Stabilization under local and global constraints

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Stabilisation sous contraintes locales et globales

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I thank my wife, and family for always support the idea of a thesis, and for providing me inspiration when I was lost. I also thank my friends for making me miss these years.

Dedico esta tese a Carolina Pereira Gomes.

"Bold ideas, unjustified anticipations, and speculative thought, are our only means for interpreting nature: our only organon, our only instrument, for grasping her. And we must hazard them to win our prize. Those among us who are unwilling to expose their ideas to the hazard of refutation do not take part in the scientific game."

Popper, The Logic of Scientific Discover

CONTENTS

1 | RÉSUMÉ DÉTAILLÉ

Contents

Les systèmes dynamiques jouent un rôle important dans la science comme un outil pour l'étude des phénomènes. La théorie mathématique du contrôle fournit un cadre pour faire face à des systèmes dynamiques en analysant leurs comportements qui peuvent être modélisés pour répondre à certaines contraintes réglementaires. Dans ce travail, les systèmes dynamiques considérés sont décrits par des équations différentielles ordinaires déterministes.

L'analyse de la stabilité des systèmes dynamiques est l'un des principaux problèmes de la théorie du contrôle et, dans ce travail, la notion utilisée est la théorie introduite par Lyapunov dans "Problème général de la stabilité du mouvement" en 1907. Parks présente le développement historique de la théorie de la stabilité de Lyapunov dans "A. M. Lyapunov's stability theory - 100 years on", 1992. D'autres concepts de stabilité ont été introduits, Leine présente le développement historique de quelques notions de stabilité dans "The historical development of classical stability concepts: Lagrange, Poisson and Lyapunov stability", 2010.

Grosso modo, un point $\bar{x}$ est Lyapunov stable pour un système dynamique si ses solutions issues d'une condition initiale $x(0)$ proche de $\bar{x}$ reste à proximité du point $\bar{x}$. Le point $\bar{x}$ est dit asymptotiquement stable si au delà d'être stable les solutions convergent vers $\bar{x}$. La Figure 1.1, basée sur (Luenberger, 1979, Fig. 9.3), illustre ce concept.

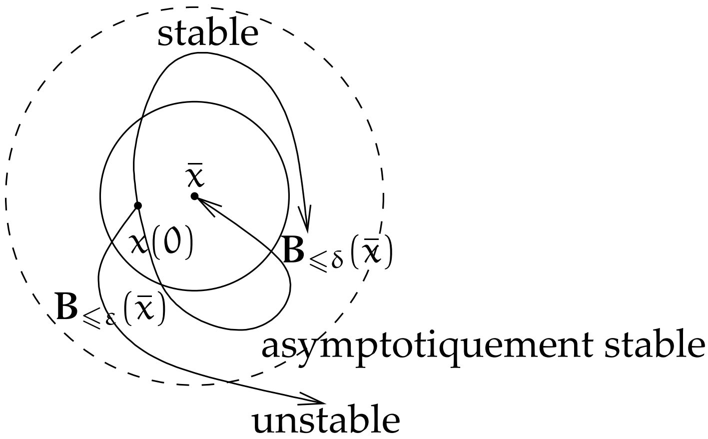


Figure 1.1: Illustration du concept de stabilité introduit par Lyapunov. Un point $\bar{x}$ du space d'état est stable si les solutions issues des conditions initiaux $x(0)$ qui sont proches $\bar{x}$ restent proches de $\bar{x}$.

Dans ce travail, deux applications de la théorie de la stabilité sont considérées: la synthèse d'une commande et l'analyse de l'interaction de deux systèmes dynamiques. Dans le premier cas, les conditions relatives à la conception de la fonction qui modifie le comportement d'un système pour atteindre un but désiré sont données, tandis que dans le second, un critère pour rendre stable l'interconnexion de deux systèmes dynamiques est fourni.

Les propriétés d'un système dynamique lorsqu'il est proche et lointain de l'ensemble souhaité sont également pris en compte pour ces applications de la théorie de la stabilité. Pour la synthèse d'une lois de commande non-linéaire, une approximation de la dynamique non-linéaire du système, autour de l'ensemble desiré, est considerée. En ce qui concerne l'interconnexion, la stabilité asymptotique globale est obtenue grâce à la stabilisation locale avec l'attractivité de différents ensembles souhaités.

Des travaux qui mélangent des comportements locaux et non locaux de systèmes sont bien connus dans la littérature. Voir (Benachour et al., 2013) pour la synthèse de lois de commande correspondant à des contraintes d'optimalité locales, (Andrieu and Prieur, 2010) dans le cadre de fonctions de commande de Lyapunov et (Chaillet, Angeli, and Ito, 2012) pour les notions de stabilité entrée-état et stabilité entrée-état intégrale.

1.1 Contributions principales

Les principales contributions de cette thèse sont résumées ci-dessous.

- Systèmes pour lesquels la méthode de *backstepping* ne peut pas être employée pour concevoir une loi de commande qui stabilise globalement l'origine, la conception d'une loi de commande non-linéaire est proposée pour stabiliser globalement un ensemble compact;

- Une approximation de la dynamique non-linéaire est fournie en utilisant des inégalités matricielles linéaires pour obtenir une inclusion différentielle linéaire;

- Une méthodologie pour la conception d'une loi de commande qui stabilise localement l'origine et satisfait des contraintes sur le bassin d'attraction est fournie;

- En utilisant le cadre des systèmes hybrides [1], une loi de commande hybride mélangeant des commandes stabilisantes locales et globales est proposée. La loi de commande résultante stabilise globalement l'origine;

- Une nouvelle condition suffisante est presentée pour des systèmes dont le théorème des petits gains ne peut pas être utilisé pour conclure sur la stabilité de l'interconnexion;

- Il est introduit une notion de gain entrée-état serré;

- Il est montré, pour des systèmes unidimensionnels, que si la condition des petits gains ne vaut pas pour la composition des gains entrée-état serrés, alors il n'existe pas d'autres gains entrée-état appropriés pour lesquelles elle peut être valable;

[1] Systèmss avec une dynamique discrète et continues.

- Il est considéré le cas où la condition du petit gain est valable dans deux régions différentes de l'espace d'état: l'intérieur d'un ensemble compact contenant l'origine et à l'extérieur d'un autre ensemble compact contenant aussi l'origine;

- Les critères pour la stabilité consistent en analyser comment une fonction propre et définie positive varie le long les solutions d'un système, dans les régions où le petit gain est valable, et de fournir une condition suffisante pour que les ensembles de conditions initiales et des solutions qui ne convergent pas vers un attracteur compact aient une mesure de Lebesgue égale à zéro, en dehors de cette régions;

- Une généralisation du critère de Bendinxson pour l'absence des ensembles ω-limite est fourni pour les systèmes planaires et pour les régions qui ne sont pas simplement connexes.

1.2 Plan

Cette thèse est organisée comme suit:

- Le Chapitre 2 considère le problème de la conception d'une loi de commande stabilisante pour un système non-linéaire, lorsque la méthode par *backstepping* ne peut pas être employée. La stratégie est de rendre un système dynamique continu hybride et de mélanger une loi de commande stabilisante locale, conçue pour satisfaire des contraintes sur le bassin de l'attraction, avec un contrôleur qui stabilise globalement asymptotiquement un ensemble qui appartient au bassin d'attraction;

- Le Chapitre 3 fournit des critères pour analyser l'interconnexion des systèmes dynamiques lorsque le petit gain n'est pas valable dans une région de l'espace d'état. Plus précisément, dans les régions où le petit état de gain est valable, il est analysé comment une fonction définie positive varie le long les solutions du système interconnecté, tandis que dans les régions où le petit gain ne pas valable, il est analysé comment la mesure de l'ensemble des solutions change le long les solutions du système interconnecté;

- Le Chapitre 4 propose la suite des travaux présentés dans les chapitres précédents et recueille quelque observations finales;

- Le Chapitre A rappelle le contexte mathématique et les sujets de la théorie de la stabilité de Lyapunov.

1.3 Problématique considérée

1.3.1 Chapitre 2

Au fil des années, la recherche de la stabilité des systèmes dynamiques non-linéaires a conduit à des nombreux outils pour la conception des lois de commandes asymptotiquement stabilisantes. Ces techniques nécessitent des structures particulières sur les systèmes. Selon les hypothèses, le concepteur peut utiliser différentes approches telles que le grand gain (Grognard, Sepulchre, and Bastin, 1999), *backstepping* (Freeman and Kokotović, 2008) ou *forwarding* (Mazenc and Praly, 1996). Cependant, en présence des dynamiques non structurées, certaines de ces méthodes ne peuvent pas être applicables.

Pour des systèmes où la technique de synthèse par *backstepping* ne peut pas être appliquée pour rendre l'origine globalement asymptotiquement stable, les approches proposées dans (Stein Shiromoto, Andrieu, and Prieur, 2011, 2012, 2013b) peuvent résoudre le problème en mélangeant une loi de commande par *backstepping* qui rend un ensemble compact globalement attractif avec une commande localement stabilisante. Par hypothèse, cet ensemble est contenu dans le bassin d'attraction du système en boucle fermée du contrôleur local. Le principal résultat est une conception de loi commande pour les systèmes hybrides qui *a priori* ne disposent pas d'une loi de commande qui stabilise globalement l'origine. Cette méthodologie de commandes hybrides est maintenant bien connue (Prieur, 2001), et elle a également été appliquée pour des systèmes qui ne satisfont pas la condition dit de Brockett ((Goebel, Prieur, and Teel, 2009) et (Hespanha, Liberzon, and Morse, 2004)). Lois de commandes hybrides peuvent avoir l'avantage de rendre l'équilibre du système robuste asymptotiquement stable par rapport au bruit de la mesure et des erreurs de actionneurs ((Prieur, Goebel, and Teel, 2007) et (Goebel and Teel, 2006)) en boucle fermée. Il est également présenté une procédure pour concevoir une loi de commande continue qui satisfait des contraintes locales sur le bassin d'attraction du système en boucle fermée.

Des travaux similaires existent dans le contexte des lois de commandes continues ((Andrieu and Prieur, 2010) et (Pan et al., 2001)). Contrairement à eux, pour la classe des systèmes considérés dans ce chapitre, *a priori* aucune commande globalement stabilisante continue existe. Notez qu'ici il est adressé un problème différent que (Mayhew, Sanfelice, and Teel, 2011), où une fonction synergique Lyapunov et une loi de commande sont conçues par backstepping.

Idée de la solution. La classe de systèmes considérés est donnée par l'équation

$$\Sigma_h(u(t)) \quad \begin{cases} \dot{x}_1(t) &= f_1(x_1(t), x_2(t)) + h_1(x_1(t), x_2(t), u(t)) \\ \dot{x}_2(t) &= f_2(x_1(t), x_2(t))u(t) + h_2(x_1(t), x_2(t), u(t)), \end{cases} \tag{2.1}$$

dont, pour tous les $t \in \mathbb{R}$, $(x_1(t), x_2(t)) \in \mathbb{R}^{n-1} \times \mathbb{R}$ est la variable d'état et $u(t) \in \mathbb{R}$ est l'entrée. À partir d'ici, les arguments t seront omies.

Les fonctions qui décrivent la dynamique de (2.1) sont

$$\begin{array}{llll} f_1 \in \mathcal{C}^1(\mathbb{R}^{n-1} \times \mathbb{R}, \mathbb{R}^{n-1}), & h_1 \in \mathcal{C}^1(\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^{n-1}), \\ f_2 \in \mathcal{C}^1(\mathbb{R}^{n-1} \times \mathbb{R}, \mathbb{R}), & h_2 \in \mathcal{C}^1(\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}). \end{array}$$

Par ailleurs, elles satisfont $f_1(0,0) = h_1(0,0,0) = 0$, $h_2(0,0,0) = 0$ et $f_2(x_1, x_2) = 0 \Leftrightarrow (x_1, x_2) = (0,0)$.

Avec une notation plus compacte, le vecteur $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$ est noté par x, la i-éme composante de $x_1 \in \mathbb{R}^{n-1}$ est notée par $x_{1,i}$. Quand $h_1(x_1, x_2, u) \equiv 0$ et $h_2(x_1, x_2, u) \equiv 0$, le système (2.1) est noté par $\Sigma(u)$.

En considérant le système $\Sigma(u)$, en supposant la stabilisabilité du sous-système x_1 avec une loi de commande differentiable qui vaut zéro lorsque son argument est aussi zéro, l'application de la procédure pour la synthèse par *backstepping* (voir la Section 2.7.1 pour plus de renseignements) donne une lois de commande $\varphi_b \in \mathcal{C}^0(\mathbb{R}^n_{\neq 0}, \mathbb{R})$ qui rend l'origine globalement asymptotiquement stable pour $\Sigma(\varphi_b)$ (le lecteur intéressé peut consulter (Krstić, Kanellakopoulos, and Kokotović, 1995) pour avoir plus de détail sur cette conclusion). Cependant, parce que les fonctions h_1 et h_2 dépendent de la variable u, la synthèse d'une lois de commande pour (2.1) conduit à une équation implicite dans la variable d'entrée (l'Équation (2.27) fournit un exemple). Par ailleurs, depuis que $f_2(0, 0) = 0$, la procédure proposée pourrait donner une loi de commande telle que $\lim_{|x| \to 0} \varphi_b(x) = \infty$.

Ces faits motivent l'introduction d'une loi de commande hybride pour assurer la stabilité globale asymptotique de l'origine pour (2.1) en boucle fermée. Pour ce faire, il a été nécessaire d'introduire les trois hypothèses listées ci-dessous.

Une lois de commande hybride est composée par un ensemble de indices $\mathbf{Q}$ telle que, pour chaque $q \in \mathbf{Q}$,

- Il existe une famille des ensembles fermés $\mathbf{C}_q, \mathbf{D}_q \subset \mathbb{R}^n$ telle que $\mathbf{C}_q \cup \mathbf{D}_q = \mathbb{R}^n$ et
$$\bigcup_{q \in \mathbf{Q}} \mathbf{C}_q = \mathbb{R}^n;$$
- Une famille de fonctions continues $\varphi_q : \mathbb{R}^n \to \mathbb{R}$;
- Des applications multivaluées $g_q : \mathbb{R}^n \rightrightarrows \mathbf{Q}$ avec proprieté de continuité appropriée.

Le système (2.1) en boucle fermée avec une commande hybride aura des dynamiques discrètes et continues, selon les régions déterminées par $\mathbf{C}_q$ et $\mathbf{D}_q$:

$$\Sigma_h(\mathscr{K}) \quad \begin{cases} \dot{x} = f_h(x, \varphi_q(x)), & x \in \mathbf{C}_q, \\ q^+ \in g_q(x), & x \in \mathbf{D}_q, \end{cases} \qquad (2.2)$$

Les solutions de (2.2) auront un comportement discret, dans les régions $\mathbf{D}_q$, et continus, dans les régions $\mathbf{C}_q$.

Stabilisation locale. Dans cette hypothèse (voir l'hypothèse (2.8)), il est supposé l'existence d'une loi de commande hybride pour laquelle l'origine est localement asymptotiquement stable pour le système (2.1) bouclé. Par ailleurs, il est supposé l'existence d'une famille de fonctions de Lyapunov qui décroissent le long les solutions du système bouclé.

Stabilisation du sous-système x_1 et bornes sur les fonction h_1 et h_2. Dans cette hypothèse (Voir l'hypothèse 2.10) il est supposé l'existence d'une loi de commande pour rendre l'origine globalement asymptotiquement stable pour le sous-système x_1 de Σ bouclé. Il est aussi supposé que les fonctions h_1, h_2 et $\partial h_1/\partial x_2$ sont bornées par une fonction continue Ψ qui ne dépende pas de la variable de l'entrée u. Par ailleurs, la dérivée de la fonction de Lyapunov du sous-système x_1 de Σ_h bouclé dans la direction de h_1 est supposée bornée et telle borne ne dépende pas de l'entrée u;

Inclusion. Dans cette hypothèse (voir l'hypothèse 2.12), il est supposé qu'un ensemble compact est inclus dans le bassin d'attraction de (2.1) bouclé avec la commande hybride localement stabilisante.

D'après l'hypothèse de la stabilisation du sous-système x_1 et les bornes sur les fonctions h_1 et h_2, il est obtenu le premier résultat. Ce résultat est énoncé dans la Proposition 2.14 et il concerne l'existence d'une loi de commande pour lequel un ensemble compact est pratiquement globalement asymptotiquement stable pour le système (2.1) bouclé.

Le résultat qui suit concerne le mélange entre cette loi de commande avec la commande hybride localement stabilisante. Le résultat concerne la construction d'une loi de commande hybride. Cette construction est illustrée dans la figure ci-dessous (voir aussi 2.1).

Avec cette approche, la commande qui stabilise globalement l'ensemble compact est utilisée en dehors de la région rouge et la commande qui stabilise localement l'origine est utilisée dans la région bleu. Par conséquence, pour chaque condition initial la composante x de la solution convergera vers l'origine. Dont la stabilité asymptotique globale suit.

Dans la section suivante, il est aussi proposé une approche pour la

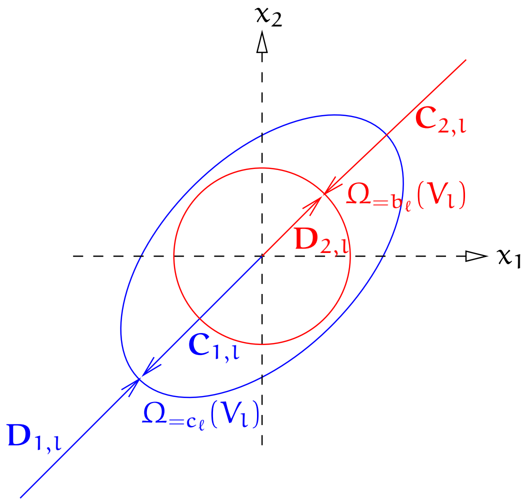


Figure 1.2: Illustration de l'approche proposée.

synthèse d'une commande qui stabilise localement l'origine sous des contraintes sur le bassin d'attraction.

Les non-linéarités de f_h sont approximées par un ensemble de matrices et le système Σ_h est reformulé en fonction d'une inclusion différentielle linéaire, dans un voisinage de l'origine. Cette approche simplifie la synthèse mais cette méthode croit exponentiellement en fonction du nombre de variables et des équations.

Le résultat principal de cette approche est présenté dans la Proposition 2.23 où il est donné des conditions sur la forme d'inégalités matricielles pour la synthèse d'une commande locale qui satisfait des contraintes sur le bassin d'attraction. Plus précisément, cette contrainte est l'inclusion de l'ensemble pratiquement globalement asymptotiquement stable pour (2.1) bouclé, donné par la Proposition 2.14.

1.3.2 Chapitre 3

L'utilisation des gains d'entrée-sortie non-linéaires pour l'analyse de la stabilité des systèmes interconnectés a été introduite par Zames dans (Zames, 1966) en considérant un système comme un opérateur d'entrée-sortie. La condition qui assure la stabilité, appelée théorème des petits gains, est basée sur le principe de la contraction.

Sontag a introduit dans (Sontag, 1989) un nouveau concept de gain liant l'entrée à l'état du système (ISS). Cette notion lie les approches de Zames et Lyapunov (Sontag, 2001). Caractérisations en termes de dissipation et fonctions de Lyapunov sont aussi présentées dans (Sontag and Wang, 1995).

Dans (Jiang, Teel, and Praly, 1994), le principe de la contraction est utilisé avec la notion de stabilité entrée-état pour obtenir un théorème des petits gains pour des systèmes ISS. Une formulation de ce critère en termes de fonctions de Lyapunov peut être trouvée dans (Jiang, Mareels, and Wang, 1996).

Au delà de l'analyse de la stabilité, le théorème des petits gains peut également être utilisé pour la conception de lois de commandes qui satisfont des contraintes de robustesse. Le lecteur intéressé est invité à voir (Freeman and Kokotović, 2008; Sastry, 1999) et les références qui s'y trouvent. D'autres versions du théorème des petits gains existent dans la littérature, voir (Angeli and Astolfi, 2007; Astolfi and Praly, 2012; Ito, 2006; Ito and Jiang, 2009) pour l'interconnexion de systèmes qui ne sont pas nécessairement ISS.

Pour appliquer le théorème des petits gains, il est nécessaire que la composition de ces gains non-linéaires soit plus petite que l'argument pour toutes les valeurs positives de son argument. Cette condition, appelée condition des petits gains, limite l'application du théorème des petits gains à une composition bien choisie des gains.

Les approches présentées dans (Stein Shiromoto, Andrieu, and Prieur, 2013a,c) fournissent un critère alternatif pour la stabilisation de systèmes interconnectés, quand la condition des petits gains n'est pas valable globalement. Il consiste à montrer que si localement (resp. non-localement) la condition du petit gain est valable dans une région locale (resp. non-locale) de l'espace d'état, et l'intersection de la région locale et non-locale est vide. D'ailleurs, si à l'extérieur de l' union de ces régions, l'ensemble des conditions initiales à partir des quelles les trajectoires associées ne convergent pas à la région locale ont une mesure zéro, le système interconnecté résultant est presque asymptotiquement stable (cette notion est précisément définie dans le Chapitre 3). Dans ce chapitre, une condition suffisante garantissant cette propriété est présentée. De plus, pour les systèmes planaires, une extension du critère de Bendixson aux régions qui ne sont pas simplement connexes est donnée. Ceci permet d'obtenir la stabilité asymptotique globale de l'origine.

Cette approche peut être considérée comme un mélange de deux conditions des petits gains qui valent dans différentes régions: une locale et un non-locale.

IDÉE DE LA SOLUTION. La classe de systèmes considéré est donnée par

$$\begin{cases} \dot{x} &= f(x,z), \\ \dot{z} &= g(x,z) \end{cases} \tag{3.8}$$

dont $f, g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, les sous-systèmes dans le variables x et z sont ISS avec V et W, respectivement, des fonctions de Lyapunov de type ISS. Avec un notation vectoriel, le système (3.8) est noté par $\dot{y} = h(y)$.

La dérivée de Lie de chaque fonction satisfait

$$\begin{aligned} V(x) &\geqslant \gamma(W(z)) &\Rightarrow \quad D_f^+ V(x,z) &\leqslant -\lambda_x(x), \\ W(z) &\geqslant \delta(V(x)) &\Rightarrow \quad D_g^+ W(x,z) &\leqslant -\lambda_z(z), \end{aligned} \tag{1.1}$$

avec γ et δ fonctions de classe $\mathcal{K}_\infty$, λ_x et λ_z fonctions positive définies et continues.

Le théorème du petit gain énonce que (3.8) est stable si, pour tous les $s \in \mathbb{R}_{\geqslant 0}$,

$$\gamma(s) \circ \delta(s) < s. \tag{SGC}$$

Dans le cas où (SGC) n'est plus valable, dans un'intervalle borné de $\mathbb{R}_{\geqslant 0}$ qui ne contient pas l'origine, l'approche consiste à montrer qu'il existe

- Deux gains ISS γ_ℓ et γ_g pour le sous-système x de (3.8);
- Deux gains ISS δ_ℓ et δ_g pour le sous-système z de (3.8);

- Les compositions $\gamma_\ell \circ \delta_\ell$ et $\gamma_g \circ \delta_g$ satisfont la condition du petit gain, pas pour tous les valeurs de l'argument mais pour deux intervalles différentes;

- Dans les regions hors de l'union de ces intervalles une condition pour assurer l'inexistence des ensembles ω-limite est vrai.

Alors, pour presque toutes les conditions initiales, les solutions de (3.8) convergent vers l'origine. Par ailleurs, si $n = m = 1$, cette conclusion est valable pour toutes les condition initiales.

Les hypothèses nécessaires pour obtenir le résultat de la stabilisation sont présentées ci-dessous.

Stabilité régionale. La première hypothèse (voir Assumption 3.5) concerne la propreté qui satisfont les fonctions de Lyapunov V et W dans un ensemble $\mathbf{S}$. Plus précisément, il est supposé l'existence des constantes, gains ISS et un ensemble pour lesquels l'inégalité (1.1) est vraie dans cet ensemble;

Petit gain régional. La deuxième hypothèse (voir Assumption 3.6) concerne la composition des gains ISS dans l'ensemble $\mathbf{S}$, donné par l'hypothèse précédente. Plus précisement, il est supposé que la fonction qui résulte de la composition des gains est bornée supérieurement par la fonction identité, pour toutes les valeurs de l'argument qui appartiennent à un'intervalle correspondant à l'ensemble $\mathbf{S}$.

Le premier résultat concerne la convergence des solutions dans l'ensemble $\mathbf{S}$. Plus précisément, Proposition 3.8 énonce qui, sous les hypothèses 3.5 et 3.6, les solutions de (3.8) convergent vers une frontière de l'ensemble $\mathbf{S}$.

Deux corollaires suivent de la Proposition 3.8 et sont énoncés dans 3.9 et 3.11. Plus précisément, si les constantes de l'hypothèse de la stabilité régionale sont telles que $\mathbf{S}$ est compact et contient l'origine, alors elle localement asymptotiquement stable (voir Corollaire 3.9). Si les constantes de l'hypothèse de la stabilité régionale sont telles qui $\mathbf{S}$ est l'ensemble complémentaire d'un compact de $\mathbb{R}^n$ contenant l'origine, alors cet ensemble compact est globalement attractif (voir Corollaire 3.11).

Si les contantes des Corollaires 3.9 et 3.11 sont telles que le bassin d'attraction de l'origine ne contient pas l'ensemble globalement attractive, alors il pourrait exister des solutions de (3.8) qui ne convergent pas vers l'origine mais vers un ω-limite.

D'après les corollaires, la région où les ensembles ω-limite peuvent exister est un ensemble compact. Plus précisément, la région résultant de la différence entre l'ensemble attracteur global et le bassin d'attraction de l'origine. Du coup, la méthode qui suit consiste en donner des conditions suffisantes pour que ces ensembles ω-limite n'existent pas. Ce cas est illustré dans la figure ci-dessous.

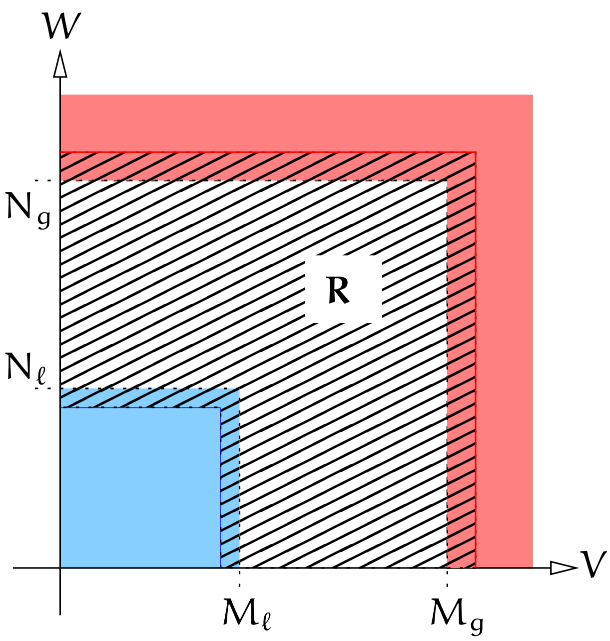


Figure 1.3: Illustration des ensembles. Le bassin d'attraction de l'origine (région blue), région d'attraction pour l'ensemble compact (région rose) qui est l'attracteur global (ligne rouge) et **R** la région résultant de la différence entre l'attracteur global et le bassin d'attraction de l'orgine.

La condition pour avoir l'absence des ensembles ω-limite pour les systèmes interconnectés avec $n > 2$ est donnée dans le Théorème 3.13. Plus précisément, sous les conditions des Corolaires 3.9 et 3.11, s'il existe un fonction de densité ρ avec support compact qui contient **R** et telle que, le divergent de $h\rho$ est strictement positive dans **R**, alors pour presque toutes les conditions initiales, les solution issues convergent vers l'origine. Par ailleurs, l'ensemble des conditions initiales et de solutions qui ne convergent pas ont une mesure de Lebesgue égale à zéro.

Pour des systèmes interconnectés avec $n = 2$ il est donné, dans le Théorème 3.14, une condition pour l'absence des ensembles ω-limite. Plus précisément, sous les conditions des Corolaires 3.9 et 3.11, si la divergence de h est différente de zéro dans $\mathbf{R}$, et h n'y a aucun point d'équilibre dans $\mathbf{R}$, alors pour toutes les conditions initiales, les solutions convergent vers l'origine.

1.4 Travaux en cours

Le Chapitre 4 présente quelques pistes pour généraliser les résultats antérieurs. Une possible généralisation serait de définit une notion de contrôlabilité presque partout et l'utiliser pour la synthèse des commandes stabilisantes avec des gains différents dans chaque région de l'espace d'état. Par ailleurs, trois lignes de recherches sont proposées

Interconnexion à large échelle. Dans ce cas, les résultats du Chapitre 3 sont en train d'être adaptés pour l'interconnexion de plus de deux systèmes;

Interconnexion de systèmes qui sont presque ISS. Dans ce cas, sont considérés des systèmes qui sont presque ISS et une approche pour montrer la stabilité de l'interconnexion est proposée;

Approximations homogènes de systèmes hybrides à l'infini. Dans ce cas, sont présentées des définitions pour que les systèmes hybrides soient approximés par des fonctions homogènes à l'infini.

2 | BLENDING NONLINEAR FEEDBACK LAWS

In this chapter, a class of nonlinear systems with structural obstacles to design nonlinear continuous feedback laws are considered. The strategy consists in designing a feedback law rendering a suitable compact set (strictly containing the origin) practically asymptotically stable. This feedback is blended with a locally stabilizing one. A constructive approach is given by employing a differential inclusion representation of the nonlinear dynamics. The results are illustrated with an example.

Contents

2.1 Introduction

Over the years, research in control of nonlinear dynamical systems has led to many different tools for the design of (globally) asymptotically stabilizing feedback laws. These techniques require particular structures on the systems. Depending on the assumptions, the designer may use different approaches such as high-gain (Grognard, Sepulchre, and Bastin, 1999), backstepping (Freeman and Kokotović, 2008) or forwarding (Mazenc and Praly, 1996). However, in the presence of unstructured dynamics, some of these methods may fail to be applied.

For systems where the classical backstepping technique can not be applied to render the origin globally asymptotically stable, the approaches proposed in (Stein Shiromoto, Andrieu, and Prieur, 2011, 2012, 2013b) may solve the problem by blending a backstepping feedback law that renders a suitable compact set globally attractive with a locally stabilizing controller. By assumption, this set is contained in the basin of attraction of the system in closed loop with the latter. The main result can be seen as a design of hybrid feedback laws for systems which *a priori* do not have a feedback law that globally stabilizes the origin. This methodology of hybrid stabilizers is by now well known (Prieur, 2001), and it has been also applied for systems that do not satisfy the Brockett's condition ((Goebel, Prieur, and Teel, 2009) and (Hespanha, Liberzon, and Morse, 2004)). Hybrid feedback laws can have the advantage of rendering the equilibrium of the closed-loop system robustly asymptotically stable with respect to measurement noise and actuators' errors ((Prieur, Goebel, and Teel, 2007) and (Goebel and Teel, 2006)). It is also presented a procedure to design a continuous local feedback law satisfying constraints on the basin of attraction of the closed-loop system.

Related works do exist in the context of continuous controllers ((Andrieu and Prieur, 2010) and (Pan et al., 2001)). In contrast to them, for the class of systems considered in this chapter, *a priori* no continuous globally stabilizing controller exists. Note that it is addressed a different problem than (Mayhew, Sanfelice, and Teel, 2011), where a synergistic Lyapunov function and a feedback law are designed by backstepping.

2.2 Background, motivation, and problem statement

2.2.1 System under consideration

Consider the class of nonlinear systems defined by

$$\Sigma_h(u) \quad \begin{cases} \dot{x}_1(t) &= f_1(x_1(t), x_2(t)) + h_1(x_1(t), x_2(t), u(t)) \\ \dot{x}_2(t) &= f_2(x_1(t), x_2(t))u(t) + h_2(x_1(t), x_2(t), u(t)), \end{cases} \tag{2.1}$$

where, for every $t \in \mathbb{R}$, $(x_1(t), x_2(t)) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is the state, and $u(t) \in \mathbb{R}$ is the input. From now on, arguments t will be omitted.

The functions describing the dynamics of (2.1) are

$$\begin{aligned} f_1 &\in \mathcal{C}^1(\mathbb{R}^{n-1} \times \mathbb{R}, \mathbb{R}^{n-1}), & h_1 &\in \mathcal{C}^1(\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^{n-1}), \\ f_2 &\in \mathcal{C}^1(\mathbb{R}^{n-1} \times \mathbb{R}, \mathbb{R}), & h_2 &\in \mathcal{C}^1(\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}). \end{aligned}$$

Moreover, they satisfy $f_1(0,0) = h_1(0,0,0) = 0$, $h_2(0,0,0) = 0$, and $f_2(x_1, x_2) = 0 \Leftrightarrow (x_1, x_2) = (0,0)$.

In a more compact notation, the vector $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is denoted by x, the i-th component of $x_1 \in \mathbb{R}^{n-1}$ is denoted by $x_{1,i}$. When $h_1(x_1, x_2, u) \equiv 0$ and $h_2(x_1, x_2, u) \equiv 0$, system (2.1) is denoted by $\Sigma(u)$.

2.2.2 Motivation

Consider system $\Sigma(u)$, assuming stabilizability of the x_1-subsystem with a smooth feedback law, and applying the *backstepping* design procedure[1], it is obtained a feedback law $\varphi_b \in \mathcal{C}^0(\mathbb{R}^n_{\neq 0}, \mathbb{R})$ that renders the origin globally asymptotically stable for $\Sigma(\varphi_b)$ (e.g. Krstić, Kanellakopoulos, and Kokotović (1995)). However, because functions h_1 and h_2 depend on u, the design of a feedback law for (2.1) leads to an implicit equation in the input variable (see (2.27) below for an example). Moreover, since $f_2(0,0) = 0$, the suggested procedure could

[1] Cf. Section 2.7.1.

yield a feedback law such that $\lim_{|x|\to 0} \varphi_b(x) = \infty$.

These facts motivate the introduction of a hybrid feedback law ensuring global asymptotic stability of the origin for (2.1) in closed loop.

2.2.3 Background

[2] Based on (Prieur, Goebel, and Teel, 2007, Definition 3.4)

Definition 2.1 (Hybrid feedback law). [2] A *hybrid feedback law*, denoted by $\mathscr{K}$, consists in a finite set $\mathbf{Q} \subset \mathbb{N}$. For every $q \in \mathbf{Q}$,

- Closed sets $\mathbf{C}_q, \mathbf{D}_q \subset \mathbb{R}^n$ such that $\mathbf{C}_q \cup \mathbf{D}_q = \mathbb{R}^n$ and
$$\bigcup_{q\in\mathbf{Q}} \mathbf{C}_q = \mathbb{R}^n;$$
- Functions $\varphi_q \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$;
- Set-valued maps $g_q \in \mathcal{OSC}(\mathbb{R}^n, \mathbf{Q})$. ○

From now on, assume the existence of a hybrid feedback law.

System (2.1) in closed loop with $\mathscr{K}$ leads to a system with continuous and discrete dynamics

$$\Sigma_h(\mathscr{K}) \quad \begin{cases} \dot{x} &= f_h(x, \varphi_q(x)), \quad x \in \mathbf{C}_q, \\ q^+ &\in g_q(x), \quad x \in \mathbf{D}_q, \end{cases} \tag{2.2}$$

with state space given by $\mathbb{R}^n \times \mathbf{Q}$.

System (2.2) is analyzed by the framework provided in (Goebel, Sanfelice, and Teel, 2012; Prieur, Goebel, and Teel, 2007).

[3] (Goebel, Sanfelice, and Teel, 2012, Definition 2.3)

Definition 2.2 (Hybrid time domain). [3] A set $\mathbf{T} \subset \mathbb{R}_{\geqslant 0} \times \mathbb{N}$ is called *compact hybrid time domain* if

$$\mathbf{T} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_J$. It is *hybrid time domain* if, for every $(T, J) \in \mathbf{T}$, $\mathbf{T} \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. ○

[4] (Prieur, Goebel, and Teel, 2007, Definition 3.5)

Definition 2.3 (Hybrid solution). [4] Let $\mathbf{T}$ be a hybrid time domain, and consider the functions $X : \mathbf{T} \to \mathbb{R}^n$ and $Q : \mathbf{T} \to \mathbf{Q}$. The pair of functions (X, Q) is called *hybrid solution* of (2.2) if

- For a fixed j, the function $t \mapsto X(t, j)$ is locally absolutely continuous, and $(t, j) \in \mathbf{T}$;
- For a fixed j, the function $t \mapsto Q(t, j)$ is constant, and $(t, j) \in \mathbf{T}$.

The time-domain where (X, Q) is defined is denoted by $\mathtt{dom}(X, Q)$. Furthermore,

S_1. $X(0,0) \in \mathbf{C}_{Q(0,0)} \cup \mathbf{D}_{Q(0,0)}$;

S_2. For every $j \in \mathbb{N}$ such that $\mathbf{I}^j := \{t \in \mathbb{R} : (t, j) \in \mathtt{dom}(X, Q)\}$, and for almost every $t \in \mathbf{I}^j$,

$$\dot{X}(t,j) = f_h(X(t,j), \varphi_{Q(t,j)}(X(t,j))), \quad X(t,j) \in \mathbf{C}_{Q(t,j)};$$

S_3. For every $(t, j) \in \mathtt{dom}(X, Q)$ such that $(t, j+1) \in \mathtt{dom}(X, Q)$,

$$Q(t, j+1) \in g_{Q(t,j)}(X(t,j)), \quad X(t,j) \in \mathbf{D}_{Q(t,j)}.$$

A hybrid solution (X, Q) is called

- *non-trivial* if $\mathtt{dom}(X, Q)$ contains at least two points;
- *complete* if $\sup(\mathtt{dom}(X, Q)) = \infty$;
- *maximal* if there exists no other hybrid solution $(\overline{X}, \overline{Q})$ of (2.2) such that $\mathtt{dom}(X, Q) \subsetneq \mathtt{dom}(\overline{X}, \overline{Q})$, and for every $(t, j) \in \mathtt{dom}(X, Q)$, $(X(t,j), Q(t,j)) = (\overline{X}(t,j), \overline{Q}(t,j))$. ◦

The basic regularity conditions for the existence of solutions of (2.2) are presented in the following

Definition 2.4 (Basic assumptions). [5] If, for every[6] $q \in \mathbf{Q}$,

1. The sets $\mathbf{C}_q$ and $\mathbf{D}_q$ are closed subsets of $\mathbb{R}^n$;
2. The map $f_h(\cdot, \varphi_q(\cdot)) : \mathbf{C}_q \to \mathbb{R}^n$ is continuous;
3. The set-valued map $g_q : \mathbf{D}_q \rightrightarrows \mathbf{Q}$ is outer semicontinuous, locally bounded, and for every $x \in \mathbf{D}_q$, $g_q(x)$ is nonempty,

then (2.2) satisfies the basic assumptions. ◦

[5] Based on (Goebel, Sanfelice, and Teel, 2009, Assumption 6.10) and (Prieur, Goebel, and Teel, 2007, pp. 2210).

[6] Items 1. and 2. have been added here for the sake of completeness of the chapter. Item 1 holds from Definition 2.1. Item 2 holds, since the functions f_h and φ_q are continuous.

Lemma 2.5. *For every initial condition* $(X(0,0), Q(0,0)) \in \mathbf{C}_{Q(0,0)} \cup \mathbf{D}_{Q(0,0)}$, *there exists a non-trivial hybrid solution* (X, Q) *for* (2.2) *such that either*

a. (X, Q) *is complete or;*

b. $\mathtt{dom}(X, Q)$ *is bounded and the interval* $\mathbf{I}^J$, *where* $J = \sup_j \mathtt{dom}(X, Q)$, *has nonempty interior and* $t \to (X(t, J), Q(t, J))$ *is a maximal solution of* $\Sigma_h(\varphi_{Q(t,J)})$, *in fact*

$$\lim_{t \to T} |X(t, J)| \to \infty,$$

where $T = \sup_t \mathtt{dom}(X, Q)$. □

Roughly speaking, the proof of Lemma 2.5 consists of showing that (2.2) satisfies Assumption 2.4. From Proposition 6.10 of (Goebel, Sanfelice, and Teel, 2012), the conclusion follows. The proof of Lemma 2.5 is provided in details in Section 2.5.1.

Definition 2.6 (Uniform local asymptotic stability). [7] Consider system (2.2). A compact invariant set $\mathbf{A} \subset \mathbb{R}^n$ is called

[7] (Goebel, Sanfelice, and Teel, 2012, Definition 7.1).

- *uniformly stable for* (2.2) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every solution (X, Q) of (2.2) with $|X(0,0)|_{\mathbf{A}} \leqslant \delta$, $|X(t,j)|_{\mathbf{A}} \leqslant \varepsilon$, for every $(t,j) \in \mathtt{dom}(X,Q)$;
- *attractive for* (2.2) if there exists $\delta_0 > 0$ such that every solution (X, Q) of (2.2) with $|X(0,0)|_{\mathbf{A}} \leqslant \delta_0$ is complete and

$$\lim_{t+j \to \infty} |X(t,j)|_{\mathbf{A}} = 0;$$

- *asymptotically stable for* (2.2) if stable and attractive.

The *uniform basin of attraction*[8] of the compact set $\mathbf{A}$ is the set of all $X(0,0) \in \mathbb{R}^n$ such that, for every $Q(0,0) \in \mathbf{Q}$, there exists a hybrid solution (X, Q) of (2.2) that is complete and

[8] Based on (Goebel, Sanfelice, and Teel, 2012, Definition 7.3) and (Prieur, Goebel, and Teel, 2007, Definition 3.3).

$$\lim_{t+j \to \infty} |X(t,j)|_{\mathbf{A}} = 0.$$

The prefix pre is dropped, when hybrid solutions are complete. ○

The uniformty in Definition 2.6 is with respect to the variable Q.

Definition 2.7 (Hybrid Lyapunov function candidate). [9] Consider the system (2.2). For every $q \in \mathbf{Q}$, a function $V_q : \mathtt{dom}(V_q) \to \mathbb{R}_{\geqslant 0}$ is called *hybrid Lyapunov function candidate* for (2.2) if,

[9] Based on (Goebel, Sanfelice, and Teel, 2012, Definition 3.16).

1. $\mathbf{C}_q \cup \mathbf{D}_q \cup g_q(\mathbf{D}_q) \subset \mathtt{dom}(V_q)$;
2. V_q is continuously differentiable on an open set containing $\mathbf{C}_q$. ○

Note that Definition 2.7 is less restrictive than Definition A.34, because in the former no requirement is made regarding positive definiteness and properness properties.

2.3 Standing assumptions and results

2.3.1 Standing assumptions

Assumption 2.8 (Locally stabilizing hybrid feedback law). [10] Given a finite discrete set $\mathbf{L} \subset \mathbb{N}$ such that, for every $l \in \mathbf{L}$, there exist

[10] (Stein Shiromoto, Andrieu, and Prieur, 2013b)

- Closed sets $\mathbf{C}_l \subset \mathbb{R}^n$ and $\mathbf{D}_l \subset \mathbb{R}^n$ with $\mathbf{C}_l \cup \mathbf{D}_l = \mathbb{R}^n$ and

$$\bigcup_{l \in \mathbf{L}} \mathbf{C}_l = \mathbb{R}^n;$$

- Feedback laws $\varphi_l \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$;
- Set-valued maps $g_l \in \mathcal{OSC}(\mathbb{R}^n, \mathbf{L})$;
- $V_l \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$, $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, and a constant value $c_\ell > 0$ satisfying,

$$\forall x \in \mathbb{R}^n, \quad \underline{\alpha}(|x|) \leqslant V_l(x) \leqslant \overline{\alpha}(|x|), \tag{2.3}$$

$$\forall x \in (\Omega_{\leqslant c_\ell}(V_l) \cap \mathbf{C}_l) \setminus \{0\}, \quad L_{f_h} V_l(x, \varphi_l(x)) < 0, \tag{2.4}$$

$$\forall x \in (\Omega_{\leqslant c_\ell}(V_l) \cap \mathbf{D}_l) \setminus \{0\}, g \in g_l(x), \quad V_g(x) - V_l(x) < 0. \tag{2.5}$$

Note that, since the set $\mathbf{L}$ is finite, it is possible to pick functions of class $\mathcal{K}_\infty$ that bound all functions V_l.

The hybrid feedback law together with the family of function V_l satisfying (2.3)-(2.5) is denoted by $\mathcal{K}_0$. ◦

Note that, for every $l \in \mathbf{L}$, V_l is a hybrid Lyapunov function candidate.

Definition 2.9. For every $l \in \mathbf{L}$, the function V_l satisfying (2.3)-(2.5) is called *hybrid Lyapunov function for system*[11] $\Sigma_h(\mathcal{K}_0)$. ◦

[11] System (2.1) in closed loop with $\mathcal{K}_0$ is analogous to (2.2).

From the proof of Lemma 2.5, system $\Sigma_h(\mathcal{K}_0)$ satisfies the basic assumptions[12] for the existence of solutions. From (Goebel, Sanfelice, and Teel, 2009, Theorem 20), Assumption 2.8 implies that the set $\{0\} \times \mathbf{L}$ is locally asymptotically stable for $\Sigma_h(\mathcal{K}_0)$. To see this claim, note that whenever a solution starts in a neighbourhood of the origin, Equation (2.4) ensures that, for every $l \in \mathbf{L}$, the Lyapunov function V_l is strictly decreasing during a flow. Equation (2.5) ensures that for a solution starting in a neighbourhood of the origin, during a transition from a feedback law l to a feedback law determined by $g \in g_l$, the value $V_l(x)$ strictly decreases to $V_g(x)$. [13]

[12] See Definition 2.4.

[13] To see that $\Sigma_h(\mathcal{K}_0)$ is stable in the sense of Definition 2.6, the ideas of (Theorem 3.18 Goebel, Sanfelice, and Teel, 2012) are used here.

Since the functions V_l are strictly decreasing along solutions of $\Sigma_h(\mathcal{K}_0)$, for every $(t,j) \in \mathtt{dom}(X, L)$, $V_{L(t,j)}(X(t,j)) \leqslant V_{L(0,0)}(X(0,0))$. From (2.3), $|X(t,j)| \leqslant \underline{\alpha}^{-1} \circ \overline{\alpha}(|X(0,0)|)$. Thus, for every $|X(0,0)| \leqslant \varepsilon$, by letting $\delta = \overline{\alpha}^{-1} \circ \underline{\alpha}(\varepsilon)$, $|X(0,0)| \leqslant \delta \Rightarrow |X(t,j)| \leqslant \varepsilon$. Hence, $\{0\} \times \mathbf{L}$ is stable for $\Sigma_h(\mathcal{K}_0)$.

It now remains to show that attractivity holds. From the continuity the functions V_l, there exists $\alpha \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ such that $L_h V_l(\cdot, \varphi_l(\cdot)) \leqslant -\alpha(\cdot)$ and $V_g(\cdot) - V_l(\cdot) \leqslant -\alpha(\cdot)$. Assume, for the purposes of contradiction, that there exist $r > 0$ such that, for every $|X(0,0)| \leqslant \delta$, $r \leqslant |X(t,j)| \leqslant \varepsilon$, for every $(t,j) \in \mathtt{dom}(X, L)$. Let $\gamma = \min\{\rho(x) : r \leqslant |x| \leqslant \varepsilon\}$, $V(X(t,j)) \leqslant V(X(0,0)) - \gamma(t+j)$. As $t+j$ increases, the value $V(X(t,j))$ will become negative, contradicting its positive definiteness. Thus, the set $\{0\} \times \mathbf{L}$ is attractive to $\Sigma_h(\mathcal{K}_0)$. Therefore it is locally asymptotically stable for $\Sigma_h(\mathcal{K}_0)$.

The second assumption provides bounds for the terms that impeach the direct application of the backstepping method. It also concerns the global stabilization of the origin for

$$\dot{x}_1 = f_1(x_1, x_2), \tag{2.6}$$

when x_2 is considered as an input.

Assumption 2.10 (Bounds). [14] There exist a proper function $V_1 \in (\mathcal{C}^1 \cap \mathcal{P})(\mathbb{R}^{n-1}, \mathbb{R}_{\geqslant 0})$, a feedback law $\psi_1 \in \mathcal{C}^1(\mathbb{R}^{n-1}, \mathbb{R})$, and a locally Lipschitz function $\alpha \in \mathcal{K}_\infty$ such that

[14] (Stein Shiromoto, Andrieu, and Prieur, 2011, 2013b)

a. Stabilizing feedback law for (2.6): for every $x_1 \in \mathbb{R}^{n-1}$,

$$L_{f_1} V_1(x_1, \psi_1(x_1)) \leqslant -\alpha(V_1(x_1));$$

In addition, there exist $\Psi \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$, and two positive constant values ε and M with $0 < \varepsilon \leqslant 1$ satisfying, for every $(x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,

b. Bounds on h_1:

$$\begin{aligned} |h_1(x_1, x_2, u)| &\leqslant \Psi(x_1, x_2), \\ \left|\frac{\partial h_1}{\partial x_2}(x_1, x_2, u)\right| &\leqslant \Psi(x_1, x_2), \\ L_{h_1} V_1(x_1, \psi_1(x_1), u) &\leqslant (1-\varepsilon)\alpha(V_1(x_1)) + \varepsilon\alpha(M); \end{aligned} \tag{2.7}$$

c. Bound on h_2:

$$|h_2(x_1, x_2, u)| \leqslant \Psi(x_1, x_2). \qquad \circ$$

The motivation of items b and c of Assumption 2.10 is explained as follows. Suppose that the item a of Assumption 2.10 holds and consider the Lyapunov function candidate given by

$$\begin{aligned} V: \quad \mathbb{R}^{n-1} \times \mathbb{R} &\to \mathbb{R}_{\geqslant 0} \\ (x_1, x_2) &\mapsto V_1(x_1) + \frac{1}{2}(x_2 - \psi_1(x_1))^2. \end{aligned}$$

Taking its Lie derivative in the f_h-direction yields

$$\begin{aligned} L_{f_h} V(x, u) &= \operatorname{grad} V(x_1, x_2) \cdot f_h(x_1, x_2, u) \\ &\leqslant -\alpha_1(V_1(x_1)) + L_{h_1} V_1(x_1, \psi_1(x_1), u) + (x_2 - \psi_1(x_1))\Big(f_2(x_1, x_2)u + h_2(x_1, x_2, u) \\ &\quad - L_{r_1}\psi_1(x_1, x_2, u) + \frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial r_1}{\partial \eta_{x_1, x_2}}(x_1, \eta_{x_1, x_2}(s), u)\Big) \end{aligned} \tag{2.8}$$

where, for every $(x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$, $r_1(x_1, x_2, u) := f_1(x_1, x_2) + h_1(x_1, x_2, u)$.

To see the role of the uniform bound Ψ, note that to compute a stabilizing feedback law, one needs to solve Equation (2.8) in the implicit variable u. Without the uniform bound provided by items b and c of Assumption 2.10, finding a feedback law φ rendering (2.8) negative definite may not be an easy task.

To see the role of the constant values ε and M, assume that Assumption 2.10 holds and consider the proper function $V_1 \in (\mathcal{C}^1 \cap \mathcal{P})(\mathbb{R}^{n-1}, \mathbb{R}_{\geqslant 0})$. Taking its Lie derivative in the r_1-direction yields, for every $(x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,

$$L_{r_1} V_1(x_1, x_2, u) = L_{f_1} V_1(x_1, x_2) + L_{h_1} V_1(x_1, x_2, u).$$

Considering feedback law $\psi_1 \in \mathcal{C}^1(\mathbb{R}^{n-1}, \mathbb{R})$ and letting, for every $x_1 \in \mathbb{R}^{n-1}$, $x_2 = \psi_1(x_2)$.

$$\begin{aligned} L_{r_1} V_1(x_1, \psi_1(x_1), u) &= L_{f_1} V_1(x_1, \psi_1(x_1)) + L_{h_1} V_1(x_1, \psi_1(x_1), u) \\ &\leqslant -\alpha_1(V_1(x_1)) + L_{h_1} V_1(x_1, \psi_1(x_1), u). \end{aligned}$$

From Equation (2.7),

$$\begin{aligned} L_{r_1} V_1(x_1, \psi_1(x_1), u) &\leqslant -\alpha_1(V_1(x_1)) + (1-\varepsilon)\alpha(V_1(x_1)) + \varepsilon\alpha(M) \\ &\leqslant \varepsilon[\alpha(M) - \alpha(V_1(x_1))]. \end{aligned}$$

The upper bound for $L_{h_1} V_1$ increases the value of $L_{r_1} V_1$. Moreover, the role of the variables M and ε as follows.

- If $M = 0$, then for every $(x_1, u) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

  $$L_{r_1} V_1(x_1, \psi_1(x_1), u) \leqslant -\varepsilon\alpha(V_1(x_1)).$$

  This implies that the feedback law ψ_1 stabilizes the x_1-subsystem, for every u. Thus, in this case the input variable u has no role in the x_1-subsystem which seems to be a strong assumption on M;

- The role of ε is related to the speed of convergence of the solutions of

  $$\dot{x}_1 = f_1(x_1, \psi_1(x_1)) + h_1(x_1, \psi_1(x_1), u)$$

  to the set $\Omega_{\leqslant M}(V_1)$. Moreover, the presence of the term h_1 decreases the speed of convergence of solutions to ε. Thus, ε must be different from zero. For the case $\varepsilon = 1$, the presence of the function h_1 has no role in the speed of convergence of the solutions.

Remark 2.11. Items b and c of Assumption 2.10 can be satisfied for a class of linear systems with scalar inputs, provided that a suitable matrix exists. To see this claim, consider the linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u,$$

supposed to be controllable, and with $b_2 \neq 0$.

Suppose that the matrix

$$T = \begin{bmatrix} c_{11} & -c_{11}\dfrac{b_1}{b_2} \\ 0 & 1 \end{bmatrix}$$

satisfying $c_{11} \neq 0$ and $-c_{11}a_{12}{}^{b_1}/_{b_2} \neq a_{22}$ exists.

Let the variable $z := Tx$. This implies that,

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} a_{11}c_{11} & -c_{11}a_{12}\dfrac{b_1}{b_2} + a_{22} \\ a_{21}a_{11} & -c_{11}a_{22}\dfrac{b_1}{b_2} + a_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} u.$$

Consider the variable $v \in \mathbb{R}$, by letting

$$u = \frac{1}{b_2}\left(v - a_{21}a_{11}z_1 + c_{11}a_{22}\frac{b_1}{b_2} - a_{22}z_2 + z_2\right)$$

it yields the linear system

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} a_{11}c_{11} & -c_{11}a_{12}\dfrac{b_1}{b_2} + a_{22} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

with input variable v. Note that,

$$\begin{aligned} f_1(z_1, z_2) &= a_{11}c_{11}z_1 + \left(a_{22} - c_{11}a_{12}\frac{b_1}{b_2}\right) z_2, & h_1(z_1, z_2, v) &\equiv 0, \\ f_2(z_1, z_2) &= z_2, & h_2(z_1, z_2, v) &\equiv 0. \end{aligned}$$

Thus, items b and c of Assumption 2.10 hold. ○

Before introducing the last assumption, consider the set $\mathbf{A} \subset \mathbb{R}^n$ given by

$$\mathbf{A} = \{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} : V_1(x_1) \leqslant M, x_2 = \psi_1(x_1)\}. \tag{2.9}$$

Since the function V_1 is proper and $\psi_1 \in \mathcal{C}^1(\mathbb{R}^{n-1}, \mathbb{R})$, the set $\mathbf{A}$ is closed and bounded. Thus, it is compact. It will be shown in Proposition 2.14 that, under Assumption 2.10, there exists a feedback law $\varphi_g \in \mathcal{C}^0(\mathbb{R}^n_{\neq 0}, \mathbb{R})$ rendering $\mathbf{A}$ globally practically attractive for $\Sigma_h(\varphi_g)$. Note that, when $M = 0$, the set $\mathbf{A}$ is the origin.

Assumption 2.12. (Inclusion)[15] For every $l \in \mathbf{L}$, the function V_l satisfies

$$\max_{x \in \mathbf{A}} V_l(x) < c_\ell,$$

where the constant c_ℓ was introduced in Assumption 2.8. ○

[15] (Stein Shiromoto, Andrieu, and Prieur, 2013b)

Assumption 2.12 states the set $\mathbf{A}$ is included in the basin of attraction of the set $\{0\} \times \mathbf{L}$.

2.3.2 Global practical stabilization

Before stating the first result, the concept of practical stabilizability is recalled.

Definition 2.13 (Global practical asymptotic stabilizability). [16] A compact invariant set $\mathbf{S} \subset \mathbb{R}^n$ containing the origin is *globally practically asymptotically stabilizable* for (2.1) if, for every $a \in \mathbb{R}_{>0}$, there exists a feedback law $\varphi_g : \mathbb{R}^n \to \mathbb{R}$ such the set

$$\mathbf{B}_{\leqslant a}(\mathbf{S}) := \{x \in \mathbb{R}^n : |x|_{\mathbf{S}} \leqslant a\}$$

contains a compact set that is globally asymptotically stable for $\Sigma_h(\varphi_g)$. ∘

[16] Based on (Isidori, 1999, pp. 126).

Proposition 2.14. [17] *Under Assumption 2.10, the set $\mathbf{A}$ is globally practically asymptotically stabilizable for (2.1).* □

[17] (Stein Shiromoto, Andrieu, and Prieur, 2011, 2013b)

A sketch of the proof of Proposition 2.14 is given as follows. Under Assumption 2.10, the terms impeaching the design of a feedback law are bounded. This fact allows the application of the *backstepping* method[18] to design a feedback law φ_g that renders $\mathbf{A}$ globally practically asymptotically stable for $\Sigma_h(\varphi_g)$. The proof of Proposition 2.14 is provided in details in Section 2.5.2.

[18] Cf. Section 2.7.1

Corollary 2.15. *Under Assumption 2.10 with $M = 0$, the origin is globally practically asymptotically stabilizable for (2.1).* □

Since $\mathbf{A}$ is globally practically asymptotically stabilizable for (2.1) and under Assumptions 2.8 and 2.12 the set $\mathbf{A}$ is included in the basin of attraction of $\Sigma_h(\mathscr{K}_0)$, it is possible to build a hybrid feedback law $\mathscr{K}$ that globally stabilizes the set $\{0\} \times \mathbf{L}$ for (2.2). This is the idea behind the following theorem whose proof is provided in details in Section 2.5.3.

Theorem 2.16. [19] *Under Assumptions 2.8-2.12, there exist*

- *A feedback law $\varphi_g \in \mathcal{C}^0(\mathbb{R}^n_{\neq 0}, \mathbb{R})$;*
- *A constant value b_ℓ satisfying $0 < b_\ell < c_\ell$;*
- *A hybrid state feedback law $\mathscr{K}$ defined by the discrete set $\mathbf{Q} := \{1, 2\} \times \mathbf{L}$ such that, for every $l \in \mathbf{L}$,*

[19] (Stein Shiromoto, Andrieu, and Prieur, 2013b)

– *The subsets of* $\mathbb{R}^n$ *are defined by*

$$\begin{aligned} \mathbf{C}_{1,l} &= \Omega_{\leqslant c_\ell}(V_l)\cap \mathbf{C}_l, & \mathbf{D}_{1,l} &= (\Omega_{\leqslant c_\ell}(V_l)\cap \mathbf{D}_l)\cup \Omega_{\geqslant c_\ell}(V_l), \\ \mathbf{C}_{2,l} &= \Omega_{\geqslant b_\ell}(V_l), & \mathbf{D}_{2,l} &= \Omega_{\leqslant b_\ell}(V_l); \end{aligned} \tag{2.10}$$

– *The feedback laws* $\varphi_{q,l} \in \mathcal{C}^0(\mathbf{C}_{q,l},\mathbb{R})$ *are defined by*

$$\varphi_{q,l}(\cdot) = \begin{cases} \varphi_l(\cdot), & \textit{if} \quad q = 1, \\ \varphi_g(\cdot), & \textit{if} \quad q = 2; \end{cases} \tag{2.11}$$

– *The set-valued maps* $g_{q,l} \in \mathcal{OSC}(\mathbf{D}_{q,l},\mathbf{Q})$ *are defined by*

$$\begin{aligned} g_{2,l}: \quad \mathbf{D}_{2,l} &\rightrightarrows \mathbf{Q} \\ x &\mapsto \{(1,l)\} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} g_{1,l}: \quad \mathbf{D}_{1,l} &\rightrightarrows \mathbf{Q} \\ x &\mapsto \begin{cases} \{(1,g_l(x))\}, & x \in \Omega_{<c_\ell}(V_l)\cap \mathbf{D}_l, \\ \{(2,l)\}, & x \in \Omega_{>c_\ell}(V_l), \\ \{(1,g_l(x)),(2,l)\}, & x \in \Omega_{=c_\ell}(V_l)\cap \mathbf{D}_l, \end{cases} \end{aligned} \tag{2.13}$$

rendering the set $\{0\}\times \mathbf{L}$ *is globally asymptotically stable for* $\Sigma_h(\mathscr{K})$. □

These results are useful for systems that do not satisfy the Brockett necessary condition for the existence of a continuous stabilizing controller and for which there exists a locally stabilizing hybrid feedback law (see e.g. (Goebel, Sanfelice, and Teel, 2009, Example 38) and (Hespanha and Morse, 1999)).

Remark 2.17. Recall the concept of robust stability of hybrid systems.[20] The compact invariant set $\mathbf{S}$ is called *robustly asymptotically stable for* (2.2) *with respect to measurement noise*, if it is asymptotically stable for (2.2), and in addition, there exists $\rho \in \mathcal{C}^0(\mathbb{R}^n,\mathbb{R}_{\geqslant 0})$ such that $\mathbf{S}$ is asymptotically stable for

$$\Sigma_h^\rho(\mathscr{K}) \quad : \quad \begin{cases} \dot{x} \in f_q^\rho(x), & x \in \mathbf{C}_q^\rho, \\ q^+ \in g_q^\rho(x), & x \in \mathbf{D}_q^\rho, \end{cases} \tag{2.14}$$

where

$$\begin{aligned} f_q^\rho(x) &:= \mathrm{co}\left\{ f_h\Big(x, \varphi_q(\mathbf{B}_{\leqslant \rho(x)}(x)\cap \mathbf{C}_q)\Big)\right\}, \\ g_q^\rho(x) &:= g_q\Big(\mathbf{B}_{\leqslant\rho(x)}(x)\cap \mathbf{D}_q\Big), \\ \mathbf{C}_q^\rho &:= \left\{x\in\mathbb{R}^n : \mathbf{B}_{\leqslant\rho(x)}(x)\cap \mathbf{C}_q \neq \emptyset\right\}, \\ \mathbf{D}_q^\rho &:= \left\{x\in\mathbb{R}^n : \mathbf{B}_{\leqslant\rho(x)}(x)\cap \mathbf{D}_q \neq \emptyset\right\}. \end{aligned}$$

[20] Based on (Goebel, Sanfelice, and Teel, 2012, Definition 6.27) and (Prieur, Goebel, and Teel, 2007, Definition 4.1).

Since ρ is continuous, the system (2.14) satisfies Assumption 2.4.[21] Thus, from Lemma 2.5, there exists a non-trivial solution of system (2.14).

[21] (Goebel, Sanfelice, and Teel, 2012, Proposition 6.28)

Robust stability of $\{0\} \times \mathbf{L}$ for (2.2) can be concluded from (Prieur, Goebel, and Teel, 2007, Theorem 4.3) as follows. If system (2.2) satisfies Assumption 2.4, then it satisfies all the necessary assumptions of (Prieur, Goebel, and Teel, 2007, Theorem 4.3). From this result, if $\{0\} \times \mathbf{L}$ is asymptotically stable for (2.2), then it is robustly asymptotically stable.

Although $\{0\}\times\mathbf{L}$ is robustly globally asymptotically stable for $\Sigma_h(\mathscr{K})$, the design of the sets $\mathbf{C}_q$ and $\mathbf{D}_q$ imposes a limitation on the perturbation ρ. More precisely, if the perturbation ρ is such that $\mathbf{D}^{\rho}_{1,l} \cap \mathbf{D}^{\rho}_{2,l} \neq \emptyset$, then the hysteresis region is empty and it might exist chattering between the feedback laws φ_g and φ_l. In this case, the x-component of the solution would remain in the region $\mathbf{D}^{\rho}_{1,l} \cap \mathbf{D}^{\rho}_{2,l} \neq \emptyset$ and never flow. ◦

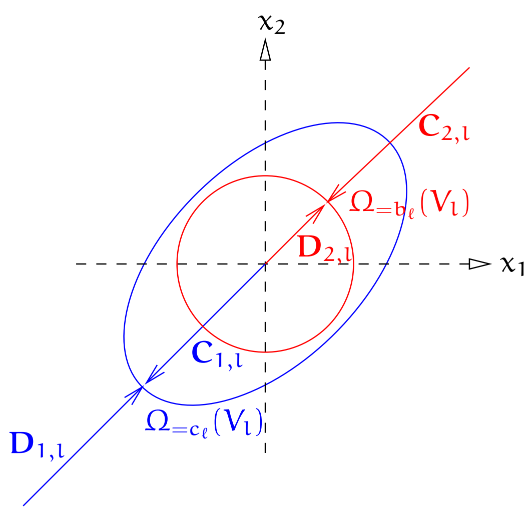


Figure 2.1: Illustration of the proposed approach.

In the next section, a local continuous feedback law satisfying Assumptions 2.8 and 2.12 is designed using a linear differential inclusion (2.1).

2.3.3 Semiglobal stabilization

So far, global practical stabilization of the set $\mathbf{A}$ has been achieved. The objective of this section is to obtain the semiglobal stabilization of the set $\{0\} \times \mathbf{L}$ under the condition that the basin of attraction contains $\mathbf{A}$. Thus, the aim is to design a feedback law satisfying Assumptions 2.8 and 2.12. To start it is necessary to introduce the concept of semiglobal asymptotic stabilizability.

Definition 2.18 (Semiglobal asymptotic stabilizability)**.** [22] The origin is called *semiglobally asymptotically stabilizable* for Σ_h if, for every compact set $\mathbf{K} \subset \mathbb{R}^n$ containing the origin, there exists a feedback law $\varphi_\ell : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{K}$ is contained in the basin of attraction of $\Sigma_h(\varphi_\ell)$. ◦

[22] Based on (Isidori, 1999, pp. 126) and (Chaillet and Loría, 2008).

Based on the approach presented on (Andrieu and Tarbouriech, 2013), the nonlinear dynamics of (2.1) is formulated in terms of a Linear Differential Inclusion. Let $\mathbf{L}$ be a singleton, the procedure starts by defining a neighborhood $\mathbf{N}_{\leqslant r}$ of the origin such that

a. There exist a feedback law $\varphi_l \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$, and a Lyapunov function $V_l \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ satisfying, for every $x \in \mathbf{N}_{\leqslant r} \backslash \{0\}$, $L_{f_h} V_l(x, \varphi_l(x)) < 0$;

b. It strictly contains an estimation of the basin of attraction of $\Sigma_h(\varphi_l)$, and a convex set that contains $\mathbf{A}$.

Under Assumption 2.10, there exist[23] a finite set $\mathbf{P} \subset \mathbb{N}$ of indexes and a set of vectors $\{x_p \in \mathbb{R}^n : p \in \mathbf{P}\}$ such that

[23] Because $\mathbf{A}$ is a compact set.

$$\mathbf{A} \subset \mathrm{co}(\{x_p \in \mathbb{R}^n : p \in \mathbf{P}\}). \tag{2.15}$$

Let $r_u > 0$ be a constant value and $r = [r_1, \ldots, r_n] \in \mathbb{R}^n$ be a vector of strictly positive values such that $\mathrm{co}(\{x_p \in \mathbb{R}^n : p \in \mathbf{P}\}) \subset \mathbf{N}_{\leqslant r} = \{x : |x_i| \leqslant r_i, 1 \leqslant i \leqslant n\}$.

Consider the function

$$\tilde{f}_h(x, u) = f_h(x, u) - Fx - Gu, \tag{2.16}$$

where F and G are the linearization of (2.1) around the origin:

$$\dot{x} = Fx + Gu := \frac{\partial f_h}{\partial x}(0)x + \frac{\partial f_h}{\partial u}(0)u. \tag{2.17}$$

Since $f_h \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, $\tilde{f}_h \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$. For each $m \in \mathbf{M} := \{m \in \mathbb{N} : 1 \leqslant m \leqslant 2^{n \times n}\}$, let $C_m \in \mathbb{R}^{n \times n}$ be a matrix with components c_{ij}, where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$, given by either

$$c^+_{ij} = \max_{\substack{x \in \mathbf{N}_{\leqslant r} \\ |u| \leqslant r}} \frac{\partial \tilde{f}_{h,i}}{\partial x_j}(x, u) \quad \text{or} \quad c^-_{ij} = \min_{\substack{x \in \mathbf{N}_{\leqslant r} \\ |u| \leqslant r}} \frac{\partial \tilde{f}_{h,i}}{\partial x_j}(x, u). \tag{2.18}$$

For each $v \in \mathbf{V} := \{v \in \mathbb{N} : 1 \leqslant v \leqslant 2^n\}$, let $D_v \in \mathbb{R}^{n \times 1}$ be a vector with components d_i, where $1 \leqslant i \leqslant n$, given by either

$$d_i^+ = \max_{\substack{x \in \mathbf{N}_{\leqslant r} \\ |u| \leqslant r}} \frac{\partial \tilde{f}_{h,i}}{\partial u}(x,u) \quad \text{or} \quad d_i^- = \min_{\substack{x \in \mathbf{N}_{\leqslant r} \\ |u| \leqslant r}} \frac{\partial \tilde{f}_{h,i}}{\partial u}(x,u). \tag{2.19}$$

Remark 2.19 (Computational cost)**.** To see the maximum number of matrices provided by the approach proposed in (2.18), consider a function $g \in \mathcal{C}^1(\mathbb{R}^p, \mathbb{R}^q)$. It has the Jacobian matrix $J \in \mathbb{R}^{q \times p}$ with elements given by

$$J_{ij} = \frac{\partial g_i}{\partial x_j},$$

where $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$ are, respectively, the number of lines and columns.

Fix two values $\bar{i}$ and $\bar{j}$ and consider a matrix having only one element of the form $J^-_{\bar{i}\,\bar{j}}$ and, consequently, all of the others of the form J^+_{ij}, for every $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$ with $i \neq \bar{i}$ and $j \neq \bar{j}$. For instance, if $\bar{i} = \bar{j} = 1$, then this matrix is represented by

$$\begin{bmatrix} - & + & \dots & + \\ + & + & \dots & + \\ \vdots & \vdots & \ddots & \vdots \\ + & + & \cdots & + \end{bmatrix}.$$

The number of matrices with only one element of the form $J^-_{\bar{i}\,\bar{j}}$ is given by the combinantion of one symbol $-$ among the all of the $(p \times q - 1)$ symbols $+$. Thus,

$$\binom{p \times q}{1} = \frac{(p \times q)!}{1!(p \times q - 1)!} = (p \times q)!$$

which is the permutation of $-$ among all $+$.

Hence, the total number of matrices is given by the sum of all of the above combinations

$$\sum_{k=1}^{p \times q} \binom{p \times q}{k} = \sum_{k=1}^{p \times q} \frac{(p \times q)!}{k!(p \times q - k)!} = 2^{p \times q}.$$

Therefore, the number of matrices grows exponentially with the dimension of the domain and image sets of g. For a system with dynamics described by g, the number of matrices increases exponentially with the number of variables and equations of the system.

Note also that, depending on the structure of the function g, the number of matrices may be smaller than $p \times q$. Example 2.20 illustrates this case. ◦

Example 2.20. Consider the nonlinear function $f_{h,1} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R})$, and the function $f_{h,2} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R})$ assumed to be linear in the arguments. Recall the definition of the function $\tilde{f}_h$,

$$\tilde{f}_h(x, u) = f_h(x, u) - Fx - Gu, \tag{2.16}$$

where $x := (x_1, x_2)$. Since $f_{h,2}$ is linear in the arguments, the function (2.16) is given by $\tilde{f}_h(x_1, x_2, u) = [\tilde{f}_{h,1}(x_1, u), 0]^T$. Following the previous definitions of the derivatives, the $2^{2^2} = 16$ matrices C_m are

$$\begin{aligned}
&\begin{bmatrix} c_{11}^+ & c_{12}^+ \\ c_{21}^+ & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^+ \\ c_{21}^+ & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^+ & c_{12}^- \\ c_{21}^+ & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^+ & c_{12}^+ \\ c_{21}^- & c_{22}^+ \end{bmatrix}, \\
&\begin{bmatrix} c_{11}^+ & c_{12}^+ \\ c_{21}^+ & c_{22}^- \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^- \\ c_{21}^+ & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^+ \\ c_{21}^- & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^+ \\ c_{21}^+ & c_{22}^- \end{bmatrix}, \\
&\begin{bmatrix} c_{11}^+ & c_{12}^- \\ c_{21}^- & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^+ & c_{12}^- \\ c_{21}^+ & c_{22}^- \end{bmatrix}, \begin{bmatrix} c_{11}^+ & c_{12}^+ \\ c_{21}^- & c_{22}^- \end{bmatrix}, \begin{bmatrix} c_{11}^+ & c_{12}^- \\ c_{21}^- & c_{22}^- \end{bmatrix}, \\
&\begin{bmatrix} c_{11}^- & c_{12}^+ \\ c_{21}^- & c_{22}^- \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^- \\ c_{21}^+ & c_{22}^- \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^- \\ c_{21}^- & c_{22}^+ \end{bmatrix}, \begin{bmatrix} c_{11}^- & c_{12}^- \\ c_{21}^- & c_{22}^- \end{bmatrix}.
\end{aligned} \tag{2.20}$$

Because of the structure of $\tilde{f}_{h,1}$ and $\tilde{f}_{h,2}$, $c_{12}^- = c_{12}^+ = 0$, and for every $i = 1, 2$, $c_{2,i}^+ = c_{2,i}^- = 0$. Thus, the matrices (2.20) are reduced to the first pair at the first line.

For the derivatives with respect to the input u, the $2^2 = 4$ vectors D_ν are

$$\begin{bmatrix} d_1^+ \\ d_2^+ \end{bmatrix}, \begin{bmatrix} d_1^- \\ d_2^+ \end{bmatrix}, \begin{bmatrix} d_1^+ \\ d_2^- \end{bmatrix}, \begin{bmatrix} d_1^- \\ d_2^- \end{bmatrix},$$

Since $d_2^- = d_2^+ = 0$, these vectors are also reduced to the first pair. ⋄

From (2.16), for every $x \in \mathbf{N}_{\leqslant r}$ and every $|u| \leqslant r_u$, the value $f_h(x, u)$ is contained in the convex set formed by (2.17), and the matrices C_m and vector D_ν, where $m \in \mathbf{M}$ and $\nu \in \mathbf{V}$. More precisely,

Claim 2.21. *For every $x \in \mathbf{N}_{\leqslant r}$, and for every $|u| \leqslant r_u$, system (2.1) satisfies the following linear differential inclusion*

$$\dot{x} \in \text{co}\{(F + C_m)x + (G + D_\nu)u\}, \tag{2.21}$$

where $m \in \mathbf{M}$ and $\nu \in \mathbf{V}$. □

The proof of Claim 2.21 is provided in Section 2.5.4.

Remark 2.22. [24] This linear differential inclusion goes further than the linearization (2.17), because the gradient of the nonlinear terms is also taken into account. Two aspects are crucial, when using this method to describe the dynamics of f_h. Namely, the size of the neighborhood $\mathbf{N}_{\leqslant r}$, and the rate of change of the nonlinear terms $\tilde{f}_h$. ◦

[24] (Stein Shiromoto, Andrieu, and Prieur, 2013b)

Consider the canonical basis in $\mathbb{R}^n$, i.e., the set of vectors $\{e_i\}_{i\in\mathbf{I}}$, $\mathbf{I} = \{i \in \mathbb{N} : 1 \leqslant i \leqslant n\}$, where the components are all 0 except the i-th one which is equals to 1.

Proposition 2.23. [25] *Assume that there exist a symmetric positive definite matrix* $W \in \mathbb{R}^{n\times n}$ *and a matrix* $H \in \mathbb{R}^{n\times 1}$ *satisfying, for every* $m \in \mathbf{M}$, *and for every* $v \in \mathbf{V}$,

[25] (Stein Shiromoto, Andrieu, and Prieur, 2013b)

$$W(F + C_m)^\top + H(G + D_v)^\top + (F + C_m)W + (G + D_v)H^\top < 0, \quad (2.22)$$

$$\forall i \in \mathbf{I}, \quad \begin{bmatrix} r_i^2 W & We_i \\ * & 1 \end{bmatrix} \geqslant 0, \quad (2.23)$$

$$\forall p \in \mathbf{P}, \quad \begin{bmatrix} 1 & x_p^\top \\ * & W \end{bmatrix} \geqslant 0, \quad (2.24)$$

and

$$\begin{bmatrix} r_u^2 W & H \\ * & 1 \end{bmatrix} \geqslant 0. \quad (2.25)$$

Then, by letting $\mathbf{L} = \{1\}$, $V_1(x) = x^\top P x$, *where* $P = W^{-1}$, $c_\ell = 1$, $\mathbf{C}_1 = \mathbb{R}^n$, $\mathbf{D}_1 = \Omega_{\geqslant 1}(V_1)$, $g_1(x) \equiv \{1\}$ *and* $\varphi_1(x) = Kx$, *where* $K = H^\top P$, *Assumptions 2.8 and 2.12 hold.* □

Since the results given by Proposition 2.23 depend on the neighbourhood chosen *a priori*, and the parameters ε and M from Assumption 2.10 must be constant and such that (2.7) holds globally, it is not possible to compute them here in Proposition 2.23.

From Schur's complement,[26] the matrices (2.23)-(2.25) are, respectively, equivalent to the following system of matrix inequalities in the variables W and H

[26] See Theorem 2.28.

$$We_i e_i^\top W^\top \leqslant r_i^2 W, \quad (2.23.\text{bis})$$

$$x_p W^{-1} x_p^\top \leqslant 1, \quad (2.24.\text{bis})$$

$$HH^\top \leqslant W r_u^2. \quad (2.25.\text{bis})$$

The proof of Proposition 2.23 is provided with details in Section 2.5.5. A sketch of the proof is given as follows. Equation (2.22) implies that V_1 is a Lyapunov function in the small[27] for $\Sigma_h(\varphi_1)$. Equation (2.23.bis) implies that $\Omega_{\leqslant 1}(V_1) \subset \mathbf{N}_{\leqslant r}$. Equation (2.24.bis) implies that $\mathrm{co}(\{x_p \in \mathbb{R}^n : p \in \mathbf{P}\}) \in \Omega_{\leqslant 1}$. Equation (2.25.bis) implies that $|u| \leqslant r_u$.

[27] See also Definition A.34.

Note that, from Proposition 2.23 and Assumption 2.8, the set $\{0\} \times$

$\{1\}$ semiglobally asymptotically stabilizable for Σ_h. It now remains to blend the practical asymptotic stability of $\mathbf{A}$ with the semiglobal asymptotic stability of the set $\{0\} \times \{1\}$ to achieve global asymptotic stabilization of $\{0\} \times \{1\}$. This is purpose of the next section.

2.3.4 Global stabilization

In this section, the global practical asymptotical stabilizability of $\mathbf{A}$ is blended with the semiglobal asymptotic stabilizability of the set $\{0\} \times \{1\}$ to achieve global asymptotic stability of the latter. More formally,

Corollary 2.24. *Under Assumption 2.10 and the hypotheses of Proposition 2.23, by defining the hybrid feedback law $\mathcal{K}$ as in (2.10)-(2.13), the set $\{0\} \times \{(1,1)\}$ is globally asymptotically stable for $\Sigma_h(\mathcal{K})$.* □

2.4 Illustration

Consider the system given by

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + 0.1[x_1^2 + (1+x_1)\sin(u)], \\ \dot{x}_2 = u. \end{cases} \quad (2.26)$$

In the presence of the term $0.1(1+x_1)\sin(u)$ in the time-derivative of x_1, it will be shown that the *backstepping* technique is difficult to apply.

Let

$$\begin{aligned} f_1(x_1,x_2) &= x_1 + x_2 + 0.1x_1^2, & h_1(x_1,x_2,u) &= 0.1(1+x_1)\sin(u), \\ f_2(x_1,x_2) &\equiv 0, & h_2(x_1,x_2,u) &\equiv 0. \end{aligned}$$

Firstly, it is checked the necessary assumptions for Proposition 2.14 and Theorem 2.16.

Assumption 2.10. To see that item a. holds, consider the Lyapunov function candidate given, for every $x_1 \in \mathbb{R}$, by $V_1(x_1) = x_1^2/2$. Taking the Lie derivative in the f_1-direction yields, for every $x_1 \in \mathbb{R}$,

$$L_{f_1}V_1(x_1,x_2) = x_1^2 + x_1x_2 + 0.1x_1^3$$

Consider the feedback law given, for every $x_1 \in \mathbb{R}$, by $\psi_1(x_1) = -(1+K_1)x_1 - 0.1x_1^2$, where $K_1 > 0$ is a constant value. Letting $x_2 = \psi_1(x_1)$ yields

$$L_{f_1}V_1(x_1,\psi_1(x_1)) = -K_1x_1^2 = -\alpha(V_1(x_1)),$$

where, for every $s \in \mathbb{R}_{\geqslant 0}$, $\alpha(s) := 2K_1 s$.

Since item a. of Assumption 2.10 holds, the *backstepping* technique could be applied to (2.26). Following the procedure described in Section 2.7.1, consider the Lyapunov function candidate

$$\begin{array}{rcl} V: \quad \mathbb{R}\times\mathbb{R} & \to & \mathbb{R}_{\geqslant 0} \\ (x_1, x_2) & \mapsto & V_1(x_1) + \frac{1}{2}(x_2 - \psi_1(x_1))^2. \end{array}$$

Taking its Lie derivative, algebraic computations yield, for every $(x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} L_{f_h} V(x_1, x_2, u) \leqslant\ & -K_1 x_1^2 + x_1 0.1(1+x_1)\cdot\sin(u) + (x_2 - \psi_1(x_1)) \\ & \cdot \left(u + \frac{x_1}{2} + (1 + K_1 + 0.2K_1 x_1)\cdot(x_1 + x_2 + 0.1[x_1^2 + (1+x_1)\cdot \sin(u)])\right). \end{aligned} \tag{2.27}$$

In order to have a term proportional to $(x_2 - \psi_1(x_1))^2$ in the right-hand side of (2.27), it is necessary to solve an implicit equation in the variable u defined as $E(x_1, x_2, u) \leqslant -K_1 x_1^2 - L(x_2 - \psi_1(x_1))^2$, where E is the right-hand side of (2.27), and $L > 0$ is a constant value. Since this procedure seems to be difficult (if not impossible), it motivates the design a hybrid feedback by applying Theorem 2.16.

To see that items b. and c. hold note that, for every $(x_1, x_2, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$\begin{array}{rcl} |h_1(x_1, x_2, u)| & = & |0.1(1+x_1)\sin(u)| \quad \leqslant \quad 0.1(1+|x_1|) \\ |h_2(x_1, x_2, u)| & \equiv & 0 \\ \left|\frac{\partial h_1}{\partial x_2}(x_1, x_2, u)\right| & = & 0 \\ L_{h_1} V_1(x_1, \psi_1(x_1), u) & \leqslant & |x_1| 0.1 + x_1^2 0.1 \\ & \leqslant & \frac{x_1^2}{2} 1.2 + \frac{0.1^2}{2}, \end{array}$$

where the last inequality is obtained by applying Young's inequality in $|x_1|\theta$.

Finally by letting, for every $(x_1, x_2) \in \mathbb{R}\times\mathbb{R}$, $\Psi(x_1, x_2) = 0.1(1+|x_1|)$, $\varepsilon \leqslant 1 - 1.2/(2K_1)$ and[28] $M \geqslant 0.1/(4K_1\varepsilon)$ items b. and c. of Assumption 2.10 are satisfied.

[28] The conditions $\varepsilon \leqslant 1 - 1.2/(2K_1)$ and $\varepsilon > 0$ imply in a lower bound for $K_1 > 0.6$.

Assumptions 2.8 and 2.12. From the definitions of V_1 and ψ_1, the set $\mathbf{A}$ is given by

$$\mathbf{A} = \left\{(x_1, x_2) \in \mathbb{R}\times\mathbb{R} : |x_1| \leqslant \sqrt{2M}, x_2 = -(1+K_1)x_1 - 0.1x_1^2\right\}.$$

Firstly, it is established the sets $\mathbf{P}$ and $\{x_p \in \mathbb{R}^n : p \in \mathbf{P}\}$ such that (2.15) holds. Since $\psi_1 \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, from the mean value theorem, for every $x_1 \in \left[-\sqrt{2M}, \sqrt{2M}\right] \setminus \{0\}$, there exits $c \in \left[-\sqrt{2M}, \sqrt{2M}\right]$ such that

$$\psi_1(x_1) = \frac{\partial \psi_1}{\partial x_1}(c) \cdot x_1 .$$

Let the constant values

$$\begin{aligned} a^+ &= \max_{|x_1| \leqslant \sqrt{2M}} \frac{\partial \psi_1}{\partial x_1}(x_1) = \max_{|x_1| \leqslant \sqrt{2M}} -(1+K_1) - 0.2x_1 = -(1+K_1) + 0.2\sqrt{2M} \\ a^- &= \min_{|x_1| \leqslant \sqrt{2M}} \frac{\partial \psi_1}{\partial x_1}(x_1) = \min_{|x_1| \leqslant \sqrt{2M}} -(1+K_1) - 0.2x_1 = -(1+K_1) - 0.2\sqrt{2M} \end{aligned}$$

and let $\mathbf{P} = \{1, 2, 3, 4\}$.

Together with the definition of the constant values a^+ and a^-, for every $|x_1| \leqslant \sqrt{2M}$, $a^- \cdot x_1 \leqslant \psi_1(x_1) \leqslant a^+ \cdot x_1$. Thus, for every $(x_1, x_2) \in \mathbf{A}$, $a^- \cdot x_1 \leqslant x_2 \leqslant a^+ \cdot x_1$. This implies that

$$\mathbf{A} \subseteq \mathrm{co}((\{\sqrt{2M}\} \times \{x_2^{+,<0}, x_2^{-,<0}\}) \cup (\{-\sqrt{2M}\} \times \{x_2^{+,>0}, x_2^{-,>0}\})), \quad (2.28)$$

where

$$\begin{aligned} x_2^{+,>0} &= -a^+\sqrt{2M}, \quad x_2^{+,<0} = a^+\sqrt{2M}, \\ x_2^{-,>0} &= -a^-\sqrt{2M}, \quad x_2^{-,<0} = a^-\sqrt{2M}. \end{aligned} \quad (2.29)$$

Figure 2.2 illustrate the inclusion (2.28).

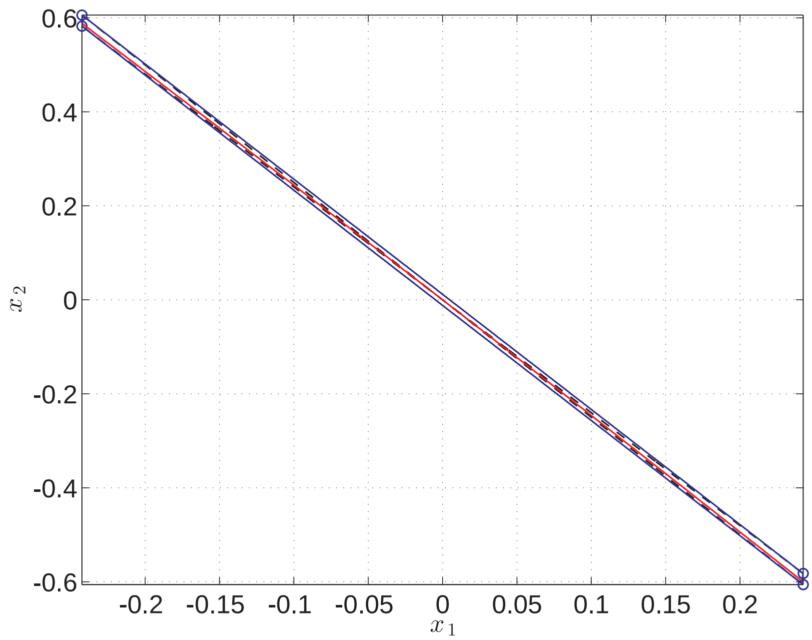


Figure 2.2: The sets $\mathbf{A}$ (in red) and the convex set defined in (2.28) (in blue) are presented in solid line. The circles are the vertexes of this set. The dashed straight lines which bound $\mathbf{A}$ are given by functions $x_1 \mapsto a^+ x_1$ and $x_1 \mapsto a^- x_1$.

A necessary condition for feasibility of the Linear Matrix Inequalities of Proposition 2.23 is $\mathbf{A} \subset \mathbf{N}_{\leqslant r}$. This inclusion holds, if $\sqrt{2M} < r_1$

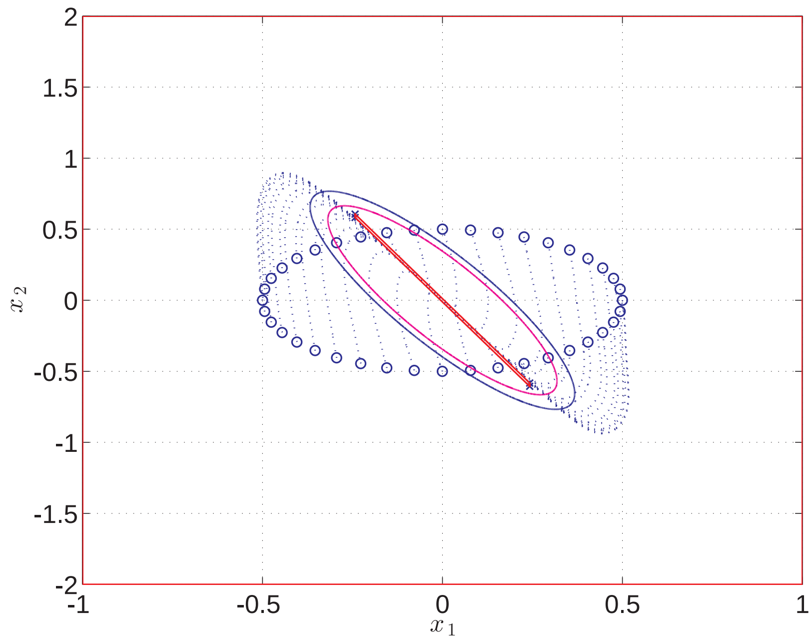


Figure 2.3: The sets $\mathbf{N}_{\leqslant r}$ (in red), $\Omega_{=1}(x^T P x)$ (in blue), and the inclusion (2.28) (in red) at the center. Initial conditions are points given in a ball of radius 0.5 and centered at the origin.

and $|a^{\pm}\sqrt{2M}| < r_2$. These inequalities imply that K_1 must satisfy

$$\frac{0.1}{2r_1^2} + 0.6 < K_1 < \frac{r_2}{r_1} - 0.2r_1 - 1. \tag{2.30}$$

Remark 2.25. Equation (2.30) imposes a limitation on the speed of response, since K_1 is lower and upper bounded. ◦

Applying the technique presented in Section 2.3.3, let $\theta = 0.1$, $r = [1, 2]$, $\mathbf{N}_{\leqslant r} = \{(x_1, x_2) \in \mathbb{R}\times\mathbb{R} : |x_1| \leqslant 1, |x_2| \leqslant 2\}$ and $|u| \leqslant 2\pi$. Moreover, Equation (2.30) holds with $K_1 = 1.45$. From Assumption 2.10, and such a choice for K_1, let $M = 0.03$, and $\varepsilon = 0.6$.

The matrices F and G defined in (2.17) are given by

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}.$$

The function $\tilde{f}_h$ is given by

$$\tilde{f}_h(x, u) = \begin{bmatrix} 0.1\sin(u)(x_1 + 1) - 0.1u + 0.1x_1^2 \\ 0 \end{bmatrix},$$

and its derivatives with respect to the state and input are

$$\frac{\partial \tilde{f}_h}{\partial x}(x_1, x_2) = \begin{bmatrix} 0.2x_1 + 0.1\sin(u) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \frac{\partial \tilde{f}_h}{\partial u}(x_1, x_2) = \begin{bmatrix} 0.1\cos(u)(x_1 + 1) - 0.1 \\ 0 \end{bmatrix}$$

The matrices $\{C_m \in \mathbb{R}^{2\times 2} : 1 \leqslant m \leqslant 2\}$ and $\{D_\nu \in \mathbb{R}^2 : 1 \leqslant \nu \leqslant 2\}$

have elements defined by (2.18) and (2.19). The matrices that are not identically zero are given by

$$C_m = \begin{bmatrix} 0.3 & 0 \\ 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} -0.3 & 0 \\ 0 & 0 \end{bmatrix}, D_1 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \text{ and } D_2 = \begin{bmatrix} -0.3 \\ 0 \end{bmatrix}.$$

Using SeDuMi 1.3 to solve the Linear Matrix Inequalities (2.22)-(2.25), it yields

$$P = \begin{bmatrix} 27.4276 & 11.2248 \\ 11.2248 & 6.2911 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} -12.9029 & -6.4736 \end{bmatrix}.$$

From Proposition 2.23, Assumptions 2.8 and 2.12 hold with $c_\ell = 1$.

Figure 2.3 shows some solutions of system (2.26) in closed loop with the feedback law φ_ℓ, the inclusions $\mathbf{A} \subset \Omega_{\leqslant 1}(V_1)$ and $\Omega_{\leqslant 1}(V_1) \subset \mathbf{N}_{\leqslant r}$.

The main result. Since Assumption 2.10 holds, from the proof of Proposition 2.14 the feedback law given, for every $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, by

$$\begin{aligned} \varphi_g(x_1, x_2) &= -(1 + K_1 + 2\theta x_1)(x_1 + \theta x_1^2 + x_2) - \frac{x_1}{2K_V} \\ &\quad - \frac{x_1 - \psi_1(x_1)}{K_V}\left[c + \frac{c}{4}\Delta(x_1, x_2)^2\right], \end{aligned}$$

where

$$\Delta(x_1, x_2) = |x_1|\theta(1 + |x_1|) + K_V\theta(1 + |x_1|)(1 + |1 + K_1 + 2\theta x_1|),$$

and with parameters $\theta = 0.1$,

$$c = 350 > 349 = \max\left\{\frac{1}{\varepsilon[\alpha(a') - \alpha(M)]}, \varepsilon K_V K_1, 1\right\} = \max\left\{\frac{1}{\varepsilon[\alpha(a') - \alpha(M)]}, \varepsilon\frac{K_V K_\alpha}{2}, 1\right\},$$

$$K_V = 350 \leqslant \frac{M + a}{a^2},$$

$a = 0.01$, and $a' = 0.04$ renders the set $\mathbf{A}$ globally practically asymptotically stable for (2.26) in closed loop with φ_g.

Theorem 2.16, provides a hybrid feedback law $\mathcal{K}$. Let $b_\ell = 0.75$, the discrete set $\mathbf{Q} = \{1, 2\} \times \{1\}$. From (2.10), the subsets of $\mathbb{R}^n$ are given by

$$\begin{aligned} \mathbf{C}_{1,1} &= \Omega_{\leqslant 1}(x^T P x), & \mathbf{D}_{1,1} &= \Omega_{\geqslant 1}(x^T P x), \\ \mathbf{C}_{2,1} &= \Omega_{\geqslant 0.75}(x^T P x), & \mathbf{D}_{2,1} &= \Omega_{\leqslant 0.75}(x^T P x). \end{aligned}$$

From (2.11)-(2.12), the maps are given by

$$\varphi_{q,1}(\cdot) = \begin{cases} \varphi_1(\cdot), & \text{if} \quad q = 1, \\ \varphi_g(\cdot), & \text{if} \quad q = 2, \end{cases}$$

$\mathbf{D}_{2,l} \ni x \mapsto g_{2,1}(x) = \{(1,1)\}$ and

$$\begin{array}{rcl} g_{1,1}: \quad \mathbf{D}_{1,1} & \rightrightarrows & \mathbf{Q} \\ x & \mapsto & \begin{cases} \{(1,1)\}, & x \in \Omega_{<1}(x^T P x) \cap \mathbf{D}_1, \\ \{(2,1)\}, & x \in \Omega_{>1}(x^T P x), \\ \{(1,1),(2,1)\}, & x \in \Omega_{=1}(V_l) \cap \mathbf{D}_1, \end{cases} \end{array}$$

Moreover, from Theorem 2.16, the origin is globally asymptotically stable for (2.26) in closed loop with $\mathscr{K}$.

A simulation of (2.26) in closed loop with $\mathscr{K}$ with initial condition $(x_1, x_2, q) = (2, 0, 1)$ is presented in Figure 2.4. It is shown the time evolution of the x_1, x_2 and q components[29]. Firstly, (2.26) is in closed loop with φ_g (for $t \in [0, 1.4]$), and after (2.26) is in closed loop with φ_1, and the solution converges to the origin.

[29] Regarding q, here it is shown only its first component, because the second one does not change.

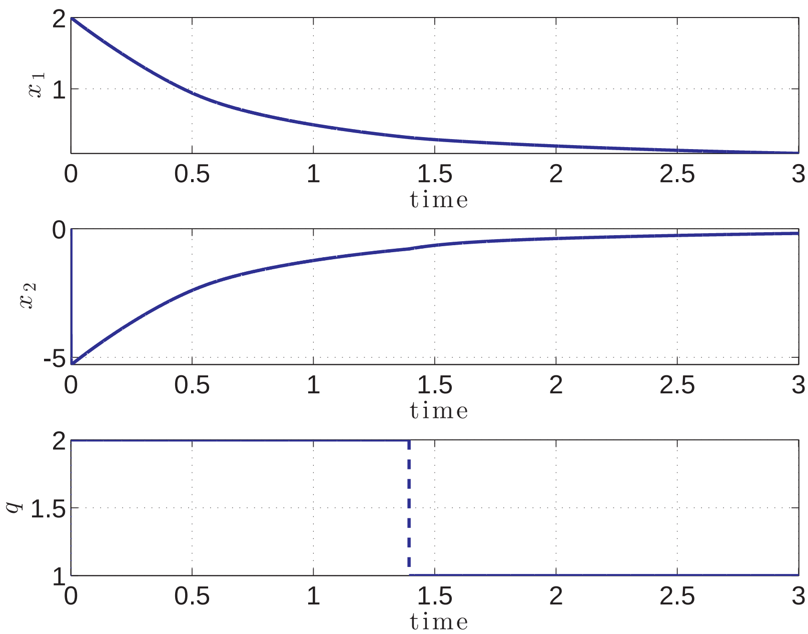


Figure 2.4: Time evolution of a solution of (2.26) in closed loop with $\mathscr{K}$ starting from $(2, 0, 1)$.

2.5 Proofs of Chapter 2

2.5.1 Proof of Lemma 2.5

Proof. The proof consists of showing that (2.2) satisfies Assumption 2.4 and use (Goebel, Sanfelice, and Teel, 2012, Proposition 6.10) to show that solutions exist.

From Definition 2.1, for every $q \in \mathbf{Q}$,

- the sets $\mathbf{C}_q$ and $\mathbf{D}_q$ are closed subsets of $\mathbb{R}^n$;
- Since $\varphi_q \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ and $f_h \in \mathcal{C}^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$, for every fixed $q \in \mathbf{Q}$, $f_h(\cdot, \varphi_q(\cdot)) \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^n)$. From Theorem A.20, $f_h(\cdot, \varphi_q(\cdot))$ is locally bounded relative to $\mathbf{C}_q$. Moreover, for every $x \in \mathbf{C}_q$, the set $\{f_h(x, \varphi_q(x))\}$ is a singleton, thus convex and nonempty. Since $\mathtt{dom}(f_h) = \mathbb{R}^n$, $\mathbf{C}_q \subset \mathtt{dom}(f_h)$;
- By definition, $g_q \in \mathcal{OSC}(\mathbb{R}^n, \mathbf{Q})$. Since $\mathbf{Q}$ is a finite discrete map, for every $x \in \mathbf{D}_q$, $g_q(x)$ is locally bounded. Moreover, $\mathtt{dom}(g_q) = \mathbf{D}_q$ implies that $\mathbf{D}_q \subset \mathtt{dom}(g_q)$.

Thus, Assumption 2.4 is satisfied. Now, it remains to show that solutions of $\Sigma_h(\mathscr{K}_0)$ exist.

For every $Q(0,0) \in \mathbf{Q}$, since $\mathbb{R}^n = \mathbf{C}_{Q(0,0)} \cup \mathbf{D}_{Q(0,0)}$, $X(0,0) \in \mathbf{D}_{Q(0,0)}$, or $X(0,0) \in \mathbf{C}_{Q(0,0)} \setminus \mathbf{D}_{Q(0,0)}$. In the latter case, system (2.2) is given by

$$\dot{x} = f_h(x, \varphi_{Q(0,0)}(x)). \tag{2.31}$$

Since $f_h \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, and $\varphi_q \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$, from Theorems A.24 and A.25, for every initial condition $X(0,0) \in \mathbf{C}_{Q(0,0)} \setminus \mathbf{D}_{Q(0,0)}$, there exists a unique solution for (2.31). Thus, for every $X(0,0) \in \mathbf{C}_{Q(0,0)} \setminus \mathbf{D}_{Q(0,0)}$, there exists a neighbourhood $\mathbf{N}(X(0,0))$ of $X(0,0)$ such that, for every $x \in \mathbf{N}(X(0,0)) \cap \mathbf{C}_{Q(0,0)}$,

$$\{f_h(x, \varphi_{Q(0,0)}(x))\} \cap \mathbf{T}_{\mathbf{C}_{Q(0,0)}}(x) \neq \emptyset,$$

where

$$\mathbf{T}_{\mathbf{C}_{Q(0,0)}}(x) = \left\{ w \in \mathbb{R}^n : \exists \{x_i\}_{i \in \mathbb{N}} \subset \mathbf{C}_{Q(0,0)}, x_i \to x, \exists \{\tau_i\}_{i \in \mathbb{N}} \subset \mathbb{R}, \tau_i \searrow 0, \text{ s.t. } \frac{x_i - x}{\tau_i} \to w \right\}.$$

In other words, the tangent cone at $x \in \mathbf{N}(X(0,0))$ has non-empty intersection with the image of the vector field f_h.

Note that, from Definition 2.3 at each jump, the function X remains constant. Also, it never leaves the set $\mathbf{C}_q \cup \mathbf{D}_q$, because $\mathbf{C}_q \cup \mathbf{D}_q = \mathbb{R}^n$.

Hence, the third case of (Goebel, Sanfelice, and Teel, 2012, Proposition 6.10) never happens.

From (Goebel, Sanfelice, and Teel, 2012, Proposition 6.10), for every $Q(0,0) \in \mathbf{Q}$, and for every $X(0,0) \in \mathbf{C}_{Q(0,0)} \cup \mathbf{D}_{Q(0,0)}$, there exists a non-trivial hybrid solution (X, Q) for (2.2) that satisfies one of the following conditions

a. (X, Q) is complete or;

b. $\mathtt{dom}(X, Q)$ is bounded and the interval I^J, where $J = \sup_j \mathtt{dom}(X, Q)$, has nonempty interior and $t \to (X(t, J), Q(t, J))$ is a maximal solution of $\Sigma_h(\varphi_{Q(t,J)})$, in fact $\lim_{t\to T} |X(t, J)| \to \infty$, where $T = \sup_t \mathtt{dom}(X, Q)$.

This concludes the proof. ∎

2.5.2 Proof of Proposition 2.14

To prove Proposition 2.14, the following lemma is needed.

Lemma 2.26. *There exist positive constant values a' and K_V, and a class $\mathcal{C}^1$, proper and positive definite function*[30]

$$\begin{aligned} V : \mathbb{R}^{n-1} \times \mathbb{R} &\to \mathbb{R} \\ (x_1, x_2) &\mapsto V_1(x_1) + \frac{K_V}{2}(x_2 - \psi_1(x_1))^2 \end{aligned} \tag{2.32}$$

such that the set $\Omega_{\leqslant a'}(V)$ satisfies the inclusion

$$\Omega_{\leqslant a'}(V) \subset \mathbf{B}_{\leqslant a}(\mathbf{A}). \tag{2.33}$$

□

In other words, Lemma 2.26 shows that it is possible to tune the gain K_V such that there exist sublevel sets of the positive definite function given by (2.32) that are contained in $\mathbf{B}_{\leqslant a}(\mathbf{A})$.

[30] Recall that the proper function $V_1 \in (\mathcal{C}^1 \cap \mathcal{P})(\mathbb{R}^{n-1}, \mathbb{R}_{\geqslant 0})$ is introduced in Assumption 2.10.

Proof (of Lemma 2.26). Consider the sequence of proper functions $\{V_k\}_{k\in\mathbb{N}} \subset (\mathcal{C}^1 \cap \mathcal{P})(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ given, for every $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, by

$$V_k(x_1, x_2) = V_1(x_1) + \frac{k}{2}(x_2 - \psi_1(x_1))^2,$$

and the sequence $\{a'_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_{\geqslant 0}$, where $a'_k = (a+1)/k$.

Assume, for purposes of contradiction that, for every $k > 0$, inclusion (2.33) does not hold. From the surjectivity and continuity of V_k, there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$, where $x_k = (x_{1,k}, x_{2,k})$, such that

$$V_k(x_{1,k}, x_{2,k}) \leqslant M + a'_k \quad \text{and} \quad (x_{1,k}, x_{2,k}) \notin \mathbf{B}_{\leqslant a}(\mathbf{A}).$$

This implies that there exists $\varepsilon \in (0,1)$ satisfying

$$\begin{cases} V_1(x_{1,k}) & \leqslant \varepsilon\left(M+\dfrac{a+1}{k}\right), \\ (x_{2,k}-\psi_1(x_{1,k}))^2 & \leqslant 2\dfrac{1-\varepsilon}{k}\left(M+\dfrac{a+1}{k}\right). \end{cases} \tag{2.34}$$

Since[31] V_1 is proper, from the first equation of (2.34), the sequence $\{x_{1,k}\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ belongs to a compact subset. Hence, there exists a subsequence $\{x_{1,k_j}\}_{j\in\mathbb{N}} \subset \{x_{1,k}\}_{k\in\mathbb{N}}$ such that $x_{1,k_j} \to x_1^*$, as $j \to \infty$.

[31] See Assumption 2.10.

It remains to show that $x_{2,k}$ is also bounded. Note that, for every $k \in \mathbb{N}$, $x_{2,k} = \psi_1(x_{1,k}) + x_{2,k} - \psi_1(x_{1,k})$. This implies that $x_{2,k}^2 \leqslant |\psi_1(x_{1,k})|^2 + |x_{2,k} - \psi_1(x_{1,k})|^2$. From the second equation of (2.34), as $k \to \infty$, $(x_{2,k} - \psi_1(x_{1,k}))^2 = 0$. Together with the existence of the converging subsequence $\{x_{1,k_j}\}_{j\in\mathbb{N}} \subset \{x_{1,k}\}_{k\in\mathbb{N}}$, as $j \to \infty$, $|x_2^*| \leqslant |\psi_1(x_1^*)|$ and the sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ is bounded.

From (2.34), $V_1(x_1^*) \leqslant \varepsilon M < M$ and $x_2^* - \psi_1(x_1^*) = 0$. Hence $(x_1^*, x_2^*) \in \mathbf{A}$. This contradicts the fact that $x_{k_j} = (x_{1,k_j}, x_{2,k_j}) \notin \mathbf{B}_{\leqslant a}(\mathbf{A})$. Consequently, there exist $a' > 0$ and $K_V > 0$ such that (2.33) holds. This concludes the proof of Lemma 2.26. ∎

Remark 2.27. It remains to estimate the values K_V and a'. By letting $a' \leqslant M + a$ and $K_v = {}^{(M+a)}/_{(2a^2)}$ the condition $\Omega_{\leqslant a'}(V) \subset \mathbf{B}_{\leqslant a}(\mathbf{A})$ is satisfied. To see this claim, pick any $(x_1^*, x_2^*) \in \Omega_{\leqslant a'}(V)$ such that[32]

[32] Note that with these constraints, $V(x_1^*, x_2^*) \leqslant a'$.

$$\begin{cases} V_1(x_1^*) & \leqslant \dfrac{a'}{2} \\ |x_2^* - \psi_1(x_1^*)| & \leqslant a. \end{cases}$$

This implies that

$$V(x_1^*, x_2^*) = V_1(x_1^*) + \frac{K_V}{2}(x_2^* - \psi_1(x_1^*))^2 \leqslant \frac{M+a}{2} + \frac{M+a}{4} = 3\frac{M+a}{4}$$

and the inclusion (2.33) holds.

Note that the gain K_V brings a dependence of V on the parameter a. In other words, Equation (2.32) is also parametrized by a. A Lyapunov stability theorem for uniform global practical stability is provided in (Chaillet, 2006, Theorem 7.5) (see Theorem 2.29, below), where an addition condition concerning the asymptotic behaviour of bounds of the parametrized Lyapunov function is introduced. To see that (2.32) also satisfies (2.44), consider the function

$$\begin{aligned} e: \quad \mathbb{R}^{n-1} \times \mathbb{R} &\to \mathbb{R}_{\geqslant 0} \\ (x_1, x_2) &\mapsto (x_2 - \psi_1(x_1))^2 \end{aligned}$$

that is positive definite with respect to the set $\{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_2 = \psi_1(x_1)\}$ and proper. From Claim A.33, there exist functions $\underline{\alpha}_e$ and $\overline{\alpha}_e$ of class $\mathcal{K}_\infty$ such that, for every $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$\underline{\alpha}_e(|(x_1, x_2)|) \leqslant e(x_1, x_2) \leqslant \overline{\alpha}_e(|(x_1, x_2)|).$$

From (2.32), consider function $K_V : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ given by $K_V(a)$ and satisfying $K_v(a) \to \infty$, as $a \to 0$,

$$\underline{\alpha}_1(|x_1|) + K_V(a)\underline{\alpha}_e(|(x_1, x_2)|) \leqslant V(x_1, x_2) \leqslant \overline{\alpha}_1(|x_1|) + K_V(a)\overline{\alpha}_e(|(x_1, x_2)|).$$

Moreover,

$$\begin{aligned} K_V(a)\underline{\alpha}_e(|(x_1, x_2)|) &\leqslant \underline{\alpha}_1(|x_1|) + K_V(a)\underline{\alpha}_e(|(x_1, x_2)|) \leqslant V(x_1, x_2) \\ &\leqslant \overline{\alpha}_1(|x_1|) + K_V(a)\overline{\alpha}_e(|(x_1, x_2)|) \leqslant \max\{1, K_V(a)\}\Big(\overline{\alpha}_1(|(x_1, x_2)|) + \overline{\alpha}_e(|(x_1, x_2)|)\Big). \end{aligned}$$

Define, for every $s \in \mathbb{R}_{\geqslant 0}$, the class $\mathcal{K}_\infty$ functions

$$\begin{aligned} \overline{\alpha}_a(s) &= \max\{1, K_V(a)\}(\overline{\alpha}_1(s) + \overline{\alpha}_e(s)) \\ \underline{\alpha}_a(s) &= \frac{1}{K_V(a)}\underline{\alpha}_e(s). \end{aligned}$$

Since[33], for every $s \in \mathbb{R}_{\geqslant 0}$,

$$\underline{\alpha}_a^{-1}(s) = \underline{\alpha}_e^{-1}\left(\frac{1}{K_V(a)}s\right),$$

[33] Because, for every $s \in \mathbb{R}_{\geqslant 0}$, $\underline{\alpha}_a^{-1} \circ \underline{\alpha}_a(s) = s$.

it follows that

$$\underline{\alpha}_a^{-1} \circ \overline{\alpha}_a(s) = \underline{\alpha}_e^{-1}\left(\frac{1}{K_V(a)}\max\{1, K_V(a)\}(\overline{\alpha}_1(s) + \overline{\alpha}_e(s))\right).$$

Thus,

$$\lim_{a \to 0} \underline{\alpha}_a^{-1} \circ \overline{\alpha}_a(a) = \lim_{a \to 0} \underline{\alpha}_e^{-1}\left(\frac{1}{K_V(a)}\max\{1, K_V(a)\}(\overline{\alpha}_1(a) + \overline{\alpha}_e(a))\right) = 0.$$

Hence, the condition (2.32) holds. ◦

Proof (of Proposition 2.14). Let $a > 0$ be a constant value. It will be shown that there exist a feedback law $\varphi_g \in \mathcal{C}^0(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ and a constant value $a' > 0$ such that the set $\Omega_{a'}(V) \subset \mathbf{B}_{\leqslant a}(\mathbf{A})$ is globally asymptotically stable for $\Sigma_h(\varphi_g)$.

Consider the function

$$\begin{aligned} r_1 : \quad \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} &\to \mathbb{R}^{n-1} \\ (x_1, x_2, u) &\mapsto f_1(x_1, x_2) + h_1(x_1, x_2, u). \end{aligned}$$

Taking the Lie derivative of V_1 yields, for every $(x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} L_{r_1} V_1(x_1, x_2, u) &= L_{f_1} V_1(x_1, x_2) + L_{h_1} V_1(x_1, x_2, u) \\ &= L_{f_1} V_1(x_1, x_2) + L_{f_1} V_1(x_1, \psi_1(x_1)) - L_{f_1} V_1(x_1, \psi_1(x_1)) + L_{h_1} V_1(x_1, x_2, u) \\ &\quad + L_{h_1} V_1(x_1, \psi_1(x_1), u) - L_{h_1} V_1(x_1, \psi_1(x_1), u) \\ &\leqslant -\alpha(V_1(x_1)) + (1-\varepsilon)\alpha(V_1(x_1)) + \varepsilon\alpha(M) + L_{f_1} V_1(x_1, x_2) + L_{h_1} V_1(x_1, x_2, u) \\ &\quad - L_{f_1} V_1(x_1, \psi_1(x_1)) - L_{h_1} V_1(x_1, \psi_1(x_1), u). \end{aligned}$$

where the last inequalitiy is due to a. and b. of Assumption[34] 2.10. Thus, for every $(x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,

[34] Recall that the inequality $L_{h_1} V_1(x_1, \psi_1(x_1), u) \leqslant (1-\varepsilon)\alpha(V_1(x_1)) + \varepsilon\alpha(M)$ is assumed.

$$\begin{aligned} L_{r_1} V_1(x_1, x_2, u) \leqslant\ & \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + L_{r_1} V_1(x_1, x_2, u) \\ & - L_{r_1} V_1(x_1, \psi_1(x_1), u). \end{aligned} \tag{2.35}$$

Given $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$ fixed, let

$$\begin{aligned} \eta_{x_1,x_2} : \quad [0,1] &\to \mathbb{R} \\ s &\mapsto s x_2 + (1-s)\psi_1(x_1) \end{aligned}$$

Since $r_1 \in \mathcal{C}^1(\mathbb{R}^{n+1}, \mathbb{R}^{n-1})$ and $\eta_{x_1,x_2} \in \mathcal{C}^1([0,1], \mathbb{R})$,

$$\frac{dr_1}{ds}(x_1, \eta_{x_1,x_2}(s), u) = \frac{\partial r_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s), u) \cdot (x_2 - \psi_1(x_1)).$$

Integrating both sides with respect to s, yields

$$r_1(x_1, x_2, u) - r_1(x_1, \psi_1(x_1), u) = (x_2 - \psi_1(x_1)) \cdot \int_0^1 \frac{\partial r_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s), u)\, ds.$$

Hence, from (2.35), for every $(x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,

$$L_{r_1} V_1(x_1, x_2, u) \leqslant \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + \frac{\partial V_1}{\partial x_1}(x_1) \cdot (x_2 - \psi_1(x_1)) \cdot \int_0^1 \frac{\partial r_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s), u)\, ds.$$

Consider the feedback law defined by

$$\begin{aligned} \tilde{\psi} : \quad ((\mathbb{R}^{n-1} \times \mathbb{R}) \setminus \{(0,0)\}) \times \mathbb{R} &\to \mathbb{R} \\ (x_1, x_2, \bar{u}) &\mapsto \frac{1}{f_2(x_1, x_2)} \left[\frac{\bar{u}}{K_V} + L_{f_1}\psi_1(x_1, x_2) - \frac{1}{K_V} \frac{\partial V_1}{\partial x_1}(x_1) \right. \\ &\qquad \left. \cdot \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds \right], \end{aligned} \tag{2.36}$$

where K_V is given by Lemma 2.26.

Let $u = \tilde{\psi}(x_1, x_2, \bar{u})$ and denote it by $\tilde{\psi}(\bar{u})$. Recall Equation (2.32)

$$\begin{aligned} V : \mathbb{R}^{n-1} \times \mathbb{R} &\to \mathbb{R} \\ (x_1, x_2) &\mapsto V_1(x_1) + \frac{K_V}{2}(x_2 - \psi_1(x_1))^2. \end{aligned} \tag{2.32}$$

By taking its Lie derivative in the f_h-direction, it yields

$$\begin{aligned} L_{f_h} V(x_1, x_2, \tilde{\psi}(\bar{u})) &= L_{r_1} V_1(x_1, x_2, \tilde{\psi}(\bar{u})) + K_V(x_2 - \psi_1(x_1))[f_2(x_1, x_2)\tilde{\psi}(\bar{u}) + h_2(x_1, x_2, \tilde{\psi}(\bar{u})) \\ &\quad - L_{r_1}\psi_1(x_1, x_2, \tilde{\psi}(\bar{u}))] \\ &\leqslant \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + \frac{\partial V_1}{\partial x_1}(x_1) \cdot (x_2 - \psi_1(x_1)) \cdot \int_0^1 \frac{\partial r_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s), \tilde{\psi}(\bar{u}))\, ds \\ &\quad + K_V(x_2 - \psi_1(x_1))[f_2(x_1, x_2)\tilde{\psi}(\bar{u}) + h_2(x_1, x_2, \tilde{\psi}(\bar{u})) - L_{r_1}\psi_1(x_1, x_2, \tilde{\psi}(\bar{u}))] \\ &\leqslant \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + (x_2 - \psi_1(x_1)) \cdot \Bigg[\frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial r_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s), \tilde{\psi}(\bar{u}))\, ds \\ &\quad + K_V f_2(x_1, x_2)\tilde{\psi}(\bar{u}) + K_V h_2(x_1, x_2, \tilde{\psi}(\bar{u})) - K_V L_{r_1}\psi_1(x_1, x_2, \tilde{\psi}(\bar{u}))\Bigg] \\ &\leqslant \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + (x_2 - \psi_1(x_1)) \cdot \Bigg[\frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial r_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s), \tilde{\psi}(\bar{u}))\, ds \\ &\quad + \bar{u} + K_V h_2(x_1, x_2, \tilde{\psi}(\bar{u})) + K_V L_{f_1}\psi_1(x_1, x_2) - \frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds \\ &\quad - K_V L_{r_1}\psi_1(x_1, x_2, \tilde{\psi}(\bar{u}))\Bigg] \\ &\leqslant \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + (x_2 - \psi_1(x_1)) \Bigg[\bar{u} + \frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial h_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s), \tilde{\psi}(\bar{u}))\, ds \\ &\quad + K_V h_2(x_1, x_2, \tilde{\psi}(\bar{u})) - K_V L_{h_1}\psi_1(x_1, x_2, \tilde{\psi}(\bar{u}))\Bigg] \\ &\leqslant \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + (x_2 - \psi_1(x_1))[\bar{u} + \Upsilon(x_1, x_2, \tilde{\psi}(\bar{u}))], \end{aligned}$$

where

$$\Upsilon(x_1, x_2, \tilde{\psi}(\bar{u})) = \frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial h_1}{\partial x_2}(x_1, \eta_{x_1,x_2}(s), \tilde{\psi}(\bar{u}))\, ds + K_V h_2(x_1, x_2, \tilde{\psi}(\bar{u})) - K_V L_{h_1}\psi_1(x_1, x_2, \tilde{\psi}(\bar{u})).$$

From items b)-d) of Assumption 2.10, for every $(x_1, x_2, \bar{u}) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,

$$|\Upsilon(x_1, x_2, \tilde{\psi}(\bar{u}))| \leqslant \Delta(x_1, x_2),$$

where

$$\begin{aligned}\Delta(x_1,x_2) &= \left|\frac{\partial V_1}{\partial x_1}(x_1)\right|\left|\int_0^1 \Psi(x_1,\eta_{x_1,x_2}(s))\,ds\right. \\ &\quad + K_V\Psi(x_1,x_2)\left(1+\left|\frac{\partial\psi_1}{\partial x_1}(x_1)\right|\right).\end{aligned}$$

From Cauchy-Schwartz inequality, and for each constant value $c > 0$, and for every $(x_1, x_2, \bar{u}) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,

$$(x_2-\psi_1(x_1))\Upsilon(x_1,x_2,\bar{u}) \leqslant \frac{1}{c}+\frac{c}{4}(x_2-\psi_1(x_1))^2\Delta(x_1,x_2)^2.$$

Letting

$$\bar{u} = -(x_2-\psi_1(x_1))\left[c+\frac{c}{4}\Delta(x_1,x_2)^2\right] \tag{2.37}$$

it yields, for every $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $c \geqslant 1$,

$$\begin{aligned}L_{f_h}V(x_1,x_2,\tilde{\psi}) &\leqslant \varepsilon[\alpha(M)-\alpha(V_1(x_1))]+(x_2-\psi_1(x_1))\bar{u}+\frac{1}{c}+\frac{c}{4}(x_2-\psi_1(x_1))^2\Delta(x_1,x_2)^2 \\ &\leqslant \varepsilon[\alpha(M)-\alpha(V_1(x_1))]-(x_2-\psi_1(x_1))^2\left[c+\frac{c}{4}\Delta(x_1,x_2)^2\right]+\frac{1}{c} \\ &\quad +\frac{c}{4}(x_2-\psi_1(x_1))^2\Delta(x_1,x_2)^2.\end{aligned}$$

Thus,

$$\begin{aligned}L_{f_h}V(x_1,x_2,\tilde{\psi}) &\leqslant \varepsilon[\alpha(M)-\alpha(V_1(x_1))]+\frac{1}{c} \\ &\quad -c(x_2-\psi_1(x_1))^2,\end{aligned} \tag{2.38}$$

where, in order to simplify the presentation, $\tilde{\psi}(x_1, x_2)$ is denoted by $\tilde{\psi}$.

Since V_1 is proper function, the set

$$\mathbf{A}_{\geqslant 0} := \left\{(x_1,x_2)\in\mathbb{R}^{n-1}\times\mathbb{R}^n : \varepsilon\alpha(V_1(x_1))+c(x_2-\psi_1(x_1))^2 \leqslant \varepsilon\alpha(M)+\frac{1}{c}\right\}$$

is compact. Note that $\mathbf{A}_{\geqslant 0}$ is the set of $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$ for which $L_{f_h}V(x_1, x_2, \tilde{\psi}) \geqslant 0$.

Let $\zeta = \max\{V(x_1, x_2) : (x_1, x_2) \in \mathbf{A}_{\geqslant 0}\}$, for every $c > 1$, and for every $(x_1, x_2) \in \Omega_{>\zeta}(V)$, $L_{f_h}V(x_1, x_2, \tilde{\psi}) < 0$. In other words, $\Omega_{\leqslant\zeta}(V)$ is globally asymptotically stable for $\Sigma_h(\tilde{\psi})$.

Let $K_\alpha > 0$ be the Lipschitz constant of α in the compact set $[0, \zeta]$. For every $(x_1, x_2) \in \Omega_{\leqslant\zeta}(V)$,

$$|\alpha(V_1(x_1))-\alpha(V(x_1,x_2))| \leqslant \frac{K_V K_\alpha}{2}(x_2-\psi_1(x_1))^2.$$

From (2.38), for every $c > 1$, and for every $(x_1, x_2) \in \Omega_{\leqslant\zeta}(V)$,

$$\begin{aligned} L_{f_h}V(x_1,x_2,\tilde{\psi}) &\leqslant \varepsilon[\alpha(M)-\alpha(V_1(x_1))]+\frac{1}{c}-c(x_2-\psi_1(x_1))^2, \\ &\leqslant \varepsilon[\alpha(M)-\alpha(V(x_1,x_2))]+\varepsilon[\alpha(V(x_1,x_2))-\alpha(V_1(x_1))]+\frac{1}{c}-c(x_2-\psi_1(x_1))^2, \\ &\leqslant \varepsilon[\alpha(M)-\alpha(V(x_1,x_2))]+\frac{1}{c}-\left(c-\varepsilon\frac{K_V K_\alpha}{2}\right)(x_2-\psi_1(x_1))^2. \end{aligned}$$

Consider the constant value a' given by Lemma 2.26 and let

$$c_g = \max\left\{\frac{1}{\varepsilon[\alpha(a')-\alpha(M)]}, \varepsilon\frac{K_V K_\alpha}{2}, 1\right\}.$$

For every $c > c_g$, and for every $(x_1,x_2) \in \Omega_{\leqslant\zeta}(V)$,

$$L_{f_h}V(x_1,x_2,\tilde{\psi}) \leqslant \varepsilon\left[\alpha(a')-\alpha(V(x_1,x_2))\right].$$

Thus, for every $c > c_g$, and for every $(x_1,x_2) \in \Omega_{>a'}(V)$,

$$L_{f_h}V(x_1,x_2,\tilde{\psi}) < 0.$$

Let $|x|_{a'} := \mathtt{dist}(x, \Omega_{\leqslant a'}(V))$. Since $\alpha \in \mathcal{K}_\infty$, defined the function note that $\alpha_{a'}(|x|) := \alpha(|(x_1,x_2)|_{a'}) = -\alpha(a') + \alpha(V(x_1,x_2))$ which is of class $\mathcal{K}_\infty$. Thus, for every $(x_1,x_2) \in \Omega_{\geqslant a'}(V)$, $L_{f_h}V(x_1,x_2,\tilde{\psi}) \leqslant -\alpha_{a'}(|x|_{a'})$ and Eq. (2.43) is satisfied.

Hence, the set $\Omega_{\leqslant a'}(V)$ is an attractor for $\Sigma_h(\tilde{\psi})$.

Since from Lemma[35] 2.26 the inclusion $\Omega_{\leqslant a'}(V) \subset \mathbf{B}_{\leqslant a}(\mathbf{A})$ holds, solutions of $\Sigma_h(\tilde{\psi})$ with initial conditions belonging to $\mathbf{B}_{>a}(\mathbf{A})$ will converge to a set contained in $\mathbf{B}_{\leqslant a}(\mathbf{A})$. Thus, $\mathbf{A}$ is practically asymptotically stable[36] for $\Sigma_h(\tilde{\psi})$.

[35] More specifically, from (2.33)

[36] This also could be concluded with Theorem 2.29 below.

Moreover, from (2.36) and (2.37), by letting, for every $(x_1,x_2) \in \mathbb{R}^{n-1}\times\mathbb{R}\setminus\{(0,0)\}$,

$$\begin{aligned} \varphi_g(x_1,x_2) \;=\; \frac{1}{K_V f_2(x_1,x_2)}\Bigg[&K_V L_{f_1}\psi_1(x_1,x_2) - \frac{\partial V_1}{\partial x_1}(x_1)\cdot\int_0^1 \frac{\partial f_1}{\partial x_2}(x_1,\eta_{x_1,x_2}(s))\,ds \\ &-(x_2-\psi_1(x_1))\cdot\left(c+\frac{c}{4}\Delta^2(x_1,x_2)\right)\Bigg], \end{aligned}$$

where $c > c_g$ the feedback law $\tilde{\psi}(\cdot) = \varphi_g(\cdot)$ renders the set $\Omega_{\leqslant a'}(V)$ globally asymptotically stable for $\Sigma_h(\varphi_g)$. This concludes the proof of Proposition 2.14. ■

2.5.3 Proof of Theorem 2.16

Proof. Let the constant values a, b_ℓ, and c_ℓ satisfy[37] $0 < b_\ell < c_\ell$ and be such that, for every $l \in \mathbf{L}$,

[37] From Assumptions 2.8 and 2.12, such values do exist because, for every $l \in \mathbf{L}$, V_l is a proper continuous function.

$$\max\{V_l(x) : x \in \mathbf{B}_{\leqslant a}(\mathbf{A})\} < b_\ell. \tag{2.39}$$

Under Assumption 2.10, Proposition 2.14 provides the feedback law $\varphi_g \in \mathcal{C}^0(\mathbb{R}^n\setminus\{0\},\mathbb{R})$. This is used to design a hybrid feedback law $\mathscr{K}$ building an hysteresis of local and non-local feedback laws on appropriate domains.[38] Define $\mathbf{Q} = \{1,2\} \times \mathbf{L}$. Consider the subsets (2.10) and the maps defined in (2.11) and (2.12). Recall that the state of system $\Sigma_h(\mathscr{K})$ is $(x, q) \in \mathbb{R}^n \times \mathbf{Q}$.

[38] see also (Goebel, Sanfelice, and Teel, 2009, Page 51) or (Prieur, 2001) for similar concepts applied to different control problems.

Case 1. Assume that $q = \{(2, l)\}$.

1.i. If $x \in \mathbf{C}_{2,l}$, then from (2.11), $\varphi_{2,l}(x) = \varphi_g(x)$. From Assumption 2.10 and Proposition 2.14, $\mathbf{A}$ is globally practically asymptotically stable for $\Sigma_h(\varphi_g)$ and, from (2.10) and (2.39), $\mathbf{A} \subset \mathbf{D}_{2,l}$. Together with the fact that solutions of (2.2) will not jump until the x component be in $\mathbf{D}_{2,l}$, they will converge to $\mathbf{D}_{2,l}$;

1.ii. If $x \in \mathbf{D}_{2,l}$, then from (2.12), $g_{2,l}(x) = \{(1,l)\}$ and, after the jump, the local hybrid feedback law is selected. Since the value of x does not change during a jump, $x \in \mathbf{D}_{2,l}$ after a jump, and from (2.10) and (2.39), $\mathbf{D}_{2,l} \subset \Omega_{<c_\ell}(V_l)$. From the local asymptotic stability of $\{0\} \times \mathbf{L}$, solutions of $\Sigma_h(\mathscr{K}_0)$ starting in $\mathbf{D}_{2,l}$ will converge to $\{0\} \times \mathbf{L}$;

To sum up Case 1, whenever $(X(0,0), Q(0,0)) \in \mathbb{R}^n \times \{(2,l)\}$, the solutions of (2.2) converge to $\{0\} \times \mathbf{L}$.

Case 2. Assume that $q = \{(1, l)\}$.

2.i. If $x \in \mathbf{C}_{1,l}$, then from (2.11), $\varphi_{1,l}(x) = \varphi_l(x)$, and the local hybrid feedback law is selected. From the local asymptotic stability of $\{0\} \times \mathbf{L}$, solutions of $\Sigma_h(\mathscr{K}_0)$ starting in $\mathbf{C}_{1,l}$ will converge to $\{0\} \times \mathbf{L}$;

2.ii. If $x \in \mathbf{D}_{1,l}$. Then from (2.10) and (2.13), either

2ii.a. $q^+ = \{(2,l)\}$ and, after the jump, φ_g is selected. Since before this jump $x \in \Omega_{\geqslant c_\ell}(V_l)$, and $\Omega_{\geqslant c_\ell}(V_l) \subset \mathbf{C}_{2,l}$, and the x-component remains constant after the jump; from Case 1.i., solutions of (2.2) converge to $\mathbf{D}_{2,l}$;

2ii.b. or $q^+ = \{(1, g_l(x))\}$ and, after the jump, a local feedback law is selected. Since before this jump, $x \in \Omega_{\leqslant c_\ell}(V_l) \cap \mathbf{D}_l$ and the x-component of the solutions remains constant just after the jump, from the local asymptotic stability of $\{0\} \times \mathbf{L}$, solutions of $\Sigma_h(\mathscr{K}_0)$ starting in $\Omega_{\leqslant c_\ell}(V_l) \cap \mathbf{D}_l$ will converge to $\{0\} \times \mathbf{L}$;

To sum up Case 2, whenever $(X(0,0), Q(0,0)) \in \mathbb{R}^n \times \{(1,l)\}$, the solutions of (2.2) converge to $\{0\} \times \mathbf{L}$.

Thus, the set $\{0\} \times \mathbf{L}$ is locally stable and globally attractive for (2.2). Hence, it is globally asymptotically stable for (2.2). This concludes the proof. ∎

2.5.4 Proof of Claim 2.21

Proof. Let $\overline{\mathbf{N}} := \mathbf{N}_{\leqslant r} \times \{u \in \mathbb{R} : |u| \leqslant r_u\}$, and fix $(x_1, u_1), (x_2, u_2) \in \overline{\mathbf{N}}$. Let also

$$\begin{aligned} g_i : \quad [0,1] &\to \mathbb{R} \\ t &\mapsto \tilde{f}_{h,i}((1-t)(x_1,u_1) + t(x_2,u_2)), \end{aligned}$$

where $1 \leqslant i \leqslant n$.

Since $\tilde{f}_h \in \mathcal{C}^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$, for every $i = 1, \ldots, n$, $g_i \in \mathcal{C}^1([0,1], \mathbb{R}^n)$. Moreover, for every $t \in [0,1]$,

$$\begin{aligned} \frac{dg_i}{dt}(t) &= \operatorname{grad} \tilde{f}_{h,i}((1-t)(x_1,u_1) + t(x_2,u_2))((x_2,u_2) - (x_1,u_1)). \\ &= \frac{\partial \tilde{f}_{h,i}}{\partial x}((1-t)x_1 + tx_2) \cdot (x_2 - x_1) + \frac{\partial \tilde{f}_{h,i}}{\partial u}((1-t)u_1 + tu_2)(u_2 - u_1) \end{aligned}$$

From the mean value theorem, there exists $c \in (0,1)$ such that

$$\frac{dg_i}{dt}(c) = \tilde{f}_{h,i}(x_2,u_2) - \tilde{f}_{h,i}(x_1,u_1).$$

Thus,

$$\tilde{f}_{h,i}(x_2,u_2) - \tilde{f}_{h,i}(x_1,u_1) = \frac{\partial \tilde{f}_{h,i}}{\partial x}((1-c)x_1 + cx_2) \cdot (x_2 - x_1) + \frac{\partial \tilde{f}_{h,i}}{\partial u}((1-c)u_1 + cu_2)(u_2 - u_1)$$

Hence, for every $(x,u) \in \overline{\mathbf{N}}$, there exists $(\bar{x}, \bar{u}) \in \overline{\mathbf{N}}$ such that

$$\tilde{f}_{h,i}(x,u) = \frac{\partial \tilde{f}_{h,i}}{\partial x}(\bar{x},\bar{u}) \cdot x + \frac{\partial \tilde{f}_{h,i}}{\partial u}(\bar{x},\bar{u})u.$$

From Equations (2.16) and (2.17), for every $(x,u) \in \overline{\mathbf{N}}$, there exists $(\bar{x},\bar{u}) \in \overline{\mathbf{N}}$ such that

$$f_{h,i}(x,u) = \left(\frac{\partial \tilde{f}_{h,i}}{\partial x}(\bar{x},\bar{u}) + \frac{\partial f_{h,i}}{\partial x}(0,0) \right) \cdot x + \left(\frac{\partial \tilde{f}_{h,i}}{\partial u}(\bar{x},\bar{u}) + \frac{\partial f_{h,i}}{\partial u}(0,0) \right) u.$$

From (2.18), there exist $\underline{m}$, $\overline{m} \in \mathbf{M}$ such that

$$\sum_{j=1}^{n} c_{\underline{m},ij} \leqslant \sum_{j=1}^{n} \frac{\partial \tilde{f}_{h,i}}{\partial x_j}(\bar{x}, \bar{u}) \leqslant \sum_{j=1}^{n} c_{\overline{m},ij}.$$

This implies that, for every $(x, u) \in \overline{\mathbf{N}}$, there exist $\underline{m}$, $\overline{m} \in \mathbf{M}$ such that

$$C_{\underline{m},i} \cdot x \leqslant \frac{\partial \tilde{f}_{h,i}}{\partial x}(\bar{x}, \bar{u}) \cdot x \leqslant C_{\overline{m},i} \cdot x.$$

Together with (2.19), for every $(x, u) \in \overline{\mathbf{N}}$, there exist $\underline{m}, \overline{m} \in \mathbf{M}$, and $\underline{v}, \overline{v} \in \mathbf{V}$ such that

$$C_{\underline{m},i} \cdot x + D_{\underline{v},i} u \leqslant \frac{\partial \tilde{f}_{h,i}}{\partial x}(\bar{x}, \bar{u}) \cdot x + \frac{\partial \tilde{f}_{h,i}}{\partial u}(\bar{x}, \bar{u}) u \leqslant C_{\overline{m},i} \cdot x + D_{\overline{v},i}.$$

Thus, for every $(x, u) \in \overline{\mathbf{N}}$, there exist $\underline{m}, \overline{m} \in \mathbf{M}$, and $\underline{v}, \overline{v} \in \mathbf{V}$

$$\left(\frac{\partial f_{h,i}}{\partial x}(0,0) + C_{\underline{m},i}\right) \cdot x + \left(\frac{\partial f_{h,i}}{\partial u}(0,0) + D_{\underline{v},i}\right) u \leqslant f_{h,i}(x,u) \leqslant \left(\frac{\partial f_{h,i}}{\partial x}(0,0) + C_{\overline{m},i}\right) \cdot x + \left(\frac{\partial f_{h,i}}{\partial u}(0,0) + D_{\overline{v},i}\right) u.$$

In other words, for every $(x, u) \in \overline{\mathbf{N}}$, there exist $\underline{m}, \overline{m} \in \mathbf{M}$, and $\underline{v}, \overline{v} \in \mathbf{V}$, and $t \in [0, 1]$ such that

$$f_{h,i}(x,u) \in (1-t)(F_i + C_{\underline{m},i}) \cdot x + (G_i + D_{\underline{v},i})u + t(F_i + C_{\overline{m},i}) \cdot x + (G_i + D_{\overline{v},i})u.$$

Therefore from (2.1), for every $(x, u) \in \overline{\mathbf{N}}$,

$$\dot{x} \in \mathrm{co}\{(F + C_m)x + (G + D_v)u\},$$

where $v \in \mathbf{V}$ and $m \in \mathbf{M}$. This concludes the proof. ■

2.5.5 Proof of Proposition 2.23

Proof. Equation (2.22) rewritten in terms of a Linear Matrix Inequality in the matrix variables (2.2) and W given by

$$W(F + C_m)^T + H(G + D_v)^T + (F + C_m)W + (G + D_v)H^T < 0.$$

Multiplying this equation at left and right by a symmetric positive definite matrix P yields, for every $m \in \mathbf{M}$, and for every $v \in \mathbf{V}$, and for every $x \in \mathbf{N}_{\leqslant r} \setminus \{0\}$,

$$x^T(F + C_m + (G + D_v)K)^T Px + x^T P(F + C_m + (G + D_v)K)x < 0.$$

From Theorem 2.28, Equation (2.23) is equivalent, for every $i \in \mathbf{I}$, to

$$r_i^2 W - We_i e_i^T W^T \geqslant 0$$

Since $W = P^{-1}$, for every $x \in \Omega_{\leqslant 1}(V_l)$,

$$x^T e_i e_i^T x \leqslant r_i^2 x^T P x \leqslant r_i^2.$$

Since $e_i \cdot x = x_i$, for every $x \in \Omega_{\leqslant 1}(V_l)$, $x_i^2 \leqslant r_i^2$. Thus, $\Omega_{\leqslant 1}(V_l) \subset \mathbf{N}_{\leqslant r}$.

From Theorem 2.28, Equation (2.24) implies that, for every $p \in \mathbf{P}$, $x_p^T P x_p \leqslant 1$. Thus $\mathrm{co}(\{x_p \in \mathbb{R}^n : p \in \mathbf{P}\}) \subset \Omega_{\leqslant 1}(x^T P x)$.

From Theorem 2.28, Equation (2.25) is equivalent to $r_u^2 W - HH^T \geqslant 0$. This implies $r_u^2 W^{-T} \geqslant K^T K$. Then, for every $x \in \Omega_{\leqslant 1}(V_l)$, $x^T K^T K x \leqslant r_u^2 x^T P x \leqslant r_u^2$. This concludes the proof. ∎

2.6 Summary

A design of hybrid feedback laws method has been presented in this chapter to combine a nonlinear feedback law that stabilizes a compact set with a local feedback law that renders the origin locally asymptotically stable. This procedure provides a stabilizing feedback law for nonlinear control systems for which the *backstepping* design procedure can not be applied to globally stabilize the origin. It has been developed a method to design a linear feedback law satisfying constraints on the basin of attraction of the closed-loop system.

2.7 Appendix of Chapter 2

2.7.1 The *backstepping* procedure

The *backstepping* is a well known method to design a feedback law rendering cascaded systems asymptotically stable see, e.g., (Isidori, 1999; Khalil, 2001; Kokotović, 1992; Krstić, Kanellakopoulos, and Kokotović, 1995).

Consider the system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2)u \end{cases} \tag{2.40}$$

where, $f_1 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^{n-1})$ and $f_2 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_{\neq 0})$.

Assume that, there exists a feedback law $\phi_1 \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ with $\phi_1(0) = 0$ for the x_1-subsystem rendering the origin globally asymptotically stable for

$$\dot{x}_1 = f_1(x_1, \phi_1(x_1)).$$

From the converse Lyapunov theorem[39], there exist proper function $V_1 \in (\mathcal{C}^1 \cap \mathcal{P})(\mathbb{R}^{n-1}, \mathbb{R}_{\geqslant 0})$ and $\alpha_1 \in \mathcal{K}_\infty$ such that, for every $x_1 \in \mathbb{R}^{n-1}$, $L_{f_1} V_1(x_1, \phi_1(x_1)) \leqslant -\alpha_1(|x_1|)$.

[39] See Theorem A.36.

Let $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$ be fixed, consider the function

$$\begin{array}{rccl} \eta_{x_1,x_2}: & [0,1] & \to & \mathbb{R} \\ & s & \mapsto & s x_2 + (1-s)\phi_1(x_1) \end{array}$$

Since for every $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $f_2(x_1, x_2) \neq 0$, by letting $u = {}^{v}/_{f_2(x_1,x_2)}$, system (2.40) can be rewritten as

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, \phi_1(x_1)) + (x_2 - \phi_1(x_1)) \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds \\ \dot{x}_2 & = & v, \end{array} \right.$$

where $v \in \mathbb{R}$.

Consider the variable change $e := x_2 - \phi_1(x_1)$. Taking the time-derivative of e yields $\dot{e} = \dot{x}_2 - L_{f_1}\phi_1(x_1, x_2)$. System (2.40) rewritten in the new variable e is given by

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, \phi_1(x_1)) + e \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds \\ \dot{e} & = & w, \end{array} \right. \tag{2.41}$$

where $w = v - L_{f_1}\phi_1(x_1, x_2)$. System (2.41) is denoted by $\dot{x} = f(x)$.

Consider the Lyapunov function candidate for system (2.41) given by

$$V(x_1, e) = V_1(x_1) + \frac{e^2}{2}.$$

Taking its Lie derivative in the f-direction yields

$$\begin{array}{rcl} L_f V(x_1, e) & = & L_{f_1} V_1(x_1, \phi_1(x_1)) + \frac{\partial V_1}{\partial x_1} e \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds + ew \\ & \leqslant & \alpha_1(|x_1|) + e \left[\frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds + w \right]. \end{array}$$

Consider the feedback law

$$\phi(x_1, e) = -\frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds - Ke,$$

where $K > 0$ is a constant value. Letting $w = \phi(x_1, e)$ yields, for every

$(x_1, e) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$L_f V(x_1, e) \leqslant -\alpha_1(|x_1|) - Ke^2.$$

Thus, the origin is globally asymptotically stable for (2.41) in closed loop with ϕ. Hence, the origin is also asymptotically stable for (2.40), because $(x_1, e) = (0, 0) \Rightarrow 0 = x_2 - \phi_1(0)$.

Since $\phi(x_1, e) = w = v - L_{f_1}\phi_1(x_1, x_2)$, it follows that

$$v = -\frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds - (x_2 - \phi_1(x_1)) + L_{f_1}\phi_1(x_1, x_2).$$

Therefore, the feedback law defined, for every $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, by

Note that, if $f_2(0, 0) = 0$ the feedback law φ_b would be discontinuous at the origin.

$$\varphi_b(x_1, x_2) = \frac{1}{f_2(x_1, x_2)} \left[-\frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial f_1}{\partial \eta_{x_1,x_2}}(x_1, \eta_{x_1,x_2}(s))\, ds - (x_2 - \phi_1(x_1)) + L_{f_1}\phi_1(x_1, x_2) \right].$$

renders the origin globally asymptotically stable for (2.40).

2.7.2 The Schur's complement

The Schur's complement is used in this chapter to design a linear feedback law rendering the origin locally asymptotically stable. For further reading on the Schur's complement, the interested reader is invited to see (Jbilou, Messaoudi, and Tabaâ, 2004; Zhang, 2005). Here, it is presented some basic concepts.

Consider the matrices $A \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{p \times q}$, $C \in \mathbb{R}^{q \times p}$, and $D \in \mathbb{R}^{q \times q}$, and the block matrix $M \in \mathbb{R}^{(p+q)\times(p+q)}$ given by

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

and assume that $\det(A) \neq 0$. Consider a vector $z = (x, y) \in \mathbb{R}^p \times \mathbb{R}^q$. The linear system $Mz^T = 0$ is equivalent to

$$\begin{cases} Ax + By = 0, \\ Cx + Dy = 0. \end{cases}$$

Multiplying the first equation by $-CA^{-1}$, in the left, and adding it to the second one, the x component of the vector is eliminated, and the linear system is given by

$$(D - CA^{-1}B)y = 0.$$

The matrix $S = D - CA^{-1}B$ is called *Schur complement* of A in M.[40]

[40] (Jbilou, Messaoudi, and Tabaâ, 2004)

Theorem 2.28. [41] *Let $M \in \mathbb{R}^{(p+q)\times(p+q)}$ be a symmetric matrix given by*

[41] Based on (Zhang, 2005, Theorem 1.12).

$$M = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix},$$

where $A \in \mathbb{R}^{p\times p}$ is square and $\det(A) \neq 0$. Then, $M > 0$ if and only if $A > 0$ and $(D - B^T A^{-1} B) > 0$. □

From (Jbilou, Messaoudi, and Tabaâ, 2004, Proposition 1), under the hypothesis of Theorem 2.28, $M > 0$ if and only if $D > 0$ and $A - BD^{-1}B^T > 0$.

2.7.3 A remark on the Lyapunov sufficient conditions for practical stability

Recall the concept of global practical asymptotic stabilizability stated in Definition 2.13.

Definition 2.13 (Global practical asymptotic stabilizability).[42] A compact invariant set $\mathbf{S} \subset \mathbb{R}^n$ containing the origin is *globally practically asymptotically stabilizable* for (2.1) if, for every $a \in \mathbb{R}_{>0}$, there exists a feedback law $\varphi_g : \mathbb{R}^n \to \mathbb{R}$ such the set

[42] Based on (Isidori, 1999, pp. 126).

$$\mathbf{B}_{\leqslant a}(\mathbf{S}) := \{x \in \mathbb{R}^n : |x|_{\mathbf{S}} \leqslant a\}$$

contains a compact invariant set that is globally asymptotically stable for $\Sigma_h(\varphi_g)$. ○

Under Assumption 2.10, for every $a \in \mathbb{R}_{>0}$, there exists[43] a feedback law $\varphi_g : \mathbb{R}^n \to \mathbb{R}$ such the set $\mathbf{B}_{\leqslant a}(\mathbf{A})$ contains a compact invariant set that is globally asymptotically stable for $\Sigma_h(\varphi_g)$, where $\mathbf{A}$ is the set given by

[43] See Proposition 2.14, above.

$$\mathbf{A} = \{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} : V_1(x_1) \leqslant M, x_2 = \psi_1(x_1)\}. \tag{2.9}$$

Because of the choice of K_V, in the proof of 2.14, the feedback law φ_b is parametrized by a. Thus, the closed-loop system $\Sigma_h(\varphi_g)$ and the candidate Lyapunov function

$$\begin{aligned} V : \mathbb{R}^{n-1} \times \mathbb{R} &\to \mathbb{R} \\ (x_1, x_2) &\mapsto V_1(x_1) + \frac{K_V}{2}(x_2 - \psi_1(x_1))^2 \end{aligned} \tag{2.32}$$

are also parametrized by a. The dependence of the the above functions on the parameter a is highlighted by adding it as subscript, for instance, $\Sigma_h(\varphi_{g,a})$

A question that arises from this dependence of a concerns the behaviour of the Lyapunov functions. A theorem introduced in (Chaillet, 2006) gives a sufficient condition to achieve stability of a compact invariant set. For the sake of completeness of this chapter, it is recalled here

Theorem 2.29 (Lyapunov sufficient conditions for global practical asymptotic stability). [44] *Let* $\mathbf{A} \subset \mathbb{R}^n$ *be a compact set. Suppose that, given any* $a > 0$*, there exist a continuous differentiable Lyapunov function* $V_a : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$*, and class* $\mathcal{K}\infty$ *functions* $\underline{\alpha}_a$*,* $\underline{\alpha}_a$*, and* α_a *such that, for every* $x \in \mathbf{B}_{\geqslant a}(\mathbf{A})$*,*

$$\underline{\alpha}_a(|x|_{\mathbf{A}}) \leqslant V_a(x) \leqslant \overline{\alpha}_a(|x|_{\mathbf{A}}), \tag{2.42}$$

$$L_{f_h} V_a(x, \varphi_{g,a}) \leqslant -\alpha_a(|x|_{\mathbf{A}}), \tag{2.43}$$

$$\lim_{a \to 0} \underline{\alpha}_a^{-1} \circ \overline{\alpha}_a(a) = 0. \tag{2.44}$$

Then, the set $\mathbf{A}$ *is globally practically asymptotically stable for system* $\Sigma_h(\varphi_g)$*.*

□

[44] Based on (Chaillet, 2006, Theorem 7.5).

3 | ANALYSIS UNDER NESTED CRITERIA

A sufficient condition for the stability of the interconnection of dynamical systems is given by the small gain theorem. Roughly speaking, to apply this theorem, it is required that the gains' composition is continuous, increasing and strictly smaller than the identity function. In this chapter, it is presented an alternative sufficient condition when such criterion fails due to either lack of continuity or the bound of the composed gain is larger than the identity function. The local (resp. global) asymptotic stability of the origin (resp. attractivity of a compact set) is ensured by a region-dependent small gain condition. Under an additional condition that implies convergence of solutions for almost all initial conditions in a suitable domain, the almost global asymptotic stability of the origin is ensured. An example illustrates the approach.

Contents

3.1 Introduction

The use of nonlinear input-output gains for stability analysis was introduced in (Zames, 1966) by considering a system as an input-output operator. The condition that ensures stability, called Small Gain Theorem, of interconnected systems is based on the contraction principle.

Sontag introduced a new concept of gain relating the input to system states (Sontag, 1989). This notion of stability links Zames' and Lyapunov's approaches (Sontag, 2001). Characterizations in terms of dissipation and Lyapunov functions are given in (Sontag and Wang, 1995).

In (Jiang, Teel, and Praly, 1994), the contraction principle is used with the input-to-state stability notion to obtain an equivalent Small Gain Theorem. A formulation of this criterion in terms of Lyapunov functions may be found in (Jiang, Mareels, and Wang, 1996).

Besides stability analysis, the Small Gain Theorem may also be used for the design of dynamic feedback laws satisfying robustness constraints. The interested reader is invited to see (Freeman and Kokotović, 2008; Sastry, 1999) and references therein. Other versions of the Small Gain theorem do exist in the literature, see (Angeli and Astolfi, 2007; Astolfi and Praly, 2012; Ito, 2006; Ito and Jiang, 2009) for not necessarily ISS systems.

In order to apply the Small Gain Theorem, it is required that the composition of the nonlinear gains is smaller than the argument for all of its positive values. Such a condition, called Small Gain Condition, restricts the application of the Small Gain Theorem to a composition of well chosen gains.

The approaches introduced in (Stein Shiromoto, Andrieu, and Prieur, 2013a,c) provide an alternative criterion for the stabilization of interconnected systems, when a single Small Gain Condition does not hold globally. It consists in showing that if a local (resp. non-local) Small Gain Condition holds in a local (resp. non-local) region of the state space, and the intersection of the local and non-local regions is empty, additionally if outside the union of these regions, the set of initial conditions from which the associated trajectories do not converge to the local region have measure zero, then the resulting interconnected system is almost asymptotically stable (this notion is precisely defined below). In this chapter, a sufficient condition guaranteeing this property to hold is presented. Moreover, for planar systems, an extension of the Bendixson criterion to regions which are not simply connected is given. This allows to obtain global asymptotic stability of the origin.

This approach may be seen as a blend of two small gain conditions that hold in different regions: a local and a non-local. The use of a unifying approach for local and non-local properties is well known in the literature see (Andrieu and Prieur, 2010) in the context of control Lyapunov functions, and (Chaillet, Angeli, and Ito, 2012) for blending iISS and ISS properties.

3.2 Background, motivation, and problem statement

3.2.1 Background

Let $f \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$\dot{x}(t) = f(x(t), u(t)), \tag{3.1}$$

where, for every $t \in \mathbb{R}_{\geqslant 0}$, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, for some positive integers n and m. A solution of (3.1) with initial condition x, and input u at time t is denoted by $X(t, x, u)$. From now on, arguments t will be omitted. Note that, from Theorems A.24 and A.25, for every $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$, and for every initial condition, there exists a unique solution of (3.1).

Definition 3.1 (Input-to-state stability). [1] The equilibrium of the origin is said to be *input-to-state stable for* (3.1) if there exist $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that, for every $x \in \mathbb{R}^n$, for every $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^m)$, and for every $t \in \mathbb{R}_{\geqslant 0}$,

$$|X(t, x, u)| \leqslant \beta(|x|, t) + \gamma\left(\left|u_{[0,t]}\right|_\infty\right), \tag{3.2}$$

[1] (Sontag, 2008, pp. 8)

As remarked in (Sontag, 2008, pp. 9), since $\gamma \in \mathcal{K}_\infty$, $\sup\{\gamma(|u(s)|) : s \in [0, t]\}$ is equivalent to $\gamma(\sup\{|u(s)| : s \in [0, t]\}) = \gamma(|u_{[0,t]}|_\infty)$.

where $\gamma\left(\left|u_{[0,t]}\right|_\infty\right)$ is called *ISS gain for* (3.1). ∘

From now on, by saying that (3.1) *is ISS* it is meant that the equilibrium of the origin is input-to-state stable for (3.1).

Note that, solutions of ISS systems converge to a ball centered at the origin and with radius given by $\gamma(|u|_\infty)$ (see (3.2)). On the other hand, for systems that are UGAS, solutions converge to the origin for every input belonging to a given set (see (A.8)).

[2] Based on (Dashkovskiy, Rüffer, and Wirth, 2010; Liberzon, Nešić, and Teel, 2013; Sontag, 2008).

Definition 3.2. [2] Let $k \geqslant 0$ be a constant integer. A function $V \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ that is locally Lipschitz is called an *ISS-Lyapunov function for* (3.1) if

- There exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that, for every $x \in \mathbb{R}^n$,

$$\underline{\alpha}(|x|) \leqslant V(x) \leqslant \overline{\alpha}(|x|); \tag{3.3}$$

- There exist $\alpha_x \in \mathcal{K}$, and a proper function $\lambda_x \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ such that, for every $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$V(x) \geqslant \alpha_x(|u|) \Rightarrow D_f^+ V(x, u) \leqslant -\lambda_x(x), \tag{3.4}$$

  where α_x is an input gain. ∘

From Remark A.40, if $k > 0$, then $D_f^+ V(x, u) = \operatorname{grad} V(x) \cdot f(x, u)$, for every $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$. The ISS property is characterized in terms of the existence of smooth ISS-Lyapunov functions. More precisely,

[3] (Sontag and Wang, 1995)

Theorem 3.3. [3] *System* (3.1) *is ISS if and only if there exists a smooth ISS-Lyapunov function for* (3.1). □

As remarked in (Dashkovskiy, Rüffer, and Wirth, 2010), the proof that (3.3) and (3.4) imply that (3.1) is ISS goes along the lines presented in (Sontag and Wang, 1995).

3.2.2 Motivation

Consider the system

$$\dot{z} = g(v, z), \tag{3.5}$$

where $g \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m)$. A solution of (3.5) with initial condition z, and input v at time t is denoted by $Z(t, z, v)$. Let $k \geqslant 0$ be a constant integer, from now on, assume that there exists a function $W \in \mathcal{C}^k(\mathbb{R}^m, \mathbb{R}_{\geqslant 0})$

$$\exists \underline{\alpha}_z \in \mathcal{K}_\infty, \exists \overline{\alpha}_z \in \mathcal{K}_\infty : \forall z \in \mathbb{R}^m, \qquad \underline{\alpha}_z(|z|) \leqslant W(z) \leqslant \overline{\alpha}_z(|z|), \tag{3.6}$$

$$\exists \alpha_z \in \mathcal{K}, \exists \lambda_z \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^m, \mathbb{R}_{\geqslant 0}) \text{ proper} : \forall (x,z) \in \mathbb{R}^n \times \mathbb{R}^m, \quad W(z) \geqslant \alpha_z(|v|) \Rightarrow D_g^+ W(v,z) \leqslant -\lambda_z(z). \tag{3.7}$$

In other words, (3.5) is ISS.

System under consideration. Interconnecting systems (3.1) and (3.5) yields

$$\begin{cases} \dot{x} &= f(x,z), \\ \dot{z} &= g(x,z). \end{cases} \tag{3.8}$$

Using vectorial notation, system (3.8) is denoted by $\dot{y} = h(y)$. A solution initiated from $y \in \mathbb{R}^{n+m}$ and evaluated at time t is denoted $Y(t,y)$. Considering the ISS-Lyapunov inequalities, after interconnection, the following inequalities

$$\begin{array}{rclcrcl} V(x) & \geqslant & \gamma(W(z)) & \Rightarrow & D_f^+ V(x,z) & \leqslant & -\lambda_x(x), \\ W(z) & \geqslant & \delta(V(x)) & \Rightarrow & D_g^+ W(x,z) & \leqslant & -\lambda_z(z), \end{array}$$

are obtained with the gains $\gamma(\cdot) = \alpha_x \circ \overline{\alpha}_z^{-1}(\cdot) \in \mathcal{K}$, and $\delta(\cdot) = \alpha_z \circ \overline{\alpha}_x^{-1}(\cdot) \in \mathcal{K}$.

A sufficient condition ensuring that the origin is globally asymptotically stable for (3.8) is given by the following

Theorem 3.4 (Small-gain theorem). [4] *Assume that V (resp. W) is an ISS-Lyapunov function for* (3.1) *(resp.* (3.5)*) satisfying* (3.3) *and* (3.4) *(resp.* (3.6) *and* (3.7)*). If,*

$$\forall s \in \mathbb{R}_{>0}, \quad \gamma \circ \delta(s) < s, \tag{SGC}$$

then the origin is globally asymptotically stable for (3.8). □

[4] Based on (Jiang, Mareels, and Wang, 1996, Theorem 3.1).

Equation (SGC) is called *Small Gain Condition*. At this point, it is possible to explain the problem that is dealt with in this chapter. Systems for which the composition of the gains does not satisfy (SGC) are considered.

In this chapter, it will be shown that, if

- There exist two gains γ_ℓ and γ_g, for the x-subsystem of (3.8);
- There exist two gains δ_ℓ and δ_g, for the z-subsystem of (3.8);
- The compositions $\gamma_\ell \circ \delta_\ell$ and $\gamma_g \circ \delta_g$ satisfy the Small Gain Condition, not for all values of the arguments, but for two different intervals;

- In the gaps where the small gain conditions do not hold, a condition ensuring the convergence of almost all solutions towards compact sets holds.

Then, for almost every initial condition, solutions of (3.8) converge to the origin. Moreover, if $n = m = 1$, the previous conclusion holds true for every initial condition.[5]

[5] See Theorems 3.13 and 3.14 below.

3.3 Standing assumptions

Assumption 3.5. [6] There exist constant values

$$0 \leqslant \underline{M} < \overline{M} \leqslant \infty \quad \text{and} \quad 0 \leqslant \underline{N} < \overline{N} \leqslant \infty,$$

[6] (Stein Shiromoto, Andrieu, and Prieur, 2013c)

functions γ and δ of class $\mathcal{K}$ such that,

$$\begin{aligned} b &= \limsup_{s\to\infty} \gamma(s) > \overline{M}, \quad \text{if} \quad \overline{M} < \infty, \\ b &= \limsup_{s\to\infty} \gamma(s) > \underline{M}, \quad \text{if} \quad \overline{M} = \infty. \end{aligned} \tag{3.9}$$

If $\min\{\overline{M}, \overline{N}\} < \infty$, assume also that

$$\max\{\gamma^{-1}(\underline{M}), \underline{N}\} < \min\{\delta(\overline{M}), \overline{N}\}. \tag{3.10}$$

Let

$$\begin{aligned} \mathbf{S} = & \left\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m : \underline{M} \leqslant V(x) \leqslant \overline{M}, W(z) \leqslant \overline{N}\right\} \\ & \cup \left\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m : V(x) \leqslant \overline{M}, \underline{N} \leqslant W(z) \leqslant \overline{N}\right\}. \end{aligned} \tag{3.11}$$

The functions V and W satisfy, for every $(x,z) \in \mathbf{S}$,

$$V(x) \geqslant \gamma(W(z)) \quad \Rightarrow \quad D_f^+ V(x,z) \leqslant -\lambda_x(x), \tag{3.12}$$

$$W(z) \geqslant \delta(V(x)) \quad \Rightarrow \quad D_g^+ W(x,z) \leqslant -\lambda_z(z). \tag{3.13}$$

○

Assumption 3.5 gives conditions for the ISS-Lyapunov inequalities (3.12) and (3.13) to hold within the set $\mathbf{S} \subset \mathbb{R}^{n+m}$, defined as the union of sublevel sets of the ISS-Lyapunov functions V and W.

Equation (3.9) suppose that, for the gain γ, its maximum value is reached outside the set $\mathbf{S}$ (for the case in which $\mathbf{S}$ is bounded) or inside it (for the case in which $\mathbf{S}$ is unbounded).

The role of Equation (3.10) becomes clear in Proposition 3.8, where it is used for the computation of an attractor sublevel set and for the estimation of the basin of attraction of this attractor.

Assumption 3.6. [7]

$$\begin{aligned} &\text{if } \overline{M} < \infty, \quad s \in [\underline{M}, \overline{M}] \setminus \{0\}, \quad \gamma \circ \delta(s) < s, \\ &\text{if } \overline{M} = \infty, \quad s \in [\underline{M}, \overline{M}) \setminus \{0\}, \quad \gamma \circ \delta(s) < s. \end{aligned} \tag{3.14}$$

○

Assumption 3.6 implies that the small-gain condition holds in an interval of $\mathbb{R}_{\geqslant 0}$ corresponding to the set $\mathbf{S}$.

Remark 3.7. Equation (3.14) is equivalent to

$$\begin{aligned} &\text{if } \overline{M} < \infty, \quad s \in [\gamma^{-1}(\underline{M}), \gamma^{-1}(\overline{M})] \setminus \{0\}, \quad \delta \circ \gamma(s) < s, \\ &\text{if } \overline{M} = \infty, \quad s \in [\gamma^{-1}(\underline{M}), \gamma^{-1}(\overline{M})) \setminus \{0\}, \quad \delta \circ \gamma(s) < s. \end{aligned} \tag{3.15}$$

To see this claim, from (3.14)

$$\begin{aligned} &\text{if } \overline{M} < \infty, \quad s \in [\underline{M}, \overline{M}] \setminus \{0\}, \quad \delta(s) < \gamma^{-1}(s), \\ &\text{if } \overline{M} = \infty, \quad s \in [\underline{M}, b) \setminus \{0\}, \quad \delta(s) < \gamma^{-1}(s). \end{aligned}$$

which is equivalent to,

$$\begin{aligned} &\text{if } \overline{M} < \infty, \quad \forall s \in [\gamma^{-1}(\underline{M}), \gamma^{-1}(\overline{M})] \setminus \{0\}, \quad \delta \circ \gamma(s) < s \\ &\text{if } \overline{M} = \infty, \quad \forall s \in [\gamma^{-1}(\underline{M}), b) \setminus \{0\}, \quad \delta \circ \gamma(s) < s \end{aligned}$$

which, consequently, is equivalent to (3.15). ○

3.4 Results

Proposition 3.8. [8] *Under Assumptions 3.5 and 3.6, let*

$$\widetilde{M} = \max\{\gamma^{-1}(\underline{M}), \underline{N}\} \quad \text{and} \quad \widehat{M} = \min\{\delta(\overline{M}), \overline{N}\}.$$

Then, there exists a proper function $U \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geqslant 0})$ *that is locally Lipschitz on* $\mathbb{R}^{n+m} \setminus \{0\}$ *and such that,*

$$\forall y \in \Omega_{\leqslant \widehat{M}}(U) \setminus \Omega_{\leqslant \widetilde{M}}(U), \quad \lim_{t \to \infty} U(Y(t,y)) \leqslant \widetilde{M}.$$

Moreover, if $\gamma, \delta \in (\mathcal{C}^1 \cap \mathcal{K}_\infty)$*, then a suitable* U *is given by*

$$U(x,z) = \max\left\{\frac{\delta(V(x)) + \gamma^{-1}(V(x))}{2}, W(z)\right\}. \tag{3.16}$$

□

A sketch of the proof of Proposition 3.8 is given as follows. Assumptions 3.5 and 3.6 provide a proper function $U \in (\mathcal{C}^0, \cap \mathcal{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geqslant 0})$ whose derivative[9] is negative definite in $\mathbf{S}$. From (3.10), $\Omega_{\leqslant \widehat{M}}(U) \setminus$

[7] (Stein Shiromoto, Andrieu, and Prieur, 2013c)

[8] (Stein Shiromoto, Andrieu, and Prieur, 2013c)

[9] In the sense of Dini derivatives (see Definition A.39 below).

$\Omega_{\leqslant \widetilde{M}}(U) \neq \emptyset$, and $\Omega_{\leqslant \widehat{M}}(U) \setminus \Omega_{\leqslant \widetilde{M}}(U) \subset \mathbf{S}$. Thus, solutions of (3.8) starting in $\Omega_{\leqslant \widehat{M}}(U) \setminus \Omega_{\leqslant \widetilde{M}}(U)$ converge to $\Omega_{\leqslant \widetilde{M}}(U)$. The proof of Proposition 3.8 is provided in details in Section 3.6.1.

Corollary 3.9. *[Local stabilization]* [10] *Consider Assumptions 3.5 and 3.6 with the constant values* $\underline{M} = \underline{N} = 0$, $M_\ell := \overline{M} < \infty$ *and* $N_\ell := \overline{N} < \infty$. *The set* $\Omega_{\leqslant \widehat{M}_\ell}(U_\ell)$ *is included in the basin of attraction of the origin of (3.8), where* U_ℓ *and* $\widehat{M}_\ell$ *are given by Proposition 3.8.* □

[10] See also (Dashkovskiy, Rüffer, and Wirth, 2010; Dashkovskiy and Rüffer, 2010; Stein Shiromoto, Andrieu, and Prieur, 2013c).

Remark 3.10. Since the function U given by Proposition 3.8 is not locally Lipschitz in the origin, the construction of the function $\sigma \in (\mathcal{K}_\infty \cap \mathcal{C}^1)$ is slightly different. The interested reader may check the construction of the function U in the proofs of (Sanfelice, 2014, Lemma 4.1 and Theorem 4.2) and (Praly, 2011, Théorème 3.109). ◦

Corollary 3.11. *[Global attractivity]* [11] *Consider Assumptions 3.5 and 3.6 with the constant values* $M_g := \underline{M} > 0$ *and* $N_g := \underline{N} > 0$, *and* $\overline{M} = \overline{N} = \infty$. *The set* $\Omega_{\leqslant \widetilde{M}_g}(U_g)$ *is globally attractive for (3.8), where* U_g *and* $\widetilde{M}_g$ *are given by Proposition 3.8.* □

[11] See also (Dashkovskiy, Rüffer, and Wirth, 2010; Dashkovskiy and Rüffer, 2010; Stein Shiromoto, Andrieu, and Prieur, 2013c).

From Corollary 3.9 (resp. 3.11), solutions of (3.8) starting in $\Omega_{\leqslant \widehat{M}_\ell}(U_\ell)$ (resp. $\Omega_{> \widetilde{M}_g}(U_g)$) converge to the origin (resp. $\Omega_{\leqslant \widetilde{M}_g}(U_g)$).

Note that if $\Omega_{\leqslant \widetilde{M}_g}(U_g) \subset \Omega_{\leqslant \widehat{M}_\ell}(U_\ell)$, then global asymptotic stability of (3.8) is achieved by the local stability of origin together with the global attraction of $\Omega_{\leqslant \widetilde{M}_g}(U_g)$. When that inclusion does not hold, solutions of (3.8) starting in $\Omega_{\leqslant \widetilde{M}_g}(U_g) \setminus \Omega_{\leqslant \widehat{M}_\ell}(U_\ell)$ may converge to a ω-limit set instead of $\Omega_{\leqslant \widehat{M}_\ell}(U_\ell)$. To avoid it, the following theorems provide sufficient conditions.

Definition 3.12. [12] Let $\mathbf{A} \subset \mathbb{R}^{n+m}$ be a compact set with respect to (3.8). It is called *almost globally asymptotically stable* if it is locally stable in the Lyapunov sense, i.e.,

$$\forall \varepsilon > 0, \exists \delta > 0 \quad : \quad |y|_{\mathbf{A}} \leqslant \delta \Rightarrow |Y(t, y)| \leqslant \varepsilon, \quad \forall t \geqslant 0,$$

and attractive for almost every initial condition. More precisely, there exists $\aleph \subset \mathbb{R}^{n+m}$ with $\mu(\aleph) = 0$ such that,

$$\forall y \in \mathbb{R}^{n+m} \setminus \aleph, \quad \lim_{t \to \infty} |Y(t, y)|_{\mathbf{A}} = 0.$$ ◦

[12] (Angeli, 2004)

In other words, an invariant set $\mathbf{A}$ is said to be almost globally asymptotically stable if it is stable in the Lyapunov sense and, for almost every initial condition, solutions will converge to $\mathbf{A}$.

Theorem 3.13. [13] *Under Assumptions 3.5 and 3.6, assume that the constant values of Corollaries 3.9 and 3.11 are such that*

$$M_\ell < M_g \quad or \quad N_\ell < N_g.$$

Let

$$\mathbf{R} = \mathtt{cl}\left\{\Omega_{\leqslant \widetilde{M}_g}(U_g) \setminus \Omega_{\leqslant \widehat{M}_\ell}(U_\ell)\right\}.$$

If there exists $\rho \in \mathcal{C}^1(\mathbb{R}^{n+m} \setminus \{0\}, \mathbb{R}_{\geqslant 0})$ *with* $\mathtt{supp}(\rho) \supseteq \mathbf{R}$ *such that,*

$$\forall y \in \mathbf{R}, \quad \mathtt{div}(h\rho)(y) > 0.$$

Then, the origin is almost globally asymptotically stable for (3.8). □

[13] (Stein Shiromoto, Andrieu, and Prieur, 2013c)

From Liouville's theorem[14], the measure of the sets of solutions and their initial conditions are related to the divergence of vector field of (3.8). Under the hypothesis of Theorem 3.13 it is shown that the measure of the set of solutions that do not converge to $\Omega_{\leqslant \widehat{M}_\ell}(U_\ell)$ is zero. Also, the measure of the set of their initial conditions is zero. Thus, the conclusion of Theorem 3.13 follows. The proof of 3.13 is based on (Angeli, 2004; Rantzer, 2001), and is provided in Section 3.6.2.

[14] See Lemma 3.33.

Theorem 3.14 (Application of the Bendixson extended criterion). [15] *Let* $n = m = 1$. *Under Assumptions 3.5 and 3.6, assume that the constant values of Corollaries 3.9 and 3.11 are such that*

$$M_\ell < M_g \quad or \quad N_\ell < N_g.$$

Let

$$\mathbf{R} = \mathtt{cl}\left\{\Omega_{\leqslant \widetilde{M}_g}(U_g) \setminus \Omega_{\leqslant \widehat{M}_\ell}(U_\ell)\right\}.$$

If,

$$\forall y \in \mathbf{R}, \quad \mathtt{div}\, h(y) \neq 0 \quad and \quad h(y) \neq 0,$$

then the origin is globally asymptotically stable for (3.8). □

[15] (Stein Shiromoto, Andrieu, and Prieur, 2013c)

The absence of ω-limit sets in $\mathbf{R}$ is shown by exhibiting a contradiction between its existence and the assumption that, for every $y \in \mathbf{R}$, $\mathtt{div}\, h(y) \neq 0$ and $h(y) \neq 0$. The proof of Theorem 3.14 is provided in Section 3.6.3.

Figure 3.1 illustrates the region $\mathbf{R}$ obtained from the hypothesis of Corollaries 3.9 and 3.11, when $M_l < M_g$ and $N_l < N_g$.

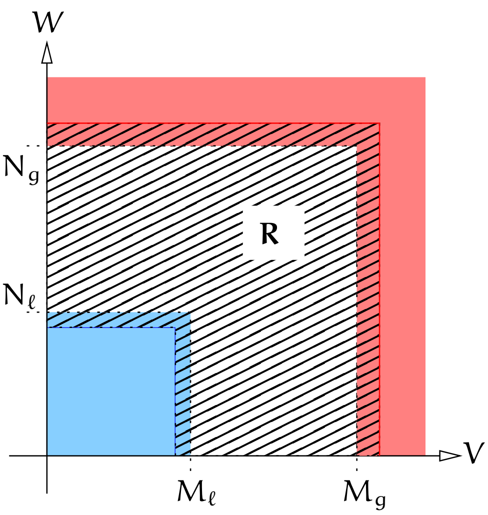


Figure 3.1: Illustration of sets $\Omega_{\leqslant M_\ell}(V) \times \Omega_{\leqslant N_\ell}(\mathcal{W})$ (blue region), $\Omega_{=\widetilde{M}_\ell}(U_\ell)$ (dark blue line), $\Omega_{\geqslant \underline{M}}(V) \times \Omega_{\geqslant \underline{N}}(\mathcal{W})$ (pink region), $\Omega_{=\widetilde{M}_g}(U_g)$ (red line), and $\mathbf{R} = \mathtt{cl}\{\Omega_{\leqslant \widehat{M}_g}(U_g) \setminus \Omega_{\leqslant \widetilde{M}_\ell}(U_\ell)\}$ (pattern filled).

3.5 Illustration

In this section, an example where the small-gain condition cannot be applied is given.[16] Corollaries 3.9 and 3.11, and Theorem 3.14 are illustrated.

[16] To conclude about the asymptotic stability of this example, one may infer from the LaSalle invariance principle together with the Lyapunov function $V + \mathcal{W}$. Other techniques also apply such as (Angeli and Astolfi, 2007).

Consider the system

$$\begin{cases} \dot{x} = f(x,z) = -\rho_x(x) + z, \\ \dot{z} = g(x,z) = -z + \rho_z(x), \end{cases} \tag{3.17}$$

where, for every $x \in \mathbb{R}$,

$$\rho_x(x) = \frac{x^3}{3} - 3\frac{x^2}{2} + 2x \quad \text{and} \quad \rho_z(x) = 0.8\rho_x(x),$$

and

$$\forall x \in \mathbb{R}, \qquad V(x) = |x|,$$
$$\forall z \in \mathbb{R}, \qquad \mathcal{W}(z) = |z|.$$

The Dini derivative[17] of V in the f-direction yields, for every $(x,z) \in \mathbb{R} \times \mathbb{R}$,

[17] see Definition A.39 below.

$$D_f^+ V(x,z) \leqslant -\rho_x(V(x)) + \mathcal{W}(z).$$

Letting, for every $x \in \mathbb{R}$, $\lambda_x(x) = \varepsilon_x \rho_x(V(x))$, where $\varepsilon_x \in (0,1)$. For every $(x,z) \in \mathbb{R} \times \mathbb{R}$,

Note that there, here it is given the system and the ISS-Lyapunov function and the problem is to find the gain γ and the decrease rate λ_x. In (Ito and Jiang, 2009, Lemma 1), the converse problem is considered: given a function $\lambda \in (\mathcal{P} \cap \mathcal{C}^0)(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$, and a gain γ find the ISS system and ISS-Lyapunov function corresponding to such a pair.

$$\rho_x(V(x)) \geqslant \frac{\mathcal{W}(z)}{1-\varepsilon_x} \Rightarrow D_f^+ V(x,z) \leqslant -\lambda_x(x). \tag{3.18}$$

Since ρ_x is strictly decreasing in $(1,2)$, it is not possible to use its inverse as a gain. From now on, fix $\varepsilon_x = 0.05$. Consider the piecewise continuous and positive definite function[18]

$$\Gamma(s) = \begin{cases} \rho_x^{-1}\left(\dfrac{s}{0.95}\right), & s \in [0, 0.6), \\ \\ \rho_{x,+}^{-1}\left(\dfrac{s}{0.95}\right), & s \in [0.6, \infty), \end{cases} \tag{3.19}$$

where the function $\rho_{x,+} : [2, \infty) \to [\rho_x(2), \infty)$ is given by $\rho_{x,+}(\cdot) = \rho_x(\cdot)$.

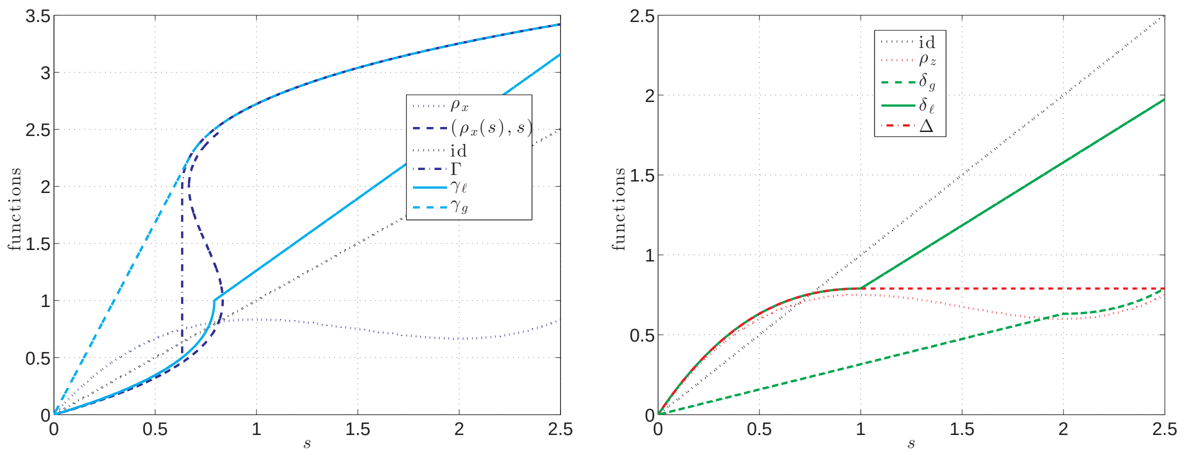
[18] Note that $0.95\rho_x(2) = 0.6$.

Claim 3.15. *The positive definite function Γ can be viewed as a non-smooth input-to-state gain of the x-subsystem of (3.17). More precisely, for every $(x, z) \in \mathbb{R} \times \mathbb{R}$,*

$$V(x) \geqslant \Gamma(W(z)) \Rightarrow D_f^+ V(x, z) \leqslant -\lambda_x(x). \tag{3.20}$$

Moreover, the function Γ is "tight". More precisely, if there exist a function $\Gamma^* : \mathbb{R} \to \mathbb{R}$, and a positive value s^* such that $\Gamma^*(s^*) < \Gamma(s^*)$, then there exists $(x^*, z^*) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ satisfying $V(x^*) \geqslant \Gamma^*(W(z^*))$ and $D_f^+ V(x^*, z^*) > 0$. □

The proof of Claim 3.15 is provided in Section 3.6.4. Note that any function $\gamma \in \mathcal{K}$ such that, for every $s \in \mathbb{R}_{\geqslant 0}$, $\Gamma(s) \leqslant \gamma(s)$ is a gain for the x-subsystem of (3.17).



a) b)

Figure 3.2: a) Graph of functions ρ_x (dotted blue), id (dotted black), Γ (dashdotted blue), γ_ℓ (solid cyan), γ_g (dashed cyan) and the parametrized curve $(\rho_x(s), s)$ (dashed blue), in the interval $[0, 2.5]$. b) Graph of functions ρ_z (dotted red), id (dotted black), Δ (dashedotted red), δ_ℓ (solid green) and δ_g (dashed green). For $\varepsilon_x = \varepsilon_z = 0.05$.

The above reasoning can be applied to the z-subsystem yielding, for all $(x, z) \in \mathbb{R} \times \mathbb{R}$,

$$W(z) \geqslant \Delta(V(x)) \Rightarrow D_g^+ W(x, z) \leqslant -\lambda_z(z), \tag{3.21}$$

where Δ is the tight input gain for the z-subsystem of (3.17) defined by

$$\Delta(s) = \begin{cases} \underline{\rho}(s), & s \in [0,1), \\ \underline{\rho}(1), & s \in [1,2.5), \\ \underline{\rho}(s), & s \in [2.5,\infty), \end{cases}$$

and, for every $s \in \mathbb{R}_{>0}$, $\underline{\rho}(s) = \rho_z(s)/0.95$. Note also that any function $\delta \in \mathcal{K}$ such that, for every $s \in \mathbb{R}_{\geqslant 0}$, $\Delta(s) \leqslant \delta(s)$ is a gain for the z-subsystem of (3.17).

The small gain condition (SGC) does not hold. More precisely,

Claim 3.16. [19] *For every* $s \in (0.65, 2.5)$, $s \leqslant \Gamma \circ \Delta(s)$. □

[19] The proof of Claim 3.16 is provided in Section 3.6.5.

Note also that the approach proposed in (Astolfi and Praly, 2012; Ito, 2006; Ito and Jiang, 2009) cannot be applied here, since they require a composition of gains to be smaller than its argument. However, the results proposed in (Angeli and Astolfi, 2007) can be applied in this example.

Illustration of Corollary 3.9. Consider a function $\gamma_\ell \in \mathcal{K}$ such that, for every $s \in [0, 0.5]$, $\gamma_\ell(s) = \overline{\rho}_x(s)$. Let $\delta_\ell \in \mathcal{K}$ be such that,

$$\delta_\ell(s) \begin{cases} = \Delta(s), & s \in [0,1), \\ \geqslant \Delta(s), & s \in [1,2.5]. \end{cases}$$

Assumption 3.5. Pick[20] $\overline{M} = M_\ell = \overline{N} = N_\ell = 0.3$, and $\underline{M} = \underline{N} = 0$. Note that, $\max\{\gamma_\ell^{-1}(\underline{M}), \underline{N}\} = 0$ and $\min\{\delta_\ell(M_\ell), N_\ell\} = 0.3$. Moreover, for every $(x,z) \in \mathbf{S}_\ell := (\Omega_{\leqslant M_\ell}(V) \times \Omega_{\leqslant N_\ell}(W))$,

[20] Note that $b = \infty$.

$$\begin{aligned} V(x) &\geqslant \gamma_\ell(W(z)) &\Rightarrow \quad D_f^+ V(x,z) &\leqslant -\lambda_x(x), \\ W(z) &\geqslant \delta_\ell(V(x)) &\Rightarrow \quad D_g^+ W(x,z) &\leqslant -\lambda_z(z). \end{aligned}$$

Assumption 3.6. From Remark 3.7, for every $s \in (0, \gamma_\ell^{-1}(M_\ell)] = (0, 0.7]$,

$$\delta_\ell \circ \gamma_\ell(s) = \frac{\rho_z \circ \rho_x^{-1}(s/0.95)}{0.95} = \frac{0.8}{0.95} \rho_x \circ \rho_x^{-1}\left(\frac{s}{0.95}\right) < s.$$

From Corollary 3.9, the set $\Omega_{\leqslant 0.3}(U_\ell)$ is included in the basin of attraction of the origin. Moreover,

$$U_\ell(x,z) = \max\left\{\frac{\delta_\ell(V(x)) + \gamma_\ell^{-1}(V(x))}{2}, W(z)\right\}.$$

Illustration of Corollary 3.11. Consider a function $\gamma_g \in \mathcal{K}$ such that, for every $s \in [0.7, \infty)$, $\gamma_g(s) = \Gamma(s)$. Let also $\delta_g \in \mathcal{K}$ be such that, for every $s \in [2, \infty)$, $\delta_g(s) = \underline{\rho}(s)$.

Assumption 3.5. Pick $\underline{M} = M_g = 4$ and $\underline{N} = N_g = 1$, and $\overline{M} = \overline{N} = \infty$. Note that, $\max\{\gamma_g^{-1}(\underline{M}), \underline{N}\} = 4$ and $\min\{\delta_g(\overline{M}), \overline{N}\} = \infty$. Moreover, for every $(x, z) \in \mathbf{S}_g := (\Omega_{\geqslant M_g}(V) \times \Omega_{\geqslant N_g}(W))$,

$$\begin{array}{rcccrcl} V(x) & \geqslant & \gamma_g(W(z)) & \Rightarrow & D_f^+ V(x,z) & \leqslant & -\lambda_x(x), \\ W(z) & \geqslant & \delta_g(V(x)) & \Rightarrow & D_g^+ W(x,z) & \leqslant & -\lambda_z(z). \end{array}$$

Assumption 3.6. From Remark 3.7, for every $s \in [\gamma_g^{-1}(M_g), \infty) = [4.5, \infty)$,

$$\delta_g \circ \gamma_g(s) = \frac{0.8}{0.95} \rho_x \circ \rho_x^{-1}\left(\frac{s}{0.95}\right) < s.$$

From Corollary 3.11 the set $\Omega_{\leqslant 4}(U_g)$ is globally attractive for (3.17). Moreover,

$$U_g(x,z) = \max\left\{\frac{\delta_g(V(x)) + \gamma_g^{-1}(V(x))}{2}, W(z)\right\}.$$

Illustration of Theorem 3.14. Note that, $M_\ell = 0.3 < 4 = M_g$ and $N_\ell = 0.3 < 1 = N_g$. Since system (3.17) is $\mathcal{C}^1$, and the only equilibrium point is the origin, and the equation

$$\frac{\partial f}{\partial x}(x,z) + \frac{\partial g}{\partial z}(x,z) = -x^2 + 3x - 3 = 0$$

has no zeros in

$$\mathbf{R} = \mathtt{cl}\{\Omega_{\leqslant 4}(U_g) \setminus \Omega_{\leqslant 0.3}(U_\ell)\},$$

from Theorem 3.14, the origin is globally asymptotically stable for (3.17).

Figure 3.3 shows a simulation of (3.17) for some initial conditions.

3.6 Proofs of Chapter 3

3.6.1 Proof of Proposition 3.8

Before proving Proposition 3.8, the following lemma is needed.

Lemma 3.17. [21] *Under Assumptions 3.5 and 3.6, there exists $\tilde{\gamma} \in \mathcal{K}_\infty$ such that,*

$$\forall s \in \mathbb{R}_{>0}, \quad \delta(s) < \tilde{\gamma}(s). \tag{3.22}$$

Moreover,

$$\begin{array}{llll} \textit{if} \;\; \overline{M} < \infty, & \textit{then} & \forall s \in [\underline{M}, \overline{M}] \setminus \{0\}, & \tilde{\gamma}(s) < \gamma^{-1}(s), \\ \textit{if} \;\; \overline{M} = \infty, & \textit{then} & \forall s \in [\underline{M}, b) \setminus \{0\}, & \tilde{\gamma}(s) < \gamma^{-1}(s). \end{array} \tag{3.23}$$

[21] Based on (Stein Shiromoto, Andrieu, and Prieur, 2013a).

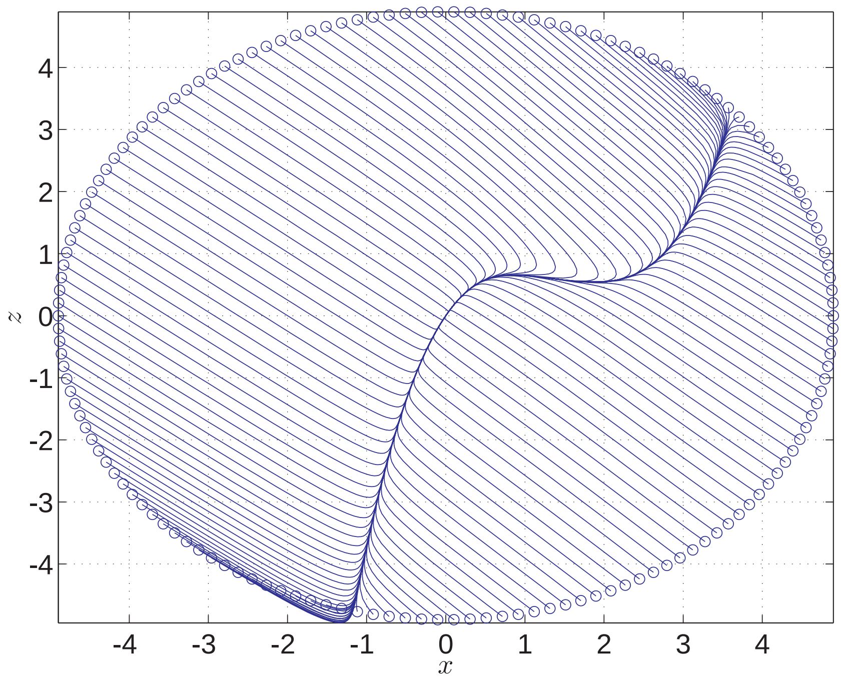


Figure 3.3: Simulation of (3.17).

□

The proof of Lemma 3.17 is based on (Jiang, Mareels, and Wang, 1996) and is provided after the proof of Proposition 3.8.

Proof (of Proposition 3.8). The proof of Proposition 3.8 is adapted from (Jiang, Mareels, and Wang, 1996, proof of Theorem 3.1). Here, it is divided into 2 parts. Firstly, it is shown that the Dini derivative of a proper function $U \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geqslant 0})$ is strictly negative in the set $\mathbf{S}$ defined in (3.11). In the second part, it is shown that solutions of (3.8) starting in $\Omega_{\leqslant \widehat{M}}(U) \setminus \Omega_{\leqslant \widetilde{M}}(U)$ converge to $\Omega_{\leqslant \widetilde{M}}(U)$.

First Part. Under Assumptions 3.5 and 3.6. Let $\widetilde{\gamma} \in \mathcal{K}_\infty$ be given by Lemma 3.17. Since δ is of class $\mathcal{K}$ and $\widetilde{\gamma}$ is of class $\mathcal{K}_\infty$ satisfying (3.23), from (Jiang, Mareels, and Wang, 1996, Lemma A.1), there exists a function $\sigma \in \mathcal{K}_\infty \cap \mathcal{C}^1$ whose derivative is strictly positive and satisfies,

$$\forall s \in \mathbb{R}_{>0}, \quad \delta(s) < \sigma(s) < \widetilde{\gamma}(s). \tag{3.24}$$

Let

$$\begin{aligned} U: \quad \mathbb{R}^n \times \mathbb{R}^m &\to \mathbb{R}_{\geqslant 0} \\ (x, z) &\mapsto \max\{\sigma(V(x)), W(z)\}. \end{aligned}$$

Note that $U \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geqslant 0})$ is a proper function. For any $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$, one of the following cases is possible:

Case 1. $\sigma(V(x)) < W(z)$;

Case 2. $W(z) < \sigma(V(x))$ or;

Case 3. $W(z) = \sigma(V(x))$.

The proof follows by showing that the Dini derivative of U is strictly negative. For each case, assume that

$$(x, z) \in \mathbf{S}_{\neq 0} := \mathbf{S} \setminus \{(0, 0)\},$$

where $\mathbf{S}$ is defined in (3.11).

Case 1. Assume that

$$\sigma(V(x)) < W(z).$$

This implies

$$U(x, z) = W(z) \quad \text{and} \quad D^+_{f,g} U(x, z) = D^+_g W(x, z).$$

From (3.24),

$$\delta(V(x)) < \sigma(V(x)) < W(z).$$

From (3.13), $D^+_g W(x, z) \leqslant -\lambda_z(z)$. Thus,

$$W(z) > \sigma(V(x)) \Rightarrow D^+_{f,g} U(x, z) \leqslant -\lambda_z(z).$$

Case 2. Assume that

$$W(z) < \sigma(V(x)).$$

This implies

$$U(x, z) = \sigma(V(x)) \quad \text{and} \quad D^+_{f,g} U(x, z) = \sigma'(V(x)) D^+_f V(x, z).$$

Since $(x, z) \in \mathbf{S}_{\neq 0}$, from (3.24),

$$W(z) < \sigma(V(x)) < \widetilde{\gamma}(V(x)).$$

If $\overline{M} < \infty$, then from (3.23),

$$W(z) < \sigma(V(x)) < \widetilde{\gamma}(V(x)) < \gamma^{-1}(V(x)). \tag{3.25}$$

Together with (3.12), $D^+_f V(x, z) \leqslant -\lambda_x(x)$.

If $\overline{M} = \infty$, then two regions of x must be analyzed: $b < V(x)$ and $\underline{M} \leqslant V(x) \leqslant b$.

Case 2.a. In the region where $b < V(x)$, Equations (3.12) and (3.9) yield

$$V(x) > b > \gamma(W(z)) \Rightarrow D_f^+ V(x,z) \leqslant -\lambda_x(x).$$

Case 2.b. In the region where $\underline{M} \leqslant V(x) \leqslant b$, from (3.23), (3.25), and (3.12),

$$V(x) > \gamma(W(z)) \Rightarrow D_f^+ V(x,z) \leqslant -\lambda_x(x).$$

Summing up Case 2,

$$W(z) < \sigma(V(x)) \Rightarrow D_{f,g}^+ U(x,z) \leqslant -\sigma'(V(x))\lambda_x(x).$$

Case 3. Assume that

$$W(z) = \sigma(V(x)).$$

Note that,

$$\begin{aligned} D_{f,g}^+ U(x,z) &= \limsup_{t \searrow 0} \frac{1}{t}\left[\max\left\{\sigma\big(V(X(t,x,z))\big), W(Z(t,z,x))\right\} - U(x,z)\right] \\ &= \limsup_{t \searrow 0} \max\left\{\frac{\sigma(V(X(t,x,z))) - \sigma(V(x))}{t}, \frac{W(Z(t,z,x)) - W(z)}{t}\right\} \\ &= \max\left\{\sigma'(V(x)) D_f^+ V(x,z), D_g^+ W(x,z)\right\}. \end{aligned}$$

The analysis of $D_{f,g}^+ U$ is divided in two sub cases. In the first one, the function $D_g^+ W$ is analyzed while in the last the function $D_f^+ V$ is analyzed.

Case 3.a. The analysis of $D_g^+ W$. From (3.24), and the fact that $x \neq 0$ and $z \neq 0$, the inequality $\delta(V(x)) < \sigma(V(x)) = W(z)$ holds. Analogously to case 1, $D_g^+ W(x,z) \leqslant -\lambda_z(z)$.

Case 3.b. The analysis of $D_f^+ V$. From (3.24), and the fact that $x \neq 0$ and $z \neq 0$, the inequality $W(z) = \sigma(V(x)) < \tilde{\gamma}(V(x))$ holds. Analogously to case 2, $D_f^+ V(x,z) \leqslant -\lambda_x(x)$.

Summing up Case 3,

$$0 \neq W(z) = \sigma(V(x)) \Rightarrow D_{f,g}^+ U(x,z) \leqslant -\min\{\sigma'(V(x))\lambda_x(x), \lambda_z(z)\}.$$

Claim 3.18. *There exists* $c > 0$ *such that* $\Omega_{\leqslant c}(U) \subset \Omega_{\leqslant \overline{M}}(V) \times \Omega_{\leqslant \overline{N}}(W)$. *Moreover, the constant values* $\widetilde{M}$ *and* $\widehat{M}$ *are such that*

$$\left(\Omega_{\leqslant \underline{M}}(V) \times \Omega_{\leqslant \underline{N}}(W)\right) \subset \Omega_{\leqslant \widetilde{M}}(U) \subset \Omega_{\leqslant \widehat{M}}(U) \subset \left(\Omega_{\leqslant \overline{M}}(V) \times \Omega_{\leqslant \overline{N}}(W)\right). \tag{3.26}$$

□

The proof of Claim 3.18 is provided after the proof of Proposition 3.8.

From the above case study and (3.26),

$$\widetilde{M} \leqslant U(x,z) \leqslant \widehat{M} \Rightarrow D^+_{f,g}U(x,z) \leqslant -E(x,z),$$

where

$$\begin{array}{rccl} E: & \mathbb{R}^n \times \mathbb{R}^m & \to & \mathbb{R}_{\geqslant 0} \\ & (x,z) & \mapsto & \min\{\sigma'(V(x))\lambda_x(x), \lambda_z(z)\}. \end{array}$$

Since $E \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}_{\geqslant 0})$, there exists[22] $\alpha \in \mathcal{K}_\infty$ such that, for every $(x,z) \in \mathbb{R}^{n+m}$, $\alpha(|(x,z)|) \leqslant E(x,z)$. Moreover, since $\Omega_{<\widetilde{M}}(U)$ contains the origin and

[22] (Sontag, 1989, pp. 13). See also the proof of Claim A.33.

$$|(x,z)|_{\widetilde{M}} := \mathtt{dist}\left((x,z), \Omega_{\leqslant \widetilde{M}}(U)\right) \leqslant |(x,z)|,$$

the inequality

$$\alpha\left(|(x,z)|_{\widetilde{M}}\right) \leqslant \alpha\left(|(x,z)|\right) \leqslant E(x,z)$$

holds. Thus,

$$\widetilde{M} \leqslant U(x,z) \leqslant \widehat{M} \Rightarrow D^+_{f,g}U(x,z) \leqslant -\alpha\left(|(x,z)|_{\widetilde{M}}\right). \tag{3.27}$$

Second part. The function U is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m \setminus \{(0,0)\}$. To see this claim, note that in the region $\sigma(V(x)) > W(z)$ (resp. $\sigma(V(x)) < W(z)$) $\sigma \circ V$ (resp. W) is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ (resp. $\mathbb{R}^m$). In the region $U^*(x,z) := \sigma(V(x)) = W(z) \neq 0$, given any (x_1,z_1) and $(x_2,z_2) \in \mathbb{R}^n \times \mathbb{R}^m \setminus \{(0,0)\}$,

$$\begin{aligned} |U^*(x_1,z_1) - U^*(x_2,z_2)| &= |\sigma(V(x_1)) - \sigma(V(x_2))| \\ &= |\sigma(V(x_1)) - \sigma(V(x_2)) + W(z_1) - W(z_2) \\ &\quad + W(z_2) - W(z_1)| \\ &\leqslant L_\sigma |V(x_1) - V(x_2)| + 2L_W|z_1 - z_2| \\ &\leqslant L_\sigma L_V |x_1 - x_2| + 2L_W|z_1 - z_2|, \end{aligned}$$

where L_V, L_σ, and L_W are, respectively, the Lipschitz constant of functions V, σ, and W. Let $L = \max\{L_V, L_\sigma, 2L_W\}$

$$|U^*(x_1,z_1) - U^*(x_2,z_2)| \leqslant L\left(|x_1 - x_2| + |z_1 - z_2|\right).$$

From Theorem A.42, for every $y \in \mathbb{R}^{n+m}$, and for every $t \in \mathbb{R}_{\geqslant 0}$, along solutions of (3.8),

$$D^+U(Y(t,y)) = D^+_h U(Y(t,y)).$$

From Remark A.43, for every y such that $\widetilde{M} \leqslant U(y) \leqslant \widehat{M}$, and for every $t \in \mathbb{R}_{\geqslant 0}$, the function

$$t \mapsto U(Y(t,y))$$

is strictly decreasing. Moreover, it will be shown that

$$U^{\infty} := \lim_{t \to \infty} U(Y(t,y)) \leqslant \widetilde{M}.$$

To see this fact suppose, by absurd, that $U^{\infty} > \widetilde{M}$. From the continuity of U, there exists $\varepsilon > 0$ such that

$$U^{\infty} - \varepsilon > \widetilde{M} \quad \text{and} \quad U^{\infty} - \varepsilon \leqslant U(y) \leqslant U^{\infty} + \varepsilon.$$

Since U is proper, the set $\{y \in \mathbb{R}^{n+m} : U^{\infty} - \varepsilon \leqslant U(y) \leqslant U^{\infty} + \varepsilon\}$ is compact. Thus, the constant

$$\xi = \min\left\{\alpha\left(|y|_{\widetilde{M}}\right) > 0 : U^{\infty} - \varepsilon \leqslant U(y) \leqslant U^{\infty} + \varepsilon\right\}$$

exists.

Since U is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m \backslash \{(0,0)\}$ and solutions of (3.8) are functions of class $\mathcal{C}^1$, the function $t \mapsto U(Y(t,y))$ is also locally Lipschitz on $\mathbb{R}_{\neq 0}$. Thus, for almost every $t \in \mathbb{R}_{\geqslant 0}$, $U(t)$ is differentiable[23] and $D^+U(t)$ is integrable[24]. From the definition of the constant ξ, for every $t \in \mathbb{R}_{\geqslant 0}$,

[23] From Rademacher's theorem (see Evans, 1998, Theorem 6).

[24] Since U is locally Lipschitz, it is absolutely continuous (Hunter, 2013, Example 3.57). Thus, D^+U is 1-integrable. See Definition A.15.

$$\begin{aligned} U(Y(t,y)) &= U(y) + \int_0^t D^+U(Y(s,y))\,ds \\ &\leqslant U(y) - \xi t. \end{aligned}$$

As t increases to infinity, the value $U(Y(t,y))$ will eventually be negative contradicting the positive definiteness of U. Thus, $U^{\infty} \leqslant \widetilde{M}$. Hence, solutions of (3.8) starting in $\Omega_{\leqslant \widehat{M}}(U) \setminus \Omega_{\leqslant \widetilde{M}}(U)$ converge to $\Omega_{\leqslant \widetilde{M}}(U)$.

To see that U can be given as in (3.16), note that U relies on the computation of σ. Let for every $s \in \mathbb{R}_{\geqslant 0}$, $\sigma(s) = (\delta(s) + \gamma^{-1}(s))/2$, this implies, for every $s \in \mathbb{R}_{>0}$,

$$2\frac{d\sigma}{ds}(s) = \frac{d\delta}{ds}(s) + \frac{1}{\frac{d\gamma}{ds}(\gamma^{-1}(s))}$$

which is positive, because[25] $d\delta(s)/ds > 0$ and[26] $d\gamma(\gamma^{-1}(s))/ds > 0$. Moreover, the function σ satisfies (3.24). This concludes the proof of

[25] due to the fact that $\delta \in \mathcal{K}_{\infty} \cap \mathcal{C}^1$.

[26] due to the fact that $\gamma^{-1}, \gamma \in \mathcal{K}_{\infty} \cap \mathcal{C}^1$.

Proposition 3.8. ■

Proof (of Claim 3.18). Let c be a positive real number[27] such that $\Omega_{\leqslant c}(U) \subset \Omega_{\leqslant \overline{M}}(V) \times \Omega_{\leqslant \overline{N}}(W)$.

In the first part, it will be shown that, for every $(x, z) \in \mathbf{S}$,

$$U(x, z) \leqslant \widehat{M} \Rightarrow \max\{V(x), W(z)\} \leqslant \min\{\overline{M}, \overline{N}\}. \quad (3.28)$$

In the second part, it will be shown that, for every $(x, z) \in \mathbf{S}$,

$$\widetilde{M} \leqslant U(x, z) \Rightarrow \max\{\underline{M}, \underline{N}\} \leqslant \min\{V(x), W(z)\}. \quad (3.29)$$

Part 1. Inequality $U(x, z) \leqslant \widehat{M}$ implies

$$\max\{\sigma(V(x)), W(z)\} = U(x, z) \leqslant \widehat{M} = \min\left\{\delta(\overline{M}), \overline{N}\right\}.$$

Assume that $\max\{\sigma(V(x)), W(z)\} = \sigma(V(x))$.

Part 1.a. Suppose that $\min\{\delta(\overline{M}), \overline{N}\} = \delta(\overline{M})$. Since $\sigma(V(x)) \leqslant \delta(\overline{M})$, from (3.24), $V(x) \leqslant \sigma^{-1} \circ \delta(\overline{M}) < \overline{M}$;

Part 1.b. Suppose that $\min\{\delta(\overline{M}), \overline{N}\} = \overline{N}$. Since $\sigma(V(x)) \leqslant \overline{N} \leqslant \delta(\overline{M})$. From part 1.a, $V(x) \leqslant \sigma^{-1} \circ \delta(\overline{M}) < \overline{M}$.

Assume now that $\max\{\sigma(V(x)), W(z)\} = W(z)$.

Part 1.c. Suppose that $\min\{\delta(\overline{M}), \overline{N}\} = \delta(\overline{M})$. This implies $W(z) \leqslant \delta(\overline{M}) \leqslant \overline{N}$;

Part 1.d. Suppose that $\min\{\delta(\overline{M}), \overline{N}\} = \overline{N}$. This implies $W(z) \leqslant \overline{N}$.

Thus, (3.28) holds and

$$\Omega_{\leqslant \widehat{M}}(U) \subset \left(\Omega_{\leqslant \overline{M}}(V) \times \Omega_{\leqslant \overline{N}}(W)\right);$$

Part 2. Inequality $\widetilde{M} \leqslant U(x, z)$ implies

$$\max\{\gamma^{-1}(\underline{M}), \underline{N}\} = \widetilde{M} \leqslant U(x, z) = \max\{\sigma(V(x)), W(z)\}.$$

Assume that $\max\{\sigma(V(x)), W(z)\} = \sigma(V(x))$.

Part 2.a. Suppose that $\max\{\gamma^{-1}(\underline{M}), \underline{N}\} = \gamma^{-1}(\underline{M})$. Since $\gamma^{-1}(\underline{M}) \leqslant \sigma(V(x))$, from (3.23) and (3.24), $\underline{M} \leqslant \gamma \circ \sigma(V(x)) < V(x)$;

Part 2.b. Suppose that $\max\{\gamma^{-1}(\underline{M}), \underline{N}\} = \underline{N}$. Since $\gamma^{-1}(\underline{M}) \leqslant \underline{N} \leqslant \sigma(V(x))$, from item 2.a., $\underline{M} \leqslant \gamma \circ \sigma(V(x)) < V(x)$;

[27] Such a positive real number always exist. Otherwise, for every $n \geqslant \mathbb{N}$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^{n+m}$ such that, for every $n \in \mathbb{N}$, $y_n \in \Omega_{\leqslant 1/n}(U)$ and $y_n \notin \Omega_{\leqslant \overline{M}}(V) \times \Omega_{\leqslant \overline{N}}(W)$. Since U is proper, $\Omega_{\leqslant 1/n}(U) \subset \Omega_{\leqslant 1}(U)$ is compact. Hence, there exists $\{y_{n_j}\}_{j\in\mathbb{N}} \subset \{y_n\}_{n\in\mathbb{N}}$ such that $y_{n_j} \xrightarrow{j\to\infty} y^*$ and $U(y^*) = 0$. From the positive definiteness of U, $y^* = 0$. Consequently, y_{n_j} is a sequence converging to zero and outside $\Omega_{\leqslant \overline{M}}(V) \times \Omega_{\leqslant \overline{N}}(W)$. This is impossible since this set is a neighborhood of the origin.

Thus, $\max\{\underline{M}, \underline{N}\} \leqslant V(x)$.

Assume now that $\max\{\sigma(V(x)), W(z)\} = W(z)$.

Part 2.c. Suppose that $\max\{\gamma^{-1}(\underline{M}), \underline{N}\} = \gamma^{-1}(\underline{M})$. Since $\gamma^{-1}(\underline{M}) \leqslant W(z)$, $\underline{N} \leqslant \gamma^{-1}(\underline{M}) \leqslant W(z)$;

Part 2.d. Suppose that $\max\{\gamma^{-1}(\underline{M}), \underline{N}\} = \underline{N}$. It yields $\underline{N} \leqslant W(z)$.

Thus, $\max\{\underline{M}, \underline{N}\} \leqslant W(z)$.

From parts 1 and 2, $\max\{\underline{M}, \underline{N}\} \leqslant V(x)$ and $\max\{\underline{M}, \underline{N}\} \leqslant W(z)$. Hence,

$$\max\{\underline{M}, \underline{N}\} \leqslant \min\{V(x), W(z)\}.$$

Therefore, (3.29) holds and

$$\left(\Omega_{\leqslant \underline{M}}(V) \times \Omega_{\leqslant \underline{N}}(W)\right) \subset \Omega_{\leqslant \widetilde{M}}(U).$$

Since (3.10) is a strict inequality, from the continuity and surjectivity of U, there exists $(x, z) \in S$ such that $\widetilde{M} \leqslant U(x, z) \leqslant \widehat{M}$. From (3.28) and (3.29),

$$\widetilde{M} \leqslant U(x, z) \leqslant \widehat{M} \Rightarrow \max\{\underline{M}, \underline{N}\} \leqslant \min\{V(x), W(z)\} \leqslant \max\{V(x), W(z)\} \leqslant \min\{\overline{M}, \overline{N}\},$$

and the inclusion (3.26) holds. This concludes the proof of Claim 3.18. ■

Proof (of Lemma 3.17). Under Assumptions 3.5 and 3.6, there exist constant values $\varepsilon > 0$ and $M > 0$ such that defining, if $\overline{M} < \infty$,

$$\tilde{\gamma}(s) = \begin{cases} \delta(s) + \min\{s, M\}, & s \in [0, \underline{M}), \\ \delta(s) + \min\left\{s, \dfrac{\gamma^{-1}(s) - \delta(s)}{2}\right\}, & s \in [\underline{M}, \overline{M}], \\ A(s) + (B(s) - A(s))\, \dfrac{s - \overline{M}}{\varepsilon}, & s \in (\overline{M}, \overline{M} + \varepsilon), \\ \delta(s) + s, & s \in [\overline{M} + \varepsilon, \infty), \end{cases} \tag{3.30}$$

where

$$A(s) = \delta(s) + \min\left\{\overline{M}, \frac{\gamma^{-1}(\overline{M}) - \delta(\overline{M})}{2}\right\},$$

$$B(s) = \delta(s) + \overline{M} + \varepsilon$$

and, if $\overline{M} = \infty$,

$$\tilde{\gamma}(s) = \begin{cases} \delta(s) + \min\{s, M\}, & s \in [0, \underline{M}), \\ \delta(s) + \min\left\{s, \dfrac{\gamma^{-1}(s) - \delta(s)}{2}\right\}, & s \in [\underline{M}, b), \\ \delta(s) + s, & s \in [b, \infty). \end{cases} \tag{3.31}$$

the function $\tilde{\gamma}$ is of class $\mathcal{K}_\infty$ and satisfies (3.22) and (3.23).

The proof of this claim consists of three parts. Firstly, it is obtained the constant values M and ε. In the second part, it is shown that $\tilde{\gamma}$ is of class $\mathcal{K}_\infty$. In the last part, it is shown that $\tilde{\gamma}$ satisfies (3.22) and (3.23).

First part. Consider the functions $\delta, \gamma \in \mathcal{K}$ given by Assumption 3.5. The function γ^{-1} is defined on $[0, b)$ and satisfies

$$\lim_{s \nearrow b} \gamma^{-1}(s) = \infty.$$

Since $\underline{M} < b$, Equation (3.9) and Assumption 3.6 imply,

$$\begin{array}{llll} \text{if} & \overline{M} < \infty, & \forall s \in [\underline{M}, \overline{M}] \setminus \{0\}, & \delta(s) < \gamma^{-1}(s), \\ \text{if} & \overline{M} = \infty, & \forall s \in [\underline{M}, \overline{M}) \setminus \{0\}, & \delta(s) < \gamma^{-1}(s). \end{array} \tag{3.32}$$

Let,

$$M = \frac{\gamma^{-1}(\underline{M}) - \delta(\underline{M})}{2} < \infty.$$

If $\overline{M} < \infty$, from the continuity of γ and δ and from (3.32), there exists a constant value $\varepsilon > 0$ such that $\overline{M} + \varepsilon < b$ and, for every $s \in [\underline{M}, \overline{M} + \varepsilon)$, $\delta(s) < \gamma^{-1}(s)$.

Second part. To see that $\tilde{\gamma}$ is continuous, the limits around each interval is analyzed. Afterwards, the increasing behavior is concluded.

Around $\underline{M}$, it yields

$$\begin{aligned} \lim_{s \nearrow \underline{M}} \tilde{\gamma}(s) &= \lim_{s \nearrow \underline{M}} [\delta(s) + \min\{s, M\}] \\ &= \delta(\underline{M}) + \min\{\underline{M}, M\} \\ &= \lim_{s \searrow \underline{M}} \left[\delta(s) + \min\left\{s, \frac{\gamma^{-1}(s) - \delta(s)}{2}\right\}\right] \\ &= \lim_{s \searrow \underline{M}} \tilde{\gamma}(s). \end{aligned}$$

Around $\overline{M} < \infty$, it yields

$$\begin{aligned}
\lim_{s \nearrow \overline{M}} \tilde{\gamma}(s) &= \lim_{s \nearrow \overline{M}} \left[\delta(s) + \min\left\{ s, \frac{\gamma^{-1}(s) - \delta(s)}{2} \right\} \right] \\
&= A(\overline{M}) \\
&= \lim_{s \searrow \overline{M}} \left[A(s) + (B(s) - A(s)) \frac{s - \overline{M}}{\varepsilon} \right] \\
&= \lim_{s \searrow \overline{M}} \tilde{\gamma}(s).
\end{aligned}$$

Around $\overline{M} + \varepsilon < \infty$, it yields

$$\begin{aligned}
\lim_{s \nearrow \overline{M}+\varepsilon} \tilde{\gamma}(s) &= \lim_{s \nearrow \overline{M}+\varepsilon} \left[A(s) + (B(s) - A(s)) \frac{s - \overline{M}}{\varepsilon} \right] \\
&= B(\overline{M} + \varepsilon) \\
&= \lim_{s \searrow \overline{M}+\varepsilon} [\delta(s) + s] \\
&= \lim_{s \searrow \overline{M}+\varepsilon} \tilde{\gamma}(s).
\end{aligned}$$

When $\overline{M} = \infty$, around b,

$$\begin{aligned}
\lim_{s \nearrow b} \tilde{\gamma}(s) &= \lim_{s \nearrow \overline{M}} \left[\delta(s) + \min\left\{ s, \frac{\gamma^{-1}(s) - \delta(s)}{2} \right\} \right] \\
&= \delta(b) + b \\
&= \lim_{s \searrow b} \delta(s) + s \\
&= \lim_{s \searrow b} \tilde{\gamma}(s).
\end{aligned}$$

Thus, $\tilde{\gamma}$ is continuous.

Third part. To see that inequality (3.22) is satisfied, note that, for every $s \in \mathbb{R}_{\geqslant 0}$, $\tilde{\gamma}$ is defined as the sum of $\delta(s)$ with a positive function or value. Thus, for every $s \in \mathbb{R}_{>0}$, $\delta(s) < \tilde{\gamma}(s)$. Moreover, $\tilde{\gamma} \in \mathcal{K}_\infty$.

To see that inequality (3.23) is satisfied, assume that $\overline{M} < \infty$ (resp. $\overline{M} = \infty$) and $s \in [\underline{M}, \overline{M}] \setminus \{0\}$ (resp. $s \in [\underline{M}, b) \setminus \{0\}$). Two cases may occur, according to the value of

$$\min\left\{ s, \frac{\gamma^{-1}(s) - \delta(s)}{2} \right\}.$$

From (3.30) (resp. (3.31)),

Case 1. $s < \frac{\gamma_\ell^{-1}(s) - \delta_\ell(s)}{2}$ implies

$$\tilde{\gamma}(s) = \delta(s) + s < \delta(s) + \frac{\gamma^{-1}(s) - \delta(s)}{2} = \frac{\gamma^{-1}(s) + \delta(s)}{2};$$

Case 2. $s \geqslant \frac{\gamma^{-1}(s) - \delta(s)}{2}$ implies

$$\tilde{\gamma}(s) = \frac{\gamma^{-1}(s) + \delta(s)}{2}.$$

From (3.32), cases 1 and 2 yield, for every $s \in [\underline{M}, \overline{M}] \setminus \{0\}$ (resp. $s \in [\underline{M}, b) \setminus \{0\}$), $\tilde{\gamma}(s) < \gamma^{-1}(s)$. Thus, (3.23) holds. This concludes the proof of Lemma 3.17. ■

3.6.2 Proof of Theorem 3.13

Before proving Theorem 3.13 the following lemma, based on (Angeli, 2004; Rantzer, 2001), is needed.

Lemma 3.19. [28] *Under the hypotheses of Theorem 3.13, if there exists* $\rho \in \mathcal{C}^1(\mathbb{R}^{n+m}, \mathbb{R}_{\geqslant 0})$ *with* $\mathtt{supp}(\rho) \supseteq \mathbf{R}$ *such that,*

$$\forall y \in \mathbf{R}, \quad \mathtt{div}(h\rho)(y) > 0, \tag{3.33}$$

then for almost every initial condition in $\mathbf{R}$*, the solutions of* (3.8) *converge to* $\Omega_{\leqslant \hat{M}_\ell}(V)$. □

[28] (Stein Shiromoto, Andrieu, and Prieur, 2013c)

The proof of Lemma 3.19 is provided after the proof of Theorem 3.13.

Proof (of Theorem 3.13). This proof is divided into 4 parts. Firstly, it is shown that every solution starting in $\Omega_{>\widetilde{M}_g}(U_g)$ converges to $\Omega_{\leqslant\widetilde{M}_g}(U_g)$. The second part shows that every solution starting in $\Omega_{\leqslant\widehat{M}_\ell}(U_\ell)$ converges to the origin. In the third part, it is shown that almost every solution starting in $\mathbf{R}$ converges to $\Omega_{\leqslant\widehat{M}_\ell}(U_\ell)$. The fourth part concludes the almost global asymptotic stability of the origin.

First part. From Corollary 3.11, the set $\Omega_{\leqslant\widetilde{M}_g}(U_g)$ is globally attractive for (3.8), where $\widetilde{M}_g = \max\{\gamma_g^{-1}(M_g), N_g\}$, M_g and N_g are defined in Corollary 3.11, and γ_g is given by Assumption 3.5.

Second part. From Corollary 3.9, the set $\Omega_{\leqslant\widehat{M}_\ell}(U_\ell)$ is contained in the basin of attraction of the origin, where $\widehat{M}_\ell = \min\{\delta_\ell(M_\ell), N_\ell\}$, M_ℓ and N_ℓ are defined in Corollary 3.9, and γ_ℓ is given by Assumption 3.5.

Third part. It remains to show that $\Omega_{\leqslant\widehat{M}_\ell}(U_\ell) \subsetneq \Omega_{\leqslant\widetilde{M}_g}(U_g)$. From the proof of Claim 3.18,

$$\begin{aligned} U_\ell(x,z) &\leqslant \widehat{M}_\ell &\Rightarrow&\quad \max\{V(x), W(z)\} &\leqslant&\quad \min\{M_\ell, N_\ell\}, \\ U_g(x,z) &\geqslant \widetilde{M}_g &\Rightarrow&\quad \min\{V(x), W(z)\} &\geqslant&\quad \max\{M_g, N_g\}. \end{aligned}$$

Since $\min\{M_\ell, N_\ell\} < \max\{M_g, N_g\}$, $\Omega_{\leqslant \widehat{M}_\ell}(U_\ell) \subsetneq \Omega_{\leqslant \widetilde{M}_g}(U_g)$ and

$$\mathbf{R} = \mathtt{cl}\left\{\Omega_{\leqslant \widetilde{M}_g}(U_g) \setminus \Omega_{\leqslant \widehat{M}_\ell}(U_\ell)\right\} \neq \emptyset.$$

Because there exists $\rho \in \mathcal{C}^1(\mathbb{R}^{n+m}, \mathbb{R}_{\geqslant 0})$ with $\mathtt{supp}(\rho) \supseteq \mathbf{R}$ and such that,

$$\forall y \in \mathbf{R}, \quad \mathtt{div}(h\rho)(y) > 0,$$

by employing Lemma 3.19, the set $\mathbf{Z}$ of initial conditions in $\mathbf{R}$ from which solutions do not converge to $\Omega_{\leqslant \widehat{M}_\ell}(U_\ell)$ and the set of solutions have Lebesgue measure zero.

Fourth part. It is shown that initial conditions from which issuing solutions converge to the set $\mathbf{Z} \subset \mathbf{R}$ have also measure zero. For every t, let

$$Y(t, \mathbf{Z}) = \{Y(t, z) : t \in \mathtt{dom}(z), z \in \mathbf{Z}\},$$

where $\mathtt{dom}(z)$ is the maximum time interval where $Y(t, z)$ exists. Since $\mathbf{Z}$ is positively invariant, for every $t_1, t_2 \in \mathtt{dom}(z)$,

$$t_1 < t_2 \leqslant 0 \Rightarrow Y(t_2, \mathbf{Z}) \subset Y(t_1, \mathbf{Z}).$$

This inclusion implies that[29]

$$\mathbf{Y} := \bigcup_{t \leqslant 0} \{Y(t, \mathbf{Z})\} = \bigcup_{l \in \mathbb{Z}_{<0}} \{Y(t, \mathbf{Z}) : t \leqslant l\}.$$

Hence, the set $\mathbf{Y}$ is a countable union of images of $\mathbf{Z}$ by the flow. Since $\mathbf{Z}$ is measurable[30] and, for every $t \in \mathtt{dom}(y)$, the map $\mathbf{Z} \ni y \mapsto Y(t, y)$ is a diffeomorphism[31], $\mathbf{Y}$ is also measurable.

Note that,

$$\forall t \in \mathtt{dom}(\mathbf{Z}), \qquad \int_{Y(t,\mathbf{Z})} dz \leqslant \int_{\mathbf{Z}} |\mathtt{grad}\, Y(t, y)|\, dy = 0,$$

because $\mathbf{Z}$ has measure zero. This implies that, for every $t \in \mathtt{dom}(\mathbf{Z})$, $Y(t, \mathbf{Z})$ have Lebesgue measure zero. Since $\mathbf{Y}$ is the countable union of sets of measure zero, it has also measure zero. Recall that $\mathbf{Y}$ is the set of initial conditions for which solutions of (3.8) do not converge to the origin.

From the above discussion, the origin is locally stable and almost globally attractive for (3.8). Thus, it is almost globally asymptotically stable for (3.8). This concludes the proof of Theorem 3.13. ■

Proof (of Lemma 3.19). It will be shown that almost all solutions of

[29] Note also that, when $Y(t, \mathbf{Z})$ does not exist, for $t \leqslant 0$, then $Y(t, \mathbf{Z}) = \emptyset$.

[30] See the proof of Lemma 3.19.

[31] Because (3.8) is of class $\mathcal{C}^1$ and solutions are unique (see also Hartman, 1982, Corollary 3.1).

(3.8) starting in $\mathbf{R}$ converge to $\Omega_{\leqslant \widehat{M}_\ell}(U_\ell)$. To do so, it is followed the same line as in (Rantzer, 2001, Theorem 1) and (Angeli, 2004, Theorem 3). However, here a less conservative condition is required, since it is considered a set that is only positively invariant and it is needed for the divergence to be positive only in a compact set.

Let $\mathbf{Z} \subset \mathbb{R}^{n+m}$ be given by

$$\mathbf{Z} = \bigcap_{l=1}^{\infty} \left\{ y \in \Omega_{\leqslant \widetilde{M}_g}(U_g) : U_\ell(Y(t,y)) > \widehat{M}_\ell, t > l \right\},$$

Since $\mathbf{Z}$ is a countable intersection of open sets, it is measurable.[32]

[32] See also Propositions A.8 and A.9.

Note that $\mathbf{Z}$ is the set of all initial conditions belonging to $\Omega_{\leqslant \widetilde{M}_g}(U_g)$ from which issuing solutions do not converge to $\Omega_{\leqslant \widehat{M}_\ell}(U_\ell)$. Since $\Omega_{\leqslant \widetilde{M}_g}(U_g)$ is positively invariant[33], the set $\mathbf{Z}$ is also positively invariant. Thus, given a fixed $\tau \in \mathbb{R}_{>0}$,

[33] Cf. Corollary 3.11.

$$\forall t \geqslant \tau, \quad Y(t, \mathbf{Z}) \subset Y(\tau, \mathbf{Z}).$$

Hence, for every $t \in \mathbb{R}_{\geqslant 0}$,

$$\int_{Y(t,\mathbf{Z})} \rho(y)\, dy - \int_{\mathbf{Z}} \rho(y)\, dY \leqslant 0. \tag{3.34}$$

From Lemma 3.33, for every $t \in \mathbb{R}_{\geqslant 0}$,

$$\int_0^t \int_{Y(s,\mathbf{Z})} \mathtt{div}(h\rho)(y)\, dy ds = \int_{Y(t,\mathbf{Z})} \rho(y)\, dy - \int_{\mathbf{Z}} \rho(y)\, dy.$$

Since, for every $y \in \mathbf{R}$, $\mathtt{div}(h\rho)(y) > 0$, and $\mathbf{Z} \subset \mathbf{R}$, for every $t \in \mathbb{R}_{\geqslant 0}$,

$$\begin{aligned} t \int_{Y(t,\mathbf{Z})} \mathtt{div}(h\rho)(y)\, dy &\leqslant \int_0^t \int_{Y(s,\mathbf{Z})} \mathtt{div}(h\rho)(y)\, dy ds \\ &\leqslant \int_{Y(t,\mathbf{Z})} \rho(y)\, dy - \int_{\mathbf{Z}} \rho(y)\, dy. \end{aligned}$$

From (3.34),

$$\forall t \in \mathbb{R}_{\geqslant 0}, \quad \int_{Y(t,\mathbf{Z})} \mathtt{div}(h\rho)(y)\, dy \leqslant 0.$$

Together with (3.33),

$$\forall t \in \mathbb{R}_{\geqslant 0}, \quad \int_{Y(t,\mathbf{Z})} \mathtt{div}(h\rho)(y)\, dy = 0.$$

Thus, for every $t \in \mathbb{R}_{\geqslant 0}$, $Y(t, \mathbf{Z})$ has Lebesgue measure zero. From the continuity of Y, $\mathbf{Z}$ has also Lebesgue measure zero. Consequently,

$$\text{for a.e. } y \in \mathbf{R}, \quad \limsup_{t\to\infty} U_\ell(Y(t,y)) \leqslant \widehat{M}_\ell.$$

This concludes the proof of Lemma 3.19. ■

3.6.3 Proof of Theorem 3.14

Lemma 3.20 (Extended Bendixson criterion). [34] *Let $n = m = 1$, under the hypotheses of Theorem 3.14 if,*

$$\forall y \in \mathbf{R}, \quad \mathtt{div}\, h(y) \neq 0, \; h(y) \neq 0, \tag{3.35}$$

then all solutions of (3.8) issuing from $\mathbf{R}$ converge to $\Omega_{\leqslant \widehat{M}_\ell}(U_\ell)$. □

[34] (Stein Shiromoto, Andrieu, and Prieur, 2013c)

The proof of Lemma 3.20 is provided after the proof of Theorem 3.14.

Proof (of Theorem 3.14). The proof of Theorem 3.14 follows the same line as the proof of Theorem 3.13. The difference consists in the third and fourth parts. Here, it is assumed that, for every $y \in \mathbf{R}$, $\mathtt{div}\, h(y) \neq 0$ and $h(y) \neq 0$, the existence of the function ρ is not needed. In the fourth part, no consideration concerning the measure of the sets of initial conditions is needed.

Similarly to the discussion of the proof of Theorem 3.13, the origin is locally stable and globally attractive for (3.8). Thus, it is globally asymptotically stable for (3.8). This concludes the proof of Theorem 3.14. ■

Before proving Lemma 3.20, some concepts regarding the asymptotic behavior of solutions are recalled. For planar systems, a closed curve $C \subset \mathbb{R}^2$ is called *closed orbit* if C is not an equilibrium point and there exists a time $T < \infty$ such that, for each $(x,z) \in C$, $(X(nT,x,z), Z(nT,x,z)) = (x,z)$, $\forall n \in \mathbb{Z}$. [35]

[35] cf. (Sastry, 1999, Definition 2.6).

Proof (of Lemma 3.20). Consider the proper function $U_\ell \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geqslant 0})$ (resp. $U_g \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geqslant 0})$) and the constant values given by Corollary 3.9 (resp. 3.11). From the proof of Claim 3.18,

$$\begin{aligned} U_\ell(x,z) &\leqslant \widehat{M}_\ell &\Rightarrow \max\{V(x), W(z)\} &\leqslant \min\{M_\ell, N_\ell\}, \\ U_g(x,z) &\geqslant \widetilde{M}_g &\Rightarrow \min\{V(x), W(z)\} &\geqslant \max\{M_g, N_g\}. \end{aligned}$$

Since $\min\{M_\ell, N_\ell\} < \max\{M_g, N_g\}$, $\Omega_{\leqslant \widehat{M}_\ell}(U_\ell) \subsetneq \Omega_{\leqslant \widetilde{M}_g}(U_g)$ and

$$\mathbf{R} = \mathtt{cl}\left\{\Omega_{\leqslant \widetilde{M}_g}(U_g) \setminus \Omega_{\leqslant \widehat{M}_\ell}(U_\ell)\right\} \neq \emptyset.$$

From Lemma 3.22, there exists a proper function $U_\infty \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geqslant 0})$ that is locally Lipschitz on $\mathbb{R}^{n+m} \setminus \{0\}$ (resp. a function $\tilde{h} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$) with $\mathrm{supp}(U_\infty)$ (resp. $\mathrm{supp}(\tilde{h})$) compact satisfying $\mathrm{supp}(U_\infty) \supset \mathbf{R}$ (resp. $\mathrm{supp}(\tilde{h}) \supset \mathbf{R}$). Moreover, for every $y \in \mathbf{R}$, $U_\infty(y) = U_g(y)$ (resp. $\tilde{h}(y) = h(y)$).

From Theorem 3.29,

- The set $\Omega_{=M_g}(U_\infty)$ has finite perimeter;
- The function U_∞ is almost everywhere[36] differentiable in $\Omega_{=M_g}(U_\infty)$;
- Let $\mathbf{S}_\infty \subset \Omega_{=M_g}(U_\infty)$ be set of points where U_g is not differentiable. There exists a Lipschitz parametrization $p_\infty : [a_\infty, b_\infty] \subset \mathbb{R} \to \Omega_{=M_g}(U_\infty)$ that is injective and satisfies, for almost[37] every $s \in [a_\infty, b_\infty]$, $p_\infty(s) \notin \mathbf{S}_\infty$ and ${}^{dp_\infty(s)}/_{ds}$ is perpendicular to $\nabla U_\infty(p_\infty(s))$.

[36] In the Hausdorff measure sense.

[37] In the Lebesgue measure sense.

From Theorem 3.31,

$$\iint\limits_{\Omega_{\leqslant M_g}(U_\infty)} \mathtt{div}\, \tilde{h}(y)\, dy = \oint\limits_{\Omega_{=M_g}(U_\infty)} \tilde{h}(y) \cdot n_\infty(y)\, dy, \tag{3.36}$$

where n_∞ is the outward normal of $\Omega_{\leqslant M_g}(U_\infty)$ defined, for every $y \in \Omega_{=M_g}(U_\infty)$, by

$$n_\infty(y) = \begin{cases} \dfrac{\mathtt{grad}\, U_\infty(y)}{|\mathtt{grad}\, U_\infty(y)|}, & \text{if} \quad \mathtt{grad}\, U_\infty(y) \text{ exists}, \\ 0, & \text{if} \quad \text{otherwise}. \end{cases}$$

Note that, from the previous paragraph, for almost every $y \in \Omega_{=M_g}(U_\infty)$, $\mathtt{grad}\, U_g(y)$ exists.

From the proof of Proposition 3.8 and Corollary 3.11, there exists $E_g \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geqslant 0})$ such that

$$\forall y \in \Omega_{=M_g}(U_\infty), \quad D^+_{\tilde{h}} U_\infty(y) \leqslant -E_g(y) < 0.$$

From the above discussions, the existence of the parametrization p_∞, and Remark A.40,

$$\text{for a.e. } s \in [a_\infty, b_\infty], \quad \frac{D^+_{\tilde{h}} U_\infty(p_\infty(s))}{|\mathtt{grad}\, U_\infty(p_\infty(s))|} = \tilde{h}(p_\infty(s)) \cdot n_\infty(p_\infty(s)) < 0.$$

Applying the generalized divergence theorem to the level curve $\Omega_{=M_g}(U_\infty)$, from (3.36)

$$\iint\limits_{\Omega_{\leqslant M_g}(U_g)} \mathtt{div}\, h(y)\, dy = \int\limits_{[a_\infty, b_\infty]} h(p_\infty(s)) \cdot n_\infty(p_\infty(s))\, ds < 0, \tag{3.37}$$

because, for every $y \in \mathbf{K}$, $U_\infty(y) = U_g(y)$ and $\tilde{h}(y) = h(y)$.

Analogously to the above and from the proof of Proposition 3.8 and Corollary 3.9, and by letting $p_\ell : [a_\ell, b_\ell] \to \Omega_{=M_\ell}(U_\ell)$ be a parametrization of $\Omega_{=M_\ell}(U_\ell)$ with outward unit normal n_ℓ, based on Equation (3.36),

$$\iint\limits_{\Omega_{\leqslant \widehat{M}_\ell}(U_\ell)} \operatorname{div} h(y)\, dy = \int\limits_{[a_\ell, b_\ell]} h(p_\ell(s)) \cdot n_\ell(p_\ell(s))\, ds < 0 \tag{3.38}$$

Suppose, by absurd, that there exists a closed orbit[38] $C \in \mathbb{R}^2$, parametrized by $p : [a, b] \to C$ and with outward unit normal n, and contained in $\mathbf{R}$. From the generalized divergence theorem,

[38] From the uniqueness of solutions with respect to initial conditions, the closed orbit C is a simple closed curve.

$$\iint\limits_{\mathbf{D}_C} \operatorname{div} h(y)\, dy = \int\limits_{[a,b]} h(p(s)) \cdot n(p(s))\, ds = 0, \tag{3.39}$$

where $\mathbf{D}_C$ is the simply connected region bounded by C.

Note that,

$$\begin{aligned} \iint\limits_{\Omega_{\leqslant M_g}(U_g)\setminus \mathbf{D}_C} \operatorname{div} h(y)\, dy &= \iint\limits_{\Omega_{\leqslant M_g}(U_g)} \operatorname{div} h(y)\, dy - \iint\limits_{\mathbf{D}_C} \operatorname{div} h(y)\, dy \\ &= \iint\limits_{\Omega_{\leqslant M_g}(U_g)} \operatorname{div} h(y)\, dy, \end{aligned}$$

where the last equality is due to (3.39). This yields with (3.37)

$$\iint\limits_{\Omega_{\leqslant M_g}(U_g)\setminus \mathbf{D}_C} \operatorname{div} h(y)\, dy < 0. \tag{3.40}$$

On the other hand,

$$\begin{aligned} \iint\limits_{\mathbf{D}_C \setminus \Omega_{\leqslant \widehat{M}_\ell}(U_\ell)} \operatorname{div} h(y)\, dy &= \iint\limits_{\mathbf{D}_C} \operatorname{div} h(y)\, dy - \iint\limits_{\Omega_{\leqslant \widehat{M}_\ell}(U_\ell)} \operatorname{div} h(y)\, dy \\ &= - \iint\limits_{\Omega_{\leqslant \widehat{M}_\ell}(U_\ell)} \operatorname{div} h(y)\, dy, \end{aligned}$$

where the last equality is also due to (3.39). From (3.38),

$$\iint\limits_{\mathbf{D}_C \setminus \Omega_{\leqslant \widehat{M}_\ell}(U_\ell)} \operatorname{div} h(y)\, dy > 0. \tag{3.41}$$

From (3.40), (3.41) and the fact that C is arbitrary, $\operatorname{div} h$ changes sign in $\mathbf{R}$. From the continuity of $\operatorname{div} h$, there exists $\bar{y} \in \mathbf{R}$ such that $\operatorname{div} h(\bar{y}) = 0$ which is a contradiction with (3.42). Thus, there exist no closed orbits C contained in $\mathbf{R}$.

From the Poincaré-Bendixson Theorem[39], the ω-limit set of a solution starting in $\mathbf{R}$ is a closed orbit or equilibria. Since equilibria do not exist by assumption and with the above analysis, there exist no ω-limit sets in $\mathbf{R}$. Thus, every solution starting in $\mathbf{R}$ will converge to

[39] (Sastry, 1999, Theorem 2.15)

$\Omega_{\leqslant M_\ell}(U_\ell)$. This concludes the proof of Lemma 3.20. ■

Corollary 3.21 (Bendixson criterion). *Let $n = m = 1$, under the hypotheses of Theorem 3.14 if* $\mathbf{R}$ *is a simply connected region such that,*

$$\forall y \in \mathbf{R}, \quad \mathtt{div}\, h(y) \neq 0, \; h(y) \neq 0, \tag{3.42}$$

then all solutions of (3.8) issuing from $\mathbf{R}$ *converge to* $\Omega_{\leqslant \hat{M}_\ell}(U_\ell)$. □

3.6.4 Proof of Claim 3.15

Assume for a fixed $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$, $V(x) \geqslant \Gamma(W(z))$. Suppose also that $z \in \Omega_{<0.6}(W)$. From (3.19),

$$\Gamma(W(z)) = \rho_x^{-1}\left(\frac{W(z)}{0.95}\right).$$

This implies

$$V(x) \geqslant \rho_x^{-1}\left(\frac{W(z)}{0.95}\right).$$

Because ρ_x^{-1} is strictly increasing in $[0, 0.6)$, it is invertible and

$$\rho_x(V(x)) \geqslant \frac{W(z)}{0.95}.$$

From (3.18), $D_f^+ V(x, z) \leqslant -\lambda_x(x)$. The above reasoning applied for $z \in \Omega_{\geqslant 0.6}(W)$ yields the same conclusion.

Now it remains to show that Γ is tight. From the surjectivity and continuity of W, there exists $z^* \in \mathbb{R}_{>0}$ such that $s^* = W(z^*)$. Thus, $\Gamma^*(W(z^*)) < \Gamma(W(z^*))$.

Assume that $z^* \in (\mathbb{R}_{>0} \cap \Omega_{<0.6}(W))$. From (3.19),

$$\Gamma(W(z^*)) = \rho_x^{-1}\left(\frac{W(z^*)}{0.95}\right).$$

Since ρ_x^{-1} is strictly increasing in the interval $[0, 0.6)$, it is invertible and $\rho_x(\Gamma^*(W(z^*)))0.95 < W(z^*)$. From the surjectivity and continuity of V, there exists $x^* \in \mathbb{R}_{>0}$ such that $\rho_x(\Gamma^*(W(z^*)))0.95 \leqslant \rho_x(V(x^*)) < W(z^*)$. Since $(x^*, z^*) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, the Dini derivative of V yields

$$D_f^+ V(x^*, z^*) = -\rho_x(V(x^*)) + W(z^*) > 0.$$

The case in which $z^* \in \Omega_{\geqslant 0.6}(W)$ is parallel. Thus, there exists $(x^*, z^*) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ such that $D_f^+ V(x^*, z^*) > 0$. This concludes the proof. ■

3.6.5 Proof of Claim 3.16

In the interval $(0.65, 2.5)$ two regions will be analyzed: $(0.65, 1)$ and $[1, 2.5)$.

Suppose that $s \in (0.65, 1)$. In this region, Δ is strictly increasing and $0.6 < \Delta(s) < 0.7$. From (3.19), $\forall s \in (0.6, 0.7)$, $\Gamma(s)$ is strictly increasing. Moreover, $\forall s \in (0.65, 0.7)$, $2.2 < \rho_x^{-1}(s) < 2.3$. Hence, $\forall s \in (0.65, 1)$, $s \leqslant \Gamma \circ \Delta(s)$. The analysis for the interval $[1, 2.5)$ is analogous. This concludes de proof. ■

3.7 Conclusion

Systems for which the small gain theorem cannot be used a sufficient condition for the stability of the resulting interconnected system is proposed. The approach consists in verifying if the small gain conditions holds in two different regions of the state space: a local and a non-local. In the gap between both regions, it must be checked if ω-limit sets exist. An approach is proposed for planar system for which Bendixson criterion does not hold. A condition is given to check the absence of ω-limits sets and trajectories that converge to them with measures larger than 0. An example is given to illustrate the results.

3.8 Appendix of Chapter 3

3.8.1 Technical lemma

Lemma 3.22. *Let* $k \geqslant 0$ *and* $p > 0$ *be constant integers. Given a function* $h \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}^p)$*, and compact set* $\mathbf{K} \subset \mathbb{R}^n$ *such that, for every* $y \in \mathbf{K}$, $h(y) \neq 0$*. Then, there exists* $\tilde{h} \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}^p)$ *such that* $\mathtt{supp}(\tilde{h}) \supset \mathbf{K}$ *with* $\mathtt{supp}(\tilde{h})$ *compact and, for every* $y \in \mathbf{K}$, $\tilde{h}(y) = h(y)$. □

Proof. This proof if based on (see Salsa, 2008, pp. 370). From the compactness of $\mathbf{K}$ and since, for every $y \in \mathbf{K}$, $h(y) \neq 0$, there exists $\varepsilon > 0$ such that, for every $y \in \mathbf{B}_{\leqslant \varepsilon}(\mathbf{K})$, $h(y) \neq 0$.

Let, for every $y \in \mathbb{R}^n$, the function

$$\eta(y) = \begin{cases} c \exp\left(\dfrac{-1}{1 - |y|^2}\right), & \text{if} \quad |y| < 1, \\ 0, & \text{if} \quad \text{otherwise}, \end{cases}$$

where c is chosen to satisfy

$$\int_{\mathbb{R}^n} \eta(y)\, dy = 1. \tag{3.43}$$

Note that, $\eta \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ and $\mathrm{supp}(\eta) = \mathbf{B}_{\leqslant 1}(0)$. Pick a constant value $\delta > 0$ and define, for every $y \in \mathbb{R}^n$, the function

$$\eta_\delta(y) = \frac{1}{\delta^n}\eta\left(\frac{y}{\delta}\right).$$

Note also that $\eta_\delta \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$, $\mathrm{supp}(\eta_\delta) = \mathbf{B}_{\leqslant \delta}(0)$, and

$$\int_{\mathbb{R}^n} \eta_\delta(y)\, dy = 1.$$

Let

$$\chi(y) = \begin{cases} 1, & \text{if} \quad y \in \mathbf{B}_{\leqslant \frac{\varepsilon}{2}}(\mathbf{K}), \\ 0, & \text{if} \quad \text{otherwise} \end{cases}$$

and consider the function[40]

$$\overline{\chi}(y) = \int_{\mathbb{R}^n} \chi(y - x)\eta_{\frac{\varepsilon}{2}}(x)\, dx.$$

[40] also known as convolution of χ and $\eta_{\varepsilon/2}$ (see Salsa, 2008, pp. 370).

Note that

$$\overline{\chi}(y) = \begin{cases} \int_{\mathbb{R}^n} \eta_{\frac{\varepsilon}{2}}(x)\, dx, & \text{if} \quad y - x \in \mathbf{B}_{\leqslant \frac{\varepsilon}{2}}(\mathbf{K}) \quad \text{and} \quad |x| < \frac{\varepsilon}{2}, \\ 0, & \text{if} \quad y - x \notin \mathbf{B}_{> \frac{\varepsilon}{2}}(\mathbf{K}) \quad \text{or} \quad |x| \geqslant \frac{\varepsilon}{2}. \end{cases}$$

Thus, $\overline{\chi} \in \mathcal{C}^\infty(\mathbb{R}^n, [0, 1])$ and satisfies

- Together with (3.43), for every $y \in \mathbf{K}$, $\overline{\chi}(y) = 1$;
- For every $y \in \mathbf{B}_{>\varepsilon}(\mathbf{K})$, $\overline{\chi}(y) = 0$;
- $\mathrm{supp}(\overline{\chi}) = \mathbf{B}_{\leqslant \varepsilon}(\mathbf{K})$.

Now, for every $y \in \mathbb{R}^n$, define $\tilde{h}(y) = \overline{\chi}(y)h(y)$. Note that $\tilde{h} \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}^p)$ and satisfies

- For every $y \in \mathbf{K}$, $\tilde{h}(y) = h(y)$;
- For every $y \in \mathbf{B}_{>\varepsilon}(\mathbf{K})$, $\tilde{h}(y) = 0$.

Moreover, $\mathbf{K} \subset \mathrm{supp}\left(\tilde{h}\right)$, the set $\mathrm{supp}\left(\tilde{h}\right)$ is compact, and for every $y \in \mathrm{supp}\left(\tilde{h}\right)$, $\tilde{h} \neq 0$. This concludes the proof. ■

3.8.2 The divergence theorem for level sets of a Lyapunov function

Definition 3.23 (Gamma function). [41] The function

$$\begin{aligned} \Gamma: \quad \mathbb{R}_{\geqslant 0} &\to \mathbb{R} \\ s &\mapsto \int_0^\infty t^{s-1} e^{-t}\,dt \end{aligned}$$

is called *gamma function*. ○

[41] (Rudin, 1976, Definition 8.17)

Definition 3.24 (Hausdorff measure). [42] Let $E \subset \mathbb{R}^n$, the *diameter* of E is the function

$$\begin{aligned} \mathtt{diam}: \quad E \times E &\to \mathbb{R}_{\geqslant 0} \\ (x, y) &\mapsto \sup\{|x - y|\}. \end{aligned}$$

Let $0 \leqslant n < \infty$. For $0 < \delta \leqslant \infty$ define

$$\mathscr{H}_\delta^n(E) = \inf\left\{ \sum_{j\in\mathbb{N}} \mathtt{diam}(E_j)^n : E \subset \bigcup_{j\in\mathbb{N}} E_j, \mathtt{diam}(E_j) < \delta, E_j \subset \mathbb{R}^n \right\}.$$

The n-dimensional *unnormalized Hausdorff measure of* E is the limit

$$\widetilde{\mathscr{H}}^n(E) = \lim_{\delta\to 0} \mathscr{H}_\delta^n(E) = \sup_{\delta>0} \mathscr{H}_\delta^n(E).$$

The n-dimensional *Hausdorff measure of* E is given by

$$\mathscr{H}^n(E) = \frac{\alpha(s)}{2^n} \widetilde{\mathscr{H}}^n(E),$$

where

$$\alpha(n) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}.$$ ○

[42] (Pfeffer, 2012, pp. 14)

Remark 3.25. [43] Note that the n-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$ is the n-fold product of unidimensional Lebesgue measures [44] while the Hausdorff measure is computed in terms of arbitrarily coverings of E with small diameters. Moreover, the Lebesgue measure in $\mathbb{R}^n$ is equivalent to the n-dimensional Hausdorff measure, i.e., $\mu = \mathscr{H}^n$. Also, if $\mathscr{H}^n(E) < \infty$, then $\mathscr{H}^{n-1}(E) = \infty$ and $\mathscr{H}^{n+1}(E) = 0$.[45] ○

[43] (Evans and Gariepy, 1992, Section 2.2)

[44] cf. Definitions A.5 and A.7.

[45] (Marzocchi, 2005)

Definition 3.26 (Essential boundaries). [46] For a set $E \subset \mathbb{R}^n$,

- The *essential exterior* is the set

$$\mathtt{ext}_*(E) = \left\{ x \in \mathbb{R}^n : \lim_{r\to 0} \frac{\mu\left(E \cap \mathbf{B}_{\leqslant r}(x)\right)}{\mu\left(\mathbf{B}_{\leqslant r}(x)\right)} = 0 \right\};$$

[46] (Pfeffer, 2012, pp. 50)

- The *essential interior* is the set $\mathtt{int}_*(E) = \mathtt{ext}_*(\mathbb{R}^n \setminus E)$;
- The *essential closure* is the set $\mathtt{cl}_*\{E\} = \mathbb{R}^n \setminus \mathtt{ext}_*(E)$;
- The *essential boundary* is the set $\partial_* E = \mathtt{cl}_*\{E\} \setminus \mathtt{int}_*(E)$. ◦

The following properties hold[47]

[47] (Pfeffer, 2012, pp. 49)

$$\mathtt{int}_*(E) \subset \mathtt{cl}_*\{E\}, \quad \mathtt{int}_*(\mathbb{R}^n \setminus E) = \mathtt{ext}_*(E),$$
$$\partial_* E = \mathtt{cl}_*(E) \cap \mathtt{cl}_*\{\mathbb{R}^n \setminus E\} = \partial_*(\mathbb{R}^n \setminus E) = \mathbb{R}^n \setminus (\mathtt{int}_*(E) \cup \mathtt{ext}_*(E)).$$

Definition 3.26 is related to the usual topological concepts as follows

$$\mathtt{int}(E) \subset \mathtt{int}_*(E), \quad \mathtt{cl}_*\{E\} \subset \mathtt{cl}\{E\}, \quad \partial_* E \subset \partial E.$$

Moreover,

$$\partial_* E = \partial E \Leftrightarrow \mathtt{int}(E) = \mathtt{int}_*(E) \quad \text{and} \quad \mathtt{cl}_*\{E\} = \mathtt{cl}\{E\},$$

and

$$\mathtt{int}_*(E) \subset \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{\mu\left(E \cap \mathbf{B}_{\leqslant r}(x)\right)}{\mu\left(\mathbf{B}_{\leqslant r}(x)\right)} = 1 \right\}$$

becomes an inequality, when E is measurable.

Example 3.27. [48] Consider the heart-shaped open set $\mathbf{C}$ illustrated by Figure 3.4. Assume that it is measurable, and points $x_1, x_2, x_5 \notin \mathbf{C}$, and that $\mathbf{C}$ is cusped in x_2 and x_5.

[48] (Marzocchi, 2005)

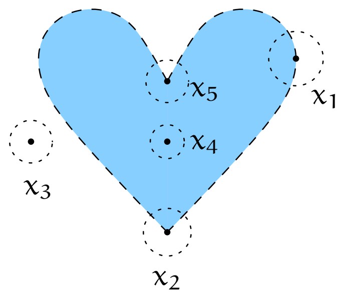


Figure 3.4: Heart-shaped set. Circumferences illustrates the set $\mathbf{B}_{=r}(x_i)$, $i = 1, \ldots, 5$.

Note that, $x_1, x_2, x_3 \notin \mathtt{int}_*(\mathbf{C})$ while $x_4, x_5 \in \mathtt{int}_*(\mathbf{C})$. In particular, $x_1 \in \partial_* \mathbf{C}$, and $x_2, x_3 \in \mathtt{ext}_*(\mathbf{C})$. Moreover, and if $\mathbf{C}$ was not cusped in x_2, then $x_2 \in \partial_* \mathbf{C}$. ◇

Definition 3.28 (Perimeter of a set). [49] The *perimeter* of a set $E \subset \mathbb{R}^n$ is the measure

$$P(E) = \mathscr{H}^{n-1}(\partial_* E).$$

The perimeter is *finite* if $\mu(E) + P(E) < \infty$. ◦

[49] (Pfeffer, 2012, Definition 4.5.1)

Theorem 3.29. [50] *Let $k \geqslant 0$ be a constant integer, and consider the Lipschitz*

[50] Adapted from (Alberti, Bianchini, and Crippa, 2013, Theorem 2.5).

map $V \in \mathcal{C}^k(\mathbb{R}^2, \mathbb{R}_{\geqslant 0})$ *with* $\mathtt{supp}(V)$ *compact. The following statements hold, for almost every* $c \in \mathbb{R}_{\geqslant 0}$.

1. $\Omega_{=c}(V)$ *is 1-rectifiable*[51] *and* $\mathscr{H}^1(\Omega_{=c}(V)) < \infty$;

[51] $\Omega_{=c}(V)$ can be $\mathscr{H}^1$-almost everywhere covered by countably many 1-dimensional curves of class $\mathcal{C}^1$.

2. *For* $\mathscr{H}^1$*-almost every* $x \in \Omega_{=c}(V)$, *the map* V *is differentiable at* x;

3. *Every connected component* C *of* $\Omega_{=c}(V)$ *is either a point or a closed simple curve with a Lipschitz parametrization* $p : [a, b] \to C$ *which is injective and satisfies, for almost every* $t \in [a, b]$,

$$\frac{dp}{dt}(t) = \tau(p(t)),$$

where, for every $x \in C$, $\tau(x)$ *is the vector tangent to* C. □

From item 2, since level set $\Omega_{=c}(V)$ is either a point or a simple closed curve of $\mathbb{R}^2$, $\partial\Omega_{=c}(V) = \mathtt{cl}\{\Omega_{=c}(V)\} = \Omega_{=c}(V)$. Moreover, $\partial_*\Omega_{=c}(V) \subset \mathtt{cl}\{\Omega_{=c}(V)\}$. Thus, the sub-level set $\Omega_{\leqslant c}(V)$ has finite perimeter and, from item 1,

$$\int_{\Omega_{=c}(V)} d\mathscr{H}^1 < \infty,$$

where, from Remark 3.25, the integral is defined in the Lebesgue sense in $\mathbb{R}^1$.

Definition 3.30. [Outward normal][52] For every $x \in \partial_* E$, denote by $n_E(x)$ the unit vector of $\mathbb{R}^n$ such that

[52] Based on (Pfeffer, 2012, pp. 127-128) and (Marzocchi, 2005, Definition 1.6).

$$H_{\pm}(E, x) = \{y \in \mathbb{R}^n : \pm n_E(x) \cdot (y - x) \geqslant 0\}.$$

The function n_E is called *outward unit normal of* $E \subset \mathbb{R}^n$ if, for every $x \in \partial_* E$,

$$\begin{aligned} \lim_{r\to 0} \frac{\mu\left(B_{\leqslant r}(x) \cap H_+(E, x) \cap E\right)}{\mu\left(B_{\leqslant r}(x)\right)} &= 0, \\ \lim_{r\to 0} \frac{\mu\left(B_{\leqslant r}(x) \cap (H_-(E, x) \setminus E)\right)}{\mu\left(B_{\leqslant r}(x)\right)} &= 0 \end{aligned} \tag{3.44}$$

hold. ∘

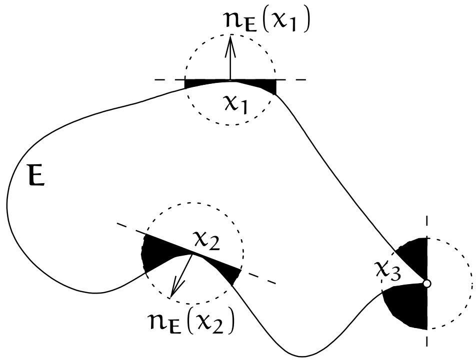


Figure 3.5: Illustration of Equation (3.44), the measure of the shaded sets must go to zero faster than the measure of the balls. Note that, in x_3 the normal is a cone. Thus, it does not satisfy (3.44). Hence, it is 0 at x_3. Figure originally presented in (Marzocchi, 2005).

From item 2 of Theorem 3.29, for $\mathscr{H}^1$-almost every $x \in \Omega_{=c}(V)$, $\operatorname{grad} V(x)$ exists. Thus, the vector field

$$\begin{aligned} n : \quad \Omega_{=c}(V) &\to \mathbb{R}^2 \\ x &\mapsto \begin{cases} \dfrac{\operatorname{grad} V(x)}{|\operatorname{grad} V(x)|}, & \text{if} \quad \operatorname{grad} V(x) \text{ exists,} \\ 0, & \text{if} \quad \text{otherwise} \end{cases} \end{aligned} \tag{3.45}$$

is $\mathscr{H}^1$-almost everywhere an outward normal to $\Omega_{\leqslant c}(V)$. Since the outward normal to sets of finite perimeter is unique[53], n satisfies (3.44).

[53] (Federer, 1945, Theorem 3.4)

For a further reading on the sets of finite perimeters and on the construction of outward normals for them, the interested reader is invited to see (Pfeffer, 2012, Chapters 5 and 6).

Theorem 3.31 (Generalized divergence theorem)**.** [54] *Under the assumptions of Theorem 3.29. Let $k \geqslant 0$ be a constant integer, and consider the map $f \in \mathcal{C}^k(\mathbb{R}^2, \mathbb{R}^2)$ with $\operatorname{supp}(f)$ compact. Then, the formula*

$$\iint\limits_{\Omega_{\leqslant c}(V)} \operatorname{div} f(x)\, dx = \oint\limits_{[a,b]} f(p(s)) \cdot n(p(s))\, ds$$

holds, where the integral of the lefthand-side (resp. righthand-side) is taken in the Lebesgue (resp. 1-dimensional Hasudorff) measure on $\mathbb{R}^2$ (resp. $\mathbb{R}$), and $p : [a, b] \to \Omega_{=c}(V)$ is a parametrization of $\Omega_{=c}(V)$. □

[54] Adapted from (Marzocchi, 2005, Theorem 1.7) or (Pfeffer, 2012, Theorem 6.5.4). In the latter, the set where the integral is computed is assumed to have *Bounded Variariation*, in (Pfeffer, 2012, Theorem 6.5.5) it is shown that a set has bounded variation if and only if it has finite perimeter.

Before showing a sketch of the proof of Theorem 3.31, the following Lemma is needed.[55]

[55] For a detailed proof in $\mathbb{R}^n$, the interested reader may consult (Pfeffer, 2012, Chapters 1 to 6).

Lemma 3.32 (Green's Theorem)**.** [56] *Let $C \subset \mathbb{R}^2$ be a positively oriented, piecewise smooth, simple closed curve with finite length, let $\mathbf{D}_C$ be the region bounded by C, and let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$. If $f_1 : \mathbb{R}^2 \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}$ are defined on an open region containing $\mathbf{D}_C$, and f is differentiable in such*

[56] Based on (Spiegel, 1959, pp. 106).

a region, then

$$\oint_C (f_1(x_1,x_2)\,dx_1 + f_2(x_1,x_2)\,dx_2) = \iint_{\mathbf{D}_C} \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1\,dx_2, \quad (3.46)$$

where the path of integration along C *is counterclockwise.* □

Proof (of Lemma 3.32). [57] Since C is a simple closed curve in the plane, the region $\mathbf{D}_C$ is bounded. The projection of the curve in the x-axis (resp. y-axis) yields an interval $[a,b]$ (resp. $[e,f]$). Consider the points of $A, B \in C$ (resp. $E, F \in C$) corresponding to the points a and b (resp. e and f) on the x-axis, the curve C can be saw as the union of the curves AEB and AFB. The Figure 3.6 illustrates the curve C, and the intervals $[a,b]$, and $[e,f]$.

[57] Based on (Spiegel, 1959, pp. 108).

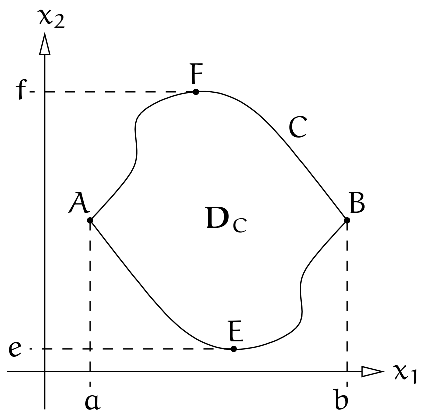


Figure 3.6: Illustration of the curve C.

Let the equation of the curve containing the points AEB (resp. AFB) be given by a piecewise continuous function $\eta_1 : [a,b] \to \mathbb{R}^2$ (resp. $\eta_2 : [a,b] \to \mathbb{R}^2$).

Integrating the partial derivative of f_1 with respect to x_2 in $\mathbf{D}_C$ yields

$$\iint_{\mathbf{D}_C} \frac{\partial f_1}{\partial x_2}(x_1,x_2)\,dx_1\,dx_2 = \int_a^b \int_{\eta_1(x_1)}^{\eta_2(x_1)} \frac{\partial f_1}{\partial x_2}(x_1,x_2)\,dx_2\,dx_1$$

where the equality is due to Fubini's theorem[58]. Moreover, since C has finite length, the equality

[58] (DiBenedetto, 2002, Theorem 14.1)

$$\begin{aligned}
\iint_{\mathbf{D}_C} \frac{\partial f_1}{\partial x_2}(x_1,x_2)\,dx_1\,dx_2 &= \int_a^b (f_1(x_1,\eta_2(x_1)) - f_1(x_1,\eta_1(x_1)))\,dx_1 \\
&= -\int_a^b f_1(x_1,\eta_1(x_1))\,dx_1 - \int_b^a f_1(x_1,\eta_2(x_1))\,dx_1 \\
&= -\oint_C f_1(x_1,x_2)\,dx_1.
\end{aligned}$$

holds.

Analogously, integrating the partial derivative of f_2 with respect to x_1 in $\mathbf{D}_C$ yields

$$\iint_{\mathbf{D}_C} \frac{\partial f_2}{\partial x_1}(x_1,x_2)\,dx_1\,dx_2 = \oint_C f_2(x_1,x_2)\,dx_2.$$

From where the conclusion follows. ■

Now, it is possible to present an idea of the proof of Theorem 3.31. From Theorem 3.29,

- The curve $\Omega_{=c}(V)$ is piecewise $\mathcal{C}^1$, because it is rectifiable. Moreover, it is also simple and closed;

- Since V is $\mathscr{H}^1$-a.e. differentiable in $\Omega_{=c}(V)$, the outward normal vector defined by (3.45) is $\mathscr{H}^1$-a.e. non null;

- The curve $\Omega_{=c}(V)$ has finite length, because $\mathscr{H}^1(\Omega_{=c}(V)) < \infty$;

- There exists a injective and Lipschitz continuous parametrization $p : [a, b] \to \Omega_{=c}(V)$ that is a.e. differentiable.

Consider the vector field $\tilde{f} = (-f_2, f_1)$ that is perpendicular to $f = (f_1, f_2)$. Since $f = (f_1, f_2) \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$, $\tilde{f} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$. Together with the above, from[59] Lemma 3.32,

[59] More specifically from (3.46).

$$\oint_{\Omega_{=c}(V)} (-f_2(x_1, x_2)\, dx_1 + f_1(x_1, x_2)\, dx_2) = \oint_{\Omega_{=c}(V)} (-f_2(x_1, x_2), f_1(x_1, x_2)) \cdot (dx_1, dx_2)$$

Consider a point $\bar{x} = (\bar{x}_1, \bar{x}_2) \in C$ for which there exists $\bar{s} \in [a, b]$ such that $p(\bar{s}) = (\bar{x}_1, \bar{x}_2)$ and $p'(\bar{s})$ is defined. The unit tangent vector to C at $\bar{x}$ is given by $T(\bar{s}) = p'(\bar{s})/_{|p'(\bar{s})|} = (\tau(\bar{s}), \sigma(\bar{s}))$. The unit normal vector at $\bar{x}$ is given by $N(\bar{s}) = n(p(s)) = (\sigma(\bar{s}), -\tau(\bar{s}))$. For almost every $s \in [a, b]$,

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = T(s)\, ds = \begin{pmatrix} \tau(s) \\ \sigma(s) \end{pmatrix} ds.$$

Thus,

$$\begin{aligned}\oint_{\Omega_{=c}(V)} (-f_2(x_1,x_2)\,dx_1 + f_1(x_1,x_2)\,dx_2) &= \oint_{[a,b]} (-f_2(p(s)), f_1(p(s)))\cdot(\tau(s),\sigma(s))\,ds \\ &= \oint_{[a,b]} (f_1(p(s)), f_2(p(s)))\cdot(\sigma(s),-\tau(s))\,ds \\ &= \oint_{[a,b]} (f_1(p(s)), f_2(p(s)))\cdot n(p(s))\,ds\end{aligned}$$

From (3.46),

$$\iint_{\Omega_{\leqslant c}(V)} \left(\frac{\partial f_1}{\partial x_1}(x_1,x_2) + \frac{\partial f_2}{\partial x_2}(x_1,x_2)\right) dx_1\,dx_2 = \oint_{[a,b]} (f_1(p(s)), f_2(p(s)))\cdot n(p(s))\,ds.$$

This concludes the sketch of the proof of Theorem 3.31.

3.8.3 Integration along solutions of an ODE

Lemma 3.33 (Liouville's Theorem). [60] *Let* $k \geqslant 1$ *and* $p \geqslant 1$ *be constant integers, the function* $\rho \in (\mathcal{C}^k \cap \mathcal{L}^p)(\mathbb{R}^{n+m}, \mathbb{R}_{\geqslant 0})$. *Let also* $Y(t,y)$ *be a solution of* (3.8) *starting in* $y \in \mathbb{R}^{n+m}$ *and computed at time* $t \in \mathbb{R}_{\geqslant 0}$. *For a measurable set* $\mathbf{Z}$, *let* $Y(\cdot,\mathbf{Z}) = \{Y(\cdot,z) : z \in \mathbf{Z}\}$. *Then,*

$$\int_{Y(t,\mathbf{Z})} \rho(y)\,dy - \int_{\mathbf{Z}} \rho(y)\,dy = \int_0^t \int_{Y(\tau,\mathbf{Z})} \mathtt{div}(\rho h)(y)\,dy\,d\tau.$$

□

[60] Based on (Rantzer, 2001, Lemma A.1)

Theorem 3.34 (Almost attractivity). [61] *Let* $k \geqslant 1$ *and* $p \geqslant 1$ *be constant integers. Suppose that there exists* $\rho \in (\mathcal{C}^k \cap \mathcal{L}^p)(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ *such that,*

$$\int_{\mathbf{B}_{\geqslant 1}(0)} \frac{(h\rho)(y)}{|y|}\,dy < \infty$$

and,

$$\text{for a.e. } y \in \mathbb{R}^{n+m}, \quad \mathtt{div}(h\rho)(y) > 0.$$

Then, for almost every initial condition $y \in \mathbb{R}^{n+m}$,

$$\limsup_{t\to\infty} |Y(t,y)| = 0.$$

Moreover, if the origin is stable, then the conclusion remains valid when ρ *takes negative values.* □

[61] (Rantzer, 2001, Theorem 1)

4 | CONCLUSIONS AND PERSPECTIVES

This chapter collects final remarks of this dissertation and discuss further development on the results presented in the previous chapters.

Contents

4.1 Conclusions

In this dissertation two applications of stability theory were considered: the synthesis of a hybrid nonlinear feedback satisfying constraints in the basin of attraction, and the analysis of the stability of an interconnected system using nested criteria.

In Chapter 2, the problem of designing a stabilizing feedback law for a nonlinear system, when the *backstepping* method may fail to be employed, were considered. By adding a discrete dynamics to the continuous system under consideration, it was possible to blend a local stabilizing feedback law, designed to satisfy constraints in the basin the attraction, with a controller that globally asymptotically stabilizes a set contained in the basin the attraction. Some drawbacks of this approach were discussed in Remarks 2.17 and 2.19. In the former, it is estimated the maximum admissible perturbation, while in the latter it is shown the computational cost to obtain a local feedback law using an approximation of the nonlinear dynamics.

In Chapter 3, when the small gain condition does not hold in a given region of the state space, some criteria to analyze the interconnection of dynamical systems were provided. More specifically, in the regions where the small gain condition holds, it is analyzed how a positive definite function varies along the solutions of the interconnected system, while in the regions where the small gain condition does not hold, it is provided a sufficient condition for the Lebesgue measure of the sets of initial conditions, and corresponding solutions that do not converge to an compact attractor to be zero.

Although this approach does not ask for the small-gain condition to hold everywhere, it is needed further regularity on the vectorial field on the same region, because of the use of differentiability of the vector field. An attempt to relax the need for regularity could generalize the presented approach. Also, a method to check the absence of ω-limit sets with Lebesgue measure zero would improve the criterion. Applications of Chapter 3 include analysis of large-scale systems (see Section 4.2.1 below), and could lead to the design of feedback laws satisfying different gains.

4.2 Perspectives and work in progress

The objective of this section is to show some possible applications and developments on the work presented on the previous chapters.

4.2.1 Analysis of large-scale systems under nested criteria

With Dashkovskiy, the author is currently working on an application of the criterion proposed on Chapter 3 for the stability analysis of systems obtained from the interconnection of several components.

The idea is to merge the criterion proposed in (Dashkovskiy, Rüffer, and Wirth, 2010; Dashkovskiy and Rüffer, 2010) which relies on a small-gain like assumption holding globally with the one presented on Chapter 3.

4.2.2 Interconnecting almost input-to-state stable systems

Background and motivation. Consider the system

$$\dot{x}_1 = f_1(x_1, u_1) \tag{4.1}$$

where, $f_1 \in \mathcal{C}^1(\mathbb{R}^{n+m}, \mathbb{R}^n)$.

Definition 4.1. [1] The system (4.1) is called *almost input-to-state stable* (aISS) with respect to an invariant compact $\mathbf{A}_1 \subset \mathbb{R}^n$, if

[1] (Angeli, 2004)

- The set $\mathbf{A}_1$ is locally asymptotically stable in the Lyapunov sense, when[2] $u = 0$;
- There exists $\gamma_1 \in \mathcal{K}$ such that, for every $u_1 \in \mathbb{R}^m$, there exists $\aleph_1(u_1) \subset \mathbb{R}^n$ with $\mu_1(\aleph_1(u_1)) = 0$ satisfying,

[2] Note that Angeli did not state whether $u \equiv 0$ or not. It is assumed that $u \equiv 0$, because of properties 2) - 3) listed just before (Angeli, 2004, Definition 2.1).

$$\forall x_1 \in \mathbb{R}^n \setminus \aleph_1(u_1), \quad \limsup_{t \to \infty} |X_1(t, x_1, u_1)|_{\mathbf{A}} \leqslant \gamma_1(\|u_1\|_\infty). \qquad \circ$$

From now on, assume that $\mathbf{A}_1 = \{0\}$, and (4.1) is aISS.

Theorem 4.2. [3] *System (4.1) is aISS if and only if there exists a mapping $\aleph_1 : \mathbb{R}^m \to 2^{\mathbb{R}^n}$ with $\mu(\aleph_1(u_1)) = 0$ for any measurable input $u_1 \in \mathbb{R}^m$ such that,*

[3] (Angeli, 2004, Theorem 1)

$$\forall u_1 \in \mathbb{R}^m, \forall x_1 \in \mathbb{R}^n \setminus \aleph_1(u_1), \quad \limsup_{t \to \infty} |X_1(t, x_1, u_1)| \leqslant \gamma_1\left(\limsup_{t \to \infty} |u_1(t)|\right). \tag{4.2}$$

□

Is the interconnection of two almost input-to-state systems almost globally asymptotically stable?

4.2.3 Homogeneous approximations of hybrid systems

Homogeneous approximations of dynamical systems allow the analysis of dynamical systems using approximation functions. The aim of this section is to analyze systems with mixed dynamics (continuous and discrete) using functions that approximate it around the origin and in the infinity.

The advantage in using this method are based in two important results. The first is concerned with the stability of the equilibrium, while the second is concerned with attractors containing the equilibrium. Roughly speaking, if this equilibrium is asymptotically stable for the homogenous approximation in the equilibrium, then it is also asymptotically stable for the nomimal system.[4] Moreover, if this equilibrium is globally asymptotically stable for the homogenous approximation in the infinity, then there exists a set that is globally asymptotically stable for the nominal system and strictly contains this equilibrium point.[5]

[4] (Goebel and Teel, 2010)

[5] (Andrieu, Praly, and Astolfi, 2008)

The aim is to extend the concepts and results of (Andrieu, Praly, and Astolfi, 2008), concerning the homogeneous approximation in the infinity using the framework of hybrid systems, and complementing (Goebel and Teel, 2010). More precisely, is it possible to generalize the following result for hybrid systems?

[6] (Andrieu, Praly, and Astolfi, 2008)

Proposition 4.3. [6] *Consider a homogeneous approximation of the vector field* $f : \mathbb{R}^n \to \mathbb{R}^n$ *in the infinity* $f_\infty : \mathbb{R}^n \to \mathbb{R}^n$*. If the origin is globally asymptotically stable for the system*

$$\dot{x} = f_\infty(x),$$

then there exists an invariant compact subset of $\mathbb{R}^n$*, denoted* $\mathbf{C}_\infty$*, which is globally asymptotically stable for the system*

$$\dot{x} = f(x). \qquad \square$$

A | ORDINARY DIFFERENTIAL EQUATIONS

Contents

A.1 Lebesgue measure and integral

Some concepts of Lebesgue measure an integral, used mainly on Chapter 3, are recalled here.

A.1.1 The Lebesgue measure

Definition A.1 (σ-algebra). [1] A collection $\mathfrak{S}$ of subsets of a set $\mathbb{R}^n$ is called *σ-algebra of sets* if

[1] Based on (Salsa, 2008, Definition B.1)

1. $\emptyset, \mathbb{R}^n \in \mathfrak{S}(\mathbb{R}^n)$;

2. if $\mathbf{A} \in \mathfrak{S}(\mathbb{R}^n)$, then $\mathbb{R}^n \setminus \mathbf{A} \in \mathfrak{S}(\mathbb{R}^n)$;

3. if $\{\mathbf{A}_i : i \in \mathbb{N}\} \subset \mathfrak{S}(\mathbb{R}^n)$, then

$$\bigcup_{i \in \mathbb{N}} \mathbf{A}_i \in \mathfrak{S}(\mathbb{R}^n) \quad \text{and} \quad \bigcap_{i \in \mathbb{N}} \mathbf{A}_i \in \mathfrak{S}(\mathbb{R}^n).$$

The pair $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n))$ is called *measurable space*, and the sets $\mathbf{A} \in \mathfrak{S}(\mathbb{R}^n)$ are called *measurable set*. ○

[2] (Hunter, 2011, Definition 2.18)

Definition A.2. [2] Let $\mathfrak{T}(\mathbb{R}^n)$ be collection of all open sets of $\mathbb{R}^n$. The *Borel σ-algebra* $\mathfrak{B}(\mathbb{R}^n)$ on $\mathbb{R}^n$ is the σ-algebra generated by the open sets, $\mathfrak{B}(\mathbb{R}^n) = \mathfrak{S}(\mathfrak{T}(\mathbb{R}^n))$. A set that belongs to $\mathfrak{B}(\mathbb{R}^n)$ is called *Borel* set. ○

[3] Based on (Kurtz and Swartz, 2004, Definition 3.32, and pp. 80)

Definition A.3 (Measure). [3] Consider a measurable space $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n))$, a mapping $\mathtt{mes} : \mathfrak{S}(\mathbb{R}^n) \to \mathbb{R}_{\geqslant 0}$ is called *measure* if

1. $\mathtt{mes}(\emptyset) = 0$;

2. $\mathtt{mes}$ is countably additive, i.e.,

$$\mathtt{mes}\left(\bigcup_{i \in \mathbb{N}} \mathbf{A}_i\right) = \sum_{i \in \mathbb{N}} \mathtt{mes}(\mathbf{A}_i),$$

   for every sequence of pairwise disjoint sets $\{\mathbf{A}_i\}_{i \in \mathbb{N}} \subset \mathfrak{S}(\mathbb{R}^n)$.

The triple $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n), \mathtt{mes})$ is called *measure space*. ○

[4] Based on (Hunter, 2011, pp. 10) and (Kurtz and Swartz, 2004, Definition 2.1, and pp. 80).

Definition A.4 (Partition, length, and volume). [4] The *length* of an interval $[a, b]$ is defined by $\ell([a, b]) = b - a$. A *rectangle* $\mathbf{R} \subset \mathbb{R}^n$ is a set of the form

$$\mathbf{R} = \bigtimes_{i=1}^{n} [a_i, b_i].$$

A rectangle is called *open* if $\mathbf{R} = \mathtt{int}(\mathbf{R})$. Two rectangles $\mathbf{R}_1$ and $\mathbf{R}_2$ are called *almost disjoint* if $\mathtt{int}(\mathbf{R}_1) \cap \mathtt{int}(\mathbf{R}_2) = \emptyset$. The set of all n-dimensional rectangles of $\mathbb{R}^n$ is denoted by $\mathfrak{R}$. The *volume* of a rectangle $\mathbf{R}$ is defined by

$$\mathtt{vol}(\mathbf{R}) = \prod_{i=1}^{n} \ell([a_i, b_i])$$

with the convention that $0 \cdot \infty = 0$. ○

[5] Based on (Hunter, 2011, Definition 2.1) and (Kurtz and Swartz, 2004, Definition 3.40).

Definition A.5 (Lebesgue outer measure). [5] Let $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n))$ be a measurable space the *Lebesgue outer measure* of $E \subset \mathbb{R}^n$ is defined by

$$\mu^*(E) = \inf\left\{\sum_{i \in \mathbb{N}} \mathtt{vol}(\mathbf{R}_i) : E \subset \bigcup_{i \in \mathbb{N}} \mathbf{R}_i, \mathbf{R}_i \in \mathfrak{R}\right\},$$

where the infimum is taken over all countable collection whose union contains E. The mapping

$$\begin{aligned} \mu^*: \quad 2^{\mathbb{R}^n} &\to \mathbb{R}_{\geqslant 0} \\ E &\mapsto \mu^*(E), \end{aligned}$$

where $2^{\mathbb{R}^n}$ denotes the collection of all subsets of $\mathbb{R}^n$, is called *outer Lebesgue measure*. ○

Proposition A.6. [6] *The Lebesgue outer measure has the following properties*

[6] (Hunter, 2011, Theorem 2.4)

P$_1$. $\mu^*(\emptyset) = 0$;

P$_2$. *if* $E \subset \mathfrak{S}(\mathbb{R}^n)$, *then* $\mu^*(E) \leqslant \mu^*(\mathfrak{S}(\mathbb{R}^n))$;

P$_3$. *if* $\{E_i \subset \mathbb{R}^n : i \in \mathbb{N}\}$ *is a countable collection of subsets of* $\mathbb{R}^n$, *then*

$$\mu^*\left(\bigcup_{i=1}^{n} E_i\right) \leqslant \sum_{i=1}^{m} \mu^*(E_i).$$

□

Definition A.7 (Lebesgue measure, Measurable set). [7] A set $E \subset \mathbb{R}^n$ is said to be *Lebesgue measurable* if, for every $\mathbf{A} \subset \mathbb{R}^n$,

[7] (Hunter, 2011, Definition 2.10)

$$\mu^*(\mathbf{A}) = \mu^*(\mathbf{A} \cap E) + \mu^*(\mathbf{A} \setminus E). \tag{A.1}$$

Let $\mathfrak{L}(\mathbb{R}^n)$ denote the σ-algebra of Lebesgue measurable sets, the restriction of the Lebesgue outer measure μ^* to the Lebesgue measurable sets, $\mu = \mu^*|_{\mathfrak{L}(\mathbb{R}^n)}$, $\mu : \mathfrak{L}(\mathbb{R}^n) \to [0, \infty]$, is called *Lebesgue measure*. ○

Proposition A.8. [8] *Every rectangle is Lebesgue measurable.* □

[8] (Hunter, 2011, Proposition 2.11)

Proposition A.9. [9] *Every open set is a countable union of almost disjoint rectangles.* □

[9] (Hunter, 2011, Proposition 2.20)

As a consequence of Propositions A.8 and A.9, every open set is Lebesgue measurable.

Definition A.10. [10] Let $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n))$ and $(\mathbb{R}^m, \mathfrak{S}(\mathbb{R}^m))$ be measurable spaces. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *measurable* if $f^{-1}(\mathbf{B}) \in \mathfrak{S}(\mathbb{R}^n)$, for every $\mathbf{B} \in \mathfrak{S}(\mathbb{R}^m)$. ○

[10] (Hunter, 2011, Definition 3.1)

A.1.2 The Lebesgue integral

Definition A.11. [11] A *characteristic function of a subset* $E \subset \mathbb{R}^n$ is the

[11] (Hunter, 2011, Definition 3.11)

function

$$\begin{aligned} \chi_E : \mathbb{R}^n &\to \mathbb{R} \\ y &\mapsto \begin{cases} 1, & \text{if } y \in E, \\ 0, & \text{if } y \notin E. \end{cases} \end{aligned}$$

A *simple function* $\phi : \mathbb{R}^n \to \mathbb{R}$ on a measurable space $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n))$ is a function of the form

$$\phi(y) = \sum_{i=1}^{N} c_i \chi_{E_i}(y),$$

where, for every $i = 1, \dots, N$, $c_i \in \mathbb{R}$, and $E_i \in \mathfrak{S}(\mathbb{R}^n)$. It is called *positive simple function* if, in addition, for every $i = 1, \dots, N$, $c_i \in \mathbb{R}_{\geqslant 0}$. ◦

Definition A.12. [12] Let $(\mathbb{R}^n, \mathfrak{S}, \mu)$ be a measure space and $\phi : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ a positive simple function the *integral of* ϕ *with respect to* μ is

$$\int \phi \, d\mu = \sum_{i=1}^{N} c_i \mu(E_i).$$

With the convention that if $c_i = 0$ and $\mu(E_i) = \infty$, then $0 \cdot \infty = 0$. ◦

[12] (Hunter, 2011, Definition 4.1)

Definition A.13. [13] Let $(\mathbb{R}^n, \mathfrak{S}, \mu)$ be a measure space and $h : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ a measurable function

$$\int h \, d\mu = \sup \left\{ \int \phi \, d\mu : 0 \leqslant \phi \leqslant h, \phi \text{ simple} \right\}. \quad ◦$$

[13] (Hunter, 2011, Definition 4.4)

Remark A.14. [14] Definitions A.12 and A.13 also applies to vector fields. In this case, the integral of a vector field simple function ϕ is defined exactly as in Definition A.12. A vector field $f : \mathbb{R}^n \to \mathbb{R}^m$ is called integrable if there exists a sequence of integrable simple functions $\{\phi_i\}_{i \in \mathbb{N}}$, where $\phi_i : \mathbb{R}^n \to \mathbb{R}^m$, such that $\phi_i \to f$ pointwise[15], where the convergence is with respect to the norm on $\mathbb{R}^m$, and

$$\int ||f - \phi_n|| \, d\mu \to 0, \quad \text{as } n \to \infty.$$

◦

[14] (Hunter, 2011, pp. 53)

[15] Let $S \subset \mathbb{R}^n$, and $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of real valued functions defined on S. The sequence $\{f_n\}_{n \in \mathbb{N}}$ is said to *converge pointwisely to* f if $\exists \bar{x} \in S$ such that, $\forall \varepsilon > 0$, $\exists N(\bar{x}, \varepsilon) > 0$ such that,

$$\forall n \geqslant N(\bar{x}, \varepsilon), |f_n(\bar{x}) - f(\bar{x})| < \varepsilon.$$

Definition A.15. [16] Let $(\mathbb{R}^n, \mathfrak{S}(\mathbb{R}^n), \mu)$, and $p \in [1, \infty)$. The Lebesgue measurable function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be *locally* p*-integrable* (or p-*summable*) on $\mathbb{R}^n$ if, for every compact set $K \subset \mathbb{R}^n$, and for $p \in [1, \infty)$,

$$\int_K |h(y)|^p \, d\mu < \infty.$$

The class of locally p-integrable functions h on $\mathbb{R}^n$ is denoted by $\mathcal{L}^p_{loc}(\mathbb{R}^n, \mathbb{R})$.

[16] Based on (Hunter, 2011, Definition 7.1) and (Vladimirov, 2002, pp. 3).

For $p = \infty$,

$$\operatorname*{ess\,sup}_{y \in \mathbf{K}} |h(y)| < \infty,$$

where

$$\operatorname*{ess\,sup}_{y \in \mathbf{K}} h(y) = \inf\{a \in \mathbb{R} : \mu\{y \in \mathbf{K} : h(y) > a\} = 0\}.$$

The class of locally ∞-integrable functions h on $\mathbb{R}^n$ is called *locally essentially bounded*, and is denoted by $\mathcal{L}^{\infty}_{\text{loc}}(\mathbb{R}^n, \mathbb{R})$. The *essential supremum norm of* h *in* $\mathbb{R}^n$ is the positive value $|h|_\infty := \operatorname{ess\,sup}\{|h(y)| : y \in \mathbb{R}^n\}$. ◦

A.2 Continuity of functions

Definition A.16 (Uniform continuity). Let $\mathbf{Y} \subset \mathbb{R}^n$ be an open set. A function $h : \mathbf{Y} \to \mathbb{R}^n$ is called *continuous* if,

$$\forall \bar{y} \in \mathbf{Y}, \forall \epsilon > 0, \exists \delta(\bar{y}, \epsilon) > 0 : \forall y \in \mathbf{Y}, |y - \bar{y}| < \delta(\bar{y}, \epsilon) \Rightarrow |h(y) - h(\bar{y})| < \epsilon.$$

The class of k-times continuously differentiable functions $h : \mathbf{Y} \to \mathbb{R}^n$ is denoted by $\mathcal{C}^k(\mathbf{Y}, \mathbb{R}^m)$. The function h is said to be *uniformly* continuous if

$$\forall \epsilon > 0, \exists \delta(\epsilon) > 0 : \forall y_1 \in \mathbf{Y}, \forall y_2 \in \mathbf{Y}, |y_2 - y_2| < \delta(\epsilon) \Rightarrow |h(y_1) - h(y_2)| < \epsilon.$$

◦

Definition A.17. A function $h \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m)$ is called *locally Lipschitz* continuous if, for every compact set $\mathbf{K} \subset \mathbb{R}^n$, there exists a constant value $M > 0$, called *Lipschitz constant* such that, for every y_1, $y_2 \in \mathbf{K}$,

$$|h(y_1) - h(y_2)| \leqslant M|y_1 - y_2|.$$

◦

Definition A.18 (Absolute continuity). Let $[a, b] \subset \mathbb{R}$ be a compact set. A function $h : [a, b] \to \mathbb{R}^n$ is called *absolutely* continuous if there exists $g \in \mathcal{L}^1([a, b], \mathbb{R}^n)$ such that, for every $t \in [a, b]$,

$$h(t) = h(a) + \int_a^t g(s)\, ds$$ ◦

Definition A.19 (Local boundedness). [17] A function map $h : \mathbb{R}^n \to \mathbb{R}^m$ is called *locally bounded* if, for all $x \in \mathbb{R}^n$, there exists an open set $\mathbf{O} \subset \mathbb{R}^n$ such that h is bounded on $\mathbf{O}$. ◦

[17] (Mimna and Mimna, 1997, Definition 1).

Theorem A.20. [18] *If* $f \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m)$, *then* f *is locally bounded.* □

[18] (Mimna and Mimna, 1997, Theorem 1).

A.2.1 Set-valued maps

[19] (Goebel, Sanfelice, and Teel, 2012, Definition 5.9)

Definition A.21 (Outer semicontinuity). [19] A set-valued map $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called *outer semicontinuous at* $x \in \mathbb{R}^n$ if, for every sequence of points $\{x_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^n$ convergent to $x \in \mathbb{R}^n$, and any convergent sequence of points $\{y_i\}_{i\in\mathbb{N}} \subset M(\{x_i\}_{i\in\mathbb{N}})$, one has $y \in M(x)$, where $y_i \to y$. The map is *outer semicontinuous* if it is outer semicontinuous for every $x \in \mathbb{R}^n$. Given $\mathbf{S} \subset \mathbb{R}^n$, $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous relative* to $\mathbf{S}$ if the set-valued map from $\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by $M(x)$, for $x \in \mathbf{S}$, and $\emptyset$, for $x \notin \mathbf{S}$, is *outer semicontinuous* at each $x \in \mathbf{S}$. ∘

Note that continuous functions are outer-semicontinuous set-valued maps.

A.3 Initial-Value Problem

A.3.1 Existence and uniqueness

Let $k \geqslant 0$ be a constant integer. Consider the functions $h \in \mathcal{C}^k(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$, and the equation

$$\begin{cases} \dot{y}(t) &= h(y(t), u(t)), \\ y(t_0) &= y_0, \end{cases} \tag{ODE}$$

where, for every $t \in \mathbb{R}$, $y(t) \in \mathbb{R}^n$. The function u is called *an input of* (ODE).

From the fundamental theorem of calculus, for almost every $t \in \mathbb{R}$, and for every fixed function $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$, Equation (ODE) is equivalent to

$$y(t) = y_0 + \int_{t_0}^{t} h(y(s), u(s))\, ds.$$

Definition A.22 (Solution). Let $y_0 \in \mathbb{R}^n$, $\mathbf{I} \subset \mathbb{R}$ with $t_0 \in \mathbf{I}$, and a fixed function $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$. A function $Y : \mathbf{I} \times \{y_0\} \times \{u\} \to \mathbb{R}^n$ is called *solution of* (ODE) *with initial condition* y_0 *and input* u if

1. $Y(t_0) = y_0$;

2. The function $\mathbf{I} \ni t \mapsto Y(t, y_0, u) \in \mathbb{R}^n$ is absolutely continuous;

3. For almost every $t \in \mathbf{I}$,

$$\frac{dY}{dt}(t, y, u) = h(Y(t, y, u)).$$

The set of solutions of (ODE) with initial condition y_0 and input u is denoted by $\mathcal{S}_h(y_0, u)$. From now on, the interval $\mathbf{I}$ is denoted by $\mathtt{dom}(Y)$. When $u \equiv 0$, the solution is denoted by $Y(t, y_0)$ and the set of solutions by $\mathcal{S}_h(y_0)$. ◦

Definition A.23 (Continuation of solution). [20] Let Y and $\bar{Y}$ be solutions of (ODE), $\bar{Y}$ is called *continuation* of Y if $\mathtt{dom}(Y) \subsetneq \mathtt{dom}(\bar{Y})$, for every $t \in \mathtt{dom}(Y)$, $Y(t, y_0, u) = \bar{Y}(t, y_0, u)$, and for almost every $t \in \mathtt{dom}(\bar{Y})$,

$$\frac{d\bar{Y}}{dt}(t, y_0, u) = h(\bar{Y}(t, y_0, u)).$$

A solution Y is called

- *complete* if $\mathtt{dom}(Y)$ is unbounded. If $\sup \mathtt{dom}(Y) = \infty$, then Equation (ODE) is called *forward complete*;
- *maximal* if cannot be continued. ◦

[20] Based on (Hale, 1980, pp. 16)

Theorem A.24 (Existence). [21] *Let $k \geqslant 0$ be a constant integer, and $u \in \mathcal{L}^{\infty}_{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$ be fixed. If $h \in \mathcal{C}^k(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, then for every $y_0 \in \mathbb{R}^n$, there exists at least one solution Y of (ODE) that is locally Lipschitz and can be continued to a maximal interval of existence. Moreover, if Y is maximal, then $Y(t, y_0, u)$ tends to ∞ as $t \to \partial\mathtt{dom}(Y)$.* □

[21] Based on (Hale, 1980, Theorems I.1.1 and I.2.1), (Teschl, 2012, Theorem 2.8) and (Praly, 2011, Théorème 1.7).

Theorem A.25 (Uniqueness). [22] *Let $u \in \mathcal{L}^{\infty}_{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$ be fixed. If $h \in \mathcal{C}^0(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ is locally Lipschitz with respect to y and uniformly with respect to u on compact sets, then for every $y_0 \in \mathbb{R}^n$, there exists a unique solution Y of (ODE).* □

[22] (Hale, 1980, Theorem I.3.1) and (Praly, 2011, Théorème 1.7).

Theorem A.26 (Regularity). [23] *Let $k \geqslant 1$ be a constant integer, and $u \in \mathcal{L}^{\infty}_{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$ be fixed. If $h \in \mathcal{C}^k(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, then for every $y_0 \in \mathbb{R}^n$, solution Y of (ODE) is of class $\mathcal{C}^k(\mathtt{dom}(Y) \times \{y_0\} \times \{u\}, \mathbb{R}^n)$ and the map $\mathbb{R}^n \ni y_0 \mapsto Y(\cdot, y_0, u) \in \mathbb{R}^n$ is a diffeomorphism of class $\mathcal{C}^k$.* □

[23] Based on (Teschl, 2012, Theorem 2.7) and (Hartman, 1982, Corollary 3.1).

A.3.2 Stability

Let $k \geqslant 0$ be a constant integer, consider a fixed function $\phi \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}^m)$, called *feedback law*. When it substitutes u in (ODE), it yields

$$\begin{cases} \dfrac{dy}{dt}(t) &= h(y(t), \phi(y(t))), \\ y(t_0) &= y_0. \end{cases} \qquad (\phi\text{-ODE})$$

From now on, with an abuse of notation, $h(y(\cdot), \phi(y(\cdot)))$ is denoted by $h(y(\cdot))$. Also, assume that (ϕ-ODE) is forward complete.

Definition A.27 (ω-limit and invariant sets)**.** The element $y_\infty \in \mathbb{R}^n$ is called[24] *ω-limit point of* Y if there exists $\{t_i\}_{i\in\mathbb{N}} \subset \mathtt{dom}(Y)$ with $t_i \nearrow \infty$ such that, for every $Y \in \mathcal{S}_h(y_0)$, $Y(t_n, y_0) \to y_\infty$. The set of all ω-limit points of Y is called ω-limit set, and it is denoted by $\omega(y_0)$.

A set $\mathbf{M} \subset \mathbb{R}^n$ is called[25] *positively invariant* with respect to (ϕ-ODE) if, for every $y_0 \in \mathbf{M}$, and for every $Y \in \mathcal{S}_h(y_0)$, $Y(t, y_0) \in \mathbf{M}$, for every $t \geqslant t_0$. ○

[24] (Sastry, 1999, Definition 2.11).

[25] Based on (Khalil, 2001, pp. 127).

Definition A.28 (Equilibrium point)**.** [26] Let $\bar{Y}$ be a solution of (ϕ-ODE) with initial condition $\bar{y} \in \mathbb{R}^n$. The element $\bar{y}$ is called an *equilibrium point of* (ϕ-ODE) if, for almost every $t \in \mathtt{dom}(Y)$, $h(\bar{Y}(t, \bar{y})) = 0$. ○

[26] Based on (Sastry, 1999, Definition 5.2).

Note that, if (ϕ-ODE) have only one equilibrium point in $\mathbf{Y}$, then there exists a coordinate change rendering the origin an equilibrium point. To see this claim, assume that $\bar{y}$ is an equilibrium point, and for every $t \in \mathbb{R}$, consider the coordinate change $c(t) = y(t) - \bar{y}$. Taking its derivative, for almost every $t \in \mathbb{R}$, it yields $\dot{c}(t) = \dot{y}(t)$, and $\dot{c}$ is described by the same vector field as $\dot{y}$. Thus, the properties referring to the origin as an equilibrium point are equivalent to refer to any other single equilibrium point.

Definition A.29. Let $\mathbf{A} \subset \mathbb{R}^n$ be a compact set, assume that it is positively invariant with respect to (ϕ-ODE), it is called

- *Stable*[27] *for* (ϕ-ODE) if, for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that, for every $y_0 \in \mathbb{R}^n$, and for every $Y \in \mathcal{S}_h(y_0)$,

$$|y_0|_{\mathbf{A}} \leqslant \delta(\epsilon) \Rightarrow |Y(t, y_0)|_{\mathbf{A}} \leqslant \epsilon, \quad \forall t \geqslant t_0;$$

- *Unstable for* (ϕ-ODE) if not stable;[28]
- *Locally attractive for* (ϕ-ODE)[29] if, there exists $\delta_0 > 0$ such that, for every $y_0 \in \mathbb{R}^n$, and for every $Y \in \mathcal{S}_h(y_0)$,

$$|y_0|_{\mathbf{A}} \leqslant \delta_0 \Rightarrow \lim_{t\to\infty} |Y(t, y_0)|_{\mathbf{A}} = 0.$$

- *Locally asymptotically stable for* (ϕ-ODE)[30] if it is stable and locally attractive. The adjective locally is replaced by *globally*, when the choice of δ_0 can be taken as large as desired. ○

[27] Based on (Bacciotti and Rosier, 2001, Definition 2.1).

[28] (Bhatia and Szegö, 1967, Definition 1.5)

[29] Based on (Sastry, 1999, Definition 5.6)

[30] Based on (Bacciotti and Rosier, 2001, Definition 2.3).

Example A.30. Note that stability and attractivity are different concepts. The following example is analyzed in details in (Hahn, 1967, Paragraph 40) and illustrates a case where the origin is unstable and attractive.

$$\begin{cases} \dot{x} = \dfrac{x^2(y-x)+y^5}{(x^2+y^2)(1+(x^2+y^2))^2} \\ \dot{y} = \dfrac{y^2(y-2x)}{(x^2+y^2)(1+(x^2+y^2))^2}. \end{cases} \quad (A.2)$$

The phase portrait is presented in Figure A.1. ◇

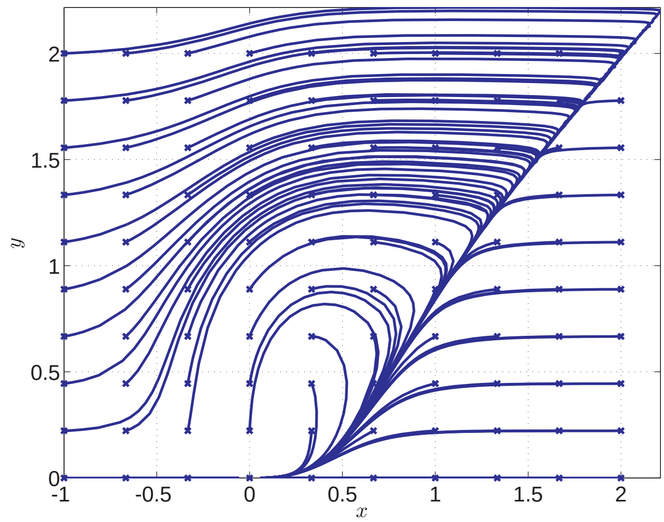


Figure A.1: Phase portrait of system (A.2).

A.4 Differentiation along solutions of an ODE

Definition A.31 (Comparison functions). Let $k \geqslant 0$ be a constant integer,

- A function $\alpha \in \mathcal{C}^k([0,a),\mathbb{R}_{\geqslant 0})$ is called *strictly increasing* if, for every $s_1, s_2 \in [0,a)$ with $s_1 < s_2$, $\alpha(s_1) < \alpha(s_2)$;
- A function $V \in \mathcal{C}^k(\mathbb{R}^n,\mathbb{R}_{\geqslant 0})$, $p \geqslant 1$, is called *locally positive definite with respect to a set* $\mathbf{S} \subset \mathbb{R}^n$ if[31] there exists a constant value $r > 0$ such that, for every $y \in \mathbf{B}_{\leqslant r}(\mathbf{S}) \setminus \{\mathbf{S}\}$, $V(y) > 0$, and $V(y) = 0$ if and only if $y \in \mathbf{S}$. The class of such functions is denoted $\mathcal{P}_{loc}(\mathbb{R}^n,\mathbb{R}_{\geqslant 0})$. It is called *positive definite* if r can be taken as large as desired and $\mathbf{S} = \{0\}$, in this case, the class of functions is denoted by $\mathcal{P}(\mathbb{R}^n,\mathbb{R}_{\geqslant 0})$;
- A function $V \in \mathcal{C}^k(\mathbb{R}^n,\mathbb{R}_{\geqslant 0})$ and $p \geqslant 1$, is called *proper*[32] if, as $|y| \to \infty$, $V(y) \to \infty$;
- A function[33] $\alpha \in (\mathcal{C}^k \cap \mathcal{P})([0,a),\mathbb{R}_{\geqslant 0})$ is called *of class* $\mathcal{K}([0,a),\mathbb{R}_{\geqslant 0})$ if it is strictly increasing. It is denoted by $\mathcal{K}$, if a can be taken as large as desired. It is called of *class* $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and proper;
- A function[34] $\beta \in \mathcal{C}^k(\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0},\mathbb{R}_{\geqslant 0})$ is called *of class* $\mathcal{KL}$ if, for

[31] Based on (Sastry, 1999, Definition 5.12).

[32] This property is also called radially unbounded. This definition is based on (Schwartz, 1970, pp. 111) for a definition in terms of pre-images of compact sets see (Bourbaki, 1966, Proposition I.3.7).

[33] Based on (Khalil, 2001, Definitions 4.2).

[34] Based on (Khalil, 2001, Definitions 4.3).

a fixed $t \in \mathbb{R}_{\geqslant 0}$, the function $s \mapsto \beta(s,t)$ is of class $\mathcal{K}$ and, for a fixed $s \in \mathbb{R}_{\geqslant 0}$, the function $t \mapsto \beta(s,t) \in \mathbb{R}_{\geqslant 0}$ is non-increasing and satisfies $\beta(s,t) \to 0$, as $t \to \infty$. ∘

Figure A.2 illustrates the functions presented in Definition A.31.

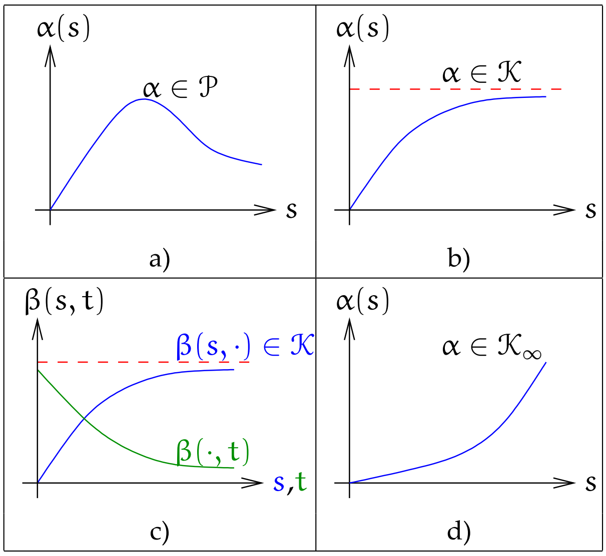


Figure A.2: Illustration of some of the functions described in Definition A.31. a) a positive definite function, b) a function of class $\mathcal{K}$, c) a function of class $\mathcal{KL}$, and d) a function of class $\mathcal{K}_\infty$.

Proposition A.32 (Properties of comparison functions). [35] *Let α_1, $\alpha_2 \in \mathcal{K}$, α_3, $\alpha_4 \in \mathcal{K}_\infty$, and $\beta \in \mathcal{KL}$. Then,*

- $\alpha_1^{-1} \in \mathcal{K}$ *is defined on* $[0, \alpha_1(a)]$;
- $\alpha_3^{-1} \in \mathcal{K}_\infty$;
- $\alpha_1 \circ \alpha_2 \in \mathcal{K}$;
- $\alpha_3 \circ \alpha_4 \in \mathcal{K}_\infty$;
- $\beta(s,t) = \alpha_1(\beta(\alpha_2(s),t)) \in \mathcal{KL}$;
- $\beta(s,t) \leqslant \alpha_1(\alpha_2(s)e^{-t}) \in \mathcal{KL}$.[36] □

[35] (Khalil, 2001, Lemma 4.2)

[36] (Sontag, 1998, Proposition 7)

Claim A.33. *Let $k \geqslant 0$ be a constant integer, the function $V \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ is locally positive definite with respect to the origin if and only if there exist a constant value $r > 0$, and $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}([0,r), \mathbb{R}_{\geqslant 0})$ such that*

$$\forall y \in \mathbf{B}_{\leqslant r}(0), \quad \underline{\alpha}(|y|) \leqslant V(y) \leqslant \overline{\alpha}(|y|). \tag{A.3}$$

Additionally, V is proper if and only if $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$. □

The proof of Claim A.33 is provided in Section A.5.1

Definition A.34 (Lie derivative, Lyapunov function). Let $k \geqslant 1$ be a constant integer. The function $V \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$. It is called *Lyapunov function candidate* (resp. *in the small*) if there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ (resp.

$\underline{\alpha}, \overline{\alpha} \in \mathcal{K}([0,r),\mathbb{R}_{\geqslant 0})$) such that, for every $y \in \mathbb{R}^n$ (resp. for every $y \in \mathbf{B}_{\leqslant r}(0)$),

$$\underline{\alpha}(|y|) \leqslant V(y) \leqslant \overline{\alpha}(|y|). \tag{A.4}$$

The function

$$\begin{array}{rrcl} L_h V: & \mathbb{R}^n & \to & \mathbb{R} \\ & y & \mapsto & \langle \mathtt{grad}\, V(y), h(y) \rangle. \end{array}$$

is called *Lie derivative of* V *in direction of* h. If $-L_h V \in (\mathcal{C}^{k-1} \cap \mathcal{P})(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ (resp. $-L_h V \in (\mathcal{C}^{k-1} \cap \mathcal{P}_{\text{loc}})(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$), then V is called *Lyapunov function* (resp. in the small) for (ϕ-ODE). ∘

From Definition A.34 and Claim A.33, $-L_h V \in (\mathcal{C}^{k-1} \cap \mathcal{P}_{\text{loc}})(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ implies that there exists $\alpha \in \mathcal{K}([0,r),\mathbb{R}_{\geqslant 0})$ such that, for every $y \in \mathbf{B}_{\leqslant r}(0)$, $L_h V(y) \leqslant -\alpha(|y|)$. In (Lin, Sontag, and Wang, 1996, Remark 4.1) it is shown that it is not restrictive to choose the functions $\underline{\alpha}$, $\overline{\alpha}$ given by Claim A.33, and α as being of class $(\mathcal{C}^\infty \cap \mathcal{K})([0,r),\mathbb{R}_{\geqslant 0})$.

Theorem A.35 (Basic Lyapunov theorems). [37] *If*

[37] (Sastry, 1999, Theorem 5.16)

1. $V \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$, *and for every* $y \in \mathbf{Y}$, $L_h V(y) \leqslant 0$, *then the origin is stable for* (ϕ-ODE);
2. $V \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ *and proper, and for every* $y \in \mathbf{Y}$, $L_h V(y) \leqslant 0$, *then the origin is uniformly stable for* (ϕ-ODE);
3. $V \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ *and proper, and* $-L_h V \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$, *then the origin is locally uniformly asymptotically stable for* (ϕ-ODE);
4. $V \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ *and proper, and* $-L_h V \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$, *then the origin is globally uniformly asymptotically stable for* (ϕ-ODE). □

Theorem A.36 (Converse Lyapunov Theorem). [38] *If the origin is globally (resp. locally) asymptotically stable for* (ϕ-ODE), *then there exists a Lyapunov function (resp. in the small).* □

[38] (Bacciotti and Rosier, 2001, Theorem 2.4)

Proposition A.37. [39] *Consider the scalar differential equation*

[39] (Khalil, 2001, Lemma 4.4)

$$\begin{cases} \dot{y} &= -\alpha(y) \\ y(t_0) &= y_0, \end{cases}$$

where α *is a locally Lipschitz function of class* $\mathcal{K}([0,a),\mathbb{R}_{\geqslant 0})$. *For every* $0 \leqslant y_0 < a$, *this equation has a unique solution* Y *defined on* $[t_0,\infty)$. *Moreover, there exists* $\beta \in \mathcal{KL}(([0,a),\mathbb{R}_{\geqslant 0}) \times \mathbb{R}_{\geqslant 0}, \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0})$ *such that, for every* $t \in [t_0,\infty)$, $y(t) \leqslant \beta(y_0, t-t_0)$. □

[40] (Khalil, 2001, Lemma 4.5)

Proposition A.38. [40] *Assume that the origin is an equilibrium point of* (ϕ-ODE). *Then,*

- *It is uniformly stable if and only if there exist* $\alpha \in \mathcal{K}$, *and a constant value* $c > 0$, *independent of* t_0, *such that for every* $|y_0| \leqslant c$,

$$|Y(t, y_0)| \leqslant \alpha(|y_0|), \quad t \geqslant t_0;$$

- *It is locally uniformly asymptotically stable if and only if there exist* $\beta \in \mathcal{KL}$, *and a constant value* $c > 0$ *independent of* t_0 *such that for every* $|y_0| \leqslant c$,

$$|Y(t, y_0)| \leqslant \beta(|y_0|, t - t_0), \quad t \geqslant t_0; \tag{A.5}$$

- *It is globally uniformly asymptotically stable if and only if the constant value* c *can be taken as large as desired, in inequality* (A.5). □

Sometimes, ask for continuously differentiable Lyapunov function candidate may be quite restrictive. Inspired by works such as (Praly, 2011), (Liberzon, Nešić, and Teel, 2013), and (Dashkovskiy, Rüffer, and Wirth, 2010), relaxed notions of derivative and the existence of sufficient conditions ensuring asymptotic stability are recalled.

[41] Based on (Rouche, Habets, and Laloy, 1977, pp. 345) and (McShane, 1947, pp. 188).

Definition A.39 (Dini Derivatives). [41] Consider a function $f : [a, b) \to \mathbb{R}$, the limits at $t \in [a, b)$

$$\begin{aligned} D^+ f(t) &= \limsup_{\tau \searrow 0} \frac{f(t+\tau) - f(t)}{\tau}, \\ D_+ f(t) &= \liminf_{\tau \searrow 0} \frac{f(t+\tau) - f(t)}{\tau}, \\ D^- f(t) &= \limsup_{\tau \nearrow 0} \frac{f(t+\tau) - f(t)}{\tau}, \\ D_- f(t) &= \liminf_{\tau \nearrow 0} \frac{f(t+\tau) - f(t)}{\tau}, \end{aligned}$$

if they exist, then they are called *Dini derivatives*. Let $n > 0$ be a positive value, $y, \nu \in \mathbb{R}^n$, $V : \mathbb{R}^n \to \mathbb{R}$. The limit

$$D^+_\nu V(y) = \limsup_{\tau \searrow 0} \frac{V(y + \tau\nu) - V(y)}{\tau} \tag{A.6}$$

(if it exists) is called *Dini derivative of* V *in the* ν*-direction at* y. The other three Dini derivatives can be analogously defined in the ν-direction. The set

Note that, in (Clarke et al., 1998, pp. 4), the Dini derivative of a several variable function $V : \mathbb{R}^n \to \mathbb{R}$ is defined, for every $y \in \mathbb{R}^n$, by

$$D^+_\nu V(y) = \limsup_{\substack{w \to \nu \\ \tau \searrow 0}} \frac{V(y + \tau w) - V(y)}{\tau}.$$

In the case where V is locally Lipschitz, the above limit reduces to (A.6) (see Clarke, 2013, Exercise 11.19).

$$\mathtt{grad}_D\, V(y) = \{\xi \in \mathbb{R}^n : D_{+,\nu} V(y) \geqslant \xi \cdot \nu, \forall \nu \in \mathbb{R}^n\}$$

is called *Dini subdifferential*[42] *of* V and each ξ is called *Dini subgradient*. ◦

[42] (see Clarke, 2013, Definition 11.42)

Remark A.40. [43] Let $V \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ be locally Lipschitz. Then, for every $y \in \mathbb{R}^n$, $D_v^+V(y)$ exists. Moreover, if V is of class $\mathcal{C}^1$, then, for every $y \in \mathbb{R}^n$,

$$D_v^+V(y) = \mathtt{grad}\, V(y) \cdot v.$$

To see the the first statement. Since V is locally Lipschitz, for every $\tau > 0$, and for every compact set $\mathbf{K} \subset \mathbb{R}^n$, there exists $L > 0$ such that, for every $y, v \in \mathbf{K}$ with $y + \tau v \in \mathbf{K}$,

$$D_v^+V(y) \leqslant \limsup_{\tau \searrow 0} \frac{|V(y+\tau v) - V(y)|}{\tau} \leqslant \limsup_{\tau \searrow 0} \frac{L|\tau v|}{\tau} \leqslant L|v|.$$

For the second statement. Since $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, for every $y, v \in \mathbb{R}^n$, and for every $\tau > 0$, its Taylor first order expansion yields $V(y + \tau v) = V(y) + \mathtt{grad}\, V(y) \cdot \tau v + \vartheta(y, v, \tau)$. Thus, for every $y \in \mathbb{R}^n$,

$$D_v^+V(y) = \limsup_{\tau \searrow 0} \frac{\mathtt{grad}\, V(y) \cdot \tau v + \vartheta(y, v, \tau)}{\tau} = \mathtt{grad}\, V(y) \cdot v. \qquad \circ$$

[43] Based on (Praly, 2011, Proposition 1.26) and (Clarke et al., 1998, pp. 4).

Proposition A.41. [44] *Let $f \in \mathcal{C}^0((a, b), \mathbb{R})$. The function f is increasing in (a, b) if and only if, on the interval (a, b), the four Dini derivatives are larger or equal to zero.* □

[44] See (Rouche, Habets, and Laloy, 1977, Theorems 2.1 and 2.3, and Corollary 2.4).

The following theorem is credited to (Yoshizawa, 1966), and it is presented and proved in (Rouche, Habets, and Laloy, 1977, Theorem 4.3). It states that the Dini derivative with respect to time of a locally Lipschitz function computed along solutions of (ϕ-ODE) is equal to its Dini derivative computed along solutions of (ϕ-ODE) in the direction of the vector field.

Theorem A.42. [45] *Let Y be a solution of (ϕ-ODE), and $V \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ be locally Lipschitz. Then, for every $t \in \mathtt{dom}(Y)$,*

$$D^+V(Y(t, y)) = D_h^+V(Y(t, y)). \qquad \square$$

[45] See also (Praly, 2011, Lemme 1.28).

A consequence[46] of Theorem A.42 and Proposition A.41 is that if, for every $y \in \mathbb{R}^n$, $D_h^+V(y) \leqslant 0$, then V is non-increasing along the solutions of (ϕ-ODE). Moreover, the above statements and consequences remain true for the other three Dini derivatives.

[46] Based on (Rouche, Habets, and Laloy, 1977, Remark 4.4)

Remark A.43. Let the proper function $V \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ be locally Lipschitz. If there exists $\alpha \in \mathcal{K}_\infty$ such that, for every $y \in \mathbb{R}^n$,

$$D_f^+V(y) \leqslant -\alpha(|y|),$$

then V is strictly decreasing along solutions of (ϕ-ODE).

To see this claim, consider a solution Y of (ϕ-ODE) with initial condition $y \in \mathbb{R}^n$. Since $V \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ is locally Lipschitz, for almost every $y \in \mathbb{R}^n$, $D_h^+ V(y)$ exists, and V is absolutely continuous[47]. Moreover, for every $t_1, t_2 \in \mathtt{dom}(Y)$ with $t_1 < t_2$,

[47] Thus, $D_h^+ V$ is 1-integrable.

$$\begin{aligned} V(Y(t_2, y)) - V(Y(t_1, y)) &= \int_{t_1}^{t_2} D^+ V(Y(s, y))\, ds \\ &\leqslant -\int_{t_1}^{t_2} \alpha(|Y(s, y)|)\, ds \\ &\leqslant -\alpha(|Y(t_1, y)|)(t_2 - t_1) \\ &< 0. \end{aligned}$$

Thus, $V(Y(t_2, y)) < V(Y(t_1, y))$.

Note that, for a function $W \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ the condition $D_h^+ W(y) < 0$, for every $y \in \mathbb{R}^n$, is not enough to ensure that $D_h^+ W$ is strictly decreasing. Since $D_h^+ W$ may not exist in a given $\bar{y}$, the condition $D_h^+ W(\bar{y}) < 0$ implies $\sup D_h^+ W(\bar{y}) = 0$, and the above integral inequality no longer holds. ◦

Definition A.44 (Clarke derivative). [48] The *Clarke upper and lower derivatives* of a function $V : \mathbb{R}^n \to \mathbb{R}$ are given, respectively, by

[48] (Clarke, 2013, pp. 194)

$$V_v^\circ(y) = \limsup_{\substack{x \to y \\ \tau \searrow 0}} \frac{V(x + \tau v) - V(x)}{\tau},$$

$$V_{\circ,v}(y) = \liminf_{\substack{x \to y \\ \tau \searrow 0}} \frac{V(x + \tau v) - V(x)}{\tau}.$$

The set [49]

[49] (Clarke, 1983, pp. 27). The generalized gradient is defined as a subset of the space of continuous linear functionals defined on the same space as v and that are upper-bounded by the upper Clarke derivative.

$$\mathtt{grad}_C\, V(y) = \{\xi \in \mathbb{R}^n : V_v^\circ(y) \geqslant \xi \cdot v, \forall v \in \mathbb{R}^n\}$$

is called *Clarke generalized gradient of* V at $y \in \mathbb{R}^n$. ◦

Since the Clarke generalized gradient has the following property[50]

[50] See also (Ceragioli, 2000, pp. 22)

$$\mathtt{grad}_C\, V(y) = \{\xi \in \mathbb{R}^n : V_{\circ,v}(y) \leqslant \xi \cdot v \leqslant V_v^\circ(y)\},$$

the Clarke derivatives can be reconstructed as follows

$$\begin{aligned} V_{\circ,v}(y) &= \inf \{\xi \cdot v : \xi \in \mathtt{grad}_C\, V(y)\}, \\ V_v^\circ(y) &= \sup \{\xi \cdot v : \xi \in \mathtt{grad}_C\, V(y)\}. \end{aligned}$$

Proposition A.45. *The following properties hold for the Clarke generalized gradient*

- *If* $V \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ *with* $k \geqslant 1$ *a constant integer, then* $\mathtt{grad}_C\, V(y) = \{\mathtt{grad}\, V(y)\}$;[51]
- *Let* $x \in \mathbb{R}^n$, $V \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ *be locally Lipschitz. Let* $\aleph$ *be any subset of measure zero in* $\mathbb{R}^n$, *and let* $\aleph_V$ *be the set of points at which* V *fails to be differentiable. Then,*[52]

$$\mathtt{grad}_C\, V(y) = \mathtt{co}\left\{\lim_{i\to\infty} \mathtt{grad}\, V(y_i) : y_i \to y, y_i \notin \aleph \cup \aleph_V\right\}. \quad \square$$

[51] (Clarke, 2013, Theorem 10.8)

[52] (Clarke, 2013, Theorem 10.27)

The result that gives sufficient a condition for the monotonicity of a function $V : \mathbb{R}^n \to \mathbb{R}$ is given by the following

Theorem A.46. [53] *Let the proper function* $V \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ *be locally Lipschitz. If, for every* $y \in \mathbb{R}^n$, *and for every* $\xi \in \mathtt{grad}_C\, V(y)$,

$$\max\{h(y) \cdot \xi\} \leqslant 0,$$

then V *is non-increasing along solutions of* (ϕ-ODE). $\square$

[53] Based on (Clarke, 2013, Theorem 12.17).

Remark A.47. Under the conditions of Theorem A.46. If there exists $\alpha \in \mathcal{K}_\infty$ such that, for every $y \in \mathbb{R}^n$, and for every $\xi \in \mathtt{grad}_C\, V(y)$,

$$\max\{h(y) \cdot \xi\} \leqslant -\alpha(|y|), \tag{A.7}$$

then V is strictly decreasing, along solutions of (ϕ-ODE).

From (Clarke et al., 1998, Proposition 5.3), Equation (A.7) is equivalent to

$$\inf_{x\in\mathbb{R}^n} D_{+,h}V(y) \leqslant -\alpha(|y|).$$

The conclusion follows from Remark A.43. ○

Trade-offs. Note that[54], for every $\tau \in \mathbb{R}_{>0}$,

$$\liminf_{x\to y} \frac{V(x+\tau v) - V(x)}{\tau} \leqslant \frac{V(y+\tau v) - V(y)}{\tau} \leqslant \limsup_{x\to y} \frac{V(x+\tau v) - V(x)}{\tau}.$$

[54] See also (Ceragioli, 2000, pp. 22).

From Definitions A.39 and A.44, and the continuity of V, the last inequalities imply

$$V_{\circ,v}(y) \leqslant D_{+,v}V(y) \leqslant D_v^+V(y) \leqslant V_v^\circ(y).$$

Consequently,

Proposition A.48. [55] *If* $V \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ *is locally Lipschitz, then for every* $y \in \mathbb{R}^n$,

$$\mathtt{grad}_D\, V(y) \subset \mathtt{grad}_C\, V(y). \quad \square$$

[55] Based on (Clarke, 2013, Proposition 11.49).

From Definition A.39, the Dini subdifferential reduces to the derivative, when it exists. This is not necessarily the case for the Clarke generalized gradient, because it is defined on the dual set. On the other hand, the Clarke derivatives can be reconstruct directly from the Clarke generalized gradient while the four Dini derivatives can not be directly reconstruct from the Dini subdifferential.[56]

[56] This discussion was originally made in (Clarke, 2013, pp. 252).

From Proposition A.48, requiring a property to hold with respect to the Clarke generalized gradient is more restrictive than require for it to hold with respect to Dini subdifferential.

A.4.1 The system with inputs

Recall system (ODE) defined by

$$\begin{cases} \dot{y}(t) &= h(y(t), u(t)), \\ y(t_0) &= y_0. \end{cases} \tag{ODE}$$

From now on assume that (ODE) is forward complete and that, for every $t \in \mathbb{R}_{>0}$, $u(t) \in \mathbf{K}_m$, where $\mathbf{K}_m \subset \mathbb{R}^m$ is a compact set

The objective of this section is to recall the existing results that extends the stability analysis of (ODE) to systems with inputs $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}, \mathbf{K}_m)$ in a given compact set. The concepts of stability and Lyapunov functions[57] are slightly different.

[57] See, respectively, Definitions A.29 and A.34.

Definition A.49 (Invariant set, stability). [58] A closed set $\mathbf{M} \subset \mathbb{R}^n$ is called an *invariant set with respect to* (ODE) if

[58] Based on (Lin, Sontag, and Wang, 1996, Definition 2.2).

$$\forall y_0 \in \mathbf{M}, \forall u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geqslant 0}, \mathbf{K}_m), \forall t \geqslant 0, \quad Y(t, y_0, u) \in \mathbf{M}.$$

System (ODE) is uniformly globally asymptotically stable (UGAS) with respect to $\mathbf{M}$ if it is

- Uniformly stable: there exists $\delta \in \mathcal{K}_\infty$ such that, for every $\varepsilon > 0$, and for every $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geqslant 0}, \mathbf{K}_m)$, and for every $t \geqslant 0$,

$$|y_0|_{\mathbf{M}} \leqslant \delta(\varepsilon) \Rightarrow |Y(t, y_0, u)| \leqslant \varepsilon;$$

- Uniform attraction: for any $r, \varepsilon > 0$, there exists $T > 0$ such that, for every $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geqslant 0}, \mathbf{K}_m)$, and for every $t \geqslant T$,

$$|y_0| \leqslant r \Rightarrow |Y(t, y_0, u)| \leqslant \varepsilon. \qquad \circ$$

Analogously to Proposition A.38, the UGAS of (ODE) is also characterized in terms of comparison and Lyapunov functions. Namely,

Proposition A.50. [59] *System (ODE) is UGAS with respect to a closed invariant set* $\mathbf{M} \subset \mathbb{R}^n$ *if and only if there exists* $\beta \in \mathcal{KL}$ *such that, for every* $y_0 \in \mathbb{R}^n$*, for every* $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}, \mathbf{K}_m)$*, and for every* $t \in \mathbb{R}_{\geqslant 0}$*,*

$$|Y(t, y_0, u)|_{\mathbf{M}} \leqslant \beta(|y_0|, t). \tag{A.8}$$

□

[59] (Lin, Sontag, and Wang, 1996, Proposotion 2.5).

Definition A.51. [60] Let $V \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$ be a Lyapunov function candidate for (ODE) with respect to a nonempty, closed invariant set $\mathbf{M} \subset \mathbb{R}^n$ that is smooth on $\mathbb{R}^n \setminus \mathbf{M}$. It is called smooth *Lyapunov function for* (ODE) *with respect to* $\mathbf{M}$ if there exists $\alpha \in \mathcal{K}_\infty$ such that, for every $y \in \mathbb{R}^n \setminus \mathbf{M}$, and for every $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}, \mathbf{K}_m)$,

$$L_h V(y, u) \leqslant -\alpha\left(|y|_{\mathbf{M}}\right).$$

∘

[60] Based on (Lin, Sontag, and Wang, 1996, Definition 2.6).

Theorem A.52. [61] *Let* $\mathbf{M} \subset \mathbb{R}^n$ *be a nonempty compact invariant set with respect to (ODE). Then, system (ODE) is UGAS with respect to* $\mathbf{M}$ *if and only if there exists a smooth Lyapunov function* V *for (ODE) with respect to* $\mathbf{M}$. □

[61] (Lin, Sontag, and Wang, 1996, Theorem 2.9).

A.5 Proof of Chapter A

A.5.1 Proof of Claim A.33

This proof is based on the proof of (Khalil, 2001, Lemma 4.3).

- The upper bound. Let, for every $s \in [0, r]$,

$$\phi(s) = \sup_{|y| \leqslant s} V(y).$$

  Since $V \in (\mathcal{C}^k \cap \mathcal{P}_{\text{loc}})(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$, $\phi \in (\mathcal{C}^k \cap \mathcal{P}_{\text{loc}})(\mathbb{R}_{\geqslant 0}, \mathbb{R}_{\geqslant 0})$ and it is non-decreasing. Let $\bar{c} > 1$ be a constant value, and $\overline{\alpha} \in \mathcal{K}([0, r), \mathbb{R}_{\geqslant 0})$ be such that, for every $s \in [0, r]$, $\bar{c}\phi(s) \leqslant \overline{\alpha}(s)$. Then, for every $y \in \mathbf{B}_{\leqslant r}(0)$,

$$V(y) \leqslant \phi(|y|) \leqslant \overline{\alpha}(|y|);$$

- The lower bound. Let, for every $s \in [0, r]$,

$$\psi(s) = \inf_{s \leqslant |y| \leqslant r} V(y).$$

  Since, for every $y \in \mathbf{B}_{\leqslant r}(0) \setminus \{0\}$, $V(y) > 0$, it follows that, for every $s \in (0, r]$, $\psi(s) > 0$. Moreover, the function ψ is non-decreasing.

From the continuity of V, $\psi \in \mathcal{C}^0(\mathbb{R}_{\geqslant 0}, \mathbb{R}_{\geqslant 0})$. Let $\underline{c} < 1$ be a strictly positive constant value, and $\underline{\alpha} \in \mathcal{K}([0, r), \mathbb{R}_{\geqslant 0})$ be such that, for every $s \in [0, r]$, $\underline{\alpha}(s) \leqslant \underline{c}\psi(s)$. Then, for every $y \in \mathbf{B}_{\leqslant r}(0)$,

$$\underline{\alpha}(|y|) \leqslant \psi(|y|) \leqslant V(y).$$

If $V \in (\mathcal{C}^k \cap \mathcal{P})(\mathbb{R}^n, \mathbb{R}_{\geqslant 0})$, then the above reasoning is repeated with $r = \infty$. Additionally, when V is proper, $\phi(s) \to \infty$ (resp. $\psi(s) \to \infty$), as $s \to \infty$. In such a case, it is sufficient for $\underline{\alpha}$ and $\overline{\alpha}$ to be taken from $\mathcal{K}_\infty$. To see see this claim for the upper bound, note that from Definition A.31, $V(x) \to \infty$, as $|x| \to \infty$. Thus, it must hold $\overline{\alpha}(|x|) \to \infty$, $V(x) \to \infty$, as $|x| \to \infty$, in order to have V upper bounded by $\overline{\alpha}$.

Concerning the lower bound, V is proper if and only[62] if $\underline{\alpha} \in \mathcal{K}_\infty$. To see the sufficiency, note that for every constant value $c > 0$, $x \in \Omega_{\leqslant c}(V)$ implies $|x| \leqslant \underline{\alpha}^{-1}(V(x)) \leqslant \underline{\alpha}^{-1}(c)$. Consequently, $\Omega_{\leqslant c}(V)$ is bounded[63]. Since it is also closed by definition, and $\Omega_{\leqslant c}(V) \subset \mathbb{R}^n$, this set is compact. Thus V is proper. To see the necessity, assume that $\Omega_{\leqslant c}(V)$ is compact, and let for every $c \in \mathbb{R}_{>0}$, $\rho(c) = \max\{|x| : x \in \Omega_{\leqslant c}(V)\}$. Since this function is positive definite, and from the continuity of V, it is strictly increasing and proper. Thus it is of class $\mathcal{K}_\infty$. Now pick $x \in \mathbb{R}^n$ such that $V(x)$, note that $|x| \leqslant \max\{|x| : x \in \Omega_{\leqslant c}(V)\}$. Then, $\underline{\alpha}(|x|) \leqslant \underline{\alpha}(\max\{|x| : x \in \Omega_{\leqslant c}(V)\}) = \underline{\alpha}(\rho(c)) = V(x)$. This concludes the proof.

[62] This discussion is made in (Isidori, 1999, Remark 10.1.3).

[63] Note that, $\Omega_{\leqslant c}(V)$ is pre-image of $[0, c]$ by the map V.

BIBLIOGRAPHY

Alberti, G., S. Bianchini, and G. Crippa (2013). "Structure of level sets and Sard-type properties of Lipschitz maps". In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 12.4, pp. 863–902.

Andrieu, V., L. Praly, and A. Astolfi (2008). "Homogeneous approximation, recursive observer design and output feedback". In: *SIAM J. Control and Optimization* 47.4, pp. 1814–1859.

Andrieu, V. and C. Prieur (2010). "Uniting two control Lyapunov functions for affine systems". In: *IEEE Trans. Autom. Control* 55.8, pp. 1923–1927.

Andrieu, V. and S. Tarbouriech (2013). "Global asymptotic stabilization of some bilinear systems by hybrid state and output feedback". In: *IEEE Trans. Autom. Control* 58.6, pp. 1602–1608.

Angeli, D. (2004). "An Almost Global Notion of Input-to-State Stability". In: *IEEE Trans. Autom. Control* 49.6, pp. 866–874.

Angeli, D. and A. Astolfi (2007). "A tight small-gain theorem for not necessarily ISS systems". In: *Syst. & Contr. Lett.* 56.1, pp. 87–91.

Astolfi, A. and L. Praly (2012). "A weak version of the small-gain theorem". In: *IEEE 51st Conference on Decision and Control (CDC), 2012*, pp. 4586–4590.

Bacciotti, Andrea and Lionel Rosier (2001). *Liapunov Functions and Stability in Control Theory*. 2nd. Communications and Control Engineering. Springer.

Benachour, S. et al. (2013). "Forwarding design with prescribed local behavior". In: *IEEE Trans. Autom. Control* 58.12, pp. 3011–3023.

Bhatia, N. P. and G. P. Szegö (1967). *Dynamical Systems: Stability Theory and Applications*. Ed. by A. Dold and B. Eckmann. Lecture Notes in Mathematics 35. Springer.

Bourbaki, N. (1966). *Elements of Mathematics General Topology Part 1*. Addison-Wesley.

Ceragioli, F. M. (2000). "Discontinuous ordinary differential equations and stabilization". PhD thesis. Università degli studi di Firenze.

Chaillet, A. (2006). "On stability and robustness of nonlinear systems: applications to cascaed systems". PhD thesis. Université Paris XI Orsay.

Chaillet, A., D. Angeli, and H. Ito (2012). "Strong iISS: combination of iISS and ISS with respect to small inputs". In: *Proceedings of the 51st IEEE Conference on Decision and Control*, pp. 2256–2261.

Chaillet, A. and A. Loría (2008). "Uniform semiglobal practical asymptotic stability for non-autonomous cascaded systems and applications". In: *Automatica* 44, pp. 337–347.

Clarke, F. H. (1983). *Optimization and Nonsmooth Analysis*. Interscience.

– (2013). *Functional Analysis, Calculus of Variations and Optimal Control*. Vol. 264. Graduate Texts in Mathematics. Springer.

Clarke, F. H. et al. (1998). *Nonsmooth Analysis and Control Theory*. Graduate Texts in Mathematics. Springer. ISBN: 978-0-387-98336-3.

Dashkovskiy, S., B. Rüffer, and F. Wirth (2010). "Small gain theorems for large scale systems and construction of ISS Lyapunov functions". In: *SIAM J. Control Optim.* 48.6, pp. 4089–4118.

Dashkovskiy, S. and B. S. Rüffer (2010). "Local ISS of large-scale interconnections and estimates for stability regions". In: *Syst. & Contr. Lett.* 59, pp. 241–247.

DiBenedetto, E. (2002). *Real analysis*. Advanced Texts: Basler Lehrbücher. Boston, MA: Birkhäuser.

Evans, L. C. (1998). *Partial differential equations*. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society.

Evans, L. C. and R. F. Gariepy (1992). *Measure Theory and Fine Properties of Functions*. CRC Press.

Federer, H. (1945). "The Gauss-Green Theorem". In: *Trans. Amer. Math. Soc.* 58, pp. 44–76.

Freeman, R. A. and P. V. Kokotović (2008). *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. Birkhäuser.

Goebel, R., C. Prieur, and A. R. Teel (2009). "Smooth patchy control Lyapunov functions". In: *Automatica* 45.3, pp. 675–683.

Goebel, R., R. G. Sanfelice, and A. R. Teel (2012). *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press.

Goebel, R., R. G. Sanfelice, and A. W. Teel (2009). "Hybrid Dynamical Systems". In: *IEEE Contr. Syst. Mag.* 29.2, pp. 28–93.

Goebel, R. and A. R. Teel (2006). "Solutions to hybrid inclusions via set and graphical convergence with stability theory applications". In: *Automatica* 42, pp. 573–587.

– (2010). "Pre-asymptotic stability and homogeneous approximation of hybrid dynamical systems". In: *SIAM Review* 52.1, pp. 87–109.

Grognard, F., R. Sepulchre, and G. Bastin (1999). "Global stabilization of feedforward systems with exponentially unstable Jacobian linearization". In: *Syst. & Contr. Lett.* 37.2, pp. 107–115.

Hahn, W. (1967). *Stability of Motion*. Springer.

Hale, J. K. (1980). *Ordinary Differential Equations*. 2nd ed. Wiley.

Hartman, P. (1982). *Ordinary Differential Equations*. 2nd. SIAM.

Hespanha, J., D. Liberzon, and A. S. Morse (2004). "Hysteresis-based switching algorithms for supervisory control of uncertain systems". In: *Automatica* 39.2, pp. 263–272.

Hespanha, J. P. and A. S. Morse (1999). "Stabilization of Nonholonomic Integrators via Logic-Based Switching". In: *Automatica,* Special Issue on Hybrid Systems 35.3, pp. 385–393.

Hunter, J. K. (2011). "Measure Theory". University of California at Davis.

– (2013). "Notes on Partial Differential Equations". University of California at Davis.

Isidori, A. (1999). *Nonlinear Control Systems II*. Communications and Control Engineering. Springer.

Ito, H. (2006). "State-Dependent Scaling Problems and Stability of Interconnected iISS and ISS Systems". In: *IEEE Trans. Autom. Control* 51.10, pp. 1626–1643.

Ito, H. and Z.-P. Jiang (2009). "Necessary and Sufficient Small Gain Conditions for Integral Input-to-State Stable Systems: A Lyapunov Perspective". In: *IEEE Trans. Autom. Control* 54.10, pp. 2389–2404.

Jbilou, K., A. Messaoudi, and K. Tabaâ (2004). "Some Schur complement identities and applications to matrix extrapolation methods". In: *Linear Algebra and its Applications* 392, pp. 195–210.

Jiang, Z.-P., I. M. Y. Mareels, and Y. Wang (1996). "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems". In: *Automatica* 32.8, pp. 1211–1215.

Jiang, Z.-P., A. R. Teel, and L. Praly (1994). "Small-gain theorem for ISS systems and applications". In: *Mathematics of Control, Signals and Systems* 7, pp. 95–120.

Khalil, H. K. (2001). *Nonlinear Systems*. 3rd. Prentice Hall.

Kokotović, P. V. (1992). "The Joy of Feedback: Nonlinear and Adaptative". In: *IEEE Trans. Autom. Control* 12.3, pp. 7–17.

Krstić, M., I. Kanellakopoulos, and P. V. Kokotović (1995). *Nonlinear and Adaptive Control Design*. Wiley.

Kurtz, D. S. and C. W. Swartz (2004). *Theories of Integration: The integrals of Riemann, Lebesgue, Henstock-Kurzweil, and McShane*. Ed. by Series in Real Analysis. World Scientific Publishing Company.

Leine, R. I. (2010). "The historical development of classical stability concepts: Lagrange, Poisson and Lyapunov stability". In: *Nonlinear Dyn.* 59, pp. 173–182.

Liberzon, D., D. Nešić, and A. R. Teel (2013). "Lyapunov-based small-gain theorems for hybrid systems". In: *To appear in IEEE Trans. Autom. Control*.

Lin, Y., E. D. Sontag, and Y. Wang (1996). "A smooth converse Lyapunov theorem for robust stability". In: *SIAM J. Control Optim.* 34.1, pp. 124–160.

Luenberger, D. G. (1979). *Introduction to Dynamic Systems - Theory, Models, and Applications*. John Wiley & Sons.

Lyapunov, A. M. (1907). "Problème général de la stabilité du mouvement". In: *Annales de la faculté des sciences de Toulouse* 9.2e série, pp. 203–474.

Marzocchi, A. (2005). "Singular Stresses and Nonsmooth Boundaries in Continuum Mechanics". In: *XXX Ravello Summer School*.

Mayhew, C. G., R. G. Sanfelice, and A. R. Teel (2011). "Further results on synergistic Lyapunov functions and hybrid feedback design through backstepping". In: *Joint Conference on Decision and Control and European Control Conference (CDC-ECC)*. Orlando, Florida, pp. 7428–7433.

Mazenc, F. and L. Praly (1996). "Adding integrations, saturated controls, and stabilization for feedforward systems". In: *IEEE Trans. Autom. Control* 41.11, pp. 1559–1578.

McShane, E. J. (1947). *Integration*. Princeton University Press.

Mimna, R. A. and R. A. Mimna (1997). "Locally Bounded Functions". In: *Real Analysis Exchange* 23.1, pp. 251–251.

Pan, Z. et al. (2001). "Backstepping design with local optimality matching". In: *IEEE Trans. Autom. Control* 46.7, pp. 1014–1027.

Parks, P. C. (1992). "A. M. Lyapunov's stability theory - 100 years on". In: *IMA Journal of Mathematical Control & Information* 9, pp. 275–303.

Pfeffer, W. F. (2012). *The Divergence Theorem and Sets of Finite Perimeter*. CRC Press.

Popper, K. (2002). *The Logic of Scientific Discover*. Ed. by 2nd. Routledge Classics. Routledge.

Praly, L. (2011). "Fonctions de Lyapunov, Stabilité, Stabilisation et Atténuation de Perturbations". École Nationale Supérieure des Mines de Paris.

Prieur, C. (2001). "Uniting local and global controllers with robustness to vanishing noise". In: *Mathematics of Control, Signals, and Systems* 14, pp. 143–172.

Prieur, C., R. Goebel, and A. R. Teel (2007). "Hybrid Feedback Control and Robust Stabilization of Nonlinear Systems". In: *IEEE Trans. Autom. Control* 52.11, pp. 2103–2117.

Rantzer, A. (2001). "A dual Lyapunov's stability theorem". In: *Syst. & Contr. Lett.* 42, pp. 161–168.

Rouche, N., P. Habets, and M. Laloy (1977). *Stability theory by Liapunov's direct methods*. Vol. 22. Applied mathematical sciences. Springer.

Rudin, W. (1976). *Principles of Mathematical Analysis*. McGraw-Hill.

Salsa, S. (2008). *Partial Differential Equations in Action: from Modelling to Theory*. Springer.

Sanfelice, R. G. (2014). "Input-Output-to-State Stability Tools for Hybrid Systems and Their Interconnections". In: *IEEE Trans. Autom. Control* 59.5, pp. 1360–1366.

Sastry, S. (1999). *Nonlinear Systems*. Vol. 10. Interdisciplinary Applied Mathematics. Springer.

Schwartz, L. (1970). *Analyse Topologie générale et analyse fonctionelle*. Vol. 2. Hermann.

Sontag, E. D. (1989). "Smooth stabilization implies coprime factorization". In: *IEEE Trans. Autom. Control* 34.4, pp. 435–443.

– (1998). "Comments on integral variants of ISS". In: *Syst. & Contr. Lett.* 34.25, pp. 93–100.

– (2001). "The ISS philosophy as a unifying framework for stability-like behavior". In: *Nonlinear control in the year 2000*. Vol. 259. Springer London, pp. 443–467.

– (2008). "Input to State Stability: Basic Concepts and Results". In: *Nonlinear and Optimal Control Theory*. Vol. 1932. Lecture Notes in Mathematics. Springer, pp. 163–220.

Sontag, E. D. and Y. Wang (1995). "On characterizations of the input-to-state stability property". In: *Syst. & Contr. Lett.* 24.5, pp. 351–359.

Spiegel, M. R. (1959). *Vector analysis and an introduction to tensor analysis*. McGraw-Hill.

Stein Shiromoto, H., V. Andrieu, and C. Prieur (2011). "Combining a backstepping controller with a local stabilizer". In: *Proceedings of the American Control Conference (ACC)*. San Francisco, CA, United States of America, pp. 3197–3202.

– (2012). "Union d'une commande par backstepping avec une commande locale". In: *Conférence Internationale Francophone d'Automatique (CIFA)*. Grenoble, France.

– (2013a). "Interconnecting a System Having a Single Input-to-State Gain With a System Having a Region-Dependent Input-to-State Gain". In: *Proceedings of the 52nd IEEE Conference on Decision and Control (CDC)*. Florence, Italy, pp. 624–629.

– (2013b). "Relaxed and hybridized backstepping". In: *IEEE Trans. Autom. Control* 58.12, pp. 3236–3241.

– (2013c). "Stabilization of interconnected systems through local stabilization of sets". In: *Submitted to Automatica*.

Teschl, G. (2012). *Ordinary differential equations and dynamical systems*. Vol. 1XX. American Mathematical Society.

Vladimirov, V. S. (2002). *Methods of the theory of generalized functions*. Taylor & Francis.

Yoshizawa, T. (1966). *Stability Theory by Liapunov's Second Method*. The Mathematical Society of Japan.

Zames, G. (1966). "On the input-output stability of time-varying nonlinear feedback systems Part I: Conditions derived using concepts of loop gain, conicity, and positivity". In: *IEEE Trans. Autom. Control* 11.2, pp. 228–238.

Zhang, F. (2005). *The Schur complement and its applications*. Springer.