158998666;\ -0.125380457463403;\ -0.000626087842643780;\ 0.0132950937076164;\ 0.700396685271733;\ 8.42683064202764\big\}.$$

- For $l=2$,

  $$\begin{aligned} P(\lambda) = {} & -\lambda^9 + \frac{16832}{429}\,\lambda^8 + \frac{44413933}{61347}\,\lambda^7 - \frac{265540817419}{78953589}\,\lambda^6 \\ & - \frac{142454834483644}{33871089681}\,\lambda^5 + \frac{491274202605820}{440324165853}\,\lambda^4 + \frac{32813887416889600}{692629912886769}\,\lambda^3 \\ & - \frac{21885077272352000}{99046077542807967}\,\lambda^2 - \frac{328029614080000}{4721196362873846427}\,\lambda + \frac{153664000000}{2250436932969866796 87}, \end{aligned}$$

  $$\begin{aligned} \lambda = \big\{ & -16.4090158553303;\ -1.21664842364105;\ -0.0410912036615348; \\ & -0.304809798952985\cdot 10^{-3};\ 0.953876315121353\cdot 10^{-5}; \\ & 0.00451593430789748;\ 0.257946507230370; \\ & 4.73477437002391;\ 51.9052451775378\big\}. \end{aligned}$$

- For $l=3$,

  $$\begin{aligned} P(\lambda) = {} & \lambda^{12} - \frac{529712}{1989}\,\lambda^{11} - \frac{1026820496219}{31648968}\,\lambda^{10} + \frac{42299672112710069}{41966531568}\,\lambda^9 \\ & + \frac{35886035693433588426433}{4006628701860096}\,\lambda^8 - \frac{1622011514181231495062939 21}{796918448799973094 4}\,\lambda^7 \\ & - \frac{6753968984307693483705852343 69}{634028317865258593904 64}\,\lambda^6 + \frac{2229331847986064089534151989 7575}{2101803873723332238793 8816}\,\lambda^5 \\ & + \frac{1755742642755573607690655933861875}{111479677462285541945627480064}\,\lambda^4 \\ & - \frac{4948229405979849980110706764 0625}{189515451685885421307566716 1088}\,\lambda^3 \\ & - \frac{3947847741012772032837813671875}{115983456431761877840230830258 5856}\,\lambda^2 \\ & + \frac{3243513218823187594238281 25}{1478789069504963942462943085796 9664}\,\lambda \\ & + \frac{249333413680688476 5625}{754182425447531610656100973756452864}, \end{aligned}$$

$$
\begin{aligned}
\lambda = \big\{ & -109.021110505677;\ -8.76311466576166;\ -0.512995593556982;\\
& -0.0144843744241151;\ -0.000127270500865400;\\
& -1.47362079285162 \cdot 10^{-7};\ 0.629528638696634 \cdot 10^{-5};\\
& \ \ 0.00161756480976645;\ 0.0972336447618827;\\
& \ \ 2.24988045780048;\ 31.8048139856412;\ 350.479044812100 \big\}.
\end{aligned}
$$

- For $l = 4$,

$$
\begin{aligned}
P(\lambda) = & -\lambda^{15} + \frac{153915712}{81719}\lambda^{14} + \frac{10694803264138168}{6677994961}\lambda^{13}\\
& - \frac{190868554687904199799580}{545719070217959}\lambda^{12} - \frac{89759337888986393496314570243}{4054146972649217411}\lambda^{11}\\
& + \frac{1359380660283855893971859015629481772}{3644309201037135373704599}\lambda^{10}\\
& + \frac{482356605347949026025930462849050584823136}{297809303599553665603766125681}\lambda^{9}\\
& - \frac{40721250122678804518165435937760153960980378550}{24336678480851925999474164024525639}\lambda^{8}\\
& - \frac{2876749324090325961411313466640579163534576324806825}{7955076115106954163004116839680842773764}\lambda^{7}\\
& + \frac{18412423802929710604763028261511814432548586121354000}{1343142283162035510839945090954295021961199}\lambda^{6}\\
& + \frac{8131880487657461728640937007731193458068254852659266250}{109760244237718379910329472887694034899647221081}\lambda^{5}\\
& - \frac{36784810236230133008256640107560915497166067038232900000}{815408854442009844353837654082678985269479205410749}\lambda^{4}\\
& - \frac{838706204922087700378827130919540501692200159545062500}{356333669391158301982627054834130716562762412764497313}\lambda^{3}\\
& + \frac{1171353663033922564820421671021247448432306177500000}{155717813523936177966408022962515123137927174378085325781}\lambda^{2}\\
& + \frac{63830892951437176098493254084804726394068750000}{68048684509960109771320306034619108811274175203223287366297}\lambda\\
& - \frac{5907729072476296803984974687795062500000}{2703388648259324360915179430648050050047892233073506961733799},
\end{aligned}
$$

$$
\begin{aligned}
\lambda = \big\{ & -763.452736730147;\ -62.5761772084498;\ -4.30656002342561;\\
& -0.212272989274719;\ -0.00529752374979785;\\
& -0.511667825901402 \cdot 10^{-4};\ -1.22395662833629 \cdot 10^{-7};\\
& \ \ 2.28776278431758 \cdot 10^{-9};\ 0.313547654297345 \cdot 10^{-5};\\
& \ \ 0.596943653029120 \cdot 10^{-3};\ 0.0371690989859762;\\
& \ \ 1.01933310185248;\ 16.8131363773735;\\
& \ \ 223.631432116748;\ 2472.52665106747 \big\}.
\end{aligned}
$$

Observation 5.3.2. *For any nonnegative integer l, if letting*

$$\kappa = \frac{2}{2l+1} \quad , \quad p = q = \frac{l(3l+5)+2}{2(2l+1)},$$

then, the corresponding coefficient matrix $(d_{n,m})$ is a square matrix of order $(2l+2)$, and we obtain

$$G(z,\xi) = (1-z)^{l+1}(1-\xi)^{l+1}\sum_{m=0}^{l}|d_{2l+1-m,l+1+m}|(1-z)^{l-m}(1-\xi)^{m}, \tag{5.3.3}$$

$$G(z) := \lim_{\xi\to z} G(z,\xi) = \mathbb{E}\left((f'(z))^{2p-q}\right) = (1-z)^{3l+2}.$$

In the specific matrices below, the coefficients $d_{2l+1-m,l+1+m}$ in (5.3.3) are in bold.

$$(l=0)\quad \begin{bmatrix} 1 & -1 \\ -1 & \mathbf{1} \end{bmatrix},$$

$$G(z,\xi) = (1-z)(1-\xi), \quad G(z) = (1-z)^2.$$

$$(l=1)\quad \begin{bmatrix} 1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\ -\frac{5}{2} & 6 & -\frac{9}{2} & 1 \\ 2 & -\frac{9}{2} & 3 & \mathbf{-\frac{1}{2}} \\ -\frac{1}{2} & 1 & \mathbf{-\frac{1}{2}} & 0 \end{bmatrix},$$

$$G(z,\xi) = (1-z)^2(1-\xi)^2\left[\frac{1}{2}(1-z) + \frac{1}{2}(1-\xi)\right], \quad G(z) = (1-z)^5.$$

$$(l=2)\quad \begin{bmatrix} 1 & -4 & \frac{44}{7} & -\frac{34}{7} & \frac{13}{7} & -\frac{2}{7} \\ -4 & \frac{108}{7} & -\frac{162}{7} & \frac{118}{7} & -6 & \frac{6}{7} \\ \frac{44}{7} & -\frac{162}{7} & \frac{228}{7} & -\frac{152}{7} & \frac{48}{7} & -\frac{6}{7} \\ -\frac{34}{7} & \frac{118}{7} & -\frac{152}{7} & \frac{88}{7} & -\frac{22}{7} & \mathbf{\frac{2}{7}} \\ \frac{13}{7} & -6 & \frac{48}{7} & -\frac{22}{7} & \mathbf{\frac{3}{7}} & 0 \\ -\frac{2}{7} & \frac{6}{7} & -\frac{6}{7} & \mathbf{\frac{2}{7}} & 0 & 0 \end{bmatrix},$$

$$G(z,\xi) = (1-z)^3(1-\xi)^3\left[\frac{2}{7}(1-z)^2 + \frac{3}{7}(1-z)(1-\xi) + \frac{2}{7}(1-\xi)^2\right],$$
$$G(z) = (1-z)^8.$$

$$
(l=3)\quad \begin{bmatrix}
1 & -\frac{11}{2} & \frac{77}{6} & -\frac{33}{2} & \frac{38}{3} & -\frac{35}{6} & \frac{3}{2} & -\frac{1}{6} \\
-\frac{11}{2} & \frac{88}{3} & -66 & \frac{244}{3} & -\frac{119}{2} & 26 & -\frac{19}{3} & \frac{2}{3} \\
\frac{77}{6} & -66 & 142 & -\frac{497}{3} & \frac{227}{2} & -46 & \frac{31}{3} & -1 \\
-\frac{33}{2} & \frac{244}{3} & -\frac{497}{3} & 180 & -\frac{225}{2} & \frac{122}{3} & -8 & \frac{2}{3} \\
\frac{38}{3} & -\frac{119}{2} & \frac{227}{2} & -\frac{225}{2} & \frac{185}{3} & -\frac{37}{2} & \frac{17}{6} & \boldsymbol{-\frac{1}{6}} \\
-\frac{35}{6} & 26 & -46 & \frac{122}{3} & -\frac{37}{2} & 4 & \boldsymbol{-\frac{1}{3}} & 0 \\
\frac{3}{2} & -\frac{19}{3} & \frac{31}{3} & -8 & \frac{17}{6} & \boldsymbol{-\frac{1}{3}} & 0 & 0 \\
-\frac{1}{6} & \frac{2}{3} & -1 & \frac{2}{3} & \boldsymbol{-\frac{1}{6}} & 0 & 0 & 0
\end{bmatrix},
$$

$$
\begin{aligned}
&G(z,\xi) \\
&= (1-z)^4(1-\xi)^4\left[\frac{1}{6}(1-z)^3 + \frac{1}{3}(1-z)^2(1-\xi) + \frac{1}{3}(1-z)(1-\xi)^2 + \frac{1}{6}(1-\xi)^3\right], \\
&G(z) = (1-z)^{11}.
\end{aligned}
$$

$$(l=4)$$

$$
\begin{bmatrix}
1 & -7 & \frac{238}{11} & -\frac{427}{11} & \frac{6384}{143} & -\frac{4893}{143} & \frac{2506}{143} & -\frac{829}{143} & \frac{161}{143} & -\frac{14}{143} \\
-7 & \frac{525}{11} & -\frac{1575}{11} & \frac{35525}{143} & -\frac{39375}{143} & \frac{29001}{143} & -\frac{1295}{13} & \frac{4515}{143} & -\frac{840}{143} & \frac{70}{143} \\
\frac{238}{11} & -\frac{1575}{11} & \frac{59325}{143} & -\frac{98875}{143} & \frac{104685}{143} & -\frac{73269}{143} & \frac{34055}{143} & -\frac{10185}{143} & \frac{1785}{143} & -\frac{140}{143} \\
-\frac{427}{11} & \frac{35525}{143} & -\frac{98875}{143} & \frac{157045}{143} & -\frac{157045}{143} & \frac{102851}{143} & -\frac{44345}{143} & \frac{12215}{143} & -\frac{1960}{143} & \frac{140}{143} \\
\frac{6384}{143} & -\frac{39375}{143} & \frac{104685}{143} & -\frac{157045}{143} & \frac{146265}{143} & -\frac{87759}{143} & \frac{34055}{143} & -\frac{8295}{143} & \frac{105}{13} & -\frac{70}{143} \\
-\frac{4893}{143} & \frac{29001}{143} & -\frac{73269}{143} & \frac{102851}{143} & -\frac{87759}{143} & \frac{4263}{13} & -\frac{1421}{13} & \frac{3129}{143} & -\frac{336}{143} & \boldsymbol{\frac{14}{143}} \\
\frac{2506}{143} & -\frac{1295}{13} & \frac{34055}{143} & -\frac{44345}{143} & \frac{34055}{143} & -\frac{1421}{13} & \frac{4165}{143} & -\frac{595}{143} & \boldsymbol{\frac{35}{143}} & 0 \\
-\frac{829}{143} & \frac{4515}{143} & -\frac{10185}{143} & \frac{12215}{143} & -\frac{8295}{143} & \frac{3129}{143} & -\frac{595}{143} & \boldsymbol{\frac{45}{143}} & 0 & 0 \\
\frac{161}{143} & -\frac{840}{143} & \frac{1785}{143} & -\frac{1960}{143} & \frac{105}{13} & -\frac{336}{143} & \boldsymbol{\frac{35}{143}} & 0 & 0 & 0 \\
-\frac{14}{143} & \frac{70}{143} & -\frac{140}{143} & \frac{140}{143} & -\frac{70}{143} & \boldsymbol{\frac{14}{143}} & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
\begin{aligned}
G(z,\xi) = (1-z)^5(1-\xi)^5\bigg[&\frac{14}{143}(1-z)^4 + \frac{35}{143}(1-z)^3(1-\xi) + \frac{45}{143}(1-z)^2(1-\xi)^2 \\
&+ \frac{35}{143}(1-z)(1-\xi)^3 + \frac{14}{143}(1-\xi)^4\bigg], \\
G(z) = (1-z)^{14}.&
\end{aligned}
$$

$$(l=5)$$

$$\begin{bmatrix}
1 & -\frac{17}{2} & \frac{425}{13} & -\frac{1955}{26} & \frac{1496}{13} & -\frac{493}{4} & \frac{2455}{26} & -\frac{2695}{52} & 20 & -\frac{269}{52} & \frac{21}{26} & -\frac{3}{52} \\
-\frac{17}{2} & \frac{918}{13} & -\frac{6885}{26} & \frac{7701}{13} & -\frac{3519}{4} & \frac{11862}{13} & -\frac{35145}{52} & \frac{9315}{26} & -\frac{6939}{52} & \frac{433}{13} & -\frac{261}{52} & \frac{9}{26} \\
\frac{425}{13} & -\frac{6885}{26} & \frac{12546}{13} & -2091 & \frac{77931}{26} & -\frac{155529}{52} & \frac{27630}{13} & -\frac{56079}{52} & \frac{9981}{26} & -\frac{4757}{52} & \frac{171}{13} & -\frac{45}{52} \\
-\frac{1955}{26} & \frac{7701}{13} & -2091 & \frac{56778}{13} & -\frac{312279}{52} & \frac{148695}{26} & -\frac{200709}{52} & \frac{24078}{13} & -\frac{32307}{52} & \frac{3617}{26} & -\frac{75}{4} & \frac{15}{13} \\
\frac{1496}{13} & -\frac{3519}{4} & \frac{77931}{26} & -\frac{312279}{52} & \frac{102240}{13} & -\frac{92016}{13} & \frac{58218}{13} & -\frac{25983}{13} & \frac{8046}{13} & -\frac{6605}{52} & \frac{405}{26} & -\frac{45}{52} \\
-\frac{493}{4} & \frac{11862}{13} & -\frac{155529}{52} & \frac{148695}{26} & -\frac{92016}{13} & \frac{77364}{13} & -\frac{45129}{13} & \frac{18306}{13} & -\frac{20295}{52} & \frac{916}{13} & -\frac{387}{52} & \frac{9}{26} \\
\frac{2455}{26} & -\frac{35145}{52} & \frac{27630}{13} & -\frac{200709}{52} & \frac{58218}{13} & -\frac{45129}{13} & \frac{23772}{13} & -\frac{8490}{13} & \frac{309}{2} & -\frac{1187}{52} & \frac{24}{13} & \boldsymbol{-\frac{3}{52}} \\
-\frac{2695}{52} & \frac{9315}{26} & -\frac{56079}{52} & \frac{24078}{13} & -\frac{25983}{13} & \frac{18306}{13} & -\frac{8490}{13} & \frac{2556}{13} & -\frac{1917}{52} & \frac{101}{26} & \boldsymbol{-\frac{9}{52}} & 0 \\
20 & -\frac{6939}{52} & \frac{9981}{26} & -\frac{32307}{52} & \frac{8046}{13} & -\frac{20295}{52} & \frac{309}{2} & -\frac{1917}{52} & \frac{63}{13} & \boldsymbol{-\frac{7}{26}} & 0 & 0 \\
-\frac{269}{52} & \frac{433}{13} & -\frac{4757}{52} & \frac{3617}{26} & -\frac{6605}{52} & \frac{916}{13} & -\frac{1187}{52} & \frac{101}{26} & \boldsymbol{-\frac{7}{26}} & 0 & 0 & 0 \\
\frac{21}{26} & -\frac{261}{52} & \frac{171}{13} & -\frac{75}{4} & \frac{405}{26} & -\frac{387}{52} & \frac{24}{13} & \boldsymbol{-\frac{9}{52}} & 0 & 0 & 0 & 0 \\
-\frac{3}{52} & \frac{9}{26} & -\frac{45}{52} & \frac{15}{13} & -\frac{45}{52} & \frac{9}{26} & \boldsymbol{-\frac{3}{52}} & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$G(z,\xi) = (1-z)^6(1-\xi)^6\Big[\frac{3}{52}(1-z)^5 + \frac{9}{52}(1-z)^4(1-\xi) + \frac{7}{26}(1-z)^3(1-\xi)^2 + \frac{7}{26}(1-z)^2(1-\xi)^3 + \frac{9}{52}(1-z)(1-\xi)^4 + \frac{3}{52}(1-\xi)^5\Big],$$

$$G(z) = (1-z)^{17}.$$

Besides, in this case, note that $\lambda = 0$ is a root of multiplicity $l+1$ of the characteristic polynomial $P(\lambda)$ of the coefficient matrix $(d_{n,m})_{n,m=0,1,\dots,2l+1}$. Using MAPLE, we obtain approximate values of eigenvalues λ. For instance:

- For $l = 0$,
$$P(\lambda) = \lambda\,[\lambda - 2],\ \lambda = \{0;\, 2\}.$$
- For $l = 1$,
$$P(\lambda) = \lambda^2\left[\lambda^2 - 10\,\lambda - 5\right],\ \lambda = \left\{-0.4772255751;\, 0^{(2)};\, 10.47722558\right\}.$$
- For $l = 2$,
$$\begin{aligned}P(\lambda) &= \frac{1}{7}\,\lambda^3\left[7\lambda^3 - 434\,\lambda^2 - 1928\,\lambda + 240\right],\\ \lambda &= \left\{-4.27669821438978;\, 0^{(3)};\, 0.121182116828131;\, 66.1555160975617\right\}.\end{aligned}$$
- For $l = 3$,
$$\begin{aligned}P(\lambda) &= \frac{1}{3}\,\lambda^4\left[3\,\lambda^4 - 1254\,\lambda^3 - 43285\,\lambda^2 + 66836\,\lambda + 2156\right],\\ \lambda &= \Big\{-33.4355724222944;\, -0.0316114894591856;\, 0^{(4)};\\ &\qquad 1.51113646850890;\, 449.956047443245\Big\}.\end{aligned}$$
- For $l = 4$,
$$\begin{aligned}P(\lambda) &= \frac{1}{24167}\,\lambda^5\big[24167\,\lambda^5 - 71099314\,\lambda^4 - 18613476973\,\lambda^3\\ &\qquad + 270251709000\,\lambda^2 + 137725578900\,\lambda - 1166886000\big],\\ \lambda &= \Big\{-254.650361638484;\, -0.500984761369627;\, 0^{(5)};\\ &\qquad 0.00833625918889979;\, 14.2641387138316;\, 3182.87887142683\Big\}.\end{aligned}$$
- For $l = 5$,
$$\begin{aligned}P(\lambda) &= (1/832)\,\lambda^6\big[832\,\lambda^6 - 17680000\,\lambda^5 - 34999011392\,\lambda^4 + 4352009336400\,\lambda^3\\ &\qquad + 24485667910704\,\lambda^2 - 3968526359952\,\lambda - 8890849737\big],\\ \lambda &= \Big\{-1931.17205159984;\, -5.54419489136658;\, -0.00221021169275717;\\ &\qquad 0^{(6)};\, 0.159813871185902;\, 122.511434787320;\, 23064.0472080444\Big\}.\end{aligned}$$
- For $l = 6$, setting $N := 12165074387$,
$$\begin{aligned}P(\lambda) &= (1/N)\,\lambda^7\big[12165074387\,\lambda^7 - 1899673686125146\,\lambda^6\\ &\qquad - 28501179335024060199\,\lambda^5 + 29133804061664982944264\,\lambda^4\\ &\qquad + 1572201809578397651593940\,\lambda^3 - 3228651447196418488605360\,\lambda^2\\ &\qquad - 162246287526486885321600\,\lambda + 96480960960480096000\big],\\ \lambda &= \Big\{-14667.3115363778;\, -53.2761804199220;\, -0.0496543405839516;\\ &\qquad 0^{(7)};\, 0.000587784249828495;\, 2.02835995819537\\ &\qquad 1009.28359609128;\, 1.69867324827305\cdot 10^5\Big\}.\end{aligned}$$

Observation 5.3.3. *For any integer $l \geq 1$, if letting*

$$\kappa = \frac{2}{2l+1} \quad , \quad p = \frac{3l(l+1)}{2(2l+1)} \quad , \quad q = \frac{l(3l+1)-2}{2(2l+1)},$$

then, the corresponding coefficient matrix $(d_{n,m})$ is a square matrix of order $(2l+3)$, and we obtain

$$G(z,\xi) = (1-z)^l(1-\xi)^l \sum_{m=0}^{l+2} |d_{2l+2-m,l+m}|(1-z)^{l+2-m}(1-\xi)^m, \tag{5.3.4}$$

$$G(z) := \lim_{\xi\to z} G(z,\xi) = \mathbb{E}\left((f'(z))^{2p-q}\right) = (1-z)^{3l+2}.$$

In the specific matrices below, the coefficients $d_{2l+2-m,l+m}$ in (5.3.4) are in bold.

$$(l=1) \quad \begin{bmatrix} 1 & -\frac{5}{2} & \frac{12}{5} & -\frac{11}{10} & \frac{1}{5} \\ -\frac{5}{2} & \frac{26}{5} & -\frac{39}{10} & \frac{7}{5} & \boldsymbol{-\frac{1}{5}} \\ \frac{12}{5} & -\frac{39}{10} & \frac{9}{5} & \boldsymbol{-\frac{3}{10}} & 0 \\ -\frac{11}{10} & \frac{7}{5} & \boldsymbol{-\frac{3}{10}} & 0 & 0 \\ \frac{1}{5} & \boldsymbol{-\frac{1}{5}} & 0 & 0 & 0 \end{bmatrix},$$

$$G(z,\xi) = (1-z)(1-\xi)\left[\frac{1}{5}(1-z)^3 + \frac{3}{10}(1-z)^2(1-\xi) + \frac{3}{10}(1-z)(1-\xi)^2 + \frac{1}{5}(1-\xi)^3\right],$$
$$G(z) = (1-z)^5.$$

$$(l=2) \quad \begin{bmatrix} 1 & -4 & \frac{47}{7} & -\frac{43}{7} & \frac{137}{42} & -\frac{20}{21} & \frac{5}{42} \\ -4 & \frac{102}{7} & -\frac{153}{7} & \frac{53}{3} & -\frac{58}{7} & \frac{15}{7} & -\frac{5}{21} \\ \frac{47}{7} & -\frac{153}{7} & \frac{197}{7} & -\frac{394}{21} & \frac{99}{14} & -\frac{10}{7} & \boldsymbol{\frac{5}{42}} \\ -\frac{43}{7} & \frac{53}{3} & -\frac{394}{21} & \frac{28}{3} & -\frac{7}{3} & \boldsymbol{\frac{5}{21}} & 0 \\ \frac{137}{42} & -\frac{58}{7} & \frac{99}{14} & -\frac{7}{3} & \boldsymbol{\frac{2}{7}} & 0 & 0 \\ -\frac{20}{21} & \frac{15}{7} & -\frac{10}{7} & \boldsymbol{\frac{5}{21}} & 0 & 0 & 0 \\ \frac{5}{42} & -\frac{5}{21} & \boldsymbol{\frac{5}{42}} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G(z,\xi) = (1-z)^2(1-\xi)^2\left[\frac{5}{42}(1-z)^4 + \frac{5}{21}(1-z)^3(1-\xi) + \frac{2}{7}(1-z)^2(1-\xi)^2 \right.$$
$$\left. + \frac{5}{21}(1-z)(1-\xi)^3 + \frac{5}{42}(1-\xi)^4\right],$$
$$G(z) = (1-z)^8.$$

$$(l = 3)$$

$$\begin{bmatrix}
1 & -\frac{11}{2} & \frac{239}{18} & -\frac{37}{2} & \frac{180}{11} & -\frac{623}{66} & \frac{229}{66} & -\frac{49}{66} & \frac{7}{99} \\
-\frac{11}{2} & \frac{256}{9} & -64 & \frac{908}{11} & -\frac{4445}{66} & \frac{394}{11} & -\frac{401}{33} & \frac{238}{99} & -\frac{7}{33} \\
\frac{239}{18} & -64 & \frac{1454}{11} & -\frac{5089}{33} & \frac{225}{2} & -\frac{584}{11} & \frac{143}{9} & -\frac{91}{33} & \frac{7}{33} \\
-\frac{37}{2} & \frac{908}{11} & -\frac{5089}{33} & \frac{5228}{33} & -\frac{6535}{66} & \frac{3862}{99} & -\frac{104}{11} & \frac{14}{11} & -\boldsymbol{\frac{7}{99}} \\
\frac{180}{11} & -\frac{4445}{66} & \frac{225}{2} & -\frac{6535}{66} & \frac{4975}{99} & -\frac{995}{66} & \frac{5}{2} & -\boldsymbol{\frac{35}{198}} & 0 \\
-\frac{623}{66} & \frac{394}{11} & -\frac{584}{11} & \frac{3862}{99} & -\frac{995}{66} & \frac{100}{33} & -\boldsymbol{\frac{25}{99}} & 0 & 0 \\
\frac{229}{66} & -\frac{401}{33} & \frac{143}{9} & -\frac{104}{11} & \frac{5}{2} & -\boldsymbol{\frac{25}{99}} & 0 & 0 & 0 \\
-\frac{49}{66} & \frac{238}{99} & -\frac{91}{33} & \frac{14}{11} & -\boldsymbol{\frac{35}{198}} & 0 & 0 & 0 & 0 \\
\frac{7}{99} & -\frac{7}{33} & \frac{7}{33} & -\boldsymbol{\frac{7}{99}} & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\begin{aligned}
G(z,\xi) = (1-z)^3(1-\xi)^3 \Big[& \frac{7}{99}(1-z)^5 + \frac{35}{198}(1-z)^4(1-\xi) + \frac{25}{99}(1-z)^3(1-\xi)^2 \\
& + \frac{25}{99}(1-z)^2(1-\xi)^3 + \frac{35}{198}(1-z)(1-\xi)^4 + \frac{7}{99}(1-\xi)^5 \Big],
\end{aligned}$$

$$G(z) = (1-z)^{11}.$$

$$(l = 4)$$

$$\begin{bmatrix}
1 & -7 & \frac{243}{11} & -\frac{457}{11} & \frac{7399}{143} & -\frac{6393}{143} & \frac{299}{11} & -\frac{1643}{143} & \frac{42}{13} & -\frac{6}{11} & \frac{6}{143} \\
-7 & \frac{515}{11} & -\frac{1545}{11} & \frac{35755}{143} & -\frac{42025}{143} & \frac{34215}{143} & -\frac{19595}{143} & \frac{7805}{143} & -\frac{2070}{143} & \frac{30}{13} & -\frac{24}{143} \\
\frac{243}{11} & -\frac{1545}{11} & \frac{56835}{143} & -\frac{94725}{143} & \frac{103575}{143} & -\frac{78075}{143} & \frac{41265}{143} & -\frac{15135}{143} & \frac{3690}{143} & -\frac{540}{143} & \frac{36}{143} \\
-\frac{457}{11} & \frac{35755}{143} & -\frac{94725}{143} & \frac{146075}{143} & -\frac{146075}{143} & \frac{7665}{11} & -\frac{47215}{143} & \frac{15385}{143} & -\frac{300}{13} & \frac{420}{143} & -\frac{24}{143} \\
\frac{7399}{143} & -\frac{42025}{143} & \frac{103575}{143} & -\frac{146075}{143} & \frac{131075}{143} & -\frac{78645}{143} & \frac{32095}{143} & -\frac{8785}{143} & \frac{1530}{143} & -\frac{150}{143} & \boldsymbol{\frac{6}{143}} \\
-\frac{6393}{143} & \frac{34215}{143} & -\frac{78075}{143} & \frac{7665}{11} & -\frac{78645}{143} & \frac{3087}{11} & -\frac{1029}{11} & \frac{2823}{143} & -\frac{342}{143} & \boldsymbol{\frac{18}{143}} & 0 \\
\frac{299}{11} & -\frac{19595}{143} & \frac{41265}{143} & -\frac{47215}{143} & \frac{32095}{143} & -\frac{1029}{11} & \frac{3395}{143} & -\frac{485}{143} & \boldsymbol{\frac{30}{143}} & 0 & 0 \\
-\frac{1643}{143} & \frac{7805}{143} & -\frac{15135}{143} & \frac{15385}{143} & -\frac{8785}{143} & \frac{2823}{143} & -\frac{485}{143} & \boldsymbol{\frac{35}{143}} & 0 & 0 & 0 \\
\frac{42}{13} & -\frac{2070}{143} & \frac{3690}{143} & -\frac{300}{13} & \frac{1530}{143} & -\frac{342}{143} & \boldsymbol{\frac{30}{143}} & 0 & 0 & 0 & 0 \\
-\frac{6}{11} & \frac{30}{13} & -\frac{540}{143} & \frac{420}{143} & -\frac{150}{143} & \boldsymbol{\frac{18}{143}} & 0 & 0 & 0 & 0 & 0 \\
\frac{6}{143} & -\frac{24}{143} & \frac{36}{143} & -\frac{24}{143} & \boldsymbol{\frac{6}{143}} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\begin{aligned}
G(z,\xi) = (1-z)^4(1-\xi)^4 \Big[& \frac{6}{143}(1-z)^6 + \frac{18}{143}(1-z)^5(1-\xi) + \frac{30}{143}(1-z)^4(1-\xi)^2 \\
& + \frac{35}{143}(1-z)^3(1-\xi)^3 + \frac{30}{143}(1-z)^2(1-\xi)^4 \\
& + \frac{18}{143}(1-z)^1(1-\xi)^5 + \frac{6}{143}(1-\xi)^6 \Big],
\end{aligned}$$

$$G(z) = (1-z)^{14}.$$

$$(l = 5)$$

$$\begin{bmatrix}
1 & -\frac{17}{2} & \frac{431}{13} & -\frac{2045}{26} & \frac{1647}{13} & -\frac{585}{4} & \frac{54915}{442} & -\frac{69245}{884} & \frac{16085}{442} & -\frac{10741}{884} & \frac{1223}{442} & -\frac{341}{884} & \frac{11}{442} \\
-\frac{17}{2} & \frac{906}{13} & -\frac{6795}{26} & \frac{7727}{13} & -\frac{3663}{4} & \frac{223110}{221} & -\frac{723415}{884} & \frac{217545}{442} & -\frac{192861}{884} & \frac{15367}{221} & -\frac{13371}{884} & \frac{891}{442} & -\frac{55}{442} \\
\frac{431}{13} & -\frac{6795}{26} & \frac{12192}{13} & -2032 & \frac{1313019}{442} & -\frac{2738175}{884} & \frac{523565}{221} & -\frac{1186461}{884} & \frac{247419}{442} & -\frac{148283}{884} & \frac{7578}{221} & -\frac{3795}{884} & \frac{55}{221} \\
-\frac{2045}{26} & \frac{7727}{13} & -2032 & \frac{920942}{221} & -\frac{5065181}{884} & \frac{2469365}{442} & -\frac{3514371}{884} & \frac{461082}{221} & -\frac{709773}{884} & \frac{97763}{442} & -\frac{36605}{884} & \frac{1045}{221} & -\frac{55}{221} \\
\frac{1647}{13} & -\frac{3663}{4} & \frac{1313019}{442} & -\frac{5065181}{884} & \frac{1622385}{221} & -\frac{2920293}{442} & \frac{950292}{221} & -\frac{452007}{221} & \frac{312363}{442} & -\frac{152895}{884} & \frac{12555}{442} & -\frac{2475}{884} & \frac{55}{442} \\
-\frac{585}{4} & \frac{223110}{221} & -\frac{2738175}{884} & \frac{2469365}{442} & -\frac{2920293}{442} & \frac{1195884}{221} & -\frac{697599}{221} & \frac{292518}{221} & -\frac{349515}{884} & \frac{18024}{221} & -\frac{9621}{884} & \frac{363}{442} & \mathbf{-\frac{11}{442}} \\
\frac{54915}{442} & -\frac{723415}{884} & \frac{523565}{221} & -\frac{3514371}{884} & \frac{950292}{221} & -\frac{697599}{221} & \frac{356328}{221} & -\frac{127260}{221} & \frac{62433}{442} & -\frac{20013}{884} & \frac{469}{221} & \mathbf{-\frac{77}{884}} & 0 \\
-\frac{69245}{884} & \frac{217545}{442} & -\frac{1186461}{884} & \frac{461082}{221} & -\frac{452007}{221} & \frac{292518}{221} & -\frac{127260}{221} & \frac{37044}{221} & -\frac{27783}{884} & \frac{1519}{442} & \mathbf{-\frac{147}{884}} & 0 & 0 \\
\frac{16085}{442} & -\frac{192861}{884} & \frac{247419}{442} & -\frac{709773}{884} & \frac{312363}{442} & -\frac{349515}{884} & \frac{62433}{442} & -\frac{27783}{884} & \frac{882}{221} & \mathbf{-\frac{49}{221}} & 0 & 0 & 0 \\
-\frac{10741}{884} & \frac{15367}{221} & -\frac{148283}{884} & \frac{97763}{442} & -\frac{152895}{884} & \frac{18024}{221} & -\frac{20013}{884} & \frac{1519}{442} & \mathbf{-\frac{49}{221}} & 0 & 0 & 0 & 0 \\
\frac{1223}{442} & -\frac{13371}{884} & \frac{7578}{221} & -\frac{36605}{884} & \frac{12555}{442} & -\frac{9621}{884} & \frac{469}{221} & \mathbf{-\frac{147}{884}} & 0 & 0 & 0 & 0 & 0 \\
-\frac{341}{884} & \frac{891}{442} & -\frac{3795}{884} & \frac{1045}{221} & -\frac{2475}{884} & \frac{363}{442} & \mathbf{-\frac{77}{884}} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{11}{442} & -\frac{55}{442} & \frac{55}{221} & -\frac{55}{221} & \frac{55}{442} & \mathbf{-\frac{11}{442}} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\begin{aligned}
G(z,\xi) = (1-z)^5(1-\xi)^5 \Big[& \frac{11}{442}(1-z)^7 + \frac{77}{884}(1-z)^6(1-\xi) + \frac{147}{884}(1-z)^5(1-\xi)^2 + \frac{49}{221}(1-z)^4(1-\xi)^3 \\
& + \frac{49}{221}(1-z)^3(1-\xi)^4 + \frac{147}{884}(1-z)^2(1-\xi)^5 + \frac{77}{884}(1-z)(1-\xi)^6 + \frac{11}{442}(1-\xi)^7 \Big],
\end{aligned}$$

$$G(z) = (1-z)^{17}.$$

In addition, in this case, note that $\lambda = 0$ is *a root of multiplicity* l of the characteristic polynomial $P(\lambda)$ of the coefficient matrix $(d_{n,m})_{n,m=0,1,\dots,2l+2}$. Using MAPLE, we get approximate values of eigenvalues λ. For instance:

- For $l = 1$, $P(\lambda) = -(1/500)\,\lambda\,[500\,\lambda^4 - 4000\,\lambda^3 - 7100\,\lambda^2 + 1200\,\lambda + 9]$,

$$\lambda = \big\{ -1.62742614692069;\ -0.00719495255354571;\ 0;\ 0.162287966346161;\ 9.47233313312807\big\}.$$

- For $l = 2$, setting $N := 27783$,

$$P(\lambda) = -\frac{1}{N}\,\lambda^2\big[27783\,\lambda^5 - 1481760\,\lambda^4 - 14089950\,\lambda^3 + 13067250\,\lambda^2 + 752500\,\lambda - 1250\big],$$
$$\lambda = \big\{ -8.97492212870802, -0.0559421034347205;\ 0^{(2)};\ 0.00161586776450401;\ 0.902321238068622;\ 61.4602604596429\big\}.$$

- For $l = 3$, setting $N := 12679867332$,

$$\begin{aligned} P(\lambda) = -(1/N)\,\lambda^3\big[&12679867332\,\lambda^6 - 4733817137280\,\lambda^5 - 281944534895100\,\lambda^4 \\ &+ 1651970595429000\,\lambda^3 + 691542051440625\,\lambda^2 \\ &- 13028876437500\,\lambda - 5252187500\big], \end{aligned}$$
$$\lambda = \big\{ -56.9534576360948;\ -0.409444533297716;\ -0.000394851578279199;\ 0^{(3)};\ 0.0184422272384619;\ 5.74034083133930;\ 424.937847295726\big\}.$$

- For $l = 4$, setting $N := 494190983$,

$$\begin{aligned} P(\lambda) = -\,(1/N)\,\lambda^4\big[&494190983\,\lambda^7 - 1328385362304\,\lambda^6 - 532354498059384\,\lambda^5 \\ &+ 21076866188265888\,\lambda^4 + 64421994743778672\,\lambda^3 \\ &- 10888016701161600\,\lambda^2 - 65082380126400\,\lambda + 6613488000\big], \end{aligned}$$
$$\lambda = \big\{ -386.073540317202;\ -3.00228847130902;\ -0.00587683289047773;\ 0^{(4)};\ 0.0000999469835051707;\ 0.166100767964351;\ 38.9534817870098;\ 3037.96202311944\big\}.$$

- For $l = 5$, setting $N := 230787600532736$,

$$\begin{aligned} P(\lambda) = -(1/N)\,\lambda^5\big[&230787600532736\,\lambda^8 - 4549285181701292032\,\lambda^7 \\ &- 12772751297221978270208\,\lambda^6 + 3551620916003369797812224\,\lambda^5 \\ &+ 80085686812398032552698912\,\lambda^4 - 112698292125376973991399424\,\lambda^3 \\ &- 7166982949539552041048608\,\lambda^2 + 1327609514296657368268 8\,\lambda \\ &+ 346792539273426477\big], \end{aligned}$$
$$\lambda = \big\{ -2717.79332022905;\ -22.2130354765117;\ -0.0626880018584103;\ -0.257634008830183 \cdot 10^{-4};\ 0^{(5)};\ 0.00182649779303946;\ 1.38680121956311;\ 274.361067547899;\ 22176.3193742056\big\}.$$

Observation 5.3.4. *For any integer $l \geq 2$, if letting*

$$\kappa = \frac{2}{2l+1} \quad , \quad p = \frac{l(3l+7)+2}{2(2l+1)} \quad , \quad q = \frac{l(3l+9)+2}{2(2l+1)},$$

then, the corresponding coefficient matrix $(d_{n,m})$ is a square matrix of order $(2l+1)$, and we obtain

$$G(z,\xi) = (1-z)^{l+2}(1-\xi)^{l+2} \sum_{m=0}^{l-2} |d_{2l-m,l+2+m}|(1-z)^{l-2-m}(1-\xi)^{m}, \qquad (5.3.5)$$

$$G(z) := \lim_{\xi \to z} G(z,\xi) = \mathbb{E}\left((f'(z))^{2p-q}\right) = (1-z)^{3l+2}.$$

In the specific matrices below, the coefficients $d_{2l-m,l+2+m}$ in (5.3.5) are in bold.

$$(l=2) \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 16 & -24 & 16 & -4 \\ 6 & -24 & 36 & -24 & 6 \\ -4 & 16 & -24 & 16 & -4 \\ 1 & -4 & 6 & -4 & \mathbf{1} \end{bmatrix},$$

$$G(z,\xi) = (1-z)^4(1-\xi)^4, \;\; G(z) = (1-z)^8.$$

$$(l=3) \begin{bmatrix} 1 & -\frac{11}{2} & \frac{25}{2} & -15 & 10 & -\frac{7}{2} & \frac{1}{2} \\ -\frac{11}{2} & 30 & -\frac{135}{2} & 80 & -\frac{105}{2} & 18 & -\frac{5}{2} \\ \frac{25}{2} & -\frac{135}{2} & 150 & -175 & \frac{225}{2} & -\frac{75}{2} & 5 \\ -15 & 80 & -175 & 200 & -125 & 40 & -5 \\ 10 & -\frac{105}{2} & \frac{225}{2} & -125 & 75 & -\frac{45}{2} & \frac{5}{2} \\ -\frac{7}{2} & 18 & -\frac{75}{2} & 40 & -\frac{45}{2} & 6 & \mathbf{-\frac{1}{2}} \\ \frac{1}{2} & -\frac{5}{2} & 5 & -5 & \frac{5}{2} & \mathbf{-\frac{1}{2}} & 0 \end{bmatrix},$$

$$G(z,\xi) = (1-z)^5(1-\xi)^5\left[(1/2)(1-z) + (1/2)(1-\xi)\right], \;\; G(z) = (1-z)^{11}.$$

$$(l=4) \begin{bmatrix} 1 & -7 & \frac{234}{11} & -\frac{403}{11} & \frac{430}{11} & -\frac{291}{11} & \frac{122}{11} & -\frac{29}{11} & \frac{3}{11} \\ -7 & \frac{533}{11} & -\frac{1599}{11} & \frac{2713}{11} & -\frac{2845}{11} & \frac{1887}{11} & -\frac{773}{11} & \frac{179}{11} & -\frac{18}{11} \\ \frac{234}{11} & -\frac{1599}{11} & \frac{4725}{11} & -\frac{7875}{11} & 735 & -\frac{5229}{11} & 189 & -\frac{465}{11} & \frac{45}{11} \\ -\frac{403}{11} & \frac{2713}{11} & -\frac{7875}{11} & \frac{12845}{11} & -\frac{12845}{11} & \frac{8043}{11} & -\frac{3073}{11} & \frac{655}{11} & -\frac{60}{11} \\ \frac{430}{11} & -\frac{2845}{11} & 735 & -\frac{12845}{11} & \frac{12425}{11} & -\frac{7455}{11} & 245 & -\frac{535}{11} & \frac{45}{11} \\ -\frac{291}{11} & \frac{1887}{11} & -\frac{5229}{11} & \frac{8043}{11} & -\frac{7455}{11} & \frac{4221}{11} & -\frac{1407}{11} & \frac{249}{11} & -\frac{18}{11} \\ \frac{122}{11} & -\frac{773}{11} & 189 & -\frac{3073}{11} & 245 & -\frac{1407}{11} & \frac{413}{11} & -\frac{59}{11} & \mathbf{\frac{3}{11}} \\ -\frac{29}{11} & \frac{179}{11} & -\frac{465}{11} & \frac{655}{11} & -\frac{535}{11} & \frac{249}{11} & -\frac{59}{11} & \mathbf{\frac{5}{11}} & 0 \\ \frac{3}{11} & -\frac{18}{11} & \frac{45}{11} & -\frac{60}{11} & \frac{45}{11} & -\frac{18}{11} & \mathbf{\frac{3}{11}} & 0 & 0 \end{bmatrix},$$

$$G(z,\xi) = (1-z)^6(1-\xi)^6\left[(3/11)(1-z)^2 + (5/11)(1-z)(1-\xi) + (3/11)(1-\xi)^2\right], \; G(z) = (1-z)^{14}.$$

$$(l=5)$$

$$\begin{bmatrix}
1 & -\frac{17}{2} & \frac{420}{13} & -\frac{940}{13} & \frac{1372}{13} & -105 & \frac{938}{13} & -\frac{440}{13} & \frac{135}{13} & -\frac{49}{26} & \frac{2}{13} \\
-\frac{17}{2} & \frac{928}{13} & -\frac{3480}{13} & \frac{7672}{13} & -847 & \frac{10752}{13} & -\frac{7238}{13} & \frac{3320}{13} & -\frac{153}{2} & \frac{176}{13} & -\frac{14}{13} \\
\frac{420}{13} & -\frac{3480}{13} & \frac{12852}{13} & -2142 & \frac{39186}{13} & -\frac{37422}{13} & 1890 & -\frac{10962}{13} & \frac{3186}{13} & -42 & \frac{42}{13} \\
-\frac{940}{13} & \frac{7672}{13} & -2142 & \frac{59136}{13} & -\frac{81312}{13} & \frac{75600}{13} & -\frac{48132}{13} & \frac{20736}{13} & -\frac{5796}{13} & \frac{952}{13} & -\frac{70}{13} \\
\frac{1372}{13} & -847 & \frac{39186}{13} & -\frac{81312}{13} & \frac{108780}{13} & -\frac{97902}{13} & \frac{59976}{13} & -\frac{24696}{13} & 504 & -\frac{1015}{13} & \frac{70}{13} \\
-105 & \frac{10752}{13} & -\frac{37422}{13} & \frac{75600}{13} & -\frac{97902}{13} & \frac{84672}{13} & -\frac{49392}{13} & \frac{19152}{13} & -\frac{4725}{13} & \frac{672}{13} & -\frac{42}{13} \\
\frac{938}{13} & -\frac{7238}{13} & 1890 & -\frac{48132}{13} & \frac{59976}{13} & -\frac{49392}{13} & \frac{27048}{13} & -\frac{9660}{13} & \frac{2142}{13} & -\frac{266}{13} & \frac{14}{13} \\
-\frac{440}{13} & \frac{3320}{13} & -\frac{10962}{13} & \frac{20736}{13} & -\frac{24696}{13} & \frac{19152}{13} & -\frac{9660}{13} & \frac{3072}{13} & -\frac{576}{13} & \frac{56}{13} & \mathbf{-\frac{2}{13}} \\
\frac{135}{13} & -\frac{153}{2} & \frac{3186}{13} & -\frac{5796}{13} & 504 & -\frac{4725}{13} & \frac{2142}{13} & -\frac{576}{13} & \frac{81}{13} & \mathbf{-\frac{9}{26}} & 0 \\
-\frac{49}{26} & \frac{176}{13} & -42 & \frac{952}{1} & -\frac{1015}{13} & \frac{672}{13} & -\frac{266}{13} & \frac{56}{13} & \mathbf{-\frac{9}{26}} & 0 & 0 \\
\frac{2}{13} & -\frac{14}{13} & \frac{42}{13} & -\frac{70}{13} & \frac{70}{13} & -\frac{42}{13} & \frac{14}{13} & \mathbf{-\frac{2}{13}} & 0 & 0 & 0
\end{bmatrix},$$

$$G(z,\xi) = (1-z)^7(1-\xi)^7\left[\frac{2}{13}(1-z)^3 + \frac{9}{26}(1-z)^2(1-\xi) + \frac{9}{26}(1-z)(1-\xi)^2 + \frac{2}{13}(1-\xi)^3\right],$$

$$G(z) = (1-z)^{17}.$$

$$(l=6)$$

$$\begin{bmatrix}
1 & -10 & \frac{228}{5} & -\frac{627}{5} & \frac{39387}{170} & -\frac{25764}{85} & \frac{24486}{85} & -\frac{17046}{85} & \frac{8634}{85} & -\frac{3106}{85} & \frac{754}{85} & -\frac{111}{85} & \frac{3}{34} \\
-10 & \frac{494}{5} & -\frac{2223}{5} & \frac{102429}{85} & -\frac{186276}{85} & \frac{239544}{85} & -\frac{13146}{5} & \frac{152526}{85} & -\frac{75654}{85} & \frac{26626}{85} & -\frac{6319}{85} & \frac{909}{85} & -\frac{12}{17} \\
\frac{228}{5} & -\frac{2223}{5} & \frac{33516}{17} & -\frac{89376}{17} & \frac{159642}{17} & -\frac{1006362}{85} & \frac{918792}{85} & -\frac{122508}{17} & \frac{59256}{17} & -\frac{20307}{17} & \frac{23436}{85} & -\frac{3276}{85} & \frac{42}{17} \\
-\frac{627}{5} & \frac{102429}{85} & -\frac{89376}{17} & \frac{234024}{17} & -\frac{409542}{17} & \frac{2523318}{85} & -\frac{2245908}{85} & \frac{291180}{17} & -\frac{136599}{17} & \frac{45297}{17} & -\frac{50484}{85} & \frac{6804}{85} & -\frac{84}{17} \\
\frac{39387}{170} & -\frac{186276}{85} & \frac{159642}{17} & -\frac{409542}{17} & \frac{700182}{17} & -\frac{4201092}{85} & \frac{3628422}{85} & -\frac{454752}{17} & \frac{410859}{34} & -\frac{65352}{17} & \frac{69636}{85} & -\frac{8946}{85} & \frac{105}{17} \\
-\frac{25764}{85} & \frac{239544}{85} & -\frac{1006362}{85} & \frac{2523318}{85} & -\frac{4201092}{85} & \frac{4888296}{85} & -\frac{814716}{17} & \frac{2449332}{85} & -\frac{1055268}{85} & \frac{318192}{85} & -\frac{63882}{85} & \frac{7686}{85} & -\frac{84}{17} \\
\frac{24486}{85} & -\frac{13146}{5} & \frac{918792}{85} & -\frac{2245908}{85} & \frac{3628422}{85} & -\frac{814716}{17} & \frac{3252816}{85} & -\frac{1858752}{85} & \frac{753858}{85} & -\frac{211722}{85} & \frac{39144}{85} & -\frac{252}{5} & \frac{42}{17} \\
-\frac{17046}{85} & \frac{152526}{85} & -\frac{122508}{17} & \frac{291180}{17} & -\frac{454752}{17} & \frac{2449332}{85} & -\frac{1858752}{85} & \frac{199440}{17} & -\frac{74790}{17} & \frac{19062}{17} & -\frac{15636}{85} & \frac{1476}{85} & -\frac{12}{17} \\
\frac{8634}{85} & -\frac{75654}{85} & \frac{59256}{17} & -\frac{136599}{17} & \frac{410859}{34} & -\frac{1055268}{85} & \frac{753858}{85} & -\frac{74790}{17} & \frac{25299}{17} & -\frac{5622}{17} & \frac{3834}{85} & -\frac{279}{85} & \mathbf{\frac{3}{34}} \\
-\frac{3106}{85} & \frac{26626}{85} & -\frac{20307}{17} & \frac{45297}{17} & -\frac{65352}{17} & \frac{318192}{85} & -\frac{211722}{85} & \frac{19062}{17} & -\frac{5622}{17} & \frac{1022}{17} & -\frac{511}{85} & \mathbf{\frac{21}{85}} & 0 \\
\frac{754}{85} & -\frac{6319}{85} & \frac{23436}{85} & -\frac{50484}{85} & \frac{69636}{85} & -\frac{63882}{85} & \frac{39144}{85} & -\frac{15636}{85} & \frac{3834}{85} & -\frac{511}{85} & \mathbf{\frac{28}{85}} & 0 & 0 \\
-\frac{111}{85} & \frac{909}{85} & -\frac{3276}{85} & \frac{6804}{85} & -\frac{8946}{85} & \frac{7686}{85} & -\frac{252}{5} & \frac{1476}{85} & -\frac{279}{85} & \mathbf{\frac{21}{85}} & 0 & 0 & 0 \\
\frac{3}{34} & -\frac{12}{17} & \frac{42}{17} & -\frac{84}{17} & \frac{105}{17} & -\frac{84}{17} & \frac{42}{17} & -\frac{12}{17} & \mathbf{\frac{3}{34}} & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$G(z,\xi) = (1-z)^8(1-\xi)^8\left[\frac{3}{34}(1-z)^4 + \frac{21}{85}(1-z)^3(1-\xi) + \frac{28}{85}(1-z)^2(1-\xi)^2 + \frac{21}{85}(1-z)(1-\xi)^3 + \frac{3}{34}(1-\xi)^4\right],$$

$$G(z) = (1-z)^{20}.$$

In addition, in this case, note that $\lambda = 0$ is *a root of multiplicity* $l+2$ of the characteristic polynomial $P(\lambda)$ of the coefficient matrix $(d_{n,m})_{n,m=0,1,\dots,2l}$. Using MAPLE, we obtain approximate values of eigenvalues λ. For instance:

- For $l = 2$,

$$P(\lambda) = -\lambda^4\,[\lambda - 70]\,,\ \lambda = \big\{0^{(4)};\,70\big\}.$$

- For $l = 3$,

$$P(\lambda) = -\lambda^5\,\big[\lambda^2 - 462\,\lambda - 4851\big]\,,\ \lambda = \big\{-10.27163115;\ 0^{(5)};\ 472.2716312\big\}.$$

- For $l = 4$,

$$\begin{aligned} P(\lambda) &= -\lambda^6\,\big[\lambda^3 - 3198\,\lambda^2 - 396279\,\lambda + 821340\big]\,,\\ &\quad \lambda = \big\{-121.419850584612;\ 0^{(6)};\ 2.03909734478645;\ 3317.38075323983\big\}. \end{aligned}$$

- For $l = 5$,

$$\begin{aligned} P(\lambda) &= -\lambda^7\,\big[\lambda^4 - 22814\,\lambda^3 - 26142413\,\lambda^2 + 874035756\,\lambda + 403213140\big]\\ &\quad \lambda = \big\{-1124.52007490399;\ -0.455130190587252;\ 0^{(7)};\ 32.9552447197475;\\ &\qquad\qquad 23906.0199603748\big\}. \end{aligned}$$

- For $l = 6$,

$$\begin{aligned} P(\lambda) &= -\frac{1}{2125}\,\lambda^8\big[2125\,\lambda^5 - 352925950\,\lambda^4 - 3442903220445\,\lambda^3\\ &\qquad + 1297186309131804\,\lambda^2 + 12023236369144728\,\lambda\\ &\qquad - 1310828169819168\big],\\ &\quad \lambda = \big\{-9585.20521130524;\ -9.15673328716147;\ 0^{(8)};\\ &\qquad 0.107771811816883;\ 372.001966142034;\ 1.75305052206639\cdot 10^5\big\}. \end{aligned}$$

- For $l = 7$,

$$\begin{aligned} P(\lambda) &= -\frac{1}{19}\,\lambda^9\big[19\,\lambda^6 - 23305210\,\lambda^5 - 1865382914425\,\lambda^4\\ &\qquad + 6862029786001500\,\lambda^3 + 837937538085037500\,\lambda^2\\ &\qquad - 2177250042747780000\,\lambda - 580819040802000000\big],\\ \lambda &= \big\{-78723.3474876778;\ -120.779114558606;\ -0.0264083818971425;\\ &\qquad 0^{(9)};\ 2.57120092294019;\ 3637.23752000905;\ 1.30179434428969\cdot 10^6\big\}. \end{aligned}$$

Observation 5.3.5. *For any nonnegative integer l, if letting*

$$\kappa = \frac{2}{2l+1} \ , \ p=1 \ , \ q = \frac{2-3l(l-1)}{2(2l+1)},$$

then, the corresponding coefficient matrix $(d_{n,m})$ is a square matrix of order $(3l+2)$, and we obtain

$$G(z,\xi) = (1-z)(1-\xi)\sum_{m=0}^{3l} |d_{3l+1-m,m+1}|(1-z)^{3l-m}(1-\xi)^m, \tag{5.3.6}$$

$$G(z) := \lim_{\xi\to z} G(z,\xi) = \mathbb{E}\left((f'(z))^{2p-q}\right) = (1-z)^{3l+2}.$$

In the specific matrices below, the coefficients $d_{3l+1-m,m+1}$ in (5.3.6) are in bold.

$$(l=0) \quad \begin{bmatrix} 1 & -1 \\ -1 & \mathbf{1} \end{bmatrix},$$

$$G(z,\xi) = (1-z)(1-\xi), \ \ G(z) = (1-z)^2.$$

$$(l=1) \quad \begin{bmatrix} 1 & -\frac{5}{2} & \frac{12}{5} & -\frac{11}{10} & \frac{1}{5} \\ -\frac{5}{2} & \frac{26}{5} & -\frac{39}{10} & \frac{7}{5} & \mathbf{-\frac{1}{5}} \\ \frac{12}{5} & -\frac{39}{10} & \frac{9}{5} & \mathbf{-\frac{3}{10}} & 0 \\ -\frac{11}{10} & \frac{7}{5} & \mathbf{-\frac{3}{10}} & 0 & 0 \\ \frac{1}{5} & \mathbf{-\frac{1}{5}} & 0 & 0 & 0 \end{bmatrix},$$

$$G(z,\xi) = (1-z)(1-\xi)\left[\frac{1}{5}(1-z)^3 + \frac{3}{10}(1-z)^2(1-\xi) + \frac{3}{10}(1-z)(1-\xi)^2 + \frac{1}{5}(1-\xi)^3\right],$$
$$G(z) = (1-z)^5.$$

$$(l=2) \quad \begin{bmatrix} 1 & -4 & \frac{51}{7} & -\frac{55}{7} & \frac{75}{14} & -\frac{16}{7} & \frac{37}{66} & -\frac{2}{33} \\ -4 & \frac{94}{7} & -\frac{141}{7} & \frac{125}{7} & -10 & \frac{269}{77} & -\frac{23}{33} & \mathbf{\frac{2}{33}} \\ \frac{51}{7} & -\frac{141}{7} & \frac{165}{7} & -\frac{110}{7} & \frac{965}{154} & -\frac{108}{77} & \mathbf{\frac{3}{22}} & 0 \\ -\frac{55}{7} & \frac{125}{7} & -\frac{110}{7} & \frac{1700}{231} & -\frac{425}{231} & \mathbf{\frac{15}{77}} & 0 & 0 \\ \frac{75}{14} & -10 & \frac{965}{154} & -\frac{425}{231} & \mathbf{\frac{50}{231}} & 0 & 0 & 0 \\ -\frac{16}{7} & \frac{269}{77} & -\frac{108}{77} & \mathbf{\frac{15}{77}} & 0 & 0 & 0 & 0 \\ \frac{37}{66} & -\frac{23}{33} & \mathbf{\frac{3}{22}} & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{33} & \mathbf{\frac{2}{33}} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$
\begin{aligned}
G(z,\xi) = (1-z)(1-\xi)\Big[&\frac{2}{33}(1-z)^6 + \frac{3}{22}(1-z)^5(1-\xi) + \frac{15}{77}(1-z)^4(1-\xi)^2 \\
&+ \frac{50}{231}(1-z)^3(1-\xi)^3 + \frac{15}{77}(1-z)^2(1-\xi)^4 \\
&+ \frac{3}{22}(1-z)(1-\xi)^5 + \frac{2}{33}(1-\xi)^6\Big], \\
G(z) = (1-z)^8.&
\end{aligned}
$$

$$(l=3)$$

$$
\begin{bmatrix}
1 & -\frac{11}{2} & \frac{29}{2} & -24 & \frac{301}{11} & -\frac{245}{11} & \frac{1862}{143} & -\frac{768}{143} & \frac{213}{143} & -\frac{1}{4} & \frac{1}{52} \\
-\frac{11}{2} & 26 & -\frac{117}{2} & \frac{908}{11} & -\frac{882}{11} & \frac{7938}{143} & -\frac{3934}{143} & \frac{1368}{143} & -\frac{1269}{572} & \frac{4}{13} & -\mathbf{\frac{1}{52}} \\
\frac{29}{2} & -\frac{117}{2} & \frac{1212}{11} & -\frac{1414}{11} & \frac{1323}{13} & -\frac{8043}{143} & \frac{280}{13} & -\frac{783}{143} & \frac{477}{572} & -\mathbf{\frac{3}{52}} & 0 \\
-24 & \frac{908}{11} & -\frac{1414}{11} & \frac{17360}{143} & -\frac{10850}{143} & \frac{4564}{143} & -\frac{1253}{143} & \frac{204}{143} & -\mathbf{\frac{15}{143}} & 0 & 0 \\
\frac{301}{11} & -\frac{882}{11} & \frac{1323}{13} & -\frac{10850}{143} & \frac{5145}{143} & -\frac{3087}{286} & \frac{49}{26} & -\mathbf{\frac{21}{143}} & 0 & 0 & 0 \\
-\frac{245}{11} & \frac{7938}{143} & -\frac{8043}{143} & \frac{4564}{143} & -\frac{3087}{286} & \frac{294}{143} & -\mathbf{\frac{49}{286}} & 0 & 0 & 0 & 0 \\
\frac{1862}{143} & -\frac{3934}{143} & \frac{280}{13} & -\frac{1253}{143} & \frac{49}{26} & -\mathbf{\frac{49}{286}} & 0 & 0 & 0 & 0 & 0 \\
-\frac{768}{143} & \frac{1368}{143} & -\frac{783}{143} & \frac{204}{143} & -\mathbf{\frac{21}{143}} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{213}{143} & -\frac{1269}{572} & \frac{477}{572} & -\mathbf{\frac{15}{143}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & \frac{4}{13} & -\mathbf{\frac{3}{52}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{52} & -\mathbf{\frac{1}{52}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
\begin{aligned}
G(z,\xi) = (1-z)(1-\xi)\Big[&\frac{1}{52}(1-z)^9 + \frac{3}{52}(1-z)^8(1-\xi) + \frac{15}{143}(1-z)^7(1-\xi)^2 \\
&+ \frac{21}{143}(1-z)^6(1-\xi)^3 + \frac{49}{286}(1-z)^5(1-\xi)^4 \\
&+ \frac{49}{286}(1-z)^4(1-\xi)^5 + \frac{21}{143}(1-z)^3(1-\xi)^6 \\
&+ \frac{15}{143}(1-z)^2(1-\xi)^7 + \frac{3}{52}(1-z)(1-\xi)^8 + \frac{1}{52}(1-\xi)^9\Big], \\
G(z) = (1-z)^{11}.&
\end{aligned}
$$

$$(l=4)$$

$$\begin{bmatrix}
1 & -7 & 24 & -53 & \frac{1085}{13} & -\frac{1278}{13} & \frac{1152}{13} & -\frac{801}{13} & \frac{14553}{442} & -\frac{5885}{442} & \frac{1276}{323} & -\frac{525}{646} & \frac{67}{646} & -\frac{2}{323} \\
-7 & 43 & -129 & \frac{3239}{13} & -\frac{4460}{13} & \frac{4590}{13} & -\frac{3609}{13} & \frac{37107}{221} & -\frac{34353}{442} & \frac{227315}{8398} & -\frac{4433}{646} & \frac{771}{646} & -\frac{41}{323} & \mathbf{\frac{2}{323}} \\
24 & -129 & \frac{4365}{13} & -\frac{7275}{13} & \frac{8595}{13} & -\frac{7533}{13} & \frac{84861}{221} & -\frac{42615}{221} & \frac{304425}{4199} & -\frac{166485}{8398} & \frac{2409}{646} & -\frac{279}{646} & \mathbf{\frac{15}{646}} & 0 \\
-53 & \frac{3239}{13} & -\frac{7275}{13} & \frac{10365}{13} & -\frac{10365}{13} & \frac{129087}{221} & -\frac{70287}{221} & \frac{537675}{4199} & -\frac{312345}{8398} & \frac{62095}{8398} & -\frac{583}{646} & \mathbf{\frac{33}{646}} & 0 & 0 \\
\frac{1085}{13} & -\frac{4460}{13} & \frac{8595}{13} & -\frac{10365}{13} & \frac{147000}{221} & -\frac{88200}{221} & \frac{728385}{4199} & -\frac{225330}{4199} & \frac{94545}{8398} & -\frac{6050}{4199} & \mathbf{\frac{55}{646}} & 0 & 0 & 0 \\
-\frac{1278}{13} & \frac{4590}{13} & -\frac{7533}{13} & \frac{129087}{221} & -\frac{88200}{221} & \frac{801738}{4199} & -\frac{267246}{4199} & \frac{59562}{4199} & -\frac{8019}{4199} & \mathbf{\frac{495}{4199}} & 0 & 0 & 0 & 0 \\
\frac{1152}{13} & -\frac{3609}{13} & \frac{84861}{221} & -\frac{70287}{221} & \frac{728385}{4199} & -\frac{267246}{4199} & \frac{64134}{4199} & -\frac{9162}{4199} & \mathbf{\frac{594}{4199}} & 0 & 0 & 0 & 0 & 0 \\
-\frac{801}{13} & \frac{37107}{221} & -\frac{42615}{221} & \frac{537675}{4199} & -\frac{225330}{4199} & \frac{59562}{4199} & -\frac{9162}{4199} & \mathbf{\frac{630}{4199}} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{14553}{442} & -\frac{34353}{442} & \frac{304425}{4199} & -\frac{312345}{8398} & \frac{94545}{8398} & -\frac{8019}{4199} & \mathbf{\frac{594}{4199}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{5885}{442} & \frac{227315}{8398} & -\frac{166485}{8398} & \frac{62095}{8398} & -\frac{6050}{4199} & \mathbf{\frac{495}{4199}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1276}{323} & -\frac{4433}{646} & \frac{2409}{646} & -\frac{583}{646} & \mathbf{\frac{55}{646}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{525}{646} & \frac{771}{646} & -\frac{279}{646} & \mathbf{\frac{33}{646}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{67}{646} & -\frac{41}{323} & \mathbf{\frac{15}{646}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{323} & \mathbf{\frac{2}{323}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\begin{aligned}
G(z,\xi) = (1-z)(1-\xi)\Big[&\frac{2}{323}(1-z)^{12} + \frac{15}{646}(1-z)^{11}(1-\xi) + \frac{33}{646}(1-z)^{10}(1-\xi)^2 + \frac{55}{646}(1-z)^9(1-\xi)^3 + \frac{495}{4199}(1-z)^8(1-\xi)^4 \\
&+ \frac{594}{4199}(1-z)^7(1-\xi)^5 + \frac{630}{4199}(1-z)^6(1-\xi)^6 + \frac{594}{4199}(1-z)^5(1-\xi)^7 + \frac{495}{4199}(1-z)^4(1-\xi)^8 \\
&+ \frac{55}{646}(1-z)^3(1-\xi)^9 + \frac{33}{646}(1-z)^2(1-\xi)^{10} + \frac{15}{646}(1-z)(1-\xi)^{11} + \frac{2}{323}(1-\xi)^{12}\Big],
\end{aligned}$$

$$G(z) = (1-z)^{14}.$$

In addition, in this case, note that $\lambda = 0$ is *a simple root* of the characteristic polynomial $P(\lambda)$ of the coefficient matrix $(d_{n,m})_{n,m=0,1,\ldots,3l+1}$. Using MAPLE, we obtain approximate values of eigenvalues λ. For instance:

- For $l = 0$, $P(\lambda) = \lambda\,[\lambda - 2]\,,\ \lambda = \big\{0;\, 2\big\}$.
- For $l = 1$,

$$\begin{aligned} P(\lambda) &= -\frac{1}{500}\,\lambda\,\big[500\,\lambda^4 - 4000\,\lambda^3 - 7100\,\lambda^2 + 1200\,\lambda + 9\big]\,,\\ \lambda &= \big\{ -1.62742614692069;\ -0.00719495255354571;\ 0;\\ &\qquad 0.162287966346161;\ 9.47233313312807\big\}. \end{aligned}$$

- For $l = 2$, setting $N := 180470690631$,

$$\begin{aligned} P(\lambda) = \frac{1}{N}\,\lambda\,\big[&180470690631\,\lambda^7 - 8225088445728\,\lambda^6 - 117481897771854\,\lambda^5\\ &+ 294819642392506\,\lambda^4 + 109274360821164\,\lambda^3\\ &- 4207128178770\,\lambda^2 - 10565299800\,\lambda + 810000\big],\\ \lambda = \big\{ &-13.1431759904417;\ -0.359768102086899;\ -0.00243704073314005;\\ &0;\ 0.0000744624443253249;\ 0.0373708187242558;\\ &2.47423323719932;\ 56.5694601906514\big\}. \end{aligned}$$

- For $l = 3$, setting $N := 227351019220718581 76$,

$$\begin{aligned} P(\lambda) = -\frac{1}{N}\,\lambda\,\big[&22735101922071858176\,\lambda^{10} - 674358100088531424 0512\,\lambda^9\\ &- 70050655683905556897 0368\,\lambda^8 + 1518478461563118921819 7824\,\lambda^7\\ &+ 670127205520885458226366 20\,\lambda^6 - 50278194926692522633942956\,\lambda^5\\ &- 52613426646631599337640 61\,\lambda^4 + 59251052720785923546822\,\lambda^3\\ &+ 51432297880700998989\,\lambda^2 - 2180469020290470\,\lambda - 2144153025\big],\\ \lambda = \big\{ &-95.55894001;\ -4.330622277;\ -0.1024039702;\ -0.0008480418200;\\ &-9.615598837\cdot 10^{-7};\ 0;\ 0.4143096726\cdot 10^{-4};\ 0.01100586548;\\ &0.7365285907;\ 21.71964277;\ 374.1409812\big\}. \end{aligned}$$

Observation 5.3.6. *For any nonnegative integer l, if letting*

$$\kappa = \frac{6}{6l+1}\quad,\quad p = 0\quad,\quad q = -\frac{2 + 9l(l+1)}{2(6l+1)},$$

then, the corresponding coefficient matrix $(d_{n,m})$ is a square matrix of order $(3l+2)$, and we obtain

$$G(z,\xi) = \sum_{m=0}^{3l+1} |d_{3l+1-m,m}|(1-z)^{3l+1-m}(1-\xi)^m, \tag{5.3.7}$$

$$G(z) := \lim_{\xi\to z} G(z,\xi) = \mathbb{E}\left((f'(z))^{2p-q}\right) = (1-z)^{3l+1}.$$

In the specific matrices below, the coefficients $d_{3l+1-m,m}$ in (5.3.7) are in bold.

$$(l=0)\quad \begin{bmatrix} 1 & \mathbf{-\frac{1}{2}} \\ \mathbf{-\frac{1}{2}} & 0 \end{bmatrix},\quad G(z,\xi)=\frac{1}{2}\big((1-z)+(1-\xi)\big),\quad G(z)=1-z.$$

$$(l=1)\quad \begin{bmatrix} 1 & -2 & \frac{24}{13} & -\frac{11}{13} & \mathbf{\frac{77}{494}} \\ -2 & \frac{30}{13} & -\frac{15}{13} & \mathbf{\frac{55}{247}} & 0 \\ \frac{24}{13} & -\frac{15}{13} & \mathbf{\frac{60}{247}} & 0 & 0 \\ -\frac{11}{13} & \mathbf{\frac{55}{247}} & 0 & 0 & 0 \\ \mathbf{\frac{77}{494}} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} G(z,\xi) &= \frac{77}{494}(1-z)^4+\frac{55}{247}(1-z)^3(1-\xi)+\frac{60}{247}(1-z)^2(1-\xi)^2 \\ &\quad +\frac{55}{247}(1-z)(1-\xi)^3+\frac{77}{494}(1-\xi)^4, \\ G(z) &= (1-z)^4. \end{aligned}$$

$$(l=2)\quad \begin{bmatrix} 1 & -\frac{7}{2} & \frac{231}{38} & -\frac{245}{38} & \frac{833}{190} & -\frac{357}{190} & \frac{2737}{5890} & \mathbf{-\frac{299}{5890}} \\ -\frac{7}{2} & \frac{168}{19} & -\frac{210}{19} & \frac{784}{95} & -\frac{357}{95} & \frac{2856}{2945} & \mathbf{-\frac{322}{2945}} & 0 \\ \frac{231}{38} & -\frac{210}{19} & \frac{924}{95} & -\frac{462}{95} & \frac{3927}{2945} & \mathbf{-\frac{462}{2945}} & 0 & 0 \\ -\frac{245}{38} & \frac{784}{95} & -\frac{462}{95} & \frac{4312}{2945} & \mathbf{-\frac{539}{2945}} & 0 & 0 & 0 \\ \frac{833}{190} & -\frac{357}{95} & \frac{3927}{2945} & \mathbf{-\frac{539}{2945}} & 0 & 0 & 0 & 0 \\ -\frac{357}{190} & \frac{2856}{2945} & \mathbf{-\frac{462}{2945}} & 0 & 0 & 0 & 0 & 0 \\ \frac{2737}{5890} & \mathbf{-\frac{322}{2945}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{-\frac{299}{5890}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} G(z,\xi) &= \frac{299}{5890}(1-z)^7+\frac{322}{2945}(1-z)^6(1-\xi)+\frac{462}{2945}(1-z)^5(1-\xi)^2 \\ &\quad +\frac{539}{2945}(1-z)^4(1-\xi)^3+\frac{539}{2945}(1-z)^3(1-\xi)^4+\frac{462}{2945}(1-z)^2(1-\xi)^5 \\ &\quad +\frac{322}{2945}(1-z)(1-\xi)^6+\frac{299}{5890}(1-\xi)^7, \\ G(z) &= (1-z)^7. \end{aligned}$$

$$(l=3)$$

$$\begin{bmatrix}
1 & -5 & \frac{63}{5} & -\frac{102}{5} & \frac{714}{31} & -\frac{2898}{155} & \frac{12558}{1147} & -\frac{26013}{5735} & \frac{312156}{246605} & -\frac{10556}{49321} & \mathbf{\frac{28652}{1726235}} \\
-5 & \frac{99}{5} & -\frac{198}{5} & \frac{7854}{155} & -\frac{1386}{31} & \frac{31878}{1147} & -\frac{69069}{5735} & \frac{858429}{246605} & -\frac{149292}{246605} & \mathbf{\frac{16588}{345247}} & 0 \\
\frac{63}{5} & -\frac{198}{5} & \frac{9702}{155} & -\frac{9702}{155} & \frac{48510}{1147} & -\frac{111573}{5735} & \frac{1450449}{246605} & -\frac{261261}{246605} & \mathbf{\frac{149292}{1726235}} & 0 & 0 \\
-\frac{102}{5} & \frac{7854}{155} & -\frac{9702}{155} & \frac{54978}{1147} & -\frac{27489}{1147} & \frac{1896741}{246605} & -\frac{357357}{246605} & \mathbf{\frac{211497}{1726235}} & 0 & 0 & 0 \\
\frac{714}{31} & -\frac{1386}{31} & \frac{48510}{1147} & -\frac{27489}{1147} & \frac{412335}{49321} & -\frac{82467}{49321} & \mathbf{\frac{7293}{49321}} & 0 & 0 & 0 & 0 \\
-\frac{2898}{155} & \frac{31878}{1147} & -\frac{111573}{5735} & \frac{1896741}{246605} & -\frac{82467}{49321} & \mathbf{\frac{38709}{246605}} & 0 & 0 & 0 & 0 & 0 \\
\frac{12558}{1147} & -\frac{69069}{5735} & \frac{1450449}{246605} & -\frac{357357}{246605} & \mathbf{\frac{7293}{49321}} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{26013}{5735} & \frac{858429}{246605} & -\frac{261261}{246605} & \mathbf{\frac{211497}{1726235}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{312156}{246605} & -\frac{149292}{246605} & \mathbf{\frac{149292}{1726235}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{10556}{49321} & \mathbf{\frac{16588}{345247}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{\frac{28652}{1726235}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\begin{aligned}
G(z,\xi) = {} & \frac{28652}{1726235}(1-z)^{10} + \frac{16588}{345247}(1-z)^{9}(1-\xi) + \frac{149292}{1726235}(1-z)^{8}(1-\xi)^2 + \frac{211479}{1726235}(1-z)^{7}(1-\xi)^3 \\
& + \frac{7293}{49321}(1-z)^{6}(1-\xi)^4 + \frac{38709}{246605}(1-z)^{5}(1-\xi)^5 + \frac{7293}{49321}(1-z)^{4}(1-\xi)^6 + \frac{211479}{1726235}(1-z)^{3}(1-\xi)^7 \\
& + \frac{149292}{1726235}(1-z)^{2}(1-\xi)^8 + \frac{16588}{345247}(1-z)(1-\xi)^9 + \frac{28652}{1726235}(1-\xi)^{10}, \\
G(z) = {} & (1-z)^{10}.
\end{aligned}$$

$$(l=4)$$

$$\begin{bmatrix}
1 & -\frac{13}{2} & \frac{663}{31} & -\frac{1430}{31} & \frac{82225}{1147} & -\frac{384813}{4588} & \frac{3719859}{49321} & -\frac{2587728}{49321} & \frac{9703980}{345247} & -\frac{3939650}{345247} & \frac{1174732}{345247} & -\frac{243048}{345247} & \frac{1903876}{21060067} & -\mathbf{\frac{915325}{168480536}} \\
-\frac{13}{2} & \frac{1092}{31} & -\frac{3003}{31} & \frac{200200}{1147} & -\frac{1036035}{4588} & \frac{10774764}{49321} & -\frac{7924917}{49321} & \frac{31052736}{345247} & -\frac{1866150}{49321} & \frac{573040}{49321} & -\frac{121524}{49321} & \frac{972192}{3008581} & -\mathbf{\frac{475969}{24068648}} & 0 \\
\frac{663}{31} & -\frac{3003}{31} & \frac{255255}{1147} & -\frac{765765}{2294} & \frac{17612595}{49321} & -\frac{13936923}{49321} & \frac{8248383}{49321} & -\frac{25379640}{345247} & \frac{1153620}{49321} & -\frac{251940}{49321} & \frac{2065908}{3008581} & -\mathbf{\frac{516477}{12034324}} & 0 & 0 \\
-\frac{1430}{31} & \frac{200200}{1147} & -\frac{765765}{2294} & \frac{20420400}{49321} & -\frac{17867850}{49321} & \frac{11377080}{49321} & -\frac{5287425}{49321} & \frac{12305280}{345247} & -\frac{397800}{49321} & \frac{3359200}{3008581} & -\mathbf{\frac{860795}{12034324}} & 0 & 0 & 0 \\
\frac{82225}{1147} & -\frac{1036035}{4588} & \frac{17612595}{49321} & -\frac{17867850}{49321} & \frac{12580425}{49321} & -\frac{12580425}{98642} & \frac{2211105}{49321} & -\frac{3659760}{345247} & \frac{4574700}{3008581} & -\mathbf{\frac{2414425}{24068648}} & 0 & 0 & 0 & 0 \\
-\frac{384813}{4588} & \frac{10774764}{49321} & -\frac{13936923}{49321} & \frac{11377080}{49321} & -\frac{12580425}{98642} & \frac{2378844}{49321} & -\frac{594711}{49321} & \frac{38061504}{21060067} & -\mathbf{\frac{2973555}{24068648}} & 0 & 0 & 0 & 0 & 0 \\
\frac{3719859}{49321} & -\frac{7924917}{49321} & \frac{8248383}{49321} & -\frac{5287425}{49321} & \frac{2211105}{49321} & -\frac{594711}{49321} & \frac{5748873}{3008581} & -\mathbf{\frac{5748873}{42120134}} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2587728}{49321} & \frac{31052736}{345247} & -\frac{25379640}{345247} & \frac{12305280}{345247} & -\frac{3659760}{345247} & \frac{38061504}{21060067} & -\mathbf{\frac{5748873}{42120134}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{9703980}{345247} & -\frac{1866150}{49321} & \frac{1153620}{49321} & -\frac{397800}{49321} & \frac{4574700}{3008581} & -\mathbf{\frac{2973555}{24068648}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3939650}{345247} & \frac{573040}{49321} & -\frac{251940}{49321} & \frac{3359200}{3008581} & -\mathbf{\frac{2414425}{24068648}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1174732}{345247} & -\frac{121524}{49321} & \frac{2065908}{3008581} & -\mathbf{\frac{860795}{12034324}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{243048}{345247} & \frac{972192}{3008581} & -\mathbf{\frac{516477}{12034324}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1903876}{21060067} & -\mathbf{\frac{475969}{24068648}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mathbf{\frac{915325}{168480536}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\begin{aligned}
G(z,\xi) = {} & \frac{915325}{168480536}(1-z)^{13} + \frac{475969}{24068648}(1-z)^{12}(1-\xi) + \frac{516477}{12034324}(1-z)^{11}(1-\xi)^2 + \frac{860795}{12034324}(1-z)^{10}(1-\xi)^3 \\
& + \frac{2414425}{24068648}(1-z)^9(1-\xi)^4 + \frac{2973555}{24068648}(1-z)^8(1-\xi)^5 + \frac{5748873}{42120134}(1-z)^7(1-\xi)^6 + \frac{5748873}{42120134}(1-z)^6(1-\xi)^7 \\
& + \frac{2973555}{24068648}(1-z)^5(1-\xi)^8 + \frac{2414425}{24068648}(1-z)^4(1-\xi)^9 + \frac{860795}{12034324}(1-z)^3(1-\xi)^{10} + \frac{516477}{12034324}(1-z)^2(1-\xi)^{11} \\
& + \frac{475969}{24068648}(1-z)(1-\xi)^{12} + \frac{915325}{168480536}(1-\xi)^{13}, \quad G(z) = (1-z)^{13}.
\end{aligned}$$

Furthermore, in this case, note that $\lambda = 0$ is not a root of the characteristic polynomial $P(\lambda)$ of the coefficient matrix $(d_{n,m})_{n,m=0,1,\dots,3l+1}$. Using MAPLE, we get approximate values of eigenvalues λ. For instance:

- For $l = 0$, $P(\lambda) = \lambda^2 - \lambda - 1/4$, $\lambda = \big\{ -0.2071067812;\ 1.207106781\big\}$.
- For $l = 1$,

$$\begin{aligned} P(\lambda) &= -\lambda^5 + \frac{877}{247}\lambda^4 + \frac{1566301}{244036}\lambda^3 - \frac{68809675}{30138446}\lambda^2 - \frac{812209475}{7444196162}\lambda \\ &\quad + \frac{269028375}{919358226007}, \\ \lambda &= \big\{ -1.54202326620578;\ -0.0450668283322828;\ 0.00254722709857864; \\ &\qquad 0.345110887775521;\ 4.79003926711336\big\}. \end{aligned}$$

- For $l = 2$,

$$\begin{aligned} P(\lambda) &= \lambda^8 - \frac{61941}{2945}\lambda^7 - \frac{7332124923}{34692100}\lambda^6 + \frac{26947562058263}{51084117250}\lambda^5 \\ &\quad - \frac{1709744016292823742}{44305382601218125}\lambda^3 - \frac{3701437192909697214 82}{65239675880293689 0625}\lambda^2 \\ &\quad + \frac{1229491381247004011136}{19213084546746491428906 25}\lambda + \frac{143699616860956160004}{5658253399016841725812890625}, \\ \lambda &= \big\{ -8.89919529100385;\ -0.608797098869804;\ -0.0141753406007436; \\ &\quad -0.383838438028620 \cdot 10^{-4};\ 0.00108908312652974;\ 0.112041749879426; \\ &\qquad 2.52949165158050;\ 27.9121812528217\big\}. \end{aligned}$$

- For $l = 3$,

$$\begin{aligned} P(\lambda) &= -\lambda^{11} + \frac{6897184}{49321}\lambda^{10} + \frac{268404263255438 44}{2979887275225}\lambda^9 \\ &\quad - \frac{1548864853014583612 7368}{104979300215265875}\lambda^8 - \frac{608110699868203478432400921 6352}{887972817304787489880 0625}\lambda^7 \\ &\quad + \frac{171870946075120807502255509402134 2304}{2189785366114471189418728128125}\lambda^6 \\ &\quad + \frac{5050546028579077439674478691524234879515392}{264605889903222992156636670517770 15625}\lambda^5 \\ &\quad - \frac{136465392813252515527771555029658886924628 7488}{176359825620498124272398340900093809140625}\lambda^4 \\ &\quad - \frac{6144704704409054853564304021176397332642070715312 64}{1576991448544403002086423189019028401701270007812 5}\lambda^3 \\ &\quad + \frac{265113432283704478762461762318167845541555805005107 2}{1555575904673170009318089562112149996006166761106406 25}\lambda^2 \\ &\quad + \frac{1551470540135622255735795642559160819227948621920460 8}{383612795971927090147887476464666749765100754122645313 28125}\lambda \\ &\quad - \frac{463664464564272399221177092732045368162894982434816}{1892016671013141601318395822671382876516453429408298949634453125}, \end{aligned}$$

$$\begin{aligned}\lambda = \big\{ & -57.5501857582972;\ -4.56514829471194;\ -0.237275856004134;\\ & -0.00490197451053624;\ -0.231342439115920\cdot 10^{-4};\\ & 5.91228291709660\cdot 10^{-7};\ 0.426049897607914\cdot 10^{-3};\\ & 0.0394707782265108;\ 1.12881543508430;\\ & 16.7621517568565;\ 184.269414876376\big\}.\end{aligned}$$

Observation 5.3.7. *For any integer $l \geq 1$, if letting*

$$\kappa = \frac{6}{6l+1} \quad , \quad p = 1 \quad , \quad q = \frac{2-3l(3l-5)}{2(6l+1)},$$

then, the corresponding coefficient matrix $(d_{n,m})$ is a square matrix of order $(3l+1)$, and we obtain

$$G(z,\xi) = (1-z)(1-\xi)\sum_{m=0}^{3l-1} |d_{3l-m,m+1}|(1-z)^{3l-1-m}(1-\xi)^m, \tag{5.3.8}$$

$$G(z) := \lim_{\xi\to z} G(z,\xi) = \mathbb{E}\left((f'(z))^{2p-q}\right) = (1-z)^{3l+1}.$$

In the specific matrices below, the coefficients $d_{3l-m,m+1}$ in (5.3.8) are in bold.

$$(l=1)\quad \begin{bmatrix} 1 & -2 & \frac{17}{13} & -\frac{4}{13} \\ -2 & \frac{44}{13} & -\frac{22}{13} & \mathbf{\frac{4}{13}} \\ \frac{17}{13} & -\frac{22}{13} & \mathbf{\frac{5}{13}} & 0 \\ -\frac{4}{13} & \mathbf{\frac{4}{13}} & 0 & 0 \end{bmatrix},$$

$$G(z,\xi) = (1-z)(1-\xi)\left[(4/13)(1-z)^2 + (5/13)(1-z)(1-\xi) + (4/13)(1-\xi)^2\right],$$
$$G(z) = (1-z)^4.$$

$$(l=2)\quad \begin{bmatrix} 1 & -\frac{7}{2} & \frac{205}{38} & -\frac{90}{19} & \frac{469}{190} & -\frac{68}{95} & \frac{17}{190} \\ -\frac{7}{2} & \frac{194}{19} & -\frac{485}{38} & \frac{862}{95} & -\frac{727}{190} & \frac{17}{19} & \mathbf{-\frac{17}{190}} \\ \frac{205}{38} & -\frac{485}{38} & \frac{1132}{95} & -\frac{566}{95} & \frac{151}{95} & \mathbf{-\frac{17}{95}} & 0 \\ -\frac{90}{19} & \frac{862}{95} & -\frac{566}{95} & \frac{176}{95} & \mathbf{-\frac{22}{95}} & 0 & 0 \\ \frac{469}{190} & -\frac{727}{190} & \frac{151}{95} & \mathbf{-\frac{22}{95}} & 0 & 0 & 0 \\ -\frac{68}{95} & \frac{17}{19} & \mathbf{-\frac{17}{95}} & 0 & 0 & 0 & 0 \\ \frac{17}{190} & \mathbf{-\frac{17}{190}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} G(z,\xi) = (1-z)(1-\xi)\bigg[&\frac{17}{190}(1-z)^5 + \frac{17}{95}(1-z)^4(1-\xi) + \frac{22}{95}(1-z)^3(1-\xi)^2 \\ &+ \frac{22}{95}(1-z)^2(1-\xi)^3 + \frac{17}{95}(1-z)(1-\xi)^4 + \frac{17}{190}(1-\xi)^5\bigg], \end{aligned}$$

$$G(z) = (1-z)^7.$$

$$(l=3)$$

$$\begin{bmatrix}
1 & -5 & \frac{296}{25} & -\frac{434}{25} & \frac{13328}{775} & -\frac{1834}{155} & \frac{6440}{1147} & -\frac{50531}{28675} & \frac{407537}{1233025} & -\frac{34684}{1233025} \\
-5 & \frac{533}{25} & -\frac{1066}{25} & \frac{40866}{775} & -\frac{34118}{775} & \frac{29218}{1147} & -\frac{290283}{28675} & \frac{3258203}{1233025} & -\frac{502918}{1233025} & \mathbf{\frac{34684}{1233025}} \\
\frac{296}{25} & -\frac{1066}{25} & \frac{54362}{775} & -\frac{54362}{775} & \frac{265958}{5735} & -\frac{590051}{28675} & \frac{7319543}{1233025} & -\frac{1246531}{1233025} & \mathbf{\frac{95381}{1233025}} & 0 \\
-\frac{434}{25} & \frac{40866}{775} & -\frac{54362}{775} & \frac{315854}{5735} & -\frac{157927}{5735} & \frac{10733107}{1233025} & -\frac{1971123}{1233025} & \mathbf{\frac{161161}{1233025}} & 0 & 0 \\
\frac{13328}{775} & -\frac{34118}{775} & \frac{265958}{5735} & -\frac{157927}{5735} & \frac{2409869}{246605} & -\frac{2409869}{1233025} & \mathbf{\frac{210749}{1233025}} & 0 & 0 & 0 \\
-\frac{1834}{155} & \frac{29218}{1147} & -\frac{590051}{28675} & \frac{10733107}{1233025} & -\frac{2409869}{1233025} & \mathbf{\frac{9163}{49321}} & 0 & 0 & 0 & 0 \\
\frac{6440}{1147} & -\frac{290283}{28675} & \frac{7319543}{1233025} & -\frac{1971123}{1233025} & \mathbf{\frac{210749}{1233025}} & 0 & 0 & 0 & 0 & 0 \\
-\frac{50531}{28675} & \frac{3258203}{1233025} & -\frac{1246531}{1233025} & \mathbf{\frac{161161}{1233025}} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{407537}{1233025} & -\frac{502918}{1233025} & \mathbf{\frac{95381}{1233025}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{34684}{1233025} & \mathbf{\frac{34684}{1233025}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\begin{aligned}
G(z,\xi) = (1-z)(1-\xi)\Big[& \frac{34684}{1233025}(1-z)^8 + \frac{95381}{1233025}(1-z)^7(1-\xi) + \frac{161161}{1233025}(1-z)^6(1-\xi)^2 \\
& + \frac{210749}{1233025}(1-z)^5(1-\xi)^3 + \frac{9163}{49321}(1-z)^4(1-\xi)^4 + \frac{210749}{1233025}(1-z)^3(1-\xi)^5 \\
& + \frac{161161}{1233025}(1-z)^2(1-\xi)^6 + \frac{95381}{1233025}(1-z)(1-\xi)^7 + \frac{34684}{1233025}(1-\xi)^8\Big],
\end{aligned}$$

$$G(z) = (1-z)^{10}.$$

$$(l = 4)$$

$$\left[\begin{array}{ccccccccccccc}
1 & -\frac{13}{2} & \frac{638}{31} & -\frac{2585}{62} & \frac{68475}{1147} & -\frac{144969}{2294} & \frac{2486484}{49321} & -\frac{2998281}{98642} & \frac{4727580}{345247} & -\frac{1547150}{345247} & \frac{348232}{345247} & -\frac{1558}{11137} & \frac{3116}{345247} \\
-\frac{13}{2} & \frac{1142}{31} & -\frac{6281}{62} & \frac{204325}{1147} & -\frac{507705}{2294} & \frac{10015764}{49321} & -\frac{13822809}{98642} & \frac{584727}{8029} & -\frac{1387650}{49321} & \frac{2706280}{345247} & -\frac{513418}{345247} & \frac{59204}{345247} & \mathbf{-\frac{3116}{345247}} \\
\frac{638}{31} & -\frac{6281}{62} & \frac{8855}{37} & -\frac{26565}{74} & \frac{18623220}{49321} & -\frac{28641921}{98642} & \frac{8162583}{49321} & -\frac{48007905}{690494} & \frac{7248840}{345247} & -\frac{213190}{49321} & \frac{26828}{49321} & \mathbf{-\frac{1558}{49321}} & 0 \\
-\frac{2585}{62} & \frac{204325}{1147} & -\frac{26565}{74} & \frac{22383900}{49321} & -\frac{39171825}{98642} & \frac{12316755}{49321} & -\frac{11205975}{98642} & \frac{12675030}{345247} & -\frac{2771850}{345247} & \frac{1700}{1591} & \mathbf{-\frac{3230}{49321}} & 0 & 0 \\
\frac{68475}{1147} & -\frac{507705}{2294} & \frac{18623220}{49321} & -\frac{39171825}{98642} & \frac{13982925}{49321} & -\frac{13982925}{98642} & \frac{1530}{31} & -\frac{7925145}{690494} & \frac{555900}{345247} & \mathbf{-\frac{5100}{49321}} & 0 & 0 & 0 \\
-\frac{144969}{2294} & \frac{10015764}{49321} & -\frac{28641921}{98642} & \frac{12316755}{49321} & -\frac{13982925}{98642} & \frac{2672094}{49321} & -\frac{1336047}{98642} & \frac{695589}{345247} & \mathbf{-\frac{46920}{345247}} & 0 & 0 & 0 & 0 \\
\frac{2486484}{49321} & -\frac{13822809}{98642} & \frac{8162583}{49321} & -\frac{11205975}{98642} & \frac{1530}{31} & -\frac{1336047}{98642} & \frac{106743}{49321} & \mathbf{-\frac{15249}{98642}} & 0 & 0 & 0 & 0 & 0 \\
-\frac{2998281}{98642} & \frac{584727}{8029} & -\frac{48007905}{690494} & \frac{12675030}{345247} & -\frac{7925145}{690494} & \frac{695589}{345247} & \mathbf{-\frac{15249}{98642}} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4727580}{345247} & -\frac{1387650}{49321} & \frac{7248840}{345247} & -\frac{2771850}{345247} & \frac{555900}{345247} & \mathbf{-\frac{46920}{345247}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1547150}{345247} & \frac{2706280}{345247} & -\frac{213190}{49321} & \frac{1700}{1591} & \mathbf{-\frac{5100}{49321}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{348232}{345247} & -\frac{513418}{345247} & \frac{26828}{49321} & \mathbf{-\frac{3230}{49321}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1558}{11137} & \frac{59204}{345247} & \mathbf{-\frac{1558}{49321}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3116}{345247} & \mathbf{-\frac{3116}{345247}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],$$

$$\begin{aligned}
G(z,\xi) = (1-z)(1-\xi)\Big[&\frac{3116}{345247}(1-z)^{11} + \frac{1558}{49321}(1-z)^{10}(1-\xi) + \frac{3230}{49321}(1-z)^{9}(1-\xi)^2 + \frac{5100}{49321}(1-z)^{8}(1-\xi)^3 \\
&+ \frac{46920}{345247}(1-z)^{7}(1-\xi)^4 + \frac{15249}{98642}(1-z)^{6}(1-\xi)^5 + \frac{15249}{98642}(1-z)^{5}(1-\xi)^6 + \frac{46920}{345247}(1-z)^{4}(1-\xi)^7 \\
&+ \frac{5100}{49321}(1-z)^{3}(1-\xi)^8 + \frac{3230}{49321}(1-z)^{2}(1-\xi)^9 + \frac{1558}{49321}(1-z)(1-\xi)^{10} + \frac{3116}{345247}(1-\xi)^{11}\Big],
\end{aligned}$$

$$G(z) = (1-z)^{13}.$$

Furthermore, in this case, note that $\lambda = 0$ is *a simple root* of the characteristic polynomial $P(\lambda)$ of the coefficient matrix $(d_{n,m})_{n,m=0,1,\dots,3l}$. Using MAPLE, we get approximate values of eigenvalues λ. For instance:

- For $l = 1$,

$$\begin{aligned} P(\lambda) &= (1/2197)\,\lambda\,\big[2197\,\lambda^3 - 10478\,\lambda^2 - 8112\,\lambda + 320\big]\,, \\ \lambda &= \big\{ -0.711111180491101;\ 0;\ 0.0376328658944030;\ 5.44270908382747\big\}. \end{aligned}$$

- For $l = 2$, setting $N := 2940367562500$,

$$\begin{aligned} P(\lambda) = -\frac{1}{N}\,\lambda\,\big[&2940367562500\,\lambda^6 - 73447286587500\,\lambda^5 \\ &- 536778108041875\,\lambda^4 + 613580056937750\,\lambda^3 \\ &+ 83531381609725\,\lambda^2 - 833056632100\,\lambda - 282969148\big], \end{aligned}$$

$$\begin{aligned} \lambda = \big\{ &-6.71908029571401;\ -0.131562669586427;\ -0.000328857911389764; \\ &0;\ 0.00964640672192552;\ 1.11775628056074;\ 30.7025165043502\big\}. \end{aligned}$$

- For $l = 3$, $N := 527031928563498579473843707007456625293664855957031 25$,

$$\begin{aligned} P(\lambda) = \frac{1}{N}\,\lambda\,\big[&5270319285634985794738437070074566252936648559570312 5\,\lambda^9 \\ &- 8300609792668264459482464091106643611433296496582031250\,\lambda^8 \\ &- 4454946308694508310250546355277113141474905818620605468 75\,\lambda^7 \\ &+ 4794778996142210900232623988837260417742766309970898437500\,\lambda^6 \\ &+ 9756465613003402089003884988756390021440062708106251171875\,\lambda^5 \\ &- 3017384501505283273071518606155848657382720371906916875000\,\lambda^4 \\ &- 1101363659384553212119922455059856289135099302468743125 00\,\lambda^3 \\ &+ 3394075057288876855511435179506603279030745610699625 00\,\lambda^2 \\ &+ 557896009180612713491431658302140647927160891555 60\,\lambda \\ &- 23136663608718360279628551715845337675643273 6\big], \end{aligned}$$

$$\begin{aligned} \lambda = \big\{ &-49.3913171271021;\ -1.99073754558809; \\ &-0.0356150192571145;\ -0.000160321576336723; \\ &0;\ 0.00000404759002436705;\ 0.00300504302403685; \\ &0.302505715828807;\ 10.7402583250749;\ 197.869342014100\big\}. \end{aligned}$$

- For $l = 4$, $N := 497478873035110874035149889519235117915098040306388138248641$,

$$
\begin{aligned}
P(\lambda) = -\frac{1}{N}\,\lambda\,\big[&497478873035110874035149889519235117915098040306388138248641\,\lambda^{12}\\
&- 532727542350127989718544701200095388951136642800569321024466020\,\lambda^{11}\\
&- 20780582478628473492426622152724321766323572440443993343834517 2524\,\lambda^{10}\\
&+ 17849005796821652589787822322841407132312910655798157087370931340240\,\lambda^{9}\\
&+ 34429786428813825131920022996150844625843349569133077827815481060 8976\,\lambda^{8}\\
&- 1305809037984406113653123425917082362518950072864121831298194147971584\,\lambda^{7}\\
&- 857459709928706636281627128531536748933260220065754221494033396979968\,\lambda^{6}\\
&+ 82350349827169436218268442272600235900268467554762816633894717952000\,\lambda^{5}\\
&+ 9373128665946616326546392041184127795289202514401279573281451520 00\,\lambda^{4}\\
&- 97329955912865260758434997748308261441164634462730740865228 8000\,\lambda^{3}\\
&- 661969386060967516354776161754510660021474068095981977 60000\,\lambda^{2}\\
&+ 187891311363831342261122696760078459289425226752000000\,\lambda\\
&+ 1064038392112832616725825172556345274880000000 0\big],
\end{aligned}
$$

$$
\begin{aligned}
\lambda = \big\{ &- 358.467163208234;\ -19.0072656658816;\ -0.646157263047298;\\
&- 0.0111599266641292;\ -0.0000666533778004677;\\
&- 5.55444566928419 \cdot 10^{-8};\ 0;\ 0.0000027825956777 6007;\\
&\ \ 0.00101759218979804;\ 0.0943002426171891;\\
&\ \ 3.74532560247919;\ 86.5431005254185;\ 1358.60267157073\big\}.
\end{aligned}
$$

Observation 5.3.8. *For any nonnegative integer l, if letting*

$$
\kappa = \frac{6}{6l+5} \quad , \quad p = 0 \quad , \quad q = -\frac{3\,(3l+4)\,(l+1)}{2\,(6l+5)},
$$

then, the corresponding coefficient matrix $(d_{n,m})$ is a square matrix of order $(3l+4)$, and we obtain

$$
G(z,\xi) = \sum_{m=0}^{3l+3} |d_{3l+3-m,m}|(1-z)^{3l+3-m}(1-\xi)^m, \tag{5.3.9}
$$

$$
G(z) := \lim_{\xi \to z} G(z,\xi) = \mathbb{E}\left((f'(z))^{2p-q}\right) = (1-z)^{3l+3}.
$$

In the specific matrices below, the coefficients $d_{3l+3-m,m}$ in (5.3.9) are in bold.

$$
(l=0) \quad \begin{bmatrix} 1 & -\frac{3}{2} & \frac{21}{22} & -\boldsymbol{\frac{5}{22}} \\ -\frac{3}{2} & \frac{12}{11} & -\boldsymbol{\frac{3}{11}} & 0 \\ \frac{21}{22} & -\boldsymbol{\frac{3}{11}} & 0 & 0 \\ -\boldsymbol{\frac{5}{22}} & 0 & 0 & 0 \end{bmatrix},
$$

$$G(z,\xi) = \frac{5}{22}(1-z)^3+\frac{3}{11}(1-z)^2(1-\xi)+\frac{3}{11}(1-z)(1-\xi)^2+\frac{5}{22}(1-\xi)^3, \quad G(z) = (1-z)^3.$$

$$(l=1) \quad \begin{bmatrix} 1 & -3 & \frac{75}{17} & -\frac{65}{17} & \frac{780}{391} & -\frac{228}{391} & \mathbf{\frac{836}{11339}} \\ -3 & \frac{105}{17} & -\frac{105}{17} & \frac{1365}{391} & -\frac{420}{391} & \mathbf{\frac{1596}{11339}} & 0 \\ \frac{75}{17} & -\frac{105}{17} & \frac{1575}{391} & -\frac{525}{391} & \mathbf{\frac{2100}{11339}} & 0 & 0 \\ -\frac{65}{17} & \frac{1365}{391} & -\frac{525}{391} & \mathbf{\frac{2275}{11339}} & 0 & 0 & 0 \\ \frac{780}{391} & -\frac{420}{391} & \mathbf{\frac{2100}{11339}} & 0 & 0 & 0 & 0 \\ -\frac{228}{391} & \mathbf{\frac{1596}{11339}} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{\frac{836}{11339}} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} G(z,\xi) = {} & \frac{836}{11339}(1-z)^6 + \frac{1596}{11339}(1-z)^5(1-\xi) + \frac{2100}{11339}(1-z)^4(1-\xi)^2 \\ & + \frac{2275}{11339}(1-z)^3(1-\xi)^3 + \frac{2100}{11339}(1-z)^2(1-\xi)^4 \\ & + \frac{1596}{11339}(1-z)(1-\xi)^5 + \frac{836}{11339}(1-\xi)^6, \quad G(z) = (1-z)^6. \end{aligned}$$

$$(l=2)$$

$$\begin{bmatrix} 1 & -\frac{9}{2} & \frac{234}{23} & -\frac{336}{23} & \frac{9576}{667} & -\frac{13167}{1334} & \frac{3135}{667} & -\frac{990}{667} & \frac{15345}{54694} & \mathbf{-\frac{2635}{109388}} \\ -\frac{9}{2} & \frac{360}{23} & -\frac{630}{23} & \frac{20160}{667} & -\frac{29925}{1334} & \frac{7524}{667} & -\frac{2475}{667} & \frac{19800}{27347} & \mathbf{-\frac{6975}{109388}} & 0 \\ \frac{234}{23} & -\frac{630}{23} & \frac{24570}{667} & -\frac{20475}{667} & \frac{11115}{667} & -\frac{3861}{667} & \frac{32175}{27347} & \mathbf{-\frac{2925}{27347}} & 0 & 0 \\ -\frac{336}{23} & \frac{20160}{667} & -\frac{20475}{667} & \frac{12480}{667} & -\frac{4680}{667} & \frac{41184}{27347} & \mathbf{-\frac{3900}{27347}} & 0 & 0 & 0 \\ \frac{9576}{667} & -\frac{29925}{1334} & \frac{11115}{667} & -\frac{4680}{667} & \frac{44460}{27347} & \mathbf{-\frac{4446}{27347}} & 0 & 0 & 0 & 0 \\ -\frac{13167}{1334} & \mathbf{\frac{7524}{667}} & -\frac{3861}{667} & \frac{41184}{27347} & \mathbf{-\frac{4446}{27347}} & 0 & 0 & 0 & 0 & 0 \\ \frac{3135}{667} & -\frac{2475}{667} & \frac{32175}{27347} & \mathbf{-\frac{3900}{27347}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{990}{667} & \frac{19800}{27347} & \mathbf{-\frac{2925}{27347}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{15345}{54694} & \mathbf{-\frac{6975}{109388}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{-\frac{2635}{109388}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} G(z,\xi) = {} & \frac{2635}{109388}(1-z)^9 + \frac{6975}{109388}(1-z)^8(1-\xi) + \frac{2925}{27347}(1-z)^7(1-\xi)^2 \\ & + \frac{3900}{27347}(1-\xi)^6(1-\xi)^3 + \frac{4446}{27347}(1-\xi)^5(1-\xi)^4 + \frac{4446}{27347}(1-\xi)^4(1-\xi)^5 \\ & + \frac{3900}{27347}(1-\xi)^3(1-\xi)^6 + \frac{2925}{27347}(1-z)^2(1-\xi)^7 \\ & + \frac{6975}{109388}(1-z)(1-\xi)^8 + \frac{2635}{109388}(1-\xi)^9, \quad G(z) = (1-z)^9. \end{aligned}$$

$$(l = 3)$$

$$\begin{bmatrix}
1 & -6 & \frac{528}{29} & -\frac{1045}{29} & \frac{20691}{406} & -\frac{10890}{203} & \frac{50820}{1189} & -\frac{30690}{1189} & \frac{1304325}{111766} & -\frac{214489}{55883} & \frac{2573868}{2961799} & -\frac{2515371}{20732593} & \frac{\mathbf{19284511}}{\mathbf{2446445974}} \\
-6 & \frac{858}{29} & -\frac{2145}{29} & \frac{24453}{203} & -\frac{28314}{203} & \frac{141570}{1189} & -\frac{90090}{1189} & \frac{1994850}{55883} & -\frac{678249}{55883} & \frac{8365071}{2961799} & -\frac{8365071}{20732593} & \frac{\mathbf{32699823}}{\mathbf{1223222987}} & 0 \\
\frac{528}{29} & -\frac{2145}{29} & \frac{30888}{203} & -\frac{41184}{203} & \frac{226512}{1189} & -\frac{154440}{1189} & \frac{3603600}{55883} & -\frac{1276704}{55883} & \frac{16277976}{2961799} & -\frac{16730142}{20732593} & \frac{\mathbf{66920568}}{\mathbf{1223222987}} & 0 & 0 \\
-\frac{1045}{29} & \frac{24453}{203} & -\frac{41184}{203} & \frac{260832}{1189} & -\frac{195624}{1189} & \frac{4890600}{55883} & -\frac{1825824}{55883} & \frac{24257376}{2961799} & -\frac{25773462}{20732593} & \frac{\mathbf{105957566}}{\mathbf{1223222987}} & 0 & 0 & 0 \\
\frac{20691}{406} & -\frac{28314}{203} & \frac{226512}{1189} & -\frac{195624}{1189} & \frac{5379660}{55883} & -\frac{2151864}{55883} & \frac{30126096}{2961799} & -\frac{33353892}{20732593} & \frac{\mathbf{141754041}}{\mathbf{1223222987}} & 0 & 0 & 0 & 0 \\
-\frac{10890}{203} & \frac{141570}{1189} & -\frac{154440}{1189} & \frac{4890600}{55883} & -\frac{2151864}{55883} & \frac{32277960}{2961799} & -\frac{5379660}{2961799} & \frac{\mathbf{166769460}}{\mathbf{1223222987}} & 0 & 0 & 0 & 0 & 0 \\
\frac{50820}{1189} & -\frac{90090}{1189} & \frac{3603600}{55883} & -\frac{1825824}{55883} & \frac{30126096}{2961799} & -\frac{5379660}{2961799} & \frac{\mathbf{25105080}}{\mathbf{174746141}} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{30690}{1189} & \frac{1994850}{55883} & -\frac{1276704}{55883} & \frac{24257376}{2961799} & -\frac{33353892}{20732593} & \frac{\mathbf{166769460}}{\mathbf{1223222987}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1304325}{111766} & -\frac{678249}{55883} & \frac{16277976}{2961799} & -\frac{25773462}{20732593} & \frac{\mathbf{141754041}}{\mathbf{1223222987}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{214489}{55883} & \frac{8365071}{2961799} & -\frac{16730142}{20732593} & \frac{\mathbf{105957566}}{\mathbf{1223222987}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2573868}{2961799} & -\frac{8365071}{20732593} & \frac{\mathbf{66920568}}{\mathbf{1223222987}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2515371}{20732593} & \frac{\mathbf{32699823}}{\mathbf{1223222987}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\mathbf{19284511}}{\mathbf{2446445974}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\begin{aligned}
G(z,\xi) &= \frac{19284511}{2446445974}(1-z)^{12} + \frac{32699823}{1223222987}(1-z)^{11}(1-\xi) + \frac{66920568}{1223222987}(1-z)^{10}(1-\xi)^2 \\
&+ \frac{105957566}{1223222987}(1-z)^9(1-\xi)^3 + \frac{141754041}{1223222987}(1-z)^8(1-\xi)^4 + \frac{166769460}{1223222987}(1-z)^7(1-\xi)^5 \\
&+ \frac{25105080}{174746141}(1-z)^6(1-\xi)^6 + \frac{166769460}{1223222987}(1-z)^5(1-\xi)^7 + \frac{141754041}{1223222987}(1-z)^4(1-\xi)^8 \\
&+ \frac{105957566}{1223222987}(1-z)^3(1-\xi)^9 + \frac{66920568}{1223222987}(1-z)^2(1-\xi)^{10} + \frac{32699823}{1223222987}(1-z)(1-\xi)^{11} + \frac{19284511}{2446445974}(1-\xi)^{12}, \\
G(z) &= (1-z)^{12}.
\end{aligned}$$

In addition, in this case, note that $\lambda = 0$ is not a root of the characteristic polynomial $P(\lambda)$ of the coefficient matrix $(d_{n,m})_{n,m=0,1,\dots,3l+3}$. Using MAPLE, we obtain approximate values of eigenvalues λ. For instance:

- For $l = 0$, $P(\lambda) = \lambda^4 - \frac{23}{11}\lambda^3 - \frac{1063}{484}\lambda^2 + \frac{915}{2662}\lambda + \frac{225}{58564}$,

$$\lambda = \Big\{ -0.872410099681405; -0.0104823056623044; 0.148710574869710; 2.82509092138309\Big\}.$$

- For $l = 1$,

$$\begin{aligned} P(\lambda) = &-\lambda^7 + \frac{129324}{11339}\lambda^6 + \frac{8154457680}{128572921}\lambda^5 - \frac{123930929733856}{1457888351219}\lambda^4 \\ &- \frac{4254434035624335 36}{16530996014472241}\lambda^3 + \frac{216314217693329011200}{187444963808100740699}\lambda^2 \\ &+ \frac{979913709606142080000 0}{2125438444620054298785961}\lambda - \frac{1786069356501830400000 0}{24100346523546795693934011779}, \end{aligned}$$

$$\begin{aligned} \lambda = \Big\{ &-4.89151642807025; -0.289696689153893; -0.00383808197097602; \\ &0.000154769333729907; 0.0430592403171764; \\ &1.34499077667683; 15.2020849700594\Big\}. \end{aligned}$$

- For $l = 2$,

$$\begin{aligned} P(\lambda) = \lambda^{10} &- \frac{2018897}{27347}\lambda^9 - \frac{15153922021517}{5982867272}\lambda^8 + \frac{7205196349154806905}{327226942574768}\lambda^7 \\ &+ \frac{7631083434607330239828225}{1431788031774 74887936}\lambda^6 \\ &- \frac{5881847838619518209995894687 5}{1957755365247202880192896}\lambda^5 \\ &- \frac{3421795010340397312870147227656 25}{107077471946830514329270253824}\lambda^4 \\ &+ \frac{3479318482068054124428390997148437 5}{73206190633249351884063840783 1232}\lambda^3 \\ &+ \frac{1372962348193138422381526358583984375}{2001969695247470025973493853896070150 4}\lambda^2 \\ &- \frac{4071799843805739972852226735839843 75}{6843483156991570350037142052811978800373 6}\lambda \\ &- \frac{135765542902152693296733947753906 25}{93574366947124237181232861859124592126908419 6}, \end{aligned}$$

$$\begin{aligned} \lambda = \Big\{ &-30.5940849366430; -2.37011219910940; -0.103335485364294; \\ &-0.00140174633994014; -0.0000023736677755757 0; \\ &0.0000843613738068656; 0.0144401176649458; \\ &0.551764238292528; 8.88527546267667; 97.4425453405804\Big\}. \end{aligned}$$

Observation 5.3.9. *For any nonnegative integer l, if letting*

$$\kappa = \frac{6}{6l+5} \quad , \quad p = 1 \quad , \quad q = 2 - \frac{3(3l+4)(l+1)}{2(6l+5)},$$

then, the corresponding coefficient matrix $(d_{n,m})$ is a square matrix of order $(3l+3)$, and we obtain

$$G(z,\xi) = (1-z)(1-\xi)\sum_{m=0}^{3l+1} |d_{3l+2-m,m+1}|(1-z)^{3l+1-m}(1-\xi)^m, \tag{5.3.10}$$
$$G(z) := \lim_{\xi \to z} G(z,\xi) = \mathbb{E}\left((f'(z))^{2p-q}\right) = (1-z)^{3l+3}.$$

In the specific matrices below, the coefficients $d_{3l+2-m,m+1}$ in (5.3.10) are in bold.

$$(l=0) \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & 2 & \mathbf{-\frac{1}{2}} \\ \frac{1}{2} & \mathbf{-\frac{1}{2}} & 0 \end{bmatrix},$$

$$G(z,\xi) = (1-z)(1-\xi)\left[\frac{1}{2}(1-z) + \frac{1}{2}(1-\xi)\right], \quad G(z) = (1-z)^3.$$

$$(l=1) \begin{bmatrix} 1 & -3 & \frac{64}{17} & -\frac{43}{17} & \frac{351}{391} & -\frac{52}{391} \\ -3 & \frac{127}{17} & -\frac{127}{17} & \frac{1563}{391} & -\frac{26}{23} & \mathbf{\frac{52}{391}} \\ \frac{64}{17} & -\frac{127}{17} & \frac{2037}{391} & -\frac{679}{391} & \mathbf{\frac{91}{391}} & 0 \\ -\frac{43}{17} & \frac{1563}{391} & -\frac{679}{391} & \mathbf{\frac{105}{391}} & 0 & 0 \\ \frac{351}{391} & -\frac{26}{23} & \mathbf{\frac{91}{391}} & 0 & 0 & 0 \\ -\frac{52}{391} & \mathbf{\frac{52}{391}} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G(z,\xi) = (1-z)(1-\xi)\left[\frac{52}{391}(1-z)^4 + \frac{91}{391}(1-z)^3(1-\xi) + \frac{105}{391}(1-z)^2(1-\xi)^2 + \frac{91}{391}(1-z)(1-\xi)^3 + \frac{52}{391}(1-\xi)^4\right],$$
$$G(z) = (1-z)^6.$$

$$
(l=2)\quad \begin{bmatrix}
1 & -\frac{9}{2} & \frac{217}{23} & -\frac{553}{46} & \frac{6720}{667} & -\frac{15029}{2668} & \frac{2717}{1334} & -\frac{1155}{2668} & \frac{55}{1334} \\
-\frac{9}{2} & \frac{394}{23} & -\frac{1379}{46} & \frac{21231}{667} & -\frac{59255}{2668} & \frac{6878}{667} & -\frac{8217}{2668} & \frac{715}{1334} & -\mathbf{\frac{55}{1334}} \\
\frac{217}{23} & -\frac{1379}{46} & \frac{28140}{667} & -\frac{23450}{667} & \frac{24865}{1334} & -\frac{573}{92} & \frac{803}{667} & -\mathbf{\frac{275}{2668}} & 0 \\
-\frac{553}{46} & \frac{21231}{667} & -\frac{23450}{667} & \frac{14690}{667} & -\frac{22035}{2668} & \frac{2327}{1334} & -\mathbf{\frac{429}{2668}} & 0 & 0 \\
\frac{6720}{667} & -\frac{59255}{2668} & \frac{24865}{1334} & -\frac{22035}{2668} & \frac{1300}{667} & -\mathbf{\frac{130}{667}} & 0 & 0 & 0 \\
-\frac{15029}{2668} & \frac{6878}{667} & -\frac{573}{92} & \frac{2327}{1334} & -\mathbf{\frac{130}{667}} & 0 & 0 & 0 & 0 \\
\frac{2717}{1334} & -\frac{8217}{2668} & \frac{803}{667} & -\mathbf{\frac{429}{2668}} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1155}{2668} & \frac{715}{1334} & -\mathbf{\frac{275}{2668}} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{55}{1334} & -\mathbf{\frac{55}{1334}} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
\begin{aligned}
G(z,\xi) = (1-z)(1-\xi)\Big[&\frac{55}{1334}(1-z)^7 + \frac{275}{2668}(1-z)^6(1-\xi) + \frac{429}{2668}(1-z)^5(1-\xi)^2 \\
&+ \frac{130}{667}(1-z)^4(1-\xi)^3 + \frac{130}{667}(1-z)^3(1-\xi)^4 + \frac{429}{2668}(1-z)^2(1-\xi)^5 \\
&+ \frac{275}{2668}(1-z)(1-\xi)^6 + \frac{55}{1334}(1-\xi)^7\Big], \quad G(z) = (1-z)^9.
\end{aligned}
$$

Besides, in this case, note that $\lambda = 0$ is *a simple root* of the characteristic polynomial $P(\lambda)$ of the coefficient matrix $(d_{n,m})_{n,m=0,1,\dots,3l+2}$. Using MAPLE, we get approximate values of eigenvalues λ. For instance:

- For $l = 0$,

$$
P(\lambda) = -\frac{1}{4}\lambda\big[4\lambda^2 - 12\lambda - 3\big], \ \ \lambda = \big\{-0.2320508076;\ 0;\ 3.232050808\big\}.
$$

- For $l = 1$, setting $N := 9138686662951$,

$$
\begin{aligned}
P(\lambda) = \frac{1}{N}\lambda\big[&9138686662951\,\lambda^5 - 127474161278094\,\lambda^4 - 468590087451369\,\lambda^3 \\
&+ 224236344353658\,\lambda^2 + 9070727237208\,\lambda - 14106849120\big], \\
\lambda = \big\{&-3.37458314603931;\ -0.0389335016002556;\ 0; \\
&0.00149977501807599;\ 0.463618278988831;\ 16.8972476984920\big\}.
\end{aligned}
$$

- For $l = 2$, setting $N = 10028746107459051168522496$,

$$
\begin{aligned}
&P(\lambda) \\
&= -\frac{1}{N}\lambda\big[10028746107459051168522496\,\lambda^8 - 84534661529185942106122982 4\,\lambda^7 \\
&- 2343377708189153441350596921 6\,\lambda^6 + 12336161254824249461384018352 0\,\lambda^5 \\
&+ 11088182969878500385413517080 0\,\lambda^4 - 12964010361790242372682779000\,\lambda^3 \\
&- 14136520824023558188826812 5\,\lambda^2 + 9049009674436066603125 0\,\lambda \\
&+ 16009382232035156 25\big],
\end{aligned}
$$

$$\begin{aligned}\lambda = \big\{ &-25.5147759584886;\ -0.876131063858002;\ -0.0106098153021541;\\ &-0.0000172288744522820;\ 0;\ 0.000622903526212357;\\ &\ 0.113920834224742;\ 5.22550233298296;\ 105.353841818877\big\}.\end{aligned}$$

Remark 5.3.1. *In all the above observations, it is interesting to note that one always has*

$$4p - 2q = p(\kappa) := \frac{(6+\kappa)(2+\kappa)}{8\kappa},$$

which implies that the point $(4p - 2q; 0)$ *coincides the intersection point* $(p(\kappa), 0)$ *of the red parabola* $\mathcal{R}$ (3.4.19) *with the p-axis (Fig. 2.1). From Theorem 2.2.1, the function* $G(z) := \mathbb{E}\left((f'(z))^{2p-q}\right)$ *has the integrable form* $(1-z)^{\gamma}$.

Appendix A

The generalized hypergeometric function, Hausdorff dimension and Minkowski dimension

A.1 Generalized hypergeometric function

Let $n = [n_1, n_2, ...]$ (list of upper parameters, may be empty), $p = \mathrm{nops}(n)$ (the number of elements in list n), $d = [d_1, d_2, ...]$ (list of lower parameters, may be empty), $q = nops(d)$ (the number of elements in list d) and complex number z. The **hypergeom(n, d, z)** calling sequence is the generalized hypergeometric function $F(n, d, z)$. This function is frequently denoted by $pFq(n, d, z)$.

Formally, $F(n, d, z)$ is defined by the series

$$\sum_{k=0}^{\infty} \frac{z^k \prod_{i=1}^{p} \mathrm{pochhammer}(n_i, k)}{k! \prod_{j=1}^{q} \mathrm{pochhammer}(d_j, k)},$$

where the **pochhammer symbol** is defined for the positive integer n and complex number z as

$$\mathbf{pochhammer}(z, n) = z(z+1)\cdots(z+n-1) = \prod_{j=0}^{n-1}(z+j),$$

with $\mathbf{pochhammer}(z, 0) = 1$.

One notes that:

- If some n_i is a non-positive integer, the series is finite (that is, $F(n, d, z)$ is a polynomial in z). If some d_j is a non-positive integer, the function is undefined for all non-zero z, unless there is also a negative upper parameter of smaller absolute value, in which case the previous rule applies.
- For the remainder of this description, assume that no n_i or d_j is a non-positive integer. When $p \leq q$, this series converges for all complex z, and hence defines

$F(n, d, z)$ everywhere.
When $p = q + 1$, the series converges for $|z| < 1$. $F(n, d, z)$ is then defined for $|z| \geq 1$ by analytic continuation. The point $z = 1$ is a branch point, and the interval $(1, +\infty)$ is the branch cut.
When $q + 1 < p$, the series diverges for all $z \neq 0$. In this case, the series is interpreted as the asymptotic expansion of $F(n, d, z)$ around $z = 0$. The positive real axis is the branch cut.

A.2 Hausdorff dimension and Minkowski dimension

Let $\alpha > 0$. The *α-dimensional Hausdorff measure* of a Borel set $E \subset \mathbb{C}$ is defined by

$$\Lambda_\alpha(E) = \lim_{\epsilon \longrightarrow 0} \inf_{(B_k)} \sum_k (\mathrm{diam} B_k)^\alpha,$$

where the infimum is taken over the covers (B_k) of E with $\mathrm{diam}\, B_k \leq \epsilon$ for all k. The **Hausdorff dimension** is defined by

$$\mathrm{H.\,dim}(E) = \inf\{\alpha : \Lambda_\alpha(E) = 0\}.$$

Sets of non-integer number dimension are called "fractals".

Proposition A.2.1. *The Hausdorff dimension $H.\mathrm{dim}(E)$ is the unique real $d \geq 0$ such that*

$$\Lambda_\alpha(E) = +\infty \ \ if \ \ 0 < \alpha < H.\dim(E)$$

and

$$\Lambda_\alpha(E) = 0 \ \ if \ \ \alpha > H.\dim(E).$$

Proof. see [19]. □

Let E be a bounded set in $\mathbb{C}$ and let $N(\epsilon, E)$ denote the minimal numbers of disks of diameter ϵ that are needed to cover E. Up to bounded multiplies it is the same as the number of squares of grid of mesh size ϵ that intersect E. We define the **Minkowski dimension** of E by

$$\mathrm{M.\,dim}(E) = \limsup_{\epsilon \longrightarrow 0} \frac{\log N(\epsilon, E)}{\log(1/\epsilon)}.$$

Proposition A.2.2. *If E is any bounded set in $\mathbb{C}$ then*

$$H.\dim E \leq \liminf_{\epsilon \longrightarrow 0} \frac{\log N(\epsilon, E)}{\log(1/\epsilon)} \leq M.\dim E.$$

Proof. see [19]. □

Appendix B

The proof of Theorem 1.2.1

This section will present the proof of Theorem 1.2.1 in Section 1.2. This is the SLE_κ case of Theorem 3.1 in [6], with the Lévy process $L_t := \sqrt{\kappa} B_t$.

Proof. [6] Let us first recall that

$$f_t(z) = e^t \Big(z + \sum_{n \geq 2} a_n(t) z^n \Big).$$

By expanding both sides of Loewner's equation (0.0.1) as power series, and identifying coefficients, leads one to the set of equations

$$\begin{aligned} \dot{a}_n(t) - (n-1)a_n(t) &= 2 \sum_{p=1}^{n-1} (n-p) a_{n-p}(t) \overline{\lambda(t)}^p \\ &= 2 \sum_{k=1}^{n-1} k a_k(t) \overline{\lambda(t)}^{n-k}, \quad n \geq 2; \end{aligned} \tag{B.0.1}$$

where $a_1 = 1$; the dot means a t-derivative, and $\overline{\lambda(t)} = 1/\lambda(t)$. Specifying for $n = 2, 3$ gives

$$\dot{a}_2 - a_2 = 2\bar{\lambda}, \tag{B.0.2}$$

$$\dot{a}_3 - 2a_3 = 4a_2\bar{\lambda} + 2\bar{\lambda}^2. \tag{B.0.3}$$

The first differential equation (B.0.2) (together with the uniform bound, $\forall t \geq 0$, $|a_2(t)| \leq C_2 < +\infty$; see Remark 5.2 in [6]) yields

$$a_2(t) = -2e^t \int_t^{+\infty} e^{-s} \overline{\lambda(s)} ds, \tag{B.0.4}$$

and it follows that

$$\mathbb{E}(a_2(0)) = -\frac{4}{\kappa + 2}. \tag{B.0.5}$$

The differential recursion (B.0.1) then becomes, for $\lambda(t) := e^{i\sqrt{\kappa}B_t}$, and in terms of the auxiliary function $u_n(t)$,

$$u_n(t) := a_n(t)e^{-(n-1)t}, \tag{B.0.6}$$

$$\dot{u}_n(t) = 2\sum_{k=1}^{n-1} kX_t{}^{n-k}u_k(t), \tag{B.0.7}$$

where X_t is defined as

$$X_t := e^{-t-i\sqrt{\kappa}B_t}. \tag{B.0.8}$$

The recursion (B.0.7) can be rewritten under the simpler form:

$$\dot{u}_n = X_t[\dot{u}_{n-1} + 2(n-1)u_{n-1}]. \tag{B.0.9}$$

Recall that $u_1 = a_1 = 1$, while the next term of this recursion, as already seen in Eq. (B.0.4), is

$$u_2(t) = -2\int_t^{+\infty} X_s ds. \tag{B.0.10}$$

Similarly, we can write the general solution u_n, for $n \geq 2$, under the form

$$u_n(t) = -2\int_t^{+\infty} X_s v_n(s)ds, \tag{B.0.11}$$

with $v_2(s) = 1$, and rewrite the differential equation (B.0.9) as an integral equation:

$$v_n(t) = X_t v_{n-1}(t) - 2(n-1)\int_t^{+\infty} X_s v_{n-1}(s)ds. \tag{B.0.12}$$

Define then the multiplicative and integral operators $\mathcal{X}$ and $\mathcal{J}$ such that

$$\mathcal{X}v(t) := X_t v(t), \tag{B.0.13}$$

$$\mathcal{J}v(t) := -2\int_t^{+\infty} X_s v(s)ds. \tag{B.0.14}$$

The solutions to (B.0.10), (B.0.11) and (B.0.12) can then be written as the operator product

$$\begin{aligned} u_n &= \mathcal{J} \circ (\mathcal{X} + (n-1)\mathcal{J}) \circ \cdots \circ (\mathcal{X} + 2\mathcal{J})\mathbb{1} \\ &= \mathcal{J}\prod_{k=1}^{n-2} \circ(\mathcal{X} + (k+1)\mathcal{J})\mathbb{1}. \end{aligned} \tag{B.0.15}$$

where $\mathbb{1}$ $(= v_2)$ is the constant function equal to 1 on $\mathbb{R}_+$,

Next, recall the strong Markov property of the Brownian motion, which implies the *identity in law*: $\forall s \geq t$, $B_s \overset{\text{(law)}}{=} B_t + \tilde{B}_{s-t}$, where $\tilde{B}_{s'}$ is an independent copy of the Brownian motion, also started at $\tilde{B}_0 = 0$. Therefore, the process X_t (B.0.8) is, in law,

$$X_s \overset{\text{(law)}}{=} X_t \tilde{X}_{s-t}, \quad \forall s \geq t, \tag{B.0.16}$$

where $\tilde{X}_{s'} := e^{-s'-i\sqrt{\kappa}\tilde{B}_{s'}}$, $s' \geq 0$, is an independent copy of that process, with $\tilde{X}_0 = 1$. The operator $\mathcal{J}$ (B.0.14) can then be written as

$$\begin{aligned} \mathcal{J}v(t) &\overset{\text{(law)}}{=} -2X_t \int_0^{+\infty} \tilde{X}_s v(s+t) ds & \text{(B.0.17)} \\ &\overset{\text{(law)}}{=} \mathcal{X} \circ \tilde{\mathcal{J}} v(t), & \text{(B.0.18)} \end{aligned}$$

with $\tilde{\mathcal{J}}v(t) := -2\int_0^{+\infty} \tilde{X}_s v(s+t)ds$. By iteration of the use of the Markov property, Eq (B.0.15) can be rewritten as

$$\begin{aligned} u_n &\overset{\text{(law)}}{=} \mathcal{J} \circ [\mathcal{X}(1+(n-1)\tilde{\mathcal{J}}^{[n-2]})] \circ \cdots \circ [\mathcal{X}(1+2\tilde{\mathcal{J}}^{[1]})]\mathbb{1} \\ &\overset{\text{(law)}}{=} \mathcal{J} \prod_{k=1}^{n-2} \circ [\mathcal{X}(1+(k+1)\tilde{\mathcal{J}}^{[k]})]\mathbb{1}, \end{aligned} \tag{B.0.19}$$

where the integral operators $\tilde{\mathcal{J}}^{[k]}$, $k = 1, \cdots, n-2$, involve successive *independent* copies, $\tilde{X}^{[k]}_{s_k}$, $k = 1, \cdots, n-2$, of the original process X_s. We therefore arrive at the following explicit representation of the solution (B.0.15):

$$u_n(t) \overset{\text{(law)}}{=} -2\int_t^{+\infty} X_s^{n-1} \prod_{k=1}^{n-2} \left(1 - 2(k+1)\int_0^{+\infty} (\tilde{X}^{[k]}_{s_k})^k ds_k\right) ds. \tag{B.0.20}$$

As mentioned in Theorem 1.2.1, the *conjugate* whole-plane Schramm Loewner evolution $e^{-i\sqrt{\kappa}B_t} f_t(e^{i\sqrt{\kappa}B_t} z)$ shold have the *same law* as $f_0(z)$. At order n, we are thus interested in the stochastically rotated coefficients:

$$e^{i(n-1)\sqrt{\kappa}B_t} a_n(t) = (X_t)^{-(n-1)} u_n(t).$$

Using again the identity in law (B.0.16) in (B.0.20), we arrive at

$$\begin{aligned} e^{i(n-1)\sqrt{\kappa}B_t} a_n(t) &\overset{\text{(law)}}{=} -2\int_0^{+\infty} \tilde{X}_s^{n-1} \prod_{k=1}^{n-2} \left(1 - 2(k+1)\int_0^{+\infty} (\tilde{X}^{[k]}_{s_k})^k ds_k\right) ds \\ &\overset{\text{(law)}}{=} a_n(0), \end{aligned} \tag{B.0.21}$$

which, as it must, no longer depends on t.

All factors in (B.0.21) involve successive independent copies of the process X_s, and their expectations can now be taken *independently*. Recall that $\mathbb{E}[(\tilde{X}_s)^k] = e^{-(\frac{\kappa}{2}k^2+k)s}$.

Thus,

$$\begin{aligned}
\mathbb{E}[a_n(0)] &= -2\int_0^{+\infty}\mathbb{E}[\tilde{X}_s^{n-1}]\prod_{k=1}^{n-2}\left(1-2(k+1)\int_0^{+\infty}\mathbb{E}[(\tilde{X}_{s_k}^{[k]})^k]ds_k\right)ds \\
&= -\frac{2}{\frac{\kappa}{2}(n-1)^2+n-1}\prod_{k=1}^{n-2}\left(1-\frac{2(k+1)}{\frac{\kappa}{2}k^2+k}\right) \\
&= -2\frac{\prod_{k=1}^{n-2}(\frac{\kappa}{2}k^2-k-2)}{\prod_{k=1}^{n-1}(\frac{\kappa}{2}k^2+k)}. \qquad \text{(B.0.22)}
\end{aligned}$$

The proof of Theorem 1.2.1 is completed. □

Appendix C

Matlab Code for the logarithmic coefficient problem

In this section, we will present Matlab code to calculate expectation of coefficients' absolute values squared of SLE process. In the first part, we construct structure of a "term" with B contains values of (α, β) in integrals (1.4.1) and C contains coefficients front of B.

Function *mySLE* uses Loewner method to compute terms of γ_n.

```
function [B C] = mySLE(n)
    if n==1
        C = -1;  B = [1 0];
        return;
    end
% compute alpha, beta
tempb = [1,0];
tempc = -1;
for   k = 2:n
      nC = 2^(k-1);
      nB = 2*length(tempb) + 2^(k-2);
      mC = zeros(1,nC);
      mB = zeros(1,nB);
      for   i = 1:2:nC-1
            mC(i) = tempc(ceil(i/2));
            mC(i+1) = -tempc(ceil(i/2))*2*(k-1);
      end
      tempc = mC;
      iB = 1;
      count = 0;
      for   i = 1:length(tempb)
            fl = 0;
            if   tempb(i)==0
                 fl = 1;
            else
                 count = count+1;
            end
            if   fl==1
                 mB(iB) = tempb(i-count)+1;
                 mB(iB+1:count+iB-1) = tempb(i-count+1:i-1);
                 iB = count+iB;
                 mB(iB) = 0;
                 mB(iB+1) = 1;
                 mB(iB+2:count+iB+1) = tempb(i-count:i-1);
                 mB(count+iB+2) = 0;
                 iB = count+iB+3;
                 count = 0;
            end
      end
      tempb = mB;
end
C = mC;   B = mB;
```

```
function r = eta(x,a,K)
% auxiliary function eta, a is alpha, K is kappa
r = (abs(x))^a*K/2;
```

Function *expbn2* uses dynamic programing to compute $\mathbb{E}\big(|\gamma_n|^2\big)$.

```
function R = expbn2(n, K, a)       % function computes Expectation of |gamma_n|^2, a is alpha, K is kappa
% syms K;
R = 0;
[B C] = mySLE(n);  % coeff of gamma_n
nC = length(C);
iB0 = find(B==0);

for   i=1:nC
      if   i==1
            a1 = B(1:iB0(1)-1);
      else
            a1 = B(iB0(i-1)+1:iB0(i)-1);
      end
      m = length(a1);
      for   j=1:i
            if   j==1
                  a2 = B(1:iB0(1)-1);
            else
                  a2 = B(iB0(j-1)+1:iB0(j)-1);
            end
            l = length(a2);
            if   isnumeric(K)==1 &&  isnumeric(a)==1
                  mT = zeros(m+1, l+1);
                  e = zeros(n-1, 1);
                  for   k=1:n-1
                        e(k) = eta(k, a, K);
                  end
            else
                  mT = sym(zeros(m+1,l+1));
                  syms e1 e2 e3 e4 e5 e6 e7;
                  e = sym(zeros(7,1));
                  e(1) = e1; e(2) = e2; e(3) = e3; e(4) = e4; e(5) = e5; e(6) = e6; e(7) = e7;
            end
            mT(m+1, l+1) = 1;
            s1 = 0;
            for  ii=m:-1:1
                  s1 = s1 + a1(ii);
                % mT(ii, l+1) = mT(ii+1,l+1)/(s1+e(s1));
                  mT(ii,l+1) = mT(ii+1,l+1)/(s1+(s1^a)*K/2);
                  s2 = 0;
                  for  jj=l:-1:1
                        s2 = s2 + a2(jj);
                      % mT(m+1, jj) = mT(m+1,jj+1)/(s2+e(s2));
                        mT(m+1,jj) = mT(m+1,jj+1)/(s2+(s2^a)*K/2);
                        if  abs(s2-s1)==0
                            mT(ii,jj) = (mT(ii+1,jj)+mT(ii,jj+1))/(s1+s2);
                        else
                        % mT(ii,jj) = (mT(ii+1,jj)+mT(ii,jj+1))/(s1+s2+e(abs(s2-s1)));
                            mT(ii,jj) = (mT(ii+1,jj)+mT(ii,jj+1))/(s1+s2+(abs(s2-s1))^a*K/2);
                        end
                      % mT(ii,jj) = (mT(ii+1,jj)+mT(ii,jj+1))/(s1+s2+(abs(s2-s1))^a*K/2);
                  end
            end
            if  i==j
                  R = R + C(i)*C(i)*mT(1,1);
            else
                  R = R + 2*C(i)*C(j)*mT(1,1);
            end
      end
end
if  isnumeric(K)==0
     R = factor(simplify(R));
end
```

Appendix D

Maple Code for computing coefficients of some functions

Function *Coeffs_Gzz* for the coefficients of $G(z,\bar{z}) := \mathbb{E}\left(|z|^q \frac{|f'(z)|^p}{|f(z)|^q}\right)$.

$$
\begin{array}{l}
\mathit{Coeffs\_Gzz} := \mathbf{proc}(k,n,m,p,q) \quad \sharp \text{ function computes the coefficients of } G(z,\bar{z}) := \mathbb{E}\left(|f'(z)|^p/|f(z)/z|^q\right)\\
\mathbf{option}\ \ \mathit{remember}; \quad \sharp \text{ parameters } p,q\in\mathbb{R}\\
\mathbf{if}\ \min(n,m)<0\ \mathbf{then\ return}\ 0\ \mathbf{elif}\ (n,m)=(0,0)\ \mathbf{then\ return}\ 1;\\
\mathbf{else\ return}\\
\mathit{factor}\Bigg(\mathit{simplify}\Bigg(\frac{1}{\frac{k}{2}\cdot(n-m)^2+n+m}\cdot\Bigg(\Big(-\frac{k}{2}\cdot(n-m)^2+n+m+2\cdot(p-q-2)\Big)\cdot \mathit{Coeffs\_Gzz}(k,n-2,m-2,p,q)\\
\quad +\Big(k\cdot(n-m-1)^2-2\cdot(n-2)-4\cdot p+3\cdot q\Big)\cdot\mathit{Coeffs\_Gzz}(k,n-2,m-1,p,q)\\
\quad +\Big(-\frac{k}{2}\cdot(n-m-2)^2+(n-m-2)+p-q\Big)\cdot\mathit{Coeffs\_Gzz}(k,n-2,m,p,q)\\
\quad +\Big(k\cdot(n-m+1)^2-2\cdot(m-2)+3\cdot q-4\cdot p\Big)\cdot\mathit{Coeffs\_Gzz}(k,n-1,m-2,p,q)\\
\quad +\Big(-2\cdot k\cdot(n-m)^2+8\cdot p-4\cdot q\Big)\cdot\mathit{Coeffs\_Gzz}(k,n-1,m-1,p,q)\\
\quad +\Big(k\cdot(n-m-1)^2+2\cdot m+q-2\cdot p\Big)\cdot\mathit{Coeffs\_Gzz}(k,n-1,m,p,q)\\
\quad +\Big(-\frac{k}{2}\cdot(n-m+2)^2+m-n+p-q-2\Big)\cdot\mathit{Coeffs\_Gzz}(k,n,m-2,p,q)\\
\quad +\Big(k\cdot(n-m+1)^2+2\cdot n+q-2\cdot p\Big)\cdot\mathit{Coeffs\_Gzz}(k,n,m-1,p,q)\Bigg)\Bigg)\Bigg);\\
\mathbf{end\ if};\\
\mathbf{end\ proc};
\end{array}
$$

Function *Coeffs_Gzξ* for the coefficients of $G(z,\xi) := \mathbb{E}\left((z-\xi)^q \frac{f'^p(z)f'^p(\xi)}{(f(z)-f(\xi))^q}\right)$.

$$
\begin{array}{l}
\mathit{Coeffs\_Gz\xi} := \mathbf{proc}(k,n,m,p,q) \quad \sharp \text{ function computes the coefficients of } G(z,\xi) := \mathbb{E}\left((z-\xi)^q\frac{f'^p(z)f'^p(\xi)}{(f(z)-f(\xi))^q}\right)\\
\mathbf{option}\ \ \mathit{remember}; \quad \sharp\ \mathit{parameters}\ p,\ q\in\mathbb{R}\\
\mathbf{if}\ \min(n,m)<0\ \mathbf{then\ return}\ 0\ \mathbf{elif}\ (n,m)=(0,0)\ \mathbf{then\ return}\ 1;\\
\mathbf{else\ return}\\
\mathit{factor}\Bigg(\mathit{simplify}\Bigg(\frac{-2}{k\cdot(n+m)^2+2\cdot(n+m)}\cdot\Bigg(\Big(\frac{k}{2}\cdot(n+m-4)^2-(n+m-4)+2\cdot(q-2\cdot p)\Big)\cdot\mathit{Coeffs\_Gz\xi}(k,n-2,m-2,p,q)\\
\quad +\Big(-k\cdot(n+m-3)^2+2\cdot(n-2)+4\cdot(2\cdot p-q)\Big)\cdot\mathit{Coeffs\_Gz\xi}(k,n-2,m-1,p,q)\\
\quad +\Big(\frac{k}{2}\cdot(n+m-2)^2+(m-n+2)+2\cdot(q-p)\Big)\cdot\mathit{Coeffs\_Gz\xi}(k,n-2,m,p,q)\\
\quad +\Big(-k\cdot(n+m-3)^2+2\cdot(m-2)+4\cdot(2\cdot p-q)\Big)\cdot\mathit{Coeffs\_Gz\xi}(k,n-1,m-2,p,q)\\
\quad +\Big(2\cdot k\cdot(n+m-2)^2+6\cdot q-16\cdot p\Big)\cdot\mathit{Coeffs\_Gz\xi}(k,n-1,m-1,p,q)\\
\quad +\Big(-k\cdot(n+m-1)^2-2\cdot m+2\cdot(2\cdot p-q)\Big)\cdot\mathit{Coeffs\_Gz\xi}(k,n-1,m,p,q)\\
\quad +\Big(\frac{k}{2}\cdot(n+m-2)^2+(n-m+2)+2\cdot(q-p)\Big)\cdot\mathit{Coeffs\_Gz\xi}(k,n,m-2,p,q)\\
\quad +\Big(-k\cdot(n+m-1)^2-2\cdot n+2\cdot(2\cdot p-q\Big)\cdot\mathit{Coeffs\_Gz\xi}(k,n,m-1,p,q)\Bigg)\Bigg)\Bigg);\\
\mathbf{end\ if};\\
\mathbf{end\ proc};
\end{array}
$$

Function *Coeffs_ Loga* for the coefficients of $G(z,\bar z) := \mathbb{E}\left(|\log f'(z)|^2\right)$ (for SLE$_2$).

Coeffs_ Loga := **proc**(n,m) ♯ function computes the coefficients of $G(z,\bar z) := \mathbb{E}(|\log f'|^2)$ *when* $\kappa = 2$
option *remember*;
if $n<0$ **then return** 0 **elif** $m<0$ **then return** 0
elif $(n,m)=(0,0)$ **then return** 0 **elif** $(n,m)=(1,1)$ **then return** 8
elif $(n,m)=(1,0)$ **then return** 0 **elif** $(n,m)=(0,1)$ **then return** 0
elif $(n,m)=(2,0)$ **then return** 0 **elif** $(n,m)=(0,2)$ **then return** 0
elif $(n,m)=(2,1)$ **then return** $-\frac{2}{3}$ **elif** $(n,m)=(1,2)$ **then return** $-\frac{2}{3}$
elif $(n,m)=(2,2)$ **then return** $\frac{5}{3}$
elif $(n,m)=(3,0)$ **then return** 0 **elif** $(n,m)=(0,3)$ **then return** 0
elif $(n,m)=(3,1)$ **then return** $\frac{4}{9}$ **elif** $(n,m)=(1,3)$ **then return** $\frac{4}{9}$
elif $(n,m)=(3,2)$ **then return** $-\frac{10}{27}$ **elif** $(n,m)=(2,3)$ **then return** $-\frac{10}{27}$

elif $n\ge 4$ **and** $m=2$ **then return**
$$\mathit{factor}\Big(\mathit{simplify}\Big(\tfrac{1}{n+m+(n-m)^2}\cdot\Big((n+m-4-(n-m)^2)\cdot \mathit{Coeffs\_Loga}(n-2,m-2)$$
$$+\,2\cdot((n-m-1)^2+2-n)\cdot \mathit{Coeffs\_Loga}(n-2,m-1)+(n-m-2)\cdot(m-n+3)\cdot \mathit{Coeffs\_Loga}(n-2,m)$$
$$+\,2\cdot((n-m+1)^2+2-m)\cdot \mathit{Coeffs\_Loga}(n-1,m-2) - 4\cdot(n-m)^2\cdot \mathit{Coeffs\_Loga}(n-1,m-1)$$
$$+\,2\cdot((n-m-1)^2+m)\cdot \mathit{Coeffs\_Loga}(n-1,m)+(n-m+2)\cdot(m-n-3)\cdot \mathit{Coeffs\_Loga}(n,m-2)$$
$$+\,2\cdot((n-m+1)^2+n)\cdot \mathit{Coeffs\_Loga}(n,m-1)-\tfrac{8}{3\cdot(n-2)\cdot(n-1)\cdot n}\Big)\Big)\Big)$$

elif $n\ge 4$ **and** $m=1$ **then return**
$$\mathit{factor}\Big(\mathit{simplify}\Big(\tfrac{1}{n+m+(n-m)^2}\cdot\Big((n+m-4-(n-m)^2)\cdot \mathit{Coeffs\_Loga}(n-2,m-2)$$
$$+\,2\cdot((n-m-1)^2+2-n)\cdot \mathit{Coeffs\_Loga}(n-2,m-1)+(n-m-2)\cdot(m-n+3)\cdot \mathit{Coeffs\_Loga}(n-2,m)$$
$$+\,2\cdot((n-m+1)^2+2-m)\cdot \mathit{Coeffs\_Loga}(n-1,m-2) - 4\cdot(n-m)^2\cdot \mathit{Coeffs\_Loga}(n-1,m-1)$$
$$+\,2\cdot((n-m-1)^2+m)\cdot \mathit{Coeffs\_Loga}(n-1,m)+(n-m+2)\cdot(m-n-3)\cdot \mathit{Coeffs\_Loga}(n,m-2)$$
$$+\,2\cdot((n-m+1)^2+n)\cdot \mathit{Coeffs\_Loga}(n,m-1)+\tfrac{16}{3\cdot(n-2)\cdot(n-1)\cdot n}\Big)\Big)\Big)$$

elif $n=2$ **and** $m\ge 4$ **then return**
$$\mathit{factor}\Big(\mathit{simplify}\Big(\tfrac{1}{n+m+(n-m)^2}\cdot\Big((n+m-4-(n-m)^2)\cdot \mathit{Coeffs\_Loga}(n-2,m-2)$$
$$+\,2\cdot((n-m-1)^2+2-n)\cdot \mathit{Coeffs\_Loga}(n-2,m-1)+(n-m-2)\cdot(m-n+3)\cdot \mathit{Coeffs\_Loga}(n-2,m)$$
$$+\,2\cdot((n-m+1)^2+2-m)\cdot \mathit{Coeffs\_Loga}(n-1,m-2) - 4\cdot(n-m)^2\cdot \mathit{Coeffs\_Loga}(n-1,m-1)$$
$$+\,2\cdot((n-m-1)^2+m)\cdot \mathit{Coeffs\_Loga}(n-1,m)+(n-m+2)\cdot(m-n-3)\cdot \mathit{Coeffs\_Loga}(n,m-2)$$
$$+\,2\cdot((n-m+1)^2+n)\cdot \mathit{Coeffs\_Loga}(n,m-1)-\tfrac{8}{3\cdot(m-2)\cdot(m-1)\cdot m}\Big)\Big)\Big)$$

elif $n=1$ **and** $m\ge 4$ **then return**
$$\mathit{factor}\Big(\mathit{simplify}\Big(\tfrac{1}{n+m+(n-m)^2}\cdot\Big((n+m-4-(n-m)^2)\cdot \mathit{Coeffs\_Loga}(n-2,m-2)$$
$$+\,2\cdot((n-m-1)^2+2-n)\cdot \mathit{Coeffs\_Loga}(n-2,m-1)+(n-m-2)\cdot(m-n+3)\cdot \mathit{Coeffs\_Loga}(n-2,m)$$
$$+\,2\cdot((n-m+1)^2+2-m)\cdot \mathit{Coeffs\_Loga}(n-1,m-2) - 4\cdot(n-m)^2\cdot \mathit{Coeffs\_Loga}(n-1,m-1)$$
$$+\,2\cdot((n-m-1)^2+m)\cdot \mathit{Coeffs\_Loga}(n-1,m)+(n-m+2)\cdot(m-n-3)\cdot \mathit{Coeffs\_Loga}(n,m-2)$$
$$+\,2\cdot((n-m+1)^2+n)\cdot \mathit{Coeffs\_Loga}(n,m-1)+\tfrac{16}{3\cdot(m-2)\cdot(m-1)\cdot m}\Big)\Big)\Big)$$

else return $\mathit{factor}\Big(\mathit{simplify}\Big(\tfrac{1}{n+m+(n-m)^2}\cdot\Big((n+m-4-(n-m)^2)\cdot \mathit{Coeffs\_Loga}(n-2,m-2)$
$$+\,2\cdot((n-m-1)^2+2-n)\cdot \mathit{Coeffs\_Loga}(n-2,m-1)+(n-m-2)\cdot(m-n+3)\cdot \mathit{Coeffs\_Loga}(n-2,m)$$
$$+\,2\cdot((n-m+1)^2+2-m)\cdot \mathit{Coeffs\_Loga}(n-1,m-2) - 4\cdot(n-m)^2\cdot \mathit{Coeffs\_Loga}(n-1,m-1)$$
$$+\,2\cdot((n-m-1)^2+m)\cdot \mathit{Coeffs\_Loga}(n-1,m)+(n-m+2)\cdot(m-n-3)\cdot \mathit{Coeffs\_Loga}(n,m-2)$$
$$+\,2\cdot((n-m+1)^2+n)\cdot \mathit{Coeffs\_Loga}(n,m-1)\Big)\Big)\Big)$$
end if;
end proc;

Function *Coeffs_* $F\overline{F}$ for the coefficients of $\mathbb{E}(\log f'(z))\cdot\overline{\mathbb{E}(\log f'(z))}$ (for SLE$_2$).

*Coeffs_* $F\overline{F}$:= **proc**(n,m)
♯ auxiliary function computes the coefficients of $F(z)\overline{F(z)} := \mathbb{E}(\log f'(z))\,\overline{\mathbb{E}(\log f'(z))}$ *when* $\kappa=2$
♯ $G(z,\bar z) = F(z)\overline{F(z)} + R(z,\bar z)$ with $G(z,\bar z) = \mathbb{E}\left(|\log f'(z)|^2\right)$
option *remember*;
if $n=0$ **then return** 0 **elif** $m=0$ **then return** 0 **elif** $(n,m)=(1,1)$ **then return** 4
elif $n=1$ **and** $m\ge 2$ **then return** $\frac{4}{3\cdot m}$ **elif** $n\ge 2$ **and** $m=1$ **then return** $\frac{4}{3\cdot n}$
elif $n\ge 2$ **and** $m\ge 2$ **then return** $\frac{4}{9\cdot n\cdot m}$
end if;
end proc;

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List of Figures

Thanh Binh LE

SUR LE PROBLÈME DE COEFFICIENT ET LA MULTIFRACTALITÉ DE WHOLE-PLANE SLE

Résumé :

Le point de départ de cette thèse est la conjecture de Bieberbach : sa démonstration par De Branges utilise deux ingrédients, à savoir la théorie de Loewner des domaines plans croissants et une inégalité de Milin qui concerne les coefficients logarithmiques. Nous commençons par étudier les coefficients logarithmiques du whole-plane SLE en utilisant une méthode combinatoire, assistée par ordinateur.
Nous retrouvons les résultats en utilisant une équation au dérivées partielles analogue à celle obtenue par Beliaev et Smirnov. Nous généralisons ces résultats en définissant le spectre généralisé du whole-plane SLE, que nous calculons par la même méthode, à savoir en dérivant, par le calcul d'Itô, une EDP parabolique satisfaite par les quantités que nous moyennons. Cette famille à deux paramètres d'EDP admet une riche structure algébrique que nous étudions en détail.
La dernière partie de la thèse concerne l'opérateur de Grunsky et ses généralisations. Plus expérimentale, nous y mettons à jour, grâce à un logiciel de calcul formel, une structure assez complexe dont nous avons commencé l'exploration.

Mots clés : Whole-plane SLE, moments logarithmiques, équation de Beliaev-Smirnov, spectre généralisé, variance asymptotique de McMullen, coefficients de Grunsky.

ON THE COEFFICIENT PROBLEM AND MULTIFRACTALITY OF WHOLE-PLANE SLE

Abstract :

The starting point of this thesis is Bieberbach's conjecture: its proof, given by De Branges, uses two ingredients, namely Loewner's theory of increasing plane domains and an inequality from Milin about the logarithmic coefficients. We start with a study of the logarithmic coefficients of the whole-plane SLE by using a combinatorial method, assisted by computer.
We find the results by using a partial differential equation similar to that obtained by Beliaev and Smirnov. We generalize these results by defining the generalized spectrum of the whole-plane SLE, that we calculate by the same method, namely by deriving, thanks to Itô calculus, a parabolic PDE satisfied by the quantities of which we take the average. This two-parameter family of PDEs admits a rich algebraic structure that we study in detail.
The last part of this thesis is about the Grunsky operator and its generalizations. In this part that is more experimental we update, thanks to a computer algebra system, a rather complex structure of which we began the exploration.

Keywords : Whole-plane SLE, logarithmic moments, Beliaev-Smirnov equation, generalized spectrum, McMullen's asymptotic variance, Grunsky coefficients.

MAPMO UMR 6628 - CNRS
Rue de Chartres - BP6759
45067 ORLÉANS CEDEX