Optimal transport problems with gradient penalization

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Problèmes de transport optimal avec pénalisation en gradient

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Résumé

Le probléme du transport optimal, désormais classique en calcul des variations et originellement introduit par Monge au 18ème siècle, consiste à minimiser l'énergie nécessaire au déplacement d'une masse dont la répartition est donnée vers une autre masse dont la répartition est elle aussi donnée, ce qui se traduit mathématiquement par

$$\inf \left\{ \int c(x, T(x)) \, \mathrm{d}\mu(x) \, : \, T_{\#}\mu = \nu \right\}$$

où $c(x, T(x))$ est le coût nécessaire pour déplacer x vers $T(x)$ et les mesures μ et ν représentent les répartitions de masse de départ et d'arrivée.

Cette thèse est consacrée à l'étude de problèmes variationnels similaires où l'on fait intervenir la matrice jacobienne de la fonction de transport, du type

$$\inf \left\{ \int L(x, T(x), DT(x)) \, \mathrm{d}\mu(x) \, : \, T_{\#}\mu = \nu \right\} ;$$

il s'agit typiquement de rajouter $\int |DT(x)|^2 \, \mathrm{d}x$ à la fonctionnelle $\int c(x, T(x)) \, \mathrm{d}\mu(x)$ afin d'obtenir une pénalisation Sobolev. Ce type de problème trouve ses motivations en mécanique des milieux continus, élasticité incompressible ou en analyse de forme et appelle d'un point de vue mathématique une approche totalement différente de celle du problème de transport usuel.

Les questions suivantes sont envisagées :

- bonne définition du problème, notamment du terme $\int |DT|^2 \, \mathrm{d}\mu$, *via* les espaces de Sobolev par rapport à une mesure, et résultats d'existence de minimiseurs ;

- caractérisation de ces minimiseurs : optimalité du transport croissant sur la droite réelle, et approche du type équation d'Euler-Lagrange en dimension quelconque ;

- sélection d'un minimiseur *via* une procédure de pénalisation du type Γ-convergence (on ajoute $\varepsilon \int |DT|^2$ et on fait tendre ε vers 0) lorsque le coût de transport est le coût de Monge $c(x, y) = |x - y|$, pour lequel l'application de transport optimale n'est pas unique ;

- autres approches du problème et perspectives : formulation dynamique du type Benamou-Brenier, et formulation duale similaire à celle de Kantorovitch dans le cas du problème du transport optimal usuel.

Abstract

The optimal transportation problem, henceforth classical in the calculus of variations, was originally introduced by Monge in the 18th century; it consists in minimizing the total energy of the displacement of a given repartition of mass onto another given repartition of mass. This is mathematically expressed as follows:

$$\inf\left\{\int c(x,T(x))\,\mathrm{d}\mu(x) \;:\; T_{\#}\mu=\nu\right\}$$

where $c(x,T(x))$ is the cost to send x onto $T(x)$, and the measure μ and ν represent the repartitions of source and target masses.

This thesis is devoted to similar variational problems, which involve the Jacobian matrix of the transport map, namely

$$\inf\left\{\int L(x,T(x),DT(x))\,\mathrm{d}\mu(x) \;:\; T_{\#}\mu=\nu\right\}\;;$$

we typically add $\int |DT|^2$ to the transport functional $\int c(x,T(x))\,\mathrm{d}\mu(x)$ in order to obtain a Sobolev-type penalization. This kind of constraints finds its motivations in continuum mechanics, incompressible elasticity or shape analysis, and a quite different mathematical approach than in the usual theory of optimal transportation is needed.

We consider the following questions:

- proper definition of the problem, in particular of the term $\int |DT|^2\,\mathrm{d}\mu$, thanks to the theory of Sobolev spaces with respect to a measure, and existence results;
- characterizations of these minimizers: optimality of the monotone transport map on the real line, and Euler-Lagrange-like approach in any dimension;
- selection of a minimizer *via* a Γ-convergence-like penalization procedure (namely adding $\varepsilon\int |DT|^2$ to the transport cost, where ε is a vanishing positive parameter) where the transport cost is the Monge cost $c(x,y)=|x-y|$ (for which the optimal transport map is not unique);
- other related problems and perspectives: dynamic Benamou-Brenier-like formulation, and dual Kantorovich-like formulation.

Table des matières

Introduction

L'objectif de cette thèse est d'étudier les problèmes variationnels du type « transport optimal » (minimiser l'énergie totale du déplacement parmi les fonctions vectorielles à mesure image prescrite) auxquels on ajoute un terme dépendant du gradient de l'application de transport. Il s'agit d'une généralisation naturelle et qui trouve ses motivations en mécanique des milieux continus, élasticité ou analyse de forme, mais qui a pourtant été peu ou pas étudiée jusqu'ici. Cette thèse est consacrée à l'analyse mathématique de ce type de problèmes, sous l'angle des questions usuelles du calcul de variations.

0.1 Transport optimal

Le problème du transport optimal a été introduit par Gaspard Monge en 1781 dans son célèbre article *Mémoire sur la théorie des déblais et des remblais* [62]. Il s'exprime dans sa motivation originelle de la façon suivante : imaginons que l'on ait d'une part un ou plusieurs tas de sable (les *déblais*) dont la répartition est connue et représentant une masse considérable, destinés à combler des trous ou des renfoncements (les *remblais*) dont le volume total est exactement celui du tas de sable à déplacer et dont la profondeur, variable en fonction du point où on se place, est elle aussi connue. Quelle est alors la façon *optimale*, c'est-à-dire la plus économe en énergie, de déplacer le tas de sable ?

Ce problème se formalise mathématiquement comme suit. Le tas de sable de départ est représenté par une mesure μ sur un espace X (la mesure $\mu(A)$ d'un ensemble A représentant la masse de sable contenu dans l'ensemble A). Le trou à combler est lui aussi représenté par une mesure ν sur un espace Y ($\nu(B)$ représentant la place disponible dans l'ensemble B, par exemple son volume). Le déplacement de sable sera simplement une fonction $T : X \to Y$ (le sable présent au point x étant envoyé au point $T(x)$), avec la contrainte suivante : étant donné un ensemble B dans l'espace d'arrivée, la masse de sable envoyée dans B doit remplir exactement le volume disponible dans B, c'est-à-dire

$$\mu(A) = \nu(B) \qquad \text{si } A = T^{-1}(B)$$

Enfin, l'on se donne une *fonction de coût* c sur $X \times Y$ à valeurs réelles (éventuellement infinies) : $c(x, y)$ sera l'énergie dépensée pour envoyer le sable présent au point x sur le point y. Étant donné un transport T, l'énergie totale dépensée pour envoyer les déblais X sur les remblais Y par ce transport-là sera donc

$$\int_X c(x, T(x))\, \mathrm{d}\mu(x)$$

et il s'agit de minimiser cette énergie parmi toutes les fonctions $T : X \to Y$ qui sont des transports. Le problème introduit par Monge est donc :

$$\inf \left\{ \int_X c(x, T(x))\, \mathrm{d}\mu(x) \;:\; T : X \to Y,\, T_{\#}\mu = \nu \right\}$$

D'un point de vue mathématique, il s'agit d'un problème variationnel (minimisation d'un critère intégral dans une classe de fonctions). La difficulté réside dans la contrainte définissant les fonctions admissibles, puisque même lorsque les mesures μ, ν sont données par des densités régulières sur des sous-ensembles eux aussi réguliers de $\mathbb{R}^d$, la contrainte $T_\#\mu = \nu$ est hautement non-linéaire et compliquée à exprimer. Dans le meilleur des cas, ici une fonction T régulière et injective, on peut utiliser la formule du changement de variable pour l'exprimer sous la forme

$$\det DT(x) = \frac{f(x)}{g(T(x))}$$

Cette équation est appelée *équation de Monge-Ampère* ; même si la densité g est constante, il s'agit d'une équation multi-linéaire par rapport aux composantes de DT.

En particulier, la méthode la plus naturelle pour démontrer ne serait-ce que l'existence d'une solution optimale, à savoir la méthode directe du calcul des variations (étant donné une suite $(T_n)_n$ dont l'énergie totale approche l'énergie optimale, on cherche à extraire une suite $(T_{n_k})_k$ ayant une limite qui sera un candidat à l'optimalité) ne fonctionne pas ici : en effet, dans le cas par exemple de mesures définies sur $\mathbb{R}^d$ et à support borné, la seule borne sur une telle suite minimisante est une borne en norme L^∞ (chaque T_n étant un transport, il doit nécessairement prendre ses valeurs dans le support de la mesure d'arrivée qui est borné) qui induit seulement une convergence faible, et ce mode de convergence n'est pas suffisant pour faire passer à la limite la contrainte sur la mesure image.

L'approche qui aboutit à l'existence d'un minimiseur est celle introduite par Kantorovitch dans les années 1940 [52, 53]. Elle se base sur la remarque suivante : étant donné un transport T, si l'on note γ_T la mesure sur l'espace produit $X \times Y$ définie par

$$\int_{X\times Y} \varphi(x,y)\,\mathrm{d}\gamma_T(x,y) = \int_{X\times Y} \varphi(x,T(x))\,\mathrm{d}\mu(x)$$

(c'est donc la mesure image de μ par l'application $x \mapsto (x, T(x))$) alors l'égalité suivante vient immédiatement :

$$\int_X c(x,T(x))\,\mathrm{d}\mu(x) = \int_{X\times Y} c(x,y)\,\mathrm{d}\gamma_T(x,y)$$

En utilisant la contrainte sur la mesure image pour T, on voit qu'une telle mesure γ_T doit vérifier

$$(\pi_1)_\#\gamma_T = \mu \qquad \text{et} \qquad (\pi_2)_\#\gamma_T = \nu \qquad \text{où} \quad \begin{cases} \pi_1(x,y) = x \\ \pi_2(x,y) = y \end{cases}$$

c'est-à-dire que ses *marginales* (ses mesures images par les projections π_1 et π_2) sont prescrites et égales aux données μ, ν. Un γ ayant μ et ν pour marginales est appelé *plan de transport* entre μ et ν, et l'on dit qu'il est *induit par* T lorsqu'il est effectivement de la forme γ_T. Si l'on note $\Pi(\mu,\nu)$ l'ensemble des plans de transport entre μ et ν, le problème proposé par Kantorovitch est alors

$$\inf\left\{\int_{X\times Y} c(x,y)\,\mathrm{d}\gamma(x,y) \;:\; \gamma\in\Pi(\mu,\nu)\right\}$$

Par rapport à la motivation du problème originel, là où la donnée de la fonction de transport T répond à la question « en quel point (unique) de l'espace d'arrivée déplace-t-on la quantité de déblai située en chaque x », la donnée d'un plan de transport γ correspondrait à la question « comment répartit-on *dans tout l'espace* Y (et non plus en un seul point $T(x)$) la même quantité de déblai » : informellement, à chaque x, on n'associe plus un point unique $T(x)$ mais une répartition qui serait $\gamma(x,\cdot)$ (et qui, mathématiquement, correspond à la *désintégration* de la mesure γ sur $X \times Y$ par rapport à la mesure μ sur X).

Mathématiquement, le problème de Kantorovitch possède un certain nombre de grands avantages :

- l'ensemble $\Pi(\mu,\nu)$ sur lequel on minimise n'est jamais vide puisqu'il contient la mesure produit $\mu\otimes\nu$ (là où l'on peut facilement construire des mesures μ, ν pour laquelle il n'existe pas de fonctions de transport : il suffit que μ soit égale à une masse de Dirac et pas ν) et est même un sous-ensemble convexe de l'ensemble des mesures positives sur l'espace produit ;
- la fonctionnelle que l'on minimise, $\gamma \mapsto \displaystyle\int c\,\mathrm{d}\gamma$, est linéaire par rapport à γ.

Ces arguments permettent de déduire assez rapidement l'existence d'un minimiseur au problème de Kantorovitch, et du fait que ce dernier est une généralisation du problème de Monge vient immédiatement l'inégalité

$$\min(K) \leq \inf(M)$$

où (K) est le problème de Kantorovitch et (M) est le problème de Monge.

On montre alors que, sous des bonnes hypothèses sur les mesures μ et ν et sur la fonction de coût c, l'optimum du problème de Kantorovitch est concentré (en tant que mesure) sur le graphe d'une fonction $y = T(x)$, et T est alors optimal pour le problème de Monge. Le cas le plus classique du coût égal au carré de la distance a été résolu par Brenier [20, 22], qui donne de plus une caractérisation du transport optimal comme étant le gradient d'une fonction convexe.

La régularité des transports optimaux a elle aussi fait l'objet de travaux importants. Toujours dans le cas quadratique, Caffarelli [24, 25, 26] a donné des conditions suffisantes pour que le transport optimal soit de classe $C^{k,\alpha}$; plus récemment, De Philippis et Figalli [34] ont prouvé sa régularité $W^{1,1}_{loc}$ sous des hypothèses très faibles (mesures à densités bornées inférieurement et supérieurement). Ces résultats sont en fait directement liés à la structure de T dans le cas quadratique et au fait qu'il résout l'équation de Monge-Ampère citée plus haut tout en étant le gradient d'une fonction convexe.

On renvoie aux notes de cours de Filippo Santambrogio [65, à paraître] et au livre de Cédric Villani [69], ainsi qu'à l'annexe B, pour plus de détails et les preuves formelles sur la théorie classique du transport optimal.

0.2 Pénalisation en gradient

Dans cette thèse, on introduit une généralisation naturelle du problème du transport optimal : que se passe-t-il si, dans la fonctionnelle $\int_X c(x,T(x))\,\mathrm{d}\mu(x)$, on ajoute un terme comprenant la matrice jacobienne ? On est alors amenés à minimiser, par exemple

$$\int_\Omega L(x, T(x), DT(x))\,\mathrm{d}x$$

parmi l'ensemble des fonctions T à mesure image $T_\#\mu$ prescrite. On peut voir ce problème simplement comme le fait d'*imposer* au transport T d'être régulier, plutôt que de déduire sa régularité de ses propriétés d'optimalité, mais le but est aussi de relier la théorie du transport optimal à d'autres disciplines du calcul des variations. Par exemple, en mécanique des milieux continus, la matrice jacobienne DT représente la déformation et les contraintes de volume sont des contraintes standard (voir [2, 47]) exprimées en général sous forme de déterminant, mais que l'on peut aussi écrire sous la forme de la mesure image. La quantité $\int_\Omega |DT|^2$ représente la déformation totale ; en élasticité incompressible, on peut aussi le remplacer par $\int_\Omega |{}^tDT + DT|^2$ qui représente l'intégrale du tenseur des contraintes, et la contrainte d'incompressibilité peut s'exprmier comme une contrainte de volume (voir [6]).

D'autres questions équivalentes apparaissent en analyse de forme, où l'on étudie comment une forme peut être *déformée* de manière continue : classiquement, on cherche une famille $(\varphi_t)_t$ de difféomorphismes indexée par un paramètre continu $t \in [0,1]$ telle que $\varphi_0 = \mathrm{id}$, que φ_1 transforme une configuration initiale donnée en une configuration finale donnée (s'agissant de mesures, on imposera donc $(\varphi_1)_\# \mu = \nu$) et minimisant l'énergie cinétique

$$\int_0^1 ||v_t||_{L^2}^2 \,\mathrm{d}t \qquad \text{où } v_t = (\varphi_t)' \circ (\varphi_t)^{-1}$$

Remplacer un tel champ de vitesse par $v_t \circ \varphi_t$ dans la norme L^2 conduit à la formulation du problème de Monge-Kantorovitch introduite par Benamou et Brenier [9] :

$$\min\left\{\int |T(x) - x|^2 \,\mathrm{d}\mu(x) \;:\; T_\# \mu = \nu\right\} = \min\left\{\int |v_t|^2 \,\mathrm{d}\rho_t\right\}$$

où le minimum du membre de droite est pris sur l'ensemble des coupes (ρ_t, v_t) satisfaisant à l'équation de continuité avec conditions initiales

$$\begin{cases} \partial_t \rho_t + \operatorname{div}(\rho_t v_t) = 0 \\ \rho_0 = \mu,\; \rho_1 = \nu \end{cases}$$

Dans cet esprit, on peut s'attendre à ce que la minimisation de $||DT - \mathrm{id}||_{H^1(\Omega)}$ parmi les transports Sobolev corresponde à la minimisation de $\int (|v_t|^2 + |\nabla v_t|^2) \,\mathrm{d}\rho_t \,\mathrm{d}t$ (cette formulation est en fait informelle si la mesure ρ_t est singulière pour un certain t, voir plus loin). Khesin *et al.* [54] se sont intéressés à une formulation similaire faisant intervenir la divergence plutôt que la norme totale de la matrice jacobienne.

Un autre problème analogue consiste à chercher le transport T « le plus isométrique possible » en introduisant le problème

$$\inf\left\{\int_\Omega W(DT(x)) \,\mathrm{d}\mu(x) \;:\; T_\# \mu = \nu\right\}$$

où W quantifie le « défaut d'isométrie » de la matrice jacobienne, par exemple :

$$W(p) = d(p, SO_d(\mathbb{R})) \qquad \text{ou} \qquad W(p) = |||p||| + |||p|||^{-1} - 2$$

(ici $|||\cdot|||$ est la norme d'algèbre usuelle sur $M_d(\mathbb{R})$). Cette approche a été introduite par Granieri et Maddalena [46] sous le terme de « reformation élastique ». La valeur minimale permet de mesurer à quel point les mesures μ et ν (c'est-à-dire les formes qu'elles représentent) sont « proches d'être isométriques ».

Dans la même classe de problèmes de « transport régularisé », un article récent de F. Santambrogio *et al.* [56] étudie la minimisation de l'oscillation des applications de transport :

$$\inf\{\omega_\delta(T) \;:\; T_\# \mu = \nu\} \qquad \text{où} \qquad \omega_\delta(T) = \sup_{\substack{|x-y|<\delta \\ x,y \in \operatorname{supp}\mu}} |T(y) - T(x)|$$

Ce problème-ci trouve ses motivations dans le domaine de la sécurité informatique [60] : il s'agit de dire à chaque utilisateur d'un réseau social quels sont ses amis qui sont situés près de lui (*i.e.* dans un rayon fixé) sans que le réseau soit informé de la position de *tous* ses utilisateurs pour des raisons de vie privée. La solution actuelle consiste à ce qu'un serveur indépendant choisisse une rotation R au hasard et la communique aux appareils des utilisateurs, qui ne transmet au réseau central que leur position rotationnée $R(x)$; ce n'est cependant pas satisfaisant, car le serveur pourrait retrouver les positions initiales à partir des positions rotationnées en comparant la densité des points $R(x)$ à celle, connue, de la Terre (dans l'hypothèse où un très grand nombre d'êtres humains utilise le réseau social). L'idée alternative consiste à envoyer la densité de population de la Terre sur une densité ν fixée (par exemple, la densité uniforme) tout en préservant l'information sur les distances : avec la quantité ω_δ, on a l'implication

$$|T(x) - T(y)| \geq \omega_\delta(T) \Rightarrow |x - y| \geq \delta$$

de sorte que le T minimisant ω_δ est bien celui qui donne le plus d'informations. L'article [56] analyse mathématiquement le problème ; l'existence d'un T optimal n'est pour le moment claire que dans le cas unidimensionnel et le cas général pour l'existence ainsi que les approximations numériques sont des problèmes ouverts à l'heure actuelle.

Dans cette thèse, on se concentre essentiellement sur les pénalisations données directement par la norme quadratique de la matrice jacobienne $\int_\Omega |DT|^2$ (éventuellement intégrée par rapport à la mesure source ou multipliée par un petit paramètre ε, *cf.* plus loin). D'un point de vue mathématique, le grand avantage des fonctionnelles faisant intervenir un terme d'ordre supérieur (le gradient, ou la matrice jacobienne pour les fonctions à valeurs vectorielles) est qu'elles permettent par définition de travailler dans des espaces de fonctions plus régulières. Dans notre cadre, ceci facilite grandement la preuve de l'existence d'un T optimal, comme on va le voir sur l'exemple suivant : supposons que Ω soit un ouvert borné à frontière lipschitzienne, que μ soit une mesure ayant une densité f telle que $f \geq c > 0$ sur Ω pour une certaine constante c, et intéressons-nous au problème

$$\inf \left\{ \int_\Omega (|T(x) - x|^2 + |DT(x)|^2) f(x) \,\mathrm{d}x \,:\, T_\# \mu = \nu \right\} \tag{1}$$

Supposons de plus qu'il existe au moins une fonction T qui soit un transport de μ à ν et qui appartienne à l'espace de Sobolev $H^1(\Omega)$. On prend alors une suite minimisante $(T_n)_n$; en particulier, il existe une constante C telle que

$$\text{pour tout } n \in \mathbb{N}, \quad C \geq \int_\Omega |DT_n|^2(x) f(x) \,\mathrm{d}x \geq c \int_\Omega |DT_n(x)|^2 \,\mathrm{d}x$$

et la suite $(T_n)_n$ n'est plus seulement bornée dans $L^\infty(\Omega)$ mais dans $H^1(\Omega)$. Ceci assure l'existence d'une application T et d'une sous-suite $(T_{n_k})_k$ telle que la convergence $T_{n_k} \underset{k}{\to} T$ ait cette fois-ci lieu *fortement* dans $L^2(\Omega)$, et ce type de convergence est suffisant pour garantir que la limite T satisfasse toujours la contrainte sur la mesure image. En effet, l'on peut aussi supposer que $T_{n_k}(x) \underset{k}{\to} T(x)$ pour presque tout x dans Ω et alors, pour toute fonction φ continue et bornée, on a

$$\int \varphi(y) \,\mathrm{d}\nu(y) = \int_\Omega \varphi(T_{n_k}(x)) \,\mathrm{d}\mu(x) \underset{k}{\to} \int_\Omega \varphi(T(x)) \,\mathrm{d}\mu(x)$$

et l'égalité $\displaystyle\int \varphi \circ T \,\mathrm{d}\mu = \int \varphi \,\mathrm{d}\nu$ pour toute telle fonction φ suffit à assurer que T envoie bien μ sur ν, donc est admissible pour (1) ; on conclut en utilisant la semi-continuité de la fonctionnelle par rapport à la convergence faible dans $H^1(\Omega)$.

L'inconvénient réside en fait dans le caractère non étudié jusqu'ici de ce type de problème, avec les inconnues qui en résultent : en particulier, il n'existe pas à notre connaissance de résultats « structurel » sur l'optimum d'une telle fonctionnelle, contrairement au cas du transport optimal où l'on sait caractériser T en utilisant la formulation duale et où, dans le cas quadratique, on sait qu'il est égal au gradient d'une fonction convexe (avec les propriétés de monotonie que cela implique). Les difficultés dans le cas du coût « avec gradient » sont donc très différentes et il s'agit d'étudier les frontières entre ces deux problèmes.

0.3 Contributions et organisation de la thèse

0.3.1 Espaces de Sobolev par rapport à une mesure et existence de minimiseurs

Le premier chapitre de cette thèse est consacrée d'une façon générale à l'existence de minimiseurs à la fonctionnelle

$$T \mapsto \int_\Omega L(x, T(x), DT(x))\,\mathrm{d}\mu(x)$$

parmi l'ensemble des fonctions T envoyant la mesure μ sur une mesure image ν prescrite. En fait, on s'aperçoit que si l'analyse précédente a un sens lorsque μ est la mesure de Lebesgue, ou a une densité bornée inférieurement, sur un ouvert de $\mathbb{R}^d$ (la question principale étant de savoir s'il y a une injection compacte de l'espace de Sobolev $H^1(\Omega)$ ou $H^1_f(\Omega)$ dans $L^2(\Omega)$), la définition même de la fonctionnelle que l'on minimise lorsque μ est quelconque n'est pas claire : par exemple, si μ possède une partie singulière, comment définit-on la quantité

$$\int_\Omega |DT(x)|^2\,\mathrm{d}\mu(x)$$

pour une fonction T dont on suppose seulement qu'elle est dans $L^\infty(\Omega)$ et dont la matrice jacobienne $DT(x)$ est seulement définie pour presque tout x ? Le problème n'est donc plus seulement le « manque de compacité » de l'espace fonctionnel dans lequel on se place mais bien la définition même de la quantité qui nous intéresse.

Afin de pallier cette difficulté, on utilise la notion d'espaces de Sobolev par rapport à une mesure qui donnent un sens au *gradient* d'une fonction (ou à sa matrice jacobienne s'il s'agit d'une fonction à valeurs vectorielles) *par rapport à cette mesure*. Bien que cette notion soit définie dans le cadre général d'un espace métrique mesuré quelconque (X, d, μ) (*cf.* les travaux de Hajłasz *et al.* [48, 49, 50] ou plus récemment Ambrosio, Gigli et Savaré [5]), nous nous concentrons dans cette thèse sur le cas de l'espace eucldien $\mathbb{R}^d$. Quelle que soit la formalisation précise, le résultat sera toujours le suivant dans le cas de mesures et de fonctions régulières : si

- M est un sous-espace vectoriel strict de $\mathbb{R}^d$;
- on se donne un système de coordonnées $(e_i)_{1\leq i\leq d}$ tel que $M = \mathrm{Vect}(e_1, \dots, e_p)$;
- μ est uniforme sur un ouvert borné non vide de M

alors, pour une fonction u de classe C^1 sur $\mathbb{R}^d$, le gradient de u par rapport à μ est défini par

$$\text{pour } \mu\text{-presque tout } x, \quad \nabla_\mu u(x) = (\partial_1 u(x), \dots, \partial_p u(x), 0, \dots, 0)$$

Autrement dit, le gradient de u par rapport à μ ne prend en compte que les dérivées *dans les directions tangentes au support de* μ, et la norme Sobolev de u est alors définie naturellement par

$$||u||^2_{H^1_\mu} = ||u||^2_{L^2_\mu} + ||\nabla_\mu u||^2_{L^2_\mu}$$

Cette définition est d'autant plus naturelle que, étant donnée une fonction T définie seulement sur M et régulière, on peut définir plusieurs extensions régulières de T à $\mathbb{R}^d$ dont les gradients ne sont pas égaux mais dont les dérivées dans les directions appartenant à M, elles, coïncident sur M, et il s'agit de prendre la plus économique possible.

Toutes les définitions étudiées sont faites pour que ce calcul soit aussi valable lorsqu'on remplace le sous-espace vectoriel M par une variété (et le gradient par rapport à μ au point x sera donné par les dérivées dans la direction de l'espace tangent à M au point x).

Cette démarche conduit donc à définir l'*espace tangent à une mesure* μ, et pour une fonction régulière, le gradient par rapport à μ sera donné ponctuellement en μ-presque tout point x par

$$\nabla_\mu u(x) = p_{T_\mu(x)}(\nabla u(x))$$

où $p_{T_\mu(x)}$ représente la projection orthogonale dans $\mathbb{R}^d$ sur $T_\mu(x)$, l'espace tangent à μ au point x.

Dans le premier chapitre de cette thèse, on donne donc un résumé des notions déjà connues sur les espaces de Sobolev par rapport à une mesure (en commençant de façon naturelle par les espaces de Sobolev à poids $W_f^{1,p}$) puis on cherche à les appliquer à notre type de problème variationnel. Pour les espaces de Sobolev du type H^1_μ, la première version vectorielle et variationnelle du gradient par rapport à une mesure a été introduite par Bouchitté, Buttazzo et Seppecher dans [14] (voir aussi la thèse de Fragalà [41] ou l'article [16] pour un panorama complet de ces notions, et [17] pour les définitions d'espaces de Sobolev d'ordre 2 dans ce cadre) et d'autres définitions ont été données par Zhikov [71, 70]. Une fois défini proprement le terme en gradient $D_\mu T$, la question centrale sera toujours la suivante : existe-t-il un résultat de compacité (du type du théorème de Rellich dans l'espace de Sobolev classique H^1), qui permet d'extraire d'une suite minimisante $(T_n)_n$ une sous-suite $(T_{n_k})_k$ ayant une limite T pour un mode de convergence garantissant que cette limite soit toujours un transport ?

La réponse est connue (et positive) dans certains cas particuliers en dimension quelconque (mesures *doublantes*) pour lesquels des résultats analogues au théorème de Rellich existent [50]. Nous montrons dans ce chapitre qu'elle est reste positive en dimension 1 sans hypothèse sur la mesure , grâce au résultat suivant qui donne une description complète de l'espace tangent et de l'espace de Sobolev par rapport à μ sur la droite réelle :

Théorème (1.3.1, paragraphe 1.3). *Soit μ une mesure positive finie sur un intervalle I de $\mathbb{R}$. On note $\mu = \mu_a + \mu_s$ la décomposition de Lebesgue de μ, où μ_a est absolument continue par rapport à la mesure de Lebesgue et a pour densité f, et μ_s est concentrée sur un sous-ensemble A de I de mesure de Lebesgue nulle. Soit*

$$M = \left\{ x \in I \,:\, \forall \varepsilon > 0,\ \int_{x-\varepsilon}^{x+\varepsilon} \frac{\mathrm{d}t}{f(t)} = +\infty \right\}$$

Alors l'espace tangent à μ est donné en μ-presque tout point x par

$$T_\mu(x) = \begin{cases} 0 & si\ x \in M \cup A \\ \mathbb{R} & sinon \end{cases}$$

avec le corollaire suivant sur la structure de l'espace $H^1_\mu(\mathbb{R})$:

Corollaire (1.3.1, paragraphe 1.3). *On adopte les mêmes notations et hypothèses que dans le théorème précédent.*

- *L'ensemble M est un fermé de I. De plus, pour tout élément u de L^2_μ, la fonction $u|_{I-M}$ est un élément de $L^1_{loc}(I-M)$ et admet donc une dérivée au sens des distributions.*
- *Un élément u de $L^2_\mu(I)$ est dans $H^1_\mu(I)$ si et seulement si cette dérivée au sens des distributions est une fonction de $L^2_f(I-M)$. Dans ce cas, la norme de u dans $H^1_\mu(I)$ est donnée par*

$$||u||^2_{H^1_\mu(I)} = ||u||^2_{L^2_\mu(I)} + ||(u|_{I-M})'||^2_{L^2_f(I-M)}$$

Notons que le fait que l'espace H^1_μ donne un rôle particulier aux points autour desquels $1/f$ est intégrable n'est pas surprenant : si x est un tel point, l'inégalité de Cauchy-Schwarz donne pour toute fonction u telle que $u^2 f$ soit intégrable

$$\int_{B(x,\varepsilon)} |u(t)|\,\mathrm{d}t = \int_{B(x,\varepsilon)} |u(t)|\sqrt{f(t)}\,\frac{\mathrm{d}t}{f(t)} \le \left(\int_\Omega u(t)^2 f(t)\,\mathrm{d}t\right)^{1/2} \left(\int_{B(x,\varepsilon)} \frac{\mathrm{d}t}{f(t)}\right)^{1/2}$$

et donc u est intégrable autour de x. Ce calcul justifie d'ailleurs que toute fonction de L^2_μ est localement intégrable en dehors de l'ensemble M qui constitue une sorte d'ensemble critique pour l'espace de Sobolev à poids H^1_f. En ce qui concerne les espaces de Sobolev à poids qui constituent maintenant un outil classique, l'article de A. Kufner et B. Opic, au titre explicite *How to define reasonably weighted Sobolev spaces* [55], donne un panorama global des définitions et propriétés de ces espaces (voir aussi le paragraphe 1.2.1 de cette thèse).

Pour notre problème variationnel, on déduit de cette description le résultat suivant : de toute suite $(T_n)_n$ qui est bornée dans $H^1_\mu(I)$, on peut extraire une suite $(T_{n_k})_k$ qui converge vers une fonction T en *presque tout point où l'espace tangent à μ est $\mathbb{R}$*. Ce n'est pas suffisant pour dire directement que, si chaque T_{n_k} est un transport, la limite T est elle aussi un transport, mais on parvient tout de même à construire une solution de notre problème variationnel en modifiant la suite minimisante là où elle ne converge pas. On obtient ainsi l'existence d'un minimiseur pour le problème

$$\inf\left\{\int_I ((T(x)-x)^2 + (\nabla_\mu T(x))^2)\,\mathrm{d}\mu(x) \ : \ T_\#\mu = \nu\right\}$$

On donne également deux exemples de densités f_1, f_2 définies sur le plan, strictement positives sur le carré $(0,1)^2$, telles que

$$\int_{B(x,\varepsilon)} \frac{\mathrm{d}t}{f_1(t)} = \int_{B(x,\varepsilon)} \frac{\mathrm{d}t}{f_2(t)} = +\infty$$

pour tout $x \in (0,1)^2$ et $\varepsilon > 0$, et telles que les mesures associées μ_1, μ_2 verifient :

$$\text{pour } \mu_1\text{-presque tout } x, \quad T_{\mu_1}(x) = \mathbb{R}\cdot e_1$$

et

$$\text{pour } \mu_2\text{-presque tout } x, \quad T_{\mu_2}(x) = \mathbb{R}^2$$

Autrement dit :

- l'espace tangent à une mesure peut être un sous-espace strict et non trivial de $\mathbb{R}^d$ bien que cette mesure soit absolument continue par rapport à la mesure de Lebesgue (premier exemple) ;
- on peut avoir un espace tangent égal à l'espace entier $\mathbb{R}^d$ pour une mesure à densité dont l'inverse n'est pourtant intégrable sur aucun ouvert (deuxième exemple).

La description précise de l'espace tangent à une mesure quelconque μ en toute dimension reste pour le moment inconnue, mais nous sommes néanmoins capables dans les deux exemples cités de démontrer l'existence de solutions à certains problèmes variationnels faisant intervenir le gradient (par rapport à μ) avec mesure image prescrite dans ces deux exemples. Précisément, dans le cas de μ_1 pour laquelle l'espace tangent est de dimension 1, on démontre que

$$\inf\left\{||\nabla_\mu T(x)||^2_{L^2_{\mu_1}} \ : \ T \in H^1_{\mu_1}(\Omega), T_\#\mu_1 = \nu\right\}$$

admet une solution (toujours sous réserve que la classe de transports Sobolev de μ à ν soit non vide) ; dans le cas de μ_2, on arrive même à montrer que l'injection $H^1_{\mu_2} \subset L^2_{\mu_2}$ est compacte (Théorème 1.4.3) ce qui suffit à démontrer l'existence de solutions à des problèmes de transport avec gradient plus généraux.

Une partie conséquente de ce chapitre a été publiée dans [58] (mais cette thèse comporte quelques améliorations et nouveaux résultats absents dans l'article).

0.3.2 Conditions d'optimalité et caractérisation des minimiseurs

Le deuxième chapitre de la thèse est consacrée à la question suivante : sous des hypothèses garantissant les résultats d'existence, quelles sont effectivement les solutions de nos problèmes variationnels ?

Optimalité du transport croissant en 1D pour le coût Sobolev. Dans le cas du problème du transport optimal avec coût quadratique, on sait depuis les travaux de Brenier que la solution a la structure particulière suivante : si la mesure μ est absolument continue par rapport à la mesure de Lebesgue, le transport optimal est l'unique application T qui envoie effectivement μ sur ν et qui est égal au gradient d'une fonction convexe. En particulier, dans le cas unidimensionel, cela signifie que T est l'unique fonction croissante qui transporte μ sur ν, résultat qui est vrai en dimension 1 pour tout coût de transport qui est une fonction convexe de la distance ($c(x, y) = h(x - y)$ avec h convexe). La première question est donc de savoir si cette propriété persiste pour des problèmes variationnels prenant en compte la dérivée de la fonction. La réponse est positive sous de bonnes hypothèses sur la mesure de départ et négative sinon (et l'on peut fournir des contre-exemples). Si la mesure de départ est égale à la mesure de Lebesgue, le point important est l'inégalité suivante (publiée avec F. Santambrogio dans [59]) :

Théorème (2.1.2, paragraphe 2.1.1)**.** *Soit I un intervalle de $\mathbb{R}$ et f une fonction convexe croissante $\mathbb{R} \to \mathbb{R}$. Soit U une fonction à valeurs réelles appartenant à l'espace de Sobolev $W^{1,1}(I)$. Soit T une fonction croissante $I \to \mathbb{R}$. On suppose que U et T envoient la mesure de Lebesgue restreinte à I sur la même mesure image. Alors l'inégalité suivante est vérifiée :*

$$\int_I f(|U'(x)|)\,\mathrm{d}x \geq \int_I f(n(x)T'(x))\,\mathrm{d}x \tag{2}$$

où l'on a noté, pour $x \in I$,

$$n(x) = \#(U^{-1}(\{T(x)\}))$$

le nombre d'antécédents de $T(x)$ par U.

Dans le cas d'une mesure différente de la mesure de Lebesgue, on peut obtenir un résultat un peu plus faible si sa densité « n'oscille pas trop » :

Théorème (2.1.3, paragraphe 2.1.1)**.** *Soit I un intervalle de $\mathbb{R}$ et f une fonction convexe croissante $\mathbb{R} \to \mathbb{R}$. Soit μ une mesure de probabilité sur I, absolument continue par rapport à la mesure de Lebesgue, dont la densité g vérifie*

$$\frac{\inf g}{\sup g} \geq \frac{1}{2}$$

Soit $U \in W^{1,1}(I)$. Soient $T_1 : I \to \mathbb{R}$ croissante et $T_2 : I \to \mathbb{R}$ décroissante, telles que $(T_1)_{\#}\mu = (T_2)_{\#}\mu = U_{\#}\mu$. Alors l'une des deux égalités suivantes est vraie :

$$\int_I f(|U'(x)|)g(x)\,\mathrm{d}x \geq \int_I f(T_1'(x))g(x)\,\mathrm{d}x \qquad \textit{ou} \qquad \int_I f(|U'(x)|)g(x)\,\mathrm{d}x \geq \int_I f(-T_2'(x))g(x)\,\mathrm{d}x$$

Commentaires sur ces deux résultats. Le théorème 2.1.2 exprime non seulement que la fonction croissante envoyant la mesure de Lebesgue sur une mesure image prescrite est alors optimal pour $\int_I f(|T'|)$ avec contrainte sur la mesure image (si U est un compétiteur, il doit avoir le même ensemble image que T ce qui implique nécessairement $n \geq 1$ et $\int_I f(|U'|) \geq \int_I f(T')$ par croissance de f) mais donne en plus uns estimation de ce que l'on gagne à choisir cette fonction croissante : la présence de la fonction n exprime que l'on a intérêt à choisir un transport tel que $n(x)$ soit le plus petit possible, c'est-à-dire qui oscille le moins possible sur I, en étant le plus injectif possible. En fait, on peut démontrer une inégalité un peu plus forte que l'inégalité (2) sans l'hypothèse que μ est la mesure de Lebesgue et si U assez régulière (monotone par morceaux et C^1 sur ces morceaux), à savoir : si T est monotone sur I et si $T_{\#}\mu = U_{\#}\mu$ alors

$$\int_I f(|U'(x)|)g(x)\,\mathrm{d}x \geq \int_I f\left(\frac{\inf g}{\sup g} n(y)|T'(y)|\right)\,\mathrm{d}y$$

(*cf.* proposition 2.1.4, paragraphe 2.1.2). Cette inégalité implique directement le théorème 2.1.2 dans ce cas, ainsi que le théorème 2.1.3 (en se restreignant à un sous-intervalle de I sur lequel $n \geq 2$, de sorte que $n \frac{\inf g}{\sup g} \geq 1$). Le point important pour la démontrer est qu'on peut dans ce cas donner une formule explicite pour la densité de la mesure image : si T est strictement monotone, alors

$$(U_{\#}g)(y) = \sum_{z : U(z) = y} \frac{g(z)}{|U'(z)|} = (T_{\#}g)(y) = \frac{g(T^{-1}(y))}{|T'(T^{-1}(y))|}$$

où la somme comporte $n(x)$ termes si $y = T(x)$; informellement, ceci suggère déjà que, si les valeurs de g aux points considérés sont proches (en particulier si elle est constante), U' « est d'autant plus grand que n est grand ». La démonstration est en fait assez simple dans ce cas, le point technique étant de généraliser à une fonction U quelconque en l'approximant par une suite de fonctions $(U_k)_k$ et en faisant passer à la limite l'inégalité (2) vraie pour tout k (mais avec les fonctions correspondantes T_k et n_k).

Dans le cas où l'hypothèse sur g n'est plus vérifiée, on peut trouver un contre-exemple très simple à l'optimalité du transport croissant : en considérant la fonction U « triangle » sur $[0, 1]$, *i.e.*

$$U(x) = \begin{cases} 2x & \text{si } 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \text{si } \frac{1}{2} \leq x \leq 1 \end{cases}$$

il est facile de construire une mesure μ dont la densité prend deux valeurs sur $[0, 1]$ et telle que, si T est le transport croissant envoyant μ sur $U_{\#}\mu$, on ait par exemple l'inégalité stricte

$$\int_0^1 |T'|^2 \, \mathrm{d}\mu > \int_0^1 |U'|^2 \, \mathrm{d}\mu$$

Conséquences pour nos problèmes variationnels. On déduit des théorèmes 2.1.2 et 2.1.3, ainsi que de l'optimalité déjà connue du T croissant pour le critère $\int_I h(T(x) - x) \, \mathrm{d}\mu(x)$ si h est convexe, les résultats d'optimalité suivants :

Proposition. *Soit f une fonction convexe croissante et h une fonction convexe. Soit I un intervalle de $\mathbb{R}$. Soit ν une mesure positive sur $\mathbb{R}$ telle qu'il existe au moins une fonction de $W^{1,1}(I)$ envoyant la mesure de Lebesgue restreinte à f sur ν. Alors le transport croissant envoyant la mesure de Lebesgue restreinte à I sur ν est optimal pour le problème*

$$\inf \left\{ \int_I h(U(x) - x) \, \mathrm{d}x + \int_I f(|U'(x)|) \, \mathrm{d}x \; : \; U_{\#}\mathcal{L}^1 = \nu \right\}$$

De plus :

- *T est l'unique optimiseur si h est strictement convexe ;*
- *si h est seulement convexe mais si de plus f est strictement croissante, alors le seul autre optimiseur possible est le transport décroissant entre la mesure de Lebesgue restreinte à I et ν.*

La dimension 2 et supérieure : approche de type équation aux dérivées partielles. La fin de ce chapitre est consacré à l'écriture des conditions d'optimalité lorsqu'on se place sur un ouvert de $\mathbb{R}^d$ avec $d \geq 2$. En calcul des variations, la méthode usuelle (appelée *méthode classique* par Dacorogna [28]) pour écrire les équations que satisfont tous les minimiseurs consiste à le « perturber » en général linéairement puis à exploiter le fait qu'il est optimal ; précisément, étant donné un minimiseur $\overline{u}$ d'un problème du type

$$\inf\{J(u) \; : \; u \in X\}$$

où X est un espace affine (par exemple un espace de Sobolev avec condition au bord), on remarque que, pour toute fonction φ appartenant à la direction vectorielle de X, la fonction $t \mapsto J(\overline{u}+t\varphi)$ est minimale en $t=0$ de sorte que

$$\text{pour toute telle fonction } \varphi, \ \frac{\mathrm{d}}{\mathrm{dt}} J(\overline{u}+t\varphi)\Big|_{t=0} = 0$$

ce qui fournit les conditions d'optimalité pour $\overline{u}$. Dans notre cas, comme on l'a dit, l'ensemble des fonctions admissibles (c'est-à-dire des transports Sobolev entre deux mesures données) n'a pas de structure linéaire ; il faut donc considérer un autre type de perturbation.

La méthode réside dans le lemme suivant, assez classique en théorie des équations différentielles ordinaires : étant donnée une mesure μ sur un ouvert borné Ω, absolument continue par rapport à la mesure de Lebesgue et dont la densité f est assez régulière (lipschitzienne), si v est un champ vectoriel lipschitzien tel que fv soit à divergence nulle et tel que $v(x)$ est parallèle à $\partial\Omega$ pour tout $x \in \partial\Omega$, alors la solution de l'équation différentielle

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{dt}} X(t,x) = v(X(t,x)) \\ X(0,x) = x \end{cases} \tag{3}$$

préserve la mesure μ, c'est-à-dire :

$$\forall t \in \mathbb{R}, \ X(t,\cdot)_{\#}\mu = \mu$$

Par conséquent, pour tout transport T entre μ et ν et pour tout t, l'application $x \mapsto T(X(t,x))$ est encore un transport, et c'est en particulier si T est un minimiseur. Puisque de plus $T(X(0,x)) = T(x)$, on aura alors, en notant J est notre fonctionnelle :

$$\frac{\mathrm{d}}{\mathrm{dt}} J(T(X(t,\cdot)))\Big|_{t=0} = 0$$

et ce calcul est valable pour toute fonction $X(t,\cdot)$ obtenue comme solution de (3), donc pour tout champ vectoriel v lipschitzien tel que $\mathrm{div}(fv) = 0$ et parallèle au bord de Ω.

Les calculs aboutissent au résultat suivant :

Proposition. *Soit Ω un ouvert borné régulier de $\mathbb{R}^d$ à frontière C^1, et μ une mesure positive sur Ω ayant une densité f régulière et bornée inférieurement et supérieurement. Soit ν une mesure positive sur $\mathbb{R}^d$ telle que l'ensemble*

$$\{T \in H^1(\Omega) \ T_{\#}\mu = \nu\}$$

soit non vide. Soit $L : \overline{\Omega} \times \mathbb{R}^d \times M_d(\mathbb{R}) \to \mathbb{R}$, différentiable par rapport à ses deuxième et troisième variables, et soit T un minimiseur pour le problème

$$\inf \left\{ \int_\Omega L(x, T(x), DT(x)) f(x)\,\mathrm{d}x \ : \ T_{\#}\mu = \nu \right\}$$

Alors, pour tout champ de vecteurs $v : \Omega \to \mathbb{R}^d$ lipschitzien et tel que

$$\begin{cases} \mathrm{div}(fv) = 0 & \textit{dans } \Omega \\ v \cdot n = 0 & \textit{sur } \partial\Omega \end{cases}$$

(où n désigne la normale extérieure à $\partial\Omega$), on a l'égalité

$$\int_\Omega \left(-\nabla_1 L(x,T(x),DT(x)) \cdot v(x) + \nabla_3 L(x,T(x),DT(x)) \cdot (DT(x) \times Dv(x))\right) f(x)\,\mathrm{d}x = 0$$

Cette égalité est la formulation faible d'une équation aux dérivées partielles avec condition au bord. Dans le cas quadratique, c'est-à-dire si la fonctionnelle que l'on minimise est

$$T \mapsto \int_\Omega |T(x) - x|^2 f(x)\,\mathrm{d}x + \int_\Omega |DT(x)|^2\,\mathrm{d}x$$

cette équation aux dérivées partielles est précisément

$$\begin{cases} T(x) - \dfrac{2}{f(x)}\,{}^tDT(x) \times \Delta T(x) = \nabla\psi(x), & x \in \Omega \\ {}^tDT(x) \times DT(x) \times n(x) = |DT(x) \times n(x)|^2 n(x), & x \in \partial\Omega \end{cases}$$

pour une fonction à valeurs réelles ψ qui joue le rôle d'une pression.

Cette équation aux dérivées partielle est non-linéaire, semble compliquée et n'a pas fait l'objet d'étude spécifique. Néanmoins, notons aussi qu'on peut la retrouver *via* des méthodes semblables à la dualité de Kantorovich (voire plus loin dans l'introduction et le chapitre 4 de cette thèse).

0.3.3 La pénalisation en gradient pour approximer le problème de Monge

Dans le troisième chapitre de cette thèse (à paraître dans un article commun avec L. De Pascale et F. Santambrogio [33]), on s'intéresse au problème suivant : sous des hypothèses telle que le minimiseur existe (mesure source à densité f lipschitzienne et bornée inférieurement), que se passe-t-il si l'on considère la fonctionnelle

$$J_\varepsilon : T \mapsto \int_\Omega |T(x) - x| f(x)\,\mathrm{d}x + \varepsilon \int_\Omega |DT(x)|\,\mathrm{d}x$$

(définie sur l'ensemble des transports T de la mesure $f \cdot \mathcal{L}^d$ à une mesure ν prescrite) lorsque ε est un paramètre strictement positif qui tend vers 0 ?

Le fait de considérer des pénalisations « de plus en plus petites » est classique en calcul de variations (dans l'étude de phénomènes de transition de phase [61], en homogénéisation [3, 10], ...) et le bon cadre fonctionnel est la Γ-convergence, introduite par De Giorgi et pour laquelle on renvoie aux livre d'Attouch [8], Dal Maso [31] et Braides [19] pour les définitions, propriétés et commentaires. Ici, on part du coût originel introduit par Monge et qui est la norme L^1 de la distance :

$$\int_\Omega |T(x) - x|\,\mathrm{d}\mu(x)$$

où $|\cdot|$ est la norme euclidienne sur $\mathbb{R}^d$. Ce coût n'est pas strictement convexe en fonction de la distance et fait l'objet d'une étude spécifique dans la théorie du transport optimal (voir annexe B.3), en particulier *le transport optimal T n'est pas unique*; la question est donc de savoir quelle solution du problème originel de Monge est « selectionnée » par la procédure consistant à choisir un minimiseur T_ε de J_ε (ou de fonctionnelles définies à partir de J_ε, *cf.* plus loin).

Dans la théorie L^1 du transport optimal, la structure des solutions est très particulière et peut être résumée de la manière suivante :

- la *formule de dualité* de Kantorovitch s'écrit
$$\inf\left\{\int_\Omega |T(x) - x|\,\mathrm{d}\mu(x) \,:\, T_\#\mu = \nu\right\} = \sup\left\{\int_\Omega u(y)\,\mathrm{d}\nu(y) - \int_\Omega u(x)\,\mathrm{d}\mu(x) \,:\, u \in \mathrm{Lip}_1(\mathbb{R}^d)\right\}$$
- étant donné un optimiseur u pour le membre de droite de l'égalité précédente, on définit alors les *rayons de transport* comme étant les segments maximaux $[x, y]$ tels que $u(y) - u(x) = |y - x|$. Alors les rayons de transports ne s'intersectent qu'à leurs extrémités, qui forment un ensemble Lebesgue-négligeable de $\mathbb{R}^d$.

- On dispose ainsi d'un ensemble de segments joignant les points du support Ω de μ aux point du support Ω' de ν. Alors, une fonction vectorielle T définie sur Ω est optimale pour le problème de Monge si
  - T envoie bien la mesure μ sur la mesure ν ;
  - T respecte les rayons de transport, c'est-à-dire que $T(x)$ appartient au même rayon de transport que x (ce rayon de transport étant bien défini pour presque tout x) ;
  - T respecte également l'orientation de ces rayons (définie de la manière suivante : $[x,y]$ est orienté de sorte que $u(y)-u(x)=|y-x|$).

Parmi ces transport optimaux, dont on note $\mathcal{O}_1(\mu,\nu)$ l'ensemble, l'un d'entre eux joue un rôle spécial, à savoir celui qui est *monotone* le long de chaque rayon de transport (au sens suivant : si x,x' appartiennent au même rayon de transport, alors $[x,x']$ et $[T(x),T(x')]$ ont même orientation) : il est l'unique solution de

$$\inf\left\{\int_\Omega |T(x)-x|^2\,\mathrm{d}\mu(x)\ :\ T \text{ est optimal pour le problème de Monge}\right\}$$

c'est-à-dire l'unique minimiseur du coût de transport quadratique parmi les transports qui sont déjà optimaux pour le problème de Monge. Notons que ce transport monotone est obtenu lui-même avec une procédure d'approximation de l'énergie de Monge, à savoir

$$\int_\Omega |T(x)-x|\,\mathrm{d}\mu(x)+\varepsilon\int_\Omega |T(x)-x|^2\,\mathrm{d}\mu(x)$$

et notons aussi les résultats de régularité suivants sur ce transport monotone : on sait qu'il est continu en dimension 2 sous de bonnes hypothèses sur les domaines et les mesures (domaines convexes et disjoints, mesures à densités continues et bornées inférieurement, *cf.* [42]) et on a aussi des estimations *a priori* pour une suite $(T_\varepsilon)_\varepsilon$ de minimiseurs d'un coût approximé ayant le transport monotone pour limite, à savoir [57] :

- T_ε minimise $\int_\Omega c_\varepsilon(x,T(x))\,\mathrm{d}\mu(x)$ parmi les transports de μ à ν, où $c_\varepsilon(x,y)=\sqrt{\varepsilon+|x-y|^2}$;
- on a des bornes locales et uniformes en ε sur les valeurs propres de la matrice jacobienne DT_ε.

Dans le cadre de l'approximation par une pénalisation en gradient, la première étape consiste à montrer que la Γ-limite de l'énergie pénalisée J_ε est bien l'énergie originelle de Monge (notée J dans la suite). La démonstration de cette convergence « à l'ordre zéro » est directe mais nécessite un résultat de densité de l'ensemble des transports entre deux mesures (assez régulières) qui appartiennent à l'espace de Sobolev $H^1(\Omega)$ (en fait, qui sont lipschitziens sur $\overline{\Omega}$) parmi l'ensemble des applications de transport entre ces deux mesures. Ce résultat semble naturel et peut bien sûr être utilisé dans d'autres contextes que celui de la Γ-convergence ; cependant, et de manière assez étonnante, il semble à notre connaissance n'avoir pas été étudié jusqu'à maintenant :

Théorème (3.2.1, paragraphe 3.2.2)**.** *Soient Ω, Ω' deux ouverts de $\mathbb{R}^d$, bornés, étoilés et à frontières lipschitziennes. Soient $\mu\in\mathcal{P}(\Omega)$, $\nu\in\mathcal{P}(\Omega')$ deux mesures, absolument continues par rapport à la mesure de Lebesgue et à densités respectives notées f et g. On suppose que, pour un certain $\alpha>0$, f et g sont de classe $C^{0,\alpha}$ sur $\overline{\Omega}$, $\overline{\Omega'}$ respectivement, et qu'elles sont toutes deux bornées inférieurement par une constante strictement positive. Alors l'ensemble*

$$\left\{T\in \mathrm{Lip}(\overline{\Omega})\ :\ T_\#\mu=\nu\right\}$$

est non vide, et est un sous-ensemble dense (pour la convergence $L^2(\Omega)$) de l'ensemble des applications de transport entre μ et ν.

Ce résultat de densité a pour conséquence directe la Γ-convergence de J_ε vers J. La question naturelle suivante est alors le premier ordre de convergence, à savoir la Γ-limite de la fonctionnelle $\frac{J_\varepsilon - W_1}{\varepsilon}$ (où W_1 est l'infimum de J, c'est-à-dire le minimum de l'énergie de Monge). Là encore le résultat est naturel, avec cette fois-ci une preuve directe et presque triviale :

$$\frac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon} \xrightarrow{\Gamma} \mathcal{H} \tag{4}$$

$$\text{où} \qquad \mathcal{H}(T) = \begin{cases} \int_\Omega |DT|^2 & \text{si } T \in \mathcal{O}_1(\mu,\nu) \cap H^1(\Omega) \\ +\infty & \text{sinon} \end{cases}$$

où $\mathcal{O}_1(\mu,\nu)$ est l'ensemble des transports entre μ et ν qui sont optimaux pour le problème de Monge ; ainsi, si parmi ces derniers il existe au moins un élément de l'espace $H^1(\Omega)$, alors le transport sélectionné par la procédure de Γ-convergence est l'élément de $\mathcal{O}_1(\mu,\nu)$ qui minimise l'énergie de Dirichlet $\int_\Omega |DT|^2$.

Étant donné les résultats de régularité déjà connus concernant le transport monotone, on pouvait naïvement s'attendre à ce que ce soit précisément celui-ci qui minimise cette énergie. En fait, on peut s'apercevoir qu'il n'en est rien grâce aux contre-exemples donnés dans la deuxième section. Supposons en effet que F et G sont deux mesures unidimensionnelles, supportées respectivement sur les segments $[0,1]$ et $[2,3]$, et telles que la solution de

$$\inf\left\{\int_0^1 t(x_1)\,\mathrm{d}x_1 \;:\; t_\# F = G\right\}$$

ne soit pas le transport croissant de F vers G. Il est facile de voir que si l'on pose

$$f(x_1,x_2) = F(x_1) \qquad \text{et} \qquad g(x_1,x_2) = G(x_1)$$

sur les carrées $[0,1]^2$ et $[2,3]\times[0,1]$ respectivement, la solution de

$$\inf\left\{\int_{[0,1]^2} |DT(x)|^2\,\mathrm{d}x \;:\; T_\# f = g\right\}$$

est de la forme

$$T \;:\; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} t(x_1) \\ x_2 \end{pmatrix}$$

où t optimise la norme Sobolev en dimension 1 parmi les transports de F vers G (et n'est donc pas croissant). Puisque les densités f et g ne dépendent que d'une variable, on peut voir que les rayons de transport sont les segments horizontaux de $\{0\}\times[0,1]$ vers $\{3\}\times[0,1]$, de sorte que tout transport de la forme $(x_1,x_2)\mapsto(t(x_1),x_2)$ est également optimal pour le coût de Monge ; autrement dit l'infimum de J_ε est *toujours* atteint par ce T (indépendamment de ε), qui n'est pas le transport monotone.

Par ailleurs, la convergence (4) a aussi pour conséquence immédiate que la valeur du minimum de J_ε approxime l'énergie minimale de Monge W_1 à l'ordre 1 en ε (précisément, $\inf J_\varepsilon = W_1 + \varepsilon \inf \mathcal{H} + o(\varepsilon)$) dans le cas où $\mathcal{H}$ n'est pas identiquement $+\infty$, *i.e.* dans le cas où il existe au moins un élément de $\mathcal{O}_1(\mu,\nu)$ qui est bien un élément de $H^1(\Omega)$. Dans le cas contraire apparaissent les questions concernant l'ordre de convergence de $\inf J_\varepsilon$ vers W_1, la limite du « reste » et le transport qui est sélectionné par la Γ-convergence.

Le cœur du chapitre est ensuite consacré à l'étude d'un exemple particulier qui présente un certain nombre de propriétés surprenantes à ce sujet. On prend deux domaines écrits en coordonnées polaires :

$$\Omega = \left\{0 < r < 1, 0 < \theta < \frac{\pi}{2}\right\} \qquad \text{et} \qquad \Omega' = \left\{R_1(\theta) < r < R_2(\theta), 0 < \theta < \frac{\pi}{2}\right\}$$

où R_1, R_2 définissent deux courbes régulières en coordonnées polaires ; ainsi, Ω est le « quart de disque unité » et Ω' est une couronne comprise entre deux courbes en polaires (on impose aussi $1 < \inf R_1$ et $\sup R_1 < \inf R_2$ afin d'avoir une vraie couronne et des domaines disjoints). On munit aussi Ω et Ω' de densités régulières et bornées inférieurement f et g, auxquelles on impose la contrainte

$$\forall \theta \in \left(0, \frac{\pi}{2}\right), \quad \int_0^1 f(r,\theta) r \, \mathrm{d}r = \int_0^1 g(r,\theta) r \, \mathrm{d}r$$

c'est-à-dire que la masse (relativement à f) du segment d'angle θ joignant 0 à la frontière de Ω est égale à celle (relativement à g) du segment joignant la « frontière inférieure » de Ω_1 à sa « frontière supérieure ». En particulier, on peut construire des transports T de f à g tels que

pour presque tout x, $T(x)$ est sur le même segment passant par l'origine que x

(il suffit de construire, angle par angle, une fonction $\varphi(\cdot, \theta)$ envoyant la densité $rf(r,\theta)$ sur la densité $rg(r,\theta)$). Par conséquent, *les rayons de transport sont les segments joignant Ω à Ω' et alignés avec l'origine*, c'est-à-dire que

$$T \text{ est optimal pour le problème de Monge } \Longleftrightarrow T(x) = \varphi(x)\frac{x}{|x|}$$

Il vient immédiatement qu'*il n'existe pas dans ce cadre de transport optimal pour le problème de Monge qui soit dans l'espace de Sobolev $H^1(\Omega)$*. Cette propriété est importante car elle suggère aussi que la valeur minimale de J_ε n'a pas un comportement du type

$$\inf J_\varepsilon = W_1(\mu,\nu) + O(\varepsilon)$$

(où $W_1(\mu,\nu)$ est la valeur minimale de l'énergie de Monge) : en effet, dans ce cas, en prenant une suite $(T_\varepsilon)_\varepsilon$ avec T_ε minimisant J_ε, on aurait informellement

$$T_\varepsilon \to T \qquad \text{et} \qquad \frac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon} = \int_\Omega |DT_\varepsilon|^2 \text{ borné}$$

ce qui entre en contradiction avec le fait que T soit optimal (puisque, pour un optimum, $\int_\Omega |DT|^2$ est infini).

On remarque au passage une analogie, formelle mais profonde, avec certaines questions issues de la théorie de l'approximation de Ginzburg-Landau [12, 51, 64]. En effet, si l'on cherche par exemple à minimiser

$$u \mapsto \frac{1}{\varepsilon^2}\int_\Omega (1 - |u|^2) + \int_\Omega |\nabla u|^2$$

avec des conditions au bord comme « u doit être parallèle à la frontière », on voit que deux termes s'opposent dans la fonctionnelle : l'un tend à sélectionner les vecteurs unitaires (avec condition au bord), l'autre cherche une régulatité H^1 ce qui est impossible puisqu'il y a forcément la création d'un vortex.

Dans notre cas, les deux phénomènes qui s'opposent sont le fait que T respecte les rayons de transport pour être optimal pour le problème de Monge (donc que T doit « envoyer l'origine sur une courbe en coordonnées polaires ») et son caractère Sobolev. Il y a donc en commun la création d'un « vortex » (ici l'origine) et le fait que le coût minimal pour J_ε n'admet pas de développement limité d'ordre 1 en ε.

Il fait en fait apparaître un terme logarithmique ; précisément, on prouve que

$$\inf J_\varepsilon = W_1 + \frac{K}{3}\varepsilon|\log\varepsilon| + O(\varepsilon)$$

où K est défini par

$$K = \min \left\{ \int_0^{\pi/2} (\varphi^2 + \varphi'^2) \ : \ R_1(\theta) \leq \varphi(\theta) \leq R_2(\theta) \right\}$$

c'est-à-dire que K est le meilleur « coût » possible, en terme de norme H^1 unidimensionnelle sur l'intervalle $(0, \pi/2)$, pour envoyer l'origine sur une courbe $r = \varphi(\theta)$ comprise dans le domaine d'arrivée. Ce comportement peut être justifié informellement par un calcul qui mène à l'approximation

$$J_\varepsilon(T_\varepsilon) = W_1 + \frac{1}{3}\varepsilon|\log\varepsilon| ||\varphi(0,\cdot)||_{H^1}^2 + O(\varepsilon)$$

pour $T(x) = \varphi(x)\frac{x}{|x|}$ et $T_\varepsilon \to T$. Autrement dit, non seulement le « développement limité » de J_ε fait apparaître un terme d'ordre $\varepsilon|\log\varepsilon|$, mais la quantité que l'on « récupère » fait intervenir le comportement de T *au voisinage de l'origine seulement*, c'est-à-dire au voisinage de l'unique intersection des rayons de transport (qui est aussi le point singulier pour tous les transports optimaux). Notons également que *cette quantité ne favorise pas le transport monotone*, au sens où la fonction R_1 peut, selon la forme du domaine Ω', ne pas être optimale au sens de la norme Sobolev.

Toutes ces remarques sont des conséquences du théorème suivant :

Théorème (3.3.1, paragraphe 3.3.3). *On note*

$$F_\varepsilon : T \mapsto \frac{1}{\varepsilon}\left(J_\varepsilon(T) - W_1 - \frac{K}{3}\varepsilon|\log\varepsilon| \right)$$

et

$$F(T) = \begin{cases} +\infty & \text{si } T \notin \mathcal{O}_1(\mu,\nu) \\ \displaystyle\int_0^1 (||\varphi(r,\cdot)||^2 - K)\frac{\mathrm{d}r}{r} + \int_0^1 ||\partial_r\varphi(r,\cdot)||_{L^2}^2 \, r\,\mathrm{d}r & \text{si } T \in \mathcal{O}_1(\mu,\nu), T(x) = \varphi(r,\theta)\dfrac{x}{|x|} \end{cases}$$

1. *Soit $(T_\varepsilon)_\varepsilon$ une famille de fonctions telle que $(F_\varepsilon(T_\varepsilon))_\varepsilon$ soit bornée. Alors on peut trouver une suite $\varepsilon_k \to 0$ et une fonction T telles que $T_{\varepsilon_k} \to T$ dans $L^2(\Omega)$.*

2. *Il existe une constante C qui ne dépend que des domaines Ω, Ω' et des mesures f, g telle que, pour toute famillle de fonctions $(T_\varepsilon)_{\varepsilon>0}$ vérifiant $T_\varepsilon \to T$ dans $L^2(\Omega)$, ont ait l'inégalité*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(T_\varepsilon) \geq F(T) - C$$

3. *De plus, il existe une famille de fonctions $(T_\varepsilon)_{\varepsilon>0}$ telle que $(F_\varepsilon)_\varepsilon$ est en effet bornée.*

Notons qu'il s'agit presque d'un résultat de Γ-convergence, au sens où on a une inégalité de type Γ-liminf mais qu'il resterait à montrer la Γ-limsup ; on s'attend à ce que

$$F_\varepsilon \xrightarrow[\varepsilon]{\Gamma} F - C'$$

où C' est une constante à préciser.

0.3.4 Autres formulations et perspectives

Les trois premiers chapitres de cette thèse sont donc consacrées aux questions classiques en calcul des variations (donner une formulation précise du problème dans les bons espaces fonctionnels, étudier l'existence de minimiseurs et les conditions d'optimalité, et étudier la « sélection » d'un minimiseur par une procédure de type Γ-convergence) autour de cette nouvelle classe de problèmes. Ces questions étaient naturelles et leur étude nécessaire, mais chacune d'entre elles était exprimée d'une façon « basique » du point de vue du problème de transport, au sens suivant : il s'agit de l'étude d'un problème « statique » (où seuls apparaissent le point de départ, le point d'arrivée et la déformation représentée par la matrice jacobienne) et on n'a pas cherché à exploiter une forme de relaxation comme il est nécessaire de le faire dans la théorie usuelle du transport optimal ; d'une certaine façon, on est vraiment dans le cas étudié par Monge, qui date maintenant de plus de deux siècles !

Le dernier chapitre donne les idées essentielles, les résultats espérés et certaines justifications informelles ou calculs purement formels dans des cas particuliers pour l'étude de ces deux approches alternatives du problème de « transport avec pénalisation en gradient ».

Formulations dynamiques. La formule qui vient le plus naturellement à l'esprit consiste simplement à reprendre celle donnée par Benamou et Brenier en remplaçant toutes les normes L^2 par des normes H^1 : il s'agirait donc d'étudier les liens entre

$$\widetilde{d}(\mu,\nu) = \inf\left\{ \int_\Omega (|T(x)-x|^2 + |D_\mu(T-\mathrm{id})|^2(x))\,\mathrm{d}\mu(x) \;:\; T\in H^1_\mu(\Omega), T_\#\mu=\nu \right\}$$

$$\text{et} \qquad \inf\left\{ \int_0^1 ||v_t||^2_{H^1_{\rho_t}}\,\mathrm{d}t \;:\; \partial_t\rho_t + \mathrm{div}(\rho_t v_t) = 0,\ \rho_0=\mu, \rho_1=\nu \right\}$$

En fait, on s'aperçoit assez vite que cette formule ne convient pas telle quelle du fait, par exemple, que la première quantité n'a *a priori* aucune raison de définir une distance entre μ et ν. On peut corriger cela en considérant l'application

$$d : (\mu,\nu) \mapsto \inf\left\{ \sum_i \widetilde{d}(\rho_i,\rho_{i+1}) \;:\; n\in\mathbb{N},\ \mu=\rho_0,\rho_1,\ldots,\rho_n=\nu\in\mathcal{P}(\mathbb{R}^d) \right\}$$

Dans ce cas, on parvient à démontrer au moins une inégalité, à savoir

$$\int_0^1\int_{\mathbb{R}^d} (|v(x,t)| + |Dv(x,t)|^2)\,\mathrm{d}\rho_t(x)\,\mathrm{d}t \geq d(\mu,\nu)$$

pour une famille (v,ρ) assez régulière satisfaisant l'équation de continuité et reliant μ à ν ; l'inégalité inverse semble elle difficile à établir, du fait qu'une courbe de mesures peut avoir des valeurs non régulières (ρ_t peut être une mesure pour laquelle la strcuture de l'espace $H^1_{\rho_t}$ n'est pas claire) et aussi du fait qu'un optimiseur pour le problème de transport avec gradient n'a pas, en règle générale, de forme « connue » comme c'est le cas pour le transport de Brenier usuel (gradient d'une fonction convexe).

La question de savoir si cette inégalité est une égalité est pour le moment ouverte ; dans le cas de $\widetilde{d}$, c'est-à-dire si l'on revient au problème statique originel, on donne aussi un exemple de calcul dans un cas très particulier qui suggère une formulation différente, où intervient l'inverse de la densité de la mesure ρ_t, à savoir

$$||T-\mathrm{id}||_{H^1} = \int_0^1 (||v_t||^2_{L^2_{\rho_t}} + ||v_t'||^2_{L^2_{1/\rho_t}})\,\mathrm{d}t$$

si T est un transport sur la droite réelle et si la mesure de départ μ est la mesure de Lebesgue.

Formulation relaxée « à la Kantorovitch ». Comme on l'a déjà dit dans le paragraphe consacré, le point de départ de la démarche de Kantorovitch est l'égalité

$$\int_\Omega c(x,T(x))\,\mathrm{d}\mu(x) = \int_{\Omega\times\Omega'} c(x,y)\,\mathrm{d}\gamma_T(x,y)$$

vraie dès lors que γ_T est le plan de transport induit par l'application T. Pour trouver une formulation équivalente dans le cas de la pénalisation en gradient, la difficulté réside dans le fait que le terme ajouté $\int |DT|^2$ ne fait pas seulement intervenir le point de départ x et le point d'arrivée $T(x)$.

La solution choisie, suggérée Alessio Figalli (et faisant l'objet d'un travail en cours avec lui), consiste à approximer le carré de la norme L^2 de la matrice jacobienne (à une constante multiplicative près) par

$$(1-s)\iint_{\Omega^2} \frac{|T(x)-T(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

qui est en fait le carré de la norme Sobolev fractionnaire d'exposant s de T (s est un paramètre positif tendant vers 1 par valeurs inférieures). Dans ce cas, il vient immédiatement que

$$||T||_{H^s}^2 = \iint_{(\Omega\times\Omega')^2} (1-s)\frac{|z-w|^2}{|x-y|^{d+2s}}\,\mathrm{d}\gamma_T(x,z)\,\mathrm{d}\gamma_T(y,w)$$

c'est-à-dire qu'on intègre par rapport à la mesure produit $\gamma\otimes\gamma$ un coût dépendant de *quatre* variables, deux variables de départ x, y et deux variables d'arrivée z, w. Contrairement au cas du transport optimal sans gradient, le fait que le plan de transport optimisant, par exemple,

$$\int_{\Omega\times\Omega} |z-x|^2\,\mathrm{d}\gamma(x,z) + \iint_{(\Omega\times\Omega')^2} (1-s)\frac{|z-w|^2}{|x-y|^{d+2s}}\,\mathrm{d}\gamma(x,z)\,\mathrm{d}\gamma(y,w)$$

est induit par une application de transport vient rapidement puisque si un point x_0 est « envoyé » sur deux points z_1, z_2 distincts, l'intégrale de $\dfrac{|z-w|^2}{|x-y|^{d+2s}}$ au voisinage de $x=y=x_0$ est automatiquement infinie. En revanche, la fonctionnelle considérée n'est plus linéaire par rapport à γ et il faut donc commencer par considérer *l'enveloppe convexe* dans l'espace des mesures de l'ensemble

$$\{\gamma\otimes\gamma\,:\,\gamma\in\Pi(\mu,\nu)\}$$

Le fait que l'infimum de notre coût en quatre variables sur cette enveloppe convexe (que l'on cherche à décrire par des contraintes simples) vient d'un élément de la forme $\gamma\otimes\gamma$ est encore une conjecture pour le moment. Sous cette conjecture, on est néanmoins capables de mener un raisonnement similaire à la dualité de Monge-Kantorovitch et d'obtenir formellement (éventuellement en considérant une régularisation du coût « Sobolev fractionnaire », qui a aussi l'inconvénient de présenter une singularité en $x=y$) des conditions d'optimalité qui font apparaître, notamment, l'opérateur laplacien fractionnaire. Lorsque s tend vers 1 (et sous réserve que les analogues des potentiels de Kantorovitch, qui dépendent de s, aient une limite), l'une de ces conditions d'optimalité devient

$$\nabla\Phi(x) + C_d({}^tDT(x)\Delta T(x) + \nabla(|DT|^2)(x)) = T(x)$$

Notons qu'il s'agit *exactement de la même équation que celle obtenue comme équation d'Euler-Lagrange* par la méthode de « perturbation ».

Les questions ouvertes sur ce sujet sont variées - à commencer la démonstration des étapes manquantes, notamment l'étude de l'enveloppe convexe des $\gamma\otimes\gamma$ - mais on insiste sur le fait que les conditions d'optimalité obtenues coïncident en partie avec celles provenant de la méthode « classique ».

Chapter 1

Existence of minimizers

This chapter is devoted to the existence of minimizers of the problem

$$\inf \left\{ \int_\Omega (c(x, T(x)) + |DT(x)|^2) \, \mathrm{d}\mu(x) : T_\# \mu = \nu \right\}$$

where:

- Ω is a bounded open set of $\mathbb{R}^d$, $d \geq 1$;
- $\mu \in \mathcal{P}(\Omega)$, $\nu \in \mathcal{P}(\mathbb{R}^d)$;
- c is a transport cost (in general we will simply take $c(x, y) = |y - x|^2$)

As we will see below, this formulation is actually quite informal in general, so we will precise at each time in which functional space we choose to work.

1.1 The general strategy

1.1.1 Method and existence for regular measures

The general strategy to prove the existence of a minimizer is the following: first, we assume that the set of transport maps T from μ to ν such that $\int_\Omega |DT|^2 \, \mathrm{d}\mu < +\infty$ is non-empty; it is true in many cases, for instance if μ, ν are absolutely continuous with Lipschitz density on two domains Ω, Ω' with regular boundary, and we in general will not focus on this assumption. Then, we use the direct method of the calculus of variations and take a minimizing sequence $(T_n)_n$ for our functional. The fact that $\int |DT_n|^2 \leq C < +\infty$ for each n gives some compactness to the sequence $(T_n)_n$, and we can extract a converging subsequence to a certain map T, where the mode of convergence is enough to have the implication:

$$\begin{cases} T_n \to T \\ \forall n, \ (T_n)_\# \mu = \nu \end{cases} \implies T_\# \mu = \nu$$

so that the limit T is still admissible, and we conclude by semi-continuity of the functional.

The mode of convergence that we need is typically the pointwise μ-a.e. convergence, as shows the following easy lemma:

Lemma 1.1.1. *Let $(T_n)_n$ be a sequence of measurable maps $\Omega \to \mathbb{R}^d$ such that*

$$\forall n \in \mathbb{N},\ (T_n)_{\#}\mu = \nu$$

Let us assume that there exists $T : \Omega \to \mathbb{R}^d$ such that $T_n(x) \to T(x)$ for μ-a.e. $x \in \Omega$. Then $T_{\#}\mu = \nu$.

Proof. The fact that $(T_n)_{\#}\mu = \nu$ for any n can be translated with

$$\forall n \in \mathbb{N}, \forall \varphi \in C_b(\mathbb{R}^d),\ \int_\Omega \varphi(T_n(x))\,\mathrm{d}\mu(x) = \int_{\mathbb{R}^d} \varphi(y)\,\mathrm{d}\nu(y)$$

In the left-hand side of this equality, since φ is continuous and bounded and $T_n \to T$ μ-a.e. on Ω, the dominated convergence theorem provides

$$\int_\Omega \varphi(T_n(x))\,\mathrm{d}\mu(x) \to \int_\Omega \varphi(T(x))\,\mathrm{d}\mu(x)$$

so that

$$\int_\Omega \varphi(T(x))\,\mathrm{d}\mu(x) = \int_\Omega \varphi(y)\,\mathrm{d}\nu(y)$$

for any $\varphi \in C_b(\mathbb{R}^d)$. This exactly means that $T_{\#}\mu = \nu$ and completes the proof. □

Of course, the direct method of the calculus of variations is not enough in the classical optimal transport theory (without "gradient term"). Indeed, the unique bound on a minimizing sequence for the transport cost, *e.g.*

$$\forall n \in \mathbb{N},\ \int_\Omega |T_n(x) - x|^2\,\mathrm{d}\mu(x) \leq C < +\infty$$

involves the L^2-norm of each T_n, and we can only extract a sequence $(T_{n_k})_k$ which has a limit T for the weak convergence in L^2, which does not implies the μ-a.e. convergence.

For us, the existence result comes from this method under some assumptions on Ω on μ. One can formulate it as follows:

Proposition 1.1.1. *Let $\mu \in \mathcal{P}(\Omega)$ absolutely continuous with respect to the Lebesgue measure. Assume that Ω has Lipschitz boundary and that the density f of μ satisfies*

$$0 < c \leq f(x)$$

for a.e. $x \in \Omega$. Let $\nu \in \mathcal{P}(\mathbb{R}^d)$ such that

$$\exists\, T \in H^1(\Omega),\ T_{\#}\mu = \nu$$

Let L be a function $\overline{\Omega} \times \mathbb{R}^d \times M_d(\mathbb{R}) \to \mathbb{R}$ such that:

- *for any (u, ξ), the function $x \mapsto L(x, u, \xi)$ is measurable, and for any x, the function $(u, \xi) \mapsto L(x, u, \xi)$ is continuous;*
- *there exists $\alpha > 0$ such that $L(x, u, \xi) \geq \alpha|\xi|^2$, for any (x, u, ξ);*
- *for any (x, u), the function $\xi \mapsto L(x, u, \xi)$ is convex on $M_d(\mathbb{R})$.*

Then the problem

$$\inf \left\{ \int_\Omega L(x, T(x), DT(x))\, f(x)\,\mathrm{d}x \right\}$$

admits at least one solution.

Proof. Let us denote by J the functional that we are trying to minimize:

$$J : T \in H^1(\Omega) \mapsto \int_\Omega L(x, T(x), DT(x))\, f(x)\, \mathrm{d}x$$

As announced, we take a minimizing sequence $(T_n)_n$; in particular, there exists a positive constant C such that

$$\forall n \in \mathbb{N}, \int_\Omega |DT_n(x)|^2\, \mathrm{d}x \leq C$$

and $(T_n)_n$ is bounded in $H^1(\Omega)$. By applying the Rellich theorem (see for instance [38], section 5.7 or [1], chap. 6), we extract from it a convergence subsequence $(T_{n_k})_k$ which has a limit T for the *strong* convergence in $L^2(\Omega)$, and up to a second extraction we can assume that $T_{n_k}(x) \underset{k}{\rightarrow} T(x)$ for a.e. $x \in \Omega$. The lemma 1.1.1 gives us that $T_\#\mu = \nu$. Moreover, the assumptions on L guarantee that J is lower semi-continuous with respect to the weak convergence (see the Theorem A.2.1 in the Appendix). This completes the proof. □

1.1.2 Difficulties for a generic measure

In a first analysis, the key point in the previous proof was the fact that we work in a functional space where any bounded sequence admits a pointwise convergent subsequence a.e. for the source measure. Actually, the assumption on f helps also to simply give a sense to our functional: if T belongs to $L^2(\Omega)$ and admits a Jacobian matrix DT in the weak sense, we have the equivalence

$$T \in H^1(\Omega) \Leftrightarrow \int_\Omega |DT(x)|^2 f(x)\, \mathrm{d}x < +\infty$$

This fact does not subsist if this assumption on f fails, and we are even not able to define the energy $\int_\Omega |DT(x)|^2\, \mathrm{d}\mu(x)$: for instance, the Jacobian matrix of a generic $T \in L^2(\Omega)$ can be not well-defined μ-a.e. if μ admits a non-null singular part. This leads us to focus on the theory of the Sobolev spaces with respect to a generic measures and applications to our kind of variational problems. The central questions will still be the following:

1. if $T \in L^2_\mu(\Omega)^d$, how to give a sense (for instance, if $L^2_\mu(\Omega)$ is not included in $L^1_{loc}(\Omega)$) to the Sobolev norm of T with respect to μ and to its Jacobian matrix?
2. does there exist a compactness result on the space $H^1_\mu(\Omega)$ which allows us to apply the direct method of the calculus of variations?

1.2 Sobolev spaces with respect to a measure

1.2.1 Weighted Sobolev spaces (case $\mu \ll \mathcal{L}^d$)

Let us assume again that $\mu \ll \mathcal{L}^d$, and denote by f its density. In general, a first idea to define the Sobolev space H^1_μ consists in studying the set of (scalar) functions $u \in L^2_\mu$ such that ∇u exists in the weak sense and belongs to L^2_μ. Here, if f is positive on Ω, we have for any $u \in L^2_\mu(\Omega)$

$$\int_\Omega |u(x)|\, \mathrm{d}x = \int_\Omega \left(|u(x)|\sqrt{f(x)}\right) \frac{\mathrm{d}x}{\sqrt{f(x)}} \leq \left(\int_\Omega |u(x)|^2 f(x)\, \mathrm{d}x\right)^{1/2} \left(\int_\Omega \frac{\mathrm{d}x}{f(x)}\right)^{1/2} \tag{1.1}$$

This shows that in this case, the assumption

$$1/f \in L^1_{loc}(\Omega) \tag{1.2}$$

implies the continuous embedding

$$L^2_\mu(\Omega) \hookrightarrow L^1_{loc}(\Omega).$$

and any element of $L^2_\mu(\Omega)$ is a regular distribution and thus admits a gradient in the weak sense. We can then define the Sobolev space with respect to f:

$$H^1_f(\Omega) = \left\{ u \in L^2_f(\Omega) \;:\; \nabla u \in L^2_f(\Omega)^d \right\}.$$

If we replace the power 2 with a generic exponent $p \in (1, +\infty)$, by applying the Hölder inequality instead of (1.1), the condition (1.2) is replaced with

$$\left(\frac{1}{f}\right)^{1/(p-1)} \in L^1_{loc}(\Omega) \tag{1.3}$$

which is a well-known sufficient condition to define the weighted Sobolev space $W^{1,p}_f(\Omega)$ with respect to f. For a generic density $f \in L^\infty$, if we set

$$M = \left\{ x \in \Omega \;:\; \forall \varepsilon > 0, \int_{B(x,\varepsilon)\cap\Omega} \left(\frac{1}{f}\right)^{1/(p-1)} = +\infty \right\}$$

then M is the minimal closed subset of Ω such that $(1/f)^{1/(p-1)} \in L^1_{loc}(\Omega - M)$, and the definition above hold on the set $\Omega - M$. One can for instance take for definition of the weighted Sobolev space with respect to f

$$W^{1,p}(\Omega) := W^{1,p}(\Omega - M)$$

$$\text{or} \qquad W^{1,p}(\Omega) := \left\{ u \in L^p(\Omega) \;:\; u|_{\Omega - M} \in W^{1,p}(\Omega - M) \right\} \tag{1.4}$$

We refer to [55] for more details about the definitions of weighted Sobolev spaces. For a compact embedding of this space into L^1 or L^1_f (which is enough for us), the Theorem 3 of [45] (applied with $q = s = 1$) gives a sufficient condition on the weight f:

Theorem 1.2.1 ([45], Theorem 3). *Assume that Ω has Lipschitz boundary and that the condition* (1.3) *holds. Assume moreover that $f \in L^r(\Omega)$ for a certain $r > n$. Then the embedding*

$$W^{1,p}_f(\Omega) \hookrightarrow L^1_f(\Omega)$$

is a compact operator.

In this case, for our variational problem, we can make the same reasoning as if f was bounded from below.

1.2.2 General definitions of the Sobolev spaces with respect to a measure

This section is devoted to an overview of the definitions and general results about tangent spaces to a generic Borel measure μ and Sobolev spaces with respect to this measure (we henceforth do not make any assumption on μ). There exists several definitions to these Sobolev spaces in the framework of metric measure spaces (X, d, μ), for instance in the papers by Shanmugalingam [66], Hajłasz [48] or Hajłasz and Koskela [50] (see [49] for a global summary of these notions). In the Euclidean space, an usual method is the following:

- we give a sense to the tangent space to μ, which is a function defined μ-a.e. and taking values in the set of linear subspaces of $\mathbb{R}^d$;

- the gradient with respect to μ for a regular function u is then defined by

$$\nabla_\mu u(x) = p_{T_\mu(x)}(\nabla u(x)) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d,$$

  where $p_{T_\mu(x)}$ is the orthogonal projection on $T_\mu(x)$ in $\mathbb{R}^d$;

- finally, we consider for the Sobolev space H^1_μ the closure of $C^\infty\left(\overline{\Omega}\right)$ for the norm

$$u \in C^\infty\left(\overline{\Omega}\right) \mapsto ||u||_{L^p_\mu} + ||\nabla_\mu u||_{L^p_\mu}.$$

There exist several ways to define the tangent space of a generic measure μ. Preiss [63] gives a method based on the idea of blow-up: a k-dimensional subspace P_μ is said to be an approximate tangent space of μ at x if we have, for some $\theta > 0$, the following convergence in the vague topology of measure when ρ goes to 0:

$$\mu(x + \rho\,\cdot\,) \rightharpoonup \theta\mathcal{H}^k|_{P_\mu}.$$

In order to examine variational problems, the first work which presents this notion in a vector framework is the paper of Bouchitté, Buttazzo and Seppecher [14], originally done for applications on studying of energies which are concentrated on low-dimensional structures (we also refer to the survey paper [16] for an overview of these notions in this framework, and notice that [17] investigates similar notions with H^2-order energies). Their approach is based on a duality argument, which consists in calling "tangent fields to μ" the vector fields belonging to

$$X^{p'}_\mu = \{\varphi \in (L^{p'}_\mu)^d : \operatorname{div}(\mu\varphi) \in L^{p'}_\mu\},$$

where the operator $\operatorname{div}(\mu v)$ is defined in the distributional sense. The tangent space $x \mapsto Q_\mu(x)$ to μ at x is then defined thanks to the following notion of "μ-essential union" (see [27]):

Proposition 1.2.1. *There exists a unique (up to a μ-negligible set) multi-valued function $Q_\mu : \Omega \to \mathcal{P}(\mathbb{R}^d)$ such that:*

1. *for μ-a.e. $x \in \Omega$, $Q_\mu(x)$ is a linear subspace of $\mathbb{R}^d$;*
2. *for any $\varphi \in X^{p'}_\mu$, for μ-a.e. $x \in \Omega$, $\varphi(x) \in Q_\mu(x)$;*
3. *if Q'_μ is a multi-valued function satisfying the two above properties, then $Q_\mu(x) \subseteq Q'_\mu(x)$ for μ-a.e. $x \in \Omega$.*

In [43], Fragalà and Mantegazza have noticed that, with these notations, we have the inclusion $Q_\mu(x) \subseteq P_\mu(x)$ for μ-a.e. of $\mathbb{R}^d$ (*cf.* also [14, Lemma 5.2]). We refer to Ilaria Fragalà's PhD. thesis [41] and, again, the survey [16] for a complete overview and more details about the definitions above.

1.2.3 Technical definitions of tangential gradient and Sobolev spaces with respect to μ

In this paragraph, we recall the definitions and basic properties of Sobolev spaces with respect to a measure that we use below. For technical reasons, since our goal consists only in applying these definitions on transport problems, we have chosen to present some definitions coming from the papers [71, 70] (which were motivated by some homogenization problems similar to [15]) or [17, Section 2]. Notice that these two approaches have not to be opposed and are actually pretty equivalent (and this equivalence is here rigorously showed in the one-dimensional case, see below).

Definition 1.2.1. *We define the space $H^1_\mu(\Omega)$ as the set of functions $u \in L^2_\mu(\Omega)$ which can be approximated by a sequence of smooth functions whose gradients have a limit in the space L^2_μ:*

$$u \in H^1_\mu \iff \exists (u_n)_n \in C^\infty\left(\overline{\Omega}\right)^{\mathbb{N}},\ v \in (L^2_\mu(\Omega))^d : \begin{cases} u_n \to u \\ \nabla v_n \to v \end{cases} \text{for the } L^2_\mu\text{-norm.}$$

The set of such limits v is denoted by $\Gamma(u)$, and its elements are called gradients of u with respect to μ *(or simply* gradients of u *if there is no ambiguity)*

In general, u can have many gradients (see below the example of a measure supported on a segment of $\mathbb{R}^2$). However, if $v_1, v_2 \in \Gamma(u)$, $v_1 = \lim_n \nabla\varphi_n$ and $v_2 = \lim_n \nabla\psi_n$ with $\varphi_n, \psi_n \to u$, it follows

$$v_1 - v_2 = \lim_n \nabla(\varphi_n - \psi_n) \quad \text{with } \varphi_n - \psi_n \to 0$$

so that $v_1 - v_2 \in \Gamma(0)$; therefore, $\Gamma(u)$ is a closed affine subspace of $L^2_\mu(\Omega)^d$, with direction $\Gamma(0)$.

We will define "the" gradient of u as being its gradient of minimal L^2_μ-norm:

Definition 1.2.2. *Let $u \in H^1_\mu(\Omega)$. We call* tangential gradient of u *the projection (in the Hilbert space L^2_μ) of the null function onto the subspace $\Gamma(u)$. If v is this tangential gradient, we define the H^1_μ-norm of u by*

$$||u||_{H^1_\mu(\Omega)} = \left(||u||^2_{L^2_\mu(\Omega)} + ||v||^2_{L^2_\mu(\Omega)^d}\right)^{1/2}$$

The following lemma will be "technically" useful in the definition of the tangent space:

Lemma 1.2.1. *The three following assertions are true:*

1. *if $g \in \Gamma(0)$ and $a \in L^\infty(\Omega)$ then the function $x \mapsto a(x)g(x)$ belongs to $\Gamma(0)$;*
2. *if $g \in L^2_\mu(\Omega)^d$ and $a \in L^\infty(\Omega)$, if p_Γ denotes the projection onto $\Gamma(0)$ in the Hilbert space L^2_μ, then*
$$p_\Gamma(ag) = a \cdot p_\Gamma(g)$$
3. *the space $L^\infty(\Omega)^d \cap \Gamma(0)$ is dense in $\Gamma(0)$.*

Proof. For the first assertion, if $a \in C^1(\overline{\Omega})$ and $g \in \Gamma(0)$, by writing $g = \lim_n \nabla\varphi_n$ with $\varphi_n \to 0$, we have for each n

$$\nabla(a\varphi_n) = \nabla a\, \varphi_n + a\, \nabla\varphi_n \to ag$$

so that ag is the limit of the sequence $(\nabla(a\varphi_n))_n$ where $a\varphi_n \to 0$, and $ag \in \Gamma(0)$. The result is shown if $a \in C^1(\overline{\Omega})$, and can be extended to any $a \in L^\infty(\Omega)$ by density.

To prove the second assertion, let us set $\overline{g} = p_\Gamma(g)$; by definition of the projection onto a linear subspace, we have

$$\begin{cases} \overline{g} \in \Gamma(0) \\ g - \overline{g} \in \Gamma(0)^\perp \end{cases}$$

Thanks to the first point, $a\overline{g} \in \Gamma(0)$ and for any $h \in \Gamma$ we get

$$\langle a(g - \overline{g}), h\rangle = \langle g - \overline{g}, ah\rangle = 0$$

since $ah \in \Gamma(0)$ (again thanks to the first point). Finally:

$$\begin{cases} a\overline{g} \in \Gamma(0) \\ a(g - \overline{g}) \in \Gamma(0)^\perp \end{cases}$$

which exactly means that $a\overline{g} = p_\Gamma(ag)$.

The third point can be shown by remarking that if $g \in \Gamma(0)$, if we set $a_n(x) = \mathbb{1}_{|g(x)| \le n}$, then $a_n g \in \Gamma(0)$ (still thanks to the first point) and $a_n g \to g$ as $n \to +\infty$. ☐

One can see the set $\Gamma(0)$ as the set of gradient vector fields which are pointwisely orthogonal to the "direction" of the measure μ. This suggests the following definition of the tangent space to μ:

Definition 1.2.3. *We denote by $(e_1, \ldots, e_d)$ the canonical basis of $\mathbb{R}^d$, and set $\xi_i = p_{\Gamma(0)}(e_i)$ where the projection is taken in the Hilbert space L^2_μ (i.e. e_i is seen as a constant function on Ω). For $x \in \Omega$, we denote by*

$$T_\mu(x) = (\mathrm{Vect}(\xi_1(x), \ldots, \xi_d(x)))^\perp$$

and call $T_\mu(x)$ (which is defined for μ-a.e. $x \in \Omega$) the tangent space to μ at x.

The following proposition allows to make the link between tangent space and gradient with respect to μ:

Proposition 1.2.2. *Let $v \in L^2_\mu(\Omega)^d$. Then $v \in \Gamma(0)$ if and only if, for μ-a.e. $x \in \Omega$, $v(x) \perp T_\mu(x)$.*

Proof. For the direct implication, let us take an element $v \in \Gamma(0) \cap L^\infty(\Omega)^d$; we write

$$v(x) = \lambda_1(x)e_1 + \cdots + \lambda_d(x)e_d$$

with $\lambda_1, \ldots, \lambda_d \in L^\infty(\Omega)$. Taking the projection onto $\Gamma(0)$ and using the first sentence of Lemma 1.2.1, we get

$$p_\Gamma(v)(x) = v(x) = \lambda_1(x)\xi_i(x) + \cdot + \lambda_d(x)\xi_i(x)$$

which, by definition of $T_\mu(x)$, belongs to its orthogonal for μ-a.e. x. The above lemma allows to generalize for any $v \in \Gamma(0)$ by density of $\Gamma(0) \cap L^\infty$ into $\Gamma(0)$.

For the converse implication, let us denote by B the set of functions $v \in L^2_\mu(\Omega)^d$ such that, for μ-a.e. x, $v(x) \perp T_\mu(x)$. We will show that the functions with form

$$x \mapsto a(x)\left(\sum_{i=1}^d \lambda_i \xi_i(x)\right)$$

where $a \in L^\infty(\Omega)$ and $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$, which of course belong to B, belong also to $\Gamma(0)$ and form a dense subset of B. This will provide $B \subset \overline{\Gamma(0)}$, and the result follows since $\Gamma(0)$ is a closed subspace of L^2_μ.

The fact that these functions belong to $\Gamma(0)$ comes again from the first sentence of Lemma 1.2.1. Moreover, if $g \in B$ verifies

$$\forall a \in L^\infty(\Omega),\ \forall \lambda_1, \ldots, \lambda_d \in \mathbb{R},\ g \perp a(x)\left(\sum_{i=1}^d \lambda_i \xi_i(x)\right)$$

then

$$0 = \left\langle g(x), a(x)\left(\sum_{i=1}^d \lambda_i \xi_i(x)\right)\right\rangle = \left\langle a(x), g(x) \cdot \left(\sum_{i=1}^d \lambda_i \xi_i(x)\right)\right\rangle$$

Since a is arbitrary, it follows that

$$g(x) \cdot \sum_{i=1}^d \lambda_i \xi_i(x) = 0$$

for any $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ and μ-a.e. $x \in \Omega$, thus $g = 0$. This achieves the proof. □

This result, combined to the orthogonality property of $\nabla_\mu u$ in L^2_μ, implies a pointwise description of the tangential gradient:

Proposition 1.2.3. *Let $u \in H^1_\mu$. Then, for $v \in \Gamma(u)$, the function*

$$x \in \Omega \mapsto p_{T_\mu(x)}(v(x))$$

is independent of the function v and only depends on u. This function coincides μ-a.e. on Ω with the tangential gradient $\nabla_\mu u$.

Proof. Let us take $v_1 = \lim \nabla\varphi_n$, $v_2 = \lim \nabla\psi_n$ with $\varphi_n, \psi_n \to u$. Then $v_1 - v_2 = \lim \nabla(\varphi_n - \psi_n) \in \Gamma(0)$, so that $(v_1 - v_2)(x) \perp T_\mu(x)$ and $p_{T_\mu(x)}(v_1(x)) = p_{T_\mu(x)}(v_2(x))$ for μ-a.e. $x \in \Omega$; this proves that $p_{T_\mu(x)}(v(x))$ does not depend on the choice of the element $v \in \Gamma(u)$. We now set $\overline{v}(x) = p_{T_\mu(x)}(v(x))$ for a generic $v \in \Gamma(u)$ and denote by w the vector field such that

$$v(x) = \overline{v}(x) + w(x) \quad \text{with } w(x) \perp T_\mu(x)$$

and since $\overline{v} \perp w$ μ-a.e. on Ω we obtain

$$||v||^2_{L^2_\mu} = ||\overline{v}||^2_{L^2_\mu} + ||w||^2_{L^2_\mu}$$

so that $\overline{v}$ is the minimal element of $\Gamma(u)$ for the L^2_μ-norm. □

In particular, we have for $u \in C^\infty(\Omega)$ the formula

$$||u||^2_{H^1_\mu(\Omega)} = ||u||^2_{L^2_\mu(\Omega)} + ||p_{T_\mu}(\nabla_u)||^2_{L^2_\mu(\Omega)^d}$$

which implies the following:

Corollary 1.2.1. *The space $H^1_\mu(\Omega)$ coincides with the completion $C^\infty\left(\overline{\Omega}\right)$ for the norm*

$$u \in C^\infty\left(\overline{\Omega}\right) \mapsto \left(\int_\Omega (|u(x)|^2 + |p_{T_\mu(x)}(\nabla u(x))|^2)\,\mathrm{d}\mu(x)\right)^{1/2}$$

Some natural examples. We can see that the words "*tangential* gradients" are quite natural in the following cases:

- if μ is the Lebesgue measure $\mathcal{L}^1$ concentrated on the segment $I = [0,1] \times \{0\} \times \cdots \times \{0\}$, then T_μ is the line $\mathbb{R} \times \{0\} \times \cdots \times \{0\}$ a.e. on I and
$$H^1_\mu = \left\{u \in L^2_\mu : \frac{\partial u}{\partial x_1} \in L^2_\mu\right\} \quad \text{and} \quad \nabla_\mu u = \left(\frac{\partial u}{\partial x_1}, 0, \ldots, 0\right);$$
- more generally, if μ is the uniform Hausdorff measure supported on a k-dimensional manifold M, then T_μ is the tangent space to M in the sense of the differential geometry.

Link between the two above definitions. Let us remark that, if v is a tangent field as defined above, *i.e.* the operator $\mathrm{div}(\varphi\mu)$ is continuous for the L^2_μ-norm on $\mathcal{D}(\Omega)$, we have for any sequence $(u_n)_n$ of smooth functions having 0 for limit in L^2_μ:

$$\left|\int_\Omega \nabla u_n \cdot \varphi \,\mathrm{d}\mu\right| \le C\,||u_n||_{L^2_\mu} \to 0.$$

Then, if $v \in \Gamma(0)$, we have $v \cdot \varphi$ in L^2_μ for any element $\varphi \in X^2_\mu$. This implies that, up to a μ-negligible set, we have the following inclusion between the two definitions of the tangent space

$$Q_\mu(x) \subseteq T_\mu(x).$$

We are not able for the moment to prove the inverse inclusion in general (but it is true in dimension one, see below), but the equality between these linear spaces holds for all the examples that we have studied; the links between these both notions are the subject (among others) to current works by S. Di Marino [35].

1.3 The one-dimensional case

In this section, we give a precise description of the tangent space to a generic measure μ (in the sense of the definition 1.2.3) defined on the real line. We set $d = 1$, $\Omega = I$ is a bounded interval of $\mathbb{R}$ and $\mu \in \mathcal{P}(I)$.

1.3.1 Structure of T_μ and compactness result in H^1_μ

The structure of the tangent space is given by the following theorem:

Theorem 1.3.1. *We denote by:*

- $\mu = \mu_a + \mu_s$ *the Lebesgue decomposition of* μ*;*
- A *a Lebesgue-negligible set on which is concentrated* μ_s*;*
- f *the density of* μ_a*, and*

$$M = \left\{ x \in I \,:\, \forall \varepsilon > 0, \int_{I \cap (x-\varepsilon, x+\varepsilon)} \frac{\mathrm{d}t}{f(t)} = +\infty \right\}$$

Then

$$T_\mu(x) = \begin{cases} \{0\} & \text{if } x \in M \cup A \\ \mathbb{R} & \text{otherwise} \end{cases}$$

for μ*-a.e.* $x \in I$*.*

As in the case of "classical" weighted Sobolev spaces, the set M appears as a critical set for the absolutely continuous part of μ. Notice also that the same set appears in the relaxation of convex functional on the space of measures (see [18], where a similar result to the above theorem can be seen, in the case of absolutely continuous measures, as a consequence of the Example 3).

Structure of the space H^1_μ. We know that M, which can of course be empty (for instance if f is bounded from below by a positive constant on I), is the maximal closed subset of I such that

$$\frac{1}{f} \in L^1_{loc}(I - M)$$

i.e. such that the condition (1.2) is satisfied for the weighted Lebesgue space $L^2_f(I - M)$. Therefore, if $u \in L^2_\mu$, then $u|_{I-M}$ admits a weak derivative.

On the other hand, as a consequence of Prop. 1.2.3, saying that T_μ is identically equal to the whole space $\mathbb{R}^d$ (with here $d = 1$) on a set B means exactly that, for $u \in H^1_\mu$, all the elements of $\Gamma(u)$ coincide on B. Here we then expect that, on the set $I - (M \cup A)$, the unique element of $\Gamma(u)$ will coincide with the weak derivative of $u|_{I-M}$.

Indeed, let us take φ a smooth function with compact support included into $I \cap M$ and verifying

$$\frac{1}{f} \in L^1(\mathrm{Supp}(\varphi))$$

Let moreover $v \in \Gamma(u)$ and $(u_n)_n$ be a sequence of smooth functions with $(u_n, u_n') \to (u, v)$ for the L^2_μ-norm. We get

$$\langle v - u', \varphi \rangle_{\mathcal{D}'(I-M), \mathcal{D}(I-M)} = - \lim_{n \to +\infty} \left(\int_I (u_n - u)\varphi' \right)$$

with, by Hölder inequality,

$$\int_I |(u_n - u)\,\varphi'| \leq ||u_n - u||_{L^2_f} \left(\int_I \frac{\varphi'}{f}\right)^{1/2} \to 0$$

since, thanks to the assumption on $1/f$, the integral $\int_I \varphi'/f$ is finite. This shows that $u'_n \to u'$ in $\mathcal{D}(I-M)$; but we also know that $u'_n \to v$ in $L^2_f(I-M)$, which leads to

$$u' \in L^2_f(I-M) \quad \text{and} \quad u' = v \ \mu\text{-a.e. on } I-M$$

so that, in particular, $u|_{I-M} \in H^1_f(I-M)$.

Conversely, if $u \in C^\infty\left(\overline{\Omega}\right)$ then the Theorem 1.3.1 gives

$$||u||^2_{H^1_\mu(I)} = ||u||^2_{L^2_\mu(I)} + ||u'||^2_{L^2_f(I-M)}$$

and H^1_μ is the closure of $C^\infty\left(\overline{\Omega}\right)$ for this norm. This achieves the proof of:

Proposition 1.3.1. *A function $u \in L^2_\mu(I)$ belongs to $H^1_\mu(I)$ if and only if $u|_{I-M} \in H^1_f(I-M)$. Moreover, for μ-a.e. $x \in I$,*

$$\nabla_\mu u(x) = \begin{cases} 0 & \text{if } x \in M \cup A \\ u'(x) & \text{otherwise} \end{cases}$$

and $||u||^2_{H^1_\mu(I)} = ||u||^2_{L^2_\mu(I)} + ||u'||^2_{L^2_f(I-M)}$.

In particular, in dimension one, if $\mu \ll \mathcal{L}^1$, then this definition of H^1_μ coincides with the definition of the weighted Sobolev space H^1_f given by the formula 1.4, and for a generic measure μ whose f is the density of the absolutely continuous part, H^1_μ is exactly the set of functions of L^2_μ which also belongs to H^1_f.

Compactness result in $H^1_\mu(I)$. In order to examine variational problems in these Sobolev spaces, the following proposition will be useful; it is already know, as a consequence of the Rellich theorem, if $\mu_s = 0$ and f is bounded from below.

Proposition 1.3.2. *Let $(u_n)_n$ be a bounded sequence of $H^1_\mu(I)$. Then there exists a subsequence $(u_{n_k})_k$ which admits a pointwise limit u on μ-a.e. every point on which T_μ is $\mathbb{R}$.*

Proof. The sequence (u_n) is bounded in $H^1_\mu(I)$, thus the sequence $(u_n|_{I-M})_n$ is bounded in the weighted Sobolev space $H^1_f(I-M)$. But since $I-M$ is exactly the set of points around which $1/f$ is integrable, we know that $L^2_f(I-M) \hookrightarrow L^1_{loc}(I-M)$; this gives that the sequence of the weak derivatives of u_n (which are functions of $L^2_f(I-M)$) is bounded in $L^1_{loc}(I-M)$, thus $(u_n|_{I-M})_n$ is bounded in the space $BV(I-M)$ and admits a $\mathcal{L}^1$-a.e. pointwise convergent subsequence. Consequently, $(u_n)_n$ converges pointwisely on μ_a-a.e. point on which T_μ is $\mathbb{R}$, and we know that T_μ is μ_s-a.e. null, which achieves the proof □

Equivalence between Q_μ and T_μ in 1D. We have already seen that, with the above notations, the tangent space Q_μ originally defined by Bouchitté *et al.* and the space T_μ satisfy the inclusion

$$\text{for } \mu\text{-a.e. } x,\ Q_\mu(x) \subseteq T_\mu(x)$$

Let us show that in the case of the real line the Theorem 1.3.1 implies that the converse inclusion is also true. Of course, we only have to show it on the points where T_μ is not null, *i.e.* we have to check that

$$\text{for } \mathcal{L}^1\text{-a.e. } x \in I-M,\ Q_\mu(x) = \mathbb{R}$$

For this, let $x \in I - M$ and $a < b$ such that $a < x < b$ and $\int_a^b 1/f < +\infty$. We now consider

$$\varphi : t \mapsto = \begin{cases} \dfrac{1}{f(t)} \displaystyle\int_a^t \frac{1}{\sqrt{f}} & \text{if } t \in (a,b) \text{ and } t \notin A \\ 0 & \text{otherwise} \end{cases}$$

We have

$$\int_I \varphi^2 \,\mathrm{d}\mu = \int_a^b \left(\frac{1}{f(t)}\right)^2 \left(\int_a^t \frac{1}{\sqrt{f}}\right)^2 f(t)\,\mathrm{d}t \leq (b-a) \left(\int_a^b \frac{1}{f}\right)^3 < +\infty$$

so that $\varphi \in L^2_\mu(I)$. Moreover, for $\psi \in \mathcal{D}(a,b)$,

$$\int_I \psi' \varphi \,\mathrm{d}\mu = -\int_a^b \psi(\varphi f)' = -\int_a^b (\psi\sqrt{f}) \left(\frac{(\varphi f)'}{\sqrt{f}}\right)$$

$$\text{thus} \qquad \left|\int_I \psi' \varphi \,\mathrm{d}\mu\right| \leq \left(\int_a^b \psi^2 f\right)^{1/2} \left(\int_a^b \frac{((\varphi f)')^2}{f}\right)^{1/2}$$

with by definition $(\varphi f)' = 1/\sqrt{f}$, which leads to

$$\left|\int_I \psi' \varphi \,\mathrm{d}\mu\right| \leq \sqrt{b-a}||\psi||_{L^2_\mu(I)}$$

This proves that $\varphi \in X_\mu$, so that $\varphi(x) \in Q_\mu(x)$ for a.e. point x; but φ is positive a.e. on (a,b). This implies $Q_\mu(x) = \mathbb{R}$ a.e. on (a,b) for the Lebesgue measure.

1.3.2 Proof of Theorem 1.3.1

We divide this proof into three parts, which correspond to the decomposition of I into its singular set A, the "nice set" $I - (M \cup A)$ for its regular part, and the "critical set" M. We will use the following characterization of the tangent space of a generic measure μ in dimension 1, which is a direct corollary of the definitions: if $B \subset I$ is a Borel set with $\mu(B) > 0$, we have the following implications:

1. if any $v \in \Gamma(0)$ is μ-a.e. null on B, then $T_\mu = \mathbb{R}$ μ-a.e. on B;
2. if, for any $u \in H^1_\mu$, there exists a gradient of u which is μ-a.e. null on B, then $T_\mu = 0$ μ-a.e. on B;
3. if there exists a gradient of 0 which is positive μ-a.e. on B, then $T_\mu = 0$ μ-a.e. on B

The singular part of μ. Let us first prove that $T_\mu = \{0\}$ for the singular part of μ. We use the third characterization of the tangent space and build a sequence of functions $(u_n)_n$ such that

$$u_n \to 0 \quad \text{and} \quad u_n' \to \mathbb{1}_A \quad \text{in } L^2_\mu$$

which will prove that $\mathbb{1}_A \in \Gamma(0)$ and imply the result.

For $n \in \mathbb{N}$, let Ω_n be an open set recovering A with $\mu(\Omega_n - A) + \mathcal{L}^1(\Omega_n) \leq 1/n$. By the Lusin theorem, there exists v_n continuous with $0 \leq v_n \leq 1$ on I and

$$(\mu + \mathcal{L}^1)(\{x \in I : v_n(x) \neq \mathbb{1}_{\Omega_n}(x)\}) \leq 1/n$$

Let us consider $u_n(x) = \int_a^x v_n(x)\,\mathrm{d}x$, where a is the lower bound of I. Then we have:

- for any $x \in I$,

$$|u_n(x)| \leq \int_I (|v_n - \mathbb{1}_{\Omega_n}|(t) + \mathbb{1}_{\Omega_n}(t))\, \mathrm{d}t \leq \mathcal{L}^1(\{v_n \neq \mathbb{1}_{\Omega_n}\}) + \mathcal{L}^1(\Omega_n) \leq 2/n$$

thus u_n goes to 0 uniformly and also for the L^2_μ-norm;

- on the other hand, since $u_n' = v_n$ coincides with $\mathbb{1}_A$ except on a set E_n such that $\mu(E_n) \leq 1/n$, we have

$$\int_I |u_n'(x) - \mathbb{1}_A(x)|^2 \, \mathrm{d}\mu(x) \leq ||v_n - \mathbb{1}_A||^2_\infty \, \mu(E_n) \leq 4/n$$

thus $u_n' \to \mathbb{1}_A$ for the L^2_μ-norm.

We obtain that $\mathbb{1}_A \in \Gamma(0)$, which guarantees that $T_\mu = 0$ on A.

The regular part, outside of the critical set. Second, we prove that $T_\mu = \mathbb{R}$ outside of $M \cup A$. Using the first characterization of the tangent space, we take an element g of $\Gamma(0)$ and we want to show that $g = 0$ μ-a.e. outside of $M \cup A$; by definition of A, it is enough to show that $g = 0$ $\mathcal{L}^1$-a.e. on U. As in the above remark, taking a sequence of smooth functions $u_n \to 0$ with $u_n' \to g$ and a test function φ such that $1/f$ is integrable on the support of φ, we obtain

$$\left|\int_U u_n'\varphi\right| = \left|\int_U u_n\varphi'\right| \leq \int_U \left|u_n\sqrt{f}\right| \left|\frac{\varphi'}{\sqrt{f}}\right| \leq \left(\int_U u_n^2 f\right)^{\frac{1}{2}} \left(\int_U \frac{\varphi'^2}{f}\right)^{\frac{1}{2}}$$

which goes to 0 as $n \to +\infty$. The same computation gives $\int_U u_n'\varphi \to \int_U g\varphi$. We deduce that $g = 0$ a.e. on U for the Lebesgue measure.

The critical set. This part is more difficult. Given a function $u \in C^1(I)$, we build a sequence $(u_n)_n$ of regular functions (say, C^1) such that $u_n \to u$ and $u_n' \to v$ for the L^2_μ-norm, with $v = 0$ on M. The strategy is the following:

- given a set Ω_n which is "almost" M, we start from a function u_n which coincides with u outside of Ω_n and is piecewise constant on Ω_n (so that its derivative is null on M);
- then, using the fact that the discontinuity points of u_n belong to the set M, we regularize u_n around this points so that its derivative stays small for the L^2_μ-norm.

The first step consists in building our set Ω_n:

Lemma 1.3.1. *Let us denote by $(x_n)_n$ a sequence containing all the atoms of μ. For $n \in \mathbb{N}$, there exists Ω_n such that:*

- $\Omega_n = \bigcup_{i=1}^{p_n} (a_i, b_i)$, *with $b_i < a_{i+1}$ for each i, and $(a_i, b_i) \cap M \neq \emptyset$;*
- $\Omega_n \supset M - \{x_1, \dots, x_n\}$;
- $\mu(\Omega_n - (M - \{x_1, \dots, x_n\})) \leq 1/n$

Proof. Let U_n be a finite union of open intervals recovering M and with $\mu(U_n - M) \leq 1/n$ (such a set exists by regularity of μ and compactness of M). We denote by $\Omega_n = U_n - \{x_1, \dots, x_n\}$. It is still a finite union of open intervals, containing $M - \{x_1, \dots, x_n\}$ and with $\mu(\Omega_n - (M - \{x_1, \dots, x_n\})) \leq 1/n$. Moreover, we may assume that all these intervals contain an element of M: it is enough to remove from Ω_n the intervals which do not contain any element of M (if after that we obtain $\Omega_n = \emptyset$, it means that $M \subset \{x_1, \dots, x_n\} \subset A$ and we already know that $T_\mu = \{0\}$ on A, so there is nothing to prove). □

Let us thus take a sequence $(w_n)_n$ of piecewise constant functions such that $w_n \to u$ in L^2_μ (it is possible since u is continuous, thus can be approached uniformly on I by a sequence of piecewise constant functions) and $||u_n||_\infty \leq C$, where C only depends on u; we replace g_n by u outside of the set Ω_n (the new function will still be called g_n), so that we have now

- $g_n \to u$ for the L^2_μ-norm;
- g_n coincides with u outside of Ω_n;
- g_n coincides on Ω_n with a piecewise constant function.

We begin by regularizing g_n around the endpoints of the intervals forming Ω_n. Let $\varepsilon_n > 0$ be small enough so that:

- $a_i + \varepsilon_n < b_i - \varepsilon_n$, for each i (we will denote $a'_i = a_i + \varepsilon_n$ and $b'_i = b_i - \varepsilon_n$);
- (a'_i, b'_i) contains an element of M, for each i;
- on (a_i, b_i), g_n has not any discontinuity point outside of (a'_i, b'_i);
- if we denote by Ω'_n the union of the intervals (a'_i, b'_i), we have $\mu(\Omega_n - \Omega'_n) \leq 1/n$.

Lemma 1.3.2. *There exists a function w_n which coincides with g_n outside of $\Omega_n - \Omega'_n$ and such that, on each interval (a_i, a'_i) and (b'_i, b_i),*

- *w_n and w'_n are bounded by constants depending only on u and u';*
- *$w_n(a_i) = u(a_i)$, $w'_n(a_i) = u'(a_i)$ and $w'_n = 0$ on a small interval before a'_i*
- *$w_n(b_i) = u(b_i)$, $w'_n(b_i) = u'(b_i)$ and $w'_n = 0$ on a small interval after b'_i.*

Proof. It is enough to replace g_n on the interval $]a_i, a_i + \varepsilon_n$ by the function $x \mapsto Q(a_i + x)$ where

$$Q(t) = -\frac{u'(a_i)}{2\varepsilon_n} t^2 + u'(a_i)t + u(a_i)$$

and to scale the new function on the interval (a_i, a'_i) by replacing it by

$$x \mapsto \begin{cases} w_n(a_i + 2(x - a_i)) & \text{if } a_i \leq x \leq a_i + \varepsilon_n/2 \\ w_n(a_i'^{-}) & \text{otherwise} \end{cases} \qquad \square$$

Since g_n and w_n are bounded uniformly in n and coincide outside of the set $\Omega_n - \Omega'_n$ whose measure is at most $1/n$, the sequence $(w_n)_n$ still has u for limit in L^2_μ; moreover, we have

$$||w'_n||_{L^2_\mu(\Omega_n - \Omega'_n)} \leq (2/n)||u'||_\infty$$

and for any discontinuity point y of w_n, w_n is piecewise constant on a (small) neighborhood of y. We now have to regularize w_n around its discontinuity points, which are on Ω'_n; this is possible only if these points belong to the set M. Then we are interested by the following "displacement" procedure of the discontinuity points:

Lemma 1.3.3. *For any n, there exists a function v_n such that*

- *$v_n = w_n$ outside of Ω'_n;*
- *v_n is still piecewise constant on Ω'_n;*

- *any discontinuity point of v_n belongs to M;*
- *$v_n \to u$ in L^2_μ.*

Proof. We have to modify w_n only on each interval (a'_i, b'_i). On this interval, w_n admits a finite number of jumps which we denote by $a'_i \leq x_1 < \cdots < x_n = b'_i$. We make the following construction:

- Let $m = \inf([a'_i, b'_i] \cap M)$. We define v_n on the interval $[a'_i, m[$ (if it is nonempty) by setting $v_n = w_n({a'_i}^+)$.
- We then reiterate the construction starting from m:
  - if $(m, b'_i) \cap M = \emptyset$, we set $v_n = w_n({b'_i}^-)$ on this interval, and we are done;
  - otherwise, let $m' = \inf((m, b'_i) \cap M)$. We have naturally $m' \geq m$. If $m \geq x_n$, then we set $v_n = w_n({b'_i}^-)$ on (m', b'_i),w_n on $[m, m')$ and we are done;
  - if $m = m' < x_n$, then we denote by j the smallest index such that $x_j > m$, we set $v_n = w_n$ on $[m, x_j)$ and we reiterate this construction starting from x_j;
  - finally, if $m < m' < x_n$, we set $v_n = w_n$ on $[m, m')$ and we reiterate this construction starting from m'.

With this construction, $w_n - v_n \neq 0$ only on $\Omega'_n - M$. Since $\mu(\Omega'_n - M) \leq 1/n$ and w_n, v_n are uniformly bounded, we get $||v_n - w_n||_{L^2_\mu} \leq C/n$, and we thus still have $v_n \to u$. Moreover, by construction, v_n is still piecewise constant on the set Ω'_n and all its jumps belong to M. □

To finish, we have to modify v_n around each discontinuity point, so that the new function u_n is regular and admits a derivative which is small for the L^2_μ-norm. This is allowed by the following result about embeddings between functional spaces:

Lemma 1.3.4. *Let J be a bounded interval of $\mathbb{R}$, and μ a finite measure on J with density $f > 0$. The following assertions are equivalent:*

1. *The function $1/f$ belongs to $L^1(J)$*
2. *The space $L^2_\mu(J)$ is continuously embedded into $L^1(J)$*

Proof. We have already seen the direct implications, which comes immediately from the Cauchy-Schwarz inequality. For the inverse implication, let us assume that $\int_J 1/f = +\infty$ and set

$$E_n = \left\{ t \in J : \frac{1}{n+1} \leq f(t) < \frac{1}{n} \right\} \quad \text{and} \quad l_n = \mathcal{L}^1(E_n)$$

We know that $\sum_n l_n < +\infty$ (it is the length of J) and

$$\sum_n n l_n = \sum_n \int_J n\, \mathbb{1}_{\{n-1 \leq 1/f \leq n\}} \geq \sum_n \int_J \frac{1}{f}\, \mathbb{1}_{\{n-1 \leq 1/f \leq n\}} \geq \int_J \frac{1}{f} = +\infty$$

thus $\sum_n n l_n = +\infty$. We will build a function U which is constant on each set E_n, belongs to L^2_μ and does not belong to L^1. If we denote by u_n the value of U on E_n, it is equivalent to find a sequence $(u_n)_n$ verifying

$$\sum_n u_n^2\,(n l_n) < +\infty \quad \text{and} \quad \sum_n |u_n|\, l_n = +\infty$$

If we resume, we want to prove the following statement: for any sequence $(l_n)_n$ of positive numbers such that $\sum_n nl_n = +\infty$ and $\sum_n l_n < +\infty$, there exists a sequence $(u_n)_n$ of positive numbers such that $\sum_n u_n^2(nl_n) < +\infty$ and $\sum_n nu_n = +\infty$. By contradiction, it is equivalent to the following: for any $(l_n)_n \subset \mathbb{R}_+^*$ such that $\sum_n l_n < +\infty$ and the following implication holds:

$$\left(\sum_n u_n^2\,(nl_n) < +\infty\right) \Rightarrow \left(\sum_n l_n|u_n| < +\infty\right)$$

we have $\sum_n nl_n < +\infty$. This result can be seen as a corollary of the Banach-Steinhaus theorem. Denoting by $\ell^2_{nl_n}$ the space of sequences $(u_n)_n$ such that $\sum_n u_n^2\,(nl_n) < +\infty$, the operator

$$T_N : u \in \ell^2_{nl_n} \longmapsto \sum_{n=0}^{N} l_n u_n$$

is linear continuous with norm $\left(\sum\limits_{n=0}^{N} nl_n\right)^{1/2}$ and the assumption about $(l_n)_n$ is equivalent to

$$\forall u \in \ell^2_{nl_n} \quad \sup_{N\in\mathbb{N}} |T_N(u)| < +\infty$$

By Banach-Steinhaus theorem, we get $\sup\limits_{N\in\mathbb{N}} ||T_N|| < +\infty$ and $\sum\limits_{n\in\mathbb{N}} nl_n < +\infty$; the proof is complete. □

We now have the arguments to transform w_n into a C^1 function u_n which will give us our approximation. Let us recall that v_n is equal to u outside of Ω_n, piecewise constant on Ω'_n, all its jumps are located in the set M and the derivative of v_n is null on a neighborhood of this jumps. Denoting by $y_1 < \cdots < y_p$ the jumps of v_n, we find ε_n such that, for each j, v_n is constant on $(y_j - \varepsilon_n, y_j)$ and $(y_j, y_j + \varepsilon_n)$. Moreover, we know that $y_1, \ldots, y_p$ are distinct of the "big atoms" $x_1, \ldots, x_n$ of μ, thus we can assume, if ε_n is small enough, that

$$\sum_{j=1}^{p}(\mu((y_j - \varepsilon_n, y_j + \varepsilon_n)) \leq 1/n + \sum_{k\geq n} \mu(x_k)$$

and we denote by μ_n this last term, which goes to 0 as $n \to +\infty$.

On the interval $(y_j - \varepsilon_n, y_j + \varepsilon_n)$, thanks to the last lemma, L^2_μ is not embedded into L^1, thus we can find a regular function g_j such that

$$\int_{y_j-\varepsilon_n}^{y_j+\varepsilon_n} g_j = v_n(y_j^+) - v_n(y_j^-) \quad \text{and} \quad \int_{y_j-\varepsilon_n}^{y_j+\varepsilon_n} g_j^2\,\mathrm{d}\mu \leq \frac{1}{nq}$$

Then, we set

$$u_n(x) = \begin{cases} \widetilde{v}_n(y_j - \varepsilon_n) + \int_{y_j-\varepsilon_n}^{x} g_j & \text{if } y_j - \varepsilon_n \leq x \leq y_j + \varepsilon_n \\ v_n(x) & \text{otherwise} \end{cases}$$

and we finish with the following:

Proposition 1.3.3. *This sequence $(u_n)_n$ satisfies $u_n \to u$ and $u'_n \to v$ for the L^2_μ-norm, where*

$$v(x) = \begin{cases} u'(x) & \text{if } x \notin M \text{ or is an atom of } \mu \\ 0 & \text{otherwise} \end{cases}$$

Consequently, $T_\mu = \{0\}$ μ-a.e. on M.

Proof. We know that $v_n \to u$, thus $u_n \to u$ for the L^2_μ-norm outside of the intervals $]y_j - \varepsilon_n, y_j + \varepsilon_n[$. But since the total mass of this intervals goes to 0 and $(u_n)_n$ is uniformly bounded, we get $u_n \to u$. For the derivative, since $u_n = u$ outside of Ω_n, we have

$$||u_n' - v||^2_{L^2_\mu} = ||u_n' - v||^2_{L^2_\mu(\Omega_n)} = ||u_n' - v||^2_{L^2_\mu(\Omega_n - M)} + ||u_n' - v||^2_{L^2_\mu(M - \{x_1, \dots, x_n\})}$$

where the first term goes to 0 (since $(u_n)_n$ is uniformly bounded and $\mu(\Omega_n - M)$ goes to 0); for the second one, we have $v = 0$ on M, thus it is enough to prove that $||u_n'||$ goes to 0 for the L^2_μ-norm on $M - \{x_1, \dots, x_n\}$; this term is bounded by

$$||u_n'||^2_{L^2_\mu(\Omega_n' - \{y_1, \dots, y_p\})} + \sum_{j=1}^{p} u_n'(y_j)\mu(\{y_j\})$$

Since $(u_n')_n$ is uniformly bounded, we know that the second term goes to 0, and since u_n is constant outside of the intervals $]y_j - \varepsilon_n, y_j + \varepsilon_n[$ the first one is equal to

$$\sum_{j=1}^{p} \int_{y_j - \varepsilon_n}^{y_j + \varepsilon_n} g_j^2 d\mu$$

which, by definition of g_j, is smaller than $1/n$. This completes the proof. □

1.3.3 Consequences for our variational problem

Theorem 1.3.2. *Let us assume that the set*

$$\{T \in H^1_\mu(I) : T_\# \mu = \nu\}$$

is non-empty. Then the problem

$$\inf \left\{ \int_I (|T(x) - x|^2 + |\nabla_\mu T(x)|^2) \, \mathrm{d}\mu(x) \ : \ T \in H^1_\mu(I), T_\# \mu = \nu \right\}$$

admits at least one solution.

Proof. Let us begin by rewriting precisely the functional that we consider in this case: we know that $T_\mu = \{0\}$ on $M \cup A$ and $\mathbb{R}$ elsewhere, so now we are minimizing

$$J : U \in H^1_\mu(I) \longmapsto \int_{I-M} ((U(x) - x)^2 + U'(x)^2) f(x) \, \mathrm{d}x + \int_{M \cup A} (U(x) - x)^2 \, \mathrm{d}\mu(x)$$

Let us moreover denote by $(x_i)_i$ the sequence of the atoms of μ (which can be empty), $(\mu_i)_i$ the sequence of their mass, and

$$\mu|_{M \cup A} = \widetilde{\mu} + \sum_i \mu_i \delta_{x_i}$$

with $\widetilde{\mu}$ is atom-less. Now, for $U \in H^1_\mu(I)$, we have

$$J(U) = \int_{I-M} ((U(x) - x)^2 + U'(x)^2) f(x) \, \mathrm{d}x + \int_{M \cup A} (U(x) - x)^2 \, \mathrm{d}\widetilde{\mu}(x) + \sum_i \mu_i (U(x_i) - x_i)^2$$

Let $(U_n)_n$ be a minimizing sequence. On $I - M$, which is exactly the set of points on which T_μ is $\mathbb{R}$, we can extract from $(U_n)_n$ a μ-a.e. (which means $\mathcal{L}^1$-a.e. wherever $f \neq 0$) pointwise convergent subsequence, whose limit is denoted by U; let us remark that U is the weak limit of $(U_n)_n$ (up to a subsequence) in the space H^1_f, and by semi-continuity, we have

$$\int_{I-M} ((U(x)-x)^2 + U'(x)^2) f(x)\,\mathrm{d}x \leq \liminf \left(\int_{I-M} ((U_n(x)-x)^2 + U_n'(x)^2) f(x)\,\mathrm{d}x \right)$$

Now we set, for $n \in \mathbb{N}$, $\nu_n = (U_n)_\# \widetilde{\mu}$ and $\widetilde{U}_n$ the optimal transport map for the Monge-Kantorovich quadratic cost between $\widetilde{\mu}$ and ν_n. We know that $\widetilde{U}_n$ is the unique non-decreasing transport map between $\widetilde{\mu}$ and ν_n, so that we can assume that $(\widetilde{U}_n)_n$ admits, for the $\widetilde{\mu}$-a.e. convergence, a limit $\widetilde{U}$. For any n, thanks to the optimality of $\widetilde{U}_n$, we have

$$\int_{M\cup A} (\widetilde{U}_n(x)-x)^2\,\mathrm{d}\widetilde{\mu}(x) \leq \int_{M\cup A} (U_n(x)-x)^2\,\mathrm{d}\widetilde{\mu}(x)$$

and by semi-continuity

$$\int_{M\cup A} (\widetilde{U}(x)-x)^2\,\mathrm{d}\widetilde{\mu}(x) \leq \liminf \left(\int_{M\cup A} (U_n(x)-x)^2\,\mathrm{d}\widetilde{\mu}(x) \right)$$

Finally, by a diagonal argument, one can find a sequence $(n_k)_k$ such that $(U_{n_k}(x_i))_k$ has a limit for each i.

Thus, if we denote by

$$T_n(x) = \begin{cases} U_n(x) & \text{if } x \in V \\ \widetilde{U}_n(x) & \text{if } x \in (M \cup A) \end{cases} \qquad \text{and} \qquad T(x) = \begin{cases} U_(x) & \text{if } x \in V \\ \widetilde{U}(x) & \text{if } x \in M \cup A \end{cases}$$

we have $T_n \to T$ μ-a.e. on I, and

$$\begin{aligned} J(T) &\leq \liminf \left(\int_{I-M} ((U_n(x)-x)^2 + U_n'(x)^2) f(x)\,\mathrm{d}x \right) + \liminf \left(\int_{M\cup A} (U_n(x)-x)^2\,\mathrm{d}\mu(x) \right) \\ &\leq \liminf J(U_n) \end{aligned}$$

where $(U_n)_n$ is a minimizing sequence for J on the set of H^1_μ transport maps between μ and ν. It is thus enough to prove that T satisfies the constraint on image measure to conclude. But for each n, by construction, $(T_n)_\#\mu = \nu$ and the μ-a.e. convergence allows to obtain that this is also true for the limit T; the proof is complete. □

1.4 Some surprising examples in dimension 2

In this section, we study some examples of "surprising" measures in dimension 2, which are absolutely continuous with respect to the Lebesgue measure but have a density such that the critical set is the whole domain Ω and the tangent space is a.e. non-null. However, we are able to show compactness results in these particular cases which are enough for our variational problem.

1.4.1 A case where $\dim T_\mu = 1$ a.e.

We begin by building a density in dimension 1 which is an example of measure for which the critical set is the whole interval, a tool that will be used several times in the following to construct examples in dimension 2 and more. This example is due to Guy David.

Proposition 1.4.1. *There exist a function $f \in L^\infty(0,1)$ which is positive a.e. (for the Lebesgue measure) on $(0,1)$ and such that, for any $x \in (0,1)$,*

$$\int_{(0,1)\cup(x-\varepsilon,x+\varepsilon)} \frac{dt}{f(t)} = +\infty$$

In other words, the set M corresponding to this function is the whole $(0,1)$.

Proof. Let $(q_n)_n$ be an enumeration of $\mathbb{Q}\cap(0,1)$. For $\varepsilon > 0$, we set

$$U_\varepsilon = \bigcup_{n\in\mathbb{N}} \left(q_n - \frac{\varepsilon}{2^n}, q_n + \frac{\varepsilon}{2^n}\right) \cap (0,1)$$

The set U_ε is open, contains $\mathbb{Q}$ and we have $2\varepsilon \leq \mathcal{L}^1(U_\varepsilon) \leq \sum_{n\in\mathbb{N}} (\varepsilon/2^n) = 4\varepsilon$. Moreover, the family $(U_\varepsilon)_\varepsilon$ decreases when $\varepsilon \to 0$.

We then take a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ which decreases to 0 as $k \to +\infty$ and verifies $\sum_k \varepsilon_k = +\infty$. Let us show that the function

$$f = \frac{1}{1 + \sum_k \mathbb{1}_{U_{\varepsilon_k}}}$$

satisfies the required property:

- f is of course bounded (by 1) on $(0,1)$.
- $f(x)$ is null if and only if x belongs to an infinite number of U_{ε_k}; but since the sequence $(U_{\varepsilon_k})_k$ is decreasing for the inclusion, this means that x belongs to U_{ε_k} for any k. On the other hand,

$$\mathcal{L}^1\left(\bigcap_{k\geq k_0} U_{\varepsilon_k}\right) = \mathcal{L}^1(U_{\varepsilon_{k_0}}) \leq 4\varepsilon_{k_0} \to 0 \text{ as } k_0 \to +\infty$$

so that the intersection of all the U_{ε_k} has zero Lebesgue measure; this implies that $f(x)$ is positive for a.e. x;

- finally, if $a < b \in (0,1)$, then

$$\int_a^b \frac{1}{f} = (b-a) + \sum_k \mathcal{L}^1(U_{\varepsilon_k} \cap (a,b))$$

Let n_0 be a large enough integer so that $a' = a + \frac{1}{2^{n_0}} < b - \frac{1}{2^{n_0}} = b'$. We have for $\varepsilon < 1$:

$$U_\varepsilon \cap (a,b) = \bigcup_{n\in\mathbb{N}} \left(q_n - \frac{\varepsilon}{2^n}, q_n + \frac{\varepsilon}{2^n}\right) \cap (a,b) \supset \bigcup_{n\geq n_0 q_n\in(a',b')} \left(q_n - \frac{\varepsilon}{2^n}, q_n + \frac{\varepsilon}{2^n}\right)$$

We deduce that

$$\mathcal{L}^1(U_{\varepsilon_k} \cap (a,b)) \geq \sum_{n\geq n_0 q_n\in(a',b')} \frac{\varepsilon}{2^{n-1}} = C\varepsilon \qquad \text{where } C = \sum_{n\geq n_0 q_n\in(a',b')} \frac{1}{2^{n-1}}$$

and C only depends on a, b and does not depend on ε. Thus, we have

$$\int_a^b \frac{1}{f} = (b-a) + \sum_k \mathcal{L}^1(U_{\varepsilon_k} \cap (a,b)) \geq 1 + C\sum_{k\in\mathbb{N}} \varepsilon_k = +\infty \quad \square$$

By construction, the density f satisfies, with the notations of the above section, $M = (0,1)$ and the tangent space to the measure with density f is null a.e. $(0,1)$. We are interested in the behavior of the Sobolev space with respect to the measure

$$\mathrm{d}\mu(x_1, x_2) = f(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

on the square $\Omega = (0,1)^2$ in $\mathbb{R}^2$. Setting $g(x_1, x_2) = f(x_1)$, this example is precisely built so that the critical set

$$M = \left\{ x \in \Omega \,:\, \forall \varepsilon > 0, \int_{B(x,\varepsilon)\cap\Omega} \frac{\mathrm{d}t}{g(t)} \right\}$$

coincides with the whole domain Ω.

The space $L^2_\mu(\Omega)$ is described as follows:

$$u \in L^2_\mu(\Omega) \iff \iint_\Omega |u(x_1,x_2)|^2 f(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 < +\infty$$

$$\iff \text{for } \mu\text{-a.e. } x_1, u(x_1,\cdot) \in L^2(0,1) \text{ and } x_1 \mapsto ||u(x_1,\cdot)||_{L^2} \in L^2_f(0,1)$$

In particular, for $u \in L^2_\mu$, the partial derivative $\partial_2 u$ can be defined in the weak sense: it is, for μ-a.e. x_1, the weak derivative of the function $u(x_1,\cdot)$ which belongs to $L^2(0,1)$. This suggests the following proposition:

Proposition 1.4.2. *For μ-a.e. $x \in \Omega$, the space $T_\mu(x)$ is the vertical line $\mathbb{R}\cdot e_2$.*

In particular, although μ is absolutely continuous with respect to the Lebesgue measure, the tangent space to μ is non-trivial on almost all the domain Ω.

Proof of Prop. 1.4.2. We first prove that $e_1 \perp T_\mu(x)$ for μ-a.e. x; thanks to the Prop. 1.2.2, it is enough to show that $e_1 \in \Gamma(0)$ *i.e.* to produce a sequence $(u_n)_n$ of smooth functions having 0 for limit and such that $\nabla u_n \to e_1$ for the L^2_μ-norm. But we already know that, in dimension one, the tangent space to the measure with density f is identically $\{0\}$: this implies $\mathbb{1} \in \Gamma(0)$, *i.e.*

$$\exists v_n \in C^\infty(0,1)^{\mathbb{N}} : \begin{cases} v_n \to 0 \\ v_n' \to \mathbb{1} \end{cases} \text{for the } L^2_f - \textit{norm}$$

and we conclude by verifying that $u_n(x_1,x_2) = v_n(x_1)$ is convenient.

Let us now show that $e_2 \in T_\mu(x)$. In this aim we show the implication

$$v = (v^1, v^2) \in \Gamma(0) \Longrightarrow v^2 = 0 \ \mu\text{−}a.e.$$

Indeed, if $u_n \to 0$ and $\nabla u_n \to v = (v^1, v^2)$, then for $\varphi \in \mathcal{D}(\Omega)$ we have

$$\left| \iint_\Omega \partial_2 u_n(x_1,x_2)\varphi(x_1,x_2) f(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \right| = \left| \iint_\Omega u_n(x_1,x_2)\partial_2\varphi(x_1,x_2) f(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \right| \le ||u_n||_{L^2_f} ||\varphi||_{L^2_f}$$

which goes to 0 as $n \to +\infty$; on the other hand

$$\iint_\Omega \partial_2 u_n(x_1,x_2)\varphi(x_1,x_2) f(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \to \iint_\Omega v^2\varphi\,\mathrm{d}\mu$$

and we obtain $\int v^2\varphi\,\mathrm{d}\mu = 0$ for any test function φ, thus $\varphi = 0$. □

This implies the following description of the Sobolev space H^1_μ:

Proposition 1.4.3. *Let $u \in L^2_\mu(\Omega)$. Then $u \in H^1_\mu(\Omega)$ if and only if the two following conditions are satisfied:*

- *for a.e. x_1, the function $x_2 \mapsto u(x_1, x_2)$ belongs to $H^1(0,1)$;*
- *the function $x_1 \mapsto ||u(x_1,\cdot)||^2_{H^1(0,1)}$ belongs to $L^1_f(0,1)$.*

In this case,

$$\nabla_\mu u = (0, \partial_2 u)$$

where, for μ-a.e. x_1, $\partial_2 u(x_1,\cdot)$ is the weak derivative of $u(x_1,\cdot)$, which belongs to $L^2(0,1)$.

Proof. For the direct implication, let us take $u = \lim_n u_n$ with $\nabla u_n \to v$ in L^2_μ. Let $x_1 \in (0,1)$ such that $u_n(x_1,\cdot) \to u(x_1,\cdot)$ and $\partial_2 u_n(x_1,\cdot) \to v^2(x_1,\cdot)$ (this is true up to a negligible set of $(0,1)$). Then, for $\varphi \in \mathcal{D}(0,1)$, we have

$$\int_0^1 v^2(x_1,\cdot)\varphi = \lim_n \int_0^1 \partial_2 u_n(x_1,\cdot)\varphi = -\lim_n \int_0^1 u_n(x_1,\cdot)\varphi' = -\int u_n(x_1,\cdot)\varphi'$$

which shows that $\partial_2 u = v^2$. In particular

$$\int_0^1 ||u(x_1,\cdot)||^2_{H^1(0,1)} f(x_1)\,\mathrm{d}x_1 = \int_0^1 \left(\int_0^1 (|u(x_1,x_2)|^2 + |v(x_1,x_2)|^2)\,\mathrm{d}x_2\right) f(x_1)\,\mathrm{d}x_1 = ||u||^2_{L^2_\mu} + ||v||^2_{L^2_\mu}$$

which is finite since $u, v \in L^2_\mu(\Omega)$. Moreover, the Prop. 1.4.2 implies that, for any $v \in \Gamma(u)$,

$$\nabla_\mu u(x) = p_{T_\mu(x)}(v(x)) = (0, v^2(x)) = (0, \partial_2 u(x))$$

For the converse implication, we again remark that, if $u \in C^\infty\left(\overline{\Omega}\right)$,

$$||u||^2_{H^1_\mu(\Omega)} = \int_0^1 \left(||u(x_1,\cdot)||^2_{L^2(0,1)} + ||(\partial_2 u)(x_1,\cdot)||^2_{L^2(0,1)}\right) f(x_1)\,\mathrm{d}x_1$$

and conclude by the fact that $H^1_\mu(\Omega)$ is the closure of $C^\infty\left(\overline{\Omega}\right)$ for this norm. □

In particular:

- in this case, the space H^1_μ does not coincide with the set

$$\left\{u \in L^2_g(\Omega) : u|_{\Omega - M} \in H^1_g(\Omega - M)^2\right\}$$

  since this last set is actually the whole space $L^2_g(\Omega)$. The definition of H^1_μ is thus in general finer than the definition of the weighted Sobolev space H^1_g.
- there cannot exist a compactness result from H^1_μ to the a.e. convergence on Ω: it is enough to take a sequence $(u_n)_n$ of functions depending only on the first variable x_1, which is bounded in $L^2_f(0,1)$ but which is non-compact for the a.e. convergence to get a counter-example (since for a such sequence we have $||u_n||_{H^1_\mu} = ||u_n||_{L^2_\mu}$).

We are however able to show that the problem "without transport" where only the gradient term is considered admits a solution:

Theorem 1.4.1. *For this measure μ, provided that there exists a μ-Sobolev transport from μ to ν, the problem*

$$\inf\left\{||\nabla_\mu T||^2_{L^2_\mu(\Omega)} : T \in H^1_\mu(\Omega), T_\# \mu = \nu\right\}$$

admits at least one solution.

Proof. We begin again by taking a minimizing sequence $(T_n)_n$. Our strategy is to build a sequence ξ_n of "rearrangement" functions which will transform T_n into a pointwisely convergent minimizing sequence: ξ_n is a function from $(0,1)$ to itself such that, if we set $U_n(x_1,x_2) = T_n(\xi_n(x_1),x_2)$, we have

- for any n, $(U_n)_\#\mu = \nu$;
- each U_n belongs again to $H^1_\mu(\Omega)$ and $||\nabla_\mu U_n||_{L^2_\mu} \leq ||\nabla_\mu T_n||_{L^2_\mu}$;
- $(U_n)_n$ admits a limit for the μ-a.e. convergence.

We start by setting, for $n \in \mathbb{N}$ and $x_1 \in (0,1)$,

$$S_n(x_1) : x_2 \mapsto T_n(x_1,x_2)$$

For a.e. x_1, $S_n(x_1)$ is a function of $H^1(0,1) \subset L^2(0,1)$; we endow $L^2(0,1)$ with the Borel σ-algebra associated to its natural norm. Then we set

$$\rho_n = (S_n)_\# f$$

which is a finite positive measure on $L^2(0,1)$.

Lemma 1.4.1. *The sequence $(\rho_n)_n$ is tight.*

Proof of Lemma 1.4.1. For $L > 0$, we set

$$K_L = \left\{u \in L^2(0,1) : u \in H^1(0,1) \text{ and } ||u||^2_{H^1(0,1)} \leq L\right\}$$

K_L is a compact subset of $L^2(0,1)$ and for each n:

$$\rho_n(L^2(0,1) - K_L) = \int_0^1 \mathbb{1}_{L^2 - K_L}\, \mathrm{d}\rho_n = \int_0^1 \mathbb{1}_{S_n(x)\notin K_L}\, f(x)\, \mathrm{d}x$$

with

$$\int_{S_n(x)\notin K_L} = \mathbb{1}_{||S_n(x)||^2_{H^1(0,1)} \geq L} \leq \frac{||S_n(x)||^2_{H^1(0,1)}}{L}$$

so that

$$\int_0^1 \mathbb{1}_{S_n(x)\notin K_L}\, f(x)\, \mathrm{d}x \leq \frac{1}{L}\int_0^1 ||S_n(x)||^2_{H^1(0,1)} f(x)\, \mathrm{d}x = \frac{||T_n||^2_{H^1_\mu}}{L} \geq \frac{C}{L}$$

where C is an upper bound for $(T_n)_n$ in H^1_μ. For each $\varepsilon > 0$, taking $L = C/\varepsilon$ and $K = K_L$, we obtain

$$\forall n \in \mathbb{N},\ \rho_n(L^2(0,1) - K) \leq \varepsilon$$

which is exactly the result. □

Applying the Prokhorov theorem, we find a subsequence of $(S_n)_n$ which has a limit for the weak-* convergence of measures. We then apply the following formulation of the Skorokhod theorem; it for instance can be found in Dudley [37], and has been used in this form for an optimal transport problem in the Marc Bernot's PhD. thesis ([11], Theorem 4.2.8):

Theorem 1.4.2. *Let (K, d) be a totally bounded metric space equipped with the σ-algebra of its Borel sets. Let $(\mu_n)_n$ be a sequence of probability measures on (K, d). Then $(\mu_n)_n$ weakly-* converges to μ if and only if there exits a family $(\chi_n)_n$ of functions $(0,1) \to K$ and $\chi : (0,1) \to K$ such that*

- $\mu_n = (\chi_n)_\# \mathcal{L}^1$ *for each* n*, and* $\mu = \chi_\# \mathcal{L}^1$*;*
- $\chi_n \to \chi$ *a.e. in* $(0,1)$*.*

By replacing χ_n with $\chi_n \circ t$, where t is the monotone transport from $f \cdot \mathcal{L}^1$ to $\mathcal{L}^1$, we get that the same results holds with $f \cdot \mathcal{L}^1$ instead of $\mathcal{L}^1$. We apply it with $K = L^2(0,1)$ and $\mu_n = \rho_n$; the sequence $(\chi_n)_n$ that we obtain verifies then:

- for each n, χ_n and S_n send $f \cdot \mathcal{L}^1$ onto the same image measure, thus they have the same image set and in particular $\chi_n(x_1) = S_n(\xi_n(x_1))$ for a certain $\xi_n(x_1)$, which means

$$\chi_n(x_1)(x_2) = T_n(\xi_n(x_1), x_2) \quad \text{for } \mu\text{-a.e. } (x_1, x_2)$$

- as a consequence of $(S_n)_\#(f \cdot \mathcal{L}^1) = (\chi_n)\#(f \cdot \mathcal{L}^1)$, if we denote by F the function

$$u \in L^2(0,1) \mapsto ||u'||_{L^2}$$

  we have $\int_0^1 F(S_n(x_1)) f(x_1)\,\mathrm{d}x_1 = \int_0^1 F(\chi_n(x_1)) f(x_1)\,\mathrm{d}x_1$ *i.e.*

$$||D\mu T_n||^2_{L^2_\mu} = \int_\Omega |\partial_2 T_n(x_1, x_2)|^2 f(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_\Omega |\partial_2 T_n(\xi_n(x_1), x_2)|^2 f(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

  which implies that the function $(x_1, x_2) \mapsto T_n(\xi_n(x_1), x_2)$ still defines a minimizing sequence;

- $||\chi_n(x_1) - \chi(x_1)||_{L^2(0,1)} \to 0$ for a.e. x_1, so that, if we set $T(x_1, x_2) = \chi(x_1)(x_2)$,

$$T_n(\xi_n(x_1), x_2) \to T(x_1, x_2)$$

  which achieves the proof. □

1.4.2 A case where $M = \Omega$ but $T_\mu = \mathbb{R}^2$ a.e.

The example is provided by the following density in dimension 1, which is also due to Guy David.

Proposition 1.4.4. *There exists a function $f \in L^\infty(0,1)$ which is a.e. positive on $(0,1)$ and such that, for a.e. $x \in (0,1)^2$ and for any $\varepsilon > 0$,*

$$\iint_{B(x,\varepsilon)\cap(0,1)^2} \frac{dx_1\,\mathrm{d}x_2}{f(x_1) + f(x_2)} = +\infty$$

Proof. Starting from the inequality

$$\frac{1}{f(x_1) + f(x_2)} \geq \frac{1}{2} \min\left(\frac{1}{f(x_1)}, \frac{1}{f(x_2)}\right)$$

we are looking for a function h bounded from below, a.e. finite on $(0,1)$ and such that

$$\iint_{B(x,\varepsilon)\cap(0,1)^2} \min(h(x_1), h(x_2))\,\mathrm{d}x_1\,\mathrm{d}x_2 = +\infty$$

for a.e. $x \in (0,1)^2$ and any $\varepsilon > 0$, and we will take $f = 1/h$.

Let $(n_k)_k$ be the increasing sequence of integers of the form

$$n_k = 2^l + i,\ 0 \le i \le 2^l - 1 \text{ and } i = 1 \bmod l^2$$

We then denote by

$$F_k = \left(\frac{i}{2^l}, \frac{i+1}{2^l}\right)$$

where (i,l) is defined by $n_k = 2^l + i$, $0 \le i \le 2^l - 1$.

Claim 1: $\sum_k \mathcal{L}^1(F_k)$ *is finite.* Indeed, for each l, the integers n_k such that $2^l \le n_k < 2^{l+1}$ are

$$2^l + 1,\ 2^l + l^2 + 1, \dots, 2^l + (p-1)l^2 + 1$$

where p verifies

$$2^l + (p-1)l^2 + 1 \le 2^{l+1} \le 2^l + pl^2 + 1$$

$$\text{thus} \qquad p \le \frac{2^{l+1}}{l^2} + 1$$

The numbers of intervals F_k corresponding to such integers n_k is thus smaller than $2^{l+1}/l^2 + 1$. On the other hand, these intervals have radius $1/2^l$. Therefore their union has Lebesgue measure smaller than

$$\frac{1}{l^2} + \frac{1}{2^l + 1}$$

which defines a summable sequence.

Let us now take a sequence $(a_k)_k$ of positive numbers. The function

$$h : x \mapsto \sum_k a_k \mathbb{1}_{F_k}(x)$$

is finite at any point x only belonging to a finite number of F_k. But since $\sum_k \mathcal{L}^1(F_k) < +\infty$, by the Borel-Cantelli lemma, the set of points x belonging to infinitely many F_k has zero Lebesgue measure; therefore $h(x) < +\infty$ for a.e. x. We will now build a sequence $(a_k)_k$ such that, for the corresponding function h,

$$\iint_\Omega \min(h(x_1), h(x_2))\, \mathrm{d}x_1\, \mathrm{d}x_2 = +\infty$$

Claim 2: for any $a < b \in (0,1)$*, there exists* l_0 *such that, for any* $l \ge l_0$*, there exists* $0 \le i \le 2^l - 1$ *verifying*

$$i = 1 \bmod l^2 \qquad \text{and} \qquad a < \frac{i}{2^l} < \frac{i+1}{2^l} < b$$

Indeed, taking l_0 such that $l_0^2/2^{l_0 - 1} > b - a$ and $1/2^{l_0} < a$, if

$$p_0 = \max\{p : (l^2 p + 1)/2^l \ge a\}$$

we have $(l^2(p_0+1)+1)/2^l > a$ and

$$\frac{l^2(p_0+2)+1}{2^l} = \frac{l^2 p_0}{2^l} + \frac{1}{2^l}$$

with $l^2 p_0/2^l \le a$ and $1/2^l < b - a$.

Therefore, if $a < b$ and $a' < b' \in (0,1)$, for l_0 corresponding both to a, b and a', b' with $l \geq l_0$, we have

$$\iint_{(a,b)\times(a',b')} \min(h(x_1), h(x_2))\,\mathrm{d}x_1\,\mathrm{d}x_2 \geq \min(a_k, a_{k'})\mathcal{L}^1(F_k))$$

where $k = i/2^l$ is such that $a < i/2^l < (i+1)/2^l < b$ and idem for k', and $\mathcal{L}^1(F_k) = \mathcal{L}^1(F_{k'}) = 1/2^l$. It is then enough to take a sequence $a_k = b_l$ with $b_l/2^l \to +\infty$ to get the result. □

We will now study the Sobolev space with respect to the measure

$$\mathrm{d}\mu(x_1, x_2) = (f(x_1) + f(x_2))\,\mathrm{d}x_1\,\mathrm{d}x_2$$

By construction, the critical set M corresponding to the density of μ is the whole domain $\Omega = (0,1)^2$. However:

Proposition 1.4.5. *We have $T_\mu = \mathbb{R}^2$ μ-a.e. on $\mathbb{R}^2$.*

Proof. As above, let us take $v = (v^1, v^2). \in \Gamma(0)$ and show first that $v^1 = 0$. If $u_n \to 0$ and $\nabla u_n \to v$ in L^2_μ, in particular,

$$\int_0^1 \left(\int_0^1 u_n^2(x)\,\mathrm{d}x_1\right) f(x_2)\,\mathrm{d}x_2 \quad \text{and} \quad \int_0^1 \left(\int_0^1 (\partial_1 u_n(x) - v^1(x))^2\,\mathrm{d}x_1\right) f(x_2)\,\mathrm{d}x_2$$

which implies, for μ-a.e. x_2,

$$u_n(\cdot, x_2) \to 0 \text{ and } \partial_1 u_n(\cdot, x_2) \to v^1(\cdot, x_2) \text{ in } L^2(0,1)$$

and $v^1 = 0$ a.e. on Ω. Since the two variables x_1, x_2 play symmetric roles, the proof that $v^2 = 0$ is similar. □

This gives an example of absolutely continuous measure with a density such that the critical set is the whole domain Ω but for which the tangent space is a.e. the whole space $\mathbb{R}^2$. The spaces L^2_μ and H^1_μ are described here:

Proposition 1.4.6. *A measurable function $u : \Omega^2 \to \mathbb{R}$ belongs to L^2_μ if and only if the two following assumptions are satisfied:*

- *for a.e. x_1, $x_2 \mapsto u(x_1, x_2)$ is a function of $L^2(0,1)$, and the function $x_1 \mapsto ||u(x_1, \cdot)||^2_{L^2(0,1)}$ belongs to $L^1_f(0,1)$;*
- *for a.e. x_2, $x_1 \mapsto u(x_1, x_2)$ is a function of $L^2(0,1)$, and the function $x_2 \mapsto ||u(\cdot, x_2)||^2_{L^2(0,1)}$ belongs to $L^1_f(0,1)$.*

It belongs to H^1_μ if and only if the same assumptions are satisfied with the $H^1(0,1)$-norm instead of the $L^2(0,1)$-norm.

We can now state our compactness result in H^1_μ, which is enough to our variational problem:

Theorem 1.4.3. *Let F be a bounded subset of $H^1_\mu(\Omega) \cap L^\infty(\Omega)$. Then F is precompact in $L^2(\Omega)$.*

Proof. We follow the proof of the original Rellich-Kondrakov theorem in H^1, that can be found in [1], Theorems 2.32 and 6.3. We fix a family of smooth convolution kernels $(\eta_h)_{h>0}$ such that, for each $h > 0$,

- $\eta_h = 0$ outside of $(-h, h)$;

- $\int_{\mathbb{R}} \eta_h = 1$;
- $0 \leq \eta_h \leq C/h$ on $\mathbb{R}$;
- η_h has a Lipschitz constant which is smaller than C/h^2;

where C is a positive constant independent of h. We extend each function $u \in F$ to the whole $\mathbb{R}^2$ by setting $u = 0$ outside of Ω and we set, for each $h > 0$,

$$u_h(x_1, x_2) = \int_{\mathbb{R}} u(x_1, y_2)\eta_h(x_2 - y_2)\,\mathrm{d}y_2 = \int_{\mathbb{R}} u(x_1, x_2 - y_2)\eta_h(y_2)\,\mathrm{d}y_2$$

the convolution of u and η_h with respect to the second variable (notice that this definition makes sense for a.e. x_1 since $u(x_1, \cdot) \in L^2(0,1)$ for a.e. x_1).

Lemma 1.4.2. *Let $h > 0$ be fixed. Then the family*

$$F_h = \{u_h \,:\, u \in F\}$$

is a precompact subset of $(C^0(\Omega), ||\cdot||_\infty)$.

Proof. We verify the assumption of the Arzelà-Ascoli theorem. We already know that F_h is bounded for the L^∞-norm and only have to check its equicontinuity. For $\delta_2 \in \mathbb{R}$, we have

$$u_h(x_1, x_2 + \delta_2) - u_h(x_1, x_2) = \int_0^1 u(x_1, y_2)(\eta_h(y_2 - x_2 - \delta_2) - \eta_h(y_2 - x_2))\,\mathrm{d}y_2$$

so that, thanks to the bound on $\mathrm{Lip}\,\eta_h$,

$$|u_h(x_1, x_2 + \delta_2) - u_h(x_1, x_2)| \leq \int_0^1 |u(x_1, y_2)|\frac{C|\delta_2|}{h^2} \leq C'\frac{|\delta_2|}{h^2} \tag{1.5}$$

where C' takes account of the bound on $||u||_\infty$. This estimation is uniform with respect to (x_1, x_2).

On the other hand, for $\delta_1 \in \mathbb{R}$

$$u_h(x_1 + \delta_1, x_2) - u_h(x_1, x_2) = \int_0^1 (u(x_1 + \delta_1, y_2) - u(x_1, y_2))\eta_h(x_2 - y_2)\,\mathrm{d}y_2 \tag{1.6}$$

where, for any y_2 such that $u(\cdot, y_2) \in H^1(0,1)$, thus for a.e. y_2,

$$|u(x_1 + \delta_1, y_2) - u(x_1, y_2)| = \left|\int_{x_1}^{x_1+\delta_1} \partial_1 u(y_1, y_2)\,\mathrm{d}y_1\right| \leq ||u(\cdot, y_2)||_{H^1(0,1)}\sqrt{|\delta_1|}$$

the last estimate coming from the Cauchy-Schwarz inequality.

Since $u \in H^1_\mu(\Omega)$, the function

$$y_2 \mapsto ||u(\cdot, y_2)||_{H^1(0,1)}$$

belongs to $L^1_f(0,1)$ (and even to $L^2_f(0,1)$), and since $f > 0$ and $f \in L^\infty(0,1)$ the Lebesgue measure and the measure with density f are equivalent; we deduce that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $||u(\cdot, y_2)||_{H^1(0,1)} \leq C_\varepsilon$ for a.e. y_2 outside of a set $N_\varepsilon \subset (0,1)$ with $\mathcal{L}^1(N_\varepsilon) \leq \varepsilon$. In particular, for such a y_2,

$$|u(x_1 + \delta_1, y_2) - u(x_1, y_2)| \leq C_\varepsilon\sqrt{|\delta_1|}$$

$$\text{and} \int_{R - N\varepsilon} |u(x_1 + \delta_1, y_2) - u(x_1, y_2)|\eta_h(x_2 - y_2)\,\mathrm{d}y_2 \leq C_\varepsilon\sqrt{|\delta_1|}\int_{\mathbb{R}} \eta_h = C_\varepsilon\sqrt{|\delta_1|} \tag{1.7}$$

On the other hand,

$$\int_{N\varepsilon} |u(x_1+\delta_1,y_2)-u(x_1,y_2)|\eta_h(x_2-y_2)\,\mathrm{d}y_2 \le \mathcal{L}^1(N_\varepsilon)2||u||_\infty||\eta_h||_\infty \le \frac{C\varepsilon}{h} \tag{1.8}$$

where C does not depend on u, h, δ_1, ε. Using (1.7) and (1.8), we deduce from (1.6) that

$$|u_h(x_1+\delta_1,x_2)-u_h(x_1,x_2)| \le \frac{C\varepsilon}{h} + C_\varepsilon\sqrt{|\delta_1|} \tag{1.9}$$

By combining (1.5) and (1.9), we finally obtain the following: for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, for any $u \in F$ and $\delta = (\delta_1,\delta_2) \in \mathbb{R}^2$,

$$||u_h(x_1+\delta_1,x_2+\delta_2)-u_h(x_1,x_2)| \le \frac{C|\delta_2|}{h} + C_\varepsilon\sqrt{|\delta_1|} + \frac{C\varepsilon}{h}$$

This is enough to get the equicontinuity of the family F_h. □

For $u \in F$ and $h > 0$, we have

$$u(x_1,x_2)-u_h(x_1,x_2) = \int_{\mathbb{R}} (u(x_1,x_2)-u(x_1,x_2-y_2))\eta_h(y_2)\,\mathrm{d}y_2 = \int_R \left(\int_{x_2-y_2}^{x_2} \partial_2 u_h(x_1,t_2)\,\mathrm{d}t_2\right)\eta_h(y_2)\,\mathrm{d}y_2$$

so that

$$\begin{aligned}|u-u_h|^2(x_1,x_2) &\le \left(\int_{-h}^{h}\left(\int_{x_2-y_2}^{x_2} |\partial_2 u_h(x_1,t_2)|\,\mathrm{d}t_2\right)\eta_h(y_2)\,\mathrm{d}y_2\right)^2 \\ &\le 2h\cdot\int_{-h}^{h}\left(\int_{x_2-y_2}^{x_2} |\partial_2 u_h(x_1,t_2)|\,\mathrm{d}t_2\right)^2 \eta_h(y_2)^2\,\mathrm{d}y_2 \\ &\le 2h^2\left(\int_{\mathbb{R}} |\partial_2 u_2(x_1,t_2)|^2\,\mathrm{d}t_2\right)\left(\int_{-h}^{h}\eta_h(y)^2\,\mathrm{d}y\right)\end{aligned}$$

If we set $\mu_1(x) = f(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2$, the above inequality gives by integrating it with respect to μ_1

$$||u-u_h||^2_{L^2_{\mu_1}} \le 2h^2||\eta_h||^2_{L^2}||u||^2_{H^1_\mu}$$

But since $||\eta_h||_\infty \le C/h$ we have

$$||\eta_h||_{L^2} = \int_{-h}^{h}\eta_h^2 \le \frac{C}{h}$$

$$\text{thus} \qquad ||u-u_h||_{L^2_{\mu_1}} \le C\sqrt{h}$$

where C depends only on the bounds on the family F.

Let $\varepsilon > 0$. We choose $h > 0$ such that

$$\forall u \in F,\ ||u-u_h||_{L^2_{\mu_1}} \le \frac{\varepsilon}{2}$$

Since the family $\{u_h : u \in F\}$ is precompact in $\left(C^0\left(\overline{\Omega}\right)\right), ||\cdot||_\infty)$, which is continuously embedded into $L^2_{\mu_1}$, there exists a finite family $\{u_h^1,\dots,u_h^p\}$ of $\left(C^0\left(\overline{\Omega}\right)\right)$ such that, for any $u \in F$,

$$||u_h-u_h^i||_{L^2_{\mu_1}} \le \frac{\varepsilon}{2}$$

for some $i, 1 \leq i \leq p$. Finally, the family $\{u_h^1, \ldots, u_h^p\}$ satisfies

$$\forall u \in F, \exists\, 1 \leq i \leq p, \; ||u - u_h^i||_{L^2_{\mu_1}} \leq \varepsilon$$

This shows that the family F is precompact in $L^2_{\mu_1}$. Since the two variables x_1, x_2 play symmetric roles, the same reasoning allows also to prove that it is precompact in $L^2_{\mu_2}$, where μ_2 has the density $(x_1, x_2) \mapsto f(x_2)$ on $(0, 1)$. It is thus precompact in the space L^2_{μ}. □

Chapter 2

Optimality conditions and characterizations of the optimizers

In this chapter, we study optimality conditions for the class of problems

$$\inf\left\{\int_\Omega |T(x)-x|^2\,\mathrm{d}\mu(x)x + \int_\Omega |DT(x)|^2\,\mathrm{d}x \,:\, T\in H^1(\Omega), T_\#\mu=\nu\right\}$$

where, according to the results of the previous section, Ω is an bounded open subset of $\mathbb{R}^d$ $\mu\in\mathcal{P}(\Omega)$ admits a density which is bounded from below by a positive constant and $\nu\in\mathcal{P}(\mathbb{R}^d)$ is such that the set of admissible functions

$$\left\{T\in H^1(\Omega) : T_\#\mu=\nu\right\}$$

is non-empty.

2.1 The one-dimensional case

We recall the following classical theorem of optimal transportation on the real line, whose proof can be found in the Appendix:

Theorem 2.1.1. *Let $\mu\in\mathcal{P}(\mathbb{R})$ and assume that μ has no atom. Let $\nu\in\mathcal{P}(\mathbb{R})$. Then there exists a unique non-decreasing map $T_1:\mathbb{R}\to\mathbb{R}$ and a unique non-increasing map $T_2:\mathbb{R}\to\mathbb{R}$ such that $(T_1)_\#\mu=(T_2)_\#\mu=\nu$. Moreover, if $h:\mathbb{R}\to\mathbb{R}$ is convex then T_1 is optimal for the problem*

$$\inf\left\{\int_\mathbb{R} h(|T(x)-x|)\,\mathrm{d}\mu(x) : T_\#\mu=\nu\right\}$$

with uniqueness if h is strictly convex.

2.1.1 Statement of the main results

When the source measure is uniform, the main result is the following, which implies the optimality of these two monotone transport maps for the Sobolev cost:

Theorem 2.1.2. *Let I a bounded interval of $\mathbb{R}$. Let $T,U\in W^{1,1}(I)$ such that $T_\#\mathcal{L}^1=U_\#\mathcal{L}^1$, with T monotone. Let $f:[0,+\infty)\to[0,+\infty)$ be a convex and non-decreasing function. Assume that $\int_I f(|U'|)\,\mathrm{d}x<+\infty$ and let us set, for $x\in I$,*

$$n(x)=\#U^{-1}(\{T(x)\})$$

the number of points whose image by U is $T(x)$. Then $x \mapsto n(x)|T'(x)|$ is integrable on I and we have the following inequality:

$$\int_I f(|U'(x)|)\,\mathrm{d}x \geq \int_I f(n(x)|T'(x)|)\,\mathrm{d}x \tag{2.1}$$

(where we use the convention $n(x)|T'(x)| = 0$ if $T'(x) = 0$ and $n(x) = +\infty$).

If μ is not uniform, we have a result a little weaker under some assumption on its density:

Theorem 2.1.3. *Let I a bounded interval of $\mathbb{R}$, and $\mu \in \mathcal{P}(I)$. Assume that μ has a density g verifying*

$$\frac{\inf g}{\sup g} \geq \frac{1}{2} \tag{2.2}$$

Let $f : [0, +\infty) \to [0, +\infty)$ be a convex and non-decreasing function. Let $U \in W^{1,1}(I)$ and $\nu = U_\#\mu$. Let T_1, T_2 be respectively the unique non-decreasing and the unique non-increasing transport map from μ to ν. Then one of the following inequalities is true:

$$\int_I f(|U'(x)|)\,\mathrm{d}\mu(x) \geq \int_I f(T_1'(x))\,\mathrm{d}\mu(x) \quad \text{or} \quad \int_I f(|U'(x)|)\,\mathrm{d}\mu(x) \geq \int_I f(-T_2'(x))\,\mathrm{d}\mu(x) \tag{2.3}$$

Consequences for our variational problem. Let us give some remarks about the function $n(x)$ which appears in Theorem 2.1.2. First, since U and T send $\mathcal{L}^1$ onto the same image measure, they have same image set in $\mathbb{R}$ so that $n(x) \geq 1$ for any x, and the inequality (2.1) and the monotonicity of f lead to

$$\int_I f(|U'(x)|)\,\mathrm{d}x \geq \int_I f(n(x)|T'(x)|)\,\mathrm{d}x \geq \int_I f(|T'(x)|)\,\mathrm{d}x$$

This proves that T is optimal for the problem

$$\inf\left\{\int_I f(|U'(x)|)\,\mathrm{d}x : U_\#\mathcal{L}^1 = \nu\right\}$$

where $\nu = U_\#\nu$. In particular, this problem admits at least the two monotone maps from $\mathcal{L}^1|_I$ to ν as solutions.

One can actually show a more precise result: if T is monotone and U is fixed with $T_\#\mathcal{L}^1 = U_\#\mathcal{L}^1$, then U is monotone if and only if $n(x) = 1$ for almost every x such that $|T'(x)| > 0$. Indeed:

- if U is monotone and $n(x) \geq 2$, it means that there exists $x_1 < x_2$ such that $U(x_1) = U(x_2) = T(x)$, and the monotonicity of U implies that U is constant with value $T(x)$ on (x_1, x_2). In particular,

$$\mathcal{L}^1(T^{-1}(\{T(x)\})) = \mathcal{L}^1(U^{-1}(\{T(x)\})) \geq x_2 - x_1 > 0$$

  so that T is also equal to $T(x)$ in at least to points; since it is monotone, it means that x belongs to an interval where T is constant, so that $T'(x) = 0$ provided that T has a derivative at x (which is true for a.e. x since it is monotone);

- conversely, assume that U is not monotone and let us take $x_1 < x_2 < x_3$ so that, for instance, $U(x_2) > U(x_1)$ and $U(x_2) > U(x_3)$; we then can assume that $U(x_1) = U(x_3) < U(x_2)$. Taking x_1', x_2' such that $T(x_1') = U(x_1)$ and $T(x_2') = U(x_2)$, we get $n \geq 2$ on (x_1', x_2'). Finally, since $T \in W^{1,1}(I)$, we have

$$|T(x_2') - T(x_2')| = \left|\int_{x_1'}^{x_2'} T'(t)\,\mathrm{d}t\right| = \int_{x_1'}^{x_2'} |T'(t)|\,\mathrm{d}t > 0$$

  (the last equality comes from the fact that T' has a constant sign on i). This implies that $|T'| > 0$ on a non-negligible subset B of (x_1', x_2'), and $n \geq 2$ on B.

In this last case, if we assume moreover f to be increasing, we get for U non-monotone:

$$\int_I f(|U'(x)|)\,\mathrm{d}x \geq \int_I f(n(x)T'(x))\,\mathrm{d}x \geq \int_{I-B} f(n(x)|T'(x)|)\,\mathrm{d}x + \int_B f(n(x)|T'(x)|)\,\mathrm{d}x$$

where

$$\int_B f(n(x)|T'(x)|)\,\mathrm{d}x \geq \int_B f(2|T'(x)|)\,\mathrm{d}x > \int_B f(|T'(x)|)\,\mathrm{d}x$$

$$\text{and} \qquad \int_{I-B} f(n(x)|T'(x)|)\,\mathrm{d}x \geq \int_{I-B} f(|T'(x)|)\,\mathrm{d}x$$

This achieves the proof of:

Proposition 2.1.1. *If $f : [0, +\infty) \to [0, +\infty)$ is convex and non-decreasing and $\nu \in \mathcal{P}(\mathbb{R})$, then the two monotone transport maps from $\mathcal{L}^1$ to ν are both optimal for the problem*

$$\inf\left\{\int_I f(|U'(x)|)\,\mathrm{d}x : U \in W^{1,1}(U), U_\#\mathcal{L}^1 = \nu\right\}$$

with uniqueness if moreover f is increasing.

This has for consequence:

Proposition 2.1.2. *Let $f : [0, +\infty) \to [0, +\infty)$ be convex and non-decreasing and $h : \mathbb{R} \to \mathbb{R}$ be convex. Then the non-decreasing transport map T from $\mathcal{L}^1$ to ν is optimal for the problem*

$$\inf\left\{\int_I h(|U(x) - x|)\,\mathrm{d}x + \int_I f(|U'(x)|)\,\mathrm{d}x : U \in W^{1,1}(I), U_\#\mathcal{L}^1 = \nu\right\}$$

Moreover:

- *if h is strictly convex, then there is uniqueness;*
- *if h is only convex but f is increasing, then the only possible competitor for T is the unique non-increasing map from $\mathcal{L}^1$ to ν.*

The rest of this section is devoted to the proof of the theorems 2.1.2 and 2.1.3.

2.1.2 The piecewise monotone case

In this paragraph, we introduce the notations and give a proof of Theorem 2.1.2 and a partial proof of Theorem 2.1.3 when U has some regularity. The assumption on U is the following: we will say that $U \in W^{1,1}$ satisfies the condition **(PM)** (for "piecewise monotone") if there exists a subdivision $x_1 < \cdots < x_l$ of I such that, for each i, $U|_{(x_i, x_{i+1})}$ is C^1 and $|U'|_{(x_i,x_{i+1})}$ is bounded from below by a positive constant (in particular, $U' \neq 0$ and U is monotone on (x_i, x_{i+1})). Moreover, we still assume that μ has a density g which is bounded from above and below by positive constants. In this case, we will adopt the following notations:

- $y_1 < \cdots < y_k$ are the elements of $U(I)$ such that $\{y_1, \ldots, y_j\} = U(\{x_1, \ldots, x_l\})$. Moreover, up to adding finitely many elements to $\{x_1, \ldots, x_l\}$, we can assume that $U^{-1}(\{y_1, \ldots, y_j\}) = \{x_1, \ldots, x_k\}$.
- for each i, $1 \leq i \leq l$, φ_i is the inverse map of $U|_{(x_i,x_{i+1})}$ (which is strictly monotone thus bijective onto its image interval);

- conversely, for each k, $1 \le k \le j$, we denote by A_k the set of indexes i, $1 \le i \le l$, such that U sends (x_i, x_{i+1}) on (y_k, y_{k+1}), and $n_k = \#A_k$. Notice that the function $y \mapsto \#U^{-1}(y)$ is constant on (y_k, y_{k+1}) with value n_k.

In this case, we are able to give an explicit formula of the image measure $U_\#\mu$:

Proposition 2.1.3. *The measure $\nu = U_\#\mu$ is absolutely continuous with respect to the Lebesgue measure. Moreover, for $y \in U(I)\backslash\{y_1, \ldots, y_j\}$, if k is such that $y_k < y < y_{k+1}$, the density of ν at y is given by*

$$(U_\# g)(y) = \sum_{i \in A_k} \frac{g(\varphi_i(x))}{|U'(\varphi_i(x))|} \tag{2.4}$$

In particular, if T is monotone and $T_\#\mu = U_\#\mu$, then

$$\frac{g(x)}{|T'(x)|} = \sum_{i \in A_k} \frac{g(u_i(x))}{|U'(u_i(x))|} \tag{2.5}$$

where $u_i = \varphi_i \circ T$.

The density of ν is in particular positive on a.e. point of the interval $U(I)$, which implies that such a T is strictly monotone, thus bijective, from I into $U(I)$.

Proof of Prop. 2.1.3. Let $c < d$ such that $y_k < c < d < y_{k+1}$. We compute $\nu((c,d))$: we have $\nu((c,d)) = \mu(U^{-1}(c,d))$ with $U^{-1}((c,d)) = \bigcup_{i \in A_k} \varphi_i((c,d))$ and this is a disjoint union, so that

$$\int_c^d \mathrm{d}\nu = \sum_{i \in A_k} \pm \int_{\varphi_i(c)}^{\varphi_i(d)} g(x)\,\mathrm{d}x = \sum_{i \in A_k} \int_c^d g(\varphi_i(y))|\varphi_i'(y)|\,\mathrm{d}x \tag{2.6}$$

The density of μ on a.e. $y \in (c,d)$ is thus given by $\sum_{A_k} g(\varphi(y))\varphi'(y)$, but since φ is the inverse of U in (c,d) we have $\varphi' = 1/(U' \circ \varphi)$. This leads to (2.4). On the other hand, if T is monotone and sends μ to ν, then

$$\int_c^d \mathrm{d}\nu = \pm \int_{T^{-1}(c)}^{T^{-1}(d)} g(x)\,\mathrm{d}x = \int_c^d \frac{g(T^{-1}(y))}{|T'(T^{-1}(y))|}\,\mathrm{d}y$$

This combined to (2.6) gives

$$\frac{g(T^{-1}(y))}{|T'(T^{-1}(y))|} = \sum_{i \in A_k} g(\varphi_i(y))|\varphi_i'(y)|$$

and (2.5) follows since $\varphi_i' = 1/(U' \circ \varphi_i)$ and $u_i = \varphi_i \circ T$. □

We can now show our results under this assumption on U:

Proposition 2.1.4. *Let T be monotone such that $T_\#\mu = U_\#\mu$. Then*

$$\int_I f(|U'(x)|)\,\mathrm{d}\mu(x) \ge \int_I f\left(\frac{\inf g}{\sup g} n(x)|T'(x)|\right)\mathrm{d}\mu(x)$$

In particular:

- *if g is constant, i.e. if μ is uniform, then the inequality* (2.1) *holds;*
- *if g only satisfies* (2.2) *and if $n \ge 2$ on I, then the inequality* (2.3) *holds.*

Proof. We have the following equalities :

$$\int_I f(|U'(x)|)\,\mathrm{d}\mu(x) = \sum_{i=1}^{l-1} \int_{x_i}^{x_{i+1}} f(|U'(x)|) g(x)\,\mathrm{d}x = \sum_{k=1}^{j-1} \left(\sum_{i\in A_k} \int_{x_i}^{x_{i+1}} f(|U'(x)|) g(x)\,\mathrm{d}x \right),$$

where the sum over i has n_k terms. On the interval (x_i, x_{i+1}), we set $x = u_i(y)$, and since the derivative of u_i is $u_i' = T'/(U'\circ u_i)$, we obtain

$$\int_a^b f(|U'|)\,\mathrm{d}\mu = \sum_{k=1}^{j-1} \sum_{i\in A_k} \left(\int_{z_k}^{z_{k+1}} f(|U'(u_i(y))|) \left| \frac{T'(y)}{U'(u_i(y))} \right| g(u_i(y))\,\mathrm{d}y \right). \tag{2.7}$$

In the equation (2.7), Prop. 2.1.3 gives $\sum_{i\in A_k} \left|\frac{T'(y)}{U'(u_i(y))}\right| \frac{g(u_i(y))}{g(y)} = 1$ and, since f is convex:

$$\int_a^b f(|U'|)\,\mathrm{d}\mu \ge \sum_{k=1}^{j-1} \int_{z_k}^{z_{k+1}} f\left(\sum_{i\in A_k} |U'(u_i(y))| \left| \frac{T'(y)}{U'(u_i(y))} \right| \frac{g(u_i(y))}{g(y)} \right) g(y)\,\mathrm{d}y;$$

We use the monotonicity of f to get

$$\int_a^b f(|U'|)\,\mathrm{d}\mu \ge \sum_{k=1}^{j-1} \int_{z_k}^{z_{k+1}} f\left(\sum_{i\in A_k} |U'(u_i(y))| \left| \frac{T'(y)}{U'(u_i(y))} \right| \frac{\inf g}{\sup g} \right) g(y)\,\mathrm{d}y$$

In the second sum, the terms $|U'(u_i(y))|$ disappear and there only remains $T'(y)\frac{\inf g}{\sup g}$ which occurs n_k times; since n_k is exactly the constant value of n on (z_k, z_{k+1}), we obtain the integral on this interval of $n(y)T'(y)\frac{\inf g}{\sup g}$; and the first sum gives the integral of this function on the whole interval I, *i.e.*

$$\int_I f(|U'|)\,\mathrm{d}\mu \ge \int_I f\left(\frac{\inf g}{\sup g} n(y) |T'(y)| \right) \mathrm{d}\mu(y) \quad \square$$

2.1.3 Proof of Theorem 2.1.2: approximation

In this paragraph, given $U \in W^{1,1}(I)$, we build an approximation $(U_k)_k$ of U by functions satisfying the condition **(PM)** and we show that the inequality (2.1) at the index k (with the increasing transport T_k - the proof would be exactly the same with the decreasing transport - and the function n_k corresponding to the map U_k) passes to the limit as $k \to +\infty$. We fix a super-linear non-decreasing function f and we first show the following:

Lemma 2.1.1. *For every $U \in W^{1,1}(I)$ with $\int f(|U'|) < +\infty$ there exists a sequence $(U_k)_{k\in\mathbb{N}}$ in $W^{1,1}(I)$ such that :*

- $U_k \xrightarrow[k\to+\infty]{} U$ *in* $W^{1,1}(I)$;
- $f \circ |U_k'| \xrightarrow[k\to+\infty]{} f \circ |U'|$ *in* $L^1(I)$;
- *for each k, U_k is piecewise affine with $U_k' \neq 0$ a.e..*

Proof. First notice that if the the thesis is true when replacing f with $x \mapsto f(x) + x$, then it stays true for the original function f. This allows to assume that f' is bounded from below by a positive constant; then $f : \mathbb{R}^+ \to \mathbb{R}^+$ is bijective and its inverse function is Lipschitz. Let $(h_k)_k$ be a sequence of positive and piecewise constant functions (say, on dyadic intervals of length $(b-a)/2^k$), such that $h_k \to f \circ |U'|$ in $L^1(I)$. We define U'_k by

$$U'_k(x) = \operatorname{sgn}(U(x_k^+) - U(x_k^-)) f^{-1}(h_k(x))$$

where x_k^+, x_k^- are the dyadic numbers around x (this is a non-ambiguous definition for a.e. $x \in I$).

We have $f \circ |U'_k| \to f \circ |U'|$ in $L^1(I)$ and we want to prove $U'_k \to U'$. First, notice that, since f^{-1} is Lipschitz, we easily get $|U'_k| \to |U'|$ in L^1. Moreover, up to subsequences the convergence also holds a.e. on I. Thus, it is enough to manage the sign and prove that $U'_k \to U'$ a.e. on I. This convergence holds on any non-dyadic point where U is differentiable with $U' \neq 0$ (which imposes the sign of $U(x_k^+) - U(x_k^-)$). These points, together with those where $|U'| = U' = 0$ cover almost all the interval ; this gives $U'_k \to U'$ a.e. on I and, by dominated convergence, we obtain $U'_k \to U'$ in L^1.

Then, if we take for U_k the primitive of U'_k which has the same value of U at a, we obtain a sequence $(U_k)_k$ of piecewise affine functions such that, by construction, $U_k \to U$ in $W^{1,1}(I)$ and $f \circ |U'_k| \to f \circ |U'|$ in $L^1(I)$. □

In particular, each U_k verifies the condition **(PM)**, thus the inequality (2.1) is true with U_k, T_k (the non-decreasing function with same image measure) and $n_k = \#U_k^{-1} \circ T_k$.

Remark. Thanks to the inequality in the piecewise affine case and since f is non-decreasing and super-linear, $n_k \geq 1$ and $\int f(|U'_k|)$ has a limit, we infer that the sequence $(f(T'_k))_k$ is bounded in $L^1(I)$, with f super-linear; thus $(T'_k)_k$ is an equi-integrable family, which implies:

- the sequence $(T_k)_k$ is equi-continuous, thus it admits, up to subsequences, a uniform limit T; this limit is obviously a non-decreasing function;
- from the strong convergence $U_k \to U$ in $W^{1,1}$ we infer a.e. pointwise convergence, which implies that $(U_k)_\#\mathcal{L}^1 \rightharpoonup U_\#\mathcal{L}^1$; analogously, we have $(T_k)_\#\mathcal{L}^1 \rightharpoonup T_\#\mathcal{L}^1$, which implies $U_\#\mathcal{L}^1 = T_\#\mathcal{L}^1$. Hence, the function T is exactly the monotone function corresponding to the original function U;
- the sequence $(T'_k)_k$ is weakly relatively compact in L^1, which, together with the uniform convergence $T_k \to T$, gives $T \in W^{1,1}(I)$ and $T_k \rightharpoonup T$ in $W^{1,1}(I)$.

Asymptotics of $n_k(x)$ as $k \to \infty$ To look at the limits of n_k, let us define the function m given by

$$m(x) = \inf\{\liminf_{k \to +\infty} n_k(x_k) \ : \ x_k \to x\}.$$

This function is actually the Γ-lim inf of the functions n_k (see [19]). A general result on Γ-lim inf functions gives that m is lower semi-continuous on I (it is easy to check it via a sort of diagonal sequence).

We are interested in the following.

Lemma 2.1.2. *For almost every $x \in I$ such that $T'(x) \neq 0$, we have $m(x) \geq n(x)$.*

Proof. Let us first show that this inequality holds if $y = T(x)$ is not a local extremum of U; thus, if we take $x' \in I$ and $\delta > 0$ with $U(x') = y$, there exist x^-, $x^+ \in]x' - \delta, x' + \delta[$ with $U(x^-) < U(x') < U(x^+)$. Let $(x_k)_k$ be a sequence of I converging to x, and $y_k = T_k(x_k)$. Thanks to the uniform convergence of $(T_k)_k$ to T, $y_k \to y$. Let p be a finite integer such that $p \leq n(x)$; we will show that we can find p distinct points having $T_k(x_k)$ as image by U, for k large enough (depending on p).

Let $z_1 < ... < z_p \in U^{-1}(y)$, and $\delta < \min_j(z_{j+1} - z_j)$. By the assumption on y, we can find $\varepsilon > 0$ and some points z_j^+ and z_j^- in each interval $]z_j - \delta, z_j + \delta[$ such that $U(z_j^-) + \varepsilon < U(z_j) < U(z_j^+) - \varepsilon$.

Since $U_k \to U$ pointwisely on I, the sequence $(U_k(z_j^-))_k$ (resp. $(U_k(z_j^+))_k$, $(U_k(z_j))_k$) converges to $U(z_j^-)$ (resp. $U(z_j^+)$, $U(z_j)$). There exists $k_0 \in \mathbb{N}$ such that, for $k \geq k_0$, we have $U_k(z_j^-) \leq U(z_j^-) + \varepsilon/2 \leq$

$y - \varepsilon/2$ and $U_k(z_j^+) \geq U(z_j^+) - \varepsilon/2 \geq y + \varepsilon/2$; moreover, since $y_k \to y$, we can assume that, for $k \geq k_0$, $y - \varepsilon/2 \leq y_k \leq y + \varepsilon/2$; combining these two points, we have

$$\text{for all } k \geq k_0, \quad U_k(z_j^-) \leq y_k \leq U_k(z_j^+);$$

then by the intermediate value theorem, since U_k is continuous, for any j, there exist z_j^k between z_j^+ and z_j^-, such that $U_k(z_j^k) = y_k$. The points z_j^k, $1 \leq j \leq p$, are distinct, since they belong to disjoint intervals $]z_j - \delta, z_j + \delta[$. Hence, $n_k(x_k) \geq p$ for $k \geq k_0$. The proof is complete if $T(x)$ is not a local extremum of U.

The last step consists in showing that the set A of points x such that $T(x)$ is a local extremum for U and verifying furthermore $T'(x) > 0$ is negligible for the Lebesgue measure. Indeed, y is a local maximum of U if, and only if, $y = \max\limits_{J_{q,r}} U$ with $q \in \mathbb{Q} \cap I$, $r \in \mathbb{Q}_+^*$ and $J_{q,r} =]q-r, q+r[\cap I$; therefore,

$$A = \bigcup_{q \in \mathbb{Q} \cap I, r \in \mathbb{Q}_+^*} \left\{ T^{-1}(\max_{J_{q,r}} U) \cap \{T' > 0\}, T^{-1}(\min_{J_{q,r}} U) \cap \{T' > 0\} \right\}$$

This equality proves that A is measurable, and it is enough to prove that for each level t we have $\mathcal{L}^1 \left(T^{-1}(t) \cap \{T' > 0\} \right) = 0$; this is true since $T' = 0$ a.e. on any level set of T (which is an interval). □

Conclusion by semi-continuity We denote for $x \in I$ and $k, j \in \mathbb{N}$:

$$n_k^j(x) = \min\left(j, \inf_{y \in I} \{ j|x-y| + n_k(y) \} \right); \quad h_j(x) = \lim_{k \to +\infty} n_k^j(x);$$

(h_j exists since the family $(n_k^j)_k$ is, for each j, uniformly bounded and equi-Lipschitz, thus we can assume that it admits a uniform limit up to subsequences). Let us notice that for any j, $n_k^j \leq n_k$ (take $x = y$ in the definition of n_k^j). Moreover, by Lemma 2.1.2, $m \geq n$ on I. Let us show the following lemma:

Lemma 2.1.3. *For any $j \in \mathbb{N}$, we have $h_j \geq m_j$ on I, where m_j is defined as*

$$m_j(x) = \min\left(j, \inf_{y \in I} \{ j|x-y| + m(y) \} \right).$$

Proof. Set $n_k^j = \min(j, \widetilde{n}_k^j)$ and $m_j = \min(j, \widetilde{m}_j)$, where

$$\widetilde{n}_k^j(x) = \inf_{y \in I} \{ j|x-y| + n_k(y) \} \text{ and } \widetilde{m}_j(x) = \inf_{y \in I} \{ j|x-y| + m(y) \}.$$

By definition, there is a sequence $(y_k)_k$ such that $\widetilde{n}_k^j(x) \leq j|x - y_k| + n_k(y_k) \leq \widetilde{n}_k^j(x) + 1/k$ for any k; taking the minimum with j, we obtain

$$n_k^j(x) \leq \min\left(j, j|x-y_k| + n_k(y_k)\right) \leq n_k^j(x) + \frac{1}{k}.$$

We may assume by compactness that $y_k \to y \in I$, and, by definition of h_j, we have $\min(j, j|x-y_k| + n_k(y_k)) \xrightarrow[k \to +\infty]{} h_j(x)$. Moreover, by definition of m, we have $\liminf_{k \to +\infty} n_k(y_k) \geq m(y)$, which gives

$$\min\left(j, j|x-y| + m(y)\right) \leq h_j(x)$$

for $y = \lim_k y_k \in I$; since $m_j(x)$ is the infimum over y of the left-hand side, we obtain $h_j(x) \geq m_j(x)$. □

The functions m_j that we just introduced are the usual Lipschitz "regularization" of the l.s.c. function m, and we will use (without proving) the following standard lemma.

Lemma 2.1.4. *The sequence of functions $(m_j)_{j\in\mathbb{N}}$ is non-decreasing, and has m for pointwise limit.*

We now return to our main result:

Theorem 2.1.4. *Let f be convex and non-decreasing, $U \in W^{1,1}$ such that $\int f(|U'|) < +\infty$ and T monotone non-decreasing such that $T_\# \mathcal{L}^1 = U_\# \mathcal{L}^1$. Then the inequality (2.1) holds.*

Proof. First of all suppose that f is superlinear. We use the approximation defined in this section. Section 2 proves that, for any k :

$$\int_I f(|U_k'(x)|)\,\mathrm{d}x \geq \int_I f(n_k(x)T_k'(x))\,\mathrm{d}x$$

and thanks to the non-decreasing behavior of f and to the remarks about n_k, n_k^j and h_j, we have the following inequalities, which are true for $k \geq k_0$ for every $\delta > 0$ and j ($k_0 = k_0(\delta, j)$):

$$\int_I f(n_k T_k') \geq \int_I f(n_k^j T_k') \geq \int_I f((h_j - \delta)T_k') \geq \int_I f((m_j - \delta)T_k'). \tag{2.8}$$

For some fixed $\delta > 0$ and $j \in \mathbb{N}$, the functional

$$T \in W^{1,1} \mapsto \int_I f((m_j(x) - \delta)T'(x))\,\mathrm{d}x$$

is lower semi-continuous with respect to the weak convergence in $W^{1,1}$ (see [28]). Thus, taking the limit $k \to +\infty$ in (2.8) gives

$$\liminf_{k\to+\infty} \int_I f(n_k T_k') \geq \int_I f((m_j - \delta)T');$$

and by monotone convergence, taking the limit $j \to +\infty$ and $\delta \to 0$ in the right-hand side gives

$$\int_I f((m_j - \delta)T') \xrightarrow[j\to+\infty]{} \int_I f((m - \delta)T') \xrightarrow[\delta\to 0]{} \int_I f(mT').$$

Since $m \geq n$ a.e. on the set $\{T' \neq 0\}$ and f is non-decreasing, the proof is complete for f super-linear.

If f has linear growth, it is sufficient to select a positive, convex, increasing and super-linear function $\widetilde{f}$ such that $\int \widetilde{f}(|U'|) < +\infty$; if we fix $\varepsilon > 0$, $f + \varepsilon\widetilde{f}$ is super-linear and non-decreasing, thus

$$\text{for all } \varepsilon > 0 \text{ we have } \int f(|U'|) + \varepsilon \int \widetilde{f}(|U'|) \geq \int f(nT') + \varepsilon \int \widetilde{f}(nT') \geq \int f(nT')$$

and passing to the limit as $\varepsilon \to 0$ gives the result. □

2.1.4 Proof of Theorem 2.1.3

Again, we begin by taking U satisfying the condition **(PM)**. The proof in this case is decomposed in several lemmas. We denote by $m(x) = \#U^{-1}(\{U(x)\})$; since U satisfies the condition **(PM)**, m is piecewise constant on I, and we denote by $x_1 < \cdots < x_p$ a subdivision of I such that m is alternatively equal to 1 and everywhere greater than 1 on (x_l, x_{k+1}) (*i.e.* $m|_{(x_k,x_{k+1})} = 1$ if k is odd and ≥ 2 if k even or the converse, depending on the value of m at the beginning of I).

Lemma 2.1.5. *The function U is strictly monotone on any interval (x_k, x_{k+1}) on which m is constant with value 1. Moreover:*

- *if $U|_{(x_k,x_{k+1})}$ is increasing, then $U(x) \leq U(x_k)$ for any $x \leq x_k$ and $U(x) \geq U(x_{k+1})$ for any $x \geq x_{k+1}$;*

- *conversely, if* $U|_{(x_k,x_{k+1})}$ *is decreasing, then* $U(x) \geq U(x_k)$ *for any* $x \leq x_k$ *and* $U(x) \leq U(x_{k+1})$ *for any* $x \geq x_{k+1}$.

Proof. Since $m = 1$ on (x_k, x_{k+1}), for any $x \in (x_k, x_{k+1})$, x is the unique point of I having $U(x)$ as image by U. In particular, $U|_{(x_k,x_{k+1})}$ is injective, thus monotone since it is continuous.

Assume now that it is increasing. If there exists $x \leq x_k$ such that $U(x) > U(x_k)$, let us take $y \in (U(x_k), U(x)) \cap U((x_k), U(x_{k+1}))$; by the intermediate value theorem, there exists $z_1 \in (x, x_k)$ and $z_2 \in (x_k, x_{k+1})$ such that $y = U(z_1) = U(z_2)$; in particular, $U(z_2) = 2$, which is impossible.

The same reasoning shows also that $U(x) \geq U(x_{k+1})$ for $x \geq x_{k+1}$. The proof is similar if $U|_{(x_k,x_{k+1})}$ is decreasing. □

Lemma 2.1.6. *U has the same monotonicity on any* (x_k, x_{k+1}) *where* $m = 1$.

Proof. Assume by contradiction that, for instance, $U|_{(x_k,x_{k+1})}$ is increasing and $U|_{(x_{k'},x_{k'+1})}$ is decreasing with $k < k'$. We then have

$$U(x_k) < U(x_{k+1}) \qquad \text{and} \qquad U(x_{k'}) > U(x_{k'+1})$$

Moreover, thanks to the lemma 2.1.5,

$$U(x_{k+1}) \leq U(x_{k'+1}) \qquad \text{and} \qquad U(x_k) \geq U(x_{k'})$$

These four equalities lead to $U(x_k) < U(x_k)$. The proof is the same if $U|_{(x_k,x_{k+1})}$ is decreasing and $U|_{(x_{k'},x_{k'+1})}$ is increasing. □

We will now say that U is "globally increasing" if it is increasing on any interval of which $m = 1$, and "globally decreasing" in the converse case.

Lemma 2.1.7. *If U is globally increasing, then, for each k,*

$$U(x) \leq x_k \text{ for } x \leq x_k \qquad \text{and} \qquad U(x) \geq x_{k+1} \text{ for } x \geq x_{k+1}$$

Conversely, if U is globally decreasing, then, for each k,

$$U(x) \geq x_k \text{ for } x \leq x_k \qquad \text{and} \qquad U(x) \leq x_{k+1} \text{ for } x \geq x_{k+1}$$

Proof. The case where $m|_{(x_k,x_{k+1})} = 1$ comes directly from the lemma 2.1.5. Assume for instance that U is globally increasing, and let us take k such that $m|_{(x_k,x_{k+1})} \geq 2$.

- Assume that there is $z < x_k$ such that $U(z) > U(x_k)$; this implies $k \geq 2$, $m|_{(x_{k-1},x_k)} = 1$ and $U_{(x_{k-1},x_k)}$ is increasing. Therefore, the Lemma 2.1.5 gives $U(x) \leq U(x_{k-1})$ for $x \leq x_{k-1}$, so that $z \in (x_{k-1}, x_k)$, but in this case the growth of U on this interval leads to $U(z) \leq U(x_k)$ which is impossible.

- Similarly, we prove that there is no $z > x_{k+1}$ with $U(z) < U(x_{k+1})$.

The proof is the same if U is globally decreasing. □

Let us now denote by $y_k = \inf_{(x_k,x_{k+1})} U$, $y_{k+1} = \sup_{(x_k,x_{k+1})} U$ if U is globally increasing and the converse if U is globally decreasing. The above lemma shows that $y_k = U(x_k)$ for each $2 \leq k \leq p-1$.

Lemma 2.1.8. *Assume that U is globally increasing. Let* T_1 *the non-decreasing transport map from* μ *to* $\nu = U|_{\#}\mu$. *Then, for each k:*

$$T_1(x_k) = y_k \qquad \text{and} \qquad (T_1)|_{\#}(\mu|_{(x_k,x_{k+1})}) = U_{\#}(\mu|_{(x_k,x_{k+1})}) = \nu_{(y_k,y_{k+1})}$$

If U is globally decreasing, then the same equality holds for each k, with T_2 *(the non-increasing transport map from* μ *to* ν*) instead of* T_1.

Proof. Assume U to be globally increasing and let us first show the result for $k = 1$. If $p = 2$ then $(x_1, x_2) = I$ and there is nothing to prove, so we also assume that $p \geq 3$. We know that $x_1 = \inf I$ by definition, and the last lemma provides $y_1 = \inf U(I)$, which implies $T_1(x_1) = y_1$, and

$$y_2 = \sup_{(x_1,x_2)} U = \inf_{(x_2,x_p)} U \quad \text{so that} \quad U(x_2) = y_2$$

On the other hand, the fact that T_1 and U have same image measure gives

$$\int \mathbb{1}_{y_1 \leq U(x) \leq c} \, \mathrm{d}\mu(x) = \int \mathbb{1}_{y_1 \leq T_1(x) \leq c} \, \mathrm{d}\mu(x)$$

In the above equality, we take $c = y_2$. Then we have $U(x) \geq y_2$ for a.e. $x \geq x_2$ with equality at most for a finite number of points x since U satisfies the condition **(PM)**, so that, for a.e. $x \in I$,

$$y_1 \leq U(x) \leq y_2 \iff x_1 \leq x \leq x_2 \tag{2.9}$$

We know that T_1 is characterized by

$$\int_{y_1}^{T_1(x)} \mathrm{d}\nu = \int_{x_1}^{x} \mathrm{d}\mu$$

On the other hand, the equivalence (2.9) provides

$$\int_{x_1}^{x_2} \mathrm{d}\mu = \int \mathbb{1}_{y_1 \leq U(x) \leq y_2} \, \mathrm{d}\mu(x) = \int_{y_1}^{y_2} \mathrm{d}\nu$$

so that $\int_{y_1}^{T_1(x_2)} \mathrm{d}\nu = \int_{y_1}^{y_2} \mathrm{d}\nu$, but ν is absolutely continuous with respect to the Lebesgue measure with a a.e. positive density so that the function $y \mapsto \int_{x_1}^{y} \mathrm{d}\nu$ is increasing. We conclude that $T_1(x_2) = y_2$. The second equality of the lemma comes from the fact that $(T_1)_{\#}\mu = \nu$ and from the equivalence

$$x_1 \leq x \leq x_2 \iff y_1 \leq T_1(x) \leq y_2 \iff y1 \leq U(x) \leq y_2$$

The lemma is proved on (x_1, x_2). We prove it similarly on (x_2, x_3) by considering (x_2, x_p) as global source interval and (y_2, y_p) as global target interval, and we make the same reasoning successively on each (x_k, x_p), $k \leq p - 1$. □

Lemma 2.1.9. *Assume U to be globally increasing. Let f be convex and non-decreasing. Then, for each k:*

- *if $m|_{(x_k,x_{k+1})} = 1$, then $U = T_1$ on (x_k, x_{k+1});*
- *if $m|_{(x_k,x_{k+1})} \geq 2$, then $\int_{x_k}^{x_{k+1}} f(|U'|) \, \mathrm{d}\mu \geq \int_{x_k}^{x_{k+1}} f(|T_1'|) \, \mathrm{d}\mu$*

In particular, $\int_I f(|U'|) \, \mathrm{d}\mu \geq \int_I f(|T_1'|) \, \mathrm{d}\mu$. If U is globally decreasing, the same results hold with T_1 instead of T_2.

Proof. If $m|_{(x_k,x_{k+1})} = 1$, then U and T_1 have the same monotonicity and send $\nu|_{(x_k,x_{k+1})}$ onto the same image measure, thus they coincide on this interval. If not, the result comes directly from Prop. 2.1.4 applied on (x_k, x_{k+1}) (the fact that $m \geq 2$ on this interval guarantees, with the notations of Prop. 2.1.4, that $n \geq 2$). □

Conclusion when U is not regular. Let $U \in W^{1,1}(I)$; we approximate U by the same sequence of piecewise affine functions $(U_n)_n$ (see Lemma 2.1.1). Each U_n is either globally increasing or globally decreasing; up to a subsequence, one can assume that their "global monotonicity" is the same. If each U_n is globally increasing, we consider the corresponding increasing transport T_1^n sending μ onto the same image measure. We know that, for each n,

$$\forall n \in \mathbb{N}, \quad \int_I f(|U_n'|)\,\mathrm{d}\mu \geq \int_I f((T_1^n)')\,\mathrm{d}\mu$$

and that:

- $f \circ |U_n'| \to f \circ U$ in $L^1(I)$;
- $T_1^n \rightharpoonup T_1$ in $W^{1,1}(I)$ (see the remarks after the Lemma 2.1.1);
- the function $T \mapsto \int_I f(|T'|)\,\mathrm{d}\mu$ is l.s.c. with respect to the weak convergence in $W^{1,1}$.

This allows to pass to the limit and get $\int_I f(|U'|)\,\mathrm{d}\mu \geq \int_I f(T_1')\,\mathrm{d}\mu$. The same inequality with T_2 follows if each U_n is globally decreasing.

2.1.5 A counter-example when the condition on μ fails

We conclude this section by giving an easy example of measure for which the inequality (2.3) fails. The strategy is the following: starting from the triangle function

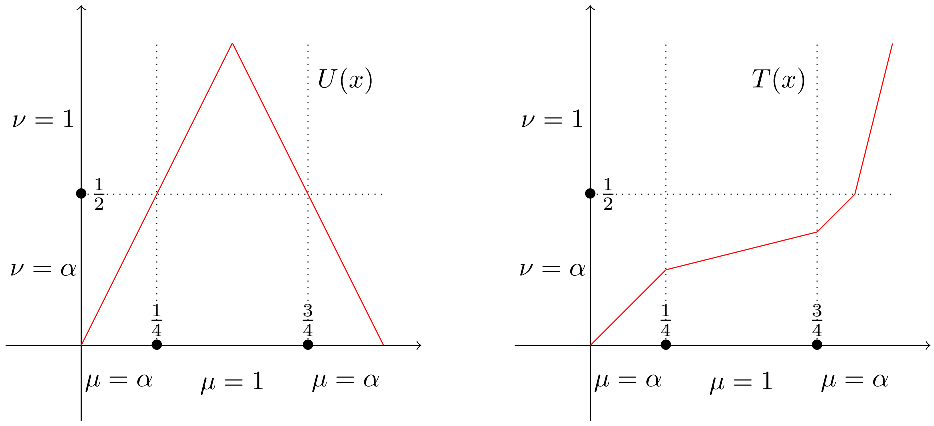
$$U : x \in (0,1) \mapsto \begin{cases} 2x & \text{if } x \leq \frac{1}{2} \\ 2 - 2x & \text{if } x \geq \frac{1}{2} \end{cases}$$

we construct a density g for which the associated increasing transport T verifies $\int_I |T'|^2\,\mathrm{d}\mu \geq \int_I |U'|^2\,\mathrm{d}\mu$. Let $\alpha > 2$ to be fixed below and

$$g(x) = \begin{cases} \alpha & \text{if } 0 < x < \frac{1}{4} \\ 1 & \text{if } \frac{1}{4} < x < \frac{3}{4} \\ \alpha & \text{if } \frac{1}{4} < x < 1 \end{cases}$$

(g is not a probability density, but it would enough to divide it by $(1+\alpha)/2$ to get it; however, we keep these formulas for the sake of simplicity of computations). We compute the increasing transport T sending g onto $U_{\#}g$ (since μ is symmetric with respect to the middle of I, the cost of its derivative is the same than for the decreasing transport from g to $U_{\#}g$), for instance thanks to the formula (2.5): it is

$$T : x \mapsto \begin{cases} x & \text{if } x \leq \dfrac{1}{4} \\ \dfrac{1}{4} + \dfrac{1}{\alpha}\left(x - \dfrac{1}{4}\right) & \text{if } \dfrac{1}{4} \leq x \leq \dfrac{3}{4} \\ x + \dfrac{1}{2}\left(\dfrac{1}{\alpha} - 1\right) & \text{if } \dfrac{3}{4} \leq x \leq 1 - \dfrac{1}{2\alpha} \\ \alpha(x-1) + 1 & \text{if } 1 - \dfrac{1}{2\alpha} \leq x \leq 1 \end{cases}$$



The graphs of U and T (here for $\alpha = 4$)

We now compute for a generic function f:

$$\int_I f(|U'|)\,\mathrm{d}\mu = \frac{1+\alpha}{2} f(2) \quad \text{and} \quad \int_I f(T')\,\mathrm{d}\mu = \frac{f(\alpha)}{2\alpha} + \frac{\alpha}{2} f(1) - \frac{f(1)}{2} + \frac{1}{2} f\left(\frac{1}{\alpha}\right)$$

Thus, the inequality 2.3 fails provided that

$$\frac{f(\alpha)}{2} - \alpha \frac{f(2) - f(1)}{2} - \frac{f(1) + f(2)}{2} + \frac{1}{2} f\left(\frac{1}{\alpha}\right) > 0$$

One see directly that, for f super-linear, one can find such an α (it is enough to take it large enough since this quantity goes to $+\infty$ as $\alpha \to +\infty$). For instance, for $f = |\cdot|^2$, this condition becomes

$$\alpha^4 - 3\alpha^3 - 5\alpha^2 + 1 > 0$$

which is true provided that $\alpha \geq \alpha_0 \simeq 4,1819$, the greatest zero of this polynomial function. This construction has also been used to build counter-example to the optimality of the monotone transport map for another type of variational problems (see [56], fourth paragraph). Notice however that:

- this does not prove directly the sharpness of the condition 2.2;
- also, this does not prove that this triangle function U is optimal for the functional $U \mapsto \int_I f(|U'|)\,\mathrm{d}\mu$ for such a μ, this proves only that this U is better than the monotone functions from μ to $U_{\#}\mu$ for this cost.

2.2 A PDE approach in dimension 2 and more

In this section, we forget the case of the real line and assume $d \geq 2$. Let Ω be an open subset of $\mathbb{R}^d$ and $L : \mathbb{R}^d \times \mathbb{R}^d \times M_d(\mathbb{R}) \to \mathbb{R}$; our goal is to characterize the minimizers of

$$J : T \mapsto \int_\Omega L(x, T(x), DT(x)) f(x)\,\mathrm{d}x$$

among the maps T such that:

- T sends $\mu = f \cdot \mathcal{L}^1$ onto a fixed non-negative measure ν on $\mathbb{R}^d$, with $\nu(\mathbb{R}^d) = \int_\Omega f$;
- T belongs to $H^1(\Omega)$.

We still assume that the class of such functions T is non-empty, and make the sufficient assumptions on Ω, f, L so that the existence of a minimizer can be proved by the direct method of the calculus of variations: Ω is a bounded open set with Lipschitz boundary, f is a density on Ω verifying $0 < c \leq f \leq C < +\infty$ and $L : \overline{\Omega} \times \mathbb{R}^d \times M_d(\mathbb{R}) \to [0, +\infty)$ is a function such that:

- there exists $\alpha > 0$ such that, for any x, u, p, $L(x, u, p) \geq \alpha |p|^2$;
- for any x and u, the function $p \mapsto L(x, u, p)$ is convex on $M_d(\mathbb{R})$.

We now fix an optimal T. Our strategy is to introduce a family $(T_t)_t$, defined for t small enough around 0, of "perturbations" of T such that $T_0 = T$; the optimality conditions will come from

$$\frac{\mathrm{d}}{\mathrm{dt}} J(T_t) \Big|_{t=0} = 0$$

2.2.1 Measure-preserving flow and perturbation of the optimal T

The key point for building our approximation is the following proposition, which constructs a measure-preserving time-parametrized flow $(X(t, \cdot))_t$ starting from a generic divergence-free vector field v with Neumann boundary conditions on v:

Proposition 2.2.1. *Assume the boundary of Ω to be C^1 and the density f to be Lipschitz. Let $v : \overline{\Omega} \to \mathbb{R}^d$ be a Lipschitz vector field satisfying*

$$\begin{cases} \operatorname{div}(fv) = 0 & \textit{in } \Omega \\ v \cdot n = 0 & \textit{on } \partial\Omega \end{cases}$$

where $x \mapsto n(x)$ denotes the outward normal of $\partial\Omega$. For $x \in \Omega$, we denote by $t \mapsto X(t, x)$ the maximal solution of the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(t, x) = v(X(t, x)) \\ X(0, x) = x \end{cases} \tag{2.10}$$

Then the following assertions are true:

- *each $X(\cdot, x)$, $x \in \Omega$, is defined on $\mathbb{R}$;*
- *each $X(t, \cdot)$, $t \in \mathbb{R}$, is Lipschitz and one-to-one $\Omega \to \Omega$ and we have*

  $$X(t, X(s, x)) = X(t + s, x)$$

  for any $s, t \in \mathbb{R}$ and $x \in \Omega$;
- *each $x \mapsto X(t, \cdot)$, $t \in \mathbb{R}$, sends the measure μ onto itself.*

As a consequence, for each Sobolev transport map T from μ to ν and for each $t \in \mathbb{R}$, the function $T \circ X(t, \cdot)$ also sends μ onto ν and is also admissible. Then, if T minimizes J among these maps, the fact that $T_0 = T$ implies our optimality condition: for any Lipschitz vector field v on $\overline{\Omega}$ such that fv is divergence-free and $v \cdot n = 0$ on $\partial\Omega$, we have

$$\frac{\mathrm{d}}{\mathrm{dt}} J(T \circ X(t, \cdot)) \Big|_{t=0} = 0$$

Let us remark that such a method can not be adapted to the one-dimensional case, since the only divergence-free vector field with Neumann boundary conditions is actually the null function. This corresponds to the fact that there exists only two measure-preserving maps on an interval which are, since such a map has to be monotone, the identity and $x \mapsto c-x$ (c being the constant such that the source and interval coincide); thus, any time-parametrized family of perturbation $(X_t)_t$ which preserves the measure is actually independent of the time and still equal to one of these maps. Notice that the problem that the one-dimensional measure-preserving maps are in some sense "isolated" also appears in variational models for fluid-mechanics (see for instance [7, 21] for the incompressible Euler equations) while, in higher dimension, as shown in [67, 68], there is more possibilities to connect any diffeomorphism to the identity maps *via* a time-parametrized family of diffeomorphisms.

Proof of Prop. 2.2.1. This proof is standard (it can be found, for instance, in [4], chap. 8), but we reproduce it here under our own assumptions for the sake of completeness.

The first point consists in showing that any maximal solution of (2.10) is global in time and does not cross the boundary of Ω. Assume that, for some $x \in \Omega$, the maximal solution of (2.10) is defined on an interval I with, for instance, $\sup I = a < +\infty$. Notice that since v is bounded on Ω, $X(\cdot, x)$ is time-Lipschitz thus admits a limit as $t \to a$; this limit cannot belong to Ω since $X(\cdot, x)$ is maximal, thus we have $X(t,x) \xrightarrow[t\to a]{} x_a \in \partial\Omega$. Since $\partial\Omega$ is C^1, there exists an open set U containing x_a such that, up to a change of coordinates,

$$\Omega \cap U = \{(x_1, \dots, x_{d-1}, \varphi(x_1, \dots, x_{d-1}))\} \cap U$$

where φ is a C^1 function defined on a neighborhood V of $(x_a^1, \dots, x_a^{d-1})$. In this system of coordinates, we denote by $\widetilde{x} = (x_1, \dots, x_{d-1})$ and $\widetilde{v} = (v_1, \dots, v_{d-1})$. The fact that $v \cdot n = 0$ on $\partial\Omega$ can be written as

$$v_d(\widetilde{x}, \varphi(\widetilde{x})) = \nabla\varphi(\widetilde{x}) \cdot \widetilde{v}(\widetilde{x}, \varphi(\widetilde{x})) \tag{2.11}$$

for $\widetilde{x}$ close enough to $\widetilde{x}_a$. Let us define $\widetilde{y}$ as being the maximal solution of the ordinary differential equation

$$\begin{cases} \widetilde{y}'(t) = \widetilde{v}(\widetilde{y}(t), \varphi(\widetilde{y}(t))) \\ \widetilde{y}(a) = x_a \end{cases}$$

and set $y_d(t) = \varphi(\widetilde{y}(t))$; $\widetilde{y}$, y_d are defined on an open interval J around a. Thanks to the equality (2.11), we have

$$y_d'(t) = \nabla\varphi(\widetilde{y}(t)) \cdot v(\widetilde{y}(t), \varphi(\widetilde{y}(t))) = v_d(\widetilde{y}(t), \varphi(\widetilde{y}(t)))$$

for t close enough to a. This implies that if we set $y = (\widetilde{y}, y_d)$, we have $y' = v(y)$ on $(a-\varepsilon, a+\varepsilon)$ for ε small enough, and $y(a) = x_a$. If we now consider the function

$$z : t \mapsto \begin{cases} X(t,x) & \text{if } t \in I \\ y(t) & \text{if } a < t < a+\varepsilon \end{cases}$$

then z is an extension of $X(\cdot, x)$ to the interval $I \cap (a-\varepsilon, a+\varepsilon)$, which strictly contains I, into a solution of (2.10). This is a contradiction since $X(\cdot, x)$ is supposed to be maximal.

The fact that $X(t, \cdot)$ is Lipschitz for any t comes directly from the classical theory of ordinary differential equations. To show that $X(t, X(s, \cdot)) = X(t+s, \cdot)$, one can see that, for each x, $t \mapsto X(t, X(s,x))$ and $t \mapsto X(t+s, x)$ both satisfy

$$\begin{cases} Y'(t) = Y(t) \\ Y(s) = X(s,x) \end{cases}$$

so that they are equal by the Cauchy-Lipschitz theorem.

For the third point, a usual method would consist in showing that the curve of measures $(X(t,\cdot)_{\#}\mu)_t$ satisfies the so-called "continuity equation" with initial condition

$$\begin{cases} \partial_t \rho_t + \operatorname{div}(\rho_t v) = 0 \\ \rho_0 = \mu \end{cases}$$

whose the time-independent measure μ is also a solution and for which there is uniqueness. This is done under very general assumptions on $(\rho_t)_t$ and v (which can also depend on the time) in the chap. 8 of [4]. In our case, one can proceed directly by computing the density of $X(t,\cdot)_{\#}\mu$: since $X(t,\cdot)$ is a bi-Lipschitz map from Ω into itself and has $X(-t,\cdot)$ as inverse map, for any test function $\varphi \in C_b(\Omega)$ and $t \in \mathbb{R}$, the change of variables formula gives

$$\int_\Omega \varphi(X(t,x)) f(x)\,\mathrm{d}x = \int_\Omega \varphi(y) f(X(-t,y)) |\det DX(-t,y)| f(X(-t,y))\,\mathrm{d}y$$

with $\det DX(t,x) > 0$ for any t and x (it cannot vanish since $X(t,\cdot)$ is a bi-Lipschitz map, is time-continuous thus has a constant sign, and is equal to 1 when $t=0$ since $X(0,\cdot)$ is the identity map). We deduce that the measure $X(t,\cdot)_{\#}\mu$ has a density on Ω which is given by

$$f_t(x) = f(X(-t,x)) \det DX(-t,x)$$

We compute the time-derivative of this density:

$$\frac{\mathrm{d}}{\mathrm{dt}} f_t(x) = \nabla f(X(-t,x)) \cdot (-\partial_t X(t,x)) + f(X(-t,x)) \operatorname{tr}(\operatorname{Com} DX(-t,x) \times -\partial_t DX(-t,x))$$

where $\operatorname{Com} DX$ denotes the matrix of cofactors of DX, and $\partial_t DX(t,x) = Dv(X(t,x)) \times DX(t,x)$, so that

$$\begin{aligned} \operatorname{tr}(\operatorname{Com} DX(-t,x) \times -\partial_t DX(-t,x)) &= -\det DX(-t,x) \times \operatorname{tr}(Dv(X(-t,x))) \\ &= -\det DX(-t,x) \times \operatorname{div} v(X(-t,x)) \end{aligned}$$

$$\text{and} \qquad \begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} f_t(x) &= -\det DX(-t,x) \times (\nabla f \cdot v + f \operatorname{div} v)(X(-t,x)) \\ &= -\det DX(-t,x) \operatorname{div}(fv)(X(-t,x)) = 0 \end{aligned}$$

Therefore the density f_t does not depend on t and its equal to its value at $t=0$, which is f since $X(0,\cdot)$ is the identity map. This proves that $X(t,\cdot)_{\#}\mu = \mu$ for any t. □

2.2.2 Derivation of the optimality conditions

The weak formulation in the general case. We make the above assumptions on $\partial\Omega$ and f, fix $v \in \operatorname{Lip}(\overline{\Omega})$ and compute the time-derivative of $J(T \circ X(t,\cdot))$ where X is defined by (2.10). Let us first notice that since $T \in H^1(\Omega)$ and $X(t,\cdot)$ is bi-Lipschitz for each t, each map $T \circ X(t,\cdot)$ still belongs to $H^1(\Omega)$ and its weak derivative is given by the usual formulas of derivation of a composition. Denoting by T_t this map, we thus have, for each t,

$$DT_t = DT \circ X(t,\cdot) \times D_x X(t,\cdot)$$

$$\text{and} \qquad J(T_t) = \int_\Omega L(x, T(X(t,x)), DT(X(t,x)) \times D_x X(t,x)) f(x)\,\mathrm{d}x$$

By using the fact that $X(t,\cdot)$ preserves μ and has $X(-t,\cdot)$ for inverse, we get

$$J(T_t) = \int_\Omega L(X(-t,x), T(x), DT(x) \times D_x X(t, X(-t,x))) f(x)\,\mathrm{d}x$$

where $D_x X(t,x)$ denotes the Jacobian matrix of $X(t,\cdot)$ at x. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}} J(T_t) &= \int_\Omega \nabla_x L(X(-t,x), T(x), DT(x) \times (D_x X)(t, X(-t,x))) \cdot \frac{\mathrm{d}}{\mathrm{dt}}(X(-t,x)) f(x)\,\mathrm{d}x \\ &+ \int_\Omega \nabla_p L(X(-t,x), T(x), DT(x) \times (D_x X)(t, X(-t,x))) \cdot DT(x) \times \frac{\mathrm{d}}{\mathrm{dt}}(D_x X(t, X(-t,x))) f(x)\,\mathrm{d}x\end{aligned} \tag{2.12}$$

We now have to compute the time-derivative of $D_x X(t, X(-t,x))$. We have

$$\frac{\mathrm{d}}{\mathrm{dt}}(D_x X(t, X(-t,x))) = (\partial_t D_x X)(t, X(-t,x)) + \sum_{i=1}^{d} (\partial_i D_x X)(t, X(-t,x)) \frac{\mathrm{d}}{\mathrm{dt}}(X(-t,x))$$

with, for $y \in \Omega$,

$$\partial_t D_x X(t,y) = D(\partial_t X(t,\cdot))(y) = D(v \circ X(t,\cdot)))(y) = Dv(X(t,y)) \times D_x X(t,y)$$

which, by taking account of $X(0,x) = x$, leads to

$$\frac{\mathrm{d}}{\mathrm{dt}}(D_x X(t, X(-t,x)))\Big|_{t=0} = Dv(x) \times D_x X(0,x) - \sum_{i=1}^{d} \partial_i D_x X(0,x) v_i(x)$$

But, since $X(0,x) = x$ for each $x \in \Omega$, it follows

$$D_x X(0,x) = I_d \qquad \text{and} \qquad \partial_i D_x X(0,x) = 0$$

$$\text{so that} \qquad \frac{\mathrm{d}}{\mathrm{dt}}(D_x X(t, X(-t,x)))\Big|_{t=0} = Dv(x)$$

We reinsert it into (2.12):

$$\frac{\mathrm{d}}{\mathrm{dt}} J(T_t)\Big|_{t=0} = \int_\Omega \left(-\nabla_x L(x, T(x), DT(x)) \cdot v(x) + \nabla_p L(x, T(x), DT(x)) \cdot (DT(x) \times Dv(x))\right) f(x)\,\mathrm{d}x$$

and the optimality of T implies that this time-derivative is null. This computation holds for any Lipschitz vector field v on $\overline{\Omega}$ such that $\mathrm{div}(fv) = 0$ with boundary condition $v \cdot n = 0$ on $\partial\Omega$. To summarize:

Proposition 2.2.2. *Under the above assumptions on f, ν and Ω, let T be an optimal map for the problem*

$$\inf \left\{ \int_\Omega L(x, T(x), DT(x)) f(x)\,\mathrm{d}x : T \in H^1(\Omega), T_\# \mu = \nu \right\}$$

Let $v \in \mathrm{Lip}\left(\overline{\Omega}\right)$ such that

$$\begin{cases} \mathrm{div}(fv) = 0 & \textit{in } \Omega \\ v \cdot n = 0 & \textit{on } \partial\Omega \end{cases}$$

Then

$$\int_\Omega \left(-\nabla_x L(x, T(x), DT(x)) \cdot v(x) + \nabla_p L(x, T(x), DT(x)) \cdot (DT(x) \times Dv(x))\right) f(x)\,\mathrm{d}x = 0 \tag{2.13}$$

An explicit PDE in the quadratic case. We will now focus on the case

$$J(T) = \int_\Omega |T(x) - x|^2 f(x)\,\mathrm{d}x + \int_\Omega |DT(x)|^2\,\mathrm{d}x$$

corresponding to $L(x,u,p) = |u-x|^2 + |p|^2/f(x)$. We compute

$$\nabla_x L(x,u,p) = 2(x-u) - \frac{|p|^2}{f(x)^2}\nabla f(x) \quad \text{and} \quad \nabla_p L(x,u,p) = \frac{2p}{f(x)}$$

The equation (2.13) becomes

$$\int_\Omega \left(2(T(x)-x) + \frac{|DT(x)|^2}{f(x)^2}\nabla f(x)\right)\cdot v(x)f(x)\,\mathrm{d}x + 2\int_\Omega DT(x)\cdot(DT(x)\times Dv(x))\,\mathrm{d}x = 0 \qquad (2.14)$$

for any $v \in \mathrm{Lip}\left(\overline{\Omega}\right)$ such that fv is divergence-free and $v\cdot n = 0$ on $\partial\Omega$.

Our goal is now to write an explicit PDE having (2.14) as weak formulation. We first assume that v is moreover compactly supported on Ω and that $T \in H^2(\Omega)$, so that we can integrate by parts the second term of (2.14):

$$\begin{aligned}\int_\Omega DT(x)\cdot(DT(x)\times Dv(x))\,\mathrm{d}x &= \int_\Omega \sum_{1\le i,j,k\le d} \partial_j T_i \partial_k T_i \partial_j v_k \\ &= -\sum_{1\le i,j,k\le d}\int_\Omega (\partial^2_{j,j}T_i\partial_k T_i + \partial_j T_i \partial^2_{j,k}T_i)v_k\end{aligned} \qquad (2.15)$$

$$\text{with} \quad \sum_{1\le i,j,k\le d} \partial^2_{j,j}T_i\partial_k T_i v_k = \Delta T\cdot(DT\times v) = ({}^t DT\times \Delta T)\cdot v \qquad (2.16)$$

$$\text{and} \quad \sum_{1\le i,j,k\le d} \partial_j T_i \partial^2_{j,k}T_i v_k = \nabla\left(\frac{1}{2}|DT|^2\right)\cdot v \qquad (2.17)$$

By inserting (2.16) and (2.17) into (2.14), we obtain

$$\int_\Omega \left(2(T(x)-x) + \frac{|DT(x)|^2}{f(x)^2}\nabla f(x) - \frac{2}{f(x)}{}^t DT(x)\times\Delta T(x) - \frac{1}{f(x)}\nabla(|DT|^2)(x)\right)\cdot(fv)(x)\,\mathrm{d}x = 0$$

But

$$\frac{|DT|^2}{f^2}\nabla f - \frac{1}{f}\nabla(|DT|^2) = \nabla\left(\frac{|DT|^2}{f}\right)$$

which, integrated against fv, gives 0 since fv is divergence-free. Similarly, $\int_\Omega x\cdot f(x)v(x)\,\mathrm{d}x = 0$. The optimality condition becomes

$$\int_\Omega \left(T(x) - \frac{2}{f(x)}{}^t DT(x)\times\Delta T(x)\right)\cdot f(x)v(x)\,\mathrm{d}x = 0 \qquad (2.18)$$

for any vector field $v\in\mathrm{Lip}\left(\overline{\Omega}\right)$ compactly supported in Ω and such that $\mathrm{div}(fv) = 0$; in other words,

$$\mathrm{curl}\left(T(x) - \frac{2}{f(x)}{}^t DT(x)\times\Delta T(x)\right) = 0$$

where the operator curl is defined in the weak sense. This is equivalent to

$$T(x) - \frac{2}{f(x)}{}^t DT(x)\times\Delta T(x) = \nabla\psi \qquad (2.19)$$

for some distribution $\psi \in \mathcal{D}'(\Omega)$.

Let us now write the boundary conditions associated to (2.19) for an optimizer T of J which also belongs to $H^2(\Omega)$. If v is not anymore supposed to be compactly supported into Ω, the integration by parts formula (2.15) involves a boundary term:

$$\int_\Omega DT(x)\cdot(DT(x)\times Dv(x))\,\mathrm{d}x$$
$$= -\int_\Omega \left({}^tDT\times\Delta T+\nabla\left(\frac{1}{2}|DT|^2\right)\right)\cdot v \;+\; \sum_{1\le i,j,k\le d}\int_{\partial\Omega}\partial_j T_i\partial_k T_i v_k n_j$$
$$\text{with} \qquad \sum_{1\le i,j,k\le d}\partial_j T_i\partial_k T_i v_k n_j = ({}^tDT\times DT\times v)\cdot n$$

We make the same computations than above with this boundary term. The equation (2.18) is now

$$\int_\Omega\left(T(x)-\frac{2}{f(x)}{}^tDT(x)\times\Delta T(x)\right)\cdot f(x)v(x) \;+\; \int_{\partial\Omega}({}^tDT(x)\times DT(x)\times v(x))\cdot n(x)\,\mathrm{d}S(x)=0 \quad (2.20)$$

for any Lipschitz vector field v on $\overline{\Omega}$ satisfying

$$\begin{cases}\operatorname{div}(fv)=0 & \text{in } \Omega\\ v\cdot n=0 & \text{on } \partial\Omega\end{cases}$$

By taking account of (2.19), the first term is equal to

$$\int_\Omega\nabla\psi(x)\cdot v(x)f(x)\,\mathrm{d}x=-\int_\Omega\psi(x)\operatorname{div}(fv)(x)\,\mathrm{d}x+\int_{\partial\Omega}\psi(x)f(x)v(x)\cdot n(x)\,\mathrm{d}S(x)$$

and the two terms are null thanks to the conditions on v. We obtain

$$\int_{\partial\Omega}({}^tDT(x)\times DT(x)\times v(x))\cdot n(x)\,\mathrm{d}S(x)=0 \quad (2.21)$$

This boundary conditions holds for any $v\in\operatorname{Lip}\left(\overline{\Omega}\right)$ such that fv is divergence-free and $v\cdot n=0$ on $\partial\Omega$. We conclude thanks to the following:

Proposition 2.2.3. *Assume that $\partial\Omega$ is $C^{3,\alpha}$ for some $\alpha>0$. Let v be a vector field $\partial\Omega\to\mathbb{R}^d$ such that $v\in C^{1,\alpha}(\partial\Omega)$ and $v\cdot n=0$ on $\partial\Omega$. Then v can be extended into a $C^{1,\alpha}$ vector field $\overline{\Omega}\to\mathbb{R}^d$ such that* $\operatorname{div} v=0$ *in Ω.*

Proof. We begin by considering an extension $\widetilde{v}$ of v into the whole $\overline{\Omega}$ belonging to $C^{1,\alpha}\left(\overline{\Omega}\right)$. Let us denote by $g=\operatorname{div}\widetilde{v}$ and define w one the solution of

$$\begin{cases}\operatorname{div} w=g & \text{in } \Omega\\ w=0 & \text{on } \partial\Omega\end{cases}$$

belonging to $C^{1,\alpha}\left(\overline{\Omega}\right)$ (see [30], Theorem 2). The function $u=\widetilde{v}-w$ is then convenient. □

We henceforth assume that $\partial\Omega$ is $C^{3,\alpha}$ and $f\in C^{1,\alpha}\left(\overline{\Omega}\right)$. The Prop. 2.2.3 and the equation (2.21) lead to

$$\int_{\partial\Omega}({}^tDT(x)\times DT(x)\times n(x))\cdot(w(x)n(x))\,\mathrm{d}S(x)=0$$

for any scalar function $w\in C^{1,\alpha}(\partial\Omega)$. We deduce that ${}^tDT(x)\times DT(x)\times n(x)$ is parallel to $n(x)$ on a.e. point x of $\partial\Omega$. If $\lambda(x)$ is thus defined by ${}^tDT(x)\times DT(x)\times n(x)=\lambda(x)n(x)$, it follows immediately $\lambda(x)=|DT(x)\times n(x)|^2$. This achieves the proof of the following:

Theorem 2.2.1. *Assume Ω to be a bounded open subset of $\mathbb{R}^d$ with C^1 boundary. Let f be a positive Lipschitz density on $\overline{\Omega}$, and $\mu = f \cdot \mathcal{L}^d$. Let ν be a positive measure on $\mathbb{R}^d$ with $\nu(\mathbb{R}^d) = \int_\Omega f$ and such that the set*

$$\{U \in H^1(\Omega) : U_\# \mu = \nu\}$$

is non-empty. Let T be a minimizer of the functional

$$J : U \mapsto \int_\Omega |U(x) - x|^2 f(x)\, \mathrm{d}x + \int_\Omega |DU(x)|^2\, \mathrm{d}x$$

among all the transport maps from μ to ν belonging to $H^1(\Omega)$. Then:

- *For any Lipschitz vector field v on $\overline{\Omega}$ such that*

$$\begin{cases} \operatorname{div}(fv) = 0 & \text{in } \Omega \\ v \cdot n = 0 & \text{on } \partial\Omega \end{cases}$$

  we have

$$\int_\Omega \left(2(T(x) - x) + \frac{|DT(x)|^2}{f(x)^2} \nabla f(x) \right) \cdot v(x) f(x)\, \mathrm{d}x + 2 \int_\Omega DT(x) \cdot (DT(x) \times Dv(x))\, \mathrm{d}x = 0$$

- *If moreover $T \in H^2(\Omega)$, $\partial\Omega$ is $C^{3,\alpha}$ and $f \in C^{1,\alpha}\left(\overline{\Omega}\right)$, then T satisfies the partial differential equation with boundary condition*

$$\begin{cases} T(x) - \dfrac{2}{f(x)}\, {}^t DT(x) \times \Delta T(x) = \nabla \psi(x), & x \in \Omega \\ {}^t DT(x) \times DT(x) \times n(x) = |DT(x) \times n(x)|^2 n(x), & x \in \partial\Omega \end{cases}$$

  for some scalar distribution ψ.

This partial differential equation is nonlinear and seems complicated to study. However, one of its interests is that we are able to find it through other ways which are close to the dual Kantorovich's approaches of the classical optimal transport problem (see Chapter 4).

Chapter 3

The Monge problem with vanishing gradient penalization

In this chapter, we investigate the following minimization problem: given μ, ν two - smooth enough - probability densities on $\mathbb{R}^d$ with μ supported in a domain Ω, we study

$$\inf \{J_\varepsilon(T) \,:\, T_\#\mu = \nu\} \qquad \text{where} \qquad J_\varepsilon(T) = \int_\Omega |T(x) - x| \,\mathrm{d}\mu(x) + \varepsilon \int_\Omega |DT|^2 dx$$

where ε is a vanishing positive parameter. Our goal is to understand the behavior of the functional J_ε in the sense of Γ-convergence and of characterize the limits of the minimizers T_ε.

Multiplying the gradient penalization with a vanishing parameter ε is motivated by the particular structure of the Monge problem (see below the paragraph 3.1 and the Appendix B.3). It is known that the minimizers of J_0 (that we will denote by J in the following) are not unique and are exactly the transport maps from μ to ν which also send almost any source point x to a point $T(x)$ belonging to the same *transport ray* as x (see below the precise definition). Among these transport maps, selection results are particularly useful, and approximating with strictly convex transport costs $c_\varepsilon(x, y) = |x - y| + \varepsilon|x - y|^2$ brings to the *monotone transport* (*i.e.* the unique transport map which is non-decreasing along each transport ray). Some regularity properties are known about it (continuity in the case of regular measures with disjoint and convex supports in the plane [42], uniform estimates on an approximating sequence under more general assumptions [57]). The question is to know which of these transport map is selected by the approximation through the gradient penalization that we propose here. This is a natural question, and, due to its higher regularity than other maps, one could wonder if it is again the monotone map in this case.

Up to very weak modifications, this chapter can fully be found as a paper in [33].

3.1 Motivations and known facts about the Monge problem

In this paragraph, we recall some known facts and useful tools about the optimal transportation problem with the Monge cost $c(x, y) = |x - y|$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. We only state here the tools that we will use in that follows; more details can be found in the Appendix B.3.

Let Ω, Ω' be two bounded open set on $\mathbb{R}^d$, $\mu \in \mathcal{P}(\Omega)$, $\nu \in \mathcal{P}(\Omega')$ and assume that μ is absolutely continuous with respect to the Lebesgue measure; we then denote by f its density. Then we set

$$W_1(\mu, \nu) = \inf \left\{ \int_\Omega |T(x) - x| \,\mathrm{d}\mu(x) \;:\; T : \Omega \to \mathbb{R}^d,\, T_\#\mu = \nu \right\}$$

the minimal value of the Monge transport cost from μ to ν, and

$$\mathcal{O}_1(\mu,\nu) = \left\{T : \Omega \to \mathbb{R}^d \ : \ T_\#\mu = \nu \text{ and } \int_\Omega |T(x) - x| \,\mathrm{d}\mu(x) = W_1(\mu,\nu)\right\}$$

the set of optimal transport maps for the Monge cost (if there is no ambiguity, we will use simply the notations W_1 and $\mathcal{O}_1$). Notice that the above Monge problem is a particular issue of its Kantorovich formulation, i.e.

$$\min\left\{\int_{\Omega\times\Omega'} |y-x| \,\mathrm{d}\gamma(x,y) \ : \ \gamma \in \mathcal{P}(\Omega\times\Omega') : (\pi_x)_\#\gamma = \mu,\ (\pi_y)_\#\gamma = \nu\right\},$$

and the minimal value of this problem coincides with $W_1(\mu,\nu)$ as well.

Theorem 3.1.1 (Duality formula for the Monge problem)**.** *We have the equality*

$$W_1(\mu,\nu) = \sup\left\{\int_{\Omega'} u(y)\,\mathrm{d}\nu(y) - \int_\Omega u(x)\,\mathrm{d}\mu(x) : u \in \mathrm{Lip}_1(\mathbb{R}^d)\right\}$$

where $\mathrm{Lip}_1(\mathbb{R}^d)$ *denotes the set of* 1*-Lipschitz functions* $\mathbb{R}^d \to \mathbb{R}$*. Such optimal Lipschitz functions are called* Kantorovich potentials*. Moreover, the Kantorovich potential is unique provided that* Ω *is a connected set and that the set* $\{f > 0\} \setminus \Omega'$ *is a dense subset of* Ω*.*

As a direct consequence of the duality formula, if $T : \Omega \to \mathbb{R}^d$ sends μ to ν and $u \in \mathrm{Lip}_1(\mathbb{R}^d)$, we have the equivalence:

$$\left\{\begin{array}{l} T \in \mathcal{O}_1(\mu,\nu) \\ u \text{ is a Kantorovich potential} \end{array}\right. \iff \text{for } \mu\text{-a.e. } x \in \Omega,\ u(T(x)) - u(x) = |T(x) - x|$$

We now introduce the following crucial notion of *transport ray*:

Definition 3.1.1 (Transport rays)**.** *Let* u *be a Kantorovich potential and* $x \in \Omega$, $y \in \Omega'$*. Then:*

- *the open oriented segment* (x,y) *is called* transport ray *if* $u(y) - u(x) = |y - x|$*;*
- *the closed oriented segment* $[x,y]$ *is called* maximal transport ray *if any point of* (x,y) *is contained into at least one transport ray with same orientation as* $[x,y]$*, and if* $[x,y]$ *is not strictly included into any segment with the same property.*

Proposition 3.1.1 (Geometric properties of transport rays)**.** *The set of maximal transport rays does not depend on the choice of the Kantorovich potential* u *and only depends on the source and target measures* μ *and* ν*. Moreover:*

- *any intersection point of two different maximal transport rays is an endpoint of these both maximal transport rays;*
- *the set of the endpoints of all the maximal transport rays is Lebesgue-negligible.*

These notions allow to prove the existence and to characterize the optimal transports:

Proposition 3.1.2 (Existence and characterization of optimal transport maps)**.** *The solutions of the Monge problem exist and are not unique; precisely, a map* T *sending* μ *to* ν *is optimal if and only if:*

- *for a.e.* $x \in \Omega$, $T(x)$ *belongs to the same maximal transport ray as* x*;*
- *the oriented segment* $[x, T(x)]$ *has the same orientation as this transport ray.*

These notions allow to prove the existence and to characterize the optimal transports:

Proposition 3.1.3 (Existence and characterization of optimal transport maps)**.** *The solutions of the Monge problem exist and are not unique; precisely, a map T sending μ to ν is optimal if and only if:*

- *for a.e. $x \in \Omega$, $T(x)$ belongs to the same maximal transport ray as x;*
- *the oriented segment $[x, T(x)]$ has the same orientation as this transport ray.*

We finish by recalling that, among all maps in $\mathcal{O}_1(\mu,\nu)$, there is a special one which has received much attention so far, and by quickly reviewing its properties.

Proposition 3.1.4 (The monotone map and a secondary variational problem)**.** *If μ is absolutely continuous, there exists a unique transport map T from μ to ν such that, for each maximal transport ray S, T is non-decreasing from the segment $S \cap \Omega$ to the segment $S \cap \Omega'$ (meaning that if $x, x' \in \Omega$ belong to the same transport ray, then $[x, x']$ and $[T(x), T(x')]$ have the same orientation).Moreover, T solves the problem*

$$\inf\left\{\int_\Omega |T(x)-x|^2 \,\mathrm{d}\mu(x) : T \in \mathcal{O}_1(\mu,\nu)\right\}.$$

Notice that this solution is already obtained as limit of minimizers of a perturbed variational problem, namely

$$\inf\left\{\int_\Omega |T(x)-x| \,\mathrm{d}\mu(x) + \varepsilon \int_\Omega |T(x)-x|^2 \,\mathrm{d}\mu(x) \;:\; T_\#\mu = \nu\right\}$$

This very particular transport map is probably one of the most natural in $\mathcal{O}_1(\mu,\nu)$ and one of the most regular. In particular, under some assumptions on the densities f, g and their supports Ω, Ω' (convex and disjoint supports in the plane and continuous and bounded by above and below densities), this transport has also be shown to be continuous [42], and, under more general assumptions, [57] gives also some regularity results for the minimizer T_ε of an approximated problem where c is replaced with $c_\varepsilon(x,y) = \sqrt{\varepsilon^2 + |x-y|^2}$ (local uniform bounds on the eigenvalues of the Jacobian matrix T_ε).

In particular, due to higher regularity of the monotone map, we can expect at first glance that adding a vanishing Dirichlet energy to the Monge cost again leads to select this particular map among the optimal transport maps.

3.2 Generalities and density of Sobolev transport maps

In that follows, we consider two bounded and star-shaped domains Ω, Ω' with Lipschitz boundaries and two measures $\mu \in \mathcal{P}(\Omega)$, $\nu \in \mathcal{P}(\Omega')$ with positive and bounded from below densities f, g; we assume moreover that the class of maps $T \in H^1(\Omega)$ sending μ onto ν is non-empty (this is guaranteed if for instance f, g are Hölder continuous, see below). The functional that we will study is defined, for $\varepsilon > 0$, by

$$J_\varepsilon \,:\, T \mapsto \int_\Omega |T(x)-x| \,\mathrm{d}\mu(x) + \varepsilon \int_\Omega |DT(x)|^2 \,\mathrm{d}x$$

and we denote by J the corresponding functional when $\varepsilon = 0$ (which is thus the classical Monge's transport energy); moreover, we extend J_ε, J to the whole $L^2(\Omega)$ by setting $J_\varepsilon(T) = J(T) = +\infty$ for a map T which is not a transport map from μ to ν. We will study the behavior of the family of functionals $(J_\varepsilon)_\varepsilon$, when ε vanishes, in the sense of Γ-convergence; the definitions and basic properties of Γ-convergence that we will use are recalled in the Appendix A.

As usual in transport theory, we consider as a setting for our variational problems the set of transport plans γ which are probabilities on the product space $\Omega \times \Omega'$ with given marginals $(\pi_x)_\#\gamma = \mu$ and

$(\pi_y)_{\#}\gamma = \nu$ and all the Γ-limits that we consider in that follows are considered with respect to the weak convergence of plans as probability measures. However, due to our choices of the functionals that we minimize, most of the transport plan that we consider will be actually induced by transport maps, i.e. $\gamma_T = (\mathrm{id} \times T)_{\#}\mu$ with $T_{\#}\mu = \nu$. These maps are valued in Ω', which is bounded, and are hence bounded. We could also consider different notions of convergence, in particular based on the pointwise convergence of these plans, and we will actually do it often. For simplicity, we will use the convergence in $L^2(\Omega;\Omega')$ (but, since these functions are bounded, this is equivalent to any other L^p convergence with $p < \infty$). As the following lemma (which will be also technically useful later) shows, this convergence is equivalent to the weak convergence in the sense of measures of the transport plans:

Lemma 3.2.1. *Assume that Ω, Ω' are compact domains and μ is a finite non-negative measure on Ω. Let $(T_n)_n$ be a sequence of maps $\Omega \to \Omega'$. Assume that there exists a map T such that $\gamma_{T_n} \rightharpoonup \gamma_T$ in the weak sense of measures. Then $T_n \to T$ in $L^2_\mu(\Omega)$.*

Conversely, if $T_n \to T$ in $L^2_\mu(\Omega)$, then we have $\gamma_{T_n} \rightharpoonup \gamma_T$ in the weak sense of measures.

Proof. If $\gamma_{T_n} \rightharpoonup \gamma_T$ and $\varphi \in C_b(\Omega)$ is a vector-valued function, we have

$$\int_\Omega \varphi(x)\cdot T_n(x)\,\mathrm{d}\mu(x) = \int_{\Omega\times\Omega'} \varphi(x)\cdot y\,\mathrm{d}\gamma_{T_n}(x,y) \to \int_{\Omega\times\Omega'} \varphi(x)\cdot y\,\mathrm{d}\gamma_T(x,y) = \int_\Omega \varphi(x)\cdot T(x)\,\mathrm{d}\mu(x)$$

which proves that $T_n \rightharpoonup T$ weakly in $L^2_\mu(\Omega)$. On the other hand,

$$\int_\Omega |T_n(x)|^2\,\mathrm{d}\mu(x) = \int_{\Omega\times\Omega'} |y|^2\,\mathrm{d}\gamma_{T_n}(x,y) \to \int_{\Omega\times\Omega'} |y|^2\,\mathrm{d}\gamma_T(x,y) = \int_\Omega |T(x)|^2\,\mathrm{d}\mu(x)$$

$$\text{thus} \qquad ||T_n||_{L^2_\mu} \to ||T||_{L^2_\mu}$$

and the convergence $T_n \to T$ is actually strong.

Conversely, assume that $T_n \to T$ in $L^2_\mu(\Omega)$ and let $(n_k)_k$ be such that the convergence $T_{n_k}(x) \to T(x)$ holds for a.e. $x \in \Omega$, then for any $\varphi \in C_b(\Omega\times\Omega')$ we have

$$\int_{\Omega\times\Omega'} \varphi(x,y)\,\mathrm{d}\gamma_{T_{n_k}}(x,y) = \int_\Omega \varphi(x,T_{n_k}(x))\,\mathrm{d}\mu(x) \to \int_\Omega \varphi(x,T(x))\,\mathrm{d}\mu(x) = \int_{\Omega\times\Omega'} \varphi(x,y)\,\mathrm{d}\gamma_T(x,y).$$

This proves $\gamma_{T_{n_k}} \rightharpoonup \gamma_T$, but, the limit being independent of the subsequence we easily get full convergence of the whole sequence. □

Since the set of transport plans between μ to ν is compact for the weak topology in the set of measures on $\Omega\times\Omega'$, a consequence of lemma 3.2.1 is that the equi-coercivity needed to deduce the convergence of minima and minimizer from the Γ-convergence of functionals (Theorem A.2.2) will be satisfied in all the Γ-convergence results of the following. Therefore, we will not focus on it anymore and still will consider that these results imply the convergence of minima and of minimizers.

3.2.1 Statement of the zeroth and first order Γ-convergences

Zeroth order Γ-limit. The first step is to check that $J_\varepsilon \xrightarrow{\Gamma} J$. Here we must consider that J_ε is extended to transport plan by setting $+\infty$ on those transport plans which are not of the form $\gamma = \gamma_T$ for $T \in H^1$, and that J is defined as usual as $J(\gamma) = \int |x-y|\,\mathrm{d}\gamma$ for transport plans. This Γ-onvergence actually requires a non-trivial result which states that the set of Sobolev transport maps is a dense subset of the set of transport maps from μ to ν. We prove this result in the next paragraph (see theorem 3.2.1 below) for Hölder densities and a large class of domains given by the following definition:

Definition 3.2.1. *We call* Lipschitz polar domain *any open bounded subset* Ω *of* $\mathbb{R}^d$ *having form*

$$\Omega = \left\{ x \in \mathbb{R}^d \,:\, |x - x_0| < \gamma\left(\frac{x - x_0}{|x - x_0|}\right)\right\}$$

for some $x_0 \in \Omega$ *and a Lipschitz function* $\gamma : S^{d-1} \to (0, +\infty)$. *In particular, such a domain* Ω *is star-shaped with Lipschitz boundary.*

Proposition 3.2.1 (Zeroth order Γ-limit). *Assume that* Ω, Ω' *are both Lipschitz polar domains and that* f, g *are both* $C^{0,\alpha}$ *and bounded from below. Then* $J_\varepsilon \xrightarrow{\Gamma} J$ *as* $\varepsilon \to 0$.

Proof. The Γ-liminf inequality is trivial (we have $J_\varepsilon \geq J$ by definition, and J is continuous for the weak convergence of plans), and the Γ-limsup inequality is a direct consequence of the Prop. A.2.2 and of the density of the set of Sobolev transports for the L^2-convergence. □

First order Γ-limit. We state it at follows, with this time a short proof:

Proposition 3.2.2 (First order Γ-limit). *Assume simply that* f, g *are bounded from below on* Ω, Ω' *and that* Ω *has Lipschitz boundary. Then the functional* $\dfrac{J_\varepsilon - W_1}{\varepsilon}$ *Γ-converges, when* $\varepsilon \to 0$, *to*

$$\mathcal{H} : T \mapsto \begin{cases} \int_\Omega |DT(x)|^2 \,\mathrm{d}x & \text{if } T \in \mathcal{O}_1(\mu,\nu) \cap H^1(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where, again, $\mathcal{H}$ *is extend to plans which are not induced by maps by* $+\infty$.

Proof. Γ-*limsup inequality.* If $T \in \mathcal{O}_1(\mu,\nu)$, then by choosing $T_\varepsilon = T$ for any ε we obtain automatically $\dfrac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon} = \int_\Omega |DT|^2$ for each ε. It remains to show that if $T \notin \mathcal{O}_1(\mu,\nu)$, then we have $\dfrac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon} \to +\infty$ for any sequence $(T_\varepsilon)_\varepsilon$ converging to T; but since the map

$$T \mapsto \int_\Omega |T(x) - x| \,\mathrm{d}\mu(x)$$

is continuous for the L^2-convergence, we have for such a $(T_\varepsilon)_\varepsilon$

$$\liminf_\varepsilon \frac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon} \geq \liminf_\varepsilon \frac{1}{\varepsilon}\left(\int_\Omega |T(x) - x| \,\mathrm{d}\mu(x) - W_1\right)$$

which is $+\infty$ since T is not optimal for the Monge problem.

Γ-*liminf inequality.* We can concentrate on sequence of maps T_ε with equibounded values for $\dfrac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon}$, which provides a bound on $\int_\Omega |DT_\varepsilon|^2$. This implies that we can assume, up to subsequences, that $(T_\varepsilon)_\varepsilon$ converges weakly in H^1 to T. Assuming the liminf to be finite, from

$$C \geq \frac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon} = \frac{1}{\varepsilon}\left(\int_\Omega |T_\varepsilon(x) - x| \,\mathrm{d}\mu(x) - W_1\right) + \int_\Omega |DT_\varepsilon|^2 \geq \int_\Omega |DT_\varepsilon|^2$$

we deduce as above that T must belong to $\mathcal{O}_1(\mu,\nu)$ and, since the last term is lower semi-continuous with respect to the weak convergence in $H^1(\Omega)$ (which is guaranteed up to subsequences since $(J_{\varepsilon_k}(T_{\varepsilon_k}))_k$ is bounded) we get the inequality we look for. □

Since the Γ-convergence implies the convergence of minima, we then have

$$\inf J_\varepsilon = \inf J + \varepsilon \inf \mathcal{H} + o(\varepsilon) = W_1 + \varepsilon \inf \mathcal{H} + o(\varepsilon)$$

provided that the infimum is finite, which means that there exists at least one transport map T also belonging to the Sobolev space $H^1(\Omega)$. In the converse case, and under the assumptions of the zeroth order Γ-convergence, we have

$$\inf J_\varepsilon \to W_1 \qquad \text{and} \qquad \frac{\inf J_\varepsilon - W_1}{\varepsilon} \to +\infty$$

which means that the lowest order of convergence of $\inf J_\varepsilon$ to J is smaller than ε. The study of a precise example where this order is $\varepsilon|\log\varepsilon|$ is the object of the section 3.3.

What about the selected map ? The first-order Γ-convergence and the basic properties of Γ-limits imply that, if T_ε minimizes J_ε, then $T_\varepsilon \to T$ which minimizes the Sobolev norm among the set $\mathcal{O}_1(\mu,\nu)$ of optimal transport maps from μ to ν. This gives a selection principle, via secondary variational problem (minimizing something in the class of minimizers), in the same spirit of what we presented for the monotone transport map along each transport ray. A natural question is to find which is this new "special" selected map, and whether it can coincide with the monotone one. Thanks to the non-optimality results of this map for the Sobolev cost on the real line, the answer is that they are in general different. We cal look at the following explicit counter-example (where we have however $\mathcal{O}_1(\mu,\nu) \cap H^1(\Omega) \neq \emptyset$).

Let us set $\Omega = (0,1)^2$, $\Omega' = (2,3) \times (0,1)$ in $\mathbb{R}^2$. Let F, G be two probability densities on the real line, supported in $(0,1)$ and $(2,3)$ respectively; we now consider the densities defined by

$$f(x_1,x_2) = F(x_1) \qquad \text{and} \qquad g(x_1,x_2) = G(x_1)$$

Then, if t is a transport map from F to G on the real line and $T(x_1,x_2) = (t(x_1),x_2)$, it is easy to check that T sends the density f onto g and if $u(x_1,x_2) = x_1$ we have

$$|T(x) - x| = |t(x_1) - x_1| = t(x_1) - x_1 = u(T(x_1,x_2)) - u(x_1,x_2)$$

This proves that T is optimal and u, which is of course 1-Lipschitz, is a Kantorovich potential, so that the maximal transport rays are exactly the segments $[0,3] \times \{x_2\}$, $0 < x_2 < 1$. As a consequence,

$$T \in \mathcal{O}_1(\mu,\nu) \qquad \Longleftrightarrow \qquad T(x_1,x_2) = (t(x_1,x_2),x_2) \quad \text{with} \quad \begin{array}{c} t(\cdot,x_2)_\# F = G \\ \text{for a.e. } x_2 \end{array}$$

In particular, the monotone transport map along the maximal transport rays is $x \mapsto (t(x_1),x_2)$, where t is the non-decreasing transport map from F to G on the real line. For this transport map T, we have

$$\int_\Omega |DT(x)|^2 \,\mathrm{d}x = \int_0^1 t'(x_1)^2 \,\mathrm{d}x_1$$

Now, one can choose F, G such that the solution of

$$\inf \left\{ \int_0^1 U'(x_1)^2 \,\mathrm{d}x_1 \ : \ U_\# F = G \right\} \tag{3.1}$$

is not attained by the increasing transport map from F, to G. Thus, if $\widetilde{t}$ minimizes (3.1) and $\widetilde{T}(x_1,x_2) = \widetilde{t}(x_1)$, we have

$$\int_\Omega |D\widetilde{T}(x)|^2 \,\mathrm{d}x = \int_0^1 \widetilde{t}'(x_1)^2 \,\mathrm{d}x_1 < \int_0^1 t'(x_1)^2 \,\mathrm{d}x_1 = \int_\Omega |DT(x)|^2 \,\mathrm{d}x$$

and $\widetilde{T}$ is also an optimal transport map for the Monge problem.

3.2.2 Proof of the density of Lipschitz transports

In this section, we prove that, under natural assumptions (domains which are diffeomorphics to the unit ball, Hölder and bounded from above and below densities), the transport maps from μ to ν which are Lipschitz continuous form a dense subset of the set of transport maps from μ to ν. Notice that, for the sake of the applications to the zero-th order Γ-convergence of the previous section, we only need the density of those maps belonging to the Sobolev space $H^1(\Omega)$; also, we needed density in the set of plans, but since it is well known that transport maps are dense in the set of transport plans, for simplicity we will prove density in the set of maps. This result, that we need in this chapter to study the Γ-convergence of the functional J_ε, could of course be useful for other aims; however, it was, to the best of our knowledge, not investigated until now.

Theorem 3.2.1. *Assume that Ω, Ω' are both Lipschitz polar domains and that f, g are both $C^{0,\alpha}$ and bounded from below. Then the set*

$$\{T \in \mathrm{Lip}(\Omega)\ :\ T_\#\mu = \nu\}$$

is non-empty, and is a dense subset of the set

$$\{T : \Omega \to \Omega'\ :\ T_\#\mu = \nu\}$$

endowed with the norm $||\cdot||_{L^2(\Omega)}$.

Notice that, exactly as for the density of transport maps inside the set of transport plans, the best possible result would be to show the density under the same assumptions guaranteeing the existence of at least such a map (for the density of maps into plans the assumption is that μ must be atomless, which is the same as for the existence of at least a map T). Here we are not so far since the already known results about regularity of some transport maps (by Caffarelli *et al.* or Dacorogna-Moser) deal with at least Hölder densities. Also, one can see from the proof below that we do not use much more than the existence of Lipschitz maps. Concerning the assumption on the domains to be star-shaped, this is used to send them onto the unit ball in a Lipschitz-diffeomorphic way (in order to get sufficiently regular measures), as shown in the following lemma which is used several times:

Lemma 3.2.2. *If U is a Lipschitz polar domain with*

$$U = \left\{x \in \mathbb{R}^d\ :\ |x - x_0| < \gamma\left(\frac{x - x_0}{|x - x_0|}\right)\right\}$$

then there exists a map $\alpha : U \to B(0,1)$ such that:

- *α is a bi-Lipschitz diffeomorphism from U to $B(0,1)$;*
- *$\det D\alpha$ is Lipschitz and bounded from below (thus, $\det D\alpha^{-1}$ is also Lipschitz);*
- *if U is star-shaped around x_0 then, for any $x \neq x_0$, $\dfrac{\alpha(x)}{|\alpha(x)|} = \dfrac{x - x_0}{|x - x_0|}$*

(We will not use the third property in the proof of Theorem 3.2.1, but it will be useful later in section 3.4.2).

Proof. Up to a translation and a dilation, we can assume $x_0 = 0$ and $\gamma \le 1$ on S^{d-1}. Now we set

$$\alpha(x) = \begin{cases} x & \text{if } |x| \le \dfrac{1}{2}\gamma\left(\dfrac{x}{|x|}\right) \\ \lambda(x)\dfrac{x}{|x|} & \text{otherwise} \end{cases}.$$

for a suitable choice of $\lambda : U \to [0,+\infty)$. Assuming that λ is Lipschitz, we compute $D\alpha$ on the region $\left\{|x| > \frac{1}{2}\gamma(x/|x|)\right\}$. Here we have $D\alpha = x \otimes \nabla\left(\frac{\lambda(x)}{x}\right) + \frac{\lambda}{|x|}I_d$ thus, in an orthonormal basis whose first vector is $e = \frac{x}{|x|}$,

$$D\alpha = \begin{pmatrix} |x|\partial_e\left(\frac{\lambda(x)}{|x|}\right) & |x|\partial_{e_2}\left(\frac{\lambda(x)}{|x|}\right) & \dots & |x|\left(\partial_{e_n}\frac{\lambda(x)}{|x|}\right) \\ 0 & 0 & \dots 0 & \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots 0 & \end{pmatrix} + \frac{\lambda}{|x|}I_n$$

$$\text{and} \qquad \det D\alpha = \left(|x|\partial_e\left(\frac{\lambda(x)}{|x|}\right) + \frac{\lambda(x)}{|x|}\right)\left(\frac{\lambda(x)}{|x|}\right)^{n-1}$$

We write $\lambda = \lambda(r,e)$ where $r = |x|$, which leads to

$$\det D\alpha = \left(r\partial_r\left(\frac{\lambda}{r}\right) + \frac{\lambda}{r}\right)\left(\frac{\lambda}{r}\right)^{n-1} = \partial_r\lambda\left(\frac{\lambda}{r}\right)^{n-1}$$

At this time, we see that the following conditions on λ allow to conclude:

- λ is Lipschitz on the domain $\left\{\frac{\gamma(x/|x|)}{2} \le |x| \le \gamma(x/|x|)\right\}$;
- if we fix $e \in S^{d-1}$, then $\lambda(\cdot,e)$ is increasing on the interval $\left[\frac{\gamma(e)}{2}, \gamma(e)\right]$;
- $\lambda\left(\frac{\gamma(e)}{2}, e\right) = \frac{\gamma(e)}{2}$ and $\lambda(\gamma(e),e) = 1$;
- $\partial_r\lambda\left(\frac{\lambda}{r}\right)^{n-1}$ is Lipschitz and is equal to 1 for $r = \frac{\gamma(e)}{2}$, which means $\partial_r\lambda\left(\frac{\gamma(e)}{2}, e\right) = 2^{n-1}$.

To satisfy these conditions, it is enough to choose for $\lambda(\cdot,e)$ a second-degree polynomial function with prescribed values at $\frac{\gamma(e)}{2}$, $\gamma(e)$ and prescribed first derivative at $\frac{\gamma(e)}{2}$. □

Proof of Theorem 3.2.1. First, we assume $\Omega = \Omega' = \left[-\frac{1}{2}, \frac{1}{2}\right]^d$ but we will also identify (for convolution purposes, see below) the cube Ω with a Torus. We will generalize later to any pair of Lipschitz star-shaped domains. Thus, let $T : \Omega \to \Omega$ sending f to g, and let us build a sequence $(T_n)_n$ of Lipschitz maps such that

$$T_n \xrightarrow[n\to+\infty]{L^2(\Omega)} T \qquad \text{and} \qquad \forall n \in \mathbb{N},\ (T_n)_\# \mu = \nu$$

Step 1: regularization of the transport plan. We denote by γ the transport plan associated to T; let us recall that it is defined by

$$\iint_{\Omega^2} \varphi(x,y)\,\mathrm{d}\gamma(x,y) = \int_\Omega \varphi(x,T(x))f(x)\,\mathrm{d}x$$

for any continuous and bounded function φ on $\Omega\times\Omega'$. We use the disintegration of measures (see Theorem 5.3.1 in [4]) to write

$$\iint_{\Omega^2} \varphi\,\mathrm{d}\gamma = \int_\Omega \left(\int_\Omega \varphi(x,y)\,\mathrm{d}\gamma_x(y)\right)\mathrm{d}\mu(x)$$

for a family of measures $(\gamma_x)_{x\in\Omega}$ on Ω. Now we take a family of periodic convolution kernels $(\rho_k)_k$, we define, for each x, the measure γ_x^k on Ω by $\gamma_x^k = \rho_k \star \gamma_x$, where the convolution product is taken in the distributional sense on the torus Ω, and set $\mathrm{d}\gamma_k(x,y) = \mathrm{d}\gamma_x^k(y) \otimes \mathrm{d}\mu(x)$; the fact that $(\rho_k)_k$ is an approximate identity guarantees that $\gamma_k \rightharpoonup \gamma$ for the weak convergence of measures.

On the other hand, notice that

$$\begin{aligned}\iint_{\Omega^2} \varphi(x,y)\,\mathrm{d}\gamma_k(x,y) &= \iiint_{\Omega^3} \varphi(x,y)\rho_k(y-z)\,\mathrm{d}\gamma_x(z) f(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega \left(\int_{\Omega^2} \varphi(x,y)\rho_k(y-z)\,\mathrm{d}\gamma(x,z)\right)\mathrm{d}y \\ &= \iint_{\Omega^2} \varphi(x,y)\rho_k(y-T(x))f(x)\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

for any test function φ. Thus, γ_k is absolutely continuous with respect to the Lebesgue measure on Ω^2 and its density is $\rho_k(y-T(x))f(x)$. In particular, this density is smooth with respect to y and bounded from below by a positive constant c_k. Moreover, if we denote by ν_k the second marginal of γ_k, then ν_k has also a Lipschitz and bounded from below density.

Step 2: construction of a transport map corresponding to the regularized transport plan. The goal of this step is to build, for each k, a family of regular maps $(T_h^k)_h$ each sending the first marginal of γ_k onto the second one and such that, for a suitable diagonal extraction, $T_{h_k}^k \to T$ for the L^2-convergence.

We fix $k \in \mathbb{N}$ and a dyadic number h, and decompose $\Omega = \bigcup_i Q_i$ into a union of cubes Q_i, each with size-length $2h$, via a regular grid. For each i, we set

$$\mu_i = \mu|_{Q_i} \qquad \text{and} \qquad \nu_i^k = (\pi_2)_\#(\gamma_k|_{(Q_i\times\Omega)})$$

Notice that γ_k has μ as first marginal, thus we have also $\mu_i = (\pi_1)_\#(\gamma_k|_{(Q_i\times\Omega)})$; in particular, μ_i (which is a measure on the cube Q_i) and ν_i^k (which is a measure on the whole cube Ω) have both same mass. Moreover, μ_i still has f as density and the density of ν_i^k is

$$y \mapsto \int_{Q_i} \rho_k(T(x)-y)f(x)\,\mathrm{d}x$$

which is smooth and bounded from below by a positive constant. We finish this step by building a map $U_{i,h}^k : Q_i \to \Omega$ such that

- $U_{i,h}^k$ is Lipschitz continuous;
- $U_{i,h}^k(x) = x$ for $x \in \partial Q_i$ (this is allowed since the source Q_i is included in the target Ω);
- $U_{i,h}^k$ sends μ_i onto ν_i^k.

Let us notice that the two first points guarantee that the global map defined by $T_h^k(x) = U_{i,h}^k(x)$ (which depends on the radius h and the original integer k) for $x \in Q_i$ will be globally Lipschitz on Ω and the third point implies that T_h^k sends globally the measure μ onto the measure $\nu_k = (\pi_2)_\#\gamma_k$ (which is equal to the sum of the ν_i^k).

Now we explain very briefly the construction of such a $U_{i,h}^k$. For the sake of simplicity, we give the details only in the case where Q_i is not an extremal cube of Ω, *i.e.* where $\partial Q_i \cap \partial\Omega$ is empty, and claim that the extreme case can be treated in a similar way.

We denote by x_i the center of Q_i which has radius h (for the infinite norm in $\mathbb{R}^d$), and by δ_1 and δ_2 the two positive numbers such that

$$\mu(\{x : \delta_1 \le ||x-x_i||_\infty \le h\}) = \mu(\{x : \delta_2 \le ||x-x_i||_\infty \le \delta_1\}) = \frac{1}{2}\nu_i^k(\Omega \setminus Q_i)$$

(such positive numbers exist by the intermediate value theorem since $\delta \mapsto \mu(B_\infty(x_i,\delta))$ is continuous). Now we consider:

- a first Lipschitz map V_1 from $\{x : \delta_1 \le ||x-x_i||_\infty \le h\}$ to $\Omega \setminus Q_i$, with $(V_1)_\#\mu = \frac{1}{2}\nu_i^k(\Omega \setminus Q_i)$ and such that $V_1(x) = x$ for $||x-x_i|| = \delta_1$;
- a second Lipschitz map V_2 from $\{x : \delta_2 \le ||x-x_i||_\infty \le \delta_1\}$ to $\Omega \setminus Q_i$, such that $(V_2)_\#\mu = \frac{1}{2}\nu_i^k(\Omega\setminus Q_i)$ and $V_2(x) = V_1(x)$ for $||x-x_i|| = \delta_2$;
- a third Lipschitz map V_3 from $\{0 : ||x-x_i||_\infty \le \delta_2\}$ to Q_i, such that $V_3(x) = V_2(x)$ on the boundary and $(V_3)_\#\mu = \nu_i^k(Q_i)$.

and we set $U_{i,h}^k(x) = V_1(x)$ if $\delta_1 \le ||x-x_i||_\infty \le h$, $V_2(x)$ if $\delta_2 \le ||x-x_i||_\infty \le \delta_1$ and $V_3(x)$ if $||x-x_i||_\infty \le \delta_2$. The constraints on $(V_1)_\#\mu$, $(V_2)_\#\mu$, $(V_3)_\#\mu$ imply that $U_{i,h}^k$ defined on the whole Q_i sends globally $\mu|_{Q_i}$ onto ν_i^k, and the constraints on the boundary on each domain guarantee that $U_{i,h}^k$ is Lipschitz on Q_i. We construct V_1, V_2, V_3 using Dacorogna-Moser's result (a simple adaptation is needed, since the source and target domains are not the same here, but are diffeomorphic to each other: either they are both cubes, or they both have the form "large cube minus small cube").

In particular, γ_k and $\gamma_{T_h^k}$ give both same mass to each $Q_i \times \Omega$ which have radius h. Now if u is a 1-Lipschitz function $\Omega \times \Omega' \to \mathbb{R}$, we compute

$$\int_{\Omega^2} u(x,y)\,\mathrm{d}\gamma_{T_h^k}(x,y) - \int_{\Omega^2} u(x,y)\,\mathrm{d}\gamma_k(x,y) = \sum_i \int_{Q_i\times\Omega} u(x,y)(\,\mathrm{d}\gamma_{T_h^k} - \,\mathrm{d}\gamma_k)(x,y)$$

Since γ_k and $\gamma_{T_h^k}$ have both the same second marginal on each $Q_i \times \Omega$,

$$\int_{Q_i\times\Omega} u(x_i,y)(\,\mathrm{d}\gamma_{T_h^k}(x,y) - \,\mathrm{d}\gamma_k(x,y)) = 0$$

where, for each i, x_i denotes the center of Q_i. Thus

$$\begin{aligned}\int_{\Omega^2} u(x,y)\,\mathrm{d}\gamma_{T_h^k}(x,y) - \int_{\Omega^2} u(x,y)\,\mathrm{d}\gamma_k(x,y) &= \sum_i \int_{Q_i\times\Omega} (u(x,y)-u(x_i,y))(\,\mathrm{d}\gamma_{T_h^k} - \,\mathrm{d}\gamma_k)(x,y)\\ &\le \sum_i \int_{Q_i\times\Omega} |x-x_i|\;|\,\mathrm{d}\gamma_{T_h^k} - \,\mathrm{d}\gamma_k|(x,y)\end{aligned}$$

where in the last inequality we used the fact that u is supposed to be 1-Lipschitz. We deduce

$$\int_{\Omega^2} u(x,y)\,\mathrm{d}\gamma_{T_h^k}(x,y) - \int_{\Omega^2} u(x,y)\,\mathrm{d}\gamma_k(x,y) \le h \iint_{\Omega^2} |\,\mathrm{d}\gamma_{T_h^k} - \,\mathrm{d}\gamma_k| \le 2h$$

This inequality holds for any 1-Lipschitz function $u : \Omega^2 \to \mathbb{R}$. Thus, if W_1 denotes the classical Wasserstein distance (on Ω^2: this is just a technical trick to metrize the weak convergence of plans), we deduce $W_1(\gamma_k, \gamma_{T_h^k}) \le h$ for any k. Since we know that $\gamma_k \to \gamma_T$ as $k \to +\infty$, we also have $W_1(\gamma_k, \gamma_T) \to 0$ and if we set $h = 1/2^k$ we have $W_1(\gamma_{T_h^k}, \gamma_T) \to 0$. Thanks to the lemma 3.2.1, we deduce $T_h^k \to T$ for the L^2-norm in Ω.

Step 3: rearranging T_h^k. It remains to compose each map T_h^k with a transport map U_k from ν_k to ν, which will ensure the obtained map to send μ onto ν as well. We use the following classical result of Caffarelli ([69], Theorem 4.14): if

- X, Y are two bounded open sets of $\mathbb{R}^d$, uniformly convex and with C^2 boundary;

- $f \in C^{0,\alpha}(\overline{X})$, $g \in C^{0,\alpha}(\overline{Y})$ are two probability densities

then the optimal transport map from f to g belongs to $C^{1,\alpha}(\overline{X})$. The assumption on the source domain is of course not satisfied here, thus we use the lemma 3.2.2 to get a bi-Lipschitz map $\alpha : \Omega \to B(0,1)$ such that $\det D\alpha$ is also Lipschitz. We then denote by

$$\overline{g} = \alpha_{\#}g \qquad \text{and} \qquad \overline{g}_k = \alpha_{\#}g_k$$

where g_k is the density of ν_k (we know that g_k is Lipschitz and bounded from below). The fact that $\det D\alpha$ is Lipschitz and bounded guarantees that $\overline{g}$ is Hölder and $\overline{g}_k$ is Lipschitz as well.

We know that $g_k \rightharpoonup g$ in the weak sense of measures, thus $\overline{g}_k \rightharpoonup \overline{g}$ also, and

$$W_2(\overline{g}_k, \overline{g}) = \int_{B(0,1)} |U_k(x) - x|^2 \overline{g}_k(x)\, \mathrm{d}x \to 0$$

where U_k is the optimal transport map from $\overline{g}_k$ to $\overline{g}$ for the quadratic cost; the Caffarelli's regularity result guarantees U_k to belong to $C^{1,\alpha}(\overline{B}(0,1))$, and we deduce from the convergence $W_2(\overline{g}_k, g) \to 0$ that $U_k(x) \to x$ for almost any $x \in B(0,1)$. Now if we consider

$$\widetilde{T}_k = \alpha^{-1} \circ U_k \circ \alpha \circ T_h^k$$

then we can check that $\widetilde{T}_k \to T$ in $L^2(\Omega)$, and we have $(\widetilde{T}_k)_{\#}\mu = \nu$ for each k by construction. The proof is complete in the case $\Omega = \Omega' = \left[-\frac{1}{2}, \frac{1}{2}\right]^d$.

Step 4: generalization to any pair of domains. If Ω, Ω' are now two Lipschitz polar domains and T is a transport maps from μ to ν, we consider two Lipschitz diffeomorphisms $\alpha_1 : \Omega \to [0,1]^d$, $\alpha_2 : \Omega' \to [0,1]^d$ as in the Lemma 3.2.2. The regularity of $\det D\alpha_1$, $\det D\alpha_2$ guarantees that the image measures $(\alpha_1)_{\#}\mu$, $(\alpha_2)_{\#}\nu$ have both $C^{0,\alpha}$ regularity, thus we are able to find a sequence $(U_k)_k$ of transport maps from $(\alpha_1)_{\#}\mu$ to $(\alpha_2)_{\#}\nu$ converging for the $L^2([0,1]^d)$-norm to $(\alpha_2) \circ T \circ (\alpha_1)^{-1}$, and it is now easy to check that $T_k = (\alpha_2)^{-1} \circ U_k \circ \alpha_1$ sends μ to ν and converges to T as well. □

3.3 An example of approximation of order $\varepsilon|\log\varepsilon|$

As we said in the above section, the convergence $\frac{J_\varepsilon - W_1}{\varepsilon} \to \mathcal{H}$ allows to know the behavior of $\inf J_\varepsilon$ and of any family $(T_\varepsilon)_\varepsilon$ of minimizers in the case where $\inf \mathcal{H} < +\infty$. This paragraph is devoted to the study of an example where this assumption fails (*i.e.* where any optimal transport map for the Monge problem is not Sobolev).

3.3.1 Notations and first remarks

In the rest of this chapter, we set $d = 2$ and we will denote by (r, θ) the usual system of polar coordinates in $\mathbb{R}^2$. Our source and target domains will be respectively

$$\Omega = \left\{ x = (r,\theta) : 0 < r < 1, 0 < \theta < \frac{\pi}{2} \right\}$$

$$\text{and} \qquad \Omega' = \left\{ x = (r,\theta) : R_1(\theta) < r < R_2(\theta), 0 < \theta < \frac{\pi}{2} \right\}$$

where R_1, R_2 are two Lipschitz functions $\left[0, \frac{\pi}{2}\right] \to (0, +\infty)$ with $\inf R_1 > 1$ and $\inf R_2 > \sup R_1$. We also suppose R_1 to be such that R_1' is a function of bounded variation. Notice also that we can choose R_1, R_2 such that the target domain Ω' is convex (for instance if the curves $r = R_1(\theta)$, $r = R_2(\theta)$ are actually two lines in the quarter plane). We assume that f and g are two Lipschitz densities on $\overline{\Omega}$, $\overline{\Omega'}$, bounded from above and below by positive constants, with the following hypothesis:

$$\forall\theta \in \left(0, \frac{\pi}{2}\right), \int_0^1 f(r,\theta)\, r\,\mathrm{d}r = \int_{R_1(\theta)}^{R_2(\theta)} g(r,\theta)\, r\,\mathrm{d}r \tag{3.2}$$

which means that, for any θ, the mass (with respect to f) of the segment joining the origin to the boundary of Ω and with angle θ is equal to the mass (with respect to g) of the segment with same angle joining the "above" and "below" boundaries of Ω' (*i.e.* the curves $r = R_1(\theta)$ and $r = R_2(\theta)$).

Then the structure of the optimal maps for the Monge cost is given by the following:

Proposition 3.3.1. *Under these assumptions on Ω, Ω', f, g, the Euclidean norm is a Kantorovich potential and the maximal transport rays are the segments joining 0 to $(R_2(\theta), \theta)$. Consequently,*

$$T \in \mathcal{O}_1(\mu,\nu) \iff T_\# f = g \text{ and } T(x) = \varphi(x)\frac{x}{|x|}$$

for some function $\varphi \in \Omega \to (\inf R_1, +\infty)$.

Proof. For $\theta \in \left(0, \frac{\pi}{2}\right)$, we denote by $t(\cdot,\theta)$ a one-dimensional transport map from the measure $r \mapsto rf(r,\theta)$ to the measure $r \mapsto rg(r,\theta)$ (such a transport map exists since these two measures have same mass thanks to the equality (3.2)). It is then easy to check that the map

$$T : x = (r,\theta) \in \Omega \to t(r,\theta)\frac{x}{|x|}$$

is a transport map from f to g, and if we set $u = |\cdot|$, u is of course 1-Lipschitz and, for any x,

$$u(T(x)) - u(x) = \left| t(r,\theta)\frac{x}{|x|}\right| - |x| = (|t(r,\theta)| - |x|)\frac{x}{|x|} = \left|(t(r,\theta) - |x|)\frac{x}{|x|}\right| = |T(x) - x| \qquad \text{for any } x \in \Omega.$$

We deduce that that $u = |\cdot|$ is a Kantorovich potential. Consequently, a segment $[x,y]$ is a transport ray if and only if

$$u(y) - u(x) = |y - x| \qquad i.e. \qquad |y - x| = |y| - |x|$$

Thus, we have $y = \lambda x$ for a positive λ. In particular, y and x belong to the same line passing through the origin. In other words, the transport rays are included in the lines passing through the origin. Moreover, a transport map T belongs to $\mathcal{O}_1(\mu,\nu)$ if and only if, for a.e. $x \in \Omega$, $|T(x) - x| = |T(x)| - |x|$ which again means that $T(x) = \varphi(x)\frac{x}{|x|}$ for some positive function φ. □

Corollary 3.3.1. *Under the above assumptions on Ω, Ω', μ, ν, we have $\mathcal{O}_1(\mu,\nu) \cap H^1(\Omega) = \emptyset$*

Proof. Let $T \in \mathcal{O}_1(\mu,\nu)$. By Prop. 3.3.1, we have $T(x) = \varphi(r,\theta)\frac{x}{|x|}$ for $x = (r,\theta)$, where φ is a real-valued function; the fact that T sends μ onto ν implies $T(x) \in \Omega'$ for any x, thus φ is bounded from below on Ω by the lower bound of R_1. We now compute the Jacobian matrix of T. Denoting by $x^\perp$ the image of x by the rotation with angle $\pi/2$, we have in the basis $\left(\frac{x}{|x|}, \frac{x^\perp}{|x|}\right)$:

$$DT(x) = \frac{x}{|x|} \otimes \nabla\varphi(x) - \frac{\varphi(x)}{|x|} I_d = \begin{pmatrix} \partial_r\varphi & 0 \\ \frac{\partial_\theta\varphi}{r} & \frac{\varphi}{r} \end{pmatrix}$$

$$\text{thus} \qquad \int_\Omega |DT(x)|^2\,\mathrm{d}x \geq \int_0^1 \int_0^{\frac{\pi}{2}} \frac{\varphi(r,\theta)^2}{r^2}\, r\,\mathrm{d}\theta\mathrm{d}r \geq \int_0^1 \frac{\pi}{2}\left(\inf R_1\right)^2 \frac{\mathrm{d}r}{r} = +\infty \quad \square$$

In particular, the first order Γ-limit $\mathcal{H}$ has here the constant value $+\infty$, which implies that the minimal value of J_ε cannot approach W_1 with order 1 in ε. This is mainly due to the blow-up of the Sobolev norm at a single point, which suggests a formal but deep analogy with the Ginzburg-Landau theory (see for instance [12, 51, 64]), where we look at the minimization of

$$u \mapsto \frac{1}{\varepsilon^2}\int_\Omega (1-|u|^2) + \int_\Omega |\nabla u|^2$$

with Neumann boundary conditions. Here two terms are contradictory in the functional: the first one suggests to select unit vector fields parallel to the boundary, and the second requires H^1-regularity, which is impossible since the previous point creates a vortex. In our case, up to a 90° rotation, the situation is similar. The two contradictory phenomenas are the fact that T has to preserve the transport rays (to optimize the Monge problem) and that it has a finite Sobolev norm; this also leads to the creation of an explosion (a vortex, rotated by 90°; here the origin, which is sent to a whole curve belonging to the target domain).

Moreover, the two next paragraphs will show that the excess in the approximation of W_1 by $\inf J_\varepsilon$ is of order $\varepsilon|\log\varepsilon|$, which is also a common feature with the Ginzburg-Landau theory (but the analogy is likely to stop here).

3.3.2 Heuristics

In this paragraph, we give a preliminary example of analysis of the behavior of $J_\varepsilon(T_\varepsilon)$ when $\varepsilon \to 0$ and T_ε approaches an optimal map T for the Monge problem; this will not lead directly to a rigorous proof of the general result, but gives an idea of which quantities will appear.

Assume that $T \in \mathcal{O}_1(\mu,\nu)$ with $T(x) = \varphi(r,\theta)\dfrac{x}{|x|}$, and let us build an approximation $(T_\varepsilon)_\varepsilon$ defined by

$$T_\varepsilon(x) = \begin{cases} S(x) & \text{if } x \in \Omega_\delta \\ T(x) & \text{otherwise} \end{cases}$$

where δ will be fixed (depending on ε), $\Omega_\delta = B(0,\delta)\cap\Omega$ and S will be build to send $(f\cdot\mathcal{L}^d)|_{\Omega_\delta}$ onto the same image measure that the original T has on Ω_δ. Notice that S depends indeed both on δ and on ε, but we omit this dependence. In this case, we have

$$J_\varepsilon(T_\varepsilon) - W_1 = \int_\Omega |T_\varepsilon(x)-x|f(x)\,\mathrm{d}x - W_1 + \int_{\Omega_\delta}|DS|^2 + \int_{\Omega\setminus\Omega_\delta}|DT|^2$$

Since T is optimal for the Monge problem and coincides with T_ε outside of Ω_δ, we have

$$\int_\Omega |T_\varepsilon(x)-x|f(x)\,\mathrm{d}x - W_1 = \int_\Omega (|T_\varepsilon(x)-x| - |T(x)-x|)f(x)\,\mathrm{d}x = \int_{\Omega_\delta}(|S(x)-x|-|T(x)-x|)f(x)\,\mathrm{d}x$$

We now claim that

$$\int_{\Omega_\delta}|T(x)-x|f(x)\,\mathrm{d}x = \int_{\Omega_\delta}(|S(x)|-|x|)f(x)\,\mathrm{d}x$$

Indeed, we still have the equality $|T(x)-x| = |T(x)|-|x|$, and the image measures of $(f\cdot\mathcal{L}^d)|_{\Omega_\delta}$ by T and S are the same. As a consequence,

$$\int_\Omega |T_\varepsilon(x)-x|f(x)\,\mathrm{d}x - W_1 = \int_{\Omega_\delta}(|S(x)-x|-|S(x)|+|x|)f(x)\,\mathrm{d}x$$

and, by the triangle inequality, $|S(x)-x|-|S(x)|+|x| \leq |(S(x)-x)-S(x)|+|x| \leq 2|x|$ so that

$$\int_\Omega |T_\varepsilon(x)-x|f(x)\,\mathrm{d}x - W_1 \leq \int_{\Omega_\delta} 2|x|\,\mathrm{d}x \leq 2\int_0^{\frac{\pi}{2}}\int_0^\delta r^2\,\mathrm{d}r\,\mathrm{d}\theta \leq \frac{\pi\delta^3}{3}$$

In order to estimate the norm of the Jacobian matrix DT outside of Ω_δ, we recall that, in the basis $\left(\frac{x}{|x|}, \frac{x}{|x|}^\perp\right)$,

$$DT(x) = \begin{pmatrix} \partial_r\varphi & 0 \\ \dfrac{\partial_\theta\varphi}{r} & \dfrac{\varphi}{r} \end{pmatrix}$$

so that

$$\int_{\Omega\setminus\Omega_\delta} |DT(x)|^2\,\mathrm{d}x = \int_\delta^1\int_0^{\frac{\pi}{2}} \left(\frac{\varphi(r,\theta)^2+\partial_\theta\varphi(r,\theta)^2}{r} + r\partial_r\varphi(r,\theta)^2\right)\mathrm{d}\theta\,\mathrm{d}r$$

Now we note that, for $\theta \in (0,\pi/2)$, the one-dimensional map $\varphi(\cdot,\theta)$ sends the density $rf(\cdot,\theta)$ onto the density $rg(r,\theta)$. The first density is bounded from above (but vanishes around $r=0$), and the second one from below (also from above). Thus, one can assume that $\partial_r\varphi(\cdot,\theta)$ is bounded (by the way, if T is the monotone map along the transport rays, we also have $\partial_r\varphi(r,\theta)\to 0$ as $r\to 0$), so that

$$\int_0^{\frac{\pi}{2}}\int_\delta^1 \partial_r\varphi(r,\theta)^2 r\,\mathrm{d}r\,\mathrm{d}\theta \leq C$$

where C is a constant independent of δ. On the other hand,

$$\int_\delta^1\int_0^{\frac{\pi}{2}} \frac{\varphi(r,\theta)^2+\partial_\theta\varphi(r,\theta)^2}{r}\,\mathrm{d}\theta\,\mathrm{d}r = \int_\delta^1 ||\varphi(r,\cdot)||^2_{H^1(0,\pi/2)}\frac{\mathrm{d}r}{r}$$

In this last integral, we make the change of variable $r=\delta^t$, which gives

$$\int_\delta^1\int_0^{\frac{\pi}{2}} \frac{\varphi(r,\theta)^2+\partial_\theta\varphi(r,\theta)^2}{r}\,\mathrm{d}\theta\,\mathrm{d}r = |\log\delta|\int_0^1 ||\varphi(\delta^t,\cdot)||^2_{H^1(0,\pi/2)}\,\mathrm{d}t$$

We moreover assume that $||\varphi(\delta^t,\cdot)||^2_{H^1(0,\pi/2)} \to ||\varphi(0,\cdot)||^2_{H^1(0,\pi/2)}$ as $\delta\to 0$. This leads to

$$\varepsilon\int_{\Omega\setminus\Omega_\delta}|DT|^2 \underset{\varepsilon,\delta\to 0}{\sim} \varepsilon|\log\delta|||\varphi(0,\cdot)||^2_{H^1(0,\pi/2)} + O(\delta^3+\varepsilon)$$

It remains to estimate the L^2-norm of the Jacobian matrix of S on Ω_δ. We recall that S has to be built so that T_ε, defined on the whole Ω, is still a transport map from μ to ν with finite Sobolev norm; thus, the map S, defined on Ω_δ, must send Ω_δ onto its original image $S(\Omega_\delta)$ in a regular way and keep the constraint on the image measures:

$$S_\#(\mu|_{\Omega_\delta}) = T_\#(\mu|_{\Omega_\delta})$$

Moreover, the regularity of the global map T_ε implies a compatibility condition at the boundary:

$$S(x) = T(x) \qquad \text{for } |x| = \delta$$

Thanks to the Dacorogna-Moser's result (Theorem A.3.1 in the Appendix, see below for its precise use), we are indeed able to build such a map S. Yet, the diameter of Ω_δ is $\sqrt{2}\delta$; on the other hand, $T(\Omega_\delta)$

contains the whole curve $\theta \mapsto \varphi(0,\theta)$, so that its diameter is bounded from below by a positive constant independent of δ. Thus, the best estimate that one can expect is

$$\operatorname{Lip} S \leq \frac{C}{\delta}$$

For a reasonable transport map T, one can show that such a map S can be found with moreover $S(x) = T(x)$ for $|x| = \delta$ (see the paragraph 3.4.2 below). In this case, the global map T_ε still sends μ to ν and we have

$$\int_{\Omega_\delta} |DT|^2 \leq \int_{\Omega_\delta} \left(\frac{C}{\delta}\right)^2 \leq C^2$$

Finally,

$$J_\varepsilon(T_\varepsilon) - W_1 = \varepsilon|\log\delta| ||\varphi(0,\cdot)||_{H^1(0,\pi/2)} + O(\delta^3 + \varepsilon)$$

If we choose $\delta = \varepsilon^{1/3}$, we obtain

$$J_\varepsilon(T_\varepsilon) = W_1 + \varepsilon|\log\varepsilon|\frac{1}{3}||\varphi(0,\cdot)||^2_{H^1(0,\pi/2)} + O(\varepsilon)$$

In particular:

- the first order of convergence of $J_\varepsilon(T_\varepsilon)$ to W_1 is not anymore ε, but $\varepsilon|\log\varepsilon|$;
- the first significant term only involves the behavior of φ around 0, which is the crossing point of all the transport rays (and also the singularity point of the measure restricted to any transport ray, since it vanishes at 0). This suggests that $T_\varepsilon \to T$, where T at $r = 0$ minimizes $||\varphi(r,\cdot)||_{H^1}$.

3.3.3 Main result and consequences

The analysis in the above paragraph suggests to introduce the minimal value of $||\varphi(0,\cdot)||_{H^1(0,\pi/2)}$ among the functions φ such that

$$x \mapsto \varphi(r,\theta)\frac{x}{|x|}$$

is a transport map from μ to ν. In particular, for such a φ and for any $x \in \Omega$, the point $\varphi(r,\theta)\frac{x}{|x|}$ still belongs to the target domain Ω'; thus, its value at $r = 0$ verifies

$$\text{for a.e } \theta \in (0,\pi/2), \quad R_1(\theta) \leq \varphi(0,\theta) \leq R_2(\theta)$$

We will thus set

$$K = \min\left\{\int_0^{\frac{\pi}{2}} (\varphi(\theta)^2 + \varphi'(\theta)^2)\,\mathrm{d}\theta \,:\, \varphi \in H^1\left(0,\frac{\pi}{2}\right),\, R_1(\theta) \leq \varphi(\theta) \leq R_2(\theta)\right\}$$

and, following the ideas of the last sub-section,

$$F_\varepsilon : T \mapsto \frac{1}{\varepsilon}\left(J_\varepsilon(T) - W_1 - \frac{K}{3}\varepsilon|\log\varepsilon|\right)$$

Notice here that the assumption $\inf R_2 > \sup R_1$ allows to remove the constraint $\varphi(\theta) \leq R_2(\theta)$ in the minimization problem defining K (indeed, if $\varphi > R_2$ on a subset of $(0,\pi/2)$, one can replace it with $\min(\varphi, \inf R_2)$ on this subset and this will decrease its Sobolev norm): in other words, we have

$$K = \min\left\{\int_0^{\frac{\pi}{2}} (\varphi(\theta)^2 + \varphi'(\theta)^2)\,\mathrm{d}\theta \,:\, \varphi \in H^1\left(0,\frac{\pi}{2}\right),\, R_1(\theta) \leq \varphi(\theta)\right\}$$

The main result of this chapter is the following:

Theorem 3.3.1. *Let us denote by*

$$G : \varphi \in H^1(0,\pi/2) \mapsto ||\varphi||^2_{H^1(0,\pi/2)} - K$$

and

$$F(T) = \begin{cases} +\infty & \text{if } T \notin \mathcal{O}_1(\mu,\nu) \\ \displaystyle\int_0^1 G(\varphi(r,\cdot))\frac{\mathrm{d}r}{r} + \int_0^1 ||\partial_r \varphi(r,\cdot)||^2_{L^2}\, r\,\mathrm{d}r & \text{if } T \in \mathcal{O}_1(\mu,\nu), T(x) = \varphi(r,\theta)\dfrac{x}{|x|} \end{cases}$$

1. *For any family of maps $(T_\varepsilon)_\varepsilon$ such that $(F_\varepsilon(T_\varepsilon))_\varepsilon$ is bounded, there exists a sequence $\varepsilon_k \to 0$ and a map T such that $T_{\varepsilon_k} \to T$ in $L^2(\Omega)$.*
2. *There exists a constant C, depending only on the domains Ω, Ω' and of the measures f, g, so that, for any family of maps $(T_\varepsilon)_{\varepsilon>0}$ with $T_\varepsilon \to T$ as $\varepsilon \to 0$ in $L^2(\Omega)$, we have*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(T_\varepsilon) \geq F(T) - C$$

3. *Moreover, there exists at least one family $(T_\varepsilon)_{\varepsilon>0}$ such that $(F_\varepsilon(T_\varepsilon))_\varepsilon$ is indeed bounded.*

Notice that we have not stated here a true $\Gamma-$convergence result, but we only provide an estimate on the Γ-liminf, and the existence of a sequence with equibounded energy. We conjecture indeed that the Γ-limit of the sequence F_ε is of the form $F - C$ for a suitable constant C depending on the shape of Ω' and $f(0)$ (again, the main important region is that around $x = 0$ in Ω, which must be sent on the curve Φ); precisely, the constant C that we expect is

$$C = \lim_{\lambda\to+\infty}\left(\inf\left\{\lambda^3 f(0)\int_\Omega \left(|y| - y\cdot\frac{S(y)}{|S(y)|}\right)\mathrm{d}y + \int_\Omega |DS(y)|^2\,\mathrm{d}y - K\log\lambda \,:\, S:\Omega_1\to\Omega'\right\}\right)$$

with may be some additional constraint in the minimization problem for each λ (see below a justification at the end of the paragraph 3.4.1).

However, we do not prove it here and we only use the estimate we prove to get some consequences on the minima and the minimizers of F_ε.

Consequences on the minimal value of J_ε. If we apply the Theorem 3.3.1 to a sequence $(T_\varepsilon)_\varepsilon$ where each T_ε minimizes J_ε (which is equivalent with minimizing F_ε), we obtain that the sequence $(F_\varepsilon(T_\varepsilon))_\varepsilon$ is bounded and

$$\inf J_\varepsilon = F_\varepsilon(T_\varepsilon) = W_1(\mu,\nu) + \frac{K}{3}\varepsilon|\log\varepsilon| + O(\varepsilon)$$

Consequences on the behavior of $(T_\varepsilon)_\varepsilon$. We first note the following:

Lemma 3.3.1. *The problem which defines the constant K, namely*

$$\inf\left\{\int_0^{\frac{\pi}{2}} (\varphi(\theta)^2 + \varphi'(\theta)^2)\,\mathrm{d}\theta \,:\, \varphi \in H^1\left(0,\frac{\pi}{2}\right),\, R_1(\theta) \leq \varphi(\theta) \leq R_2(\theta)\right\}, \tag{3.3}$$

admits a unique minimizer Φ. Moreover, if $\varphi \in H^1(0,\pi/2)$ verifies $R_1 \leq \varphi$, then

$$G(\varphi) \geq ||\varphi - \Phi||^2_{H^1}$$

Proof. Let us denote by $\mathcal{C}$ the set of functions $\varphi \in H^1(0,\pi/2)$ verifying the constraint $R_1 \le \varphi$ on $(0,\pi/2)$. Notice that $\mathcal{C}$ is a convex closed subset of $H^1(0,\pi/2)$, so that (3.3) admits as well a unique minimizer Φ, which is the orthogonal projection of 0 onto $\mathcal{C}$ in the Hilbert space $H^1(0,\pi/2)$. This implies

$$\forall \varphi \in \mathcal{C},\ \langle \Phi, \varphi - \Phi \rangle \ge 0 \tag{3.4}$$

If now $\varphi \in \mathcal{C}$, then

$$G(\varphi) - ||\varphi - \Phi||^2_{H^1} = ||\varphi||^2_{H^1} - ||\Phi||^2_{H^1} - ||\varphi - \Phi||^2_{H^1} = 2\,\langle \Phi, \varphi - \Phi \rangle$$

which is non-negative thanks to the inequality (3.4). □

As a consequence, we obtain:

Proposition 3.3.2. *Let $T \in \mathcal{O}_1(\mu,\nu)$, $T(x) = \varphi(r,\theta)\dfrac{x}{|x|}$, such that $F(T) < +\infty$. Then $r \mapsto \varphi(r,\cdot)$ is continuous from $[0,1]$ to $L^2(0,\pi/2)$, and we have $\varphi(0,\cdot) = \Phi$.*

This, combined with Theorem 3.3.1, implies that that if $(T_\varepsilon)_\varepsilon$ has an equi-bounded energy (meaning that $F_\varepsilon(T_\varepsilon)$ is uniformly bounded in ε), then it has, up to a subsequence, a limit $T = \varphi(r,\theta)\frac{x}{|x|}$ where $\varphi(r,\theta)$ is continuous with respect to r and has $\Phi(\theta)$ as limit as $r \to 0$. In other words, T sends 0 onto the curve $r = \Phi(\theta)$ which has the best H^1-norm among the curves with values in the target domain Ω'. This is in particular true if each T_ε minimizes J_ε (thus F_ε).

Proof of Prop. 3.3.2. The assumption on T implies that the integrals

$$\int_0^1 ||\partial_r \varphi(r,\cdot)||^2_{L^2(0,\pi/2)}\, r\,\mathrm{d}r \qquad \text{and} \qquad \int_0^1 ||\partial_r \varphi(r,\cdot)||^2_{L^2(0,\pi/2)}\, r\,\mathrm{d}r$$

are both controlled by some finite constant A. Now we have for $\theta \in (0,\pi/2)$:

$$|\varphi(r_1,\theta) - \varphi(r_2,\theta)| = \int_{r_1}^{r_2} \partial_r\varphi(r,\theta)\,\mathrm{d}r \le \left(\int_{r_1}^{r_2} \partial_r\varphi(r,\theta)^2\, r\,\mathrm{d}r\right)^{1/2} \left(\int_{r_1}^{r_2} \frac{\mathrm{d}r}{r}\right)^{1/2}$$

$$\text{thus} \qquad \int_0^{\pi/2} |\varphi(r_1,\theta) - \varphi(r_2,\theta)|^2\,\mathrm{d}\theta \le \left(\int_{r_1}^{r_2} ||\partial_r\varphi(r,\cdot)||^2_{L^2(0,\pi/2)}\, r\,\mathrm{d}r\right)\left(\int_{r_1}^{r_2} \frac{\mathrm{d}r}{r}\right)$$

$$\text{so that} \qquad ||\varphi(r_1,\cdot) - \varphi(r_2,\cdot)||^2_{L^2(0,\pi/2)}\,\mathrm{d}\theta \le A\log\frac{r_2}{r_1} \tag{3.5}$$

This proves the continuity of $r \mapsto \varphi(r,\cdot)$.

On the other hand, thanks to the Lemma 3.3.1, we have

$$A \ge \int_0^1 G(\varphi(r,\cdot))\frac{\mathrm{d}r}{r} \ge \int_0^1 ||\varphi(r,\cdot) - \Phi||^2_{L^2(0,\pi/2)}\frac{\mathrm{d}r}{r}$$

By setting $r = e^{-t}$, we obtain

$$A \ge \int_0^{+\infty} ||\varphi(e^{-t},\cdot) - \Phi||^2_{L^2}\,\mathrm{d}t$$

But, for $t_1 < t_2 \in (0,+\infty)$, we have

$$\begin{aligned} ||\varphi(e^{-t_1},\cdot) - \Phi||^2_{L^2} - ||\varphi(e^{-t_2},\cdot) - \Phi||^2_{L^2}| &= |\langle \varphi(e^{-t_1},\cdot) - \varphi(e^{-t^2},\cdot), \Phi\rangle_{L^2}| \\ &\le ||\varphi(e^{-t_1},\cdot) - \varphi(e^{-t_2},\cdot)||\,||\Phi|| \\ &\le A\log\frac{e^{-t_2}}{e^{-t_1}} = A(t_2 - t_1) \end{aligned}$$

where the last inequality comes from (3.5). Thus, the function $t \mapsto ||\varphi(e^{-t},\cdot) - \Phi||^2_{L^2}$ is Lipschitz and belongs to $L^1(0,+\infty)$. This implies that it vanishes at $+\infty$, so that $\varphi(r,\cdot) \to \Phi$ in $L^2(0,\pi/2)$ as $r \to 0$. □

3.4 Proof of Theorem 3.3.1

3.4.1 Γ-liminf estimate

Let $(T_\varepsilon)_\varepsilon$ be a family of maps and let us begin by writing precisely the expression of $F_\varepsilon(T_\varepsilon)$ for any $\varepsilon > 0$ and any transport map T_ε. We have

$$F_\varepsilon(T_\varepsilon) = \frac{1}{\varepsilon}\left(\int_\Omega |T_\varepsilon(x) - x| f(x)\,\mathrm{d}x - W_1\right) + \int_\Omega |DT_\varepsilon(x)|^2\,\mathrm{d}x - \frac{K}{3}|\log\varepsilon|$$

We set $\delta = \varepsilon^{1/3}$ and notice that

$$\frac{K}{3}|\log\varepsilon| = K|\log\delta| = K\int_\delta^1 \frac{\mathrm{d}r}{r}$$

We decompose T_ε into radial and tangential components:

$$T_\varepsilon(x) = \varphi_\varepsilon(r,\theta)\frac{x}{|x|} + \psi_\varepsilon(r,\theta)\frac{x^{\perp}}{|x|}$$

and compute

$$DT_\varepsilon = \frac{x}{|x|}\otimes\nabla\varphi_\varepsilon(x) + \frac{\varphi_\varepsilon(x)}{|x|}\left(I_d - \frac{x}{|x|}\otimes\frac{x}{|x|}\right) + \frac{x^{\perp}}{|x|}\otimes\nabla\psi_\varepsilon(x) + \frac{\psi_\varepsilon(x)}{|x|}\left(R - \frac{x^{\perp}}{|x|}\otimes\frac{x}{|x|}\right)$$

where R denotes the rotation with angle $\pi/2$ and we still set $x^\perp = Rx$. Thus, the matrix of DT_ε in the basis $\left(\frac{x}{|x|}, \frac{x^{\perp}}{|x|}\right)$ is

$$DT_\varepsilon(x) = \begin{pmatrix} \partial_r\varphi_\varepsilon & \partial_r\psi_\varepsilon \\ \dfrac{\partial_\theta\varphi_\varepsilon - \psi_\varepsilon}{r} & \dfrac{\varphi_\varepsilon + \partial_\theta\psi_\varepsilon}{r} \end{pmatrix}$$

so that

$$|DT_\varepsilon|^2 = \partial_r\varphi_\varepsilon^2 + \partial_r\psi_\varepsilon^2 + \frac{(\partial_\theta\varphi_\varepsilon - \psi_\varepsilon)^2}{r^2} + \frac{(\varphi_\varepsilon + \partial_\theta\psi_\varepsilon)^2}{r^2}$$

We obtain

$$\int_\Omega |DT_\varepsilon|^2 = \int_{\Omega_\delta} |DT_\varepsilon|^2 + \int_\delta^1 (||\partial_r\varphi_\varepsilon(r,\cdot)||_{L^2}^2 + ||\partial_r\psi_\varepsilon||_{L^2}^2)\, r\,\mathrm{d}r + \int_\delta^1 (||\partial_\theta\varphi_\varepsilon - \psi_\varepsilon||_{L^2}^2 + ||\varphi_\varepsilon + \partial_\theta\psi_\varepsilon||_{L^2}^2)\,\frac{\mathrm{d}r}{r}$$

On the other hand, we already know that

$$\int_\Omega |T_\varepsilon(x) - x| f(x)\,\mathrm{d}x - W_1 = \int_\Omega (|T_\varepsilon(x) - x| - |T_\varepsilon(x)| + |x|) f(x)\,\mathrm{d}x$$

Finally, the complete expression of F_ε is the following:

$$\begin{aligned} F_\varepsilon(T_\varepsilon) = &\frac{1}{\varepsilon}\int_\Omega (|T_\varepsilon(x) - x| - |T_\varepsilon(x)| + |x|) f(x)\,\mathrm{d}x + \int_{\Omega_\delta} |DT_\varepsilon|^2 \\ &+ \int_\delta^1 (||(\partial_\theta\varphi_\varepsilon - \psi_\varepsilon)(r,\cdot)||_{L^2}^2 + ||(\varphi_\varepsilon + \partial_\theta\psi_\varepsilon)(r,\cdot)||_{L^2}^2 - K)\,\frac{\mathrm{d}r}{r} + \int_\delta^1 (||\partial_r\varphi_\varepsilon(r,\cdot)||_{L^2}^2 + ||\partial_r\psi_\varepsilon||_{L^2}^2)\, r\,\mathrm{d}r \end{aligned}$$

thus, if we denote by $H(\varphi,\psi) = \int_0^{\pi/2} (\varphi'(\theta) - \psi(\theta))^2 + (\varphi(\theta) + \psi'(\theta))^2 \,\mathrm{d}\theta$ for $\varphi,\psi \in H^1(0,\pi/2)$, we have

$$F_\varepsilon(T_\varepsilon) = \frac{1}{\varepsilon}\int_\Omega (|T_\varepsilon(x) - x| - |T_\varepsilon(x)| + |x|) f(x)\,\mathrm{d}x + \int_{\Omega_\delta} |DT_\varepsilon|^2 + \int_\delta^1 (H(\varphi_\varepsilon(r,\cdot),\psi_\varepsilon(r,\cdot)) - K)\,\frac{\mathrm{d}r}{r} + \int_\delta^1 (||\partial_r\varphi_\varepsilon(r,\cdot)||_{L^2}^2 + ||\partial_r\psi_\varepsilon||_{L^2}^2)\, r\,\mathrm{d}r \quad (3.6)$$

The following lemma collects some properties of the function H.

Lemma 3.4.1. *We recall that*

$$H : (\varphi,\psi) \in H^1(0,\pi/2) \times H^1(0,\pi/2) \mapsto ||\varphi' - \psi||_{L^2}^2 + ||\varphi + \psi'||_{L^2}^2$$

for $\varphi,\psi \in H^1(0,\pi/2)$. Then:

- *The function H is lower semi-continuous with respect to the strong L^2-convergence.*
- *Assume that (φ,ψ) satisfies, for any θ,*

$$\varphi(\theta)\hat{x}(\theta) + \psi(\theta)\hat{x}^\perp(\theta) \in \Omega'$$

where $\hat{x}(\theta) = (\cos\theta, \sin\theta)$. We denote by $\widetilde{\varphi}(\theta) = \max(\varphi(\theta), R_1(\theta))$. Then we have the inequality

$$H(\varphi,\psi) \geq K + \frac{1}{2}||\widetilde{\varphi} - \Phi||_{L^2}^2 - B||\psi||_{L^2(0,\pi/2)}^{2/3} \quad (3.7)$$

for some positive constant B which only depends on Ω'.

Proof. Step 1: the semi-continuity of H. We take a sequence $(\varphi_n,\psi_n)_n$ converging to some (φ,ψ) for the L^2-norm. Up to subsequences, we can assume that

$$\liminf_{n\to+\infty} H(\varphi_n,\psi_n) = \lim_{n\to+\infty} H(\varphi_n,\psi_n)$$

and we also assume that $(H(\varphi_n,\psi_n))_n$ is bounded. Now we remark that

$$H(\varphi_n,\psi_n) = ||\varphi_n' - \psi_n||_{L^2}^2 + ||\varphi_n + \psi_n'||_{L^2}^2 \geq (||\varphi_n'||_{L^2} - ||\psi_n||_{L^2})^2 + (||\varphi_n'||_{L^2} - ||\psi_n||_{L^2}^2)^2$$

$$\text{thus} \qquad ||\varphi_n'||_{L^2} \leq \sqrt{H(\varphi_n,\psi_n)} + ||\psi_n||_{L^2} \qquad \text{and} \qquad ||\psi_n'||_{L^2} \leq \sqrt{H(\varphi_n,\psi_n)} + ||\varphi_n||_{L^2}$$

We deduce that $(\varphi_n)_n$, $(\psi_n)_n$ are bounded in $H^1(0,\pi/2)$ so that the convergence $(\varphi_n,\psi_n) \to (\varphi,\psi)$ actually holds, up to a subsequence, weakly in $H^1(0,\pi/2)$. Now the convexity of $(\varphi,\varphi',\psi,\psi') \mapsto (\varphi' - \psi)^2 + (\varphi+\psi')^2$ implies that H is lower semi-continuous with respect to the weak convergence in $H^1(0,\pi/2)$, which allows to conclude.

Now we pass to the proof of the inequality (3.7). We begin by a kind of "sub-lemma" which will be useful several times in the proof.

Step 2: preliminary estimates. We recall that $\widetilde{\varphi} = \max(R_1,\varphi)$ and denote by $h = \widetilde{\varphi} - \varphi \geq 0$. Now we claim that:

- for any $t \in (0,\pi/2)$, we have the inequality

$$0 \leq h(t) \leq B_1|\psi(t)| \quad (3.8)$$

for some constant B_1 depending only on Ω'.

- we have the inequality
$$|\langle\widetilde{\varphi}, h\rangle_{H^1}| \leq B_2||h||_\infty \tag{3.9}$$
for some constant B_2 depending only on Ω';
- the above both inequalities lead to the estimate
$$||\varphi||^2_{H^1} \geq K + ||\widetilde{\varphi} - \Phi||^2_{H^1} - B_3||\psi||_\infty \tag{3.10}$$
for some constant B_3 depending only on Ω'.

First, we remark that the constraint $\varphi(\theta)\hat{x}(\theta) + \psi(\theta)\hat{x}^\perp(\theta) \in \Omega'$ implies

$$R_1(\theta')^2 < \varphi(\theta)^2 + \psi(\theta)^2 < R_2(\theta')^2 \qquad \text{where} \qquad \theta' = \theta + \arcsin\frac{\psi(\theta)}{\sqrt{\varphi(\theta)^2+\psi(\theta)^2}}$$

Thus, we have
$$\begin{aligned} h(\theta) &= R_1(\theta) - \varphi(\theta) \\ &= R_1(\theta) - R_1(\theta') + R_1(\theta') - \varphi(\theta) \\ &\leq (\operatorname{Lip} R_1)|\theta - \theta'| + \sqrt{\varphi^2(\theta)+\psi^2(\theta)} - \varphi(\theta) \\ &\leq (\operatorname{Lip} R_1)\arcsin\frac{|\psi(\theta)|}{R_1(\theta')} + |\psi(\theta)| \\ &\leq \left(\frac{\pi}{2}\frac{\operatorname{Lip} R_1}{\inf R_1} + 1\right)|\psi(\theta)| \end{aligned}$$
which is (3.8) with $B_1 = \dfrac{\pi}{2}\dfrac{\operatorname{Lip} R_1}{\inf R_1} + 1$.

Second, we recall that $h = (R_1 - \varphi)^+$, thus $\varphi + h = R_1$ on any point where $h \neq 0$. This leads to

$$\left|\int_0^{\pi/2}(\varphi+h)h\right| = \left|\int_0^{\pi/2} R_1 h\right| \leq \frac{\pi}{2}(\sup R_1)||h||_\infty$$

and $$\left|\int_0^{\pi/2}(\varphi+h)'h'\right| = \left|\int_0^{\pi/2} R_1'h'\right| = \left|[R_1'h]_0^{\pi/2} - \int_0^{\pi/2} R_1''h\right| \leq (2\sup R_1' + ||R_1''||_1)||h||_\infty$$

We get (3.9) with $B_2 = \left(\dfrac{\pi}{2}\sup R_1 + 2\operatorname{Lip} R_1 + ||R_1''||_{L^1}\right)$.

Third, we write
$$||\varphi||^2_{H^1} = ||\widetilde{\varphi}||^2_{H^1} + ||h||^2_{H^1} + 2\langle\widetilde{\varphi}, h\rangle \tag{3.11}$$
Since $\widetilde{\varphi} \geq R_1$ on $(0, \pi/2)$ and thanks to the Lemma 3.3.1, we have $||\widetilde{\varphi}||^2_{H^1} \geq ||\widetilde{\varphi} - \Phi||^2_{H^1} + K$. On the other hand, by using (3.8) and (3.9), we have

$$\langle\widetilde{\varphi}, h\rangle \geq -B_2||h||_\infty \geq -B_1B_2||\psi||_\infty$$

We insert into (3.11) and skip $||h||^2_{H^1}$ since it is nonnegative to get

$$||\varphi||^2_{H^1} \geq K + ||\widetilde{\varphi} - \Phi||^2_{H^1} - 2B_1B_2||\psi||_\infty$$

thus (3.10) holds with $B_3 = 2B_1B_2$.

Step 3: the inequality (3.7) *holds if* $||\varphi'||_{L^2}$ *is large enough.* We start from

$$H(\varphi,\psi) = ||\varphi||^2_{H^1} + ||\psi||^2_{H^1} + 2\int_0^{\pi/2}\varphi\psi' - 2\int_0^{\pi/2}\psi\varphi' = ||\varphi||^2_{H^1} + ||\psi||^2_{H^1} - 4\int_0^{\pi/2}\psi\varphi' - 2[\varphi\psi]_0^{\pi/2}$$

First, the condition on (φ,ψ) implies that $||\varphi||_\infty, ||\psi||_\infty \leq \sup R_2$ so that

$$|[\varphi\psi]_0^{\pi/2}| \leq 2(\sup R_2)^2$$

On the other hand,

$$\left|\int_0^{\pi/2} \psi\varphi'\right| \leq ||\psi||_\infty\sqrt{\pi/2}||\varphi'||_{L^2} \leq \sup R_2\sqrt{\pi/2}||\varphi'||_{L^2}$$

This leads to

$$\begin{aligned} H(\varphi,\psi) &\geq ||\varphi||_{H^1}^2 + ||\psi||_{H^1}^2 - 4\sup R_2\sqrt{\pi/2}||\varphi'||_{L^2} - 4(\sup R_2)^2 \\ &\geq \frac{1}{2}||\varphi||_{H^1}^2 + \left(\frac{1}{2}||\varphi'||_{L^2}^2 - 4\sup R_2\sqrt{\pi/2}||\varphi'||_{L^2} - 4(\sup R_2)^2\right) \end{aligned}$$

By using (3.10), we obtain

$$H(\varphi,\psi) \geq \frac{1}{2}(K + ||\widetilde{\varphi} - \Phi||_{H^1}^2 - B_3||\psi||_\infty) + \left(\frac{1}{2}||\varphi'||_{L^2}^2 - 4\sup R_2\sqrt{\pi/2}||\varphi'||_{L^2} - 4(\sup R_2)^2\right)$$

and, since $|\psi| \leq \sqrt{\varphi^2+\psi^2} \leq R_1$, we have

$$H(\varphi,\psi) \geq \frac{1}{2}(K + ||\widetilde{\varphi} - \Phi||_{H^1}) + \left(\frac{1}{2}||\varphi'||_{L^2}^2 - 4\sup R_2\sqrt{\pi/2}||\varphi'||_{L^2} - \left(4(\sup R_2)^2 + \frac{B_3}{2}\sup R_2\right)\right)$$

The announced estimate (3.7) holds provided that the term in brackets is greater that $\frac{K}{2}$, which is true provided that $||\varphi'||_{L^2} \geq B_4$ where B_4 is the largest root of the polynom

$$\frac{1}{2}X^2 - 4\sup\ R_2\sqrt{\pi/2}X - \left(4(\sup R_2)^2 + \frac{B_3}{2}\sup R_2 + \frac{K}{2}\right)$$

and B_4 only depends of Ω'.

Step 4: case $||\varphi'||_{L^2} \leq B_4$. In this case, we still have

$$H(\varphi,\psi) = ||\varphi||_{H^1}^2 + ||\psi||_{H^1}^2 - 4\int_0^{\pi/2}\psi\varphi' - 2[\varphi\psi]_0^{\pi/2}$$

$$\text{with} \quad \left|\int_0^{\pi/2}\psi\varphi'\right| \leq ||\psi||_\infty\sqrt{\pi/2}||\varphi'||_{L^2} \leq \sqrt{\pi/2}B_4||\psi||_\infty$$

$$\text{and} \quad |[\varphi\psi]_0^{\pi/2}| \leq 2||\varphi||_\infty||\psi||_\infty \leq 2\sup R_2||\psi||_\infty$$

This leads to

$$\begin{aligned} H(\varphi,\psi) &\geq ||\varphi||_{H^1}^2 + ||\psi||_{H^1}^2 - \left(\sqrt{\pi/2}B_4 + 2\sup R_2\right)||\psi||_\infty \\ &\geq K + ||\widetilde{\varphi} - \Phi||_{H^1}^2 + ||\psi||_{H^1}^2 - B_5||\psi||_\infty \end{aligned}$$

where we have again used (3.10) and set $B_5 = \left(\sqrt{\pi/2}B_4 + 2\sup R_2\right) + B_3$, which only depends on Ω'.

It now remains to estimate $||\psi||_{H^1}^2 - B_5||\psi||_\infty$ from below with $-||\psi||_{L^2}^{2/3}$. The condition on (φ,ψ) implies that $\psi(0) \geq 0$ and $\psi(\pi/2) \leq 0$, so that there exists t_0 such that $\psi(t_0) = 0$. We then have

$$\psi^2(t) = \int_{t_0}^t \frac{\mathrm{d}}{\mathrm{dt}}(\psi^2) = \int_{t_0}^t 2\psi\psi' \leq 2||\psi||_{L^2}||\psi'||_{L^2} \quad \text{thus} \quad ||\psi||_\infty \leq \sqrt{2}\sqrt{||\psi||_{L^2}||\psi'||_{L^2}}$$

We use the Young inequality

$$ab \leq \frac{(\varepsilon a)^p}{p} + \frac{(b/\varepsilon)^q}{q} \qquad \text{for } \frac{1}{p} + \frac{1}{q} = 1 \text{ and } \varepsilon > 0$$

with $p = 4$, $q = 4/3$, $a = \sqrt{||\psi'||_{L^2}}$ and $b = \sqrt{||\psi||_{L^2}}$, to get

$$||\psi||_\infty \leq \frac{\sqrt{2}\varepsilon^4}{4}||\psi'||_{L^2}^2 + \frac{3\sqrt{2}}{4\varepsilon^{4/3}}||\psi||_{L^2}^{2/3}$$

We deduce

$$||\psi'||_{H^1}^2 - B_5||\psi||_\infty \geq \left(1 - \frac{\sqrt{2}B_5\varepsilon^4}{4}\right)||\psi'||_{L^2} - \frac{3\sqrt{2}B_5}{4\varepsilon^{4/3}}||\psi||_{L^2}^{2/3}$$

By choosing ε such that $\frac{\sqrt{2}B_5\varepsilon^4}{4} = 1$, we obtain

$$||\psi||_{H^1}^2 - B_5||\psi||_\infty \geq -B||\psi||_{L^2}^{2/3}$$

where $B = \frac{3\sqrt{2}B_5}{4\varepsilon^{4/3}}$ only depends on Ω' and K. This achieves the proof. □

We also will need the following estimate on the first term of the expression (3.6).

Lemma 3.4.2. *Let T be a transport map from μ to ν. We write*

$$T(x) = \varphi(x)\frac{x}{|x|} + \psi(x)\frac{x^{\perp}}{|x|}$$

Then, for a.e. x,

$$|T(x) - x| - |T(x)| + |x| \geq A|x|\psi^2(x) \tag{3.12}$$

for some constant A which only depends on Ω'.

Proof. We compute:

$$|T(x) - x| - |T(x)| + |x| = \frac{|T(x) - x|^2 - |T(x)|^2}{|T(x) - x| + |T(x)|} + |x|$$

We have $|T(x) - x|^2 = (\varphi(x) - |x|)^2 + \psi(x)^2$ and $|T|^2 = \varphi^2 + \psi^2$, so that

$$|T(x) - x| - |T(x)| + |x| = \frac{|x|^2 - 2|x|\varphi(x)}{|T(x) - x| + |T(x)|} + |x| = |x|\,\frac{|x| + |T(x) - x| + |T(x)| - 2\varphi(x)}{|T(x) - x| + |T(x)|}$$

We remark that

$$|T(x) - x| - \varphi(x) + |x| = \sqrt{(\varphi(x) - |x|)^2 + \psi(x)^2} - (\varphi(x) - |x|) \geq 0$$

$$\text{thus} \quad |T(x) - x| - |T(x)| + |x| \geq |x|\frac{|T(x)| - \varphi(x)}{|T(x) - x| + |T(x)|} = |x|\frac{|T(x)|^2 - \varphi(x)^2}{(|T(x) - x| + |T(x)|)(|T(x)| + \varphi(x))}$$

Since $x \in \Omega$ and $T(x) \in \Omega'$, we have

$$|T(x) - x| + |T(x)| \leq 2|T(x)| + |x| \leq 2\sup R_2 + 1$$

$$\text{and} \qquad |T(x)| + \varphi(x) \leq 2|T(x)| \leq 2\sup R_2$$

On the other hand, $|T(x)|^2 - \varphi(x)^2 = \psi(x)^2$. This leads to the result with $A = \frac{1}{(2\sup R_2 + 1)(2\sup R_2)}$

□

The estimate (3.12) leads to

$$\int_\Omega (|T_\varepsilon(x) - x| - |T_\varepsilon(x)| + |x|) f(x)\,\mathrm{d}x \geq A \inf f \int_0^1 ||\psi_\varepsilon(r,\cdot)||_{L^2}^2\, r\,\mathrm{d}r$$

and the estimate (3.7) to

$$\int_\delta^1 (H(\varphi_\varepsilon(r,\cdot),\psi_\varepsilon(r,\cdot)) - K)\frac{\mathrm{d}r}{r} \geq \int_\delta^1 \left(-B||\psi_\varepsilon(r,\cdot)||_{L^2}^{2/3} + \frac{1}{2}||\widetilde{\varphi}_\varepsilon(r,\cdot) - \Phi||_{H^1}^2\right)\frac{\mathrm{d}r}{r}$$

where we again have set $\widetilde{\varphi}_\varepsilon = \max(R_1, \varphi_\varepsilon)$. By inserting into (3.6), we have

$$\begin{aligned} F_\varepsilon(T_\varepsilon) \geq \frac{A \inf f}{\varepsilon} \int_0^1 ||\psi_\varepsilon(r,\cdot)||_{L^2}^2\, r^2\,\mathrm{d}r - B\int_\delta^1 ||\psi_\varepsilon(r,\cdot)||_{L^2}^{2/3}\frac{\mathrm{d}r}{r} \\ + \frac{1}{2}\int_\delta^1 (H(\varphi_\varepsilon(r,\cdot),\psi_\varepsilon(r,\cdot)) - K + B||\psi_\varepsilon(r,\cdot)||_{L^2}^{2/3})\frac{\mathrm{d}r}{r} + \int_\delta^1 ||\partial_r\varphi_\varepsilon(r,\cdot)||_{L^2}^2\, r\,\mathrm{d}r \end{aligned} \tag{3.13}$$

Let us denote by $X_\varepsilon = \dfrac{1}{\varepsilon}\displaystyle\int_0^1 ||\psi_\varepsilon(r,\cdot)||_{L^2}^2 r^2\,\mathrm{d}r$. By the Hölder inequality applied with respect to the measure with density $1/r$ on $(\delta, 1)$, we have

$$\int_\delta^1 ||\psi_\varepsilon(r,\cdot)||_{L^2}^{2/3}\frac{\mathrm{d}r}{r} = \int_\delta^1 (||\psi_\varepsilon(r,\cdot)||_{L^2}^2 r^3)^{1/3}\frac{1}{r}\frac{\mathrm{d}r}{r} \leq \left(\int_\delta^1 ||\psi_\varepsilon||_{L^2}^2 r^3\frac{\mathrm{d}r}{r}\right)^{1/3}\left(\int_\delta^1 \frac{1}{r^{4/3}}\frac{\mathrm{d}r}{r}\right)^{3/4}$$

$$\text{with} \quad \int_\delta^1 ||\psi_\varepsilon||_{L^2}^2 r^3\frac{\mathrm{d}r}{r} \leq \varepsilon X_\varepsilon \quad \text{and} \quad \int_\delta^1 \frac{1}{r^{4/3}}\frac{\mathrm{d}r}{r} = \frac{3}{4}\left(\frac{1}{\delta^{4/3}} - 1\right) \leq \frac{3}{2\delta^{4/3}}$$

which leads to

$$\int_\delta^1 ||\psi_\varepsilon(r,\cdot)||_{L^2}^{2/3}\frac{\mathrm{d}r}{r} \leq (\varepsilon X_\varepsilon)^{1/3}\left(\frac{3}{2\delta^{4/3}}\right)^{3/4} = \sqrt{\frac{3\sqrt{3}}{2\sqrt{2}}}\,X_\varepsilon^{1/3}$$

since $\delta = \varepsilon^{1/3}$. We insert into (3.13) to obtain

$$\begin{aligned} F_\varepsilon(T_\varepsilon) \geq (A\inf f)X_\varepsilon - B'\,X_\varepsilon^{1/3} \\ + \int_\delta^1 (H(\varphi_\varepsilon(r,\cdot),\psi_\varepsilon(r,\cdot)) - K + B||\psi_\varepsilon(r,\cdot)||_{L^2}^{2/3})\frac{\mathrm{d}r}{r} + \int_\delta^1 ||\partial_r\varphi_\varepsilon(r,\cdot)||_{L^2}^2\, r\,\mathrm{d}r \end{aligned} \tag{3.14}$$

where $B' = \sqrt{\frac{3\sqrt{3}}{2\sqrt{2}}}\,B$.

Let us assume that $(F_\varepsilon(T_\varepsilon))_\varepsilon$ is bounded by a positive constant M. This implies that $(X_\varepsilon)_\varepsilon$ is bounded by some constant M' (otherwise the term $(A\inf f)X_\varepsilon - B'\,X_\varepsilon^{1/3}$ would tend to $+\infty$, and the other term is positive), thus

$$\int_0^1 ||\psi_\varepsilon(r,\cdot)||_{L^2}^2 r^2\,\mathrm{d}r \leq M'\varepsilon$$

and $\psi_\varepsilon \to 0$ a.e. on Ω. Since $(X_\varepsilon)_\varepsilon$ and $(F_\varepsilon(T_\varepsilon))_\varepsilon$ are bounded, (3.14) provides

$$\int_\delta^1 (H(\varphi_\varepsilon(r,\cdot),\psi_\varepsilon(r,\cdot)) - K + B||\psi_\varepsilon(r,\cdot)||_{L^2}^{2/3}\frac{\mathrm{d}r}{r} + \int_\delta^1 ||\partial_r\varphi_\varepsilon(r,\cdot)||_{L^2}^2\, r\,\mathrm{d}r \leq M''$$

for some constant M'' which does not depend on ε. We now use the estimate (3.7) to get

$$\frac{1}{2}\int_\delta^1 ||\widetilde{\varphi}_\varepsilon - \Phi||_{H^1}^2 \,\frac{\mathrm{d}r}{r} + \int_\delta^1 ||\partial_r\varphi_\varepsilon(r,\cdot)||_{L^2}^2\, r\,\mathrm{d}r \le M''$$

We thus have a L^2_{loc}-loc bound $\partial_r\varphi_\varepsilon$ and on $\partial_\theta\widetilde{\varphi}_\varepsilon$, but since $\widetilde{\varphi}(r,\theta) = \max(R_1(\theta), \varphi(r,\theta))$, the bound on $\partial_r\varphi_\varepsilon$ implies a bound on $\partial_r\widetilde{\varphi}_\varepsilon$. Thus, the family $(\widetilde{\varphi}_\varepsilon)_\varepsilon$ is bounded in $H^1_{loc}(\Omega)$ and there exists $\varepsilon_k \to 0$ and $\widetilde{\varphi}$ such that $\widetilde{\varphi}_\varepsilon \to \widetilde{\varphi}$ a.e. on Ω. But we recall that the estimation (3.8) still holds and provides

$$|\varphi_{\varepsilon_k} - \widetilde{\varphi}_{\varepsilon_k}| \le B_1|\psi_\varepsilon| \to 0$$

which leads $\varphi_{\varepsilon_k} \to \widetilde{\varphi}$ a.e. on Ω. If we set now $T(x) = \widetilde{\varphi}\dfrac{x}{|x|}$, we have proved that $T_{\varepsilon_k} \to T$ a.e. on Ω. This convergence also holds in $L^2(\Omega)$ since $|T_\varepsilon| \le \sup R_2$ for any ε. This proves the first statement of the Theorem 3.3.1.

Assume now that $(T_\varepsilon)_\varepsilon$ is a family of transport maps converging to some T for the L^2-norm on Ω. We deduce from (3.14) that, if we set $C = -\inf_{X>0}(A \inf f X^3 - BX)$, which only depends on Ω', we have

$$F_\varepsilon(T_\varepsilon) \ge -C + \int_\delta^1 (H(\varphi_\varepsilon(r,\cdot),\psi_\varepsilon(r,\cdot)) - K + B||\psi_\varepsilon(r,\cdot)||_{L^2}^{2/3})\,\frac{\mathrm{d}r}{r} + \int_\delta^1 ||\partial_r\varphi_\varepsilon(r,\cdot)||_{L^2}^2\, r\,\mathrm{d}r$$

Assuming that $(F_\varepsilon(T_\varepsilon))_\varepsilon$ is bounded, the above computations give a H^1-loc bound for $(\varphi_\varepsilon)_\varepsilon$, thus

$$\liminf_{\varepsilon\to 0} \int_\delta^1 ||\partial_r\varphi_\varepsilon(r,\cdot)||_{L^2}^2\, r\,\mathrm{d}r \ge \int_0^1 ||\partial_r\varphi(r,\cdot)||_{L^2}^2\, r\,\mathrm{d}r$$

since this functional is lower semi-continuous for the weak convergence in $H^1(\Omega)$. On the other hand, the estimate (3.7) shows also that

$$H(\varphi_\varepsilon(r,\cdot),\psi_\varepsilon(r,\cdot)) - K + B||\psi_\varepsilon(r,\cdot)||_{L^2}^{2/3} \ge 0$$

for any ε and r, thus we can apply the Fatou lemma since the semi-continuity of H provides

$$\liminf_{k\to+\infty}(H(\varphi_\varepsilon(r,\cdot),\psi_\varepsilon(r,\cdot)) - K + B||\psi_\varepsilon(r,\cdot)||_{L^2}^{2/3}) \ge G(\varphi(r,\cdot))$$

for a.e. $r \in (0,1)$. Thus

$$\liminf_{\varepsilon\to 0} F_\varepsilon(T_\varepsilon) \ge -C + \int_0^1 G(\varphi(r,\cdot))\,\frac{\mathrm{d}r}{r} + \int_0^1 ||\partial_r\varphi(r,\cdot)||_{L^2}^2\, r\,\mathrm{d}r$$

as announced.

Remark. If we choose to set $\delta = \lambda\varepsilon^{1/3}$, where λ is to be fixed depending on ε, the precise expression (3.6) becomes

$$\begin{aligned} F_\varepsilon(T_\varepsilon) = \frac{1}{\varepsilon}\int_{\Omega_\delta} (|T_\varepsilon(x) - x| - |T(x)| + |x|) f(x)\,\mathrm{d}x + \int_{\Omega_\delta} |DT_\varepsilon|^2 - K\log\lambda \\ + \frac{1}{\varepsilon}\int_{\Omega\setminus\Omega_\delta} (|T_\varepsilon(x) - x| - |T(x)| + |x|) f(x)\,\mathrm{d}x - B\int_\delta^1 ||\psi_\varepsilon(r,\cdot)||_{L^2}^{2/3}\, r\,\mathrm{d}r \\ + \int_\delta^1 (H(\varphi_\varepsilon(r,\cdot),\psi_\varepsilon(r,\cdot)) - K + B||\psi_\varepsilon(r,\cdot)||_{L^2}^{2/3})\frac{\mathrm{d}r}{r} + \int_\delta^1 ||\partial_r\varphi_\varepsilon(r,\cdot)||_{L^2}^2\, r\,\mathrm{d}r \end{aligned} \tag{3.15}$$

By using the above estimates (3.7) and (3.12), we get that the second line is this time bounded from below by

$$A \inf f X_\varepsilon - \frac{B}{\lambda}(X_\varepsilon)^{1/3}$$

which is itself bounded from below by $-C_\lambda = \inf\{A \inf f X - \frac{B}{\lambda} X^{1/3}\}$, which goes to 0 as $\lambda \to +\infty$. Thus, these terms disappear if λ goes to $+\infty$ as $\varepsilon \to 0$. Let us now compute the first line of (3.15):

$$\frac{1}{\varepsilon}\int_{\Omega_\delta}(|T_\varepsilon(x)-x| - |T(x)| + |x|)f(x)\,\mathrm{d}x + \int_{\Omega_\delta}|DT_\varepsilon|^2 - K\log\lambda$$
$$= \frac{\delta^2}{\varepsilon}\int_\Omega(|U_\varepsilon(y)-\delta y| - |U_\varepsilon(y)| + |\delta y|)f(\delta y)\,\mathrm{d}y + \int_\Omega |DU_\varepsilon|^2 - K\log\lambda$$

where we have set $x = \delta y$ and $U_\varepsilon(y) = T_\varepsilon(\delta y)$. Now we use the following expansion

$$|U_\varepsilon(y) - \delta y| - |U_\varepsilon(y)| = \delta\left(|y| - y\cdot\frac{U_\varepsilon(y)}{|U_\varepsilon(y)|}\right) + o(\delta^2)$$

and recall that $\delta = \lambda\varepsilon^{1/3}$, to get

$$\frac{\delta^2}{\varepsilon}\int_\Omega |U_\varepsilon(y)-\delta y| - |U_\varepsilon(y)| + |\delta y| f(\delta y)\,\mathrm{d}y + \int_\Omega |DU_\varepsilon|^2 - K\log\lambda$$
$$= \int_\Omega \lambda^3\left(|y| - y\cdot\frac{U_\varepsilon(y)}{|U_\varepsilon(y)|}\right)f(\delta y)\,\mathrm{d}y + \int_\Omega |DU_\varepsilon|^2 - K\log\lambda + o\left(\lambda^4\varepsilon^{4/3}\right)$$

We can bound this term from below (up to the rest with order $\lambda^4\varepsilon^{1/3}$, that we can make going to 0 with a good choice of λ, and another rest taking account of $f(\delta y) \to f(0)$) with

$$-C'_\lambda = \inf\left\{f(0)\int_\Omega \lambda^3\left(|y| - y\cdot\frac{U(y)}{|U(y)|}\right)\mathrm{d}y + \int_\Omega |DU|^2 - K\log\lambda\right\}$$

where the constraint on U has to be precised and is satisfied by U_ε. For a good choice of λ, as $\varepsilon \to 0$, we skip the first constant $-C_\lambda$ and get, since the third line of (3.15) is lower semi-continuous,

$$\liminf_{\varepsilon\to 0} F_\varepsilon \geq F(T) - \lim_{\lambda\to+\infty} C'_\lambda$$

which correspond to the result that we conjectured in order to make more precise the theorem 3.3.1.

3.4.2 Construction of family of transport maps with equi-bounded energy

The last point of the proof of Theorem 3.3.1 consists in building a family of maps $(T_\varepsilon)_\varepsilon$ such that $(F_\varepsilon(T_\varepsilon))_\varepsilon$ is as well bounded. The sketch of the proof is the following: we start from a transport map $T = \varphi\dfrac{x}{|x|}$ with $\varphi(0,\cdot) = \Phi$ (that we call "the original T" in the following), assume that φ is regular except around the origin, and modify T only on Ω_δ.

Step 1: construction of the original transport map. We set $T(x) = \varphi(r,\theta)\dfrac{x}{|x|}$, where φ is built as follows:

- $\varphi(0,\theta) = \Phi(\theta)$, and $\varphi(\cdot,\theta)$ is increasing and sends the one-dimensional measure μ_θ (the starting measure μ concentrated on the transport ray with angle θ) onto $\nu_\theta/2$ (where ν_θ is the target measure on the same transport ray), until the radius ρ_1 such that $\varphi(\rho_1,\theta) = R_2(\theta)$;

- starting from this radius ρ_1, $\varphi(\cdot,\theta)$ is decreasing with the same source and target measure, until the radius ρ_2 such that, again, $\varphi(\rho_2,\theta)=\Phi(\theta)$. Therefore, on the interval (ρ_1,ρ_2), $\varphi(\cdot,\theta)$ sends μ_θ onto $\nu_\theta|_{(\Phi(\theta),R_2(\theta))}$;
- on the last interval (if it is non-empty, which corresponds to $\Phi(\theta)>R_1(\theta)$), φ is still decreasing and sends μ_θ onto $\nu_\theta|_{(R_1(\theta),\Phi(\theta)}$

Precisely, we fix θ and the expressions of μ_θ, ν_θ are

$$\mathrm{d}\mu_\theta(r)=rf(r,\theta)\,\mathrm{d}r \qquad \text{and} \qquad \mathrm{d}\nu_\theta(r)=rg(r,\theta)\,\mathrm{d}r$$

which have both same mass on $(0,1)$ and $(R_1(\theta),R_2(\theta))$ respectively. Now we define successively $\rho_1(\theta)$ and $\rho_2(\theta)$ by

$$\int_0^{\rho_1(\theta)}\mathrm{d}\mu_\theta=\int_{\Phi(\theta)}^{R_2(\theta)}\frac{1}{2}\,\mathrm{d}\nu_\theta \qquad \text{and} \qquad \int_{\rho_1(\theta)}^{\rho_2(\theta)}\mathrm{d}\mu_\theta=\int_{\Phi(\theta)}^{R_2(\theta)}\frac{1}{2}\,\mathrm{d}\nu_\theta$$

which are proper definitions thanks to the intermediate value theorem, and imply

$$\int_{\rho_2(\theta)}^{1}\mathrm{d}\mu_\theta=\int_0^{\Phi(\theta)}\mathrm{d}\nu_\theta$$

Thus, we have the equality between masses:

$$\mu_\theta(0,\rho_1(\theta))=\mu_\theta(\rho_1(\theta),\rho_2(\theta))=\frac{1}{2}\nu_\theta(\Phi(\theta),1) \quad \text{and} \quad \mu_\theta(\rho_2(\theta),1)=\nu_\theta(0,\Phi(\theta))$$

and the measures μ_θ, ν_θ are absolutely continuous on these intervals. We now define the function $\varphi(\cdot,\theta)$ as being:

- on the interval $(0,\rho_1(\theta))$, the unique increasing map $(0,\rho_1(\theta)\to(\Phi(\theta),1)$ sending μ_θ onto $\frac{1}{2}\nu_\theta$;
- on the interval $(\rho_1(\theta),\rho_2(\theta))$, the unique decreasing map $(\rho_1(\theta),\rho_2(\theta))\to(\Phi(\theta),1)$ sending μ_θ onto $\frac{1}{2}\nu_\theta$;
- on the interval $(\rho_2(\theta),1)$ (if this interval is not empty), the unique decreasing map $(\rho_2(\theta),1)\to(0,\Phi(\theta))$ sending μ_θ to ν_θ

It is easy to check that $\varphi(\cdot,\theta)$, defined on the whole $(0,1)$, sends globally μ_θ onto ν_θ. As a consequence, the two-dimensional valued function

$$T:x=(r,\theta)\in\Omega\mapsto\varphi(r,\theta)\frac{x}{|x|}\in\Omega'$$

is a transport map from μ to ν.

Step 2: estimates on φ around the origin. As above, we set $\delta=\varepsilon^{1/3}$ and we aim to modify T only on $\Omega_\delta=\Omega\cap B(0,\delta)$. In order to obtain as well a transport map from μ to ν, we have to guarantee that the new map S sends the domain Ω_δ on its image $T(\Omega_\delta)$, with the constraint of image measure. For this, the following estimates on the original transport T will be useful:

Proposition 3.4.1. *The above function φ has Lipschitz regularity on $(0,1)\times(0,\pi/2)$. Moreover, there exists some positive constants c,C depending only on Ω', f, g such that*

$$cr^2\le\varphi(r,\theta)-\Phi(\theta)\le Cr^2 \tag{3.16}$$

for any r small enough and $\theta\in(0,\pi/2)$, and

$$\mathrm{Lip}(\varphi(r,\cdot)-\Phi)\le Cr^2 \tag{3.17}$$

for any $r\in(0,1)$.

Proof. The Lipschitz regularity of φ is actually a consequence of its definition and of the inverse function theorem. First, let us recall that $\rho_1(\theta)$ is defined by

$$\widetilde{F}(\rho_1(\theta),\theta) = \frac{1}{2}(G(R_2(\theta),\theta) - \widetilde{G}(\Phi(\theta),\theta))$$

where $\widetilde{F}(R,\theta) = \int_0^R \mathrm{d}\mu_\theta = \int_0^R rf(r,\theta)\,\mathrm{d}r$ and $\widetilde{G}(R,\theta) = \int_{R_1(\theta)}^R \mathrm{d}\nu_\theta = \int_{R_1(\theta)}^R rg(r,\theta)\,\mathrm{d}r$

Notice that we have necessary $\rho_1(\theta) \geq \delta_0 > 0$, where δ_0 verifies, for instance,

$$\int_{B(0,\delta_0)\cap\Omega} \mathrm{d}\mu = \int_{\sup R_1 \leq r \leq \inf R_2} \mathrm{d}\nu$$

(such a δ_0 exists since $\delta \mapsto \mu(B(0,\delta)\cap\Omega)$ is continuous and increasing). Thus, $\widetilde{F}$ is C^1 with respect to its first variable and its derivative is Lipschitz and bounded from below by $\delta_0 \inf f$, and the same holds for $\widetilde{G}$; moreover, $\widetilde{F}$ and $\widetilde{G}$ are both Lipschitz with respect to θ. As a consequence of the inverse function theorem, ρ_1 is well-defined and Lipschitz. The same reasoning shows that ρ_2 is well-defined and Lipschitz.

Then we know that φ is defined if $0 \leq r \leq \rho_1(\theta)$ by

$$\int_0^r \mathrm{d}\mu_\theta = \int_{\Phi(\theta)}^{\varphi(r,\theta)} \mathrm{d}\nu_\theta \qquad i.e. \qquad \widetilde{F}(r,\theta) = \widetilde{G}(\varphi(r,\theta),\theta) - G(\Phi(\theta),\theta)$$

Again, by the inverse function theorem, φ is C^1 with respect to r and Lipschitz with respect to θ. The same reasoning can be applied on the intervals $(\rho_1(\theta),\rho_2(\theta))$ and, if it is non-empty, $(\rho_2(\theta),1)$. We get that the function φ is Lipschitz with respect to its both variables (r,θ).

Notice also the following estimate on $\varphi(r,\theta)$ for r small enough: we have $\int_0^r \mathrm{d}\mu_\theta = \int_{\Phi(\theta)}^{\varphi(r,\theta)} \mathrm{d}\nu_\theta$, with the inequalities

$$r^2 \inf f \leq \int_0^r \mathrm{d}\mu_\theta \leq r^2 \sup f \qquad \text{and} \qquad (\inf g)(\varphi(r,\theta) - \Phi(\theta)) \leq \int_{\Phi(\theta)}^{\varphi(r,\theta)} \mathrm{d}\nu_\theta \leq (\sup g)(\varphi(r,\theta) - \Phi(\theta))$$

This leads immediately to (3.16). On the other hand, we know that $\varphi(r,\theta)$ has Lipschitz regularity with respect to θ and

$$\int_{R_1(\theta)}^{\varphi(R,\theta)} rg(r,\theta)\,\mathrm{d}r = \int_0^R rf(r,\theta)\,\mathrm{d}r \tag{3.18}$$

If θ is such that the equality (3.18) is differentiable with respect to θ (which is true for a.e. θ since all the considered functions are at least Lipschitz), we get

$$\partial_2\varphi(R,\theta)\varphi(R,\theta)G(\varphi(R,\theta),\theta) - R_1'(\theta)R_1(\theta)G(\varphi(R,\theta),\theta) = \int_0^R r\partial_2 f(r,\theta)\,\mathrm{d}r$$

thus

$$\partial_\theta(\varphi(R,\theta) - R_1(\theta))\varphi(R,\theta)G(\varphi(R,\theta),\theta) + R_1'(\theta)(\varphi(R,\theta) - R_1(\theta))G(\varphi(R,\theta),\theta) = \int_0^R r\partial_2 f(r,\theta)\,\mathrm{d}r$$

and

$$|\partial_\theta(\varphi(R,\theta) - R_1(\theta))| \leq \frac{(\operatorname{Lip} R_1)(\sup G)}{(\inf R_1)(\inf G)}|\varphi(R,\theta) - R_1(\theta)| + \frac{R^2}{2}\operatorname{Lip} F$$

Thanks to the inequality (3.16), we get

$$|\partial_\theta(\varphi(R,\theta) - R_1(\theta))| \leq CR^2$$

for some constant C depending only on f, g, Ω', which proves (3.17). □

Step 3: perturbation of the optimal T. In that follows, we denote by

$$\Omega_\delta = \Omega_1 \cap B(0,\delta) = \{x = (r,\theta) : 0 < r < \delta \text{ and } 0 < \theta < \pi/2\}$$

$$\text{and} \qquad \Omega'_\delta = T(\Omega_\delta) = \{x = (r,\theta) : \Phi(\theta) < r < \varphi(\delta,\theta) \text{ and } 0 < \theta < \pi/2\}$$

We now denote by:

- $S_1 : x \in \Omega_\delta \mapsto \dfrac{x}{\delta} \in \Omega_1$ and $f_\delta(x) = f(\delta x)$. Notice that $f_\delta = \dfrac{1}{\delta^2}(S_1)_\#(f|_{\Omega_\delta})$;
- Ω_2 is the rectangle $(0,1) \times (0,\pi/2)$ and

  $$S_2 : (\lambda,\theta) \in \Omega_2 \mapsto x = (\Phi(\theta) + \lambda(\varphi(\delta,\theta) - \Phi(\theta)),\theta) \in \Omega'_\delta$$

  where x is here written in polar coordinates. We also denote by $g_\delta = \dfrac{1}{\delta^2}\Big((S_2)^{-1}{}_\# g|_{\Omega'_\delta}\Big)$.

As above, S_1 and S_2 actually depend on δ but we forget the index δ for the sake of simplicity of the notations. We have of course $\inf f \le f_\delta(x) \le \sup f$ for any $x \in \Omega_1$, and $\operatorname{Lip} f_\delta \le \delta \operatorname{Lip} f$. On the other hand, since S_2 is Lipschitz and one-to-one, the Monge-Ampère equation provides

$$\det DS_2(\lambda,\theta) = \frac{g_\delta(\lambda,\theta)}{g(S_2(\lambda,\theta))}$$

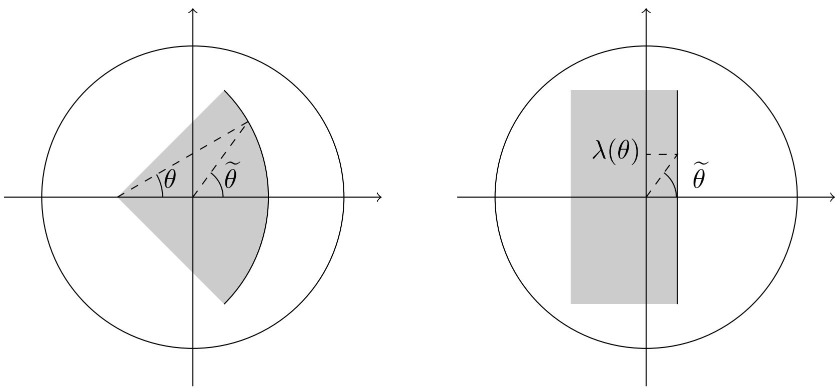
$$\text{thus} \qquad g_\delta(\lambda,\theta) = (\varphi(\delta,\theta) - \Phi(\theta))\frac{1}{\delta^2} g(S_2(\lambda,\theta)) \tag{3.19}$$

We deduce from (3.16) and (3.19) that $c \le g_\delta \le C$ for c, C positive and independent of δ. Moreover, (3.19) provides

$$\operatorname{Lip} g_\delta \le \sup(\varphi_\delta - \varphi)\frac{1}{\delta^2}\operatorname{Lip}(g \circ S_2) + \operatorname{Lip}(\varphi_\delta - \Phi)\frac{1}{\delta^2}\sup g$$

Again, thanks to (3.16) and (3.19) and using that $\operatorname{Lip} S_2$ is uniformly bounded with respect to δ, we get $\operatorname{Lip} g_\delta \le C$ independent of δ.

We would like to use the Theorem A.3.1, but this result works *a priori* only to link two measures defined on the same domain. Thus, an intermediate step consists in sending Ω_δ, Ω'_δ onto the unit ball, which is directly allowed in a regular enough thanks to the Lemma 3.2.2. Moreover, let us notice that Ω_1, Ω_2 are infinitely diffeomorphic to the following domains (which are obtained directly thanks to rotations/translations/dilations):



In the above picture, the upper right and lower right corners of the second domains are equal to the upper right and lower right corners of the first domains. Since the maps provided by the Lemma 3.2.2 from these two domains to the unit disk both preserve angles (with respect to the origin, see the angle $\widetilde{\theta}$ on the picture), if we denote by α, β the corresponding maps starting from Ω_1, Ω_2, we have

$$\beta^{-1} \circ \alpha(x) = (1, \lambda(\theta)) \qquad \text{if} \qquad |x| = 1 \text{ and } x \text{ has } \theta \text{ for angle.}$$

where λ is a bi-Lipschitz map from the interval $(0, \pi/2)$ onto $(0, 1)$; by composing the second coordinate with λ^{-1}, one can actually assume that $\beta^{-1} \circ \alpha(x) = (1, \theta)$. On the other hand, we deduce from the regularity of $\det D\alpha$, $\det D\beta$ that, if

$$\overline{f_\delta} = \alpha_\# f_\delta \qquad \text{and} \qquad \overline{g_\delta} = \beta_\# g_\delta$$

then $\overline{f_\delta}$, $\overline{g_\delta}$ have also infimum, supremum and Lipschitz constant bounded independently of δ. Now the Dacorogna-Moser result provides the existence of a bi-Lipschitz diffeomorphism u_δ, whose Lipschitz constant is bounded independently of δ, sending the density $\overline{f_\delta}$ onto $\overline{g_\delta}$, and with $u_\delta(x) = x$ for $|x| = 1$.

Finally, we consider

$$S = S_2 \circ \beta^{-1} \circ u_\delta \circ \alpha \circ S_1$$

If $|x| = \delta$, we have $|S_1(x)| = 1$, thus $\alpha(S(x)) \in \partial B(0, 1)$ and

$$\beta^{-1}(u_\delta(\alpha(S_1(x)))) = \beta^{-1}(\alpha(S_1(x)))$$

so that $\beta^{-1}(u_\delta(\alpha(S_1(x)))) = (1, \theta)$, where θ is the angle of x. Consequently, $S_2(\beta^{-1}(u_\delta(\alpha(S_1(x)))))$ has $(\varphi(\delta, \theta), \theta)$ as polar coordinates, so it is equal to $T(x)$.

To summarize, S and T coincide on the line $\{|x| = \delta\}$, S is Lipschitz on Ω_δ and, thanks to the estimates on φ, T is Lipschitz on $\Omega \setminus \Omega_\delta$. This implies that T_ε is globally Lipschitz on Ω, thus it belongs to $H^1(\Omega)$. Moreover, we have

$$\operatorname{Lip} u_\delta \leq C, \quad \operatorname{Lip} S_2 \leq \operatorname{Lip} \Phi \quad \text{and} \quad \operatorname{Lip} S_1 = \frac{1}{\delta}$$

$$\text{thus} \qquad \operatorname{Lip} S \leq \frac{C}{\delta}$$

for some constant C which does not depend on δ.

Step 4: estimates on $F_\varepsilon(T_\varepsilon)$. We restart from the expression (3.6), and use the facts that $\psi_\varepsilon = 0$ and that $T_\varepsilon = T$ outside of Ω_δ:

$$F_\varepsilon(T_\varepsilon) = \frac{1}{\varepsilon} \int_{\Omega_\delta} (|S(x) - x| - |S(x)| + |x|) f(x) \, \mathrm{d}x + \int_{\Omega_\delta} |DS(x)|^2 \, \mathrm{d}x$$
$$+ \int_\delta^1 (||\varphi(r, \cdot)||_{L^2}^2 + ||\partial_\theta \varphi(r, \cdot)||_{L^2}^2 - K) \frac{\mathrm{d}r}{r} + \int_\delta^1 ||\partial_r \varphi(r, \cdot)||_{L^2}^2 \, r \, \mathrm{d}r$$

We still have $|S(x) - x| - |S(x)| + |x| \leq 2|x|$ and $|DS(x)| \leq C/\delta$, so that

$$\frac{1}{\varepsilon} \int_{\Omega_\delta} (|S(x) - x| - |S(x)| + |x|) f(x) \, \mathrm{d}x + \int_{\Omega_\delta} |DS(x)|^2 \, \mathrm{d}x \leq \frac{\pi \sup f}{3} \frac{\delta^3}{\varepsilon} + \frac{C^2 \pi}{4}$$

which is bounded since $\delta = \varepsilon^{1/3}$. On the other hand,

$$||\varphi(r, \cdot)||_{L^2}^2 + ||\partial_\theta \varphi(r, \cdot)||_{L^2}^2 - K = (||\varphi(r, \cdot)||_{L^2}^2 - ||\Phi||_{L^2}^2) + (||\partial_\theta \varphi(r, \cdot)||_{L^2}^2 - ||\Phi'||_{L^2}^2)$$

$$= \langle \varphi(r,\cdot) - \Phi, \varphi(r,\cdot) + \Phi \rangle_{L^2} + \langle \partial_\theta \varphi(r,\cdot) - \Phi', \partial_\theta \varphi(r,\cdot) + \Phi' \rangle_{L^2}$$
$$\leq ||\varphi(r,\cdot) - \Phi'||_{L^1}(||\varphi(r,\cdot)||_\infty + ||\Phi||_\infty) + ||\partial_\theta \varphi(r,\cdot) - \Phi'||_{L^1}(\mathrm{Lip}\,\varphi + \mathrm{Lip}\,\Phi)$$

Since Φ, $\varphi(r,\cdot)$ are valued in Ω', their L^∞-norm are controlled by $\sup R_2$. By combining this and the estimates (3.16) and (3.17), we obtain

$$0 \leq ||\varphi(r,\cdot)||_{L^2}^2 + ||\partial_\theta \varphi(r,\cdot)||_{L^2}^2 - K \leq Cr^2$$

for r small enough (and where C does not depend on r). On the other hand, we know that φ, Φ and their derivatives are globally bounded on $(0,1) \times (0, \pi/2)$. This proves that

$$\int_0^1 (||\varphi(r,\cdot)||_{L^2}^2 + ||\partial_\theta \varphi(r,\cdot)||_{L^2}^2 - K)\,\frac{\mathrm{d}r}{r} < +\infty$$

$$\text{and} \qquad \int_0^1 ||\partial_r \varphi(r,\cdot)||_{L^2}^2\, r\,\mathrm{d}r < +\infty$$

and we conclude that $(F_\varepsilon(T_\varepsilon))_\varepsilon$ is bounded as well.

Chapter 4

Related models and perspectives

In the three first chapters of this thesis, we examined the most natural questions that we face in any variational problem (well-posedness, existence, optimality conditions, selection of a minimizer via a limit procedure). Although we considered that it was necessary to start from these questions, they have all been expressed in terms of the most basic formulation of transport problem, which is "static" (in sense that only appear in the functional the starting point x and the arrival point $T(x)$) and does not exploit a relaxation formula (which, in the classical Monge-Kantorovich problem, is necessary to show the existence in the nice cases of starting measure μ).

In optimal transport, the two following alternative formulations are now well-known:

- the Kantorovich's formulation of the optimal transport problem, namely

$$\inf\left\{\int c(x,y)\,\mathrm{d}\gamma(x,y)\;:\;(\pi_1)_\#\gamma=\nu,(\pi_2)_\#\gamma=\nu\right\}$$

  was the basic tool to show the existence of a minimizing transport map T, but also provides several characterizations of it *via* the duality formula

$$\inf\left\{\int c(x,T(x))\,\mathrm{d}\mu(x)\;:\;T_\#\mu=\nu\right\}=\sup\left\{\int\varphi\,\mathrm{d}\mu+\int\psi\,\mathrm{d}\nu\;:\;\varphi(x)+\psi(y)\leq c(x,y)\right\}$$

  For instance the fact that T is the gradient of a convex function in the quadratic case $c(x,y)=|y-x|^2$ is typically obtained thanks to this formulation;

- In a well-known paper [9], Benamou and Brenier have shown the equality between the optimal quadratic transport cost and the quantity

$$\inf\left\{\int_0^1 ||v_t||^2_{L^2_{\rho_t}}\,\mathrm{d}t\right\}$$

  where the infimum is taken among the family of velocity fields and measure $(\rho_t,v_t)_t$ satisfying the continuity equation and which connect μ to ν; this corresponds to the minimal kinetic energy of the geodesics from μ to ν in the space of measures (see the details below), and this formulation is also important for its numerical applications because it can be expressed as a convex minimization problem.

It is thus natural to look at:

- similar *dynamics* formulations (since they involve a time parameter) for transport problems also involving the gradient; in particular, we expect to relate

$$\inf\left\{||T-\mathrm{id}||^2_{H^1_\mu}\ :\ T_\#\mu=\nu\right\}$$

$$\text{to}\qquad \inf\left\{\int_0^1 ||v_t||^2_{H^1_{\rho_t}}\,\mathrm{d}t\right\}$$

which just corresponds to replace in the two terms the L^2-norm with the H^1-norm (with respect to the corresponding measures in both cases);

- dual Kantorovich-like formulations, *e.g.* expressed in terms of a measure γ which corresponds to a transport plan (but with something more corresponding to the gradient).

This chapter is devoted to some proposal and formal ideas on what could be done on these two problems. There is actually not any very precise result for the moment, but we give here the main ideas, the expected results and the formal computations that could be a guideline for possible future researches (some of them being currently in progress).

4.1 Dynamic formulations of the Sobolev transport problem

4.1.1 The original Benamou-Brenier result and its implications

We begin this section by recalling the main result of Benamou and Brenier, which presented a fluid-mechanics approach of the Monge-Kantorovich problem with important numerical applications. Their main result is the following ([9], Prop. 1.1, proof in the 4th paragraph):

Proposition 4.1.1. *Assuming μ, ν to be two probability densities on $\mathbb{R}^d$, both compactly supported and smooth enough on their respective supports. Then the minimal value of the Monge-Kantorovich problem with quadratic cost*

$$\inf\left\{\int_\Omega |T(x)-x|^2\,\mathrm{d}\ :\ T_\#\mu=\nu\right\}$$

is equal to the minimal value of the kinetic energy

$$\int_0^1\int_{\mathbb{R}^d}|v(t,x)|^2\,\mathrm{d}\rho_t(x)\,\mathrm{d}t$$

among all the vector fields $(t,x)\mapsto v(t,x)$ and the curves of measures $(\rho_t)_{0\le t\le 1}$ satisfying the continuity equation with initial and final conditions

$$\begin{cases}\partial_t\rho_t+\mathrm{div}(\rho_t v(t,\cdot))=0\\ \rho_0=\mu,\rho_1=\nu\end{cases}\tag{4.1}$$

Sketch of the proof. We begin by showing the inequality

$$\int_0^1\int_{\mathbb{R}^d}|v(t,x)|^2\,\mathrm{d}\rho_t(x)\,\mathrm{d}t\ge W_2(\mu,\nu)^2\tag{4.2}$$

where $W_2(\mu,\nu)^2$ denotes the minimal value of the quadratic transport energy from μ to ν, and for any (v,ρ) satisfying (4.1). For such a (v,ρ), we define X as being the solution of the ordinary differential equation

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}X(t,x)=v(X(t,x))\\ X(0,x)=x\end{cases}\tag{4.3}$$

We claim that, for each t, $X(t,\cdot)$ sends μ onto ρ_t. To prove it, we verify that $(X(t,\cdot)_{\#}\mu, v)$ verifies also the continuity equation (4.1) for which, given the vector field v and the initial condition ρ_0, the solution $(\rho_t)_t$ is unique (see the chap. 8 of [4]). Indeed, we have for a test function φ depending on $(t,x) \in (0,1)\times\mathbb{R}^d$ and compactly supported into this space:

$$\int_0^1 \int_{\mathbb{R}}^d (\partial_t\varphi + \nabla_x\varphi\cdot v)\,\mathrm{d}((X_t,\cdot)_{\#}\mu)\,\mathrm{d}t = \int_0^1 \int_{\mathbb{R}^d} (\partial_t\varphi(X(t,x)) + \nabla_x\varphi(X(t,x))\cdot v(t,x))\,\mathrm{d}\mu(x)\,\mathrm{d}t$$

We integrate by parts the last term and use the fact that $\frac{\mathrm{d}}{\mathrm{dt}}X(t,\cdot) = v(X(t,\cdot))$ to get

$$\begin{aligned}\int_0^1 \int_{\mathbb{R}}^d (\partial_t\varphi + \nabla_x\varphi\cdot v)\,\mathrm{d}((X_t,\cdot)_{\#}\mu)\,\mathrm{d}t &= \int_0^1 \frac{\mathrm{d}}{\mathrm{dt}}\left(\int_{\mathbb{R}^d}\varphi(t,X(t,\cdot))\,\mathrm{d}\mu(x)\right)\mathrm{d}t\\ &= \int_{\mathbb{R}^d}(\varphi(1,X(1,,x)) - \varphi(0,X(0,x)))\,\mathrm{d}\mu(x)\end{aligned}$$

which is null for any φ compactly supported in time. Consequently, $(X(t,\cdot)_{\#}\mu)_t$ satisfies (4.1) in the distributional sense and with initial condition $X(0,\cdot)_{\#}\mu = \mathrm{id}_{\#}\mu = \mu$, which implies $X(t,\cdot)_{\#}\mu = \rho_t$.

This implies that $X(1,\cdot)$ sends μ onto $\rho_1 = \nu$. On the other hand, we compute

$$\int_0^1\int_{\mathbb{R}^d}|v(t,x)|^2\,\mathrm{d}\rho_t(x)\,\mathrm{d}t = \int_{\mathbb{R}^d}\int_0^1|v(t,X(t,x))|^2\,\mathrm{d}t\,\mathrm{d}\mu(x) \geq \int_{\mathbb{R}^d}\left|\int_0^1 v(t,X(t,x))\,\mathrm{d}t\right|^2\mathrm{d}\mu(x)$$

where the last inequality comes from the Jensen's inequality. By noticing that $v(X(t,x)) = \frac{\mathrm{d}}{\mathrm{dt}}X(t,x)$, we deduce

$$\int_0^1\int_{\mathbb{R}^d}|v(t,x)|^2\,\mathrm{d}\rho_t(x)\,\mathrm{d}t \geq \int_{\mathbb{R}^d}|X(1,x)-X(0,x)|^2\,\mathrm{d}\mu(x) \geq \int_{\mathbb{R}^d}|X(1,x)-x|^d\,\mathrm{d}\mu(x)$$

and this is greater than $W_2(\mu,\nu)^2$ since $X(1,\cdot)$ sends μ onto ν.

Now it is enough to find a velocity field v and a curve of measures ρ which realize the equality in (4.2). For this, we define

$$X(t,x) = x + t(T(x)-x)$$

where T is the optimal map from μ to ν for the Monge-Kantorovich problem. We recall that $T(x) = \nabla\psi(x)$ for some convex function $\psi:\Omega\to\mathbb{R}$; thus the Jacobian matrix of $X(t,\cdot)$ is

$$DX(t,x) = I_d + t(D^2\psi(x) - I_d) = (1-t)I_d + tD^2\psi(x)$$

Since $D^2\psi$ is positive, this Jacobian matrix is positive for any (t,x) and we deduce that the map $X(t,\cdot)$ is invertible for any t. We then set

$$v(t,x) = \partial_t X(t,(X(t,\cdot))^{-1}(x))$$

By construction of v, X and v satisfy the ordinary differential equation (4.3). If we denote by $\rho_t = X(t,\cdot)_{\#}\mu$, we thus know that $(\rho_t, v(t,\cdot))_t$ satisfy the continuity equation; moreover, we have $X(0,\cdot) = \mathrm{id}$ and $X(1,\cdot) = T$ thus the initial and final condition $\rho_0 = \mu$, $\rho_1 = \nu$ also hold. It remains to compute:

$$\begin{aligned}\int_0^1\int_{\mathbb{R}^d}|v(t,x)|^2\,\mathrm{d}\rho_t(x)\,\mathrm{d}t &= \int_0^1\int_{\mathbb{R}^d}|T(X(t,\cdot)^{-1}(x)) - X(t,\cdot)^{-1}(x)|^2\,\mathrm{d}(X(t,\cdot)_{\#}\mu)(x)\,\mathrm{d}t\\ &= \int_0^1\int_{\mathbb{R}^d}|T(x)-x|^2\,\mathrm{d}\mu(x)\,\mathrm{d}t\\ &= W_2(\mu,\nu)^2\end{aligned}$$

as announced. □

In particular, the proof also shows that, in the optimal pair $(\rho_t, v_t)_t$, the curve of measures $(\rho_t)_t$ is

$$t \mapsto (X_t)_{\#}\mu \qquad \text{where} \qquad X_t(x) = (1-t)x + tT(x)$$

Actually, one can show that if we endow the space $\mathcal{P}_2(\mathbb{R}^d)$ of the probability measures with finite moment of order 2 with the Wasserstein distance given by the Monge-Kantorovich problem

$$\begin{aligned} W_2(\mu,\nu) &= \left(\inf\left\{\int_{\mathbb{R}^d} |T(x)-x|^2 \,\mathrm{d}\mu(x) \;:\; T_{\#}\mu = \nu\right\}\right)^{1/2} \\ &= \left(\inf\left\{\int_{\mathbb{R}^d} |y-x|^2 \,\mathrm{d}\gamma(x,y) \;:\; \gamma \in \Pi(\mu,\nu)\right\}\right)^{1/2} \end{aligned}$$

then the constant-speed geodesic from μ to ν is precisely the curve $t \mapsto (X_t)_{\#}\mu$ described above. Thus, the Benamou-Brenier formula and its proof make the link between the Wasserstein distance (with order 2) and the kinetic energy of the velocity fields associated to the corresponding curve.

Our question is: does there exist such a link if we look at the H^1-order transport cost? and what is the behavior of the space $\mathcal{P}_2(\mathbb{R}^d)$ if we replace the Wasserstein distance with, for instance,

$$(\mu,\nu) \mapsto \inf\left\{||T - \mathrm{id}||_{H^1_\mu} \;:\; T_{\#}\mu = \nu\right\}$$

or its Benamou-Brenier version with

$$\inf\left\{\int_0^1 ||v_t(x)||^2_{H^1_{\rho_t}} \,\mathrm{d}t\right\} ?$$

Notice also that in this formulation we need to consider Sobolev spaces w.r.t. varying measures (since ρ_t depends on the time). In particular, some properties could depend on the regularity of ρ_t and, when it has a density, on the summability of $1/\rho_t$. Yet, we will see in the next section that $1/\rho_t$ plays a special role, at least in a very particular case

4.1.2 Possible approaches for a H^1-order distance between measures

Motivations and related problems. A typical question in shape analysis is to know how much a shape can be deformed during the time, starting from a given configuration (represented by the source measure μ) and in order to obtain a given target configuration at $t = 1$ (also represented by a measure ν), *via* a continuous family of diffeomorphsims $(\varphi_t)_t$, with $\varphi_0 = \mathrm{id}$ and the constraint $(\varphi_1)_{\#}\mu = \nu$. Notice that this last constraint is close to the classical constraint of optimal transportation, but we add to it the fact that φ_t has to be a diffeomorphism. Under this constraint, it is usual to look at the minimum of

$$\int_0^1 ||v_t||^2 \,\mathrm{d}t \qquad \text{where} \qquad \partial_t \varphi_t = v_t \circ \varphi_t$$

and the norm can be L^p or W^{1-p}-like and can involve or not the measure $(\varphi_t)_{\#}\mu$ (e.g. $||v_t||_{L^2}$, $||v_t \circ \varphi_t||_{L^2}$ or their Sobolev equivalents). For instance, the quantity

$$\int_0^1 ||v_t \circ \varphi_t||^2_{L^2_\mu} \,\mathrm{d}t$$

is exactly the Benamou-Brenier "norm" of the family $(v_t, (\varphi_t)_{\#}\mu)_t$.

Partial analysis of a pure H^1-Benamou-Brenier-like formula. The most natural idea consists in looking at the links between the two pure H^1 versions of the L^2 Benamou-Brenier formula, namely

$$\inf\left\{\int_{\mathbb{R}^d}|T(x)-x|^2\,\mathrm{d}\mu(x)+\int_{\mathbb{R}^d}|D_\mu(T-\mathrm{id})(x)|^2\,\mathrm{d}\mu(x)\ :\ T\in H^1_\mu, T_\#\mu=\nu\right\}$$

$$\text{and}\qquad \inf\left\{\int_0^1||v_t||^2_{H^1_{\rho_t}}\,\mathrm{d}t\ :\ \partial_t\rho_t+\mathrm{div}(\rho_t v_t)=0,\ \rho_0=\mu,\rho_1=\nu\right\}$$

where the H^1-norm is *a priori* in both cases taken in the sense of the Sobolev spaces with respect to a measure as described in the Chapter 1. The first remark is that, by contrast to the Monge-Kantorovich problem, the fact that

$$\widetilde{d}:(\mu,\nu)\mapsto\inf\left\{\int_{\mathbb{R}^d}|T(x)-x|^2\,\mathrm{d}\mu(x)+\int_{\mathbb{R}^d}|D_\mu(T-\mathrm{id})(x)|^2\,\mathrm{d}\mu(x)\ :\ T\in H^1_\mu, T_\#\mu=\nu\right\}$$

defines a distance on $\mathcal{P}(\mathbb{R}^d)$ (even if we allow a distance to take an infinite value) seems seems false: neither the symmetry, nor the triangle inequality, are likely to be true in general (even if we did not produce explicit examples where this is the case). To solve this problem, the idea consists in considering the application

$$d:(\mu,\nu)\mapsto\inf\left\{\sum_i\widetilde{d}(\rho_i,\rho_{i+1})\ :\ n\in\mathbb{N},\ \mu=\rho_0,\rho_1,\dots,\rho_n=\nu\in\mathcal{P}(\mathbb{R}^d)\right\}$$

In this case, we can at least prove an inequality in a very smooth framework:

Proposition 4.1.2. *Let $(\rho_t,v_t)_t$ be a regular enough curve of measures satisfying the continuity equation* (4.1) *and connecting μ to ν. Then the following inequality holds:*

$$\int_0^1\int_{\mathbb{R}^d}(|v(x,t)|+|Dv(x,t)|^2)\,\mathrm{d}\rho_t(x)\,\mathrm{d}t\geq\widetilde{d}(\mu,\nu)$$

Sketch of the proof. Let $0=t_0<t_1<\dots<t_n=1$ be a subdivision of $[0,1]$, and let us set, for each i, $\rho_i=\rho_{t_i}$ and $h_i=t_{i+1}-t_i$. For each i, we denote by X_i the flow associated to the vector field v and equal to the identity map at the time t_i:

$$\begin{cases}\partial_t X_i(t,x)=v(t,X(t,x))\\ X_i(t_i,x)=x\end{cases}$$

We again verify that $X_i(t,\cdot)_\#\rho_i=\rho_t$, for each i and $t\in[t_i,t_{i+1}]$. Now we compute, starting from the left-hand side of the inequality that we want to show:

$$\int_0^1\int_{\mathbb{R}^d}|v(t,.)|^2+|Dv(t,.)|^2\,\mathrm{d}\rho_t\,\mathrm{d}t=\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}^d}|v(t,X_i(t,.))|^2+|Dv(t,X_i(t,.))|^2\,\mathrm{d}\rho_{t_i}\,\mathrm{d}t$$

with, by definition, $v(t,X_i(t,.))=\partial_t X_i(t,.)$ and $Dv(t,X_i(t,.))\times DX_i(t,.)=\partial_t DX_i(t,.)$. We obtain

$$\begin{aligned}\int_0^1\int_{\mathbb{R}^d}(|v|^2+|Dv|^2)\,\mathrm{d}\rho_t\,\mathrm{d}t=&\sum_{i=0}^{n-1}\int_{\mathbb{R}^d}\int_{t_i}^{t_{i+1}}\left(|\partial_t X_i|^2+|\partial_t DX_i|^2\right)\,\mathrm{d}t\,\mathrm{d}\rho_{t_i}\\&+\sum_{i=0}^{n-1}\int_{\mathbb{R}^d}\int_{t_i}^{t_{i+1}}\left(|Dv(t,X_i)|^2-|Dv(t,X_i)\times DX_i(t,x)|^2\right)\,\mathrm{d}t\,\mathrm{d}\rho_{t_i}\end{aligned}$$

As in the original proof, the Jensen inequality provides

$$\int_{t_i}^{t_{i+1}} \left(|\partial_t X_i|^2 + |\partial_t DX_i|^2\right) \mathrm{d}t \geq \frac{1}{t_{i+1}-t_i}\left(|X_i(t_{i+1},x)-x|^2 + |DX_i(t_{i+1},x)-\mathrm{id}|^2\right) \mathrm{d}t$$

and, since $(X_i(t_{i+1}))_\# \rho_i = \rho_{i+1}$, it follows

$$\begin{aligned}\sum_{i=0}^{n-1}\int_{\mathbb{R}^d}\int_{t_i}^{t_{i+1}} \left(|\partial_t X_i|^2 + |\partial_t DX_i|^2\right) \mathrm{d}t\,\mathrm{d}\rho_i &\geq \sum_{i=0}^{n-1}\frac{1}{t_{i+1}-t_i}\int_{\mathbb{R}^d}\left(|X_i(t_{i+1})-x|^2 + |DX_i(t_{i+1})-\mathrm{id}|^2\right)\mathrm{d}\rho_i \\ &\geq \sum_{i=0}^{n-1}\frac{1}{t_{i+1}-t_i}\widetilde{d}(\rho_i,\rho_{i+1})^2 \geq \left(\sum_{i=0}^{n-1}\widetilde{d}(\rho_i,\rho_{i+1})\right)^2 \geq d(\mu,\nu)^2\end{aligned}$$

We finally get

$$\int_0^1\int_{\mathbb{R}^d}(|v|^2+|Dv|^2)\,\mathrm{d}\rho_t\,\mathrm{d}t \geq d(\mu,\nu)^2 + \sum_{i=1}^{n-1}\int_{\mathbb{R}^d}\int_{t_i}^{t_{i+1}}\left(|Dv(t,X_i)|^2 - |Dv(t,X_i)\times DX_i|^2\right)\mathrm{d}t\,\mathrm{d}\rho_i$$

which is precisely the inequality that we are looking for with an "error term"; it remains to make this error term arbitrary small by estimate it with respect to the step of the subdivision $(t_0,\dots,t_n)$. We first remark that, by properties of the norms on $M_d(\mathbb{R})$ and since the derivatives of v are bounded, we have

$$\sum_{i=1}^{n-1}\int_{\mathbb{R}^d}\int_{t_i}^{t_{i+1}}\left(|Dv(t,X_i)|^2 - |Dv(t,X_i)\times DX_i|^2\right)\mathrm{d}t\,\mathrm{d}\rho_i \leq C\sum_{i=0}^{n-1}\int_{\mathbb{R}^d}\int_{t_i}^{t_{i+1}}|DX_i - id|\,\mathrm{d}t\,\mathrm{d}\rho_i$$

for some positive constant C. To estimate this term, we differentiate with respect to x the equality $\partial_t X_i = v(t,X_i)$ and compute the inner product in $M_d(\mathbb{R})$ with DX_i:

$$\frac{1}{2}\partial_t|DX_i(t,x)|^2 = \langle({}^tDX_i, Dv(t,X_i)\times DX_i\rangle \leq C|DX_i|^2$$

Then the Grönwall inequality gives $|DX_i(t,x)| \leq \exp(At)$ for some constant A. In particular, $|DT| \leq B = \exp A$ for any $t\in[0,1]$. We deduce that

$$|DX_i - \mathrm{id}| \leq \int_{t_i}^{t}|v(s,X_i(s,x))||DX_i(s,x)|\,\mathrm{d}s \leq M(t-t_i)$$

for some positive constant M.

$$\sum_{i=0}^{n-1}\int_{\mathbb{R}^d}\int_{t_i}^{t_{i+1}}|DX_i-\mathrm{id}|\,\mathrm{d}t\,\mathrm{d}\rho_i \leq C\sum_{i=0}^{n-1}(t_{i+1}-t_i)^2$$

which goes to 0 when $\max\limits_{1\leq i\leq n}(t_{i+1}-t_i)\to 0$. This achieves the proof. □

An alternative formula in a particular case. We finish this paragraph by producing an example of computation which suggests an equivalent formula with H^1-norm in a very particular case; this example is due to François-Xavier Vialard.

In one dimension and if the starting measure is the Lebesgue measure, let φ_1 be a bijective map sending $\mathcal{L}^1$ onto ν and let us define

$$\varphi_t : x \mapsto (1-t)x + t\varphi_1(x)$$

For monotonicity reasons this map is also a diffeomorphism and we can define $v_t(x) = \partial_t \varphi_t(\varphi_t^{-1}(x))$ as velocity field. Now

$$||\varphi_1 - \mathrm{id}||^2_{H^1(I)} = \int_0^1 ||\partial_t \varphi_t||^2_{H^1(I)} \,\mathrm{d}t$$

$$= \int_0^1 (||v_t \circ \varphi_t||^2_{L^2(I)} + ||(v_t \circ \varphi_t)'||^2_{L^2(I)}) \,\mathrm{d}t$$

$$\text{with} \qquad ||v_t \circ \varphi_t||_{L^2(I)} = ||v_t||^2_{L^2_{\mu_t}}$$

$$\text{with} \qquad ||(v_t \circ \varphi_t)'||^2_{L^2(I)} = \int_I v_t'(\varphi_t(x))^2 \varphi_t'(x) \,\mathrm{d}x = \int v_t'(y) \varphi_t'(\varphi_t^{-1}(y))^2 \frac{\mathrm{d}y}{\varphi_t'(\varphi_t^{-1}(y))}$$

where the last equality comes by changing of variables; but we know that $\dfrac{1}{\varphi_t'(\varphi_t^{-1}(y))}$ is actually the density of $(\varphi_t)_\# \mu$; thus, if we again denote by μ_t this density, we finally obtain

$$||(v_t \circ \varphi_t)'||^2_{L^2(I)} = \int_0^1 v_t'(x)^2 \frac{\mathrm{d}x}{\mu_t(x)}$$

So the obtained formula is finally

$$||\varphi_1 - \mathrm{id}||_{H^1(I)} = \int_0^1 (||v_t||^2_{L^2_{\mu_t}} + ||v_t'||^2_{L^2_{1/\mu_t}}) \,\mathrm{d}t$$

The surprising fact is that we obtain the density $1/\mu_t$ (and not anymore μ_t) for the gradient of the velocity field.

4.2 A tentative Kantorovich-like formulation

The goal of this section is to express the problem "transport plus gradient"

$$\inf \left\{ \int_\Omega |T(x) - x|^2 \,\mathrm{d}\mu(x) + \int_\Omega |DT(x)|^2 \,\mathrm{d}x \,:\, T_\# \mu = \nu \right\} \tag{4.4}$$

under a Kantorovich-like formulation, in order to obtain an equivalent of the duality formula in the classical optimal transport problem (see Appendix B) and to deduce properties of the optimizer. Let us recall that the starting point of the transition from the Monge problem to the Kantorovich problem is the remark

$$\int_\Omega c(x, T(x)) \,\mathrm{d}\mu(x) = \int_{\Omega \times \Omega'} c(x, y) \,\mathrm{d}\gamma_T(x, y)$$

where $\gamma_T = (\mathrm{id} \times T)_\# \mu$. In the case of the gradient penalization, the integral that we aim to minimize involves the Jacobian matrix of T and not only anymore the starting point x and the arrival point $T(x)$.

The strategy to correct this is due to Alessio Figalli. It consists - in case where the source measure μ is the Lebesgue measure on Ω, or at least where the gradient term which appears is integrated with respect to the Lebesgue measure - in using the following approximation of the L^2-norm of the Jacobian matrix:

$$(1 - s) \iint_{\Omega^2} \frac{|T(x) - T(y)|^2}{|x - y|^{d+2s}} \,\mathrm{d}x \,\mathrm{d}y \to C_d ||DT||^2_{L^2(\Omega)}$$

for T regular enough and where C_d only depends on the dimension (see below the summary of notions on fractional Sobolev spaces that we will need here). Thus, we consider the approximated problem

$$\inf\left\{\int_\Omega |T(x)-x|^2\,\mathrm{d}\mu(x) + (1-s)\iint_{\Omega^2}\frac{|T(x)-T(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \;:\; T\in H^s(\Omega),\, T_\#\mathcal{L}^d=\nu\right\}$$

where $H^s(\Omega)$ denotes the set of (vector) functions T such that the integral $\iint_{\Omega^2}\frac{|T(x)-T(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y$ is finite. In this case, we can write the quantity that we minimize in the following form involving the measure $\gamma_T=(\mathrm{id}\times T)_\#\mu$:

$$\int_{\Omega\times\Omega'}|z-x|^2\,\mathrm{d}\gamma_T(x,z)+\iint_{\Omega\times\Omega'}\frac{|z-w|^2}{|x-y|^{d+2s}}\,\mathrm{d}\gamma_T(x,z)\,\mathrm{d}\gamma_T(y,w)$$

The strategy is thus to study this problem (in particular, the main difficulties are that this function is not linear with respect to γ_T and the cost function involves a singular part $\frac{|z-w|^2}{|x-y|^{d+2s}}$) and to try to deduce a dual formulation (involving a cost function with 4 variables) that we aim to pass to the limit $s\to 1$.

We begin by a brief overview of the functional tools that we use (since they only appear in this chapter, we include they here and not in the Appendix).

4.2.1 Fractional Sobolev norms, fractional Laplacian and regularization

Almost all the definitions below come from [36] where can also be found more details, proofs and references. in what follows, Ω is a bounded open set of $\mathbb{R}^d$ with Lipschitz boundary.

Fractional Sobolev spaces.

Definition 4.2.1. *For $s\in(0,1)$ and $u\in L^2(\Omega)$, we define*

$$||u||^2_{H^s(\Omega)}=||u||^2_{L^2(\Omega)}+s(1-s)\iint_\Omega\frac{|u(x)-u(y)|^2}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y$$

and denote by $H^s(\Omega)$ the vector subspace of $L^2(\Omega)$ formed with functions u such that $||u||_{H^s(\Omega)}$ is finite.

Proposition 4.2.1. *We have the inclusion $H^1(\Omega)\subset H^{s'}(\Omega)\subset H^s(\Omega)$ if $1>s'>s>0$.*

Proposition 4.2.2. *The following convergences hold:*

- *for $u\in H^1(\Omega)$,*
$$\lim_{s\to1}(1-s)\iint_\Omega\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y=C_d||\nabla u||^2_{L^2}$$
where C_d depends only on the dimension and is given by $C_d=\int_{S^{d-1}}u_1^2\,\mathrm{d}u$;

- *for $u\in\bigcup_s(\Omega)$,*
$$\lim_{s\to0}s\iint_\Omega\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y=C'_d||u||_{L^2(\Omega)}$$
where C'_d only depends on the dimension.

Fractional Laplacian operator. It can be defined through a Fourier transform, but we choose the following definition since we work in bounded domains and not in $\mathbb{R}^d$:

Proposition 4.2.3. *For $x \in \Omega$, we have the convergence:*

$$\lim_{s \to 1} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{d+2s}} \,\mathrm{d}y = -\frac{1}{2} C_d \Delta u(x)$$

if u is regular enough and where C_d is the constant defined above. Thus, we will set

$$\Delta^s u(x) = -\frac{2}{C_d} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{d+2s}} \,\mathrm{d}y$$

Also, this choice of definition implies that $\Delta^s u$ is the first variation of $||\nabla u||^2_{H^s}$. We also verify and give a sketch of the proof of the following convergence:

Proposition 4.2.4. *If T is a regular enough vector-valued function, if*

$$K_s(x) = (1-s) \frac{1}{|x|^{d+2s}}$$

then

$$\lim_{s \to 1} \int_\Omega |T(x) - T(y)|^2 \nabla K_s(x-y) \,\mathrm{d}y = C_d({}^t DT(x) \Delta T(x) + \nabla(|DT|^2)(x))$$

Sketch of the proof. We compute the integral on a small bound around x (the integral for y far away from x will vanish as $s \to 1$). We start from a Taylor expansion of T: if $y = x + h$,

$$\begin{aligned} |T(x) - T(y)|^2 \nabla K_s(x-y) = |DT(x)(h)|^2 \frac{h}{|h|^{d+2s+2}} \\ + 2(\,DT(x)(h) \cdot D^2T(x)(h,h)) \frac{h}{|h|^{d+2s+2}} + O(|h|^{5-(d+2s+2)}) \end{aligned}$$

The first term is odd with respect to h, thus its integral on a ball centered in 0 is null, and the integral of $|h|^{5-(d+2s+2)}$ is bounded with respect to s thus the last term (that we multiply with $(1-s)$) vanishes when $s \to 1$. Thus, it remains to compute

$$\lim_{s \to 1} 2(d+2s)(1-s) \int_{B(x,\varepsilon)} (DT(x)(h) \cdot D^2T(x)(h,h)) \frac{h}{|h|^{d+2s+2}} \,\mathrm{d}h$$

We have

$$DT(x)(h) \cdot D^2T(x)(h,h) h_i = \sum_{j,k,l,m} \partial_k T_j(x) \partial^2_{l,m} T_j(x) h_i h_k h_l h_m$$

and when we integrate it on a ball centered in 0, all the terms which are odd with respect to h again disappear by symmetry, thus they are null except if

$$\begin{cases} l = m \\ i = k \end{cases} \quad \text{or} \quad \begin{cases} l \neq m \\ l = i \text{ and } m = k \end{cases} \quad \text{or} \quad \begin{cases} l \neq m \\ l = k \text{ and } m = i \end{cases}$$

It remains

$$\begin{aligned} 2(d+2s)(1-s) \sum_{1 \le j,l \le d} \partial_i T_j(x) \partial^2_{l,l} T_j(x) \int_{B(x,\varepsilon)} \frac{h_i^2 h_l^2}{|h|^{d+2s+2}} \,\mathrm{d}h \\ + 4(d+2s)(1-s) \sum_{\substack{1 \le j,k \le d \\ k \neq i}} \partial_k T_j(x) \partial^2_{i,k} T_j(x) \int_{B(x,\varepsilon)} \frac{h_i^2 h_k^2}{|h|^{d+2s+2}} \,\mathrm{d}h \end{aligned}$$

The first term (with $\partial^2_{l,l}T_j$) will give ${}^tDT\Delta T\ T$ and the second one will give $\nabla|DT|^2$.. It only remains to compute the limits

$$\lim_{s\to 1}(1-s)\int_{B(0,\varepsilon)}\frac{h_i^2h_j^2}{|h|^{d+2s+2}}\,\mathrm{d}h$$

in function of the constant C_d (this limit depends on the equality $i=j$, and can be computed by using the classical hyperspherical coordinates and Wallis integrals). □

A regularizing convolution kernel. We also will use the following regularizing kernel, that we define as a Fourier transform:

$$\mathcal{F}(K_{s,\varepsilon})(\xi) = -\frac{C_d}{2}|\xi|^{2s}\exp(-\varepsilon|\xi|^{2s})$$

and we verify that, for any regular enough function u,

$$\text{for any fixed } s\in(0,1),\qquad \int_\Omega (u(x)-u(y))K_{s,\varepsilon}(x-y)\,\mathrm{d}y \underset{\varepsilon\to 0}{\longrightarrow} -\frac{1}{2}\Delta^s u(x)$$

$$\text{and for any fixed } \varepsilon>0,\qquad \int_\Omega (u(x)-u(y))K_{s,\varepsilon}(x-y)\,\mathrm{d}y \underset{s\to 1}{\longrightarrow} -\frac{1}{2}C_d\Delta(u\star\rho_\varepsilon)$$

where ρ_ε is the classical Gauss function

$$\rho_\varepsilon(x) = \frac{1}{(2\sqrt{\pi\varepsilon})^d}\exp\left(-\frac{|x|^2}{4\varepsilon}\right)$$

This kernel will used to regularize our cost function, since our minimization problem involves

$$\iint_{(\Omega\times\Omega')^2}\frac{|z-w|^2}{|x-y|^{d+2s}}\,\mathrm{d}\gamma(x,z)\,\mathrm{d}\gamma(y,w)$$

where the cost integrated with respect to $\gamma\otimes\gamma$ admits here a singularity along the line $x=y$.

4.2.2 A sort of Kantorovich problem with 4 variables

Following the introduction of this section, we begin by studying, for a fixed $s\in(0,1)$,

$$\inf\left\{\iint_{(\Omega\times\Omega')^2}\left(\frac{1}{2}(|z-x|^2+|w-y|^2)+(1-s)\frac{|z-w|^2}{|x-y|^{d+2s}}\right)\mathrm{d}\gamma(x,z)\,\mathrm{d}\gamma(y,w)\right\} \tag{4.5}$$

where the infimum is taken among the transport plans $\gamma\in\Pi(\mu,\nu)$ whose first and second marginal are prescribed (and we still assume that μ is actually the Lebesgue measure on Ω). Thus, we will set

$$c_s(x,z,y,w) = \frac{1}{2}(|z-x|^2+|y-w|^2)+(1-s)\frac{|z-w|^2}{|x-y|^{d+2s}}$$

(we have replaced the transport cost $|z-x|^2$ with respect to its "symmetrized" version in order to have a cost function depending on the 4 variables x,z,y,w and symmetric with respect to the transformation $(x,z,y,w)\mapsto(y,w,x,z)$).

The first remark is the following, which guarantees that the infimum (if it exists) comes from a transport map:

Proposition 4.2.5. *Assume that $\gamma \in \Pi(\mu,\nu)$ is such that*

$$\iint_{(\Omega\times\Omega')^2} \frac{|z-w|^2}{|x-y|^{d+2s}}\,\mathrm{d}\gamma(x,z)\,\mathrm{d}\gamma(y,w) < +\infty$$

Then γ is induced by a transport map.

Proof. Indeed, if γ is not supported onto a graph then there exists (x_0,z_1), (x_0,z_2) belonging to the support of γ with $z_1 \neq z_2$; we choose $\varepsilon > 0$ such that the closed balls $\overline{B}(z_1,\varepsilon)$ and $\overline{B}(z_2,\varepsilon)$ are disjoints and note that, for some $\delta > 0$,

$$\int_{B((x_0,z_1,x_0,z_2),\varepsilon)} \frac{|z-w|^2}{|x-y|^{d+2s}}\,\mathrm{d}\gamma(x,z)\,\mathrm{d}\gamma(y,w) \geq \delta \int_{B((x_0,z_1,x_0,z_2),\varepsilon)} \frac{\mathrm{d}\gamma(x,z)\,\mathrm{d}\gamma(y,w)}{|x-y|^{d+2s}}$$

$$= \delta\nu(B(z_1,\varepsilon))\nu(B(z_2,\varepsilon)) \iint_{B(x_0,\varepsilon)^2} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{d+2s}} = +\infty \quad \square$$

The fact that the optimal plan from μ to ν for this fractional Sobolev cost comes from a transport map is thus elementary, but this fact does not hold anymore if we replace c_s by its regularized version

$$c_{s,\varepsilon} = \frac{1}{2}(|z-x|^2 + |w-y|^2) + |z-w|^2 K_{s,\varepsilon}(x-y)$$

since the integral $\iint K_{s,\varepsilon}(x-y)\,\mathrm{d}x\,\mathrm{d}y$ is then finite. Moreover, the functional that we consider is not convex with respect to γ since it involves a term with $\gamma\otimes\gamma$ (while the classical Monge-Kantorovich cost is linear with respect to γ).

Thus, the next step is to study the convex hull of the set

$$\{\gamma\otimes\gamma \,:\, \gamma \in \Pi(\mu,\nu)\}$$

The result that we expect is the following:

Conjecture 4.2.1. *Let Σ be the set of positive measures σ on $(\Omega\times\Omega')^2$ satisfying the following conditions:*

- $(\pi_{1,3})_\#\sigma = \mu\otimes\mu$, *where* $\pi_{1,3}(x,z,y,w) = (x,y)$;
- $(\pi_{2,4})_\#\sigma = \nu\otimes\nu$, *where* $\pi_{2,4}(x,z,y,w) = (z,w)$;
- $\tau_\#\sigma = \sigma$, *where* $\tau(x,z,y,w) = (y,w,x,z)$;
- $\forall f \in C_b((\Omega\times\Omega')^2)$, $\int f(x,z)f(y,w)\,\mathrm{d}\sigma(x,z,y,w) \geq 0$

Then $\left\{\int c\,\mathrm{d}\sigma \,:\, \sigma\in\Sigma\right\}$ *is attained by an element with form $\gamma\otimes\gamma$, $\gamma\in\Pi(\mu,\nu)$.*

Let us explain briefly the origin of the four constraints above: the first and the second one express the fact that the first and second marginal of $\gamma\in\Pi(\mu,\nu)$ are prescribed and the third and the fourth one express that $\gamma\otimes\gamma$ is still symmetric and "positive" among the functions with form $f(x,z)f(y,w)$. Since these four constraints are all preserved by convex combinations, the set Σ is a convex subset of the set of positive measures on $(\Omega\times\Omega')^2$.

We hope that these four constraints are "maximal" (meaning that Σ is actually the convex hull of $\{\gamma\otimes\gamma : \gamma\in\Pi(\mu,\nu)\}$) and a possible strategy to show this conjecture consists in studying the extremal points of Σ

4.2.3 Duality formula for 4-variables costs in the class Σ

We forget in this paragraph the dependence of the cost c on s and ε, just assume that it is symmetric (*i.e.* invariant by the transformation $(x,z,y,w)\mapsto(y,w,x,z)$) and only focus on the constraint "$\sigma\in\Sigma$". Following the original proof of the duality formula in the optimal transport (see the Appendix B), we remark that, for a positive measure σ on $(\Omega\times\Omega')^2$:

$$\sup\left\{\int\varphi(x,y)(\,\mathrm{d}(\mu\otimes\mu)(x,y)-\,\mathrm{d}\sigma(x,z,y,w))\ :\ \varphi\in C_b(\Omega^2)\right\}=\begin{cases}0 & \text{if } (\pi_{1,3})_\#\sigma=\mu\otimes\mu\\ +\infty & \text{otherwise}\end{cases}$$

$$\sup\left\{\int\psi(x,y)(\,\mathrm{d}(\nu\otimes\nu)(z,w)-\,\mathrm{d}\sigma(x,z,y,w))\ :\ \psi\in C_b(\Omega'^2)\right\}=\begin{cases}0 & \text{if } (\pi_{2,4})_\#\sigma=\nu\otimes\nu\\ +\infty & \text{otherwise}\end{cases}$$

$$\sup\left\{\int(\chi\circ\tau-\chi)\,\mathrm{d}\sigma\ :\ \tau\in C_b((\Omega\times\Omega')^2)\right\}=\begin{cases}0 & \text{if } \tau_\#\sigma=\chi\\ +\infty & \text{otherwise}\end{cases}$$

and that $\sup\left\{-\int\omega(x,z)\omega(y,w)\,\mathrm{d}\sigma(x,z,y,w)\right\}$ is 0 if the fourth constraint defining σ is satisfied and $+\infty$ otherwise. Notice also that we have the equality

$$\sup\left\{-\int\omega(x,z)\omega(y,w)\,\mathrm{d}\sigma(x,z,y,w)\right\}=\sup\left\{-\int\left(\sum_i\omega_i(x,z)\omega_i(y,w)\right)\,\mathrm{d}\sigma(x,z,y,w)\right\}$$

where the sum inside the right hand-side is taken in sense of the uniform convergence (and this allows to consider a supremum among a convex class of functions). Thus, we have the equality:

$$\begin{aligned}\inf\left\{\int_{(\Omega\times\Omega')^2}c\,\mathrm{d}\sigma\ :\ \sigma\in\Sigma\right\}=\inf\Bigg\{\sup\Bigg\{&\int\varphi(\,\mathrm{d}\mu\otimes\mu-\,\mathrm{d}\sigma)+\int\psi(\,\mathrm{d}\nu\otimes\nu-\,\mathrm{d}\sigma)\\&+\int(\chi-\chi\circ\tau)\,\mathrm{d}\sigma-\int\left(\sum_i\omega_i(x,z)\omega_i(y,w))\right)\,\mathrm{d}\sigma+\int c\,\mathrm{d}\sigma\Bigg\}\Bigg\}\end{aligned}$$

Assume that we are able to invert infimum and supremum. It remains

$$\begin{aligned}\inf\left\{\int_{(\Omega\times\Omega')^2}c\,\mathrm{d}\sigma\ :\ \sigma\in\Sigma\right\}=&\sup_{\varphi,\psi,\chi,\omega}\Bigg\{\int(\varphi\,\mathrm{d}(\mu\otimes\mu)+\psi\,\mathrm{d}(\nu\otimes\nu))\\&+\inf_\sigma\Bigg\{\int(c(x,z,y,w)-\varphi(x,z)-\varphi(y,w)\\&\quad-(\chi-\chi\circ\tau)(x,z,y,w)-\sum_i\omega_i(x,z)\omega_i(y,w))\,\mathrm{d}\sigma(x,z,y,w)\Bigg\}\Bigg\}\end{aligned}$$

The above infimum is 0 if

$$c(x,z,y,w)\geq\varphi(x,y)+\psi(z,w)+\chi(x,z,y,w)-\chi(y,w,x,z)+\sum_i\omega_i(x,z)\omega_i(y,w)$$

for any $(x,z,y,w)\in(\Omega\times\Omega')^2$, and $+\infty$ otherwise. Thus, we look at the problem

$$\sup\left\{\int\varphi\,\mathrm{d}\mu\otimes\mu+\int\psi\,\mathrm{d}\nu\otimes\nu\right\}$$

where the supremum is taken among the pairs of functions (φ, ψ) such that

$$\exists \chi, (\omega_i)_i : \quad \varphi(x,y) + \psi(z,w) + \chi(x,z,y,w) - \chi(y,w,x,z) + \sum_i \omega_i(x,z)\omega_i(y,w) \leq c(x,z,y,w)$$

Notice that if, in the last equality, we exchange (x, y) and (z, w) and compute the half-sum, it follows since c is symmetric that

$$\frac{1}{2}(\varphi(x,y) + \varphi(y,x)) + \frac{1}{2}(\psi(z,w) + \psi(w,z)) + \sum_i \omega_i(x,z)\omega_i(y,w) \leq c(x,z,y,w)$$

and by noticing the equalities

$$\int \frac{1}{2}(\varphi(x,y) + \varphi(y,x)) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) = \int \varphi(x,y) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y)$$

$$\text{and} \qquad \int \frac{1}{2}(\psi(x,y) + \psi(y,x)) \, \mathrm{d}\nu(x) \, \mathrm{d}\nu(y) = \int \psi(x,y) \, \mathrm{d}\nu(x) \, \mathrm{d}\nu(y)$$

we deduce that one can consider

$$\sup \left\{ \int \varphi \, \mathrm{d}\mu \otimes \mu + \int \psi \, \mathrm{d}\nu \otimes \nu \right\}$$

where the supremum is taken among the pairs of functions (φ, ψ) which are both symmetric with respect to to their variables (x, y) and (z, w) and such that

$$\exists (\omega_i)_i \ : \ \varphi(x,y) + \psi(z,w) + \sum_i \omega_i(x,z)\omega_i(y,w) \leq c(x,z,y,w)$$

The duality formula would be written as follows:

Conjecture 4.2.2. *We have the equality*

$$\inf \left\{ \int c \, \mathrm{d}\sigma : \sigma \in \Sigma \right\} = \sup \left\{ \int \varphi \, \mathrm{d}\mu \otimes \mu + \psi \, \mathrm{d}\nu \otimes \nu : (\varphi, \psi) \in \mathcal{S} \right\}$$

where $\mathcal{S}$ is the set of pairs (φ, ψ) such that

- $\varphi \in C_b(\Omega^2)$, $\psi \in C_b(\Omega')^2$;
- $\varphi(x,y) = \varphi(y,x)$ *and* $\psi(z,w) = \psi(w,z)$ *for any* x, y, z, w;
- *there exists a sequence* $(\omega_i)_i$ *of functions of* $C_b(\Omega \times \Omega')$ *such that* $\sum_i \omega_i(x,z)\omega_i(y,w)$ *uniformly converges and the inequality*

$$\varphi(x,y) + \psi(z,w) + \sum_i \omega_i(x,z)\omega_i(y,w) \leq c(x,z,y,w) \tag{4.6}$$

  holds for any $(x, z, y, w) \in (\Omega \times \Omega')^2$.

Properties of the optimizers. Now we take an optimal $\sigma \in \Sigma$, an optimal pair $(\varphi, \psi) \in \mathcal{S}$ and sequence $(\omega_i)_i$ as in the above conjecture. Assuming that $\sigma = \gamma \otimes \gamma$ for some γ, we integrate the inequality (4.6) with respect to σ. It follows

$$\int \varphi \,\mathrm{d}(\mu \otimes \mu) + \int \psi \,\mathrm{d}(\nu \otimes \nu) + \sum_i \left(\int \omega_i \,\mathrm{d}\gamma \right)^2 \leq \int c \,\mathrm{d}\sigma$$

On the other hand, we know if the duality formula is true that $\int \varphi \,\mathrm{d}\mu \otimes \nu + \psi \,\mathrm{d}\nu \otimes \nu = \int c \,\mathrm{d}\sigma$. This implies that $\int \omega_i \,\mathrm{d}\gamma = 0$ for any i. Thus, we have

$$\begin{cases} \varphi(x,y) + \psi(z,w) + \sum_i \omega_i(x,z)\omega_i(y,w) \leq c(x,z,y,w) \quad \text{for any } x,z,y,w \in (\Omega \times \Omega')^2 \\ \int \left(\varphi(x,y) + \psi(z,w) + \sum_i \omega_i(x,z)\omega_i(y,w) \right) \mathrm{d}\sigma(x,z,y,w) = \int c \,\mathrm{d}\sigma \end{cases}$$

This implies (4.6) is actually an equality on the support of σ. We also notice that integrating (4.6) with respect to $\mathrm{d}\gamma(y,w)$ leads, since $\int \omega_i(y,w) \,\mathrm{d}\gamma(y,w) = 0$ for any i, to

$$\int \varphi(x,y) \,\mathrm{d}\mu(y) + \int \psi(z,w) \,\mathrm{d}\nu(w) \leq \int c(x,z,y,w) \,\mathrm{d}\gamma(y,w)$$

and the integral of the two hand-sides of this inequality with respect to $\mathrm{d}\gamma(x,z)$ are equal, which implies that these hand-sides are actually equal for (x,z) belonging to the support of γ. We summarize:

Proposition 4.2.6. *Assume that the above conjectures hold. Then, if σ and (φ, ψ) are optimal for their respective problems, the inequality*

$$\varphi(x,y) + \psi(z,w) + \sum_i \omega_i(x,z)\omega_i(y,w) \leq c(x,z,y,w)$$

is an equality on the support of σ. Moreover, if $\sigma = \gamma \otimes \gamma$ with $\gamma \in \Pi(\mu,\nu)$, then $\int \omega_i \,\mathrm{d}\gamma = 0$ for any i which implies that the following inequality holds:

$$\int \varphi(x,y) \,\mathrm{d}\mu(y) + \int \psi(z,w) \,\mathrm{d}\nu(w) \leq \int c(x,z,y,w) \,\mathrm{d}\gamma(y,w)$$

with equality if (x,z) belongs to the support of γ.

4.2.4 Back to the Sobolev cost and optimality conditions

We aim to exploit the analysis in the above paragraph in the case of regularized (or not) fractional Sobolev cost with 4 variables:

$$c_{s,\varepsilon} = \frac{1}{2}(|z-x|^2 + |w-y|^2) + h_{s,\varepsilon}(x,z,y,w)$$

where $h_{s,\varepsilon}(x,z,y,w) = |z-w|^2 K_{s,\varepsilon}(x-y)$ (which gives, for $\varepsilon = 0$, $h_s(x,z,y,w) = (1-s)\dfrac{|x-y|^2}{|z-w|^{d+2s}}$). If s and ε are fixed, we take an optimal pair $(\varphi_{s,\varepsilon}, \psi_{s,\varepsilon})$ for this cost $c_{s,\varepsilon}$ and expect that the duality formula

$$\varphi_{s,\varepsilon}(x,y) + \psi_{s,\varepsilon}(z,w) + \sum_i \omega^i_{s,\varepsilon}(x,z)\omega^i_{s,\varepsilon}(y,w) \leq c_{s,\varepsilon}(x,z,y,w)$$

holds with equality on the support of the optimal $\sigma_{s,\varepsilon}$ and with

$$\int \sum_i \omega^i_{s,\varepsilon}(x,z)\omega^i_{s,\varepsilon}(y,w)\,\mathrm{d}\sigma_{s,\varepsilon}(x,z,y,w) = 0$$

$$\text{which means} \qquad \int \omega^i_{s,\varepsilon}(x,z)\,\mathrm{d}\gamma_{s,\varepsilon}(x,z) = 0 \qquad \text{for any } i$$

where, according to the Conjecture 4.2.1, $\gamma_{s,\varepsilon}$ is defined by $\sigma_{s,\varepsilon} = \gamma_{s,\varepsilon} \otimes \gamma_{s,\varepsilon}$. Notice that by setting $\widetilde{\varphi}_{s,\varepsilon} = \dfrac{1}{2}(|x|^2 + |y|^2) - \varphi_{s,\varepsilon}(x,y)$ and $\widetilde{\psi}_{s,\varepsilon} = \dfrac{1}{2}(|z|^2 + |w|^2) - \psi_{s,\varepsilon}(z,w)$, we can transform (4.6) into

$$\widetilde{\varphi}_{s,\varepsilon}(x,y) + \widetilde{\psi}_{s,\varepsilon}(z,w) + h_{s,\varepsilon}(x,z,y,w) - \sum_i \omega^i_{s,\varepsilon}(x,z)\omega^i_{s,\varepsilon}(y,w) \geq \langle x,z\rangle + \langle y,w\rangle$$

with equality if (x,z,y,w). Since we do not have any information about $\omega^i_{s,\varepsilon}$ except that its integral with respect to $\gamma_{s,\varepsilon}$ is null (where we assume that $\gamma_{s,\varepsilon} \otimes \gamma_{s,\varepsilon} = \sigma_{s,\varepsilon}$), our idea consists to integrate this inequality with respect to $\gamma_{s,\varepsilon}(y,w)$ which leads to

$$\int_\Omega \widetilde{\varphi}_{s,\varepsilon}(x,y)\,\mathrm{d}\mu(y) + \int_{\Omega'} \widetilde{\psi}_{s,\varepsilon}(z,w)\,\mathrm{d}\nu(w) + \int_{\Omega\times\Omega'} h_{s,\varepsilon}(x,z,y,w)\,\mathrm{d}\gamma_{s,\varepsilon}(y,w)$$
$$\geq \langle x,z\rangle + \int_{\Omega\times\Omega'} \langle y,w\rangle\,\mathrm{d}\gamma_{s,\varepsilon}(y,w) \tag{4.7}$$

with equality if (x,z) belongs to the support of $\gamma_{s,\varepsilon}$. Now the problem is the following:

- assume $\varepsilon = 0$. Then we know that γ_s is induced by a transport map T, and the integral

$$\int_{\Omega\times\Omega'} h_s(x,z,y,w)\,\mathrm{d}\gamma_{s,\varepsilon}(y,w) = \int_\Omega \frac{|z - T(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x$$

  is actually $+\infty$ except if $z = T(x)$. Thus, a reasoning like "if (z,y) belongs to the support of γ, then z is the critical point of some function of which we compute the gradient" does not hold here;

- if $\varepsilon > 0$, then all the functions considered in the inequality (4.7) are regular as well and we obtain

$$\Phi_{s,\varepsilon}(x) + \Psi_{s,\varepsilon}(y) + \int_{\Omega\times\Omega'} h_{s,\varepsilon}(x,z,y,w)\,\mathrm{d}\gamma_{s,\varepsilon}(y,w) \geq \langle x,z\rangle + \int_{\Omega\times\Omega'} \langle y,w\rangle\,\mathrm{d}\gamma_{s,\varepsilon}(y,w)$$

  for some functions $\Phi_{s,\varepsilon}$, $\Psi_{s,\varepsilon}$ and with equality on the support of $\gamma_{s,\varepsilon}$; but we do not have any information on $\gamma_{s,\varepsilon}$ (and the fact that it is induced by a transport map is not guaranteed).

Formal computations and back to a PDE. Let us notice however that a formal computation in the case $\varepsilon = 0$ allows to make the link with the partial differential equation that we obtained at the end of Chapter 2. Indeed, we have in this case

$$\Phi_s(x) + \Psi_s(z) + \int_{\Omega\times\Omega'} (1-s)\frac{|z-T(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}y \geq \langle x,z\rangle + \int_{\Omega\times\Omega'} \langle y,w\rangle\,\mathrm{d}\gamma_s(y,w)$$

with equality if (x,z) belongs to the support of $\gamma_s \otimes \gamma_s$, and where we have set

$$\Phi_s(x) = \int_\Omega \widetilde{\varphi}_s(x,y)\,\mathrm{d}y \qquad \text{and} \qquad \Psi_s(z) = \int_{\Omega'} \widetilde{\psi}_s(z,w)\,\mathrm{d}\nu(w)$$

If we were allowed to take the derivative of this equality, and since we know that in this case γ_s is induced by a transport map T, we would obtain that x a critical point of the function

$$\widetilde{x} \mapsto \Phi_s(\widetilde{x}) + \int_\Omega (1-s)\frac{|T(x)-T(y)|^2}{|\widetilde{x}-y|^{d+2s}}\,\mathrm{d}y - \langle \widetilde{x}, T(x)\rangle$$

This leads to the equality

$$\nabla\Phi_s(x) + \int_\Omega |T(x)-T(y)|^2 \nabla K_s(x-y)\,\mathrm{d}y = T(x)$$

By using the Lemma 4.2.4, we obtain if $\nabla\Psi_s(x)$ has a limit as $s \to 1$:

$$\nabla\Phi(x) + C_d({}^tDT(x)\Delta T(x) + \nabla(|DT|^2)(x)) = T(x) \tag{4.8}$$

This is the equation which appears in the Theorem 2.2.1. Notice also that if we use the "fact" that $T(x)$ is the critical point of the function

$$z \mapsto \Phi_s(x) + \Psi_s(z) + \int_\Omega (1-s)\int_\Omega \frac{|z-T(y)|^2}{|\widetilde{x}-y|^{d+2s}}\,\mathrm{d}y - \langle x, z\rangle$$

we obtain

$$\nabla\Psi(z) + 2(1-s)\int_\Omega \frac{z-T(y)}{|x-y|^{d+2s}}\,\mathrm{d}y = x$$

for $z = T(x)$, thus

$$\nabla\Psi_s(T(x)) - C_d\Delta^s T(x) = x \tag{4.9}$$

where Δ^s is the fractional Laplacian defined in the paragraph 4.2.2. Thus, we have actually at the limit $s \to 1$ a couple of equation

$$\left\{\begin{array}{l} \nabla\Phi(x) + C_d({}^tDT(x)\Delta T(x) + \nabla(|DT|^2)(x)) = T(x) \\ \nabla\Psi(T(x)) - C_d\Delta T(x) = x \end{array}\right. \tag{4.10}$$

Notice also that this system leads, for instance, to

$$\nabla\Phi(x) + {}^tDT(x)\nabla\Psi(T(x)) + C_d\nabla(|DT|^2)(x) = T(x) + {}^tDT(x)x$$

In other words,

$$\nabla(\Phi + \Psi\circ T + C_d|DT|^2 - \langle x, T(x)\rangle) = 0$$

thus this is a constant function on Ω; we are even able to compute this constant since the integral of $\Phi + \Psi\circ T$ (which are defined starting from the optimal pair (φ,ψ) for the dual problem with 4 variables) is known, and we find that this constant is null. Thus, we obtain formally the relation

$$\Phi(x) + \Psi\circ T(x) + C_d|DT(x)|^2 = \langle x, T(x)\rangle$$

At this point, it would be interesting to make the link between our "potentials" Φ, Ψ and the original Kantorovich potentials which appear when we study the classical Monge-Kantorovich problem (which verify, with the above notations, $\widetilde{\varphi}(x) + \widetilde{\psi}(T(x)) = \langle x, T(x)\rangle$).

The case of the regularized problem. We now return to the equation (4.7) in the case $\varepsilon > 0$ and we assume that $\gamma_{s,\varepsilon}$ is yet induced by a transport map T. In what follows, we set

$$H_{s,\varepsilon}(x,z) = \int_\Omega |z - T(y)|^2 K_{s,\varepsilon}(x-y)\,\mathrm{d}y$$

We first notice that the function $\Psi_{s,\varepsilon}$ can be replaced with

$$z \mapsto \sup_z \left\{ H_{s,\varepsilon}(x,z) - \Phi_{s,\varepsilon}(x) - \int_\Omega \langle y, T(y)\rangle\,\mathrm{d}y - \langle x, z\rangle \right\}$$

Since we have $D^2_{z,z}H_{s,\varepsilon}(x,y) = 2I_d \int_\Omega K_{s,\varepsilon}(x-y)\,\mathrm{d}y \leq C_{s,\varepsilon} I_d$ for some constant $C_{s,\varepsilon}$ (which goes to $+\infty$ as $\varepsilon \to 0$), the function whose we are taking the supremum are all semi-concave with respect to z. Thus, the obtained function is itself semi-concave and the constraints and the value of the integral $\int(\Phi_{s,\varepsilon}\,\mathrm{d}\mu + \Psi_{s,\varepsilon}\,\mathrm{d}\nu)$ did not change. The same reasoning holds for $\Psi_{s,\varepsilon}$. We conclude that *we can assume the "potentials" $\Phi_{s,\varepsilon}$, $\Psi_{s,\varepsilon}$ to be semi-concave* (this is similar to the classical transport theory where the potential functions are concave; however, the semi-concavity here does not hold at the limit $\varepsilon \to 0$).

Now we know that any $x \in \Omega$ is a critical point of the function

$$\widetilde{x} \mapsto \Phi_{s,\varepsilon}(\widetilde{x}) + \Psi_\varepsilon(T(x)) + H_{s,\varepsilon}(\widetilde{x}, T(x)) - \langle \widetilde{x}, T(x)\rangle - \int_\Omega \langle y, T(y)\,\mathrm{d}y$$

and that $T(x)$ is a critical point of the function

$$z \mapsto \Phi_{s,\varepsilon}(x) + \Psi_{s,\varepsilon}(z) + H_{s,\varepsilon}(x,z) - \langle x, z\rangle - \int_\Omega \langle y, T(y)\,\mathrm{d}y$$

The optimality conditions are then the following couple of equations:

$$\begin{cases} \nabla\Phi_{s,\varepsilon}(x) + \displaystyle\int_\Omega |T(x) - T(y)|^2 \nabla K_{s,\varepsilon}(x-y)\,\mathrm{d}y = T(x) \\ \nabla\Psi_{s,\varepsilon}(x) + 2\displaystyle\int_\Omega (T(x) - T(y)) K_{s,\varepsilon}(x-y)\,\mathrm{d}y = x \end{cases}$$

where we know moreover that our potential functions $\Phi_{s,\varepsilon}$, $\Psi_{s,\varepsilon}$ are both semi-concave. We can verify that (4.8), (4.9) both appear at the limit $\varepsilon \to 0$ and for a fixed s (still provided that the functions $\Phi_{s,\varepsilon}, \Psi_{s,\varepsilon}$ have a limit as $\varepsilon \to 0$). Also, one can write the equations where $\varepsilon > 0$ stay fixed but $s \to 1$:

$$\begin{cases} \nabla\Phi_\varepsilon(x) + \dfrac{1}{2}\Delta(\nabla|T|^2 \star \rho_\varepsilon)(x) - C_d\Delta({}^t DT \star \rho_\varepsilon)(x)T(x) = T(x) \\ \nabla\Psi_\varepsilon(T(x)) - C_d\Delta(T \star \rho_\varepsilon)(x) = x \end{cases}$$

and again, we verify that (4.10) follows as a limit of this system as $\varepsilon \to 0$.

The open question are several:

- how to show that the infimum among Σ is attained by a tensor product $\gamma \otimes \gamma$? (we suggest the following tracks: study the extremal points or the set Σ; or examine the case of finite measures, which is reduced to a matrix problem);
- how to show that, in the regularized problem with $\varepsilon > 0$, the optimal γ (in the problem $\inf c\,\mathrm{d}\gamma \otimes \gamma$ among the measures with prescribed marginals) is induced by a transport map? (one could envisage to establish a "twist condition" on the function $H_{s,\varepsilon}$ but this seems not clear for the moment)

However, we emphasize the fact that the optimality condition that we obtain formally at the limit coincides with the partial differential equation coming from the measure-preserving perturbation of the optimal T.

Appendix A

Tools of measure theory and calculus of variations

A.1 Basis of measure theory

A.1.1 Image measure

In this paragraph, we recall the definition and characterizations of the image measure:

Definition A.1.1. *Let (X, Σ), (Y, Σ') be two measurable spaces. If $Y : X \to Y$ is a map and μ is a positive measure on X, we define the image measure of μ by T, or push-forward of μ by T, as*

$$T_{\#}\mu(B) = \mu(T^{-1}(B))$$

for any $B \in \Sigma'$. We also say that T sends the measure μ onto the measure $T_{\#}\mu$.

As a consequence of the definition, the measures μ and $T_{\#}\mu$ still have same mass. The following characterization is used frequently:

Proposition A.1.1. *Let (X, Σ), (Y, Σ') be two measurable spaces, $T : X \to Y$ be a map and μ be a positive measure on X. Let $\varphi : Y \to \mathbb{R}$ be a measurable function. Then φ is integrable with respect to $T_{\#}\mu$ if and only if $\varphi \circ T$ is integrable with respect to μ and in this case the following equality holds:*

$$\int_Y \varphi(y)\,\mathrm{d}(T_{\#}\mu)(y) = \int_X \varphi(T(x))\,\mathrm{d}\mu(x)$$

On the other hand, if ν is a positive measure on Y and the equality

$$\int_Y \varphi\,\mathrm{d}\nu = \int_X \varphi \circ T\,\mathrm{d}\mu$$

holds for any function $\varphi \in L^1_\mu(X)$ (or, if X, Y are metric spaces and μ, ν are Borel measures, for any continuous and bounded function φ on X), then $\nu = T_{\#}\mu$.

In this context, a natural question is the following: given a measure μ and a map T, how to compute the image measure $T_{\#}\mu$? The following proposition gives the answer if T is regular and one-to-one:

Proposition A.1.2. *Assume $X = \mathbb{R}^d$ and μ to be absolutely continuous with respect to the Lebesgue measure, with density f. Assume $\operatorname{supp} f = \overline{\Omega}$ for a bounded open set $\Omega \subset \mathbb{R}^d$. Let $T : \Omega \to \mathbb{R}^d$ be a measurable map; assume that T is C^1 and injective on Ω with non-null Jacobian determinant; we set $\Omega' = T(\Omega)$. Then the image measure $T_\# \mu$ has a density g on Ω'; moreover, we have for any $x \in \Omega$ the identity*

$$|\det DT(x)| = \frac{f(x)}{g(T(x))} \tag{A.1}$$

Proof. Given φ a continuous and bounded function on Ω', we compute the integral of φg (with respect to the Lebesgue measure) by changing of variable:

$$\int_{\Omega'} \varphi(y) g(y)\,\mathrm{d}y = \int_\Omega \varphi(T(x)) g(T(x)) |\det DT(x)|\,\mathrm{d}x$$

On the other hand, we know since $\nu = T_\# \mu$ that

$$\int_{\Omega'} \varphi\,\mathrm{d}\nu = \int_\Omega \varphi \circ T\,\mathrm{d}\mu \qquad \textit{i.e.} \qquad \int_{\Omega'} \varphi(y) g(y)\,\mathrm{d}y = \int_\Omega \varphi(T(x)) f(x)\,\mathrm{d}x$$

Thus, the equality

$$\int_\Omega \varphi(T(x)) g(T(x)) |\det DT(x)|\,\mathrm{d}x = \int_\Omega \varphi(T(x)) f(x)\,\mathrm{d}x$$

holds for any continuous and bounded function φ on Ω. This implies $g(T(x))|\det DT(x)| = f(x)$ for any $x \in \Omega$. □

Notice that one can also obtain the equality (A.1) by applying the coarea formula, which more generally implies

$$g(y) = \sum_{x \in \Omega : y = T(x)} \frac{f(x)}{|\det DT(x)|}$$

for any Lipschitz function $\Omega \to \mathbb{R}^d$ and $y \in T(\Omega)$ (see for instance [39], paragraph 3.4.2). In this thesis, we give an elementary proof of a similar formula in a particular case on the real line (piecewise monotone maps, see Chapter 2, Prop. 2.1.3).

Finally, we also state here the definitions of the *marginals* of a measure definied on a finite product of measure spaces (which is used frequentely in optimal transport theory):

Definition A.1.2. *Let $X_1, \ldots, X_p$ be finitely many measure spaces and γ a measure of the product space $X_1 \times \cdots \times X_p$. For $1 \le i \le p$, we call i-th marginal of γ the measure on X_i denoted by μ_i defined by*

$$\mu_i = (\pi_i)_\# \gamma \qquad \textit{where} \qquad \pi_i : (x_1, \ldots, x_p) \in X_1 \times \cdots \times X_p \mapsto x_i \in X_i$$

A.1.2 Other useful results

We here give statements of some classical theorems of measure theory which are used in this thesis.

Theorem A.1.1 (Lusin's theorem)**.** *Let μ be a Borel measure on $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ be a measurable function. Let $A \subset \mathbb{R}^d$ be a Borel set with $\mu(A) < +\infty$. Then, for any $\varepsilon > 0$, there exists a compact set $K \subset A$ such that $\mu(A - K) \le \varepsilon$ and $f|_K$ is continuous.*

The proof can be found for instance in [39], paragraph 1.2, or in [40], Theorem 2.3.5.

Tight families of measures and Prokhorov's theorem.

Definition A.1.3. *Let (X, d, Σ) be a metric measure space.*

- *We say that a sequence $(\mu_n)_n$ of positive measures on X weakly converges to a measure μ if the convergence*
$$\int_X \varphi \,\mathrm{d}\mu_n \xrightarrow[n\to+\infty]{} \int_X \varphi \,\mathrm{d}\mu$$
holds for any continuous and bounded function φ on X

- *A family $\mathcal{F}$ of probability measures on X is* tight *if, for any $\varepsilon > 0$, there exists a compact subset K of X such that $\mu(K) \geq 1 - \varepsilon$ for any $\mu \in \mathcal{F}$.*

Theorem A.1.2 (Prokhorov's theorem, see [37], Th. 9.3.3)**.** *Let $(\mu_n)_n$ be a sequence of probability measures on X. If the family $\{\mu_n : n \in N\}$ is tight, then there exists a subsequence $(\mu_{n_k})_k$ which weakly converges to a probability measure μ.*

Skorokhod's theorem.

Theorem A.1.3 (Skorokhod's theorem, see [13], Th. 25.6, or [11], Th. 4.2.8, for an application of optimal transport)**.** *If (X, d) is a metric space and $(\mu_n)_n$ is a sequence of probability measures on X having a weak limit μ, then there exists a sequence $(\chi_n)_n$ of measurable functions $(0,1) \to X$ and a function $\chi : (0,1) \to X$ such that*

- *$\chi_n(x) \to \chi(x)$ for a.e. $x \in (0,1)$;*

- *$\mu = \chi_\# \mathcal{L}^1$ and, for any n, $\mu_n = (\chi_n)_\# \mathcal{L}^1$.*

Disintegration of measure. We state this result under sufficient assumptions for us:

Theorem A.1.4 (Disintegration, see [4], Th. 5.3.1 or [32], III-70)**.** *Let X, Y be two complete separable metric spaces and let γ be a finite measure on $X \times Y$. We denote by μ, ν the first and second marginals of γ respectively. Then there exists two measurable families of probability measures $(\gamma_x)_{x\in X}$ and $(\gamma_y)_{y\in Y}$ such that*
$$\mathrm{d}\gamma = \mathrm{d}\gamma_x(y) \otimes \mathrm{d}\mu(x) = \mathrm{d}\gamma_y(x) \otimes \mathrm{d}\nu(y)$$
which means that, for any continuous and bounded function $\varphi : X \times Y \to \mathbb{R}$,
$$\iint_{X\times Y} \varphi \,\mathrm{d}\gamma = \int_X \left(\int_Y \varphi(x,y) \,\mathrm{d}\gamma_x(y) \right) \mathrm{d}\mu(x) = \int_Y \left(\int_X \varphi(x,y) \,\mathrm{d}\gamma_y(x) \right) \mathrm{d}\nu(y)$$

A.2 Calculus of variations and basis of Γ-convergence.

We use several times several results of semicontinuity with respect to the weak convergence, which are all consequences of this general result (see [29], Theorem 3.4):

Theorem A.2.1. *Let Ω be a bounded open set of $\mathbb{R}^d$ with Lipschitz boundary, and $p \geq 1$. Let $L : \overline{\Omega} \times \mathbb{R}^d \times M_d(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ be a map satisfying the two following conditions:*

- *L is a Caratheodory function, i.e. $L(\cdot, u, \xi)$ is measurable for any (u, ξ) and $L(x, \cdot, \cdot)$ is continuous for any x;*

- *for a.e. x and any u, ξ, we have the inequality*

$$L(x, u, \xi) \geq \langle a(x), \xi \rangle + b(x)$$

where b is a function of $L^p(\Omega)$ and a is a function of $L^{p'}(\Omega)$ with matrix values;

- *for a.e. $x \in \Omega$ and any $u \in \mathbb{R}^d$, the map $\xi \mapsto L(x, u, \xi)$ is convex on $M_d(\mathbb{R})$.*

Then, the map

$$T \in W^{1,p}(\Omega) \mapsto \int_\Omega L(x, T(x), DT(x))\,\mathrm{d}x$$

is lower semi-continuous with respect to the weak convergence in $W^{1,p}(\Omega)$.

Γ-convergence: definitions and basic properties. All the following definitions and properties of Γ-convergence can be found in [31] or [19] with very much more details. In that follows, (X, d) is a metric space.

Definition A.2.1. *Let $(F_n)_n$ be a sequence of functions $X \mapsto \overline{\mathbb{R}}$. We say that $(F_n)_n$ Γ-converges to F, and we write $F_n \xrightarrow[n]{\Gamma} F$ if, for any $x \in X$, we have*

- *for any sequence $(x_n)_n$ of X converging to x,*

$$\liminf_n F_n(x_n) \geq F(x) \qquad (\Gamma\text{-liminf inequality});$$

- *there exists a sequence $(x_n)_n$ converging to x and such that*

$$\limsup_n F_n(x_n) \leq F(x) \qquad (\Gamma\text{-limsup inequality}).$$

This definition is actually equivalent to the following equalities for any $x \in X$:

$$F(x) = \inf\left\{\liminf_n F_n(x_n) : x_n \to x\right\} = \inf\left\{\limsup_n F_n(x_n) : x_n \to x\right\}$$

The function $x \mapsto \inf\left\{\liminf_n F_n(x_n) : x_n \to x\right\}$ is called Γ-liminf of the sequence $(F_n)_n$, and the other one its Γ-limsup. A useful result is the following (which, for instance, implies that a constant sequence of functions does not Γ-converge to itself in general):

Proposition A.2.1. *The Γ-liminf and the Γ-limsup of a sequence of functions $(F_n)_n$ are both lower semi-continuous on X.*

Proof. Let us denote it by F^- this Γ-liminf , and let $(x_n)_n$ be a sequence of X having a limit x. We will show that, for any $\varepsilon > 0$, we have $F_n(x_n) \geq F^-(x) - 2\varepsilon$ for n large enough (depending on ε), which implies $\liminf_n F_n(x_n) \geq F(x)$.

We recall that $F^-(x_n) = \inf\{\liminf_k F_k(x_{n,k}) : x_{n,k} \to x_n\}$. Thus, for any $n \in \mathbb{N}$, we take a sequence $(x_{n,k})_k$ with $\liminf_k F_k(x_{n,k}) \leq F^-(x_n) + \varepsilon$; in particular, we have

$$F_k(x_{n,k}) \leq F^-(x_n) + \frac{1}{n} + \varepsilon \qquad \text{for infinitely many } k \in \mathbb{N} \tag{A.2}$$

Now we find an extraction φ such that $(x_{n,\varphi(n)})_n$ converges to x as $n \to +\infty$ and (A.2) is true with $k = \varphi(n)$. We then can write $x_{n,\varphi(n)} = y_{\varphi(n)}$ for a sequence $(y_m)_m$ having x for limit (for instance, by setting $y_m = x_{n,\varphi(n)}$ with $\varphi(n) \leq m < \varphi(n+1)$), so that

$$F_{\varphi(n)}(x_{n,\varphi(n)}) \geq F^-(x) - \varepsilon \tag{A.3}$$

for n large enough, since $\liminf_n (F_{\varphi(n)}(x_{n,\varphi(n)})) \geq F^-(x)$. By combining (A.2) for $k = \varphi(n)$ and (A.3), we get

$$F^-(x_n) \geq F^-(x) - 2\varepsilon + \frac{1}{n}$$

for n large enough, which achieves the proof. □

The main interest of Γ-convergence is its consequences in terms of convergence of minima:

Theorem A.2.2. *Let $(F_n)_n$ be a sequence of functions $X \to \overline{\mathbb{R}}$ and assume that $F_n \xrightarrow[n]{\Gamma} F$. Assume moreover that there exists a compact and non-empty subset K of X such that*

$$\forall n \in N,\ \inf_X F_n = \inf_K F_n$$

(we say that $(F_n)_n$ is equi-mildly coercive on X). Then F admits a minimum on X and the sequence $(\inf_X F_n)_n$ converges to $\min F$. Moreover, if $(x_n)_n$ is a sequence of X such that

$$\lim_n F_n(x_n) = \lim_n (\inf_X F_n)$$

and if $(x_{\varphi(n)})_n$ is a subsequence of $(x_n)_n$ having a limit x, then $F(x) = \inf_X F$.

We finish with the following result, which allows to focus on the Γ-limsup inequality only on a dense subset of X under some assumptions:

Proposition A.2.2. *Let $(F_n)_n$ be a sequence of functionals and F be a functional $X \to \overline{\mathbb{R}}$. Assume that there exists a dense subset $Y \subset X$ such that:*

- *for any $x \in X$, there exists a sequence $(x_n)_n$ of Y such that $x_n \to x$ and $F(x_n) \to F(x)$;*
- *the Γ-limsup inequality holds for any $x \in Y$.*

Then it holds for any x belonging to the whole x.

Proof. This comes directly from the lower semi-continuity of the Γ-limpsup: denoting by

$$F^+(x) = \inf\{\limsup F_n(x_n) : x_n \to x\}$$

we chose, given $x \in X$, a sequence $(x_n)_n$ converging to x and with $F(x_n) \to F(x)$. We know that $F^+(x_n) \leq F(x_n)$ for any x, and that F^+ is lower semi-continuous on X, which leads directly to $F^+(x) \leq F(x)$ by passing to the limit $n \to +\infty$; this is the Γ-limsup inequality. □

A.3 The Dacorogna and Moser's result

We finish by giving a "transportation" result (which is not "optimal" transportation, but could be called "regular" transportation) by Dacorogna and Moser (see [30]). They gave sufficient conditions to send a density f onto a density g by a regular diffeomorphism, where f, g satisfy regularity assumptions and are defined on the same domain Ω whose boundary also satisfies regularity assumptions.

More precisely, the statement that we use in this thesis is the following:

Theorem A.3.1. *Let $U \subset \mathbb{R}^d$ be a bounded open set with $C^{3,\alpha}$ boundary ∂U. Let f_1, f_2 be two positive Lipschitz functions on $\overline{U}$ such that*

$$\int_U f_1 = \int_U f_2$$

Then there exists a Lipschitz diffeomorphism $u : \overline{U} \to \overline{U}$ satisfying

$$\begin{cases} \det \nabla u(x) = \dfrac{f_1(x)}{f_2(u(x))}, & x \in U \\ u(x) = x, & x \in \partial U \end{cases}$$

Moreover we have the following estimate:

$$\operatorname{Lip} u \leq \exp\left(C(||f_1 - f_2||_\infty + \operatorname{Lip} f_1 + \operatorname{Lip} f_2) \left(\frac{1}{\inf f_1 \wedge f_2} + \frac{\operatorname{Lip} f_1 + \operatorname{Lip} f_2}{(\inf f_1 \wedge f_2)^2} \right) \right) \tag{A.4}$$

The result given in the original paper ([30], Theorem 1') deals with more general assumptions on the density f_1 ($f \in C^{k+3,\alpha}(\overline{U})$ and the result is a $C^{k+1,\alpha}$ diffeomorphism) but only with $f_2 = 1$, which corresponds to the Lebesgue measure (up to a normalizing constant). We adapt it in a more general case for the target density and give moreover an estimate on the regularity of the transport that we obtained which is vacant in the original paper (but not so difficult to obtain). This estimate is probably not "sharp", but sufficient in the case which we are interested in (see paragraph 3.4.2).

Sketch of the proof. Following the original paper, we begin by finding a solution $v \in C^{1,\alpha}(\overline{U})$ of

$$\begin{cases} \operatorname{div} v(x) = f_1(x) - f_2(x), & x \in U \\ v(x) = 0, & x \in \partial U \end{cases}$$

with the estimate (coming from the standard theory of elliptic equations, see [44])

$$||v||_{C^{1,\alpha}} \leq K||f_1 - f_2||_{C^{0,\alpha}} \leq C(||f_1 - f_2||_\infty + \operatorname{Lip}(f_1 - f_2)) \tag{A.5}$$

where C only depends on α, f_1, f_2, U. Then we set

$$w(t,x) = \frac{v(x)}{(1-t)f_1(x) + tf_2(x)}$$

and consider the solution Φ of the Cauchy problem.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x) = w(t, \Phi(t,x)) \\ \Phi(0,x) = x \end{cases}$$

We define $u(x) = \Phi(1,x)$. Then it remains to check that u satisfies the announced equation. Moreover, a standard Grönwall argument provides

$$||\Phi(t,x) - \Phi(t,y)|| \leq ||x - y|| \exp\left(\int_0^t \operatorname{Lip}(w(s,\cdot))\, \mathrm{d}s \right)$$

where the Lipschitz constant of $w(s,\cdot)$, which is a quotient of Lipschitz and bouded from below functions, is estimated as follows:

$$\operatorname{Lip} w(s,\cdot) \leq \frac{\operatorname{Lip} v}{\inf((1-s)f_1 + sf_2)} + ||v||_\infty \operatorname{Lip}\left(\frac{1}{(1-s)f_1 + sf_2} \right)$$

The estimate (A.5) allows immediately to control the Lipschitz cnstant of v. On the other hand, v is valued in U, thus $||v||_\infty$ is also bounded. This leads to the announced estimate (A.4). □

Appendix B

Some facts about optimal transport

B.1 General theory: Monge and Kantorovich problems, duality formula and existence results

Monge problem. The original formulation of the optimal transport problem is the following: if X, Y are two metric spaces (say complete and separable, but in this thesis we were essentially focused on the case of compact subsets of $\mathbb{R}^d$) and $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$, we study the so-called *Monge problem*

$$\inf \left\{ \int_X c(x, T(x))\, \mathrm{d}\mu(x) : T : X \to Y, T_{\#}\mu = \nu \right\}$$

where μ, ν are two finite measures on X, Y with same mass (one can assume that they are both probability measures). Depending on the spaces X, Y and on the measures μ, ν, the existence of transport maps from μ to ν is not guaranteed in general; moreover, even under this assumption and in regular case, the existence of a minimizer cannot be obtained, for instance, thanks to the direct method in the calculus of variations. The henceforth well-known method, due to Kantorovich, consists in introducing the following relaxation.

Kantorovich problem. We study

$$\inf \left\{ \int_{X\times Y} c(x, y)\, \mathrm{d}\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\} \tag{B.1}$$

where $\Pi(\mu, \nu)$ denotes the set of probability measures γ on the product space $X \times Y$ such that

$$(\pi_1)_{\#}\gamma = \mu \qquad \text{and} \qquad (\pi_2)_{\#}\gamma = \nu \qquad \text{where} \quad \begin{cases} \pi_1(x, y) = x \\ \pi_2(x, y) = y \end{cases}$$

An immediate advantage of this minimization problem is that the set $\Pi(\mu, \nu)$ is never empty (it contains $\mu \otimes \nu$); moreover, the functional that we consider is here linear with respect to the variable γ. The existence result follows:

Proposition B.1.1. *Assume that μ, ν are two probability measures on $\mathbb{R}^d$ and that $c : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ is lower semi-continuous. Then there exists at least a solution to the Kantorovich minimization problem*

$$\inf \left\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} c(x, y)\, \mathrm{d}\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\}$$

Dual problem. Now we remark that, if γ is a positive measure on $\mathbb{R}^d \times \mathbb{R}^d$, then

$$\sup\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x)\,\mathrm{d}\gamma(x,y) - \int_{\mathbb{R}^d} \varphi\,\mathrm{d}\mu : \varphi \in C_b(\mathbb{R}^d)\right\} = \begin{cases} 0 & \text{if } (\pi_1)_\#\gamma = \mu \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\sup\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} \psi(y)\,\mathrm{d}\gamma(x,y) - \int_{\mathbb{R}^d} \psi\,\mathrm{d}\nu : \psi \in C_b(\mathbb{R}^d)\right\} = \begin{cases} 0 & \text{if } (\pi_2)_\#\gamma = \nu \\ +\infty & \text{otherwise} \end{cases}$$

Thus, the Kantorovich problem can be written

$$\inf_{\gamma\in\mathcal{M}^+(\mathbb{R}^d\times\mathbb{R}^d)} \Bigg\{ \sup_{(\varphi,\psi)\in C_b(\mathbb{R}^d)^2} \Bigg\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} c(x,y)\,\mathrm{d}\pi(x,y) - \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x) dx\gamma(x,y) - \int_{\mathbb{R}^d} \varphi\,\mathrm{d}\mu\right) - \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \psi(y)\,\mathrm{d}\gamma(x,y) - \int_{\mathbb{R}^d} \psi\,\mathrm{d}\nu\right)\Bigg\}\Bigg\}$$

Let us assume that the infimum and the supremum can be inverted. Notice that the quantity

$$\inf_{\gamma\in\mathcal{M}^+(\mathbb{R}^d\times\mathbb{R}^d)} \left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} (c(x,y) - (\varphi(x) + \psi(y)))\,\mathrm{d}\gamma(x,y)\right\}$$

is 0 if $c(x,y) - \varphi(x) - \psi(y) = 0$ and $-\infty$ otherwise. This suggests to look at the minimization problem

$$\sup\left\{\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}\mu(x) + \int_{\mathbb{R}^d} \psi(y)\,\mathrm{d}\nu(y) : \varphi,\psi \in C_b(\mathbb{R}^d), \varphi(x) + \psi(y) \le c(x,y)\right\} \tag{B.2}$$

The problem (B.2) is called *dual problem.*

Convexity, c-concavity, cyclical monotonicity and duality formula. Notice that if a pair (φ,ψ) is a candidate for (B.2), then $(\overline{\varphi},\overline{\psi})$ defined by

$$\overline{\psi}(y) = \inf_x\{c(x,y) - \varphi(x)\} \qquad \text{and} \qquad \overline{\varphi}(x) = \inf_y\{c(x,y) - \overline{\psi}(y)\}$$

is also admissible and satisfies $\overline{\psi} \ge \psi$ and $\overline{\varphi} \ge \varphi$. This suggests to study the problem (B.2) among the such pairs.

We henceforth assume that μ, ν are supported into bounded open subsets Ω, Ω' of $\mathbb{R}^d$ and that c is defined on $\overline{\Omega} \times \overline{\Omega}'$.

Definition B.1.1. *We say that a function $\varphi : \overline{\Omega} \to \mathbb{R}$ is c-concave if there exists a function $\chi : \overline{\Omega}' \to \mathbb{R}$ such that*

$$\varphi(x) = \inf_{y\in\overline{\Omega}'}\{c(x,y) - \chi(y)\}$$

for any $x \in \Omega$. Similarly, a function $\psi : \overline{\Omega} \to \overline{\mathbb{R}}$ is said to be c-concave if

$$\psi(y) = \inf x \in \overline{\Omega}\{c(x,y) - \xi(x)\}$$

for some function $\xi : \overline{\Omega} \to \overline{\mathbb{R}}$ and for any $y \in \overline{\Omega}'$.

Moreover, for two function $\varphi : \overline{\Omega} \to \mathbb{R}$ and $\psi : \overline{\Omega}' \to \mathbb{R}$ (no necessary c-concave), we denote by

$$\varphi^c(y) = \inf\{c(x,y) - \varphi(x) : x \in \overline{\Omega}\} \qquad \text{and} \qquad \psi^{\overline{c}}(x) = \inf\{c(x,y) - \psi(y) : y \in \overline{\Omega}\}$$

The above analysis suggest the following method to solve (B.2): starting from a maximizing sequence $(\varphi_n, \psi_n)_n$, we notice that the sequence $((\varphi_n^c)^{\overline{c}}, \varphi_n^c)_n$ is still formed by admissible functions and, for any n, $(\varphi_n^c)^{\overline{c}} + (x)\varphi_n^c(y) \geq \varphi_n(x) + \psi_n(y)$. This leads to the following existence result:

Proposition B.1.2. *Assume μ, ν to be compactly supported and c to be continuous on the product of their supports. Then the problem* (B.2) *admits a solution(φ, ψ), with φ is a c-concave function and $\psi = \varphi^c$.*

We now introduce, for a subset of $\mathbb{R}^d$, the notion of *c-cyclical monotonicity*:

Proposition B.1.3. *Let γ be an optimizer for the Kantorovich problem* (B.1)*. Assume c to be continuous. Then, for any finite family of pairs $(x_1, y_1), \ldots, (x_p, y_p)$ belonging to the support of γ and for any permutation σ of $\{1, \ldots, p\}$, the following inequality holds:*

$$\sum_{i=1}^{p} c(x_i, y_i) \leq \sum_{i=1}^{p} c(x_i, y_{\sigma(i)})$$

We say that the support of γ is a c-cyclically monotone subset of $(\mathbb{R}^d)^2$.

The following property of c-cyclically monotone subsets of $(\mathbb{R}^d)^2$ will imply the equality between Kantorovich and Monge problems:

Lemma B.1.1. *Assume that Γ is a c-cyclically monotone subset of $(\mathbb{R}^d)^2$. Then there exists a c-concave function φ such that*

$$\Gamma \subset \{(x, y) : \varphi(x) + \overline{\varphi}(y) = c(x, y)\}$$

By taking (φ, φ^c) as candidate for the dual problem, we obtain:

Proposition B.1.4. *With the above assumptions on μ, ν, the optimal values of the problems* (B.1) *and* (B.2) *are equal:*

$$\inf_{\gamma \in \Pi(\mu,\nu)} \left\{ \int c(x, y) \,\mathrm{d}\gamma(x, y) \right\} = \sup_{\substack{(\varphi, \psi) \in C_b(\Omega) \times C_b(\Omega') \\ \varphi(x) + \psi(y) \leq c(x, y)}} \left\{ \int \varphi(x) \,\mathrm{d}\mu(x) + \int \psi(y) \,\mathrm{d}\nu(y) \right\}$$

Moreover, the optimal value of (B.2) *is attained by a pair (φ, φ^c) for a c-concave function φ . Finally, if γ is the optimizer of the Kantorovich problem* (B.1)*, the equality*

$$\varphi(x) + \psi(y) = c(x, y)$$

holds for any (x, y) in the support of γ.

Back to the Monge problem in the strictly convex case. We conclude with the existence and uniqueness result for the Monge and the Kantorovich problem in the case of strictly convex costs. Given a Kantorovich potential φ and an optimal transport plan γ, we start from the inequality

$$\varphi(x) + \overline{\varphi}(y) \leq c(x, y)$$

for any $(x, y) \in (\mathbb{R}^d)^2$ with equality if (x, y) belongs to the support of γ. Thus, for such a (x, y), the function

$$\widetilde{x} \mapsto c(\widetilde{x}, y) - \varphi(\widetilde{x})$$

is maximal for $\widetilde{x} = x$. Assume now that $c(x, y) = h(y - x)$ for a strictly convex function h; in this case, it can be proven that any c-concave function is actually Lipschitz continuous and x is a critical point of the function $h(\cdot - y) - \varphi$, which means

$$\nabla h(x - y) - \nabla \varphi(x) = 0$$

(provided that x is a differentiability point for φ, thus for a.e. $x \in \Omega$). Since h is strictly convex, its gradient can be inverted and we obtain $y = x - (\nabla h)^{-1}(\nabla \varphi(x))$, thus y is well-defined in function of x. To summarize:

Theorem B.1.1. *We make the above assumptions on μ, ν and assume moreover that μ is absolutely continuous with respect to the Lebesgue measure. Assume that $c(x, y) = h(x - y)$ for a strictly convex function h. Then there exists a unique solution γ of the Kantorovich problem* (B.1) *and a unique map T such that $\gamma = (id \times T)_{\#}\mu$. Moreover, there exists at least one Kantorovich potential φ which is a Lipschitz function, and we have the equality*

$$T(x) = x - (\nabla h)^{-1}(\nabla \varphi(x))$$

for a.e. x belonging to the support of μ.

B.2 Optimal transport on the real line

We here summarize some well-known results about the optimal transportation problem in dimension 1.

Theorem B.2.1. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$. Assume μ to be atomlesss. Then there exists a unique non-decreasing map $T_1 : \mathbb{R} \to \mathbb{R}$ and a unique non-increasing map $T_2 : \mathbb{R} \to \mathbb{R}$ sending μ to ν.*

Proof. Step 1: existence of the non-decreasing map. We define

$$F_\mu(x) = \mu((-\infty, x]) \qquad \text{and} \qquad F_\nu(y) = \nu((-\infty, y])$$

It is clear that F_μ, F_ν are both non-decreasing and right-continuous, and have 0 as limit in $-\infty$ and 1 as limit in $+\infty$; moreover, the fact that μ is atomlesss guarantees F_μ to be continuous on $\mathbb{R}$. Now we set

$$T_1(x) = \inf \{y : F_\nu(y) \geq F_\mu(x)\} \tag{B.3}$$

We moreover claim that the infimum in the right-hand-side of (B.3) is attained, since, if $(y_n)_n$ is a minimizing sequence, one can assume it to be decreasing since F_ν is non-decreasing so that $y_n \to y$ and, since F_ν is right-continuous, $F_\nu(y) = \lim_n F_\nu(y_n) \geq F_\mu(x)$.

Since F_μ and F_ν are both non-decreasing, it is clear that T_1 is also non-decreasing, and it remains to check that it sends μ onto ν; for this, it is enough to show the equality

$$\mu(T^{-1}((-\infty, b])) = \nu((-\infty, b]))$$

for any $b \in \mathbb{R}$. We begin by noticing the equivalence

$$x \in T_1^{-1}((-\infty, b]) \iff T_1(x) \leq b \iff F_\mu(x) \leq F_\nu(b) \tag{B.4}$$

If $F_\nu(b) = 1$, then the last inequality is satisfied for any $x \in \mathbb{R}$, so that $T^{-1}((-\infty, b]) = \mathbb{R}$ and

$$\mu(T^{-1}((-\infty, b])) = 1 = F_\nu(b) = \nu((-\infty, b])$$

Assume now that $F_\nu(b) < 1$. In this case, since F_μ is continuous and non-decreasing, we deduce from (B.4) that $T_1^{-1}((\infty, b])$ is an interval with form $(-\infty, d]$ with $F_\mu(d) = F_\nu(b)$; this provides

$$\mu(T^{-1}(-\infty, b])) = \mu((-\infty, d]) = F_\mu(d) = F_\nu(b) = \nu((-\infty, b])$$

which is the announced result.

Step 2: uniqueness of the non-decreasing map. We now define for $y \in \mathbb{R}$:

$$G_\nu(y) = \nu((-\infty, y))$$

G_ν is also non-decreasing and is moreover left-continuous; moreover, we have $G_\nu \leq F_\nu$ with equality everywhere except on the atoms of ν. We then claim the following: if U is a non-decreasing map sending μ to ν, then, for any $x \in \mathbb{R}$,

$$G_\nu(U(x)) \leq F_\mu(x) \leq F_\nu(U(x)) \tag{B.5}$$

Indeed, the monotonicity of U implies

$$U^{-1}((-\infty, U(x))) \subseteq (-\infty, x] \subseteq U^{-1}((-\infty, U(x)])$$

and (B.5) follows by computing the measure μ of these third sets. Since (B.5) holds for any map U which is non-decreasing and sends μ onto ν, if U, $\widetilde{U}$ are two such maps and x verifies $U(x) < \widetilde{U}(x)$, we have

$$G_\nu(\widetilde{U}(x)) \leq F_\mu(x) \leq F_\nu(U(x))$$

On the other hand,

$$(-\infty, U(x)] \subseteq (-\infty, \widetilde{U}(x)) \qquad \text{thus} \qquad F_\nu(U(x)) \leq G_\nu(\widetilde{U}(x))$$

By combining the two inequalities above, we get that, for such a x,

$$U(x) < \widetilde{U}(x) \qquad \text{and} \qquad G_\nu(U(x)) = G_\nu(\widetilde{U}(x))$$

Since G_ν is non-decreasing, this proves that G_ν is constant on the interval $[U(x), \widetilde{U}(x)]$ with value $F_\mu(x)$. In particular, $F_\mu(x)$ belongs to

$$L = \{y \in (0,1) : \exists a < b, G_\nu = l \text{ on } (a,b)\}$$

i.e. to the image by G_ν of the union of all the intervals with positive length on which G_ν is constant. The maximal such intervals form a countable set, thus L is a countable set of $\mathbb{R}$. Now we claim that

$$\mu(F_\mu^{-1}(\{l\})) = 0 \qquad \text{for any } l \in [0,1]$$

and this will imply that U, $\widetilde{U}$ are equal except on a μ-negligible set of $\mathbb{R}$ (which is $F_\mu^{-1}(L)$).

Indeed, for $l \in [0,1]$, the pre-image of $\{l\}$ by F_μ is an interval I with $\mu(I) = \mu\left(\mathring{I}\right)$ since μ has no atom. Now, by regularity of μ,

$$\mu(I) = \sup\left\{\mu([a,b]) : (a,b) \subset \mathring{I}\right\} = \sup\left\{F_\mu(b) - F_\mu(a) : (a,b) \subset \mathring{I}\right\}$$

and this supremum is null since F_μ is constant on $\mathring{I}$ with value l.

Step 3: the non-increasing case. We remark that if we set $\mu^-([a,b]) = \mu^-([-b,-a])$ then μ^- is also an atomless probability measure on $\mathbb{R}$, and the existence and uniqueness of a non-decreasing transport map from μ^- to ν, which we just have shown, is equivalent to the existence and uniqueness of a non-increasing transport map from μ to ν. □

This proof gives also explicit formulas and elementary properties for these two maps T_1, T_2. For instance, assume that μ, ν are compactly supported on intervals $[a,b]$, $[c,d]$ respectively, and have densities f, g which are a.e. positive on these intervals; then T_1, T_2 are both continuous and given by the formulas

$$\int_a^x f(t)\,\mathrm{d}t = \int_c^{T_1(x)} g(t)\,\mathrm{d}t \qquad \text{and} \qquad \int_a^x f(t)\,\mathrm{d}t = \int_{T_2(x)}^d g(t)\,\mathrm{d}t$$

On the other hand, under the same assumptions, assume moreover that g is symmetric with respect to the center of $[c,d]$, then we have the formula $T_2(x) = d - (T_1(x) - c)$ for any x; one can show it by verifying that, in such a case,

$$\int_c^{T_1(x)} g(t)\,\mathrm{d}t = \int_{d-(T_1(x)-c)}^d g(t)\,\mathrm{d}t$$

In particular we get, for instance, $T_1(x) = -T_2(x)$ on any point where T_1, T_2 admit both derivatives (thus a.e. on $[c,d]$ since both of them are monotone).

Now we state the optimality result about the non-decreasing transport map:

Theorem B.2.2. *Let μ, ν be two probability measures on $\mathbb{R}$, both compactly supported, with μ non-atomic. Let $h : \mathbb{R} \to \mathbb{R}$ be a convex function. Then the above non-decreasing transport map T_1 from μ to ν is optimal for the problem*

$$\inf\left\{\int_{\mathbb{R}} h(T(x) - x)\,\mathrm{d}\mu(x) : T_\#\mu = \nu\right\}$$

with uniqueness if h is strictly convex.

Sketch of proof. In the strictly convex case, the existence and uniqueness of a minimizer T is already known and it remains only to prove that it is non-decreasing. This comes from the c-cyclical monotonicity of the support of $(\mathrm{id} \otimes T)_\#\mu$: it implies

$$h(T(x) - x) + h(T(x') - x') \le h(T(x') - x) + h(T(x) - x') \tag{B.6}$$

for μ-a.e. $x \in \mathbb{R}$.

Assuming now by contradiction that $x < x'$ and $T(x) > T(x')$, we set

$$a = T(x) - x, \qquad b = T(x') - x' \qquad \text{and } c = x' - x$$

The inequality (B.6) can be written as

$$h(a) + h(b) \le h(a-c) + h(b+c) \tag{B.7}$$

Now, the assumptions on $x, x', T(x), T(x')$ imply $b < b+c < a$. We write $b + c = (1-t)b + ta$ and deduce from the strict convexity of h that

$$h(b+c) < (1-t)h(b) + th(a)$$

Similarly, we note that $b < a - c < a$ and write $a - c = (1-t')b + t'a$ so that

$$h(a-c) < (1-t')h(b) + t'h(a)$$

Now we compute $t = c/(a-b)$ and $t' = 1 - t$ so that

$$h(b+c) + h(a-c) < (2 - (t+t'))h(b) + (t+t')h(a) = h(b) + h(a)$$

which combined with (B.7) leads to $h(a) - h(b) < h(a) - h(b)$ which is a contradiction.

The proof in the non-strictly convex case is obtained by approximation of h by the family of strict convex functions $h_\varepsilon = h + \varepsilon|\cdot|^2$. □

B.3 Optimal transport with distance cost

We here give the main existence results and characterizations of optimal transport maps when the cost is actually equal to the Euclidean distance. In all this section, we thus fix $c(x,y) = |y-x|$ for $(x,y) \in (\mathbb{R}^d)^2$.

The first result characterizes the c-concave functions:

Proposition B.3.1. *If $c(x,y) = |y-x|$, then a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is c-concave if and only if it is Lipschitz continuous on $\mathbb{R}^d$ with Lipschitz constant at most 1. Moreover, in this case, with the notations of the section B.1, we have $\varphi^c = -\varphi$.*

As a consequence, if μ, $\nu \in \mathcal{P}(\mathbb{R}^d)$ are both compactly supported with $\mu \ll \mathcal{L}^d$, the duality formula can be written as

$$\inf_{\gamma \in \Pi(\mu,\nu)} \left\{ \int |y-x| \,\mathrm{d}\gamma(x,y) \right\} = \sup_{u \in \mathrm{Lip}_1(\mathbb{R}^d)} \left\{ \int u(y)\,\mathrm{d}\nu(y) - \int u(x)\,\mathrm{d}\mu(x) \right\}$$

We now fix a Kantorovich potential u. The following notion of transport ray is crucial.

Definition B.3.1. *Let x, y, $x \neq y$, be two points of $\mathbb{R}^d$. Then:*

- *the oriented open segment (x,y) is a transport ray if and only if $u(y) - u(x) = |y-x|$;*
- *the oriented closet segment $[x,y]$ is a maximal transport ray if and only if*
  - *any $z \in (x,y)$ belongs to a transport ray (x',y') with same orientation than $[x,y]$;*
  - *any closed segment satisfying the above property and containing x and y is equal to $[x,y]$.*

The maximal transport rays are well-defined up to a Lebesgue-negligible (thus μ-negligible) set of $\mathbb{R}^d$, as states the following proposition:

Proposition B.3.2. *Let $[x,y]$ and $[x',y']$ be two maximal transport rays. Then they can intersect only on a point which is an extremity of both of them. In such a case, the Kantorovich potential u is non-differentiable at this point.*

In particular, since u is Lipschitz, the set of points of $\mathbb{R}^d$ belonging to at least two transport rays is Lebesgue-negligible.

Now let us examine the consequence of this structure for the duality formula and for an optimal transport T. Assume that such a T exists; the duality formula provides

$$\int_\Omega |T(x)-x|\,\mathrm{d}\mu(x) = \int_{\mathbb{R}^d} u(y)\,\mathrm{d}\nu(y) - \int_{\mathbb{R}^d} u(x)\,\mathrm{d}\mu(x) = \int_\Omega (u(T(x)) - u(x))\,\mathrm{d}\mu(x)$$

Moreover, the fact that u is 1-Lipschitz implies the equality

$$u(T(x)) - u(x) \leq |T(x) - x|$$

for μ-a.e. $x \in \Omega$. Thus, since the integral of the both hand-sides of this inequality are equal, it is actually an equality:

$$\text{for } \mu - a.e.\ x \in \Omega, \qquad u(T(x)) - u(x) = |T(x) - x|$$

This implies that $(x, T(x))$ is a transport ray. In particular, x and $T(x)$ belong to the same maximal transport ray and are oriented in the same sense than this transport ray.

One can verify converse implication is also true. To summarize:

Proposition B.3.3. *The solutions of the Monge problem*

$$\inf\left\{\int_\Omega |T(x)-x|\,\mathrm{d}\mu(x) : T_\#\mu = \nu\right\}$$

exist and are not unique. A map $T:\Omega\to\mathbb{R}^d$ is a solution if and only if:

- *T sends μ onto ν;*
- *for μ-a.e. $x\in\Omega$,*
  - *$T(x)$ and x belong to the same maximal transport ray;*
  - *the orientation of the segment $(x,T(x))$ is the same than that one of this transport rays.*

We finish by some words about the *monotone transport along the transport rays.* We define it as being the only map T verifying the two following properties:

- $T\in\mathcal{O}_1(\mu,\nu)$, and
- for a.e. $x,x'\in\Omega$, if x,x' belong both to the same transport ray, then $T(x)-x$ and $T(x')-x'$ have same direction.

The uniqueness of such a map T can be seen as a consequence of the uniqueness of the non-decreasing transport map in the one-dimensional case since, given a transport ray S, T has to send the measure μ "concentrated on S" onto the measure ν "concentrated on S" in a monotone way. The formal justification would use the disintegration of measures, and we will not enter into the details here.

Notice also the following remarks:

- T is the unique solutions of the minimization problem

  $$\inf\left\{\int_\Omega |T(x)-x|^2\,\mathrm{d}\mu(x) : T\in\mathcal{O}_1(\mu,\nu)\right\}$$

  i.e. the unique optimizer of the quadratic costs among the maps which are already optimal for the Monge cost with L^1-distance;
- also, and this is actually the tool to obtain the above property, T can be obtained by solving

  $$\inf\left\{\int_\Omega (|T(x)-x|+\varepsilon|T(x)-x|^2)\,\mathrm{d}\mu(x) : T_\#\mu = \nu\right\}$$

  and passing to the limit as $\varepsilon\to 0$;
- finally, let us notice some regularity properties of this monotone T. First, it has been shown T to be continuous under some strong assumptions on the domains and measures (convex and disjoint domains in the plane and positive densities, see [42]). Also, [57] gives also some regularity results for the minimizer T_ε of an approximated problem where c is replaced with $c_\varepsilon(x,y)=\sqrt{\varepsilon^2+|x-y|^2}$ (local uniform bounds on the eigenvalues of the Jacobian matrix T_ε).

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