Tensor products in p-adic Hodge theory

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Résumé et introduction

Contenu. Ce texte est consacré à l'étude de produits tensoriels cristallins (ou semi-stables, ou de de Rham, ou de Hodge-Tate) de représentations p-adiques de $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$, où $K/\mathbf{Q}_p$ est une extension finie, ainsi que de produits tensoriels triangulins de représentations p-adiques de G_K. On étudie également la situation où l'image d'une représentation p-adique par un foncteur de Schur (tel, par exemple, Sym^n ou $\bigwedge^n$) est cristalline (ou semi-stable, ou de de Rham, ou de Hodge-Tate). Les résultats présentés dans cette thèse sont des énoncés pour les B-paires; ils s'appliquent en particulier aux représentations p-adiques de G_K.

Cette introduction résume les questions que nous avons abordées, ainsi que leurs réponses et quelques éléments de demonstration. Dans la partie I de cette thèse, nous rappelons quelques éléments fondamentaux de la théorie de Hodge p-adique et des foncteurs de Schur. La partie II est consacrée à l'étude de produits tensoriels et objets de Schur cristallins (ou semi-stables, ou de de Rham, ou de Hodge-Tate), tandis que la partie III est consacrée à l'étude de produits tensoriels triangulins de représentations p-adiques de G_K.

Mots-clefs : Représentation p-adique d'un groupe de Galois absolu d'un corps p-adique, B-paire, théorie de Hodge p-adique, représentation cristalline, semi-stable, de de Rham, de Hodge-Tate, représentation trianguline, produit tensoriel, foncteur de Schur.

Produits tensoriels et objets de Schur admissibles en théorie de Hodge p-adique. Soient $\overline{\mathbf{Q}}_p$ une clôture algébrique de $\mathbf{Q}_p$, $K/\mathbf{Q}_p$ et $E/\mathbf{Q}_p$ des extensions finies contenues dans $\overline{\mathbf{Q}}_p$, et $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ le groupe de Galois absolu de K. On note $\mathrm{Rep}_E(G_K)$ la catégorie des représentations E-linéaires continues de G_K. En utilisant les anneaux $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$, et $\mathbf{B}_{\mathrm{dR}}$ (voir [**Fon94b**]), Fontaine a défini la notion de représentation E-linéaire de G_K cristalline, semi-stable et de de Rham, et il a montré que les sous-catégories correspondantes de $\mathrm{Rep}_E(G_K)$ sont stables par sous-quotient, somme directe et produit tensoriel. L'un des buts de la partie II de cette thèse est de répondre à la question suivante : si V et V' sont des représentations E-linéaires non nulles de G_K dont le produit tensoriel $V \otimes_E V'$ est cristalline (ou semi-stable, ou de Rham, ou de Hodge-Tate), alors que peut-on dire de V et V' ? On répond aussi à la question suivante :

si $\mathscr{F} : \mathrm{Rep}_E(G_K) \to \mathrm{Rep}_E(G_K)$ est un foncteur de Schur (comme, par exemple, Λ^n ou Sym^n) et si $\mathscr{F}(V)$ est cristalline (ou semi-stable, ou de de Rham, ou de Hodge-Tate), alors que peut-on dire de V (sous certaines hypothèses sur $\dim_E V$) ?

Les résultats obtenus dans cette thèse portent sur les B-paires, ce qui permet d'en déduire des résultats pour les représentations galoisiennes p-adiques. On explique maintenant plus en détail les énoncés obtenus. On note $\mathbf{B}_{\mathrm{e}}$ l'anneau $\mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. Berger a défini dans [**Ber08**] la catégorie des B-paires. Une $B_{|K}^{\otimes E}$-paire est une paire $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^+)$, où W_{e} est une $\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} E$-représentation de G_K (c'est-à-dire un $\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} E$-module libre de rang fini muni d'une action $\mathbf{B}_{\mathrm{e}}$-semi-linéaire et E-linéaire de G_K) et W_{dR}^+ est un $\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} E$-réseau de $W_{\mathrm{dR}} = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} E) \otimes_{(\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} E)} W_{\mathrm{e}}$ stable par l'action de G_K. Si $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^+)$ est une $B_{|K}^{\otimes E}$-paire, alors le rang de W est défini par $\mathrm{rang}(W) := \mathrm{rang}_{(\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} E)} W_{\mathrm{e}}$. Par exemple, si V est une représentation E-linéaire de G_K, alors $W(V) = ((\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} E) \otimes_E V, (\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} E) \otimes_E V)$ est une $B_{|K}^{\otimes E}$-paire de rang $d = \dim_E V$, et la catégorie $\mathrm{Rep}_E(G_K)$ s'identifie par le foncteur $W(-)$ à une sous-catégorie tensorielle de la catégorie des $B_{|K}^{\otimes E}$-paires. Les notions d'objets cristallins, semi-stables, de de Rham, et de Hodge-Tate s'étendent aux $B_{|K}^{\otimes E}$-paires de telle façon qu'une représentation E-linéaire V de G_K est cristalline (ou semi-stable, ou de de Rham, ou de Hodge-Tate) si et seulement si la $B_{|K}^{\otimes E}$-paire $W(V)$ l'est.

La théorie de Sen des représentations $\mathbf{C}_p$-semi-linéaires de G_K (voir [**Sen80**]) et la théorie de Fontaine des representations $\mathbf{B}_{\mathrm{dR}}$-semi-linéaires de G_K (voir [**Fon04**]) nous permettent de montrer le résultat suivant (c'est le théorème 5.1.0.15 de la partie II).

Théorème A. *Soient W et W' des $B_{|K}^{\otimes E}$-paires non nulles. Si la $B_{|K}^{\otimes E}$-paire $W \otimes W'$ est de Hodge-Tate, alors il existe une extension finie F/E et un caractère $\mu : G_K \to F^\times$ tels que les $B_{|K}^{\otimes F}$-paires $W(\mu^{-1})$ et $W'(\mu)$ soient de Hodge-Tate. De plus, si $W \otimes W'$ est de de Rham, alors $W(\mu^{-1})$ et $W'(\mu)$ le sont.*

Il est connu que tout $B_{|K}^{\otimes E}$-paire de de Rham est potentiellement semi-stable, grâce aux résultats de [**And02**], [**Ber02**], [**Ked00**], et [**Meb02**]. Les propriétés des $(\varphi, N, \mathrm{Gal}(L/K))$-modules (ici, L/K est une extension finie galoisienne) nous permettent comprendre la situation où W et W' sont toutes les deux potentiellement semi-stables : c'est le théorème 6.2.0.21 de la partie II.

Théorème B. *Soient W et W' des $B_{|K}^{\otimes E}$-paires non nulles potentiellement semi-stables. Si la $B_{|K}^{\otimes E}$-paire $W \otimes W'$ est semi-stable, alors il existe une extension finie F/E et un caractère $\mu : G_K \to F^\times$ tels que les $B_{|K}^{\otimes F}$-paires $W(\mu^{-1})$ et $W'(\mu)$ sont semi-stables. De plus, si $W \otimes W'$ est cristalline, alors $W(\mu^{-1})$ et $W'(\mu)$ le sont.*

En particulier, les deux théorèmes ci-dessus s'appliquent lorsque W et W' sont des $B_{|K}^{\otimes E}$-paires de la forme $W = W(V)$ et $W' = W(V')$ avec $V, V' \in \mathrm{Rep}_E(G_K)$ (voir les corollaires 5.1.0.16 et 6.2.0.22 de la partie II).

Corollaire. *Soient V et V' des représentations E-linéaires non nulles de G_K. Si $V \otimes_E V'$ est cristalline (resp. semi-stable, ou de de Rham, ou de Hodge-Tate), alors il existe une extension finie F/E et un caractère $\mu : G_K \to F^\times$ tels que $V(\mu^{-1})$ et $V'(\mu)$ soient cristallines (resp. semi-stables, ou de de Rham, ou de Hodge-Tate).*

Les méthodes utilisées pour démontrer les théorèmes A et B peuvent être utilisées pour comprendre la situation où l'image par un foncteur de Schur d'une B-paire est cristalline (ou semi-stable, ou de de Rham, ou de Hodge-Tate), ce que l'on explique dans la suite.

Si $n \in \mathbf{N}_{>0}$ et si $n = u_1 + \ldots + u_r$ est une partition de n en entiers avec $u_1 \geq \ldots \geq u_r \geq 1$, alors le r-uplet $u = (u_1, \ldots, u_r)$ définit un foncteur de Schur, noté $\mathrm{Schur}^u(-)$, qui envoie une $B_{|K}^{\otimes E}$-paire W sur une $B_{|K}^{\otimes E}$-paire $\mathrm{Schur}^u(W)$. Si $r = 1$ ou si $u_1 = u_2 = \ldots = u_r$, alors on pose $r(u) = r + 1$; sinon, on pose $r(u) = r$. Par exemple, si $u = (n)$, alors $r(u) = 2$ et le foncteur de Schur associé à u est $\mathrm{Sym}^n(-)$; si $u = (1, \ldots, 1)$, alors $r(u) = n + 1$ et le foncteur de Schur associé à u est $\Lambda^n(-)$.

Les deux théorèmes suivants correspondent respectivement aux théorèmes 5.2.0.18 et 6.3.0.24 de la partie II.

Théorème C. *Soit W une $B_{|K}^{\otimes E}$-paire telle que* $\mathrm{rang}(W) \geq r(u)$. *Si la $B_{|K}^{\otimes E}$-paire* $\mathrm{Schur}^u(W)$ *est de Hodge-Tate, alors il existe une extension finie F/E et un caractère $\mu : G_K \to F^\times$ tels que la $B_{|K}^{\otimes F}$-paire $W(\mu^{-1})$ soit de Hodge-Tate. Si, de plus,* $\mathrm{Schur}^u(W)$ *est de de Rham, alors $W(\mu^{-1})$ l'est.*

Théorème D. *Soit W une $B_{|K}^{\otimes E}$-paire potentiellement semi-stable avec* $\mathrm{rang}(W) \geq r(u)$. *Si la $B_{|K}^{\otimes E}$-paire* $\mathrm{Schur}^u(W)$ *est semi-stable, alors il existe une extension finie F/E et un caractère $\mu : G_K \to F^\times$ tels que la $B_{|K}^{\otimes F}$-paire $W(\mu^{-1})$ soit semi-stable. Si, de plus,* $\mathrm{Schur}^u(W)$ *est cristalline, alors $W(\mu^{-1})$ l'est.*

Les théorèmes C et D impliquent à leur tour des résultats analogues pour les représentations p-adiques de G_K (voir les corollaires 5.2.0.19 et 6.3.0.25 de la partie II). On montre après le corollaire 5.2.0.19 que la borne sur $\mathrm{rang}(X)$ donnée dans les théorèmes C et D est optimale.

Corollaire. *Soit V une représentation E-linéaire de G_K avec* $\dim_E(V) \geq r(u)$. *Si* $\mathrm{Schur}^u(V)$ *est cristalline (resp. semi-stable, ou de de Rham, ou de Hodge-Tate), alors* *il existe une extension finie F/E et un caractère $\mu : G_K \to F^\times$ tels que $V(\mu^{-1})$ soit cristalline (resp. semi-stable, ou de de Rham, ou de Hodge-Tate).*

Dans son étude de la compatibilité de Langlands locale pour les formes modulaires de Hilbert, Skinner a montré que si V est une représentation p-adique telle que $\mathrm{Sym}^2(V)$ soit cristalline, alors les méthodes de Wintenberger présentées dans [**Win95**] et [**Win97**] peuvent être appliquées pour montrer l'existence d'un caractère quadratique μ tel que $V(\mu)$ soit cristalline (voir §2.4.1 de [**Ski09**]). Récemment, Chenevier et Harris ont utilisé des arguments analogues pour le foncteur $\Lambda^2(-)$ dans la démonstration de la partie (b) du théorème 3.2.3 de [**CheHar13**]. On s'attend à ce que les méthodes de Wintenberger puissent être utilisées de la même manière pour redémontrer les théorèmes A, B, C, et D.

Produits tensoriels triangulins de représentations galoisiennes p-adiques. Si W est une $B_{|K}^{\otimes E}$-paire, on dit que W est *triangulable* si elle est une extension successive de $B_{|K}^{\otimes E}$-paires de rang 1, et l'on dit que W est *potentiellement triangulable* s'il existe une extension finie L/K telle que la $B_{|L}^{\otimes E}$-paire $W|_{G_L}$ soit triangulable. Si V est une représentation E-linéaire de G_K, on dit que V est *trianguline deployée* si la $B_{|K}^{\otimes E}$-paire $W(V)$ est triangulable, et on dit que V est *trianguline* s'il existe une extension finie E'/E telle que la représentation E'-linéaire $E' \otimes_E V$ soit trianguline deployée. On dit que $V \in \mathrm{Rep}_E(G_K)$ est *potentiellement trianguline* s'il existe une extension finie L/K telle que $V|_{G_L}$ est trianguline. Par exemple, si W est une $B_{|L}^{\otimes E}$-paire semi-stable, alors il existe une extension finie F/E telle que la $B_{|K}^{\otimes F}$-paire $F \otimes_E W$ est triangulable (c'est la proposition 7.1.4.1). Ces notions ont a été introduites par Colmez dans le cadre de son travail sur la correspondance de Langlands p-adique pour $\mathrm{GL}_2(\mathbf{Q}_p)$ (voir [**Col08c**]).

Dans la partie III de cette thèse, on s'intéresse à la question suivante : si V et V' sont des représentations E-linéaires de G_K dont le produit tensoriel $V \otimes_E V'$ est trianguline, alors que peut-on dire de V et V' ?

Pour toute extension finie $E/\mathbf{Q}_p$, l'anneau $\mathbf{B}_{\mathrm{e},E} = \mathbf{B}_\mathrm{e} \otimes_{\mathbf{Q}_p} E$ est un anneau principal (voir le paragraphe 2.5.1 pour des références) et on note $F_E = \mathrm{Frac}(\mathbf{B}_{\mathrm{e},E})$. Si W est une $B_{|K}^{\otimes E}$-paire, alors W est triangulable si et seulement si la représentation F_E-semi-linéaire $F_E \otimes_{\mathbf{B}_{\mathrm{e},E}} W_\mathrm{e}$ est une extension successive d'objets de dimension 1 (c'est la corollaire 7.1.3.2). Ce résultat, combiné avec des résultats sur l'algèbre semi-linéaire de F_E-représentations de G_K et le théorème A plus haut nous permettent de montrer le résultat principal de la partie III (c'est le théorème 7.3.1.2).

Théorème E. *Si W et W' sont deux $B_{|K}^{\otimes E}$-paires telles que $W \otimes W'$ soit triangulable, alors il existe des extensions finies F/E et L/K telles que les $B_{|L}^{\otimes F}$-paires $(F \otimes_E W)|_{G_L}$ et $(F \otimes_E W')|_{G_L}$ soient triangulables.*

En particulier, le théorème ci-dessus s'applique lorsque W et W' sont des $B_{|K}^{\otimes E}$-paires de la forme $W = W(V)$ et $W' = W(V')$ avec $V, V' \in \mathrm{Rep}_E(G_K)$.

Corollaire. *Si V et V' sont des représentations E-linéaires de G_K telles que $V \otimes_E V'$ soit trianguline, alors V et V' sont potentiellement triangulines.*

Dans le paragraphe 7.3.2 de la partie III, on donne l'exemple d'une représentation E-linéaire V de dimension 2 de $G_{\mathbf{Q}_p}$ qui est potientiellement trianguline sans être trianguline, tandis que $V \otimes_E V$ est trianguline.

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Synopsis and introduction

Contents. This text is devoted to the study of crystalline (resp. semi-stable, de Rham, and Hodge-Tate) tensor products of p-adic representations of $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$, where $K/\mathbf{Q}_p$ is a finite extension, as well as trianguline tensor products of p-adic representations of G_K. We also study the situation where the image of a p-adic representation of G_K by a Schur functor (for example, Sym^n(-) or Λ(-)) is crystalline, semi-stable, de Rham, or Hodge-Tate. The results presented in this thesis are statements about B-pairs; they apply in particular to p-adic representations.

This synopsis states the questions studied, the statements of our theorems, and elements of the methods used to prove them. Part I of this thesis introduces various fundamental results from p-adic Hodge theory and properties of Schur functors. Part II is devoted to the study of crystalline (resp. semi-stable, de Rham, and Hodge-Tate) tensor products and Schur objects in p-adic Hodge theory, and in part III we study trianguline tensor products of p-adic Galois representations of G_K.

Keywords: p-adic Galois representations of absolute Galois groups of p-adic fields, B-pairs, crystalline representations, trianguline representations, tensor products, Schur functors, p-adic Hodge theory.

Admissible tensor products and Schur objects in p-adic Hodge theory. Let $\overline{\mathbf{Q}}_p$ be an algebraic closure of $\mathbf{Q}_p$, let K and E be finite extensions of $\mathbf{Q}_p$ contained in $\overline{\mathbf{Q}}_p$, and let $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ be the absolute Galois group of K. We let $\mathrm{Rep}_E(G_K)$ denote the category of continuous linear E-representations of G_K. Using the rings $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$, and $\mathbf{B}_{\mathrm{dR}}$ (see, for example, [**Fon94b**]), Fontaine has defined the notions of crystalline, semi-stable and de Rham continuous E-linear representations of G_K and he has proved that the corresponding categories are stable under sub-quotient, direct sum and tensor product. One of the goals of part II is to answer the following question: if V and V' are nonzero p-adic representations of G_K whose tensor product is crystalline (or semi-stable or de Rham or Hodge-Tate), then what can be said about V and V'?

We also answer the following question: if $\mathscr{F} : \mathrm{Rep}_E(G_K) \to \mathrm{Rep}_E(G_K)$ is a Schur functor (for example, Λ^n or Sym^n) and if $\mathscr{F}(V)$ is crystalline (or semi-stable or de Rham or Hodge-Tate), then what can be said about V (under suitable constraints on $\dim_E(V)$)?

The results obtained in this thesis are stated for B-pairs, which allows us to deduce analogous results for p-adic Galois representations as corollaries. We now explain the results we have obtained in more detail. Let $\mathbf{B}_{\mathrm{e}}$ denote the ring $\mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. Berger has defined the tensor category of $B_{|K}^{\otimes E}$-pairs, in which the objects are pairs $W=(W_{\mathrm{e}},W_{\mathrm{dR}}^{+})$ such that W_{e} is a $\mathbf{B}_{\mathrm{e}}\otimes_{\mathbf{Q}_p}E$-representation of G_K (i.e., W_{e} is a free $\mathbf{B}_{\mathrm{e}}\otimes_{\mathbf{Q}_p}E$-module of finite rank endowed with a $\mathbf{B}_{\mathrm{e}}$-semi-linear, E-linear action of G_K) and W_{dR}^{+} is a G_K-stable $\mathbf{B}_{\mathrm{dR}}^{+}\otimes_{\mathbf{Q}_p}E$-lattice of $W_{\mathrm{dR}}=(\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbf{Q}_p}E)\otimes_{(\mathbf{B}_{\mathrm{dR}}^{+}\otimes_{\mathbf{Q}_p}E)}W_{\mathrm{e}}$. If $W=(W_{\mathrm{e}},W_{\mathrm{dR}}^{+})$ is a $B_{|K}^{\otimes E}$-pair, then the rank of W is defined to be $\mathrm{rank}_{(\mathbf{B}_{\mathrm{e}}\otimes_{\mathbf{Q}_p}E)}W_{\mathrm{e}}=\mathrm{rank}_{(\mathbf{B}_{\mathrm{dR}}^{+}\otimes_{\mathbf{Q}_p}E)}W_{\mathrm{dR}}^{+}$. For example, if V is an E-linear representation of G_K, then $W(V)=((\mathbf{B}_{\mathrm{e}}\otimes_{\mathbf{Q}_p}E)\otimes_E V,(\mathbf{B}_{\mathrm{dR}}^{+}\otimes_{\mathbf{Q}_p}E)\otimes_E V)$ is a $B_{|K}^{\otimes E}$-pair of rank $d=\dim_E V$, and the category $\mathrm{Rep}_E(G_K)$ identifies with a full tensor subcategory of the category of $B_{|K}^{\otimes E}$-pairs by the functor $W(\text{-})$. The notions of crystalline, semi-stable, de Rham, and Hodge-Tate objects extend to objects in the category of $B_{|K}^{\otimes E}$-pairs in such a way that an E-linear representation V of G_K is crystalline (or semi-stable or de Rham or Hodge-Tate) if and only if the $B_{|K}^{\otimes E}$-pair $W(V)$ is.

Using Sen's theory of $\mathbf{C}_p$-representations (as in [**Sen80**]) and Fontaine's theory of $\mathbf{B}_{\mathrm{dR}}$-representations (as in [**Fon04**]), one can show the following result (which appears as theorem 5.1.0.15 of part II).

Theorem A. *Let W and W' be nonzero $B_{|K}^{\otimes E}$-pairs. If the $B_{|K}^{\otimes E}$-pair $W\otimes W'$ is Hodge-Tate, then there is a finite extension F/E and a character $\mu:G_K\to F^{\times}$ such that the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are Hodge-Tate. If, moreover, $W\otimes W'$ is de Rham, then so are $W(\mu^{-1})$ and $W'(\mu)$.*

It is known that every de Rham $B_{|K}^{\otimes E}$-pair is potentially semi-stable, due to the results of [**And02**], [**Ber02**], [**Ked00**], and [**Meb02**]. The properties of $(\varphi,N,\mathrm{Gal}(L/K))$-modules (where L/K is a finite Galois extension) allow us to understand the situation when W and W' are both potentially semi-stable. The following is theorem 6.1.0.20 in part II.

Theorem B. *Let W and W' be nonzero potentially semi-stable $B_{|K}^{\otimes E}$-pairs. If the $B_{|K}^{\otimes E}$-pair $W\otimes W'$ is semi-stable, then there is a finite extension F/E and a character $\mu:G_K\to F^{\times}$ such that the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are semi-stable. If, moreover, $W\otimes W'$ is crystalline, then so are $W(\mu^{-1})$ and $W'(\mu)$.*

In particular, the above two theorems may be used to deduce analogous results for p-adic representations (see corollaries 5.1.0.16 and 6.2.0.22 in part II).

Corollary. *Let V and V' be non-zero linear E-representations of G_K. If $V\otimes_E V'$ is crystalline (resp. semi-stable, de Rham, or Hodge-Tate), then there is a finite extension*

F/E and a character $\mu : G_K \to F^\times$ such that $V(\mu^{-1})$ and $V'(\mu)$ are crystalline (resp. semi-stable, de Rham, or Hodge-Tate).

The methods used to prove theorems A and B above may be used to understand the situation when the image of a B-pair by a Schur functor is crystalline (or semi-stable or de Rham or Hodge-Tate), which we now explain.

If $n \in \mathbf{N}_{>0}$ and if $n = u_1 + \ldots + u_r$ is an integer partition with $u_1 \geq \ldots \geq u_r \geq 1$, then the r-tuple $u = (u_1, \ldots, u_r)$ gives rise to a Schur functor, denoted $\mathrm{Schur}^u(-)$, which sends $B_{|K}^{\otimes E}$-pairs to $B_{|K}^{\otimes E}$-pairs. If $r = 1$ or if $u_1 = u_2 = \ldots = u_r$, then we put $r(u) = r+1$ and we put $r(u) = r$ when this is not the case. For example, if $n \geq 1$ and if $u = (n)$, then $r(u) = 2$ and the associated Schur functor is $\mathrm{Sym}^n(-)$ and if $u = (1, \ldots, 1)$, then $r(u) = n+1$ and the associated Schur functor is $\Lambda^n(-)$.

The following theorems correspond to theorems 5.2.0.18 and 6.3.0.24, respectively, in part II.

Theorem C. *Let W be a $B_{|K}^{\otimes E}$-pair such that $\mathrm{rank}(W) \geq r(u)$. If the $B_{|K}^{\otimes E}$-pair $\mathrm{Schur}^u(W)$ is Hodge-Tate, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is Hodge-Tate. If, moreover, $\mathrm{Schur}^u(W)$ is de Rham, then $W(\mu^{-1})$ is de Rham.*

Theorem D. *Let W be a potentially semi-stable $B_{|K}^{\otimes E}$-pair such that $\mathrm{rank}(W) \geq r(u)$. If the $B_{|K}^{\otimes E}$-pair $\mathrm{Schur}^u(W)$ is semi-stable, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is semi-stable. If, moreover, $\mathrm{Schur}^u(W)$ is crystalline, then so is $W(\mu^{-1})$.*

Theorems C and D above imply the analogous results for p-adic Galois representations (see corollaries 5.2.0.19 and 6.3.0.25 in part II). In the discussion following corollary 5.2.0.19 in part II, we show that the bounds on $\mathrm{rank}(W)$ in theorems C and D are optimal.

Corollary. *Let $V \in \mathrm{Rep}_E(G_K)$ be a representation such that $\dim_E(V) \geq r(u)$. If the linear E-representation $\mathrm{Schur}^u(V)$ is crystalline (resp. semi-stable, de Rham, or Hodge-Tate), then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that $V(\mu^{-1})$ is crystalline (resp. semi-stable, de Rham, or Hodge-Tate).*

Special cases of theorems C and D have been used in the study of Galois representations attached to certain automorphic forms. For example, in his study of local Langlands compatibility for Hilbert modular forms, Skinner showed and used the fact that if V is a p-adic representation and if $\mathrm{Sym}^2(V)$ is crystalline, then Wintenberger's methods of [**Win95**] and [**Win97**] may be applied to show that there exists a quadratic character μ

such that $V(\mu)$ is crystalline (see §2.4.1 of [**Ski09**]). Recently, similar arguments for the functor $\Lambda^2(-)$ were applied by Chenevier and Harris to prove part (b) of theorem 3.2.3 of [**CheHar13**]. It is expected that Wintenberger's methods can be used in the same fashion to give alternate proofs of theorems A, B, C, and D.

Trianguline tensor products of p-adic representations of G_K. If W is a $B_{|K}^{\otimes E}$-pair, then we say that W is *triangulable* if it is a successive extension of $B_{|K}^{\otimes E}$-pairs of rank 1, and we say that W is *potentially triangulable* if there is a finite extension L/K such that the $B_{|L}^{\otimes E}$-pair $W|_{G_L}$ is triangulable. If V is a linear E-representation of G_K, then V is said to be *split trianguline* if the $B_{|K}^{\otimes E}$-pair $W(V)$ is triangulable, and V is said to be *trianguline* if there is a finite extension F/E such that $F \otimes_E V$ is split trianguline. One says that $V \in \mathrm{Rep}_E(G_K)$ is *potentially trianguline* if there is a finite extension L/K such that $V|_{G_L}$ is trianguline. For example, if W is a semi-stable $B_{|K}^{\otimes E}$-pair, then there is a finite extension F/E such that the $B_{|K}^{\otimes F}$-pair $F \otimes_E W$ is triangulable (see proposition 7.1.4.1). These notions were introduced by Colmez in [**Col08c**] in the context of his work on the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$.

In part III of this thesis, we study the following question: if V and V' are linear E-representations of G_K such that $V \otimes_E V'$ is trianguline, then what can be said of V and V'? We now explain our results in this direction.

The category of E-(φ, Γ_K)-modules is equivalent to the category of $B_{|K}^{\otimes E}$-pairs (see theorem A of [**Ber08**]); in particular, the notion of a trianguline representation may be translated in terms of B-pairs.

If $E/\mathbf{Q}_p$ is a finite extension, then $\mathbf{B}_{\mathrm{e},E} = \mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} E$ is a principal ideal domain (see paragraph 2.5.1 for references) and we define $F_E = \mathrm{Frac}(\mathbf{B}_{\mathrm{e},E})$. If W is a $B_{|K}^{\otimes E}$-pair, then W is triangulable if and only if the semi-linear F_E-representation $F_E \otimes_{\mathbf{B}_{\mathrm{e},E}} W_{\mathrm{e}}$ is a successive extension of 1-dimensional semi-linear F_E-representations of G_K (see corollary 7.1.3.2). Using this and several other results on semi-linear algebra of F_E-representations of G_K, and our theorem A, we may show the following, which is theorem 7.3.1.2 of part III.

Theorem E. *If W and W' are $B_{|K}^{\otimes E}$-pairs such that $W \otimes W'$ is triangulable, then there are finite extensions F/E and L/K such that the $B_{|L}^{\otimes F}$-pairs $(F \otimes_E W)|_{G_L}$ and $(F \otimes_E W')|_{G_L}$ are triangulable.*

In particular, the theorem above applies to $B_{|K}^{\otimes E}$-pairs of the form $W = W(V)$ and $W' = W(V')$ for representations $V, V' \in \mathrm{Rep}_E(G_K)$.

Corollary. *If V and V' are linear E-representations of G_K such that $V \otimes_E V'$ is trianguline, then V and V' are potentially trianguline.*

In section 7.3.2 we give an example of a potentially trianguline 2-dimensional E-linear representation V of $G_{\mathbf{Q}_p}$ which is not trianguline and such that $V \otimes_E V$ is trianguline.

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Part 1

General background

CHAPTER 1

Galois representations

1.1. Notation

If $p > 0$ is a prime integer, then let $\mathbf{Q}_p$ denote the completion of $\mathbf{Q}$ for the topology defined by the p-adic absolue value $|\text{-}|_p$ (normalized so that $|p|_p = 1/p$), and let $\overline{\mathbf{Q}}_p$ denote an algebraic closure of $\mathbf{Q}_p$. The field $\overline{\mathbf{Q}}_p$ is endowed with its p-adic topology, which is defined by the absolute value arising from putting $|\text{-}| = \sqrt[n]{|\operatorname{Nm}_{E/\mathbf{Q}_p}(\text{-})|_p}$ for every finite extension $E/\mathbf{Q}_p$ of degree $n = [E : \mathbf{Q}_p]$. The field $\overline{\mathbf{Q}}_p$ is not complete for the p-adic topology, and its completion is denoted by $\mathbf{C}_p$; the field $\mathbf{C}_p$ is algebraically closed.

If $E/\mathbf{Q}_p$ is a sub-extension of $\mathbf{C}_p/\mathbf{Q}_p$, then we let $\mathcal{O}_E$ denote the valuation ring of E. If $E/\mathbf{Q}_p$ is a sub-extension of $\overline{\mathbf{Q}}_p/\mathbf{Q}_p$, then $\mathcal{O}_E$ is the ring of integers of E over $\mathbf{Z}_p$ and it is a discrete valuation ring; we denote its maximal idea by $\mathfrak{m}_E$, and its residue field $\mathcal{O}_E/\mathfrak{m}_E$ by k_E. When we choose a uniformizer of E, we will denote it by π_E or simply π.

In this document, if K is a field and if $\overline{K}/K$ is an algebraic closure, then K^{sep}/K denotes the separable closure of K in $\overline{K}/K$ and G_K denotes the absolute Galois group $\operatorname{Gal}(K^{\text{sep}}/K)$ endowed with its profinite topology, which is compact, Hausdorff, and totally disconnected. If F/K is a separable sub-extension of $\overline{K}/K$, then F^{Gal} denotes the Galois closure of F in $\overline{K}/K$. The group $G_{\mathbf{Q}_p} = \operatorname{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ acts on $\overline{\mathbf{Q}}_p$ by $\mathbf{Q}_p$-linear isometries, so that the action of $G_{\mathbf{Q}_p}$ extends to an action on $\mathbf{C}_p$. Continuity estimates on the action of $G_{\mathbf{Q}_p}$ on $\mathbf{C}_p$ due to Ax, Sen, and Tate imply that if $K/\mathbf{Q}_p$ is a sub-extension of $\overline{\mathbf{Q}}_p/\mathbf{Q}_p$, then the inclusion $\widehat{K} \subset \mathbf{C}_p^{G_K}$ is an equality.

Let $F/\mathbf{Q}_p$ be a finite extension. If k/k_F is a finite extension, then there is a unique non-ramified sub-extension $F(k)/F$ of $\overline{\mathbf{Q}}_p/F$ with residual extension k/k_F, and there is a unique continuous F-automorphism $\sigma_F : F(k) \to F(k)$ lifting $x \mapsto x^{|k_F|}$ on k/k_F. If, for example, L/F is a finite extension and $k = k_L$, then $F(k_L) \subset L$, and $L/F(k_L)$ is totally ramified. We let F^{nr}/F denote the maximal non-ramified sub-extension of $\overline{\mathbf{Q}}_p/F$, and we let (abusing notation) $\sigma_F : \widehat{F^{\text{nr}}} \to \widehat{F^{\text{nr}}}$ denote the unique continuous F-automorphism extending the maps σ_F on finite sub-extensions $F(k)/F$ of F^{nr}/F. If $F = \mathbf{Q}_p$ (so that $k_F = \mathbf{F}_p$) and if $K/\mathbf{Q}_p$ is a finite extension with residual extension $k/\mathbf{F}_p$, then we write K_0 instead of $\mathbf{Q}_p(k)$ and we write σ instead of $\sigma_{\mathbf{Q}_p}$.

1.2. Galois representations: definitions and examples

Let $E/\mathbf{Q}_p$ be a finite extension, and let K be a field. We let $\mathrm{Rep}_E(G_K)$ denote the category of continuous linear E-representations of G_K, where morphisms of objects are G_K-equivariant E-linear transformations. When E is unspecified, we simply refer to such objects as *p-adic representations of G_K*. Every linear E-representation of G_K is also a linear $\mathbf{Q}_p$-representation of G_K.

If $\eta : G_K \to E^\times$ is any continuous linear character and if $W = E \cdot e$ is a 1-dimensional E-vector space with basis (e), then defining $g.e = \eta(g) \cdot e$ makes W into a 1-dimensional linear E-representation of G_K, which we denote by $E(\eta)$. More generally, if $V \in \mathrm{Rep}_E(G_K)$ and if $\eta : G_K \to E^\times$ is a continuous linear character, then we define $V(\eta) = V \otimes_E E(\eta)$.

Let $F \in \{\mathbf{Q}, \mathbf{Q}_p\}$. If $\overline{F}/F$ is an algebraic closure, and if $(\zeta_{p^n})_{n\geq 1}$ is a sequence of primitive p^n-th roots of 1 in $\overline{F}$ such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$, then for each $n \geq 1$ and $g \in G_F$, $g(\zeta_{p^n})$ is a primitive p^n-th root of 1, and we have $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi_n(g)}$ for some $\chi_n(g) \in \{1, \ldots, p^n - 1\}$ prime to p, defining a character $\chi_n : G_F \to (\mathbf{Z}/p^n\mathbf{Z})^\times$. This gives rise in the limit to the cyclotomic character $\chi : G_F \to \mathbf{Z}_p^\times$, which is surjective and continuous. If K/F is finite, if $V \in \mathrm{Rep}_E(G_K)$, and if $k \in \mathbf{Z}$, then we will write $V(k)$ instead of $V(\chi^k|_{G_K})$.

Let $\ell > 0$ be a prime number, and let $\overline{\mathbf{Q}}$ and $\overline{\mathbf{Q}}_\ell$ be algebraic closures of $\mathbf{Q}$ and $\mathbf{Q}_\ell$, respectively. Giving a maximal ideal $\mathfrak{l} \subset \mathcal{O}_{\overline{\mathbf{Q}}}$ lying over ℓ is equivalent to giving an embedding $\tau_\mathfrak{l} : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_\ell$ lifting $\mathbf{Q} \subset \mathbf{Q}_\ell$. Such an embedding gives an injective group morphism $\varphi_{\tau_\mathfrak{l}} : \mathrm{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell) \to \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ defined by sending $g \in \mathrm{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)$ to its restriction $g|_{\overline{\mathbf{Q}},\tau_\mathfrak{l}}$ via $\tau_\mathfrak{l}$. The inclusion $\overline{\mathbf{Q}} \subset \mathbf{C}$ corresponds to the infinite place, and allows one to view $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ as a sub-group of $G_\mathbf{Q}$. The group $G_\mathbf{Q}$ acts transitively on the set of primes $\mathfrak{l}|\ell$ of $\mathcal{O}_{\overline{\mathbf{Q}}}$ lying over ℓ, and the image of the map $\varphi_{\tau_\mathfrak{l}}$ is equal to the *decomposition group* $D(\mathfrak{l}|\ell) = \{g \in G_\mathbf{Q} | g(\mathfrak{l}) = \mathfrak{l}\} = \mathrm{Stab}_{\mathfrak{l}|\ell}$; in particular, $D(\mathfrak{l}|\ell) \simeq G_{\mathbf{Q}_\ell}$. For each $\mathfrak{l}|\ell$, we have a surjective group morphism $D(\mathfrak{l}|\ell) \to \mathrm{Gal}(\overline{\mathbf{F}}_\ell/\mathbf{F}_\ell)$. Its kernel is denoted by $I(\mathfrak{l}|\ell)$, and is called the *inertia group of* $\mathfrak{l}|\ell$. A *Frobenius element at* $\mathfrak{l}|\ell$ is any elemenet $\mathrm{Frob}_{\mathfrak{l}|\ell} \in D(\mathfrak{l}|\ell)$ such that $\mathrm{Frob}_{\mathfrak{l}|\ell}(x) = x^\ell (\mathrm{mod}\, \mathfrak{l})$ for all $x \in \mathcal{O}_{\overline{\mathbf{Q}}}$. If $\mathfrak{l}, \mathfrak{l}'|\ell$ are maximal ideals of $\mathcal{O}_{\overline{\mathbf{Q}}}$ lying over ℓ, then decomposition groups $D_{\mathfrak{l}|\ell}$ and $D_{\mathfrak{l}'|\ell}$ (resp. inertia groups $I_{\mathfrak{l}|\ell}$ and $I_{\mathfrak{l}'|\ell}$) are conjugate in $G_\mathbf{Q}$; we let D_ℓ (resp. I_ℓ) denote any such decomposition group (resp. inertia group) at ℓ when speaking about properties that only depend on ℓ. One has analogous notions and notation over a general number field $K/\mathbf{Q}$.

In particular, if $V \in \mathrm{Rep}_E(G_\mathbf{Q})$, and if ℓ is a prime, then we may consider the restricted representation $\rho|_{D_\ell} : D_\ell \to \mathrm{GL}_{\mathbf{Q}_p}(V)$ (which only depends on ℓ up to isomorphism); we sometimes write, somewhat abusively, $\rho|_{G_{\mathbf{Q}_\ell}} = \rho|_{D_\ell}$. We say that ρ is *unramified at* ℓ if

$I_\ell \subset \ker \rho$; in this case, $\rho|_{D_\ell}$ is determined by the image of a Frobenius element of a place over ℓ. For example, if $\chi : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ is the cyclotomic character for $F = \mathbf{Q}$ and if $\ell \neq p$, then $\chi|_{D_\ell}$ is unramified and $\chi|_{D_\ell}(\mathrm{Frob}_\ell) = \ell$. After identifying D_p with $G_{\mathbf{Q}_p}$, $\chi|_{G_{D_p}}$ is equal to the cyclotomic character for $F = \mathbf{Q}_p$.

1.2.1. Lubin-Tate characters. Let $K/\mathbf{Q}_p$ be a finite extension. In this paragraph, we summarize some constructions used in [**LubTat65**] to describe G_K^{ab}.

A *commutative* 1*-parameter formal group over* $\mathcal{O}_K$ is a formal series $F \in \mathcal{O}_K[[X,Y]]$ satisfying the following conditions:

(1) $F(X,Y) = X + Y (\mathrm{mod}\deg 2)$,
(2) $F(X, F(Y,Z)) = F(F(X,Y),Z)$,
(3) $F(X,Y) = F(Y,X)$.

If $F \in \mathcal{O}_K[[X,Y]]$ is a formal group as above, then there is a unique $\lambda \in \mathcal{O}_K[[X]]$ such that $\lambda(X) = -X (\mathrm{mod}\deg 2)$ and $F(X, \lambda(X)) = F(\lambda(X), X) = 0$. If $F, G \in \mathcal{O}_K[[X,Y]]$ are formal groups as above, then a *morphism* $f : F \to G$ is a formal series $f \in \mathcal{O}_K[[X]]$ such that $f(F(X,Y)) = G(f(X), f(Y))$; the set $\mathrm{End}_{\mathcal{O}_K}(F)$ of endomorphisms of F is a ring with respect to addition and composition. A *formal* $\mathcal{O}_K$*-module* is a formal group F together with a ring homomorphism $[\text{-}] : \mathcal{O}_K \to \mathrm{End}_{\mathcal{O}_K}(F)$ such that $[a](X) = aX (\mathrm{mod}\deg 2)$.

A formal group (resp. formal $\mathcal{O}_K$-module) F may be used to endow a group (resp. $\mathcal{O}_K$-module) structure upon domains of convergence for F (resp. F and $[a](X)$ for all $a \in \mathcal{O}_K$). For example, if $\mathfrak{m}_{\overline{\mathbf{Q}}_p}$ is the maximal ideal of the valuation ring $\mathcal{O}_{\overline{\mathbf{Q}}_p}$ of $\overline{\mathbf{Q}}_p$, then for all power series $F \in \mathcal{O}_K[[X_1, \ldots, X_n]]$ with constant term 0 and all $x_1, \ldots, x_n \in \mathfrak{m}_{\overline{\mathbf{Q}}_p}$, $F(x_1, \ldots, x_n)$ converges to an element in $\mathfrak{m}_{\overline{\mathbf{Q}}_p}$. In particular, if $F \in \mathcal{O}_K[[X,Y]]$ is a formal group (resp. formal $\mathcal{O}_K$-module) then $x +_F y := F(x,y)$ defines a map $\mathfrak{m}_{\overline{\mathbf{Q}}_p} \times \mathfrak{m}_{\overline{\mathbf{Q}}_p} \to \mathfrak{m}_{\overline{\mathbf{Q}}_p}$ (resp. and $a.x = [a](x)$ defines a map $\mathcal{O}_K \times \mathfrak{m}_{\overline{\mathbf{Q}}_p} \to \mathfrak{m}_{\overline{\mathbf{Q}}_p}$) which makes $\mathfrak{m}_{\overline{\mathbf{Q}}_p}$ into a commutative group with additive inverses given by λ (resp. $\mathcal{O}_K$-module). If F is a formal $\mathcal{O}_K$-module and if $a \in \mathcal{O}_K$, then let $F[a] = \{x \in \mathfrak{m}_{\overline{\mathbf{Q}}_p} | [a](x) = 0\}$.

Let $q = |k_K|$, and let $\pi \in \mathcal{O}_K$ be a uniformizer. If $f \in \mathcal{O}_K[[X]]$ satisfies

(i) $f(X) = \pi X (\mathrm{mod}\deg 2)$, and
(ii) $f(X) = X^q (\mathrm{mod}\,\pi)$,

then there is a unique commutative 1-parameter formal group $F_f \in \mathcal{O}_K[[X,Y]]$ such that $f \in \mathrm{End}(F_f)$. Moreover, F_f is a formal $\mathcal{O}_K$-module with structural map $[\text{-}]_f : \mathcal{O}_K \to \mathrm{End}(F_f)$ satisfying $[\pi]_f(X) = f(X)$. If $g \in \mathcal{O}_K[[X]]$ is another power series satisfying conditions (i) and (ii), then F_f and F_g are isomorphic as formal $\mathcal{O}_K$-modules; in particular, the isomorphism class of F_f depends only on π. In particular, while the set of π^n-torsion points in $\mathfrak{m}_{\overline{\mathbf{Q}}_p}$ of F_f depends on f, the extension $K_{n,\pi} = K(F_f[\pi^n])$ of K depends only on π and n.

EXAMPLE 1.2.1.1. *Let* $K = \mathbf{Q}_p$, *so that* $\mathcal{O}_K = \mathbf{Z}_p$, *and let* $\pi = p$. *If* $f(X) = (X+1)^p - 1 = \sum_{k=1}^{p} \binom{p}{k} X^k$, *then* $F_f = X + Y + XY$, *and* $[\text{-}]_f : \mathbf{Z}_p \to \mathrm{End}(F_f)$ *is given by* $a \mapsto (1+X)^a - 1 := \sum_{k=1}^{\infty} \binom{a}{k} X^k$. *The set* $F_f[p^n]$ *consists of* $\zeta - 1$ *with* ζ *a* p^n*-th root of* 1.

In what follows, let $\pi \in \mathcal{O}_K$ be a uniformizer and let $\mathrm{LT} \in \mathcal{O}_K[[X]]$ denote the formal $\mathcal{O}_K$-module attached to $f(X) = \pi X + X^q$. For $n \geq 1$, let $K_\pi = \bigcup_{n \geq 1} K_{n,\pi}$, where $K_{n,\pi} = K(\mathrm{LT}[\pi^n])$. By Galois theory, the following theorem of Lubin and Tate gives an explicit description of G_K^{ab}.

THEOREM 1.2.1.2. *For each* $n \geq 1$, *the extension* $K_{n,\pi}/K$ *is totally ramified of degree* $q^{n-1}(q-1)$, *where* $q = |k_K|$, *and we have an isomorphism* $\mathrm{Gal}(K_{n,\pi}/K) \xrightarrow{\sim} (\mathcal{O}_K/\pi^n\mathcal{O}_K)^\times$. *The sub-extensions* K_π/K *and* K^{nr}/K *of* $\overline{\mathbf{Q}}_p/K$ *are linearly disjoint, and* $K_\pi K^{\mathrm{nr}} = K^{\mathrm{ab}}$, *the maximal abelian sub-extension of* $\overline{\mathbf{Q}}_p/K$.

For $n \geq 1$, $\mathrm{LT}[\pi^n]$ is a free $\mathcal{O}_K/\pi^n\mathcal{O}_K$-module of rank 1 which is stable by the natural action of G_K on $\mathfrak{m}_{\overline{\mathbf{Q}}_p}$. For $n \geq 1$, we therefore have a system of surjective characters $\chi_{n,\pi} : G_K \to (\mathcal{O}_K/\pi^n\mathcal{O}_K)^\times$ which give rise in the limit to a character $\chi_\pi : G_K \to \mathcal{O}_K^\times$ with $\ker(\chi_\pi) = G_{K_\pi}$. The character χ_π is called the *Lubin-Tate* character attached to π and satisfies $g(x) = [\chi(g)](x)$ for all $x \in \bigcup_{n\geq 1} \mathrm{LT}[\pi^n]$ and $g \in G_K$.

EXAMPLE 1.2.1.3. *If* $K = \mathbf{Q}_p$ *and* $\pi = p$ *as in 1.2.1.1, then the Lubin-Tate character* $\chi_p : G_{\mathbf{Q}_p} \to \mathbf{Z}_p^\times$ *is the cyclotomic character.*

1.2.2. Galois representations coming from geometry. Let $\mathcal{E}/\mathbf{Q}$ be an elliptic curve. The set $\mathcal{E}(\overline{\mathbf{Q}})$ of $\overline{\mathbf{Q}}$-rational points of $\mathcal{E}/\mathbf{Q}$ is endowed with the structure of an abelian group. If $\ell > 0$ is a prime number, then we let $\overline{\mathcal{E}}_\ell$ denote the reduction modulo ℓ of $\mathcal{E}$, which is a curve over $\mathbf{F}_\ell$; $\mathcal{E}/\mathbf{Q}$ is said to be of *good reduction at* ℓ if $\overline{\mathcal{E}}_\ell$ is smooth (i.e., if it is an elliptic curve over $\mathbf{F}_\ell$), and $\mathcal{E}/\mathbf{Q}$ is said to be of *bad reduction at* ℓ otherwise. If $\mathcal{E}/\mathbf{Q}$ is of bad reduction at ℓ, then $\overline{\mathcal{E}}_\ell$ has a singularity which is either a node (in which case, we say that $\mathcal{E}/\mathbf{Q}$ is of multiplicative reduction at ℓ) or a cusp (in which case, we say that $\mathcal{E}/\mathbf{Q}$ is of additive reduction at ℓ). Attached to $\mathcal{E}/\mathbf{Q}$ is an integer $N = N_{\mathcal{E}}$ called the *conductor of* $\mathcal{E}/\mathbf{Q}$ whose prime divisors are the finitely many primes ℓ at which $\mathcal{E}$ has bad reduction, and the ℓ-adic valuation of N is defined in terms of the reduction type of $\mathcal{E}$ at ℓ. If $\mathcal{E}/\mathbf{Q}$ is of good or multiplicative reduction for every prime ℓ (in this case, $\mathcal{E}/\mathbf{Q}$ is said to be of *semistable reduction*), then N is equal to the product of the primes of bad reduction.

If $p > 0$ is a prime number, then for each $n \geq 1$, the set $\mathcal{E}[p^n] \subset \mathcal{E}(\overline{\mathbf{Q}})$ of p^n-torsion points is a free $\mathbf{Z}/p^n\mathbf{Z}$-module of rank 2 and is stable by the natural action of $G_{\mathbf{Q}}$. Passing to the limit with respect to the multiplication by p maps $x \mapsto p.x : \mathcal{E}[p^{n+1}] \to \mathcal{E}[p^n]$, we may consider $V_p(\mathcal{E}) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \varprojlim \mathcal{E}(\overline{\mathbf{Q}})[p^n]$, which is a 2-dimensional $\mathbf{Q}_p$-vector space endowed with a continuous linear action of $G_{\mathbf{Q}}$. We denote this representation by $(V_p(\mathcal{E}), \rho_{\mathcal{E},p})$; the basic properties of these representations are developed, for example, in [**Ser68**] and [**Sil**].

Proposition 1.2.2.1. *If $\mathcal{E}/\mathbf{Q}$ is an elliptic curve of conductor N, then* $\det \rho_{\mathcal{E},p} = \chi$ *and for all prime numbers $\ell > 0$ such that $\ell \nmid pN$, we have*

(1) $\rho_{\mathcal{E},p}|_{D_\ell} : D_\ell \to \mathrm{GL}_2(\mathbf{Q}_p)$ is non-ramified, and
(2) the characteristic polynomial of $\rho_{\mathcal{E},p}(\mathrm{Frob}_\ell)$ is equal to $X^2 - (\ell + 1 - |\overline{\mathcal{E}}_\ell(\mathbf{F}_\ell)|)X + \ell$.

Note in particular that if $c \in G_{\mathbf{Q}}$ is complex conjugation, then $\det \rho_{\mathcal{E},p}(c) = \chi(c) = -1$ since $c(\zeta) = \zeta^{-1}$ for every p^n-th root of unity ζ; the representation $\rho_{\mathcal{E},p}$ is therefore said to be *odd*. Results of Faltings imply that the Galois representation $\rho_{\mathcal{E},p}$ determines $\mathcal{E}/\mathbf{Q}$ up to isogeny.

More generally, if $X/\mathbf{Q}$ is an algebraic variety, then the i-th étale cohomology $H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)$ (here, $X_{\overline{\mathbf{Q}}}$ denotes the base change of $X/\mathbf{Q}$ to $\overline{\mathbf{Q}}$) is a finite dimensional $\mathbf{Q}_p$-vector space endowed with a continuous $\mathbf{Q}_p$-linear action of $G_{\mathbf{Q}}$ which comes from functoriality. If $\mathcal{E}/\mathbf{Q}_p$ is an elliptic curve, then $H^1_{\text{ét}}(\mathcal{E}_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p) = V_p(\mathcal{E})^*$.

1.2.3. Galois representations coming from modular forms. Let $\mathcal{H} = \{z \in \mathbf{C} |\, \mathrm{Im}(z) > 0\}$ and let $\mathcal{O}(\mathcal{H})$ denote the $\mathbf{C}$-vector space of holomorphic functions $f : \mathcal{H} \to \mathbf{C}$. For $k \in \mathbf{Z}$, the *right weight-k action* of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathcal{O}(\mathcal{H})$ is defined as follows: if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ and if $f \in \mathcal{O}(\mathcal{H})$, then $(f|_k\gamma)(z) := (cz+d)^{-k} f(\gamma.z)$, where $\gamma.z = \frac{az+b}{cz+d}$, defines a holomorphic function on $\mathcal{H}$; this definition makes sense since $\mathrm{Im}(\gamma.z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$.

For all $N \geq 1$, $\Gamma_1(N) = \{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) | \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod N\}$ is a sub-group of $\mathrm{SL}_2(\mathbf{Z})$. A holomorphic function $f : \mathcal{H} \to \mathbf{C}$ is said to be a *modular form of weight k and level* $\Gamma_1(N)$ if it is invariant by $\Gamma_1(N)$ for the weight-k action and if $\lim_{y \in \mathbf{R} \to \infty} (f|_k\gamma)(iy)$ exists and is finite for all $\gamma \in \mathrm{SL}_2(\mathbf{Z})$; the $\mathbf{C}$-vector space $M_k(\Gamma_1(N))$ of modular forms of weight k and level $\Gamma_1(N)$ is known to be finite dimensional. Elements of the sub-space $S_k(\Gamma_1(N)) \subset M_k(\Gamma_1(N))$ of f such that $\lim_{y \in \mathbf{R} \to \infty} (f|_k\gamma)(iy) = 0$ are called *cusp forms*. If $k < 0$ then $M_k(\Gamma_1(N)) = \{0\}$, and $M_0(\Gamma_1(N))$ consists of constant functions.

If $f \in \mathcal{O}(\mathcal{H})$ is a modular form of weight k and level $\Gamma_1(N)$, then $f(z+1) = f(z)$ and therefore f has a Fourier series expansion (also called the q-expansion of f) of the form $f(z) = \sum_{n=0}^{\infty} c_n(f) q^n$ where $q = e^{2\pi i z}$; if f is a cusp form, then $c_0(f) = 0$.

EXAMPLE 1.2.3.1. *(1) If* $\Delta(z) = q\prod_{n>0}(1-q^n)^{24}$ *with* $q = e^{2\pi iz}$ *for* $z \in \mathcal{H}$*, then* Δ *is a cusp form of weight* 12 *and level* $\Gamma_1(1)$ *and* $S_{12}(\Gamma_1(1))$ *is generated by* $\Delta(z)$ *as a* $\mathbf{C}$*-vector space.*

(2) The space $M_{12}(\Gamma_1(1))$ *decomposes as* $\mathcal{E}_{12} \oplus S_{12}(\Gamma_1(1))$*, where* $\mathcal{E}_{12}$ *is the* $\mathbf{C}$*-vector space generated by the Eisenstein series* $E_{12}(z) = \frac{691}{32760} + \sum_{n=1}^{\infty}(\sum_{d|n} d^{11})q^n$ *with* $q = e^{2\pi iz}$ *for* $z \in \mathcal{H}$*.*

The $\mathbf{C}$-vector space $S_k(\Gamma_1(N))$ is stable by the action of the sub-group $\Gamma_0(N) = \{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) | \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) (\mathrm{mod}\, N)\}$. The sub-group $\Gamma_1(N)$ is normal in $\Gamma_0(N)$, and $\Gamma_0(N)$ acts on $S_k(\Gamma_1(N))$ through the quotient $\Gamma_0(N)/\Gamma_1(N) \xrightarrow{\sim} (\mathbf{Z}/N\mathbf{Z})^\times$ (this latter isomorphism is given by $\overline{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \mapsto \overline{d}$); if d is prime to N, then we let $\langle d \rangle$ denote the corresponding operator on $S_k(\Gamma_1(N))$ (one calls it a *diamond operator*). If f is a simultaneous eigenform for the operators $\langle d \rangle$ (with $d \in (\mathbf{Z}/N\mathbf{Z})^\times$), then $\langle d \rangle(f) = \epsilon(d)f$ for some character $\epsilon : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$. Moreover, for each prime $q \nmid N$, there is a so-called Hecke operator T_q on $S_k(\Gamma_1(N))$. One has $T_q \circ T_{q'} = T_{q'} \circ T_q$ and $\langle d \rangle \circ T_q = T_q \circ \langle d \rangle$ for all primes $q, q' \nmid N$ and $d \in (\mathbf{Z}/N\mathbf{Z})^\times$. The space $S_k(\Gamma_1(N))$ is therefore a direct sum of generalized simultaneous eigenspaces for the Hecke and diamond operators; a simultaneous eigenvector $f \in S_k(\Gamma_1(N))$ is called a *Hecke eigenform*. It is known that if $f \in S_k(\Gamma_1(N))$ is a Hecke eigenform, then the eigenvalues of f generate a finite extension $E_f/\mathbf{Q}$; we will call this the *coefficient field of* f.

The following theorem is due to Eichler [**Eic54**], Shimura [**Shi58**], Igusa [**Igu59**] in the weight $k = 2$ case, to Deligne [**Del71**] in the weight $k > 2$ case, and to Deligne-Serre [**DelSer74**] in the weight $k = 1$ case. The irreducibility statement is due to Ribet [**Rib77**].

THEOREM 1.2.3.2. *Let* $k, N \geq 1$*. If* $f \in S_k(\Gamma_1(N))$ *is a normalized eigenform with character* ϵ *and coefficient field* $E_f/\mathbf{Q}$ *and if* p *is a prime number, then for every maximal ideal* $\mathfrak{p} \subset \mathcal{O}_{E_f}$ *lying over* p*, there is an irreducible Galois representation* $\rho_{f,\mathfrak{p}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O}_{E_f,\mathfrak{p}})$ *such that* $\det \rho_{f,\mathfrak{p}} = \epsilon\chi^{k-1}$ *(where* $\chi : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ *is the cyclotomic character and* ϵ *is viewed as a character of* $G_{\mathbf{Q}}$ *via the quotient* $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q}) \xrightarrow{\sim} (\mathbf{Z}/N\mathbf{Z})^\times$*) such that for all* ℓ *with* $\ell \nmid pN$*,*

(1) $\rho_{f,\mathfrak{p}}|_{D_\ell} : D_\ell \to \mathrm{GL}_2(\mathcal{O}_{E_f,\mathfrak{p}})$ *is non-ramified, and*
(2) The characteristic polynomial of $\rho_{f,\mathfrak{p}}|_{D_\ell}(\mathrm{Frob}_\ell))$ *is* $X^2 - c_\ell(f)X + \epsilon(\ell)\ell^{k-1}$*.*

The representation $\rho_{f,\mathfrak{p}}$ is odd.

1.2.4. Galois representations coming from Hilbert modular forms. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$ and if $z \in \mathcal{H}$, then putting $\gamma.z = \frac{az+b}{cz+d}$ defines an action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathcal{H}$. For $n \geq 1$, let $\mathrm{SL}_2(\mathbf{R})^n = \mathrm{SL}_2(\mathbf{R}) \times \ldots \times \mathrm{SL}_2(\mathbf{R})$ act componentwise on $\mathcal{H}^n =$

$\mathcal{H} \times \ldots \times \mathcal{H}$. If $k = (k_1, \ldots, k_n) \in \mathbf{N}^n_{>0}$, then the *weight k-action* of $\mathrm{SL}_2(\mathbf{R})^n$ on the space $\mathcal{O}(\mathcal{H}^n)$ of holomorphic functions $f : \mathcal{H}^n \to \mathbf{C}$ is defined as follows: for $f \in \mathcal{O}(\mathcal{H}^n)$ and $\gamma = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)_{1 \le i \le n} \in \mathrm{SL}_2(\mathbf{R})^n$, put

$$(f|_k \gamma) := (c_1 z_1 + d_1)^{-k_1} \cdot \ldots \cdot (c_n z_n + d_n)^{-k_n} f(\gamma.z)$$

for all $z = (z_1, \ldots, z_n) \in \mathcal{H}^n$.

If $F/\mathbf{Q}$ is totally real of degree $[F : \mathbf{Q}] = n > 1$ then the set $I = \{\tau_1, \ldots, \tau_n\}$ of embeddings of F into $\mathbf{R}$ allows us to view $\mathrm{SL}_2(\mathcal{O}_F)$ as a sub-group of $\mathrm{SL}_2(\mathbf{R})^n$ via the embedding $\gamma \mapsto (\tau_i(\gamma))_{1 \le i \le n} : \mathrm{SL}_2(\mathcal{O}_F) \to \mathrm{SL}_2(\mathbf{R})$.

If $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_F)$ is a congruence sub-group and if $k \in \mathbf{N}^n_{>0}$, then a *Hilbert modular form of level Γ and weight k* is a holomorphic function $f \in \mathcal{O}(\mathcal{H}^n)$ such that $(f|_k \gamma) = f$ for all $\gamma \in \Gamma$. In contrast with the $n = 1$ situation as in paragraph 1.2.3, no additional condition on cusps is necessary. The $\mathbf{C}$-vector space $S_k(\Gamma)$ of Hilbert modular forms of weight k and level Γ is finite dimensional.

Suppose now that $k = (k_1, \ldots, k_n)$ satisfies $k_i \ge 2$ and $k_i \equiv k_j \bmod 2$ for all $1 \le i, j \le n$. For a non-zero ideal $\mathfrak{n} \subset \mathcal{O}_F$, we have a congruence sub-group $\Gamma(\mathfrak{n}) \subset \mathrm{SL}_2(\mathcal{O}_F)$. If $\mathfrak{q} \subset \mathcal{O}_F$ is a prime ideal and if $\mathfrak{a} \subset F$ is a fractional ideal prime to $\mathfrak{n}$, then there are notions of *Hecke operators* $T_\mathfrak{q}$ and $S_\mathfrak{a}$ on $S_k(\Gamma(\mathfrak{n}))$. If $f \in S_k(\Gamma(\mathfrak{n}))$ and if f is an eigenform for all $T_\mathfrak{q}$ and all $S_\mathfrak{a}$, then Shimura showed in [**Shi78**] that the extension $E_f/\mathbf{Q}$ generated by the Hecke eigenvalues $\theta(T_q)$ and $\theta(S_\mathfrak{a})$ of f is finite.

The following theorem was proven in various cases (including when $[F : \mathbf{Q}]$ is odd) by Carayol [**Car86**] and Wiles [**Wil88**], in the case when $[F : \mathbf{Q}]$ is even by Taylor [**Tay89**], and in general by Blasius-Rogawski [**BlaRog93**], following work of Eichler, Shimura, Deligne, Ohta, Tunnel, and others.

THEOREM 1.2.4.1. *Let k and $\mathfrak{n}$ be as above. If $f \in S_k(\Gamma(\mathfrak{n}))$ is a Hilbert eigenform with field of definition $E_f/\mathbf{Q}$ and if $\mathfrak{p}|p$ is a non-zero prime of $\mathcal{O}_{E_f}$, then one may associate to f a continuous representation*

$$\rho_{f,\mathfrak{p}} : G_F \to \mathrm{GL}_2(\mathcal{O}_{E_f,\mathfrak{p}})$$

such that

(1) $\rho_{f,\mathfrak{p}}$ is unramified outside $\mathfrak{n}p$,
(2) $\mathfrak{l}$ is a prime of F not dividing $\mathfrak{n}p$, then the characteristic polynomial of $\rho_{f,\mathfrak{p}}(\mathrm{Frob}_\mathfrak{l})$ is $X^2 - \theta(T_\mathfrak{l})X + \theta(S_\mathfrak{l})\,\mathrm{Nm}(\mathfrak{l})$

CHAPTER 2

p-adic Hodge theory

2.1. Rings of periods and admissible representations

In this section, we summarize some fundamental notions from p-adic Hodge theory.

2.1.1. The rings $\widetilde{\mathbf{E}}^+$ and $\widetilde{\mathbf{B}}^+$. In this paragraph, we summarize several results on the rings defined in §1 of [**Fon94a**]. The articles [**Fon82**], [**FonWin**], and [**Win83**] are also good references, as well as sections §4.3 and §5.1 of [**Col08a**] are also good references for the statements presented in this paragraph. We signal to the reader that the notation used for these rings differs in the literature; for example, the ring $\widetilde{\mathbf{E}}^+$ is sometimes denoted by $\mathcal{R}$ or by $\mathcal{R}(\mathbf{C}_p)$.

Let $\widetilde{\mathbf{E}}^+$ denote the set

$$\varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbf{C}_p} = \{(x^{(i)})_{i\in\mathbf{Z}_{\geq 0}} \,|\, x^{(i)} \in \mathcal{O}_{\mathbf{C}_p} \text{ and } (x^{(i+1)})^p = x^{(i)}\ \forall i \in \mathbf{Z}_{\geq 0}\}$$

If $x = (x^{(i)})$ and $y = (y^{(i)})$ are elements of $\widetilde{\mathbf{E}}^+$, then the operations $x + y$ defined by putting $(x+y)^{(i)} = \lim_{j\to\infty}(x^{(i+j)} + y^{(i+j)})^{p^j}$ (this limit converges for the p-adic topology) and $x \cdot y$ defined by $(x \cdot y)^{(i)} = x^{(i)} \cdot y^{(i)}$ make $\widetilde{\mathbf{E}}^+$ into a perfect ring of characteristic p.

If $x = (x^{(i)}) \in \widetilde{\mathbf{E}}^+$, then putting $\mathrm{val}_{\mathbf{E}}(x) = v_p(x^{(0)})$ defines a discrete valuation on $\widetilde{\mathbf{E}}^+$, and $\widetilde{\mathbf{E}}^+$ is complete for the topology defined by $\mathrm{val}_{\mathbf{E}}$. The componentwise action of $G_{\mathbf{Q}_p}$ on $\widetilde{\mathbf{E}}^+$ is an action by ring endomorphisms.

Let $\{\zeta_{p^k}\}_{\mathbf{Z}_{>0}}$ denote a sequence of primitive p^k-th roots of 1 such that $\zeta_{p^{k+1}}^p = \zeta_{p^k}$ for all $k \geq 1$. Let $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \widetilde{\mathbf{E}}^+$ and let $\overline{\pi} = \epsilon - 1$. Since $v_p(\zeta_{p^n} - 1) = \frac{1}{p^{n-1}(p-1)}$, we have $\mathrm{val}_{\mathbf{E}}(\overline{\pi}) = \frac{p}{p-1}$, and $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}^+[\overline{\pi}^{-1}]$. The field $\widetilde{\mathbf{E}}$ is algebraically closed (see (iv) of proposition 4.10 of [**Col08b**]).

If R is a perfect ring of prime characteristic $p > 0$, then we let $W(R)$ denote the ring of Witt vectors of R; it is a ring of characteristic 0 which is p-adically separated and complete, and such that $W(R)/pW(R) = R$, and it comes equipped with a multiplicative map [-] : $R \to W(R)$ (called the *Teichmüller map*) such that $\overline{[x]} = x$ for all $x \in R$ (here, $\overline{\cdot}$ denotes the reduction mod p map). In particular, every element $x \in W(R)$ may be written as a p-adically convergent series $x = \sum_{k=0} p^i[x_i]$ (with $x_i \in R$). For example, $W(\mathbf{F}_p) = \mathbf{Z}_p$. More generally, if $K/\mathbf{Q}_p$ is finite, then $W(k_K)[1/p] = K_0$ is the maximal non-ramified sub-extension of $K/\mathbf{Q}_p$ and $W(k_K) = \mathcal{O}_{K_0}$ as in paragraph 1.1. See chapter II of [**Ser68b**] for details on the construction of the ring of Witt vectors.

Let $\overline{\mathbf{F}}_p$ be an algebraic closure of $\mathbf{F}_p$, and let $[\text{-}] : \overline{\mathbf{F}}_p \to \widehat{\mathcal{O}}_{\mathbf{Q}_p^{\mathrm{nr}}}$ denote the Teichmüller map. The map $\overline{\mathbf{F}}_p \to \widetilde{\mathbf{E}}^+$ given by $x \mapsto ([x^{1/p^k}])_{k \in \mathbf{Z}_{\geq 0}}$ allows us to view $\overline{\mathbf{F}}_p$ as a sub-ring of $\widetilde{\mathbf{E}}^+$.

The ring $\widetilde{\mathbf{A}}^+ = W(\widetilde{\mathbf{E}}^+)$ and the field $\widetilde{\mathbf{B}}^+ = W(\widetilde{\mathbf{E}}^+)[1/p]$ are p-adically separated and complete and inherit the actions of $G_{\mathbf{Q}_p}$ and φ on $\widetilde{\mathbf{E}}^+$: more precisely, if $x = \sum_{k \gg -\infty} p^k [x_k] \in \widetilde{\mathbf{B}}^+$ (such an expression is unique, and x is in $\widetilde{\mathbf{A}}^+$ if $x_k = 0$ for all $k < 0$), then we have:

$$g(x) = \sum_{k \gg -\infty} p^k [g(x_k)] \text{ for all } g \in G_{\mathbf{Q}_p} \text{ and } \varphi(x) = \sum_{k \gg -\infty} p^k [x_k^p]$$

Note that $(\widetilde{\mathbf{A}}^+)^{\varphi=1} = \mathbf{Z}_p$ and $(\widetilde{\mathbf{B}}^+)^{\varphi=1} = \mathbf{Q}_p$.

The universal property of Witt vectors gives rise to a surjective ring morphism $\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbf{C}_p}$ given by $\sum_{k \gg -\infty} p^k [x_k] \mapsto \sum_{k \gg -\infty} p^k x_k^{(0)}$, which extends to a morphism $\theta : \widetilde{\mathbf{B}}^+ \to \mathbf{C}_p$. The following can be found, for example, as proposition 2.4 of [**Fon82**].

PROPOSITION 2.1.1.1. *If* $\omega \in \ker(\theta) \subset \widetilde{\mathbf{A}}^+$*, then* ω *generates* $\ker\theta$ *if and only if* $\mathrm{val}_{\mathbf{E}}(\overline{\omega}) = 1$*. Every generator of* $\ker(\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbf{C}_p})$ *is a generator of* $\ker(\theta : \widetilde{\mathbf{B}}^+ \to \mathcal{O}_{\mathbf{C}_p})$*.*

For example, the above proposition shows that if $\widetilde{p} \in \widetilde{\mathbf{E}}^+$ is any element such that $\widetilde{p}^{(0)} = p$, then the element $\xi = [\widetilde{p}] - p \in \widetilde{\mathbf{A}}^+$ generates $\ker\theta$.

The *strong topology* (resp. *weak topology*) on $\widetilde{\mathbf{A}}^+$ is the finest topology making the projection map $\widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{E}}^+$ continuous, when $\widetilde{\mathbf{E}}^+$ is endowed with the discrete topology (resp. the topology of $\mathrm{val}_{\mathbf{E}}$). The strong topology on $\widetilde{\mathbf{A}}^+$ is the same as the p-adic topology, and the weak topology on $\widetilde{\mathbf{A}}^+$ is the same as the topology given by the family of semi-valuations $\{\omega_k(-)\}_{k \in \mathbf{Z}}$ defined by $\omega_k(\sum_{i=0}^{\infty} p^i [x_i]) = \inf_{i \leq k} \mathrm{val}_{\mathbf{E}}(x_i)$. The ring $\widetilde{\mathbf{B}}^+$ may be endowed with the strong (resp. weak) topology by giving it the inductive limit topology from $\widetilde{\mathbf{B}}^+ = \bigcup_{n \geq 0} p^{-n} \widetilde{\mathbf{A}}^+$. The following can be found, for example, as proposition 5.2 and remark 5.3 of [**Col08a**].

PROPOSITION 2.1.1.2. *The map* φ *acts continuously on* $\widetilde{\mathbf{B}}^+$ *for the weak and strong topologies. The action of* $G_{\mathbf{Q}_p}$ *on* $\widetilde{\mathbf{B}}^+$ *is continuous for the weak topology, but is not continuous for the* p*-adic topology.*

2.1.2. The rings $\mathbf{B}_{\mathrm{dR}}^+$ and $\mathbf{B}_{\mathrm{HT}}$. Let $\theta : \widetilde{\mathbf{B}}^+ \to \mathbf{C}_p$ be the surjective G_K-equivariant ring homomorphism given by $\sum_{k \gg -\infty} p^k [x_k] \mapsto \sum_{k \gg -\infty} p^k x_k^{(0)}$ as in the previous paragraph. The $(\ker\theta)$-adic completion of $\widetilde{\mathbf{B}}^+$ is denoted by $\mathbf{B}_{\mathrm{dR}}^+$. The kernel of $\theta : \widetilde{\mathbf{B}}^+ \to \mathbf{C}_p$ is generated, for example, by the element $[\epsilon] - 1$, so that the series $t = \log([\epsilon]) =$

$\sum_{k\geq 1}(-1)^{k-1}\frac{([\epsilon]-1)^k}{k}$ converges in $\mathbf{B}_{\mathrm{dR}}^+$ for the $(\ker\theta)$-adic topology. Moreover, $\mathbf{B}_{\mathrm{dR}}^+$ is a discrete valuation ring and t is a uniformizer; in this document, we let $v_t(\text{-})$ denote the t-adic valuation on the field $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[1/t]$. For all $g \in G_K$, $g(t) = \chi(g)t$ (where χ is the cyclotomic character). There is a G_K-equivariant section of θ over $\overline{\mathbf{Q}}_p \subset \mathbf{C}_p$, so that we have a G_K-equivariant inclusion $\overline{\mathbf{Q}}_p \subset \mathbf{B}_{\mathrm{dR}}^+$ (see, for example, §1.2 and §2 of [**Col12**] for a proof of this fact and a detailed topological discussion).

The associated graded ring $\mathrm{gr}_t(\mathbf{B}_{\mathrm{dR}}^+) = \bigoplus_{i\in\mathbf{Z}}(t^i\mathbf{B}_{\mathrm{dR}}^+/t^{i+1}\mathbf{B}_{\mathrm{dR}}^+)$ is denoted by $\mathbf{B}_{\mathrm{HT}}$, and is isomorphic in a $G_{\mathbf{Q}_p}$-equivariant way to the graded ring $\mathbf{C}_p[T,T^{-1}]$ of Laurent polynomials, where $\mathbf{C}_p[T,T^{-1}]$ is endowed with the action of $G_{\mathbf{Q}_p}$ given by the natural action on $\mathbf{C}_p$ and by defining $g(T) = \chi(g)T$ for all $g \in G_{\mathbf{Q}_p}$).

2.1.3. The rings $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{st}}$. Let $\widetilde{p} \in \widetilde{\mathbf{E}}^+$ be an element with $\widetilde{p}^{(0)} = p$, so that $\widetilde{p} = (p, p^{1/p}, p^{1/p^2}, \ldots)$ and so that $\xi = [\widetilde{p}] - p$ generates the ideal $\ker(\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbf{C}_p})$. The sub-ring

$$\mathbf{B}_{\mathrm{cris}}^+ = \{\, x = \sum_{n\geq 0} a_n\frac{\xi^n}{n!} \mid a_n \in \widetilde{\mathbf{B}}^+ \text{ and } a_n \to 0 \ p\text{-adically} \,\}$$

of $\mathbf{B}_{\mathrm{dR}}^+$ is stable by the action of $G_{\mathbf{Q}_p}$ and contains the element $t = \log([\epsilon])$. Put $\mathbf{B}_{\mathrm{cris}} := \mathbf{B}_{\mathrm{cris}}[1/t] \subset \mathbf{B}_{\mathrm{dR}}$. The Frobenius φ on $\widetilde{\mathbf{B}}^+$ extends by continuity to $\mathbf{B}_{\mathrm{cris}}^+$, and thus to a map $\varphi : \mathbf{B}_{\mathrm{cris}} \to \mathbf{B}_{\mathrm{cris}}$ which is additive and injective, and such that for all $\lambda \in \widehat{\mathbf{Q}}_p^{\mathrm{nr}}$ and $x \in \mathbf{B}_{\mathrm{cris}}$, one has $\varphi(\lambda x) = \sigma(\lambda)\varphi(x)$, where $\sigma : \widehat{\mathbf{Q}}_p^{\mathrm{nr}} \to \widehat{\mathbf{Q}}_p^{\mathrm{nr}}$ is the absolute frobenius for $F = \mathbf{Q}_p$ as defined in paragraph 1.1.

The action of $G_{\mathbf{Q}_p}$ on $\mathbf{B}_{\mathrm{cris}}$ extends to the polynomial ring $\mathbf{B}_{\mathrm{st}} = \mathbf{B}_{\mathrm{cris}}[X]$ by defining $g(X) = X + c(g)t$ where $c : G_{\mathbf{Q}_p} \to \mathbf{Z}_p^\times$ is the cocycle defined by $g(p^{1/p^n}) = p^{1/p^n} \cdot \zeta_{p^n}^{c(g)}$ for all $n \geq 1$ and primitive p^n-th roots ζ_{p^n} of 1, and one extends φ on $\mathbf{B}_{\mathrm{cris}}$ to $\mathbf{B}_{\mathrm{st}}$ by defining $\varphi(X) = pX$. The map $N = -\partial_X$ on $\mathbf{B}_{\mathrm{st}}$ is referred to as the *monodromy operator*. By sending the variable X to an element $\log([\widetilde{p}]/p) = -\sum_{n=1}^{\infty}\frac{(1-[\widetilde{p}]/p)^{n-1}}{n}$, which converges in $\mathbf{B}_{\mathrm{dR}}$, the ring $\mathbf{B}_{\mathrm{st}}$ may be equipped with an injective $G_{\mathbf{Q}_p}$-equivariant morphism $\mathbf{B}_{\mathrm{st}} \hookrightarrow \mathbf{B}_{\mathrm{dR}}$ of $\mathbf{B}_{\mathrm{cris}}$-algebras, which allows us to view $\mathbf{B}_{\mathrm{st}}$ as a $G_{\mathbf{Q}_p}$-stable sub-ring of $\mathbf{B}_{\mathrm{dR}}$ (this is theorem 4.2.4 of [**Fon94a**]); this inclusion corresponds to choosing an extension of $\log_p : \mathcal{O}_{\overline{\mathbf{Q}}_p}^\times \to \overline{\mathbf{Q}}_p$ to $\overline{\mathbf{Q}}_p^\times$ by putting $\log_p(p) = 0$, while choosing other values for $\log_p(p)$ gives rise to different embeddings of $\mathbf{B}_{\mathrm{st}}$ in $\mathbf{B}_{\mathrm{dR}}$. If $K/\mathbf{Q}_p$ is finite, then $\mathbf{B}_{\mathrm{st}}^{G_K} = \mathbf{B}_{\mathrm{cris}}^{G_K} = K_0$. The properties of $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{st}}$ are developed in §2.3.3 of [**Fon94a**]; see also III §2 of [**Col98b**] for a technical discussion of the topology on $\mathbf{B}_{\mathrm{cris}}$, as well for a discussion of the ring $\mathbf{B}_{\max}$, which is sometimes used as an alternative to the ring $\mathbf{B}_{\mathrm{cris}}$.

2.1.4. Admissibility. In this paragraph, let $\mathbf{B}$ denote any of the rings $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$, $\mathbf{B}_{\mathrm{dR}}$, or $\mathbf{B}_{\mathrm{HT}}$ from paragraphs 2.1.2 or 2.1.3, or any G_K-stable sub-extension of $\mathbf{C}_p/\mathbf{Q}_p$.

A *semi-linear* $\mathbf{B}$*-representation of* G_K is a free $\mathbf{B}$-module of finite rank together with an action of G_K by semi-linear operators. If W and W' are semi-linear $\mathbf{B}$-representations of G_K, then a *morphism* $f : W \to W'$ is defined to be a morphism of $\mathbf{B}$-modules such that $f(g(w)) = g(f(w))$ for all $w \in W$ and $g \in G_K$. If W is a semi-linear $\mathbf{B}$-representation of G_K, then W is said to be *trivial* if it admits a basis of G_K-invariant elements; this is equivalent to saying $W \simeq \mathbf{B}^{\operatorname{rank}_{\mathbf{B}} W}$ as semi-linear representations. We will sometimes denote the category of semi-linear $\mathbf{B}$-representations of G_K by $\operatorname{Rep}_{\mathbf{B}}(G_K)$.

For example, if $V \in \operatorname{Rep}_E(G_K)$, then $W = \mathbf{B} \otimes_{\mathbf{Q}_p} V$ is a free $\mathbf{B}$-module of rank $d = [E : \mathbf{Q}_p] \dim_E V$, and putting $g(b \otimes v) = g(b) \otimes g(v)$ for all $g \in G_K$ defines an action of G_K on W by semi-linear operators. One says that V is $\mathbf{B}$-*admissible* if the semi-linear $\mathbf{B}$-representation $\mathbf{B} \otimes_{\mathbf{Q}_p} V$ is trivial. One says that V is *potentially* $\mathbf{B}$-*admissible* if there is a finite extension L/K such that the restriction of V to G_L is $\mathbf{B}$-admissible.

To be more precise, we say that $V \in \operatorname{Rep}_E(G_K)$ is (potentially) crystalline if it is (potentially) $\mathbf{B}_{\text{cris}}$-admissible, and similarly we speak of (potentially) semi-stable, de Rham, or Hodge-Tate representations.

If $V \in \operatorname{Rep}_E(G_K)$, then $D_{\mathbf{B}}(V) = (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{G_K}$ is a vector space over the field $\mathbf{B}^{G_K}$; we often abreviate $D_{\mathbf{B}_{\text{cris}}}$(-) by D_{cris}(-), and we use similar notation for $D_{\mathbf{B}}$(-) when $\mathbf{B} \in \{\mathbf{B}_{\text{st}}, \mathbf{B}_{\text{dR}}, \mathbf{B}_{\text{HT}}\}$. The following can be found as propositions 1.4.2, 5.1.2, and 3.6 of [**Fon94b**].

PROPOSITION 2.1.4.1. *Let* $\mathbf{B} \in \{\mathbf{B}_{\text{cris}}, \mathbf{B}_{\text{st}}, \mathbf{B}_{\text{dR}}, \mathbf{B}_{\text{HT}}\}$. *If* $V \in \operatorname{Rep}_E(G_K)$, *then the morphism* $f : \mathbf{B} \otimes_{\mathbf{B}^{G_K}} D_{\mathbf{B}}(V) \to \mathbf{B} \otimes_{\mathbf{Q}_p} V$ *of semi-linear* $\mathbf{B}$*-representations of* G_K *is injective, and* $\dim_{\mathbf{B}^{G_K}} D_{\mathbf{B}}(V) \leq \dim_{\mathbf{Q}_p} V$. *Moreover, the following conditions are equivalent:*

(1) f *is an isomorphism,*
(2) V *is* $\mathbf{B}$*-admissible,*
(3) $\dim_{\mathbf{B}^{G_K}} D_{\mathbf{B}}(V) = \dim_{\mathbf{Q}_p} V$

Note that the above proposition implies that for $V \in \operatorname{Rep}_E(G_K)$, the notion of $\mathbf{B}$-admissibility is encoded by the dimension of $D_{\mathbf{B}}(V)$ as a $\mathbf{B}^{G_K}$-vector space. On the other hand, the object $D_{\mathbf{B}}(V)$ inherits additional structures from the structures on $\mathbf{B}$ (such as filtration, φ, or monodromy operator N, etc.).

For example, if V is Hodge-Tate (i.e., $\mathbf{B}_{\text{HT}}$-admissible), then we define $i \in \mathbf{Z}$ to be a *Hodge-Tate weight of* V *of multiplicity* m if $(\mathbf{C}_p(-i) \otimes_{\mathbf{Q}_p} V)^{G_K} \neq 0$ and $\dim_K(\mathbf{C}_p(-i) \otimes_{\mathbf{Q}_p} V)^{G_K} = m$. If $k \in \mathbf{Z}$, then $V = \mathbf{Q}_p(k)$ is Hodge-Tate, with unique Hodge-Tate weight k of multiplicity 1. If V is a p-adic representation of G_K, then the K-vector space $D_{\text{dR}}(V)$ is endowed with the filtration of sub-K-vector spaces $\operatorname{Fil}^i D_{\text{dR}}(V) := (t^i \mathbf{B}_{\text{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{G_K}$. If V is de Rham, then it is also Hodge-Tate; in this case, the Hodge-Tate weights of V

are the integers $i \in \mathbf{Z}$ such that $\operatorname{Fil}^{-i} D_{\mathrm{dR}}(V) \neq \operatorname{Fil}^{-i+1} D_{\mathrm{dR}}(V)$, and the multiplicity of i is equal to $\dim_K(\operatorname{Fil}^{-i} D_{\mathrm{dR}}(V)/\operatorname{Fil}^{-i+1} D_{\mathrm{dR}}(V))$. In the same spirit, it turns out that a semi-stable representation V can be completely described by simple invariants on $D_{\mathrm{st}}(V)$; more will be said about this in paragraph 2.1.5 below.

If $\mathbf{B}$ contains $\overline{\mathbf{Q}}_p$ as a G_K-stable sub-ring (for example if $\mathbf{B}$ is $\mathbf{B}_{\mathrm{dR}}$ or $\mathbf{B}_{\mathrm{HT}}$), then the notion of $\mathbf{B}$-admissibility is equivalent to the notion of potential $\mathbf{B}$-admissibility by Galois descent. The various $G_{\mathbf{Q}_p}$-equivariant inclusions and morphisms between the period rings from paragraphs 2.1.2 and 2.1.3 give the following relationships:

$$\begin{array}{ccccccc}
\text{pot. crystalline} & \Longrightarrow & \text{pot. semi-stable} & \Longrightarrow & \text{de Rham} & \Longrightarrow & \text{Hodge-Tate} \\
\Uparrow & & \Uparrow & & & & \\
\text{crystalline} & \Longrightarrow & \text{semi-stable} & & & &
\end{array}$$

It is also known that every de Rham representation is potentially semi-stable, due to the results of Y. André [**And02**], L. Berger [**Ber02**], K. Kedlaya [**Ked00**], and Z. Mebkhout [**Meb02**]. This was formerly known as the p-adic local monodromy conjecture of Fontaine. Aside from this, the other implications in the above diagram are strict. For example, the following appears as proposition 3.10 of [**Fon03**].

EXAMPLE 2.1.4.2. *If* $0 \to \mathbf{Q}_p \to V \to \mathbf{Q}_p(1) \to 0$ *is a non-split extension in* $\operatorname{Rep}_{\mathbf{Q}_p}(G_K)$, *then* V *is Hodge-Tate but not de Rham.*

The following describes the $*$-admissible (for $* \in \{\mathbf{B}_{\mathrm{cris}}, \mathbf{B}_{\mathrm{st}}, \mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{HT}}\}$) 1-dimensional linear $\mathbf{Q}_p$-representations of G_K (see, for example, propositions 4.3 and 5.6 of [**Fon04**]).

PROPOSITION 2.1.4.3. *Let* $\eta : G_K \to \mathbf{Q}_p^\times$ *be a continuous character and let* $V = \mathbf{Q}_p(\eta)$.

(1) The following conditions are equivalent:
   - *(a)* V *is crystalline,*
   - *(b)* V *is semi-stable,*
   - *(c)* $\eta = \chi^h \eta'$ *for some* $h \in \mathbf{Z}$ *and some non-ramified character* η'.

(2) The following conditions are equivalent:
   - *(a)* V *is de Rham,*
   - *(b)* V *is Hodge-Tate,*
   - *(c)* $\eta = \chi^h \eta'$ *for some* $h \in \mathbf{Z}$ *and some finitely-ramified character* η'.

Propostion 1.5.2 of [**Fon94b**] implies the following.

PROPOSITION 2.1.4.4. *The full sub-category of* $\operatorname{Rep}_E(G_K)$ *of crystalline (resp. semi-stable, de Rham, Hodge-Tate) representations is stable by direct sum, sub-objects, quotients, tensor product, and duals.*

The notion of (potential) $\mathbf{B}$-admissibility for $\mathbf{B} \in \{\mathbf{B}_{\mathrm{cris}}, \mathbf{B}_{\mathrm{st}}, \mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{HT}}\}$ and the basic properties of the functors $D_{\mathbf{B}}(\text{-})$ attached to period rings are developed in §1 of [**Fon94b**].

If $V \in \mathrm{Rep}_E(G_K)$ and if L/K is finite, then the $\mathbf{B}^{G_L}$-vector space $D_{\mathbf{B},L}(V) = (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{G_L}$ is also a $\mathbf{B}^{G_L} \otimes_{\mathbf{Q}_p} E$-module. The following appears in §6.3 of [**Col08b**].

PROPOSITION 2.1.4.5. *Let $V \in \mathrm{Rep}_E(G_K)$ and let L/K be a finite extension.*

(1) The $L_0 \otimes_{\mathbf{Q}_p} E$-modules $D_{\mathrm{cris},L}(V)$ and $D_{\mathrm{st},L}(V)$ are free.
(2) If V is de Rham, then $D_{\mathrm{dR},L}(V)$ is a free $L \otimes_{\mathbf{Q}_p} E$-module.

2.1.5. Semi-stable representations and admissible filtered (φ, N)-modules. Let $E/\mathbf{Q}_p$ and $K/\mathbf{Q}_p$ be finite sub-extensions of $\overline{\mathbf{Q}}_p/\mathbf{Q}_p$. In this section, we let $K_{0,E}$ denote $K_0 \otimes_{\mathbf{Q}_p} E$. A *filtered E-(φ, N)-module over K* is a free $K_{0,E}$-module D of finite rank endowed with the following structures:

(1) a bijective additive map $\varphi : D \to D$ which is E-linear and K_0-semilinear for the Frobenius $\sigma : K_0 \to K_0$,
(2) a $K_{0,E}$-linear operator $N : D \to D$ such that $N\varphi = p\varphi N$,
(3) a family $\{\mathrm{Fil}^i(K \otimes_{K_0} D)\}_{i \in \mathbf{Z}}$ of sub-$(K \otimes_{\mathbf{Q}_p} E)$-modules of $K \otimes_{K_0} D$ such that
   (a) $\mathrm{Fil}^{i+1}(K \otimes_{K_0} D) \subset \mathrm{Fil}^i(K \otimes_{K_0} D)$ for all $i \in \mathbf{Z}$,
   (b) $\mathrm{Fil}^i(K \otimes_{K_0} D) = \{0\}$ for $i \gg 0$ and $\mathrm{Fil}^i(K \otimes_{K_0} D) = K \otimes_{K_0} D$ for $i \ll 0$

The relation $N\varphi = p\varphi N$ implies that N is nilpotent. Every filtered E-(φ, N)-module over K may also be viewed as a filtered $\mathbf{Q}_p$-(φ, N)-module over K by forgetting the E-linear structure.

If D and D' are filtered E-(φ, N)-modules over K, then a *morphism* $f : D \to D'$ is a morphism of $K_{0,E}$-modules such that f commutes with φ and N on D and D', and such that the K-linearisation $f : K \otimes_{K_0} D \to K \otimes_{K_0} D'$ satisfies $f(\mathrm{Fil}^i(K \otimes_{K_0} D)) \subset \mathrm{Fil}^i(K \otimes_{K_0} D')$ for all $i \in \mathbf{Z}$.

If D and D' are filtered E-(φ, N)-modules over K, then the free $K_{0,E}$-module $D \otimes_{K_{0,E}} D'$ together with

(1) the map $\varphi : D \otimes_{K_{0,E}} D' \to D \otimes_{K_{0,E}} D'$ defined by $\varphi : d \otimes d' \mapsto \varphi(d) \otimes \varphi(d')$,
(2) the map $N : D \otimes_{K_{0,E}} D' \to D \otimes_{K_{0,E}} D'$ defined by $N : d \otimes d' \mapsto N(d) \otimes 1 + 1 \otimes N(d')$,
(3) a filtration defined by $\mathrm{Fil}^i((K \otimes_{K_0} D) \otimes_{K_{0,E}} (K \otimes_{K_0} D')) = \sum_{a+b=i} \mathrm{Fil}^a(K \otimes_{K_0} D) \otimes_{K_{0,E}} \mathrm{Fil}^a(K \otimes_{K_0} D')$ for all $i \in \mathbf{Z}$.

is a filtered E-(φ, N)-module, which we refer to as the tensor product of D and D'.

The basic properties of filtered E-(φ, N)-modules over K are developed in chapter 4 of [**Fon94b**], in [**ColFon00**], §5.1-5.3 of [**Fon04**], §3.1 of [**BreMez**], and §2 of [**Col10e**], among other places. We refer to §5.1 of [**Fon04**] for the notions of exact sequences and quotients in the category of filtered E-(φ, N)-modules over K.

EXAMPLE 2.1.5.1. *If $V \in \mathrm{Rep}_E(G_K)$ and if L/K is finite, then $D_{\mathrm{st},L}(V) = (\mathbf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{G_L}$ is a filtered E-(φ, N)-module over L. If $V|_{G_L}$ is semi-stable, then $\mathrm{rank}_{L_{0,E}} D_{\mathrm{st},L}(V) = \dim_E V$.*

The following example appears in remark 3.1.1.4 of [**BreMez**].

EXAMPLE 2.1.5.2. *If $p \not\equiv 1 \bmod 4$ is a prime, then let $E := \mathbf{Q}_p(\sqrt{-1})$ denote the unramified quatratic extension of $\mathbf{Q}_p$. If $D = (E \otimes_{\mathbf{Q}_p} E) \cdot e$, then defining*

$$\varphi(e) = (\tfrac{p+1}{2} \otimes 1 + \tfrac{p-1}{2}\sqrt{-1} \otimes \sqrt{-1}) \cdot e, \;\; N(e) = 0, \;\; \mathrm{Fil}^i D = \begin{cases} D & \text{for } i \leq 0 \\ (E \otimes_{\mathbf{Q}_p} E) \cdot [(1 \otimes 1 + \sqrt{-1} \otimes \sqrt{-1}) \cdot e] & \text{for } i = 1 \\ \{0\} & \text{for } 2 \leq i \end{cases}$$

makes D into a filtered E-(φ, N)-module over E.

In the above example, the $(E \otimes_{\mathbf{Q}_p} E)$-module $\mathrm{Fil}^1 D$ is annihilated by the element $1 \otimes \sqrt{-1} + \sqrt{-1} \otimes 1$, and thus is not free.

Admissible filtered E-(φ, N)-modules. Let D be a filtered E-(φ, N)-module over K. When viewed as a filtered $\mathbf{Q}_p$-(φ, N)-module, the dimension of D as a K_0-vector space is $d = [E : \mathbf{Q}_p] \, \mathrm{rank}_{K_{0,E}}(D)$ and $\bigwedge^d_{K_0} D = K_0 \cdot e$ is a filtered $\mathbf{Q}_p$-(φ, N)-module of rank 1. We let $t_H(D)$ denote the maximal integer $i \in \mathbf{Z}$ such that $\mathrm{Fil}^i(K \otimes_{K_0} \bigwedge^d_{K_0} D) \neq 0$, and we let $t_N(D) = v_p(\lambda)$, where $\lambda \in K_0$ is defined by $\varphi(e) = \lambda \cdot e$ (the integer $t_N(D)$ depends only on D). One says that D is *weakly admissible* if

(i) $t_H(D') \leq t_N(D')$ for all sub-filtered $\mathbf{Q}_p$-(φ, N)-modules $D' \subset D$ over K,
(ii) and $t_H(D) = t_N(D)$.

If $V \in \mathrm{Rep}_E(G_K)$ is semi-stable, then $D_{\mathrm{st},K}(V)$ as in 2.1.5.1 is weakly admissible. It is shown in proposition 3.1.1.5 of [**BreMez**] that if D is a filtered E-(φ, N)-module over K, then D is weakly admissible if and only if $t_H(D) = t_N(D)$ and $t_H(D') \leq t_N(D')$ for all sub-filtered E-(φ, N)-modules $D' \subset D$ over K.

The following theorem of Colmez and Fontaine (see [**ColFon00**]) allows one to translate questions about the category $\mathrm{Rep}^{\mathrm{st}}_E(G_K)$ of semi-stable representations in $\mathrm{Rep}_E(G_K)$ into questions about filtered E-(φ, N)-modules over K.

THEOREM 2.1.5.3. *The functor $D_{\mathrm{st},K}$ induces an equivalence of categories between $\mathrm{Rep}^{\mathrm{st}}_E(G_K)$ and the full sub-category of weakly admissible filtered E-(φ, N)-modules over K; crystalline representations in $\mathrm{Rep}_E(G_K)$ correspond to weakly admissible filtered E-(φ, N)-modules over K for which $N = 0$. On $\mathrm{Rep}^{\mathrm{st}}_E(G_K)$, the functor $D_{\mathrm{st},K}$ is compatible with direct sums, tensor products, and exact sequences.*

2.2. Examples

Here are some examples of admissible linear $\mathbf{Q}_p$-representations of dimension > 1.

2.2.1. Lubin-Tate characters. Let $K/\mathbf{Q}_p$ be a finite extension and let $\pi \in \mathcal{O}_K$ be a uniformizer. One may attach to π a formal Lubin-Tate $\mathcal{O}_K$-module $\mathrm{LT}_\pi \in \mathcal{O}_K[[X,Y]]$ and the Lubin-Tate character $\chi_\pi : G_K \to \mathcal{O}_K^\times$ as in 1.2.1.

PROPOSITION 2.2.1.1. *The linear K-representation $V = K(\chi_\pi)$ of G_K is Hodge-Tate with Hodge-Tate weights 1 with multiplicity 1 and 0 with multiplicity $[K : \mathbf{Q}_p] - 1$*

The above proposition is a special case of more general results on Hodge-Tate decompositions for p-divisible groups given by Tate in [**Tat66**]. A proof along these lines is given, for example, in lemma 2 in appendix A5 to chapter III of [**Ser68**]. See also [**Col93**] and [**Fou09**] for another perspective. More precisely, what is shown in [**Ser68**] is that the proposition is true after restriction to $G_{K^{\mathrm{Gal}}}$, but $\mathbf{B}_{\mathrm{HT}}$-admissibility and Hodge-Tate weights are invariant by restriction to an open sub-group.

2.2.2. Representations coming from geometry. If $X/\mathbf{Q}_p$ is an algebraic variety, then $V = H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$ is a continuous linear $\mathbf{Q}_p$-representation of $G_{\mathbf{Q}_p}$. If $X/\mathbf{Q}_p$ is proper and smooth, then the i-th hypercohomology $H^i_{\mathrm{dR}}(X/\mathbf{Q}_p)$ of the de Rham complex $\Omega^\bullet_{X/\mathbf{Q}_p}$ has a natural structure as a filtered $\mathbf{Q}_p$-vector space of finite dimension. The following was formerly a conjecture of Fontaine and Jannsen (see §6.2 of [**Fon94b**] and [**Tsu02**] for a survey), and has been proven in various cases and generalities by Faltings, Fontaine, Fontaine-Messing, Hyodo, Nizioł, Tsuji, and more recently by Beilinson and Bhatt.

THEOREM 2.2.2.1. *If $X/\mathbf{Z}_p$ is a proper smooth variety, then there is an isomorphism of semi-linear $\mathbf{B}_{\mathrm{dR}}$-representations of $G_{\mathbf{Q}_p}$*

$$\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} H^i_{\acute{e}t}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p) \simeq \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} H^i_{\mathrm{dR}}(X/\mathbf{Q}_p)$$

which is compatible with filtrations, so that $V = H^i_{\acute{e}t}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$ is de Rham and $D_{\mathrm{dR}}(V) \simeq H^i_{\mathrm{dR}}(X/\mathbf{Q}_p)$ as filtered $\mathbf{Q}_p$-vector spaces.

(1) If $X/\mathbf{Z}_p$ is of good reduction, then V is crystalline and in this case $D_{cris}(V) = H^i_{\mathrm{cris}}(X/\mathbf{Z}_p)$.
(2) If $X/\mathbf{Z}_p$ is of bad semi-stable reduction, then V is semi-stable and $D_{\mathrm{st}}(V) = H^i_{\mathrm{log\text{-}cris}}(X)$.

2.2.3. Representations coming from modular forms. Let $k, N \geq 1$ and let $f \in S_k(\Gamma_1(N))$ be an eigenform with character $\epsilon : (\mathbf{Z}/N\mathbf{Z}) \to \mathbf{C}^\times$. Let p be a prime and let $E/\mathbf{Q}_p$ be the finite extension, of degree $d = [E : \mathbf{Q}_p]$ say, generated by the images of the Hecke eigenvalues by a fixed embedding $\iota : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. Let $\rho_{f,p} : G_{\mathbf{Q}} \to \mathrm{GL}_2(E)$ be the p-adic representation attached to f with underlying E-vector space $V_{f,p}$, as described in section 1.2.3.

PROPOSITION 2.2.3.1. *The representation $\rho_{f,p}|_{D_p}$ is potentially semi-stable, and if $p \nmid N$, then it is crystalline and $D_{\mathrm{cris}}(V_{f,p}^*) = D_{k,a_p}$ is the 2 dimensional E-vector space $E \cdot e_1 \oplus E \cdot e_2$ with filtered E-φ-module structure given by*

$$\mathrm{Mat}(\varphi) = \begin{pmatrix} 0 & -1 \\ \epsilon(p)p^{k-1} & a_p \end{pmatrix} \quad \text{and} \quad \mathrm{Fil}^i D = \begin{cases} D_{k,a_p} & \text{for } i \leq 0 \\ Ee_1 & \text{for } 1 \leq i \leq k-1 \\ \{0\} & \text{for } i \geq k \end{cases}$$

In particular, the Hodge-Tate weights of $\rho_{f,p}|_{D_p}$ are 0 (with multiplicity d) and $k-1$ (with multiplicity d).

The fact that $\rho_{f,p}|_{D_p}$ is potentially semi-stable is is due to Saito (see [**Sai97**]) and the description of $D_{\mathrm{cris}}(V_{f,p}^*)$ when $p \nmid N$ is due to Scholl (see [**Sch90**]). Saito also established local Langlands compatibility of Weil-Deligne representations for f at p.

2.3. Sen's theory of $\mathbf{C}_p$-representations

2.3.1. Sen-Tate theory for $\mathbf{C}_p$-representations. Let $K/\mathbf{Q}_p$ be a finite extension and let $\chi : G_{\mathbf{Q}_p} \to \mathbf{Z}_p^\times$ be the cyclotomic character. If $H_K = \ker(\chi|_{G_K} : G_K \to \mathbf{Z}_p^\times)$, then $H_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$, where $K_\infty = \bigcup_{n\geq 1} K(\zeta_{p^n}) \subset \overline{\mathbf{Q}}_p$. The group $\Gamma_K = G_K/H_K$ is isomorphic to an open subgroup of $\mathbf{Z}_p^\times$ via χ. The following theorem is due to Sen, and appears as theorem 2 of [**Sen80**]:

THEOREM 2.3.1.1. *If X is a semi-linear $\mathbf{C}_p$-representation of G_K of dimension d, then X^{H_K} is a d-dimensional semi-linear $\widehat{K}_\infty$-representation of Γ_K and $X = \mathbf{C}_p \otimes_{\widehat{K}_\infty} X^{H_K}$ as semi-linear $\mathbf{C}_p$-representations of G_K.*

If $W \in \mathrm{Rep}_{\widehat{K}_\infty}(G_K)$, then let W^{fini} denote the set of $w \in W$ such that the Γ_K-orbit of w in W generates a finite-dimensional K-vector space. The set W^{fini} is a K_∞-vector space and is stable by the action of Γ_K. The following is theorem 3 of [**Sen80**]:

THEOREM 2.3.1.2. *If $W \in \mathrm{Rep}_{\widehat{K}_\infty}(\Gamma_K)$, then $W = \widehat{K}_\infty \otimes_{K_\infty} W^{\mathrm{fini}}$.*

Theorems 2.3.1.1 and 2.3.1.2, taken together, give the following.

THEOREM 2.3.1.3. *The functor $D_{\mathrm{sen}} : \mathrm{Rep}_{\mathbf{C}_p}(G_K) \to \mathrm{Rep}_{K_\infty}(\Gamma_K)$ defined by $X \mapsto D_{\mathrm{sen}}(X) := (X^{H_K})^{\mathrm{fini}}$ is an equivalence of categories, with quasi-inverse given by the extension of scalars functor $\mathbf{C}_p \otimes_{\mathbf{Q}_p}$ - : $\mathrm{Rep}_{K_\infty}(\Gamma_K) \to \mathrm{Rep}_{\mathbf{C}_p}(\Gamma_K)$.*

Proposition 2.3.2.5 below gives some properties of the functor D_{sen}(-). Theorem 2.3.1.1 is applicable in particular when $X = \mathbf{C}_p \otimes_{\mathbf{Q}_p} V$ for some $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, and theorem 2.3.1.2 may be applied to $(\mathbf{C}_p \otimes_{\mathbf{Q}_p} V)^{H_K}$. In this note, we also write (abusively) $D_{\mathrm{sen}}(V)$ for $(\mathbf{C}_p \otimes_{\mathbf{Q}_p} V)^{H_K}$. We signal to the reader that the notation used in this text differs from the notation of in [**Sen80**].

2.3.2. Sen's operator and generalized Hodge-Tate weights. The cyclotomic character induces an inclusion $\Gamma_K = G_K/H_K \hookrightarrow \mathbf{Z}_p^\times$, so that we may write $\Gamma_K = \Sigma \times \Gamma_0$ where Σ is a finite abelian group and $\Gamma_0 \simeq \mathbf{Z}_p$. Let $\gamma \in \Gamma_0$ be a topological generator. If $D \in \mathrm{Rep}_{K_\infty}(\Gamma_K)$ and if $\mathcal{E} = (e_1, \dots, e_d)$ is a K_∞-basis, then lemma 2 of [**Sen80**] shows that there is a finite sub-extension K'/K of K_∞/K such that $\mathrm{Mat}(g|\mathcal{E}) \in \mathrm{GL}_d(K')$ for all $g \in \Gamma_K$; i.e., $W' = \bigoplus_{i=1}^d K' \cdot e_i$ is stable by Γ_K. If $\Gamma' \subset \Gamma_K$ is the open sub-group such that $K' = K_\infty^{\Gamma'}$, then we have a linear representation $\rho' : \Gamma' \to \mathrm{GL}_d(K') = \mathrm{Aut}_{K'}(W')$ and there is an operator $T \in \mathrm{End}_{K'}(W')$ such that $\rho'(\gamma^z) = \exp(zT)$ for all $z \in \mathbf{Z}_p$ sufficiently close to 0. Put $\Theta = \frac{1}{\log_p \chi(\gamma)} T$.

In this way, to each $D \in \mathrm{Rep}_{K_\infty}(\Gamma_K)$, Sen associates a K_∞-linear operator $\Theta_D : D \to D$. The following is theorem 4 of [**Sen80**], and shows how Θ (locally) describes the action of Γ_K on D.

THEOREM 2.3.2.1. *If $D \in \mathrm{Rep}_{K_\infty}(\Gamma_K)$, then $\Theta_D : D \to D$ is the unique K_∞-linear operator such that for all $x \in D$, there is an open sub-group $\Gamma_{K,x} \subset \Gamma_K$ such that $\gamma(x) = \exp(\log_p(\chi(\gamma)) \cdot \Theta)(x)$ for all $\gamma \in \Gamma_{K,x}$.*

In particular, Θ_D commutes with the action of Γ_K on D, and therefore its characteristic polynomial P_Θ has coefficients in $K = K_\infty^{\Gamma_K}$. The roots of P_Θ in $\overline{\mathbf{Q}}_p$ are called the *Sen weights of D*, or sometimes the *generalized Hodge-Tate weights of D*.

EXAMPLE 2.3.2.2. *If $X = \mathbf{C}_p(i)$, then $D_{\mathrm{sen}}(X) = K_\infty(i)$ and $\Theta : D_{\mathrm{sen}}(X) \to D_{\mathrm{sen}}(X)$ is given by multiplication by i.*

If $\langle \chi \rangle$ denotes the projection of χ onto the second factor of $\mathbf{Z}_p^\times = [\mathbf{F}_p^\times] \times (1 + p\mathbf{Z}_p)$ for p odd (resp. the second factor of $\mathbf{Z}_2^\times = \{\pm 1\} \times (1 + 4\mathbf{Z}_2)$ if $p = 2$) and if $s \in \mathbf{Z}_p$, then $\langle \chi(g) \rangle^s$ converges in $\mathbf{Z}_p^\times$ for all $g \in G_{\mathbf{Q}_p}$ and therefore defines a character $\langle \chi \rangle^s : G_{\mathbf{Q}_p} \to \mathbf{Z}_p^\times$.

EXAMPLE 2.3.2.3. *If $\eta = \mu \langle \chi \rangle^s$ for some $s \in \mathbf{Z}_p$ and a finitely ramified character $\mu : G_{\mathbf{Q}_p} \to \mathbf{Z}_p^\times$, then the unique generalized Hodge-Tate weight of $\mathbf{C}_p(\eta)$ is s.*

The following is the corollary in §2.3 of [**Sen80**], and it gives a re-interpretation of the notion of Hodge-Tate representation in terms of Sen's theory.

PROPOSITION 2.3.2.4. *If $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, then V is Hodge-Tate with Hodge-Tate weights $h_1, \dots, h_d$ if and only if Θ_V is semi-simple with integer eigenvalues $h_1, \dots, h_d$.*

The above proposition explains the teminology "generalized Hodge-Tate weights." Here are some properties of Sen's operator and generalized Hodge-Tate weights, which are proven in §2.2 of [**Sen80**].

PROPOSITION 2.3.2.5. *Let $D, D' \in \mathrm{Rep}_{K_\infty}(\Gamma_K)$.*

(1) If L/K is a finite extension, then $D_{\mathrm{sen}}(D|_{G_L}) = L_\infty \otimes_{K_\infty} D_{\mathrm{sen}}(D)$ and $\Theta_{D|_{G_L}}$ is the L_∞-linearization of Θ_D. In particular, the Sen weights of D are the same as the Sen weights of $D|_{G_L}$.

(2) $D_{\mathrm{sen}}(D \oplus D') = D_{\mathrm{sen}}(D) \oplus D_{\mathrm{sen}}(D')$ and $\Theta_{D\oplus D'} = \Theta_D \oplus \Theta_{D'}$.

(3) $D_{\mathrm{sen}}(D \otimes D') = D_{\mathrm{sen}}(D) \otimes D_{\mathrm{sen}}(D')$ and $\Theta_{D\otimes D'} = \Theta_D \otimes \mathrm{Id} + \mathrm{Id} \otimes \Theta_{D'}$. In particular, the Sen weights of $D \otimes D'$ are the elements of the form $\alpha + \beta$ where α is a Sen weight of D and β is a Sen weight of D'.

(4) If D' is a sub-object of D, then $\Theta_{D'} = \Theta_D|_{D'}$, and $\Theta_{D/D'}$ is the canonical operator induced by Θ_D.

Sen's theory has been generalized to apply to different rings than $\mathbf{C}_p$ (see §3.3 of [**Col03**] and §3 of [**BerCol02**]). We also briefly mention that Colmez has given a reinterpretation of Sen's theory more in the spirit of the period ring formalism (see [**Col94**]).

2.4. Fontaine's theories of $\mathbf{C}_p$-representations and $\mathbf{B}_{\mathrm{dR}}$-representations

2.4.1. Fontaine's theory of semilinear $\mathbf{C}_p$-representations. Let $\mathrm{Rep}_{\mathbf{C}_p}(G_K)$ denote the category of semi-linear $\mathbf{C}_p$-representations of G_K. We will say that $W \in \mathrm{Rep}_{\mathbf{C}_p}(G_K)$ is *Hodge-Tate* if there is a $\mathbf{C}_p$-basis $(e_1, \ldots, e_d)$ of W and integers $h_1, \ldots, h_d$ such that $g(e_i) = \chi(g)^{h_i} e_i$ for all $i \in \{1, \ldots, d\}$. When $W = \mathbf{C}_p \otimes_{\mathbf{Q}_p} V$ for some $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, W is Hodge-Tate if and only if V is $\mathbf{B}_{\mathrm{HT}}$-admissible; one also says that V is *Hodge-Tate* in this case.

We now describe Fontaine's classification of $\mathbf{C}_p$-representations as in §2.6 of [**Fon04**]. A G_K*-orbit* $A \subset \overline{\mathbf{Q}}_p$ is a sub-set of $\overline{\mathbf{Q}}_p$ of the form $O_x = \{gx | g \in G_K\}$ for some $x \in \overline{\mathbf{Q}}_p$; note that O_x is necessarily finite and stable by the action of G_K. If $W \in \mathrm{Rep}_{\mathbf{C}_p}(G_K)$ and if $A \subset \overline{\mathbf{Q}}_p$, then Fontaine says that W is *of type* S_A if every element of the multiset $\mathrm{Wt}(W)$ of generalized Hodge-Tate weights of W is an element of A. If $W \in \mathrm{Rep}_{\mathbf{C}_p}(G_K)$ is indecomposible, then the set $\mathrm{Wt}(W)$ of Sen weights of W is a single G_K-orbit A in $\overline{\mathbf{Q}}_p$. If $d \geq 0$, then let $\mathbf{Z}_p(0; d)$ denote the $\mathbf{Z}_p$-module of polynomials of degree[1] $\leq d$ in a formal variable $X = \log t$, on which $G_{\mathbf{Q}_p}$ acts by $g(\log t) = \log t + \log_p(\chi(g))$. If $W \in \mathrm{Rep}_{\mathbf{C}_p}(G_K)$ is irreducible, then $W \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(0; d)$ is indecomposible; it is irreducible if and only if $d = 0$.

Fontaine shows in proposition 2.13 of [**Fon04**] that if $A \subset \overline{\mathbf{Q}}_p$ is a G_K-orbit, then there is a unique irreducible object $\mathbf{C}_p[A] \in \mathrm{Rep}_{\mathbf{C}_p}(G_K)$ of type S_A. For example, if $A = \{i\}$ for some integer $i \in \mathbf{Z}$, then $\mathbf{C}_p[A]$ is just $\mathbf{C}_p(i)$. If V is a 2-dimensional $\mathbf{Q}_p$-vector space

[1] In [**Fon04**], $\mathbf{Z}_p(0; d)$ denotes the $\mathbf{Z}_p$-module of polynomials of degree $< d$, but we have shifted d to make some of our notation later work out more cleanly.

on which $g \in G_{\mathbf{Q}_p}$ acts on a basis $\mathcal{E} = (e_1, e_2)$ by the matrix

$$\mathrm{Mat}(g|\mathcal{E}) = \begin{pmatrix} 1 & \log_p(\chi(g)) \\ 0 & 1 \end{pmatrix}$$

then $\mathbf{C}_p \otimes_{\mathbf{Q}_p} V = \mathbf{C}_p[\{0\}; 1]$.

The following is theorem 2.14 of [**Fon04**]:

THEOREM 2.4.1.1. *If $W \in \mathrm{Rep}_{\mathbf{C}_p}(G_K)$ is non-zero, then there are unique G_K-orbits $A_1, \ldots, A_r \subset \overline{\mathbf{Q}}_p$ and integers $d_1, \ldots, d_r \in \mathbf{N}$ and $h_i = h_{A_i,d_i}(W)$ such that $W = \bigoplus_{i=1}^r \mathbf{C}_p[A_i; d_i]^{h_i}$.*

If $W \in \mathrm{Rep}_{\mathbf{C}_p}(G_K)$ admits a decomposition as in the above theorem, then W is Hodge-Tate if and only if for all i, $A_i = \{\alpha_i\}$ with $\alpha_i \in \mathbf{Z}$ and $d_i = 0$ (see p. 45 of loc. cit.). In this case, $h_{\{\alpha_i\},0}(W)$ is the multiplicity of α_i as a generalized Hodge-Tate weight of W.

2.4.2. Fontaine's theory of $\mathbf{B}_{\mathrm{dR}}$-representations. Let $\mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}}(G_K)$ denote the category of semi-linear $\mathbf{B}_{\mathrm{dR}}$-representations of G_K. We will say that $W \in \mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}}(G_K)$ is *de Rham* if it is trivial as a semi-linear $\mathbf{B}_{\mathrm{dR}}$-representation of G_K. If $W \in \mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}}(G_K)$, then a *$G_K$-stable lattice of* W is a G_K-stable sub-$\mathbf{B}_{\mathrm{dR}}^+$-module $W^+ \subset W$ of finite type such that $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{dR}}^+} W^+$. For example, if $V \in \mathrm{Rep}_E(G_K)$, then $\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$ is a G_K-stable lattice of $W = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$. More generally, if $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^+)$ is a $B_{|K}^{\otimes E}$-pair, then W_{dR}^+ is (by definition) a G_K-stable lattice of $W_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR},E} \otimes_{\mathbf{B}_{\mathrm{e},E}} W_{\mathrm{e}}$.

Let W be a $\mathbf{B}_{\mathrm{dR}}$-representation of G_K and let $\mathcal{W} \subset W$ be a G_K-stable $\mathbf{B}_{\mathrm{dR}}^+$-lattice. The quotient $\overline{\mathcal{W}} := \mathcal{W}/t\mathcal{W}$ is a $\mathbf{C}_p$-representation of G_K, and we may therefore associate to it the set $\mathrm{Wt}(\overline{\mathcal{W}})$ of its Sen weights, which is a set of elements of $\overline{\mathbf{Q}}_p$ of cardinal $\dim_{\mathbf{B}_{\mathrm{dR}}} W$ which is stable by the action of G_K. The following proposition shows that all lattices of W have the same Sen weights up to integers, so that the set of Sen weights modulo $\mathbf{Z}$ of a lattice $\mathcal{W}$ is an invariant of W.

PROPOSITION 2.4.2.1. *Let W be a $\mathbf{B}_{\mathrm{dR}}$-representation of G_K. If $\mathcal{W}$ and $\mathcal{W}'$ are two G_K-stable $\mathbf{B}_{\mathrm{dR}}^+$-lattices of W, then each Sen weight of $\overline{\mathcal{W}'}$ may be written in the form $\alpha + i$ where α is a Sen weight of $\overline{\mathcal{W}}$ and $i \in \mathbf{Z}$.*

PROOF. Let $c \geq 0$ be an integer such that the lattice $t^c\mathcal{W}'$ is contained in $\mathcal{W}$ and let $c' \geq 0$ be an integer such that the lattice $t^{c'}\mathcal{W}$ is contained in $t^c\mathcal{W}'$.

Consider the sequence of G_K-stable lattices :

$$t^c\mathcal{W}' = t^c\mathcal{W}' + t^{c'}\mathcal{W} \subset t^c\mathcal{W}' + t^{c'-1}\mathcal{W} \subset \ldots \subset t^c\mathcal{W}' + t\mathcal{W} \subset t^c\mathcal{W}' + \mathcal{W} = \mathcal{W},$$

and let $\mathcal{X}_k$ denote the lattice $t^c\mathcal{W}' + t^{c'-k}\mathcal{W}$ (for $0 \leq k \leq c'$). We have G_K-equivariant inclusions $t\mathcal{X}_{k+1} \subset \mathcal{X}_k \subset \mathcal{X}_{k+1}$ for $k = 0, 1, \ldots, c' - 1$; we therefore have exact sequences of $\mathbf{C}_p$-representations :

$$\mathcal{X}_{k+1}/t\mathcal{X}_{k+1} \to \mathcal{X}_{k+1}/\mathcal{X}_k \to 0 \quad \text{and} \quad 0 \to t\mathcal{X}_{k+1}/t\mathcal{X}_k \to \mathcal{X}_k/t\mathcal{X}_k \to \mathcal{X}_{k+1}/t\mathcal{X}_{k+1}$$

which, taken together with (i) and (iii) of proposition 4.2.1.2, and since $x \mapsto tx$ induces an isomorphism of $(\mathcal{X}_{k+1}/\mathcal{X}_k)(1)$ onto $t\mathcal{X}_{k+1}/t\mathcal{X}_k$, implies that $\mathrm{Wt}(\overline{\mathcal{X}_k}) \subset \mathrm{Wt}(\overline{\mathcal{X}_{k+1}}) \cup (\mathrm{Wt}(\overline{\mathcal{X}_{k+1}}) + 1)$. By recurrence, the Sen weights of $\mathcal{X}_0 = t^c\mathcal{W}'$ are all of the form $\alpha + i$, where α is a Sen weight of $\overline{\mathcal{X}_{c'}} = \overline{\mathcal{W}}$ and i is an integer. Again by (iii) of proposition 4.2.1.2, the Sen weights of $\mathcal{W}'$ are of the form $\alpha + i$ where α is a Sen weight of $\overline{\mathcal{W}}$. $\square$

If W is a $\mathbf{B}_{\mathrm{dR}}$-representation of G_K and if $\mathcal{W} \subset W$ is a G_K-stable lattice, then the multiset $\mathrm{Wt}_{\mathrm{dR}}(W)$ of *de Rham weights* of W is the multiset of images of elements of $\mathrm{Wt}(\overline{\mathcal{W}})$ modulo $\mathbf{Z}$; by proposition , this multiset depends only on W and not on the lattice $\mathcal{W}$.

For each G_K-orbit $A \subset \overline{\mathbf{Q}}_p$, Fontaine constructed a simple object $\mathbf{C}_p[A] \in \mathrm{Rep}_{\mathbf{C}_p}(G_K)$ (see the previous paragraph), which corresponds (via Sen's theory) to a simple object $K_\infty[A] \in \mathrm{Rep}_{K_\infty}(\Gamma_K)$. Fontaine defines $\mathbf{B}_{\mathrm{dR}}[A] = \mathbf{B}_{\mathrm{dR}} \otimes_{K_\infty} K_\infty[A]$; the set of de Rham weights of W is the image of A modulo $\mathbf{Z}$. Fontaine showed in proposition 3.18 of [**Fon04**] that for G_K-orbits $A, A' \subset \overline{\mathbf{Q}}_p$, one has $\mathbf{B}_{\mathrm{dR}}[A] \simeq \mathbf{B}_{\mathrm{dR}}[A']$ if and only if $A = A' \bmod \mathbf{Z}$, and (ii) of Theorem 3.19 of loc. cit. asserts that for each $d \geq 0$, the $\mathbf{B}_{\mathrm{dR}}$-representation $\mathbf{B}_{\mathrm{dR}}[A; d] = \mathbf{B}_{\mathrm{dR}}[A] \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(0; d)$ is indecomposible, and it is simple if and only if $d = 0$. The following is part of theorem 3.19 of [**Fon04**].

THEOREM 2.4.2.2. *If $W \in \mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}}(G_K)$ is non-zero, then there are G_K-orbits $A_1, \ldots, A_r \subset \overline{\mathbf{Q}}_p$ (unique modulo $\mathbf{Z}$) and unique integers $d_1, \ldots, d_r \in \mathbf{N}$ and $h_i = h_{A_i,d_i}(W)$ such that $W = \bigoplus_{i=1}^r \mathbf{B}_{\mathrm{dR}}[A_i; d_i]^{h_i}$.*

In light of the above, $W \in \mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}}(G_K)$ is de Rham (i.e., isomorphic to $\mathbf{B}_{\mathrm{dR}}^d$ as an object in $W \in \mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}}(G_K)$) if and only the decomposition of W as in the above theorem is such that for all $1 \leq i \leq r$, $A_i = \{z_i\}$ for some $z_i \in \mathbf{Z}$ and $d_i = 1$.

2.5. The category of B-pairs

In this section we recall several basic properties of B-pairs developed in [**Ber08**], [**BerChe10**], and [**Nak09**].

2.5.1. The ring $\mathbf{B}_{\mathrm{e},E}$. Let $\mathbf{B}_{\mathrm{e}} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. If $E/\mathbf{Q}_p$ is a finite extension, then let G_K act on the ring $\mathbf{B}_{\mathrm{e},E} = \mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} E$ by defining $g(b \otimes e) = g(b) \otimes e$. It is known that $\mathbf{B}_{\mathrm{e},E}$ is a Bézout domain; for $E = \mathbf{Q}_p$, this is shown in proposition 1.1.9 of [**Ber08**], and the same method is used to show general case in lemma 1.6 [**Nak09**]. In fact, it is now known that $\mathbf{B}_{\mathrm{e}}$ is a principal ideal domain (see theorem 10.2 of [**FarFon**]), and therefore $\mathbf{B}_{\mathrm{e},E}$ is principal as well since it is a quotient of the polynomial ring $\mathbf{B}_{\mathrm{e}}[X]$, and thus noetherian.

PROPOSITION 2.5.1.1. *If $E/\mathbf{Q}_p$ is finite, then the ring $\mathbf{B}_{\mathrm{e},E}$ is a principal ideal domain.*

2.5.2. The category of B-pairs. A $B_{|K}^{\otimes E}$*-pair* $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^+)$ is a $\mathbf{B}_{\mathrm{e},E}$-representation W_{e} of G_K together with a G_K-stable $\mathbf{B}_{\mathrm{dR},E}^+$-lattice W_{dR}^+ of $W_{\mathrm{dR}} = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} E) \otimes_{(\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} E)} W_{\mathrm{e}}$; saying that W_{dR}^+ is a $\mathbf{B}_{\mathrm{dR},E}^+$-lattice of W_{dR} means that $W_{\mathrm{dR}}^+ \subset W_{\mathrm{dR}}$ is a free sub-$\mathbf{B}_{\mathrm{dR},E}^+$-module such that $\mathbf{B}_{\mathrm{dR},E} \otimes_{\mathbf{B}_{\mathrm{dR},E}^+} W_{\mathrm{dR}}^+ = W_{\mathrm{dR}}$. If W is a $B_{|K}^{\otimes E}$-pair, then $\mathrm{rank}(W)$ is defined to be the rank of W_{e} as a $\mathbf{B}_{\mathrm{e},E}$-module, which is equal to the rank of W_{dR}^+ as a $\mathbf{B}_{\mathrm{dR},E}^+$-module. If W and W' are $B_{|K}^{\otimes E}$-pairs, then $W \otimes W' = (W_{\mathrm{e}} \otimes_{\mathbf{B}_{\mathrm{e},E}} W'_{\mathrm{e}}, W_{\mathrm{dR}}^+ \otimes_{\mathbf{B}_{\mathrm{dR},E}^+} W'^+_{\mathrm{dR}})$ is a $B_{|K}^{\otimes E}$-pair. If F/E and L/K are finite extensions and if W is a $B_{|K}^{\otimes E}$-pair, then $F \otimes_E W|_{G_L}$ is a $B_{|L}^{\otimes F}$-pair.

For example, if V is a linear E-representation of G_K, then

$$W(V) = ((\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} E) \otimes_E V, (\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} E) \otimes_E V)$$

is a $B_{|K}^{\otimes E}$-pair of rank $d = \dim_E V$.

The notions of crystalline, semi-stable, de Rham, and Hodge-Tate objects in $\mathrm{Rep}_E(G_K)$ may be extended to objects in the category of $B_{|K}^{\otimes E}$-pairs in such a way that an E-linear representation V of G_K is crystalline (or semi-stable or de Rham or Hodge-Tate) if and only if the associated $B_{|K}^{\otimes E}$-pair $W(V)$ is. More precisely, if $\mathbf{B}$ is any of the period rings $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$, or $\mathbf{B}_{\mathrm{dR}}$ and if $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^+)$ is a $B_{|K}^{\otimes E}$-pair, then W is said to be $\mathbf{B}$-*admissible* if the semi-linear $\mathbf{B}$-representation ($\mathbf{B} \otimes_{\mathbf{B}_{\mathrm{e}}} W_{\mathrm{e}}$ of G_K is trivial and W is Hodge-Tate if the $\mathbf{C}_p$-representation $\overline{W} = W_{\mathrm{dR}}^+/tW_{\mathrm{dR}}^+$ is Hodge-Tate.

One can also construct $B_{|K}^{\otimes E}$-pairs from filtered E-(φ, N)-modules over K: if D is a filtered E-(φ, N)-module over K, and if $W_{\mathrm{e}}(D) = (\mathbf{B}_{\mathrm{st},E} \otimes_{K_{0,E}} D)^{\varphi=1,N=0}$ and $W_{\mathrm{dR}}^+(D) = \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR},E} \otimes_{K_{0,E}} D)$, then $W(D) = (W_{\mathrm{e}}(D), W_{\mathrm{dR}}^+(D))$ is a semi-stable $B_{|K}^{\otimes E}$-pair of rank $r = \mathrm{rank}_{K_{0,E}} D$ (see proposition 2.3.3 of [**Ber08**]).

Quotients in the category of B-pairs. If $W' \subset W$ are $B_{|K}^{\otimes E}$-pairs, then W' is said to be *saturated* in W if $W'^+_{\mathrm{dR}} = W'_{\mathrm{dR}} \cap W_{\mathrm{dR}}^+$. In this case, $W_{\mathrm{dR}}^+/W'^+_{\mathrm{dR}}$ is a free $\mathbf{B}_{\mathrm{dR},E}^+$-module and $W/W' = (W_{\mathrm{e}}/W'_{\mathrm{e}}, W_{\mathrm{dR}}^+/W'^+_{\mathrm{dR}})$ is therefore a $B_{|K}^{\otimes E}$-pair of rank $r = \mathrm{rank}(W) - \mathrm{rank}(W')$.

Rank one B-pairs. The following is shown for $E = \mathbf{Q}_p$ in lemma 2.1.3 of [**Ber08**] and for $E \supset K^{\mathrm{Gal}}$ in theorem 1.45 of [**Nak09**]; the same proof as in lemma 2.1.3 of [**Ber08**] together with proposition 7.1.2.3 implies the following:

LEMMA 2.5.2.1. *If W is a $B_{|K}^{\otimes E}$-pair of rank 1, then there is a linear character $\eta : G_K \to E^\times$ and a G_K-stable fractional ideal*[2] *$W^+ \subset \mathbf{B}_{\mathrm{dR},E}$ such that $W = (\mathbf{B}_{\mathrm{e},E}(\eta), W^+(\eta))$.*

[2] I.e., $W^+ \subset \mathbf{B}_{\mathrm{dR},E}$ is a G_K-stable free sub-$\mathbf{B}_{\mathrm{dR},E}^+$-module of rank 1. If $E = \mathbf{Q}_p$, then the fractional ideals of $\mathbf{B}_{\mathrm{dR}}$ are of the form $t^i\mathbf{B}_{\mathrm{dR}}^+$ with $i \in \mathbf{Z}$.

PROOF. If $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^+)$ is a $B_{|K}^{\otimes E}$-pair of rank 1, then proposition 7.1.2.3 implies that there is a character $\eta : G_K \to E^\times$ such that $W_{\mathrm{e}} = \mathbf{B}_{\mathrm{e},E}(\eta)$, so that $W^+ = W_{\mathrm{dR}}^+(\eta^{-1}) \subset \mathbf{B}_{\mathrm{dR},E}$ is free sub-$\mathbf{B}_{\mathrm{dR}}^+$-module of rank 1 on which G_K acts trivially. □

2.5.3. Trianguline representations. Let $E/\mathbf{Q}_p$ be a finite extension. In the context of the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$, Colmez has defined the notion of a trianguline E-linear representation of $G_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.

A $B_{|K}^{\otimes E}$-pair W is said to be *triangulable* if there are sub-$B_{|K}^{\otimes E}$-pairs $\{0\} = W_0 \subset W_1 \subset \ldots \subset W_n = W$ such that W_i/W_{i-1} is a $B_{|K}^{\otimes E}$-pair of rank 1 for all $i \in \{1, \ldots, n\}$. If $V \in \mathrm{Rep}_E(G_K)$, then V is said to be *split trianguline* if the $B_{|K}^{\otimes E}$-pair $W(V)$ is triangulable, and V is said to be *trianguline* if there is a finite extension E'/E such that the linear E'-representation $E' \otimes_E V$ of G_K is split trianguline. For example, if $V \in \mathrm{Rep}_E(G_K)$ is semi-stable, then V is trianguline (see proposition 7.1.4.1). The following is an easy consequence of corollary 7.1.3.2.

PROPOSITION 2.5.3.1. *The sub-category of split trianguline representations in* $\mathrm{Rep}_E(G_K)$ *is stable by sub-quotients, extensions, tensor product, and duals.*

Here is another example of a trianguline representation. If f is an overconvergent p-adic modular form of finite slope in the sense of Katz (see [**Kat73**]), then one may attach to f a 2-dimensional p-adic representation $V_p f$ of $G_{\mathbf{Q}}$, and results of Kisin (theorem 6.3 of [**Kis03**]) and Colmez (prop 4.3 of [**Col08c**]) show that $V_p f|_{G_{\mathbf{Q}_p}}$ is trianguline. Representations attached to finite slope overconvergent p-adic modular forms need not be Hodge-Tate at p in general (indeed, they need not have Hodge-Tate weights in $\mathbf{Z}$).

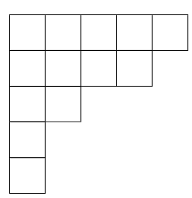
CHAPTER 3

Schur functors

In this chapter, we present some fundamental properties of Schur functors which are used in this document. We signal to the reader that we have chosen to follow the construction of Schur functors as given in [**Ful97**], which is dual to the construction given in §6.1 of [**FulHar**] for example.

3.1. Young diagrams and tableaux

3.1.1. Basic definitions. Let $n \geq 1$ be an integer and let $n = u_1 + \ldots + u_r$ be an integer partition such that $u_i \geq u_{i+1} \geq 1$ for all $i \in \{1, \ldots, r-1\}$. We denote this partition by $u = (u_1, \ldots, u_r)$ and we may represent u by its *Young diagram* Y_u, which is a diagram of n-many boxes arranged into left-justified rows such that the i-th row from the top contains u_i-many boxes. We say that Y_u has *shape* u. For example, the partition $13 = 5 + 4 + 2 + 1 + 1$ is represented by the following diagram:



If $d \geq 1$ is an integer, then a *tableau on* Y_u *with values in* $\{1, \ldots, d\}$ is a labeling of the boxes of Y_u with elements in $\{1, \ldots, d\}$ such that the labeling is weakly increasing from left to right and strongly increasing from top to bottom. Note that if d is strictly less than the length of the left-most column of Y_u, then Y_u has no tableau with values in $\{1, \ldots, d\}$. For example, the following is a tableau with values in $\{1, \ldots, 7\}$ on the above Young diagram:

1	1	2	3	3
2	3	3	4	
4	5			
5				
7				

We will write $T = (t_{ij})$ to denote a tableau with the integer $t_{ij} \in \{1, \ldots, d\}$ in the i-th row (from the top) of the j-th column (from the left) of Y_u.

Let v_j denote the length of the j-th column from the left of Y_u. Put $r(u) = r + 1$ if Y_u is a rectangle (i.e., if $u_1 = \ldots = u_r$) and put $r(u) = r$ if Y_u is not a rectangle. For example, if $n \geq 1$ and if $u = (n)$, then we have $r(u) = 2$ and for $u = (1, \ldots, 1)$ we have $r(u) = n + 1$. If $d \geq r$, then there is a tableau on Y_u with values in $\{1, \ldots, d\}$ which has i in each box of the i-th row from the top; we refer to this tableau as the standard tableau, and we denote it by T_1. For example, here is T_1 on the Young diagram for $u = (5, 4, 2, 1, 1)$:

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\ \hline 2&2&2&2\\ \cline{1-4} 3&3\\ \cline{1-2} 4\\ \cline{1-1} 5\\ \cline{1-1}\end{array}$$

Proposition 3.1.1.1. *If $d \geq r(u)$, then there are tableaux $T_1, T_2, \ldots, T_d$ on Y_u with values in $\{1, \ldots, d\}$ such that for all $i \in \{1, \ldots, d-1\}$, there is an integer $j \in \{1, \ldots, d-1\}$ such that T_j and T_{j+1} have the same entries in all but one box, and this box of T_j contains i while this box in T_{j+1} contains $i + 1$.*

For example, for the partition $u = (2, 2, 2, 1, 1)$ of $n = 8$, we have $r(u) = 5$ and if $d = 5$, then we have $T_1, \ldots, T_5$ as follows:

$$\begin{array}{|c|c|}\hline 1&1\\ \hline 2&2\\ \hline 3&3\\ \hline 4\\ \cline{1-1} 5\\ \cline{1-1}\end{array}\qquad \begin{array}{|c|c|}\hline 1&1\\ \hline 2&2\\ \hline 3&4\\ \hline 4\\ \cline{1-1} 5\\ \cline{1-1}\end{array}\qquad \begin{array}{|c|c|}\hline 1&1\\ \hline 2&2\\ \hline 3&5\\ \hline 4\\ \cline{1-1} 5\\ \cline{1-1}\end{array}\qquad \begin{array}{|c|c|}\hline 1&1\\ \hline 2&3\\ \hline 3&5\\ \hline 4\\ \cline{1-1} 5\\ \cline{1-1}\end{array}\qquad \begin{array}{|c|c|}\hline 1&2\\ \hline 2&3\\ \hline 3&5\\ \hline 4\\ \cline{1-1} 5\\ \cline{1-1}\end{array}$$

Proof of the proposition. Let T_1 denote the trivial tableau on Y_u with values in $\{1, \ldots, d\}$; this is the tableau having i in every box in the i-th row. Let v_{u_1} denote the length of the rightmost column in Y_u. From T_1 obtain T_2 by adding 1 to the bottom-most entry in the right-most column of T_1 (i.e., in coordinate (u_1, v_{u_1}), and therefore it is equal to v_{u_1}). If $i \geq 2$ and if the entry in the (u_1, v_{u_1}) coordinate of T_i is less than d, then let T_{i+1} be the tableau obtained from T_i by adding 1 to the entry in its (u_1, v_{u_1}) coordinate. Repeat this process until we obtain a tableau T_r with d in the bottom-most cell of the right-most column of Y_u. To obtain T_{r+1}, add 1 to the entry just above the bottom-most cell in the right-most column of T_r. To obtain T_{r+1} from T_r, add 1 to the entry of T_r in the $(u_1, v_{u_1} - 1)$ coordinate. To obtain T_{r+2} from T_{r+1}, add 1 to the entry of T_{r+1} in the $(u_1, v_{u_1} - 2)$ coordinate, and so on until we end with a tableau T_d having 2 the rightmost column of the first row of Y_u. The tableaux $T_1, T_2, \ldots, T_d$ satisfy the desired condition. $\square$

The row bumping operation. Let Y_u be a Young diagram of shape u and let $T = (t_{ij})$ be a tableau on Y_u with values in $\{1, \ldots, d\}$. If $x \in \{1, \ldots, d\}$, then we define a Tableau on a Young diagram having one more box than Y_u (denoted by $T \leftarrow x$) by the following procedure. If $t_{1j} \leq x$ for all j, then append one box to the end of row 1 and label it with the number x (the result is a tableau). If this is not the case, then there is some j such that $t_{1,j} < x$ and $x' := t_{1,j+1} \leq x$, and we replace $t_{1,j+1}$ with x (the result is a Tableau). At this point, we do the same for x' in the second row of T and so on. Here are two examples:

$$\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2&3&3\\\hline 3&4\\\cline{1-2} 4\\\cline{1-1}\end{array} \leftarrow 2 = \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\\hline 2&3&3\\\cline{1-3} 3&4\\\cline{1-2} 4\\\cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2&3&3\\\hline 3&4\\\cline{1-2} 4\\\cline{1-1}\end{array} \leftarrow 1 = \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&2&3\\\hline 3&3\\\cline{1-2} 4&4\\\cline{1-2}\end{array}$$

Products of tableaux. The row bumping operation is used to define the product of two tableaux. Let Y_u and Y_v be Young diagrams of shape u and v respectively. If T is a tableau on Y_u and if T' is a tableau on Y_v, both with values in $\{1, \ldots, d\}$, then we may define a *product tableau* $T \cdot T'$ as a series of row-bumping operations: let x be left-most number in the bottom-most column of T', construct $T \leftarrow x$, and repeat this process with $T \leftarrow x$ and T' (ignoring x in T'), continuing until there are no more entries in T' left to bump. Here are two examples of products of tableaux :

Example 1:

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array} \cdot \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&3\\\hline 2&2&2\\\cline{1-3} 3\\\cline{1-1}\end{array}$$

Example 2:

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&3\\\hline 3&3\\\cline{1-2} 4\\\cline{1-1}\end{array} \cdot \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline 4\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline 2&2&2&3\\\cline{1-4} 3&3&3&4\\\cline{1-4} 4\\\cline{1-1}\end{array}$$

If λ, μ, ν are partitions of integers σ_λ, σ_μ, and σ_ν, then we denote by $c^\nu_{\lambda,\mu}$ the number of tableaux T on Y_λ and U on Y_μ such that $T \cdot U$ is equal to the standard tableau on Y_ν; this number is referred to as the *Littlewood-Richardson number*. Note that $c^\nu_{\lambda,\mu} = 0$ unless $|\lambda| + |\mu| = |\nu|$ and Y_ν can contain both Y_λ and Y_μ. There are many equivalent definitions of the Littlewood-Richardson number; we have chosen that which has been easiest to work with in our situation (see proposition 2 of §5 and §5.3 of [**Ful97**]).

3.2. Schur functors

3.2.1. The universal property of Schur modules. Let $u = (u_1, \ldots, u_r)$ be a partition of an integer $n \geq 1$ as in 3.1.1 and let v_j denote the length of the j-th column from the left of Y_u.

Let R be a commutative ring with 1. If M is an R-module, then we let $M^{\times u}$ denote the n-fold direct product $M \times \ldots \times M$ indexed by the squares of Y_u; i.e., elements of $M^{\times u}$ are tuples of the form $m = (m_{ij})$ ($1 \leq j \leq u_1$ and $1 \leq i \leq v_j$), so that $m_{ij} \in M$ is the coordinate of m corresponding to the box of Y_u in the i-th row of the j-th column. If $m = (m_{ij}) \in M^{\times u}$, then $m' \in M^{\times u}$ is said to be an $(a, b; k)$-*exchange of* m if there exist $1 \leq a < b \leq u_1$, $1 \leq k \leq v_b$, and integers $1 \leq i_1 < \ldots i_k \leq v_a$ and $1 \leq i'_1 < \ldots < i'_k \leq v_b$ such that

(1) $m'_{ij} = m_{ij}$ for all $j \notin \{a, b\}$,
(2) $m'_{ia} = m_{ia}$ if $i \notin \{i_1, \ldots, i_k\}$, and $m'_{i_t,a} = m_{i'_t,b}$ for all $t \in \{1, \ldots, k\}$, and
(3) $m'_{ib} = m_{ib}$ if $i \notin \{i'_1, \ldots, i'_k\}$, and $m'_{i'_t,b} = m_{i_t,a}$ for all $t \in \{1, \ldots, k\}$, and

Note that there are a total of $\binom{v_a}{k}$-many $(a, b; k)$-exchanges of m. We let $\mathcal{E}_m \subset M^{\times u}$ denote the set of all exchanges of m (i.e., for all a, b, and k).

EXAMPLE 3.2.1.1. *Let* $u = (4, 3, 2, 1)$, *and* $(a, b; k) = (2, 3; 2)$. *If* $m = (m_{ij})$ *is given by*

a	b	c	d
e	f	g	
h	i		
j			

then there are three $(2, 3; 2)$*-exchanges of* m*, and they are as follows:*

a	c	b	d
e	g	f	
h	i		
j			

a	c	b	d
e	f	i	
h	g		
j			

a	b	f	d
e	c	i	
h	g		
j			

If M is an R-module, then there is an R-module $\mathrm{Schur}^u(M)$ together with a morphism $\varphi : M^{\times u} \to \mathrm{Schur}^u(M)$ of R-modules such that

(1) φ is R-multilinear,
(2) φ is alternating with respect to the columns of Y_u; i.e., if $m = (m_{ij}) \in M^{\times u}$ and if there is some j and some $1 \leq i < i' \leq v_j$ such that $m_{ij} = m_{i'j}$, then $\varphi(m) = 0$.
(3) for all $m \in M^{\times u}$, $\varphi(m) = \sum_{m' \in \mathcal{E}_m} \varphi(m')$.

and if M' is an R-module with $\phi : M^{\times u} \to M'$ a morphism of R-modules satisfying (1), (2), and (3), then there is a unique morphism of R-modules $\phi' : \mathrm{Schur}^u(M) \to M'$ such that $\phi = \phi' \circ \varphi$.

By the universal properties for alternating product and tensor product, conditions (1) and (2) imply that $\mathrm{Schur}^u(M)$ is a quotient of of the R-module $\Lambda^{v_1}(M)\otimes_R \ldots \otimes_R \Lambda^{v_{u_1}}(M)$. In particular, if $\{m_1, \ldots, m_k\} \subset M$ and if $T = (t_{ij})$ is a tableau on Y_u with values in $\{1, \ldots, k\}$, then we may consider the image of the element $(m_{t_{11}} \wedge \ldots \wedge m_{t_{v_1 1}}) \otimes \ldots \otimes (m_{t_{1u_1}} \wedge \ldots \wedge m_{t_{v_{u_1} u_1}})$ in $\mathrm{Schur}^u(M)$; we denote this element by m_T.

EXAMPLE 3.2.1.2. *(1) When $u = (n)$ (so that Y_u consists of a single row of n-many boxes), condition (2) imposes no restrictions and condition (3) says that φ is symmetric; in this case,* $\mathrm{Schur}^u(M) = \mathrm{Sym}^n(M)$.

(2) When $u = (1, \ldots, 1)$ (i.e., Y_u consists of a single column of n-many boxes), condition (3) imposes no restrictions and condition (2) says that φ is alternating; in this case, $\mathrm{Schur}^u(M) = \Lambda^n(M)$.

(3) We now describe the Schur functor associated to the partition $u = (2,1)$ of $n = 3$. In this situation, the exchange condition (3) is the following: for $m \in M^{\times u}$ with $m = (m_{11}, m_{21}; m_{12})$, $\varphi(m) = \varphi(m_{12}, m_{21}; m_{11}) + \varphi(m_{11}, m_{12}; m_{21})$. Concretely, $\mathrm{Schur}^{(2,1)}(M)$ *is the quotient of $\Lambda^2(M) \otimes_R M$ by the sub-R-module generated by elements of the form $(a \wedge b) \otimes c - (c \wedge b) \otimes a - (a \wedge c) \otimes b$.*

The following is theorem 1 of §8.1 in [**Ful97**]:

PROPOSITION 3.2.1.3. *If M is a free R-module of finite rank with basis $(e_1, \ldots, e_d)$, then* $\mathrm{Schur}^u(M)$ *is a free R-module (which is nonzero if $d \geq r$) with basis $(e_T)_T$, where T ranges over all tableaux on Y_u with values in $\{1, \ldots, d\}$.*

The universal property of Schur^u(-) implies that $\mathrm{Schur}^u : R\text{-}\mathrm{Mod} \to R\text{-}\mathrm{Mod}$ is functorial, and we just saw that it sends free modules to free modules.

PROPOSITION 3.2.1.4. *If E is a field of characteristic* 0 *and if V is a finite-dimensional E-vector space together with an operator $f : V \to V$, and if $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of T counted with multiplicity, then the eigenvalues of the operator induced by f on* $\mathrm{Schur}^u(V)$ *are the elements of the form $\lambda_T = \sum_{T=(t_{ij})} \lambda_{t_{ij}}$, where T ranges over all tableaux on u with values in* $1, \ldots, d$.

The functor $\mathrm{Schur}^u(-)$ is compatible with extension of scalars:

PROPOSITION 3.2.1.5. *If $R \to R'$ is a ring morphism and if M is an R-module, then* $\mathrm{Schur}^u(R' \otimes_R M) = R' \otimes_R \mathrm{Schur}^u(M)$.

3.2.2. Direct sums. If $M = M' \oplus M''$ is a direct sum of free R-modules, then $\mathrm{Sym}^n(M) \simeq \bigoplus_{p+q=n} \mathrm{Sym}^p(M') \otimes_R \mathrm{Sym}^q(M'')$ and $\bigwedge^n(M) \simeq \bigoplus_{p+q=n} \bigwedge^p(M') \otimes_R \bigwedge^q(M'')$; these two isomorphisms are special cases of the following more general result, which appears in (20) of §8.3 of [**Ful97**]:

Proposition 3.2.2.1. *If W and W' are free R-modules, then we have a functorial decomposition*

$$\operatorname{Schur}^u(W \oplus W') \simeq \bigoplus_{\lambda,\mu} (\operatorname{Schur}^\lambda(W) \otimes_R \operatorname{Schur}^\mu(W'))^{\oplus c^u_{\lambda,\mu}}$$

where $c^u_{\lambda,\mu} \geq 0$ denotes the Littlewood-Richardson number.

The above, together with the following lemma, will be used in part 2:

Lemma 3.2.2.2. *If u is a partition of an integer $n \geq 1$ and if $R \geq r(u)$, then for all $1 < d \leq R$, there are sub-shapes Y_λ and Y_μ of Y_u such that $d \geq r(\lambda)$ and $R - d \geq r_\mu$ and such that $c^u_{\lambda,\mu} \neq 0$.*

Note that the above lemma is obvious for the partitions $u = (n)$ or $u = (1, \ldots, 1)$ of n. Here are two more examples.

Example 3.2.2.3.

(1) *For $R = 5$, $u = (5, 4, 4, 2, 1)$, and $d = 3$, the following factorisation shows that $\lambda = (2, 1)$ and $\mu = (5, 4, 4)$ satisfy the conditions of the lemma:*

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline 2&2&2&2\\\cline{1-4} 3&3&3&3\\\cline{1-4} 4&4\\\cline{1-2} 5\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 4&4\\\hline 5\\\cline{1-1}\end{array} \cdot \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline 2&2&2&2\\\cline{1-4} 3&3&3&3\\\cline{1-4}\end{array}$$

(2) *For $R = 4$, $u = (3, 3, 2, 2)$, and $d = 2$, the following factorisation shows that $\lambda = (3, 2)$ and $\mu = (3, 2)$ satisfy the conditions of the lemma:*

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&2\\\hline 3&3\\\cline{1-2} 4&4\\\cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline 2&3&3\\\hline 4&4\\\cline{1-2}\end{array} \cdot \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2}\end{array}$$

Proof of the lemma. Label Y_u with the standard tableau with values in $\{1, \ldots, R\}$. Draw a horizontal line L between row d and row $d + 1$ in Y_u.

If the shape above this line is not a rectangle, then denote it by λ (so that $d = r(\lambda)$) and denote the shape below this line by μ. If T_λ (resp. T_μ) denotes the tableau of entries on λ (resp. μ) inherited from the standard tableau on Y_u, then $T_\mu \cdot T_\lambda$ is the standard tableau on Y_u.

Suppose now that the shape above the line between row d and row $d+1$ is a rectangle. Let x denote the entry in the rightmost column of row d of the standard tableau on Y_u. If T_μ is the tableau obtained from bumping x (see the previous section for the definition of the bumping operation) into the diagram below the line L, and if T_λ is the tableau on the shape λ obtained by removing the rightmost box in row d from the shape above the line L, then $d = r(\lambda)$ and $T_\mu \cdot T_\lambda$ is equal to the standard tableau T_1 on Y_u. □

3.2.3. Schur functors applied to B-pairs. If $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^+)$ is a $B_{|K}^{\otimes E}$-pair, then $\mathrm{Schur}^u(W) = (\mathrm{Schur}^u(W_{\mathrm{e}}), \mathrm{Schur}^u(W_{\mathrm{dR}}^+))$ is a $B_{|K}^{\otimes E}$-pair by 3.2.1.3 and 3.2.1.5. If V is an E-linear representation of G_K, then we have an isomorphism of $B_{|K}^{\otimes E}$-pairs $\mathrm{Schur}^u(W(V)) \xrightarrow{\sim} W(\mathrm{Schur}^u(V))$.

Part 2

Admissible tensor products & Schur objects

CHAPTER 4

Notation and generalities

4.1. Notation and generalities

4.1.1. Notation. Let $\overline{\mathbf{Q}}_p$ be an algebraic closure of $\mathbf{Q}_p$ and let $\mathbf{C}_p$ be the p-adic completion of $\overline{\mathbf{Q}}_p$. Let $\mathbf{Q}_p^{\mathrm{nr}}$ denote the maximal non-ramified sub-extension of $\overline{\mathbf{Q}}_p/\mathbf{Q}_p$. If $F/\mathbf{Q}_p$ is a finite extension, then we let F^{Gal} denote the Galois closure of F in $\overline{\mathbf{Q}}_p$. Let $\mathbf{B}_{\mathrm{dR}}$, $\mathbf{B}_{\mathrm{dR}}^+$, $\mathbf{B}_{\mathrm{cris}}$, and $\mathbf{B}_{\mathrm{st}}$ denote Fontaine's rings as in [**Fon94a**] and let $\mathbf{B}_{\mathrm{e}} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. In this part, $E/\mathbf{Q}_p$ and $K/\mathbf{Q}_p$ denote finite extensions. If $\mathbf{B}$ is any of the above rings or any Galois sub-extension of $\overline{\mathbf{Q}}_p/\mathbf{Q}_p$, then $\mathbf{B}_E$ will denote the ring $\mathbf{B} \otimes_{\mathbf{Q}_p} E$ endowed with the action of $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ defined by $g(b \otimes e) = g(b) \otimes e$ for all $g \in G_K$. If W is a free $\mathbf{B}_E$-module of finite rank endowed with a semi-linear action of G_K, then we refer to W as a $\mathbf{B}_E$-representation of G_K.

4.1.2. Decomposing representations with coefficients. If $F/\mathbf{Q}_p$ is a finite extension and if $\mathbf{B} \in \{\mathbf{C}_p, \mathbf{B}_{\mathrm{dR}}\}$ or if $\mathbf{B}$ is any Galois sub-extension of $\overline{\mathbf{Q}}_p/\mathbf{Q}_p$ containing F, then the map

$$\begin{aligned} \mathbf{B} \otimes_{\mathbf{Q}_p} F &\simeq \bigoplus_{h : F \to \overline{\mathbf{Q}}_p} \mathbf{B} \\ (b \otimes f) &\mapsto (b \cdot h(f))_h \end{aligned} \tag{1}$$

(where h runs over the embeddings of F into $\overline{\mathbf{Q}}_p$) is an isomorphism of $\mathbf{B}$-algebras, and it is G_K-equivariant if $K \supset F^{\mathrm{Gal}}$.

If W is a $\mathbf{B}_F$-module, then for each embedding $h : F \to \overline{\mathbf{Q}}_p$, let W_h denote the sub-$\mathbf{B}$-module of W coming from the h-factor map $(b \otimes f) \mapsto b \cdot h(f) : \mathbf{B} \otimes_{\mathbf{Q}_p} F \to \mathbf{B}$ in the isomorphism (1) above.

PROPOSITION 4.1.2.2. *Let W be a $\mathbf{B}_F$-module and let W_h denote the $\mathbf{B}$-module corresponding to the embedding $h : F \to \overline{\mathbf{Q}}_p$.*

(1) We have a direct sum decomposition $W = \bigoplus_h W_h$ of $\mathbf{B}$-modules.
(2) If W is free of rank d as a $\mathbf{B}_F$-module, then W_h is free of rank d as a $\mathbf{B}$-module.
(3) If W' is another $\mathbf{B}_F$-module and if $T : W \to W'$ is a morphism of $\mathbf{B}_F$-modules, then T (viewed as a morphism of $\mathbf{B}$-modules) sends W_h to W'_h for all $h : F \to \overline{\mathbf{Q}}_p$.

In particular, if $K \supset F^{\mathrm{Gal}}$, then a $\mathbf{B}_F$-representation W of G_K decomposes into a direct sum $W = \bigoplus_{h:F\to\overline{\mathbf{Q}}_p} W_h$ as a $\mathbf{B}$-representation of G_K, and $\mathrm{rank}_{\mathbf{B}} W_h = \mathrm{rank}_{\mathbf{B}_F} W$.

Note that if $K \supset F^{\mathrm{Gal}}$, then a $\mathbf{B}_{\mathrm{dR},F}$-representation W of G_K is de Rham if and only if the $\mathbf{B}_{\mathrm{dR}}$-representations W_h are de Rham for each embedding $h : F \to \overline{\mathbf{Q}}_p$ and a $\mathbf{C}_{p,F}$-representation W of G_K is Hodge-Tate if and only if the $\mathbf{C}_p$-representations W_h are Hodge-Tate for all embeddings $h : F \to \overline{\mathbf{Q}}_p$.

PROPOSITION 4.1.2.3. *Let W and W' be $\mathbf{B}_F$-representations of G_K with $K \supset F^{\mathrm{Gal}}$ and let $W = \bigoplus_h W_h$ and $W' = \bigoplus_h W'_h$ be the decompositions as described above.*

(1) The $\mathbf{B}_F$-representation $W \otimes_{\mathbf{B}_F} W'$ decomposes as $\bigoplus_{h:F\to\overline{\mathbf{Q}}_p}(W_h \otimes_{\mathbf{B}} W'_h)$.
(2) The $\mathbf{B}_F$-representation $W \oplus W'$ decomposes as $\bigoplus_{h:F\to\overline{\mathbf{Q}}_p}(W_h \oplus W'_h)$.
(3) If $W' \subset W$ is a sub-$\mathbf{B}_F$-module, then the $\mathbf{B}_F$-representation W/W' decomposes as $W/W' = \bigoplus_{h:F\to\overline{\mathbf{Q}}_p}(W_h/W'_h)$

COROLLARY 4.1.2.4. *Let $n \geq 1$ and let $u = (u_1, \ldots, u_r)$ be a partition of n as defined in paragraph 3.1.1 of chapter 3. If W is a $\mathbf{B}_F$-representation of G_K and if $W = \bigoplus_h W_h$, then the $\mathbf{B}_F$-module* $\mathrm{Schur}^u(W)$ *decomposes into* $\mathrm{Schur}^u(W) = \bigoplus_{h:F\to\overline{\mathbf{Q}}_p} \mathrm{Schur}^u(W_h)$ *as a $\mathbf{B}$-representation of G_K.*

4.2. Sen's theory for representations with coefficients

4.2.1. Generalized Hodge-Tate weights. Let E and K denote finite extensions of $\mathbf{Q}_p$. In what follows, a *$\mathbf{C}_{p,E}$-representation of G_K* is a free finite rank $\mathbf{C}_{p,E}$-module W endowed with a $\mathbf{C}_p$-semi-linear E-linear action of G_K, such that the action of G_K is continuous when W is viewed as a $\mathbf{C}_p$-representation of rank $[E : \mathbf{Q}_p] \cdot \mathrm{rank}_{\mathbf{C}_{p,E}}(W)$. For example, if $V \in \mathrm{Rep}_E(G_K)$, then $\mathbf{C}_p \otimes_{\mathbf{Q}_p} V$ is a $\mathbf{C}_{p,E}$-representation of G_K. Similarly, if W is a $B^{\otimes E}_{|K}$-pair, then $\overline{W} := W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}$ is a $\mathbf{C}_{p,E}$-representation of G_K.

If W is a $\mathbf{C}_{p,E}$-representation of G_K of rank d, then $D_{\mathrm{sen}}(W) = (W^{H_K})^{\mathrm{fini}}$ (see §2.3.1) is a $K_{\infty,E}$-module, and 2.3.2.1 implies that the Γ_K-equivariant operator $\Theta : D_{\mathrm{sen}}(W) \to D_{\mathrm{sen}}(W)$ is $K_{\infty,E}$-linear. It turns out that $D_{\mathrm{sen}}(W)$ is in fact free of rank d as a $K_{\infty,E}$-module; by 4.1.2 of [**BerCol02**], there is a finite extension L/K such that $D_{\mathrm{sen}}(W|_{G_L}) = L_\infty \otimes_{K_\infty} D_{\mathrm{sen}}(W)$ is free as a $L_{\infty,E}$-module and since the ring extension $L_{\infty,E}/K_{\infty,E}$ is faithfully flat, $D_{\mathrm{sen}}(W)$ is necessarily free as a $K_{\infty,E}$-module.

PROPOSITION 4.2.1.1. *Let W be a $\mathbf{C}_{p,E}$-representation of G_K.*

(1) $D_{\mathrm{sen}}(W)$ is a free $K_{\infty,E}$-module of rank $d = \mathrm{rank}_{\mathbf{C}_{p,E}}(W)$ on which Γ_K acts K_∞-semi-linearly and E-linearly.
(2) The operator $\Theta_{D_{\mathrm{sen}}(W)}$ is $K_{\infty,E}$-linear.

The characteristic polynomial $P_{\Theta,E}(X) = \det(X\cdot\mathrm{Id} - \Theta)$ has coefficients in $(K\otimes_{\mathbf{Q}_p} E)$ since Θ commutes with the action of Γ_K on D. Let $P_\Theta \in K[X]$ denote the characteristic polynomial of $\Theta : D \to D$ when viewed as a K_∞-linear operator (as in paragraph 2.3.1 of chapter 2).

The following is a more precise version of proposition 2.3.2.5 for representations with coefficients:

PROPOSITION 4.2.1.2. *Let W and W' be $\mathbf{C}_{p,E}$-representations of G_K.*

(i) If W' is a sub-representation of W, then $\Theta_W|_{W'} = \Theta_{W'}$ and $\Theta_{W/W'}$ is the canonical operator induced by Θ_W. In particular, if $0 \to W' \to W \to W'' \to 0$ is an exact sequence of $\mathbf{C}_{p,E}$-representations, then $P_{\Theta_W} = P_{\Theta_{W'}}P_{\Theta_{W''}}$.
(ii) If F/E is a finite extension, then $D_{\mathrm{sen}}(F\otimes_E W) = F\otimes_E D_{\mathrm{sen}}(W)$ and $\Theta_{F\otimes W}$ is the F-linearisation of Θ_W.
(iii) We have a natural isomorphism $D_{\mathrm{sen}}(W\otimes_{\mathbf{C}_{p,E}} W') = D_{\mathrm{sen}}(W)\otimes_{K_{\infty,E}} D_{\mathrm{sen}}(W')$ of $K_{\infty,E}$-representations of Γ_K and the Sen operator on $D_{\mathrm{sen}}(W\otimes_{\mathbf{C}_{p,E}} W')$ is $\Theta_W\otimes\mathrm{Id} + \mathrm{Id}\otimes\Theta_{W'}$.
(iv) If L/K is a finite Galois extension, then $D_{\mathrm{sen}}(W|_{G_L}) = L_\infty\otimes_{K_\infty} D_{\mathrm{sen}}(W)$ as an $L_{\infty,E}$-representation of Γ_L, and $\Theta_{W|_{G_L}}$ is the L_∞-linearization of Θ_W.

Recall that the set of *generalized Hodge-Tate weights of W* is the set of roots of P_Θ in $\overline{\mathbf{Q}}_p$ counted with multiplicity. In what follows, we explain two ways to recover the generalized Hodge-tate weights of W from the polynomial $P_{\Theta,E}(X)$.

4.2.1.3. *Weight combinatorics I.* Let $D \in \mathrm{Rep}_{K_{\infty,E}}(\Gamma_K)$ be a nonzero object, let Θ be Sen's $K_{\infty,E}$-linear operator on D, and let $P_{\Theta,E} \in (K\otimes_{\mathbf{Q}_p} E)[X]$ be its characteristic polynomial. For all $j : E \to \overline{\mathbf{Q}}_p$, let $P_{\Theta,E,j}(X) \in (K\cdot E^{\mathrm{Gal}})[X]$ be the polynomial obtained from $P_{\Theta,E}$ by applying the map $K\otimes E \to K\cdot E^{Gal}$ defined by $x\otimes e \mapsto xj(e)$ to the coefficients of $P_{\Theta,E}(X)$. We now explain how to recover the generalized Hodge-Tate weights of D from $P_{\Theta,E}(X)$.

PROPOSITION 4.2.1.4. *Let $L \supset E^{\mathrm{Gal}}$ with $L/\mathbf{Q}_p$ finite. If $D \in \mathrm{Rep}_{L_{\infty,E}}(\Gamma_L)$ and if $D = \bigoplus_{j:E\to\overline{\mathbf{Q}}_p} D_h$ is the decomposition of D as an L_∞-representation of Γ_L, then*

(1) $\Theta|_{D_j} = \Theta_{D_j}$
(2) $P_\Theta(X) = \prod_{j:E\to\overline{\mathbf{Q}}_p} P_{\Theta,E,j}$ in $L[X]$.

PROOF. Assertion (1) is an immediate consequence of proposition 2.3.2.5. Assertion (2) is an immediate consequence of assertion (1) since $P_{L_\infty,E,j}$ is the characteristic polynomial of Θ_{D_j}. ☐

Let K and E be finite extensions of $\mathbf{Q}_p$ as before (i.e., we are not assuming that $K \supset E^{\mathrm{Gal}}$) and let $W \in \mathrm{Rep}_{\mathbf{C}_{p,E}}(G_K)$. Part (2) of the above proposition shows that the generalized Hodge-Tate weights of $D = D_{\mathrm{sen}}(W)$ may be decomposed into subsets corresponding to the embeddings $j : E \to \overline{\mathbf{Q}}_p$; more precisely, if $L = K \cdot E^{\mathrm{Gal}}$, then the Sen operator on $D' = D_{\mathrm{sen}}(W|_{G_L}) = (L_\infty \otimes_{K_\infty} D)|_{\Gamma_L}$ is just the L_∞-linearization of Θ on D; in particular, $P_{\Theta_D} = P_{\Theta_{D'}}$ and $P_{\Theta_D,E} = P_{\Theta_{D'},E}$. By the above proposition, we therefore have $P_{\Theta_D} = \prod_{j:E\to\overline{\mathbf{Q}}_p} P_{\Theta_D,E,j}$ in $L[X]$, where $L = K \cdot E^{\mathrm{Gal}}$.

With this in mind, if $W \in \mathrm{Rep}_{\mathbf{C}_{p,E}}(G_K)$, then we may partition the multiset $\mathrm{Wt}(W) = \bigsqcup_{j:E\to\overline{\mathbf{Q}}_p} \mathrm{Wt}_j(W)$, where $\mathrm{Wt}_j(W)$ is the set of roots of $P_{\Theta,E,j}$ in $\overline{\mathbf{Q}}_p$ counted with multiplicity. If L/K is a finite extension such that $L \supset E^{\mathrm{Gal}}$ and if $W = \bigoplus_j W_j$ is the decomposition as a $\mathbf{C}_p$-representation of G_L as in proposition 4.1.2.2, then $\mathrm{Wt}_j(W)$ may be interpreted as precisely the set of generalized Hodge-Tate weights of $W_j \in \mathrm{Rep}_{\mathbf{C}_p}(G_K)$.

Here is a description of the sets $\mathrm{Wt}_j(W)$ for the example in paragraph 2.2.1.

Example 4.2.1.5. *Let $K/\mathbf{Q}_p$ be a finite extension and let $\pi \in \mathcal{O}_K$ be a uniformizer. If $\chi_\pi : G_K \to \mathcal{O}_K^\times$ is the character attached to a Lubin-Tate formal module for π, then* $\mathrm{Wt}_j(K(\chi_\pi)) = \{1\}$ *if j is the inclusion $K \subset \overline{\mathbf{Q}}_p$ and* $\mathrm{Wt}_j(K(\chi_\pi)) = \{0\}$ *for any of the other $[K : \mathbf{Q}_p] - 1$ embeddings $j : K \hookrightarrow \overline{\mathbf{Q}}_p$.*

Propositions 4.1.2.3 and 4.2.1.2 tell us how the sets $\mathrm{Wt}_j(W)$ behave with respect to various operations:

Proposition 4.2.1.6. *Let W', W be nonzero $\mathbf{C}_{p,E}$-representations of G_K and let $j : E \to \overline{\mathbf{Q}}_p$ be an embedding.*

(1) *If $0 \to W' \to W \to W'' \to 0$ is an extension of $\mathbf{C}_{p,E}$-representations of G_K, then* $\mathrm{Wt}_j(W) = \mathrm{Wt}_j(W') \sqcup \mathrm{Wt}_j(W'')$. *In particular,* $\mathrm{Wt}_j(W'') = \mathrm{Wt}_j(W) - \mathrm{Wt}_j(W')$ *(this is a multiset difference with multiplicity).*

(2) *If* $\mathrm{Wt}_j(W) = \{\alpha_1, \ldots, \alpha_d\}$ *and* $\mathrm{Wt}_j(W') = \{\alpha'_1, \ldots, \alpha'_{d'}\}$ *(enumerated with multiplicity), then* $\mathrm{Wt}_j(W \otimes_{\mathbf{C}_{p,E}} W') = \{\alpha_i + \alpha'_{i'} | 1 \le i \le d, 1 \le i' \le d'\}$.

(3) *If F/E is a finite extension, then for all embeddings $j' : F \to \overline{\mathbf{Q}}_p$ such that $j'|_E = j$,* $\mathrm{Wt}_{j'}(F \otimes_E W) = \mathrm{Wt}_j(W)$.

(4) *If L/K is a finite extension, then* $\mathrm{Wt}_j(W|_{G_L}) = \mathrm{Wt}_j(W)$.

Similarly, propositions 4.1.2.3 and 4.2.1.2 together with 3.2.1.4 from part I imply the following:

Corollary 4.2.1.7. *Let W be a $\mathbf{C}_{p,E}$-representation of G_K of rank d. If $j : E \to \overline{\mathbf{Q}}_p$ is an embedding and if $a_{1,j}, \ldots, a_{d,j}$ denote the elements of* $\mathrm{Wt}_j(W)$, *then the elements of* $\mathrm{Wt}_j(\mathrm{Schur}^u(W))$ *are the elements $a_T = \sum_{i,k} a_{t_{ik},j}$ for any tableau $T = (t_{ik})$ on the Young diagram of u with values in $\{1, \ldots, d\}$.*

4.2.1.8. *Weight combinatorics II.* Suppose now that $E/\mathbf{Q}_p$ is finite Galois and $K \subset E$. Let $D \in \mathrm{Rep}_{K_{\infty,E}}(\Gamma_K)$ be a nonzero object, let Θ be Sen's $K_{\infty,E}$-linear operator on D, and let $P_{\Theta,E} \in (K \otimes_{\mathbf{Q}_p} E)[X]$ be its characteristic polynomial. For each embedding $h : K \to E$, let $P^h_{\Theta,E} \in E[X]$ be the polynomial obtained by applying the map $K \otimes_{\mathbf{Q}_p} E \to E$ defined by $\lambda \otimes e \mapsto h(\lambda)e$ to the coefficients of $P_{\Theta,E}$. Let $\mathrm{Wt}^h(W)$ denote the multiset of roots of $P^h_{\Theta,E}$ in $\overline{\mathbf{Q}}_p$, counted with multiplicity.

The following proposition relates the sets $\mathrm{Wt}^h(W)$ to the sets $\mathrm{Wt}_j(W)$ from the previous paragraph.

PROPOSITION 4.2.1.9. *Let $h : K \to E$ and let $j : E \to E$ be an embedding lifting h. We have an equality $j(P_{\Theta,E,j^{-1}}) = P^h_{\Theta,E}$.*

In particular, we see that if $h : K \to E$ and if $j : E \to E$ is an embedding lifting h, then we have a bijection $\mathrm{Wt}^h(W) \xrightarrow{\sim} \mathrm{Wt}_{j^{-1}}(W)$ of multisets given by $\alpha \mapsto j^{-1}(\alpha)$. For example, the above proposition and 4.2.1.5 show the following:

EXAMPLE 4.2.1.10. *Let $K/\mathbf{Q}_p$ be a finite extension and let $\chi_\pi : G_K \to \mathcal{O}_K^\times$ be the Lubin-Tate character coming from π. If $E/\mathbf{Q}_p$ is finite Galois and if $E \supset K$, then the h-weight of $E(\chi_\pi)$ is 1 for h the inclusion $K \subset E$, and the h-weight is 0 for all other embeddings.*

Here are some formal properties of h-weights.

PROPOSITION 4.2.1.11. *Let $W, W' \in \mathrm{Rep}_{\mathbf{C}_{p,E}}(G_K)$ and let $h : K \to E$ be an embedding*

(1) *If $0 \to W' \to W \to W'' \to 0$ is an exact sequence in $\mathrm{Rep}_{\mathbf{C}_{p,E}}(G_K)$, then $\mathrm{Wt}^h(W) = \mathrm{Wt}^h(W') \sqcup \mathrm{Wt}^h(W'')$ as multisets.*
(2) $\mathrm{Wt}^h(W \otimes W') = \{\alpha + \alpha' | \alpha \in \mathrm{Wt}^h(W), \alpha' \in \mathrm{Wt}^h(W')\}$
(3) *If F/E is a finite extension, then $\mathrm{Wt}^h(F \otimes_E W) = \mathrm{Wt}^h(W)$.*
(4) *If L/K is a finite sub-extension of E/K, then $\mathrm{Wt}^h(W|_{G_L}) = \mathrm{Wt}^h(W)$.*

PROOF. Assertions (1) and (2) are immediate consequences of propositions 4.2.1.6 and 4.2.1.9. Assertions (3) and (4) follow immediately from 4.2.1.2. □

COROLLARY 4.2.1.12. *Let W be a $\mathbf{C}_{p,E}$-representation of G_K of rank d. If $h : K \to E$ is an embedding and if $a_{1,h}, \ldots, a_{d,h}$ denote the elements of $\mathrm{Wt}^h(W)$, then the elements of $\mathrm{Wt}^h(\mathrm{Schur}^u(W))$ are the elements $a_T = \sum_{i,k} a_{t_{ik},h}$ for any tableau $T = (t_{ik})$ on the Young diagram of u with values in $\{1, \ldots, d\}$.*

Characters with prescribed weights. Let $K/\mathbf{Q}_p$ be a finite extension and let $\chi_\pi : G_K \to \mathcal{O}_K^\times$ be the Lubin-Tate character associated to a uniformizer $\pi \in \mathcal{O}_K$. Let $E/\mathbf{Q}_p$ be a finite Galois extension with $K \subset E$. For each embedding $h : K \hookrightarrow E$, the h-weight of $E(\chi_K)$ is 1 if h is the inclusion of K in E, and 0 otherwise.

THEOREM 4.2.1.13. *Let $h_1, \ldots, h_r$ denote the embeddings of K into E and let $\omega_1, \ldots, \omega_r$ be elements of E. There exists a finite Galois extension F/E and a character $\mu : G_K \to F^\times$ such that $\mathrm{Wt}^{h_i}(F(\mu)) = \{\omega_i\}$ for $i = 1, \ldots, r$.*

PROOF. Let $\chi_K : G_K \to \mathcal{O}_K^\times$ be the character associated to a Lubin-Tate module over $\mathcal{O}_K$.

If $\omega \in E$, then $\omega = p^{-n}\omega'$ for some $\omega' \in \mathcal{O}_E$, and some integer $n \geq 0$. Consider the topological factorisation $\mathcal{O}_K^\times = [k_K^\times] \times (1 + \mathfrak{m}_K)$. Consider a topological factorisation of the $\mathbf{Z}_p$-module $1 + \mathfrak{m}_K$ into $\mathbf{Z}/p^a\mathbf{Z} \times \mathbf{Z}_p^r$, where $a \geq 0$ and $r = [K : \mathbf{Q}_p]$. Let $\langle \chi_K \rangle$ denote the projection of χ_K onto the submodule $\mathbf{Z}_p^r$ in this factorisation. If $\{y_1, \ldots, y_r\}$ is a $\mathbf{Z}_p$-basis of $\mathbf{Z}_p^r$, and if F/E is a finite extension which is Galois over $\mathbf{Q}_p$ and contains $z_1, \ldots, z_r \in 1 + \mathfrak{m}_F$ such that $z_i^{p^n} = y_i$, then the map $\mu(y_1^{a_1} \cdot \ldots \cdot y_r^{a_r}) := z_1^{\omega' a_1} \cdot \ldots \cdot z_r^{\omega' a_r}$ composed with $\langle \chi_K \rangle$ is a character whose h-weight is $p^{-n}\omega' = \omega$ when $h = id$ and 0 otherwise. We denote this character by $\langle \chi_K \rangle^\omega : G_K \to \mathcal{O}_F^\times$.

Given $\omega_1, \ldots, \omega_r \in E$, the product of characters $\prod \langle h_i^{-1}(\chi_K) \rangle^{\omega_i}$ has h_i-weight equal to ω_i for each $1 \leq i \leq r$, where $h_i^{-1} : F \to F$ is the inverse of an automorphism $h_i : F \to F$ extending $h_i : K \to E \subset F$. □

CHAPTER 5

Tensor products and Schur B-pairs of Hodge-Tate type

5.1. Hodge-Tate and de Rham tensor products of B-pairs

Let $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^+)$ be a $B_{|K}^{\otimes E}$-pair. We say that W is *de Rham* if the $\mathbf{B}_{\mathrm{dR}}$-representation W_{dR} of G_K is de Rham. We say that W is *Hodge-Tate* if the $\mathbf{C}_{p,E}$-representation $\overline{W} = W_{\mathrm{dR}}^+/tW_{\mathrm{dR}}^+$ of G_K is Hodge-Tate.

Proposition 5.1.0.14. *If W and W' are $\mathbf{C}_p$-representations of G_K with Sen weights in $\mathbf{Z}$ such that $W \otimes_{\mathbf{C}_p} W'$ is Hodge-Tate, then W and W' are Hodge-Tate.*

If W and W' are $\mathbf{B}_{\mathrm{dR}}$-representations of G_K with de Rham weights in $\mathbf{Z}$ such that $W \otimes_{\mathbf{B}_{\mathrm{dR}}} W'$ is de Rham, then W and W' are de Rham.

Proof. Let W and W' be $\mathbf{B}_{\mathrm{dR}}$-representations of G_K with de Rham weights in $\mathbf{Z}$. By Fontaine's theorem [**Fon04**, 3.19], W and W' admit unique decompositions $W \simeq \bigoplus_{i=1}^{r} \mathbf{B}_{\mathrm{dR}}[\{0\}; d_i]^{e_i}$ and $W' \simeq \bigoplus_{j=1}^{r'} \mathbf{B}_{\mathrm{dR}}[\{0\}; d'_j]^{e'_j}$. The $\mathbf{B}_{\mathrm{dR}}$-representations W and W' are de Rham if and only if all of the d_i and d'_j are equal to zero. If $W \otimes_{\mathbf{B}_{\mathrm{dR}}} W'$ is de Rham, then $\mathbf{B}_{\mathrm{dR}}[\{0\}; d_i] \otimes_{\mathbf{B}_{\mathrm{dR}}} \mathbf{B}_{\mathrm{dR}}[\{0\}; d'_j]$ is de Rham for every $1 \le i \le r$ and $1 \le j \le r'$. Suppose, for example, that W is not de Rham, so that we may assume $d_1 > 0$. Let $U = \mathbf{B}_{\mathrm{dR}}[\{0\}; d_1] \otimes_{\mathbf{B}_{\mathrm{dR}}} \mathbf{B}_{\mathrm{dR}}[\{0\}; d'_1]$, let $v_1 = 1 \otimes 1$, and let $(v_1, v_2, \ldots, v_f)$ be a K-basis of $D_{\mathrm{dR}}(U) = U^{G_K}$, where $f = (d_1+1)(d'_1+1)$. If U is de Rham, then the element $X \otimes 1 \in U$ (where $X = \log t$ in $\mathbf{B}_{\mathrm{dR}}[\{0\}; d_1]$) may be written as a sum $X \otimes 1 = b_1(1 \otimes 1) + \sum_{i=2}^{f} b_i e_i$ with $b_i \in \mathbf{B}_{\mathrm{dR}}$ for all $1 \le i \le f$. Since $g(X \otimes 1) = X \otimes 1 + \log(\chi(g))(1 \otimes 1)$ for all $g \in G_K$, we have $g(b_1) - b_1 = \log(\chi(g))$ for all $g \in G_K$. If $b_1 \in \mathbf{B}_{\mathrm{dR}}^+$, then $g(\theta(b_1)) - \theta(b_1) = \log \chi(g)$ for all $g \in G_K$, which is impossible since $g \mapsto \log \chi(g)$ is a generator of the one-dimensional K-vector space $H^1(G_K, \mathbf{C}_p)$. If $b_1 \in t^h \mathbf{B}_{\mathrm{dR}}^+ \backslash t^{h+1} \mathbf{B}_{\mathrm{dR}}^+$ for some $h < 0$, then $b_1 = t^h b'$ for a unique $b' \in \mathbf{B}_{\mathrm{dR}}^+ \backslash t \mathbf{B}_{\mathrm{dR}}^+$ and $\chi(g)^h g(b') - b' \in t^{-h} \mathbf{B}_{\mathrm{dR}}^+ \subset t \mathbf{B}_{\mathrm{dR}}^+$, so that reducing modulo t would imply that $\theta(b') \in \mathbf{C}_p(h)^{G_K} = \{0\}$, a contradiction. We therefore see that W and W' must be de Rham.

The same arguments together with Fontaine's theorem [**Fon04**, 2.14] show that if W and W' are $\mathbf{C}_p$-representations of G_K with Sen weights in $\mathbf{Z}$ such that $W \otimes_{\mathbf{C}_p} W'$ is Hodge-Tate, then W and W' are Hodge-Tate. $\square$

THEOREM 5.1.0.15. *Let W and W' be nonzero $B_{|K}^{\otimes E}$-pairs. If the $B_{|K}^{\otimes E}$-pair $W\otimes W'$ is Hodge-Tate, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are Hodge-Tate. If, moreover, $W\otimes W'$ is de Rham, then so are $W(\mu^{-1})$ and $W'(\mu)$.*

PROOF. Let W and W' be $B_{|K}^{\otimes E}$-pairs and suppose that the $B_{|K}^{\otimes E}$-pair $W\otimes W'$ is Hodge-Tate. By extending scalars if necessary, we may suppose that $E/\mathbf{Q}_p$ is finite Galois and contains K, so that the methods of paragraph 4.2.1.8 apply.

Let $r = \mathrm{rank}(W)$ and let $r' = \mathrm{rank}(W')$. For each embedding $h : K \to E$, let $a_{1,h}, \dots, a_{r,h}$ denote the h-weights of the $\mathbf{C}_{p,E}$-representation $\overline{W}$ and let $a'_{1,h}, \dots, a'_{r',h}$ denote the h-weights of $\overline{W'}$. Part (iii) of proposition 4.2.1.2 implies that if $h : K \to E$ is an embedding, then the h-weights of $\overline{W\otimes W'}$ are the elements $a_{i,h} + a'_{j,h}$ for $1 \le i \le r$ and $1 \le j \le r'$, which are integers since the $\mathbf{C}_{p,E}$-representation $\overline{W\otimes W'} = \overline{W} \otimes_{\mathbf{C}_{p,E}} \overline{W'}$ is Hodge-Tate. By lemma 4.2.1.13, there is a finite Galois extension F/E and a character $\mu : G_K \to F^\times$ such that for all embeddings $h : K \to E \subset F$, the h-weight of the $\mathbf{C}_{p,F}$-representation $\overline{W(F(\mu))}$ is $a_{1,h}$.

We now show that the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are Hodge-Tate. If $h : K \to E \subset F$ is an embedding, then (ii) and (iii) of proposition 4.2.1.2 imply that the h-weights of $W(\mu^{-1})$ are the integers $a_{i,h} - a_{1,h}$ (for $1 \le i \le r$) and the h-weights of $W'(\mu)$ are the integers $a_{1,h} + a'_{j,h}$ for $1 \le j \le r'$. Since being Hodge-Tate is the same as being potentially Hodge-Tate, it suffices to show that the $B_{|F}^{\otimes F}$-pairs $W(\mu^{-1})|_{G_F}$ and $W'(\mu)|_{G_F}$ are Hodge-Tate. Let $\overline{W(\mu^{-1})} = \bigoplus_{h:F\to F} \overline{W(\mu^{-1})}_h$ and $\overline{W'(\mu)} = \bigoplus_{h:F\to F} \overline{W'(\mu)}_h$ be the decompositions of $\mathbf{C}_{p,F}$-representations of G_F as described in paragraph 4.1.2. The $\mathbf{C}_p$-representations $\overline{W(\mu^{-1})}_h$ and $\overline{W'(\mu)}_h$ have weights in $\mathbf{Z}$ for every h. The isomorphism

$$\overline{W(\mu^{-1})\otimes W'(\mu)} \simeq \bigoplus_{h:F\to F} \overline{W(\mu^{-1})}_h \otimes_{\mathbf{C}_p} \overline{W'(\mu)}_h$$

of $\mathbf{C}_p$-representations of G_F as in lemma 4.1.2.3 implies that $\overline{W(\mu^{-1})}_h \otimes_{\mathbf{C}_p} \overline{W'(\mu)}_h$ is Hodge-Tate for each embedding $h : F \to F$. By proposition 5.1.0.14, $\overline{W(\mu^{-1})}_h$ and $\overline{W'(\mu)}_h$ are Hodge-Tate for each embedding $h : F \to F$, and therefore $\overline{W(\mu^{-1})}$ and $\overline{W'(\mu)}$ are Hodge-Tate. Therefore, the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are Hodge-Tate.

Suppose now that $E/\mathbf{Q}_p$ is a finite Galois extension and that W and W' are $B_{|K}^{\otimes E}$-pairs such that the $B_{|K}^{\otimes E}$-pair $W\otimes W'$ is de Rham. By the above, there is a finite Galois extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are Hodge-Tate. We now show that $W(\mu^{-1})$ and $W'(\mu)$ are de Rham. It suffices to show that the restrictions of $W(\mu^{-1})$ and $W'(\mu)$ to G_F are de Rham. Let $W(\mu^{-1})_{\mathrm{dR}} = \bigoplus_{h:F\to F} W(\mu^{-1})_{\mathrm{dR},h}$ and $W'(\mu)_{\mathrm{dR}} = \bigoplus_{h:F\to F} W'(\mu)_{\mathrm{dR},h}$ be the decompositions of $\mathbf{B}_{\mathrm{dR}}$-representations of G_F as in paragraph 4.1.2. For each embedding $h : F \to F$, the $\mathbf{B}_{\mathrm{dR}}$-representations $W(\mu^{-1})_{\mathrm{dR},h}$ and $W'(\mu)_{\mathrm{dR},h}$ have de Rham weights in $\mathbf{Z}$. By lemma 4.1.2.3, the $\mathbf{B}_{\mathrm{dR}}$-representation $W(\mu^{-1})_{\mathrm{dR},h} \otimes_{\mathbf{B}_{\mathrm{dR}}} W'(\mu)_{\mathrm{dR},h}$ is de Rham for each embedding $h : F \to F$, and therefore so are $W(\mu^{-1})_{\mathrm{dR},h}$ and $W'(\mu)_{\mathrm{dR},h}$ by proposition 5.1.0.14. Therefore, the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are de Rham. ☐

COROLLARY 5.1.0.16. *Let $E/\mathbf{Q}_p$ and $K/\mathbf{Q}_p$ be finite extensions, and let V and V' be nonzero E-linear representations of G_K. If $V \otimes_E V'$ is Hodge-Tate, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that $V(\mu^{-1})$ and $V'(\mu)$ are Hodge-Tate. If, moreover, $V \otimes_E V'$ is de Rham, then so are $V(\mu^{-1})$ and $V'(\mu)$.*

5.2. Hodge-Tate and de Rham Schur B-pairs

In what follows, let $n \geq 1$ be an integer and let $u = (u_1, \ldots, u_r)$ denote an integer partition $n = u_1 + \ldots + u_r$ ($u_i \geq u_{i+1} \geq 1$) of n. If $u_1 = \ldots = u_r$, put $r(u) = r + 1$. Otherwise, put $r(u) = r$.

PROPOSITION 5.2.0.17. *If W is a $\mathbf{C}_p$-representation of G_K having Sen weights in $\mathbf{Z}$ such that $\dim_{\mathbf{C}_p}(W) \geq r(u)$ and $\mathrm{Schur}^u(W)$ is Hodge-Tate, then W is Hodge-Tate.*

If W is a $\mathbf{B}_{\mathrm{dR}}$-representation of G_K having de Rham weights in $\mathbf{Z}$ such that $\dim_{\mathbf{B}_{\mathrm{dR}}}(W) \geq r(u)$ and $\mathrm{Schur}^u(W)$ is de Rham, then W is de Rham.

PROOF. Let W be a $\mathbf{B}_{\mathrm{dR}}$-representation of G_K having de Rham weights in $\mathbf{Z}$ such that $\dim_{\mathbf{B}_{\mathrm{dR}}}(W) \geq r(u)$. If W is not de Rham, then Fontaine's theorem [**Fon04**, 3.19] gives a decomposition $W = \mathbf{B}_{\mathrm{dR}}[\{0\}; d] \oplus W'$ for some $d > 0$, so that

$$\mathrm{Schur}^u(W) \simeq \bigoplus_{\lambda,\mu} (\mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\}; d]) \otimes_{\mathbf{B}_{\mathrm{dR}}} \mathrm{Schur}^\mu(W'))^{\oplus c^u_{\lambda,\mu}}$$

as a $\mathbf{B}_{\mathrm{dR}}$-representation of G_K, where $c^u_{\lambda,\mu} \geq 0$ denotes the Littlewood-Richardson number. By lemma 3.2.2.2, there are λ and μ such that $c^u_{\lambda,\mu}$ and $\mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\}; d]) \otimes_{\mathbf{B}_{\mathrm{dR}}} \mathrm{Schur}^\mu(W')$ are nonzero, and such that $d + 1 \geq r(\lambda)$.

The $\mathbf{B}_{\mathrm{dR}}$-representations $\mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\}; d])$ and $\mathrm{Schur}^\mu(W')$ have de Rham weights in $\mathbf{Z}$ by lemma 4.2.1.2. If $\mathrm{Schur}^u(W)$ is de Rham, then so is $\mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\}; d]) \otimes_{\mathbf{B}_{\mathrm{dR}}} \mathrm{Schur}^\mu(W')$ and proposition 5.1.0.14 implies that $\mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\}; d])$ is de Rham. Let $X = \log t$ in $\mathbf{B}_{\mathrm{dR}}[\{0\}; d]$, so that $(1, X, X^2, \ldots, X^d)$ is the standard $\mathbf{B}_{\mathrm{dR}}$-basis of $\mathbf{B}_{\mathrm{dR}}[\{0\}; d]$. If T_1 is the standard tableau defined in 3.1.1, then the element $e_{T_1} \in \mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\}; d])$ is such that $g(e_{T_1}) = e_{T_1}$ for all $g \in G_K$. Let T' be the tableau with values in $\{1, \ldots, d+1\}$ which is obtained from T_1 by adding 1 to the value in the bottom-most cell of the right-most column of Y_λ; this tableau T' exists since $d + 1 \geq r(\lambda)$. A calculation shows that $g(e_{T'}) = e_{T'} + \nu \log \chi(g) e_{T_1}$, where ν is the length of the right-most column of Y_λ. If

$\mathrm{Schur}^{\lambda}(\mathbf{B}_{\mathrm{dR}}[\{0\};d])$ is de Rham, then it admits a basis of the form $(e_{T_1}, e_2, \ldots, e_f)$ (recall that e_{T_1} is an element of $\mathrm{Schur}^{\lambda}(\mathbf{B}_{\mathrm{dR}}[\{0\};d])$ defined by the standard tableau on Y_λ, as in paragraph 3.2.1), such that for all $i = 2, \ldots, f$, $g(e_i) = e_i$ for all $g \in G_K$. If $b_1, \ldots, b_f \in \mathbf{B}_{\mathrm{dR}}$ are elements such that $e_{T'} = b_1 e_T + \sum_{i \geq 2} b_i e_i$, then $g(b_1) - b_1 = \nu \log \chi(g)$ for all $g \in G_K$, which is impossible. Therefore, W and W' must be de Rham.

One can prove the claim for $\mathbf{C}_p$-representations by using Fontaine's theorem [**Fon04**, 2.14] and applying the same arguments. □

THEOREM 5.2.0.18. *Let W be a $B_{|K}^{\otimes E}$-pair such that* $\mathrm{rank}(W) \geq r(u)$. *If the $B_{|K}^{\otimes E}$-pair* $\mathrm{Schur}^u(W)$ *is Hodge-Tate, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is Hodge-Tate. If, moreover,* $\mathrm{Schur}^u(W)$ *is de Rham, then $W(\mu^{-1})$ is de Rham.*

PROOF. Let W be a $B_{|K}^{\otimes E}$-pair such that $d = \mathrm{rank}(W) \geq r(u)$ and suppose that $\mathrm{Schur}^u(W)$ is Hodge-Tate. By extending scalars if necessary, we may suppose that $E/\mathbf{Q}_p$ is finite Galois and contains K.

If $h : K \to E$ is an embedding, then let $a_{1,h}, \ldots, a_{d,h}$ denote the h-weights of $\overline{W}$. By corollary 4.2.1.7, the h-weights of the $\mathbf{C}_{p,E}$-representation $\overline{\mathrm{Schur}^u(W)} = \mathrm{Schur}^u(\overline{W})$ are the elements of the form $a_{T,h} = \sum a_{t_{ij},h}$ for any tableau $T = (t_{ij})$ with values in $\{1, \ldots, d\}$ on the Young diagram of u. Since $\mathrm{Schur}^u(W)$ is Hodge-Tate, the elements $a_{T,h}$ are in $\mathbf{Z}$. Considering the tableaux $T_1, \ldots, T_d$ as in proposition 3.1.1.1, we see that for all $i \in \{2, \ldots, d\}$, there is a $j \in \{1, \ldots, d\}$ such that $a_{T_j,h} - a_{T_{j-1},h} = a_{i,h} - a_{i-1,h} \in \mathbf{Z}$, and therefore $a_{i,h} - a_{1,h} \in \mathbf{Z}$ for all $1 \leq i \leq d$. By lemma 4.2.1.13, there is a finite Galois extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes F}$-pair $W(F(\mu))$ has $a_{1,h}$ as its h-weight for each embedding $h : K \to E \subset F$.

We now show that the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is Hodge-Tate. It suffices to show that the restriction of $W(\mu^{-1})$ to G_F are Hodge-Tate. Let $\overline{W(\mu^{-1})} = \bigoplus_{h:F\to F} \overline{W(\mu^{-1})}_h$ be the decomposition as a $\mathbf{C}_p$-representation of G_F as described in paragraph 4.1.2. The $\mathbf{C}_p$-representation $\overline{W(\mu^{-1})}_h$ has Sen weights in $\mathbf{Z}$ for each embedding $h : F \to F$. By lemma 4.1.2.4, the $\mathbf{C}_p$-representation $\mathrm{Schur}^u(\overline{W(\mu^{-1})}_h)$ of G_F is Hodge-Tate for each embedding $h : F \to F$. Since $\dim_{\mathbf{C}_p} \overline{W(\mu^{-1})}_h = \mathrm{rank}(W) \geq r(u)$, proposition 5.2.0.17 implies that $\overline{W(\mu^{-1})}_h$ is Hodge-Tate for each embedding $h : F \to F$. The $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is therefore Hodge-Tate.

Suppose now that W is a $B_{|K}^{\otimes E}$-pair such that $\mathrm{rank}(W) \geq r(u)$ and $\mathrm{Schur}^u(W)$ is de Rham. There is a finite Galois extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes E}$-pair $W(\mu^{-1})$ is Hodge-Tate. We now show that $W(\mu^{-1})$ is de Rham. Let $W(\mu^{-1})_{\mathrm{dR}} \simeq \bigoplus_{h:F\to F} W(\mu^{-1})_{\mathrm{dR},h}$ be the decomposition as a $\mathbf{B}_{\mathrm{dR}}$-representation of G_F as described in paragraph 4.1.2. The $\mathbf{B}_{\mathrm{dR}}$-representation $W(\mu^{-1})_{\mathrm{dR},h}$ has de Rham weights in $\mathbf{Z}$ for each embedding $h : F \to F$. By lemma 4.1.2.4, $\mathrm{Schur}^u(W(\mu^{-1})_{\mathrm{dR},h})$ is a de Rham $\mathbf{B}_{\mathrm{dR}}$-representation of G_F for each embedding $h : F \to F$ and therefore proposition 5.2.0.17 implies that $W(\mu^{-1})_{\mathrm{dR},h}$ is de Rham for each embedding h since $\dim_{\mathbf{B}_{\mathrm{dR}}} W(\mu^{-1})_{\mathrm{dR},h} = \mathrm{rank}(W) \geq r(u)$. Therefore, the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is de Rham. □

Corollary 5.2.0.19. *Let $n \geq 1$ be an integer, let u be a partition of n, and let V be an E-linear representation of G_K such that $\dim_E(V) \geq r(u)$. If $\mathrm{Schur}^u(V)$ is Hodge-Tate, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that $V(\mu^{-1})$ is Hodge-Tate. If, moreover, $\mathrm{Schur}^u(V)$ is de Rham, then V is de Rham.*

We now show that the bound on $\mathrm{rank}(W)$ in theorem 5.2.0.18 is optimal. Suppose $r(u) = r$, so that Y_u is non-rectangular. In this case, $\mathrm{Schur}^u(W) = 0$ for any $B_{|K}^{\otimes E}$-pair W such that $\mathrm{rank}(W) < r(u) = r$, and optimality is clear in this situation. Otherwise, $r(u) = r + 1$ (i.e., Y_u is rectangular, and $u_1 = \ldots = u_r$). Let W be a $B_{|K}^{\otimes E}$-pair. If $\mathrm{rank}(W) < r$, then $\mathrm{Schur}^u(W) = 0$. If $\mathrm{rank}(W) = r$, then $\mathrm{Schur}^u(W) = \bigotimes_{i=1}^{r} \det(W)$. To show optimality in the $r(u) = r+1$ case, it therefore suffices to find a $B_{|K}^{\otimes E}$-pair which is not Hodge-Tate up to a twist, but such that $\det(W)$ is Hodge-Tate. Let V be the $\mathbf{Q}_p$-module of polynomials in $X = \log(t)$ of degree $\leq r$, viewed as a representation of $G_{\mathbf{Q}_p}$, so that $\mathbf{C}_p \otimes_{\mathbf{Q}_p} V = \mathbf{C}_p[\{0\}; r]$. Considering $W = W(V)$, we see that $\det(W)$ is the trivial 1-dimensional representation of $G_{\mathbf{Q}_p}$ and therefore $\mathrm{Schur}^u(W)$ is trivial, and therefore Hodge-Tate. On the other hand, there is no character $\mu : G_{\mathbf{Q}_p} \to E^\times$ (with $E/\mathbf{Q}_p$ finite) such that $V(\mu)$ is Hodge-Tate; such a character would necessarily have Hodge-Tate weights in $\mathbf{Z}$, and 5.2.0.17 would imply that V is Hodge-Tate (which is not the case).

CHAPTER 6

Tensor products and Schur B-pairs of semi-stable type

6.1. Semi-stable B-pairs

Let $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^{+})$ be a $B_{|K}^{\otimes E}$-pair. We say that W is *crystalline* if the $\mathbf{B}_{\mathrm{cris}}$-representation $(\mathbf{B}_{\mathrm{cris},E}) \otimes_{\mathbf{B}_{\mathrm{e},E}} W_{\mathrm{e}}$ of G_K is trivial. Similarly, we say that W is *semi-stable* if the $\mathbf{B}_{\mathrm{st}}$-representation $(\mathbf{B}_{\mathrm{st},E}) \otimes_{\mathbf{B}_{\mathrm{e},E}} W_{\mathrm{e}}$ of G_K is trivial. We say that W is *potentially crystalline* (or *potentially semi-stable*) if there is a finite extension L/K such that the $B_{|L}^{\otimes E}$-pair $W|_{G_L}$ is crystalline (or semi-stable). Note that if V is an E-linear representation of G_K, then V is crystalline (or semi-stable) if and only if the $B_{|K}^{\otimes E}$-pair $W(V)$ is crystalline (or semi-stable).

Let L/K be a finite Galois extension and let $L_0 = L \cap \mathbf{Q}_p^{\mathrm{nr}}$. If W is a $B_{|K}^{\otimes E}$-pair which is semi-stable when restricted to G_L, then $D_{\mathrm{st},L}(W) = (\mathbf{B}_{\mathrm{st},E} \otimes_{\mathbf{B}_{\mathrm{e},E}} W_{\mathrm{e}})^{G_L}$ is a free $L_{0,E}$-module such that $\mathrm{rank}_{L_{0,E}}(D_{\mathrm{st},L}(W)) = \mathrm{rank}(W)$, and it is endowed with an injective additive self-map φ that is E-linear and semi-linear for the absolute Frobenius automorphism σ on L_0, an $L_{0,E}$-linear nilpotent endomorphism N such that $N\varphi = p\varphi N$, and an E-linear and L_0-semi-linear action of $\mathrm{Gal}(L/K)$ which commutes with φ and N. The following follows from [**Fon94b**, 4.2.6, 5.1.5].

Proposition 6.1.0.20. *Let W be a potentially semi-stable $B_{|K}^{\otimes E}$-pair, semi-stable when restricted to G_L where L/K is finite and Galois. The $B_{|K}^{\otimes E}$-pair W is semi-stable if and only if the inertia group $I_{L/K}$ acts trivially on $D_{\mathrm{st},L}(W)$, and W is crystalline if and only if it is semi-stable and $N = 0$ on $D_{\mathrm{st},L}(W)$.*

6.2. Semi-stable tensor products

Theorem 6.2.0.21. *Let W and W' be nonzero potentially semi-stable $B_{|K}^{\otimes E}$-pairs. If the $B_{|K}^{\otimes E}$-pair $W \otimes W'$ is semi-stable, then there is a finite extension F/E and a character $\mu : G_K \to F^{\times}$ such that the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are semi-stable. If, moreover, $W \otimes W'$ is crystalline, then so are $W(\mu^{-1})$ and $W'(\mu)$.*

Proof. Let L/K be a finite Galois extension such that W and W' are semi-stable as $B_{|L}^{\otimes E}$-pairs. By an E-linear analogue of [**Fon94b**, 5.1.7], we have an isomorphism of E-$(\varphi, N, \mathrm{Gal}(L/K))$-modules:

$$D_{\mathrm{st},L}(W \otimes W') \xleftarrow{\sim} D_{\mathrm{st},L}(W) \otimes_{L_{0,E}} D_{\mathrm{st},L}(W').$$

Let $\mathcal{E} \subset D_{\mathrm{st},L}(W)$ and $\mathcal{E}' \subset D_{\mathrm{st},L}(W')$ be $L_{0,E}$-bases, so that the set $\mathcal{E} \otimes \mathcal{E}'$ of elementary tensors is a basis of $D_{\mathrm{st},L}(W \otimes W')$. For all $g \in G_K$, let $U_g = \mathrm{Mat}(g|\mathcal{E}) \in \mathrm{GL}_d(L_{0,E})$ and let $U'_g = \mathrm{Mat}(g|\mathcal{E}') \in \mathrm{GL}_{d'}(L_{0,E})$. By proposition 6.1.0.20, $I_{L/K}$ acts trivially on $D_{\mathrm{st},L}(W \otimes W')$, and we have $\mathrm{Mat}(g|\mathcal{E} \otimes \mathcal{E}') = U_g \otimes U'_g = \mathrm{Id}$ for all $g \in I_{L/K}$, so that $U_g = \eta_g \,\mathrm{Id}$ and $U'_g = \eta_g^{-1}\,\mathrm{Id}$ with $\eta_g \in (L_{0,E})^\times$. The relation $\varphi g = g\varphi$ on $D_{\mathrm{st},L}(W)$ translates to the matrix relation $\mathrm{Mat}(\varphi|\mathcal{E})\cdot\sigma(U_g) = U_g \cdot g(\mathrm{Mat}(\varphi|\mathcal{E}))$ for all $g \in \mathrm{Gal}(L/K)$, so that for all $g \in I_{L/K}$, we have $\eta_g \in (L_{0,E})^{\sigma=1} = E$ and therefore $\eta_g \in E^\times$.

We now show that there is a finite extension F/E such that the character $\eta : I_{L/K} \to E^\times$ can be extended to a character $\mu : \mathrm{Gal}(L/K) \to F^\times$. Let $\omega \in \mathrm{Gal}(L/K)$ be such that its residual image generates the cyclic group $\mathrm{Gal}(k_L/k_K)$. If $g \in \mathrm{Gal}(L/K)$, then we can write $g = g'\omega^i$ for a unique $g' \in I_{L/K}$ and unique $0 \le i \le f-1$, where $f = [k_L : k_K]$. Let $\xi \in \overline{\mathbf{Q}}_p$ be an f^{th} root of $\eta(\omega^f)$. Since $\eta(\omega g' \omega^{-1}) = \eta(g')$ for all $g' \in I_{L/K}$, putting $F = E(\xi)$ and $\mu(g) := \eta(g')\xi^i$ defines a homomorphism $\mu : G_K \to F^\times$.

The $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are semi-stable, by proposition 6.1.0.20. If, moreover, $W \otimes W'$ is crystalline, then the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1}) \otimes W'(\mu)$ is crystalline as well and by the isomorphism of F-$(\varphi, N, \mathrm{Gal}(L/K))$-modules recalled above, we have :

$$D_{\mathrm{st},L}(W(\mu^{-1}) \otimes W'(\mu)) \xleftarrow{\sim} D_{\mathrm{st},L}(W(\mu^{-1})) \otimes_{L_{0,F}} D_{\mathrm{st},L}(W'(\mu)).$$

The monodromy operator $N \otimes \mathrm{Id} + \mathrm{Id} \otimes N'$ is zero, and therefore the matrices of N and N' are scalar multiples of the identity. Since N and N' are nilpotent, these scalars are necessarily zero since $L_{0,F}$ is reduced, and thus $W(\mu^{-1})$ and $W'(\mu)$ are crystalline by 6.1.0.20. $\square$

Corollary 6.2.0.22. *Let V and V' be nonzero potentially semi-stable E-linear representations of G_K. If $V \otimes_E V'$ is semi-stable, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that the F-linear representations $V(\mu^{-1})$ and $V'(\mu)$ are semi-stable. If, moreover, $V \otimes_E V'$ is crystalline, then so are $V(\mu^{-1})$ and $V'(\mu)$.*

6.3. Semi-stable Schur B-pairs

In this paragraph, $n \ge 1$ is an integer and $u = (u_1, \ldots, u_r)$ denotes an integer partition $n = u_1 + \ldots + u_r$ such that $u_i \ge u_{i+1} \ge 1$ for each $i \in \{1, \ldots, r-1\}$.

Lemma 6.3.0.23. *Let L/K be a finite Galois extension and let D be an E-$(\varphi, N, \mathrm{Gal}(L/K))$-module such that $\mathrm{rank}(D) \ge r(u)$. If $I_{L/K}$ acts trivially on $\mathrm{Schur}^u(D)$, then there is a finite extension[1] E'/E such that $I_{L/K}$ acts on D via a character $\eta : I_{L/K} \to E'^\times$. If $N = 0$ on $\mathrm{Schur}^u(D)$, then $N = 0$ on D.*

[1]In [**DiM13**], we mistakenly neglect to point out that one should extend scalars.

PROOF. By extending scalars if necessary, we may suppose that $E \supset L$. We have an isomorphism of rings, $L_{0,E} \xrightarrow{\sim} \bigoplus_{h:L_0 \to \overline{\mathbf{Q}}_p} E$ on which $I_{L/K}$ acts trivially on both sides. We therefore see that D decomposes as an E-linear representation of $I_{L/K}$ into $D \simeq \bigoplus_h D_h$ where D_h is the E-linear representation of $I_{L/K}$ coming from the h-factor map $(\lambda, e) \mapsto h(\lambda)e : L_{0,E} \to E$. The corresponding decomposition of $\mathrm{Schur}^u(D)$ is given by $\mathrm{Schur}^u(D) \simeq \bigoplus_h \mathrm{Schur}^u(D_h)$, and by assumption $I_{L/K}$ acts trivially on each E-linear representation $\mathrm{Schur}^u(D_h)$. Let $I_{L/K}$ act $\overline{\mathbf{Q}}_p$-linearly on $\overline{D}_h = \overline{\mathbf{Q}}_p \otimes_E D_h$. Let $g \in I_{L/K}$. Since $I_{L/K}$ is finite, there is a $\overline{\mathbf{Q}}_p$-basis $\mathcal{E}_h^g = (e_{1,h}^g, \ldots, e_{d,h}^g)$ of $\overline{D}_h$ and elements $\lambda_{1,h}^g, \ldots, \lambda_{d,h}^g \in \overline{\mathbf{Q}}_p$ such that $g(e_{i,h}^g) = \lambda_{i,h}^g e_{i,h}^g$ for all $i \in \{1, \ldots, d\}$. Consider the $\overline{\mathbf{Q}}_p$-basis of $\mathrm{Schur}^u(\overline{D}_h)$ consisting of elements $e_{T,h}^g$, where T ranges over all tableaux on Y_u with values in $\{1, \ldots, d\}$. By proposition 3.2.1.4, one has $g(e_{T,h}^g) = \lambda_{T,h}^g e_{T,h}^g$, where $\lambda_{T,h}^g = \prod_{i=1}^d (\lambda_{i,h}^g)^{m_T(i)}$ and $m_T(i)$ denotes the number of times that i appears in the tableau T. Since $\dim_{\overline{\mathbf{Q}}_p} \overline{D}_h = \mathrm{rank}(D) \geq r(u)$, there are tableaux $T_1, \ldots, T_d$ as in proposition 3.1.1.1. Since $I_{L/K}$ acts trivially on $\mathrm{Schur}^u(D)$, one sees that $\lambda_{T,h}^g = 1$ for all g and all tableaux T and in particular, for all $i \in \{2, \ldots, d\}$, there is a $j \in \{1, \ldots, d\}$ such that $1 = \lambda_{T_j,h}^g \cdot (\lambda_{T_{j+1},h}^g)^{-1} = \lambda_{i,h}^g (\lambda_{i+1,h}^g)^{-1}$ (again, by proposition 3.1.1.1). In particular, $\lambda_{1,h}^g = \lambda_{2,h}^g = \ldots = \lambda_{d,h}^g = \lambda_h^g$, and therefore $g(z) = \lambda_h^g z$ for all $z \in \overline{D}_h$. If E'/E denotes the extension generated by the λ_h^g for all g and all h, then we see that for each embedding $h : L_0 \to E$, $I_{L/K}$ acts on $\overline{D}_h$ by a character $\eta_h : I_{L/K} \to E'^{\times}$, which translates to saying that $I_{L/K}$ acts on $\overline{D}$ (and therefore on $E' \otimes_E D$) by a character $\eta : I_{L/K} \to (L_{0,E'})^{\times}$, which takes values in $E'^{\times}$ since $\varphi g = g \varphi$ for all $g \in I_{L/K}$ and since $(L_{0,E'})^{\sigma=1} = E'$.

Moreover, since N is an $L_{0,E}$-linear map, the factors in the decomposition $D \simeq \bigoplus_h D_h$ are N-stable. We let N again denote the E-linear nilpotent map induced on D_h. Since $N = 0$ on $\mathrm{Schur}^u(\overline{D}) = \bigoplus_h \mathrm{Schur}^u(\overline{D}_h)$, we see that $N = 0$ on $\mathrm{Schur}^u(\overline{D}_h)$ for each embedding $h : L_0 \to \overline{\mathbf{Q}}_p$. Let $(e'_{1,h}, \ldots, e'_{d,h})$ denote a Jordan canonical basis for N on $\overline{D}_h$. Suppose that $N \neq 0$, so that we may suppose $N(e'_{2,h}) = e'_{1,h}$. If T is the tableau on Y_u in which i appears in all boxes of the i-th row, except in the right most column where $i+1$ appears, then a calculation shows that $N(e_{T,h}) = e_{T',h}$, where T' is another tableau, therefore contradicting the fact that $N = 0$ on $\overline{D}_h$. We therefore see that $N = 0$ on each $\overline{D}_h$, so that $N = 0$ on $\overline{D}$ and thus $N = 0$ on D. □

THEOREM 6.3.0.24. *Let W be a potentially semi-stable $B_{|K}^{\otimes E}$-pair such that* $\mathrm{rank}(W) \geq$ *$r(u)$. If the $B_{|K}^{\otimes E}$-pair* $\mathrm{Schur}^u(W)$ *is semi-stable, then there is a finite extension F/E and a character $\mu : G_K \to F^{\times}$ such that the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is semi-stable. If, moreover,* $\mathrm{Schur}^u(W)$ *is crystalline, then so is $W(\mu^{-1})$.*

Proof. Let L/K be a finite Galois extension such that W is semi-stable as a $B_{|L}^{\otimes E}$-pair, so that [**Fon94b**, 5.1.7] implies that we have an isomorphism of E-$(\varphi, N, \mathrm{Gal}(L/K))$-modules

$$\mathrm{Schur}^u(D_{\mathrm{st},L}(W)) \xrightarrow{\sim} D_{\mathrm{st},L}(\mathrm{Schur}^u(W))$$

If $\mathrm{Schur}^u(W)$ is semi-stable, then proposition 6.1.0.20 implies that $I_{L/K}$ acts trivially on $\mathrm{Schur}^u(D_{\mathrm{st},L}(W))$. Lemma 6.3.0.23 implies that there is a finite extension E'/E such that $I_{L/K}$ acts on $D_{\mathrm{st},L}(E' \otimes_E W)$ via a character $\eta : I_{L/K} \to E'^{\times}$. By the same reasoning as in the proof theorem 6.2.0.21, there is a finite extension F/E and a character μ : $\mathrm{Gal}(L/K) \to F^{\times}$ such that $\mu|_{I_{L/K}} = \eta$. By proposition 6.1.0.20, $W(\mu^{-1})$ is semi-stable.

If $\mathrm{Schur}^u(W)$ is crystalline, then $N = 0$ on $\mathrm{Schur}^u(D_{\mathrm{st},L}(W))$. Lemma 6.3.0.23 implies that $N = 0$ on $D_{\mathrm{st},L}(W)$, which implies the same for $D_{\mathrm{st},L}(W(\mu^{-1}))$, so that $W(\mu^{-1})$ is crystalline. □

Theorem 6.3.0.24 implies the following.

Corollary 6.3.0.25. *Let V be a potentially semi-stable E-linear representation of G_K such that $\dim_E V \geq r(u)$. If the E-linear representation* $\mathrm{Schur}^u(V)$ *of G_K is semi-stable, then there is a finite extension F/E and a character $\mu : G_K \to F^{\times}$ such that the F-linear representation $V(\mu^{-1})$ of G_K is semi-stable. If, moreover,* $\mathrm{Schur}^u(V)$ *is crystalline, then so is $V(\mu^{-1})$.*

Part 3

Triangulable tensor products

CHAPTER 7

Triangulable tensor products

7.1. Notation and generalities

7.1.1. Notation. Let $\overline{\mathbf{Q}}_p/\mathbf{Q}_p$ be an algebraic closure and let $\mathbf{C}_p$ be the p-adic completion of $\overline{\mathbf{Q}}_p$. Let $\mathbf{Q}_p^{\mathrm{nr}}$ denote the maximal non-ramified sub-extension of $\overline{\mathbf{Q}}_p/\mathbf{Q}_p$. Let $\mathbf{B}_{\mathrm{dR}}$, $\mathbf{B}_{\mathrm{dR}}^+$, $\mathbf{B}_{\mathrm{cris}}$, and $\mathbf{B}_{\mathrm{st}}$ denote Fontaine's rings as in [**Fon94a**] and let $\mathbf{B}_{\mathrm{e}} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. In this chapter, $E/\mathbf{Q}_p$ and $K/\mathbf{Q}_p$ denote finite extensions. If $\mathbf{B}$ is any of the above rings or any Galois sub-extension of $\overline{\mathbf{Q}}_p/K$, then $\mathbf{B}_E$ denotes the ring $\mathbf{B} \otimes_{\mathbf{Q}_p} E$ endowed with an action of G_K defined by $g(b \otimes e) = g(b) \otimes e$ for all $g \in G_K$. If W is a free $\mathbf{B}_E$-module of finite rank endowed with a semi-linear action of G_K, then we refer to W as a $\mathbf{B}_E$*-representation of* G_K.

7.1.2. The ring $\mathbf{B}_{\mathrm{e},E}$. The ring $\mathbf{B}_{\mathrm{cris}}$ may be viewed as a G_K-stable sub-ring of $\mathbf{B}_{\mathrm{dR}}$, and therefore the same is true for $\mathbf{B}_{\mathrm{e}}$. We therefore have a G_K-equivariant map $\mathbf{B}_{\mathrm{e},E} \to \mathbf{B}_{\mathrm{dR},E}$, where the actions of G_K on these two rings are defined as in paragraph 7.1.1 above. On the other hand, there is an injective morphism of rings $\mathbf{B}_{\mathrm{cris}} \otimes_{E_0} E \to \mathbf{B}_{\mathrm{dR}}$ which is G_K-equivariant if $K \supset E^{\mathrm{Gal}}$, and therefore we have an injective morphism $\mathbf{B}_{\mathrm{e},E} \to \mathbf{B}_{\mathrm{dR}}$ which is G_K-equivariant if $K \supset E^{\mathrm{Gal}}$. One has $\mathbf{B}_{\mathrm{e}} \cap \mathbf{B}_{\mathrm{dR}}^+ = \mathbf{Q}_p$ and $\mathbf{B}_{\mathrm{e}} \cap t\mathbf{B}_{\mathrm{dR}}^+ = \{0\}$. We will use the following generalization (see §8.7 of [**Col02**]).

PROPOSITION 7.1.2.1. *If $E/\mathbf{Q}_p$ is finite, then $\mathbf{B}_{\mathrm{e},E} \cap \mathbf{B}_{\mathrm{dR}}^+ = E$.*

As pointed out in paragraph 2.5.1 of chapter 2, we also have the following.

PROPOSITION 7.1.2.2. *If $E/\mathbf{Q}_p$ is a finite extension, then $\mathbf{B}_{\mathrm{e},E}$ is a principal ideal domain.*

In this chapter, F_E denotes the field $\mathrm{Frac}(\mathbf{B}_{\mathrm{e},E})$.

PROPOSITION 7.1.2.3. *If $E/\mathbf{Q}_p$ is finite, then $\mathbf{B}_{\mathrm{e},E}^\times = E^\times$, and if $\lambda \in \mathbf{B}_{\mathrm{e},E}$ generates a G_K-stable $\mathbf{B}_{\mathrm{e},E}$-module of rank 1, then $\lambda \in E^\times$.*

PROOF. For $E = \mathbf{Q}_p$, this is lemma 1.1.8 of [**Ber08**]. Suppose now $d = [E : \mathbf{Q}_p] > 1$. For each embedding $\tau : E \to \overline{\mathbf{Q}}_p$, we have an injective morphism $\iota_\tau : \mathbf{B}_{\mathrm{e},E} \to \mathbf{B}_{\mathrm{dR}}$. If $\lambda \in \mathbf{B}_{\mathrm{e},E}^\times$, then $\mathrm{Nm}_{E/\mathbf{Q}_p}(\lambda) = \prod_\tau \iota_\tau(\lambda) \in \mathbf{B}_{\mathrm{e}}^\times = \mathbf{Q}_p^\times$. Therefore, since $v_t(\lambda) = v_t(\iota_\tau(\lambda))$

for all τ, we have $v_t(\lambda) = 0$ (here v_t denotes the t-adic valuation of $\mathbf{B}_{\mathrm{dR}}$). Therefore $\lambda \in \mathbf{B}_{\mathrm{e},E} \cap (\mathbf{B}_{\mathrm{dR}}^+ - t\mathbf{B}_{\mathrm{dR}}^+) = E^\times$ and thus $\mathbf{B}_{\mathrm{e},E}^\times = E^\times$.

Suppose now that $\lambda \in \mathbf{B}_{\mathrm{e},E}$ generates a G_K-stable $\mathbf{B}_{\mathrm{e},E}$-module of rank 1. For all $g \in G_K$, $g(\lambda)/\lambda \in \mathbf{B}_{\mathrm{e},E}^\times = E^\times$ and $\mu : G_K \to E^\times$ given by $g \mapsto g(\lambda)/\lambda$ is a linear character. Let L/K be a finite extension such that $L \supset E^{\mathrm{Gal}}$. The element $\lambda' = \mathrm{Nm}(\lambda) \in \mathbf{B}_{\mathrm{e}} - \{0\}$ is a period for the character $\eta = \mathrm{Nm}_{E/\mathbf{Q}_p}(\mu) : G_L \to \mathbf{Q}_p^\times$. Applying lemma 1.1.8 of [**Ber08**], we deduce that $\lambda' \in \mathbf{Q}_p^\times$, so that we again have $v_t(\lambda)$ by the same reasoning as above, and therefore $\lambda \in \mathbf{B}_{\mathrm{e},E} \cap (\mathbf{B}_{\mathrm{dR}}^+ - t\mathbf{B}_{\mathrm{dR}}^+) = E^\times$. □

In particular, the above proposition immediately implies the following.

Corollary 7.1.2.4. *If X is a semi-linear $\mathbf{B}_{\mathrm{e},E}$-representation of G_K, then there is a linear character $\eta : G_K \to E^\times$ such that $X = \mathbf{B}_{\mathrm{e},E}(\eta)$.*

Proposition 7.1.2.5. *If $E/\mathbf{Q}_p$ is a finite extension, then* $\mathrm{Frac}(\mathbf{B}_{\mathrm{e}}) \otimes_{\mathbf{Q}_p} E$ *is a field.*

Proof. It suffices to show that if $P \in \mathbf{Q}_p[T]$ is monic and irreducible, then P remains irreducible when viewed as a polynomial in $\mathrm{Frac}(\mathbf{B}_{\mathrm{e}})[T]$. To that end, let $\mathrm{Frac}(\mathbf{B}_{\mathrm{e}})$ be viewed as a sub-field of $\mathbf{B}_{\mathrm{dR}}$ and suppose that $P = AB$ with $A, B \in \mathrm{Frac}(\mathbf{B}_{\mathrm{e}})[T] \subset \mathbf{B}_{\mathrm{dR}}[T]$ monic and non-constant. The relations between roots and coefficients of a polynomial imply that there is a finite extension $E'/\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p \subset \mathbf{B}_{\mathrm{dR}}^+$ such that $A, B \in E'[T]$. We now show $E' \cap \mathrm{Frac}(\mathbf{B}_{\mathrm{e}}) = \mathbf{Q}_p$. Let $\lambda = \frac{a}{b} \in E' \cap \mathrm{Frac}(\mathbf{B}_{\mathrm{e}}) - \{0\}$ with $a, b \in \mathbf{B}_{\mathrm{e}}$, $b \neq 0$, and $(a, b) = 1$. Let $K/\mathbf{Q}_p$ be a finite extension containing ${E'}^{\mathrm{Gal}}$. We have $g(a) = \lambda g(b)$ for all $g \in G_K$ and therefore, since $\mathbf{B}_{\mathrm{e}}$ is principal, $b|g(b)$ in $\mathbf{B}_{\mathrm{e}}$ for all $g \in G_K$ so that $g(a) = (g(b)/b)a$ for all $g \in G_K$, and thus a generates a G_K-stable $\mathbf{B}_{\mathrm{e}}$-module of rank 1, so that 7.1.2.3 implies $a \in \mathbf{Q}_p^\times$. Similarly, $b \in \mathbf{Q}_p^\times$ and therefore, we see that $\mathrm{Frac}(\mathbf{B}_{\mathrm{e}}) \cap E' = \mathbf{Q}_p$. In particular, both A and B have coefficients in $\mathbf{Q}_p$, and therefore one of them is a nonzero constant since P is irreducible in $\mathbf{Q}_p[T]$, which contradicts the assumption that both A and B are non-constant. □

In particular, $F_E := \mathrm{Frac}(\mathbf{B}_{\mathrm{e},E}) = \mathrm{Frac}(\mathbf{B}_{\mathrm{e}}) \otimes_{\mathbf{Q}_p} E$ and F_E may be interpreted as the compositum of $\mathrm{Frac}(\mathbf{B}_{\mathrm{e}})$ and E inside $\mathbf{B}_{\mathrm{dR}}$, and this interpretation is compatible with the action of G_K on $\mathbf{B}_{\mathrm{dR}}$ when $K \supset E^{\mathrm{Gal}}$.

7.1.3. Triangulability. Recall that $F_E := \mathrm{Frac}(\mathbf{B}_{\mathrm{e},E}) = \mathrm{Frac}(\mathbf{B}_{\mathrm{e}}) \otimes_{\mathbf{Q}_p} E$. If $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^+)$ is a $B_{|K}^{\otimes E}$-pair of rank r, then $X = F_E \otimes_{\mathbf{B}_{\mathrm{e},E}} W_{\mathrm{e}}$ is a semi-linear F_E-representation of G_K and $\dim_{F_E} X = r$.

Proposition 7.1.3.1. *Let W be a $B_{|K}^{\otimes E}$-pair and let $X = F_E \otimes_{\mathbf{B}_{\mathrm{e},E}} W_{\mathrm{e}}$ be the associated semi-linear F_E-representation of G_K.*

(1) *If $X' \subset X$ is a G_K-stable sub-F_E-vector space, then there is a saturated sub-$B_{|K}^{\otimes E}$-pair $W' \subset W$ such that $X' = F_E \otimes_{\mathbf{B}_{e,E}} W'_e$. In particular,* $\mathrm{rank}(W') = \dim_{F_E} X'$.

(2) *If X'' is a quotient of X in the category of semi-linear F_E-representations of G_K, then there is a $B_{|K}^{\otimes E}$-pair W'' such that W'' is a quotient of W and $X'' = F_E \otimes_{\mathbf{B}_{e,E}} W''_e$*

PROOF. The sub-$\mathbf{B}_{e,E}$-module $W'_e = X' \cap W_e$ of W_e is stable by the action of G_K. Since $\mathbf{B}_{e,E}$ is a principal ideal domain, W'_e is a free $\mathbf{B}_{e,E}$-module, and a basis of W'_e may be extended to a basis of W_e. In particular, $\mathrm{rank}_{\mathbf{B}_{e,E}} W'_e = \dim_{F_E} X'$.

The sub-$\mathbf{B}^+_{\mathrm{dR},E}$-module $W'^+_{\mathrm{dR}} = W'_{\mathrm{dR}} \cap W^+_{\mathrm{dR}}$ is a G_K-stable $\mathbf{B}^+_{\mathrm{dR},E}$-lattice of W'_{dR}, and $W' = (W'_e, W'^+_{\mathrm{dR}})$ is a saturated sub-$B_{|K}^{\otimes E}$-pair of W. In particular, the quotient $W/W' = (W_e/W'_e, W^+_{\mathrm{dR}}/W'^+_{\mathrm{dR}})$ is a $B_{|K}^{\otimes E}$-pair of rank equal to $\mathrm{rank}(W) - \mathrm{rank}(W')$.

If $f : X \to X''$ is a surjective morphism of F_E-representations of G_K, then (2) follows from (1) by considering $X' = \ker(f : X \to X'')$ and taking $W'' = W/W'$. □

If $\mathbf{B} \in \{F_E, \mathbf{B}_{e,E}\}$, then we say that a semi-linear $\mathbf{B}$-representation W of G_K is *triangulable* if it is a successive extension of rank 1 semi-linear $\mathbf{B}$-representations.

COROLLARY 7.1.3.2. *Let $W = (W_e, W^+_{\mathrm{dR}})$ be a $B_{|K}^{\otimes E}$-pair. The following conditions are equivalent.*

(1) *W is triangulable in the category of $B_{|K}^{\otimes E}$-pairs.*
(2) *W_e is triangulable in the category of semi-linear $\mathbf{B}_{e,E}$-representations of G_K.*
(3) *$X = F_E \otimes_{\mathbf{B}_{e,E}} W_e$ is triangulable in the category of semi-linear F_E-representations of G_K.*

PROOF. Showing that (1) implies (2) and that (2) implies (3) is straightforward. Proposition 7.1.3.1 allows one to construct a triangulation of W from a triangulation of X. □

In particular, if $V \in \mathrm{Rep}_E(G_K)$, then V is split trianguline if and only if the semi-linear F_E-representation $F_E \otimes_E V$ of G_K is triangulable.

7.1.4. Semi-stable B-pairs. In §2.4 of [**BelChe09**], it was shown that if $V \in \mathrm{Rep}_E(G_{\mathbf{Q}_p})$ is crystalline, then V is trianguline. Using similar arguments, we can show the following.

PROPOSITION 7.1.4.1. *If W is a semi-stable $B_{|K}^{\otimes E}$-pair, then there is a finite extension F/E such that the $B_{|K}^{\otimes F}$-pair $F \otimes W$ is triangulable.*

PROOF. If E'/E is a finite extension, then $E' \otimes_E W$ is a semi-stable $B_{|K}^{\otimes E'}$-pair; we may therefore assume without loss of generality that $E \supset K_0$. If $f = [K_0 : \mathbf{Q}_p]$ and if

$\sigma : K_0 \to K_0$ denotes the absolute frobenius, then $\{\mathrm{Id}, \sigma, \sigma^2, \dots, \sigma^{f-1}\}$ are the distinct embeddings of K_0 into $\overline{\mathbf{Q}}_p$. The E-(φ, N)-module $D = D_{\mathrm{st},K}(W)$ decomposes as an E-vector space into $D_{\mathrm{st},K}(W) = \bigoplus_{i=0}^{f-1} D_{\sigma^i}$, where D_{σ^i} is the sub-E-vector space coming from the i-th projection in the isomorphism of E-algebras $K_{0,E} \xrightarrow{\sim} \bigoplus_{\sigma^i : K_0 \to K_0} E$. One has $D_{\sigma^i} = \varphi^i(D_{\mathrm{Id}})$.

The operators φ^f and N on $D_{\mathrm{st},K}(W)$ are both $K_{0,E}$-linear, and they therefore both stabilize the sub-E-vector spaces D_{σ^i} (for each $i \in \{0, \dots, f-1\}$) when viewed as E-linear operators. The relation $N\varphi^f = p^f \varphi^f N$ implies that there is a finite extension F/E and an F-basis $\mathcal{E} = (e_1, \dots, e_d)$ of D_{Id} such that $\mathrm{Mat}(\varphi^f|\mathcal{E})$ and $\mathrm{Mat}(N|\mathcal{E})$ are simultaneously upper-triangular. If $i \in \{1, \dots, d\}$ and if $v_i = e_i \oplus \varphi(e_i) \oplus \dots \oplus \varphi^{f-1}(e_i)$, then $\mathcal{V} = (v_1, \dots, v_d)$ is a $K_{0,F}$-basis of $D_{\mathrm{st},K}(W)$. For each $i \in \{1, \dots, d\}$, the sub-$K_{0,F}$-module $D_i = \bigoplus_{j=1}^{i} K_{0,F} \cdot v_i$ is stable by φ and by N, and $K \otimes_{K_0} D_i$ inherits a filtration from the filtration on $K \otimes_{K_0} D_{\mathrm{st},K}(W)$; in particular, D_i is a filtered F-(φ, N)-module over K.

For each $i \in \{1, \dots, d\}$, the sub-$B_{|K}^{\otimes F}$-pair $W(D_{i-1}) \subset W(D_i)$ is saturated, and $W(D_i)/W(D_{i-1})$ is a $B_{|K}^{\otimes F}$-pair of rank 1. By proposition 2.3.3 of [**Ber08**], we have $W(D_{\mathrm{st},K}(W)) = W$, and therefore we have a triangulation

$$0 \subset W(D_1) \subset \dots \subset W(D_{\mathrm{st},K}(W)) = W$$

in the category of $B_{|K}^{\otimes F}$-pairs. □

If W is a de Rham $B_{|K}^{\otimes E}$-pair, then it is potentially semi-stable (see theorem 2.3.5 of [**Ber08**]). We therefore have the following.

Corollary 7.1.4.2. *If W is a de Rham $B_{|K}^{\otimes E}$-pair, then there are finite extensions F/E and L/K such that the $B_{|L}^{\otimes F}$-pair $(F \otimes W)|_{G_L}$ is triangulable.*

7.2. Semi-linear algebra

In this section, F denotes a field and G denotes a group that acts on F by field automorphisms.

7.2.1. Semi-linear representations. We say that a semi-linear F-representation V of G is *irreducible* if its only G-stable sub-F-vector spaces are $\{0\}$ and V.

If X is a semi-linear F-representation of G, then X is said to be *triangulable* if there is a filtration of G-stable sub-F-vector spaces

$$\{0\} = X_0 \subset X_1 \subset \dots \subset X_d = X$$

such that X_i/X_{i-1} is of dimension 1 for all $1 \leq i \leq d$. We say that X is *triangulable by characters* if there are linear characters $\eta_1, \dots, \eta_d : G \to (F^G)^\times \subset F^\times$ such that $X_i/X_{i-1} = F(\eta_i)$ for all i.

LEMMA 7.2.1.1. *If $0 \to X' \to X \to X'' \to 0$ is an exact sequence of semi-linear F-representations of G, then X is triangulable if and only if X' and X'' are triangulable.*

7.2.2. The trace form. If V is a semi-linear F-representation of G, then the F-vector space $\mathrm{End}_F(V)$ equipped with the action $g.f : x \mapsto g(f(g^{-1}.x))$ is a semi-linear F-representation of G of dimension $(\dim_F V)^2$. If $\mu : G \to F^\times$ is a 1-cocycle (for example, a linear character $\mu : G \to (F^G)^\times$), then let $\mathrm{End}_F(G)(\mu)$ denote $\mathrm{End}_F(G) \otimes_F F(\mu)$, where $F(\mu) = F \cdot e$ with $g(e) = \mu(g)e$ is the 1-dimensional semi-linear F-representation of G defined by μ. For all $f \in \mathrm{End}_F(V)(\mu)$ and all $g \in G$, we have $\mathrm{Tr}(g.f) = \mu(g)g(\mathrm{Tr}(f))$. In particular, $W_0 = \{f \mid \mathrm{Tr}(f) = 0\} \subset \mathrm{End}_F(V)(\mu)$ is a G-stable sub-F-vector space.

The map

$$\begin{aligned} \langle \text{-}, \text{-} \rangle : \mathrm{End}_F(V)(\mu) \times \mathrm{End}_F(V)(\mu) &\to F(\mu^2) \\ (h, h') &\mapsto \mathrm{Tr}(h \circ h') \end{aligned}$$

satisfies the following properties:

(1) $\langle \text{-}, \text{-} \rangle$ is F-bilinear,
(2) $\langle \text{-}, \text{-} \rangle$ is symmetric,
(3) $\langle g.h, g.h' \rangle = \mu^2(g)g(\langle h, h' \rangle)$.

In particular, if $W \subset \mathrm{End}_F(V)(\mu)$ is a G-stable sub-F-vector space, then $W^\perp = \{h \mid \mathrm{Tr}(h \circ h') = 0 \text{ for all } h' \in W\} \subset \mathrm{End}_F(V)(\mu)$ is a G-stable sub-F-vector space.

The trace form $\langle \text{-}, \text{-} \rangle$ is, in particular, a bilinear form; therefore, if $W_0 \cap W_0^\perp = \{0\}$, then $\mathrm{End}_F(V) = W_0 \oplus W_0^\perp$ as F-vector spaces, and in this case $\mathrm{End}_F(V)(\mu) = W_0 \oplus W_0^\perp$ is a decomposition into G-stable sub-F-vector spaces.

LEMMA 7.2.2.1. *If $\mathrm{char}(F) = 0$, then $W_0 \cap W_0^\perp = \{0\}$.*

PROOF. If $h \in W_0 \cap W_0^\perp$, then $\mathrm{Tr}(h \circ h) = 0$ and thus $h^2 \in W_0$. We therefore have $\mathrm{Tr}(h^k) = 0$ for all $k \geq 1$. Since $\mathrm{char}(F) = 0$, h is therefore nilpotent. Let $\mathcal{E} = (e_1, \dots, e_d)$ be a Jordan canonical basis for h, so that $\mathrm{Mat}(h|\mathcal{E})$ is a direct sum of $r \times r$ blocks of the form $J_r(0) = (a_{ij})_{1 \leq i,j \leq r}$ with $a_{ij} = 1$ if $j = i + 1$ and $a_{ij} = 0$ otherwise. If $h \neq 0$, then we may suppose that $h(e_1) = 0$ and $h(e_2) = e_1$. Let $h' : V \to V$ be the F-linear map defined by $h'(e_1) = e_2$ and $h'(e_j) = 0$ for all $j \neq 1$. Note that $\mathrm{Tr}(h') = 0$, but that $\mathrm{Tr}(h \circ h') = 1 \neq 0$, which contradicts $h \in W_0^\perp$. We must therefore have $h = 0$. □

7.2.3. Triangulable tensor products.

LEMMA 7.2.3.1. *If F is of characteristic 0, if X and X' are irreducible semi-linear F-representations of G, and if $\eta : G \to (F^G)^\times \subset F^\times$ is a linear character, then every short exact sequence $0 \to \ker\varphi \to X \otimes_F X' \xrightarrow{\varphi} F(\eta) \to 0$ is split in the category of semi-linear F-representations of G.*

PROOF. The G-equivariant F-linear map φ is non-zero, and therefore gives an isomorphism of semi-linear representations $x \mapsto (x' \mapsto \varphi(x \otimes x')) : X \xrightarrow{\varphi'} (X')^*(\eta)$. We therefore have an isomorphism $\Phi : X \otimes_F X' \simeq (X'^* \otimes_F X')(\eta) \simeq \mathrm{End}_F(X')(\eta)$ of semi-linear F-representations of G which sends $a \otimes b$ to the endomorphism $f : w \mapsto \varphi'(a \otimes w)b$. If $x \in X \otimes_F X'$, then a calculation reveals that $x \in \ker\varphi$ if and only if $\mathrm{Tr}(\Phi(x)) = 0$. The sub-F-vector space $W_0 \subset \mathrm{End}_F(X')(\eta)$ is G-stable. The map $\langle \text{-}, \text{-} \rangle : \mathrm{End}_F(X')(\eta) \times \mathrm{End}_F(X')(\eta) \to F(\eta^2)$ given by $(h, h') \mapsto \mathrm{Tr}(h \circ h')$ is a G-equivariant symmetric F-bilinear form, so that $W_0^\perp = \{h \in \mathrm{End}_F(X')(\eta) | \mathrm{Tr}(h \circ h') = 0 \text{ for all } h \in W_0\}$ is G-stable. Moreover, by lemma 7.2.2.1 we have $W_0 \cap W_0^\perp = \{0\}$, so that $\mathrm{End}_F(X')(\eta) = W_0 \oplus W_0^\perp$ and therefore $X \otimes_F X' = \ker\varphi \oplus f^{-1}(W_0^\perp) = \ker\varphi \oplus F(\eta)$ as a semi-linear F-representation of G. □

COROLLARY 7.2.3.2. *If F is of characteristic 0, if X and X' are irreducible semi-linear F-representations of G, and if $X \otimes_F X'$ is triangulable by characters $\eta_1, \ldots, \eta_d : G \to (F^G)^\times \subset F^\times$, then $X \otimes_F X' \simeq \bigoplus_{i=1}^d F(\eta_i)$.*

PROOF. Let $\{0\} = X_0 \subset X_1 \subset \ldots \subset X_d = X \otimes_F X'$ be G-stable sub-F-vector spaces such that $X_i/X_{i-1} \simeq F(\eta_i)$ for all $i \in \{1, \ldots, d\}$. For each i, let $\varphi_i : X_i \to F(\eta_i)$ denote the quotient morphism. By lemma 7.2.3.1, the exact sequence $0 \to X_{d-1} \to X \otimes_F X' \to F(\eta_d) \to 0$ is split, and therefore $X \otimes_F X' \simeq X_{d-1} \oplus F(\eta_d)$ as semi-linear F-representations of G.

Suppose that we have an isomorphism $X \otimes_F X' \simeq X_j \oplus F(\eta_{j+1}) \oplus \ldots \oplus F(\eta_d)$ of semi-linear F-representations of G, with $j \in \{1, \ldots, d-1\}$. If $p_j : X \otimes_F X' \to X_j$ is the natural G-equivariant F-linear projection and if $\phi_j = \varphi_j \circ p_j$, then lemma 7.2.3.1 implies that the exact sequence $0 \to \ker(\phi_j) \to X \otimes_F X' \xrightarrow{\phi_j} F(\eta_j) \to 0$ is split. Since $\ker(\varphi_j \circ p_j) = X_{j-1} \oplus F(\eta_{j+1}) \oplus \ldots \oplus F(\eta_d)$, we therefore see that $X \otimes_F X' \simeq X_{j-1} \oplus F(\eta_j) \oplus F(\eta_{j+1}) \oplus \ldots \oplus F(\eta_d)$. The claim therefore follows by induction. □

7.3. Triangulable tensor products

7.3.1. Triangulable tensor products. In this section, let $E/\mathbf{Q}_p$ and $K/\mathbf{Q}_p$ be finite extensions. By proposition 7.1.2.5, $\mathrm{Frac}(\mathbf{B}_\mathrm{e}) \otimes_{\mathbf{Q}_p} E$ is a field and therefore $\mathrm{Frac}(\mathbf{B}_\mathrm{e}) \otimes_{\mathbf{Q}_p}$

$E = F_E$. In particular, if E'/E is a finite extension, then $F_E \otimes_E E' = F_{E'}$ and if $K \supset E^{\mathrm{Gal}}$, then F_E is isomorphic as a G_K-ring to the compositum of $\mathrm{Frac}(\mathbf{B}_{\mathrm{e}})$ and E inside $\mathbf{B}_{\mathrm{dR}}$.

LEMMA 7.3.1.1. *Let Y, Y' be irreducible semi-linear F_E-representations of G_K. If there are linear characters $\eta_1, \ldots, \eta_n : G_K \to E^\times$ such that $Y \otimes_{F_E} Y' = \bigoplus_{i=1}^n F_E(\eta_i)$, then $\eta_1^{-1}\eta_i$ is of finite order for all $i \in \{1, \ldots, n\}$.*

PROOF. For each $i \in \{1, \ldots, n\}$, let $\phi_i : Y \otimes_{F_E} Y' \to F_E(\eta_i)$ denote the surjective G_K-equivariant projection coming from the direct sum decomposition. Since Y and Y' are irreducible, we have isomorphisms of semilinear F_E-representations $\sigma_i : Y \to (Y')^*(\eta_i)$ sending $y \mapsto (y' \mapsto \phi_i(y \otimes y'))$ and $\tau_i : Y' \to Y^*(\eta_i)$ sending $y' \mapsto (y \mapsto \phi_i(y \otimes y'))$. Therefore, for each $j \in \{1, \ldots, n\}$ we have a composite isomorphism $\tau_1^* \circ \sigma_j : Y \to Y(\mu_j)$ for each j, where $\mu_j = \eta_1^{-1}\eta_j : G_K \to E^\times$. In particular, taking determinants gives rise to an element $\lambda = \lambda_j \in F_E - \{0\}$ such that $g(\lambda) = \mu_j^r(g)\lambda$ for all $g \in G_K$, where $r = \dim_{F_E} Y$.

In particular, taking determinants gives rise to an element $\lambda = \lambda_j \in F_E - \{0\}$ such that $g(\lambda) = \mu_j^r(g)\lambda$ for all $g \in G_K$, where $r = \dim_{F_E} Y$. Since $\mathbf{B}_{\mathrm{e},E}$ is a principal ideal domain, we may write $\lambda = x/y$ with $x, y\mathbf{B}_{\mathrm{e},E}$ and $(x, y) = 1$, so that $y|g(y)$ in $\mathbf{B}_{\mathrm{e},E}$ and the relation $g(x) = \mu_j^r(g)\frac{g(y)}{y}x$ shows that x generates a G_K-stable $\mathbf{B}_{\mathrm{e},E}$-module of rank 1, so that $x \in E^\times$ by proposition 7.1.2.3. Similarly $y \in E^\times$, and therefore we have $\mu_j^r(g) = 1$ for all $g \in G_K$. $\square$

THEOREM 7.3.1.2. *If W and W' are $B_{|K}^{\otimes E}$-pairs such that the $B_{|K}^{\otimes E}$-pair $W \otimes W'$ is triangulable, then there are finite extensions E'/E and L/K such that the $B_{|L}^{\otimes E'}$-pairs $(E' \otimes_E W)|_{G_L}$ and $(E' \otimes_E W)|_{G_L}$ are triangulable.*

PROOF. The semi-linear F_E-representations $X = F_E \otimes_{\mathbf{B}_{\mathrm{e},E}} W_{\mathrm{e}}$ and $X' = F_E \otimes_{\mathbf{B}_{\mathrm{e},E}} W'_{\mathrm{e}}$ of G_K admit filtrations $\{0\} = X_0 \subset X_1 \subset \ldots X_d = X$ and $\{0\} = X'_1 \subset X'_2 \subset \ldots \subset X'_{d'} = X'$ by sub-F_E-representations of G_K such that the quotients X_i/X_{i-1} and X'_j/X'_{j-1} are irreducible for all $1 \le i \le d$ and $1 \le j \le d'$. The semi-linear F_E-representation $X \otimes_{F_E} X'$ is triangulable by 7.1.3.2, and therefore lemma 7.2.1.1 implies that $X_i/X_{i-1} \otimes_{F_E} X'_j/X'_{j-1}$ is triangulable for all $1 \le i \le d$ and $1 \le j \le d'$.

Fix $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, d'\}$. Let $Y = X_i/X_{i-1}$ and $Y' = X'_j/X'_{j-1}$, and let $r = \dim_{F_E} Y$ and $r' = \dim_{F_E} Y'$. By lemma 7.1.3.1, there are $B_{|K}^{\otimes E}$-pairs U and U' such that $Y = F_E \otimes_{\mathbf{B}_{\mathrm{e},E}} U_{\mathrm{e}}$ and $Y' = F_E \otimes_{\mathbf{B}_{\mathrm{e},E}} U'_{\mathrm{e}}$.

By corollary 7.2.3.2, $Y \otimes_{F_E} Y' = \bigoplus_{j=1}^{rr'} Y_j$, where Y_j is a 1-dimensional semi-linear F_E-representation of G_K for each j. By lemmas 7.1.3.1 and 2.5.2.1, there are characters $\eta_1, \ldots, \eta_{rr'} : G_K \to E^\times$ such that $Y_j = F_E(\eta_j)$, and lemma 7.3.1.1 implies that $\eta_1^{-1}\eta_j$ is of finite order for each j. If K'/K is a finite extension such that $\eta_1^{-1}\eta_j|_{G_{K'}} = \mathbf{1}$ for all j and $K' \supset E^{\mathrm{Gal}}$, then $Y \otimes_{F_E} Y'(\eta_1^{-1}) = \bigoplus_{j=1}^{rr'} F_E$ as semilinear F_E-representations of $G_{K'}$ and extending scalars by the $G_{K'}$-equivariant map $\mathbf{B}_{\mathrm{e},E} \to F_E \to \mathbf{B}_{\mathrm{dR},E}$ shows that $\mathbf{B}_{\mathrm{dR},E} \otimes_{\mathbf{B}_{\mathrm{e},E}} U \otimes U'(\eta_1^{-1})$ admits a $\mathbf{B}_{\mathrm{dR},E}$-basis of $G_{K'}$-invariants, and therefore the $B_{|K'}^{\otimes E}$-pair $U \otimes U'(\eta_1^{-1})$ is de Rham, and thus potentially semi-stable. Let L/K' be a finite extension such that $U \otimes U'(\eta_1^{-1})$ is a semi-stable $B_{|L}^{\otimes E}$-pair. By theorem 5.1.0.15, there is a finite extension E'/E and a character $\mu : G_L \to E'^{\times}$ such that the $B_{|L}^{\otimes E'}$-pairs $(E' \otimes U)|_{G_L}(\mu^{-1})$ and $(E' \otimes U'(\eta_1^{-1}))|_{G_L}(\mu)$ are semi-stable, and therefore triangulable by corollary 7.1.4.2. Since triangulability is insensitive to twisting by characters, we see that $F_{E'} \otimes_{F_E} Y|_{G_L}$ and $F_{E'} \otimes_{F_E} Y'|_{G_L}$ are triangulable by corollary 7.1.3.2.

Therefore, for finite extensions E'/E and L/K big enough, $(F_{E'} \otimes_{F_E} X)|_{G_L}$ and $(F_{E'} \otimes_{F_E} X')|_{G_L}$ are successive extensions of triangulable $F_{E'}$-representations of G_L, and therefore are themselves triangulable by lemma 7.2.1.1. By corollary 7.1.3.2, the $B_{|L}^{\otimes E'}$-pairs $(E' \otimes_E W)|_{G_L}$ and $(E' \otimes_E W')|_{G_L}$ are triangulable.

□

COROLLARY 7.3.1.3. *If V and V' are linear E-representations of G_K such that $V \otimes_E V'$ is trianguline, then V and V' are potentially trianguline.*

7.3.2. Quaternions. Let Q_8 denote the group of quaternions. If $p > 0$ is a prime congruent to $3 \bmod 4$, then there is an octic Galois extension $K/\mathbf{Q}_p$ such that $\mathrm{Gal}(K/\mathbf{Q}_p) = Q_8$; for the construction of such extensions, see p. 466 of [**Jen89**]. There are six non-isomorphic Galois extensions $K/\mathbf{Q}_2$ with $\mathrm{Gal}(K/\mathbf{Q}_2)$ isomorphic to Q_8, four of which are totally ramified; for the construction of such extensions, see chapter 3, section 1 of [**HSVT09**], or [**JenYui88**].

let $E/\mathbf{Q}_p$ be a finite extension containing $\mathbf{Q}_p(\sqrt{-1})$, and let $K/\mathbf{Q}_p$ be an octic Galois extension such that $\mathrm{Gal}(K/\mathbf{Q}_p)$ is isomorphic to the group Q_8. If $Q_8 = \{\pm \mathrm{Id}, \pm\mathfrak{i}, \pm\mathfrak{j}, \pm\mathfrak{k}\} \subset \mathrm{GL}_2(E)$ is the sub-group generated by the matrices $\mathfrak{i} = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $\mathfrak{j} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and $\mathfrak{k} = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, then we have an isomorphism $\mathrm{Gal}(K/\mathbf{Q}_p) \xrightarrow{\sim} Q_8$. Let (V, ρ) denote the E-vector space $V = E \cdot e_1 \oplus E \cdot e_2$ endowed the E-linear action of $G_{\mathbf{Q}_p}$ defined as follows: if $g \in G_{\mathbf{Q}_p}$ and if $\mathfrak{g}$ denotes the image of g in Q_8, then the matrix of g acting on (e_1, e_2) is $\mathfrak{g}$.

The representation (V, ρ) is potentially trivial, and thus potentially trianguline. On the other hand, the relation $\mathfrak{i}^2 = \mathfrak{j}^2 = \mathfrak{k}^2 = \mathfrak{i}\mathfrak{j}\mathfrak{k} = -\mathrm{Id}$ implies that that the semi-linear F_E-representation $X = F_E \otimes_E V$ of $G_{\mathbf{Q}_p}$ is irreducible (for any $E \supset \mathbf{Q}_p(\sqrt{-1})$), and therefore V is not trianguline by corollary 7.1.3.2.

On the other hand, $(e_1 \otimes e_1 + e_2 \otimes e_2, e_1 \otimes e_1 - e_2 \otimes e_2, e_1 \otimes e_2 + e_2 \otimes e_1, e_1 \otimes e_2 - e_2 \otimes e_1)$ is a basis of $V \otimes_E V$ consisting of simultaneous eigenvectors for the $\rho(g)$ with eigenvalues in $\{\pm 1\}$, so that $V \otimes_E V$ is a direct sum of characters of G_K, and therefore trianguline.

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