Two dimensional Maximal Supergravity, Consistent Truncations and Holography

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Supergravité maximale bidimensionnelle, troncatures cohérentes et holographie

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Résumé de la thèse

À la fin des années 1990, Juan Maldacena a fait une découverte majeure qui a révolutionné le domaine de la théorie des cordes. Il s'agit d'une correspondance ou plutôt d'une équivalence entre une théorie de la gravité et une théorie semblable à celle des interactions fortes. Cette correspondance établit un pont entre deux domaines de la physique des hautes énergies, en apparence très différents : la gravité quantique d'une part et les modèles de Yang-Mills de la théorie quantique des champs.

Enfin, elle a émergé de deux éléments pionniers. Le premier concerne le travail de Gerard 't Hooft, en 1973, sur la limite d'un grand nombre de couleurs des modèles de Yang-Mills, et leur ressemblance avec les modèles de résonance duale construits par Gabriele Veneziano dans son approche de l'interaction forte. Ces modèles préfigurent les premières théories des cordes proposées par Yoichiro Nambu, Holger Bech Nielsen et Leonard Susskind au début des années 1970.

Le deuxième élément clef repose sur la découverte en 1995 par Joseph Polchinski d'objets dynamiques étendus, appelés branes de Dirichlet ou D-Branes, pouvant être décrits par une théorie des super-cordes. En particulier, la donnée de N D3-branes coïncidentes, décrites par la théorie des cordes de type IIB, sert de cadre pour la découverte de Maldacena. Dans ce contexte, en 1998, une équivalence a pu être établie entre la dynamique des cordes super-symétriques de type IIB sur un espace-temps anti de Sitter (AdS) et une théorie de Yang-Mills super-conforme (sCFT) avec groupe de jauge SU(N). Plus précisément, on peut penser la théorie des cordes contenant l'interaction gravitationnelle, comme « vivant » à l'intérieur de l'espace-temps anti de Sitter, et la théorie des champs de Yang-Mills (semblable à la théorie de l'interaction forte) comme « résidant » au bord de l'espace anti de Sitter. Dans cette configuration, la correspondance apparait comme holographique, dans le sens où toute l'information d'un objet se trouvant à l'intérieur d'un espace peut être « encodée » dans la surface constituant la bordure de cet espace.

Pour mieux comprendre la facette gravitationnelle ou anti de Sitter de cette correspondance AdS/CFT, il faut pouvoir décrire la théorie des super-cordes de type IIB sur un état fondamental assez compliqué : un espace-temps produit entre anti de Sitter à cinq dimensions et une sphère de dimension cinq.

Heureusement, il existe un régime de la correspondance qui est plus faible, mais plus simple à manipuler. Il s'agit de la limite à basses énergies de la théorie des cordes. Dans cette limite, la théorie effective décrivant les super-cordes est une théorie de supergravité maximale qui de plus possède comme groupe de jauge SO(6), provenant du groupe des isométries de la sphère à cinq dimensions. Cette idée, de rechercher la théorie effective de supergravité maximale pertinente pour l'étude d'une correspondance de type

gravité/théorie de Yang-Mills, constitue le point de départ de ce travail de thèse.

En effet, dans cette thèse nous nous sommes intéressés à une version généralisée de la correspondance AdS/CFT, concernant la théorie des super-cordes de type IIA sur un espace-temps produit entre anti de Sitter à deux dimensions et la sphère de dimensions huit. Cette théorie rend compte de la dynamique de N D0-branes coïncidentes, et sa théorie de Yang-Mills duale n'est pas invariante conforme. Cette dualité est intéressante car d'une part, dans sa version à basse énergie, elle met en jeu, du côté de la gravité, la théorie de supergravité maximale à deux dimensions avec groupe de jauge SO(9). D'autre part, la théorie de Yang-Mills duale n'est rien d'autre que le modèle de matrice BFSS, proposé comme une formulation de la théorie M sous-jacente aux cinq théories des super-cordes.

Dans un premier temps, avec mon directeur de thèse le professeur Henning Samtleben, nous avons construit la théorie de supergravité maximale à deux dimensions avec groupe de jauge SO(9). En effet, cette construction n'avait jamais été réalisée et ce résultat est venu compléter le tableau des supergravités maximales jaugées décrivant la dynamique effective de l'ensemble des Dp-branes impliquées dans la correspondance AdS/CFT, ainsi que sa généralisation aux cas non-conformes. Par ailleurs ce travail est intéressant du point de vue de la supergravité, car il constitue une première déformation non-triviale de la supergravité maximale à deux dimensions, reconnue pour ses propriétés de symétrie étendue, organisée par une algèbre de Kac-Moody exceptionnelle.

Le deuxième résultat de cette thèse renoue le lien entre la supergravité maximale jaugée à deux dimensions et son origine dans la théorie des cordes de type IIA. Dans ce cadre, il a été montré qu'un sous-secteur de la supergravité à deux dimensions pouvait être élevé à dix dimensions dans la supergravité maximale de type IIA, reconnue comme une version basse énergie de la super-corde de type IIA. Cette inclusion à dix dimensions est cohérente, elle permet ainsi de « plonger » plusieurs solutions des équations du mouvement, de deux dimensions à dix. Dès lors, on peut remonter à onze dimensions, le nombre maximum où l'on peut écrire une théorie de supergravité, et où celle-ci est unique. Ce travail effectué avec le professeur Henning Samtleben et le professeur Andrés Anabalón Dupuy de l'Université Adolfo Ibáñez du Chili, constitue le deuxième volet de ma thèse.

Enfin, avec les professeurs Henning Samtleben et Dimitrios Tsimpis de l'IPNL Université Lyon 1, nous avons étudié d'un point de vue holographique, des excitations autour de solutions super-symétriques de la supergravité maximale SO(9) à deux dimensions, et nous en avons extrait des informations sur des fonctions de corrélation dans les modèles matriciels duaux. Ceci résume le troisième volet de ma thèse et conclut l'exposé de l'ensemble des résultats que nous avons obtenus.

Summary

A complete non trivial supersymmetric deformation of the maximal supergravity in two dimensions is achieved by the gauging of a $SO(9)$ group. The resulting theory describes the reduction of type IIA supergravity on an $AdS_2 \times S^8$ background and is of first importance in the Domain-Wall / Quantum Field theory correspondence for the D0-brane case. To prepare the construction of the $SO(9)$ gauged maximal supergravity, we focus on the eleven dimensional supergravity and the maximal supergravity in three dimensions since they give rise to important off-shell inequivalent formulations of the ungauged theory in two dimensions. The embedding tensor formalism is presented, allowing for a general description of the gaugings consistent with supersymmetry. The $SO(9)$ supergravity is explicitly constructed and applications are considered. In particular, an embedding of the bosonic sector of the two-dimensional theory into type IIA supergravity is obtained. Hence, the Cartan truncation of the $SO(9)$ supergravity is proved to be consistent. This motivated holographic applications. Therefore, correlation functions for operators in dual Matrix models are derived from the study of gravity side excitations around half BPS backgrounds. These results are fully discussed and outlooks are presented.

Key words: Maximal supergravities, Gauging, Embedding tensor, Consistent Truncations, Kaluza-Klein reduction, AdS/CFT, Holography, Matrix models, Branes, String Theory, Supergravity, Supersymmetry.

Contents

Chapter 1

Introduction

Supergravity is a wide and fascinating domain of investigation. It stands at the crossroads of General Relativity and Particle Physics, and was first designed for unifying all known fundamental interactions, at the quantum level. It includes a new symmetry, which was not present in Particle Physics: supersymmetry.

This symmetry acts differently, on a new type of coordinates, but when acting twice it surprisingly reproduces a spacetime symmetry. Therefore, its algebraic structure triggered great interest from Mathematics, and an important work was dedicated to Lie superalgebras and their representations. From the physics perspective, supersymmetry turned out to be one of the very restricted possibilities to extend the symmetries of the Standard Model of Particle Physics [1] [2]. If the supersymmetry is promoted to a local one (gauged), which means that any transformation can act independently on each point of spacetime and leave the action (or the equations of motion) invariant, then the theory is automatically invariant under general coordinate transformations (diffeomorphisms). Consequently, a field theory invariant under local supersymmetry can describe gravity. Reciprocally, if a theory of gravity contains supersymmetry, it is automatically realized locally. Therefore, a field theory with local supersymmetry is named: supergravity (see [3] for a detailed review).

These theories have been intensively studied, because they were candidate for a quantum theory of gravity. However, such a perspective was later abandoned, because most of them were non-renormalizable (ill-defined) and because the field met another important one: String theory. Indeed, it was demonstrated that particular supergravity theories in ten dimensions described the low-energy effective dynamics of supersymmetric strings. This generated new interests in the maximally supersymmetric theories of gravity and their possible deformations. In this context, deformations were studied by promoting global symmetry groups to local ones. Among them, maximal supergravities with local symmetry (gauge) group $SO(n)$, coming from the isometry groups of spheres, were involved in the revolutionary proposal of Maldacena [4]: the gauge/gravity correspondence called AdS/CFT and its extensions.

This chapter aims to provide a brief review of the birth of maximal gauged supergravities, their importance for String theory and our contribution to this field.

Supersymmetry was born in 1971 from the suggestion of Gol'fand and Likhtman that the Poincaré algebra could be non-trivially extended to a super-algebra containing fermionic conserved charges [5]. It has the structure of a $\mathbb{Z}_2$-graded Lie algebra mixing with the bosonic spacetime symmetries :

$$[B\,,\,B] = B\,, \qquad [B\,,\,F] = F\,, \qquad \{F\,,\,F\} = B\,. \tag{1.0.1}$$

The fermionic supercharges are constant spinors whose number is denoted by $\mathcal{N}$. When $\mathcal{N} > 1$, the supersymmetric theory is called extended. Eventually, the irreducible representations of the superalgebra account for the (super) particles.

The most popular quantum field theory including supersymmetry in four dimensions was constructed three years later by Wess and Zumino [6]. In 1974 again, the $\mathcal{N} = 1$, $D = 4$ Wess-Zumino model was extended to include internal symmetries, and the framework of superfields was developed to construct supersymmetric field theories in four dimensions [7].

At that time, two elegant features filled the community with enthusiasm. Firstly, the field theoretical divergences appeared to be softer when global supersymmetry is included [8]. For instance, the Wess-Zumino model was shown to be renormalizable [9].

Secondly, Haag, Lopuszański and Sohnius classified all the possible conformal and Poincaré superalgebras compatible with the Quantum Field Theory assumptions in four dimensions [2]. This generalized the famous no-go theorem of Coleman and Mandula [1] and opened the path to the study of the superalgebras and their representations [10] [11] [12]. A consequence of this work states that any particle has a super-partner of the same mass, and their spins must differ by 1/2. Furthermore, in any supersymmetric quantum field theory, there must be an equal number of bosonic and fermionic degrees of freedom. As an illustration, a supersymmetric extension of the Standard Model would contain the following particle content [13]

Particle	Spin	Spartner	Spin
quark: q	$\frac{1}{2}$	squark: $\tilde{q}$	0
lepton: l	$\frac{1}{2}$	slepton: $\tilde{l}$	0
photon: γ	1	photino: $\tilde{\gamma}$	$\frac{1}{2}$
W	1	wino: $\tilde{W}$	$\frac{1}{2}$
Z	1	zino: $\tilde{Z}$	$\frac{1}{2}$
Higgs: H	0	higgsino: $\tilde{H}$	$\frac{1}{2}$

Table 1.1: Particles in the Standard Model and their supersymmetric partners.

However, no superpartner particle has been discovered at the scales of energy investigated, therefore supersymmetry must be broken. Fortunately, spontaneous supersymmetry breaking mechanisms have been elaborated, for example in [14].

Later, it was shown that the three gauge coupling constants of the standard model do not converge to the same value when the energy grows, whereas they do meet each other at an energy of 10^{16} GeV if supersymmetry is included, see Figure 1.1 and [16], [17], [18] and [19] for details.

This may be a clue that supersymmetry is the right framework for unifying the fundamental interactions.

Supergravity The last known fundamental force: gravity, is maybe the most intriguing one, but it has been excluded from the previous discussion. Indeed, only globally supersymmetric quantum field theories were considered. Actually, finding a quantum description of gravity is one of the most difficult challenge of Theoretical Physics. It would thereby allow for a complete understanding of the early universe and the black holes.

However, as we saw before, when supersymmetry is gauged, gravity is described. The idea has soon been put in practice, when in 1976, Freedman, Ferrara, van Nieuwenhuizen

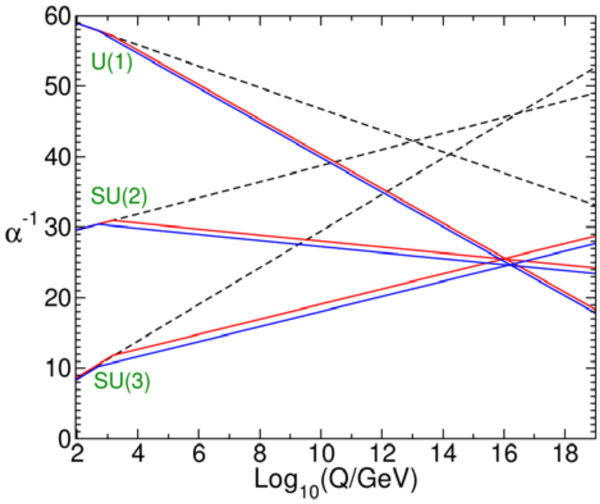

Figure 1.1: Two-loop renormalization group evolution of the inverse gauge couplings α^{-1} in the Standard Model (dashed lines) and in the Minimal Supersymmetric Standard Model (solid lines). The sparticle masses are treated as a common threshold varied between 500 GeV and 1.5 TeV. [15]

[20], and independently Deser and Zumino [21], constructed the first $\mathcal{N} = 1$ supergravity in four dimensions. It is a theory of pure gravity (General relativity) extended by supersymmetry. It thus describes a spin two graviton and its superpartner: a spin 3/2 fermion called gravitino [22], [23].

Coupling to matter was then realized [24] as well as increasing the number of supersymmetries [25]. Actually, Nahm already noticed that the number of charges generating supersymmetry is bounded from the top if the interacting theory is to be restricted to spins lower than two. Namely, the total number of real components of the supersymmetry generators Q should not exceed 32

$$\mathcal{N} \times \dim_{\mathbb{R}} Q \leq 32 \ . \tag{1.0.2}$$

This is a necessary requirement since no consistent interaction is known for spins $s \geq 5/2$ [26] [27] [28] [29]. The bound is lowered to 16 real components for globally supersymmetric field theories (without gravity) where the maximal spin involved is equal to one. Eventually, a general Yang-Mills-matter-supergravity system with $\mathcal{N} = 1$ local supersymmetry and arbitrary gauge group G was constructed in [30], paving the way to phenomenological applications.

Nevertheless, the lack of renormalizability property forbade to establish the supergravities at the quantum level. Therefore, the investigation was directed towards extended supergravities, provided that supersymmetry can be further broken [30]. Indeed, the more supercharges are present, the softer are field theoretical divergences. For instance, maximal supergravity in four dimensions might be finite [31]. This $\mathcal{N} = 8$, $D = 4$ supergravity [32] [33] was built by dimensional reduction of the unique supergravity in eleven dimensions, discovered in 1978 by Cremmer, Julia and Scherk [34].

Eleven dimensional spacetime is particular for supergravity, since, according to the Nahm bound on supercharges, supergravities can only be constructed in $D \leq 11$. This is why eleven dimensional supergravity is called "maximal". Besides, Witten showed that eleven is the minimum number of dimensions if the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ is to be recovered in four dimensions by dimensional reductions

[35]. Therefore, several questions arose whether a unique (higher dimensional) theory had been found and could lead to the Standard Model in some limit while providing on another hand a quantum description of gravity. This renewed also the interest for Kaluza-Klein ideas of compactifying dimensions on such small scales that they are hidden from the current experiments [36]. The simplest and most widely studied example, is the reduction of eleven-dimensional supergravity on tori. It leads directly to maximal supergravities in lower dimensions. The massless sector of the non-chiral $\mathcal{N} = 2$, $D = 10$ (named Type IIA) supergravity was thus constructed [37]. Not all maximal supergravities can be generated from the eleven-dimensional one, so is the case of the chiral $\mathcal{N} = 2$, $D = 10$ (Type IIB) supergravity [38], [39] and [40]. Furthermore, the maximal supergravities obtained from the eleven dimensional one or the type IIB, by Kaluza-Klein reduction on tori, do not have a non-abelian gauge group. They are called "ungauged" maximal supergravities. The only known way to further deform these theories while preserving supersymmetry is done by gauging global symmetries. A general framework for accomplishing this task has been developed in [41]. Let us note that the compactification on more complicated manifold leads to low-dimensional theories whose gauge group contains the isometry group of the compactifying space and thus may be non-abelian. For example, the reduction of eleven-dimensional or type IIB supergravities on n-dimensional spheres gives rise to maximal supergravities with gauge group $SO(n + 1)$ corresponding to the isometry group of the sphere. Nonetheless, it is hard to perform the reduction explicitly, even if the consistency has been demonstrated for several cases [42] [43].

Despite the appealing properties of eleven-dimensional supergravity, it was soon realized that it could not stand for a unifying quantum field theory of all fundamental interactions. Indeed, the supergravity was plagued by severe problems: above all, Kaluza-Klein reduction of eleven-dimensional supergravity does not lead to a chiral theory in four dimensions [35], thus the Standard Model of Particle Physics cannot be reproduced. Moreover it contains gravitational anomalies [44] (where by anomaly we mean a classical symmetry that does not hold at the quantum level, see [45], [46] and [47] for more details) and the dimensionally reduced theory might not be renormalizable. The search for anomaly cancellation in a possibly unique high dimensional chiral theory is what brought the community to a major result which deeply transformed the motivations for understanding supergravities.

This started from the discovery that the chiral type IIB supergravity is anomaly free [44]. Then, Green and Schwarz showed that in ten dimensions, the $\mathcal{N} = 1$ Einstein-Yang-Mills supergravity is also anomaly free, provided that the Yang-Mills gauge group is chosen to be $SO(32)$ or $E_8 \times E_8$ [48]. For the $SO(32)$ group, the anomaly cancellation implied to add new higher-derivative terms that precisely match the low-energy expansion of a string theory: the Heterotic type $SO(32)$ supersymmetric string. Following this work, the Heterotic type $E_8 \times E_8$ superstring theory was built [49], and it was fully recognized that the type I, IIA, IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$ supergravities in ten dimensions are low-energy limits of supersymmetric string theories. As a consequence, supergravity can be thought of as an effective description of more fundamental objects: elementary strings. In addition, the discovery of the heterotic superstrings made conceivable the connection with the Standard Model [50] and thus triggered great enthusiasm.

String theory and the AdS/CFT correspondence The first theory of relativistic strings was considered way before the birth of supergravity. It begins in 1968 with the construction of dual resonance models by Gabriele Veneziano [51], in an attempt to understand the Strong interaction. Later overcome by the more predictive Quantum Chromo

Dynamics [52] [53], the beauty of the Veneziano model prevented it from disappearing, and it was soon realized that relativistic strings, whose vibrating modes represent elementary particles, were actually described [54] [55] [56]. Eventually, Schwarz, Scherk [57] and Toneya [58] understood that a relativistic theory of strings could contain gravitons, the mediating particle of the gravitational interaction. Thus, from a model of the strong interaction it became a theory of quantum gravity. Indeed, strings generalize point particles because they are one-dimensionally extended, and this simple fact remarkably allows for a quantum description that is prevented from divergences [59].

The coupling to fermions was done in a theory that incorporates, as a key ingredient, supersymmetry [60] [61]. The supersymmetric string theories are required to live in ten dimensions by consistency, and an important work including the study of anomalies enabled to draw the picture of all possible superstrings. There are only five of them, known as: Type I, IIA and IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. Furthermore, Witten realized that due to the dualities relating each others [62] [63] [64] , they may come from a unique eleven-dimensional theory: the so-called M-theory [65], whose low-energy limit is eleven-dimensional supergravity. This revived the idea of a unifying quantum theory underlying all elementary interactions, and gave to the field new motivations.

Finally, one may wonder whether more (spatially) extended relativistic objects can be considered, such as membranes or volumes moving through spacetime, and whether they could be linked with String theory. The idea has first been put in practice by Dirac [66], still it is very hard in general to formulate a quantum theory for p-dimensional extended objects with $p > 1$. Fortunately, such objects, called p-branes arise as particular solutions (solitons) of supergravities. For example, eleven-dimensional supergravity admits a 2-brane and a 5-brane solution. Several other branes can be derived from type IIA and IIB supergravities. Since, the supergravities are low-energy limit of superstring theories, non-perturbative p-brane solutions are present in String theory. In Table 1.2 are referenced the possible supersymmetric branes of M-theory and type IIA and IIB superstrings [67], with the names they have been given, for $0 \leq p \leq 8$. The cases $p = 0$ and $p = 1$ correspond respectively to a point-like and a one dimensional object. In particular, F1 stands for the type IIA and IIB (Fundamental) strings.

Theory	Branes								
M-theory			M2			M5			
IIA	D0	F1	D2		D4	NS5	D6		D8
IIB		F1 + D1		D3		NS5 + D5		D7	

Table 1.2: The possible superbranes with $p \leq 8$.

Branes are intrinsically non-perturbative objects, however some of them can be viewed as sub-manifolds of spacetime, parametrized by the end-points of open strings. Said differently, by imposing particular boundary conditions on open strings, the Dirichlet condition, their end points can be attached to some hypersurfaces of spacetime. In turn, the hypersurface is interpreted as a brane, called D-brane. Following this interpretation [68], [69], Polchinski demonstrated that the complicated dynamics of the D-branes could be simply described by the interactions of open strings using string perturbation theory [70]. Thus, the dynamics of the D-branes became manageable . This deep result led to an extraordinary discovery which motivates many of the investigation in String theory today: The AdS/CFT correspondence.

The idea came from the observation that a stack of N coinciding parallel D3-branes has a world volume theory that possesses a $U(N)$ Yang-Mills symmetry. On the other hand, as we saw before, the D-brane admits a string theory description. Consequently, the same object, a D-brane, could be described by two different theories: a gauge theory on one side, and a theory of quantum gravity on the other one. In this context, Maldacena proposed in 1998, a correspondence between a particular Yang-Mills gauge theory and a string theory. Namely, he conjectured that the $\mathcal{N}=4$, $D=4$ super-conformal Yang-Mills theory with gauge group $SU(N)$ is equivalent, in the large N limit, to type IIB string theory on the spacetime background $AdS_5 \times S^5$. The gauge theory would live on the boundary of AdS spacetime, providing all the information about string (or supergravity) excitations propagating in the bulk. As a result, this correspondence is often named Holography. Maldacena's conjecture [4], offers a new dimension to 't Hooft first intuition [71] and had huge consequences because it connects two different domains of High energy physics: quantum gravity and Yang-Mills theory. Soon, a precise framework has been developed, allowing for concrete tests, [72] [73] and a generalization to the non-conformal case was considered [74].

The same idea can be applied to a general stack of N coinciding and parallel Dp-branes, whose description is provided by either type IIA or IIB superstring theory on an $AdS_{p+2} \times S^{8-p}$ background [75]. The corresponding $SU(N)$ super-Yang-Mills theory is not conformal invariant for $p \neq 3$. Computations are more easily done in the supergravity low-energy approximation of superstrings, to which Maldacena's conjecture extends. All effective supergravities on $AdS_{p+2} \times S^{8-p}$ have been constructed so far, except for $p=0$. They are the maximal supergravities in $(p+2)$ dimensions, with gauge group $SO(8-p+1)$ coming from the reduction over the S^{8-p} spheres. Having at hand the supergravity enables to explore deeply the gauge theory side by applying holographic techniques. Since for $p=0$, the corresponding gauge theory is the BFSS Matrix model proposed as a formulation of M-theory [76], its full characterization through holography would be of great interest.

The construction of the maximal $SO(9)$ gauged supergravity in two dimensions was among the objectives of our thesis. It is the first non-trivial supersymmetry preserving deformation of the two-dimensional maximal supergravity, and it is not only interesting from the gravity/gauge correspondence point of view, but also for the mathematical structure of its symmetries, organized in an exceptional Kac-Moody algebra.

Outline of the thesis To this end, we will begin with the presentation of essential features about maximal supergravities, in chapter 2. Our interests lie in the eleven, three and two-dimensional maximal supergravities. As a key ingredient for going from eleven to two dimensions, the Kaluza-Klein method of dimensional reduction will be described. Then, we will explain how to gauge a global symmetry in maximal supergravity, using the embedding tensor formalism. This method is extremely useful to determine the possible gaugings consistent with supersymmetry and provides a concrete way to do it. In chapter 3, the ungauged $\mathcal{N}=16$, $D=2$ maximal supergravity is derived from dimensional reduction of three and eleven-dimensional maximal supergravities. Thus, two on-shell equivalent formulations of the theory are obtained, and each one reveals interesting insights in the maximal two-dimensional supergravity. However, only one frame allows for the consistent $SO(9)$ gauging. The main result of the chapter is the explicit construction of the $SO(9)$ gauged maximal supergravity in two dimensions and is presented in full details. Finally, applications are considered, such as supersymmetric solutions of the BPS equations. Chapter 4 aims at reconciling the two-dimensional theory with its higher dimensional origins. Hence, an explicit framework is established to uplift a sub-sector of the $SO(9)$ gauged maximal supergravity, from two dimensions to the ten-dimensional type

IIA supergravity and then to eleven dimensions. The sub-sector corresponds to a $U(1)^4$ Cartan truncation of the bosonic theory and the full non-linear Kaluza-Klein ansatz is constructed. Then, the truncation is demonstrated to be consistent by computing all the ten-dimensional equations of motion. Eventually, interesting two-dimensional solutions, such as a half-supersymmetric domain-wall, are embedded in ten and eleven dimensions. To conclude, an extension of the uplift to the case of non-vanishing axions is envisaged.

In chapter 5, holographic techniques, devised for the non-conformal cases of the gauge-gravity correspondence [77] will be applied. Thus, after recalling important features about the AdS/CFT correspondence, we will focus on the BFSS model holography. Hence, correlation functions will be computed from the gravity side, by studying excitations around (half-BPS) backgrounds of the $SO(9)$ gauged supergravity. To this end, the holographic renormalization techniques developed in [78] will be presented and put in practice. Previous results about BFSS holography will be recovered [79] [80] as well as new insights in the matrix model holography. Finally, the BMN model, arising as a deformation of the BFSS matrix theory, will be studied. In particular, one and two-points correlation functions will be computed from the gravity side by examining gravity and scalar excitations around a $SO(3) \times SO(6)$ preserving half-supersymmetric background.

Some of the results presented here were already published in [81], [82]. Other results are work in progress [83].

Chapter 2

Maximal Supergravities and its Gauging

2.1 Introduction

Since its discovery in 1998, The AdS/CFT correspondence conjectured in [4] has led to a tremendous number of work and applications. It is based on the properties of a stack of N coinciding parallel D3-branes, in the large N limit, which is described by type IIB superstring theory on an $AdS_5 \times S^5$ background. The isometry group of the AdS spacetime acts as the $SO(2,4)$ conformal group on the four-dimensional AdS boundary. Then, the type IIB string theory on AdS_5 is thought to be equivalent to a conformal $\mathcal{N}=4$, $D=4$ Super Yang-Mills theory living on the AdS boundary. Eventually, the equivalence extends to the supergravity regime of string theory.

This conjecture has been generalized to Dp-branes of type IIA or IIB superstrings, for $\{p=0,1,2,4,5,6\}$ in [75] [74]. However, in this ($p \neq 3$) cases, the near-horizon limit of the brane leads to an $AdS_{p+2} \times S^{8-p}$ spacetime coupled to a non-trivial dilaton. The presence of the dilaton breaks the scale invariance of the AdS isometry group and it gets reduced to the Poincaré group. Thus, the quantum field theory (QFT) living on the boundary is no longer conformal invariant but still is a maximally supersymmetric Yang-Mills theory in $(p+1)$ dimensions. Eventually, because AdS spacetimes with linear dilaton are conformally equivalent to domain-wall (DW) spacetimes [84], the correspondence is named after DW/QFT in this context.

After compactification on the S^{8-p} spheres, the low-energy effective theories that describe the relevant superstring theories on the (warped) AdS_{p+2} background, are $SO(9-p)$ gauged maximal supergravities in $(p+2)$ dimensions [75], where the gauge group comes from the isometries of the spheres. This theories have often been studied well before the AdS/CFT correspondence, in the view of classifying all the possible deformations of maximal supergravities, see Table 2.1. For example, in four dimensions, the $SO(8)$ gauged maximal supergravity was first constructed in [85]. It corresponds to the reduction of eleven-dimensional supergravity on the seven sphere S^7 and from the brane point of view, it accounts for the M2-brane dynamics. The last remaining supergravity in this picture was the $SO(9)$ gauged maximal supergravity in two dimensions. Its construction was indeed difficult because of the very large symmetry structure of the ungauged theory. Indeed, unlike the higher dimensions, the theory has an infinite dimensional symmetry group: the exceptional Lie group E_9. Thus, an $SO(9)$ subgroup needed to be selected inside E_9, and promoted to a local group of symmetry compatible with maximal supersymmetry. The task was huge and the result came in essentially three steps. Firstly, the

Brane	D=10	Background	Effective SUGRA	Ref
D6	IIA	$AdS_8 \times S^2$	$\mathcal{N} = 2\ ,\ D = 8\ ,\ SO(3)$	[86]
D5	IIB	$AdS_7 \times S^3$	$\mathcal{N} = 4\ ,\ D = 7\ ,\ SO(4)$	[87]
D4	IIA	$AdS_6 \times S^4$	$\mathcal{N} = 8\ ,\ D = 6\ ,\ SO(5)$	[88]
D3	IIB	$AdS_5 \times S^5$	$\mathcal{N} = 8\ ,\ D = 5\ ,\ SO(6)$	[89]
D2	IIA	$AdS_4 \times S^6$	$\mathcal{N} = 8\ ,\ D = 4\ ,\ SO(7)$	[90]
D1	IIB	$AdS_3 \times S^7$	$\mathcal{N} = 16\ ,\ D = 3\ ,\ SO(8)$	[91] [92]
D0	IIA	$AdS_2 \times S^8$	$\mathcal{N} = 16\ ,\ D = 2\ ,\ SO(9)$	[81]

Table 2.1: Gauge/Gravity correspondence

maximal ungauged supergravity was constructed and its infinite dimensional symmetries were identified [93]. Then, all the possible gaugings compatible with supersymmetry were determined group theoretically [94]. Finally, the $SO(9)$ gauged maximal supergravity in two dimension has been explicitly constructed in [81] and it constitutes the first result of this thesis. Preparing this discussion, is the goal of the present chapter. Therefore we will present the maximal supergravities in eleven and three dimensions, because they will originate two formulations of the ungauged maximal supergravity in two dimensions, that will be important for understanding the $SO(9)$ gauging. As a key ingredient for dimensional reduction, the Kaluza-Klein torus reduction will be presented as well as the enhancement of symmetries which explains why the target space differs in the two formulations. The ungauged supergravities being discussed, we will study the embedding tensor formalism an explain how a maximal supergravity can be gauged while preserving supersymmetry.

2.2 Maximal Ungauged Supergravities

Maximal ungauged supergravites can all be obtained by Kaluza-Klein reduction of eleven-dimensional or type IIB supergravity, on tori. Here we present the most important of them for our thesis work.

2.2.1 The $\mathcal{N} = 1$, $D = 11$ Supergravity

As explained in the Introduction, the eleven-dimensional supergravity is a very particular one. Indeed, eleven dimensions arises as the maximal number of dimensions for formulating a supergravity theory since if $D > 11$, there is no possibility of matching bosonic and fermionic degrees of freedom in a field theory containing spins less than 2 ($s \leq 2$). Were it possible to construct such a theory in $D > 11$, then by dimensional reduction the resulting four dimensional theory would be a $\mathcal{N} > 8$, $D = 4$ supergravity violating the upper bound of supersymmetries. This is why $\mathcal{N} = 1$ in $D = 11$ and the supergravity is maximal. Moreover, the eleven-dimensional supergravity constructed in [34] is unique. These are certainly among the reasons why the $\mathcal{N} = 1$, $D = 11$ supergravity occupies such an important place in the landscape of maximal supergravities. The next subsections will be devoted to a detailed presentation of this theory.

Field content of the $\mathcal{N} = 1$, $D = 11$ supergravity

As an eleven dimensional gravity theory, the bosonic sector contains the metric $g_{\mu\nu}$. The (quantum) excitations of this gravitational field belong to the traceless symmetric tensor representation of the little group $SO(9)$. Thus it represents

$$\frac{11\,(11-3)}{2}=44 \qquad \text{bosonic on-shell degrees of freedom.} \tag{2.2.1}$$

The supersymmetry partner of the metric is a Majorana spinor gravitino Ψ_μ and transforms as a γ-traceless ($\gamma^\mu\,\Psi_\mu=0$) vector-spinor under the little group. This amounts to

$$(11-3)\,2^{[\frac{11-3}{2}]}=128 \qquad \text{fermionic on-shell degrees of freedom.} \tag{2.2.2}$$

Because of supersymmetry, the theory must contain the same number of on-shell bosonic and fermionic degrees of freedom. This leads to the introduction of a three-form gauge fields $A_{\mu\nu\rho}$ whose excitations transform in the third rank antisymmetric tensor representation of $SO(9)$ accounting for

$$\binom{9}{3}=84=128-44 \qquad \text{bosonic on-shell degrees of freedom.} \tag{2.2.3}$$

This is why E. Cremmer, B. Julia and J. Scherk started from the hypothesis [34] that the field content of the $D=11$ maximal supergravity should be : the metric $g_{\mu\nu}$, the 3-form gauge field $A_{\mu\nu\rho}$ and the Majorana vector-spinor Ψ_μ.

The eleven dimensional Lagrangian

In this section, we review the eleven dimensional action constructed in [34] and present it (see [95] and [96] for reviews).

$$\begin{aligned}\mathcal{S}_{11}=\frac{1}{2\kappa_{11}^2}\int d^{11}x\,e_{11}\Big[&R(\Omega(e,\Psi))-\frac{1}{48}\,F_{\mu_1\ldots\mu_4}F^{\mu_1\ldots\mu_4}-\bar\Psi_\mu\gamma^{\mu\nu\rho}D_\nu\Big(\frac{1}{2}(\Omega+\hat\Omega)\Big)\Psi_\rho\\ &-\frac{1}{192}\left(\bar\Psi_{\mu_1}\gamma^{\mu_1\cdots\mu_6}\Psi_{\mu_2}+12\bar\Psi^{\mu_3}\gamma^{\mu_4\mu_5}\Psi^{\mu_6}\right)\left(F_{\mu_3\ldots\mu_6}+\hat F_{\mu_3\ldots\mu_6}\right)\Big]\\ &-\frac{1}{(12)^4}\,\epsilon^{\mu_1\ldots\mu_{11}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}A_{\mu_9\mu_{10}\mu_{11}}\,,\end{aligned} \tag{2.2.4}$$

in Minkowskian signature (-1,1,. . . ,1) with field strength and supercovariant field strength

$$F_{\mu_1\ldots\mu_4}=4\,\partial_{[\mu_1}A_{\mu_2\mu_3\mu_4]}\,, \quad \text{and} \quad \hat F_{\mu_1\ldots\mu_4}=F_{\mu_1\ldots\mu_4}+3\,\bar\Psi_{[\mu_1}\gamma_{\mu_2\mu_3}\Psi_{\mu_4]} \tag{2.2.5}$$

and

$$\begin{aligned}\Omega_{\mu ab}&=\hat\Omega_{\mu ab}+\frac{1}{8}\,\bar\Psi_\nu\gamma_{\mu ab}{}^{\nu\lambda}\Psi_\lambda\,,\\ \hat\Omega_{\mu ab}&=\Omega^{(0)}_{\mu ab}(e_{11})-\frac{1}{4}\left(\bar\Psi_\mu\gamma_b\Psi_a-\bar\Psi_\mu\gamma_a\Psi_b+\bar\Psi_b\gamma_\mu\Psi_a\right).\end{aligned} \tag{2.2.6}$$

Here, the $[\mu_1\ldots\mu_4]$ means completely anti-symmetric in those indices with weight one, and $\Omega^{(0)}_{\mu ab}(e_{11})$ stands for the torsion-less spin connection associated with the eleven dimensional vielbein $e_{11\,\mu}{}^a$. Moreover, the covariant derivative $D(\Omega)$ is defined by

$$D_\mu(\Omega)\Psi_\nu=\big(\partial_\mu+\frac{1}{4}\Omega_\mu{}^{ab}\gamma_{ab}\big)\Psi_\nu \quad \text{and} \quad [D_\mu,D_\nu]=\frac{1}{4}\,R_{\mu\nu}{}^{ab}\gamma_{ab}\,, \tag{2.2.7}$$

were $R_{\mu\nu}{}^{ab}$ is the curvature tensor of the connection $D(\Omega)$ and $R(\Omega)$ is the Ricci scalar. Finally,

$$\epsilon^{\mu_1\ldots\mu_{11}}\equiv e_{11}\,\epsilon^{a_1\ldots a_{11}}\,e_{11\,a_1}{}^{\mu_1}\ldots e_{11\,a_{11}}{}^{\mu_{11}} \tag{2.2.8}$$

and $\epsilon^{a_1\ldots a_{11}} = -\epsilon_{a_1\ldots a_{11}}$ where the latter Levi-Civita symbol is defined by

$$\epsilon_{a_1\ldots a_{11}} = \begin{cases} +1, \text{ if } a_1\ldots a_{11} \text{ is an even permutation of } 0\ldots 10, \\ \text{-1, if } a_1\ldots a_{11} \text{ is an odd permutation of } 0\ldots 10, \\ 0, \text{ otherwise.} \end{cases} \tag{2.2.9}$$

Supersymmetry

The Lagrangian (2.2.4) is invariant under the local supersymmetry transformations

$$\begin{aligned} \delta_\epsilon\, e_{11\mu}{}^a &= \frac{1}{2}\,\bar\epsilon\,\gamma^a\Psi_\mu\,, \quad \delta_\epsilon A_{\mu_1\mu_2\mu_3} = -\frac{3}{2}\,\bar\epsilon\,\gamma_{[\mu_1\mu_2}\Psi_{\mu_3]}\,, \\ \delta_\epsilon\Psi_\mu &= D_\mu\big(\hat\Omega\big)\epsilon + \frac{1}{2(12)^2}\big(\gamma_\mu{}^{\nu_1\ldots\nu_4} - 8\,\delta_\mu^{\nu_1}\gamma^{\nu_2\nu_3\nu_4}\big)\hat F_{\nu_1\ldots\nu_4}\epsilon\,. \end{aligned} \tag{2.2.10}$$

The commutator of two supersymmetry transformations closes on local symmetries of the theory, provided that the field equations are satisfied. The resulting local symmetries are: a general coordinate transformation (gct), plus a field dependent local Lorentz, supersymmetry and 3-form gauge transformations.

$$\big[\delta_Q(\epsilon_1)\,,\,\delta_Q(\epsilon_2)\big] = \delta_{\text{gct}}\big(\xi^\mu\big) + \delta_L\big(\lambda^{ab}\big) + \delta_Q\big(\epsilon_3\big) + \delta_A\big(\theta_{\mu\nu}\big) \tag{2.2.11}$$

The parameters are given by

$$\begin{aligned} \xi^\mu &= \frac{1}{2}\,\bar\epsilon_2\gamma^\mu\epsilon_1\,, \\ \lambda^{ab} &= -\xi^\mu\,\hat\Omega_\mu^{ab} + \frac{1}{2(12)^2}\,\bar\epsilon_1\big(\gamma^{ab\mu\nu\rho\sigma}\hat F_{\mu\nu\rho\sigma} + 24\,\gamma_{\mu\nu}\hat F^{ab\mu\nu}\big)\epsilon_2\,, \\ \epsilon_3 &= -\xi^\mu\Psi_\mu\,, \\ \theta_{\mu\nu} &= -\xi^\rho A_{\rho\mu\nu} + \frac{1}{2}\,\bar\epsilon_1\gamma_{\mu\nu}\epsilon_2\,. \end{aligned} \tag{2.2.12}$$

The spinors bilinears on the right hand side of λ^{ab} deserve some comments. They come from the global eleven dimensional supersymmetry algebra of the theory. The fermionic charges are represented by Majorana spinors Q_α with 32 real components and their commutator is given by

$$\{Q_\alpha\,,\,Q_\beta\} = \big(\gamma^a\,C^{-1}\big)_{\alpha\beta}P_a + \big(\gamma^{ab}\,C^{-1}\big)_{\alpha\beta}\,Z_{ab} + \big(\gamma^{abcde}\,C^{-1}\big)_{\alpha\beta}\,Z_{abcde}\,, \tag{2.2.13}$$

where C is the charge conjugation matrix defined by

$$\gamma_a{}^T = -\,C\gamma_a C^{-1}\,. \tag{2.2.14}$$

Using the fact that $\Gamma^{(5)}$ can be related to $\Gamma^{(6)}$ in eleven dimensions, notice that the bilinears $\bar\epsilon_1\Gamma^{(2)}\epsilon_2$ and $\bar\epsilon_1\Gamma^{(5)}\epsilon_2$ come from the central charges terms of the global supersymmetry algebra where the corresponding central charges are Z_{ab} and Z_{abcde}. For a general review on this issue, one may look at [96] and [67]. The structure of the theory (2.2.4) is highly constrained by supersymmetry since for example, no free parameter appears inside the Lagrangian.

2.2.2 Low-dimensional effective theories

The eleven-dimensional supergravity ($SUGRA_{11}$) can be used to generate lower dimensional supergravities by dimensional reduction. In particular, toroidal compactification leads to $D < 11$ extended supergravity theories with 32 real supersymmetries, and global symmetry group $GL(11-D) \ltimes \mathbb{R}^q$ with

$$q = \frac{1}{6}(11-D)(10-D)(9-D)\,. \tag{2.2.15}$$

The scalar sector is thus described by a non-linear σ-model with target space $\frac{GL(11-D)}{SO(11-D)} \ltimes \mathbb{R}^q$.

Toroidal compactification

Let us recall precisely the Kaluza-Klein reduction of eleven dimensional supergravity on an arbitrary torus. In the following, we will focus on the bosonic sector. The dimensionally reduced theory will live on a D-dimensional spacetime according to the splitting

$$\mathcal{M}_{11} = \mathcal{M}_D \times T^p\,, \qquad 1 \leq D \leq 10\,, \qquad p \equiv 11 - D\,. \tag{2.2.16}$$

The coordinates on $\mathcal{M}_D$ are denoted by x^μ, ($\mu = 0, \ldots, D-1$) and the coordinates on T^{11-D} are denoted by y^m ($m = 1, \ldots, 11-D$). This decomposition is allowed by assuming the existence of a set of p mutually commuting Killing vector fields on $\mathcal{M}_D$ [97]. An ansatz for the eleven-dimensional vielbein is

$$E_M{}^A = \begin{pmatrix} e_\mu{}^\alpha & \rho^{\frac{1}{p}}\, A_\mu{}^m \mathcal{V}_m{}^a \\ 0 & \rho^{\frac{1}{p}}\, \mathcal{V}_m{}^a \end{pmatrix} \tag{2.2.17}$$

where the local Lorentz invariance is used to set the $e_m{}^\alpha$ coefficients to zero. Following the Kaluza-Klein condition, no dependence on the internal coordinates is assumed. Furthermore, the vielbein $\rho^{\frac{1}{p}}\, \mathcal{V}_m{}^a$ on the internal space has been decomposed into

- its determinant part: $\rho(x)$ which can be interpreted as a scalar field named the "dilaton".
- the determinant one matrix $\mathcal{V}_m{}^a$ which represents $SL(p, \mathbb{R})$-valued scalar fields.

Therefore, the eleven-dimensional vielbein gives rise to: a vielbein, plus scalar fields and vector fields.

$$E_M{}^A \quad \xrightarrow{\;T^p\;} \quad \{\, e_\mu{}^\alpha\,,\, \rho\,,\, \mathcal{V}_m{}^a\,,\, A_\mu{}^m \,\}\,. \tag{2.2.18}$$

The Kaluza-Klein ansatz enables us to give the line element and thus the metric

$$\begin{aligned} ds^2 &= (e_\mu{}^\alpha\, \eta_{\alpha\beta}\, e_\nu{}^\beta)\, dx^\mu dx^\nu \\ &\quad + \rho^{\frac{2}{p}} \left(dy^m + A_\mu{}^m dx^\mu\right) (\mathcal{V}_m{}^a \delta_{ab} \mathcal{V}_n{}^b) \left(dy^n + A_\nu{}^n dx^\nu\right) \\ &= g_{\mu\nu}\, dx^\mu dx^\nu + \rho^{\frac{2}{p}}\, M_{mn} \left(dy^m + A_\mu{}^m dx^\mu\right)\left(dy^n + A_\nu{}^n dx^\nu\right), \end{aligned} \tag{2.2.19}$$

with

$$M \equiv \mathcal{V}\mathcal{V}^T\,. \tag{2.2.20}$$

The Einstein-Hilbert $\mathcal{L}_{\mathrm{EH}} = e_{11} R^{(11)}$ Lagrangian becomes

$$\begin{aligned} \mathcal{L}^{(D)}_{\mathrm{EH}} = \; & e_D\, \rho\, R^{(D)} - \frac{1}{4}\, e_D\, \rho^{1+\frac{2}{p}}\, M_{mn}\, F^m_{\mu\nu} F^n_{\mu\nu} - \frac{1}{4}\, e_D\, \rho\, \mathrm{tr}\big((M^{-1}\partial_\mu M)(M^{-1}\partial^\mu M)\big) \\ & + \frac{p-1}{p}\, e_D\, \rho\, \big((\rho^{-1}\partial_\mu \rho)(\rho^{-1}\partial^\mu \rho)\big)\,. \end{aligned} \tag{2.2.21}$$

When $D \neq 2$, by applying a Weyl rescaling

$$g_{\mu\nu} \longrightarrow \rho^{-\frac{2}{D-2}} g_{\mu\nu}\,, \tag{2.2.22}$$

the Einstein-Hilbert term can be rescaled to go to the Einstein frame (see Appendix A for details about Weyl rescaling). We will not make explicit the reduction of the three-form kinetic and FFA term since our main interest lies in the symmetry structure of the dimensionally reduced theory. Let us mention however the lower-dimensional fields originating from the three-form. In flat indices, the three-form decomposes into

$$A_{BCD} \quad \xrightarrow{T^p} \quad \{ A^{(3)}_{\alpha\beta\gamma}\,,\, A^{(2)}_{\alpha\beta\, c}\,,\, A^{(1)}_{\alpha\, bc}\,,\, A^{(0)}_{abc} \}\,. \tag{2.2.23}$$

The several n-form potentials are independent of the internal coordinates y^m . Notice in particular that $\binom{11-D}{3} = \frac{1}{6}(11-D)(10-D)(9-D)$ scalar fields (axions) $A^{(0)}_{ijk}$ are generated.

Symmetries

The symmetries of the eleven-dimensional supergravity have implications for the reduced theory. They indeed translate into particular symmetries which are detailed now.

Diffeomorphisms The eleven-dimensional general coordinate transformations (diffeomorphisms)

$$\delta x^M = -\xi^M(x) \tag{2.2.24}$$

transforms the fields according to

$$\begin{aligned} \delta_\xi\, e_M{}^A &= \xi^P \partial_P\, e_M{}^A + \big(\partial_M \xi^P\big) e_P{}^A\,, \\ \delta_\xi A_{MNP} &= \xi^Q\, \partial_Q\, (A_{MNP}) + 3\big(\partial_{[M}\xi^Q\big)\, A_{NP]Q}\,. \end{aligned} \tag{2.2.25}$$

Notice that the three-form is written in world (curved) space indices. After dimensional reduction, the Kaluza-Klein condition imposes that none of the fields shall depend on the internal coordinates y^m. This is also true for the variation of fields under a general coordinate transformation. Therefore, the parameterizing functions ξ^M are submitted to the constraints

$$\partial_m \xi^\mu = 0\,, \qquad \partial_m\partial_\mu \xi^k = 0\,, \qquad \partial_m\partial_n \xi^k = 0\,. \tag{2.2.26}$$

These equations are solved by

$$\xi^\mu = \xi^\mu(x^\nu)\,, \qquad \xi^m = y^n\, C_n{}^m + \xi^m(x^\nu)\,, \tag{2.2.27}$$

where $C_n{}^m$ are constant $\mathcal{M}_p(\mathbb{R})$ matrices. Thus in D dimensions the diffeomorphisms split into

- $\delta x^\mu = -\xi^\mu(x^\nu)$: diffeomorphism in D dimensions,
- $\delta y^m = -\xi^m(x^\nu)$: local $\mathbb{R}^p$ invariance,
- $\delta y^m = -y^n\, C_n{}^m$: global $GL(p,\mathbb{R})$ invariance.

The local $\mathbb{R}^p$ invariance generates gauge transformations of the Kaluza-Klein vectors descending from the metric

$$\delta A_\mu{}^m = \partial_\mu \xi^m\,. \tag{2.2.28}$$

Moreover, the $GL(p,\mathbb{R})$ invariance can be split into a global $\mathbb{R}\times SL(p,\mathbb{R})$ invariance as we identify

$$\lambda \equiv (\mathrm{tr}C)^p\,, \qquad \Lambda_m{}^n = C_m{}^n - \frac{\mathrm{tr}C}{p}\,\delta_m^n \;\in\; \mathfrak{sl}(p,\mathbb{R})\,. \tag{2.2.29}$$

Then, the $\mathbb{R}$ part acts on the dilaton

$$\delta\rho = \lambda\,\rho\,, \tag{2.2.30}$$

and the $SL(p,\mathbb{R})$ part acts on internal world indices on fields according to

$$\delta\mathcal{V}_m{}^a = \Lambda_m{}^k\,\mathcal{V}_k{}^a\,, \qquad \delta A_\mu{}^m = -A_\mu{}^k\,\Lambda_k{}^m\,, \qquad \text{etc.} \tag{2.2.31}$$

To put it in a nutshell, diffeomorphism invariance of the eleven dimensional theory leads to a diffeomorphism invariant theory in d-dimensions, together with $U(1)^p$ invariant Maxwell fields and a global $\mathbb{R}\times SL(p,\mathbb{R})$ symmetry.

Local Lorentz symmetry The use of the vielbein as a fundamental field implies that the theory is invariant under local Lorentz transformations. Indeed, the vielbein (or frame field) $e_\mu{}^\alpha(x)$ defines locally non-coordinate bases in which the metric is Minkowski [98]

$$\theta^\alpha \equiv e_\mu{}^\alpha(x)\,dx^\mu\,, \qquad g = g_{\mu\nu}\,dx^\mu\otimes dx^\nu = \eta_{\alpha\beta}\,\theta^\alpha\otimes\theta^\beta\,. \tag{2.2.32}$$

However, there are many non-coordinate bases that can be chosen this way, each of which is related to the other by local Lorentz transformation

$$e_\mu{}^\alpha(x) \quad\longrightarrow\quad e'_\mu{}^\alpha(x) = e_\mu{}^\beta(x)\,\Lambda_\beta{}^\alpha(x)\,. \tag{2.2.33}$$

After Kaluza-Klein reduction, this local symmetry splits into

- local Lorentz invariance in D-dimensions.
- local $SO(p)$ invariance acting on the scalar fields $\mathcal{V}$

$$\mathcal{V}_m{}^a \longrightarrow \mathcal{V}'_m{}^a = \mathcal{V}_m{}^b\,K(x)_b{}^a\,, \qquad K(x)\in SO(p)\,. \tag{2.2.34}$$

Remember that the lower-dimensional fields and parameters only depend on the D-dimensional coordinates. The latter symmetry is also manifest through the fact that the metric depends on $M\equiv\mathcal{V}\mathcal{V}^T$. It enables to reduce the degrees of freedom carried by the scalar fields $\mathcal{V}$

$$\text{d.o.f} = \dim\big(SL(p,\mathbb{R})\big) - \dim\big(SO(p)\big)\,, \tag{2.2.35}$$

and by defining the equivalence relation

$$\mathcal{V}'\sim\mathcal{V} \qquad \text{iff} \qquad \mathcal{V}' = \mathcal{V}\cdot K\,, \qquad K\in SO(p)\,, \tag{2.2.36}$$

the scalars fields are shown to live in the left coset $SL(p,\mathbb{R})/SO(p)$. This local symmetry can be implemented in the Lagrangian by introducing a covariant derivative. First consider the $\mathfrak{sl}(p,\mathbb{R})$ current

$$J_\mu \equiv \mathcal{V}^{-1}\partial_\mu\mathcal{V} = P_\mu + Q_\mu\,, \qquad P_\mu^T \equiv P_\mu\,, \qquad Q_\mu^T \equiv -Q_\mu\,, \tag{2.2.37}$$

an divide it into an $\mathfrak{so}(p)$-valued vector Q_μ which belongs to the "'compact"' (with respect to the Cartan-Killing form) subalgebra of $\mathfrak{sl}(p,\mathbb{R})$ and a vector P_μ which belongs to the non-compact subspace of $\mathfrak{sl}(p,\mathbb{R})$. Then, under an infinitesimal $\mathfrak{so}(p)$ transformation, the fields transforms as

$$\delta Q_\mu = \partial_\mu k + [Q_\mu\,,\, k]\,, \qquad \delta P_\mu = [P_\mu\,,\, k]\,, \qquad k(x) \in \mathfrak{so}(p)\,. \tag{2.2.38}$$

Consequently, the Q_μ field transforms as a $\mathfrak{so}(p)$ connection which enables to define the $SO(p)_{\text{coset}}$ covariant derivative

$$D_\mu = \partial_\mu + Q_\mu\,. \tag{2.2.39}$$

Moreover, the kinetic term for the Lagrangian can be written

$$\mathrm{tr}\big((M^{-1}\partial_\mu M)(M^{-1}\partial^\mu M)\big) = 4\,\mathrm{tr}\big(P_\mu P^\mu\big)\,, \tag{2.2.40}$$

and it is invariant under the global $SL(p,\mathbb{R})$ and local $SO(p)$ symmetries.

Local gauge symmetries Finally, there is a local gauge symmetry of the three-form in eleven dimensions

$$\delta A_{MNP} = 3\,\partial_{[M}\lambda_{NP]}\,. \tag{2.2.41}$$

This generates local gauge symmetries for the dimensionally-reduced gauge potentials $\{\,A^{(3)}_{\mu\nu\rho}\,,\, A^{(2)}_{\mu\nu i}\,,\, A^{(1)}_{\mu ij}\,\}$. Let us focus on the axionic scalars $A^{(0)}_{ijk}$. The assumption of internal coordinate independence translates into

$$\partial_m\,\delta A_{npq} = 0 = 3\,\partial_m\partial_{[n}\lambda_{pq]} \tag{2.2.42}$$

which is solved by

$$\lambda_{mn} = c_{mnp}(x^\mu)\,y^p\,, \qquad c_{mnp} = c_{[mnp]}\,. \tag{2.2.43}$$

Furthermore, given a gauge transformation parametrized by c_{mnp}, the following equality

$$\begin{aligned}\partial_m\,\delta A_{\mu pq} = 0 &= 3\,\partial_m\partial_{[\mu}\big(c_{pq]r}\,y^r\big)\\ &= \partial_m\partial_\mu\big(c_{pqr}\,y^r\big)\\ &= \partial_\mu c_{pqm}\end{aligned} \tag{2.2.44}$$

shows that the parameters c_{mnp} are constant. Consequently, the eleven-dimensional gauge transformation yields an $\mathbb{R}^q$ global shift symmetry on the axions

$$\delta A_{mnp} = c_{mnp}\,, \tag{2.2.45}$$

with

$$q = \binom{11-D}{3} = \frac{1}{6}(11-D)(10-D)(9-D)\,. \tag{2.2.46}$$

The q different $\mathbb{R}$ symmetries commute with each other but do not with the $GL(p,\mathbb{R})$ ones. This is due to the fact that the gauge potentials with one or more internal indices are charged under $GL(p,\mathbb{R})$, because of the eleven-dimensional diffeomorphisms. The commutator of two transformations (that can be checked on the axions) is given by [99]

$$[\delta_c\,,\,\delta_\Lambda] = \delta_{c'}\,, \qquad c'_{mnp} \equiv 3\,\Lambda_{[m}{}^k\,c_{np]k}\,. \tag{2.2.47}$$

Consequently, the theory is now invariant under semi-direct product $GL(p,\mathbb{R}) \ltimes \mathbb{R}^q$. Notice for example that the toroidal Kaluza-Klein reduction of eleven-dimensional supergravity leads to

- a global $GL(8,\mathbb{R}) \ltimes \mathbb{R}^{56}$ symmetry in three-dimensions.
- a global $GL(9,\mathbb{R}) \ltimes \mathbb{R}^{84}$ symmetry in two dimensions.

Dualisation

The previous symmetry structure may extend to a bigger non-compact one. The underlying mechanism is called *enhancement* and it relies on the dualisation of the p-forms of the supergravity. By dualisation we mean an on-shell relation between p-form potentials $A^{(p)}$ with associated field strength $F^{(p+1)} \equiv dA^{(p)}$ and additional "dual" $D-p-2$-forms $B^{(D-p-2)}$ with associated field strength $G^{(D-p-1)} \equiv dB^{(D-p-2)}$. The duality equation can be schematically written

$$* F^{(p+1)} = \big(\text{scalar prefactor}\big)\big(G^{(D-p-1)} + \text{Chern-Simons contributions } + \text{fermions}\,\big)\,. \tag{2.2.48}$$

It is a first order equation that can be derived from the action. When all the forms are dualized into lowest possible degree, the $GL(11-D) \ltimes \mathbb{R}^q$ global symmetry enlarges to an E_{11-D} symmetry [99] [100]. It is a non-trivial mechanism, because, for $6 \le D \le 8$, $GL(11-D) \ltimes \mathbb{R}^q$ is a subgroup of E_{11-D}, for $D = 9$ they coincide and for $D \in \{5\,,4\,,3\}$, the semi-direct product is not contained in E_{11-D} since $\mathbb{R}^q \in \{\mathbb{R}^{20}\,,\mathbb{R}^{35}\,,\mathbb{R}^{56}\}$ whereas the maximal abelian subgroups of $\{E_6\,,E_7\,,E_8\}$ are isomorphic to $\{\mathbb{R}^{16}\,,\mathbb{R}^{27}\,,\mathbb{R}^{36}\}$. Hence, the scalar sector of the dualized theory is described by a sigma model on the symmetric space $E_{(11-D,11-D)}/K(E_{11-D})$, where

- the Lie groups $E_{(11-D,11-D)}$ are the maximally non-compact form of E_{11-D},
- $K(E_{11-D})$ is the maximal compact subgroup of E_{11-D}.

In our conventions:

$$\begin{aligned} &E_1 \equiv \mathbb{R}\,, && E_2 \equiv GL(2,\mathbb{R})\,, && E_3 \equiv SL(3,\mathbb{R}) \times SL(2,\mathbb{R})\,, \\ &E_4 \equiv SL(5,\mathbb{R})\,, && E_5 \equiv O(5,5)\,, && E_{6,7,8} \equiv \text{exceptional Lie groups}\,. \end{aligned} \tag{2.2.49}$$

For example, after dualisation of all forms to the lowest possible degree, the toroidal dimensional reduction of SUGRA_{11} to four dimensions leads to $\mathcal{N}=8\,,\,D=4$ supergravity with global symmetry group $E_{(7,7)}$ (the real non-compact form of the Lie group E_7) , as shown in [33]. If we go to three dimensions the resulting theory is the $\mathcal{N}=16\,,\,D=3$ supergravity with global symmetry group $E_{(8,8)}$ [101] and in two dimensions, the $\mathcal{N}=16\,,\,D=2$ supergravity has an infinite number of symmetries realized on-shell and described by the affine Kac-Moody group $E_{(9,9)}$ [93].

Let us mention that the $(D-2)$-forms are dual to scalars, since this will be important for the three-dimensional supergravity. Let us illustrate, how it works in this case. The duality is realized by imposing the Bianchi identity of the field strengths associated with the $D-2$-forms, at the level of the Lagrangian

$$dF_{(D-1)} = 0\,, \qquad F_{(D-1)} = dA_{(D-2)}\,. \tag{2.2.50}$$

This is done by introducing the "dual" scalar field ϕ which plays the role of a Lagrange multiplier. Thus a new term is added to the Lagrangian

$$\mathcal{L}_{\text{Bianchi}} = \phi\, dF_{(D-1)}\,. \tag{2.2.51}$$

It can be integrated by part and because there are no more terms in the Lagrangian that contains derivatives on $F_{(d-1)}$, the field strength will satisfy an algebraic equation of motion solved by

$$F_{(D-1)} \sim \text{(scalars)}\, *\, (d\phi + \dots)\,. \tag{2.2.52}$$

Plugging it back to the Lagrangian leads to a kinetic term for the "dual" scalars plus additional couplings and no more $D-2$ forms. In three dimensions, this mechanism is responsible for the enhancement of the symmetries leading to the scalar target space $E_{(8,8)}/SO(16)$.

In the following, we will focus on the three-dimensional maximal supergravity as a starting point for building the $\mathcal{N}=16$, $D=2$ supergravity. The construction of these two theories will be reviewed, preparing the ground for the $SO(9)$ gauged maximal supergravity in two dimensions.

2.2.3 The $\mathcal{N}=16$, $D=3$ Supergravity

Maximal supergravity in three dimensions is interesting in several aspects. While pure extended supergravity is topological [102], the coupling to matter leads to a unique target space $E_{(8,8)}/SO(16)$ for $\mathcal{N}=16$, where $SO(16)$ is also the R-symmetry group of the theory [101]. The occurrence of the maximal exceptional Lie group $E_{(8,8)}$ is a manifestation of the extremely rich symmetry structure of the theory. Then, deformations can be considered through the gauging of global symmetries. This is important in the AdS/CFT context since maximal gauged supergravities can admit AdS_3 ground states which are dual to CFTs in two dimensions. A very detailed classification of gauged $\mathcal{N}=16$, $D=3$ supergravities can be found in [103], [104] and [92]. This section is devoted to the presentation of the maximal ungauged theory. The issue of gauging symmetries will be treated afterwards, as an application of the embedding tensor formalism.

Field content

The $\mathcal{N}=16$ supergravity multiplet contains

- the "dreibeins" $e_\mu{}^\alpha$.
- Its superpartners, the 16 gravitino fields $\psi_\mu{}^I$. They transform as a vector under $SO(16)$.
- 128 scalar fields $\mathcal{V}$ belonging to the non-compact coset $E_{(8,8)}/SO(16)$
- 128 Majorana spinors $\chi^{\dot{A}}$ transforming in one of the two inequivalent real spinor representation of $SO(16)$.

The ungauged maximal supergravity can be derived from eleven dimensions by reduction on the eight torus T^8, however in this case the scalar target space is $GL(8,\mathbb{R}) \ltimes \mathbb{R}^{56}$. As we saw before, the coset $E_{(8,8)}/SO(16)$ is obtained when all the vector fields have been dualized into scalars. The propagating degrees of freedom are carried by the 128 scalar fields and 128 fermions which are the physical fields of the theory. They transform in the two inequivalent real fundamental spinor representations of $SO(16)$. They will be labeled by the indices $A=1\,,\,\ldots\,,\,128$ and $\dot{A}=1\,,\,\ldots\,,\,128$ respectively. Let us check the degrees of freedom carried by the fermions. In odd dimensions, the Majorana spinors account for $2^{\frac{D-1}{2}-1}$ real dof on-shell, which means one in our case. However they are charged under $SO(16)$, and

$$\dim\big(R_{\mathrm{spin}(SO(16))}\big) = 128\,. \tag{2.2.53}$$

Hence the 128 fermionic degrees of freedom. Now let us focus on the bosonic sector and make the coset space construction explicit. If we decompose $E_{(8,8)}$ under $SO(16)$: the 248 generators of $E_{(8,8)}$ split into 120 compact generators $X^{IJ} = -X^{JI}$ and 128 non-compact generators Y^A

$$248 \xrightarrow{SO(16)} 120 \oplus 128\,, \tag{2.2.54}$$

where the indices $I, J = 1\,,\,\dots\,,\,16$ label the vector representation of $SO(16)$. Then, the commutation relations of the $E_{(8,8)}$ generators are given by

$$\begin{aligned} \left[X^{IJ}\,,\,X^{KL}\right] &= \delta^{IL}X^{JK} + \delta^{JK}X^{IL} - \delta^{IK}X^{JL} - \delta^{JL}X^{IK}\,, \\ \left[X^{IJ}\,,\,Y^{A}\right] &= -\frac{1}{2}\Gamma^{IJ}_{AB}Y^{B}\,, \quad \left[Y^{A}\,,\,Y^{B}\right] = \frac{1}{4}\Gamma^{IJ}_{AB}X^{IJ}\,. \end{aligned} \tag{2.2.55}$$

where Γ^{IJ}_{AB} is the anti-symmetric product of two $SO(16)$ Γ-matrices defined by

$$\Gamma^{I}_{A\dot{A}}\Gamma^{J}_{\dot{A}B} = \delta^{IJ}_{AB} + \Gamma^{IJ}_{AB}\,. \tag{2.2.56}$$

The scalar fields $\mathcal{V}$ are described by elements of the non-compact coset space $E_{(8,8)}/SO(16)$ with linearly realized global symmetry acting (for example) on the left and local $SO(16)$ invariance acting on the right

$$\mathcal{V} \longrightarrow \Lambda\,\mathcal{V}\,K(x)\,, \qquad \Lambda \in E_{(8,8)}\,,\; K(x) \in SO(16)\,. \tag{2.2.57}$$

As seen before, the local coset symmetry is insured by a “composite” $SO(16)$ gauge field Q_μ obtained from the $\mathfrak{e}_{(8,8)}$ Lie algebra decomposition

$$\mathcal{V}^{-1}\partial_\mu\mathcal{V} = Q_\mu + P_\mu = \frac{1}{2}\,Q^{IJ}_\mu X^{IJ} + P^A_\mu Y^A\,. \tag{2.2.58}$$

The scalar fields parametrizing $\mathcal{V}$ account for the bosonic degrees of freedom of the theory. The corresponding number is given by: $\dim\big(E_8\big) - \dim\big(SO(16)\big) = 248 - 120 = 128$. Let us mention that the local $SO(16)$ symmetry can be fixed so that $\mathcal{V}$ is generated by the non compact generators Y transforming in the spinor representation of $SO(16)$,

$$\mathcal{V} = \exp\left(b^A\,Y^A\right) \tag{2.2.59}$$

with some parameterizing fields b^A. This choice would correspond to the so called “unitary gauge”. Maybe one of the first discussion of the higher-dimensional origin of coset-space formulations of the scalar sector was done in [105] while the $E_{8(8)}/SO(16)$ three-dimensional case is treated in [106].

To conclude, let us summarize the field content

$$\{e_\mu{}^\alpha\,,\,\Psi_\mu{}^I\,,\,\mathcal{V}\,,\,\chi^{\dot{A}}\}\,, \tag{2.2.60}$$

and write down a Lagrangian for the theory.

The Lagrangian

The Lagrangian of $\mathcal{N} = 16$, $D = 3$ Supergravity will be given up to quadratic order in fermions, but the quartic terms can be found in [101].

$$\begin{aligned} \mathcal{L}_{3D} = &-\frac{1}{4}eR + \frac{1}{2}\epsilon^{\mu\nu\rho}\,\bar{\Psi}^I_\mu D_\nu \Psi^I_\rho + \frac{1}{4}\,e\,P^A_\mu P^{\mu\,A} \\ &- \frac{i}{2}\,e\,\bar{\chi}^{\dot{A}}\gamma^\mu D_\mu\chi^{\dot{A}} - \frac{1}{2}\,e\,\bar{\chi}^{\dot{A}}\gamma^\rho\gamma^\mu\Psi^I_\rho\,\Gamma^I_{A\dot{A}}P^A_\mu + \dots\,. \end{aligned} \tag{2.2.61}$$

The dots indicate higher order fermionic terms. We recognize in the Lagrangian: the Einstein-Hilbert term, plus the kinetic terms for the gravitino, the scalar fields and the matter fermions. Finally, the last term represents a Fermion coupling where P^A_μ has been defined in (2.2.58). The spacetime gamma matrices conventions are the following:

- The metric signature is $(+--)$.
- The spacetime gamma matrices are represented in terms of the Pauli matrices

$$\gamma^0 = \sigma_2\,, \quad \gamma^1 = i\,\sigma_3 \quad \text{and} \quad \gamma^2 = i\,\sigma_1\,,$$
$$\text{so that} \qquad \gamma^{\mu\nu\rho} = -i\,\epsilon^{\mu\nu\rho} \quad \text{with} \quad \epsilon^{012} = 1\,. \tag{2.2.62}$$

- The spinor adjoint is defined by

$$\bar{\chi} \equiv \chi^t\, i\,\gamma^0\,. \tag{2.2.63}$$

Then, the covariant derivatives are given by

$$D_\mu \Psi^I_\nu = \big(\partial_\mu + \frac{1}{4}\omega_{\mu\,\alpha\beta}\gamma^{\alpha\beta}\big)\Psi^I_\nu + Q^{IJ}_\mu \Psi^J_\nu\,,$$
$$D_\mu \chi^{\dot{A}} = \big(\partial_\mu + \frac{1}{4}\omega_{\mu\,\alpha\beta}\gamma^{\alpha\beta}\big)\chi^{\dot{A}} + \frac{1}{4}Q^{IJ}_\mu \Gamma^{IJ}_{\dot{A}\dot{B}}\chi^{\dot{B}}\,. \tag{2.2.64}$$

Supersymmetry

The Lagrangian (2.2.61) is invariant under the linearized supersymmetry transformations

$$\delta_\epsilon e_\mu{}^\alpha = i\,\bar{\epsilon}^I\gamma^\alpha\Psi^I_\mu\,, \qquad \mathcal{V}^{-1}\delta_\epsilon\mathcal{V} = \bar{\chi}^{\dot{A}}\epsilon^I\Gamma^I_{A\dot{A}}Y^A\,,$$
$$\delta_\epsilon\Psi^I_\mu = D_\mu\epsilon^I\,, \qquad \delta_\epsilon\chi^{\dot{A}} = \frac{i}{2}\gamma^\mu\epsilon^I\Gamma^I_{A\dot{A}}P^A_\mu\,, \tag{2.2.65}$$

up to higher order fermionic terms that have been checked in [101] and [107]. The commutator of two local supersymmetry transformations leads to local symmetry transformations composed of: a general coordinate, local Lorentz, local $SO(16)$ and local supersymmetry transformations

$$\big[\delta_Q(\epsilon_1)\,,\,\delta_Q(\epsilon_2)\big] = \delta_{\text{gct}}\big(\xi\big) + \delta_L\big(\lambda\big) + \delta_Q\big(\epsilon_3\big) + \delta_{SO(16)}\big(K\big) \tag{2.2.66}$$

with parameters

$$\xi^\mu = i\,\bar{\epsilon}^I_2\gamma^\mu\epsilon^I_1\,, \qquad \epsilon^I_3 = -\xi^\mu\,\psi^I_\mu\,,$$
$$\lambda^{\alpha\beta} = -\xi^\mu\,\omega_\mu{}^{\alpha\beta}\,, \qquad K^{IJ} = -\frac{1}{2}\,\xi^\mu\,Q^{IJ}_\mu\,. \tag{2.2.67}$$

Now that the ungauged maximal supergravity in three dimensions has been presented, the path is open to its dimensional reduction. This is the way we will get the $\mathcal{N}=16$, $D=2$ supergravity and it is discussed in the next section.

2.2.4 The $\mathcal{N}=16$, $D=2$ Supergravity

The two-dimensional theory shares a lot of interesting properties. Maybe the most important one is the fact that the non-compact $E_{(8,8)}$ group of global symmetries enlarges to its infinite dimensional affine extension $E_{(9,9)}$, realized on-shell on the scalar sector. This is a consequence of the presence of an infinite set of independent on-shell duality equations that can be generated recursively [94]. This symmetry structure has been analyzed in [93] and [108] and in particular, it leads to the integrability of the classical theory [109].

Our starting point will be the derivation of the theory from dimensional reduction of maximal supergravity in three dimensions.

Dimensional reduction

In the following, we will compactify the spacetime of $\mathcal{N} = 16$, $D = 3$ supergravity on a circle

$$\mathcal{M}_3 = \mathcal{M}_2 \times S^1 . \tag{2.2.68}$$

This will be done according to the Kaluza-Klein procedure discussed above. The world and tangent space indices are split into

$$\begin{aligned} m\,,\, n\,,\, \ldots &= (\mu\,,\, \dot{2})\,,\ (\mu\,,\, \dot{2})\,,\ \ldots \\ a\,,\, b\,,\, \ldots &= (\alpha\,,\, 2)\,,\ (\beta\,,\, 2)\,,\ \ldots \end{aligned} \tag{2.2.69}$$

where $\mu\,,\, \alpha\ \in \{0,1\}$. Since we are interested only in the massless modes, no dependence on the third coordinate $x^{\dot{2}}$ will be assumed. Consequently, the three-dimensional vielbein reduces as in (2.2.17) with $p = 1$, the gravitino splits into

$$\Psi^I_a = \left(\psi^I_\alpha\,,\, \psi^I_2\right) \quad \text{in flat indices}\,, \tag{2.2.70}$$

and $\{\mathcal{V}\,,\, \chi^{\dot{A}}\}$ remain the same. Even if the irreducible spinors in two dimensions are Majorana-Weyl, we will write them as two-components Majorana spinors. Moreover, the two-dimensional gamma matrices are built from the three-dimensional ones

$$\gamma^0 = \sigma_2\,, \quad \gamma^1 = i\,\sigma_3 \tag{2.2.71}$$

with a $\gamma^3 \equiv -i\,\gamma^2 = \sigma_1$ defined so that

$$\gamma_\alpha \gamma_\beta = \eta_{\alpha\beta} + \epsilon_{\alpha\beta}\,\gamma^3\,, \qquad \epsilon_{01} \equiv 1\,. \tag{2.2.72}$$

Now we are in position to reduce the three-dimensional Lagrangian to two dimensions. Let us recall the Lagrangian,

$$\begin{aligned} \mathcal{L}_{3D} = &- \frac{1}{4} e_3 R + \frac{1}{2}\epsilon^{mnp}\,\bar{\Psi}^I_m D_n \Psi^I_p + \frac{1}{4}\,e_3\,P^A_m P^{m\,A} \\ &- \frac{i}{2}\,e_3\,\bar{\chi}^{\dot{A}}\gamma^m D_m \chi^{\dot{A}} - \frac{1}{2}\,e_3\,\bar{\chi}^{\dot{A}}\gamma^p\gamma^m \Psi^I_p\,\Gamma^I_{A\dot{A}} P^A_m\,. \end{aligned} \tag{2.2.73}$$

The easiest part comes from the scalar fields. Indeed, $\partial_{\dot{2}}\mathcal{V} = 0$, so $P_{\dot{2}} = 0 = Q_{\dot{2}}$. Then,

$$\frac{1}{4}\,e_3\,P^A_m P^{m\,A} = \frac{1}{4}\,e_2\,\rho\,P^A_\mu P^{\mu\,A}\,. \tag{2.2.74}$$

Let us focus on the last term. Its reduction is straightforward

$$\begin{aligned} -\frac{1}{2}\,e_3\,\bar{\chi}^{\dot{A}}\gamma^p\gamma^m\Psi^I_p\,\Gamma^I_{A\dot{A}}P^A_m = &-\frac{i}{2}\,e_2\,\rho\,\bar{\chi}^{\dot{A}}\gamma^3\gamma^\mu\psi^I_2\,\Gamma^I_{A\dot{A}}P^A_\mu \\ &- \frac{1}{2}\,e_2\,\rho\,\bar{\chi}^{\dot{A}}\gamma^\rho\gamma^\mu\psi^I_\rho\,\Gamma^I_{A\dot{A}}P^A_\mu \end{aligned} \tag{2.2.75}$$

where we have used the fact that $\gamma^2 = i\,\gamma^3$. After that, the computations are more technical because the spin connection is involved. Let us begin with the kinetic term for the fermion χ:

$$\bar{\chi}^{\dot{A}}\gamma^m D_m \chi^{\dot{A}} = \bar{\chi}^{\dot{A}}\gamma^{\dot{2}} D_{\dot{2}} \chi^{\dot{A}} + \bar{\chi}^{\dot{A}}\gamma^\mu D_\mu \chi^{\dot{A}} \tag{2.2.76}$$

with

$$
\begin{aligned}
\bar{\chi}^{\dot{A}}\gamma^{\dot{2}}D_{\dot{2}}\chi^{\dot{A}} &= \frac{1}{4}\,\bar{\chi}^{\dot{A}}\gamma^{2}\,\omega_{2ab}\gamma^{ab}\,\chi^{\dot{A}} \\
&= \frac{1}{4}\,\bar{\chi}^{\dot{A}}\gamma^{2}\,\omega_{2\alpha\beta}\gamma^{\alpha\beta}\,\chi^{\dot{A}} - \frac{1}{2}\,\bar{\chi}^{\dot{A}}\omega_{22\beta}\,\gamma^{\beta}\,\chi^{\dot{A}}\,, \\
&= \frac{i}{4}\,\omega_{2\alpha\beta}\,\epsilon^{\alpha\beta}\,\bar{\chi}^{\dot{A}}\,\chi^{\dot{A}}\,,
\end{aligned}
\tag{2.2.77}
$$

because $\partial_{\dot{2}}\chi = 0 = Q_{\dot{2}}$ and we used the Majorana flip relations [95]

$$
\begin{aligned}
\bar{\lambda}\gamma_{\mu_1\ldots\mu_r}\chi &= t_r\,\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\lambda\,, \\
t_r &= -1 \quad \text{for} \quad r\in\{1\,,\,2\}\,, \\
t_r &= 1 \quad \text{for} \quad r\in\{0\,,\,3\}\,, \\
t_r &\equiv t_{r+4}\,,
\end{aligned}
\tag{2.2.78}
$$

to eliminate the $\bar{\chi}^{\dot{A}}\,\gamma^{\beta}\,\chi^{\dot{A}}$ term. Thus, one needs to compute the coefficients of the spin connection. There are two equivalent ways to do it. First, one can compute them directly from the vielbein by using the torsionless condition

$$
\omega_m{}^{ab} = 2\,e^{p[a}\partial_{[m}e_{p]}{}^{b]} - e^{p[a}e^{b]s}e_{mc}\,\partial_p e_s{}^{c}\,. \tag{2.2.79}
$$

Or one can start from the anholonomic coefficients

$$
\Omega_{abc} \equiv 2e_a{}^{m}e_b{}^{n}\partial_{[n}e_{m]c} = \Omega_{[ab]c} \tag{2.2.80}
$$

and then compute the torsionless spin connection

$$
\omega_{a[bc]} = -\frac{1}{2}\big(-\Omega_{bca} + \Omega_{cab} + \Omega_{abc}\big)\,. \tag{2.2.81}
$$

We will choose the second option since the two-dimensional Ricci scalar can be obtained straightforwardly from the anholonomic coefficients. Consequently, we find

$$
\omega_{2\alpha\beta} = \frac{1}{2}\rho\,e_\alpha{}^{\mu}e_\beta{}^{\nu}\,F_{\mu\nu}\,, \quad \text{with} \qquad F_{\mu\nu} \equiv 2\partial_{[\mu}A_{\nu]}\,. \tag{2.2.82}
$$

Thus,

$$
\bar{\chi}^{\dot{A}}\gamma^{\dot{2}}D_{\dot{2}}\chi^{\dot{A}} = \frac{i}{8}\,\rho\,\epsilon^{\mu\nu}\,F_{\mu\nu}\,\bar{\chi}^{\dot{A}}\,\chi^{\dot{A}}\,. \tag{2.2.83}
$$

Now the kinetic term for the gravitino can be reduced. After integrating by part, this term decomposes into

$$
\begin{aligned}
\epsilon^{mnp}\,\bar{\Psi}^I_m D_n \Psi^I_p &= 2\epsilon^{\mu\nu 2}\bar{\psi}^I_2 D_\mu \psi^I_\nu - \epsilon^{\mu\nu\dot{2}}\bar{\psi}_\mu D_{\dot{2}}\psi_\nu \\
&= \epsilon^{\mu\nu 2}\Big(2\bar{\psi}^I_2 D_\mu\psi^I_\nu - \frac{1}{4}\,\bar{\psi}^I_\mu\,\omega_{2\alpha\beta}\gamma^{\alpha\beta}\,\psi^I_\nu - \frac{1}{2}\,\bar{\psi}^I_\mu\,\omega_{22\alpha}\gamma^{2}\gamma^{\alpha}\,\psi^I_\nu\Big)\,.
\end{aligned}
\tag{2.2.84}
$$

Knowing the spin connection

$$
\omega_{22\alpha} = -\,e^{\mu}{}_{\alpha}\,\rho^{-1}\partial_\mu\rho\,, \tag{2.2.85}
$$

we get

$$
\epsilon^{mnp}\,\bar{\Psi}^I_m D_n\Psi^I_p = \epsilon^{\mu\nu 2}\Big(2\bar{\psi}^I_2 D_\mu\psi^I_\nu - \frac{1}{8}\,\bar{\psi}^I_\mu\gamma^3\psi^I_\nu\,\epsilon^{\sigma\lambda}F_{\sigma\lambda} + \frac{i}{2}\,\bar{\psi}^I_\mu\gamma^3\gamma^\sigma\psi^I_\nu\,\rho^{-1}\partial_\sigma\rho\Big)\,. \tag{2.2.86}
$$

Eventually, the expression can be simplified by considerations on the Levi-Civita symbol

$$\epsilon^{\mu\nu\dot{2}} = e_3\, e_\alpha{}^\mu e_\beta{}^\nu\, \epsilon^{\alpha\beta 2} = -e_3\, e_2^{-1}\, \epsilon^{\mu\nu} = -\rho\, \epsilon^{\mu\nu}\,, \qquad (\epsilon^{\alpha\beta 2} = -\epsilon^{\alpha\beta} \quad \text{with our conventions})$$

$$\text{and also} \qquad \epsilon^{\mu\nu}\epsilon_{\sigma\lambda} = -2\, \delta_\sigma^{[\mu}\delta_\lambda^{\nu]}\,, \qquad \gamma^3\gamma_\mu = e_2\, \epsilon_{\mu\nu}\gamma^\nu\,. \tag{2.2.87}$$

This leads to

$$\epsilon^{mnp}\, \bar\Psi^I_m D_n \Psi^I_p = -2\rho\, \epsilon^{\mu\nu}\, \bar\psi^I_2 D_\mu \psi^I_\nu - \frac{1}{4}\, \rho\, \bar\psi^I_\mu \gamma^3 \psi^I_\nu\, F^{\mu\nu} + i\, e_2\, \bar\psi^I_\mu \gamma^\nu \psi^I_\nu\, \partial^\mu \rho\,. \tag{2.2.88}$$

so finally

$$\frac{1}{2}\epsilon^{mnp}\, \bar\Psi^I_m D_n \Psi^I_p = -\rho\, \epsilon^{\mu\nu}\, \bar\psi^I_2 D_\mu \psi^I_\nu - \frac{1}{8}\, \rho\, \bar\psi^I_\mu \gamma^3 \psi^I_\nu\, F^{\mu\nu} - \frac{i}{2}\, e_2\, \bar\psi^I_\nu \gamma^\nu \psi^I_\mu\, \partial^\mu \rho\,. \tag{2.2.89}$$

Let us conclude this analysis by computing the Ricci scalar in two dimensions. It is given in terms of the anholonomic coefficients of (2.2.80) by

$$R^{(3)} = -\frac{1}{4}\Big(\Omega_{abc}\Omega^{abc} - 2\Omega_{abc}\Omega^{cab} - 4\Omega_{ca}{}^a\Omega^c{}_b{}^b\Big) \tag{2.2.90}$$

see [37] for example. Thus,

$$R^{(3)} = R^{(2)} + \frac{1}{4}\rho^2\, F_{\mu\nu}F^{\mu\nu} \tag{2.2.91}$$

Consequently, the two-dimensional Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_{2D} = &-\frac{1}{4}\, e_2\, \rho\, R^{(2)} - \frac{1}{16}\, e_2\, \rho^3\, F_{\mu\nu}F^{\mu\nu} - \rho\, \epsilon^{\mu\nu}\, \bar\psi^I_2 D_\mu \psi^I_\nu \\
&- \frac{1}{8}\, \rho\, \bar\psi^I_\mu \gamma^3 \psi^I_\nu\, F^{\mu\nu} - \frac{i}{2}\, e_2\, \bar\psi^I_\nu \gamma^\nu \psi^I_\mu\, \partial^\mu \rho \\
&+ \frac{1}{4}\, e_2\, \rho\, P^A_\mu P^{\mu\, A} - \frac{i}{2}\, e_2\, \rho\, \bar\chi^{\dot A} \gamma^\mu D_\mu \chi^{\dot A} + \frac{1}{16}\, e_2\, \rho^2\, \epsilon^{\mu\nu}\, F_{\mu\nu}\, \bar\chi^{\dot A}\, \chi^{\dot A} \\
&- \frac{i}{2}\, e_2\, \rho\, \bar\chi^{\dot A} \gamma^3 \gamma^\mu \psi^I_2\, \Gamma^I_{A\dot A} P^A_\mu - \frac{1}{2}\, e_2\, \rho\, \bar\chi^{\dot A} \gamma^\rho \gamma^\mu \psi^I_\rho\, \Gamma^I_{A\dot A} P^A_\mu\,.
\end{aligned} \tag{2.2.92}$$

This is the result obtained by Kaluza-Klein reduction of the $\mathcal{N} = 16$, $D = 3$ supergravity on a circle, when only massless terms have been kept. This Lagrangian deserves some comments that are collected in the next section.

The Lagrangian

In two dimensions the vector field A_μ is auxiliary, so its equation of motion can be used to integrate it at the level of the Lagrangian. By doing so, quartic terms in fermions are generated together with a scalar term proportional to the constant of integration. Since in the following, we will only work up to quadratic order in fermions and we will stick to the undeformed Lagrangian, we can just drop out the vector fields by setting $A_\mu = 0$. Therefore, the two dimensional Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_{2D} = &-\frac{1}{4}\, e_2\, \rho\, R^{(2)} + \frac{1}{4}\, e_2\, \rho\, P^A_\mu P^{\mu\, A} - \rho\, \epsilon^{\mu\nu}\, \bar\psi^I_2 D_\mu \psi^I_\nu - \frac{i}{2}\, e_2\, \bar\psi^I_\nu \gamma^\nu \psi^I_\mu\, \partial^\mu \rho \\
&- \frac{i}{2}\, e_2\, \rho\, \bar\chi^{\dot A} \gamma^\mu D_\mu \chi^{\dot A} - \frac{i}{2}\, e_2\, \rho\, \bar\chi^{\dot A} \gamma^3 \gamma^\mu \psi^I_2\, \Gamma^I_{A\dot A} P^A_\mu - \frac{1}{2}\, e_2\, \rho\, \bar\chi^{\dot A} \gamma^\nu \gamma^\mu \psi^I_\nu\, \Gamma^I_{A\dot A} P^A_\mu\,.
\end{aligned} \tag{2.2.93}$$

The field content is composed of

- the zweibein $e_\mu{}^\alpha$

- its superpartner the gravitino ψ_μ^I: a two-dimensional Majorana vector-spinor, transforming in the **16** vector representation of $SO(16)$.
- There is also the dilaton field ρ
- and its superpartner: the dilatino ψ_2^I which is a Majorana spinor also transforming in the **16** of $SO(16)$.
- the 128 bosonic degrees of freedom are again mediated by group valued matrices $\mathcal{V}$ which belong to the coset space $\frac{E_{(8,8)}}{SO(16)}$.
- The corresponding 128 superpartner are Majorana fermions $\chi^{\dot{A}}$ that transform in the $\mathbf{128}_c$ conjugate spinor representation of $SO(16)$.

Now it remains to check that maximal supersymmetry is preserved.

Supersymmetry

Indeed, by examining the reduction of (2.2.65), one can show that the Lagrangian is invariant (up to total derivatives and quartic terms in fermions) under the following supersymmetry transformations

$$\begin{aligned} \delta_\epsilon e_\mu{}^\alpha &= i\,\bar{\epsilon}^I\gamma^\alpha\psi_\mu^I\,, & \delta_\epsilon\psi_\mu^I &= D_\mu\epsilon^I\,, \\ \delta_\epsilon\rho &= -\rho\,\bar{\epsilon}^I\gamma^3\psi_2^I\,, & \delta_\epsilon\psi_2^I &= -\frac{i}{2}\,\gamma^3\gamma^\mu\epsilon^I\rho^{-1}\partial_\mu\rho\,, \\ \mathcal{V}^{-1}\delta_\epsilon\mathcal{V} &= \bar{\epsilon}^K\Gamma^K_{A\dot{A}}\chi^{\dot{A}}Y^A\,, & \delta_\epsilon\chi^{\dot{A}} &= \frac{i}{2}\,\Gamma^I_{A\dot{A}}\gamma^\mu\epsilon^I P_\mu^A\,. \end{aligned} \tag{2.2.94}$$

The commutator of two supersymmetry transformations closes again on

$$\big[\delta_Q(\epsilon_1)\,,\,\delta_Q(\epsilon_2)\big] = \delta_{\rm gct}\big(\xi\big) + \delta_L\big(\lambda\big) + \delta_Q\big(\epsilon_3\big) + \delta_{SO(16)}\big(K\big) \tag{2.2.95}$$

with parameters

$$\begin{aligned} \xi^\mu &= i\,\bar{\epsilon}_2^I\gamma^\mu\epsilon_1^I\,, & \epsilon_3^I &= -\xi^\mu\,\psi_\mu^I\,, \\ \lambda^{\alpha\beta} &= -\xi^\mu\,\omega_\mu{}^{\alpha\beta}\,, & K^{IJ} &= -\frac{1}{2}\,\xi^\mu\,Q_\mu^{IJ}\,. \end{aligned} \tag{2.2.96}$$

Now that we have the $\mathcal{N}=16$, $D=2$ supergravity action, let us review the basics of gauging a subgroup of the global symmetry group with the embedding tensor formalism.

2.3 Gauging Maximal Supergravities

As we saw in Section 2.2.2, the scalar fields of maximal supergravities parametrize a symmetric space G/H, where G is a non-compact group, and H (or $K(G)$) denotes its maximal compact subgroup. The different groups for $2 \leq D \leq 10$ are collected in Table 2.2. So far, all gauge symmetries considered were abelian. However, for previously mentioned reasons, we may want to introduce a non abelian gauge group while preserving supersymmetry. This is precisely the point of the embedding tensor formalism. In this scheme, a subgroup G_0 of G is selected and promoted to a local symmetry of the supergravity. More precisely, this formalism aims at encoding all the possible deformations of the ungauged supergravity into a so-called "embedding" tensor. After that, the classification is done group-theoretically. In the following, we will present the general framework developed in [103], [107], [41], [110], [111] and [94] for maximal supergravities, and reviewed in [112] and [113]. Finally, we will describe the main ingredients that will be employed to get the $SO(9)$ gauged maximal supergravity in two dimensions.

D	G		H	
10	$O(1,1)$ (IIA)	$SL(2)$ (IIB)	— (IIA)	$SO(2)$ (IIB)
9	$GL(2)$		$SO(2)$	
8	$SL(2) \times SL(3)$		$SO(2) \times SO(3)$	
7	$SL(5)$		$SO(5)$	
6	$SO(5,5)$		$SO(5) \times SO(5)$	
5	$E_{(6,6)}$		$USp(8)$	
4	$E_{(7,7)}$		$SU(8)$	
3	$E_{(8,8)}$		$SO(16)$	
2	$E_{(9,9)}$		$K(E_9)$	

Table 2.2: Maximal supergravities symmetric spaces

2.3.1 The Embedding Tensor formalism

Covariantization

Let us focus on the ungauged maximal supergravity in $2 \leq D \leq 9$ dimensions. Given a subgroup G_0 of the global symmetry group G, our goal is to promote it to a gauge group of the theory. The theory is by construction G_0 globally invariant, but giving local dependence on the group parameters will break the invariance because the derivatives no longer transform covariantly. Thus, the first step in the gauging process consist in introducing covariant derivatives with respect to the group G_0. This is done formally according to

$$\partial_\mu \longrightarrow D_\mu \equiv \partial_\mu - g\, A_\mu{}^M\, X_M\,. \tag{2.3.1}$$

Here

- g is the gauge coupling,
- $A_\mu{}^M$ represents the set of n_v vector fields available in the supergravity,
- t_α is a given set of generators of the Lie algebra $\mathfrak{g}$ of G,
- $\Theta_M{}^\alpha$ is the *embedding tensor* which selects a family of n_v elements X_M of $\mathfrak{g}$

$$X_M \equiv \Theta_M{}^\alpha t_\alpha \;\in\; \mathfrak{g}\,, \tag{2.3.2}$$

  that will generate the gauge group G_0. In this sense, the embedding tensor can be seen as a constant $(n_v \times \text{dim}G)$ matrix whose rank is equal to the dimension of the gauge group G_0. As a consequence, the dimension of the gauge group must satisfy $\text{dim}G_0 \leq n_v$. In particular the family $\{X_M\}$ may be a spanning set of $\mathfrak{g}_0$ but not a linearly independent one.

Because, the vector fields transform in some representation of G

$$\delta_\Lambda A_\mu^M = -\Lambda^\alpha (t_\alpha)_N{}^M A_\mu^N\,, \qquad M\,,\, N = 1, \ldots, n_v \tag{2.3.3}$$

imposed by supersymmetry and listed in Table 2.3, we get a manifestly G-covariant formalism which is broken only when the embedding tensor takes a particular value. Moreover, the embedding tensor transforms as the tensor product of two representations: the dual of the representation $\mathcal{R}_v$ in which the vector fields transform, from the left, and the adjoint representation of G, from the right

$$\Theta_M{}^\alpha \;:\qquad \mathcal{R}_{v^*} \otimes \mathcal{R}_{\text{adj}}\,. \tag{2.3.4}$$

D	G	Scalars	Vectors
8	$SL(2) \times SL(3)$	$(\mathbf{3}-1,\, \mathbf{1}) + (\mathbf{1},\, \mathbf{8}-3)$	$(\mathbf{2},\, \mathbf{3'})$
7	$SL(5)$	$\mathbf{24} - 10$	$\mathbf{10'}$
6	$SO(5,5)$	$\mathbf{45} - 20$	$\mathbf{16}_c$
5	$E_{(6,6)}$	$\mathbf{78} - 36$	$\mathbf{27'}$
4	$E_{(7,7)}$	$\mathbf{133} - 63$	$\mathbf{56}$
3	$E_{(8,8)}$	$\mathbf{248} - 120$	$\mathbf{248}$
2	$E_{(9,9)}$	$\mathcal{R}_{\text{adj}}(E_9)$	$\mathcal{R}_{\text{basic}}(E_9)$

Table 2.3: Vectors and Scalars in maximal supergravities in $2 \leq D \leq 8$

In general, this tensor product decomposes into several irreducible representations of G to which Θ may belong. Nonetheless, consistency relations constrain the possible representations of the embedding tensor. These constraints do not depend on the selected gauge group G_0 at this stage. They come from the general requirement of the theory to

- be covariant under the yet arbitrary gauge group G_0,
- remain supersymmetric after covariantization of the action.

In the following sections we will explain the origin of the constraints and then apply this formalism to the maximal supergravity in three dimensions. This will open the path to the $SO(9)$ gauging of the maximal supergravity in two dimensions which will be discussed in details in the next chapter.

Constraints

The Quadratic constraint Now that covariant derivatives have been introduced, an ansatz for local gauge transformations under G_0 can be formulated from the global G invariance

$$\begin{aligned} \delta_\Lambda \mathcal{V} &= \Lambda^\alpha t_\alpha \cdot \mathcal{V}\,, \\ \delta_\Lambda A_\mu^M &= -\Lambda^\alpha \left(t_\alpha\right)_N{}^M A_\mu^N\,, \\ &\vdots \;\; \text{other fields charged under } G\,, \end{aligned} \tag{2.3.5}$$

with $\Lambda^\alpha =$ constant , and $\alpha = 1, \ldots, \dim G$. Then, by substituting

$$\Lambda^\alpha t_\alpha \longrightarrow \Lambda^M(x)\; X_M \tag{2.3.6}$$

with a local parameter $\Lambda^M(x)$ and $M = 1, \ldots, n_v$, and introducing covariant derivatives in the action, local gauge invariance under G_0 can be imposed by

$$\begin{aligned} \delta_\Lambda \mathcal{V} &= g\, \Lambda^M X_M \cdot \mathcal{V}\,, \\ \delta_\Lambda A_\mu^M &= \partial_\mu \Lambda^M + g\, A_\mu^N X_{NP}{}^M \Lambda^P = D_\mu \Lambda^M\,, \\ &\vdots \;\; \text{other fields}\,. \end{aligned} \tag{2.3.7}$$

with $X_{NP}{}^M \equiv \Theta_N{}^\alpha (t_\alpha)_P{}^M$. However we see that the covariance of quantities such as

$$\delta_\Lambda \left(D_\mu \mathcal{V}\right) = g\, \Lambda^M X_M \cdot \left(D_\mu \mathcal{V}\right) \tag{2.3.8}$$

requires that Θ is invariant under the gauge group G_0

$$0 \stackrel{!}{=} \delta_P \Theta_M{}^\alpha \,. \tag{2.3.9}$$

More generally, the consistency of the gauged theory demands the invariance of Θ under the action of the gauge group. This translates into a *quadratic constraint* on the embedding tensor

$$\begin{aligned} 0 \stackrel{!}{=} \delta_P \Theta_M{}^\alpha &= \Theta_P{}^\beta \delta_\beta \Theta_M{}^\alpha \\ &= \Theta_P{}^\beta (t_\beta)_M{}^N \Theta_N{}^\alpha + \Theta_P{}^\beta f_{\beta\gamma}{}^\alpha \Theta_M{}^\gamma \end{aligned} \tag{2.3.10}$$

where $f_{\alpha\beta}{}^\gamma$ are the structure constants associated to the generators of G,

$$\big[t_\alpha \,,\, t_\beta\big] = f_{\alpha\beta}{}^\gamma t_\gamma \,. \tag{2.3.11}$$

When contracted with a generator t_α, the quadratic constraint implies the closure of the generators X_M into a subalgebra of $\mathfrak{g}$

$$\big[X_M \,,\, X_N\big] = -X_{MN}{}^P X_P \,. \tag{2.3.12}$$

The deformed tensor gauge algebra Furthermore, the proper covariantization of the field strengths and the higher p-forms set another problem that can be fixed in the embedding tensor formalism. It deals with the deformation of the tensor gauge algebra that is needed to account for the fact that the standard non abelian field strength

$$\mathcal{F}^M_{\mu\nu} = \partial_\mu A^M_\nu - \partial_\nu A^M_\mu + g\, X_{[NP]}{}^M A^N_\mu A^P_\nu \,. \tag{2.3.13}$$

is not in general a covariant object. Indeed,

$$\delta_\Lambda \mathcal{F}^M_{\mu\nu} = -g \Lambda^P X_{PQ}{}^M \mathcal{F}^Q_{\mu\nu} + 2g\, X_{(PQ)}{}^M \big(\Lambda^P \mathcal{F}^Q_{\mu\nu} - A^P_{[\mu} \delta A^Q_{\nu]}\big) \tag{2.3.14}$$

where the last term is anomalous. As a result, the field strengths need to be deformed in Θ, by the addition of a 2-form. For a detailed account of the deformed tensor gauge algebra, see [112] and [111]. However, in three dimensions, we will not focus on this issue since in the gauging process, the vector fields will enter the action via a Chern-Simons term [103] [107]. There, the gauge invariance of the CS term translates into a quadratic constraint on the embedding tensor. For a complete discussion on this topic, see [92]. Finally in two dimensions, the tensor hierarchy is rather trivial because there are no p-forms for $p > 2$. Moreover, the field strength that will enter the two-dimensional gauged action will be contracted with the embedding tensor, thus, the resulting term will be covariant provided that the quadratic constraint is satisfied.

Supersymmetry and the Linear constraint The supersymmetry variation of the vector fields generates terms coupled to fermionic currents that have not been taken into account yet. These contributions violate supersymmetry, but they can be canceled by following a *Noether procedure* [85]. It consist in

- adding fermionic mass terms

$$\mathcal{L}_{\text{ferm-mass}} \sim g\, \bar{f}\, (\text{scalars})\, f \tag{2.3.15}$$

  to the Lagrangian which are linear in the deformation parameter Θ, in order to compensate the previous contributions.

- Introducing additional linear in Θ terms, the so-called *fermion-shift*, in the fermionic supersymmetry transformations in order to compensate the new contributions from the variation of the fermion mass terms

$$\delta_\epsilon f \sim b\,\epsilon + g\,(\dots)\,\epsilon\,. \tag{2.3.16}$$

- Adding a scalar potential to the Lagrangian which is quadratic in Θ and aims at canceling all the contributions of order g^2.

$$\mathcal{L}_{\text{pot}} \sim g^2\,bb\,. \tag{2.3.17}$$

No further contributions of order g^n with $n \geq 3$ can appear, so the procedure stops here.

Let us focus on the fermion mass terms. They take the schematic form

$$\mathcal{L}_{\text{ferm-mass}} = g\left(\bar{\psi}^i\,A_{ij}\,\psi^j + \bar{\chi}^A\,B_{Ai}\,\psi^i + \bar{\chi}^A\,C_{AB}\,\chi^B\right) \quad + \quad \text{h.c.} \tag{2.3.18}$$

where ψ^i and χ^A denote the gravitini and spin-1/2 fermions which live in some representation of H labeled by i and A. Remember that H stands for the maximal compact subgroup of G in the coset space construction G/H, and it is also the R-symmetry group of the supersymmetric theory, because we are dealing with maximal supergravities. Thus, A_{ij}, B_{Ai} and C_{AB} are tensors, depending on the scalar fields, which transform in the tensor product of some representations of H. However, they are proportional to the embedding tensor Θ since it is the deformation parameter of the gauge theory. To take into account this dependence, let us define the *T-tensor*:

$$T_{\underline{N}}{}^{\underline{\beta}} \equiv \Theta_M{}^\alpha\,\mathcal{V}^M{}_{\underline{N}}\,\mathcal{V}_\alpha{}^{\underline{\beta}}\,, \tag{2.3.19}$$

as the embedding tensor multiplied from the left and right by the scalar group matrix $\mathcal{V}$ evaluated in the fundamental and adjoint representation of G respectively. Now this tensor lives in the same G-representation than Θ, and it can be decomposed into irreducible part under H. These H-irreducible representations

$$T_{\underline{M}}{}^{\underline{\alpha}} \xrightarrow{\;H\;} \left(A_{ij}\,,\,B_{Ai}\,,\,C_{AB}\right) \tag{2.3.20}$$

precisely correspond to the fermionic mass tensors and to the fermion shifts

$$\delta_\epsilon\psi^i = \left(\delta_\epsilon\psi^i\right)\big|_{g=0} - g\,A^{ij}\epsilon_j\,, \qquad \delta_\epsilon\chi^A = \left(\delta_\epsilon\chi^A\right)\big|_{g=0} - g\,B^{Ai}\epsilon_i\,. \tag{2.3.21}$$

They must match the tensor product of the H-representation of ψ^i and χ^A in the mass terms

$$\bar{\psi}^i\psi^j\,, \quad \bar{\psi}^i\chi^A\,, \quad \bar{\chi}^A\chi^B\,. \tag{2.3.22}$$

This results in a *linear constraint* on the embedding tensor Θ, imposed by supersymmetry requirements. Furthermore, the commutator of two covariant derivatives is now proportional to the embedding tensor

$$\left[D_\mu\,,\,D_\nu\right] = -g\,\mathcal{F}^M_{\mu\nu}\,X_M\,. \tag{2.3.23}$$

Hence, for instance, by varying the kinetic term for the gravitino, supersymmetry violating terms of the form[1]

$$\mathcal{F}\,\Theta\,(\bar{\epsilon}\psi) \tag{2.3.24}$$

[1] The spacetime and internal indices have been dropped for simplicity.

are generated. Fortunately, these contributions can be canceled by introducing a covariant topological term into the Lagrangian $\mathcal{L}_{\text{top}}$, as we will see for the $\mathcal{N}=16$, $D=3$ supergravity, but it imposes also a linear constraint on the embedding tensor. This linear constraint projects out irreducible G-representations in Θ, that are not allowed by supersymmetry

$$\mathbb{P}_l \Theta = 0 \; . \tag{2.3.25}$$

Finally, the consistent cancellation of all supersymmetry variations in order g^2 implies quadratic algebraic identities. Nevertheless, all these identities can be viewed as a consequence of the quadratic constraint (2.3.10) on the embedding tensor. At this stage, the possibility to consistently gauge the theory relies only on the resolution of the linear and quadratic constraints on the embedding tensor.

Strategy The consistent gauging is constructed as follows:

- First, supersymmetry imposes a linear constraint on the embedding tensor (2.3.25). It enables to select *allowed* and *forbidden* irreducible G-representations in the tensor product $\mathcal{R}_{v^*} \otimes \mathcal{R}_{\text{adj}}$ to which the embedding tensor belongs. A classification has been done in [113] for maximal supergravities[2] in $2 \le D \le 8$. It is reproduced in Table 2.4.
- Secondly, solve the quadratic constraint, coming from consistency of the covariance under the gauge group G_0, to fully determine the embedding tensor and solve all the algebraic identities imposed by supersymmetry.
- Finally, select a gauging group among all the possible consistent gaugings provided by the embedding tensor.

D	G	$\mathcal{R}_{\text{adj}} \otimes \mathcal{R}_{v^*}$	=	Allowed	$\oplus$	Forbidden
8	$SL(2)\times SL(3)$	$((\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{8}))\otimes(\mathbf{2},\mathbf{3'})$	=	$(\mathbf{2},\mathbf{3'})\oplus(\mathbf{2},\mathbf{6})$	$\oplus$	$(\mathbf{2},\mathbf{3'})\oplus(\mathbf{2},\mathbf{15'})\oplus(\mathbf{4},\mathbf{3'})$
7	$SL(5)$	$\mathbf{24}\otimes\mathbf{10'}$	=	$\mathbf{15}\oplus\mathbf{40'}$	$\oplus$	$\mathbf{10}\oplus\mathbf{175}$
6	$SO(5,5)$	$\mathbf{45}\otimes\mathbf{16}_c$	=	$\mathbf{144}_s$	$\oplus$	$\mathbf{16}_c\oplus\mathbf{560}_c$
5	$E_{(6,6)}$	$\mathbf{78}\otimes\mathbf{27'}$	=	$\mathbf{351'}$	$\oplus$	$\mathbf{27}\oplus\mathbf{1728}$
4	$E_{(7,7)}$	$\mathbf{133}\otimes\mathbf{56}$	=	$\mathbf{912}$	$\oplus$	$\mathbf{56}\oplus\mathbf{6480}$
3	$E_{(8,8)}$	$\mathbf{248}\otimes\mathbf{248}$	=	$\mathbf{1}\oplus\mathbf{3875}$	$\oplus$	$\mathbf{248}\oplus\mathbf{27000}\oplus\mathbf{30380}$
2	$E_{(9,9)}$	$\mathcal{R}_{\text{adj}} \otimes \mathcal{R}_{v^*}$	=	$\mathcal{R}_{v^*}$	$\oplus$	rest

Table 2.4: Tensor product representations for $2 \le D \le 8$

In the following section, the embedding tensor formalism is illustrated on the gauging of $\mathcal{N}=16$, $D=3$ maximal supergravities, with particular emphasis on the linear and quadratic constraints.

2.3.2 Gauging the $\mathcal{N}=16$, $D=3$ supergravity

Gauging maximal supergravity in three dimensions is somewhat different than in higher dimensions, because no vector field enters the Lagrangian when the scalar coset space is described by $E_{(8,8)}/SO(16)$. Indeed, if the theory was to be obtained by Kaluza-Klein reduction from $D=11$ supergravity on an 8-torus, all the vector fields would need to be dualized into scalars in order to make the global $E_{(8,8)}$ symmetry manifest. This

[2] The last line of Table 2.4 has been conjectured from the higher dimensional cases, since very few is known about gaugings in this infinite dimensional context. It turns out that the decomposition allows for the $SO(9)$ gauging, which stands for a non-trivial test of the group-theoretical framework.

feature will be also encountered in two dimensions where no vector fields can propagate. Actually, gauging the $\mathcal{N}=16$, $D=3$ maximal supergravity exemplifies the last step before understanding the gauging of maximal supergravity in two dimensions. This is why it will be described in this section, as an illustration of the embedding tensor formalism.

Nevertheless, if there are no vector fields, one may wonder how to use the embedding tensor formalism. Fortunately there is a way to introduce vector fields in the three-dimensional theory so that they do not carry additional physical degrees of freedom. This is done by means of a Chern-Simons term [92].

The embedding tensor

The embedding tensor transforms in the tensor product of the dual representation of the vector fields, labeled by indices $M=1,\ldots,n_v$, and the adjoint representation of $E_{(8,8)}$ whose generators are denoted by t^α .

$$\Theta \longrightarrow \Theta_M{}^\alpha\,. \tag{2.3.26}$$

However, since the number of vector fields involved in the gauging is for the moment arbitrary but less than the dimension of the global symmetry group

$$n_v \leq \mathrm{dim}E_{(8,8)}\,, \tag{2.3.27}$$

we can label them with the adjoint representation indices, keeping in mind that the embedding tensor will act as a projector on the gauge subalgebra

$$A_\mu{}^\alpha \Theta_{\alpha\beta} t^\beta \equiv A_\mu{}^M X_M\,. \tag{2.3.28}$$

Then, following [103], and as we saw before, we define the $G_0 \subset E_{(8,8)}$ covariant derivative by

$$\mathcal{D}_\mu \equiv \partial_\mu + g\, A_\mu{}^M X_M\,. \tag{2.3.29}$$

The gauge invariant Lagrangian result from (2.2.61) after introducing covariant derivatives

$$\begin{aligned} \mathcal{L}^{(0)} = & -\frac{1}{4}eR + \frac{1}{2}\epsilon^{\mu\nu\rho}\,\overline{\Psi}{}^I_\mu D_\nu \Psi^I_\rho + \frac{1}{4}\,e\,\mathcal{P}^A_\mu \mathcal{P}^{\mu\,A} \\ & -\frac{i}{2}\,e\,\overline{\chi}^{\dot{A}}\gamma^\mu D_\mu \chi^{\dot{A}} - \frac{1}{2}\,e\,\overline{\chi}^{\dot{A}}\gamma^\rho\gamma^\mu \Psi^I_\rho \Gamma^I_{A\dot{A}}\mathcal{P}^A_\mu \end{aligned} \tag{2.3.30}$$

where the covariant scalar current is given by

$$\begin{aligned} \mathcal{V}^{-1}\mathcal{D}_\mu \mathcal{V} &\equiv \mathcal{V}^{-1}\partial_\mu \mathcal{V} + g\, A_\mu{}^M \mathcal{V}^{-1} X_M \mathcal{V} \\ &= \mathcal{P}^{\mu\,A} Y^A + \frac{1}{2}\mathcal{Q}^{IJ}_\mu X^{IJ}\,. \end{aligned} \tag{2.3.31}$$

Nonetheless, the supersymmetry variation of terms involving the commutator of two covariant derivatives leads to supersymmetry violating terms proportional to the field strength

$$F_{\mu\nu}{}^M = \partial_\mu A_\nu{}^M - \partial_\nu A_\mu{}^M + g\, X_{[NP]}{}^M A_\mu{}^N A_\nu{}^P\,, \tag{2.3.32}$$

where $X_{NP}{}^M$ are the "structure constants" of the gauge group defined in (2.3.7). These contributions are precisely canceled by a Chern-Simons term for the vector fields

$$\mathcal{L}^{(1)}_{\mathrm{CS}} = -\frac{1}{4}\,g\,\epsilon^{\mu\nu\rho} A_\mu{}^M\,\Theta_{MQ}\left(\partial_\nu A_\rho{}^Q + \frac{1}{3}\,g\,X_{NP}{}^Q A_\nu{}^N A_\rho{}^P\right), \tag{2.3.33}$$

provided that

- the embedding tensor $\Theta_{\alpha\beta}$ is symmetric [104] ,
- under local supersymmetry the vector fields transform as [114]

$$\delta_\epsilon A_\mu{}^M = -2\mathcal{V}^M{}_{IJ}\bar{\epsilon}^I\Psi^I_\mu + i\Gamma^I_{A\dot{A}}\mathcal{V}^M{}_A\bar{\epsilon}^I\gamma_\mu\chi^{\dot{A}} \,. \tag{2.3.34}$$

The Chern-Simons term also enables to introduce vector fields without changing the number of propagating degrees of freedom. The fact that the embedding tensor is symmetric restricts its possible content. Indeed, as an $E_{(8,8)}$ tensor it decomposes into

$$\Theta \;:\qquad \mathbf{248} \otimes_{\text{sym}} \mathbf{248} = \mathbf{1} \oplus \mathbf{3875} \oplus \mathbf{27000} \,. \tag{2.3.35}$$

Among these irreducible parts, some are allowed and other are forbidden by supersymmetry. In order to derive them, let us analyze the Noether procedure applied to the three-dimensional Lagrangian.

Additional terms and identities

As we saw before, from the supersymmetry variation of the covariant Lagrangian $\mathcal{L}^{(0)}$, contributions linear in g (or Θ) appear from the variation of the vector fields. Apart from the supersymmetry violating terms canceled by the variation of the Chern-Simons Lagrangian, other contributions occurs which are coupled to Noether terms. These supersymmetry violating terms are canceled by the following fermionic mass terms

$$\mathcal{L}^{(1)}_{\text{ferm-mass}} = ge\,\big(\frac{1}{2}\,A_1^{IJ}\,\overline{\Psi}^I_\mu\gamma^{\mu\nu}\Psi^J_\nu + iA_2^{I\dot{A}}\,\overline{\chi}^{\dot{A}}\gamma^\mu\Psi^I_\mu + \frac{1}{2}\,A_3^{\dot{A}\dot{B}}\,\overline{\chi}^{\dot{A}}\chi^{\dot{B}}\big) \tag{2.3.36}$$

together with the fermion shift

$$\begin{aligned} \delta_\epsilon\Psi^I_\mu &= D_\mu\epsilon^I + i\,g\,A_1^{IJ}\gamma_\mu\epsilon^J \,, \\ \delta_\epsilon\chi^{\dot{A}} &= \frac{i}{2}\gamma^\mu\epsilon^I\Gamma^I_{A\dot{A}}P^A_\mu + \,g\,A_2^{I\dot{A}}\epsilon^I \,. \end{aligned} \tag{2.3.37}$$

For example, notice that A_1^{IJ} in the fermion shift enables to cancel the order g spinorial variation of the first fermion mass term, by varying the kinetic Rarita-Schwinger term in $\mathcal{L}^{(0)}$. Eventually, the addition of a scalar potential of quadratic order in g, will end the Noether procedure and provide the framework to get a gauge invariant maximal supersymmetric theory

$$\mathcal{L}^{(2)}_{\text{pot}} = \frac{1}{8}\,g^2\,e\,\big(A_1^{IJ}A_1^{IJ} - \frac{1}{2}A_2^{I\dot{A}}A_2^{I\dot{A}}\big)\,. \tag{2.3.38}$$

Local supersymmetry of the resulting Lagrangian

$$\mathcal{L}_{\text{gauged}} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)}_{\text{CS}} + \mathcal{L}^{(1)}_{\text{ferm-mass}} + \mathcal{L}^{(2)}_{\text{pot}} \tag{2.3.39}$$

imposes linear and quadratic identities on the tensors $\{A_1^{IJ}\,,\,A_2^{I\dot{A}}\,,\,A_3^{\dot{A}\dot{B}}\}$, like for example

$$\begin{aligned} &\Gamma^{[I}_{A\dot{A}}A_2^{J]\dot{A}} = \mathcal{V}^\gamma{}_{IJ}\,\Theta_{\gamma\delta}\,\mathcal{V}^\delta{}_A\,, \qquad \mathcal{D}_\mu A_1^{IJ} = \mathcal{P}^A_\mu\Gamma^{(I}_{A\dot{A}}A_2^{J)\dot{A}}\,, \\ &3A_1^{IJ}A_2^{J\dot{A}} - A_2^{I\dot{A}}A_3^{\dot{A}\dot{B}} = \frac{1}{16}\Gamma^I_{A\dot{A}}\Gamma^J_{A\dot{B}}\big(3A_1^{JK}A_2^{K\dot{B}} - A_2^{J\dot{C}}A_3^{\dot{B}\dot{C}}\big)\,, \end{aligned} \tag{2.3.40}$$

see [103] for a more general account. These identities translate into identities on the T tensor[3]

$$T_{\alpha\beta} \equiv \mathcal{V}^\gamma{}_\alpha\mathcal{V}^\delta{}_\beta\Theta_{\gamma\delta} \tag{2.3.41}$$

[3] where the adjoint indices can be further split into $[IJ]$ and A which respectively accounts for the compact X^{IJ} generators of $E_{(8,8)}$ and the non-compact Y^A ones.

which imply a linear and quadratic constraint on Θ. In the following, the constraints will be discussed and a non trivial solution for Θ and the fermionic mass tensor $\{A_1^{IJ}\,,\,A_2^{I\dot{A}}\,,\,A_3^{\dot{A}\dot{B}}\}$ will be presented.

The linear and quadratic constraints

Owing to the $SO(16)$ index structure of the fermions, and their commutation properties

$$\overline{\chi}^{\dot{A}}\chi^{\dot{B}} = \overline{\chi}^{\dot{B}}\chi^{\dot{A}}\,, \qquad \overline{\Psi}^I_\mu\gamma^{\mu\nu}\Psi^J_\nu = -\overline{\Psi}^J_\nu\gamma^{\mu\nu}\Psi^I_\mu = \overline{\Psi}^J_\nu\gamma^{\nu\mu}\Psi^I_\mu\,, \tag{2.3.42}$$

the fermionic mass tensors that enter the Lagrangian can be decomposed into the following irreducible representations of $SO(16)$

$$\begin{aligned} A_1^{IJ} &: \qquad \mathbf{16}\otimes_s\mathbf{16} = \mathbf{1}\oplus\mathbf{135}\,,\\ A_2^{I\dot{A}} &: \qquad \mathbf{16}\otimes\mathbf{128} = \mathbf{128}\oplus\mathbf{1920}\,,\\ A_3^{\dot{A}\dot{B}} &: \qquad \mathbf{128}\otimes_s\mathbf{128} = \mathbf{1}\oplus\mathbf{1820}\oplus\mathbf{6435}\,. \end{aligned} \tag{2.3.43}$$

Since the fermion mass tensor are proportional to the embedding tensor, we can discard the $E_{(8,8)}$ irrep of Θ whose decomposition under $SO(16)$ does not fit in the $\{A_{1,2,3}\}$ tensors $SO(16)$ irreps. Under $SO(16)$, the **3875** of (2.3.35) decomposes into

$$\mathbf{3875} = \mathbf{135}\oplus\mathbf{1820}\oplus\mathbf{1920} \tag{2.3.44}$$

which belong to the irreps of $\{A_{1,2,3}\}$. So is the case of the $E_{(8,8)}$ singlet **1** of (2.3.35), but it is not the case for the **27000**. Consequently, the linear constraint can be written

$$\mathbb{P}_l\left(\mathbf{248}\otimes_{\text{sym}}\mathbf{248}\right) = \mathbf{1}\oplus\mathbf{3875}\,. \tag{2.3.45}$$

Then, the embedding tensor is parametrized by

$$\Theta_{\alpha\beta} = \theta\,\eta_{\alpha\beta} + \Theta^{\mathbf{3875}}_{\alpha\beta} \tag{2.3.46}$$

where $\eta_{\alpha\beta}$ is the Cartan-Killing form of $\mathfrak{e}_8$. It can be shown [103],[104] that Θ given in (2.3.46) automatically solves the quadratic constraint (2.3.10) imposed by the covariance of the theory. The fermion mass tensors are thus given by

$$\begin{aligned} A_1^{IJ} &= -\theta\,\delta_{IJ} - \frac{1}{7}\,\mathcal{V}^\alpha{}_{IK}\mathcal{V}^\beta{}_{KJ}\,\Theta^{\mathbf{3875}}_{\alpha\beta}\,,\\ A_2^{I\dot{A}} &= -\frac{1}{7}\,\Gamma^J_{A\dot{A}}\mathcal{V}^\alpha{}_{IJ}\mathcal{V}^\beta{}_A\,\Theta^{\mathbf{3875}}_{\alpha\beta}\,,\\ A_3^{IJ} &= -\theta\,\delta_{IJ} - \frac{1}{7}\,\mathcal{V}^\alpha{}_{IK}\mathcal{V}^\beta{}_{KJ}\,\Theta^{\mathbf{3875}}_{\alpha\beta}\,. \end{aligned} \tag{2.3.47}$$

As a result, the linear constraint (2.3.45) fully determines the possible gaugings of the $\mathcal{N}=16$, $D=3$ maximal supergravity. Every gauge group embodied in the embedding tensor (2.3.46) defines a consistent gauged maximal supergravity in three dimensions. They are all detailed in [103] and [104].

Summary

Maximal supergravities, dimensional reduction and gauging were at the core of this chapter. Starting from the eleven dimensional supergravity, we studied the structure of the three and two-dimensional ungauged maximal supergravities. The embedding tensor of the general gaugings of maximal supergravity were presented and illustrated in three dimensions.

The next chapter focuses on the explicit $SO(9)$ gauging of $\mathcal{N} = 16$, $D = 2$ maximal supergravity. As explained before, it is of first importance for the DW/QFT correspondence.

Chapter 3

$SO(9)$ supergravity in two dimensions

3.1 Introduction

This chapter is devoted to the construction of maximal supergravity in two dimensions, with gauge group $SO(9)$. By this, we intend to fill the gap in the effective supergravities available for the DW/QFT correspondence in various dimensions, see Table 2.1. The non-trivial deformation is performed from the ungauged theory in two dimensions, thus it will be our starting point.

Two-dimensional ungauged maximal supergravity has many interesting features. One of them is the symmetry structure underlying the theory. Indeed, if we look at the bosonic sector of the theory, all the degrees of freedom reside within the scalar sector (since in two dimensions the vector fields do not propagate), and their dynamics is described by a dilaton-coupled non-linear sigma model with target space $E_{(8,8)}/SO(16)$, directly inherited from maximal supergravity in three dimensions. The bosonic Lagrangian can be found by reducing the bosonic sector of $\mathcal{N}=16\,,\ D=3$ Supergravity on a circle:

$$\mathcal{L}_0 = -\frac{e}{4}\,\rho\left(R^{(2)} - \mathrm{tr}(P_\mu P^\mu)\right) . \tag{3.1.1}$$

The theory admits $E_{(8,8)}$ as a global symmetry group of isometries of the target space. In addition, the integrability structure of the reduction of four-dimensional Einstein gravity to two dimensions extends to maximal supergravity [108]. As a result, the theory admits an infinite number of conserved charges that generates an infinite dimensional global group of symmetry realized on-shell: $E_{(9,9)}$, the centrally extended affine extension of $E_{(8,8)}$. The group acts on an infinite tower of scalar fields, related by first order on-shell duality equations. Integrating step by step the duality equations, all the scalar fields can be determined in terms of the "physical" fields parametrizing the bosonic Lagrangian [94].

This symmetry structure enables to formulate the theory in different off-shell inequivalent frames. More specifically, in two dimensions, the different off-shell formulations of maximal supergravity are described by σ-models with different target-space geometry and Wess-Zumino term, related by T-duality. Within the $E_{(9,9)}$ picture, different formulations correspond to choose particular sets of physical scalar fields from the infinite tower of available scalars. The first formulation of the theory we will use, is the one obtained from the Kaluza-Klein reduction of the $\mathcal{N}=16,\ D=3$ supergravity on a circle. In this "frame", the scalar target space is $E_{(8,8)}/SO(16)$, hence it will be called: the "E_8" frame. Moreover, the theory can also be derived from the torus reduction of eleven-dimensional supergravity. The resulting ungauged maximal supergravity in two dimensions has a scalar sector described by the coset space $\mathbb{R} \times \frac{SL(9)}{SO(9)} \ltimes \mathbb{R}^{84}$. Therefore, this formulation will be named: the "$SL(9)$" frame. There are two inequivalent embeddings of $SO(9)$ into $E_{(9,9)}$

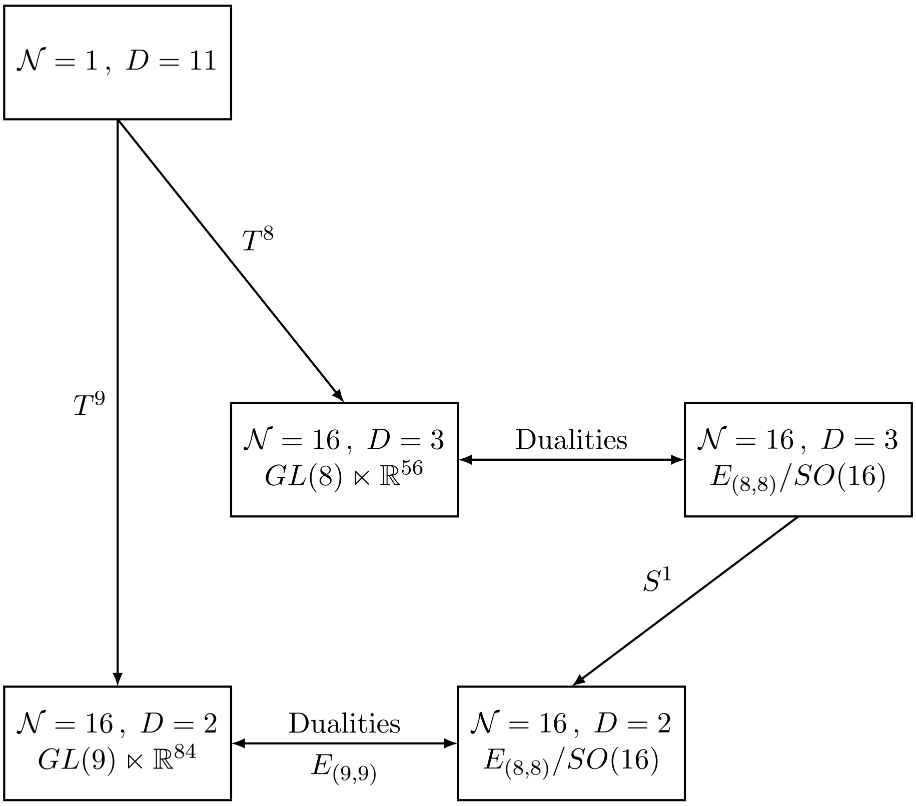

Figure 3.1: Maximal supergravities and Torus reduction.

$$\begin{aligned} & SO(9) \subset E_{(8,8)} \subset E_{(9,9)}\,, \\ \text{and}\quad & SO(9) \subset SL(9) \subset E_{(9,9)}\,. \end{aligned} \tag{3.1.2}$$

but only one leads to a consistent supergravity [94]: it corresponds to the $SL(9)$ frame. In the following section we first recall the formulation of the $\mathcal{N}=16\,,\ D=2$ supergravity in the $E_{(8,8)}$ frame. The field content, the Lagrangian and the supersymmetry transformations will be given. Then we will describe the theory in the $SL(9)$ frame. Its bosonic sector coming from dimensional reduction of eleven-dimensional supergravity will be presented. Moreover the fermion coupling and the supersymmetry transformations will be given and finally, the symmetry structure will be discussed.

3.2 $\mathcal{N}=16\,,\ D=2$ supergravity

3.2.1 Reduction from $3D$: The $E_{(8,8)}$ frame

The most compact formulation of maximal supergravity in two dimensions is obtained by dimensional reduction of the maximal three-dimensional theory [101] on a circle. Since it has been presented in the previous chapter, we will just recall the main features.

The Lagrangian of $\mathcal{N}=16\,,\ D=2$ supergravity in the $E_{(8,8)}$ frame is given by

$$\begin{aligned}e^{-1}\,\mathcal{L}_0 = &-\frac{1}{4}\rho R^{(2)} + \frac{1}{4}\rho\, P^{\mu A}P^A_\mu - \rho\, e^{-1}\varepsilon^{\mu\nu}\bar{\psi}^I_2 D_\mu\psi^I_\nu - \frac{i}{2}\,(\partial^\mu\rho)\,\bar{\psi}^I_\nu\gamma^\nu\psi^I_\mu \\ &- \frac{i}{2}\,\rho\,\bar{\chi}^{\dot{A}}\gamma^\mu D_\mu\chi^{\dot{A}} - \frac{1}{2}\,\rho\,\bar{\chi}^{\dot{A}}\gamma^\nu\gamma^\mu\psi^I_\nu\,\Gamma^I_{A\dot{A}}P^A_\mu - \frac{i}{2}\,\rho\,\bar{\chi}^{\dot{A}}\gamma^3\gamma^\mu\psi^I_2\,\Gamma^I_{A\dot{A}}P^A_\mu\,. \qquad (3.2.1)\end{aligned}$$

$E_{8(8)}$ acts by left multiplication on the matrices $\mathcal{V}$ and gives rise to the algebra-valued (conserved) Noether current

$$J_\mu \equiv \rho\, P^A_\mu\,(\mathcal{V}\,Y^A\mathcal{V}^{-1}) \in \mathfrak{e}_{8(8)}\,. \qquad (3.2.2)$$

In general dimensions, we saw that p-forms are dual to $(D-p-2)$-forms. Therefore, in two dimensions, the duality relates scalars. In particular, the current (3.2.2) can be employed to define a dual scalar field Y by

$$\partial_\mu Y = e\,\varepsilon_{\mu\nu}J^\nu\ . \qquad (3.2.3)$$

Then, Schwarz integrability condition holds provided the current is conserved.

$$\begin{aligned}0 &= \varepsilon^{\mu\nu}\partial_\mu\partial_\nu Y = \varepsilon^{\mu\nu}\partial_\mu(e\,\varepsilon_{\nu\sigma}J^\sigma) = \varepsilon^{\mu\nu}\,\varepsilon_{\nu\sigma}\,\partial_\mu(eJ^\sigma) \\ &= \delta^\mu_\sigma\,\partial_\mu(eJ^\sigma) = \partial_\mu(eJ^\mu) \\ &= e\,\nabla_\mu J^\mu \qquad (3.2.4)\end{aligned}$$

More generally, one can prove the existence of an infinite tower of dual scalar fields Y_m, due to the integrability structure of the two-dimensional equations of motion [94],

$$\begin{aligned}&\partial_\pm Y_2 = (\pm\rho\tilde{\rho} + \frac{1}{2}\rho^2)\mathcal{V}P_\pm\mathcal{V}^{-1} + \frac{1}{2}[Y,\partial_\pm Y]\,, \\ &\partial_\pm Y_3 = (\mp\frac{1}{2}\rho^3 \mp \rho\tilde{\rho}^2 - \rho^2\tilde{\rho})\mathcal{V}P_\pm\mathcal{V}^{-1} + [Y,\partial_\pm Y_2] - \frac{1}{6}[Y,[Y,\partial_\pm Y]]\,, \\ &\partial_\pm Y_4 = \ldots \qquad (3.2.5)\end{aligned}$$

where $x^\pm = (x^0 \pm x^1)/\sqrt{2}$ and the fermionic contribution have been neglected. Although a finite set of scalar fields enters the action, it is just a subset of an infinite tower of scalars defined on-shell. The off-shell fields transform under $E_{(8,8)}$, but the full tower organizes into a representation of the infinite dimensional $E_{(9,9)}$, which is the actual symmetry group of the theory, and is realized only on-shell [108]. This symmetry structure is characteristic of two dimensions and will play an important role in the general gauging.

3.2.2 Reduction from $11D$: The $SL(9)$ frame

The formulation of $\mathcal{N}=16\,,\ D=2$ supergravity which will turn out to be relevant for the gauging of $SO(9)$ is obtained by direct dimensional reduction of the eleven-dimensional theory [34], on a nine-dimensional torus T^9. As we saw in Section 2.2.2, the theory exhibits a $GL(9)\ltimes\mathbb{R}^{84}$ global symmetry group, and the scalar target space is $\frac{GL(9)}{SO(9)}\ltimes\mathbb{R}^{84}$. Let us present the Kaluza-Klein ansatz that will be relevant for our construction.

Kaluza-Klein reduction

First, we split the eleven-dimensional coordinates according to $x^M \to (x^\mu, y_m)$ with $\{\mu = 1,2\}$ and $\{m = 1,\ldots,9\}$. Then, we start with the compactification ansatz of (2.2.17) for the vielbein, and the three-form is written in curved indices

$$A_{MNK} = \Big(0\,,0\,,A_\mu{}^{mn} + A_{\mu\,l}\,\phi^{lmn}\,,\ \phi^{mnk}\Big)\ . \qquad (3.2.6)$$

Here, we have chosen to define the lower-dimensional components of A_{MNK} with internal indices upstairs. It is a a pure convention. Thus, the Kaluza-Klein vector has its internal indice below and the internal vielbein involves the matrix $\mathcal{V}^{-1}$. Then, as discussed previously, $\mathcal{V} \in SL(9)$ and ρ is a dilaton field. They both come from the torus vielbein. $A_{\mu\,k}$ is the Kaluza-Klein vector and it transforms in the **9** of $SL(9)$. For what concerns the three-form, it splits into: a vector field in the $\overline{\mathbf{36}}$ of $SL(9)$, ${A_\mu}^{mn} = {A_\mu}^{[mn]}$. Plus axions $\phi^{klm} = \phi^{[klm]}$ in the **84** of $SL(9)$. In the reduction ansatz, fields of the form $B_{\mu\nu\,k} = B_{[\mu\nu]\,k}$ would have not contributed to the lower-dimensional action and a parameter of the form $C_{\mu\nu\rho} = C_{[\mu\nu\rho]}$ would have been identically zero in two dimensions.

A Weyl rescaling is performed on the bosonic lower-dimensional Lagrangian, see Appendix A,

$$ {e_\mu}^\alpha \longrightarrow \rho^s\, {e_\mu}^\alpha \, . \tag{3.2.7} $$

This leaves the possibility to eliminate the kinetic term for the dilaton by a clever choice of s.

$$ \begin{aligned} e^{-1}\mathcal{L}_{2d} = &-\frac{1}{4}\rho R^{(2)} + \frac{1}{4}\rho\, P^{\mu\,ab} P_\mu^{ab} + \frac{1}{12}\,\rho^{1/3}\,\varphi^{\mu\,abc}\varphi_\mu^{abc} - \left(\frac{s}{2}+\frac{2}{9}\right)\rho^{-1}\partial_\mu\rho\,\partial^\mu\rho \\ &+\frac{1}{648}\,e^{-1}\varepsilon^{\mu\nu}\varepsilon_{klmnpqrst}\,\phi^{klm}\,\partial_\mu\phi^{npq}\,\partial_\nu\phi^{rst} - \frac{1}{16}\,\rho^{11/9-2s}\,\mathcal{M}^{-1kl}\,F_{\mu\nu\,k}{F^{\mu\nu}}_l \\ &-\frac{1}{8}\rho^{5/9-2s}\left({F_{\mu\nu}}^{kl} + \phi^{klp}F_{\mu\nu\,p}\right)\mathcal{M}_{km}\mathcal{M}_{ln}\left(F^{\mu\nu mn} + \phi^{mnq}{F^{\mu\nu}}_q\right) \end{aligned} \tag{3.2.8} $$

where, $\mathcal{M} = \mathcal{V}\mathcal{V}^T$. Moreover, we have introduced the currents[1]

$$ \varphi_\mu^{abc} \equiv {\mathcal{V}_{[klm]}}^{abc}\,\partial_\mu\phi^{klm} \, . \tag{3.2.9} $$

Here, and in the following we use the notation ${\mathcal{V}_{[klm]}}^{abc} \equiv {\mathcal{V}_{[k}}^a {\mathcal{V}_l}^b {\mathcal{V}_{m]}}^c$, for the group-valued $SL(9)$ matrix evaluated on tensor products. We chose to eliminate the kinetic term for the dilaton in (3.2.8), for simplicity. This selects $s = -4/9$. Then, since the vector fields in two dimensions do not carry any propagating degrees of freedom, we can eliminate them from the Lagrangian by using their equation of motion. These equations can always be integrated in two dimensions since

$$ D^\mu F_{\mu\nu} = 0 \qquad \text{implies} \qquad F_{\mu\nu} = \text{constant} \, . \tag{3.2.10} $$

In our case, the first-order equations are

$$ \begin{aligned} \mathcal{M}^{-1kl}\,F_{\mu\nu\,l} + 2\,\rho^{-2/3}\phi^{kmn}\mathcal{M}_{mp}\mathcal{M}_{nq}\left({F_{\mu\nu}}^{pq} + \phi^{pqr}F_{\mu\nu\,r}\right) &= e\,\varepsilon_{\mu\nu}\,\rho^{-19/9}\,\theta^k \, , \\ \mathcal{M}_{km}\mathcal{M}_{ln}\left({F_{\mu\nu}}^{mn} + \phi^{mnp}F_{\mu\nu\,p}\right) &= e\,\varepsilon_{\mu\nu}\,\rho^{-13/9}\,\tilde{\theta}_{kl} \, , \end{aligned} \tag{3.2.11} $$

with integration constants θ^l and $\tilde{\theta}_{mn} = \tilde{\theta}_{[mn]}$. Keeping non-zero values for this constants will lead to massive deformation of the two-dimensional supergravity. These deformations are treated on the same footing as gaugings in the embedding tensor formalism, therefore at the level of the ungauged theory, they will not be relevant for us. This is why we will set $\theta^l = 0 = \tilde{\theta}_{mn}$. Consequently, the field strengths vanish and we can set the vector fields to zero. As a result, the bosonic Lagrangian is given by

$$ \begin{aligned} e^{-1}\mathcal{L}_{2d} = &-\frac{1}{4}\rho R^{(2)} + \frac{1}{4}\rho\, P^{\mu\,ab} P_\mu^{ab} + \frac{1}{12}\,\rho^{1/3}\,\varphi^{\mu\,abc}\varphi_\mu^{abc} \\ &+\frac{1}{648}\,e^{-1}\varepsilon^{\mu\nu}\varepsilon_{klmnpqrst}\,\phi^{klm}\,\partial_\mu\phi^{npq}\,\partial_\nu\phi^{rst} \, . \end{aligned} \tag{3.2.12} $$

[1]In our conventions for this chapter, we reserve letters $a, b, c, \ldots$ from the beginning of the alphabet for 'flat' $SO(9)$ indices which are raised and lowered with δ_{ab}. In contrast, the letters $k, l, m, \ldots$ indicate $SL(9)$ vector indices which transform under the global $SL(9)$ of the ungauged theory. Both indices run from 1 to 9.

It is a dilaton-gravity coupled non-linear σ-model with target space $\frac{SL(9)}{SO(9)} \ltimes \mathbb{R}^{84}$ and topological term.

Full Lagrangian

The fermions come from the reduction of the eleven-dimensional gravitino Ψ_M. When going from eleven to two dimensions, the vector and spinor representations of the Lorentz group $SO(1,10)$ split into vector and spinor representations of $SO(1,1)$ and $SO(9)$

$$\mathbf{11} \otimes \mathbf{32} \longrightarrow (\underbrace{\mathbf{2} \oplus \mathbf{9}}_{\text{vector}}) \otimes (\underbrace{\mathbf{2} \otimes \mathbf{16}}_{\text{spinor}}) . \tag{3.2.13}$$

Thus, Ψ_M gives rise to: a two-dimensional gravitino ψ_μ^I transforming in the $\mathbf{16}$ (spinor representation) of $SO(9)$, and a vector spinor of $SO(9)$, named χ^{aI}. Because the tensor product of the vector and spinor representations of $SO(9)$ splits into

$$\mathbf{9} \otimes \mathbf{16} = \mathbf{16} \oplus \mathbf{128} , \tag{3.2.14}$$

a traceless condition with respect to the $SO(9)$ Γ-matrices is assumed. This enables to select the irreducible $\mathbf{128}$:

$$\Gamma^a_{IJ} \chi^{aJ} \equiv 0 . \tag{3.2.15}$$

The remaining trace part contributes to the last two-dimensional spinor: the dilatino ψ_2^I which transforms in the $\mathbf{16}$ of $SO(9)$. The fermionic content may be summarized as follows

$$\Psi_M \xrightarrow{T^9} \{\psi_\mu^I , \chi^{aI}, \psi_2^I\} . \tag{3.2.16}$$

Accordingly, the covariant derivatives on fermions are defined by

$$\begin{aligned} D_\mu \psi_\nu^I &= \partial_\mu \psi_\nu^I + \frac{1}{4} \omega_\mu{}^{\alpha\beta} \gamma_{\alpha\beta}\, \psi_\nu^I + \frac{1}{4} Q_\mu^{ab}\, \Gamma^{ab}_{IJ}\, \psi_\nu^J , \\ D_\mu \chi^{aI} &= \partial_\mu \chi^{aI} + \frac{1}{4} \omega_\mu{}^{\alpha\beta} \gamma_{\alpha\beta}\, \chi^{aI} + Q_\mu^{ab}\, \chi^{bI} + \frac{1}{4} Q_\mu^{bc}\, \Gamma^{bc}_{IJ}\, \chi^{aJ} , \end{aligned} \tag{3.2.17}$$

with the $SO(9)$ connection Q_μ^{ab} from (2.2.37). Eventually, the full Lagrangian is obtained by imposing supersymmetry, rather than performing the dimensional reduction of the fermionic sector. We have then completely determined the supersymmetry transformations by imposing the on-shell closure of the supersymmetry algebra.

$$\begin{aligned} \delta_\epsilon e_\mu{}^\alpha &= i\, \bar\epsilon^I \gamma^\alpha \psi_\mu^I , & \delta_\epsilon \psi_\mu^I &= D_\mu \epsilon^I - \frac{1}{24}\, \rho^{-1/3}\, \Gamma^{abc}_{IJ} \left(\frac{1}{3} \gamma_\mu \gamma^\nu + \gamma^\nu \gamma_\mu \right) \gamma^3 \epsilon^J\, \varphi_\nu^{abc} , \\ \delta_\epsilon \rho &= -\rho\, \bar\epsilon^I \gamma^3 \psi_2^I , & \delta_\epsilon \psi_2^I &= -\frac{i}{2}\, \gamma^3 \gamma^\mu \epsilon^I\, \rho^{-1} \partial_\mu \rho , \\ \delta_\epsilon \mathcal{V}_i{}^a &= \bar\epsilon^I \Gamma^{(a}_{IJ} \chi^{b)J} \mathcal{V}_i{}^b , & \delta_\epsilon \chi^{aI} &= \frac{i}{2}\, \Gamma^b_{IJ}\, \gamma^\mu \epsilon^J\, P_\mu^{(ab)} - \frac{i}{6}\, \rho^{-1/3} \big(\delta^{ab} \Gamma^{cd}_{IJ} - \frac{1}{6}\, \Gamma^{abcd}_{IJ} \big)\, \gamma^3 \gamma^\mu \epsilon^J\, \varphi_\mu^{bcd} , \end{aligned} \tag{3.2.18}$$

$$\delta_\epsilon \phi^{ijk} = \frac{3}{2}\, \rho^{1/3}\, \mathcal{V}^{-1\,[ijk]}_{abc} \Gamma^{ab}_{IJ}\, \bar\epsilon^I \gamma^3 \chi^{cJ} + \frac{1}{6}\, \rho^{1/3}\, \mathcal{V}^{-1\,[ijk]}_{abc}\, \Gamma^{abc}_{IJ}\, \bar\epsilon^I \psi_2^J .$$

Thus, the full Lagrangian with the kinetic terms for the fermions and the Noether couplings were found by hand, after lengthy calculations

$$\begin{aligned}
e^{-1}\mathcal{L}_0 = &-\frac{1}{4}\rho R^{(2)} + \frac{1}{4}\rho\, P^{\mu\, ab}P_\mu^{ab} + \frac{1}{12}\,\rho^{1/3}\,\varphi^{\mu\, abc}\varphi_\mu^{abc} \\
&+\frac{1}{648}\, e^{-1}\varepsilon^{\mu\nu}\varepsilon_{klmnpqrst}\,\phi^{klm}\,\partial_\mu\phi^{npq}\,\partial_\nu\phi^{rst} \\
&-\rho e^{-1}\varepsilon^{\mu\nu}\bar\psi_2^I D_\mu\psi_\nu^I - \frac{i}{2}\,\bar\psi_\nu^I\gamma^\nu\psi_\mu^I\,\partial^\mu\rho - \frac{i}{2}\,\rho\,\bar\chi^{aI}\gamma^\mu D_\mu\chi^{aI} \\
&-\frac{1}{2}\,\rho\,\bar\chi^{aI}\gamma^\nu\gamma^\mu\psi_\nu^J\Gamma^b_{IJ}P_\mu^{ab} - \frac{i}{2}\,\rho\,\bar\chi^{aI}\gamma^3\gamma^\mu\psi_2^J\Gamma^b_{IJ}P_\mu^{ab} \\
&-\frac{1}{4}\,\rho^{2/3}\,\bar\chi^{aI}\gamma^3\gamma^\nu\gamma^\mu\psi_\nu^J\Gamma^{bc}_{IJ}\,\varphi_\mu^{abc} - \frac{i}{12}\,\rho^{2/3}\,\bar\chi^{aI}\gamma^\mu\psi_2^J\Gamma^{bc}_{IJ}\,\varphi_\mu^{abc} \\
&+\frac{i}{54}\,\rho^{2/3}\,\bar\psi_2^I\gamma^3\gamma^\mu\psi_2^J\Gamma^{abc}_{IJ}\,\varphi_\mu^{abc} + \frac{1}{24}\,\rho^{2/3}\,\bar\psi_2^I\Big(\gamma^\mu\gamma^\nu - \frac{1}{3}\gamma^\nu\gamma^\mu\Big)\,\psi_\nu^J\,\Gamma^{abc}_{IJ}\,\varphi_\mu^{abc} \\
&+\frac{i}{2}\,\rho^{2/3}\,\bar\chi^{aI}\gamma^3\gamma^\mu\chi^{bJ}\Gamma^c_{IJ}\,\varphi_\mu^{abc} - \frac{i}{24}\,\rho^{2/3}\,\bar\chi^{aI}\gamma^3\gamma^\mu\chi^{aJ}\Gamma^{bcd}_{IJ}\,\varphi_\mu^{bcd}\,.
\end{aligned} \tag{3.2.19}$$

The Lagrangian is invariant with respect to (3.2.18). Notice also that supersymmetry does not require a scalar potential, as was expected from dimensional reduction on the torus.

Supersymmetry algebra

The supersymmetry algebra closes on diffeomorphisms, local Lorentz transformations and local $SO(9)$ transformations coming from the coset space structure.

$$\begin{aligned}
&[\delta_{\varepsilon_1},\delta_{\varepsilon_2}] = \delta_{\rm gct}(\xi) + \delta_{\rm L}(\lambda) + \delta_{\rm SO(9)}(K)\,, \\
&\text{with}\quad \xi^\mu = i\,\bar\epsilon_2^I\gamma^\mu\epsilon_1^I\,,\quad \lambda^{\alpha\beta} = -\,\xi^\mu{\omega_\mu}^{\alpha\beta}\,,\quad K^{ab} = -\,\xi^\mu{Q_\mu}^{ab}\,.
\end{aligned} \tag{3.2.20}$$

The commutator of two supersymmetries are more easily checked on the bosonic fields,

$$\begin{aligned}
[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\, e_\mu^\alpha &= \xi^\nu\,\partial_\nu e_\mu^\alpha + e_\nu^\alpha\,\partial_\mu\xi^\nu + (-\xi^\nu{\omega_\nu}^{\alpha\beta})\, e_\mu^\beta \\
[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\,\rho &= \xi^\mu\,\partial_\mu\rho + \text{quartic fermions} \\
[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\,{\mathcal{V}_m}^a &= \xi^\mu\,\partial_\mu{\mathcal{V}_m}^a + {\mathcal{V}_m}^b\,(\xi^\mu\,Q_\mu^{ba}) + \text{q.f.} \\
[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\,\phi^{ijk} &= \xi^\mu\,\partial_\mu\phi^{ijk} + \text{q.f.} \\
\xi^\mu &= i\,\bar\varepsilon_2^I\gamma^\mu\epsilon_1^I
\end{aligned} \tag{3.2.21}$$

We take the occasion to determine the supersymmetry variation of the vector fields $A_{\mu\, k}$ and ${A_\mu}^{kl}$ of (3.2.6), by closure of their supersymmetry algebra. Up to a global factor that can be absorbed by rescaling of the vector fields, they are given by

$$\begin{aligned}
\delta_\epsilon\, A_{\mu\, k} &= -\,2\rho^{-5/9}\Big(\bar\psi_\mu^I\gamma^3\epsilon^J\Gamma^a_{IJ} + \frac{5i}{9}\bar\psi_2^I\gamma_\mu\epsilon^J\Gamma^a_{IJ} - i\bar\chi^{aI}\gamma^3\gamma_\mu\epsilon^I\Big){\mathcal{V}_k}^a\,, \\
\delta_\epsilon\, {A_\mu}^{kl} &= \rho^{-2/9}\Big(\bar\psi_\mu^I\epsilon^J\Gamma^{ab}_{IJ} - \frac{2i}{9}\,\bar\psi_2^I\gamma^3\gamma_\mu\epsilon^J\Gamma^{ab}_{IJ} - 2i\,\bar\chi^{I[a}\gamma_\mu\epsilon^J\Gamma^{b]}_{IJ}\Big){\mathcal{V}^{-1}}_{[ab]}{}^{kl} \\
&\quad -{\mathcal{V}^{-1}}_{[abc]}{}^{klm}\,\varphi^{abc}\,(\delta_\epsilon\, A_{\mu\, m})\,.
\end{aligned} \tag{3.2.22}$$

The supersymmetry algebra closes on-shell provided that their field strength vanish:

$$F_{\mu\nu\, k} = 0 = {F_{\mu\nu}}^{kl}\,. \tag{3.2.23}$$

This is the case owing to the choice, $\theta^l = 0 = \tilde{\theta}_{mn}$. Then, the algebra closes into abelian gauge transformations

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] A_{\mu\, k} = \partial_\mu \Lambda_k\,, \qquad [\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] A_\mu{}^{kl} = \partial_\mu \Lambda^{kl}\,, \tag{3.2.24}$$

with gauge parameters

$$\Lambda_k = -2\rho^{-5/9}\,\bar{\epsilon}_1^I \gamma^3 \epsilon_2^J \Gamma^a_{IJ} \mathcal{V}_k{}^a\,, \qquad \Lambda^{kl} = \rho^{-2/9}\,\bar{\epsilon}_1^I \epsilon_2^J \Gamma^{ab}_{IJ}\, \mathcal{V}^{-1}{}_{[ab]}{}^{kl} + 2\,\rho^{-5/9}\,\bar{\epsilon}_1^I \gamma^3 \epsilon_2^J \Gamma^a_{IJ}\, \mathcal{V}^{-1}{}_{[bc]}{}^{kl}\,\varphi^{abc}\,. \tag{3.2.25}$$

In the following, the global internal bosonic symmetries of the action are analyzed and the associated Noether currents are given.

Noether current

The global $SL(9)$ off-shell symmetry of the Lagrangian (3.2.19) acts by left multiplication on the matrices $\mathcal{V}_m{}^a$ and by matrix multipication on the scalar fields ϕ^{kmn}:

$$\delta \mathcal{V}_m{}^a = \Lambda_m{}^n\, \mathcal{V}_n{}^a\,, \qquad \delta\phi^{klm} = -3\Lambda_n{}^{[k}\,\phi^{lm]n}\,. \tag{3.2.26}$$

All other fields are invariant under $SL(9)$. The associated $\mathfrak{sl}_9$-valued conserved Noether current can be computed from (3.2.19) and is given by

$$(J_\mu)_k{}^l = \rho\, \mathcal{V}_k{}^a P^{ab}_\mu\, \mathcal{V}^{-1bl} - \rho^{1/3}\left(\mathcal{V}_k{}^a \mathcal{V}^{-1dl}\,\varphi^{bcd}\varphi^{abc}_\mu - \frac{1}{9}\delta^l_k\,\varphi^{abc}\varphi^{abc}_\mu\right) + \frac{1}{54}\,e\varepsilon_{\mu\nu}\,\varepsilon^{abcdefghi}\mathcal{V}_k{}^a\mathcal{V}^{-1jl}\,\varphi^{bcj}\varphi^{def}\varphi^{\nu\, ghi} \;+\; \text{fermions}\,, \tag{3.2.27}$$

where in analogy to (3.2.9) we have defined the dressed scalar fields $\varphi^{abc} \equiv \mathcal{V}_{[klm]}{}^{abc}\phi^{klm}$. As we saw before, a dual scalar $Y_k{}^l$ field can be associated to the $\mathfrak{sl}_9$-valued conserved Noether current

$$\partial_\mu Y_k{}^l = -e\varepsilon_{\mu\nu}\,(J^\nu)_k{}^l\,. \tag{3.2.28}$$

Fermionic contributions are already included in the current J_μ. For later use, we need to determine the supersymmetry variation of $Y_k{}^l$. This could have been done from (3.2.28), but a more straightforward method consist in requiring the closure of the supersymmetry algebra. As a remarkable feature, the supersymmetry variation of the dual field $Y_k{}^l$ depends only on the physical fields

$$\delta_\epsilon Y_k{}^l = \bar{\chi}^{aI}\gamma^3\epsilon^J\,\mathcal{V}_k{}^b\mathcal{V}^{-1cl}\left(\frac{1}{6}\,\rho^{1/3}\left(\varphi^{agh}\varphi^{efc}\delta^{db} - \delta^{b[a}\varphi^{gh]c}\varphi^{def}\right)\Gamma^{defgh}_{IJ} - \rho\,\delta^{a(b}\,\Gamma^{c)}_{IJ}\right) + \frac{3}{2}\,\rho^{2/3}\,\bar{\chi}^{aI}\epsilon^J\,\mathcal{V}^{-1gl}\mathcal{V}_k{}^{[a}\varphi^{bc]g}\,\Gamma^{bc}_{IJ} + \frac{1}{3}\,\rho^{2/3}\,\bar{\psi}^I_2\gamma^3\epsilon^J\,\mathcal{V}^{-1gl}\mathcal{V}_k{}^a\varphi^{bcg}\,\Gamma^{abc}_{IJ} + \bar{\psi}^I_2\epsilon^J\left(\frac{1}{2}\,\rho\,\mathcal{V}^{-1al}\mathcal{V}_k{}^b\,\Gamma^{ab}_{IJ} + \frac{1}{54}\,\rho^{1/3}\,\mathcal{V}^{-1gl}\mathcal{V}_k{}^d\,\varphi^{abc}\varphi^{efg}\,\Gamma^{abcdef}_{IJ}\right). \tag{3.2.29}$$

Then, two supersymmetries closes into diffeomorphisms upon using the duality equation (3.2.28)

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,Y_k{}^l = -e\varepsilon_{\mu\nu}\,\xi^\mu\,(J^\nu)_k{}^l \;=\; \xi^\mu\,\partial_\mu Y_k{}^l\,. \tag{3.2.30}$$

From the symmetry point of view, the equation (3.2.28), defining the dual potential implies a global symmetry acting by a constant shift

$$\delta Y_k{}^l = \Lambda_k{}^l \tag{3.2.31}$$

It can be shown that ${Y_k}^l$ is just one element of an infinite tower of dual scalar fields, that we discussed in the E_8 frame. The dual fields can be generated recursively by applying on-shell duality equations which have schematically the following form,

$$\partial_\mu Y_N = e\,\varepsilon_{\mu\nu}\, J(Y_1, \ldots, Y_{N-1}, \text{off-shell fields})^\nu_N\,. \tag{3.2.32}$$

Thus, the on-shell symmetry algebra will be infinite dimensional and will contain in particular the infinite number of shift symmetries acting on the dual scalar fields. In addition to the global $SL(9)$ symmetry, there are other global off-shell symmetries of the action. They correspond to the 84 translations $\mathbb{R}^{84}$ and act on the 84 scalars ϕ^{abc} as shifts

$$\delta\phi^{klm} = \Lambda^{klm}\,. \tag{3.2.33}$$

The associated conserved Noether current define new scalar fields

$$\partial_\mu Y_{kmn} = -e\,\varepsilon_{\mu\nu} {j^\nu}_{kmn}\,, \tag{3.2.34}$$

on which the translations also act as shifts

$$\delta Y_{kmn} = \Lambda_{kmn}\,. \tag{3.2.35}$$

Finally, the last off-shell global symmetry is the two-dimensional Weyl rescaling

$$\begin{aligned} \delta_\kappa {e_\mu}^\alpha &= \kappa\, {e_\mu}^\alpha\,, & \delta_\kappa \psi^I_\mu &= \frac{\kappa}{2}\,\psi^I_\mu\,, \\ \delta_\kappa \chi^{aI} &= -\frac{\kappa}{2}\,\chi^{aI}\,, & \delta_\kappa \psi^I_2 &= -\frac{\kappa}{2}\,\psi^I_2\,, \end{aligned} \tag{3.2.36}$$

and the scalar fields are left invariant. The corresponding Noether current is given by

$$j^{\text{Weyl}}_\mu \equiv \partial_\mu \rho + \text{fermions}\,. \tag{3.2.37}$$

Again, it enables to define a dual scalar field according to

$$\partial_\mu \tilde{\rho} = -e\,\varepsilon_{\mu\nu}\, j^\nu\,. \tag{3.2.38}$$

In the next section, the general structure of the symmetries will be described in the view of the consistent gauging of an $SO(9)$ subgroup.

3.2.3 General symmetry structure

The $\mathfrak{e}_9$ symmetry algebra

As discussed in [94], the $SL(9)$ frame is better suited than the E_8 one, in order to consistently gauge a $SO(9)$ subgroup. In this frame, the Lagrangian of two-dimensional maximal supergravity has been given (3.2.19), and the Noether currents associated to the different off-shell symmetries have been presented. Nonetheless, we know that the on-shell symmetries of the two-dimensional theory extend to the infinite dimensional algebra $\mathfrak{e}_9 = \hat{\mathfrak{e}}_8$. Thus, it is interesting to identify the different symmetries, manifest off-shell or on-shell in the $SL(9)$ frame, within the full $\mathfrak{e}_9$ algebra. This is done by analyzing the decomposition of the adjoint representation of $\mathfrak{e}_9$ under $\mathfrak{sl}_9$.

The $\widehat{\mathfrak{sl}_9}$ subalgebra

There are several parts in this decomposition. First, the infinite sum

$$\ldots \oplus \mathbf{80}_{-1} \oplus (K_0 \oplus \mathbf{80}_0) \oplus \mathbf{80}_{+1} \oplus \ldots \tag{3.2.39}$$

corresponds to the centrally extended affine subalgebra $\widehat{\mathfrak{sl}_9}$ defined by

$$\begin{aligned}
[\,T_{\alpha,m}\,,\, T_{\beta,n}\,] &= f_{\alpha\beta}{}^{\gamma}\, T_{\gamma,m+n} + m\,\delta_{m+n}\,\eta_{\alpha\beta}\, K_0\,, \\
[T_{\alpha\, m}\,,\, K_0\,] &= 0\,, \qquad\qquad (3.2.40)\\
[\,\mathbf{d}\,, T_{\alpha,m}\,] &= -m\, T_{\alpha,m}\,, \\
[\,\mathbf{d}\,,\, K_0\,] &= 0\,, \qquad\qquad (3.2.41)
\end{aligned}$$

where the $T_{\alpha,0}$ are the generators of $\mathfrak{sl}_9$, the $f_{\alpha\beta}{}^{\gamma}$ stand for the structure constants and $\eta_{\alpha\beta}$ is the Cartan-Killing form. Moreover, K_0 is the central element and $\mathbf{d}$ is the derivation of $\mathfrak{sl}_9$. All the generators are realized on the fields as we shall explain. For example $\mathbf{80}_0$ corresponds to the global $SL(9)$ off shell symmetry (3.2.26),

$$\delta\mathcal{V}_m{}^a = \Lambda_m{}^n\,\mathcal{V}_n{}^a\,, \qquad \delta\phi^{klm} = -3\Lambda_n{}^{[k}\,\phi^{lm]n}\,. \tag{3.2.42}$$

Furthermore, $\mathbf{80}_{+1}$ is realized as the shift symmetry on the dual field $Y_k{}^l$, see (3.2.31). More generally, a shift symmetry $Y_m \to Y_m + \Lambda_m$ is associated to each dual fields (3.2.32) and it corresponds to the action of $T_{\alpha,m}$ (with $m > 0$) which generates the $\mathbf{80}_m$. In addition, $\mathbf{80}_{-1}$ corresponds to the action of $T_{\alpha,-1}$ on the physical fields. It is a non-linear realization which involves the dual fields and are thus non-local. Let us write them schematically

$$\Lambda^{\alpha}\delta_{\alpha,-1}\mathcal{V} = F(\Lambda, Y, \rho, \tilde{\rho}, \phi)\,\mathcal{V}\,. \tag{3.2.43}$$

It is the first example of an infinite family of on-shell symmetries generated by the $T_{\alpha,m}$, ($m < 0$) and corresponding to the $\mathbf{80}_m$, ($m < 0$) in the $\mathfrak{e}_9$ picture. They are called "hidden" symmetries and a compact way to encode them requires the use of a linear system [115, 116, 109]. Consequently, the symmetries may be summed up like this

$$\begin{array}{rl}
 & \vdots \\
\text{shift symmetries} \quad \mathbf{80}_{+1}: & \delta Y_{1k}{}^l = \Lambda^{(1)}{}_k{}^l\,, \\
\text{off-shell } SL(9) \text{ symmetries} \quad \mathbf{80}_0: & \delta\mathcal{V}_m{}^a = \Lambda^{(0)}{}_m{}^n\,\mathcal{V}_n{}^a\,, \quad \delta\phi^{klm} = -3\Lambda^{(0)}{}_n{}^{[k}\,\phi^{lm]n}\,, \\
\text{hidden symmetries} \quad \mathbf{80}_{-1}: & \delta\mathcal{V} = F(\Lambda, \tilde{\rho}, Y, \rho, \phi)\,\mathcal{V}\,. \\
 & \vdots
\end{array} \tag{3.2.44}$$

Finally the central extension K_0 is realized by the Weyl rescaling (3.2.36) and the derivation $\mathbf{d}$ by an on-shell scaling symmetry that acts on the bosonic fields, and scales the Lagrangian,

$$\begin{array}{lll}
\delta\,\rho = \lambda\,\rho\,, & \delta\,\phi^{klm} = \dfrac{\lambda}{3}\,\phi^{klm}\,, & \delta\,\mathcal{L}_0 = \lambda\,\mathcal{L}_0\,, \\
\delta\,\tilde{\rho} = \lambda\,\tilde{\rho}\,, & \delta\,Y_{klm} = \dfrac{2\lambda}{3}\,Y_{klm}\,, & \delta\,Y_k{}^l = \lambda\,Y_k{}^l\,.
\end{array} \tag{3.2.45}$$

The whole picture: hidden, off-shell and shift symmetries

In the $\mathfrak{sl}_9$ picture of $\mathfrak{e}_{(9,9)}$, the $\widehat{\mathfrak{sl}_9}$ subalgebra (3.2.41) is completed by an infinite set of generators transforming in the **84** and **84′** (its dual with respect to $\mathfrak{sl}_9$), to get the full Lie algebra $\mathfrak{e}_{(9,9)}$. The **84** generators are organized into a grading, under the action of the derivation of $\widehat{\mathfrak{sl}_9}$. For example, the fields ϕ^{klm} transform under the off-shell $\mathbb{R}^{84}$ translations, according to (3.2.33). Let us write the generators of this symmetry: $T^{[klm]}$. It belongs to the **84** of $SL(9)$. From (3.2.45), we can deduce the adjoint action of the derivation $\mathbf{d}$ on this generator

$$[\,\mathbf{d}\,,\,T^{[klm]}] = -\frac{1}{3}\,T^{[klm]}\,. \tag{3.2.46}$$

Thus, following the notation of (3.2.41), the generators of the off-shell $\mathbb{R}^{84}$ translations have a weight of $(+1/3)$ with respect to $\mathbf{d}$. Therefore, in the $\widehat{\mathfrak{sl}_9}$ grading of $\mathfrak{e}_9$, we will identify them as $\mathbf{84}_{+1/3}$. The same argument can be made for the generators of the shift symmetries acting on the dual fields Y_{kmn} in (3.2.35). According to (3.2.45), the generators are identified with the $\mathbf{84'}_{+2/3}$. Eventually, by taking successive commutators, the generators $\mathbf{84}_{+1/3}$ generate a positive half of the Kac-Moody algebra $\mathfrak{e}_{(9,9)}$. Finally, it can be shown [94], that in the $\widehat{\mathfrak{sl}_9}$ grading, the charges with respect to $\mathbf{d}$ belong to $\frac{1}{3}\,\mathbb{Z}$. In the light of this analysis, a picture of $\mathfrak{e}_{(9,9)}$ can be schematically drawn

$$\mathfrak{e}_{(9,9)} \underset{\mathfrak{sl}_9}{\longrightarrow} \underbrace{\ldots \oplus \mathbf{84}_{-2/3} \oplus \mathbf{84}'_{-1/3}}_{\text{hidden symmetries}} \oplus \underbrace{(K_0 \oplus \mathbf{80}_0) \oplus \mathbf{84}_{+1/3}}_{\text{off-shell symmetries}} \oplus \underbrace{\mathbf{84}'_{+2/3} \oplus \mathbf{80}_{+1} \oplus \ldots}_{\text{shift symmetries}}\,. \tag{3.2.47}$$

3.3 Vector fields and gauging

The gauging of a $SO(9)$ subgroup is achieved by the embedding tensor formalism. As explained above, this scheme enables to gauge a subgroup of the global symmetries in a way compatible with supersymmetry. The first step deals with the vector fields. Indeed, the embedding tensor realizes the minimal coupling between vector fields and the generators of an $SO(9)$ subgroup of the global symmetries. Then consistency requirement emanating from gauge invariance and supersymmetry must be satisfied. In this section we will first discuss the representation content of the vector fields. Then we will describe the different components of the embedding tensor, and finally we will show which coupling allows the gauging of $SO(9)$.

3.3.1 Vector fields and the embedding tensor in two dimensions

Vector fields

The vector fields of two-dimensional maximal supergravity transform in the basic representation of $\mathfrak{e}_{(9,9)}$, i.e. the unique level 1 representation of this affine algebra [94]. Thus, under $\mathfrak{sl}_9$, the representation of the vector fields decomposes as follows

$$\begin{aligned}\mathcal{R}_{\text{vectors}} \;\rightarrow\; & \mathbf{9}_{5/9} \oplus \\ & \mathbf{36}'_{2/9} \oplus \\ & \mathbf{126}_{-1/9} \oplus \\ & (\mathbf{9} \oplus \mathbf{315}')_{-4/9} \oplus \\ & (\mathbf{36}' \oplus \mathbf{45}' \oplus \mathbf{720})_{-7/9} \oplus \ldots\,.\end{aligned} \tag{3.3.1}$$

The subscripts come from the action of the derivation $\mathbf{d}$ of $\widehat{\mathfrak{sl}_9}$, see (3.2.45). For example the vector field $A_{\mu\, k}$ in the $\mathbf{9}$ carries a charge of 5/9 which can be read from its corresponding kinetic term in the two-dimensional Lagrangian (3.2.8)

$$-\frac{1}{16}\,\rho^{11/9-2s}\,\mathcal{M}^{-1kl}\,F_{\mu\nu\,k}F^{\mu\nu}{}_{l}\,. \tag{3.3.2}$$

Because the Lagrangian carries a charge of +1, and the $F\,F$ term is multiplied by $\rho^{19/9}$, each F should carry a charge of $\frac{1}{2}\,(\frac{19}{9}-1)=\frac{5}{9}$. Moreover, the vector fields $A_\mu{}^{mn}$ in the $\mathbf{36}$ carry a charge of $\frac{2}{9}$ because they have the same charge than $A_{\mu\,k}\,\phi^{kmn}$. It can be seen in the ansatz that we have made for the three-form in the toroidal compactification of eleven dimensional supergravity (3.2.6),

$$A_{MNK}=\Big(0\,,0\,,A_\mu{}^{mn}+A_{\mu\,k}\,\phi^{kmn}\,,\;\phi^{mnk}\Big)\,. \tag{3.3.3}$$

$A_{\mu\,k}$ and ϕ^{kmn} carry respectively the charges $\frac{5}{9}$ and $\frac{1}{3}$, so the vector fields $A_\mu{}^{mn}$ should carry the charge $\frac{5}{9}-\frac{1}{3}=\frac{2}{9}$. The other assignments follow by decreasing the charge in steps of $\frac{1}{3}$.

The embedding tensor

In general, for maximal supergravities, after imposing the *linear constraint* the following statement holds:

the embedding tensor transforms in the representation dual to the (D-1)-forms. (†)

In two dimensions, this statement is not straightforward because of the infinite dimensional context. This implies an important constraint on $\Theta_{\mathcal{M}}{}^{\alpha}$ since the embedding tensor transforms in $\mathcal{R}_{v^*}\times\mathcal{R}_{\text{adj}}$, whereas the dual representation of the $(2-1)$-forms (the vector fields) is $\mathcal{R}_{v^*}$. It turns out that in two dimensions, this conjecture motivated by the higher-dimensional cases, is the expression of the linear constraint [94]. According to this constraint, the embedding tensor is no longer parametrized by $\Theta_{\mathcal{M}}{}^{\alpha}$ but by a tensor $\Theta_{\mathcal{M}}$ in the $\mathcal{R}_{v^*}$ of $\mathfrak{e}_{(9,9)}$:

$$\Theta_{\mathcal{M}}{}^{\alpha}=\eta^{\alpha\beta}(t_\beta)_{\mathcal{M}}{}^{\mathcal{N}}\Theta_{\mathcal{N}}\,. \tag{3.3.4}$$

As a hint for (†), let us consider the first components of the representation of the vector fields. The corresponding fields are $A_{\mu\,k}$ and $A_\mu{}^{[mn]}$, belonging respectively to the $\mathbf{9}$ and $\mathbf{36'}$ of $SL(9)$. Then, the equations of motion for the vector fields, once integrated (3.2.11), show that the lowest components of the embedding tensor, θ^k and $\tilde\theta_{kl}$, transform respectively in the representations dual to the ones of the vector fields. It was shown in [94], that a Θ transforming in $\mathcal{R}_{v^*}$, solves the *linear constraint.* With respect to $\mathfrak{sl}_9$ it splits according to

$$\begin{aligned}\mathcal{R}_\Theta\;\rightarrow\;\; & \mathbf{9'}_{-14/9}\oplus\\ & \mathbf{36}_{-11/9}\oplus\\ & \mathbf{126'}_{-8/9}\oplus\\ & (\mathbf{9'}\oplus\mathbf{315})_{-5/9}\oplus\\ & (\mathbf{36}\oplus\mathbf{45}\oplus\mathbf{720'})_{-2/9}\;\oplus\;\ldots\,.\end{aligned} \tag{3.3.5}$$

To derive the charges of the component of $\mathcal{R}_\Theta$ under the derivation $\mathbf{d}$, we use the fact that when the subscripts of $\mathcal{R}_{\text{vectors}}$ and $\mathcal{R}_\Theta$ corresponding to the same row are added, the result is -1. The minus sign is a matter of convention, so let us focus on the absolute value of 1. It originates from the fact that the charge of the Lagrangian under the derivation equals one. As an illustration, consider the vector field $A_{\mu\,k}$. Its kinetic term (3.2.8) schematically written MFF, scales like the Lagrangian under (3.2.45). However, we saw that the equation of motion for the vector (once integrated), relates the field strength with a component of the embedding tensor (3.2.11), $MF = \Theta$. Therefore, ΘF has a charge of 1 under (3.2.45), as expected above.

The couplings between the vector fields and the generators of the internal symmetries are induced by the different components of the embedding tensor. They have been clas-si … dding

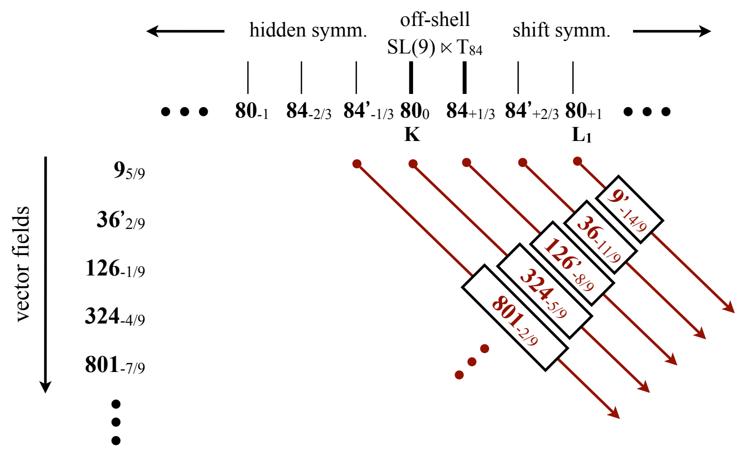

Figure 3.2: Minimal couplings induced by different components of the embedding tensor $\Theta_{\mathcal{M}}$.

tensor of the general form $\Theta_{\mathcal{M}}{}^{\alpha}$, all the cells would have been filled by an independent component of the embedding tensor. However, when the linear constraint (3.3.4) is satisfied, the different couplings are ensured by an embedding tensor of the form $\Theta_{\mathcal{M}}$ with components given by (3.3.5). Consequently, the *allowed* couplings are very restricted and the same component of the embedding tensor is involved for the couplings of the diagonals. The matching is made by ensuring that the sum of the charge of the vector field under the derivation (written in the subscripts), the component of the embedding tensor and the corresponding generator of symmetry, is equal to zero. Thus, in the light of representation theory, the picture shows how the minimal couplings must be done in order to gauge the desired symmetry. For example, the component $\mathbf{36}_{-11/9}$ of the embedding tensor is involved in the gauging of the $\mathbf{80_{+1}}$ and $\mathbf{L_1}$ on-shell symmetries through the minimal coupling with the vector fields in the $\mathbf{36'}_{2/9}$. The same component of the embedding tensor is also involved in the gauging of the $\mathbf{84'}_{+2/3}$ on-shell symmetry through the minimal coupling with the vector fields in the $\mathbf{9}_{5/9}$.

As an example, let us try to gauge the on-shell $\mathbf{L_1}$ symmetry, acting on the dual dilaton $\tilde{\rho}$ of (3.2.38) as a shift. We learn from Figure 3.2 that the vector fields in the $\mathbf{9_{5/9}}$ can be used to achieve a minimal coupling with $\mathbf{L_1}$, induced by the component $\mathbf{9'_{-14/9}}$ of the embedding tensor. The vectors fields corresponds to the $A_{\mu\,k}$ of (3.2.8) and the embedding tensor in the $\mathbf{9'_{-14/9}}$ is parametrized by constants θ^k. Thus, covariant derivatives can be introduced and the shift symmetry on $\tilde{\rho}$ is knitted with the gauge transformation of the vector field

$$\delta\tilde{\rho} \equiv g\,\Lambda_k\,\theta^k\,,$$

$$\begin{aligned} D_\mu &\equiv \partial_\mu - g\, A_{\mu\, k}\, \theta^k\, T_{\mathbf{L_1}}\,, \\ \delta A_{\mu\, k} &\equiv \partial_\mu \Lambda_k\,. \end{aligned} \tag{3.3.6}$$

The first step in the gauging, namely the covariantization, has been achieved. By construction, we known that the gauging is consistent with supersymmetry at the linear order, since the embedding tensor solves the linear constraint. Therefore, the quadratic constraint remains to be solved in order to get a consistent $\mathbf{L_1}$-gauged maximal supergravity in two dimensions.

Now let us focus on the $SO(9)$ gauging. The component of the embedding tensor that will interest us to construct the $SO(9)$ supergravity is $\theta_{kl} = \theta_{(kl)}$ transforming in the $\mathbf{45}_{-2/9}$ from the fifth level of the decomposition (3.3.5). It belongs to the $\mathbf{801}_{-2/9} = (\mathbf{36} \oplus \mathbf{45} \oplus \mathbf{720'})_{-2/9}$ described in the Figure 3.2. Thus it induces several couplings including generators of the $\mathbf{80}_0$ which describes the off-shell $SL(9)$ symmetry. More precisely, we read off that the $\mathbf{45}_{-2/9}$ component induces a coupling between the vector fields ${A_\mu}^{kl}$ of (3.3.3) in the $\mathbf{36'}_{2/9}$ and the $SL(9)$ generators of $\mathbf{80}_0$. Therefore, the representation theory fixes the coupling

$$X_{kl} \equiv \theta_{m[k}\, {T_{l]}}^m\,. \tag{3.3.7}$$

Here the traceless ${T_k}^l$ denote the generators of $\mathfrak{sl}_9$. This coupling is interesting because:

- the X_{kl} generate a $\mathfrak{cso}_{p,q,r}$ subalgebra of $\mathfrak{sl}_9$ where the integers $p+q+r=9$ characterize the signature of θ_{kl} [41]. Consequently, by choosing $\theta_{kl} \equiv \delta_{kl}$, a $\mathfrak{so}(9) \in \mathfrak{sl}(9)$ is gauged.

- It has been shown in [94] that the embedding tensor in the $\mathbf{45}_{-2/9}$ automatically satisfies its quadratic constraint (2.3.10).

- The couplings induced by an embedding tensor in the $\mathbf{45}_{-2/9}$ does not involve the generators of the $\mathbf{84'}_{-1/3}$, neither the $\mathbf{84}_{+1/3}$ nor the K_0, because no $\mathbf{45}_{-2/9}$ appears respectively in the tensor product of $\mathbf{9}_{5/9} \otimes \mathbf{84'}_{-1/3}$, $\mathbf{126}_{-1/9} \otimes \mathbf{84}_{+1/3}$ and $\mathbf{36'}_{2/9} \otimes \mathbf{1}$.

As a result, we have at hand a tool that enables to gauge a $SO(9)$ subgroup of the off-shell symmetries of maximal supergravity in two dimensions, formulated in the "$SL(9)$ frame", such that it is consistent and in particular compatible with supersymmetry. If we were working in the E_8 frame, the consistent $SO(9)$ gauging would have involved couplings with hidden symmetries generators. Thus at the level of the action, a complicated non-local topological term would have been needed to restore supersymmetry [94]. The fact that in the $SL(9)$ frame, the $SO(9)$ group (which according to the embedding tensor, can be gauged in a way consistent with supersymmetry) belongs to the off-shell symmetries of the theory, is the reason why the $SL(9)$ frame is better suited than the E_8 frame to perform the gauging. Eventually, there remains to concretely construct the gauged theory. This will be done through the Noether procedure presented in page 33, but it looks like a hard task, since the bosonic field content now involves the 84 fields ϕ^{klm} which renders the structure of the Lagrangian (3.2.19) more complicated than in the "E_8 frame" (3.2.1).

To conclude, a group theoretical analysis enables us to identify the right couplings for gauging $SO(9)$ among the off-shell symmetries of the maximal two-dimensional supergravity. Knowing the right embedding tensor, one can introduce minimal coupling via covariant derivatives

$$\mathcal{D}_\mu = \partial_\mu - g\, {A_\mu}^{kl}\, \theta_{mk}\, {T_l}^m\,. \tag{3.3.8}$$

We may recall here that the $SO(9)_{\text{gauge}}$ that we will gauge, shall not be confused with the local $SO(9)_{\text{coset}}$ symmetry coming with the $SL(9)/SO(9)$ coset structure. Indeed, in the coset space formulation, the $SL(9)$ group acts globally on the scalar fields $\mathcal{V} \in SL(9)/SO(9)$ by left multiplication

$$\mathcal{V}_k{}^a \to \Lambda_k{}^l\,\mathcal{V}_l{}^a\,, \quad \Lambda \in \mathfrak{sl}(9)\,. \tag{3.3.9}$$

We will gauge a $SO(9)$ inside these global symmetries,

$$\mathcal{V}_k{}^a \to \Lambda(x)_k{}^l\,\mathcal{V}_l{}^a\,, \quad \Lambda(x) \in \mathfrak{so}(9)_{\text{gauge}}\,. \tag{3.3.10}$$

On the contrary, the local $SO(9)_{\text{coset}}$ acts on the scalar fields $\mathcal{V}$ by right multiplication

$$\mathcal{V}_k{}^a \to \mathcal{V}_k{}^b\,K(x)_b{}^a\,, \quad K(x) \in \mathfrak{so}(9)\,. \tag{3.3.11}$$

The latter ensures that the σ-model describes the right number of physical degrees of freedom: d.o.f. $= 80 - 36 = 44$. The full construction of the $SO(9)$ gauged Lagrangian and its $\mathcal{N} = 16$ supersymmetry invariance will be detailed in the next section.

3.4 $SO(9)$ supergravity: Lagrangian

This section describes the first result of the thesis and maybe the main one. Indeed, the $SO(9)$ gauged maximal supergravity is constructed in full details.

3.4.1 General ansatz

To begin, the first step deals with the covariantization of the two-dimensional Lagrangian (3.2.19), by turning the derivatives into covariant ones

$$\begin{aligned}
Q_\mu^{[ab]} + P_\mu^{(ab)} \to \quad & \mathcal{Q}_\mu^{[ab]} + \mathcal{P}_\mu^{(ab)} \;\equiv\; \mathcal{V}^{-1ak}\left(\partial_\mu \mathcal{V}_k{}^b - A_\mu{}^{lm}\theta_{mk}\,\mathcal{V}_l{}^b\right)\,,\\
\partial_\mu\phi^{klm} \to \quad & \mathcal{D}_\mu\phi^{klm} \;\equiv\; \partial_\mu\phi^{klm} - 3\,A_\mu{}^{p[k}\theta_{pq}\,\phi^{lm]q}\,,\\
\varphi_\mu^{abc} \to \quad & \widetilde{\varphi}_\mu^{abc} \;\equiv\; \mathcal{V}_{klm}{}^{[abc]}\,\mathcal{D}_\mu\phi^{klm}\,,\\
D_\mu\psi_\nu^I \to \quad & \mathcal{D}_\mu\psi_\nu^I \;\equiv\; \partial_\mu\psi_\nu^I + \frac{1}{4}\omega_\mu{}^{\alpha\beta}\gamma_{\alpha\beta}\,\psi_\nu^I + \frac{1}{4}\mathcal{Q}_\mu^{ab}\,\Gamma_{IJ}^{ab}\,\psi_\nu^J\,,\\
D_\mu\chi^{aI} \to \quad & \mathcal{D}_\mu\chi^{aI} \;\equiv\; \partial_\mu\chi^{aI} + \frac{1}{4}\omega_\mu{}^{\alpha\beta}\gamma_{\alpha\beta}\,\chi^{aI} + \mathcal{Q}_\mu^{ab}\,\chi^{bI} + \frac{1}{4}\mathcal{Q}_\mu^{bc}\,\Gamma_{IJ}^{bc}\,\chi^{aJ}\,.
\end{aligned} \tag{3.4.1}$$

By doing so, the Lagrangian (3.2.19) becomes

$$\begin{aligned}
e^{-1}\mathcal{L}_{0\,,\text{cov}} = & -\frac{1}{4}\rho R^{(2)} + \frac{1}{4}\rho\,\mathcal{P}^{\mu\,ab}\mathcal{P}_\mu^{ab} + \frac{1}{12}\,\rho^{1/3}\,\widetilde{\varphi}^{\mu\,abc}\widetilde{\varphi}_\mu^{abc}\\
& + \frac{1}{648}\,e^{-1}\varepsilon^{\mu\nu}\varepsilon_{klmnpqrst}\,\phi^{klm}\,\mathcal{D}_\mu\phi^{npq}\,\mathcal{D}_\nu\phi^{rst}\\
& - \rho e^{-1}\varepsilon^{\mu\nu}\bar{\psi}_2^I\mathcal{D}_\mu\psi_\nu^I - \frac{i}{2}\,\bar{\psi}_\nu^I\gamma^\nu\psi_\mu^I\,\partial^\mu\rho - \frac{i}{2}\,\rho\,\bar{\chi}^{aI}\gamma^\mu\mathcal{D}_\mu\chi^{aI}\\
& - \frac{1}{2}\,\rho\,\bar{\chi}^{aI}\gamma^\nu\gamma^\mu\psi_\nu^J\Gamma_{IJ}^b\mathcal{P}_\mu^{ab} - \frac{i}{2}\,\rho\,\bar{\chi}^{aI}\gamma^3\gamma^\mu\psi_2^J\Gamma_{IJ}^b\mathcal{P}_\mu^{ab}\\
& - \frac{1}{4}\,\rho^{2/3}\,\bar{\chi}^{aI}\gamma^3\gamma^\nu\gamma^\mu\psi_\nu^J\Gamma_{IJ}^{bc}\,\widetilde{\varphi}_\mu^{abc} - \frac{i}{12}\,\rho^{2/3}\,\bar{\chi}^{aI}\gamma^\mu\psi_2^J\Gamma_{IJ}^{bc}\,\widetilde{\varphi}_\mu^{abc}\\
& + \frac{i}{54}\,\rho^{2/3}\,\bar{\psi}_2^I\gamma^3\gamma^\mu\psi_2^J\Gamma_{IJ}^{abc}\,\widetilde{\varphi}_\mu^{abc} + \frac{1}{24}\,\rho^{2/3}\,\bar{\psi}_2^I\left(\gamma^\mu\gamma^\nu - \frac{1}{3}\gamma^\nu\gamma^\mu\right)\psi_\nu^J\,\Gamma_{IJ}^{abc}\,\widetilde{\varphi}_\mu^{abc}\\
& + \frac{i}{2}\,\rho^{2/3}\,\bar{\chi}^{aI}\gamma^3\gamma^\mu\chi^{bJ}\Gamma_{IJ}^{c}\,\widetilde{\varphi}_\mu^{abc} - \frac{i}{24}\,\rho^{2/3}\,\bar{\chi}^{aI}\gamma^3\gamma^\mu\chi^{aJ}\Gamma_{IJ}^{bcd}\,\widetilde{\varphi}_\mu^{bcd}\,.
\end{aligned} \tag{3.4.2}$$

As a result, up to total derivative, the Lagrangian is invariant under the local gauge symmetry

$$\begin{aligned} \delta\mathcal{V}_m{}^a &= \Lambda(x)^{nk}\,\theta_{km}\,\mathcal{V}_n{}^a\,, \\ \delta\phi^{klm} &= -3\Lambda(x)^{m[k}\,\theta_{mn}\,\phi^{lm]n}\,, \qquad \delta A_\mu^{kl} = \mathcal{D}_\mu\Lambda(x)^{kl}\,, \end{aligned} \tag{3.4.3}$$

with the gauge parameter $\Lambda(x)^{kl} = \Lambda(x)^{[kl]}$.

Auxiliary fields

However, the construction should not stop here, or else the vector fields would behave as Lagrange multipliers and would reduce the number of degrees of freedom by generating an additional on-shell constraint on the physical fields. Indeed, taking the variation of the Lagrangian with respect to the gauge field yields, up to total derivative, the following term

$$0 = \frac{\delta\mathcal{L}_{0\,,\mathrm{cov}}}{\delta A_\mu{}^{kl}} \equiv -\frac{e}{2}\,\mathcal{J}^\mu{}_{kl}\,. \tag{3.4.4}$$

In passing, the r.h.s is just the covariantized $\mathfrak{sl}_9$ Noether current (3.2.27) projected with θ_{kl},

$$\mathcal{J}_{\mu\,kl} \equiv -\theta_{m[k}\,J^{\mathrm{cov}}_{|\mu|\,l]}{}^m\,. \tag{3.4.5}$$

A solution to the problem of degrees of freedom would be to add a kinetic term for the vector fields, of the form: $\mathcal{L}_{\mathrm{FF}} \;\propto\; \mathcal{F}\mathcal{F}$ with

$$\mathcal{F}_{\mu\nu}{}^{kl} \equiv 2\partial_{[\mu}A_{\nu]}{}^{kl} + 2\,\theta_{pq}\,A_{[\mu}{}^{p[k}A_{\nu]}{}^{l]q}\,. \tag{3.4.6}$$

The equation of motion for the gauge field would give,

$$\frac{\delta\mathcal{L}}{\delta A_\mu{}^{kl}} = 0 = \mathcal{D}_\nu\mathcal{F}^{\nu\mu}{}_{kl} - \mathcal{J}^\mu{}_{kl}\,. \tag{3.4.7}$$

However, the Yang-Mills kinetic term is not natural for the gauging since when the embedding tensor is put to zero, the ungauged theory (3.2.19) is not recovered. Another term that we can imagine to cancel the on-sell current has the following form,

$$\mathcal{L}_{\mathrm{FY}} = -\frac{1}{4}\,\varepsilon^{\mu\nu}\mathcal{F}_{\mu\nu}{}^{kl}\theta_{lm}\,Y_k{}^m\,, \tag{3.4.8}$$

where the $Y_k{}^m \in \mathfrak{sl}(9)$ are auxiliary scalar fields. Then, the equation of motion for the vector fields gives,

$$\frac{\delta\mathcal{L}}{\delta A_\mu{}^{kl}} = 0 = e^{-1}\,\varepsilon^{\mu\nu}\theta_{m[k}\mathcal{D}_{|\nu|}Y_{l]}{}^m - \mathcal{J}^\mu{}_{kl}\,. \tag{3.4.9}$$

This is nothing but the covariantization of the $\mathfrak{sl}(9)$ duality equation (3.2.28) projected on the gauge subgroup part. If we choose $\theta_{kl} \propto \delta_{kl}$, it will relate the $\mathfrak{so}(9)$ part of the Noether current with its dual potential. Nevertheless, a scalar potential of order g^2 that depends on Y has to be added, so that the variation of the Lagrangian with respect to the scalar field $Y_k{}^l$ does not lead to the cancellation of the gauge field on-shell:

$$\frac{\delta\mathcal{L}_{0\,,\mathrm{cov}}}{\delta Y_k{}^m} = 0 = \varepsilon^{\mu\nu}\mathcal{F}_{\mu\nu}{}^{kl}\theta_{lm}\,. \tag{3.4.10}$$

Consequently, both the $A_\mu{}^{kl}$ and $Y_k{}^l$ appear as auxiliary fields in the Lagrangian. It happens that the parametrization involving Y is more natural to restore supersymmetry than a Lagrangian without the ${Y_k}^l$ fields but with a kinetic term ($\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu}$) for the gauge fields. These two possible parametrizations will be related to each other and discussed at the end of this chapter. Eventually, the supersymmetry of the Lagrangian will not survive the covariantization. In particular because the supersymmetry variation of the vector fields is not canceled. To provide the most complete candidate for supersymmetry invariance, we are led to follow the Noether procedure described in Section 2.3.1. Thus, new Yukawa-type couplings $\mathcal{L}_{\text{Yuk}}$, the so-called fermionic mass terms, are added to the Lagrangian which now reads

$$\mathcal{L} = \mathcal{L}_{0,\text{cov}} - \frac{1}{4}\,\varepsilon^{\mu\nu}{\mathcal{F}_{\mu\nu}}^{kl}\theta_{lm}\,{Y_k}^m + \mathcal{L}_{\text{Yuk}}\,. \tag{3.4.11}$$

Supersymmetry

Our starting point is the Lagrangian (3.4.22), where the Yukawa couplings need to be determined. Let us assume that the bosonic supersymmetry transformation rules (3.2.18) and (3.2.22) remain unchanged. They are summarized below

$$\begin{aligned}
\delta_\epsilon {e_\mu}^\alpha &= i\,\bar\epsilon^I\gamma^\alpha\psi^I_\mu\,, \qquad \delta_\epsilon\rho = -\rho\,\bar\epsilon^I\gamma^3\psi^I_2\,, \qquad \delta_\epsilon{\mathcal{V}_i}^a = \bar\epsilon^I\Gamma^{(a}_{IJ}\chi^{b)J}{\mathcal{V}_i}^b\,,\\
\delta_\epsilon\phi^{ijk} &= \frac{3}{2}\,\rho^{1/3}\,\mathcal{V}^{-1[ijk]}_{abc}\Gamma^{ab}_{IJ}\,\bar\epsilon^I\gamma^3\chi^{cJ} + \frac{1}{6}\,\rho^{1/3}\,\mathcal{V}^{-1[ijk]}_{abc}\Gamma^{abc}_{IJ}\,\bar\epsilon^I\psi^J_2\,,\\
\delta_\epsilon\,{A_\mu}^{kl} &= \rho^{-2/9}\Big(\bar\psi^I_\mu\epsilon^J\Gamma^{ab}_{IJ} - \frac{2i}{9}\,\bar\psi^I_2\gamma^3\gamma_\mu\epsilon^J\Gamma^{ab}_{IJ} - 2i\,\bar\chi^{I[a}\gamma_\mu\epsilon^J\Gamma^{b]}_{IJ}\Big){{\mathcal{V}^{-1}}_{[ab]}}^{kl}\\
&\quad + 2\,\rho^{-5/9}\Big(\bar\psi^I_\mu\gamma^3\epsilon^J\Gamma^a_{IJ} + \frac{5i}{9}\,\bar\psi^I_2\gamma_\mu\epsilon^J\Gamma^a_{IJ} - i\bar\chi^{aI}\gamma^3\gamma_\mu\epsilon^I\Big){{\mathcal{V}^{-1}}_{[bc]}}^{kl}\,\varphi^{abc}\,.
\end{aligned} \tag{3.4.12}$$

Then, let us look for the most general ansatz for the Yukawa-type couplings,

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{Yuk}} = &-\frac{1}{2}\,e^{-1}\rho\,\varepsilon^{\mu\nu}\left(\bar\psi^I_\nu\psi^J_\mu B_{IJ} + \bar\psi^I_\nu\gamma^3\psi^J_\mu\tilde B_{IJ} - 2i\bar\psi^I_2\gamma_\nu\psi^J_\mu A_{IJ}\right) + i\rho\,\bar\psi^I_2\gamma^\mu\psi^J_\mu\tilde A_{IJ}\\
&+ i\rho\,\bar\chi^{aI}\gamma^\mu\psi^J_\mu C^a_{IJ} - i\rho\,\bar\chi^{aI}\gamma^3\gamma^\mu\psi^J_\mu\tilde C^a_{IJ} + \rho\,\bar\psi^I_2\psi^J_2 D_{IJ} + \rho\,\bar\psi^I_2\gamma^3\psi^J_2\tilde D_{IJ}\\
&+ \rho\,\bar\chi^{aI}\psi^J_2 E^a_{IJ} + \rho\,\bar\chi^{aI}\gamma^3\psi^J_2\tilde E^a_{IJ} + \rho\,\bar\chi^{aI}\chi^{bJ}F^{ab}_{IJ} + \rho\,\bar\chi^{aI}\gamma^3\chi^{bJ}\tilde F^{ab}_{IJ}\,,
\end{aligned} \tag{3.4.13}$$

Here the A, B, C, D, E, F tensors depend on the scalar fields ρ, $\mathcal{V}$, ϕ, Y, are proportional to the deformation parameter θ, and have to be determined. We shall call them the "Yukawa tensors" or the "fermionic mass tensors" in accordance with the presentation of page 33. Notice that the spinorial structure implies some symmetry properties and some constraints on these tensors:

$$B_{(IJ)} = \tilde D_{(IJ)} = 0 = \tilde B_{[IJ]} = D_{[IJ]}\,, \qquad F^{ab}_{IJ} = F^{ba}_{JI}\,, \qquad \tilde F^{ab}_{IJ} = -\tilde F^{ba}_{JI}\,, \tag{3.4.14}$$

and

$$\Gamma^a_{IJ}C^a_{IK} = \Gamma^a_{IJ}\tilde C^a_{IK} = 0\,. \tag{3.4.15}$$

The introduction of such couplings induces modifications proportional to the fermionic supersymmetry transformations: the so-called fermion shifts.

$$\begin{aligned}
\delta_\epsilon\psi^I_\mu &= \mathcal{D}_\mu\epsilon^I - \frac{1}{24}\,\rho^{-1/3}\Gamma^{abc}_{IJ}\Big(\gamma^\nu\gamma_\mu + \frac{1}{3}\gamma_\mu\gamma^\nu\Big)\gamma^3\epsilon^J\,\tilde\varphi^{abc}_\nu + i\,\Big(A_{IJ} + \tilde A_{IJ}\gamma^3\Big)\,\gamma_\mu\epsilon^J\,,\\
\delta_\epsilon\psi^I_2 &= -\frac{i}{2}\,\rho^{-1}\,(\partial_\mu\rho)\,\gamma^3\gamma^\mu\epsilon^I + \Big(B_{IJ} + \tilde B_{IJ}\,\gamma^3\Big)\,\epsilon^J\,,\\
\delta_\epsilon\chi^{aI} &= \frac{i}{2}\,\Gamma^b_{IJ}\,\gamma^\mu\epsilon^J\mathcal{P}^{ab}_\mu - \frac{i}{6}\,\rho^{-1/3}\Big(\delta^{ad}\Gamma^{bc}_{IJ} - \frac{1}{6}\,\Gamma^{abcd}_{IJ}\Big)\,\gamma^3\gamma^\mu\epsilon^J\,\tilde\varphi^{bcd}_\mu + \Big(C^a_{IJ} + \tilde C^a_{IJ}\gamma^3\Big)\,\epsilon^J\,,
\end{aligned} \tag{3.4.16}$$

This shifts involve the same undetermined tensors in a very constrained way. As a remarkable fact, the tensors B and $\tilde{B}$ which couple to the $\bar{\psi}_\mu \psi_\nu$ Yukawa terms, are present in the supersymmetry variation of ψ_2 instead of ψ_μ. This is due to the fact that in two dimensions, the Rarita-Schwinger term

$$-\rho\,\epsilon^{\mu\nu}\,\bar{\psi}_2^I\,\mathcal{D}_\mu \psi_\nu^I \tag{3.4.17}$$

of (3.4.2) mixes ψ_2 and ψ_μ.

At the linear order in θ, when we take the supersymmetry variation of the Lagrangian (3.4.22), we first get terms linear in θ coming from $\mathcal{L}_{0,\text{cov}}$:

$$\begin{aligned}\delta_\varepsilon \mathcal{L}_{0,\text{cov}} = &-\frac{1}{2}\big(\delta_\varepsilon A_\mu{}^{kl}\big)\mathcal{J}_{\mu\,kl} + \frac{1}{4}\varepsilon^{\mu\nu}\mathcal{F}_{\mu\nu}{}^{kl}\theta_{lm}\,\Xi_k{}^m \\ &+ \mathcal{O}(\theta)\,\text{shifts} \quad + \text{q.f.}\end{aligned} \tag{3.4.18}$$

The supersymmetry violating terms proportional to the field strength come from the fact that the covariant derivatives containing the gauge field, no longer commute

$$\begin{aligned}\left[\mathcal{D}_\mu, \mathcal{D}_\nu\right] \phi^{klm} &= -3\,\theta_{pq}\,\mathcal{F}_{\mu\nu}{}^{p[k}\,\phi^{lm]q}\,, \\ \mathcal{D}_{[\mu}\,\mathcal{P}_{\nu]}^{ab} &= \frac{1}{2}\,\theta_{kl}\,\mathcal{V}_{m(a}\,\mathcal{V}^{-1}{}_{b)}{}^k\,\mathcal{F}_{\mu\nu}{}^{lm}\,.\end{aligned} \tag{3.4.19}$$

It is canceled by imposing the following variation of the auxiliary field

$$\begin{aligned}\delta_\epsilon\, Y_k{}^l &= \Xi_k{}^l \\ &= \bar{\chi}^{aI}\gamma^3\epsilon^J\,\mathcal{V}_k{}^b\mathcal{V}^{-1cl}\left(\frac{1}{6}\,\rho^{1/3}\left(\varphi^{agh}\varphi^{efc}\delta^{db} - \delta^{b[a}\varphi^{gh]c}\varphi^{def}\right)\Gamma_{IJ}^{defgh} - \rho\,\delta^{a(b}\,\Gamma_{IJ}^{c)}\right) \\ &\quad + \frac{3}{2}\,\rho^{2/3}\,\bar{\chi}^{aI}\epsilon^J\,\mathcal{V}^{-1gl}\mathcal{V}_k{}^{[a}\varphi^{bc]g}\,\Gamma_{IJ}^{bc} + \frac{1}{3}\,\rho^{2/3}\,\bar{\psi}_2^I\gamma^3\epsilon^J\,\mathcal{V}^{-1gl}\mathcal{V}_k{}^a\varphi^{bcg}\,\Gamma_{IJ}^{abc} \\ &\quad + \bar{\psi}_2^I\epsilon^J\left(\frac{1}{2}\,\rho\,\mathcal{V}^{-1al}\mathcal{V}_k{}^b\,\Gamma_{IJ}^{ab} + \frac{1}{54}\,\rho^{1/3}\,\mathcal{V}^{-1gl}\mathcal{V}_k{}^d\,\varphi^{abc}\varphi^{efg}\,\Gamma_{IJ}^{abcdef}\right)\,.\end{aligned} \tag{3.4.20}$$

This is exactly the variation (3.2.29) that we have found for the ungauged theory. Moreover, the terms coming from the variation of the vector field and coupled to the Noether current (3.4.5) is to be canceled by the $\mathcal{O}(\theta)$ variation of $\mathcal{L}_{\text{Yuk}}$ together with the $\mathcal{O}(\theta)$ contributions introduced in $\delta_\varepsilon \mathcal{L}_{0,\text{cov}}$ by the fermion shifts. The same occurs with the variation

$$-\frac{1}{4}\,\varepsilon^{\mu\nu}\,\delta_\varepsilon\big(\mathcal{F}_{\mu\nu}{}^{kl}\big)\,\theta_{lm}\,Y_k{}^m\,. \tag{3.4.21}$$

All this leads to linear constraints on the Yukawa tensors. Then supersymmetry is restored at the linear level in θ, if and only if there is a solution to these Yukawa linear constraints.

As was expected, there appear new supersymmetry contributions of quadratic order in θ. More precisely, they come from the fermion shifts in the supersymmetry variation of $\mathcal{L}_{\text{Yuk}}$. As we will see, part of them are canceled by the variation of a scalar potential $\mathcal{L}_{\text{pot}}$, quadratic in θ, but the majority give rise to quadratic constraints. Here, these constraints are pure consistency checks and will be detailed below. Thus, the full Lagrangian may now be written

$$\mathcal{L}_{\text{full}} = \mathcal{L}_{0,\text{cov}} - \frac{1}{4}\,\varepsilon^{\mu\nu}\mathcal{F}_{\mu\nu}{}^{kl}\theta_{lm}\,Y_k{}^m + \mathcal{L}_{\text{Yuk}} + \mathcal{L}_{\text{pot}}\,. \tag{3.4.22}$$

According to the previous analysis, at this point, the supersymmetry only rely on the resolution of linear and quadratic identities among the Yukawa tensors. They will be detailed in the following.

3.4.2 Yukawa tensors

Linear constraints

A close examination of supersymmetry at linear order in θ reveals that all the contributions are proportional to first derivative terms of this kind: $\partial_\mu\rho$, $\tilde{\varphi}_\mu^{abc}$, $\mathcal{P}_\mu^{ab}$ and $\mathcal{D}_\mu Y_k{}^l$. The 8 linearly independent terms that we can form with $\partial_\mu\rho$ are collected here

$$\begin{array}{llll} \bar{\psi}_\mu^I\,\epsilon^J\partial^\mu\rho\,, & \bar{\psi}_\mu^I\gamma^3\epsilon^J\partial^\mu\rho\,, & \bar{\psi}_\mu^I\gamma^{\mu\nu}\epsilon^J\partial_\nu\rho\,, & \bar{\psi}_\mu^I\gamma^{\mu\nu}\gamma^3\epsilon^J\partial_\nu\rho\,, \\ \bar{\psi}_2{}^I\gamma^\mu\epsilon^J\partial_\mu\rho\,, & \bar{\psi}_2{}^I\gamma^\mu\gamma^3\epsilon^J\partial_\mu\rho\,, & \bar{\chi}^{aI}\gamma^\mu\epsilon^J\partial_\mu\rho\,, & \bar{\chi}^{aI}\gamma^\mu\gamma^3\epsilon^J\partial_\mu\rho\,. \end{array} \quad (3.4.23)$$

From this sequence, the other linearly independent terms that we can form with $\tilde{\varphi}_\mu^{abc}$, $\mathcal{P}_\mu^{ab}$ and $\mathcal{D}_\mu Y_k{}^l$ can be deduced easily. Associated to these terms are linear combinations of the Yukawa tensors. By requiring the linearly independent fermionic terms to vanish, all the sets of linear equations on the Yukawa tensors are generated. For example, the term proportional to $\bar{\psi}_\mu^I\gamma^{\mu\nu}\epsilon^J\partial_\nu\rho$ comes, on one side, from the fermion shift part

$$\tilde{B}_{IJ}\gamma^3\epsilon^J \;\in\; \delta_\epsilon\psi_2^I\,, \quad (3.4.24)$$

in the term

$$-\,\rho\,e^{-1}\varepsilon^{\mu\nu}\,\bar{\psi}_2^I\,\mathcal{D}_\mu\psi_\nu^I\;\in\mathcal{L}_{0,\text{cov}} \quad (3.4.25)$$

see (3.4.2). From the other side, it comes from the first order derivative part

$$\mathcal{D}_\mu\epsilon^I\;\in\;\delta_\epsilon\psi_\mu^I \quad\text{and}\quad -\frac{i}{2}\,\rho^{-1}\,(\partial_\mu\rho)\,\gamma^3\gamma^\mu\epsilon^I\;\in\;\delta_\epsilon\psi_2^I \quad (3.4.26)$$

in the term

$$-\,\frac{1}{2}\,e^{-1}\rho\,\varepsilon^{\mu\nu}\bar{\psi}_\nu^I\gamma^3\psi_\mu^J\tilde{B}_{IJ} + i\rho\,\bar{\psi}_2^I\gamma^\mu\psi_\mu^J\tilde{A}_{IJ}\;\in\mathcal{L}_{\text{Yuk}}\,, \quad (3.4.27)$$

see (3.4.13). Thus, the contribution reads

$$\delta_\epsilon\big(\mathcal{L}_{0,\text{cov}}+\mathcal{L}_{\text{Yuk}}\big)\;\ni\;\big(e^{-1}\bar{\psi}_\mu^I\gamma^{\mu\nu}\epsilon^J\partial_\nu\rho\big)\left(A_{IJ}-\tilde{B}_{IJ}-\rho\frac{\partial\tilde{B}_{IJ}}{\partial\rho}\right)\overset{!}{=}0\,, \quad (3.4.28)$$

and we obtain one linear equation on the Yukawa tensors

$$A_{IJ}-\tilde{B}_{IJ}-\rho\frac{\partial\tilde{B}_{IJ}}{\partial\rho}=0\,. \quad (3.4.29)$$

The full set of linear equations on the Yukawa tensors is summarized in Appendix B.1. To find an explicit solution, the tensors are decomposed into their $SO(9)_{\text{coset}}$ irreducible parts. After some lengthy calculation, it turns out that the set of linear equations provides a *unique* solution for the Yukawa tensors, in terms of the scalar fields ρ, $\mathcal{V}_k{}^a$, ϕ^{klm} and the auxiliary fields $Y_k{}^l$. The final result is[2]

$$
\begin{aligned}
A_{IJ} &= \frac{7}{9}\,\delta_{IJ}\,b - \frac{5}{9}\,\Gamma^a_{IJ}\,b^a - \frac{1}{9}\,\Gamma^{abcd}_{IJ}\,b^{abcd}\,,\\
\tilde{A}_{IJ} &= -\frac{2}{9}\,\Gamma^{ab}_{IJ}\,b^{ab} - \frac{4}{9}\,\Gamma^{abc}_{IJ}\,b^{abc}\,,\\
B_{IJ} &= -\Gamma^{ab}_{IJ}\,b^{ab} - \Gamma^{abc}_{IJ}\,b^{abc}\,,\\
\tilde{B}_{IJ} &= \delta_{IJ}\,b + \Gamma^a_{IJ}\,b^a - \Gamma^{abcd}_{IJ}\,b^{abcd}\,,\\
C^a_{IJ} &= \frac{8}{9}\,\delta_{IJ}\,b^a - \frac{1}{9}\Gamma^{ab}_{IJ}\,b^b - \frac{20}{9}\,\Gamma^{bcd}_{IJ}\,b^{abcd} + \frac{4}{9}\,\Gamma^{abcde}_{IJ}\,b^{bcde} + c^{ab}\,\Gamma^b_{IJ}\,,\\
\tilde{C}^a_{IJ} &= +\frac{14}{9}\,\Gamma^b_{IJ}\,b^{ab} - \frac{2}{9}\,\Gamma^{abc}_{IJ}\,b^{bc} - \frac{2}{3}\,\Gamma^{bc}_{IJ}\,b^{abc} + \frac{1}{9}\,\Gamma^{abcd}_{IJ}\,b^{bcd} + c^{a,bc}\,\Gamma^{bc}_{IJ}\,,\\
D_{IJ} &= \frac{14}{81}\,\delta_{IJ}\,b - \frac{70}{81}\,\Gamma^a_{IJ}\,b^a - \frac{8}{81}\,\Gamma^{abcd}_{IJ}\,b^{abcd}\,,\\
\tilde{D}_{IJ} &= -\frac{22}{81}\,\Gamma^{ab}_{IJ}\,b^{ab} + \frac{20}{81}\,\Gamma^{abc}_{IJ}\,b^{abc}\,,\\
E^a_{IJ} &= -\frac{26}{9}\,\Gamma^b_{IJ}\,b^{ab} + \frac{1}{9}\,\Gamma^{bc}_{IJ}\,b^{abc} - \frac{1}{9}\,c^{a,bc}\,\Gamma^{bc}_{IJ}\,,\\
\tilde{E}^a_{IJ} &= \frac{19}{9}\,\delta_{IJ}\,b^a - \frac{28}{9}\,\Gamma^{bcd}_{IJ}\,b^{abcd} - \frac{5}{9}\,c^{ab}\,\Gamma^b_{IJ}\,,\\
F^{ab}_{IJ} &= -\frac{1}{18}\,\delta^{ab}\delta_{IJ}\,b + \frac{1}{2}\,\delta^{ab}\,\Gamma^c_{IJ}\,b^c + \frac{1}{2}\,\delta^{ab}\,\Gamma^{cdef}_{IJ}\,b^{cdef} - 12\,\Gamma^{cd}_{IJ}\,b^{abcd} - 2\,c^{ab}\,\delta_{IJ}\,,\\
\tilde{F}^{ab}_{IJ} &= -\frac{1}{2}\,\delta^{ab}\,\Gamma^{cd}_{IJ}\,b^{cd} + \frac{1}{2}\,\delta^{ab}\,\Gamma^{cde}_{IJ}\,b^{cde} + 2\,\delta_{IJ}\,b^{ab} + 2\,\Gamma^c_{IJ}\,b^{abc} - 2\,c^{c,ab}\,\Gamma^c_{IJ}\,,
\end{aligned}
\qquad (3.4.30)
$$

and the $SO(9)$ irreducible tensors are given by

$$
\begin{aligned}
b &= \frac{1}{4}\,\rho^{-2/9}\,T\,,\\
b^a &= -\rho^{-14/9}\,T^{cd}\,\varphi^{abc}\mathcal{Y}^{bd} - \frac{1}{288}\,\rho^{-14/9}\,\varepsilon^{bcdefghij}T^{kl}\varphi^{kef}\varphi^{lgh}\varphi^{aij}\varphi^{bcd}\,,\\
b^{ab} &= -\frac{1}{2}\,\rho^{-11/9}\,T^{d[a}\mathcal{Y}^{b]d} + \frac{1}{144}\,\rho^{-11/9}\,\varepsilon^{abcdefghi}T^{jk}\varphi^{jcd}\varphi^{kef}\varphi^{ghi}\,,\\
b^{abc} &= \frac{1}{4}\,\rho^{-5/9}\,T^{d[a}\varphi^{bc]d}\,,\\
b^{abcd} &= -\frac{1}{8}\,\rho^{-8/9}\,T^{ef}\varphi^{e[ab}\varphi^{cd]f}\,,\\
c^{ab} &= -\frac{1}{2}\,\rho^{-2/9}\Big(T^{ab} - \frac{1}{9}\delta^{ab}T\Big)\,,\\
c^{a,bc} &= \frac{1}{3}\,\rho^{-5/9}\,\Big(T^{da}\varphi^{bcd} - T^{d[b}\varphi^{c]ad}\Big)\,,
\end{aligned}
\qquad (3.4.31)
$$

where we have defined

$$
T^{ab} \equiv \mathcal{V}^{-1(kl)}{}_{ab}\,\theta_{kl}\,, \qquad T \equiv T^{aa}\,,
$$
$$
\varphi^{abc} \equiv \mathcal{V}_{[klm]}{}^{abc}\phi^{klm}\,, \qquad \mathcal{Y}^{ab} \equiv \mathcal{V}^{-1ak}\,\mathcal{V}_l{}^b\,Y_k{}^l\,. \qquad (3.4.32)
$$

As a non-trivial fact, there are more linear equations in (B.1.1)–(B.1.4), than unknowns. However there is a non-zero solution.

---

[2]For "simplicity" of the expressions we have chosen to give the tensors E^a_{IJ} and F^{ab}_{IJ} (and their tilded analogues) in a form which is not yet explicitly projected onto the gamma-traceless part in the corresponding indices, e.g. $\Gamma^a_{IJ}E^a_{JK} \neq 0$, etc. Nevertheless, in the Lagrangian (3.4.13) all these tensors appear only under projection with the (gamma-traceless) fermions χ^{aI}, i.e. eventually only their gamma-traceless parts contribute to the couplings.

Quadratic constraints

The variation of $\mathcal{L}_{\rm Yuk}$ at order θ^2 and proportional to $\bar\psi^I_\mu\gamma^\mu\epsilon^J$ cannot vanish identically. This motivates the introduction of the new term: $\mathcal{L}_{\rm pot}$. Indeed, the trace part $\bar\psi^I_\mu\gamma^\mu\epsilon^I$ can be canceled by the variation of the determinant of the metric ($\delta_\varepsilon e = -ie\,\bar\psi^I_\mu\gamma^\mu\epsilon^I$) times a scalar potential quadratic in the Yukawa tensors, and of the following form

$$\mathcal{L}_{\rm pot} \equiv -eV_{\rm pot} = -\frac{1}{16}\,e\rho\left(2\tilde{A}_{IJ}B_{IJ} - 2A_{IJ}\tilde{B}_{IJ} + C^a_{IJ}C^a_{IJ} + \tilde{C}^a_{IJ}\tilde{C}^a_{IJ}\right). \qquad (3.4.33)$$

Thus, the scalar potential is entirely determined by supersymmetry. The other quadratic constraints come from the terms proportional to the traceless part of $\bar\psi^I_\mu\gamma^\mu\epsilon^J$, and the 5 remaining, linearly independent bilinear terms in fermions

$$\bar\psi^I_\mu\,\gamma^\mu\gamma^3\,\epsilon^J\,, \quad \bar\psi^I_2\,\epsilon^J\,, \quad \bar\psi^I_2\,\gamma^3\,\epsilon^J\,, \quad \bar\chi^{aI}\,\epsilon^J\,, \quad \bar\chi^{aI}\,\gamma^3\,\epsilon^J\,. \qquad (3.4.34)$$

Each of them is associated to a quadratic combination of Yukawa tensors that must vanish identically. This leads to 6 sets of quadratic equations that we have collected in Appendix B.2. They are consistency checks on the solution (3.4.30)-(3.4.31). For example, let us examine the terms proportional to $\bar\psi^I_\mu\,\gamma^\mu\gamma^3\,\epsilon^J$. The fermion shifts in the variation of

$$-\frac{1}{2}\,e^{-1}\rho\,\varepsilon^{\mu\nu}\left(\bar\psi^I_\nu\psi^J_\mu B_{IJ} + \bar\psi^I_\nu\gamma^3\psi^J_\mu\tilde{B}_{IJ} - 2i\bar\psi^I_2\gamma_\nu\psi^J_\mu A_{IJ}\right) + i\rho\,\bar\psi^I_2\gamma^\mu\psi^J_\mu\tilde{A}_{IJ}$$
$$+\,i\rho\,\bar\chi^{aI}\gamma^\mu\psi^J_\mu C^a_{IJ} \,-\, i\rho\,\bar\chi^{aI}\gamma^3\gamma^\mu\psi^J_\mu\tilde{C}^a_{IJ} \,\in\, \mathcal{L}_{\rm Yuk} \qquad (3.4.35)$$
$$(3.4.36)$$

yields the quadratic term

$$-\,i\rho\big(\bar\psi^I_\mu\,\gamma^\mu\gamma^3\,\epsilon^J\big)\left(2A_{K(I}B_{J)K} + 2\tilde{A}_{K(I}\tilde{B}_{J)K} + C^a_{KI}\tilde{C}^a_{KJ} + C^a_{KJ}\tilde{C}^a_{KI}\right) \stackrel{!}{=} 0\,. \qquad (3.4.37)$$

This implies the quadratic relation

$$2A_{K(I}B_{J)K} + 2\tilde{A}_{K(I}\tilde{B}_{J)K} + C^a_{KI}\tilde{C}^a_{KJ} + C^a_{KJ}\tilde{C}^a_{KI} = 0\,. \qquad (3.4.38)$$

Eventually, this equation is identically satisfied by the solution (3.4.30) – (3.4.31). Indeed, employing the linear constraints (B.1.1), this relation reduces to

$$(4+2\rho\partial\rho)\,(B_{K(I}\tilde{B}_{J)K}) = C^a_{KI}\tilde{C}^a_{KJ} + C^a_{KJ}\tilde{C}^a_{KI} \qquad (3.4.39)$$

where in terms of $SO(9)$, the l.h.s. and r.h.s. of this equation become respectively

$$(4+2\rho\partial\rho)\,(B_{K(I}\tilde{B}_{J)K}) = \Gamma^b_{IJ}\left(\frac{80}{3}\,b^{def}b^{defb} - \frac{28}{9}\,b^a b^{ab}\right) + \frac{2}{9}\,\Gamma^{aefg}_{IJ}\left(8\,b^{ab}b^{befg} - b^a b^{efg}\right)$$
$$-\,\frac{40}{3}\,\Gamma^{abefg}_{IJ}\,b^{abc}b^{cefg}\,,$$
$$C^a_{KI}\tilde{C}^a_{KJ} + C^a_{KJ}\tilde{C}^a_{KI} = \Gamma^b_{IJ}\left(\frac{80}{3}\,b^{def}b^{defb} - \frac{28}{9}\,b^a b^{ab}\right) + \frac{2}{9}\,\Gamma^{aefg}_{IJ}\left(8\,b^{ab}b^{befg} - b^a b^{efg}\right)$$
$$+\,\Gamma^{abefg}_{IJ}\left(\frac{8}{3}\,b^{abc}b^{cefg} - 8\,c^{c,ab}b^{cefg}\right).$$

Thus, the quadratic equation reduces to

$$c^{f,[ab}b^{cde]f} = 2\,b^{f[ab}b^{cde]f}\,, \qquad (3.4.40)$$

which is satisfied by the explicit form (3.4.31). Further computations show that all the quadratic equations collected in Appendix B.2 are identically satisfied![3] This non-trivial fact shows that the θ_{pq} gauging is consistent with supersymmetry. As a remarkable fact, the scalar potential of (3.4.33) now takes a simple quadratic form in terms of the $SO(9)_{\text{coset}}$ irreducible tensors

$$V_{\text{pot}} = \rho\Big(2\,b^a b^a + 4\,b^{ab} b^{ab} + 48\,b^{abcd} b^{abcd} + c^{ab} c^{ab} + 2\,c^{a,bc}\,c^{a,bc} - \frac{14}{9}\,bb - 4\,b^{abc} b^{abc}\Big)\,. \quad (3.4.41)$$

It is an eighth order polynomial in the scalars ϕ^{klm} and when expanded to quadratic order it is given by

$$\begin{aligned} V_{\text{pot}} &= \frac{1}{8}\,\rho^{5/9}\left(2\,\text{tr}[\mathcal{M}^{-1}\mathcal{M}^{-1}] - \left(\text{tr}[\mathcal{M}^{-1}]\right)^2\right) + \frac{1}{4}\,\rho^{-1/9}\mathcal{M}_{mp}\mathcal{M}_{nq}\mathcal{M}^{-1}{}_{kl}\,\phi^{mnk}\phi^{pql} \\ &\quad + \rho^{-13/9}\left(\mathcal{M}^{-1\,km}\mathcal{M}^{-1\,ln} + 2\,\rho^{-2/3}\phi^{klp}\mathcal{M}_{pq}\phi^{qmn}\right) Y_{kl} Y_{mn} \; + \mathcal{O}(\phi^3)\,. \quad (3.4.42) \end{aligned}$$

The first term corresponds to the standard potential of a sphere reduction [43], but with a dilaton pre-factor which comes from the warped geometry of the reduction. This allows for a domain wall background that will be derived. To conclude, we have shown that the Lagrangian (3.4.22) with the Yukawa tensors given in (3.4.30) and (3.4.31) is maximally supersymmetric (up to higher fermion terms and total derivatives), under the transformations (3.4.12), (3.4.16) and (3.4.20). This complements the group theoretical analysis of [94]. Notice however that θ_{kl} never needed to be made explicit in the previous computations. Thus, in principle, θ_{kl} allows for any $CSO(p,q,r)$ consistent gauging, depending on its signature. As we are interested in a $SO(9)$ gauged maximal supergravity, the embedding tensor will be set to $\theta_{kl} \equiv g\,\delta_{kl}$ in the following.

3.4.3 Supersymmetry algebra

To end the discussion of the $SO(9)$ gauged maximal supergravity, let us present the closure of the supersymmetry algebra. The analysis is more easily done on the bosonic fields where the commutator of two supersymmetry transformations closes on: general coordinate transformation, local Lorentz transformation, local $SO(9)_{\text{coset}}$ and $SO(9)_{\text{gauge}}$ transformations

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \xi^\mu\,\partial_\mu + \delta_\omega^{\text{Lorentz}} + \delta_\Omega^{SO(9)_{\text{coset}}} + \delta_\Lambda^{SO(9)_{\text{gauge}}}\,. \quad (3.4.43)$$

The parameters are given by

$$\begin{aligned} \xi^\mu &= i\,\bar\epsilon_2^I\gamma^\mu\epsilon_1^I\,, \\ \omega^{\alpha\beta} &= \xi^\mu\omega_\mu{}^{\alpha\beta} - 2\,\varepsilon^{\alpha\beta}\left(\bar\epsilon_2^I\gamma^3\epsilon_1^J A_{IJ} - \bar\epsilon_2^I\epsilon_1^J\tilde A_{IJ}\right)\,, \\ \Lambda^{kl} &= -\xi^\mu A_\mu{}^{kl} - \rho^{-5/9}\,\mathcal{V}^{-1}{}_{[ab]}{}^{kl}\left(\rho^{1/3}\,\bar\epsilon_2^I\epsilon_1^J\Gamma^{ab}_{IJ} + 2\,\bar\epsilon_2^I\gamma^3\epsilon_1^J\,\Gamma^c_{IJ}\,\varphi^{abc}\right)\,, \\ \Omega^{ab} &= -\xi^\mu\mathcal{Q}_\mu^{ab} + g\,\Lambda^{kl}\,\delta_{kq}\,\mathcal{V}^{-1q[a}\,\mathcal{V}_l{}^{b]}\,. \end{aligned} \quad (3.4.44)$$

In the following we will compute for every bosonic field, and up to quartic order fermionic terms, the commutator of two supersymmetry transformations, and we will show the closure on the bosonic transformations mentioned above. Sometimes the cancellation of linear combinations of Yukawa tensors are needed. These linear constraints precisely belong to the set of linear equations that were determined by maximal supersymmetry of the Lagrangian and are collected in Appendix B.1.

[3] Part of these calculations have been facilitated by use of the computer algebra system Cadabra [117] [118].

SuSy algebra on the bosonic fields

Let us begin with the dilaton :

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,\rho = \xi^\mu\partial_\mu\rho - 2\rho\,\bar\epsilon_2^I\gamma^3\epsilon_1^J\,B_{(IJ)} - 2\rho\,\bar\epsilon_2^I\epsilon_1^J\,\tilde B_{[IJ]} \quad + \text{ q.f.} \tag{3.4.45}$$

$$= \xi^\mu\partial_\mu\rho \quad + \text{ q.f.}\,, \tag{3.4.46}$$

provided that $B_{(IJ)} = 0 = \tilde B_{[IJ]}$.

The vielbein Concerning the vielbein,

$$\begin{aligned}[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,e_\mu{}^\alpha &= D_\mu\xi^\alpha - \frac{i}{18}\rho^{-1/3}\Gamma^{abc}_{(IJ)}\big(2\bar\epsilon_2^I\gamma^\alpha\gamma^3\epsilon_1^J\widetilde\varphi^{abc}_\mu - e\varepsilon_{\mu\nu}\bar\epsilon_2^I\gamma^\alpha\epsilon_1^J\widetilde\varphi^{\nu\,abc}\big) - 2\bar\epsilon_2^I\epsilon_1^J A_{[IJ]}\,e_\mu{}^\alpha \\ &\quad - 2\bar\epsilon_2^I\gamma^3\epsilon_1^J A_{(IJ)}\varepsilon^\alpha{}_\beta\,e_\mu{}^\beta + 2\bar\epsilon_2^I\gamma^3\epsilon_1^J\tilde A_{(IJ)}\,e_\mu{}^\alpha + 2\bar\epsilon_2^I\epsilon_1^J\tilde A_{[IJ]}\varepsilon^\alpha{}_\beta\,e_\mu{}^\beta \ + \text{ q.f.} \\ &= \xi^\nu\,\partial_\nu e^\alpha_\mu + e^\alpha_\nu\,\partial_\mu\xi^\nu + \omega^\alpha{}_\beta\,e^\beta_\mu \ + \text{ q.f.}\,,\end{aligned} \tag{3.4.47}$$

because: $\Gamma^{abc}_{(IJ)} = 0$ as a $SO(9)$ gamma matrix, and $A_{[IJ]} = 0 = \tilde A_{(IJ)}$ according to Appendix B.1.

Let us focus now on the scalar fields $\mathcal{V}_k{}^a$,

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,\mathcal{V}_m{}^a = \xi^\mu\,\mathcal{P}^{ab}_\mu\mathcal{V}_m{}^b + \lambda^{(ab)}\,\mathcal{V}_m{}^b + \text{ q.f.}\,, \tag{3.4.48}$$

where according to (3.4.1),

$$\begin{aligned}\xi^\mu\,\mathcal{P}^{ab}_\mu\,\mathcal{V}_m{}^b &= \xi^\mu\,\mathcal{P}^{ba}_\mu\,\mathcal{V}_m{}^b \\ &= \big(-\mathcal{Q}^{ba}_\mu + \mathcal{V}^{-1\,bk}\partial_\mu\mathcal{V}_k{}^a - g\,\mathcal{V}^{-1\,bk}A_\mu{}^{lp}\delta_{pk}\mathcal{V}_l{}^a\big)\,\mathcal{V}_m{}^b \\ &= \mathcal{V}_m{}^b\big(-\xi^\mu\mathcal{Q}^{ba}_\mu\big) + \xi^\mu\partial_\mu\mathcal{V}_m{}^a + g\big(-\xi^\mu A_\mu{}^{lp}\,\delta_{pm}\big)\mathcal{V}_l{}^a\,.\end{aligned} \tag{3.4.49}$$

We have introduced $\lambda^{(ab)}$, which can be further simplified according to Appendix B.1

$$\begin{aligned}\lambda^{(ab)} &\equiv 2\,\Big(\bar\epsilon_2^I\gamma^3\epsilon_1^J\,\tilde C^{(a}_{K(I}\Gamma^{b)}_{J)K} - \bar\epsilon_2^I\epsilon_1^J\,C^{(a}_{K[I}\Gamma^{b)}_{J]K}\Big) \\ &= 2\,\rho^{-5/9}\,\bar\epsilon_2^I\gamma^3\epsilon_1^J\,\Gamma^c_{IJ}\,T^{d(a}\varphi^{b)cd} + \rho^{-2/9}\,\bar\epsilon_2^I\epsilon_1^J\,\Gamma^{c(a}T^{b)c}\,.\end{aligned} \tag{3.4.50}$$

We then find that up to quartic order in the fermions, two supersymmetry transformations close on a diffeomorphism, a local $SO(9)_{\text{coset}}$ and a $SO(9)_{\text{gauge}}$ transformation with parameters given in (3.4.44)

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,\mathcal{V}_m{}^a = \xi^\mu\,\partial_\mu\mathcal{V}_m{}^a + \mathcal{V}_m{}^b\,\Omega^{ba} + g\,\Lambda^l{}_m\,\mathcal{V}_l{}^a\,. \tag{3.4.51}$$

Vector fields Let us mention also the closure of the supersymmetry algebra on the vector fields. When computing the commutator of two supersymmetry transformations, one gets the expected gauge transformation plus additional contributions from fermion shifts

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,A_\mu{}^{kl} = \mathcal{D}_\mu\Lambda^{kl} + 2i\rho^{-2/9}\,\mathcal{V}^{-1kl}{}_{[ab]}\Big(\bar\epsilon_2^I\gamma_\mu\epsilon_1^J\,Z^{ab}_{IJ} - \bar\epsilon_2^I\gamma^3\gamma_\mu\epsilon_1^J\,\tilde Z^{ab}_{IJ}\Big)\,. \tag{3.4.52}$$

Upon using (B.1.1), the tensors Z^{ab}_{IJ} and $\tilde Z^{ab}_{IJ}$ become

$$\begin{aligned}Z^{ab}_{IJ} &= 2C^{[a}_{K(I}\Gamma^{b]}_{J)K} - \Big(A_{K(I} - \frac{2}{9}\tilde B_{K(I}\Big)\,\Gamma^{ab}_{J)K} - 2\rho^{-1/3}\varphi^{abc}\Big[\tilde C^c_{(IJ)} + \Big(\tilde A_{K(I} + \frac{5}{9}B_{K(I}\Big)\Gamma^c_{J)K}\Big] \\ &= \Big[2C^{[a}_{K(I}\Gamma^{b]}_{J)K} - \Big(\frac{7}{9} + \rho\partial_\rho\Big)\,\tilde B_{K(I}\,\Gamma^{ab}_{J)K}\Big] - 2\rho^{-1/3}\varphi^{abc}\Big[\tilde C^c_{(IJ)} - \Big(\frac{4}{9} + \rho\partial_\rho\Big)B_{K(I}\Gamma^c_{J)K}\Big] \\ &= 0\,,\end{aligned} \tag{3.4.53}$$

and

$$
\begin{aligned}
\tilde{Z}^{ab}_{IJ} &= 2\tilde{C}^{[a}_{K(I}\Gamma^{b]}_{J)K} + \Big(\tilde{A}_{K(I} + \frac{2}{9}B_{K(I}\Big)\Gamma^{ab}_{J)K} - 2\rho^{-1/3}\varphi^{abc}\Big[C^{c}_{(IJ)} - \Big(A_{K(I} - \frac{5}{9}\tilde{B}_{K(I}\Big)\Gamma^{c}_{J)K}\Big] \\
&= \Big[2\tilde{C}^{[a}_{K(I}\Gamma^{b]}_{J)K} - \Big(\frac{7}{9} + \rho\partial_\rho\Big)B_{K(I}\Gamma^{ab}_{J)K}\Big] - 2\rho^{-1/3}\varphi^{abc}\Big[C^{c}_{(IJ)} - \Big(\frac{4}{9} + \rho\partial_\rho\Big)\tilde{B}_{K(I}\Gamma^{c}_{J)K}\Big] \\
&= 4\,\delta_{IJ}\,\rho^{-2/9}\Big(b^{ab} - \rho^{-1/3}\,\varphi^{abc}b^{c}\Big)\,. \qquad (3.4.54)
\end{aligned}
$$

As a result, the commutator on the vector fields (3.4.52) closes into the standard form

$$
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,A_\mu{}^{kl} = \mathcal{D}_\mu\Lambda^{kl} + \xi^\nu\,\mathcal{F}_{\nu\mu}{}^{kl}\,, \qquad (3.4.55)
$$

provided their field strengths satisfy the relation

$$
\mathcal{V}_{kl}{}^{ab}\,\mathcal{F}_{\mu\nu}{}^{kl} = 8\,e\varepsilon_{\mu\nu}\,\rho^{-2/9}\Big(b^{ab} - \rho^{-1/3}\,\varphi^{abc}b^{c}\Big) + \text{fermions}\,. \qquad (3.4.56)
$$

This is precisely the equations of motion obtained by varying the Lagrangian (3.4.22) with respect to the auxiliary field $Y_k{}^l$. Thus, the algebra closes on-shell.

Scalar fields ϕ^{ijk} After a lengthy computation involving the $SO(9)$ gamma matrix algebra and the equations (B.1.2), the supersymmetry algebra acting on the 84 scalar fields ϕ^{ijk} reduces to

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,\phi^{ijk} &= \xi^\mu\,\mathcal{D}_\mu\phi^{ijk} + \rho^{1/3}\mathcal{V}^{-1}{}_{[abc]}{}^{kl}\Big[-\bar{\epsilon}^{I}_2\epsilon^{J}_1(3\tilde{C}^{[a}_{K[I}\Gamma^{bc]}_{J]K} + \frac{1}{3}B_{K[I}\Gamma^{abc}_{J]K}) \\
&\quad + \bar{\epsilon}^{I}_2\gamma^3\epsilon^{J}_1(3C^{[a}_{K(I}\Gamma^{bc]}_{J)K} + \frac{1}{3}\,\tilde{B}_{K(I}\Gamma^{abc}_{J)K})\Big] \\
&= \xi^\mu\,\partial_\mu\phi^{ijk} + 3g\,\Lambda^{l[i}\phi^{jk]l}\,. \qquad (3.4.57)
\end{aligned}
$$

Finally, let us study the auxiliary fields.

Scalars Y_{kl} These scalar fields are defined by the projected subset of the auxiliary fields $Y_{kl} \equiv \delta_{p[l}\,Y_{k]}{}^{p}$. Closure of the supersymmetry algebra on these fields requires the first-order field equation

$$
g\,\mathcal{D}_\mu Y_{[kl]} = -e\,\epsilon_{\mu\nu}\,\mathcal{J}^{\nu}{}_{kl}\,. \qquad (3.4.58)
$$

obtained from the Lagrangian (3.4.22) by varying with respect to the vector fields. The commutator of two supersymmetry variations on Y gives

$$
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,Y_{kl} = \xi^\mu\partial_\mu Y_{kl} + 2g\,\Lambda_{[k}{}^{n}\,Y_{l]n}\,. \qquad (3.4.59)
$$

This yields another check for the supersymmetry transformations of these fields proposed in (3.4.20).

The entire picture

The supersymmetry algebra acting on the bosonic fields is summarized below. Up to quartic order in fermions the relations are

$$\begin{aligned}
[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,\rho &= \xi^\mu\partial_\mu\rho\,,\\
[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,e_\mu{}^\alpha &= \xi^\nu\,\partial_\nu e_\mu^\alpha + e_\nu^\alpha\,\partial_\mu\xi^\nu + \omega^\alpha{}_\beta\,e_\mu^\beta\,,\\
[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,\mathcal{V}_m{}^a &= \xi^\mu\,\partial_\mu\mathcal{V}_m{}^a + \mathcal{V}_m{}^b\,\Omega^{ba} + g\,\Lambda^l{}_m\,\mathcal{V}_l{}^a\,,\\
[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,A_\mu{}^{kl} &= \mathcal{D}_\mu\Lambda^{kl} + \xi^\nu\,\mathcal{F}_{\nu\mu}{}^{kl}\,,\\
[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,Y_{kl} &= \xi^\mu\partial_\mu Y_{kl} + 2g\,\Lambda_{[k}{}^n\,Y_{l]n}\,,\\
[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,\phi^{ijk} &= \xi^\mu\,\partial_\mu\phi^{ijk} + 3g\,\Lambda^{l[i}\phi^{jk]l}\,.
\end{aligned}\tag{3.4.60}$$

This puts an end to the construction of the $SO(9)$ gauged maximal supergravity in two dimensions. Let us summarize the main features: Vector fields have been introduced in the Lagrangian to impose the $SO(9)$ local symmetry. The degrees of freedom have been balanced by the addition of a FY term which couples the field strengths to auxiliary fields. Then, supersymmetry has been restored, following the Noether procedure. Thus, new Yukawa couplings $\mathcal{L}_{\rm Yuk}$ appeared in the Lagrangian, together with a scalar potential $\mathcal{L}_{\rm pot}$. Finally, the supersymmetry algebra has been checked to complete the picture of the theory. Now, the following section will be devoted to applications. First of all, the equations of motion will be computed and their consistency checked. This will open the path to the study of particular solutions such as a half-supersymmetric domain wall solution, and it will lead to the distinction of different on-shell equivalent formulations of the theory.

3.5 $SO(9)$ supergravity: properties

Having at hand the two-dimensional $SO(9)$ gauged maximal supergravity, let us present some properties and applications.

3.5.1 The bosonic field equations

In this section, the bosonic field equations will be discussed and their structure commented.

Gravity sector

We give here the equations of motion for the dilaton and the trace and traceless parts of Einstein' equations.

$$\begin{aligned}
\frac{1}{4}R &= \frac{1}{4}\,\mathcal{P}^{\mu\,ab}\mathcal{P}_\mu^{ab} + \frac{1}{36}\,\rho^{-2/3}\,\tilde{\varphi}^{\mu\,abc}\tilde{\varphi}_\mu^{abc} - \frac{\partial V_{\rm pot}}{\partial\rho}\,,\\
\nabla^2\rho &= 4V_{\rm pot}\,,\\
0 &= \nabla_\mu\partial_\nu\rho + \rho\,\mathcal{P}_\mu^{ab}\mathcal{P}_\nu^{ab} + \frac{1}{3}\,\rho^{1/3}\tilde{\varphi}_\mu^{abc}\tilde{\varphi}_\nu^{abc} - \frac{1}{2}\,g_{\mu\nu}\,(\text{trace})\,.
\end{aligned}\tag{3.5.1}$$

The last equation corresponds to a constraint imposed by the two unimodular degrees of freedom of the two-dimensional metric that can be viewed as Lagrange multipliers.

Scalar sector

The equation of motion for the scalar fields ${\cal V}_m{}^a$ is written covariantly so that ${\cal P}_\mu^{ab}$ appears directly. It is obtained by varying the Lagrangian with respect to a covariant scalar variation

$$\delta_\Sigma {\cal V}_m{}^a \equiv {\cal V}_m{}^c\, \Sigma^{ac}\,, \tag{3.5.2}$$

where Σ^{ab} is symmetric and traceless:

$${\cal D}^\mu \left(\rho\, {\cal P}_\mu^{ab}\right) = \left({\cal V}_{kl}{}^{ab} - \tfrac{1}{9}\delta^{ab}{\cal M}_{kl}\right) {\cal M}_{mn}{\cal M}_{pq}\, {\cal D}^\mu \phi^{kmp} {\cal D}_\mu \phi^{lnq} - 2\,\frac{\partial V_{\rm pot}}{\partial \Sigma^{ab}}\,, \tag{3.5.3}$$

where ${\cal M}_{kl} \equiv {\cal V}_k{}^a {\cal V}_l{}^a$. Besides, the equation on the ϕ^{klm} fields is given by

$$\begin{aligned}{\cal D}_\mu {\cal D}^\mu \left({\cal N}_{klm,pqr}\, \phi^{pqr}\right) = &\ \frac{1}{36}\, e^{-1} \varepsilon^{\mu\nu} \varepsilon_{klmnpqrst} \left({\cal D}_\mu \phi^{npq}\, {\cal D}_\nu \phi^{rst} - {\cal F}_{\mu\nu\,u}{}^r\, \phi^{npq} \phi^{stu}\right) \\ &- 6\,\frac{\partial V_{\rm pot}}{\partial \phi^{klm}}\,,\end{aligned} \tag{3.5.4}$$

with ${\cal N}_{klm,pqr} \equiv \rho^{1/3}\, {\cal V}_{(klm)}{}^{abc} {\cal V}_{(pqr)}{}^{abc}$.

Vector and auxiliary fields

Eventually, the vector fields and auxiliary fields satisfy the first-order equations

$${\cal V}_{kl}{}^{ab}\, {\cal F}_{\mu\nu}{}^{kl} = 8\, e\, \varepsilon_{\mu\nu}\, \rho^{-2/9} \left(b^{ab} - \rho^{-1/3}\, \varphi^{abc} b^c\right)\,, \tag{3.5.5}$$

$$\rho\, {\cal W}_{kl}{}^{ab} \left({\cal P}_\mu^{ab} - \rho^{-2/3}\, \varphi^{bcd} \tilde{\varphi}_\mu^{acd}\right) = e\, \varepsilon_{\mu\nu} \left({\cal D}^\nu Y_{[kl]} - \frac{1}{54} \varepsilon^{abcdefghi}\, {\cal W}_{kl}{}^{aj}\, \varphi^{bcj} \varphi^{def} \tilde{\varphi}^{\nu\, ghi}\right),$$

with the scalar tensor ${\cal W}_{kl}{}^{ab} = \delta_{m[k} {\cal V}_{l]}{}^a {\cal V}^{-1bm}$. Let us discuss the consistency of these first-order equations: as we are in two dimensions, the Bianchi identity associated with the first equation is trivial. However, the second equation leads to a non-trivial second order equation when contracted with a covariant derivative ${\cal D}_\mu$. In particular, it involves the scalar equations of motion. Using (3.5.4) and the first duality equation (3.5.5), we find

$$-\,2\, {\cal W}_{kl}{}^{ab}\, \frac{\partial V_{\rm pot}}{\partial \Sigma^{ab}} - 6\, \frac{\partial V_{\rm pot}}{\partial \phi^{mn[k}}\, \phi_{l]}{}^{mn} = e^{-1} \varepsilon^{\mu\nu} {\cal F}_{\mu\nu}{}^m{}_{[k} Y_{l]m} = -4\, \frac{\delta V_{\rm pot}}{\delta Y^{m[k}}\, Y_{l]}{}^m \tag{3.5.6}$$

i.e.

$${\cal W}_{kl}{}^{ab}\, \frac{\partial V_{\rm pot}}{\partial \Sigma^{ab}} + 3\, \frac{\partial V_{\rm pot}}{\partial \phi^{mn[k}}\, \phi_{l]}{}^{mn} - 2\, \frac{\delta V_{\rm pot}}{\delta Y^{m[k}}\, Y_{l]}{}^m = 0\,. \tag{3.5.7}$$

This is nothing but the gauge invariance of the potential, satisfied by construction. Thus, the set of first order and second order bosonic field equations are consistent.

3.5.2 Domain wall solution

The scalar potential exhibits a part (3.4.42) that corresponds to the S^8 sphere reduction of the pure gravity sector, in type IIA supergravity. Nevertheless, the dilaton factor suggests that the ten dimensions ground state corresponds to an $AdS_2 \times S^8$ geometry coupled to a dilaton. This implies that the two-dimensional theory supports a half-supersymmetric domain wall solution instead of a pure AdS geometry [75, 119, 120]. In order to find such a solution, we study the Killing spinor equations of the theory obtained by imposing the fermionic supersymmetry transformations (3.4.16) to vanish. As we are looking for a ground state, the Killing spinor equations will be evaluated at the origin of the scalar target space:

$$\mathcal{V}_m{}^a = \delta^a_m \,, \quad \phi^{klm} = 0 = Y_{kl} \,. \tag{3.5.8}$$

In this truncation, the fermionic supersymmetry transformations reduce to

$$\begin{aligned}
0 &\overset{!}{=} \delta_\epsilon \psi^I_\mu = \mathcal{D}_\mu \epsilon^I + \frac{7i}{4}\, g\, \rho^{-2/9} \gamma_\mu \epsilon^I \,, \\
0 &\overset{!}{=} \delta_\epsilon \psi^I_2 = -\frac{i}{2} \rho^{-1} \left(\partial_\mu \rho\right) \gamma^3 \gamma^\mu \epsilon^I + \frac{9}{4}\, g\, \rho^{-2/9} \gamma^3 \epsilon^I \,, \\
0 &\overset{!}{=} \delta_\epsilon \chi^{aI} = 0 \,.
\end{aligned} \tag{3.5.9}$$

Given the domain wall ansatz for the metric

$$ds^2 = e^{2A(r)}\, dt^2 - dr^2 \,, \tag{3.5.10}$$

and assuming a Killing spinor of the form $\epsilon^I = f(r)\, \epsilon^I_0$, the equations (5.3.10) are solved by

$$f(r) = f_0\, r^{7/4} \,, \qquad A(r) = A_0 + \frac{7}{2} \ln r \,, \qquad \rho(r) = (gr)^{9/2} \,. \tag{3.5.11}$$

Moreover, the constant spinor ϵ_0 must satisfy the projection condition

$$\gamma^1 \epsilon^I_0 = -i\, \epsilon^I_0 \,. \tag{3.5.12}$$

This implies that the solution is half-supersymmetric and it is straightforward to verify that (3.5.11) is a solution of the equations of motion (3.5.1). Setting the constant $A_0 = 0$, the metric and associated Ricci scalar are

$$\begin{aligned}
ds^2 &= r^7\, dt^2 - dr^2 \,, \\
R &= \frac{35}{2} \frac{1}{r^2} \,.
\end{aligned} \tag{3.5.13}$$

This is a two-dimensional domain wall solution corresponding to the D0-brane near-horizon geometry [75, 119].

3.5.3 Auxiliary fields

Considering the Lagrangian (3.4.22), one may consider the possibility to rewrite it using the equations of motion of the auxiliary fields. In particular, integrating out the $Y_k{}^l$ fields will lead to a kinetic term for the vector fields. This reminds us the expression of the ungauged Lagrangian obtained by dimensional reduction of the eleven-dimensional maximal supergravity on a T^9 torus. More explicitly, let us start with the duality equation (3.5.5),

$$\begin{aligned}
\mathcal{F}_{\mu\nu}{}^{kl} = &\; 4ge\, \varepsilon_{\mu\nu}\, \rho^{-13/9} \left(\mathcal{M}^{-1\,kp} \mathcal{M}^{-1\,lq} + 2\rho^{-2/3}\, \phi^{klm} \mathcal{M}_{mn} \phi^{npq} \right) Y_{pq} \\
&+ \frac{ge}{18}\, \varepsilon_{\mu\nu}\, \rho^{-13/9} \left(\mathcal{M}^{-1\,kp} \mathcal{M}^{-1\,lq} + \rho^{-2/3}\, \phi^{klm} \mathcal{M}_{mn} \phi^{npq} \right) \varepsilon_{pqrstuvxy}\, \phi^{zrs} \phi_z{}^{tu} \phi^{vxy} \\
&+ \text{fermions} \,.
\end{aligned} \tag{3.5.14}$$

Invert it in order to express the Y in terms of the field strength

$$Y_{kl} = -\frac{g^{-1}}{8}\, e^{-1}\, \rho^{13/9}\, \mathcal{O}^{-1}_{kl,pq} \epsilon^{\mu\nu} \mathcal{F}_{\mu\nu}{}^{pq} + \ldots \tag{3.5.15}$$

where the inversion matrix is given by

$$\mathcal{O}^{kl,pq} \equiv \mathcal{M}^{-1\,kp}\mathcal{M}^{-1\,lq} + 2\,\rho^{-2/3}\,\phi^{klm}\mathcal{M}_{mn}\phi^{npq}\,. \tag{3.5.16}$$

When (3.5.15) is injected into the Lagrangian (3.4.22), a two-dimensional Yang-Mills term of the form

$$\mathcal{L}_{\mathcal{F}^2} \propto e\rho^{13/9}\,\mathcal{F}_{\mu\nu}{}^{kl}\,\mathcal{O}^{-1}_{kl,mn}\,\mathcal{F}^{\mu\nu\,mn}\,, \tag{3.5.17}$$

is generated. This expression should be obtained by dimensional reduction of type IIA on a S^8 sphere. As the field strengths are defined by

$$\mathcal{F}_{\mu\nu}{}^{kl} \equiv 2\partial_{[\mu}A_{\nu]}{}^{kl} + 2\,g\,\delta_{pq}\,A_{[\mu}{}^{p[k}A_{\nu]}{}^{l]q}\,, \tag{3.5.18}$$

the limit $g \to 0$ can be considered and leads to the ungauged theory (3.2.8) involving Maxwell vector fields. Indeed, this result can be seen up to quadratic order in the ϕ^{klm} fields when the operator $\mathcal{O}^{-1}$ is approximately given by

$$\mathcal{O}^{-1}_{kl,pq} \simeq \mathcal{M}_{kp}\mathcal{M}_{lq} - 2\,\rho^{-2/3}\,\phi^{rst}\mathcal{M}_{kr}\mathcal{M}_{ls}\mathcal{M}_{tu}\mathcal{M}_{pv}\mathcal{M}_{qw}\phi^{uvw} + \ldots \tag{3.5.19}$$

Then, taking the kinetic terms for the vector fields from the Lagrangian (3.2.8),

$$\begin{aligned} e^{-1}\mathcal{L}_{\mathcal{F}\mathcal{F}} = &-\frac{1}{16}\,\rho^{19/9}\,\mathcal{M}^{-1kl}\,\mathcal{F}_{\mu\nu\,k}\mathcal{F}^{\mu\nu}{}_{l} \\ &-\frac{1}{8}\rho^{13/9}\left(\mathcal{F}_{\mu\nu}{}^{kl} + \phi^{klp}\mathcal{F}_{\mu\nu\,p}\right)\mathcal{M}_{km}\mathcal{M}_{ln}\left(\mathcal{F}^{\mu\nu mn} + \phi^{mnq}\mathcal{F}^{\mu\nu}{}_{q}\right) \end{aligned} \tag{3.5.20}$$

and inserting equation (3.2.11) with $\theta^k = 0$ leads to

$$\mathcal{L}_{\mathcal{F}\mathcal{F}} = -\frac{1}{8}e\,\rho^{13/9}\,\mathcal{F}_{\mu\nu}{}^{kl}\,\mathcal{O}^{-1}_{kl,mn}\,\mathcal{F}^{\mu\nu\,mn} + \mathcal{O}(\phi^3)\,. \tag{3.5.21}$$

Actually the formulation of the theory which involves Yang-Mills kinetic terms for the vector fields, will be of primary interest to study consistent embeddings in higher dimensions. Indeed, the Kaluza-Klein reduction gives rise to $\mathcal{F}\mathcal{F}$ terms in the Lagrangian, as we saw in (3.2.8). This will be the starting point of the construction of consistent truncations of type IIA supergravity that will be discussed in the next chapter.

3.6 Summary

The construction of the $SO(9)$ gauged maximal supergravity in two dimensions was the central point of this chapter. After recalling the main features of the ungauged theory, the vector field content transforming in the basic representation of $\mathfrak{e}_{(9,9)}$ was examined. A new difficulty but also a richness arose with the infinite dimensional structure of symmetries. Then, all consistent gaugings were classified by an embedding tensor which allowed to find the minimal coupling for gauging $SO(9)$. Explicit computations established the supersymmetry of the theory and showed for example the generation of a scalar potential, accounting for the gauging which was performed for the entire class of $SO(p, 9-p)$ and $CSO(p, q, 9-p-q)$ groups. Let us stress that it constitutes a first non-trivial gauging of maximal supergravity in two dimensions. A closer study of the potential shows the existence of a domain wall background. This was confirmed by solving the Killing spinor equations. Then, one may wonder what is the higher dimensional origin of this solution. The next chapter answers this question and provides an explicit embedding in ten and eleven dimensions. Besides, equipped with the domain wall ground state of (3.5.11), one can study fluctuations propagating around it. According to the DW/QFT correspondence, informations about matrix models, such as the BFSS model, can be extracted from the study of the gravity side excitations. Thus, the holography of domain wall solutions allows for a test of the DW/QFT correspondence and may shed a new light on the quantum matrix models. A detailed discussion of this subject will be at the core of the last chapter.

Chapter 4

Consistent truncations of supergravity

4.1 Introduction

The properties of the D3-brane solution of type IIB superstring theory is at the core of the AdS/CFT correspondence [4]. As discussed earlier, this correspondence actually extends to all the Dp-branes of IIA and IIB superstrings, whose near horizon geometry yields an $AdS_{p+2} \times S^{8-p}$ spacetime, with a non-vanishing dilaton for $p \neq 3$. The low-energy excitations on the gravity side are expected to be described by effective theories resulting from Kaluza-Klein reduction on the sphere. When restricted to the massless sector, they correspond to maximal $SO(9-p)$ gauged supergravity in $(p+2)$ dimensions. These gauged supergravities admit an AdS vacuum solution for $p = 3$ and domain-walls in the other cases. From this statement comes the generalization of AdS/CFT to the Domain-Wall/QFT correspondence. Accordingly, gravity side excitations are dual to operators on the gauge theory side, and as a remarkable fact, the computation of correlation functions of operators in the dual theory is facilitated when the lower-dimensional supergravity arises as a "consistent truncation". Indeed, then the massless modes of the effective lower-dimensional supergravity are dual to a subset of operators in the gauge theory side which is closed under operator product expansion (OPE). Thus, holography computations can be applied to the lower-dimensional $SO(9-p)$ gauged supergravity, without taking into account contributions originating from massive Kaluza-Klein modes.

Here, by a *consistent truncation* we mean that in the full non-linear lower-dimensional theory, non-vanishing solutions for the massless modes can be found, when all the massive Kaluza-Klein modes are put to zero. Then, a non-linear Kaluza-Klein ansatz can be constructed to uplift the massless sector of the lower-dimensional theory into the higher-dimensional one. For instance, the Kaluza-Klein reduction on the n-tori T^n, are always consistent, since the massless fields are singlet under the $U(1)^n$ isometry group of the lower-dimensional theory, whereas the massive fields are not. Thus, massless fields cannot appear as sources in the equations of motion of the massive fields, and the latter can be consistently put to zero. Any attempts to generalize the result to spheres, proves very difficult for an arbitrary field theory, and affordable only for some supergravities [121] [43]. In this sense, maybe the most impressive result is the demonstration that the reduction of eleven-dimensional supergravity on S^7, once restricted to its massless sector, leads to $\mathcal{N} = 8$, $D = 4$ maximal supergravity with gauge group $SO(8)$ and is consistent [42]. In general, the reduction is shown to be consistent, only for some *truncations* of the massless bosonic sector of the lower-dimensional supergravity. For example, the reduction of type

IIB supergravity on S^5 gives rise to the $\mathcal{N}=8$, $D=5$ supergravity with gauge group $SO(6)$. However, the consistency of the reduction has been proved only for the truncation to the $U(1)^3$ Cartan subgroup of $SO(6)$, where the bosonic sector of the theory contains: three Maxwell vector fields, the metric and scalar singlets under $U(1)^3$ [122].

Motivated by the DW/QFT correspondence, we aim to show the consistency of a truncation of the $SO(9)$ gauged maximal supergravity in two dimensions. Indeed, it accounts for the low energy dynamics of type IIA supergravity excitations around an $AdS_2 \times S^8$ background (coupled to a dilaton) which stands for the gravity side of the conjecture, in the $D0$-brane case. To this end, we will first truncate the bosonic sector of the $SO(9)$ supergravity to singlets under the $U(1)^4$ Cartan subgroup. Then, particular solutions of the truncated theory will be derived. This will help us to establish the non-linear Kaluza-Klein ansatz for embedding the bosonic truncated sector into type IIA supergravity. Hence, all the ten-dimensional bosonic equations of motion (including Einstein' equations) will be solved explicitly, showing that the $U(1)^4$ truncation is consistent. Finally, applications will be presented such as the uplift to ten and eleven dimensions of particular solutions of the $U(1)^4$ theory.

4.2 $U(1)^4$ truncation of $SO(9)$ supergravity

4.2.1 Bosonic sector of $SO(9)$ supergravity

Let us start with the bosonic part of Lagrangian (3.4.22). It describes a dilaton-gravity coupled non-linear sigma model with 128-dimensional target space $\left(SL(9) \ltimes \mathbb{R}^{84}\right)/SO(9)$ and Wess-Zumino term

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}e\rho\, R + \frac{1}{4}e\rho\, \mathcal{P}_\mu^{\alpha\beta}\mathcal{P}^{\mu\,\alpha\beta} + \frac{1}{12}\, e\rho^{1/3}\, \mathcal{V}_{klm}{}^{[\alpha\beta\gamma]}\mathcal{V}_{npq}{}^{[\alpha\beta\gamma]} D_\mu\phi^{klm} D_\mu\phi^{npq} \\ &+ \frac{1}{648}\varepsilon^{\mu\nu}\varepsilon_{klmnpqrst}\, \phi^{klm} D_\mu\phi^{npq} D_\nu\phi^{rst} - \frac{g}{4}\varepsilon^{\mu\nu}\mathcal{F}_{\mu\nu}^{kl}\, Y^{kl} - e\, V_{\rm pot}(\mathcal{V},\phi,Y)\,. \qquad (4.2.1)\end{aligned}$$

Here only 36 auxiliary fields $Y^{kl} \equiv Y^{[kl]}$ are present, and remember that the 36 vector fields enter the covariant derivatives defined by

$$\begin{aligned} J_\mu^{\alpha\beta} &\equiv \mathcal{V}^{-1\alpha k}\left(\partial_\mu \mathcal{V}_k{}^\beta + gA_\mu^{kl}\mathcal{V}_l{}^\beta\right) \;\equiv\; \mathcal{Q}_\mu^{[\alpha\beta]} + \mathcal{P}_\mu^{(\alpha\beta)}\,, \\ D_\mu\phi^{klm} &\equiv \partial_\mu\phi^{klm} - 3\, g\, A_\mu^{p[k}\phi^{lm]p}\,. \end{aligned} \qquad (4.2.2)$$

The scalar potential is given up to quadratic order in the ϕ^{klm} by

$$V_{\rm pot} = \frac{g^2}{8}\,\rho^{5/9}\left(2\,{\rm tr}[MM] - ({\rm tr}\, M)^2\right) + g^2\rho^{-13/9}M^{km}M^{ln}\,Y_{kl}Y_{mn} + \mathcal{O}(\phi^2)\,, \qquad (4.2.3)$$

where the matrix M is defined by $M \equiv (\mathcal{V}\mathcal{V}^T)^{-1}$. This Lagrangian is left invariant, up to total derivative, by the following local $SO(9)$ gauge transformation

$$\begin{aligned} &\forall\Lambda \in \mathfrak{so}(9)\,, \qquad \delta_\Lambda \mathcal{V}_m{}^a = \Lambda^{kl}\delta_{lm}\,\mathcal{V}_k{}^a\,, \qquad \delta_\Lambda\phi^{klm} = 3\Lambda^{p[k}\phi^{lm]p}\,, \\ &\qquad\qquad\qquad\quad \delta_\Lambda A_\mu{}^{kl} = D_\mu\Lambda^{kl}\,, \qquad\qquad \delta_\Lambda Y^{kl} = 2\Lambda^{m[k}Y^{l]m}\,. \end{aligned} \qquad (4.2.4)$$

4.2.2 Selecting the Cartan subgroup

A Cartan subgroup of $SO(9)$ is given by the maximal torus $\left(SO(2)\right)^4$ (or equivalently named $U(1)^4$). It is the group of simultaneous rotations in four pairwise orhtogonal planes of $\mathbb{R}^9$, the ninth direction being fixed. Let us present a basis of generators of the associated Lie algebra $\big(\mathfrak{so}(2)\big)^4 \subset \mathfrak{so}(9)$:

$$T_1{}^{kl} \equiv 2\delta_1^{[k}\delta_2^{l]}\,, \quad T_2{}^{kl} \equiv 2\delta_3^{[k}\delta_4^{l]}\,, \quad T_3{}^{kl} \equiv 2\delta_5^{[k}\delta_6^{l]}\,, \quad T_4{}^{kl} \equiv 2\delta_7^{[k}\delta_8^{l]}\,. \tag{4.2.5}$$

Then any gauge transformation will be parametrized by four real parameters

$$\Lambda^{kl} \equiv \Lambda^a\, T_a^{kl}\,, \quad \Lambda \in \mathbb{R}^4. \tag{4.2.6}$$

Thus, the gauge fields become Maxwell fields

$$A_\mu{}^{kl} \equiv A_\mu^a\, T_a{}^{kl}\,, \qquad \delta_\Lambda A_\mu^a = \partial_\mu \Lambda^a\,. \tag{4.2.7}$$

Following the work of [122] we will restrict to a subsector of the scalar fields where none of them are charged under $U(1)^4$.

Explicitly, the auxiliary scalars transforming in the adjoint of $SO(9)$ will reduce to

$$Y^{kl} \equiv \frac{\rho}{4}\, y^a\, T_a{}^{kl}\,, \qquad a = 1, \ldots, 4 \tag{4.2.8}$$

where the factor $\frac{\rho}{4}$ has been chosen for later computational convenience. Thus, $U(1)^4$ being abelian, the Y^{kl} are invariant under an $U(1)^4$ transformation

$$\delta_\Lambda Y^{kl} = 0\,. \tag{4.2.9}$$

Now let us focus on the coset-space scalar fields. The local $SO(9)_{\text{coset}}$ symmetry is fixed so that the scalar matrix $\mathcal{V}$ is represented by

$$\begin{aligned} &\mathcal{V} = \exp\left(v_a h^a\right)\,, \\ &h^1 \equiv \mathrm{diag}\,(1,1,0,0,0,0,0,0,-2)\,, \qquad h^2 \equiv \mathrm{diag}\,(0,0,1,1,0,0,0,0,-2)\,, \\ &h^3 \equiv \mathrm{diag}\,(0,0,0,0,1,1,0,0,-2)\,, \qquad h^4 \equiv \mathrm{diag}\,(0,0,0,0,0,0,1,1,-2)\,. \end{aligned} \tag{4.2.10}$$

It is parametrized by four scalar fields v_a. Owing to the fixation of the coset space symmetry, the scalars transform trivially under $U(1)^4$

$$\delta_\Lambda\, \mathcal{V} = [\Lambda, \mathcal{V}] = 0\,. \tag{4.2.11}$$

Finally, from the 84 scalars ϕ^{klm}, only four are chosen to survive

$$\phi^1 \equiv \phi^{129}\,, \quad \phi^2 \equiv \phi^{349}\,, \quad \phi^3 \equiv \phi^{569}\,, \quad \phi^4 \equiv \phi^{789}\,, \tag{4.2.12}$$

where all other components vanish. These four fields correspond to the axions of dimensional reduction. According to (4.2.4) they also transform trivially under $U(1)^4$

$$\delta_\Lambda\, \phi^{klm} = 0\,. \tag{4.2.13}$$

Consequently, the bosonic sector of the theory reduces to

- the two-dimensional metric $g_{\mu\nu}$,
- five dilatons $\{\rho, v_a\}$ and four axions ϕ^a,
- four auxiliary fields y^a and four Maxwell vector fields A_μ^a .

The additional dilaton ρ indicates the fact that the theory supports a domain wall solution. We are now prepared to formulate the Lagrangian of the truncated theory.

4.2.3 Truncated Lagrangians

Plugging the $U(1)^4$ truncation ansatz into the $SO(9)$ bosonic Lagrangian (4.2.1) yields the simpler Lagrangian

$$\mathcal{L} = -\frac{1}{4} e\rho\, R + \frac{1}{2} e\rho \sum_a \partial_\mu u_a\, \partial^\mu u_a + \frac{1}{2} e\rho^{1/3} X_0^{-1} \sum_{a=1}^{4} X_a^{-2} \left(\partial_\mu \phi^a\right) \left(\partial^\mu \phi^a\right) - \frac{\rho}{8}\, g\, \varepsilon^{\mu\nu} F^a_{\mu\nu}\, y^a - e\, V_{\rm pot}\,, \tag{4.2.14}$$

where we have defined

$$X_a \equiv e^{-2v_a} \equiv e^{-2\, A_{ab} u_b}\,, \qquad X_0 \equiv (X_1 X_2 X_3 X_4)^{-2}\,, \tag{4.2.15}$$

with the matrix

$$A = \begin{pmatrix} 1/6 & -1/\sqrt{2} & -1/\sqrt{6} & -1/(2\sqrt{3}) \\ 1/6 & 0 & 0 & \sqrt{3}/2 \\ 1/6 & 0 & \sqrt{2/3} & -1/(2\sqrt{3}) \\ 1/6 & 1/\sqrt{2} & -1/\sqrt{6} & -1/(2\sqrt{3}) \end{pmatrix}, \tag{4.2.16}$$

and the abelian field strengths $F^a_{\mu\nu} \equiv 2\, \partial_{[\mu} A^a_{\nu]}$. The potential can be evaluated from its expression (3.4.41). Indeed, the $SO(9)$ irreducible tensors (3.4.31) simplify, and after some computation one finds

$$\begin{aligned} V_{\rm pot} = g^2 \rho^{5/9} \Big[& \frac{1}{8} \Big({X_0}^2 - 8 \sum_{a<b} X_a X_b - 4 X_0 \sum_a X_a \Big) + \frac{1}{2} \rho^{-2/3} \sum_a X_a^{-2} \left(X_0 - 4 X_a\right) (\phi^a)^2 \\ & + 2\, \rho^{-4/3} \sum_{a<b} X_a^{-2} X_b^{-2} (\phi^a)^2 (\phi^b)^2 + \frac{1}{8} \rho^{-2} \sum_a X_a \Big(\rho\, y^a + 8 \prod_{b\neq a} \phi^b \Big)^2 \\ & + 2\, \rho^{-4/3} \sum_{a<b} X_a^{-2} X_b^{-2} (\phi^a)^2 (\phi^b)^2 + \frac{1}{8} \rho^{-2} \sum_a X_a \Big(\rho\, y^a + 8 \prod_{b\neq a} \phi^b \Big)^2 \\ & + \frac{1}{2} \rho^{-8/3} X_0^{-1} \Big(\sum_a \rho\, y^a \phi^a + 8 \prod_a \phi^a \Big)^2 \Big]\,. \end{aligned} \tag{4.2.17}$$

This is an eighth order polynomial expression in the ϕ^a. Under the field redefinition

$$X_a \equiv H_a\, X_0\,, \qquad \phi^a \equiv \frac{1}{2} \rho^{1/3}\, \eta_a\, X_a\, X_0^{1/2}\,, \qquad a = 1 \ldots 4\,, \tag{4.2.18}$$

it takes the simpler form

$$\begin{aligned} V_{\rm pot} = \frac{g^2}{8} \rho^{5/9} H_0^{-4/9} \Big[& 1 - 8 \sum_{a<b} H_a H_b - 4 \sum_a H_a + \sum_a \left(1 - 4 H_a\right) \eta_a^2 + \sum_{a<b} \eta_a^2 \eta_b^2 \\ & + \sum_a \eta_a^{-2} (y^a H_a \eta_a + \eta_0)^2 + \Big(\eta_0 + \sum_a y^a H_a \eta_a \Big)^2 \Big]\,. \end{aligned} \tag{4.2.19}$$

Here $H_0 \equiv H_1 H_2 H_3 H_4$ and $\eta_0 \equiv \eta_1 \eta_2 \eta_3 \eta_4$.

Integrating out the auxiliary fields

Another interesting formulation of the truncated theory is obtained when the auxiliary fields have been integrated out. The equations of motion for the auxiliary scalars y^a lead to

$$y^a = -\sum_b \mathcal{O}^{-1}{}_{ab} \Big(\frac{1}{2} (ge)^{-1} \rho^{4/9}\, \varepsilon^{\mu\nu} F^b_{\mu\nu} + 8\, \mathcal{O}_{bb} \prod_{c\neq b} \phi^c \Big)\,, \tag{4.2.20}$$

with the matrix $\mathcal{O}_{ab} \equiv X_a X_b \ (\delta_{ab} + \eta_a \eta_b) \equiv X_a X_b \ m_{ab}$. It follows exactly from the truncation of (3.5.5). Thus, by replacing y^a in the Lagrangian (4.2.14), a two-dimensional Maxwell term is generated together with a linear coupling in the field strengths.

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4} e\rho\, R + \frac{1}{2} e\rho \ \sum_a (\partial_\mu u_a)\, (\partial^\mu u_a) + \frac{1}{2} e\rho^{1/3} H_0^{2/3} \sum_a H_a^{-2}\, (\partial_\mu \phi^a)\, (\partial^\mu \phi^a) \\ & - \frac{e}{16}\, \rho^{13/9}\, H_0^{4/9} \sum_{a,b} H_a^{-1} H_b^{-1} m^{-1}{}_{ab}\, F_{\mu\nu}{}^a F^{\mu\nu b} \\ & + \frac{g}{8} \rho\, \eta_0 \sum_{a,b} \varepsilon^{\mu\nu} F_{\mu\nu}{}^a\, H_a^{-1} \eta_b^{-1} (1 + \eta_b^2)\, m^{-1}{}_{ab} - e\widehat{V}_{\text{pot}}\,, \end{aligned} \tag{4.2.21}$$

where the modified scalar potential is given by

$$\begin{aligned}\widehat{V}_{\text{pot}} = & \frac{g^2}{8}\, \rho^{5/9} H_0^{-4/9} \Big(1 - 8 \sum_{a<b} H_a H_b - 4 \sum_a H_a + \frac{9\, \eta_0^2}{1 + \sum_a \eta_a^2} \\ & + \sum_a (1 - 4H_a)\, \eta_a^2 + \sum_{a<b} \eta_a^2 \eta_b^2 \Big)\,. \end{aligned} \tag{4.2.22}$$

In this formulation, the H^a will be called dilatons and the ϕ^a scalars will be named axions.

4.2.4 Dilaton sector

Owing to the complicated structure of the potential when the axions are present, our work will be restricted to the subsector of vanishing axions: $\phi^a \equiv 0$. This so-called *dilaton sector* is parametrized by the two-dimensional fields $\{g_{\mu\nu},\, X_a,\, \rho,\, A_\mu^a\}$. Hence, our goal is to embed the dilaton sector into ten dimensions, with a suitable non-linear Kaluza-Klein ansatz.

Lagrangian

For $\phi^a \equiv 0$, the Lagrangian (4.2.21) takes the form

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4} e\rho\, R + \frac{1}{2} e\rho \ \sum_a \partial_\mu u_a\, \partial^\mu u_a - \frac{1}{16} e\rho^{13/9}\, H_0^{4/9} \ \sum_a H_a^{-2} F_{\mu\nu}{}^a F^{\mu\nu a} \\ & - \frac{1}{8}\, e g^2 \rho^{5/9}\, H_0^{-4/9} \Big(1 - 8 \sum_{a<b} H_a H_b - 4 \sum_a H_a \Big)\,. \end{aligned} \tag{4.2.23}$$

It is in accordance with the truncations of the maximal AdS supergravities in $(D = 4, 5, 7)$ described in [122]. Again, the particular behaviour of the fifth dilaton ρ comes from the fact that the theory supports a domain wall solution.

Equations of motion

The equations of motion are more easily solved from the Lagrangian containing the auxiliary fields. For the dilaton sector, the Lagrangian is computed by inserting $\phi^a = 0$ in (4.2.14), which leads to

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4} e\rho\, R + \frac{1}{2} e\rho \sum_a \partial_\mu u_a\, \partial^\mu u_a - \frac{\rho}{8}\, g \ \sum_a \varepsilon^{\mu\nu} F_{\mu\nu}^a \ y^a \\ & - \frac{e g^2}{8}\, \rho^{5/9} H_0^{-4/9} \Big(1 - 8 \sum_{a<b} H_a H_b - 4 \sum_a H_a + \sum_a (y^a H_a)^2 \Big)\,. \end{aligned} \tag{4.2.24}$$

As a result, the equations of motion for the vector field simply state that

$$\rho\, y^a = \text{constant}\,, \tag{4.2.25}$$

while the equations for the auxiliary fields yield

$$F^a_{\mu\nu} = g\, e\, \varepsilon_{\mu\nu}\, \rho^{-4/9} H_0^{-4/9} H_a^2\, y^a\,. \tag{4.2.26}$$

Besides, the scalar field equations are given by

$$\begin{aligned} 0 = &\sum_b \left(\rho^{-1}\nabla^\mu \left(\rho\, \partial_\mu u_b\right) A^{-1}{}_{b\,a}\right) \\ &+ g^2 \rho^{-4/9} H_0^{-4/9}\Big(1 + 2\, H_a \sum_{b\neq a} H_b + H_a - 2\sum_b H_b - \frac{1}{2}(y^a H_a)^2\Big) \end{aligned} \tag{4.2.27}$$

The traceless part of the Einstein equations writes

$$\rho^{-1}\nabla_\mu\partial_\nu\rho + 2\sum_a \partial_\mu u_a\, \partial_\nu u_a = \frac{1}{2}\, g_{\mu\nu}\left(\rho^{-1}\nabla_\rho\partial^\rho\rho + 2\sum_a \partial_\rho u_a\, \partial^\rho u_a\right), \tag{4.2.28}$$

and a combination of the dilaton and the trace part of the Einstein equations leads to

$$\begin{aligned} R = &\, 2\sum_a \left(\partial^\mu u_a\right)\left(\partial_\mu u_a\right) \\ &- \frac{5}{18}\, g^2\, \rho^{-4/9}\, H_0^{-4/9}\Big(1 - 8\sum_{a<b} H_a H_b - 4\sum_a H_a - \frac{13}{5}\sum_a (y^a H_a)^2\Big)\,, \\ \rho^{-1}\nabla^\mu\partial_\mu\rho = &\sum_{a,b}\left(\rho^{-1}\nabla^\mu\left(\rho\,\partial_\mu u_b\right) A^{-1}{}_{b\,a}\right) + \frac{9}{2}\, g^2\, \rho^{-4/9} H_0^{-4/9}\Big(1 - 2\sum_a H_a\Big)\,. \end{aligned} \tag{4.2.29}$$

Particular solutions

Assuming that the *scalars* H_a *are constant* is a natural hypothesis to compute particular solutions. Then, the scalar fields equations can be solved for y^a:

$$(y^a)^2 = 2\, H_a^{-2}(1 + H_a) - 4 + 4\sum_b \frac{H_b(H_a - 1)}{H_a^2}\,. \tag{4.2.30}$$

For definiteness of the previous expression, we recall that the H^a being exponential of the real valued fields v_a, they cannot equal zero. The other field equations reduce to

$$\begin{aligned} F^a_{\mu\nu} &= g\, e\, \varepsilon_{\mu\nu}\, \rho^{-4/9} H_0^{-4/9} H_a^2\, y^a \\ 0 &= \rho^{-1}\nabla_\mu\partial_\nu\rho - \frac{1}{2}\, g_{\mu\nu}\, \rho^{-1}\nabla_\sigma\partial^\sigma\rho \\ R &= \frac{1}{2}\, g^2\, \rho^{-4/9}\, H_0^{-4/9}\Big(11 - 18\sum_a H_a + 16\sum_{a<b} H_a H_b\Big) \\ \rho^{-1}\nabla^\mu\partial_\mu\rho &= \frac{9}{2}\, g^2\, \rho^{-4/9} H_0^{-4/9}\Big(1 - 2\sum_a H_a\Big) \end{aligned} \tag{4.2.31}$$

Now two simple cases can be identified :

- the case of vanishing field strengths, i.e. $y^a = 0$. Equations (4.2.30) then imply that all scalar fields are equal $H_1 = H_2 = H_3 = H_4 \equiv \mathrm{H}$, (recall that $H_a > 0$), with two distinct solutions
$$\mathrm{H} = 1\,, \quad \text{or} \quad \mathrm{H} = \frac{1}{6}\,. \tag{4.2.32}$$
The first choice ($H = 1$) together with a domain wall ansatz for the metric
$$ds^2 = e^{2\,A(r)} dt^2 - dr^2\,, \tag{4.2.33}$$
describes the half-supersymmetric domain-wall solution
$$\rho = (gr)^{9/2}\,, \qquad A(r) \;=\; \frac{7}{2} \ln r\,, \qquad R \;=\; \frac{35}{2}\frac{1}{r^2}\,, \tag{4.2.34}$$
corresponding to the ten-dimensional D0-brane near-horizon geometry. The second choice ($H = 1/6$) does not lead to a supersymmetric solution.

- the AdS case: imposing a constant dilaton field ρ, equation (4.2.31) implies
$$\sum_a H_a = \frac{1}{2}\,, \tag{4.2.35}$$
and the remaining equations of motion are solved by a two-dimensional AdS metric
$$\begin{aligned} \mathrm{ds}^2 &= f(r)\,\mathrm{dt}^2 - \frac{1}{f(r)}\,\mathrm{dr}^2\,, \\ f(r) &= -C + g^2\,\frac{\left(1 + 8\,\sum_{a<b} H_a H_b\right)}{2\,\rho^{4/9}\,H_0^{4/9}} r^2\,, \\ F_{\mu\nu}{}^a &= 2g\,\rho^{-4/9}\left(\frac{H_a^2\,\sqrt{H_a^{-1} - 1}}{H_0^{4/9}}\right)\epsilon_{\mu\nu}\,, \\ r_{\mathrm{AdS}} &= \frac{\sqrt{2}\,\rho^{2/9} H_0^{2/9}}{g}\left(1 + 8\sum_{a<b} H_a H_b\right)^{-1/2} = \sqrt{\frac{2}{R}}\,, \end{aligned} \tag{4.2.36}$$
where C is an integration constant. We thus obtain a three-parameter family of pure AdS_2 solutions. The Killing spinor equations (5.3.10) show that these solutions break all supersymmetries. While the metric is locally AdS, it resembles the (r-t) section of non-rotating BTZ black hole [123, 124] with C being the mass of the spacetime.

4.3 Embedding into type IIA supergravity

Type IIA supergravity comes from dimensional reduction of eleven-dimensional supergravity on a circle. The massless bosonic sector contains: a ten-dimensional vielbein, a vector field and a dilaton descending from the eleven-dimensional vielbein, plus a three-form and a two-form coming from the eleven-dimensional three-form.

$$\begin{array}{ccc} \mathrm{D} = 11 & E_{\hat{M}}{}^{\hat{A}} & A_{\hat{M}\hat{N}\hat{P}} \\ & \downarrow & \downarrow \\ \mathrm{D} = 10 & \{\,E_M{}^A,\,B_M,\,\phi\,\} & \{\,A_{MNP}\,,\,A_{MN}\,\} \end{array} \tag{4.3.1}$$

The theory described by the Lagrangian (4.2.23) will be embedded in the subsector of type IIA where the three-form and two-form are set to zero

$$A_{MNP} = 0 = A_{MN} \,. \tag{4.3.2}$$

This truncation originates from pure gravity in eleven dimensions. Given the Kaluza-Klein ansatz for the eleven-dimensional vielbein

$$E_{\hat{M}}{}^{\hat{A}} = \begin{pmatrix} e^{-\frac{\phi}{6}}\, E_M{}^A & e^{\frac{4}{3}\phi}\, B_M \\ 0 & e^{\frac{4}{3}\phi} \end{pmatrix} \tag{4.3.3}$$

the eleven-dimensional action reduces to

$$\begin{aligned} S_{11d} &= \int d^{11}x \, \big(- \frac{1}{4}\, e_{11}\, R_{11} \big) \\ &= \int d^{10}x \, \Big(- \frac{1}{4}\, e_{10}\, R_{10} + \frac{1}{2}\, e_{10}\, (\partial\, \phi)^2 - \frac{e^{3\phi}}{16}\, e_{10}\, F^2 \Big) \end{aligned} \tag{4.3.4}$$

where

$$F_{MN} \equiv 2\, \partial_{[M} B_{N]} \,. \tag{4.3.5}$$

Hence, the Lagrangian of type IIA relevant for our study is given by

$$\mathcal{L} = -\frac{1}{4} e_{10}\, R_{10} + \frac{1}{2}\, e_{10}\, \partial_M \phi\, \partial^M \phi - \frac{1}{16} e_{10}\, e^{3\phi}\, F_{MN}\, F^{MN} \,, \tag{4.3.6}$$

where the signature of the tangent space metric is

$$\eta_{AB} = \mathrm{diag}\,(1, -1, \ldots, -1) \,. \tag{4.3.7}$$

The associated ten-dimensional equations of motion come from Einstein equations in eleven dimensions ($R_{11\,\hat{M}\hat{N}} = 0$) and are given by

$$0 = R_{10MN} - \frac{1}{2} g_{MN} R_{10} - 2\, \big(\partial_M \phi \partial_N \phi - \frac{1}{2}\, g_{MN}\, (\partial \phi)^2 \big) \,, \tag{4.3.8}$$

$$+ \frac{1}{4}\, e^{3\phi}\, \big(2\, F_M{}^P\, F_{NP} - \frac{1}{2}\, g_{MN}\, FF \big) \,, \tag{4.3.9}$$

$$0 = \nabla \partial \phi + \frac{3}{16} e^{3\phi} FF \,, \tag{4.3.10}$$

$$0 = \partial_M \big(e_{10}\, e^{3\phi}\, F^{MN} \big) \,, \tag{4.3.11}$$

plus the Bianchi identity satisfied by the field strength

$$\partial_{[M} F_{NP]} = 0 \,. \tag{4.3.12}$$

By an embedding of the two-dimensional theory in type IIA we mean that to any solution of the two-dimensional equations of motion derived from the Lagrangian (4.2.23) or equivalently (4.2.24), we can associate a solution of the ten-dimensional field equations (4.3.8) and (4.3.12). If it is possible, the dilaton sector of the $U(1)^4$ truncated supergravity in two dimensions, will be a consistent truncation of type IIA supergravity.

$$\{g_{\mu\nu}\,,\, X_a\,,\, \rho\,,\, A_\mu^a\} \xrightarrow{\text{embedding}} \{g_{MN}\,,\, \phi\,,\, B_M\} \tag{4.3.13}$$

4.3.1 Non-linear Kaluza-Klein ansatz

To perform the embedding of the dilaton sector in type IIA supergravity, we construct a non-linear Kaluza-Klein ansatz. It is given by generalizing the AdS reduction ansaetze of [122] to a non-constant dilaton ρ. To make the ansatz explicit, the ten-dimensional coordinates are split into $\{x^M\} \longrightarrow \{x^\mu\,,\,\mu_a\,,\,\sigma_a\}$ with the labels $\mu = 0, 1$, and $a = 1, \ldots, 4$. Therefore, the line element on the eight-dimensional round sphere is given by

$$ds_8^2 = d\mu_0{}^2 + \sum_{a=1}^{4} \left(d\mu_a{}^2 + \mu_a^2\, d\sigma_a{}^2\right), \tag{4.3.14}$$

$$\text{with} \qquad \mu_0^2 \equiv 1 - \sum_a \mu_a^2 \qquad \text{and} \qquad 0 \le \mu_a^2 \le 1\,. \tag{4.3.15}$$

Then following [122], we perform a diagonal distortion of the sphere by introducing the four scalar fields $\{X_a\,,\ a = 1 \ldots 4\}$:

$$ds_8^2 = X_0^{-1} d\mu_0{}^2 + \sum_{a=1}^{4} X_a^{-1}\left(d\mu_a{}^2 + \mu_a^2\, d\sigma_a{}^2\right), \tag{4.3.16}$$

$$\text{with} \qquad X_0 \equiv (X_1 X_2 X_3 X_4)^{-2} \qquad \text{and} \quad X_a = X_a(x^\mu)\,. \tag{4.3.17}$$

Thus, the metric part of the non-linear Kaluza-Klein ansatz as well as the ten-dimensional dilaton and two-form field strength can be formulated in analogy with the consistent sphere reductions [43], [125] and [122]

$$ds_{10}^2 = \Delta^{7/8}\, ds_2^2 - g^{-2}\,\Delta^{-1/8}\left(X_0^{-1} d\mu_0^2 + \sum_a X_a^{-1}\left(d\mu_a^2 + \mu_a^2\,(d\sigma_a + g\,A^a)^2\right)\right), \tag{4.3.18}$$

with

$$\Delta \equiv \sum_{\alpha=0}^{4} X_\alpha\, \mu_\alpha^2\,. \tag{4.3.19}$$

If not mentioned, the sums over a will always run from 1 to 4. Notice that on the contrary, the sum over α runs over 0 to 4. The ten-dimensional dilaton and two-form field strength are given by

$$\begin{aligned} \phi &= \frac{1}{3}\log\left(\Delta^{-9/8}\right), \qquad\qquad (4.3.20) \\ F &= \left(2\,g \sum_{\alpha=0}^{4}\left(X_\alpha^2 \mu_\alpha^2 - \Delta X_\alpha\right) + g\,\Delta\, X_0\right)\varepsilon_2 \\ &\quad + \frac{1}{2g^2}\sum_a X_a^{-2} d(\mu_a^2)\wedge(d\sigma_a + gA^a)\,(*_2\,F^a) + \frac{1}{2g}\sum_{\alpha=0}^{4} X_\alpha^{-1} *_2 dX_\alpha \wedge d(\mu_\alpha^2)\,. \end{aligned}$$

These formulae are by themselves a great achievement, and the answer would have been found if we were to neglect the dilaton. Indeed, the ansatz realizes “only” an embedding of the $\rho \equiv 1$ **subsector** of the dilaton sector. We have shown that all the ten-dimensional equations of motion are satisfied, provided that the two-dimensional fields $\{g_{\mu\nu},\, X_a,\, A_\mu^a\}$ satisfy their own equation of motion, with $\rho \equiv 1$. However, we are interested in the embedding of the **whole** dilaton sector, where ρ is non-constant. The generalization of (4.3.18) is a difficult challenge since ρ can enter the ansatz almost everywhere. We have thus proceeded in several steps that we present below.

First: find the embedding for constant $\rho \neq 1$

Now we would like to generalize the ansatz (4.3.18) to the case of an arbitrary constant ρ. The ten-dimensional equations of motion must be solved after we have introduced constant ρ factors in (4.3.18). A possible way to achieve this, consist in adding ρ factors in the Kaluza-Klein ansatz, such that the ten-dimensional equations of motion do not change. This is possible by exploiting the symmetries of the equations of motion (4.3.8) and (4.3.12). Indeed, the scaling symmetries

$$g_{MN} \to \lambda^2\, g_{MN}\,, \quad F_{MN} \to \lambda\, \mu\, F_{MN}\,, \quad \phi \to \phi - \frac{2}{3}\log\mu\,, \tag{4.3.21}$$

leave the ten-dimensional equations of motion invariant. Thus, the ansatz (4.3.18) can be modified according to (4.3.21), and will be still an embedding of the $\rho \equiv 1$ dilaton sub-sector. It is also the case if we rescale the coupling constant g by a constant factor. However, this will break the structure of the S^8 line element in the Kaluza-Klein ansatz. Therefore, a better way to implement the rescaling is to accompany it with a field redefinition of the two-dimensional vector potential

$$g \longrightarrow \kappa\, g\,, \qquad A^a \longrightarrow \kappa^{-1}\, A^a\,. \tag{4.3.22}$$

These symmetries modify the ansatz (4.3.18) according to

$$\begin{aligned}
ds^2_{10} &= \lambda^2\, \Delta^{7/8}\, ds^2_2 - \lambda^2 \kappa^{-2}\, g^{-2}\, \Delta^{-1/8} \left(X_0^{-1} d\mu_0^2 + \sum_a X_a^{-1} \left(d\mu_a^2 + \mu_a^2\, (d\sigma_a + g\, A^a)^2 \right)\right) \\
\phi &= \frac{1}{3}\log\left(\Delta^{-9/8}\right) - \frac{2}{3}\log\mu\,, \\
F &= \lambda\mu \Bigg[\kappa \left(2\, g \sum_{\alpha=0}^{4} \left(X_\alpha^2 \mu_\alpha^2 - \Delta X_\alpha \right) + g\, \Delta\, X_0 \right) \varepsilon_2 \\
&\quad + \frac{\kappa^{-3}}{2g^2} \sum_a X_a^{-2} d(\mu_a^2) \wedge (d\sigma_a + g A^a)\, (*_2\, F^a) + \frac{\kappa^{-1}}{2g} \sum_{\alpha=0}^{4} X_\alpha^{-1} *_2 dX_\alpha \wedge d(\mu_\alpha^2) \Bigg]\,,
\end{aligned} \tag{4.3.23}$$

but it is still an embedding of the $\rho \equiv 1$ dilaton subsector. Now the idea is to substitute $\{\kappa,\, \lambda,\, \mu\}$ by arbitrary functions of ρ. The simplest ansatz we can imagine is

$$\kappa \equiv \rho^A\,, \qquad \lambda \equiv \rho^B\,, \qquad \mu \equiv \rho^C\,, \tag{4.3.24}$$

where ρ is by assumption a constant. Thus, the Kaluza-Klein ansatz is generalized to

$$\begin{aligned}
ds^2_{10} &= \rho^{2B} \Delta^{7/8}\, ds^2_2 - \rho^{2(B-A)} g^{-2} \Delta^{-1/8} \Big[X_0^{-1} d\mu_0^2 + \sum_a X_a^{-1} \big(d\mu_a^2 + \mu_a^2\, (d\sigma_a + g\, A^a)^2 \big) \Big] \\
\phi &= \frac{1}{3}\log\left(\rho^{-2C} \Delta^{-9/8}\right), \\
F &= \rho^{B+C} \Bigg[\rho^A \left(2\, g \sum_{\alpha=0}^{4} \left(X_\alpha^2 \mu_\alpha^2 - \Delta X_\alpha \right) + g\, \Delta\, X_0 \right) \varepsilon_2 \\
&\quad + \frac{\rho^{-3A}}{2g^2} \sum_a X_a^{-2} d(\mu_a^2) \wedge (d\sigma_a + g A^a)\, (*_2\, F^a) + \frac{\rho^{-A}}{2g} \sum_{\alpha=0}^{4} X_\alpha^{-1} *_2 dX_\alpha \wedge d(\mu_\alpha^2) \Bigg]\,.
\end{aligned} \tag{4.3.25}$$

where the coefficients $\{A,\, B,\, C\}$ remain to be determined so that (4.3.25) represents an embedding of the $\rho \equiv$ constant dilaton sector.

Second step: use particular solutions

At this point, we use the particular solutions that were derived before, to determine completely the Kaluza-Klein ansatz (4.3.25). In particular, with the AdS solutions (4.2.36), a family of constant dilaton solutions is at hand. They enable to determine two of the three dilaton powers

$$B = -7/72\,, \qquad C = 7/8\,. \tag{4.3.26}$$

To get the last power A, we try to embed the domain-wall solution (4.2.34). Being a very simple solution, it also provides a consistency check on the possibility to extend the Kaluza-Klein ansatz to the embedding of non-constant dilaton solutions. Notice that with solution (4.2.34) the fields $X_a = 1$ and $A^a = 0$. Thus, the reduction ansatz is given by

$$\begin{aligned} ds_{10}^2 &= \rho^{-7/36}\Big(ds_2^2 - g^{-2}\,\rho^{-2A}\,d\Omega_8^2\Big)\,,\\ \phi &= -\frac{7}{12}\,C\,\log\rho\,,\\ F &= -7\rho^{A+7/9}\,g\,\varepsilon_2\,, \end{aligned} \tag{4.3.27}$$

where $d\Omega_8^2$ denotes the line element of the unit 8-sphere. Notice in particular that the Bianchi identity $dF = 0$ is trivially satisfied since ρ is a function of the two-dimensional coordinates. The embedding of the domain-wall fixes the last dilaton power

$$A = -2/9\,. \tag{4.3.28}$$

In the light of the previous results, we are led to the following general claim: The Kaluza-Klein ansatz for the ten dimensional metric

$$\begin{aligned} ds_{10}^2 &= \rho^{-7/36}\Delta^{7/8}\,ds_2^2\\ &\quad - g^{-2}\,\rho^{1/4}\,\Delta^{-1/8}\Big(X_0^{-1}d\mu_0^2 + \sum_a X_a^{-1}\Big(d\mu_a^2 + \mu_a^2\,(d\sigma_a + g\,A^a)^2\Big)\Big) \end{aligned} \tag{4.3.29}$$

the dilaton and two-form field strength

$$\phi = \frac{1}{3}\log\Big(\rho^{-7/4}\Delta^{-9/8}\Big)\,, \tag{4.3.30}$$

$$\begin{aligned} F &= \Big(2\,\rho^{5/9}\,g\sum_{\alpha=0}^{4}\left(X_\alpha^2\mu_\alpha^2 - \Delta X_\alpha\right) + \rho^{5/9}\,g\,\Delta\,X_0\Big)\,\varepsilon_2\\ &\quad + \frac{\rho^{13/9}}{2g^2}\sum_a X_a^{-2}d(\mu_a^2)\wedge(d\sigma_a + gA^a)\,(*_2\,F^a) + \frac{\rho}{2g}\sum_{\alpha=0}^{4}X_\alpha^{-1}*_2\,dX_\alpha\wedge d(\mu_\alpha^2)\,, \end{aligned}$$

realize an embedding of the whole dilaton sector introduced in page 73.

Last step: perform the complete embedding computation

To prove the previous claim is a hard task, since it implies to solve all the ten-dimensional equations of motion (4.3.8)–(4.3.12) (including Einstein' equations) with ansatz (4.3.29)–(4.3.30), using the two-dimensional field equations derived from (4.2.23). This is what we have done after lengthy calculations using the software: Mathematica. Thus, the claim is true and we have an explicit embedding of the dilaton sector. Consequently, the Cartan truncation of the $SO(9)$ maximal supergravity in two dimensions, once restricted to the dilaton sector, is consistent.

4.4 Applications

4.4.1 Embedding the domain-wall

As an application, we would like to embed the bosonic two-dimensional domain-wall solution (4.2.34) into eleven dimensions. First the "ansatz$_{10}$": (4.3.29)–(4.3.30) is used to go from 2d to 10d

$$\begin{cases} ds_2^2 = r^7 \mathrm{dt}^2 - \mathrm{dr}^2 \\ \rho = (gr)^{9/2} \\ X^a = 1 \end{cases} \xrightarrow{\text{ansatz}_{10}} \begin{cases} ds_{10}^2 = (gr)^{-7/8}\big(r^7\mathrm{dt}^2 - (\mathrm{dr}^2 + r^2\, d\Omega_8^2)\big) \\ \phi = -\dfrac{21}{8}\ln(gr) \\ F = d\left(g^{7/2}\, r^7 \mathrm{dt}\right) \end{cases} \tag{4.4.1}$$

The ten-dimensional metric given by

$$ds_{10}^2 = (gr)^{-7/8} r^2 \big(r^5 \mathrm{dt}^2 - \frac{\mathrm{dr}^2}{r^2} - d\Omega_8^2\big) \tag{4.4.2}$$

describes a warped $AdS_2 \times S^8$ geometry. Furthermore, if we go to the string frame by rescaling the metric (see Appendix A) and redefining the dilaton and the field strength

$$g_{s\,\mu\nu} \equiv e^{\phi}\, g_{\mu\nu}\,, \qquad \tilde{\phi} = 2\,\phi\,, \qquad \tilde{F} \equiv \frac{1}{g_s} F\,, \qquad g_s \equiv \text{string coupling constant}\,, \tag{4.4.3}$$

the type IIA action associated to the Lagrangian (4.3.6) reads

$$S_{\mathrm{IIA}} = -\frac{1}{4}\int d^{10}x\, e_{10}\Big(e^{-2\tilde{\phi}}\big(R + 4(\partial\tilde{\phi})^2\big) + \frac{g_s^2}{4}\,\tilde{F}^2\Big)\,. \tag{4.4.4}$$

Following this parametrization, we introduce the D0-brane charge Q and radius r_0:

$$Q \equiv l_s^7\, g_s\, N = r_0^7\,, \tag{4.4.5}$$

where l_s is the string length and N is a positive integer. After performing the redefinition

$$g \to \frac{1}{r_0}\,, \qquad t \to \frac{t}{r_0^{7/2}}\,, \tag{4.4.6}$$

the solution (4.4.1) becomes

$$\begin{aligned} ds_{10}^2 &= \big(\frac{r}{r_0}\big)^{7/2}\mathrm{dt}^2 - \big(\frac{r}{r_0}\big)^{-7/2}(\mathrm{dr}^2 + r^2\, d\Omega_8^2) \\ \tilde{\phi} &= -\frac{21}{4}\ln\big(\frac{r}{r_0}\big) \\ \tilde{F} &= d\Big(g_s^{-1}\,\big(\frac{r}{r_0}\big)^7 \wedge \mathrm{dt}\Big)\,. \end{aligned} \tag{4.4.7}$$

This corresponds to the limit ($r << r_0$) of

$$\begin{aligned} ds_{10}^2 &= \Big(1 + \big(\frac{r_0}{r}\big)^7\Big)^{-\frac{1}{2}}\mathrm{dt}^2 - \Big(1 + \big(\frac{r_0}{r}\big)^7\Big)^{\frac{1}{2}}(\mathrm{dr}^2 + r^2\, d\Omega_8^2)\,, \\ \tilde{\phi} &= \frac{3}{4}\ln\Big(1 + \big(\frac{r_0}{r}\big)^7\Big)\,, \\ \tilde{B} &= g_s^{-1}\Big[\Big(1 + \big(\frac{r_0}{r}\big)^7\Big)^{-1} - 1\Big]\,, \end{aligned} \tag{4.4.8}$$

with

$$\tilde{F} = d\tilde{B}\,. \tag{4.4.9}$$

It describes a probe D0-brane feeling the influence of a stack of N source D0-branes at radial distance r, see [79], [126] and [127]. Having discussed the ten-dimensional solution (4.4.1), we are now prepared to embed it in eleven dimensions. According to (4.3.3), the eleven dimensional metric is given by

$$ds_{11}^2 = e^{-\frac{1}{3}\phi} ds_{10}^2 - e^{\frac{8}{3}\phi}\big(B + dz\big)^2\,. \tag{4.4.10}$$

So the embedding follows

$$\begin{cases} ds_{10}^2 = (gr)^{-7/8}\big(r^7 \mathrm{d}t^2 - (\mathrm{d}r^2 + r^2\, d\Omega_8^2)\big) \\ \phi = -\dfrac{21}{8}\ln(gr) \\ F = d\left(g^{7/2}\, r^7 \wedge \mathrm{d}t\right) \end{cases} \xrightarrow{\text{ansatz}_{11}} \begin{cases} ds_{11}^2 = -\,2g^{-7/2} dt dz - \big(gr\big)^{-7} dz^2 \\ \qquad\quad - \big(dr^2 + r^2 d\Omega_8^2\big) \end{cases} \tag{4.4.11}$$

with,

$$dt\, dz \equiv \frac{1}{2}(dt \otimes dz + dz \otimes dt)\,. \tag{4.4.12}$$

After making successively (and from left to right) the change of variables,

$$t \to g^{7/2}\, t\,, \qquad z \to -z\,, \qquad x^+ = \frac{t+z}{\sqrt{2}}\,, \tag{4.4.13}$$

$$r \to g^{-1}\, r\,, \qquad z \to t - z\,, \qquad x^- = \frac{t-z}{\sqrt{2}}\,, \tag{4.4.14}$$

the eleven dimensional solution takes the simple form

$$ds_{11}^2 = 2dx^+\, dx^- + 2(1 - r^{-7})(dx^-)^2 - \frac{1}{g^2}(dr^2 + r^2\, d\Omega_8^2)\,. \tag{4.4.15}$$

This is a pp-wave in eleven dimensions, see [128] and [129], and [130].

4.4.2 Application to the Rotating D0-brane

The D0-brane solution

The large brane charge limit of the rotating 0-brane [122] yields a ten-dimensional solution of the equations of motion (4.3.8) that falls into the parametrization (4.3.29)–(4.3.30) where the two-dimensional fields are given by

$$\begin{aligned} ds_2^2 &= (gr)^7 h(r)^{-7/9} f(r)\, dt^2 - h(r)^{2/9} f(r)^{-1}\, dr^2\,, \\ A^a(r) &= \frac{1 - H_a(r)}{l_a}\,\sqrt{2mg^5}\; dt\,, \\ \rho(r) &= (gr)^{9/2}\, h(r)^{-1/2}\,, \\ X_a(r) &= h(r)^{-2/9}\, H_a(r)\,, \end{aligned} \tag{4.4.16}$$

with free constants g, m, l_a , and the functions

$$h(r) \equiv \prod_a H_a(r)\,, \qquad H_a(r) \equiv \left(1 + \frac{l_a^2}{r^2}\right)^{-1}\,, \qquad f(r) \equiv 1 - \frac{2m\, h(r)}{r^7}\,. \tag{4.4.17}$$

The ansatz (4.4.16) is a solution of the two-dimensional equations of motion, but according to the Killing spinor equations obtained from the $SO(9)$ supersymmetry variations (3.4.16) in the $U(1)^4$ truncation, it breaks all supersymmetries. From a two-dimensional point of view, the full metric given in (4.4.16) describes a "domain-wall black hole" whose structure will be understood below through particular cases. The associated curvature approaches (4.2.34) for $(r \to +\infty)$:

$$R = \frac{35}{2\,r^2} + \mathcal{O}\left(r^{-23/9}\right) , \tag{4.4.18}$$

whereas at $r = 0$, it behaves like

$$R = -\frac{7}{6}\,r^{-34/9}\prod_a (l_a)^{4/9} + \mathcal{O}\left(r^{-26/9}\right) . \tag{4.4.19}$$

Case where $m = 0 = l_a$ Notice that in the limit $(m, l_a) \to 0$, the half-supersymmetric domain-wall solution (4.2.34) is recovered.

Massless case Moreover, it follows that in the massless limit $m \to 0$, with arbitrary angular momenta l_a:

$$\begin{aligned} ds_2^2 &= (gr)^7 h(r)^{-7/9}\,dt^2 - h(r)^{2/9}\,dr^2\,, \\ A^a(r) &= 0\,, \\ \rho(r) &= (gr)^{9/2}\,h(r)^{-1/2}\,, \\ X_a(r) &= h(r)^{-2/9}\,H_a(r)\,. \end{aligned} \tag{4.4.20}$$

This ansatz is also a solution of the two-dimensional BPS equations and preserves half of the supersymmetries.

Non-rotating case Let us study now the case of vanishing momenta $l_a = 0$. The ansatz (4.4.16) reduces to

$$\begin{aligned} ds_2^2 &= \rho^{14/9} f(r)\,dt^2 - f(r)^{-1}\,dr^2\,, \\ A^a(r) &= 0\,, \\ \rho(r) &= (gr)^{9/2}\,, \\ X_a(r) &= 1\,, \end{aligned} \tag{4.4.21}$$

with

$$f(r) = 1 - \frac{2m}{r^7}\,. \tag{4.4.22}$$

which also breaks all supersymmetry. Once plugged into the ten-dimensional metric of the non-linear Kaluza-Klein ansatz (4.3.29),

$$ds_{10}^2 = \rho(r)^{-7/36}\left(\rho(r)^{14/9} f(r)\,dt^2 - f(r)^{-1}\,dr^2 - r^2\,d\Omega_8^2\right) . \tag{4.4.23}$$

The solution corresponds to a ten-dimensional Schwarzschild black hole [131] coupled to a non-constant dilaton. Thus, in the general case where the physical parameters comprising the mass m and four angular momenta l_a are non-vanishing, the solution (4.4.16) is called a domain-wall black hole. In the following, the occurrence of singularities will be studied and when it is needed horizons will be characterized.

Singularity and Horizon in the rotating case

Singularity In this section the possibility for the metric (4.4.16) to be singular is examined. Non-zero angular momenta l_a will be considered for simplicity, even if the analysis may be performed in the case where some momenta (but not all) are put to zero. The only contribution in the metric that can lead to a singularity is the function $f(r)$. Indeed, $h(r) \equiv \prod_a H_a(r)$ where the behavior of the $H_a(r)$ is depicted in Figure 4.1. This is

r	0		$+\infty$
$H'_a(r)$	0	$+$	0
$H_a(r)$	0	$\nearrow$	1

Figure 4.1: Behavior of H_a

motivated by the following computations

$$
\begin{aligned}
&H_a(r) = \Big(1 + \frac{l_a^2}{r^2}\Big)^{-1}, \quad H'_a(r) = \frac{2l_a^2}{r^3}\Big(1 + \frac{l_a^2}{r^2}\Big)^{-2} > 0 \quad \forall\, r \in]0, +\infty[\,, \\
&H_a(r) \underset{r\to 0}{\sim} l_a^{-2}\, r^2 \underset{r\to 0}{\to} 0\,, \quad H_a(r) \underset{r\to +\infty}{\sim} 1 \\
&H'_a(r) \underset{r\to 0}{\sim} 2r\, l_a^{-2} \underset{r\to 0}{\to} 0\,, \quad H'_a(r) \underset{r\to +\infty}{\sim} \frac{2l_a^2}{r^3} \underset{r\to +\infty}{\to} 0\,.
\end{aligned}
\tag{4.4.24}
$$

So, the behavior of h is deduced from the following computations

r	0		$+\infty$
$h'(r)$		$+$	
$h(r)$	0	$\nearrow$	1

Figure 4.2: Behavior of h

$$
\begin{aligned}
&h(r) > 0\,, \qquad h'(r) = \frac{2h(r)}{r}\Big(4 - \sum_a H_a(r)\Big) > 0\,, \qquad \forall\, r \in]0, +\infty[\\
&h(r) \underset{r\to 0}{\sim} \big(\prod_a l_a^{-2}\big)\, r^8 \underset{r\to 0}{\to} 0\,, \qquad h(r) \underset{r\to +\infty}{\sim} 1\,.
\end{aligned}
\tag{4.4.25}
$$

On the contrary,

$$
f(r) \equiv 1 - \frac{2m\, h(r)}{r^7}\,, \qquad f'(r) = 4m \frac{h(r)}{r^8}\Big(\sum_a H_a(r) - \frac{1}{2}\Big)\,.
\tag{4.4.26}
$$

Since $\Big(\sum_a H_a(r) - \frac{1}{2}\Big)$ is strictly increasing in $]0, +\infty[$ and starts from a negative value and ends at a positive one, there is exactly one root of $f'(r) = 0$ called $\tilde{r}$. The behavior of $f(r)$ can then be derived and is summarized in Figure 4.3.

r	0		$\tilde{r}$		$+\infty$
$f'(r)$		$-$	0	$+$	0
$f(r)$	1	$\searrow$	$f(\tilde{r})$	$\nearrow$	1

Figure 4.3: Behavior of f

$$f'(0) \underset{r\to 0}{\longrightarrow} \Big(- 2m \prod_a l_a^{-2} \Big) < 0 \tag{4.4.27}$$

and the sign of $f(\tilde{r})$ depends on the sign of

$$1 - 2m \, \frac{h(\tilde{r})}{\tilde{r}^7} \, . \tag{4.4.28}$$

Since $h(\tilde{r})$ and $\tilde{r}$ depend only on the angular momenta l_a, and because the mass m is an independent positive constant, it is possible to choose the mass and the angular momenta so that $f(\tilde{r}) = 0$ (or $f(\tilde{r}) < 0$). In this case, $f(r) = 0$ has exactly one (two) root(s) which can play the role of a singularity. On the contrary, if the physical parameters are chosen so that $f(\tilde{r}) > 0$, no singularity occurs. In the following, m and the l_a are chosen in such a way that a singularity of the metric is present. The larger root will be named r_0.

Horizon Unless otherwise mentioned, the discussion will be held in two-dimensional space-time. The Ricci scalar is well defined on $]0, +\infty[$ but diverge at $r = 0$. In particular, it is well defined at $r = r_0$, as well as the volume form

$$\varepsilon_{2d} = (gr)^7 h(r)^{-5/9} \, dt \wedge dr \, . \tag{4.4.29}$$

Nonetheless, in this system of coordinates, the light cones close up at $r = r_0$ since

$$\frac{dt}{dr} = \pm \, (gr)^{7/2} h(r)^{1/2} f(r)^{-1} \underset{r\to (r_0)^+}{\longrightarrow} \pm\infty \, . \tag{4.4.30}$$

This is why in analogy to higher dimensions, the point $r = r_0$ will be considered as the "horizon" of the singularity. As a natural question one may wonder what is the behavior of the two-dimensional fields near the horizon. An answer is provided in the next section.

The near-horizon limit

In this section the behavior of the two-dimensional fields near the horizon will be described. To this end, the horizon will be explored by expanding the coordinates according to

$$r \to r_0 + \epsilon \, r \, , \qquad t \to \rho_0^{-7/9} h_0^{-1/9} \, \frac{t}{\epsilon} \, , \tag{4.4.31}$$

where

$$\rho_0 \equiv (g r_0)^{9/2} h_0^{-1/2} \, , \quad h_0 \equiv \prod_{a=1}^{4} H_{a0} \, , \quad H_{a0} \equiv \Big(1 + \frac{l_a^2}{r_0^2} \Big)^{-1} . \tag{4.4.32}$$

Then ϵ will be sent to zero. In this way, the two-dimensional volume form remains well-defined while we are taking the limit $\epsilon \to 0$. The resulting fields describe a near-horizon AdS_2 configuration

$$ds^2 = f_0\, dt^2 - \frac{1}{f_0}\, dr^2\,, \qquad F^a_{tr} = 2g\, \rho_0^{-4/9} \left(\frac{H^2_{a0} \sqrt{H^{-1}_{a0} - 1}}{h_0^{4/9}} \right)\,, \tag{4.4.33}$$

with

$$f_0 \equiv g^2 \frac{\left(1 + 8 \sum_{a<b} H_{a0} H_{b0}\right)}{2\, \rho_0^{4/9}\, h_0^{4/9}}\, r^2\,, \tag{4.4.34}$$

provided the constants H_{a0} satisfy the following further condition

$$\sum_{a=1}^{4} H_{a0} \;=\; 1/2\,. \tag{4.4.35}$$

This is exactly the $C = 0$ case of the solution (4.2.36) found above. According to the embedding (4.3.29)–(4.3.30), this solution corresponds to a ten-dimensional warped product geometry $\mathrm{AdS}_2 \times \mathcal{M}_8$.

4.5 $\mathrm{AdS}_2 \times \mathcal{M}_8$ solutions with non-vanishing axions

Looking for particular solutions of the field equations derived from (4.2.14) we restricted our study to the dilaton sector where the axions were put to zero. This led to the construction of a Kaluza-Klein ansatz for the embedding of the two-dimensional theory into type IIA supergravity. In addition, the generalization of the construction to non-vanishing axions would be of primary interest. As seen before, particular solutions of the two-dimensional equations provide important insights for finding the embedding. Thus one may wonder whether it is possible to find two-dimensional solutions with non-vanishing axions.

Owing to the form of the potential (4.2.19), finding a general solution seems difficult. As a natural idea we propose to generalize the AdS_2 solution (4.2.36) to the case of non-vanishing axions. Let us start with an AdS ansatz

$$ds_2^2 = f(r)\, \mathrm{d}t^2 - \frac{1}{f(r)}\, \mathrm{d}r^2 \tag{4.5.1}$$

and constant non-zero dilatons and axions parametrized by

$$\{\rho, H_a\} \quad \text{and} \quad \{\eta_a\}\,. \tag{4.5.2}$$

In this truncation, the two-dimensional equations of motion are listed below: the vector fields equations imply that the auxiliary fields are constant

$$\partial_\mu\, y^a = 0. \tag{4.5.3}$$

Moreover, the auxiliary field equation determines the field strengths

$$\epsilon^{\mu\nu} F^a_{\mu\nu} = -8\rho^{-1} g^{-1}\, e\, \frac{\partial V_{\mathrm{pot}}}{\partial y^a}\,. \tag{4.5.4}$$

Then, the dilaton ρ equation determines the AdS radius

$$R = f''(r) = 4\rho^{-1}\Big(\frac{\partial V_{\rm pot}}{\partial y^a} - \frac{5}{9}\,V_{\rm pot}\Big) = \text{constant}\,. \tag{4.5.5}$$

Besides, the traceless part of Einstein equations is identically satisfied and the trace part leads to

$$V_{\rm pot} = 0\,. \tag{4.5.6}$$

Finally, the scalar fields equations generate other constraints on the potential, since they are supposed to be constant

$$\frac{\partial V_{\rm pot}}{\partial H_a} = 0 = \frac{\partial V_{\rm pot}}{\partial \eta_a}\,. \tag{4.5.7}$$

In summary, the unknown of the problem are $\{\rho, H_a, y_a, \eta_a\}$ and they may be determined by the equations

$$V_{\rm pot} = 0\,, \qquad \frac{\partial V_{\rm pot}}{\partial H_a} = 0\,, \qquad \frac{\partial V_{\rm pot}}{\partial \eta_a} = 0\,. \tag{4.5.8}$$

Actually, the dilaton ρ is not constrained by these equations, so it will be considered as a free parameter and we are left with twelve unknown parameters $\{H_a, y_a, \eta_a\}$ and nine equations (4.5.8). Assuming that none of the parameters are zero, the structure of the potential

$$\begin{aligned} V_{\rm pot} = \frac{g^2}{8}\,\rho^{5/9} H_0^{-4/9}\Big[&1 - 8\sum_{a<b} H_a H_b - 4\sum_a H_a + \sum_a (1-4H_a)\,\eta_a^2 + \sum_{a<b}\eta_a^2\eta_b^2 \\ &+ \sum_a \eta_a^{-2}(y^a H_a\eta_a + \eta_0)^2 + \big(\eta_0 + \sum_a y^a H_a \eta_a\big)^2\Big]\,, \end{aligned} \tag{4.5.9}$$

can be simplified by making the following change of variables

$$y_i \to \big(Y^i - 1\big)\, H_i^{-1}\eta_i^{-1}\eta_P\,. \tag{4.5.10}$$

Then, the potential is given by

$$\begin{aligned} V = H_0^{-4/9}\Bigg[&1 - 8\sum_{a<b} H_a H_b - 4\sum_a H_a + \sum_a (1-4H_a)\,\eta_a^2 + \sum_{a<b}\eta_a^2\eta_b^2 \\ &+ \eta_0^2\Big(\sum_a \eta_a^{-2}\, Y^{a2} + \big(\sum_a Y^a - 3\big)^2\Big)\Bigg]\,, \end{aligned} \tag{4.5.11}$$

and the equation $\frac{\partial V_{\rm pot}}{\partial \eta_a} = 0$ can be integrated for H_a

$$\begin{aligned} \frac{\partial V_{\rm pot}}{\partial \eta_a} &= 0 = 2\eta_a(1-4H_a) + 2\eta_a\sum_{b\neq a}\eta_b^2 + 2\eta_a^{-1}\eta_0^2\Big(\sum_{b\neq a}\eta_b^{-2}Y^b + Y^a\big(\sum_b Y^b - 3\big)\Big) \\ 4H_a &= 1 + \sum_{b\neq a}\eta_b^2 + \eta_a^{-2}\eta_0^2\Big(\sum_{b\neq a}\eta_b^{-2}Y^b + Y^a\big(\sum_b Y^b - 3\big)\Big)\,. \end{aligned} \tag{4.5.12}$$

This leaves us with five equations for eight unknowns. The counting suggest that there are several families of solutions, however the algebraic equations (4.5.8) are too complicated to allow for finding the general explicit solution.

4.5.1 An Explicit solution

As an example we give an explicit solution found by further truncating the system

$$H_1 \equiv H_3\,, \quad H_2 \equiv H_4\,, \quad y_1 \equiv y_3\,, \quad y_2 \equiv y_4\,, \quad \eta_1 \equiv \eta_3\,, \quad \eta_2 \equiv \eta_4\,. \tag{4.5.13}$$

In this truncation, the equations reduce to quadratic equations and allow for the explicit solution

$$\begin{aligned} H_1 &= \frac{1}{128}(43 - 5\sqrt{33})\,, & H_2 &= \frac{1}{64}(25 + 9\sqrt{33})\,, \\ (y_1)^2 &= 12\,\big(6 + \sqrt{33}\big)\,, & (y_2)^2 &= 2\,\big(-1 + \sqrt{33}\big)\,, \\ (\eta_1)^2 &= \frac{1}{8}(9 + \sqrt{33})\,, & (\eta_2)^2 &= \frac{1}{16}(1 + \sqrt{33})\,. \end{aligned} \tag{4.5.14}$$

with Ricci scalar given by

$$R = \frac{2^{2/3}\,3\,\big(3815 + 759\sqrt{33}\big)}{\Big(-205 + 131\sqrt{33}\Big)^{8/9}}\,\frac{g^2}{\rho^{4/9}} \simeq 143.27\,\frac{g^2}{\rho^{4/9}}\,. \tag{4.5.15}$$

This is a solution describing an AdS geometry with non vanishing axions. It constitutes a first step towards more general solutions which may help to find the Kaluza-Klein ansatz with non-vanishing axions for the embedding in type IIA supergravity.

4.6 Summary

This chapter was dedicated to the study of the bosonic Cartan truncation of the $SO(9)$ gauged maximal supergravity in two dimensions. It has been shown that the dilaton sub-sector can be consistently embedded in type IIA supergravity. Consequently, an explicit uplift to ten and eleven dimensions is at hand, and it may be used to identify the higher-dimensional origin of particular solutions of the two-dimensional supergravity. As examples we applied the uplift to the near-horizon of the D0-brane, and also to particular AdS_2 solutions. They could be interpreted as limits of the rotating D0-brane solution of type IIA supergravity.

We have already mentioned that the domain-wall solution is important in the DW/QFT correspondence. Indeed, it is the background on the gravity side around which the excitations are encoded into a dual one dimensional quantum field theory. The next chapter deals with the holography of this solution and the computation of correlation functions on the gravity side.

Chapter 5

Holography

5.1 Introduction

5.1.1 The AdS/CFT correspondence

The properties of the D3-brane solution in supergravity and superstring theory led Maldacena to postulate a correspondence between type IIB superstring theory on an $\mathrm{AdS}_5 \times S^5$ background and $\mathcal{N} = 4$, $D = 4$ super Yang-Mills theory, see [132] and [133] for a review. More precisely, the conjecture states the equivalence between

- Type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$, with string coupling constant g_s, where: AdS_5 and S^5 have the same radius L and the self-dual 5-form F_5 has an integer flux over the five-sphere
$$N = \int_{S^5} F_5 \,. \tag{5.1.1}$$
- $\mathcal{N} = 4\,,\ D = 4$ super Yang-Mills theory with gauge group $SU(N)$ and Yang-Mills coupling constant g_{YM}.

Then the following identification is done

$$g_s = (g_{\mathrm{YM}})^2\,, \qquad L^4 = 4\pi g_s N(\alpha')^2\,, \tag{5.1.2}$$

where α' is the square of the String length: $l_s = \sqrt{\alpha'}$. This conjecture has three forms that differ in strength. Concerning the gravity side: the strongest one relates the full quantum string theory on $\mathrm{AdS}_5 \times S^5$ to the full quantum $\mathcal{N} = 4$, $D = 4$ super Yang-Mills theory with gauge group $SU(N)$. The second and weaker one deals with the classical string theory approximation ($g_s << 1$) and is dual to the super Yang-Mills theory in the 't Hooft limit, which corresponds to a topological expansion of planar Feynman diagrams [71]. Finally, on the gravity side, the last and weakest form concerns the classical supergravity approximation ($\alpha' << 1$) and corresponds to the super Yang-Mills theory after taking successively the 't Hooft limit and the large 't Hooft coupling constant limit. In this regime, the Quantum field theory is considered at strong coupling, and perturbation theory is not applicable. The three levels of the conjecture are summarized in Table 5.1.

Symmetries

As a hint for such a correspondence, one may consider that the symmetries of the type IIB theory on $\mathrm{AdS}_5 \times S^5$ and the $\mathcal{N} = 4$, $D = 4$ super Yang-Mills do match. Indeed, let us consider the bosonic global symmetries.

Quantum type IIB Superstring on $\mathrm{AdS}_5 \times S^5$ $L^4 = 4\pi g_s N(\alpha')^2$	$\Leftrightarrow$	Quantum $\mathcal{N}=4$, $D=4$ SYM with gauge group $SU(N)$ $g_{\mathrm{YM}} = \sqrt{g_s}$
Classical type IIB Superstring $g_s << 1$ Weak coupling regime	$\Leftrightarrow$	't Hooft limit of $SU(N)$ SYM $N \to \infty$, $\lambda = (g_{\mathrm{YM}})^2 N$ fixed. Topological expansion
Classical type IIB Supergravity $\alpha' << 1$ Supergravity approximation	$\Leftrightarrow$	't Hooft and Large λ limit of SYM $\lambda \to \infty$ Strong coupling regime

Table 5.1: The three levels of the AdS/CFT correspondence.

- On one hand, the isometry group of $\mathrm{AdS}_5 \times S^5$ spacetime is $SO(2,4) \times SO(6)$, and one can show that $SO(2,4)$ acts on the boundary of AdS_5 as the conformal group on a four dimensional Minkowski spacetime [132].
- On another hand, $\mathcal{N}=4$, $D=4$ super Yang-Mills has a global superconformal $SU(2,2|4)$ symmetry group, whose maximal bosonic subgroup is isomorphic to

$$SO(2,4) \times SO(6)_R \quad \sim \quad SO(2,4) \times SU(4)_R \,. \tag{5.1.3}$$

Here, the subscript R stands for the R-symmetry group of the supersymmetric theory.

Correlation Functions

Finally, let us mention that according to the correspondence, correlation functions of CFT operators on the gauge side can be generated from the gravity side. More precisely, to every single trace operators $\mathcal{O}$ on the SYM side, a boundary field $\phi_{(0)}$ can be associated in order to form a generating functional for correlation functions $\Gamma[\phi_{(0)}]$,

$$\exp\big(-\Gamma[\phi_{(0)}]\big) \equiv \langle \exp\Big(\int_{\partial H} d^4x\, \phi_{(0)}\, \mathcal{O}\Big)\rangle \,. \tag{5.1.4}$$

Here, ∂H stands for the boundary of Euclidean AdS_5, that can be viewed as

$$H = \{(r,\vec{x}) \,,\, r>0,\, \vec{x} \in \mathbb{R}^4\} \,, \tag{5.1.5}$$

with Poincaré metric given by

$$ds^2 = \frac{1}{r^2}\big(dr^2 + (d\vec{x})^2\big) \,. \tag{5.1.6}$$

This generating functional can be computed from an action for type IIB superstring on $\mathrm{AdS}_5 \times S^5$ by the following prescription. Let us consider the correspondence in the supergravity approximation for instance. The action is the type IIB supergravity compactified on the five sphere, where all the field are classified with respect to S^5 spherical harmonics. The excitations around the AdS_5 vacuum are described by an effective theory, the maximal $SO(6)$ gauge supergravity in five dimensions [89]. Fluctuations around the AdS background are encoded by fields ϕ satisfying a particular boundary condition. For example, massive scalar fields are shown to couple to CFT operators $\mathcal{O}_\Delta$ with conformal dimension Δ given by the largest root of

$$\Delta\big(\Delta-4\big) = m^2 \,. \tag{5.1.7}$$

Then, the solution of the equations of motion which is not divergent in the interior region (the "bulk") corresponds to a boundary condition

$$\phi(r,\vec{x}) \underset{r\to 0}{\sim} r^{\Delta-4}\,\phi_{(0)}(\vec{x})\,. \tag{5.1.8}$$

The associated boundary field is precisely the one that enters the generating functional in (5.1.4). Moreover, and this is a general fact valid for every field of the gravity theory, the generating functional Γ is given by the supergravity action evaluated on-shell

$$\Gamma[\phi_{(0)}] = S_{\text{on-shell}}[\phi_{(0)}]\,, \tag{5.1.9}$$

where the bulk fields are expressed in terms of the boundary data $\phi_{(0)}(\vec{x})$ by means of a boundary-to-bulk propagator. Then, n-points correlation functions are derived formally by functional derivative with respect to the boundary fields,

$$\langle \mathcal{O}(\vec{x_1})\ldots\mathcal{O}(\vec{x_n})\rangle = \left.\frac{\delta^{(n)}S_{\text{on-shell}}}{\delta\phi_{(0)}(\vec{x_1})\ldots\delta\phi_{(0)}(\vec{x_n})}\right|_{\phi_{(0)}=0}\,. \tag{5.1.10}$$

However, the on-shell action is in general divergent near $r=0$. Consequently, a renormalization scheme is needed. It is achieved by introducing a lower cut-off for r

$$0<\epsilon\leq r\,, \tag{5.1.11}$$

then the divergences are canceled by adding covariant counter-terms to the on-shell action. The determination of the boundary-to-bulk propagator and the renormalization of the gravity action are two important difficulties that have been dealt with Witten diagrams [72] and Holographic renormalization [78].

5.1.2 Domain-wall / QFT correspondence

The AdS/CFT correspondence has soon been extended to the non-conformal case [74], [75] and [134]. Hence Dp-brane ($p\neq 3$) solutions of type IIA and IIB superstrings are considered. In the near-horizon limit, they lead to domain-wall backgrounds instead of AdS ones. Thus, a non-trivial dilaton survives which breaks the conformal invariance of the boundary. Nonetheless, a gravity/gauge theory correspondence can be formulated and Holographic renormalization methods have been developed to compute correlation functions. The following work is dedicated to the holography of the D0-brane solution of type IIA superstring, in the classical supergravity approximation where the effective theory is given by the $SO(9)$ gauged maximal supergravity in two dimensions. It accounts for the spherical Kaluza-Klein modes of type IIA supergravity on the warped $\text{AdS}_2\times S^8$ spacetime, arising as the near-horizon geometry of the D0-brane. The corresponding two

Brane	Vacuum configuration	Gauged SUGRA
D0	$\text{AdS}_2\times S^8$	$D=2\quad SO(9)$

Table 5.2: D0 Brane and $SO(9)$ gauged supergravity

dimensional vacuum is a domain-wall solution which preserves sixteen supercharges [84].

$$\begin{array}{ccc} D=10 & & D=2\quad SO(9) \\ \text{D0-brane} & \xrightarrow{S^8\text{ reduction}} & \text{Half supersymmetric Domain Wall} \end{array} \tag{5.1.12}$$

Supergravity			Super Yang-Mills
$D=11$	type IIA	$D=2\,,\ \mathcal{N}=16\,,\ SO(9)$	$D=1\,,\ \mathcal{N}=16\,,\ U(N)$
pp-wave	D0-brane	domain wall ($\frac{1}{2}$ BPS)	BFSS model

Table 5.3: DW/QFT correspondence and the BFSS model

The gravity theory is conjectured to be dual to a one dimensional quantum field theory: the BFSS Matrix model proposed in [76], as was discussed in [126] and [130].

Computations of two-points functions on both sides have already been done. We recover these results using Holographic renormalization methods. The second part of the chapter concerns the holography of gravity side excitations that live on a half-supersymmetric domain-wall background which breaks the $SO(9)$ invariance down to $SO(3) \times SO(6)$. The dual theory is a quantum Matrix model which shares similarity with the BMN Matrix model [135]. Two-points functions are computed for a large class of scalar excitations. After recalling some useful features about the DW/QFT correspondence, we will turn into Matrix model Holography.

Precision holography for non-conformal branes

There exists a general method based on the non-conformal gravity/gauge correspondence, that enables to compute correlation functions of operators in the BFSS model from the gravity side. It is called "holographic renormalization" and has been developed in [78] and applied in [77] for non-conformal branes. We will follow this procedure to compute the two-points correlation functions.

The key ingredient is the supergravity action $S_{\rm SG}$ evaluated on an asymptotically AdS background with a dilaton. The fluctuations of a "bulk" field Φ around the gravity background take some value on the AdS boundary: $\Phi_{(0)}$. This value acts as a source for correlation functions of an associated "boundary" Operator $\mathcal{O}_\Phi$ in the dual QFT. The link between these correlation functions is provided by the postulated equality

$$\int_{\Phi|_{\rm boundary}=\Phi_{(0)}} D\Phi\ \exp\Big(-S_{\rm SG}[\Phi]\Big) = \langle \exp\Big(-\int_{\partial AAdS}\Phi_{(0)}\,\mathcal{O}_\Phi\Big)\rangle_{\rm QFT}\,, \tag{5.1.13}$$

where the expectation value is over the QFT path integral. Thus, in the saddle point approximation, the equality becomes

$$S_{\rm SG\ onshell}[\Phi_{(0)}] = -\Gamma_{\rm QFT}[\Phi_{(0)}]\,, \tag{5.1.14}$$

where $\Gamma_{\rm QFT}$ is the generating function of QFT connected graphs. The on-shell supergravity action shares two characteristics:

- It depends on prescribed boundary conditions of the on-shell fields. This implies to solve a Dirichlet problem associated to the field equations.

- Once evaluated on-shell, the action is in general divergent. To cure this problem, a renormalization process based on the addition of covariant counter-terms will be employed.

Indeed, another key feature of Holographic renormalization is the possibility to expand the bulk field near the boundary with respect to the radial coordinate r. For example, the fact to consider an asymptotically AdS spacetime allows to use a Fefferman-Graham parametrization for the fluctuations around the metric and dilaton background

$$\begin{aligned} ds^2 &= \frac{dr^2}{4r^2} + \frac{g_{ij}(r,\vec{x})\,dx^i dx^j}{r}\,, \\ \phi(r,\vec{x}) &= \alpha\,\ln r + \kappa(r,\vec{x})\,, \end{aligned} \tag{5.1.15}$$

where α is the dilaton power, and the functions $g_{ij}(r,\vec{x})$ and $\kappa(r,\vec{x})$ can be expanded near the boundary $r = 0$ as follows

$$\begin{aligned} g_{ij}(r,\vec{x}) &= g_{ij\,(0)}(\vec{x}) + r\,g_{ij\,(1)}(\vec{x}) + \ldots \\ \kappa(r,\vec{x}) &= \kappa_{(0)}(\vec{x}) + r\,\kappa_{(1)}(\vec{x}) + \ldots \end{aligned} \tag{5.1.16}$$

From the renormalized action, then, correlation functions are computed by functional derivation. For instance, given a classical supergravity action that depends on a scalar, a vector field and a metric $\left(\Phi\,,\ A_\mu\,, g_{\mu\nu}\right)$, the associated one-point functions are given by

$$\begin{aligned} \langle \mathcal{O}(x) \rangle &= \frac{1}{\sqrt{|g_{(0)}(x)|}}\frac{\delta S_{\text{SG ren}}}{\delta \Phi_{(0)}(x)}\,, \\ \langle J_i(x) \rangle &= \frac{1}{\sqrt{|g_{(0)}(x)|}}\frac{\delta S_{\text{SG ren}}}{\delta A^i_{(0)}(x)}\,, \\ \langle T_{ij}(x) \rangle &= \frac{2}{\sqrt{|g_{(0)}(x)|}}\frac{\delta S_{\text{SG ren}}}{\delta g^{ij}_{(0)}(x)}\,, \end{aligned} \tag{5.1.17}$$

where the $\{x^i\,,\quad i = 1,..,D-1\}$ are coordinates of the boundary and $\{\mathcal{O}\,,\ J_i\,,\ T_{ij}\}$ are operators of the QFT. Then, the n-points functions follow by further derivation

$$\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_n) \rangle = \left(\frac{1}{\sqrt{|g_{(0)}(x)|}}\frac{\delta^{(n-1)} S_{\text{SG ren}}}{\delta \Phi_{(0)}(x_1)\ldots\delta \Phi_{(0)}(x_n)}\right)\Bigg|_{\Phi_{(0)}=0\,,\,A_{i(0)}=0\,,\,g_{ij(0)}=0}\,. \tag{5.1.18}$$

In the following, after presenting the gauge theory side of the DW/QFT correspondence we are interested in, Holographic renormalization will be applied to the gravity action. We will start from the two-dimensional supergravity action that supports a domain-wall solution. Then we will derive the action in a frame where the domain-wall solution translates into an AdS metric coupled to a dilaton. In this frame we will perform, the bulk to boundary analysis, going from the renormalization of the on-shell action to the computation of two-points correlation functions.

5.2 BFSS model holography

Gauge theory side: BFSS model

The BFSS model arises from the description of D0-branes. Indeed, it has been proposed in [76], that the $(N \to +\infty)$ limit of a $\mathcal{N} = 16$ supersymmetric matrix quantum mechanics coming from the dimensional reduction of the $(\mathcal{N} = 1\,,\ D = 1+9)$ supersymmetric $U(N)$ Yang-Mills to $(D = 1+0)$ is equivalent to M-theory. The resulting supersymmetric quantum mechanics is a low energy effective theory that aims at describing the dynamics of D0-branes. The action first obtained in [136] is given by

$$S = \int dt\, \mathrm{tr}\Big(D_t X^i\, D_t X^i + 2\Psi^T D_t \Psi - \frac{1}{2}[X^i, X^j]^2 - 2\Psi^T \gamma_i [\Psi, X^i]\Big) \qquad (5.2.1)$$

and is parametrized by

- 9 $(N \times N)$ matrices $X^i_{a,b}$
- 16 $(N \times N)$ fermionic super-partners $\Psi_{a,b}$ which transform as spinors under the $SO(9)$ group of transverse rotations
- Vector fields A_t that enter the covariant derivative $\big(D_t = \partial_t + iA\big)$.

Now let us focus on the dual gravity theory.

Gravity side: Two-dimensional effective action

The two-dimensional effective action that describes the dynamics of the D0-brane compactified on the eight sphere S^8, restricted to the metric and dilaton, has been found in [75], [126]. However, if we take into account all the lowest mass fluctuations around the D0-brane geometry, they are encoded in the full $SO(9)$ gauged supergravity constructed in [81]. These fluctuations will enable us to extract information about BFSS correlation functions through the holographic procedure described above.

To begin, we derive the effective action for the D0-brane dynamics. This action will allow us to describe fluctuations around the gravity sector, so let us start from the $SO(9)$ action (3.4.22) evaluated at the origin of the target space

$$\mathcal{V}_m{}^a = \delta^a_m\,, \quad \phi^{klm} = 0 = Y_{kl}\,, \qquad (5.2.2)$$

$$S = \frac{1}{4}\int d^2x\, e\,\big(\rho R - \frac{63}{2}\, g^2 \rho^{5/9}\big)\,. \qquad (5.2.3)$$

Let us make a change of variables so that the background metric is pure AdS

$$t \to \frac{2}{5}\, g^{5/2}\, t\,, \quad r \to g^{-1}\, r^{-1/5}\,, \quad g_{\mu\nu} \to g^{-2}\, \rho^{4/9}\, g_{\mu\nu}\,, \quad \rho = e^{-\frac{6}{7}\phi}\,. \qquad (5.2.4)$$

The action yields the expression of [126]

$$S = \frac{1}{4}\int d^2x\, \sqrt{|\mathrm{det} g|}\; e^{-\frac{6}{7}\phi}\,\big(R + \frac{16}{49}(\partial\phi)^2 - \frac{63}{2}\big)\,, \qquad (5.2.5)$$

where the minus sign in front of the cosmological constant comes from the fact that our signature is $(+,-)$. Since the computation of the two-points function are more easily done in Euclidean signature $(+,+)$, we will work in this signature until the end of the chapter. After a Wick rotation, the action reads

$$S = \frac{1}{4}\int d^2x\, \sqrt{|\mathrm{det} g|}\; e^{-\frac{6}{7}\phi}\,\big(R + \frac{16}{49}(\partial\phi)^2 + \frac{63}{2}\big)\,. \qquad (5.2.6)$$

5.2.1 On-shell action and Renormalization of the gravity sector

In the following, the formalism of Holographic renormalization developed in [77] and [78], will be applied to compute correlation functions associated with the gravity sector. First, the action (5.2.6) will be evaluated on-shell, on a background solution. Then, fluctuations around the background will be considered and the on-shell action will be renormalized to define a generating functional for correlation functions. Hence, one-point and two-points correlation functions will be computed and discussed.

On-shell action

Let us focus on the effective action that describes the pure gravity sector. In order to stick to the literature we will perform a further rescaling of the metric

$$g_{\mu\nu} \to \frac{4}{25}\, g_{\mu\nu}\,, \tag{5.2.7}$$

so that the two-dimensional effective action is of the form considered in [77]

$$S = \frac{1}{4}\int d^2x\,\sqrt{|\mathrm{det}g|}\,e^{\gamma\phi}\left(R+\beta(\partial\phi)^2+C\right). \tag{5.2.8}$$

The domain-wall vacuum solution has been found in (3.5.2) and the coefficients are given by

$$\begin{cases} ds^2 = \dfrac{dt^2}{r}+\dfrac{dr^2}{4r^2} \\ e^{\phi} = r^{\alpha} \end{cases} \quad \begin{cases} \gamma = -\dfrac{6}{7}\,,\ C = \dfrac{126}{25}\,, \\ \alpha = \dfrac{21}{20}\,,\ \beta = \dfrac{16}{49}\,. \end{cases} \tag{5.2.9}$$

In this frame, the background is an AdS spacetime coupled to a dilaton. With these coordinates, the boundary of AdS is located at $r = 0$. In the next step, fluctuations around the background will be considered such that the geometry remains the one described by an *asymptotically* AdS metric [78]. That is to say, a metric that can be put into the form

$$ds^2 = \frac{f(t,r)}{r}\,dt^2 + \frac{dr^2}{4r^2}\,. \tag{5.2.10}$$

As before, r is the radial coordinate from the boundary $r = 0$ of AdS spacetime and the function $f(t,r)$ has a well defined limit when $r \to 0$. The equations of motion associated to the two-dimensional effective action are given by

$$\begin{aligned} 0 &= (\nabla_\mu\partial_\nu\phi) - \frac{g_{\mu\nu}}{2}\nabla\partial\phi - \Big(\frac{\beta}{\gamma}-\gamma\Big)\Big((\partial_\mu\phi)\,(\partial_\nu\phi) - \frac{g_{\mu\nu}}{2}\,(\partial\phi)^2\Big)\,, \\ 0 &= \gamma\nabla\partial\phi + \gamma^2\big(\partial\phi\big)^2 - C\,, \\ 0 &= R - 2\frac{\beta}{\gamma}\nabla\partial\phi - \beta\,(\partial\phi)^2 + C\,. \end{aligned} \tag{5.2.11}$$

They respectively stand for: the traceless and trace part of Einstein equations, and the dilaton field equation. Owing to the fact that a global dilaton factor enters the action, the dilaton field equation can be used to straightforwardly evaluate the action on-shell

$$e^{\gamma\phi}\big(R+\beta(\partial\phi)^2+C\big) = \frac{2\beta}{\gamma}\nabla\big(e^{\gamma\phi}\partial\phi\big)\,. \tag{5.2.12}$$

Eventually, the full on-shell action is obtained by introducing a Gibbons-Hawking term which takes into account the boundary of the spacetime background

$$\int_{\mathcal{M}} d^2x\,\sqrt{|\mathrm{det}g|}\,e^{\gamma\phi}\,R \quad\longrightarrow\quad \int_{\mathcal{M}} d^2x\,\sqrt{|\mathrm{det}g|}\,e^{\gamma\phi}\,R + \int_{\partial\mathcal{M}} ds\sqrt{h}\,e^{\gamma\phi}\,2\,K\,. \tag{5.2.13}$$

Here h is the induced metric on the boundary and K is the trace of the extrinsic curvature of the boundary that can be computed from a unit length vector n^μ normal to the boundary

$$K = \nabla_\mu n^\mu\,. \tag{5.2.14}$$

So the on shell action is given by

$$\begin{aligned} S_{\text{on shell}} &= \frac{\beta}{2\gamma}\int_{\mathcal{M}} d^2x\,\sqrt{|\det g|}\,\nabla\big(e^{\gamma\phi}\partial\phi\big) + \frac{1}{4}\int_{\partial\mathcal{M}} ds\sqrt{h}\,e^{\gamma\phi}\,2\,K \\ &= \frac{1}{2}\int_{\partial\mathcal{M}} ds\,\sqrt{h}\,e^{\gamma\phi}\,\Big(\frac{\beta}{\gamma}n^\mu\partial_\mu\phi + K\Big) \end{aligned} \tag{5.2.15}$$

where the boundary is located at $r = 0$. In general, because the integral diverges when $r \to 0$, an infinitesimal parameter ϵ will be introduced in order to control the divergences

$$S_{\text{on shell}} = \frac{1}{2}\int_{\partial\text{AAdS}\,,\,r=\epsilon} dt\,\sqrt{h}\,e^{\gamma\phi}\,\Big(\frac{\beta}{\gamma}n^\mu\partial_\mu\phi + K\Big)\,. \tag{5.2.16}$$

This action has to be evaluated on functions that parametrize the fluctuations around the background and that satisfy the equations of motion. This is the point of the next section.

Fluctuations

By fixing the diffeomorphism invariance, the most general fluctuations can be encoded in two functions depending on space-time coordinates :

$$\begin{aligned} ds^2 &= \frac{f(t,r)}{r}dt^2 + \frac{1}{4r^2}dr^2\,, \\ \phi &= \alpha\ln r + \frac{\kappa(t,r)}{\gamma}\,. \end{aligned} \tag{5.2.17}$$

Moreover, we will consider fluctuations which goes asymptotically to an AdS spacetime (coupled to a dilaton), thus the following near-boundary conditions should hold:

$$\begin{aligned} f(t,r) &= f_{(0)}(t) + \underset{r\to 0}{o}(1)\,, \\ \kappa(t,r) &= \kappa_{(0)}(t) + \underset{r\to 0}{o}(1)\,, \end{aligned} \tag{5.2.18}$$

and the fields admit a power expansion in r near $r = 0$. Then, according to the equations of motion (5.2.11), the functions $f(t,r)$ and $\kappa(t,r)$ are subject to

$$\begin{aligned} 0 &= -\frac{1}{4}\big(f^{-1}f'\big)^2 + \frac{1}{2}f^{-1}f'' + \kappa'' + \big(1-\frac{\beta}{\gamma^2}\big)\big(\kappa'\big)^2\,, \\ 0 &= \big(1-\frac{\beta}{\gamma^2}\big)\dot\kappa\kappa' + \dot\kappa' - \frac{1}{2}f'f^{-1}\dot\kappa\,, \\ 0 &= 2\alpha\gamma f' + r\big(2f'' - f^{-1}\big(f'\big)^2\big) + \ddot\kappa - \frac{1}{2}f^{-1}\dot f\dot\kappa + \big(1-\frac{\beta}{\gamma^2}\big)\big(\dot\kappa\big)^2 - 2f\big(1-rf^{-1}f'\big)\kappa'\,, \\ 0 &= 4r\big(\kappa'' + \big(\kappa'\big)^2\big) + \big(8\alpha\gamma + 2 + 2rf^{-1}f'\big)\kappa' + f^{-1}\big(\ddot\kappa - \frac{1}{2}f^{-1}\dot f\dot\kappa + \big(\dot\kappa\big)^2\big) + 2f^{-1}f'\alpha\gamma\,, \end{aligned} \tag{5.2.19}$$

where every dot means ∂_t and every prime means ∂_r. These are non linear second order partial differential equations. Instead of finding a general solution of these equations, we will focus on the behavior of the parameterizing functions near the boundary $r = 0$. A first attempt to get an idea of the behavior consists in the study of the linearized equations around the background

$$\begin{aligned} f(t,r) &= 1 + \eta(t,r)\,, \\ \kappa(t,r) &= 0 + \kappa(t,r)\,, \end{aligned} \tag{5.2.20}$$

where $\eta(t,r)$, $\kappa(t,r)$ and their derivatives are infinitesimal. The equations of motion become

$$\begin{aligned}
0 &= \frac{1}{2}\eta^{''} + \kappa^{''}\,,\\
0 &= \dot{\kappa}^{'}\,,\\
0 &= 2\alpha\gamma\,\eta^{'} + 2r\,\eta^{''} + \ddot{\kappa} - 2\kappa^{'}\,,\\
0 &= 4r\kappa^{''} + (2+8\alpha\gamma)\,\kappa^{'} + \ddot{\kappa} + 2\alpha\gamma\,\eta^{'}\,,
\end{aligned} \tag{5.2.21}$$

and a general solution is provided by

$$\begin{aligned}
\eta(t,r) &= \eta_{(0)}(t) + \eta_{(5)}(t)\,r - 2A\,r^{7/5}\,,\\
\kappa(t,r) &= \kappa_{(0)}(t) + A\,r^{7/5}\,,\\
\ddot{\kappa}_{(0)}(t) &= \frac{9}{5}\,\eta_{(5)}(t)\,,
\end{aligned} \tag{5.2.22}$$

where A is a constant of integration. The solution admits a polynomial part and a non-integer power in r. This will guide us to formulate an asymptotic ansatz when $(r \to 0)$ for the solutions of the full non-linear equations

$$\begin{aligned}
f(t,r) &= f_{(0)}(t) + r\,f_{(5)}(t) + \cdots + r^{\sigma}\left(f_{(\sigma)}(t) + \dots\right),\\
\kappa(t,r) &= \kappa_{(0)}(t) + r\,\kappa_{(5)}(t) + \cdots + r^{\sigma}\left(\kappa_{(\sigma)}(t) + \dots\right),
\end{aligned} \tag{5.2.23}$$

where σ stands for the first non-integer power in the r power series expansion. The first coefficient functions $\{f_{(0)}(t)\,,\,\kappa_{(0)}(t)\}$ will be interpreted as sources for the fluctuations on the boundary and the other ones as responses of the fields in the bulk. The equations of motion (5.2.19) impose several constraints that can be solved iteratively. Two relevant constraints are brought by the cancellation of:

- Terms in $r^{\sigma-1}$

$$\begin{aligned}
0 &= \big(1-\frac{\beta}{\gamma^2}\big)f_{(0)}\dot{\kappa}_{(0)}\kappa_{(\sigma)} + f_{(0)}\dot{\kappa}_{(\sigma)} - \frac{1}{2}\dot{\kappa}_{(0)}f_{(\sigma)}\,,\\
0 &= \big(4\alpha\gamma + 2\sigma - 1\big)f_{(0)}\kappa_{(\sigma)} + \alpha\gamma f_{(\sigma)}\,,\\
0 &= -f_{(0)}\kappa_{(\sigma)} + \big(\alpha\gamma + \sigma - 1\big)f_{(\sigma)}\,.
\end{aligned} \tag{5.2.24}$$

The last two equations imply $\sigma = \frac{1}{2} - \alpha\gamma = \frac{7}{5}$. Then,

$$\begin{aligned}
0 &= f_{(\sigma)} + 2f_{(0)}\kappa_{(\sigma)}\,,\\
0 &= \dot{\kappa}_{(\sigma)} + \frac{14}{9}\dot{\kappa}_{(0)}\kappa_{(\sigma)}\,,
\end{aligned} \tag{5.2.25}$$

- Terms in r^0

$$\begin{aligned}
0 &= \big(8\alpha\gamma + 2\big)f_{(0)}^2\kappa_{(5)} + f_{(0)}\ddot{\kappa}_{(0)} - \frac{1}{2}\dot{f}_{(0)}\dot{\kappa}_{(0)} + f_{(0)}\dot{\kappa}_{(0)}^2 + 2\alpha\gamma f_{(0)}f_{(5)}\,,\\
0 &= 2\alpha\gamma f_{(5)} + \ddot{\kappa}_{(0)} - \frac{1}{2}f_{(0)}^{-1}\dot{f}_{(0)}\dot{\kappa}_{(0)} + \big(1-\frac{\beta}{\gamma^2}\big)\dot{\kappa}_{(0)}^2 - 2f_{(0)}\kappa_{(5)}\,,\\
0 &= \big(1-\frac{\beta}{\gamma^2}\big)f_{(0)}\dot{\kappa}_{(0)}\kappa_{(5)} + f_{(0)}\dot{\kappa}_{(5)} - \frac{1}{2}f_{(5)}\dot{\kappa}_{(0)}\,,
\end{aligned} \tag{5.2.26}$$

thus,

$$\kappa_{(5)} = \frac{5}{36}\, f_{(0)}^{-1}\dot{\kappa}_{(0)}^2\,,$$
$$f_{(5)} = \frac{5}{9}\big(\ddot{\kappa}_{(0)} - \frac{1}{2} f_{(0)}^{-1}\dot{f}_{(0)}\dot{\kappa}_{(0)} + \frac{5}{18}\dot{\kappa}_{(0)}^2\big)\,. \tag{5.2.27}$$

To begin, notice that $\sigma = \frac{7}{5}$ is the first non-integral power which appears in the expansion of the fluctuations. This was expected from the linear analysis. Moreover, all the coefficients including the order r^σ are determined in terms of the functions $\{f_{(0)}\,,\,\kappa_{(0)}\}$. In this sense one can interpret them as sources for the fluctuations. As will be explained in the next section, it turns out that the renormalized action only involves terms in these expansion up to the order $r^{7/5}$. Higher order terms will disappear in the limit: $\epsilon \to 0$. Thus, we are left with the asymptotic ansatz

$$f(t,r) = f_{(0)}(t) + r f_{(5)}(t) + r^{7/5} f_{(7)}(t) + \dots$$
$$\kappa(t,r) = \kappa_{(0)}(t) + r\kappa_{(5)}(t) + r^{7/5}\kappa_{(7)}(t) + \dots \tag{5.2.28}$$

for the fluctuations. Now let us renormalize the on-shell action.

Renormalization and Correlation functions

Knowing the asymptotic behaviour of the fields near the boundary, the on-shell action (5.2.6) may be evaluated. Let us recall that n^μ is a unit vector normal to the boundary

$$n^\mu\partial_\mu = n\,\partial_r\,, \quad n^\mu n_\mu = 1 \quad \text{imply} \quad n = 2r\,, \tag{5.2.29}$$

and

$$h = \frac{f(t,r)}{r} dt^2\,,$$
$$K = \nabla_\mu n^\mu = -1 + r\,\partial_r \ln f\,. \tag{5.2.30}$$

Inserting the expansion (5.2.28) in the action (5.2.6) leads to the different contributions

$$\sqrt{h}\, e^{\gamma\phi} = |f_{(0)}|^{1/2} e^{\kappa_{(0)}}\, \epsilon^{-7/5}\left[1 + \big(\frac{1}{2} f_{(0)}^{-1} f_{(5)} + \kappa_{(5)}\big)\,\epsilon + \big(\frac{1}{2} f_{(0)}^{-1} f_{(7)} + \kappa_{(7)}\big)\,\epsilon + o(\epsilon^{7/5})\right],$$
$$K|_{r=\epsilon} = -1 + f_{(0)}^{-1}\left[f_{(5)}\,\epsilon + \frac{7}{5} f_{(7)}\,\epsilon^{7/5} + \underset{\epsilon\to 0}{o}(\epsilon^{7/5})\right],$$
$$n^\mu\partial_\mu\phi|_{r=\epsilon} = 2\alpha + \frac{2}{\gamma}\big[\kappa_{(5)}\epsilon + \frac{7}{5}\kappa_{(7)}\epsilon^{7/5}\big] + o(\epsilon^{7/5})\,. \tag{5.2.31}$$

Notice that the first contribution coming from the determinant of the induced metric times the dilaton involves a global factor of $\epsilon^{-7/5}$. This factor is due to the background, not to the fluctuations

$$\big(\sqrt{h}\, e^{\gamma\phi}\big)|_{r=\epsilon} = \big(r^{\alpha\gamma - \frac{1}{2}}\, |f(t,r)|^{1/2} e^{\kappa(t,r)}\big)|_{r=\epsilon}$$
$$= \big(|f(t,r)|^{1/2} e^{\kappa(t,r)}\big)|_{r=\epsilon}\, \epsilon^{-7/5} \tag{5.2.32}$$

and this is precisely the reason why the power series expansions for the fluctuations need only to be determined up to order $r^{7/5}$. The on-shell action is now expressed as a perturbed expansion in $r = \epsilon$ up to vanishing orders when ϵ goes to zero

$$
\begin{aligned}
S_{\text{on-shell}} = \frac{1}{2}\int dt\, |f_{(0)}|^{1/2} e^{\kappa_{(0)}} \Bigg[& \big(-1 + \frac{2\alpha\beta}{\gamma}\big)\, \epsilon^{-7/5} \\
& + \Big[\big(-1 + \frac{2\alpha\beta}{\gamma}\big)\frac{1}{2} f_{(0)}^{-1} f_{(5)} + \big(-1 + \frac{2\alpha\beta}{\gamma}\big)\kappa_{(5)} + f_{(0)}^{-1} f_{(5)} + \frac{2\beta}{\gamma^2}\kappa_{(5)}\Big]\, \epsilon^{-2/5} \\
& + \Big[\big(-1 + \frac{2\alpha\beta}{\gamma}\big)\frac{1}{2} f_{(0)}^{-1} f_{(7)} + \big(-1 + \frac{2\alpha\beta}{\gamma}\big)\kappa_{(7)} + \frac{7}{5} f_{(0)}^{-1} f_{(7)} + \frac{7}{5}\frac{2\beta}{\gamma^2}\,\kappa_{(7)}\Big] \\
& + o(1)\Bigg]\,.
\end{aligned} \tag{5.2.33}
$$

The next step deals with adding covariant counter-terms such that the divergent terms vanish.

Renormalization The first counter-term that one can imagine is a cosmological constant

$$
S_{\text{ct1}} = \frac{1}{2}\int \mathrm{d}t \sqrt{h}\, e^{\gamma\phi}\Big(1 - \frac{2\alpha\beta}{\gamma}\Big)\,. \tag{5.2.34}
$$

This kills the first divergent term in (5.2.33) and adds also higher order contributions that further simplify the expression

$$
\begin{aligned}
S_{\text{on-shell}} + S_{\text{ct1}} = \frac{1}{2}\int dt\, |f_{(0)}|^{1/2} e^{\kappa_{(0)}} \Big[& \big(f_{(0)}^{-1} f_{(5)} + \frac{2\beta}{\gamma^2}\kappa_{(5)}\big)\, \epsilon^{-2/5} \\
& + \frac{7}{5}\big(f_{(0)}^{-1} f_{(7)} + \frac{2\beta}{\gamma^2}\kappa_{(7)}\big) \;+\; o(1)\Big]\,.
\end{aligned} \tag{5.2.35}
$$

Moreover, $f_{(5)}$ and $\kappa_{(5)}$ are related to the sources by (5.2.27). This corresponds to the expansion of

$$
\begin{aligned}
\big(\nabla^t \partial_t \phi\big)(r = \epsilon) &= \frac{f_{(0)}^{-1}}{\gamma}\big(\ddot\kappa_{(0)} - \frac{1}{2} f_{(0)}^{-1} \dot f_{(0)} \dot\kappa_{(0)}\big)\, \epsilon + o(\epsilon)\,, \\
\big(\partial\phi\big)^2 (r = \epsilon) &= \frac{f_{(0)}^{-1}\dot\kappa_{(0)}^2}{\gamma^2}\, \epsilon + o(\epsilon)\,,
\end{aligned} \tag{5.2.36}
$$

and leads to a guess for a second counter-term

$$
S_{\text{ct2}} = \frac{1}{2}\int dt \sqrt{h}\, e^{\gamma\phi}\Big(\frac{10}{21}\big(\nabla^t\partial_t\phi\big) - \frac{10}{49}\big(\partial\phi\big)^2\Big)\,. \tag{5.2.37}
$$

The resulting action is now given by

$$
S_{\text{on-shell}} + S_{\text{ct1}} + S_{\text{ct2}} = \frac{1}{2}\int dt |f_{(0)}|^{1/2} e^{\kappa_{(0)}} \Big[\frac{7}{5}\big(f_{(0)}^{-1} f_{(7)} + \frac{2\beta}{\gamma^2}\kappa_{(7)}\big) + o(1)\Big] \tag{5.2.38}
$$

and contains only finite terms. Eventually, a relation between $f_{(7)}$ and $\kappa_{(7)}$ is provided by (5.2.25), so the renormalized action takes the form:

$$
\begin{aligned}
S_{\text{ren}} &\equiv \lim_{\epsilon\to\infty} \big(S_{\text{on-shell}} + S_{\text{ct1}} + S_{\text{ct2}}\big) \\
&= -\frac{7}{9}\int dt |f_{(0)}|^{1/2} e^{\kappa_{(0)}} \kappa_{(7)}\,.
\end{aligned} \tag{5.2.39}
$$

As a remarkable fact, the renormalized action can either be expressed in terms of the coefficient $f_{(7)}$ or $\kappa_{(7)}$ depending on which substitution is made from (5.2.25),

$$
S_{\text{ren}} = \frac{7}{18}\int dt |f_{(0)}|^{-1/2} e^{\kappa_{(0)}} f_{(7)}\,. \tag{5.2.40}
$$

One-point functions From the renormalized action, one can extract one-point functions by functional derivation.

$$\langle \mathcal{O}_\kappa(t)\rangle = \frac{1}{|f_{(0)}(t)|^{1/2}}\frac{\delta S_{\text{ren}}}{\delta\kappa_{(0)}(t)} = -\frac{7}{9}\,e^{\kappa_{(0)}}\,\kappa_{(7)}\,,$$
$$\langle \mathcal{O}_f(t)\rangle = \frac{2}{|f_{(0)}(t)|^{1/2}}\frac{\delta S_{\text{ren}}}{\delta f^{-1}_{(0)}(t)} = \frac{7}{18}\,e^{\kappa_{(0)}}\,f_{(7)}\,. \tag{5.2.41}$$

In particular, when evaluated on the background

$$f_{(0)} = 1\,,\quad \kappa_{(0)} = 0\,, \tag{5.2.42}$$

using again (5.2.25), which states that

$$f_{(7)} + 2f_{(0)}\kappa_{(7)} = 0\,, \tag{5.2.43}$$

we find the following relation

$$\langle \mathcal{O}_\kappa(t)\rangle = \langle \mathcal{O}_f(t)\rangle\,. \tag{5.2.44}$$

Two-points function Due to the relations between f and κ, only the two-points function associated to κ will be given. It is obtained by specifying a relation between the response and the source. Here, such a relation is given by the second equation of (5.2.25)

$$\kappa_{(7)} = A\,e^{-\frac{14}{9}\kappa_{(0)}(t)}\,, \tag{5.2.45}$$

where A is a real constant. However, as pointed out in [78], if no exact solution can be found from the non-linear equations of motion, the two-points correlation functions can be computed from exact solutions of the linearized equations of motion. Then, asymptotic conditions in the bulk need to be specified in order to select the physical solution. Here, we know exact solutions at the linearized level (5.2.22) but the physical solution is associated with a *non-divergent* behavior in the bulk. Therefore, $A = 0$ in (5.2.22). Consequently, the coefficient $\kappa_{(7)}$ vanishes and the two-points function reads

$$\langle \mathcal{O}_\kappa(t_1)\mathcal{O}_\kappa(t_2)\rangle = \left(\frac{1}{\sqrt{|f_{(0)}(t)|}}\frac{\delta^{(2)}S_{\text{SG ren}}}{\delta\kappa_{(0)}(t_1)\delta\kappa_{(0)}(t_2)}\right)\Bigg|_{f_{(0)}=1\,,\,\kappa_{(0)}=0} = 0\,. \tag{5.2.46}$$

This is a rather trivial expression and suggests that the gravity sector in two dimensions is somewhat degenerated. Nonetheless, as a toy example it allowed us to develop the tools of holographic renormalization, and we are now prepared to study more physical example by exciting scalars in the $SO(9)$ theory.

5.2.2 Correlation functions associated to scalar fields

Fluctuations around the scalar sector

Two kinds of physical scalar fields are present in the $SO(9)$ supergravity: the 44 coset space scalars encoded in $\mathcal{V} \in SL(9)/SO(9)$, and the remaining 84 scalars ϕ^{abc}. By expanding around the origin of the target space and up to quadratic terms, the Euclidean effective action takes the general form

$$S = \frac{1}{4}\int d^2x\,\sqrt{|g|}\,e^{\gamma\phi}\left(R + \beta\,(\partial\phi)^2 + C - e^{a\phi}\left((\partial y)^2 - m^2\,y^2\right)\right)\,. \tag{5.2.47}$$

The equations of motion follow

$$
\begin{aligned}
0 &= (\nabla_\mu \partial_\nu \phi) - \frac{g_{\mu\nu}}{2}\nabla\partial\phi - \Big(\frac{\beta}{\gamma} - \gamma\Big)\Big((\partial_\mu\phi)\,(\partial_\nu\phi) - \frac{g_{\mu\nu}}{2}\,(\partial\phi)^2 \Big) \\
&\quad + \frac{e^{a\phi}}{\gamma}\big(\partial_\mu y \partial_\nu y - \frac{1}{2} g_{\mu\nu}(\partial y)^2\big)\,, \\
0 &= \gamma\nabla\partial\phi + \gamma^2\big(\partial\phi\big)^2 - C - m^2 e^{a\phi} y^2\,, \\
0 &= R - 2\frac{\beta}{\gamma}\nabla\partial\phi - \beta\,(\partial\phi)^2 + C - \big(1 + \frac{a}{\gamma}\big)e^{a\phi}\,\Big((\partial y)^2 - m^2\, y^2\Big)\,, \\
0 &= \nabla^\mu\big(e^{(a+\gamma)\phi}\,\partial_\mu y\big) + m^2 e^{(a+\gamma)\phi}\, y\,.
\end{aligned}
\tag{5.2.48}
$$

We will consider two types of scalar fluctuations. Firstly the case of fluctuations in the $SL(9)/SO(9)$ coset sector, around the identity matrix $\mathcal{V}_0 \equiv \mathbb{I}_9$. This background value preserves $SO(9)$. Since in the effective action for the fluctuations we are only interested in quadratic order terms (kinetic and mass terms), and because the **44** of $SO(9)$ is irreducible, we can choose a simple representative of the fluctuations. Here, it will be encoded by a matrix:

$$
\mathcal{V} = \begin{pmatrix} e^x\,\mathbb{I}_{8\times 8} & 0 \\ 0 & e^{-8x} \end{pmatrix}
\tag{5.2.49}
$$

where $x = x(t,r)$ is a real valued scalar field. By expanding the action up to quadratic order in x, the effective action follows from (5.2.47) with parameters given by

$$
a = 0\,, \quad m^2 = \frac{8}{5}\,, \quad y \equiv 6\sqrt{2}\,x\,.
\tag{5.2.50}
$$

Secondly, we will consider fluctuations around the ϕ^{abc} fields in the **84** of $SO(9)$. It is sufficient to excite for example $\phi^{[123]}$, in order to get the mass and the global dilaton factor of these fields. Again, the effective action is described by (5.2.47) with

$$
a = \frac{4}{7}\,, \quad m^2 = \frac{12}{25}\,, \quad y \equiv \sqrt{2}\,\phi^{[123]}\,.
\tag{5.2.51}
$$

Asymptotic expansion

In order to renormalize the two-dimensional effective action we need to know the behavior of the fields near the boundary. Extending the previous parametrization for the fluctuation with the scalar fields

$$
\{f(t,r)\,,\ \kappa\,,\ y(t,r)\}
\tag{5.2.52}
$$

we may wonder what is their asymptotic expansion near ($r = 0$). From the linear analysis applied to the Einstein and dilaton equation we get the same answer as in (5.2.22), because the scalar field y disappears from these linearized equations since it enters only quadratically. Thus, a first clue for the y asymptotic expansion is provided by the scalar field equation linearized around the AdS background

$$
\begin{aligned}
ds^2 &= \frac{dt^2}{r} + \frac{dr^2}{4r^2}\,, \\
e^\phi &= r^{21/20}\,.
\end{aligned}
\tag{5.2.53}
$$

This gives a linear differential equation that can be simplified by taking the Fourier transform with respect to time:

$$r^2\,\tilde{y}''(q,r) + \big(\frac{21}{20}\,a - \frac{2}{5}\big)r\,\tilde{y}'(q,r) - \frac{1}{4}(q^2 r - m^2)\,\tilde{y}(q,r) = 0\,. \tag{5.2.54}$$

For the scalar perturbation x one finds $(a, m^2) = (0, \frac{8}{5})$, and the solution behavior near $r = 0$ is

$$\tilde{y}(r,q) = r^{2/5}(\tilde{y}_{(0)}(q) + r^{3/5}\,\tilde{y}_{(1)}(q) + r\,\tilde{y}_{(2)}(q) + ...)\,. \tag{5.2.55}$$

However, for the scalar perturbation ϕ^{123} with (5.2.51), the solution admits the following power expansion near $r = 0$:

$$\tilde{y}(r,q) = r^{1/5}(\tilde{y}_{(0)}(q) + r^{2/5}\,\tilde{y}_{(1)}(q) + r\,\tilde{y}_{(2)}(q) + ...)\,. \tag{5.2.56}$$

This suggests an expansion in $r^{1/5}$ for the ansatz of the fluctuations and power series for the y field should start from $r^{2/5}$ for the x scalar and from $r^{1/5}$ for the ϕ^{123} scalars. Where shall we stop the series? The answer follows from the on-shell action.

Correlation functions for the x field.

Let us start from the on-shell action with a scalar, in the case of an x field perturbation $(a, m^2) = (0, \frac{8}{5})$,

$$S = \frac{1}{4}\int d^2x\,\sqrt{|g|}\,e^{\gamma\phi}\Big(R + \beta(\partial\phi)^2 + C - \big((\partial y)^2 - \frac{8}{5}\,y^2\big)\Big)\,. \tag{5.2.57}$$

In this case, the dilaton enters the Lagrangian as a global factor. Consequently, the on-shell action takes a simple form when we use the dilaton field equation:

$$S_{\text{on-shell}} = \frac{1}{2}\int_{\partial AAdS, r=\epsilon} dt\sqrt{h}\,e^{\gamma\phi}\big(K + \frac{\beta}{\gamma}n^\mu\partial_\mu\phi\big)\,. \tag{5.2.58}$$

Again the term $\sqrt{h}\,e^{\gamma\phi}$ involves a global factor of $\epsilon^{-7/5}$, thus we only need to know the expansions up to the power $r^{7/5}$, because higher power contributions will vanish after taking the limit $\epsilon \to 0$ in the renormalization procedure. As a result, we are led to assume the following asymptotic ansatz for the fluctuations

$$\begin{aligned} f(t,r) &= f_{(0)}(t) + \sum_{k=1}^{7} r^{k/5} f_{(k)} + \dots \\ \kappa(t,r) &= \kappa_{(0)}(t) + \sum_{k=1}^{7} r^{k/5} \kappa_{(k)} + \dots \\ y(r,t) &= \sum_{k=2}^{7} r^{k/5} y_{(k)} + \dots \end{aligned} \tag{5.2.59}$$

The equations of motion constrain this ansatz such that,

$$\begin{aligned} f(t,r) &= f_{(0)}(t) + r^{4/5}\,f_{(4)}(t) + r\,f_{(5)}(t) + r^{7/5}\,f_{(7)}(t) + \dots \\ \kappa(t,r) &= \kappa_{(0)}(t) + r^{4/5}\,\kappa_{(4)}(t) + r\,\kappa_{(5)}(t) + r^{7/5}\,\kappa_{(7)}(t) + \dots \\ y(r,t) &= r^{2/5}\,y_{(2)}(t) + r\,y_{(5)}(t) + \dots \end{aligned} \tag{5.2.60}$$

where the sources are assumed to be : $\{f_{(0)}(t), \kappa_{(0)}(t), y_{(2)}(t)\}$. The constraints deduced from the equations of motion

$$\begin{aligned} f_{(4)} &= -\frac{5}{18} f_{(0)}\, y_{(2)}^2\,, & \kappa_{(4)} &= -\frac{1}{4}\, y_{(2)}^2\,,\\ f_{(7)} &= -2 f_{(0)}\kappa_{(7)} - \frac{80}{63} f_{(0)}\, y_{(2)} y_{(5)}\,, & \kappa_{(5)} &= \frac{5}{36}\, f_{(0)}^{-1} \dot{\kappa}_{(0)}^2\,,\\ f_{(5)} &= \frac{5}{9}\big(\ddot{\kappa}_{(0)} - \frac{1}{2} f_{(0)}^{-1} \dot{f}_{(0)} \dot{\kappa}_{(0)} + \frac{5}{18}\dot{\kappa}_{(0)}^2\big)\,, \end{aligned} \tag{5.2.61}$$

enable to find the covariant counter-terms

$$\begin{aligned} S_{\text{ct1}} &= \frac{1}{2}\int dt\sqrt{h}\, e^{\gamma\phi}\big(1 - \frac{2\alpha\beta}{\gamma}\big)\,,\\ S_{\text{ct2}} &= \frac{1}{2}\int dt\sqrt{h}\, e^{\gamma\phi}\big(\frac{2}{5} y^2\big)\,,\\ S_{\text{ct3}} &= \frac{1}{2}\int dt\sqrt{h}\, e^{\gamma\phi}\big(\frac{10}{21}(\nabla^t\partial_t\phi) - \frac{10}{49}(\partial\phi)^2\big)\,. \end{aligned} \tag{5.2.62}$$

Therefore, the action on shell is given by

$$\begin{aligned} S_{\text{ren}} &= \lim_{\epsilon\to 0}\left(S_{\text{on-shell}} + S_{\text{ct1}} + S_{\text{ct2}} + S_{\text{ct3}}\right)\\ &= \int \mathrm{dt}|f_{(0)}|^{1/2} e^{\kappa_{(0)}}\left(-\frac{22}{45} y_{(2)} y_{(5)} - \frac{7}{9}\kappa_{(7)}\right)\,. \end{aligned} \tag{5.2.63}$$

This corresponds to the renormalized action (5.2.39) supplemented by a term that accounts for the scalar field.

One-point functions The one-point functions are given by

$$\begin{aligned} \langle \mathcal{O}_\kappa(t)\rangle &= \frac{1}{|f_{(0)}(t)|^{1/2}}\frac{\delta S_{\text{ren}}}{\delta\kappa_{(0)}(t)} = e^{\kappa_{(0)}}\Big(-\frac{22}{45}\, y_{(2)}(t) y_{(5)}(t) - \frac{7}{9}\,\kappa_{(7)}\Big)\,,\\ \langle \mathcal{O}_y(t)\rangle &= \frac{1}{|f_{(0)}(t)|^{1/2}}\frac{\delta S_{\text{ren}}}{\delta y_{(2)}(t)} = -\frac{22}{45}\, e^{\kappa_{(0)}}\, y_{(5)}(t)\,, \end{aligned} \tag{5.2.64}$$

thus the following identity holds

$$\langle \mathcal{O}_\kappa(t)\rangle = y_{(2)}(t)\,\langle \mathcal{O}_y(t)\rangle - \frac{7}{9}\, e^{\kappa_{(0)}}\,\kappa_{(7)}\,. \tag{5.2.65}$$

Two-points functions The equations of motion imply that all the coefficients are completely determined by the sources except for the following responses: $\{f_{(7)}(t),\ \kappa_{(7)}(t),\ y_{(5)}(t)\}$. The near-boundary analysis is insufficient to link these responses to the sources and one has to get closer to an exact solution of the equations of motion to find the desired relation. Actually, it is enough to look for an exact solution of the linearized equations of motion around the background, to find the relation between the undetermined coefficients and the sources.

The two-points function associated to the dilaton will not change from the study of the gravity sector, so let us focus on the scalar field. As described in the linear analysis, the scalar fluctuation is a solution of the equation (5.2.54) with parameters $(a, m^2) = (0, \frac{8}{5})$:

$$r^2\,\tilde{y}'' - \frac{2}{5} r\,\tilde{y}' - \frac{1}{4}(q^2 r - \frac{8}{5})\,\tilde{y} = 0\,. \tag{5.2.66}$$

This equation (5.2.54) can be written in a more canonical form making the following change of variables and function redefinition

$$\tilde{r} = q\sqrt{r}\,, \qquad \tilde{y}(q,\tilde{r}) = \tilde{r}^{\lambda}\, s(q,\tilde{r})\,, \quad \lambda = \frac{7}{5}(1-\frac{3}{4}a)\,. \tag{5.2.67}$$

Thus, the equation becomes

$$\tilde{r}^2\, s^{''} + \tilde{r}\, s^{'} - \left(\tilde{r}^2 + \lambda^2 - m^2\right) s = 0\,. \tag{5.2.68}$$

In the present case $(a, m^2) = (0, \frac{8}{5})$, so the rescaled function s satisfies

$$\tilde{r}^2\, s^{''} + \tilde{r}\, s^{'} - \left(\tilde{r}^2 + (\frac{3}{5})^2\right) s = 0\,. \tag{5.2.69}$$

This corresponds to the modified Bessel's equation with parameter 3/5. It admits two linearly independent solutions which may be described by modified Bessel function of the first kind I or the second kind K. For example if we choose to parametrize the solution with Bessel I functions, we get the general form:

$$\tilde{y}(q,r) = q^{7/5} r^{7/10}(C_1\,\mathrm{Bessel}_I(3/5, q\sqrt{r}) + C_2 \mathrm{Bessel}_I(-3/5, q\sqrt{r}))\,. \tag{5.2.70}$$

The physical solution must be regular in the bulk which translates into the regularity condition

$$\lim_{r\to+\infty} y(t,r) = 0\,, \quad \forall t\,. \tag{5.2.71}$$

So the acceptable solution is determined up to a global constant factor

$$\begin{aligned} \tilde{y}(q,r) &= q^{7/5} r^{7/10}\, C_1\, (\mathrm{Bessel}_I(3/5, q\sqrt{r}) - \mathrm{Bessel}_I(-3/5, q\sqrt{r})) \\ &= -\, q^{7/5} r^{7/10}\, C_1'\, \mathrm{Bessel}_K(3/5, q\sqrt{r})\,, \end{aligned} \tag{5.2.72}$$

where $C_1' \equiv C_1\left(\frac{2}{\pi}\sin(\frac{3\pi}{5})\right)$. Consequently, we have now access to an asymptotic expansion near $r = 0$:

$$\begin{aligned} \tilde{y}(q,r) &= -q^{4/5}\, C_1' \left(\frac{\Gamma(\frac{3}{5})}{2^{2/5}}\, r^{2/5} + \frac{\Gamma(-\frac{3}{5})}{2^{8/5}}\, q^{6/5}\, r + \frac{5\Gamma(\frac{3}{5})}{2^{17/5}}\, q^2\, r^{7/5} + \underset{r\to 0}{o}(r^{7/5})\right) \\ &= \tilde{y}_2(q)\, r^{2/5} + \tilde{y}_5(q)\, r + \ldots \end{aligned} \tag{5.2.73}$$

Notice that the expansion is in agreement with the perturbative ansatz (5.2.60). Moreover it enable us to relate the first two coefficients in the power expansion:

$$\tilde{y}_5(q) \propto q^{6/5} \tilde{y}_2(q)\,. \tag{5.2.74}$$

So the two-points function in momentum space is

$$\langle \mathcal{O}_y(0) \mathcal{O}_y(q) \rangle \propto q^{6/5}\,, \tag{5.2.75}$$

and the correlation function in time follows to be

$$\langle \mathcal{O}_y(t_1) \mathcal{O}_y(t_2) \rangle \propto \mathrm{TF}^{-1}(q^{6/5})(t_1 - t_2) \propto \frac{1}{|t_1 - t_2|^{11/5}}\,. \tag{5.2.76}$$

In [80], correlation functions are also obtained from the gravity side. The exponent of the two-points functions is defined by

$$\langle \mathcal{O}_y(t_1) \mathcal{O}_y(t_2) \propto \frac{1}{|t_1 - t_2|^{2\nu+1}}\,. \tag{5.2.77}$$

So in our case $\nu = \frac{3}{5}$. When compared to the correlation functions obtained in [80], such an exponent corresponds to a supergravity mode with $SO(9)$ total angular momentum $l = 5$ which comes from the eleven-dimensional metric. According to [80], on the gauge theory side, the corresponding operators are

$$T^{++}_{i_1\ldots i_5} \propto \mathrm{tr}\big(X_{(i_1}\ldots X_{i_5)}\big)\,, \tag{5.2.78}$$

where the parenthesis $(i_1 \ldots i_5)$ means that the product of operators X_i is totally symmetry and traceless under contraction of any two indices. This certainly does not correspond to the gravity side scalar perturbations that we study since they do not transform in the same irreducible representation of $SO(9)$.

However, if we exchange the role of the source and the response, we find

$$\langle \mathcal{O}_y(0)\mathcal{O}_y(q)\rangle \propto q^{-6/5}\,, \tag{5.2.79}$$

and the correlation function in time is given by

$$\langle \mathcal{O}_y(t_1)\mathcal{O}_y(t_2)\rangle \propto \mathrm{TF}^{-1}(q^{-6/5})(t_1-t_2) \propto \frac{1}{|t_1-t_2|^{-1/5}}\,. \tag{5.2.80}$$

According to [80], such an exponent ($\nu = -\frac{3}{5}$) corresponds to a supergravity mode with angular momentum $l = 2$ originating from the eleven-dimensional metric. On the Matrix theory side, the corresponding operators [80] are given by

$$T^{++}_{ij} \propto \frac{1}{N}\Big(\mathrm{tr}\big(X^iX^j\big) - \frac{\delta^{ij}}{9}\sum_{k=1}^{9}\mathrm{tr}\big(X^kX^k\big)\Big)\,, \tag{5.2.81}$$

and transform in the **44** of $SO(9)$. The representations agree. Moreover, using Monte Carlo calculations, the authors of [80] showed that the two-points correlation function associated with (5.2.81) matches exactly (5.2.80). The fact that an ambiguity in the source and response was present, is related to the value of the mass square m^2 that we found in (5.2.50) which allows for two *admissible* scaling dimensions ($\Delta_- \leq \Delta_+$), in the sense of [137].

Correlation functions for the ϕ^{123} field.

In the case of the ϕ^{abc} perturbation, here we take the example of ϕ^{123}, the action is

$$S = \frac{1}{4}\int d^2x\,\sqrt{|g|}\,e^{\gamma\phi}\Big(R + \beta(\partial\phi)^2 + C - e^{\frac{4}{7}\phi}\big((\partial y)^2 - \frac{12}{25}\,y^2\big)\Big)\,. \tag{5.2.82}$$

Once evaluated on-shell we get

$$S_{\text{on-shell}} = \frac{1}{2}\int_{\partial AAdS,r=\epsilon} dt\sqrt{h}\,e^{\gamma\phi}\left(K + \frac{\beta}{\gamma}n^\mu\partial_\mu\phi + \frac{2}{7\gamma}e^{\frac{4}{7}\phi}\,y\,n^\mu\partial_\mu y\right)\,. \tag{5.2.83}$$

Again, if we assume that (f,κ) admit a power expansion in r with first non zero terms: $(f_{(0)},\kappa_{(0)})$, then $\sqrt{h}e^{\gamma\phi}$ gives a power $r^{-7/5}$, and $\sqrt{h}e^{(\gamma+\frac{4}{7})\phi}$ gives a power $r^{-4/5}$. Thus we only need to know the expansions of (f,κ) up to the $r^{7/5}$ terms and the expansion of y up to $r^{4/5}$. Eventually, the linear analysis, followed by the study of the full non-linear equations of motion lead to the perturbative ansatz:

$$\begin{aligned}
f(t,r) &= f_{(0)}(t) + r\,f_{(5)}(t) + r^{7/5}\,f_{(7)}(t) + \ldots\\
\kappa(t,r) &= \kappa_{(0)}(t) + r\,\kappa_{(5)}(t) + r^{7/5}\,\kappa_{(7)}(t) + \ldots\\
y(r,t) &= r^{1/5}\,y_{(1)}(t) + r^{3/5}\,y_{(3)}(t) + \ldots
\end{aligned} \tag{5.2.84}$$

Owing to the constraints

$$\begin{aligned} f_{(5)} &= \frac{5}{9}\big(\ddot\kappa_{(0)} - \frac{1}{2} f_{(0)}^{-1}\dot f_{(0)}\dot\kappa_{(0)} + \frac{5}{18}\dot\kappa_{(0)}^2\big) + \frac{1}{45} e^{-\frac{2\kappa_{(0)}}{3}} f_{(0)} x_{(1)}^2 \,, \\ \kappa_{(5)} &= \frac{5}{36} f_{(0)}^{-1}\dot\kappa_{(0)}^2 - \frac{1}{10} e^{-\frac{2\kappa_{(0)}}{3}} x_{(1)}^2 \,, \\ f_{(7)} &= -2 f_{(0)}\kappa_{(7)} - \frac{8}{21} e^{-\frac{2\kappa_{(0)}}{3}} f_{(0)} x_{(1)} x_{(3)} \,, \end{aligned} \tag{5.2.85}$$

satisfied by the different coefficients in the expansions, we are led to introduce the counter terms:

$$\begin{aligned} S_{\text{ct1}} &= \frac{1}{2}\int dt\sqrt{h}\, e^{\gamma\phi}(1 - \frac{2\alpha\beta}{\gamma}) \,, \\ S_{\text{ct2}} &= \frac{1}{2}\int dt\sqrt{h}\, e^{\gamma\phi}\big(\frac{10}{21}(\nabla^t\partial_t\phi) - \frac{10}{49}(\partial\phi)^2\big) \,, \\ S_{\text{ct3}} &= \frac{1}{2}\int dt\sqrt{h}\, e^{(\gamma+a)\phi}\,\big(\frac{y^2}{5}\big) \,. \end{aligned} \tag{5.2.86}$$

Thus, the renormalized action is given by,

$$\begin{aligned} S_{\text{ren}} &= \lim_{\epsilon\to 0}\left(S_{\text{on-shell}} + S_{\text{ct1}} + S_{\text{ct2}} + S_{\text{ct3}}\right) \\ &= \int \mathrm{dt} |f_{(0)}|^{1/2} e^{\kappa_{(0)}} \left(-\frac{1}{3} e^{-\frac{2\kappa_{(0)}}{3}} y_{(1)} y_{(3)} - \frac{7}{9}\kappa_{(7)}\right) \,. \end{aligned} \tag{5.2.87}$$

To compute the correlation functions we study the equation of motion for the scalar field, linearized around the background. As described in the linear analysis, the scalar fluctuation is a solution of the equation (5.2.54) with parameters $(a, m^2) = (\frac{4}{7}, \frac{12}{25})$:

$$r^2\,\tilde y'' + \frac{1}{5} r\,\tilde y' - \frac{1}{4}(q^2 r - \frac{12}{25})\,\tilde y = 0 \,, \tag{5.2.88}$$

which owing to the transformations (5.2.67) takes the canonical form

$$\tilde r^2\, s'' + \tilde r\, s' - \Big(\tilde r^2 + \big(\frac{2}{5}\big)^2\Big)\, s = 0 \,. \tag{5.2.89}$$

It is a modified Bessel equation and the general solution satisfying the regularity condition is given up to a global constant C_2 :

$$\tilde y(q, r) = q^{4/5} r^{4/10}\, C_2\, \mathrm{Bessel}_K(2/5, q\sqrt{r}) \,. \tag{5.2.90}$$

The series expansion near $r = 0$ is

$$\tilde y(q, r) = q^{2/5}\, C_2\left(\frac{\Gamma(\frac{2}{5})}{2^{3/5}}\, r^{1/5} + \frac{\Gamma(-\frac{2}{5})}{2^{7/5}}\, q^{4/5}\, r^{3/5} + \frac{5\Gamma(\frac{2}{5})}{12\ 2^{3/5}}\, q^2\, r^{6/5} + \underset{r\to 0}{o}(r^{6/5})\right) . \tag{5.2.91}$$

Thus, the first two coefficients are related by

$$\tilde y_3(q) \propto q^{4/5}\tilde y_1(q) \,, \tag{5.2.92}$$

and the two-points function is

$$\langle \mathcal{O}_y(t_1)\mathcal{O}_y(t_2)\rangle \propto \mathrm{TF}^{-1}(q^{4/5})(t_1 - t_2) \propto \frac{1}{|t_1 - t_2|^{9/5}} \,. \tag{5.2.93}$$

When compared to the correlation functions obtained in [80], such an exponent ($\nu = \frac{2}{5}$) corresponds to a supergravity mode with angular momentum given by $l = 1$, which comes from the eleven-dimensional three-form. Here, there is no ambiguity since if we exchange the role of the source and the response, no supergravity mode match in [80]. On the Matrix theory side, the corresponding operators are given by

$$\frac{1}{N}\mathrm{tr}\big([X^i,\, X^j]X^k\big)\,, \tag{5.2.94}$$

and transform in the **84** of $SO(9)$. Again, Monte Carlo calculations performed in [80] on the gauge theory side provide two-points correlation functions whose behavior matches exactly (5.2.93).

5.3 Deformed BFSS model holography

In this section, we will apply the holography techniques to extend our study to a supersymmetric deformation of the BFSS quantum mechanics. An important model which comes to mind is the BMN model [135]. It was first constructed to describe M-theory plane waves and comes from the eleven-dimensional supermembrane action

$$S[Z(\zeta)] = \int d^3\zeta \left[-\sqrt{-G(Z)} - \frac{1}{6}\epsilon^{abc}\, \Pi_a^A \Pi_b^B \Pi_c^C\, B_{CBA}\big(Z(\zeta)\big)\right], \tag{5.3.1}$$

- where $Z^A = \big(X^M(\zeta), \xi(\zeta)\big)$ are superspace embedding coordinates,
- B_{CBA} is the antisymmetric tensor gauge superfield,
- and $\Pi_a^r = \partial_a Z^A E_A^r$ is the supervielbein pullback, see [138], [139].

It is obtained when this action is evaluated on the maximally supersymmetric pp-wave background

$$\begin{aligned} ds^2 &= 2dx^+dx^- - \sum_{i=1}^{3}\left(\frac{\mu}{3}\right)^2 x^{i2}(dx^+)^2 - \sum_{i=4}^{9}\left(\frac{\mu}{6}\right)^2 x^{i2}(dx^+)^2 + \sum_{i=1}^{9}(dx^i)^2\,, \\ F^{(4)} &= \mu\, dx^+ \wedge dx^1 \wedge dx^2 \wedge dx^3\,. \end{aligned} \tag{5.3.2}$$

and further truncated to a Matrix model action by the procedure described in [140].

5.3.1 A deformation of BFSS: the BMN model

Starting from the hypothesis that the gauge theory side is represented by the BMN model, let us introduce it. The action for the BMN model is given by the BFSS term

$$S_{\mathrm{BFSS}} = \int dt\, \mathrm{tr}\Big(D_tX^i\, D_tX^i + 2\Psi^T D_t\Psi - \frac{1}{2}[X^i, X^j]^2 - 2\Psi^T\gamma_i[\Psi, X^i]\Big) \tag{5.3.3}$$

supplemented by mass and Myers terms [141]

$$S_{\mathrm{mM}} = \int dt\, \mathrm{tr}\left(-\sum_{i=1}^{3}\left(\frac{\mu}{3}\right)^2 X^{i2} - \sum_{i=4}^{9}\left(\frac{\mu}{6}\right)^2 X^{i2} + \frac{2i\mu}{3}\sum_{i,j,k=1}^{3}\epsilon_{ijk}X^iX^jX^k - \frac{i\mu}{2}\Psi^T\gamma_{123}\Psi\right) \tag{5.3.4}$$

Thus,

$$S_{\mathrm{BMN}} \equiv S_{\mathrm{BFSS}} + S_{\mathrm{mM}}\,. \tag{5.3.5}$$

The mass and Myers terms of this BMN quantum mechanics break the global $SO(9)$ symmetry to $SO(3) \times SO(6)$ whilst preserving the 16 supercharges of the BFSS model: it is an *operator* deformation.

From the gravity side, the search of interesting geometries that are dual to the BMN model, has already been investigated directly in type IIA supergravity or in M-theory. Notably, a general class of half BPS, $SO(3) \times SO(6)$ preserving solutions has been discussed, as well as their implication for the BMN model [142] [143]. In the following section we will investigate $SO(3) \times SO(6)$ preserving half supersymmetric backgrounds, but we will adopt the effective two-dimensional supergravity point of view. We will find a unique background, however as we will see, it will not correspond to an operator deformation of the BFSS model. Thus, the dual Matrix model will not be described by the BMN model but rather by a *vev* deformation of the BFSS model [78]. Eventually, one-point and two-points correlation functions will be computed, allowing for a test on the corresponding dual Matrix model.

5.3.2 $SO(3) \times SO(6)$ gravity sector

In the full two-dimensional, maximal, $SO(9)$ gauged supergravity we may try to find a supersymmetric background preserving $SO(3) \times SO(6)$ symmetry. In two dimensions, a simple ansatz for such a vacuum solution is provided by exciting the $SL(9)/SO(9)$ scalars in a diagonal way and letting the other scalar fields at the origin of the target space:

$$\mathcal{V} = \begin{pmatrix} e^{-x}\mathbb{I}_{3\times 3} & 0 \\ 0 & e^{x/2}\mathbb{I}_{6\times 6} \end{pmatrix}, \qquad \phi^{klm} = Y_{kl} = 0\,. \tag{5.3.6}$$

In this truncation, the two-dimensional bosonic effective Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} e\rho R + \frac{9}{8} e\rho\, (\partial_\mu x)(\partial^\mu x) + \frac{3}{8} e\rho^{5/9}\, e^{-2x}(8 + 12e^{3x} + e^{6x})\,.$$

The BPS equations are derived from the supersymmetry variations (3.4.16):

$$\begin{aligned}
0 &\stackrel{!}{=} \delta_\varepsilon \psi^I_\mu = \partial_\mu \epsilon^I + \frac{1}{4}\omega_\mu{}^{\alpha\beta}\gamma_{\alpha\beta}\epsilon^I + \frac{7}{12}\, i\rho^{-2/9}(e^{2x} + 2e^{-x})\,\gamma_\mu \epsilon^I\,,\\
0 &\stackrel{!}{=} \delta_\varepsilon \psi^I_2 = -\frac{i}{2}(\rho^{-1}\partial_\mu \rho)\,\gamma^\mu \epsilon^I + \frac{3}{4}\rho^{-2/9}(e^{2x} + 2e^{-x})\,\epsilon^I\,,\\
0 &\stackrel{!}{=} \delta_\varepsilon \chi^{aI} \Leftrightarrow 0 = (\partial_\mu x)\,\gamma^\mu \epsilon^I - \frac{2i}{3}\rho^{-2/9}(e^{2x} - e^{-x})\,\epsilon^I\,.
\end{aligned} \tag{5.3.7}$$

Apart from the $SO(9)$ invariant solution (3.5.11) for which $x = 0$, these equations admit a unique non-trivial solution. Part of the diffeomorphisms can be fixed to identify x with the radial coordinate, and we find

$$\begin{aligned}
ds_2^2 &= \widetilde{f}(x)^2 dt^2 - \widetilde{g}(x)^2 dx^2\,,\\
\widetilde{f}(x) &= e^{\frac{7}{2}x}(e^{3x} - 1)^{-7/4}\,, \qquad \widetilde{g}(x) = \frac{3}{2}e^{2x}(e^{3x} - 1)^{-3/2}\,,\\
\rho(r) &= e^{\frac{9}{2}x}(e^{3x} - 1)^{-9/4}
\end{aligned} \tag{5.3.8}$$

up to coordinate redefinitions and the scaling symmetry

$$\rho \to \lambda\,\rho, \quad g_{\mu\nu} \to \lambda^{4/9}\, g_{\mu\nu}, \quad \mathcal{L} \to \lambda\,\mathcal{L} \tag{5.3.9}$$

of the Lagrangian (5.3.7). The associated Killing spinors are given by

$$\epsilon^I(x) = a(x)\,\epsilon^I_0\,, \qquad \text{with} \quad \gamma^1 \epsilon^I_0 = -i\epsilon^I_0\,, \tag{5.3.10}$$

and a function $a(x)$ that is obtained from integrating the first equation of (5.3.7). This confirms that the background preserves sixteen supercharges, i.e. has the same number of supersymmetries as the $SO(9)$ domain wall (3.5.11). Since x is non-vanishing in the bulk, this deformation breaks $SO(9)$ down to $SO(3) \times SO(6)$. The Ricci scalar of (5.3.8) takes the following form

$$\begin{aligned} R &= -\frac{5}{6}\, e^{-2x}\big(e^{6x} - 12e^{3x} - 4\big)\,, \\ R &= \frac{25}{2} + \underset{x\to 0}{\mathcal{O}}(x^2)\,, \qquad R = -\frac{5}{6}\, e^{4x} + 10\, e^{x} + \underset{x\to +\infty}{o(1)}\,. \end{aligned} \tag{5.3.11}$$

It is well defined on $x \in [0\,, +\infty[$ in contrast to the metric and the dilaton which are singular at $x = 0$.

5.3.3 Higher-dimensional interpretation

Although the geometry of this solution may be obscure in this parametrization, its interpretation becomes clearer in eleven dimensions. As before, in order to uplift the solution, we go first from two to ten dimensions using the embedding of $SO(9)$ supergravity in type IIA supergravity [82]. Then we go from ten to eleven by standard techniques.

From two to ten dimensions

Using the Kaluza-Klein ansatz (4.3.29) constructed in [82], the BPS solution (5.3.8) can be uplifted to ten dimensions. Thus, we obtain a solution of type IIA bosonic equations of motion derived from the Lagrangian (4.3.6),

$$\begin{aligned} ds_{10}^2 &= \rho^{-7/36}\Delta^{7/8}\, ds_2^2 - \rho^{1/4}\,\Delta^{-1/8}\Big(\frac{\Delta}{e^x(1-\mu^2)}\, d\mu^2 + e^{-2x}(1-\mu^2)\, d\Omega_2^2 + e^x\mu^2\, d\Omega_5^2\Big)\,, \\ \phi &= \frac{1}{3}\log\big(\rho^{-7/4}\Delta^{-9/8}\big)\,, \\ F &= 2\rho^{5/9}\,\big(f_1(x) + \mu^2\, f_2(x)\big)\,\varepsilon_2 - \frac{3}{2}\rho\,(*_2 dx)\wedge d(\mu^2)\,. \end{aligned} \tag{5.3.12}$$

where,

$$\begin{aligned} &0 \le \mu^2 \le 1\,, \quad \Delta \equiv e^{2x} + \mu^2(e^{-x} - e^{2x})\,, \\ &f_1(x) \equiv -\frac{1}{2}e^{2x}(e^{2x} + 6e^{-x})\,, \quad f_2(x) \equiv -\frac{1}{2}(e^{-x} - e^{2x})(4e^{-x} + e^{2x})\,. \end{aligned} \tag{5.3.13}$$

Uplift to eleven dimensions

The uplift to eleven dimensions is realized by defining

$$ds_{11}^2 = e^{-\frac{1}{3}\phi}ds_{10}^2 - e^{\frac{8}{3}\phi}\big(A_1 + dz\big)^2\,, \tag{5.3.14}$$

with

$$A_1 = \Big(\frac{e^{9x}}{\big(e^{3x}-1\big)^{7/2}} - \frac{e^{6x}}{\big(e^{3x}-1\big)^{5/2}}\,\mu^2\Big)\, dt\,. \tag{5.3.15}$$

A_1 is defined such that $F = dA_1$. As a result we find an eleven-dimensional metric which solves Einstein's equations in 11d. An explicit and simpler form is given by

$$\begin{aligned} ds_{11}^2 = & -(dt \otimes dz + dz \otimes dt) - \frac{\left(e^{3x}-1\right)^{7/2}}{(1-\mu^2)\, e^{9x} + \mu^2 e^{6x}}\, dz^2 \\ & - \frac{9 \operatorname{csch}^2\left(\frac{3x}{2}\right)\left(1 - 2\mu^2 + \coth\left(\frac{3x}{2}\right)\right)}{32}\, dx^2 - \frac{\left(1-\mu^2\right) e^{3x} + \mu^2}{(1-\mu^2)\,(e^{3x}-1)}\, d\mu^2 \\ & - \frac{1-\mu^2}{e^{3x}-1}\, d\Omega_2^2 - \frac{\mu^2 e^{3x}}{e^{3x}-1}\, d\Omega_5^2 \,. \end{aligned} \tag{5.3.16}$$

Eventually, this expression can be considerably simplified by successively performing the coordinate transformations

$$\begin{aligned} r_2^2 &= \frac{1-\mu^2}{e^{3x}-1}\,, & r_5^2 &= \frac{\mu^2 e^{3x}}{e^{3x}-1}\,, \\ z &\to -z\,, & z &\to t - z\,, \\ x^+ &= \frac{t+z}{\sqrt{2}}\,, & x^- &= \frac{t-z}{\sqrt{2}}\,, \end{aligned} \tag{5.3.17}$$

after which the metric becomes

$$ds_{11}^2 = 2\, dx^+\, dx^- + H(r_2, r_5)(dx^-)^2 - \Big(dr_2^2 + r_2^2\, d\Omega_2^2 + dr_5^2 + r_5^2\, d\Omega_5^2\Big)\,, \tag{5.3.18}$$

where H is a harmonic function given by

$$H(r_2, r_5) \equiv 2\big(1 - F^2(r_2, r_5)\big)\,. \tag{5.3.19}$$

Moreover,

$$F^2(r_2, r_5) = \frac{(c+1-a)^{\frac{5}{2}}(c+1-b)^{-2}}{c(a-b)^{\frac{1}{2}}}\,, \tag{5.3.20}$$

and we have defined

$$a = r_2^2 + r_5^2\ , \qquad b = -r_2^2 + r_5^2\,, \qquad c = (a^2 - 2b + 1)^{\frac{1}{2}}\,. \tag{5.3.21}$$

It turns out that H satisfies the Laplace equation $\Delta H = 0$ of Euclidean space $\mathbb{E}^9$. Consequently the metric represents a pp-wave solution of the eleven-dimensional supergravity [129]. As well as the domain-wall solution (3.5.11), it is a pure gravity solution in eleven dimensions. According to the previous results, a schematic picture of the DW/QFT correspondence is drawn in Table 5.4. Now that a gravity side background has been identified, we are prepared to compute one-point and two-points correlation functions using holographic renormalization techniques. This is the point of the next two sections. As a result, we will get some informations about the dual Matrix model.

Supergravity		Super Yang-Mills
$D = 11$	$D = 2\,,\ \mathcal{N} = 16\,,\ SO(9)$	$D = 1\,,\ \mathcal{N} = 16\,,\ U(N)$
pp-wave	$SO(3) \times SO(6)$ background ($\frac{1}{2}$ BPS)	$SO(3) \times SO(6)$ Matrix model

Table 5.4: DW/QFT correspondence on a $SO(3) \times SO(6)$ background

5.3.4 On-shell action and Renormalization

Effective action

In this section we will compute an effective action from the full $SO(9)$ supergravity, that describes scalar fluctuations around the background (5.3.8) and preserves $SO(3) \times SO(6)$. The scalar fields should be expressed as the background times a perturbation and the action will be given up to quadratic order in the perturbations.

$$\mathcal{V} \equiv \mathcal{V}_{\text{background}} \left(\mathbb{I}_{9\times 9} + X + \frac{1}{2} X^2 + \dots \right) \tag{5.3.22}$$

where,

$$\mathcal{V}_{\text{background}} = \begin{pmatrix} e^{-x}\, \mathbb{I}_{3\times 3} & 0 \\ 0 & e^{x/2}\, \mathbb{I}_{6\times 6} \end{pmatrix} \tag{5.3.23}$$

is evaluated on the background solution, and $X \in \mathfrak{sl}(9)$ can be parametrized by irreducible representations of $SO(3) \times SO(6)$:

$$\begin{aligned} \mathbf{9} \otimes_s \mathbf{9} &= (3 \oplus 6) \otimes_s (3 \oplus 6) \\ &= (1,1) \oplus (5,1) \oplus (1,20) \oplus (3,6)\,. \end{aligned} \tag{5.3.24}$$

The perturbation in the singlet $(1,1)$ will not be considered in what follows, since its coupling with the background leads to possibly non-trivial linear terms in the action. For simplicity reasons, they will be put to zero. Thus, the Euclidean action is given by

$$\begin{aligned} S = -\int dx^2\, e \Bigg(&-\frac{1}{4}\rho R + \frac{9}{8} e\rho\, (\partial_\mu x)(\partial^\mu x) - \frac{3}{8} e\, \rho^{5/9}\, e^{-2x} (8 + 12 e^{3x} + e^{6x}) \\ &+ \frac{1}{2}\, e\rho (\partial x_{(5,1)})^2 + e\, \rho^{5/9}\, e^{x} (e^{3x} - 6)\, x_{(5,1)}^2 \\ &+ \frac{1}{2}\, e\rho (\partial x_{(1,20)})^2 - e\, \rho^{5/9}\, (2 e^{-2x} + 3 e^{x})\, x_{(1,20)}^2 \\ &+ \frac{1}{2}\, e\rho (\partial x_{(3,6)})^2 - e\, \rho^{5/9}\, \frac{e^{-2x}}{2} (3 + 5 e^{x} + 2 e^{3x})\, x_{(3,6)}^2 \Bigg)\,. \end{aligned} \tag{5.3.25}$$

Nonetheless, as we saw before the renormalization process is more easily done in a frame where the dilaton enters the action as a global factor. This corresponds to rescaling the metric by

$$g_{\mu\nu} \to \rho^{4/9} g_{\mu\nu}\,. \tag{5.3.26}$$

It also generates a kinetic term for the dilaton, and the effective action is

$$\begin{aligned} S = \frac{1}{4} \int d^2x\, e\rho \Big(&R + \frac{4}{9} (\rho^{-1} \partial\rho)^2 - \frac{9}{2}\, (\partial_\mu x)(\partial^\mu x) + \frac{3}{2}\, e^{-2x} (8 + 12 e^{3x} + e^{6x}) \\ &- 2\, (\partial x_{(5,1)})^2 - 4\, e^{x} (e^{3x} - 6)\, x_{(5,1)}^2 \\ &- 2\, (\partial x_{(1,20)})^2 + 4\, (2 e^{-2x} + 3 e^{x})\, x_{(1,20)}^2 \\ &- 2\, (\partial x_{(3,6)})^2 + 2\, e^{-2x} (3 + 5 e^{x} + 2 e^{3x})\, x_{(3,6)}^2 \Big)\,. \end{aligned} \tag{5.3.27}$$

Setting $I \equiv \{(5,1)\,,\,(1,20)\,,\,(3,6)\}$, the associated equations of motion are

$$
\begin{aligned}
0 &= \rho^{-1}\nabla\partial\rho - \frac{3}{2}\,e^{-2x}(8+12e^{3x}+e^{6x}) - \sum_{i\in I} F_i(x)\,x_i^2\,,\\
0 &= \rho^{-1}\big(\nabla_\mu\partial_\nu\rho - \frac{1}{2}g_{\mu\nu}\nabla\partial\rho\big) - \frac{4}{9}\,\rho^{-2}(\partial_\mu\rho\partial_\nu\rho - \frac{1}{2}g_{\mu\nu}(\partial\rho)^2) + \frac{9}{2}\,\big(\partial_\mu x\partial_\nu x - \frac{1}{2}\,g_{\mu\nu}(\partial x)^2\big)\,,\\
&\quad + 2\sum_{i\in I}\big(\partial_\mu x_i\partial_\nu x_i - \frac{1}{2}\,g_{\mu\nu}(\partial x_i)^2\big)\,,\\
0 &= R + \frac{4}{9}\,\rho^{-2}(\partial\rho)^2 - \frac{8}{9}\rho^{-1}\nabla\partial\rho - \frac{9}{2}(\partial x)^2 + \frac{3}{2}\,e^{-2x}(8+12e^{3x}+e^{6x})\,,\\
&\quad - 2\sum_{i\in I}\big((\partial x_i)^2 - \frac{F_i(x)}{2}\,x_i^2\big)\,,\\
0 &= \rho^{-1}\nabla\big(\rho\,\partial x\big) - \frac{2}{3}\,e^{-2x}(4-3e^{3x}-e^{6x}) + \frac{1}{9}\,\sum_{i\in I} F_i'(x)\,x_i^2\,,\\
0 &= \rho^{-1}\nabla(\rho\,\partial x_i) + \frac{1}{2}\,F_i(x)\,x_i\,,
\end{aligned}
\tag{5.3.28}
$$

with

$$
F_{(5,1)} = -4\,e^{x}(e^{3x}-6)\,,\quad F_{(1,20)} = 4\,(2e^{-2x}+3e^{x})\,,\quad F_{(3,6)} = 2\,e^{-2x}(3+5e^{x}+2e^{3x})\,.
\tag{5.3.29}
$$

Background

Let us consider the half-maximal supersymmetric background (5.3.8). After going to the Euclidean signature and making the Weyl rescaling (5.3.26) and coordinate change ($x = r^{2/5}$), one recovers the metric of an asymptotically AdS spacetime coupled to a dilaton

$$
\begin{aligned}
d\widehat{s}^2 &= \widehat{f}(r)^2 dt^2 + \widehat{g}(r)^2 dr^2 & \widehat{g}(r) &= \frac{3}{5}x^{-3/2}\,e^{x}(e^{3x}-1)^{-1}\,,\\
\widehat{f}(r) &= e^{\frac{5}{2}x}(e^{3x}-1)^{-5/4}\,, & \rho(r) &= e^{\frac{9}{2}x}(e^{3x}-1)^{-9/4}\,,\\
x(r) &= r^{2/5}\,, & x_i &= 0\,,\quad \forall i\in I\,.
\end{aligned}
\tag{5.3.30}
$$

Indeed, up to some numerical constants that can be absorbed, the previous metric and dilaton match the background (5.2.9) in the limit $(r\to 0)$:

$$
\begin{aligned}
ds^2 &\underset{r\to 0}{\sim} \frac{dt^2}{r} + \frac{dr^2}{4r^2}\\
\rho(t,r) &\underset{r\to 0}{\sim} r^{-9/10}
\end{aligned}
\tag{5.3.31}
$$

According to [78], in this frame, where the metric is asymptotically AdS, the near boundary behavior of the scalar field $x(r)$ allows to identify whether the dual gauge theory corresponds to an operator deformation or a vev deformation. Here,

$$
x(r) = r^{2/5}
\tag{5.3.32}
$$

corresponds to the behavior of $y_2(t)$ in (5.2.60). As we saw in the discussion of page 105, $y_2(t)$ is eventually interpreted as the response and $y_5(t)$ as the source for the coset space scalar fluctuations. Because here, the background scalar $x(r)$ behaves like the response for the coset space scalar fluctuations in the BFSS model holography, the Matrix model dual to the background (5.3.30) corresponds to a vev deformation of BFSS model [78]. As a consequence, the corresponding Matrix model of the $SO(3) \times SO(6)$ preserving, half supersymmetric background (5.3.30) is not the BMN model but another deformation of the BFSS model. Still, we can compute correlation functions from the gravity side and try to interpret them in the light of the gauge/gravity correspondence. This is the point of the following work.

On-shell action and renormalization

Again, the effective action (5.3.27) can be evaluated on-shell using the dilaton field equation. This leads to a boundary term located at the horizon of the asymptotically AdS spacetime background (5.3.30),

$$S = \frac{1}{2} \int_{r=\epsilon} dt \sqrt{|h|} \, (\frac{4}{9} n^\mu \partial_\mu \rho + \rho \, K) \,. \tag{5.3.33}$$

In the following we will treat the different irreducible representations of the scalar fluctuations separately. Here is an ansatz for the fluctuations of the gravity sector,

$$\begin{aligned} f(t,r) &= f_b(r) \, (1 + f_p(t,r)) \,, \\ \rho(t,r) &= \rho_b(r) \, (1 + \rho_p(t,r)) \,, \end{aligned} \tag{5.3.34}$$

where f_b and ρ_b stand for the background (5.3.30) and $\{f_p(t,r), \rho_p(t,r)\}$ are supposed to vanish at the horizon. No source is lit on the gravity side, they were studied in the first section. The metric is assumed to remain diagonal by fixing the diffeomorphisms. Here, let us underline the fact that the scalar $x(t,r)$ is now treated as part of the background: $x(t,r) = r^{2/5}$ according to (5.3.30). On the contrary, the dynamical scalar fields are described by what we call the fluctuations: $x_{(5,1)}$, $x_{(1,20)}$, $x_{(3,6)}$ which will be written schematically x_p. Their equation of motion,

$$\begin{aligned} 0 &= \nabla(\rho \, \partial x_{(5,1)}) - 2\rho \, e^x(-6 + e^{3x}) x_{(5,1)} \,, \\ 0 &= \nabla(\rho \, \partial x_{(1,20)}) + 4\rho \, (e^{-2x} + \frac{3}{2} e^x) x_{(1,20)} \,, \\ 0 &= \nabla(\rho \, \partial x_{3,6}) + \rho \, e^{-2x}(3 + 5e^x + 2e^{3x}) x_{(3,6)} \,, \end{aligned} \tag{5.3.35}$$

evaluated on the background, indicate that a power series expansion in r of the solution should begin with $r^{2/5}$ or r. Moreover the on-shell action (5.3.33) evaluated on the background shows that the dilaton and extrinsic curvature terms diverge as ($r^{-7/5}$ when $r \to 0$). Thus we only need to know an approximating power series expansion in r of the fields, up to order $r^{-7/5}$, because all the other orders will vanish during the renormalization process.

$$\begin{aligned} \sqrt{|h|} \, n^\mu \partial_\mu \rho &\underset{r \to 0}{\sim} r^{-7/5} \,, \\ \sqrt{|h|} \, \rho \, K &\underset{r \to 0}{\sim} r^{-7/5} \,. \end{aligned} \tag{5.3.36}$$

Under these conditions, we are led to propose the following ansatz for the fluctuations

$$
\begin{aligned}
f_p(t,r) &= \sum_{n=1}^{7} f_{(n)}(t)\, r^{n/5}\,, \\
\rho_p(t,r) &= \sum_{n=1}^{7} \rho_{(n)}(t)\, r^{n/5}\,, \\
x_p(t,r) &= \sum_{n=1}^{7} x_{p(n)}(t)\, r^{n/5}\,.
\end{aligned}
\tag{5.3.37}
$$

The expansion of x_p starts at $n = 1$ because we impose that the fluctuations vanish at the boundary $r = 0$. The equations of motion constrain the expansions to

$$
\begin{aligned}
f_p(t,r) &= f_{(4)}(t)\, r^{4/5} + f_{(6)}(t)\, r^{6/5} + f_{(7)}(t)\, r^{7/5}\,, \\
\rho_p(t,r) &= \rho_{(4)}(t)\, r^{4/5} + \rho_{(6)}(t)\, r^{6/5} + \rho_{(7)}(t)\, r^{7/5}\,, \\
x_p(t,r) &= x_{p(2)}(t)\, r^{2/5} + x_{p(4)}(t)\, r^{4/5} + x_{p(5)}(t)\, r + \dots
\end{aligned}
\tag{5.3.38}
$$

where the dots represent terms that will not be relevant in the renormalization procedure. The coefficients are related by

$$
\begin{aligned}
f_{(4)}(t) &= a_4\, x_{p(2)}(t)^2\,, & f_{(6)}(t) &= a_6\, x_{p(2)}(t)^2\,, \\
\rho_{(4)}(t) &= b_4\, x_{p(2)}(t)^2\,, & \rho_{(6)}(t) &= b_6\, x_{p(2)}(t)^2\,, \\
\rho_{(7)}(t) &= c_7\, x_{p(2)}(t) x_{p(5)}(t) + d_7\, f_{(7)}(t)\,, & x_{p(4)}(t) &= x_4\, x_{p(2)}(t)\,.
\end{aligned}
\tag{5.3.39}
$$

In particular some coefficients are left undetermined: $x_{p(2)}(t)$, $x_{p(5)}(t)$ and $f_{(7)}(t)$ or $\rho_{(7)}(t)$. In this sense, $x_{p(2)}(t)$ should be interpreted as a source for the fluctuations and the other coefficients as responses. Eventually, the numerical constants are summarized in the following table:

	a_4	b_4	x_4	a_6	b_6	c_7	d_7
$x_{p(5,1)}$	$-\frac{175}{9}$	-35	-3360	847000	1524600	$-\frac{11440}{9}$	-1
$x_{p(1,20)}$	$-\frac{175}{9}$	-35	4200	-1001000	-1801800	$-\frac{11440}{9}$	-1
$x_{p(3,6)}$	$-\frac{175}{9}$	-35	-12180	3003000	5405400	$-\frac{11440}{9}$	-1

(5.3.40)

Now we are able to evaluate the on-shell action and to renormalize the divergences. Let us recall the on-shell action (5.3.33):

$$
S = \frac{1}{2}\int_{r=\epsilon} dt\, \sqrt{|h|}\, \Big(\frac{4}{9}\, n^\mu \partial_\mu \rho + \rho\, K\Big)\,. \tag{5.3.41}
$$

The divergences that occur when we take the limit $\epsilon \to 0$ are canceled by two counter-terms

$$
\begin{aligned}
S_{\rm ct1} &= \frac{2}{9}\int_{r=\epsilon} dt\, \sqrt{|h|}\, (c_1\, \rho + c_2\, \rho^{5/9} + c_3\, \rho^{1/9} + c_4\, \rho^{-1/3})\,, \\
S_{\rm ct2} &= \frac{2}{9}\int_{r=\epsilon} dt\, \sqrt{|h|}\, (x_1\, \rho + x_2\, \rho^{5/9})\, x_p(t,\epsilon)^2\,.
\end{aligned}
\tag{5.3.42}
$$

The first one corresponds to a cosmological constant and the second is a correction to the scalar potential. The first set of numerical constants do not depend on the fluctuation we are dealing with, whereas the second set of constants do.

	c_1	c_2	c_3	c_4	x_1	x_2
x_p	$-\frac{9}{2}$	0	$-\frac{1}{2}$	$-\frac{2}{9}$	$\frac{4}{9}(9\, a_4 + 4\, b_4)$	$\frac{2}{27}(27\, a_6 + a_4(9 - 36\, x_4) + 4(3\, b_6 + b_4 - 4 x_4 b_4))$

(5.3.43)

Consequently, the renormalized action is given by

$$\begin{aligned} S_{\text{ren}} &= \lim_{\epsilon\to 0} \left(S_{\text{on-shell}} + S_{\text{ct1}} + S_{\text{ct2}}\right) \\ &\propto \int dt \,\left(x_{p(2)}(t)\, x_{p(5)}(t) + \frac{1}{2216}\,\rho_{(7)}(t)\right) . \end{aligned} \tag{5.3.44}$$

This expression for the renormalized action is in complete analogy with the previous results of section 5.2.2 so one could have guessed it. Nonetheless, it is interesting to see that the renormalization process developped in [78] works in each case. In the last step, the coefficients $x_{p(2)}(t)$ and $x_{p(5)}(t)$ should be related in order to find the two-points functions by derivation of the action.

5.3.5 Correlation Functions

Let us focus on the scalar two-points functions. They will be generated by the following action

$$\begin{aligned} S_{\text{gen}} &= \int dt\; x_{p(2)}(t)\, x_{p(5)}(t) \\ &\propto \int dq\; \tilde{x}_{p(2)}(q)\, \tilde{x}_{p(5)}(q) \,, \end{aligned} \tag{5.3.45}$$

where the functions of the momentum q stand for the coefficients of the Fourier transform of x_p. Knowing a relation between these two coefficients

$$\tilde{x}_{p(5)}(q) = C_p(q)\, \tilde{x}_{p(2)}(q) \,, \tag{5.3.46}$$

and identifying $\tilde{x}_{p(2)}(q)$ as the source, the two-points function will be given by

$$\langle \mathcal{O}(0)\mathcal{O}(q)\rangle \propto C_p(q) \,. \tag{5.3.47}$$

In the following subsection the function C_p is determined for each scalar perturbation.

Analytics

The path is well defined: one has to solve the equations of motion for the scalar perturbation, linearized on the background (5.3.30). After taking the Fourier transform with respect to time, we are left with an ordinary second order differential equation in the radial coordinate r. There exists a unique solution that is regular in the bulk (i.e. tends to zero as r goes to infinity) and we are interested in the power series expansion of this solution near the horizon $r = 0$ in order to find the ratio

$$\frac{\tilde{x}_{p(5)}(q)}{\tilde{x}_{p(2)}(q)} \,. \tag{5.3.48}$$

For computational convenience, we will make the change of variable and field redefinition

$$u = \sqrt{e^{3(r^{2/5})} - 1} \,, \qquad \tilde{x}_p(u) \to u^2\, \tilde{x}_p(u) \,. \tag{5.3.49}$$

The field equations translate into

$$
\begin{aligned}
0 &= \tilde{x}''_{(5,1)}(u) + \frac{2}{u}\Big(\frac{2u^2-1}{u^2+1}\Big)\,\tilde{x}'_{(5,1)}(u) - \frac{q^2\,u^3}{(u^2+1)^3}\,\tilde{x}_{(5,1)}(u)\,,\\
0 &= \tilde{x}''_{(1,20)}(u) + \frac{2}{u}\Big(\frac{2u^2-1}{u^2+1}\Big)\,\tilde{x}'_{(1,20)}(u) + \frac{2u^4-q^2\,u^3-2}{(u^2+1)^3}\,\tilde{x}_{(1,20)}(u)\,,\\
0 &= \tilde{x}''_{(3,6)}(u) + \frac{2}{u}\Big(\frac{2u^2-1}{u^2+1}\Big)\,\tilde{x}'_{(3,6)}(u)\\
&\quad + \frac{2u^6-q^2u^5-4u^4-11u^2-5+5(u^2+1)^{1/3}u^2+5u^2(u^2+1)^{1/3}}{u^2\,(u^2+1)^3}\,\tilde{x}_{(3,6)}(u)\,.
\end{aligned}
\tag{5.3.50}
$$

Any solution admits the following expansion at $u=0$

$$
\tilde{x}(q,u) = \alpha(q) + \beta(q)\,u^3 + \underset{u\to 0}{o}(u^3)\,,
\tag{5.3.51}
$$

and the ratio

$$
\frac{\beta(q)}{\alpha(q)} \propto \frac{\tilde{x}_{p(5)}(q)}{\tilde{x}_{p(2)}(q)}
\tag{5.3.52}
$$

is what we would like to determine.

Numerics

There is a unique solution of these equations that is regular in the bulk, and we would like to determine the ratio (up to a global constant factor) each time for this solution. To begin, let us introduce another function

$$
y(q,u) = \tilde{x}(q,u) + \frac{1}{3u}\frac{d\tilde{x}}{du}(q,u)\,,
\tag{5.3.53}
$$

which power expansion begins with

$$
y(q,u) = \alpha(q) + \beta(q)\,u + \underset{u\to 0}{o}(u^3)\,.
\tag{5.3.54}
$$

For each perturbation, the corresponding equation of motion for y, which is now well defined at $u=0$, is solved numerically. In particular, if y_1 and y_2 correspond to the unique solutions with initial conditions

$$
\{\,y_1(0)=1\,,\ y_1'(0)=0\,\}\,,\qquad \{\,y_2(0)=0\,,\ y_2'(0)=1\,\}\,,
\tag{5.3.55}
$$

then, the unique solution y_s regular at $u\to\infty$ may be written, up to a normalization factor,

$$
y_s = y_1 + \kappa(q)\,y_2 \;= 1 + \kappa(q)\,u + \underset{u\to 0}{o}(u^3) = 1 + \frac{\beta(q)}{\alpha(q)}\,u + \underset{u\to 0}{o}(u^3)\,.
\tag{5.3.56}
$$

First numerical investigation suggest that

$$
\ln\Big(\frac{\tilde{x}_{(5,1)\,(5)}(q)}{\tilde{x}_{(5,1)\,(2)}(q)}\Big) \underset{q\to\infty}{\sim} 1.19\,\ln q\,,\qquad \ln\Big(\frac{\tilde{x}_{(1,20)\,(5)}(q)}{\tilde{x}_{(1,20)\,(2)}(q)}\Big) \underset{q\to\infty}{\sim} 1.20\,\ln q\,.
\tag{5.3.57}
$$

The exponents may be compared with the ratio of $q^{6/5}$ obtained in the BFSS holography for the **44** scalar excitations (5.2.79). Accordingly, the sources correspond respectively to $\{\tilde{x}_{(5,1)\,(5)}(q),\,\tilde{x}_{(1,20)\,(5)}(q)\}$ and the responses to $\{\tilde{x}_{(5,1)\,(2)}(q),\,\tilde{x}_{(1,20)\,(2)}(q)\}$. Still, a detailed account of the asymptotic behavior of (5.3.57) and its implications for the two-points correlation functions and the dual gauge theory, is part of a work in progress [83].

5.4 Summary

This chapter was devoted to the holography of the non-conformal D0-brane. Through this procedure, two-points correlation functions associated with gravity and scalar sector excitations were computed from the gravity side. They were compared with correlation functions of operators in the dual BFSS Matrix model obtained in previous works. Although a generalization to the BMN model holography was considered, it was shown that the later found half-supersymmetric gravity background preserving $SO(3) \times SO(6)$, does not correspond to this Matrix model, but rather to a vev deformation of the BFSS model. Higher dimensional origin of the background was discussed and Holographic renormalization techniques were applied to compute two-points correlation functions. Finally, a numerical analysis gave some insights about their asymptotic behavior.

Chapter 6

Conclusion

To conclude this thesis work, let us summarize our goals and present outlooks. As an anchor point for the beginning of the thesis we decided to construct the $SO(9)$ gauged maximal supergravity in two dimensions. It was motivated by the AdS/CFT correspondence, since this theory filled the last gap in the list of the effective gravity theories accounting for the holography of the Dp-branes, see Table 2.1. On another hand, the explicit construction put an end to a work started several years ago in [108], which led to the discovery of all possible gaugings of maximal supergravity in three dimensions [103] [107]. Then, all the consistent gaugings of the two-dimensional maximal supergravity were identified group theoretically [94]. There "remained" to construct explicitly the $SO(9)$ gauged supergravity, and this constituted the first result of our thesis [81]. To account for it, we began with a general presentation of maximal supegravities in Chapter 2. A particular emphasis was put on the ungauged maximal supergravity in three dimensions and the unique eleven dimensional supergravity since they yield two important formulations of the ungauged maximal supergravity in two dimensions: the E_8 and the $SL(9)$ frames. Then, the general gaugings of maximal supergravity was presented through the embedding tensor formalism. It was applied to the three-dimensional maximal supergravity, paving the way to the more complicated structure of the gaugings in two dimensions.

The explicit construction of the $SO(9)$ gauged $\mathcal{N} = 16$, $D = 2$ supergravity was described in Chapter 3. The E_8 and the $SL(9)$ frames were obtained by dimensional reduction. The first one leads to the most compact formulation of the ungauged two-dimensional maximal supergravity and thus was used to introduce important objects. In particular, we discussed the scalar auxiliary fields associated with Noether currents, and the underlying infinite dimensional symmetry structure of the theory, realized on-shell. The infinite dimensional symmetry group E_9 was analyzed in the $SL(9)$ frame, where in the embedding tensor formalism, the right coupling between vector fields and an $SO(9)$ subgroup of the off-shell symmetries was identified. Then, the vector fields were introduced in the Lagrangian via a coupling with scalar auxiliary fields and proved useful to restore the supersymmetry of the covariantized Lagrangian. Supersymmetry was recovered by following a Noether procedure and led to the introduction of Yukawa couplings and a scalar potential. A unique explicit solution of the linear and quadratic constraints was given, and this ended the construction of the $SO(9)$ gauged maximal supergravity in two dimensions. By integrating out the auxiliary fields, another on-shell equivalent formulation of the theory was found, where a two-dimensional Yang-Mills term is generated. It would correspond to the warped sphere reduction of type IIA supergravity and this motivated our following work on consistent truncations. Moreover, a particular half BPS, domain wall vacuum solution was found in the two-dimensional theory and this opened the path to holographic applications.

In these perspectives our second and third works may be viewed as applications. To begin, we dealt with the embedding of the two-dimensional supergravity into ten and eleven dimensions. This shed light on the higher dimensional origins of the $SO(9)$ gauged supergravity. Thus we considered a Cartan truncation of the two-dimensional theory and showed that it is consistent. These results [82] were presented in Chapter 4, where the embedding of the dilaton sector was constructed explicitly. As an application, we uplifted the domain-wall solution to ten dimensions, and we recovered the D0-brane. This gave further motivations to possible holographic applications. Hence we went to eleven dimensions and we noticed that the domain-wall corresponded to a well known solution: a pp-wave. Eventually, a generalization of the embedding to non-vanishing axions was envisaged. It constitutes an important outlook, since this would be the next step towards a proof that the full spherical reduction of type IIA is consistent.

The third and last work of our thesis concerns the gravity/gauge correspondence. As described in Chapter 5, we applied holography renormalization techniques [78] [77] to study different half supersymmetric backgrounds that are expected to provide informations about dual one-dimensional Matrix models. Thus, we computed correlations functions for operators in the BFSS model, from a gravity side analysis around an $SO(9)$ preserving domain-wall background solution. Then, the procedure was generalized to a half supersymmetric gravity background which breaks $SO(9)$ to $SO(3)\times SO(6)$. This was motivated by the search for holographic dual of the BMN model. Nevertheless, after a thorough investigation of the half BPS background, we concluded that the dual Matrix model is not the BMN model but a $SO(3) \times SO(6)$ preserving, supersymmetric vev deformation of the BFSS model. Still a complete identification of the gauge theory side remains to be done and will be of great interest. Another important outlook would be the computation of the gravity background leading to the holography of the BMN model. These different issues are left to future investigations.

Appendix A

Weyl rescaling

In this chapter, we consider a real differential manifold $\mathcal{M}$ of dimension D endowed with a metric g.

A.1 Local Weyl rescaling

Let us collect some results about the behaviour of the Ricci tensor and scalar with respect to a local Weyl rescaling of the metric. The work is done at the level of the vielbein where the local Weyl recaling is performed by

$$\big(e_\mu{}^a\big)' = e^{\phi(x)}\, e_\mu{}^a \tag{A.1.1}$$

Under this transformation, the spin connection transforms as

$$\big(\omega_\mu{}^{ab}\big)' = \omega_\mu{}^{ab} - 2e^{\nu[a}e_\mu{}^{b]}\,\big(\partial_\nu\phi\big)\ , \tag{A.1.2}$$

and the Ricci tensor is given by

$$\big(R_{\mu\nu}\big)' = R_{\mu\nu} + (2-D)\nabla_\mu\partial_\nu\phi - g_{\mu\nu}\nabla\partial\phi + (D-2)\big((\partial_\mu\phi)(\partial_\nu\phi) - g_{\mu\nu}(\partial\phi)^2\big)\ . \tag{A.1.3}$$

The Ricci scalar follows directly

$$R' = e^{-2\phi}\left[R - 2(D-1)\nabla\partial\phi - (D-1)(D-2)(\partial\phi)^2\right]\ . \tag{A.1.4}$$

This enables to study the Weyl rescaling of the Einstein-Hilbert action,

$$\begin{aligned}\int d^Dx\ \big(e_D\, R\big)' &= \int d^Dx\ e_D\, e^{(D-2)\phi}\left[R - 2(D-1)\nabla\partial\phi - (D-1)(D-2)(\partial\phi)^2\right]\\ &= \int d^Dx\ e_D\, e^{(D-2)\phi}\left[R + (D-1)(D-2)(\partial\phi)^2\right]\quad +\ \text{boundary term}\ ,\end{aligned} \tag{A.1.5}$$

where $(d^Dx\ e_D)$ stands for the canonical volume form in D dimensions.

D=2 Notice that in two dimensions, the Einstein-Hilbert action is invariant under local Weyl rescaling. Actually, this two-dimensional action is trivial since, the associated equations of motion for the metric are identically satisfied. In fact, the action is proportional to the Euler characteristic of the manifold $\mathcal{M}$, which is a topological invariant.

A.2 Gravity coupled to a dilaton

Now if we are interested in Einstein gravity coupled to a dilaton, we may wonder how the Einstein-Hilbert term transforms. Let us consider the local Weyl rescaling

$$\big(e_\mu{}^a\big)' = e^{\alpha\,\phi(x)}\,e_\mu{}^a \tag{A.2.1}$$

where α is a real constant. Then,

$$\begin{aligned}\int d^Dx\,\big(e_D\,e^\phi\,R\big)' &= \int d^Dx\,e_D\,e^{(\alpha(D-2)+1)\phi}\Big[R - 2\alpha(D-1)\nabla\partial\phi - (D-1)(D-2)\alpha^2(\partial\phi)^2\Big]\\ &= \int d^Dx\,e_D\,e^{(\alpha(D-2)+1)\phi}\Big[R + \alpha(D-1)\big(2+\alpha(D-2)\big)(\partial\phi)^2\Big]\\ &\quad + \text{ boundary term },\end{aligned} \tag{A.2.2}$$

D=2 In the two-dimensional case, the action transforms as

$$\int d^2x\,\big(e_2\,e^\phi\,R\big)' = \int d^2x\,e_2\,e^\phi\Big[R + 2\alpha\,(\partial\phi)^2\Big] \quad + \text{ boundary term }, \tag{A.2.3}$$

and a kinetic term for the dilaton is generated.

Appendix B

Relations among Yukawa tensors

Supersymmetry of the Lagrangian (3.4.22) requires a number of linear, differential, and quadratic relations among the Yukawa tensors A, B, C, D, E, F introduced in (3.4.13). In this appendix we list these relations, ordered by their origin. They have been used in the main text in order to find the (unique) solution (3.4.30), (3.4.31) for the Yukawa tensors in terms of the scalar fields.

B.1 Linear relations among the Yukawa tensors

Demanding that all terms linear in space-time derivatives cancel in the supersymmetry variation of (3.4.22) implies a number of relations linear in the Yukawa tensors. The cancellation of terms carrying $\partial_\mu \rho$ induces

$$
\begin{aligned}
A_{IJ} - A_{JI} &= 0 \,, & \tilde{A}_{IJ} + \tilde{A}_{JI} &= 0 \,, \\
A_{IJ} - \tilde{B}_{IJ} - \rho \frac{\partial \tilde{B}_{IJ}}{\partial \rho} &= 0 \,, & \tilde{A}_{IJ} + B_{IJ} + \rho \frac{\partial B_{IJ}}{\partial \rho} &= 0 \,, \\
D_{IJ} + \rho \frac{\partial A_{IJ}}{\partial \rho} &= 0 \,, & \tilde{D}_{IJ} + \rho \frac{\partial \tilde{A}_{IJ}}{\partial \rho} &= 0 \,, \\
C^a_{IJ} + 2\rho \frac{\partial C^a_{IJ}}{\partial \rho} + \tilde{E}^a_{IJ} &= 0 \,, & \tilde{C}^a_{IJ} + 2\rho \frac{\partial \tilde{C}^a_{IJ}}{\partial \rho} - E^a_{IJ} &= 0 \,. \qquad \text{(B.1.1)}
\end{aligned}
$$

The cancellation of terms carrying $\widetilde{\varphi}^{abc}_\mu$ induces

$$
\begin{aligned}
0 &= 3\tilde{C}^{[a}_{K[I}\Gamma^{bc]}_{J]K} + \frac{1}{3} B_{K[I}\Gamma^{abc}_{J]K} - 3\,\rho^{-5/9} T^{de} \Gamma^{d[a}_{IJ} \varphi^{bc]e} \,, \qquad \text{(B.1.2)} \\
0 &= 3C^{[a}_{K(I}\Gamma^{bc]}_{J)K} + \frac{1}{3}\,\tilde{B}_{K(I}\Gamma^{abc}_{J)K} + 6\,\rho^{-8/9} \Gamma^{d}_{IJ} T^{ef} \varphi^{de[a} \varphi^{bc]f} \,, \\
0 &= \rho^{1/3}\, \frac{\partial \tilde{B}_{IJ}}{\partial \varphi^{abc}} + \frac{1}{9}\, B_{K(I}\Gamma^{abc}_{J)K} - \frac{1}{2}\, \tilde{C}^{[a}_{K(I}\Gamma^{bc]}_{J)K} - \frac{1}{54}\, \rho^{-11/9} \Gamma^{d}_{IJ} \varepsilon^{efghijabc} T^{kl} \varphi^{dke} \varphi^{lfg} \varphi^{hij} \,, \\
0 &= \rho^{1/3}\, \frac{\partial B_{IJ}}{\partial \varphi^{abc}} - \frac{1}{9}\, \tilde{B}_{K[I}\Gamma^{abc}_{J]K} + \frac{1}{2}\, C^{[a}_{K[I}\Gamma^{bc]}_{J]K} - \frac{2}{27}\, \rho^{-8/9} T^{i[d} \Gamma^{efghabc]}_{IJ} \varphi^{ide} \varphi^{fgh}.
\end{aligned}
$$

The cancellation of terms carrying $\mathcal{P}_\mu^{ab}$ induces

$$
\begin{aligned}
&0 = 2C^{(a}_{K[I}\Gamma^{b)}_{J]K} + \rho^{-2/9}\Gamma^{c(a}_{IJ}T^{b)c}\,, &&0 = \mathcal{P}_\mu^{ab}\Big(\frac{\partial B_{IJ}}{\partial\Sigma^{ab}} + 3\,\varphi^{acd}\frac{\partial B_{IJ}}{\partial\varphi^{bcd}} + \tilde{C}^a_{K[I}\Gamma^b_{J]K}\Big)\,,\\
&0 = 2\tilde{C}^{(a}_{K(I}\Gamma^{b)}_{J)K} - 2\,\rho^{-5/9}\Gamma^c_{IJ}T^{d(a}\varphi^{b)cd}\,, &&0 = \mathcal{P}_\mu^{ab}\Big(\frac{\partial \tilde{B}_{IJ}}{\partial\Sigma^{ab}} + 3\,\varphi^{acd}\frac{\partial \tilde{B}_{IJ}}{\partial\varphi^{bcd}} - C^a_{K(I}\Gamma^b_{J)K}\Big)\,,\\
&0 = \ \chi^{aJ}\mathcal{P}^{eb}_\mu\Big(\frac{\partial C^a_{JI}}{\partial\Sigma^{eb}} + 3\varphi^{cde}\frac{\partial C^a_{JI}}{\partial\varphi^{bcd}} - \frac12\delta^{ea}\tilde{B}_{KI}\Gamma^b_{JK} - \Gamma^e_{KI}F^{ab}_{JK} - \rho^{-2/9}\Gamma^c_{IJ}T^{b[c}\delta^{a]e}\Big)\,,\\
&0 = \ \chi^{aJ}\mathcal{P}^{eb}_\mu\Big(\frac{\partial \tilde{C}^a_{JI}}{\partial\Sigma^{eb}} + 3\varphi^{cde}\frac{\partial \tilde{C}^a_{JI}}{\partial\varphi^{bcd}} - \frac12\delta^{ea}B_{KI}\Gamma^b_{JK} + \Gamma^e_{KI}\tilde{F}^{ab}_{JK} + \rho^{-5/9}\delta_{IJ}T^{ce}\varphi^{abc}\Big)\,,
\end{aligned}
\tag{B.1.3}
$$

where the $SO(9)_{\rm coset}$ covariant variation $\partial/\partial\Sigma^{ab}$ is defined by $\delta_\Sigma\mathcal{V}_m{}^a \equiv \mathcal{V}_m{}^c\Sigma^{ac}$ with Σ^{ac} traceless. The cancellation of terms carrying $\mathcal{D}_\mu Y_k{}^l$ finally induces

$$
\begin{aligned}
&\frac{\partial A_{IJ}}{\partial Y_k{}^l} - \frac59\,\rho^{-14/9}\,\Gamma^a_{IJ}\theta_{ml}\,\mathcal{V}^{-1km}{}_{bc}\,\varphi^{abc} = 0\,, &&\frac{\partial \tilde{A}_{IJ}}{\partial Y_k{}^l} + \frac19\rho^{-11/9}\,\Gamma^{ab}_{IJ}\theta_{ml}\,\mathcal{V}^{-1km}{}_{ab} = 0\,,\\
&\frac{\partial B_{IJ}}{\partial Y_k{}^l} + \frac12\,\rho^{-11/9}\,\Gamma^{ab}_{IJ}\theta_{ml}\,\mathcal{V}^{-1km}{}_{ab} = 0\,, &&\frac{\partial \tilde{B}_{IJ}}{\partial Y_k{}^l} + \rho^{-14/9}\Gamma^a_{IJ}\theta_{ml}\,\mathcal{V}^{-1km}{}_{bc}\,\varphi^{abc} = 0\,,\\
&\frac{\partial C^a_{IJ}}{\partial Y_k{}^l} + \rho^{-14/9}\,\delta_{IJ}\theta_{ml}\,\mathcal{V}^{-1km}{}_{bc}\,\varphi^{abc} = 0\,, &&\frac{\partial \tilde{C}^a_{IJ}}{\partial Y_k{}^l} - \rho^{-11/9}\,\Gamma^b_{IJ}\theta_{ml}\,\mathcal{V}^{-1km}{}_{[ab]} = 0\,.
\end{aligned}
\tag{B.1.4}
$$

B.2 Quadratic relations among the Yukawa tensors

The remaining identities that supersymmetry imposes on the Yukawa tensors are bilinear in these tensors. They lead to the following set of equations

$$
\begin{aligned}
0 &= 2A_{K(I}B_{J)K} + 2\tilde{A}_{K(I}\tilde{B}_{J)K} + C^k_{KI}\tilde{C}^k_{KJ} + C^k_{KJ}\tilde{C}^k_{KI}\,,\\
0 &= 2\tilde{B}_{K(I}A_{J)K} + 2B_{K(I}\tilde{A}_{J)K} - C^a_{KI}C^a_{KJ} - \tilde{C}^a_{KI}\tilde{C}^a_{KJ} + \frac12\,\rho^{-1}\,\delta_{IJ}\,\frac{\partial V_{\rm pot}}{\partial\sigma}\,,\\
0 &= -4A_{K(I}\tilde{A}_{J)K} + 2D_{IK}B_{KJ} + 2\tilde{D}_{IK}\tilde{B}_{KJ} + E^a_{KI}C^a_{KJ} - \tilde{E}^a_{KI}\tilde{C}^a_{KJ}\\
&\quad - \frac{\partial V_{\rm pot}}{\partial Y_k{}^l}\Big(\frac12\,\mathcal{V}^{-1al}\mathcal{V}_k{}^b\,\Gamma^{ab}_{IJ} + \frac1{54}\,\rho^{-2/3}\,\mathcal{V}^{-1gl}\mathcal{V}_k{}^d\,\varphi^{abc}\varphi^{efg}\,\Gamma^{abcdef}_{IJ}\Big)\\
&\quad + \frac16\,\rho^{-2/3}\frac{\partial V_{\rm pot}}{\partial\varphi^{abc}}\,\Gamma^{abc}_{IJ}\,,\\
0 &= -2A_{IK}A_{KJ} + 2\tilde{A}_{IK}\tilde{A}_{KJ} + 2D_{IK}\tilde{B}_{KJ} + 2\tilde{D}_{IK}B_{KJ} + E^a_{KI}\tilde{C}^a_{KJ} - \tilde{E}^a_{KI}C^a_{KJ}\\
&\quad - \delta_{IJ}\,\frac{\partial V_{\rm pot}}{\partial\rho} - \frac13\,\rho^{-1/3}\,\mathcal{V}^{-1gl}\mathcal{V}_k{}^a\varphi^{bcg}\Gamma^{abc}_{IJ}\frac{\partial V_{\rm pot}}{\partial Y_k{}^l}\,,\\
0 &= -2C^a_{IK}A_{KJ} - 2\tilde{C}^a_{IK}\tilde{A}_{KJ} + E^a_{IK}B_{KJ} + \tilde{E}^a_{IK}\tilde{B}_{KJ} + 2F^{ab}_{IK}C^b_{KJ} + 2\tilde{F}^{ab}_{IK}\tilde{C}^b_{KJ}\\
&\quad - \rho^{-1}\,\frac{\partial V_{\rm pot}}{\partial\Sigma^{ab}}\Gamma^b_{IJ} - 3\,\rho^{-1}\,\varphi^{bc(a}\frac{\partial V_{\rm pot}}{\partial\varphi^{d)bc}}\Gamma^d_{IJ} - \frac32\,\rho^{-1/3}\,\mathcal{V}^{-1gl}\mathcal{V}_k{}^{[a}\varphi^{bc]g}\Gamma^{bc}_{IJ}\,\frac{\partial V_{\rm pot}}{\partial Y_k{}^l}\,,\\
0 &= 2C^a_{IK}\tilde{A}_{KJ} + 2\tilde{C}^a_{IK}A_{KJ} + E^a_{IK}\tilde{B}_{KJ} + \tilde{E}^a_{IK}B_{KJ} + 2F^{ab}_{IK}\tilde{C}^b_{KJ} + 2\tilde{F}^{ab}_{IK}C^b_{KJ}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\partial V_{\rm pot}}{\partial Y_k{}^l}\, \mathcal{V}_k{}^b \mathcal{V}^{-1\,cl} \left(\frac{1}{6}\, \rho^{-2/3} \left(\varphi^{agh} \varphi^{efc} \delta^{db} - \delta^{b[a} \varphi^{gh]c} \varphi^{def} \right) \Gamma^{defgh}_{IJ} - \delta^{a(b}\, \Gamma^{c)}_{IJ} \right) \\
&- \frac{3}{2}\, \rho^{-2/3}\, \frac{\partial V_{\rm pot}}{\partial \varphi^{abc}}\, \Gamma^{bc}_{IJ}\ , \qquad\qquad \text{(B.2.1)}
\end{aligned}
$$

where the last two equations should be understood as projected onto their gamma-traceless part in the indices aI. Remarkably, it turns out that all these equations are identically satisfied for the solution (3.4.30), (3.4.31) of the linear relations given in section B.1. This is a confirmation of the prediction of [94] discussed in section 3.3.1 above that any embedding tensor of the type θ_{kl} automatically satisfies the relevant quadratic constraints and thus defines a consistent gauged theory compatible with maximal supersymmetry.

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