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Stochastic Calculus With Respect to Multifractional Brownian Motion and Applications to Finance

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Sous la direction de Jacques Lévy-Véhel et Marc Yor

Soutenue publiquement le devant le jury composé de

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Je dédie cette thèse à ma mère qui m'a appris à ne pas céder sur mon désir et à la mémoire de mon père qui m'a initié à l'étrange beauté des mathématiques.

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"Sweet are the uses of adversity;
Which, like the toad, ugly and venomous,
Wears yet a precious jewel in his head."

William Shakespeare, *As you like it. Act II, scene 1.*

Stochastic Calculus With Respect to Multi-fractional Brownian Motion and Applications to Finance

Résumé en français :

L'objectif de cette thèse était de construire, développer et étudier un calcul stochastique (et plus particulièrement une intégrale stochastique) par rapport au mouvement brownien multifractionnaire (dans sa version harmonisable). Le choix de la méthode ou plus exactement des outils à employer n'étant pas fixé a priori, notre choix s'est porté sur la théorie du bruit blanc, qui généralisait, dans le cas du mouvement brownien fractionnaire (fBm), le calcul stochastique au sens de Malliavin. Le premier chapitre de cette thèse introduit les différentes notions que nous utiliserons et présente les travaux qui constituent ce mémoire. Dans le deuxième chapitre de cette thèse nous donnons une construction ainsi que les principales propriétés de l'intégrale stochastique par rapport au mBm harmonisable. Y sont également établies des formules d'Itô et une formule de Tanaka pour l'intégrale stochastique par rapport à ce mBm.. Dans le troisième chapitre nous donnons une nouvelle définition, à la fois plus simple et plus générale, du mouvement brownien multifractionnaire. Nous montrons ensuite que le mBm apparaît naturellement comme limite de suite de somme de mouvement brownien fractionnaire (fBm) d'indices de Hurst différents. Nous appliquons alors cette idée pour tenter de construire une intégrale stochastique par rapport au mouvement brownien multifractionnaire à partir d'intégrales par rapport au fBm. Cela fait nous appliquons cette définition d'intégrale par rapport au mBm pour une méthode d'intégration donnée aux deux méthodes que sont le calcul de Malliavin et la théorie du bruit blanc. Dans ce dernier cas nous comparons alors l'intégrale ainsi construite à celle obtenue au chapitre 2. Le quatrième et dernier chapitre est une application du calcul stochastique développé dans les chapitres précédents. Nous y proposons un modèle à volatilité multifractionnaire où le processus de volatilité est dirigée par un mBm. L'intérêt résidant dans le fait que l'on peut ainsi prendre en compte à la fois la dépendance à long terme des accroissements de la volatilité mais aussi le fait que la trajectoire de ces accroissements varie au cours du temps. Utilisant alors la théorie de la quantification fonctionnelle pour, entre autres, approcher la solution de certaines des équations différentielles stochastiques, nous parvenons à calculer le prix d'option à départ forward et implicitons ainsi une nappe de volatilité que l'on représente graphiquement pour différentes maturités.

Mots-clefs

Mouvement Brownien multifractionnaire, calcul stochastique par rapport aux processus gaussiens, théorie du bruit blanc, S-transform, limites de processus, chaining, quantification fonctionnelle.

Stochastic Calculus With Respect to Multi-fractional Brownian Motion and Applications to Finance

Abstract

The aim of this PhD Thesis was to build and develop a stochastic calculus (in particular a stochastic integral) with respect to multifractional Brownian motion (mBm). Since the choice of the theory and the tools to use was not fixed a priori, we chose the White Noise theory which generalizes, in the case of fractional Brownian motion (fBm) , the Malliavin calculus.
The first chapter of this thesis presents several notions we will use in the sequel. In the second chapter we present a construction as well as the main properties of stochastic integral with respect to harmonizable mBm. We also give Ito formulas and a Tanaka formula with respect to this mBm.
In the third chapter we give a new definition, simplier and generalier of multifractional Brownian motion. We then show that mBm appears naturally as a limit of a sequence of fractional Brownian motions of different Hurst index. We then use this idea to build an integral with respect to mBm as a limit of sum of integrals with respect ot fBm. This being done we particularize this definition to the case of Malliavin calculus and White Noise theory. In this last case we compare the integral hence defined to the one we got in chapter 2.
The fourth and last chapter propose a multifractional stochastic volatility model where the process of volatility is driven by a mBm. The interest lies in the fact that we can hence take into account, in the same time, the long range dependence of increments of volatility process and the fact that regularity vary along the time. Using the functional quantization theory in order to, among other things, approximate the solution of stochastic differential equations, we can compute the price of forward start options and then get and plot the implied volatility nappe that we graphically represent.

Keywords

Multifractional Brownian motion, stochastic calculus with respect to multifractional Gaussian processes, White Noise theory, S-transform, limit of processes, chaining, functional quantification.

Contents

List of Figures

Chapter 1

Introduction

Contents

1 Le mouvement brownien fractionnaire (fBm)

Définition et propriétes

Le mouvement brownien fractionnaire, terme que nous abrégerons systématiquement par fBm dans la suite de ce travail, a été introduit par A.Kolmogorov en 1940, dans [50], comme moyen d'engendrer des "spirales" gaussiennes dans un espace de Hilbert. B.Mandelbrot donna le nom de coefficient de Hurst à la constante H, qui caractérise tout mouvement brownien fractionnaire, en l'honneur de l'hydrologiste britanique Harold E. Hurst qui, tentant de modéliser les crues annuelles du Nil, mit en évidence la corrélation à long terme des amplitudes de ces dernières au cours du temps. Il en déduisit alors qu'il était impossible de modéliser ce phénomène par un processus présentant des accroissements non corrélés au cours du temps, tel que le mouvement brownien. A partir de la fin des années 1960, B.Mandelbrot et J.Van Ness (*cf.*[60]) popularisèrent le fBm en tant que modèle financier et en en étudiant les propriétés. Nous considérons fixé pour tout ce chapitre un espace de probabilité, noté $(\Omega, \mathcal{F}, \mathbf{P})$.

Définition 1.1 (Mouvement brownien fractionnaire). *Soit H une constante appartenant à l'intevalle $]0;1[$ et I un intervalle de $\mathbb{R}$. Un mouvement Brownien fractionnaire sur I d'indice de Hurst H est un processus gaussien centré, noté $B^H := (B^H_t)_{t\in I}$, dont la fonction de covariance, notée Σ_H, est donnée, pour tout couple réel (t,s) de I^2, par :*

$$\Sigma_H(t,s) := \frac{\gamma_H}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),$$

où γ_H est une constante positive ; de façon équivalente : $\mathbb{E}[(B^H_t - B^H_s)^2] = \gamma_H\ |t-s|^{2H}$.

L'existence du mouvement brownien fractionnaire résulte du théorème général d'existence des processus gaussiens centrés de fonction de covariance donnée (on pourra consulter [74, 62, 49, 47] pour plus de détails). Il est clair que lorsque H vaut $1/2$ et que $\gamma_H = 1$, le fBm n'est rien d'autre que le mouvement brownien fractionnaire standard sur I. Par souci de simplicité nous poserons $I := \mathbb{R}_+$ dans la suite de cette section. Par définition, le fBm jouit des propriétés suivantes.

1. $B^H_0 = 0$ *p.s.*
2. Les accroissements du fBm sont stationnaires *i.e* $\{B^H_{t+s} - B^H_s; t \geqslant 0\} \overset{loi}{=} \{B^H_t; t \geqslant 0\}$.

3. $B^H := (B^H_t)_{t\in\mathbb{R}_+}$ est un processus H-auto-similaire, *i.e* $\{B^H_{at}; t \geqslant 0\} \overset{loi}{=} \{a^H B^H_t; t \geqslant 0\}$, $\forall a \in \mathbb{R}_+$.

Bien des résultats sont connus concernant le fBm (on pourra consulter [63, 61, 12] ainsi que les références qui y sont indiquées). On peut ainsi montrer que, presque sûrement, $B^H := (B^H_t)_{t\in\mathbb{R}_+}$ admet une rectification Hölderienne de ses trajectoires pour tout ordre strictement inférieur à H. Par ailleurs, pas plus que le mouvement brownien le fBm n'est différentiable en temps. Pour cette raison, une bonne mesure de la régularité du fBm est l'exposant de Hölder local. Défini en tout point t_0 de $\mathbb{R}_+$, l'exposant de Hölder local d'un processus $(X_t)_{t\in\mathbb{R}_+}$ est défini par :

$$\alpha_X(t_0) := \sup \left\{ \alpha : \limsup_{\rho\to 0} \sup_{(s,t)\in B(t_0,\rho)^2} \frac{|X_t - X_s|}{|t-s|^\alpha} < +\infty \right\}. \tag{1.1}$$

Il est prouvé dans [41] que pour tout H de $]0;1[$, presque sûrement, pour tout réel t_0 de $\mathbb{R}_+$, on a $\alpha_{B^H}(t_0) = H$. Les trois figures qui suivent, obtenues grâce au logiciel libre Fraclab[1], présentent des trajectoires, $t \mapsto (t, B^H_t(\omega))$, de mouvements browniens fractionnaires B^H, pour différents coefficients de Hurst.

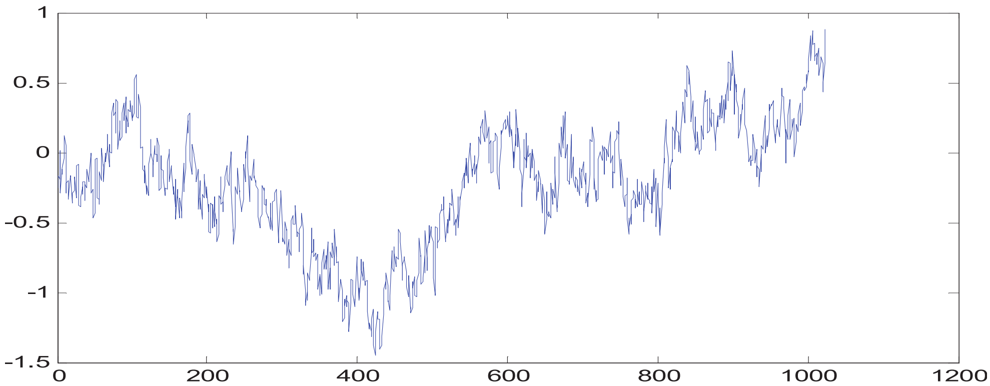

FIGURE 1.1 – Trajectoire d'un fBm de paramètre de Hurst $H = 0.3$

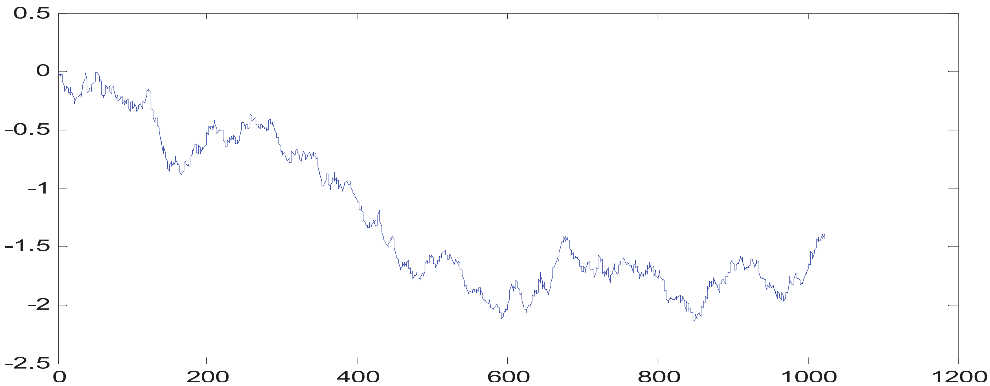

FIGURE 1.2 – Trajectoire d'un fBm de paramètre de Hurst $H = 0.5$

Comme le résultat de régularité précédent le laissait présager, on remarque que, plus la valeur de H croît, plus la régularité augmente. Inversement, plus la valeur de H décroît plus les trajectoires sont irrégulières.

1. téléchargement disponible depuis l'adresse : http ://fraclab.saclay.inria.fr/

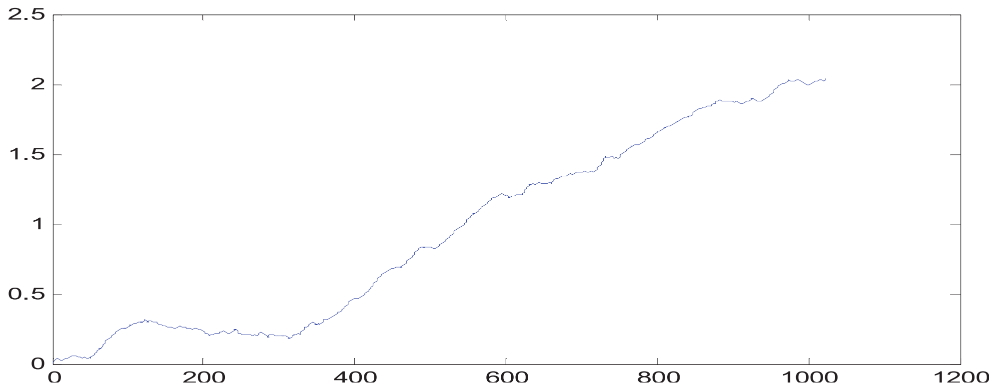

FIGURE 1.3 – Trajectoire d'un fBm de paramètre de Hurst $H = 0.8$

Corrélation des accroissements du fBm et dépendance de long terme

Le fBm ayant été, en partie, popularisé parce qu'il permettait de rendre compte de phénomènes présentant des corrélations à long terme au cours du temps il nous semble important de s'attarder un instant sur cette notion. Nous excluons d'emblée le cas $H = 1/2$ qui correspond au cas du mouvement Brownien dont les accroissements sont indépendants. Dès que H est différent de $1/2$, les accroissements du fBm sont correlés. Plus précisément, on a pour tout couple (s,t) de $\mathbb{R}_+^2$ et tout réel positif r tels que $s + r \leqslant t$ et $t - s = nr$, où n est un entier positif,

$$\rho_H(n) := Cov\left(B_{t+r}^H - B_t^H, B_{s+r}^H - B_s^H\right) = \frac{1}{2}r^{2H}\ [(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}]. \tag{1.2}$$

De la précédente égalité nous déduisons que les accroissements $B_{t+r}^H - B_t^H$ et $B_{t+2r}^H - B_{t+r}^H$ sont corrélés positivement lorsque $H > 1/2$ et négativement lorsque $H < 1/2$. Il semble donc pertinent d'utiliser un mouvement brownien fractionnaire d'indice de Hurst H strictement supérieur à $1/2$ pour modéliser les phénomènes à mémoire présentant des accroissements positivement corrélés.

Dépendance de long terme

Définition 1.2. *On dit qu'une suite de variables aléatoires stationnaires* $(Y_n)_{n\in\mathbb{N}}$ *présente une dépendance de long terme si la fonction d'autocovariance* $\rho : \mathbb{N} \to \mathbb{R}$*, définie par* $\rho(n) := Cov(Y_k, Y_{k+n})$ *satisfait la condition*

$$\lim_{n\to+\infty} \frac{\rho(n)}{cn^{-\alpha}} = 1, \tag{1.3}$$

pour une certaine constante c *et un certain réel* α *de* $]0;1[$.

Dans ce cas la dépendance entre Y_k et Y_{k+n} décroît lentement lorsque n tend vers l'infini. On a de plus

$$\sum_{n=1}^{+\infty} \rho(n) = +\infty. \tag{1.4}$$

On peut aussi prendre l'égalité (1.4) comme définition de la dépendance à long terme. Cette seconde définition de la dépendance à long terme d'une suite de variables aléatoires stationnaires est alors évidemment plus faible que la définition 1.2.
Quelle que soit la définition choisie pour la dépendance à long terme, un calcul direct montre que les accroissements $X_k := B_k^H - B_{k-1}^H$ et $X_{k+n} := B_{k+n}^H - B_{k+n-1}^H$ du fBm B^H présentent telle dépendance lorsque H est strictement supérieur à $1/2$ puisque l'on a alors :

$$\rho_H(n) := Cov\left(X_{k+n}, X_k\right) = \frac{1}{2}\left[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right] \underset{n\to+\infty}{\sim} H\,(2H-1)\; n^{2H-2}. \quad (1.5)$$

En revanche nous avons $\sum_{n=1}^{+\infty} |\rho(n)| < +\infty$ lorsque $H < 1/2$.

Les différentes représentations

Dans la suite de ce mémoire, nous utiliserons abondamment les notions de mesures gaussiennes réelles et complexes. Nous avons donc ajouté, dans l'annexe A, un court résumé de cette notion qui couvre tous les cas d'utilisation que nous en ferons dans les chapitres suivants. Nous encourageons le lecteur qui ne serait pas familier avec cette notion à s'y reporter avant de commencer la lecture de cette section.
Il existe de nombreuses représentations d'un mouvement brownien fractionnaire. Plus ou moins compliquées selon que l'on souhaite obtenir une représentation sur un compact de $\mathbb{R}$ ou sur $\mathbb{R}$ tout entier.

Représentations du fBm sur $\mathbb{R}$

Sur $\mathbb{R}$ nous avons la famille de représentations du fBm donnée, pour tout H de $(0,1)$ et (a^+, a^-) de $\mathbb{R}^2\backslash\{(0,0)\}$, par

$$Y_{(a^+,a^-)}(t) := \delta(H) \int_{\mathbb{R}} \left(a^+ f_+(t,H,u) + a^- f_-(t,H,u)\right) W(du) \quad (1.6)$$

où $\delta(H)$ est un réel non nul, W désigne une mesure gaussienne réelle et où l'on a défini :

$$f_\pm(t,H,u) := (t-u)_\pm^{H-1/2} - (-u)_\pm^{H-1/2}, \quad (1.7)$$

avec la convention $x_-^\gamma := (-x)_+^\gamma$, x, γ dans $\mathbb{R}$ et $x_+^\gamma := (-x)^\gamma$, $x > 0$, $x_+^\gamma = 0$ si $x \leqslant 0$.
On notera que si $\widetilde{W}$ est la mesure gaussienne complexe, construite à partir de W (pour plus de détails sur les mesures gaussiennes voir l'annexe A située à la fin de ce mémoire) alors on a, pour tout H de $(0,1)$ et (a^+, a^-) de $\mathbb{R}^2\backslash\{(0,0)\}$, l'égalité suivante :

$$\forall t \in \mathbb{R},\ \widetilde{Y}_{(a^+,a^-)}(t) \overset{p.s.}{=} Y_{(a^+,a^-)}(t), \quad (1.8)$$

où l'on a défini $\widetilde{Y}_{(a^+,a^-)}(t) := \frac{\Gamma(H+1/2)}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{(e^{it\xi}-1)}{|\xi|^{H+1/2}}\, U_{(a^+,a^-)}(\xi;H)\, \widetilde{W}(d\xi)$, Γ désignant la fonction Gamma d'Euler et où, pour tout réel ξ, on a posé,

$$U_{(a^+,a^-)}(\xi;H) := a^+ e^{-i\ \mathrm{sign}(\xi)(H+1/2)\pi/2} + a^- e^{i\ \mathrm{sign}(\xi)(H+1/2)\pi/2}.$$

On appelle représentation harmonisable du fBm d'indice de Hurst H, le processus défini, à une constante dépendant de H près, par

$$\widetilde{Y}(t) := \int_{\mathbb{R}} \frac{(e^{it\xi}-1)}{|\xi|^{H+1/2}}\, \widetilde{W}(d\xi),$$

qui n'est autre que $\frac{\sqrt{2\pi}}{\Gamma(H+1/2)}\, \widetilde{Y}_{(1,1)}(t)$ pour $\delta(H) := 1$. Enfin, on appelle représentation à moyenne mobile du fBm le processus $Y_{(1,0)}$ défini par (1.6) pour $(a^+, a^-) = (1,0)$ et $\delta(H) := 1$. On a donc,

$$Y_{(1,0)}(t) = \int_{\mathbb{R}} \left((t-u)_+^{H-1/2} - (-u)_+^{H-1/2}\right) W(du).$$

Le lecteur souhaitant trouver davantage de réprésentations de fBm pourra se référer par exemple à [73, 75].

Représentations du fBm sur un compact

Nous nous contentons ici de donner la représentation suivante, que l'on pourra trouver dans [63]

$$B_t^H := \int_0^T \mathbb{1}_{\{0 \leqslant u < t \leqslant T\}}(t,u)\; K_H(t,u)\; W(du), \tag{1.9}$$

$$K_H(t,u) := \begin{cases} \alpha_H \left[\left(\frac{t}{u}\right)^{H-1/2} (t-u)^{H-1/2} - (H-1/2)\; u^{1/2-H} \int_u^t (v-u)^{H-1/2} v^{H-1/2} dv \right] & \text{si } 0 < H < 1/2, \\ \left(\frac{H(2H-1)}{\beta(2-2H, H-1/2)}\right)^{1/2} u^{1/2-H} \int_u^t (v-u)^{H-3/2} v^{H-1/2}\; dv & \text{si } 1/2 < H < 1, \end{cases}$$

et où, pour tout H de $(0,1)$, $\alpha_H := \left(\frac{2H}{(1-2H)\beta(1-2H, H+1/2)}\right)^{1/2}$.

Calcul stochastique par rapport au fBm

L'indice de variation du fBm d'indice de Hurst H est défini par,

$$I(B^H, [0,T]) := \inf\{p > 0,\; \vartheta_p(B^H, [0,T]) < +\infty\},$$

où l'on a défini $\vartheta_p(B^H, [0,T]) := \sup_{\pi} \sum_{i=1}^n |B_{t_k}^H - B_{t_{k-1}}^H|^p$ et où $\pi := \{0 = t_0 < t_1 < t_2 < \cdots < t_n = T\}$ désigne une partition finie de $[0,T]$. Dans [72], il est montré que $I(B^H, [0,T]) = \frac{1}{H}$ pour tout H de $]0;1[$. Il s'ensuit que, pour tout réel positif p, la suite $(V_{n,p})_{n \in \mathbb{N}}$ définie par,

$$V_{n,p} := \sum_{i=1}^n \left| B_{\frac{i}{n}}^H - B_{(\frac{i-1}{n})}^H \right|^p$$

converge en probabilité vers 0 si $pH > 1$ et tend vers $+\infty$ si $pH < 1$, lorsque n tend vers $+\infty$. On en conclut donc que $I(B^H, [0,T]) = \frac{1}{H}$, pour tout H. Puisque l'indice de variation d'une semimartingale appartient à l'ensemble $[0;1] \cup \{2\}$, on en déduit que B^H est une semi-martingale si et seulement si $H = 1/2$.
Lorsque $H \neq 1/2$, qui est le cas qui va nous intéresser dans la suite, le calcul stochastique d'Itô développé pour les semi-martingales ne peut donc pas s'appliquer. Aussi, depuis une quinzaine d'années de nombreuses voies visant à établir un calcul stochastique relatif au fBm ont été explorées. Parmi celles-ci on peut citer

Approches probabilistes	Approches déterministes
Calcul de Malliavin L.Decreusefond, S.Ustünel, Alos, Mazet, Nualart...	Théorie des chemins rugueux Lyons, Coutin, Nourdin, Gubinelli
Théorie du bruit blanc Elliott, Van Der Hoek, Bender, Oksendal,...	Integrale de Stratonovich étendue Russo & Vallois
	Autres approches déterministes Zähle, Feyel, de la Pradelle

Nous renvoyons à [29] et aux références qui y sont données pour une présentation détaillée de ces différentes méthodes. Nous utilisons dans le présent mémoire essentiellement la théorie du bruit blanc.
Les approches déterministes sont ainsi nommées car elles correspondent à l'utilisation d'intégrales déterministes, définies ω par ω et construites en dehors de tout cadre probabiliste, pour des processus stochastiques de régularités diverses. Si l'on considère le cas de l'intégrale stochastique par rapport au fBm, cette répartition entre approches déterministes et probabilistes, pour classique qu'elle soit, semble cependant quelque peu arbitraire tant les outils de l'analyse réelle et fonctionnelle sont omniprésents dans ces différentes constructions. Nous reviendrons, au cours des prochains chapitres sur les raisons pour lesquelles nous avons choisi

d'utiliser, de préférence à toute autre, la théorie du bruit blanc. Pour motiver, *a priori*, ce choix précisons qu'il nous fallait une méthode d'intégration par rapport au mBm qui nous permette à la fois de résoudre des équations différentielles, sans conditions préalables sur les valeurs que peut prendre la fonction h dans $]0;1[$ ou sur la croissance des coefficients de l'E.D.S (ainsi nous ne voulions pas devoir exiger une croissance polynomiale,$\cdots$) mais aussi de définir une intégrale de Wiener de toute fonction ayant une régularité raisonnable (*i.e* pas nécessairement de classe C^1). La théorie des chemins rugueux satisfait la première condition mais pas la seconde alors que le calcul de Malliavin satisfait la seconde mais pas la première.

Les limites de validité des applications du mouvement Brownien fractionnaire

Très populaire et utilisé dans de nombreuses applications (financières notamment), le mouvement brownien fractionnaire présente cependant quelques inconvénients qui peuvent s'avérer gênants lorsque l'on souhaite modéliser certains phénomènes. En effet la régularité des trajectoires du fBm reste la même au cours du temps. De plus il est à accroissements stationnaires. Ceci empêche donc de modéliser, à l'aide d'un fBm, les phénomènes dont la régularité évolue au cours du temps ou dont les accroissements ne sont pas stationnaires comme par exemple le traffic internet, le traitement d'images, le relief des montagnes, la volatilité des cours de bourse $\cdots$
Ceci conduit donc naturellement à la question suivante :
Peut on généraliser le mouvement Brownien fractionnaire à un processus gaussien dont la régularité des trajectoires varie au cours du temps ?

2 Le mouvement brownien multifractionnaire (mBm)

La réponse, affirmative, à la question ci-dessus a été apportée par Jacques Lévy-Véhel et Romain Peltier dans [68].

Définition et propriétés

Dans ce premier article consacré au mouvement Brownien multifractionnaire (noté mBm dans la suite), ils ont défini le mBm à partir du fBm en remplaçant "simplement" le réel H par $h(t)$, où $h : \mathbb{R} \to (0,1)$ désigne dans toute cette section une fonction continue. Le mBm originel est donc défini par,

$$X_t^h := \frac{1}{\Gamma(h(t)+1/2)} \int_{\mathbb{R}} \left((t-s)_+^{h(t)-1/2} - (-s)_+^{h(t)-1/2} \right) W(ds). \tag{2.1}$$

Entre autres choses, il est établi dans [68], que pour tout intervalle I, le processus X^h n'est pas stationnaire et admet une rectification hölderienne de ses trajectoires de tout ordre γ strictement inférieur à $\inf_{u\in I} h(u)$. Le processus X^h n'étant pas stationnaire, il est plus intéressant de chercher des propriétés de régularité locale plutôt que globales le concernant.
L'idée qui a présidé à la définition du mBm bien équilibré est la même que celle qui a permis de définir X^h. Plus précisément, la version bien équilibrée du mBm a été définie en 1997 dans [7], en partant de la représentation harmonisable du fBm et en posant,

$$\widetilde{X}_t^h := \int_{\mathbb{R}} \frac{e^{it\xi}-1}{|\xi|^{h(t)+1/2}} \, \widetilde{W}(d\xi). \tag{2.2}$$

Même si les processus X^h et $\widetilde{X}^h$ ne sont pas équivalents (i.e n'ont pas la même loi), nous verrons à la section suivante qu'ils font partie d'un même ensemble de processus. Le fait que l'on ait cru pendant longtemps que X^h et $\widetilde{X}^h$ désignent le même processus (en loi) d'une part, le fait que $\widetilde{X}^h$ est plus facile à manipuler pour effectuer des calculs d'autre part explique que les auteurs aient préféré donner les propriétés de régularité et de dépendance à long terme pour le processus $\widetilde{X}^h$. Nous restituons ici une partie de leurs résultats.
Bien que n'étant plus auto-similaire et bien que n'étant pas stationnaire, le mBm possède une propriété plus faible appelée localisabilité. Si la fonction h est β-hölderienne, alors pour tout t tel que $h(t) < \beta$, on a $\left\{ \frac{\widetilde{X}_{t+\rho u}^h - \widetilde{X}_t^h}{\rho^{h(t)}}; u \in \mathbb{R}_+ \right\}$ converge en loi, lorsque ρ tend vers 0, vers un fBm d'indice de Hurst $h(t)$.

Régularité hölderiènne de $\widetilde{X}^h$

Il a été établi dans [38] que, pour tout réel positif t,

$$\alpha_{\widetilde{X}^h}(t) = \widetilde{\beta}(t) \wedge h(t) \quad \text{p.s}$$

où $\widetilde{\beta}(t)$ désigne l'exposant de Hölder local de la fonction h en t.
Les trois figures qui suivent, toujours obtenues grâce au logiciel Fraclab, représentent des trajectoires, $t \mapsto (t, X_t^h(\omega))$, de mouvements browniens multifractionnaires X^h ayant différents paramètres fonctionnels.

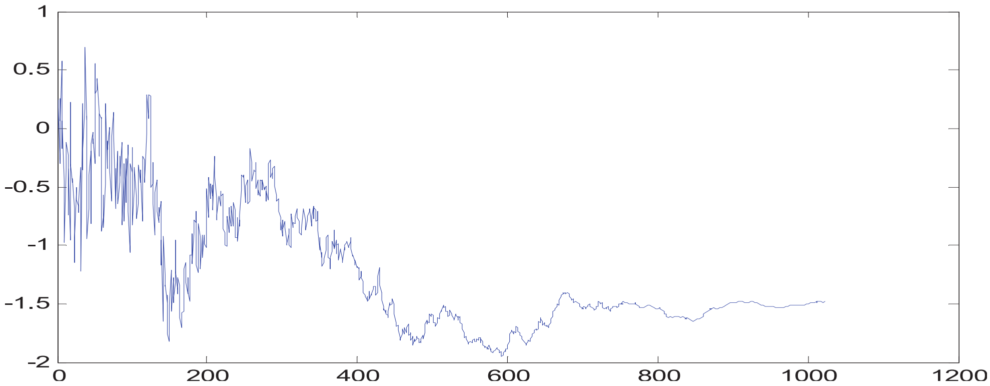

FIGURE 1.4 – Trajectoire d'un mBm de paramètre fonctionnel $h_1 : t \mapsto 0,1 + 0,8t$

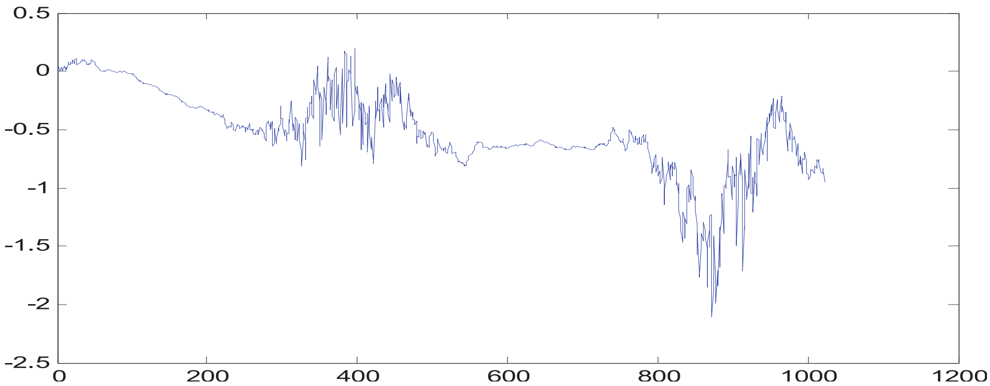

FIGURE 1.5 – Trajectoire d'un mBm de paramètre fonctionnel $h_2 : t \mapsto 0,5 + 0,3\sin(4\pi t)$

A la vue de ces trois figures, on constate là encore que plus la valeur $h(t)$ est grande plus les trajectoires sont régulières. Réciproquement, plus la valeur $h(t)$ est petite plus les trajectoires sont irrégulières.

Corrélation des accroissements du mBm et dépendance à long terme

Puisque les accroissements du mBm ne sont pas stationnaires il convient de redéfinir la notion de dépendance à long terme. Dans [5], il est proposé une nouvelle définition de dépendance à long terme que nous reprenons ici. Précisons d'abord que nous appelons processus du second ordre tout processus $X := (X_t)_{t\in\mathbb{R}}$ tel que X_t appartient à $L^2(\Omega)$ pour tout réel t.

Définition 2.1. *[5, def.6] Un processus $Y := (Y_t)_{t\in\mathbb{R}_+}$ du second ordre (stationnaire ou non) présente une dépendance à long terme s'il existe une fonction $\alpha : \mathbb{R}_+ \to]-1;0[$ telle que pour tout réel positif s,*

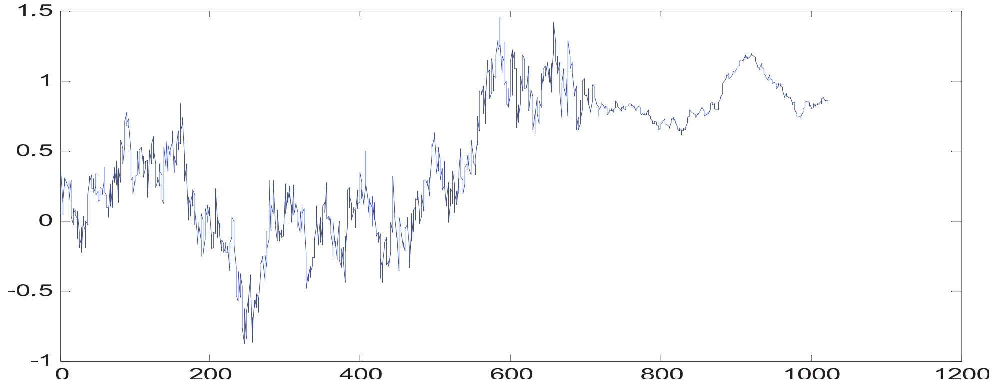

FIGURE 1.6 – Trajectoire d'un mBm de paramètre fonctionnel $h_3 : t \mapsto 0,3 + 0,3(1 + e^{-100(t-0,7)})^{-1}$

$cor_Y(s, s+r) \underset{r\to+\infty}{\approx} r^{\alpha(s)}$*, où* cor_Y *désigne la fonction de corrélation de* Y *et où* $f(r) \underset{r\to+\infty}{\approx} g(r)$ *signifie qu'il existe deux réels* c *et* d *tels que pour* r *suffisamment grand, on a* $c \leqslant \frac{f(r)}{g(r)} \leqslant d$.

Nous présentons également une seconde définition de la dépendance à long terme, plus faible que la définition 2.1.

Définition 2.2. *[5, def.7] Un processus* $Y := (Y_t)_{t\in\mathbb{R}_+}$ *du second ordre présente une dépendance à long terme si*

$$\forall\delta > 0, \forall s \geqslant 0, \sum_{k=0}^{+\infty} |cor_Y(s, s+k\delta)| = +\infty.$$

Les deux résultats principaux, portant sur $\widetilde{X}^h$ sont les suivants :

Proposition 2.1. *[5, corollaire 1] Pour toute fonction* h *non constante, le mBm* $\widetilde{X}^h$*, défini en* (2.2) *présente une dépendance à long terme au sens de la définition 2.2. De plus, si pour tout* s, $h(s) + h(t) > 1$, *pour tout* t *suffisamment grand, alors le mBm* $\widetilde{X}^h$ *présente également une dépendance à long terme au sens de la définition 2.1, le paramètre fonctionnel* α *vérifiant* $\alpha(s) = h(s) - 1$ *pour tout* s *in* $\mathbb{R}_+$.

Proposition 2.2. *[5, Proposition 8](Comportement asymptotique des accroissements du mBm) Soit* $\widetilde{X}^h$ *le mBm défini en* (2.2) *et* $Z^h := (Z^h_t)_{t\in\mathbb{R}_+}$ *le processus défini par* $Z^h_t := \widetilde{X}^h_{t+1} - \widetilde{X}^h_t$. *Lorsque* t *tend vers* $+\infty$, *et pour tout* $s \geqslant 0$ *tel que les quatre quantités* $a_{0,0}(t,s) := h(t) + h(s)$, $a_{1,0}(t,s) := h(t+1) + h(s)$, $a_{0,1}(t,s) := h(t) + h(s+1)$ *et* $a_{1,1}(t,s) := h(t+1) + h(s+1)$ *sont toutes différentes, alors :*

$$\max_{(i,j)\in\{1;2\}^2} a_{i,j}(t,s) < 1 \Rightarrow cov_{Z^h}(t,s) \underset{t\to+\infty}{\approx} 1,$$

$$\max_{(i,j)\in\{1;2\}^2} a_{i,j}(t,s) > 1 \Rightarrow cov_{Z^h}(t,s) \underset{t\to+\infty}{\approx} t^{\max_{(i,j)\in\{1;2\}^2} a_{i,j}(t,s)-1}.$$

Les différentes représentations du mBm

• **Sur** $\mathbb{R}$
Comme nous l'avons indiqué précédemment, les processus gaussiens X^h et $\widetilde{X}^h$ ne sont pas égaux en loi. Pour autant, ils ont comme point commun d'être des fBm "sur lequels on fait courir une fonction h". En 2006 S.Stoev et M.Taqqu proposèrent, dans [73], une nouvelle définition de mBm, plus générale, et qui englobait les processus originels X^h et $\widetilde{X}^h$. Une fonction β-Hölderienne h étant fixée, ils proposèrent, et partant de (1.6), de définir un mBm, pour tout (a^+, a^-) de $\mathbb{R}^2\backslash\{(0,0)\}$, par

$$Y_{(a^+,a^-)}(t) := \delta(h(t)) \int_{\mathbb{R}} \Big(a^+ f_+(t,h(t),u) + a^- f_-(t,h(t),u)\Big) \, W(du), \tag{2.3}$$

où $\delta : \mathbb{R} \to \mathbb{R}$ est une fonction de classe C^1. Nous nous permettons d'employer ici les mêmes notations qu'en (1.6) puisque, lorsque la fonction h est constante, égale à H, les deux définitions coincident. De plus, si l'on note $\widetilde{Y}_{(a^+,a^-)}$ le processus défini, pour tout réel t, par

$$\widetilde{Y}_{(a^+,a^-)}(t) := \delta(h(t)) \, \frac{\Gamma(H+1/2)}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{(e^{it\xi}-1)}{|\xi|^{h(t)+1/2}} \, U_{(a^+,a^-)}(\xi; h(t)) \, \widetilde{W}(d\xi), \tag{2.4}$$

où $\widetilde{W}$ est construite à partir de W, comme indiqué dans la section 2 de l'annexe A de ce mémoire, alors on a

$$\widetilde{Y}_{(a^+,a^-)}(t) \stackrel{p.s.}{=} Y_{(a^+,a^-)}(t).$$

Cette nouvelle famille de mBm étant définie, les considérations que nous avons développées au paragraphe précédent demeurent valables. Aussi, et plutôt que de donner la fonction de covariance des processus $\widetilde{Y}_{(a^+,a^-)}$, dont une forme intégrale se déduit immédiatement de (2.4), nous allons donner et étudier la fonction de corrélation des mBm définis par (2.4) ou, ce qui revient au même, (2.3).
Par ailleurs, il convient de se demander dans quelle mesure les éléments de cette famille de mBm sont différents. Cette question délicate a été traitée, dans le cadre de cette nouvelle définition, dans [75]. Nous rappelons ici les principaux résultats relatifs à la structure de covariance des mBm $Y_{(a^+,a^-)}$ qui y sont établis puisqu'ils permettent de mieux saisir la nature intrinsèque de ces mBm. Commençons par la fonction de covariance.

Proposition 2.3 ([75])**.** *Pour tout couple (a^+, a^-) dans $\mathbb{R}^2 \backslash \{(0,0)\}$ et tout réel t, nous avons*

$$Var(Y_{(a^+,a^-)}(t)) = C_h^2(t) \; |U_{(a^+,a^-)}(\xi; h(t))|^2 \quad \text{pour tout } \xi \neq 0, \tag{2.5}$$

où

$$C_h^2(t) := |t|^{2h(t)} \begin{cases} \frac{\cos(\pi h(t))\Gamma^2(1/2+h(t))\Gamma(2-2h(t))}{\pi h(t)(1-2h(t))} & \text{si } h(t) \neq 1/2 \\ 1 & \text{si } h(t) = 1/2 \end{cases} \tag{2.6}$$

Nous ne nous intéressons plus, à partir de maintenant, qu'aux processus $(Y_{(a^+,a^-)}(t))_{t\in\mathbb{R}}$, tels que l'on ait $a^+ \neq a^-$ ou bien $h(t) \neq 1/2$.
Définissons en outre la fonction $\alpha : \mathbb{R} \to [0, 2\pi[$, par

$$\alpha(t) := \mathrm{Arg}(a^+ e^{-i(h(t)+1/2)\pi/2} + a^- e^{i(h(t)+1/2)\pi/2}), \tag{2.7}$$

où Arg désigne la détermination principale de l'argument sur $[0; 2\pi[$. La fonction de corrélation de ce processus, notée $\rho_{(a^+,a^-)}$, est définie, pour tout couple (a^+, a^-) de $\mathbb{R}^2 \backslash \{(0,0)\}$ par

$$\rho_{(a^+,a^-)}(t,s) := \frac{\mathbf{E}[Y_{(a^+,a^-)}(t) Y_{(a^+,a^-)}(s)]}{\sqrt{Var(Y_{(a^+,a^-)}(t)) \; Var(Y_{(a^+,a^-)}(t))}}. \tag{2.8}$$

Nous avons alors le résultat suivant :

Théorème 2.1. *Les réels t et s non nuls étant fixés,*
– *si $h_{t,s} := \frac{h(t)+h(s)}{2} \neq 1/2$, alors*

$$\begin{aligned} \rho_{(a^+,a^-)}(t,s) = D_h(t,s) \Big(& \cos(\Delta\alpha_{t,s}) - \pi h_{t,s} sign(t) |t|^{2h_{t,s}} \\ & + \cos(\Delta\alpha_{t,s}) + \pi h_{t,s} sign(s) |s|^{2h_{t,s}} \\ & - \cos(\Delta\alpha_{t,s}) - \pi h_{t,s} sign(t-s) |t-s|^{2h_{t,s}} \Big), \end{aligned} \tag{2.9}$$

où $\Delta\alpha_{t,s} := \alpha(t) - \alpha(s)$, α ayant été définie en (2.7) $D_h(t,s) := \frac{\Gamma(h(t)+1/2) \; \Gamma(h(s)+1/2)}{\pi \; C_h(t) \; C_h(s)} \; \frac{\Gamma(2-2h_{t,s})}{2h_{t,s}(1-2h_{t,s})}$.

– *si* $h_{t,s} = 1/2$, *alors*

$$\rho_{(a^+,a^-)}(t,s) = \frac{\Gamma(h(t)+1/2)\ \Gamma(h(s)+1/2)}{\pi\ C_h(t)\ C_h(s)}\ \ (\pi/2\cos(\Delta\alpha_{t,s})(|t|+|s|-|t-s|))$$
$$-\sin(\Delta\alpha_{t,s})(t\ln|t| - s\ln|s| - (t-s)\ln(|t-s|)) \tag{2.10}$$

Comparaison entre mBm de paramètres (a^+, a^-) différents

Nous allons voir ici que la famille de mBm définie par (2.3) est en fait suffisamment riche pour contenir des processus de fonctions de corrélation différentes.

Proposition 2.4. *([75, corollaire 6.1]) Soient (a^+, a^-) et (b^+, b^-) deux couples de réels. On suppose que $|b^+| \neq |b^-|$. Considérons $Y_{(a^+,a^-)}$ et $Y_{(b^+,b^-)}$ deux mBm de même paramètre fonctionnel $h : \mathbb{R} \to (0,1)$. On a les résultats suivants*
(a) Si $a^+b^- = a^-b^+$, alors pour tout réel t et s dans $\mathbb{R}^*$, on a

$$\rho_{(a^+,a^-)}(t,s) = \rho_{(b^+,b^-)}(t,s)$$

(b) Si $a^+b^- \neq a^-b^+$ et si h est continue alors, pour tout $s \neq 0$ fixé, il existe, pour tout $\chi > 0$, une fonction $h_\chi : \mathbb{R} \to (0,1)$, vérifiant,

$$\sup_{t\in\mathbb{R}} |h(t) - h_\chi(t)| < \chi,$$

telle que

$$\rho_{(a^+,a^-)}(t,s;h_\chi) \neq \rho_{(a^+,a^-)}(t,s;h_\chi),\ \text{pour tout}$$

$$\rho_{(a^+,a^-)}(t,s;h_\chi) \neq \rho_{(b^+,b^-)}(t,s;h_\chi),\ \text{pour tout } t \in \begin{cases}(s, s+\varepsilon) & \text{si } s>0\\ (s-\varepsilon, s) & \text{si } s<0,\end{cases}$$

pour un certain ε strictement positif.
En particulier, les mBm $Y^{h_\chi}_{(a^+,a^-)}$ et $Y^{h_\chi}_{(b^+,b^-)}$ de paramètre fonctionnel h_χ ne peuvent pas se déduire l'un de l'autre par simple multiplication d'une fonction déterministe de t.

• Les différentes représentations du mBm sur un intervalle

Pour obtenir un mBm sur un compact, par exemple $[0,T]$, il suffit de considérer $(Y_{(a^+,a^-)}(t))_{t\in[0;T]}$. Cependant, dans [3], il est indiqué une méthode d'intégration stochastique, utilisant le calcul de Malliavin, par rapport à tous les processus gaussiens pouvant s'écrire comme intégrale sur $[0,T]$ d'un noyau $K(t,.)$ ayant certaines propriétés. Dans l'article [18] une intégrale stochastique par rapport au processus gaussien $Y^h := (Y^h_t)_{t\in[0,T]}$, défini pour tout t par $Y^h_t := \int_0^t K_{h(t)}(t,u)\ W(du)$ où $K_H(.,.)$ a été définie juste après (1.9), est étudiée, à partir des propriétés de l'intégrale stochastique établies dans [3]. Le processus Y^H (correspondant au cas où h est constante et égale à H) étant un fBm, il est naturel de se demander si le processus Y^h est un mBm au sens de la définition 2.3.
Si tel était le cas, il existerait un couple (a^+, a^-) de $\mathbb{R}^2$ et une fonction δ tels que l'on ait l'égalité

$$\int_0^T K_{h(t)}(t,u)\ K_{h(s)}(s,u)\ du =$$
$$\delta(h(t))\ \delta(h(s)) \int_{\mathbb{R}} \Big(a^+f_+(t,h(t),u) - a^-f_-(t,h(t),u)\Big)\ \Big(a^+f_+(s,h(s),u) - a^-f_-(s,h(s),u)\Big)\ du. \tag{2.11}$$

Nous donnons dans le deuxième chapitre du présent mémoire des raisons qui nous font croire qu'une telle égalité a peu de chances d'être vérifiée. De plus, que la dernière égalité soit vérifiée ou non, accepter la définition 2.3 comme définition des mBm revient à considérer qu'un mBm sur un intervalle I quelconque de $\mathbb{R}$ est un processus gaussien dont la fonction de covariance doit nécessairement appartenir à l'ensemble de fonctions

$$\{(t,s) \mapsto \mathbf{E}[Y_{(a^+,a^-)}(t)Y_{(a^+,a^-)}(s)]; (a^+,a^-) \in \mathbb{R}^2\}.$$

Ainsi, tout processus gaussien $(X(t,h(t)))_{t\in I}$ tel que $(X(t,H))_{t\in I}$ est un fBm de coefficient de Hurst H pour tout H de $]0;1[$, mais tel que $(X(t,H))_{t\in I}$ ne puisse pas s'écrire sous la forme (1.6), ne pourrait pas être considéré comme un mBm. Outre l'arbitraire de la définition 2.3, son principal inconvénient est de ne pas tenir compte du fait qu'un mBm "devrait être un fBm dans lequel on a changé H en $h(t)$", car c'est bien là l'idée originelle et originale de [68].
Ceci nous amène à penser que la définition 2.3, efficace en ce qu'elle permettait de généraliser la notion de mBm, est à la fois trop arbitraire et pas assez générale. Nous proposons au chapitre 2 du présent mémoire, qui est extrait de [40], une nouvelle définition de mBm qui pallie ces différents inconvénients.

Calcul stochastique par rapport au mBm

Le calcul stochastique par rapport au mBm faisant l'objet de cette thèse, nous renvoyons à la description des travaux du présent manuscrit, située au prochain paragraphe, pour un plan détaillé.

3 Présentation des travaux de thèse

L'objectif de cette thèse était de construire, développer et étudier un calcul stochastique (et plus particulièrement une intégrale stochastique) par rapport au mouvement brownien multifractionnaire (dans sa version harmonisable). Le choix de la méthode ou plus exactement des outils à employer n'étant pas fixé a priori, notre choix s'est porté sur la théorie du bruit blanc, qui généralisait, dans le cas du fBm, le calcul stochastique au sens de Malliavin.

Chapitre 2

Dans le deuxième chapitre de cette thèse nous donnons une construction ainsi que les principales propriétés de l'intégrale stochastique par rapport au mBm harmonisable. Y sont également établies des formules d'Itô et une formule de Tanaka pour l'intégrale stochastique par rapport à ce mBm. Plus précisément, à partir des travaux de [34], [8] et [9] nous montrons explicitement que l'ensemble des éléments déterministes qui admettent une intégrale de Wiener par rapport au fBm d'indice de Hurst H est exactement l'espace de Sobolev d'ordre $1/2 - H$, noté $L^2_H(\mathbb{R})$, et défini par

$$L^2_H(\mathbb{R}) := \{u \in \mathscr{S}'(\mathbb{R}) \ : \ \widehat{u} = T_f;\ f \in L^1_{loc}(\mathbb{R}) \text{ and } ||u||_H < +\infty\}, \tag{3.1}$$

où T_f désigne la transformée de Fourier au sens des distributions, $||u||^2{}_H := \frac{1}{c_H^2}\int_{\mathbb{R}} |\xi|^{1-2H} |\widehat{u}(\xi)|^2 \, d\xi$ est la norme issue du produit scalaire défini, sur $L^2_H(\mathbb{R}) \times L^2_H(\mathbb{R})$, par :

$$< u, v >_H := \frac{1}{c_H^2} \int_{\mathbb{R}} |\xi|^{1-2H} \ \widehat{u}(\xi) \ \overline{\widehat{v}(\xi)} \ d\xi, \tag{3.2}$$

où $c_x := \left(\frac{2\cos(\pi x)\Gamma(2-2x)}{x(1-2x)}\right)^{\frac{1}{2}}$ pour tout x de $(0,1)$. Notons que $L^2_H(\mathbb{R})$ est aussi l'image par un opérateur bijectif M_H de l'espace $L^2(\mathbb{R})$ (*cf.* théorème 3.7 du chapitre 2 pour plus de précisions).
Après avoir introduit, dans la partie 3, et étudié les principales propriétés de l'opérateur $\frac{\partial M_H}{\partial H}$, défini comme étant la dérivée, au sens de $L^2(\mathbb{R})$, de l'opérateur M_H, nous construisons par une première méthode l'intégrale de Wiener par rapport au mBm. Pour ce faire, nous établissons (*cf.* proposition 4.1) que la forme bilinéaire, notée $<,>_h$, définie sur $\mathcal{E}(\mathbb{R}) \times \mathcal{E}(\mathbb{R})$ (où $\mathcal{E}(\mathbb{R})$ désigne l'espace des fonctions en escalier) par,

$$< \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} >_h = R_h(t,s)$$

est un produit scalaire.
Nous ne considérons plus, à partir de la section 5, qu'une fonction h de classe C^1. Nous définissons d'abord la dérivée, au sens des distributions stochastiques, du mouvement Brownien multifractionnaire par :

$$W^{(h)}(t) := \sum_{k=0}^{+\infty} \left[\frac{d}{dt} \Big(\int_0^t \; M_{h(t)}(e_k)(s)\; ds \Big) \right] \; < ., e_k >, \tag{3.3}$$

où la famille $(e_k)_{k\in\mathbb{N}}$ désigne la famille orthonormée des fonctions de Hermite (définies au *cf. chapitre 2 égalité* (2.4)). Nous donnons ensuite le

Théorème-Définition 3.1. *Si la fonction h' est bornée, le processus $W^{(h)} := (W^{(h)}(t))_{t\in\mathbb{R}}$ defini par la formule* (3.3) *est un $(\mathcal{S})^*$-processus qui vérifie dans $(\mathcal{S})^*$, l'égalité suivante :*

$$W^{(h)}(t) = \sum_{k=0}^{+\infty} M_{h(t)}(e_k)(t) < ., e_k > + h'(t) \; \sum_{k=0}^{+\infty} \Big(\int_0^t \tfrac{\partial M_H}{\partial H}(e_k)(s)\big|_{H=h(t)} ds \Big) < ., e_k >. \tag{3.4}$$

De plus le processus $B^{(h)}$ est $(\mathcal{S})^*$-differentiable sur $\mathbb{R}$ et vérifie, dans $(\mathcal{S})^*$,

$$\frac{dB^{(h)}}{dt}(t) = W^{(h)}(t) = \frac{d}{dt}[\Lambda(t, h(t))]. \tag{3.5}$$

Nous définissons enfin l'intégrale stochastique par rapport au mBm de la façon suivante, $\diamond$ désignant le produit de Wick,

Définition 3.1 (Intégrale de Wick-Itô multifractionnaire)**.** *Soit $Y : \mathbb{R} \to (\mathcal{S})^*$ un processus tel que $t \mapsto Y(t) \diamond W^{(h)}(t)$ est $(\mathcal{S})^*$-integrable sur $\mathbb{R}$. On dit que le processus Y est $dB^{(h)}$-intégrable ou intégrable sur $\mathbb{R}$ par rapport au mBm $B^{(h)}$. L'integrale de Y par rapport à $B^{(h)}$ est définie par*

$$\int_{\mathbb{R}} Y(s)\; dB^{(h)}(s) := \int_{\mathbb{R}} Y(s) \diamond W^{(h)}(s) ds. \tag{3.6}$$

Pour tout intervalle I de $\mathbb{R}$, on définit $\int_I Y(s) dB^{(h)}(s) := \int_{\mathbb{R}} \mathbb{1}_I(s)\; Y(s) dB^{(h)}(s)$.

Le reste de la section est consacré à la résolution de l'équation différentielle stochastique dont la solution est un mBm géométrique ainsi qu'à la définition d'une seconde intégrale de Wiener par rapport au mBm, issue de la définition 3.1. Il appert alors que l'ensemble des intégrales de Wiener par rapport au mBm définies à la section 3 et l'ensemble des intégrales d'éléments deterministes obtenues à partir de la définition 3.1 coincident.
Le paragraphe 6 est consacré à l'obtention de formules d'Itô dans l'espace (L^2) mais aussi dans l'espace des distributions stochastiques $(\mathcal{S})^*$. Parmi celles-ci, citons les deux résultats suivants.

Théorème 3.1. *Soient $p \in \mathbb{N}$, a et b deux réels avec $0 < a < b$, et soit F un élément de $C^1([a, b], \mathscr{S}_{-p}(\mathbb{R}))$ tels que les fonctions $\frac{\partial F}{\partial x}$ et $\frac{\partial^2 F}{\partial x^2}$, de $[a, b]$ dans $\mathscr{S}_{-p}(\mathbb{R})$, sont continues. Nous avons alors, dans $(\mathcal{S})^*$, l'égalité suivante :*

$$\begin{aligned} F(b, B^{(h)}(b)) - F(a, B^{(h)}(a)) = \int_a^b \; \frac{\partial F}{\partial t}(s, B^{(h)}(s))\; ds + \int_a^b \; \frac{\partial F}{\partial x}(s, B^{(h)}(s))\; dB^{(h)}(s) \\ + \frac{1}{2} \int_a^b \; \Big(\frac{d}{ds}[R_h(s, s)] \Big) \; \frac{\partial^2 F}{\partial x^2}(s, B^{(h)}(s))\; ds. \end{aligned}$$

Théorème 3.2. *Soit $T > 0$ et $h : \mathbb{R} \to (0, 1)$ une fonction de classe C^1 telle que h' est bornée sur $\mathbb{R}$ et soit f une fonction de classe $C^{1,2}([0, T] \times \mathbb{R}, \mathbb{R})$. Supposons qu'il existe deux constantes $C \geqslant 0$ et $\lambda < \frac{1}{4 \max\limits_{t\in[0,T]} t^{2h(t)}}$ telles que pour tout (t, x) de $[0, T] \times \mathbb{R}$,*

$$\max_{t\in[0,T]} \left\{ |f(t, x)|, \left|\frac{\partial f}{\partial t}(t, x)\right|, \left|\frac{\partial f}{\partial x}(t, x)\right|, \left|\frac{\partial^2 f}{\partial x^2}(t, x)\right| \right\} \leqslant C e^{\lambda x^2}.$$

Alors pour tout t de $[0,T]$, l'égalité suivante est vérifiée dans (L^2) :

$$f(T, B^{(h)}(T)) = f(0,0) + \int_0^T \frac{\partial f}{\partial t}(t, B^{(h)}(t))\, dt + \int_0^T \frac{\partial f}{\partial x}(t, B^{(h)}(t))\, dB^{(h)}(t) + \frac{1}{2}\int_0^T \left(\frac{d}{dt}[R_h(t,t)]\right) \frac{\partial^2 f}{\partial x^2}(t, B^{(h)}(t))\, dt.$$

L'obtention d'une formule de Tanaka ainsi que l'étude de deux mBm de paramètre fonctionnel h particulier achèvent le chapitre 2.

Chapitre 3

Le point de départ du chapitre 3 est la question suivante :
Peut-on approximer un mBm de paramètre fonctionnel h par une suite de sommes de fBm de coefficient de Hurst égaux à certaines valeurs prises par la fonction h ?
Nous interrogeant alors sur ce que devrait être (compte tenu des arguments avancés dans les deux sous-sections qui précèdent) un mBm, nous rédéfinissons le mBm à partir de la notion de champ fractionnaire. Le champ fractionnaire $(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$ est un champ gaussien tel que pour tout H dans $(0,1)$, le processus $(B_t^H)_{t\in\mathbb{R}}$ défini par $B_t^H := \mathbf{B}(t,H)$ est un mouvement brownien fractionnaire de coefficient de Hurst H. Un mouvement brownien multifractionnaire de parmètre fonctionnel h étant alors défini comme étant le processus $(\mathbf{B}(t,h(t)))_{(t,H)\in\mathbb{R}\times(0,1)}$. Cette définition généralise toutes les définitions de mBm connues à ce jour.
Un intervalle $[a,b]$ de $\mathbb{R}$ étant fixé, nous construisons, à partir du champ fractionnaire $\mathbf{B}$, la suite de somme de fBm par morceaux, pour tout t dans $[a,b]$ et n dans $\mathbb{N}$, par :

$$B_t^{h_n} := \mathbf{B}(t, h_n(t)) = \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t)\ \mathbf{B}(t, h(x_k^{(n)})) + \mathbb{1}_{\{b\}}(t)\ \mathbf{B}(b, h(b)). \tag{3.7}$$

L'hypothèse de régularité suivante permet de répondre à la question posée au début de ce paragraphe.

$(\mathcal{H}_1)$: $\forall [a,b]\subset\mathbb{R}, \forall [c,d]\subset(0,1), \exists(\Lambda,\delta)\in(\mathbb{R}_+^*)^2$, such that $\mathsf{E}[(\mathbf{B}(t,H)-\mathbf{B}(t,H'))^2] \leqslant \Lambda\,|H-H'|^{\delta}$, pour tout (t,H,H') in $[a,b]\times[c,d]^2$.

En effet, on montre le théorème d'approximation suivant :

Théorème 3.3. *Soit $\mathbf{B}$ un champ fractionnaire, $h : \mathbb{R}\to(0,1)$ une fonction déterministe continue et B^h le mBm associé. Pour tout intervalle compact $[a;b]$ de $\mathbb{R}$, et toute partition $\mathcal{A}$ de $[a;b]$ (dont les éléments ont une certaine longueur), on a le résultat suivant.*

1. *Si $\mathbf{B}$ satisfait l'hypothèse de continuité de la fonction $C : (t,s,H,H') \mapsto \mathsf{E}[\mathbf{B}(t,H)\ \mathbf{B}(s,H')]$ est continue sur $\mathbb{R}^2\times(0,1)^2$, alors la suite de processus $(B^{h_n})_{n\in\mathbb{N}}$ converge dans $L^2(\Omega)$ vers B^h,i.e*

$$\forall t\in[a,b],\quad \lim_{n\to+\infty} E\left[\left(B_t^{h_n}-B_t^h\right)^2\right] = 0.$$

2. *Si $\mathbf{B}$ satisfait l'hypothèse $(\mathcal{H}_1)$ et si h is β-Hölderienne pour un certain $\beta>0$, alors la suite de processus $(B^{h_n})_{n\in\mathbb{N}^*}$ converge*
   (**i**) *en loi, i.e* $\{B_t^{h_n};\ t\in[a,b]\} \xrightarrow[n\to+\infty]{loi} \{B_t^h;\ t\in[a,b]\}.$
   (**ii**) *presque sûrement, i.e* $P\left(\{\forall t\in[a,b],\ \lim_{n\to+\infty} B_t^{h_n} = B_t^h\}\right) = 1.$

La question qui se pose ensuite naturellement est la suivante : Peut-on à partir du précédent théorème inférer une méthode d'intégration stochastique par rapport au mBm à partir des méthodes d'intégration stochastiques existant pour le fBm ?
Une méthode $(\mathcal{M})$ d'intégration par rapport au fBm étant fixée (Chemins rugueux, calcul de Malliavin, Théorie du bruit blanc, par exemple) il convient alors de donner la définition suivante :

Définition 3.2 (intégrale par morceaux par rapport à une somme de fBm)**.** *Soit $Y := (Y_t)_{t\in[0,1]}$ un processus sur $[0;1]$ intégrable par rapport à tout fBm d'indice de Hurst H appartenant à $h([0;1])$, au sens de la méthode $(\mathcal{M})$. On appelle intégrale par morceaux par rapport à une somme de fBm au sens de la méthode $(\mathcal{M})$:*

$$\int_0^1 Y_t\ d^{(\mathcal{M})} B_t^{h_n} := \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t)\ \ Y_t\ d^{(\mathcal{M})} B_t^{h(x_k^{(n)})}, \quad n \in \mathbb{N}. \tag{3.8}$$

A partir d'exemples d'intégrales de processus simples par rapport à une somme de fBm et en faisant tendre n vers $+\infty$ dans (3.8) on s'aperçoit qu'une définition raisonnable de l'intégrale stochastique d'un processus Y par rapport au mBm, au sens de la méthode $(\mathcal{M})$ est nécessairement de la forme :

$$\int_0^1 Y_t\ d^{(\mathcal{M})} B_t^{h} := \lim_{n\to\infty} \underbrace{\int_0^1 Y_t\ d^{(\mathcal{M})} B_t^{h_n}}_{:=L_n(Y)} + \Phi_h^{(\mathcal{M})}(Y), \tag{3.9}$$

où la limite dans (3.9) est prise au sens de $L^2(\Omega)$ ou de $(\mathcal{S})^*$ selon la méthode $(\mathcal{M})$ employée et où

$$\Phi_h^{(\mathcal{M})}(Y) := \int_0^1 h'(t)\ \varphi^{(\mathcal{M})}(Y_t, \frac{\partial \mathbf{B}}{\partial H}(t, h(t)))\ dt,$$

avec $\varphi^{(\mathcal{M})}(x,y) := x \diamond y$ ou xy selon la méthode employée ($\diamond$ désignant le produit de Wick). Le champ fractionnaire $\mathbf{B}$ étant, quant-à lui, supposé vérifier l'hypothèse $(\mathcal{H}_2)$ suivante :
$(\mathcal{H}_2)$: $\forall [a,b]\times[c,d] \subset \mathbb{R}\times(0,1), H \mapsto \mathbf{B}(t,H)$ est de classe C^1, au sens de $L^2(\Omega)$, de $(0,1)$ dans $L^2(\Omega)$ pour tout t de $[a,b]$ et $\exists(\Delta,\alpha,\lambda) \in (\mathbb{R}_+^*)^3$ s.t. $\mathbb{E}\Big[\Big(\frac{\partial \mathbf{B}}{\partial H}(t,H) - \frac{\partial \mathbf{B}}{\partial H}(s,H')\Big)^2\Big] \leqslant \Delta\ \Big(|t-s|^{\alpha} + |H-H'|^{\lambda}\Big)$, pour tout (t,s,H,H') de $[a,b]^2\times[c,d]^2$.
Soient $(E, ||\ ||_E)$ and $(F, ||\ ||_F)$ deux espaces vectoriels normés, on définit

$$\mathcal{H}_E := \bigcap_{\alpha\in h([0,1])} \left\{ Y \in E^{[0,1]} : \int_{[0,1]} Y_t\ d^{(\mathcal{M})} B_t^{\alpha} \text{ existe et appartient à } F \right\},$$

où B^{α} désigne un fBm de coefficient de Hurst α.
Le critère suivant permet de savoir, pour une méthode d'intégration par rapport au fBm donnée et notée $(\mathcal{M})$, si la suite de fonctionnelles $(L_n)_{n\in\mathbb{N}}$, définie en 3.9, converge simplement sur F tout entier.

Théorème 3.4. *Soit $(a_n)_{n\in\mathbb{N}}$ une suite croissante d'entiers strictement positifs telle que $2^n \leqslant \prod_{0\leqslant k\leqslant n-1} a_k \leqslant 2^{2^n}$ pour tout n in $\mathbb{N}$ et telle que $\lim_{n\to+\infty} (n(a_n-1))\ (\prod_{0\leqslant k\leqslant n-1} a_k)^{-1} = 0$.*
Supposons de plus que la suite $(q_n)_{n\in\mathbb{N}}$, definie par $q_0 = 1$ et $q_{n+1} = a_n\ q_n$ pour tout n dans $\mathbb{N}$ soit telle que l'on puisse trouver une norme sur $\mathcal{H}_E$, notée $\|\ \|_{\mathcal{H}_E}$, telle qu'il existe $M > 0$ et telle que pour toute famille de boréliens disjoints $A_1, \ldots, A_n$ de $[0,1]$,

$$\|Y.\mathbb{1}_{A_1}\|_{\mathcal{H}_E} + \cdots + \|Y.\mathbb{1}_{A_n}\|_{\mathcal{H}_E} \leqslant M\ \|Y\|_{\mathcal{H}_E}.$$

Supposons de plus que la fonction $\mathcal{I} : \mathcal{H}_E \times (0,1) \to F$, définie par

$$\forall\, Y \in \mathcal{H}_E, \forall \alpha \in (0,1), \quad \mathcal{I}(Y,\alpha) := \int_{[0,1]} Y_t\ d^{(\mathcal{M})} B_t^{\alpha},$$

est θ-Hölderienne par rapport à α uniformément en Y pour un certain réel θ, i.e. il existe $\theta > 0$ et $K > 0$ tels que

$$\forall (\alpha, \alpha') \in (0,1)^2, \qquad \sup_{\|Y\|_{\mathcal{H}_E} \leqslant 1} \|\mathcal{I}(Y,\alpha) - \mathcal{I}(Y,\alpha')\|_F \leqslant K \ |\alpha - \alpha'|^\theta.$$

Alors la suite de fonctionnelles $(L_n)_{n \in \mathbb{N}}$ définie en 3.9 converge simplement vers une fonction $L : \mathcal{H}_E \to F$.

Dans les parties 4 et 5 on considère le cas particulier où $(\mathcal{M})$ désigne le calcul de Malliavin et la théorie du bruit blanc. On y montre ainsi notamment le

Théorème 3.5. *Soit $Y = (Y_t)_{t \in [0,1]}$ un processus Bochner intégrable d'indice $p_0 \in \mathbb{N}$ (cf. chapitre 3 définition 5.1 pour une définition). Alors Y est intégrable par rapport au mBm harmonisable à la fois au sens de la méthode du bruit blanc et au sens défini dans le chapitre 2. De plus ces deux intégrales $\int_{[0,1]} Y_t \ d^{(\mathcal{M}_1)} B_t^h$ et $\int_{[0,1]} Y_t \ d^\diamond B_t^h$ coincident dans l'espace $(\mathcal{S}^*)$ des distributions stochastiques.*

Chapitre 4

Le troisième et dernier chapitre de ce mémoire, extrait de [25], concerne une application aux mathématiques financières du calcul stochastique développé dans les deux premiers chapitres. Plus précisement, nous partons du modèle à volatilité stochastique proposé dans [24], qui s'écrit,

$$\begin{cases} dF_t = \mu(t, F_t)dt + F_t \sigma_t dW_t, \\ d\ln(\sigma_t) = \theta\left(\mu - \ln(\sigma_t)\right)dt + \gamma dB_t^H, \quad \sigma > 0, \end{cases} \tag{3.10}$$

où $(F_t)_{t \in [0;T]}$ désigne le processus de prix forward d'une action, W un mouvement brownien et $(B_t^H)_{t \in [0;T]}$ un fBm de coefficient de Hurst H, independant de W.
L'introduction d'un fBm dans (3.10) tient au fait que les auteurs souhaitaient rendre compte du phénomène de dépendance à long terme du processus de volatilité observé en pratique mais dont le modèle de Hull et White standard (*i.e* (3.10) où un brownien est mis à la place de B^H) ne rend pas compte. Malheureusement, et comme nous l'avons indiqué dans le paragraphe de cette introduction consacré aux propriétés du fBm, la dépendance à long terme n'existe pour les accroissements du processus B^H que si $H > 1/2$. De plus le modèle (3.10) ne permet pas de modéliser de façon satisfaisante des smiles de volatilité de différentes maturités.
Pour pallier ces deux inconvénients l'idée est de substituer au fBm B^H un mBm B^h dans (3.10). Le modèle à volatilité stochastique devient alors

$$\begin{cases} dF_t = F_t \sigma_t dW_t, \\ d\ln(\sigma_t) = \theta\left(\mu - \ln(\sigma_t)\right)dt + \gamma_h d^\diamond B_t^h + \gamma_\sigma dW_t^\sigma, \quad \sigma_0 > 0, \\ d\langle W, W^\sigma \rangle_t = \rho dt, \end{cases} \tag{3.11}$$

où $d^\diamond B_t^h$ désigne la différentielle prise au sens de la théorie du bruit blanc.
Ceci nous permet ainsi de considérer un processus de volatilité dont les accroissements ne sont pas stationnaires et présentent une dépendance à long terme y compris lorsque la régularité Hölderienne de la volatilité est inférieure à 1/2.
La résolution de cette equation différentielle stochastique est exacte en ce qui concerne le processus de volatilité qui est donné par,

$$\sigma_t = \exp\Big(\ln(\sigma_0)e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \gamma_\sigma \int_0^t e^{\theta(s-t)} dW_s^\sigma + \gamma_h \int_0^t e^{\theta(s-t)} d^\diamond B_s^h\Big).$$

En revanche, la solution donnant $(F_t)_{t \in [0;T]}$ dans (3.11) n'étant pas explicitement connue il convient d'employer une méthode numérique, robuste, d'approximation. Pour ce faire nous utilisons la théorie de la quantification fonctionnelle pour les processus gaussiens développée par [56] ainsi qu'une méthode de cubature et de réduction de variance.
Nous donnons également une majoration de l'erreur de quantification d'un mBm B^h :

Proposition 3.1. *Soit $T > 0$ et $\beta > 0$ deux réels fixés. Soit $h : [0,T] \to (0,1)$ une fonction β-Hölderienne telle que $H_1 := \sup_{u\in[0,T]} h(u) > 0$. Soit $B^h := (B_t^h)_{t\in[0,T]}$ un mBm de paramètre fonctionnel h et de champ fractionnaire associé satisfisant l'hypothèse $(\mathcal{H})$. On a alors, pour tout (r,p) de $(\mathbb{R}_+)^2$,*

$$\mathcal{E}_N\left(B^h, |\cdot|_{L^p([0,T])}\right) = O\left(\log(N)^{-\left(H_1\wedge\frac{\beta\delta}{2}\right)}\right),$$

où $\mathcal{E}_N\left(B^h, |\cdot|_{L^p([0,T])}\right)$ désigne l'erreur de quantification, au sens de la norme $L^p(\mathbb{R})$ du mBm.

Suivant la méthode proposée, par exemple dans [66], le préconditionnement par rapport à la tribu $\mathcal{F}_T^{\sigma,h}$ nous permet d'écrire :

$$\mathsf{E}\left[\left(\frac{F_T}{F_\tau} - K\right)_+\right] = \mathsf{E}\Bigg[\underbrace{\mathsf{E}\left[\left(\frac{F_T}{F_\tau} - K\right)_+ \middle| \mathcal{F}_T^{\sigma,h}\right]}_{:=\psi(W^\sigma, B^h)}\Bigg],$$

pour une certaine fonctionnelle ψ de W^σ et B^h. Nous utilisons alors une méthode de cubature, fondée sur une quantification fonctionnelle de W^σ et B^h afin d'évaluer l'espérance $\mathsf{E}[\psi(W^\sigma, B^h)]$.
Nous évaluons ensuite le prix de forward start calls et donnons enfin, en utilisant une méthode d'extrapolation et de cubature les smiles obtenus pour différentes fonctions h et pour quatre maturités différentes.

Chapitre 2

White Noise-based Stochastic Calculus with respect to multifractional Brownian motion

Joint work with J. Lévy-Véhel

Abstract

Stochastic calculus with respect to fractional Brownian motion (fBm) has attracted a lot of interest in recent years, motivated in particular by applications in finance and Internet traffic modeling. Multifractional Brownian motion (mBm) is a Gaussian extension of fBm that allows to control the pointwise regularity of the paths of the process and to decouple it from its long range dependence properties. This generalization is obtained by replacing the constant Hurst parameter H of fBm by a function $h(t)$. Multifractional Brownian motion has proved useful in many applications, including the ones just mentioned.
In this work we extend to mBm the construction of a stochastic integral with respect to fBm. This stochastic integral is based on white noise theory, as originally proposed in [34, 11, 8, 9].
In that view, a multifractional white noise is defined, which allows to integrate with respect to mBm a large class of stochastic processes using Wick products. Itô formulas (both for tempered distributions and for functions with sub-exponential growth) are given, along with a Tanaka Formula. The cases of two specific functions h which give notable Itô formulas are presented.

keywords: multifractional Brownian motion, Gaussian processes, White Noise Theory, S-Transform, Wick-Itô integral, Itô formula, Tanaka formula, Stochastic differential equations.

Contents

1 Background and Motivations

Fractional Brownian motion (fBm) [50, 60] is a centered Gaussian process with features that make it a useful model in various applications such as financial and teletraffic modeling, image analysis and synthesis, geophysics and more. These features include self-similarity, long range dependence and the ability to match any prescribed constant local regularity. Fractional Brownian motion depends on a parameter, usually denoted H and called the Hurst exponent, that belongs to $(0, 1)$. Its covariance function R_H reads:

$$R_H(t, s) := \frac{\gamma_H}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \tag{1.1}$$

where γ_H is a positive constant. A normalized fBm is one for which $\gamma_H = 1$. Obviously, when $H = \frac{1}{2}$, fBm reduces to standard Brownian motion.
A useful representation of fBm $B^{(H)}$ of exponent H is the so-called harmonizable one:

$$B^{(H)}(t) = \frac{1}{c_H} \int_{\mathbb{R}} \frac{e^{itu} - 1}{|u|^{H+1/2}} \widetilde{W}(du), \tag{1.2}$$

where $c_x := \left(\frac{2\cos(\pi x)\Gamma(2-2x)}{x(1-2x)}\right)^{\frac{1}{2}} = \left(\frac{2\pi}{\Gamma(2x+1)\sin(\pi x)}\right)^{\frac{1}{2}}$ for x in $(0,1)$ and $\widetilde{W}$ denotes the complex-valued Gaussian measure which can be associated in a unique way to W, an independently scattered standard Gaussian measure on $\mathbb{R}$ (see [75] p.203-204 and [73] p.325-326 for more information on the meaning of $\int_{\mathbb{R}} f(u)\, \widetilde{W}(du)$ for a complex-valued function f). From (1.1) and Gaussianity, it is not hard to prove that fBm is H-self-similar.

The fact that most of the properties of fBm are governed by the single number H restricts its application in some situations. Let us give two examples. The long term correlations of the increments of fBm decay as $k^{(2H-2)}$, where k is the lag, resulting in long range dependence when $H > 1/2$ and anti-persistent behavior when $H < 1/2$. Also, almost surely, for each t, its pointwise Hölder exponent is equal to H. Since H rules both ends of the Fourier spectrum, *i.e.* the high frequencies related to the Hölder regularity and the low frequencies related to the long term dependence structure, it is not possible to have at the same time *e.g.* a very irregular local behavior (implying H close to 0) and long range dependence (implying $H > 1/2$). As a consequence, fBm is not adapted to model phenomena which display both these features, such as Internet traffic or certain highly textured images with strong global organization. Another example is in the field of image synthesis: fBm has frequently been used for generating artificial mountains. Such a modeling assumes that the regularity of the mountain is everywhere the same. This is not realistic, since it does not take into account erosion or other meteorological phenomena which smooth some parts of mountains more than others.

Multifractional Brownian motion (mBm) [68, 7] was introduced to overcome these limitations. The basic idea is to replace in (1.2) the real H by a function $h(t)$. More precisely, we will use the following definition of mBm:

Definition 1.1 (Multifractional Brownian motion)**.** *Let $h : \mathbb{R} \to (0,1)$ be a continuous function and $\alpha : (0,1) \to \mathbb{R}$ be a C^1 function. A multifractional Brownian motion with functional parameters h and α is defined as:*

$$B^{(h,\alpha)}(t) = \alpha(h(t)) \int_{\mathbb{R}} \frac{e^{itu}-1}{|u|^{h(t)+1/2}}\, \widetilde{W}(du). \tag{1.3}$$

Its covariance function reads [5]:

$$R_{(h,\alpha)}(t,s) = \alpha(h(t))\, \alpha(h(s))\, c^2_{h_{t,s}} \left[\, \frac{1}{2}\Big(|t|^{2h_{t,s}} + |s|^{2h_{t,s}} - |t-s|^{2h_{t,s}}\Big)\right], \tag{1.4}$$

where $h_{t,s} := \frac{h(t)+h(s)}{2}$ and c_x has been defined in (1.2).

It is easy to check that mBm is a zero mean Gaussian process, the increments of which are in general neither independent nor stationary.

For T in $\mathbb{R}^*_+$, we will again call (h,α)-multifractional Brownian motion on $[0,T]$ the centered Gaussian process whose covariance function is equal to $R_{(h,\alpha)}$ on $[0,T] \times [0,T]$.

When $\alpha = \alpha_c : x \mapsto \frac{1}{c_x}$, we get that:

$$R_{(h,\alpha_c)}(t,s) = \frac{c^2_{h_{t,s}}}{c_{(h(t))}c_{(h(s))}} \left[\, \frac{1}{2}\Big(|t|^{2h_{t,s}} + |s|^{2h_{t,s}} - |t-s|^{2h_{t,s}}\Big)\right]. \tag{1.5}$$

As a consequence, if h is the constant function equal to the real H, then $B^{(H,\alpha_c)}$ is a normalized fBm. For this reason, we will call $B^{(h,\alpha_c)}$ a normalized mBm. Since in the sequel we will consider only normalized mBm, we simplify the notation and write from now on $B^{(h)}$ for $B^{(h,\alpha_c)}$ and R_h for $R_{(h,\alpha_c)}$.

One can show [39, 41] that the pointwise Hölder exponent at any point t of $B^{(h)}$ is almost surely equal to $h(t) \wedge \beta_h(t)$, where $\beta_h(t)$ is the pointwise Hölder exponent of h at t. In addition, the increments of mBm display long range dependence for all non-constant $h(t)$ (the notion of long range dependence must be re-defined carefully for non-stationary increments, see [5]). Finally, at least when h is C^1, for all $u \in \mathbb{R}$, mBm locally "looks like" fBm with exponent $h(u)$ in the neighbourhood of u in the following sense [68]:

$$\lim_{r \to 0_+} \frac{B^{(h)}(u+rt) - B^{(h)}(u)}{r^{h(u)}} = B^{(h(u))}(t), \tag{1.6}$$

where the convergence holds in law. These properties show that mBm is a more versatile model that fBm: in particular, it is able to mimic in a more faithful way local properties of financial records, Internet traffic and natural landscapes [15, 55, 33] by matching their local regularity. This is important *e.g.* for purposes of detection or real-time control. The price to pay is of course that one has to deal with the added complexity brought by having a functional parameter instead of a single number.
Because of applications, in particular in finance and telecommunications, it has been an important objective in recent years to define a stochastic calculus with respect to fBm. This was not a trivial matter, as fBm is not a semi-martingale for $H \neq \frac{1}{2}$. Several approaches have been proposed, based mainly on Malliavin calculus [31, 3], pathwise approaches and rough paths ([79, 29, 36] and references therein), and white noise theory [34, 11, 9]. Since mBm seems to be a more flexible, albeit more complex, model than fBm, it seems desirable to extend the stochastic calculus defined for fBm to it. This is the aim of the current work. In that view, we will use a white noise approach, as it offers several advantages in our frame. The main task is to define a *multifractional white noise* as a Hida stochastic distribution, which generalizes the fractional white noise of, *e.g.*, [34, 11]. For that purpose, we use the properties of the Gaussian field $(B^{(H)}(t))_{(t,H)\in\mathbb{R}\times(0,1)}$. In particular, it is a crucial fact for us that the function $H \mapsto (B^{(H)}(t))$ is almost surely C^{∞}. This entails that multifractional white noise behaves essentially as fractional white noise, plus a smooth term. We obtain an Ito formula that reads:

$$f(T, B^{(h)}(T)) = f(0,0) + \int_0^T \frac{\partial f}{\partial t}(t, B^{(h)}(t))\, dt + \int_0^T \frac{\partial f}{\partial x}(t, B^{(h)}(t))\, dB^{(h)}(t) + \frac{1}{2}\int_0^T \left(\frac{d}{dt}[R_h(t,t)]\right) \frac{\partial^2 f}{\partial x^2}(t, B^{(h)}(t))\, dt,$$

where the meaning of the different terms will be explained below.
The remaining of this paper is organized as follows. In section 2, we recall basic facts about white noise theory. We study a family of operators, noted $(M_H)_{H\in(0,1)}$, which are instrumental for constructing the stochastic integral with respect to mBm in section 3. Section 4 defines the Wiener integral with respect to mBm. We build up a stochastic integral with respect to mBm in section 5. Various instances of Ito formula are proved in section 6. Finally, section 7 provides a Tanaka formula, along with the study of two particular h functions that give notable results. Readers familiar with white noise theory may skip the next section.

2 White noise theory

We recall in this section the standard set-up for classical white-noise theory. We refer *e.g.* to [52, 43] for more details.

2.1 White noise measure

Let $\mathscr{S}(\mathbb{R}) := \{f \in C^{\infty}(\mathbb{R}) \ : \ \forall (p,q) \in \mathbb{N}^2, \lim_{|x|\to+\infty} |x^p\, f^{(q)}(x)| = 0\}$ be the Schwartz space. A family of functions $(f_n)_{n\in\mathbb{N}}$ of $(\mathscr{S}(\mathbb{R}))^{\mathbb{N}}$ is said to converge to 0 as n tends to $+\infty$ if for all (p,q) in $\mathbb{N}^2$ we have $\lim_{n\to+\infty} \sup_{x\in\mathbb{R}} |x^p\, f_n^{(q)}(x)| = 0$. The topology hence given on $\mathscr{S}(\mathbb{R})$ is called the usual topology. Let $\mathscr{S}'(\mathbb{R})$ denote the space of tempered distributions, which is the dual space of $\mathscr{S}(\mathbb{R})$. The Fourier transform of a function f which belongs to $L^1(\mathbb{R}) \cup L^2(\mathbb{R})$ will be denoted $\widehat{f}$ or $\mathcal{F}(f)$:

$$\mathcal{F}(f)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x)dx, \quad \xi \in \mathbb{R}. \tag{2.1}$$

Define the probability space as $\Omega := \mathscr{S}'(\mathbb{R})$ and let $\mathcal{F} := \mathcal{B}(\mathscr{S}'(\mathbb{R}))$ be the σ-algebra of Borel sets. The Bochner-Minlos theorem ensures that there exists a unique probability measure on Ω, denoted μ, such that:

$$\int_{\mathscr{S}'(\mathbb{R})} e^{i<\omega,f>} \mu(d\omega) = e^{-\frac{1}{2}||f||^2_{L^2(\mathbb{R})}}, \qquad \forall f \in \mathscr{S}(\mathbb{R}), \tag{2.2}$$

where $<\omega, f>$ is by definition $\omega(f)$, *i.e* the action of the distribution ω on the function f. For f in $\mathscr{S}(\mathbb{R})$ the map, noted $<., f>$, from Ω to $\mathbb{R}$ defined by $<., f>(\omega) = <\omega, f>$ is thus a centered Gaussian random variable with variance equal to $||f||^2_{L^2(\mathbb{R})}$ under the probability measure μ, which is called the *white-noise probability measure*. In other words, $\mathbb{E}[<., f>] = 0$ and $\mathbb{E}[<., f>^2] = ||f||^2_{L^2(\mathbb{R})}$ for all f in $\mathscr{S}(\mathbb{R})$. Besides, for a measurable function F, from $\mathscr{S}^{'}(\mathbb{R})$ to $\mathbb{R}$, the expectation of F with respect to μ is defined, when it exists, by $E[F] := E_\mu[F] := \int_\Omega F(\omega)\mu(d\omega)$. Equality (2.2) entails that the map ζ defined on $\mathscr{S}(\mathbb{R})$ by

$$\begin{array}{rccc} \zeta: & (\mathscr{S}(\mathbb{R}), <,>_{L^2(\mathbb{R})}) & \to & (L^2(\Omega,\mathcal{F},\mu), <,>_{(L^2(\Omega,\mathcal{F},\mu)}) \\ & f & \mapsto & \zeta(f) := <., f> \end{array} \tag{2.3}$$

is an isometry. Thus, it extends to $L^2(\mathbb{R})$ and we still note ζ this extension. For an arbitrary f in $L^2(\mathbb{R})$, we then have $<., f> := \lim\limits_{n\to+\infty} <., f_n>$ where the convergence takes place in $L^2(\Omega, \mathcal{F}, \mu)$ and where $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions which belongs to $\mathscr{S}(\mathbb{R})$ and converges to f in $L^2(\mathbb{R})$. In particular, define for all t in $\mathbb{R}$, the indicator function $\mathbb{1}_{[0,t]}$ by

$$\mathbb{1}_{[0,t]}(s) = \begin{cases} 1 & if \quad 0 \leqslant s \leqslant t, \\ -1 & if \quad t \leqslant s \leqslant 0 \ \text{ except if } t = s = 0 \\ 0 & \text{otherwise,} \end{cases}$$

Then the process $(\widetilde{B}_t)_{t\in\mathbb{R}}$, defined for $t \in \mathbb{R}$, on Ω by $\widetilde{B}_t(\omega) := \widetilde{B}(t,\omega) := <\omega, \mathbb{1}_{[0,t]}>$ is a standard Brownian motion with respect to μ. It then admits a continuous version which will be denoted B. The previous equality shows that, for all functions f in $L^2(\mathbb{R})$, $I_1(f)(\omega) = <\omega, f> = \int_{\mathbb{R}} f(s) dB_s(\omega) \ \ \mu-\text{a.s.}$

2.2 Properties of Hermite functions and space $\mathscr{S}'(\mathbb{R})$

For every n in $\mathbb{N}$, define

$$e_n(x) := (-1)^n \, \pi^{-1/4}(2^n n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2}) \quad \text{the } n^{th} \text{ Hermite function.} \tag{2.4}$$

We hence have

$$e_n(x) = \pi^{-1/4}(2^n n!)^{-1/2} \; H_n(x) \; e^{-x^2/2},$$

where H_n denotes the nth Hermite polynomial, which is defined by $H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$. We will need the following properties of the Hermite functions:

Theorem 2.1. *1. The family $(e_k)_{k\in\mathbb{N}}$ belongs to $(\mathscr{S}(\mathbb{R}))^{\mathbb{N}}$ and forms an orthonormal basis of $L^2(\mathbb{R})$ endowed with its usual inner product.*

2. There exists a real constant $\widetilde{C}$ such that, for every k in $\mathbb{N}$, $\max\limits_{x\in\mathbb{R}} |e_k(x)| \leqslant \widetilde{C}\,(k+1)^{-1/12}$. More precisely, there exist positive constants C and γ such that, for every k in $\mathbb{N}$,

$$|e_k(x)| \leqslant \begin{cases} C\,(k+1)^{-1/12} & if \quad |x| \leqslant 2\sqrt{k+1}, \\ Ce^{-\gamma x^2} & if \quad |x| > 2\sqrt{k+1}. \end{cases} \tag{2.5}$$

See [77] for proofs.
In order to study precisely $\mathscr{S}(\mathbb{R})$ and its dual $\mathscr{S}'(\mathbb{R})$ it is desirable to have a family of norms on the space $\mathscr{S}(\mathbb{R})$ which gives us the usual topology.

Definition 2.1. *Let $(|\ \ |_p)_{p\in\mathbb{Z}}$ be the family of norm defined by*

$$|f|_p^2 := \sum_{k=0}^{+\infty} (2k+2)^{2p} < f, e_k >^2_{L^2(\mathbb{R})}, \qquad \forall (p, f) \in \mathbb{Z} \times L^2(\mathbb{R}). \tag{2.6}$$

For p in $\mathbb{N}$, define the spaces $\mathscr{S}_p(\mathbb{R}) := \{f \in L^2(\mathbb{R}),\ |f|_p < +\infty\}$ and $\mathscr{S}_{-p}(\mathbb{R})$ as being the completion of the space $L^2(\mathbb{R})$ with respect to the norm $|\ \ |_{-p}$.

Remark 2.2. *For a function f which is not in $L^2(\mathbb{R})$, we may still define $|f|_p$ by allowing $|f|_p$ to be infinite.*

It is well known (see [52]) that the Schwartz space $\mathscr{S}(\mathbb{R})$ is the projective limit of the sequence $(\mathscr{S}_p(\mathbb{R}))_{p\in\mathbb{N}}$ and that the space $\mathscr{S}'(\mathbb{R})$ of tempered distributions is the inductive limit of the sequence $(\mathscr{L}_p(\mathbb{R}))_{p\in\mathbb{N}}$. Since, for any p in $\mathbb{N}$, the dual space $\mathscr{S}'_p(\mathbb{R})$ of $\mathscr{S}_p(\mathbb{R})$ is $\mathscr{L}_p(\mathbb{R})$, we will write $\mathscr{L}_p(\mathbb{R})$ in the sequel to denote the space $\mathscr{S}'_p(\mathbb{R})$.
Finally one can show that the usual topology of the space $\mathscr{S}(\mathbb{R})$ and the topology given by the family of norms $(|\ |_p)_{p\in\mathbb{N}}$ are the same (see [42] appendix A.3 for example). Moreover, convergence in the inductive limit topology coincides with both convergence in the strong and the weak* topologies of $\mathscr{S}'(\mathbb{R})$.
In view of definition 2.1, it is convenient to have defined on $\mathscr{S}(\mathbb{R})$ whose eigenfunctions are the sequence $(e_n)_{n\in\mathbb{N}}$ and eigenvalues are the sequence $(2n+2)_{n\in\mathbb{N}}$. It is easy to check that the operator A, densely defined on $L^2(\mathbb{R})$, by $A := -\frac{d^2}{dx^2} + x^2 + 1$ verifies these conditions.
Note moreover that A is invertible and that its inverse A^{-1} is a bounded operator on $L^2(\mathbb{R})$. Let us note $|g|_0^2 := ||g||^2_{L^2(\mathbb{R})}$ for any g in $L^2(\mathbb{R})$. For p in $\mathbb{Z}$, let A^p denote the p^{th} iteration of the operator A, if p belongs to $\mathbb{N}$, and of A^{-1} otherwise. Then $\mathbb{D}\text{om}(A^p) = \mathscr{S}_p(\mathbb{R})$ and $\mathbb{D}\text{om}(A^{-p}) = L^2(\mathbb{R})$ where $\mathbb{D}\text{om}(U)$ denotes the domain of the operator U and p belongs to $\mathbb{N}$. Moreover, for every q in $\mathbb{Z}$ and every $f := \sum_{k=0}^{+\infty} < f, e_k >_{L^2(\mathbb{R})} e_k$ in $\mathbb{D}\text{om}(A^q)$, the equality $A^q f = \sum_{k=0}^{+\infty} (2k+2)^q < f, e_k >_{L^2(\mathbb{R})} e_k$ holds. Hence,

$$|f|_q^2 = |A^q f|_0^2 = \sum_{k=0}^{+\infty} (2k+2)^{2q} < f, e_k >^2_{L^2(\mathbb{R})}, \qquad \forall q \in \mathbb{Z}. \tag{2.7}$$

2.3 Space of Hida distributions

From now on we will denote as is customary (L^2) the space $L^2(\Omega, \mathcal{G}, \mu)$ where $\mathcal{G}$ is the σ-field generated by $(< ., f >)_{f\in L^2(\mathbb{R})}$. Neither Brownian motion nor fractional Brownian motion, whatever the value of H, are differentiable (see [60] for a proof). However, it occurs that the mapping $t \mapsto B^{(H)}(t)$ is differentiable from $\mathbb{R}$ into a space, noted $(\mathcal{S})^*$, called the space of Hida distributions, which contains (L^2). In this section we recall the construction of $(\mathcal{S})^*$.
For every random variable Φ of (L^2) there exists, according to the Wiener-Itô theorem, a unique sequence $(f_n)_{n\in\mathbb{N}}$ of functions f_n in $\widehat{L}^2(\mathbb{R}^n)$ such that Φ can be decomposed as $\Phi = \sum_{n=0}^{+\infty} I_n(f_n)$, where $\widehat{L}^2(\mathbb{R}^n)$ denotes the set of all symmetric functions f in $L^2(\mathbb{R}^n)$ and $I_n(f)$ denotes the n^{th} multiple Wiener-Itô integral of f defined by

$$I_n(f) := \int_{\mathbb{R}^n} f(t) dB^n(t) = n! \int_{\mathbb{R}} \Big(\int_{-\infty}^{t_n} \cdots \Big(\int_{-\infty}^{t_2} f(t_1, \cdots, t_n) dB(t_1) \Big) dB(t_2) \cdots dB(t_n) \Big),$$

with the convention that $I_0(f_0) = f_0$ for constants f_0. Furthermore we have the isometry

$$E[\Phi^2] = \sum_{n=0}^{+\infty} n! \, ||f_n||^2_{L^2(\mathbb{R}^n)}.$$

For any $\Phi := \sum\limits_{n=0}^{+\infty} I_n(f_n)$ satisfying the condition $\sum\limits_{n=0}^{+\infty} n! \, |A^{\otimes n} f_n|_0^2 < +\infty$, define the element $\Gamma(A)(\Phi)$ of (L^2) by $\Gamma(A)(\Phi) := \sum\limits_{n=0}^{+\infty} I_n(A^{\otimes n} f_n)$, where $A^{\otimes n}$ denotes the n^{th} tensor power of the operator A (see [47] appendix E for more details about tensor products of operators).
The operator $\Gamma(A)$ is densely defined on (L^2) and is called the second quantization operator of A. It shares a lot of properties with the operator A. In particular it is invertible and its inverse $\Gamma(A)^{-1}$ is bounded (see [52]). Let us denote $||\varphi||_0^2 := ||\varphi||^2_{(L^2)}$ for any random variable φ in (L^2) and, for n in $\mathbb{N}$, let $\mathbb{D}\text{om}(\Gamma(A)^n)$ be the domain of the n^{th} iteration of $\Gamma(A)$. The space of Hida distributions is defined in a way analogous to the one that allowed to define the space $\mathscr{S}'(\mathbb{R})$:

Definition 2.2. *Define the family of norms $(||\ ||_p)_{p\in\mathbb{Z}}$ by:*

$$||\Phi||_p := ||\Gamma(A)^p\Phi||_0 = ||\Gamma(A)^p\Phi||_{(L^2)}, \qquad \forall p \in \mathbb{Z}, \quad \forall \Phi \in (L^2) \cap \mathbb{D}om(\Gamma(A)^p). \tag{2.8}$$

For any p in $\mathbb{N}$, let $(\mathcal{S}_p) := \{\Phi \in (L^2) : \Gamma(A)^p\Phi$ exists and belongs to $(L^2)\}$ and define $(\mathcal{S}_{-p})$ as being the completion of the space (L^2) with respect to the norm $|| \ ||_{-p}$.

As in [52], we let $(\mathcal{S})$ denote the projective limit of the sequence $((\mathcal{S}_p))_{p\in\mathbb{N}}$ and $(\mathcal{S})^*$ the inductive limit of the sequence $((\mathcal{S}_{-p}))_{p\in\mathbb{N}}$. This means that we have the equalities $(\mathcal{S}) = \underset{p\in\mathbb{N}}{\cap}(\mathcal{S}_p)$ (resp. $(\mathcal{S})^* = \underset{p\in\mathbb{N}}{\cup}(\mathcal{S}_{-p})$) and that convergence in $(\mathcal{S})$ (resp. in $(\mathcal{S})^*$) means convergence in $(\mathcal{S}_p)$ for every p in $\mathbb{N}$ (resp. convergence in $(\mathcal{S}_{-p})$ for some p in $\mathbb{N}$). The space $(\mathcal{S})$ is called the space of stochastic test functions and $(\mathcal{S})^*$ the space of Hida distributions. As previously one can show that, for any p in $\mathbb{N}$, the dual space $(\mathcal{S}_p)^*$ of $\mathcal{S}_p$ is $(\mathcal{S}_{-p})$. Thus we will write $(\mathcal{S}_{-p})$, in the sequel, to denote the space $(\mathcal{S}_p)^*$. Note also that $(\mathcal{S})^*$ is the dual space of $(\mathcal{S})$. We will note $\ll , \gg$ the duality bracket between $(\mathcal{S})^*$ and $(\mathcal{S})$. If Φ belongs to (L^2) then we have the equality $\ll\Phi,\varphi\gg \ = <\Phi,\varphi>_{(L^2)} = \mathbb{E}[\Phi\ \varphi]$. Furthermore, as one can check, the family $(|f|_p)_{p\in\mathbb{Z}}$ is an increasing sequence for every f in $\mathscr{S}(\mathbb{R})$. Thus the family $(||<.,f>||_p)_{p\in\mathbb{Z}}$ is an increasing sequence for every f in $\mathscr{S}(\mathbb{R})$.

Remark 2.3. *A consequence of the previous subsection is that for every element $f := \sum_{n=0}^{+\infty} a_n e_n$ in $\mathscr{S}'(\mathbb{R})$ where $(a_n)_{n\in\mathbb{N}}$ belongs to $\mathbb{R}^{\mathbb{N}}$, there exists p_0 in $\mathbb{N}$ such that f belongs to $\mathscr{L}_{p_0}(\mathbb{R})$. Moreover if we define $\Phi := \sum_{n=0}^{+\infty} a_n <.,e_n>$, then Φ belongs to $(\mathcal{S}_{-p_0}) \subset (\mathcal{S})^*$ and we have $||\Phi||^2_{-p_0} = |f|^2_{-p_0} = \sum_{n=0}^{+\infty} \frac{a_n^2}{(2n+2)^{2p_0}} < +\infty$. Conversely, every element Φ, written as $\Phi := \sum_{n=0}^{+\infty} b_n <.,e_n>$ where $(b_n)_{n\in\mathbb{N}}$ belongs to $\mathbb{R}^{\mathbb{N}}$, belongs to $(\mathcal{S})^*$ if and only if there exists an integer p_0 such that $\sum_{n=0}^{+\infty} \frac{b_n^2}{(2n+2)^{2p_0}} < +\infty$. In this case the element $f := \sum_{n=0}^{+\infty} b_n e_n$ belongs to $\mathscr{L}_{p_0}(\mathbb{R})$ and hence is a tempered distribution which verifies $|f|^2_{-p_0} = ||\Phi||^2_{-p_0} = \sum_{n=0}^{+\infty} \frac{b_n^2}{(2n+2)^{2p_0}}$.*

Since we have defined a topology given by a family of norms on the space $(\mathcal{S})^*$ it is possible to define a derivative and an integral in $(\mathcal{S})^*$ (see [44] chapter 3 for more details about these notions). Let I be an interval of $\mathbb{R}$ (which may be equal to $\mathbb{R}$).

Definition 2.3 (stochastic distribution process). *A function $\Phi : I \to (\mathcal{S})^*$ is called a stochastic distribution process, or an $(\mathcal{S})^*$–process, or a Hida process.*

Definition 2.4 (derivative in $(\mathcal{S})^*$). *Let $t_0 \in I$. A stochastic distribution process $\Phi : I \to (\mathcal{S})^*$ is said to be differentiable at t_0 if the quantity $\lim_{r\to 0} r^{-1}(\Phi(t_0+r) - \Phi(t_0))$ exists in $(\mathcal{S})^*$. We note $\frac{d\Phi}{dt}(t_0)$ the $(\mathcal{S})^*$-derivative at t_0 of the stochastic distribution process Φ. Φ is said to be differentiable over I if it is differentiable at t_0 for every t_0 in I.*

The process Φ is said to be continuous, $C^1, \cdots, C^k, \cdots$ in $(\mathcal{S})^*$ if the $(\mathcal{S})^*$-valued function Φ is, continuous, $C^1, \cdots, C^k, \cdots$. We also say that the stochastic distribution process Φ is $(\mathcal{S})^*$-continuous and so on. It is also possible to define an $(\mathcal{S})^*$-valued integral in the following way ([52, 44]). We first recall that $L^1(\mathbb{R}, dt)$ denotes the set of measurable complex-valued functions defined on $\mathbb{R}$ such that $||f||_{L^1(\mathbb{R})} := \int_{\mathbb{R}} |f(t)|\ dt < +\infty$.

Theorem-Definition 2.1 (integral in $(\mathcal{S})^*$). *Assume that $\Phi : \mathbb{R} \to (\mathcal{S})^*$ is weakly in $L^1(\mathbb{R}, dt)$, i.e assume that for all φ in $(\mathcal{S})$, the mapping $u \mapsto \ll \Phi(u), \ \varphi\gg$ from $\mathbb{R}$ to $\mathbb{R}$ belongs to $L^1(\mathbb{R}, dt)$. Then there exists an unique element in $(\mathcal{S})^*$, noted $\int_{\mathbb{R}} \Phi(u)du$ such that*

$$\ll \int_{\mathbb{R}} \Phi(u)du, \varphi \gg = \int_{\mathbb{R}} \ll \Phi(u), \varphi \gg \ du \quad \textit{for all } \varphi \textit{ in } (\mathcal{S}). \tag{2.9}$$

We say in this case that Φ is $(\mathcal{S})^*$-integrable on $\mathbb{R}$ in the *Pettis sense*. In the sequel, when we do not specify a name for the integral of an $(\mathcal{S})^*$-integrable process Φ on $\mathbb{R}$, we always refer to the integral of Φ in Pettis' sense. See [52] p.245-246 or [44] def. 3.7.1 p.77 for more details.

2.4 S-transform and Wick product

For η in $\mathscr{S}(\mathbb{R})$, the *Wick exponential* of $< ., \eta >$, denoted $: e^{<.,\eta>} :$, is defined as the element of $(\mathcal{S})$ given by $: e^{<.,\eta>} : \overset{\text{def}}{=} \sum_{k=0}^{+\infty} k!^{-1}\, I_k(\eta^{\otimes k})$ (equality in (L^2)). More generally, for $f \in L^2(\mathbb{R})$, we define $: e^{<.,f>} :$ as the (L^2) random variable equal to $e^{<.,f>-\frac{1}{2}|f|_0^2}$ (see [47] theorem 3.33). We will sometimes note $\exp^{\diamond} < ., f >$ instead of $: e^{<.,f>} :$. This random variable belongs to $L^p(\Omega, \mu)$ for every integer $p \geqslant 1$. We now recall the definition of the S-transform of an element Φ of $(\mathcal{S}^*)$, noted $S(\Phi)$ or $S[\Phi]$. $S(\Phi)$ is defined as the function from $\mathscr{S}(\mathbb{R})$ to $\mathbb{R}$ given by

$$\forall \eta \in \mathscr{S}(\mathbb{R}), \qquad S(\Phi)(\eta) := \ll \Phi, : e^{<.,\eta>} : \gg . \tag{2.10}$$

Note that $S\Phi(\eta)$ is nothing but $\mathbb{E}[\Phi : e^{<.,\eta>} :] = e^{-\frac{1}{2}|\eta|_0^2}\, \mathbb{E}[\Phi\, e^{<.,\eta>}]$ when Φ belongs to (L^2). Following [9], formula (6) and (7), define for η in $\mathscr{S}(\mathbb{R})$ the probability measure $\mathbb{Q}_\eta$ on the space $(\Omega, \mathcal{F})$ by its Radon-Nikodym derivative given by $\frac{d\mathbb{Q}_\eta}{d\mu} \overset{\text{def}}{=} : e^{<.,\eta>} :$. The probability measures $\mathbb{Q}_\eta$ and μ are equivalent. Then, by definition,

$$\forall \Phi \in (L^2), \qquad S(\Phi)(\eta) = \mathbb{E}_{\mathbb{Q}_\eta}[\Phi]. \tag{2.11}$$

Lemma 2.4. *1. Let p be a positive integer and Φ be an element of $(\mathcal{S}_{-p})$. Then*

$$|S(\Phi)(\eta)| \leqslant ||\Phi||_{-p}\, e^{\frac{1}{2}|\eta|_p^2}, \text{ for any } \eta \text{ in } \mathscr{S}(\mathbb{R}). \tag{2.12}$$

2. Let $\Phi := \sum_{k=0}^{+\infty} a_k < ., e_k >$ belong to $(\mathcal{S})^*$. The following equality holds for every η in $\mathscr{S}(\mathbb{R})$:

$$S(\Phi)(\eta) = \sum_{k=0}^{+\infty} a_k < \eta, e_k >_{L^2(\mathbb{R})}. \tag{2.13}$$

Proof. Item 1 is proved in [52] p.79. Item 2 is an easy calculation left to the reader. □

Another useful tool in white noise analysis is the Wick product:

Theorem-Definition 2.2 ([52] p.92). *For every $(\Phi, \Psi) \in (\mathcal{S})^* \times (\mathcal{S})^*$, there exists a unique element of $(\mathcal{S})^*$, called the Wick product of Φ and Ψ and noted $\Phi \diamond \Psi$, such that, for every η in $\mathscr{S}(\mathbb{R})$,*

$$S(\Phi \diamond \Psi)(\eta) = S(\Phi)(\eta)\; S(\Psi)(\eta). \tag{2.14}$$

Lemma 2.5. *For any $(p, q) \in \mathbb{N}^2$, $X \in (\mathcal{S}_{-p})$ and $Y \in (\mathcal{S}_{-q})$,*

$$|S(X \diamond Y)(\eta)| \leqslant ||X||_{-p}\, ||Y||_{-q}\, e^{|\eta|^2_{\max\{p;q\}}}. \tag{2.15}$$

Proof. The proof is easy since, for every η in $\mathscr{S}(\mathbb{R})$, using lemma 2.4,
$|S(X \diamond Y)(\eta)| = |S(X)(\eta)|\; |S(Y)(\eta)| \leqslant ||X||_{-p}\, e^{\frac{1}{2}|\eta|_p^2}\, ||Y||_{-q}\, e^{\frac{1}{2}|\eta|_q^2} \leqslant ||X||_{-p}\, ||Y||_{-q}\, e^{|\eta|^2_{\max\{p;q\}}}$. □

For any Φ in $(\mathcal{S})^*$ and k in $\mathbb{N}^*$ let $\Phi^{\diamond k}$ denote the element $\overbrace{\Phi \diamond \cdots \diamond \Phi}^{k \text{ times}}$ of $(\mathcal{S})^*$. We can generalize the definition of $\exp^{\diamond}$ to the case where Φ belongs to $(\mathcal{S})^*$:

Definition 2.5. *For any Φ in $(\mathcal{S})^*$ such that the sum $\sum_{k=0}^{+\infty} \frac{\Phi^{\diamond k}}{k!}$ converges in $(\mathcal{S})^*$, define the element $\exp^{\diamond} \Phi$ of $(\mathcal{S})^*$ by $\exp^{\diamond} \Phi := \sum_{k=0}^{+\infty} \frac{\Phi^{\diamond k}}{k!}$.*

For f in $L^2(\mathbb{R})$ and $\Phi := < ., f >$, it is easy to verify that $\exp^{\diamond} \Phi$ given by definition 2.5 exists and coincide with $: e^{<.,f>} :$ defined at the beginning of this section.

Remark 2.6. *If Φ is deterministic then, for all Ψ in $(\mathcal{S})^*$, $\Phi \diamond \Psi = \Phi\Psi$. Moreover, let $(X_t)_{t\in\mathbb{R}}$ be a Gaussian process and let $\mathcal{H}$ be the subspace of (L^2) defined by $\mathcal{H} := \overline{vect_{\mathbb{R}}\{X_t; t \in \mathbb{R}\}}^{(L^2)}$. If X and Y are two elements of $\mathcal{H}$ then $X \diamond Y = XY - \mathbb{E}[XY]$.*

We refer to [47] chapters 3 and 16 for more details about Wick product. The following results on the S-transform will be used in the sequel. See [52] p.39 and [43] p.280-281 for proofs.

Lemma 2.7. *The S-transform verifies the following properties:*
(i) The map $S : \Phi \mapsto S(\Phi)$, from $(\mathcal{S})^*$ into $\mathcal{F}_{unction}(\mathscr{S}(\mathbb{R});\mathbb{R})$, is injective.
(ii) Let $\Phi : \mathbb{R} \to (\mathcal{S})^*$ be an $(\mathcal{S})^*$ process. If Φ is $(\mathcal{S})^*$-integrable over $\mathbb{R}$ then $S(\int_{\mathbb{R}} \Phi(u)\, du)(\eta) = \int_{\mathbb{R}} S(\Phi(u))(\eta)\, du$ *, for all η in $\mathscr{S}(\mathbb{R})$.*
(iii) Let $\Phi : \mathbb{R} \to (\mathcal{S})^*$ be an $(\mathcal{S})^*$-process differentiable at t. Then, for every η in $\mathscr{S}(\mathbb{R})$ the map $u \mapsto [S\Phi(u)](\eta)$ is differentiable at t and verifies $S[\frac{d\Phi}{dt}(t)](\eta) = \frac{d}{dt}\Big[S[\Phi(t)](\eta)\Big]$.

It is useful to have a criterion for integrability in $(\mathcal{S})^*$ in term of the S-transform. This is the topic of the next theorem (theorem 13.5 in [52]).

Theorem 2.8. *Let $\Phi : \mathbb{R} \to (\mathcal{S})^*$ be a stochastic distribution process satisfying:*
(i) The map $t \mapsto S[\Phi(t)](\eta)$, from $\mathbb{R}$ to $\mathbb{R}$, is measurable for all η in $\mathscr{S}(\mathbb{R})$.
(ii) There is a natural integer p, a real a and a function L in $L^1(\mathbb{R}, dt)$ such that for all η in $\mathscr{S}(\mathbb{R})$, $|S(\Phi(t))(\eta)| \leqslant L(t)\, e^{a|\eta|_p^2}$.
Then Φ is $(\mathcal{S})^*$-integrable over $\mathbb{R}$.

Lastly, when the stochastic distribution process is an (L^2)-valued process, the following result holds (see [9]):

Theorem 2.9. *Let $X : \mathbb{R} \to (L^2)$ be such that the function $t \mapsto S(X_t)(\eta)$ is measurable for all η in $\mathscr{S}(\mathbb{R})$ and that $t \mapsto ||X_t||_0$ is in $L^1(\mathbb{R}, dt)$. Then X is $(\mathcal{S})^*$-integrable over $\mathbb{R}$ and*

$$\Big|\Big|\int_{\mathbb{R}} X_t\, dt\Big|\Big|_0 \leqslant \int_{\mathbb{R}} ||X_t||_0\, dt.$$

3 The operators M_H and their derivatives

3.1 Study of M_H

Let us fix some notations. We will still note $\widehat{u}$ or $\mathcal{F}(u)$ the Fourier transform of a tempered distribution u and we let $L^1_{loc}(\mathbb{R})$ denote the set of measurable functions which are locally integrable on $\mathbb{R}$. We also identify, here and in the sequel, any function f of $L^1_{loc}(\mathbb{R})$ with its associated distribution, also noted T_f. We will say that a tempered distribution v is of function type if there exists a locally integrable function f such that $v = T_f$ (in particular, $< v, \phi >= \int_{\mathbb{R}} f(t)\, \phi(t)\, dt$ for ϕ in $\mathscr{S}(\mathbb{R})$).
Let $H \in (0, 1)$. Following [34], we want to define an operator, denoted M_H, which is specified in the Fourier domain by

$$\widehat{M_H(u)}(y) := \tfrac{\sqrt{2\pi}}{c_H}\, |y|^{1/2-H}\widehat{u}(y), \qquad y \in \mathbb{R}^*. \tag{3.1}$$

This operator is well defined on the homogeneous Sobolev space of order $1/2 - H$, $L^2_H(\mathbb{R})$:

$$L^2_H(\mathbb{R}) := \{u \in \mathscr{S}'(\mathbb{R}) \ : \ \widehat{u} = T_f;\ f \in L^1_{loc}(\mathbb{R}) \text{ and } ||u||_H < +\infty\}, \tag{3.2}$$

where $||u||^2{}_H := \frac{1}{c_H^2} \int_{\mathbb{R}} |\xi|^{1-2H} |\widehat{u}\,(\xi)|^2 d\xi$ derives from the inner product on $L^2_H(\mathbb{R})$, defined by:

$$< u, v >_H := \frac{1}{c_H^2} \int_{\mathbb{R}} |\xi|^{1-2H}\widehat{u}\,(\xi)\overline{\widehat{v}\,(\xi)}d\xi, \tag{3.3}$$

and c_H has been defined right after formula (1.4) (the normalization constant $\frac{\sqrt{2\pi}}{c_H}$ will be explained in remark 3.5). It is well known - see [20] p.13 for example - that $(L^2_H(\mathbb{R}), <,>_H)$ is a Hilbert space. The nature of the spaces $L^2_H(\mathbb{R})$ when H spans $(0, 1)$ is described in the following lemma, the proof of which can be found in [20] p15, theorem 1.4.1 and corollary 1.4.1.

Lemma 3.1. *If H is in $(0, 1/2]$, the space $L^2_H(\mathbb{R})$ is continuously embedded in $L^{1/H}(\mathbb{R})$. When H is in $[1/2, 1)$, the space $L^{1/H}(\mathbb{R})$ is continuously embedded in $L^2_H(\mathbb{R})$.*

Since $\widehat{M_H(u)}$ belongs to $L^2(\mathbb{R})$ for every u in $L^2_H(\mathbb{R})$, M_H is well defined as its inverse Fourier transform, *i.e.*:

$$M_H(u)(x) := \frac{1}{2\pi}\mathcal{F}\left(\widehat{M_H(u)}\right)(-x), \qquad \text{for almost every } x \text{ in } \mathbb{R}. \tag{3.4}$$

The following proposition is obvious in view of the definition of M_H:

Proposition 3.2. *M_H is an isometry from $(L^2_H(\mathbb{R}), <,>_H)$ to $(L^2(\mathbb{R}), <,>_{L^2(\mathbb{R})})$.*

Let $\mathcal{E}(\mathbb{R})$ denote the space of simple functions on $\mathbb{R}$, which is the set of all finite linear combinations of functions $\mathbb{1}_{[a,b]}(.)$ with a and b in $\mathbb{R}$. It is easy to check that both $\mathscr{S}(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$ are subsets of $L^2_H(\mathbb{R})$.
It will be useful in the sequel to have an explicit expression for $M_H(f)$ when f is in $\mathscr{S}(\mathbb{R})$ or in $\mathcal{E}(\mathbb{R})$. To compute this value, one may use the formulas for the Fourier transform of the distributions $|\ |^{\alpha}$, α in $(-1,1)$, given for instance in [21] (chapter 1, § 3). This yields, for almost every x in $\mathbb{R}$,

$$M_H(\mathbb{1}_{[a,b]})(x) = \frac{\sqrt{2\pi}}{2c_H\Gamma(H+1/2)\cos(\frac{\pi}{2}(H-1/2))}\left[\frac{b-x}{|b-x|^{3/2-H}} - \frac{a-x}{|a-x|^{3/2-H}}\right]. \tag{3.5}$$

By the same method, for f in $\mathscr{S}(\mathbb{R})$ one gets, for almost every real x:

$$M_H(f)(x) = \gamma_H < |y|^{-(3/2-H)}, f(x+y) > \quad \text{for almost every real } x, \tag{3.6}$$

with $\gamma_H := \frac{\sqrt{2\pi}}{2c_H\Gamma(H-1/2)\cos(\frac{\pi}{2}(H-1/2))} = \frac{\left(\Gamma(2H+1)\sin(\pi H)\right)^{\frac{1}{2}}}{2\Gamma(H-1/2)\cos(\frac{\pi}{2}(H-1/2))}$ and where we have written, by abuse of notation, $|y|^{-(3/2-H)}$ for the tempered distribution $y \mapsto |y|^{-(3/2-H)}$ and $f(x+y)$ for the map $y \mapsto f(x+y)$.
Note moreover the following useful equality when f belongs to $\mathscr{S}(\mathbb{R})$, given in [34](up to a constant), for almost every x in $\mathbb{R}$,

$$M_H(f)(x) = \alpha_H \frac{d}{dx}\left[\int_{\mathbb{R}}(t-x)|t-x|^{H-3/2}f(t)dt\right] \tag{3.7}$$

where $\alpha_H := -\gamma_H(H-1/2)^{-1} = -\sqrt{2\pi}\left(2c_H\Gamma(H+1/2)\cos(\frac{\pi}{2}(H-1/2))\right)^{-1}$.
In order to extend the Wiener integral with respect to fBm to an integral with respect to mBm (in section 4.2) we will need the following equality:

Proposition 3.3. $\overline{\mathcal{E}(\mathbb{R})}^{<,>_H} = L^2_H(\mathbb{R})$.

This is a straightforward consequence of the following lemma:

Lemma 3.4. *Let $\sigma : \mathbb{R} \to \mathbb{C}$ be a measurable function, continuous on $\mathbb{R}^*$, such that $|\sigma|^2$ is locally integrable at 0 and that $x \mapsto \left|\frac{\sigma(x)}{x}\right|^2$ is locally integrable at $+\infty$. Define $L^2_\sigma(\mathbb{R}) := \{u \in \mathscr{S}'(\mathbb{R}) : \widehat{u} = T_f;\ f \in L^1_{loc}(\mathbb{R})$ such that $||u||_\sigma < +\infty\}$ where $< u,v >_\sigma := \int_{\mathbb{R}} |\sigma(\xi)|^2\ \widehat{u}(\xi)\ \overline{\widehat{v}\ (\xi)}\ d\xi$. If $\mathcal{E}(\mathbb{R}) \subset L^2_\sigma(\mathbb{R})$, define $\overline{\mathcal{E}(\mathbb{R})}^{<,>_\sigma}$ as the completion of $\mathcal{E}(\mathbb{R})$ for the norm $||\ ||_\sigma$. Then, the space $(L^2_\sigma(\mathbb{R}), <,>_\sigma)$ is a Hilbert space which also verifies $\overline{\mathcal{E}(\mathbb{R})}^{<,>_\sigma} = L^2_\sigma(\mathbb{R})$.*

Proof. The fact that $(L^2_\sigma(\mathbb{R}), <,>_\sigma)$ is a Hilbert space is obvious. One needs only to show that the orthogonal space of $\mathcal{E}(\mathbb{R})$ for the norm $||\ ||_\sigma$ is equal to $\{0_{\mathcal{E}(\mathbb{R})}\}$. Let u in $L^2_\sigma(\mathbb{R})$ be such that $< u,v >_\sigma = 0$ for all v in $\mathcal{E}(\mathbb{R})$. In particular, for all t in $\mathbb{R}$, $\int_{\mathbb{R}} |\sigma(\xi)|^2\ \widehat{u}(\xi)\ \overline{\widehat{\mathbb{1}_{[0,t]}}(\xi)}\ d\xi = 0$. For all ψ in $\mathscr{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} \psi'(t)\Big(\int_{\mathbb{R}} |\sigma(\xi)|^2\ \widehat{u}(\xi)\ \overline{\widehat{\mathbb{1}_{[0,t]}}(\xi)}d\xi\Big)dt = 0,$$

where ψ' denotes the derivative of ψ. Thanks to the assumptions on $|\sigma|^2$ and $x \mapsto \left|\frac{\sigma(x)}{x}\right|^2$, Fubini theorem applies. Moreover, an integration by parts yields

$$0 = -\int_{\mathbb{R}} \frac{|\sigma(\xi)|^2}{i\xi}\ \widehat{u}(\xi)\Big(\int_{\mathbb{R}} \psi'(t)(1-e^{i\xi t})dt\Big)d\xi = \int_{\mathbb{R}} |\sigma(\xi)|^2\ \widehat{u}(\xi)\ \overline{\widehat{\psi}(\xi)}\ d\xi.$$

Thus $< |\sigma|^2\widehat{u}, \psi >= 0$ for all ψ in $\mathscr{S}(\mathbb{R})$. Since $\xi \mapsto |\sigma(\xi)|^2\widehat{u}(\xi)$ belongs to $L^1_{loc}(\mathbb{R})$, it is easy to deduce that u is equal to 0. □

Remark 3.5. *1. Because the space $\mathscr{S}(\mathbb{R})$ is dense in $L_H^2(\mathbb{R})$ for the norm $\|\ \|_H$ (see [20] p.13), it is also possible to define the operator M_H on the space $\mathscr{S}(\mathbb{R})$ and extend it, by isometry, to all elements of $L_H^2(\mathbb{R})$. This is the approach of [34] and [11] (with a different normalization constant). This clearly yields the same operator as the one defined by* (3.1)*. However this approach does not lend itself to an extension to the case where the constant H is replaced by a function h, which is what we need for mBm.*
2. For the same reasons as in 1. it is possible to define the operator M_H on the space $\mathcal{E}(\mathbb{R})$ and extend it, by isometry, to all elements of $L_H^2(\mathbb{R})$. Again, this extension coincide with (3.1)*. We will use this idea in section 4.2.*

In view of (3.3), we find that $< \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} >_H = \frac{1}{c_H^2} \int_{\mathbb{R}} \frac{(e^{it\xi}-1)(e^{-is\xi}-1)}{|\xi|^{2H+1}} d\xi = R_H(t,s)$. Thus, as in the case of standard Brownian motion, one deduces that the process $(\widetilde{B}^{(H)}(t))_{t\in\mathbb{R}}$, defined for all (t,ω) in $\mathbb{R}\times\Omega$ by:

$$\widetilde{B}^{(H)}(t)(\omega) := \widetilde{B}^{(H)}(t,\omega) := < \omega, M_H(\mathbb{1}_{[0,t]}) >, \tag{3.8}$$

is a Gaussian process which admits, as the next computation shows, a continuous version noted $B^{(H)} := (B^{(H)}(t))_{t\in\mathbb{R}}$. Indeed, under the probability measure μ, the process $B^{(H)}$ is a fractional Brownian motion since we have, using (3.9) and proposition 3.2,

$$\mathbb{E}[B^{(H)}(t)B^{(H)}(s)] = \mathbb{E}[< ., M_H(\mathbb{1}_{[0,t]}) >< ., M_H(\mathbb{1}_{[0,s]}) >] = < \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} >_H = R_H(t,s). \tag{3.9}$$

Remark 3.6. *The reason of the presence of the constant $\frac{\sqrt{2\pi}}{c_H}$ in formula* (3.1) *is now clear since this constant ensures that, for all H in $(0,1)$, the process $B^{(H)}$ defined by* (3.8) *is a normalized fBm.*

Because our operator M_H is defined on a distribution space, we can not apply the considerations of [34] p.323ff about the links between the operator M_H and Riesz potential operator. However it is crucial for our purpose that M_H is bijective from $L_H^2(\mathbb{R})$ into $L^2(\mathbb{R})$:

Theorem 3.7 (properties of M_H).

1. *For all H in $(0,1)$, the operator M_H is bijective from $L_H^2(\mathbb{R})$ into $L^2(\mathbb{R})$.*
2. *For all H in $(0,1)$ and (f,g) in $(L^2(\mathbb{R})\cap L_H^2(\mathbb{R}))^2$, we have $< f, M_H(g) >_{L^2(\mathbb{R})} = < M_H(f), g >_{L^2(\mathbb{R})}$. Moreover The last equality remains true when f belongs to $L^1_{loc}(\mathbb{R})\cap L_H^2(\mathbb{R})$ and g belongs to $\mathscr{S}(\mathbb{R})$ (in this case this equality reads $< f, M_H(g) > = < M_H(f), g >_{L^2(\mathbb{R})}$, where $<,>$ denotes the duality bracket between $\mathscr{S}'(\mathbb{R})$ and $\mathscr{S}(\mathbb{R})$).*
3. *There exists a constant D such that, for every couple (H,k) in $(0,1)\times\mathbb{N}^*$, $\max_{x\in\mathbb{R}} |M_H(e_k)(x)| \leqslant \frac{D}{c_H}\,(k+1)^{2/3}$.*

Proof. *1.* Since M_H is an isometry, we just have to establish the surjectivity of M_H, for all H in $(0,1)$. The case $H = 1/2$ being obvious, let us fix H in $(0,1)\backslash\{1/2\}$, g in $L^2(\mathbb{R})$ and define the complex-valued function w_H^g on $\mathbb{R}$ by $w_H^g(\xi) = \frac{c_H}{\sqrt{2\pi}}|\xi|^{H-1/2}\hat{g}(\xi)$ if ξ belongs to $\mathbb{R}^*$ and $w_H^g(0) := 0$. Define the tempered distribution v_H^g by $v_H^g := \frac{1}{2\pi}\check{\widehat{w_H^g}}$, where for all tempered distribution T, by definition, $< \check{T}, f > := < T, \check{f} >$ for all functions f in $\mathscr{S}(\mathbb{R})$ and where $\check{f}(x) = f(-x)$ for all x. We shall prove that v_H^g belongs to $L_H^2(\mathbb{R})$ and that $M_H(v_H^g) = g$. Note first that for all u in $\mathscr{S}'(\mathbb{R})$, $\widehat{\widehat{u}} = 2\pi\check{u}$. It is clear that $\widehat{v_H^g} = w_H^g$ and that $\widehat{v_H^g}$ belongs to $L^1_{\text{loc}}(\mathbb{R})$. Moreover, thanks to formula (3.3), we see that

$$\|v_H^g\|_H^2 = \frac{1}{c_H{}^2}\int_{\mathbb{R}} |\xi|^{1-2H}|\widehat{v_H^g}(\xi)|^2 d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{g}(\xi)|^2 d\xi = \|g\|_{L^2(\mathbb{R})}^2 < +\infty.$$

This shows that v_H^g belongs to $L_H^2(\mathbb{R})$. We can then compute $\widehat{M_H(v_H^g)}$ and obtain, for almost every ξ in $\mathbb{R}$, $\widehat{M_H(v_H^g)}(\xi) = \frac{\sqrt{2\pi}}{c_H}|\xi|^{1/2-H}\widehat{v_H^g}(\xi) = \frac{\sqrt{2\pi}}{c_H}|\xi|^{1/2-H}w_H^g(\xi) = \widehat{g}(\xi)$. This shows that $M_H(v_H^g)$ is equal to g in $L^2(\mathbb{R})$ and then establish the surjectivity of M_H.

2. See equality (3.12) of [11]. The case where f belongs to $L^1_{loc}(\mathbb{R}) \cap L^2_H(\mathbb{R})$ is obvious, in view of 2 of theorem 3.7, using the density of $\mathscr{S}(\mathbb{R})$ in $L^2_H(\mathbb{R})$.
3. is shown in lemma 4.1 of [34]. □

Of course if we just consider functions in $L^2_H(\mathbb{R})$ instead of all elements of $L^2_H(\mathbb{R})$, the map M_H is not bijective any more.

3.2 Study of $\frac{\partial M_H}{\partial H}$

We now study the operator $\frac{\partial M_H}{\partial H}$. It will prove instrumental in defining the integral with respect to mBm in section 5. Heuristically, we wish to differentiate with respect to H the expression in definition (3.1), *i.e.* differentiate the map $H \mapsto \widehat{M_H(u)}(y)$ on $(0,1)$ for (u,y) in $L^2_H(\mathbb{R}) \times \mathbb{R}^*$, assuming this is possible. By doing so, we define a new operator, denoted $\frac{\partial M_H}{\partial H}$, from a certain subset of $L^2_H(\mathbb{R})$ to $L^2(\mathbb{R})$. Of course, in order to compute the derivative at H_0 of $H \mapsto \widehat{M_H(u)}(y)$, we need to consider a neighbourhood V_{H_0} of H_0 in $(0,1)$ and thus consider only elements u which belong to $\bigcap_{H \in V_{H_0}} L^2_H(\mathbb{R})$. However, as will become apparent, the formula giving the derivative makes sense without this restriction.
In order to define in a rigorous manner the operator $\frac{\partial M_H}{\partial H}$, we shall proceed in a way analogous to the one that allowed to define M_H in the previous subsection. It will be shown in remark 3.9 that this construction effectively defines the derivative, in a certain sense, of the operator M_H.
We will note c'_H the derivative of the analytic map $H \mapsto c_H$ where c_H has been defined in (1.4) and set $\beta_H := \frac{c'_H}{c_H}$. Let H belong to $(0,1)$. Define:

$$\Gamma_H(\mathbb{R}) = \{u \in \mathscr{S}'(\mathbb{R}) \ : \ \widehat{u} = T_f; \ \ f \in L^1_{loc}(\mathbb{R}) \ \text{ and } ||u||_{\delta_H(\mathbb{R})} < +\infty\}, \tag{3.10}$$

where the norm $|| \ ||_{\delta_H(\mathbb{R})}$ derives from the inner product on $\Gamma_H(\mathbb{R})$ defined by

$$< u, v >_{\delta_H} := \frac{1}{c_H^2} \int_{\mathbb{R}} (\beta_H + \ln|\xi|)^2 \, |\xi|^{1-2H} \, \widehat{u}\,(\xi) \, \overline{\widehat{v}\,(\xi)} \, d\xi. \tag{3.11}$$

By slightly adapting lemma 3.4, it is easy to check that $(\Gamma_H(\mathbb{R}), <,>_{\delta_H(\mathbb{R})})$ is a Hilbert space which verifies the equality $\Gamma_H(\mathbb{R}) = \overline{\mathscr{S}(\mathbb{R})}^{<,>_{\delta_H}} = \overline{\mathcal{E}(\mathbb{R})}^{<,>_{\delta_H}}$. Note furthermore that, for every H in $(0,1)$, the inclusion $\Gamma_H(\mathbb{R}) \subset L^2_H(\mathbb{R})$ holds. We may now define the operator $\frac{\partial M_H}{\partial H}$ from $(\Gamma_H(\mathbb{R}), <,>_{\delta_H(\mathbb{R})})$ to $(L^2(\mathbb{R}), <,>_{L^2(\mathbb{R})})$, in the Fourier domain, by:

$$\widehat{\tfrac{\partial M_H}{\partial H}(u)}(y) := -(\beta_H + \ln|y|) \, \tfrac{\sqrt{2\pi}}{c_H} \, |y|^{1/2-H} \, \widehat{u}(y), \quad \text{for every } \ y \text{ in } \mathbb{R}^*. \tag{3.12}$$

In particular, one can check that, for f in $\mathscr{S}(\mathbb{R})$, $\widehat{\frac{\partial M_H}{\partial H}(f)}(y) = \frac{\partial}{\partial H}\widehat{M_H(f)(y)}$ for almost every real y. Since $\widehat{\frac{\partial M_H}{\partial H}(u)}$ belongs to $L^2(\mathbb{R})$ for every u in $\Gamma_H(\mathbb{R})$, $\frac{\partial M_H}{\partial H}$ is well defined and given by its inverse Fourier transform from $(\Gamma_H(\mathbb{R}), <,>_{\delta_H(\mathbb{R})})$ to $(L^2(\mathbb{R}), <,>_{L^2(\mathbb{R})})$:

$$\frac{\partial M_H}{\partial H}(u)(x) = \frac{1}{2\pi}\mathcal{F}\left(\widehat{\frac{\partial M_H(u)}{\partial H}}\right)(-x), \qquad \text{for almost every } x \text{ in } \mathbb{R}.$$

As in the previous subsection it will be useful to compute $\frac{\partial M_H}{\partial H}(f)$ for f in $\mathscr{S}(\mathbb{R})$. We summarize, in following proposition, the main properties of $\frac{\partial M_H}{\partial H}$.

Proposition 3.8. $\frac{\partial M_H}{\partial H}$ *is an isometry from* $(\Gamma_H(\mathbb{R}), <,>_{\delta_H(\mathbb{R})})$ *to* $(L^2(\mathbb{R}), <,>_{L^2(\mathbb{R})})$ *which verifies:*

$$\forall f \ \in \ \Gamma_H(\mathbb{R}), \qquad ||f||_{\delta_H} = ||\tfrac{\partial M_H}{\partial H}(f)||_{L^2(\mathbb{R})}, \tag{3.13}$$

$$\forall (f,g) \ \in \ (\Gamma_H(\mathbb{R}) \cap L^2(\mathbb{R}))^2, \qquad < \tfrac{\partial M_H}{\partial H}(f), g >_{L^2(\mathbb{R})} = < f, \tfrac{\partial M_H}{\partial H}(g) >_{L^2(\mathbb{R})}, \tag{3.14}$$

$$\forall f \ \in \ \mathscr{S}(\mathbb{R}) \cup \mathcal{E}(\mathbb{R}), \ \ \text{and for } \ \text{a.e. } x \in \mathbb{R}, \qquad \tfrac{\partial M_H}{\partial H}(f)(x) = \tfrac{\partial}{\partial H}[M_H(f)(x)]. \tag{3.15}$$

Proof. Equality (3.13) results immediately from the definition of $\frac{\partial M_H}{\partial H}$ and from (3.12). For any couple of functions (f,g) in $(\Gamma_H(\mathbb{R})\cap L^2(\mathbb{R}))^2$,

$$\begin{aligned} <\frac{\partial M_H}{\partial H}(f),g>_{L^2(\mathbb{R})} &= \frac{1}{2\pi}<\widehat{\frac{\partial M_H}{\partial H}(f)},\widehat{g}>_{L^2(\mathbb{R})} = \frac{1}{2\pi}\int_{\mathbb{R}} -(\beta_H+\ln|y|)\,\frac{\sqrt{2\pi}}{c_H}\,|y|^{1/2-H}\,\widehat{f}(y)\overline{\widehat{g}(y)}dy \\ &= \frac{1}{2\pi}<\widehat{f},\widehat{\frac{\partial M_H}{\partial H}(g)}>_{L^2(\mathbb{R})} = <f,\frac{\partial M_H}{\partial H}(g)>_{L^2(\mathbb{R})}. \end{aligned}$$

It just remains to prove (3.15). Since we will not use (3.15) in the sequel for f in $\mathcal{E}(\mathbb{R})$, we will just establish it here on $\mathscr{S}(\mathbb{R})$. Let f be in $\mathscr{S}(\mathbb{R})$ and H in $(0,1)$. Thanks to 3.6 and using theorem 5.17 which is given in subsection 5.5 below, we may write

$$\frac{\partial}{\partial H}[M_H(f)(x)] = \gamma'_H < |y|^{-(3/2-H)}, f(x+y)> \quad +\gamma_H <|y|^{-(3/2-H)}\ln|y|, f(x+y)>. \tag{3.16}$$

Furthermore, for almost every real x, we may write:

$$\begin{aligned} \frac{\partial M_H}{\partial H}(f)(x) = \frac{1}{2\pi}\widehat{\widehat{\frac{\partial M_H}{\partial H}(f)}}(-x) &= \frac{1}{2\pi}\mathcal{F}(y\mapsto -(\beta_H+\ln|y|)\frac{\sqrt{2\pi}}{c_H}|y|^{(1/2-H)}\widehat{f}(y))(-x) \\ &= -\beta_H M_H(f)(x) - \frac{1}{c_H\sqrt{2\pi}}\mathcal{F}(y\mapsto |y|^{(1/2-H)}\ln|y|\widehat{f}(y))(-x). \end{aligned}$$

Define, for every real x, $I(-x) := \mathcal{F}(y\mapsto |y|^{(1/2-H)}\ln|y|\widehat{f}(y))(-x)$ and, for every H in $(0,1)$, $\nu_H := 2\,\Gamma'(3/2-H)\sin(\frac{\pi}{2}(1/2-H)) + \pi\Gamma(3/2-H)\cos(\frac{\pi}{2}(1/2-H))$.
Using [21, p.173 – 174] we get, after some computations, $I(-x) = (-c_H\sqrt{2\pi})\gamma_H <|y|^{-(3/2-H)}\ln|y|, f(x+y)> -\nu_H <|y|^{-(3/2-H)}, f(x+y)>$. We finally have, for almost every real x,

$$\frac{\partial M_H}{\partial H}(f)(x) = (\frac{-\gamma_H\, c'_H}{c_H}+\frac{\nu_H}{c_H\sqrt{2\pi}}) <|y|^{-(3/2-H)}, f(x+y)> +\gamma_H <|y|^{-(3/2-H)}\ln|y|, f(x+y)>$$

which is nothing but (3.16) since $\frac{-\gamma_H\, c'_H}{c_H}+\frac{\nu_H}{c_H\sqrt{2\pi}} = \gamma'_H$. □

Remark 3.9. *Define* $\Sigma_{H,r}(\mathbb{R}) := \bigcup_{r\in(0,\min(H,1-H))}\Gamma_H(\mathbb{R})\bigcap\left(\bigcap_{H'\in[H-r,H+r]} L^2_{H'}(\mathbb{R})\right)$. *It is possible to show that, for all H in $(0,1)$ and f in $\Sigma_{H,r}(\mathbb{R})$ that $\frac{\partial M_H}{\partial H}(f)(.)$ (resp. $\widehat{\frac{\partial M_H}{\partial H}(f)(.)}$) is the derivative, in the $L^2(\mathbb{R})$-sense, of $M_H(f)$ (resp. of $\widehat{M_H(f)}$).*

4 Wiener integral with respect to mBm on $\mathbb{R}$

4.1 Wiener integral with respect to fBm

Similarly to what is performed in [34] and [11] (in these works this is done only for functions of $L^2_H(\mathbb{R})$), it is now easy to define a Wiener integral with respect to fractional Brownian motion. Indeed, for any element g in $L^2_H(\mathbb{R})$, define $\mathcal{J}^H(g)$ as the random variable $<.,M_H(g)>$. In other words, for all couples (ω,g) in $\Omega\times L^2_H(\mathbb{R})$:

$$\mathcal{J}^H(g)(\omega) := <\omega, M_H(g)> = \int_{\mathbb{R}} M_H(g)(s)\,dB(s)(\omega), \tag{4.1}$$

where the Brownian motion B has been defined just below formula at the end of subsection 2.1. We call the random variable $\mathcal{J}^H(g)$ the Wiener integral of g with respect to fBm. Once again, when g is a tempered distribution which is not a function, $g(s)$ does not have a meaning for a fixed real s and $\mathcal{J}^H(g)$ is just a notation for the centered Gaussian random variable $<.,M_H(g)>$.

4.2 Wiener integral with respect to mBm

We now consider a fractional Brownian field $\Lambda := \big(\Lambda(t,H)\big)_{(t,H)\in\mathbb{R}\times(0,1)}$, defined, for all (t,H) in $\mathbb{R}\times(0,1)$ and all ω in Ω, by $\Lambda(t,H)(\omega) := B^{(H)}(t,\omega) := <\omega, M_H(\mathbb{1}_{[0,t]})>$. We wish to generalize the previous construction of the Wiener integral with respect to fBm to the case of mBm. This amounts to replacing the constant H by a continuous deterministic function h, ranging in $(0,1)$. More precisely, let R_h denote the covariance function of a normalized mBm with function h (see definition 1.5). Define the bilinear form $<,>_h$ on $\mathcal{E}(\mathbb{R})\times\mathcal{E}(\mathbb{R})$ by $<\mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]}>_h = R_h(t,s)$. Our construction of the integral of deterministic elements with respect to mBm requires that $<,>_h$ be an inner product:

Proposition 4.1. *$<,>_h$ is an inner product for every function h.*

Proof. See appendix X.B. □

Define the linear map $M_h : \begin{array}{ccc} (\mathcal{E}(\mathbb{R}), <,>_h) & \to & (L^2(\mathbb{R}), <,>_{L^2(\mathbb{R})}) \\ \mathbb{1}_{[0,t]} & \mapsto & M_h(\mathbb{1}_{[0,t]}) := M_{h(t)}(\mathbb{1}_{[0,t]}) := M_H(\mathbb{1}_{[0,t]})|_{H=h(t)}, \end{array}$

and the process $\widetilde{B}^{(h)}(t) = <., M_h(\mathbb{1}_{[0,t]})>$, $t\in\mathbb{R}$. As Kolmogorov's criterion and the proof of following lemma show, this process admits a continuous version which will be noted $B^{(h)}$. A word on notation: we write $B^{(\cdot)}$ both for an fBm and an mBm. This should not cause any confusion since an fBm is just an mBm with constant h function. It will be clear from the context in the following whether the "h" is constant or not. Note that a.s., forall real t, $B^{(h)}(t) = B^{(H)}(t)_{|_{H=h(t)}}$. In view of point *2.* in remark 3.5 we may state the following lemma.

Lemma 4.2. *(i) The process $B^{(h)}$ is a normalized mBm.*
(ii) The map M_h is an isometry from $(\mathcal{E}(\mathbb{R}), <,>_h)$ to $(L^2(\mathbb{R}), <,>_{L^2(\mathbb{R})})$.

Proof. The process $B^{(h)}$ is clearly a centered Gaussian process. Moreover, for all (s,t) in $\mathbb{R}^2$,

$$\begin{aligned}\mathbb{E}[B^{(h)}(t)B^{(h)}(s)] &= \mathbb{E}[<., M_h(\mathbb{1}_{[0,t]})><., M_h(\mathbb{1}_{[0,s]})>] = \frac{1}{2\pi} < \widehat{M_{h(t)}(\mathbb{1}_{[0,t]})}, \widehat{M_{h(s)}(\mathbb{1}_{[0,s]})} >_{L^2(\mathbb{R})} \\ &= \tfrac{1}{c_{h(t)}c_{h(s)}} \int_{\mathbb{R}} \frac{(1-e^{it\xi})(1-e^{-is\xi})}{|\xi|^{1+2h_{t,s}}} d\xi = R_h(t,s) = <\mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]}>_h. \end{aligned}$$ □

By isometry, it is possible to extend M_h to the space $\overline{\mathcal{E}(\mathbb{R})}^{<,>_h}$ and we shall still note M_h this extension. Define the isometry $\mathcal{J}^h := \zeta\circ M_h$ on $\overline{\mathcal{E}(\mathbb{R})}^{<,>_h}$, *i.e*:

$$\mathcal{J}^h : \begin{array}{ccccc} (\overline{\mathcal{E}(\mathbb{R})}^{<>_h}, <,>_h) & \overset{M_h}{\to} & (L^2(\mathbb{R}), <,>_{L^2(\mathbb{R})}) & \overset{\zeta}{\to} & ((L^2), <,>_{(L^2)}) \\ \mathbb{1}_{[0,t]} & \mapsto & M_h(\mathbb{1}_{[0,t]}) & \mapsto & <., M_h(\mathbb{1}_{[0,t]})>. \end{array}$$

We can now define the Wiener integral with respect to mBm in the natural following way:

Definition 4.1. *Let $B^{(h)}$ be a normalized multifractional Brownian motion. We call Wiener integral on $\mathbb{R}$ of an element u in $\overline{\mathcal{E}(\mathbb{R})}^{<>_h}$ with respect to $B^{(h)}$, the element $\mathcal{J}^h(u)$ of (L^2) defined thanks to the isometry $\mathcal{J}^h$ given just above.*

Remark 4.3. *It follows from definition 4.1 that the Wiener integral of a finite linear combination of functions $\mathbb{1}_{[0,t]}$ is $\mathcal{J}^h(\sum_{k=1}^n \alpha_k \mathbb{1}_{[0,t_k]}) = \sum_{k=1}^n \alpha_k B^{(h)}_{t_k}$. Moreover, for any element u in $\overline{\mathcal{E}(\mathbb{R})}^{<>_h}$ (which may be a tempered distribution), the Wiener integral of u with respect to mBm, still denoted $\mathcal{J}^h(u)$, is given by $\mathcal{J}^h(u) \overset{def}{=} \lim_{n\to+\infty} \mathcal{J}^h(u_n)$, for any sequence of functions $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{E}(\mathbb{R})^{\mathbb{N}}$ which converges to u in the norm $\|\ \|_h$ and where the convergence of $\mathcal{J}^h(u_n)$ holds in (L^2).*

Since we now have a construction of the Wiener integral with respect to mBm, it is natural to ask which functions admit such an integral. In particular, we do not know so far whether $\mathscr{S}(\mathbb{R}) \subset \overline{\mathcal{E}(\mathbb{R})}^{<>_h}$. The next section contains more information about the space $\overline{\mathcal{E}(\mathbb{R})}^{<>_h}$.

5 Stochastic integral with respect to mBm

5.1 Fractional White Noise

The following theorem will allow us to give a concrete example of a derivative of an $(\mathcal{S})^*$-process.

Theorem 5.1. *1. For any real H in $(0,1)$, the map $M_H(\mathbb{1}_{[0,.]}) : \mathbb{R} \to \mathscr{S}'(\mathbb{R})$ defined by $M_H(\mathbb{1}_{[0,.]})(t) := M_H(\mathbb{1}_{[0,t]})$ is differentiable over $\mathbb{R}$ and its derivative, noted $\frac{d}{dt}[M_H(\mathbb{1}_{[0,t]})]$, is equal to $\sum_{k=1}^{+\infty} M_H(e_k)(t)\, e_k$, where the convergence is in $\mathscr{S}'(\mathbb{R})$.*

2. For any interval I of $\mathbb{R}$ and any differentiable map $F : I \to \mathscr{S}'(\mathbb{R})$, the element $< ., F(t) >$ is a differentiable stochastic distribution process which satisfies the equality $\frac{d}{dt} < ., F(t) >=< ., \frac{d}{dt}F(t) >$.

Proof. The proof of point 1 is a mere re-writing of the one of lemma 2.15 of [8] by replacing $M_{\pm}^H$ by the operator M_H. Point 2 is theorem 2.17 of [8]. □

Let $H \in (0,1)$. The process $(B^{(H)}(t))_{t\in\mathbb{R}}$ defined in (3.8) is an fBm, and $M_H(\mathbb{1}_{[0,t]})$ belongs to $L^2(\mathbb{R})$ for every real t. Hence, using equality 2 of theorem 3.7, we may write, for every real t and almost surely:

$$\begin{aligned} B^{(H)}(t) = \ & < ., M_H(\mathbb{1}_{[0,t]}) > \ = \ < ., \sum_{k=0}^{+\infty} < M_H(\mathbb{1}_{[0,t]}), e_k >_{L^2(\mathbb{R})} e_k > \\ = \ & \sum_{k=0}^{+\infty} < \mathbb{1}_{[0,t]}, M_H(e_k) >_{L^2(\mathbb{R})} \ < ., e_k > \ = \ \sum_{k=0}^{+\infty} \left(\int_0^t M_H(e_k)(u)du \right) < ., e_k > . \end{aligned} \tag{5.1}$$

(5.1) and the previous theorem lead to the definition of fractional white noise [34, 11]:

Example 5.2 (Fractional white noise). *Let:*

$$W^{(H)}(t) := \sum_{k=0}^{+\infty} M_H(e_k)(t) \ < ., e_k >. \tag{5.2}$$

Then $(W^{(H)}(t))_{t\in\mathbb{R}}$ is a $(\mathcal{S})^*$-process and is the $(\mathcal{S})^*$-derivative of the process $(B^{(H)}(t))_{t\in\mathbb{R}}$.
The proof of this fact is simple: for any integer $p \geqslant 2$, using remark 2.3, the mean value theorem and the dominated convergence theorem,

$$\begin{aligned} J_{p,r}(t) \ & := \left|\left| \tfrac{B^{(H)}(t+r)-B^{(H)}(t)}{r} - W^{(H)}(t) \right|\right|_{-p}^2 = \left|\left| \textstyle\sum_{k=0}^{+\infty} \left(\frac{1}{r}\int_t^{t+r} M_H(e_k)(u)\, du - M_H(e_k)(t) \right) < ., e_k > \right|\right|_{-p}^2 \\ & = \textstyle\sum_{k=0}^{+\infty} (2k+2)^{-2p} \left(\frac{1}{r}\int_t^{t+r} (M_H(e_k)(u) - M_H(e_k)(t))\, du \right)^2 \underset{r\to 0}{\longrightarrow} 0. \end{aligned}$$

Remark 5.3. *In particular we see that for all (t,H) in $\mathbb{R}\times(0,1)$, $W^{(H)}(t)$ belongs to $(\mathcal{S}_{-p})$ as soon as $p \geqslant 2$.*

Remark 5.4. *There are several constructions of fBm. In particular, operators different from M_H may be considered. [8] uses an operator denoted M_+^H on the grounds that fBm as defined here is not adapted to the filtration generated by the driving Brownian motion as soon as $H \neq 1/2$. While this is indeed a drawback, the crucial property for our purpose is that the same probability space $(\mathscr{S}'(\mathbb{R}), \mathcal{G}, \mu)$ is used for all parameters H in $(0,1)$. This allows to consider simultaneously several fractional Brownian motions with H taking any value in $(0,1)$, which is necessary when one deals with mBm. We choose here to work with M_H rather than with M_+^H and M_-^H of [8] as its use is simpler. M_+^H and M_-^H would nevertheless allow for a more general approach encompassing the whole family of mBm at once. This topic will be treated in a forthcoming paper.*

5.2 Multifractional White Noise

The main idea for defining a stochastic integral with respect to mBm is similar to the one used for fBm. We will relate the process $B^{(h)}$ to Brownian motion *via* the family of operators $(M_H)_{(H\in(0,1))}$. This will allow to define a multifractional white noise, analogous to the fractional white noise of example 5.2. From a heuristic point of view, multifractional white noise is obtained by differentiating with respect to t the fractional Brownian field $\Lambda(t, H)$ (defined at the beginning of section 4.2) along a curve $(t, h(t))$. Assuming that we may differentiate in the sense of $(\mathcal{S})^*$ (this will be justified below), the differential of Λ reads:

$$d\Lambda(t,H) = \frac{\partial\Lambda}{\partial t}(t,H)\,dt + \frac{\partial\Lambda}{\partial H}(t,H)\,dH = \frac{dB^{(H)}}{dt}(t)\,dt + \frac{\partial\Lambda}{\partial H}(t,H)\,dH = W^{(H)}(t)dt + \frac{\partial\Lambda}{\partial H}(t,H)\,dH, \quad (5.3)$$

where the equality will be shown to hold in $(\mathcal{S})^*$. With a differentiable function h in place of H, this formally yields

$$d\Lambda(t,h(t)) = \Big(W^{(h(t))}(t) + h'(t)\,\frac{\partial\Lambda}{\partial H}(t,H)\big|_{H=h(t)}\Big)\,dt. \quad (5.4)$$

In view of the definition of the stochastic integral with respect to fBm given in [34], [46] and [9], it then seems natural to set the following definition for the stochastic integral with respect to mBm of a Hida process $X : \mathbb{R} \to (\mathcal{S})^*$:

$$\begin{aligned}\int_{\mathbb{R}} X(s)dB^{(h)}(s) &:= \int_{\mathbb{R}} X(s)d\Lambda(s,h(s)) \\ &:= \int_{\mathbb{R}} X(s)\diamond\Big(W^{(h(s))}(s) + h'(s)\frac{\partial\Lambda}{\partial H}(s,H)|_{H=h(s)}\Big)ds. \end{aligned} \quad (5.5)$$

We shall then say that the process X is integrable with respect to mBm if the right hand side of (5.5) exists in $(\mathcal{S})^*$. Remark that when the function h is constant we recover of course the integral with respect to fBm. In order to make the above ideas rigorous, we start by writing the chaos expansion of $B^{(h)}$. Since $M_H(g)$ belongs to $L^2(\mathbb{R})$ for all (g, H) in $\mathscr{S}(\mathbb{R}) \times (0, 1)$, we may define, for all H in $(0, 1)$, $M_H : \mathscr{S}'(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$, by

$$< M_H(\omega), g > \;=\; < \omega, M_H(g) >, \quad \text{for } \mu - a.e.\ \omega \ \text{ in } \ \Omega = \mathscr{S}'(\mathbb{R}). \quad (5.6)$$

Moreover, in view of remark 3.5, we may extend (5.6) to the case where g belongs to $L^2_H(\mathbb{R})$ by writing, for all g in $L^2_H(\mathbb{R})$ and almost every ω in Ω,

$$< M_H(\omega), g >:= \lim_{n\to+\infty} < M_H(\omega), g_n > \;=\; \lim_{n\to+\infty} < \omega, M_H(g_n) > \;=\; < \omega, M_H(g) >, \quad (5.7)$$

for every sequence $(g_n)_{n\in\mathbb{N}}$ of functions of $\mathscr{S}(\mathbb{R})$ which converges to g in the norm $||\ ||_{L^2_H(\mathbb{R})}$. For all real t and integer k in $\mathbb{N}$, define the element of $L^2_{h(t)}(\mathbb{R})$:

$$d_k^{(t)} := M_{h(t)}^{-1}(e_k). \quad (5.8)$$

It is clear that, for all t in $\mathbb{R}$, the family of functions $(d_k^{(t)})_{k\in\mathbb{N}}$ forms an orthonormal basis of $L^2_{h(t)}(\mathbb{R})$. Let us now write the chaos decomposition of mBm. For almost every ω and every real t we get, (using theorem 3.7 and (5.7)),

$$\begin{aligned} B^{(h)}(t)(\omega) &=< \omega, M_h(\mathbb{1}_{[0,t]}) >=< M_{h(t)}(\omega), \mathbb{1}_{[0,t]} >=< M_{h(t)}(\omega), \sum_{k=0}^{+\infty} < \mathbb{1}_{[0,t]}, d_k^{(t)} >_{L^2_H(\mathbb{R})} d_k^{(t)} > \\ &=< M_{h(t)}(\omega), \sum_{k=0}^{+\infty} < M_{h(t)}(\mathbb{1}_{[0,t]}), e_k >_{L^2(\mathbb{R})} d_k^{(t)} >= \sum_{k=0}^{+\infty} < M_{h(t)}(\mathbb{1}_{[0,t]}), e_k >_{L^2(\mathbb{R})} < \omega, M_{h(t)}(d_k^{(t)}) > . \end{aligned}$$

We get finally:

$$a.s,\ \forall t \in \mathbb{R}, \qquad B^{(h)}(t) = \sum_{k=0}^{+\infty}\left(\int_0^t M_{h(t)}(e_k)(s)ds\right) < ., e_k >. \quad (5.9)$$

We would then like to define multifractional white noise as the $(\mathcal{S}^*)$-derivative of $B^{(h)}$, which would be formally defined by:

$$W^{(h)}(t) := \sum_{k=0}^{+\infty} \left[\frac{d}{dt} \Big(\int_0^t \ M_{h(t)}(e_k)(s)\ ds \Big) \right] \ < ., e_k >, \tag{5.10}$$

assuming h is differentiable. The following theorem states that, for all real t, the right hand side of (5.10) does indeed belong to $(\mathcal{S})^*$ and is exactly the $(\mathcal{S})^*$-derivative of $B^{(h)}$ at t.

Theorem-Definition 5.1. *Let $h : \mathbb{R} \to (0,1)$ be a C^1 function such that the derivative function h' is bounded. The process $W^{(h)} := (W^{(h)}(t))_{t\in\mathbb{R}}$ defined by formula (5.10) is an $(\mathcal{S})^*$-process which verifies the following equality in $(\mathcal{S})^*$:*

$$W^{(h)}(t) = \sum_{k=0}^{+\infty} M_{h(t)}(e_k)(t) < ., e_k > + h'(t) \ \sum_{k=0}^{+\infty} \Big(\int_0^t \tfrac{\partial M_H}{\partial H}(e_k)(s)\big|_{H=h(t)} ds \Big) < ., e_k >. \tag{5.11}$$

Moreover the process $B^{(h)}$ is $(\mathcal{S})^*$-differentiable on $\mathbb{R}$ and verifies in $(\mathcal{S})^*$

$$\frac{dB^{(h)}}{dt}(t) = W^{(h)}(t) = \frac{d}{dt}[\Lambda(t, h(t))]. \tag{5.12}$$

In order to prove this theorem, we will need two lemmas.

Lemma 5.5. *For H in $(0,1)$ and f in $\mathscr{S}(\mathbb{R})$, define $g_f : \mathbb{R} \times (0,1) \to \mathbb{R}$ by $g_f(t,H) := \int_0^t M_H(f)(x)dx$. Then*
(i) The function g_f belongs to $C^\infty(\mathbb{R} \times (0,1), \mathbb{R})$,
(ii) $\forall x \in \mathbb{R}, \quad M_H(f)(x) = \alpha_H \int_0^{+\infty} u^{H-1/2}(f'(x+u) - f'(x-u))\ du$.
where α_H has been defined right after (3.7). In particular, the function $(x,H) \mapsto M_H(f)(x)$ is differentiable on $\mathbb{R} \times (0,1)$.
(iii) Assume that $h : \mathbb{R} \to (0,1)$ is differentiable. Then, for any real t_0

$$\frac{d}{dt}[g_f(t,h(t))]\big|_{t=t_0} = M_{h(t_0)}(f)(t_0) + h'(t_0) \int_0^{t_0} \tfrac{\partial M_H}{\partial H}(f)(s)\big|_{H=h(t_0)} ds. \tag{5.13}$$

Proof. (i) Define μ_f on $\mathbb{R} \times (0,1)$ by $\mu_f(t,H) := \int_{\mathbb{R}}(u-t)|u-t|^{H-3/2} f(u)\ du$, for f in $\mathscr{S}(\mathbb{R})$. Using (3.7) we get, for all (t,H) in $\mathbb{R} \times (0,1)$, the equality

$$g_f(t,H) = \alpha_H[\mu_f(t,H) - \mu_f(0,H)]. \tag{5.14}$$

A change of variables yields

$$\begin{aligned} \mu_f(x,H) &= -\int_{-\infty}^{x} |t-x|^{H-1/2} f(t)dt + \int_x^{+\infty} |t-x|^{H-1/2} f(t)\ dt \\ &= \int_0^{+\infty} u^{H-1/2}(f(x+u) - f(x-u))\ du. \end{aligned} \tag{5.15}$$

Thanks to (5.14) and to the fact that the map $y \mapsto \alpha_y$ is C^∞ on $(0,1)$, it is sufficient to show that the function μ_f belongs to $C^\infty(\mathbb{R} \times (0,1))$. In view of applying the theorem of differentiation under the integral sign, define $j(x,H,u) := u^{H-1/2}(f(x+u) - f(x-u))$ for u in $\mathbb{R}_+^*$. Let n in $\mathbb{N}$ and (α_1, α_2) in $\mathbb{N}^2$ such that $\alpha_1 + \alpha_2 = n$. For almost every u in $\mathbb{R}_+^*$, $(x,H) \mapsto j(x,H,u)$ is C^n on $\mathbb{R} \times (0,1)$ with partial derivatives given by

$$\frac{\partial^n j}{\partial x^{\alpha_1} \partial H^{\alpha_2}}(x,H,u) = (\ln u)^{\alpha_2} u^{H-1/2}(f^{(\alpha_1)}(x+u) - f^{(\alpha_1)}(x-u)).$$

Fix (x_0, H_0) in $\mathbb{R} \times (0,1)$. Let us show that μ_f is C^n in a neighbourhood of (x_0, H_0). Choose (a,b) such that $a < x_0 < b$ and H_1, H_2 such that $0 < H_1 < H < H_2 < 1$. We have

$$\left|\frac{\partial^n j}{\partial x^{\alpha_1}\partial H^{\alpha_2}}(x,H,u)\right| \leqslant |u|^{H_1-1/2}|\ln u|^{\alpha_2}\ \mathbb{1}_{\{0<u<1\}} \sup_{(x,u)\in[a,b]\times[0,1]} |f^{(\alpha_1)}(x\pm u)| + |u||\ln u|^{\alpha_2}|f^{(\alpha_1)}(x\pm u)|\ \mathbb{1}_{\{1\leqslant|u|\}}, \quad (5.16)$$

where $f^{(\alpha_1)}(x\pm u) := f^{(\alpha_1)}(x+u) - f^{(\alpha_1)}(x-u)$. A Taylor expansion shows that there exists a real D such that, for all (u,x) in $\mathbb{R}\times(a,b)$, $|u|^4\ |f^{(\alpha_1)}(u\pm x)| \leqslant D$. As a consequence, there exists a real constant C such that, for almost every u in $\mathbb{R}_+^*$ and every (x,H) in $[a,b]\times[H_1,H_2]$,

$$\left|\tfrac{\partial^n j}{\partial x^{\alpha_1}\partial H^{\alpha_2}}(x,H,u)\right| \leqslant C\ \left[|u|^{H_1-1/2}|\ln u|^{\alpha_2}\ \mathbb{1}_{\{0<u<1\}} + |\ln u|^{\alpha_2}\ \tfrac{1}{|u|^3}\ \mathbb{1}_{\{1\leqslant u\}}\right]. \quad (5.17)$$

Since the right hand side of the previous inequality belongs to $L^1(\mathbb{R})$, the theorem of differentiation under the integral sign can be applied to conclude that the function μ_f is of class C^n in $[a,b]\times[H_1,H_2]$, for all integer n and all f in $\mathscr{S}(\mathbb{R})$. This entails (i).
(ii) (5.14) and (5.15) yield

$$M_H(f)(x) = \alpha_H \frac{\partial}{\partial x}[\mu_f(x,H)] = \alpha_H \int_0^{+\infty} u^{H-1/2}(f'(x+u) - f'(x-u))du, \quad (5.18)$$

which establishes (ii) and the fact that $(x,H)\mapsto M_H(f)(x)$ belongs to $C^\infty(\mathbb{R}\times(0,1))$.
(iii) For a differentiable function h, we have, for every real t_0,

$$\frac{d}{dt}[g_f(t,h(t))]\Big|_{t=t_0} = \frac{\partial g_f}{\partial t}(t_0,h(t_0)) + h'(t_0)\frac{\partial g_f}{\partial H}(t_0,h(t_0)).$$

For every f in $\mathscr{S}(\mathbb{R})$,(5.14), (5.18) and (3.15) show that,

$$\frac{\partial g_f}{\partial H}(t,H) = \int_0^t \tfrac{\partial}{\partial H}\ [M_H(f)(x)]\ dx = \int_0^t \tfrac{\partial M_H}{\partial H}(f)(x)\ dx.$$

Finally, we get $\frac{d}{dt}[g_f(t,h(t))]\big|_{t=t_0} = M_{h(t_0)}(f)(t_0) + h'(t_0)\int_0^{t_0} \frac{\partial M_H}{\partial H}(f)(s)\big|_{H=h(t_0)}ds.$ □

Lemma 5.6. *The following inequalities hold:*

(i) $\forall\ [a,b]\subset(0,1), \exists\rho\in\mathbb{R}$: $\forall k\in\mathbb{N}, \sup_{(H,u)\in[a,b]\times\mathbb{R}} \left|\frac{\partial M_H}{\partial H}(e_k)(u)\right| \leqslant \rho\ (k+1)^{2/3}\ \ln(k+1)$.

(ii) $\forall t\in\mathbb{R}, \forall r\in\mathbb{R}_+^*, \exists\widetilde{D_t}(r)\in\mathbb{R}, \forall k\in\mathbb{N}: \sup_{u\in[t-r,t+r]} \left|\frac{d}{du}[g_{e_k}(u,h(u))]\right| \leqslant \widetilde{D_t}(r)\ (k+1)^{2/3}$.

Proof. (i) Since $\mathscr{S}(\mathbb{R})$ is a subset of $\Gamma_H(\mathbb{R})$, (3.12) entails that $\widehat{\frac{\partial M_H}{\partial H}(e_k)}$ belongs to $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ for every k in $\mathbb{N}$. Furthermore $\widehat{e_k}(y) = (-i)^{k-1}\sqrt{2\pi}e_k(y)$ for every integer k in $\mathbb{N}^*$ and for almost every real y (see lemma 1.1.3 p.5 of [77]). Thus, for every $H\in[a,b]$ and almost every $u\in\mathbb{R}$,

$$\begin{aligned}
\tfrac{\partial M_H(e_k)}{\partial H}(u) &= \tfrac{1}{2\pi}\widehat{\widehat{\tfrac{\partial M_H(e_k)}{\partial H}}}(-u) = -\tfrac{1}{2\pi}\int_{\mathbb{R}} e^{iuy}(\beta_H + \ln|y|)\ \tfrac{\sqrt{2\pi}}{c_H}\ |y|^{1/2-H}\ \widehat{e_k}(y)\ dy \\
&= \tfrac{-\beta_H}{2\pi}\int_{\mathbb{R}} e^{iuy}\widehat{M_H(e_k)}(y)\ dy\ -\ \tfrac{1}{c_H}\int_{\mathbb{R}} e^{iuy}|y|^{1/2-H}(\ln|y|)\ (-i)^{k-1}e_k(y)\ dy \\
&= -\beta_H\ M_H(e_k)(u) + \tfrac{-(-i)^{k-1}}{c_H}\underbrace{\int_{\mathbb{R}} e^{iuy}|y|^{1/2-H}(\ln|y|)\ e_k(y)\ dy}_{=:V_k(u)}. \quad (5.19)
\end{aligned}$$

Then, using (2.5), we see that there exists a family of real constants denoted $(\rho_i)_{1\leqslant i\leqslant 11}$ such that we have, for every couple (H,k) in $[a,b]\times\mathbb{N}$ and almost every real u,

$$\begin{aligned}|V_k(u)| &\leqslant \rho_1 \Big[\textstyle\int_{|y|\leqslant 2\sqrt{k+1}} |y|^{1/2-H} |\ln|y|||e_k(y)|dy + \int_{|y|>2\sqrt{k+1}} |y|^{1/2-H}(\ln|y|)|e_k(y)|dy\Big] \\ &\leqslant \rho_2 \Big[\tfrac{1}{(k+1)^{1/12}} \underbrace{\Big(\int_0^{2\sqrt{k+1}} |\ln y|\, y^{1/2-H} dy\Big)}_{:=I_1^{(k+1)}} + \underbrace{\Big(\int_{2\sqrt{k+1}}^{+\infty} y^{1/2-H}(\ln y) e^{-\gamma y^2} dy\Big)}_{:=I_2^{(k+1)}}\Big]. \end{aligned} \quad (5.20)$$

An integration by parts yields

$$I_1^{(k)} = \rho_3((\rho_4 + \ln k)(1 + k^{3/4-H/2})) \leqslant \rho_5\, k^{3/4-H/2} \ln k. \quad (5.21)$$

Using the change of variables $u = y\sqrt{\gamma}$, we get $I_2^{(k)} \leqslant \rho_6 \int_{2\sqrt{k\gamma}}^{+\infty} |u|^{1/2-H} (\ln u) e^{-u^2} du = \rho_6\, J_{2\sqrt{k\gamma}}^{(1/2-H)}$ where $J_\delta^{(\alpha)} := \int_\delta^{+\infty} |u|^\alpha \ln u\, e^{-u^2} du$. When $\delta > 3e$, an integration by parts shows that $J_\delta^{(\alpha)} < \delta^{\alpha-1} e^{-\delta^2} \ln\delta$ for $0 < \alpha < 1/2$, and that $J_\delta^{(\alpha)} \leqslant \delta^\alpha \int_\delta^{+\infty} e^{-u^2} \ln u\, du$ for $-1/2 < \alpha < 0$. Hence we get

$$J_{2\sqrt{k\gamma}}^{(1/2-H)} \leqslant \rho_7 \begin{cases} k^{-1/4-H/2} \ln k & \text{for } \ 0 < H < 1/2 \\ k^{1/4-H/2} & \text{for } \ 1/2 < H < 1 \end{cases} \quad (5.22)$$

and we finally obtain $I_2^{(k)} \leqslant \rho_8 \frac{\ln k}{k^{H/2-1/4}}$. Using (5.19) to (5.22), the previous inequality, item 3 of theorem 3.7 and the fact that both functions $H \mapsto \beta_H$ and $H \mapsto \frac{1}{c_H}$ are continuous on $[a,b]$ we get, for every k in $\mathbb{N}$,

$$\begin{aligned}\sup_{(H,u)\in[a,b]\times\mathbb{R}} \big|\tfrac{\partial M_H}{\partial H}(e_k)(u)\big| &\leqslant \rho_9\, (k+1)^{2/3} + \rho_{10} \sup_{H\in[a,b]} \Big[((k+1)^{2/3-H/2} + (k+1)^{1/4-H/2}) \ln(k+1)\Big] \\ &\leqslant \rho_{11}\, (k+1)^{2/3} \ln(k+1).\end{aligned}$$

(ii) Let $t \in \mathbb{R}$ and $r > 0$ be fixed and define $[a,b] := [\inf_{u\in[t-r,t+r]} h(u), \sup_{u\in[t-r,t+r]} h(u)]$. Using (5.13) we have, for every k in $\mathbb{N}$,

$$\sup_{u\in[t-r,t+r]} \big|\tfrac{d}{du}[g_{e_k}(u,h(u))]\big| \leqslant \sup_{(H,u)\in[a,b]\times\mathbb{R}} |M_H(e_k)(u)| + (|t|+r) \sup_{u\in[t-r,t+r]} |h'(u)| \sup_{(H,u)\in[a,b]\times\mathbb{R}} \big|\tfrac{\partial M_H}{\partial H}(e_k)(u)\big|.$$

The result then follows from (i) above and item 3 of theorem 3.7. □

Remark 5.7. *In the sequel, we will only need the bounds of* (i) *and* (ii) *with* $(k+1)$ *in lieu of* $(k+1)^{2/3}$.

We may now proceed to the proof of theorem 5.1.
Proof of theorem 5.1. From equality (5.10) defining multifractional white noise and equality (5.13), we have formally

$$W^{(h)}(t) = \sum_{k=0}^{+\infty} M_{h(t)}(e_k)(t) < .,e_k > + h'(t) \sum_{k=0}^{+\infty} \int_0^t \frac{\partial M_H(e_k)}{\partial H}(s)\big|_{H=h(t)} ds < .,e_k >. \quad (5.23)$$

In order to establish that $W^{(h)}(t)$ is well defined in $(\mathcal{S})^*$ and that equality (5.11) holds, it is sufficient to show that both members on the right hand side of the previous equality are in $(\mathcal{S})^*$.

For t in $\mathbb{R}$, definition (5.2) of fractional white noise shows that $\sum_{k=0}^{+\infty} M_{h(t)}(e_k)(t) < .,e_k >= W^{(h(t))}(t)$ and thus belongs to $(\mathcal{S})^*$.

Let us show that $V_H(t) := \sum_{k=0}^{+\infty} \int_0^t \frac{\partial M_H(e_k)}{\partial H}(s) ds < .,e_k >$ belongs to (L^2). Using (3.14) we may write

$$\mathbb{E}[V_H^2(t)] = \sum_{k=0}^{+\infty} < \tfrac{\partial M_H}{\partial H}(e_k), \mathbb{1}_{[0,t]} >^2_{L^2(\mathbb{R})} = \sum_{k=0}^{+\infty} < e_k, \tfrac{\partial M_H}{\partial H}(\mathbb{1}_{[0,t]}) >^2_{L^2(\mathbb{R})} = ||\mathbb{1}_{[0,t]}||^2_{\delta_H} < +\infty.$$

As a consequence, $W^{(h)}(t)$ is the sum of an $(\mathcal{S})^*$ process and an (L^2) process, and thus belongs to $(\mathcal{S})^*$. We are left with proving equality (5.12), *i.e.* that $W^{(h)}(t)$ is indeed the $(\mathcal{S})^*$ derivative of $B^{(h)}(t)$ for any real t.

Let $r \neq 0$ and $t \geqslant 0$ (the case $t < 0$ follows in a similar way). The equality $W^{(h)}(t) = W^{(h(t))}(t) + h'(t)\, V_{h(t)}(t)$ and remark 5.3 entail that $W^{(h)}(t)$ belongs to $(\mathcal{S}_{-p})$ as soon as $p \geqslant 2$. For such a p, one computes:

$$\begin{aligned} J_{p,r}(t) &:= \left\|\tfrac{B^{(h)}(t+r)-B^{(h)}(t)}{r} - W^{(h)}(t)\right\|_{-p}^{2} \\ &= \left\|\sum_{k=0}^{+\infty} \left[\left(\frac{g_{e_k}(t+r,h(t+r)) - g_{e_k}(t,h(t))}{r}\right) - \frac{d}{dt}[g_{e_k}(t,h(t))]\right] < .,e_k > \right\|_{-p}^{2} \\ &= \sum_{k=0}^{+\infty} \underbrace{\frac{1}{(2k+2)^{2p}} \left[\left(\frac{g_{e_k}(t+r,h(t+r)) - g_{e_k}(t,h(t))}{r}\right) - \frac{d}{dt}[g_{e_k}(t,h(t))]\right]^{2}}_{:=J_{p,r,k}(t)}. \end{aligned} \tag{5.24}$$

Using lemma 5.6 and the Mean-Value theorem we obtain, for r in $(-1/2, 1/2)\backslash\{0\}$:

$$J_{p,r,k}(t) \leqslant \frac{4}{(2k+2)^{2p}} \sup_{u\in[t-1/2,t+1/2]} \left|\tfrac{d}{du}[g_{e_k}(u,h(u))]\right|^{2} \leqslant 4\,\widetilde{D}\,\frac{(k+1)^2}{(2k+2)^{2p}} \leqslant \frac{\widetilde{D}}{(2(k+1))^{2(p-1)}},$$

where $\widetilde{D} := \widetilde{D}_t(1/2)$. Since $J_{p,r,k}(t) \underset{r\to 0}{\longrightarrow} 0$, equality (5.12) follows from the dominated convergence theorem. □

Remark 5.8. *In (ii) of lemma 5.6, the real constant $\widetilde{D_t}(r)$ can be taken independent of t if the function $t \mapsto t\, h'(t)$ is bounded over $\mathbb{R}$.*

We note that multifractional white noise is a sum of two terms: a fractional white noise that belong to $(\mathcal{S}_{-p})$ as soon as $p \geqslant 2$, and a "smooth" term which corresponds to the derivative in the "H" direction. This is a direct consequence of the fact that the fractional Brownian field $\Lambda(t,H)$ is not differentiable in the t direction (in the classical sense) but infinitely smooth in the H direction.

Proposition 5.9. *For $p \geqslant 2$, the map $t \mapsto ||W^{(h)}(t)||_{-p}$ is continuous.*

Proof. By definition, $||W^{(h)}(t)||_{-p}^{2} = \sum_{k=0}^{+\infty} \frac{(\frac{d}{dt}[g_{e_k}(t,h(t))])^2}{(2k+2)^{2p}}$. Using the estimate given in lemma 5.6 (ii), we see that $||W^{(h)}(t)||_{-p}^{2}$ is the sum of a sequence of continuous functions that converges normally on any compact. □

5.3 Generalized functionals of mBm

In the next section, we will derive various Itô formulas for the integral with respect to mBm. It will be useful to obtain such formula for tempered distributions. In that view, we define generalized functionals of mBm as in [8].

Theorem-Definition 5.2. *Let F be a tempered distribution. For t in $\mathbb{R}_+^*$, define*

$$F(B^{(h)}(t)) := \frac{1}{\sqrt{2\pi}t^{h(t)}} \sum_{k=0}^{+\infty} (k!)^{-1} t^{-2kh(t)} < F, \xi_{(t,h(t),k)} > I_k\left(\left(M_{h(t)}(\mathbb{1}_{[0,t]}) \right)^{\otimes k}\right) \tag{5.25}$$

where the functions $\xi_{t,H,k}$ are defined for (x,H,k) in $\mathbb{R}\times(0,1)\times\mathbb{N}$ by

$$\xi_{t,H,k}(x) := (\sqrt{2})^{-k}\, t^{kH}\, h_k(x/(\sqrt{2}t^H)) \exp\{-\frac{x^2}{2\,t^{2H}}\} = \pi^{1/4}(k!)^{1/2} t^{kH} \exp\{-\frac{x^2}{4t^{2H}}\} e_k(x/(\sqrt{2}t^H)). \tag{5.26}$$

Then for all t in $\mathbb{R}_+^*$, $F(B^h(t))$ is a Hida distribution, called generalized functional of $B^{(h)}(t)$.

Proof. This is an immediate consequence of [52] p.61-64 by taking $f := M_h(\mathbb{1}_{[0,t]})$. □

Remark 5.10. *As shown in [8], when $F = f$ is of function type, $F(B^h(t))$ coincides with $f(B^h(t))$.*

The following theorem yields an estimate of $||F(B^{(h)}(t)||^2_{-p}$ which will be useful in the sequel.

Theorem 5.11. *Let $h : \mathbb{R} \to [H_1, H_2] \subset (0,1)$ be a continuous function, $B^{(h)}$ an mBm, $p \in \mathbb{N}$ and $F \in \mathscr{L}_p(\mathbb{R})$. Then there is a constant $C_p^{(H_1,H_2)}$, independent of F, such that*

$$\forall t > 0, \qquad ||F(B^{(h)}(t))||^2_{-p} \leqslant \max\{t^{-4ph(t)}, t^{4ph(t)}\}t^{-h(t)}\ C_p^{(H_1,H_2)}|F|^2_{-p}. \tag{5.27}$$

Proof. For $H \in (0,1)$ and $p \in \mathbb{N}$, Theorem 3.3 p.92 of [8] ensures that there exists $C_p^{(H)}$ such that, $\forall t > 0$,

$$||F(B^H(t))||^2_{-p} \leqslant \max\{t^{-4pH}, t^{4phH}\}t^{-H}\ C_p^{(H)}|F|^2_{-p}.$$

Now if H belongs to $[H_1, H_2]$, it is easy to show, by examining closely the iteration of (23) p.94 in [8], that one can choose a constant $C^{(H_1,H_2)}$ independent of H. We hence have

$$\forall t > 0, \quad \forall H \in [H_1, H_2], \qquad ||F(B^H(t))||^2_{-p} \leqslant \max\{t^{-4pH}, t^{4phH}\}t^{-H}\ C_p^{(H_1,H_2)}|F|^2_{-p}. \tag{5.28}$$

For $t > 0$, one only needs to set $H = h(t)$ in (5.28) to get (5.27). □

5.4 S-Transform of mBm and multifractional white noise

The following theorem makes explicit the S-transforms of mBm, multifractional white noise and generalized functionals of mBm.
We denote by γ the heat kernel density on $\mathbb{R}_+ \times \mathbb{R}$ *i.e* $\gamma(t,x) := \frac{1}{\sqrt{2\pi t}} \exp\{\frac{-x^2}{2t}\}$ if $t \neq 0$ and 0 if $t = 0$.

Theorem 5.12. *Let $h : \mathbb{R} \to (0,1)$ be a C^1 function and $(B^{(h)}(t))_{t\in\mathbb{R}}$ (resp. $(W^{(h)}(t))_{t\in\mathbb{R}}$) be an mBm (resp. multifractional white noise). For $\eta \in \mathscr{S}(\mathbb{R})$ and $t \in \mathbb{R}$,*
(i) $S[B^{(h)}(t)](\eta) = <\eta, M_h(\mathbb{1}_{[0,t]})>_{L^2(\mathbb{R})} = g_\eta(t, h(t))$, where g_η has been defined in lemma 5.5.
(ii) $S[W^{(h)}(t)](\eta) = \frac{d}{dt}[g_\eta(t, h(t))] = M_{h(t)}(\eta)(t) + h'(t)\int_0^t \frac{\partial M_H}{\partial H}(\eta)(s)\big|_{H=h(t)} ds.$
(iii) For $p \in \mathbb{N}$ and $F \in \mathscr{S}_{-p}(\mathbb{R})$, $\quad S[F(B^{(h)}(t))](\eta) = \left\langle F, \gamma\left(t^{2h(t)}, . - \int_0^t M_{h(t)}(\eta)\ (u)\ du\right)\right\rangle.$
Furthermore, there exists a constant C_p, independent of F, t and η, such that

$$|S[F(B^{(h)}(t))](\eta)|^2 \leqslant \max\{t^{-4ph(t)}, t^{4ph(t)}\}t^{-h(t)}\ C_p\ |F|^2_{-p}\ \exp\{|A^p\eta|^2_0\}. \tag{5.29}$$

Proof. (i) Thanks to (5.9), lemma 2.4 and theorem 3.7 we have, for every η in $\mathscr{S}(\mathbb{R})$ and t in $\mathbb{R}$,

$$S(B^{(h)}(t))(\eta) = \sum_{k=0}^{+\infty} < M_{h(t)}(\mathbb{1}_{[0,t]}), e_k >_{L^2(\mathbb{R})} < \eta, e_k >_{L^2(\mathbb{R})} = < M_{h(t)}(\mathbb{1}_{[0,t]}), \eta >_{L^2(\mathbb{R})} = g_\eta(t, h(t)).$$

(ii) is a straightforward consequence of lemma 2.7, (5.12) and (i).
(iii) The first equality results from theorem 7.3 p.63 in [52] with $f = M_{h(t)}(\mathbb{1}_{[0,t]})$ and from (i). Equality (5.29) results from (5.27) as in theorem 3.8 p.95 of [8]. □

Remark 5.13. *Using lemma 2.4 and (5.10) we may also write:*

$$\forall (t,\eta) \in \mathbb{R} \times \mathscr{S}(\mathbb{R}), \qquad S(W^{(h)}(t))(\eta) = \sum_{k=0}^{+\infty} \tfrac{d}{dt}[g_{e_k}(t, h(t))] < \eta, e_k >_{L^2(\mathbb{R})}. \tag{5.30}$$

5.5 The multifractional Wick-Itô integral

We are now able to define the Multifractional Wick-Itô integral, in a way analogous to the definition of the fractional Wick-Itô integral. *In the sequel of this work, we will always assume that h is a C^1 function on $\mathbb{R}$ with bounded derivative.*

Definition 5.1 (The multifractional Wick-Itô integral)**.** *Let $Y : \mathbb{R} \to (\mathcal{S})^*$ be a process such that the process $t \mapsto Y(t) \diamond W^{(h)}(t)$ is $(\mathcal{S})^*$-integrable on $\mathbb{R}$. We then say that the process Y is $dB^{(h)}$-integrable on $\mathbb{R}$ or integrable on $\mathbb{R}$ with respect to mBm $B^{(h)}$. The integral of Y with respect to $B^{(h)}$ is defined by*

$$\int_{\mathbb{R}} Y(s)\, dB^{(h)}(s) := \int_{\mathbb{R}} Y(s) \diamond W^{(h)}(s) ds. \tag{5.31}$$

For a Borel set I of $\mathbb{R}$, define $\int_I Y(s) dB^{(h)}(s) := \int_{\mathbb{R}} \mathbb{1}_I(s)\, Y(s) dB^{(h)}(s)$.

When the function h is constant, the multifractional Wick-Itô integral coincides with the fractional Itô integral defined in [34], [11], [8] and [9]. In particular, when the function h is identically $1/2$, (5.31) is nothing but the classical Itô integral with respect to Brownian motion, provided of course Y is Itô-integrable. The multifractional Wick-Itô integral verifies the following properties:

Proposition 5.14. *(i) Let (a, b) in $\mathbb{R}^2$, $a < b$. Then $\int_a^b dB^{(h)}(u) = B^{(h)}(b) - B^{(h)}(a)$ almost surely.*
(ii) Let $X : I \to (\mathcal{S}^*)$ be a $dB^{(h)}$-integrable process over I, a Borel subset of $\mathbb{R}$. Assume $\int_I X(s) dB^{(h)}(s)$ belongs to (L^2). Then $\mathbb{E}[\int_I X(s) dB^{(h)}(s)] = 0$.

Proof. (i) From (ii) of theorem 5.12, $t \mapsto S(\mathbb{1}_{[a,b]}(t)\, W^{(h)}(t))(\eta)$ is measurable on $\mathbb{R}$ for any η in $\mathscr{S}(\mathbb{R})$. Moreover, for any integer $p_0 \geqslant 2$, we have $|S(\mathbb{1}_{[a,b]}(t)\, W^{(h)}(t))(\eta)| \leqslant ||W^{(h)}(t)||_{-p_0} e^{\frac{1}{2}|\eta|^2_{p_0}}$, thanks to lemma 2.4. By proposition 5.9, $t \mapsto ||W^{(h)}(t)||_{-p_0}$ is continuous thus integrable on $[a, b]$. Theorem 2.8 then entails that $t \mapsto \mathbb{1}_{[a,b]}(t)\, W^{(h)}(t)$ is $(\mathcal{S}^*)$-integrable over $\mathbb{R}$. It is easily seen that the S-transforms of $\int_a^b dB^{(h)}(u)$ and $B^{(h)}(b) - B^{(h)}(a)$ coincide. The result then follows from the injectivity of the S-transform.
(ii) The equality $S(\int_I X(s) dB^{(h)}(s))(0) = \int_I S(X(s))(0) S(W^{(h)}(s))(0) ds = 0$ is clear since $S(W^{(h)}(s))(0) = \frac{d}{ds}[g_0(s, h(s))] = 0$. Now, when $\int_I X(s) dB^{(h)}(s)$ belongs to (L^2), $\mathbb{E}[\int_I X(s) dB^{(h)}(s)] = S(\int_I X(s) dB^{(h)}(s))(0)$. □

Theorem 5.15. *Let I be a compact subset of $\mathbb{R}$ and $X : t \mapsto X(t)$ be a process from I to (L^2) such that $t \mapsto S(X(t))(\eta)$ is measurable on I for all η in $\mathscr{S}(\mathbb{R})$ and $t \mapsto ||X(t)||_0$ belongs to $L^1(I)$. Then X is $dB^{(h)}$-integrable on I and there exist a natural integer q and a constant C_I such that,*

$$\left|\left|\int_I X(t)\, dB^{(h)}(t)\right|\right|_{-q} \leqslant C_I \int_I ||X(t)||_0\, dt. \tag{5.32}$$

Proof. For $\eta \in \mathscr{S}(\mathbb{R})$, the measurability on I of $t \mapsto S(X(t) \diamond W^{(h)}(t))(\eta)$ is clear since $S(X(t) \diamond W^{(h)}(t))(\eta) = S(X(t))(\eta)\, \frac{d}{dt}[g_\eta(t, h(t))]$. By lemma 2.5, we have, for any integer $q \geqslant 2$,

$$|S(X(t)\ \diamond\ W^{(h)}(t))(\eta)| \leqslant ||X(t)||_0\ ||W^{(h)}(t)||_{-q}\ e^{|\eta|^2_q}$$

for every t in I. Since $t \mapsto ||W^{(h)}(t)||_{-q}$ is continuous by proposition 5.9 and $t \mapsto ||X(t)||_0$ belongs to $L^1(I)$ by assumption, the result follows from theorem 2.8. We refer to theorem 13.5 of [52] for the upper bound. □

Remark 5.16. *One can show, using appendix X.A, that inequality (5.32) is true for every integer $q \geqslant 2$.*

It is of interest to have also a criterion of integrability for generalized functionals of mBm. In that view, we set up the following notation: for $p \in \mathbb{N}$, $0 < a < b$, we consider a map $F : [a, b] \to \mathscr{S}_{-p}(\mathbb{R})$ (hence $F(t)$ is a tempered distribution for all t). We then define $F(t, B^{(h)}(t)) := F(t)(B^{(h)}(t))$. Recall the following theorem (see [22], lemma 1 and 2 p.73-74):

Theorem 5.17. *Let I be an interval of $\mathbb{R}$, $t \mapsto F(t)$ be a map from I into $\mathscr{S}_{-p}(\mathbb{R})$, $t \mapsto \varphi(t, .)$ be a map from I into $\mathscr{S}(\mathbb{R})$ and $t_0 \in I$. If both maps $t \mapsto F(t)$ and $t \mapsto \varphi(t, .)$ are continuous (respectively differentiable) at t_0, then the function $t \mapsto < F(t), \varphi(t, .) >$ is continuous (respectively differentiable) at t_0.*

Theorem 5.18. *Let $p \in \mathbb{N}$, $0 < a < b$ and let $F : [a,b] \to \mathscr{S}_{-p}(\mathbb{R})$ be a continuous map. Then the stochastic distribution process $F(t, B^{(h)}(t))$ is both $(\mathcal{S})^*-$integrable and $dB^{(h)}$-integrable over $[a,b]$.*

Proof. We shall apply theorem 2.8. The measurability of $t \mapsto S[F(t, B^{(h)}(t))](\eta)$ results from (iii) of theorem 5.12, the continuity of the two maps $t \mapsto F(t)$ and $t \mapsto \gamma\left(t^{2h(t)}, . - \int_0^t M_{h(t)}(\eta)\,(u)\,du\right)$ and theorem 5.17. Since h is bounded on $[a,b]$, lemma 2.4 and (5.27) yield

$$\begin{aligned}|S[F(t, B^{(h)}(t))](\eta)| &\leqslant \max\{t^{-2ph(t)}, t^{2h(t)}\} t^{-h(t)/2} \sqrt{C_p^{(H_1,H_2)}} \max_{s\in[a,b]} |F(s)|_{-p} \exp\{\frac{1}{2}|A^p\eta|_0^2\} \\ &\leqslant ((\tfrac{1}{a})^{2pH_2} + b^{2pH_2})((\tfrac{1}{a})^{\frac{H_2}{2}} + b^{\frac{H_2}{2}}) \sqrt{C_p^{(H_1,H_2)}} \max_{s\in[a,b]} |F(s)|_{-p} \exp\{\frac{1}{2}|A^p\eta|_0^2\}, \qquad (5.33)\end{aligned}$$

where $H_1 := \min_{s\in[a,b]} h(s)$ and $H_2 := \max_{s\in[a,b]} h(s)$. This yields the second condition of theorem 2.8 and shows that $F(t, B^{(h)}(t))$ is $(\mathcal{S})^*-$integrable over $[a,b]$.
For $dB^{(h)}$-integrability, we first note that $S[F(t, B^{(h)}(t)) \diamond W^{(h)}(t)](\eta) = S[F(t, B^{(h)}(t))](\eta)\, \frac{d}{dt}[g_\eta(t, h(t))]$, thanks to (ii) of theorem 5.12. Since the function $t \mapsto \frac{d}{dt}[g_\eta(t, h(t))]$ is continuous (by lemma 5.5), the measurability of $t \mapsto S[F(t, B^{(h)}(t)) \diamond W^{(h)}(t)](\eta)$ for every function η in $\mathscr{S}(\mathbb{R})$ follows.
Moreover, for every integer $p_0 \geqslant \max\{p, 2\}$, $F(t, B^{(h)}(t))$ and $W^{(h)}(t)$ belong to $(\mathcal{S}_{-p_0})$ for all t in $[a,b]$. Using lemma 2.5 and (5.27), we may write, for all t in $[a,b]$,

$$|S[F(t, B^{(h)}(t)) \diamond W^{(h)}(t)](\eta)| \leqslant ||F(t, B^{(h)}(t))||_{-p_0}\, ||W^{(h)}(t)||_{-p_0} \exp\{|\eta|_{p_0}^2\} \leqslant K \exp\{|\eta|_{p_0}^2\}, \qquad (5.34)$$

where $K := \sup_{t\in[a,b]} ||W^{(h)}(t)||_{-p_0}\, ((\frac{1}{a})^{2p_0H_2} + b^{2p_0H_2})((\frac{1}{a})^{\frac{H_2}{2}} + b^{\frac{H_2}{2}}) \sqrt{C_{p_0}^{(H_1,H_2)}} \max_{s\in[a,b]} |F(s)|_{-p_0}$.
Theorem 2.8 applies again and shows that $t \mapsto F(t, B^{(h)}(t)) \diamond W^{(h)}(t)$ is integrable over $[a,b]$. □

Remark 5.19. *Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be of polynomial growth if there is an integer m in $\mathbb{N}$ and a constant C such that for all $x \in \mathbb{R}$, $|f(x)| \leqslant C(1 + |x|^m)$. The previous theorem entails in particular that both quantities $\int_a^b f(B^{(h)}(t))\,dt$ and $\int_a^b f(B^{(h)}(t))\,dB^{(h)}(t)$ exist in $(\mathcal{S})^*$ if f is a function of polynomial growth.*

Example 5.20 (Computation of $\int_0^T B^{(h)}(t)\,dB^{(h)}(t)$)**.** *Let $T > 0$ fixed. Then*

$$I := \int_0^T B^{(h)}(t)\,dB^{(h)}(t) = \int_0^T W^{(h)}(t) \diamond B^{(h)}(t)\,dt = \int_0^T \frac{dB^{(h)}(t)}{dt} \diamond B^{(h)}(t)\,dt. \qquad (5.35)$$

Let us prove that the last quantity is equal to $\frac{1}{2}B^{(h)}(T)^{\diamond 2} := \frac{1}{2}\,(B^{(h)}(T) \diamond B^{(h)}(T)) = \frac{1}{2}\,(B^{(h)}(T)^2 - T^{2h(T)})$ (see remark 2.6). It is sufficient to compute the S-transforms of both sides of the equality. For η in $\mathscr{S}(\mathbb{R})$,

$$\begin{aligned}S(\int_0^T B^{(h)}(t)dB^{(h)}(t))(\eta) &= \int_0^T S(B^{(h)}(t))(\eta) S(W^{(h)}(t))(\eta)dt = \int_0^T g_\eta(t, h(t)) \tfrac{d}{dt}[g_\eta(t, h(t))]dt \\ &= \tfrac{1}{2}(S(B^{(h)}(T))(\eta))^2 = \tfrac{1}{2}\, S\left(B^{(h)}(T) \diamond B^{(h)}(T)\right)(\eta) = S(\tfrac{1}{2}\,(B^{(h)}(T)^2 - T^{2h(T)}))(\eta). \qquad (5.36)\end{aligned}$$

To end this section, we present a simple but classical stochastic differential equation in the frame of mBm.

Example 5.21 (The multifractional Wick exponential)**.** *Following [34] formula (4.8) and [11] example 3.6, let us consider the multifractional stochastic differential equation*

$$\begin{cases} dX(t) = \alpha(t)X(t)dt + \beta(t)X(t)dB^{(h)}(t) \\ X(0) \in (\mathcal{S}^*), \end{cases} \qquad (5.37)$$

where t belongs to $\mathbb{R}_+$ and where $\alpha : \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R} \to \mathbb{R}$ are two deterministic continuous functions. (3.10) *is a shorthand notation for*

$$X(t) = X(0) + \int_0^t \alpha(s)\,X(s)\,ds + \int_0^t \beta(s)\,X(s)\,dB^{(h)}(s), \qquad (5.38)$$

where the previous equality holds in $(\mathcal{S})^*$. *Rewrite the previous equation in terms of derivatives in* $(\mathcal{S})^*$ *as:*

$$\begin{cases} \dfrac{dX}{dt}(t) = \alpha(t)\ X(t) + \beta(t)\ X(t)\ \diamond W^{(h)}(t) = (\alpha(t) + \beta(t)W^{(h)}(t)) \diamond X(t) \\ X(0) \in (\mathcal{S}^*). \end{cases} \tag{5.39}$$

We thus are looking for an $(\mathcal{S}^*)$*-process, noted* Z*, defined on* $\mathbb{R}_+$ *such that* Z *is differentiable on* $\mathbb{R}_+$ *and verifies equation* (3.11) *in* $(\mathcal{S}^*)$. *As in [11], it is easy to guess the solution of* (3.11) *if we replace Wick products* $\diamond$ *by ordinary products. Once we have a solution of* (3.11)*, we replace ordinary products by Wick products. This heuristic reasoning leads to defining the process* $Z := (Z(t)_{t\in\mathbb{R}_+})$ *by*

$$Z(t) := X(0) \diamond \exp^{\diamond}\Big(\int_0^t \alpha(s)ds + \int_0^t \beta(s)dB^{(h)}(s)\Big), \qquad t \in \mathbb{R}_+, \tag{5.40}$$

where $\exp^{\diamond}$ *has been defined in section* 2.4.

Theorem 5.22. *The process* Z *defined by* (3.12) *is the unique solution in* $(\mathcal{S}^*)$ *of* (3.11).

Proof. This is a straightforward application of theorem 3.1.2 in [46]. □

Remark 5.23. *[46] uses the Hermite transform in order to establish the theorem. However it is possible to start from (5.38), take* S*-transforms of both sides and solve the resulting ordinary stochastic differential equation. Besides, equation* (3.10) *may be solved with other assumptions on* α, β. *We refer to a forthcoming paper for more on stochastic differential equations driven by mBm.*

Remark 5.24. *In particular when* $X(0)$ *is deterministic, equal to* x*,* $\alpha() \equiv \alpha$ *and* $\beta() \equiv \beta$ *are constant functions, the solution* X *of* (5.38) *reads*

$$X(t) = x \exp\{\beta B^{(h)}(t) + \alpha t - \tfrac{1}{2}\beta^2 t^{2h(t)}\}, \qquad t \in \mathbb{R}_+, \tag{5.41}$$

which is analogous to formula (3.31) *given in* [*11*] *in the case of the fractional Brownian motion.*

5.6 Multifractional Wick-Itô integral of deterministic elements versus Wiener integral with respect to mBm

In section 4.2, we have defined a Wiener integral with respect to mBm. It is natural to check whether this definition is consistent with the multifractional Wick-Ito integral when the integrand is deterministic. More precisely, we wish to verify that $\int_{\mathbb{R}} f(s) \diamond W^{(h)}(s)ds = \mathcal{J}^h(f)$ for all functions f such that both members of the previous equality exist and that the left-hand side member is in (L^2). In that view we first prove the following theorem.

Theorem 5.25. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a deterministic function which belongs to* $L^1_{loc}(\mathbb{R})$. *Let* $Z := (Z(t)_{t\in\mathbb{R}})$ *be the process defined on* $\mathbb{R}$ *by* $Z(t) := \int_0^t f(s)dB^{(h)}(s)$. *Then* Z *is an* $(\mathcal{S}^*)$*-process which verifies the following equality in* $(\mathcal{S}^*)$

$$\int_0^t f(s)dB^{(h)}(s) = \sum_{k=0}^{+\infty}\Big(\int_0^t f(s)\tfrac{d}{ds}[g_{e_k}(s,h(s))]ds\Big) < .,e_k >. \tag{5.42}$$

Moreover Z *is a (centered) Gaussian process if and only if* $\sum_{k=0}^{+\infty}\big(\int_0^t f(s)\frac{d}{ds}[g_{e_k}(s,h(s))]ds\big)^2 < +\infty$*, for all* t*. In this case,*

$$Z(t) = \int_0^t f(s)dB^{(h)}(s) \rightsquigarrow \mathcal{N}\left(0, \sum_{k=0}^{+\infty}\Big(\int_0^t f(s)\tfrac{d}{ds}[g_{e_k}(s,h(s))]ds\Big)^2\right), \qquad \forall t \in \mathbb{R}. \tag{5.43}$$

In particular, the process Z *is Gaussian when the function* f *belongs to* $C^1(\mathbb{R},\mathbb{R})$ *and is such that* $\sup_{t\in\mathbb{R}}|f'(t)| < +\infty$.

Proof. We treat only the case $t \in \mathbb{R}_+^*$. The other case follows similarly. Let f be in $L^1_{\text{loc}}(\mathbb{R})$. In order to show (5.42) let us establish **a), b)** and **c)** below.

a) Let us show that $s \mapsto f(s) \diamond W^{(h)}(s)$ is $(\mathcal{S})^*$-integrable over $[0,t]$. For every $\eta \in \mathscr{S}(\mathbb{R})$ and s in $[0,t]$, we get, using lemma 2.4:

$$|S(f(s) \diamond W^{(h)}(s))(\eta)| = |f(s)| \left|S(W^{(h)}(s))(\eta)\right| \leqslant \overbrace{|f(s)| \; ||W^{(h)}(s)||_{-p_0}}^{=:L(s)} e^{\frac{1}{2}|\eta|^2_{p_0}}$$

for s in $[0,t]$ and for $p_0 \geqslant 2$. Since L is the product of a continuous function and a function of $L^1_{\text{loc}}(\mathbb{R})$, **a)** is a consequence of theorem 2.8.

b) $\Psi_f := \sum_{k=0}^{+\infty} \left(\int_0^t f(s) \frac{d}{ds}[g_{e_k}(s,h(s))]ds\right) < ., e_k >$ belongs to $(\mathcal{S}_{-p_0})$ as soon as $p_0 \geqslant 2$.

Lemma 5.6 entails that there exists a real D such that, for every $p_0 \geqslant 2$, we have

$$\begin{aligned} ||\Psi_f||^2_{-p_0} &= \sum_{k=0}^{+\infty} \left(\frac{\int_0^t f(s) \; \frac{d}{ds}[g_{e_k}(s,h(s))] \; ds}{(2k+2)^{p_0}}\right)^2 \leqslant ||f||^2_{L^1([0,t])} \sum_{k=0}^{+\infty} \frac{\sup_{s\in[0,t]} |\frac{d}{ds}[g_{e_k}(s,h(s))]|^2}{(2k+2)^{2p_0}} \\ &\leqslant D^2 \; ||f||^2_{L^1([0,t])} \sum_{k=0}^{+\infty} \frac{(k+1)^2}{(2k+2)^{2p_0}} < \; +\infty. \end{aligned}$$

c) $\int_0^t f(s) dB^{(h)}(s) = \sum_{k=0}^{+\infty} \left(\int_0^t f(s) \frac{d}{ds}[g_{e_k}(s,h(s))]ds\right) < ., e_k >$ in $(\mathcal{S}^*)$.

Denote $\Phi_f := \int_0^t f(s) dB^{(h)}(s) = \int_0^t \left(\sum_{k=0}^{+\infty} f(s) \frac{d}{ds}[g_{e_k}(s,h(s))] < ., e_k >\right) ds$ and define the $(\mathcal{S}^*)$-process $\tau : [0,t] \to (\mathcal{S}^*)$ by $\tau(s) := \sum_{k=0}^{+\infty} f(s) \frac{d}{ds}[g_{e_k}(s,h(s))] < ., e_k >$. Moreover, for N in $\mathbb{N}^*$, define on $[0,t]$, $\tau_N : s \mapsto \tau_N(s) := \sum_{k=0}^{N} f(s) \frac{d}{ds}[g_{e_k}(s,h(s))] < ., e_k >$. Obviously we have, in $(\mathcal{S}^*)$, $\Phi_f = \int_0^t \tau(s) ds$, $\Psi_f = \lim_{N\to+\infty} \int_0^t \tau_N(s) \; ds$. It then remains to show that $\Phi_f = \lim_{N\to+\infty} \int_0^t \tau_N(s) \; ds$ in $(\mathcal{S}^*)$. Let us use, for this purpose, theorem X.2. Let p_0 be an integer greater than or equal to 2. It is easily seen that τ_n and τ are weakly measurable on $[0,t]$ for every n in $\mathbb{N}$ (see definition X.1) and that, $\tau_n(s)$ and $\tau(s)$ belongs to $(\mathcal{S}_{-p_0})$ for every n in $\mathbb{N}$ and s in $[0,t]$. Moreover, both functions $s \mapsto ||\tau_n(s)||_{-p_0}$ and $s \mapsto ||\tau(s)||_{-p_0}$ belong to $L^1([0,t], du)$ since $||\tau_n(s)||_{-p_0} \leqslant ||\tau(s)||_{-p_0} \leqslant |f(s)| \; D\sqrt{\sum_{k=0}^{+\infty} (2k+2)^{-2(p_0-1)}}$ for a certain D given by lemma 5.6 (ii). We hence have shown that both functions $\tau_n(.)$ and $\tau(.)$ are Bochner integrable on $[0,t]$. Besides, for every (n,m) in $\mathbb{N}^2$ with $n \geqslant m$, we have

$$\begin{aligned} \int_0^t ||\tau_n(s) - \tau_m(s)||_{-p_0} \; ds &\leqslant \underbrace{\int_0^t \left|\left|\sum_{k=m+1}^{+\infty} f(s) \tfrac{d}{ds}[g_{e_k}(s,h(s))] < ., e_k >\right|\right|_{-p_0} ds}_{=\int_0^t \; ||\tau(s)-\tau_m(s)||_{-p_0} ds} \\ &\leqslant M \left(\sum_{k=m+1}^{+\infty} \frac{\sup_{s\in[0,t]} |\frac{d}{ds}[g_{e_k}(s,h(s))]|^2}{(2k+2)^{2p_0}}\right)^{1/2} \leqslant M \; D \left(\sum_{k=m+1}^{+\infty} \frac{1}{(2k+2)^{2(p_0-1)}}\right)^{1/2} \xrightarrow[(n,m)\to(+\infty,+\infty)]{} 0, \end{aligned}$$

where $M := ||f||_{L^1([0,t])}$ and D is again given by (ii) of lemma 5.6. Theorem X.2 then applies and allows to write that $\lim_{N\to+\infty} \int_0^t \tau_N(s) \; ds = \int_0^t \tau(s) \; ds$ in $(\mathcal{S}^*)$. We hence have shown that $\Psi_f = \lim_{N\to+\infty} \int_0^t \tau_N(s) \; ds = \int_0^t \tau(s) ds = \Phi_f$ in $(\mathcal{S}^*)$. This ends the proof of **c)** and establishes formula (5.42).

If $\sum_{k=0}^{+\infty} \left(\int_0^t f(s) \frac{d}{ds}[g_{e_k}(s,h(s))]ds\right)^2 < +\infty$, for all t, then $Z(t)$ is the (L^2)-limit of a sequence of independent Gaussian variables. Formula (5.43) is then obvious. When f is of class C^1 and such that $\sup_{t\in\mathbb{R}} |f'(t)| < +\infty$, an integration by parts yields that $\sum_{k=0}^{+\infty} \left(\int_0^t f(s) \frac{d}{ds}[g_{e_k}(s,h(s))]ds\right)^2 < +\infty$. □

It is easy to check that definitions 4.1 and 5.1 coincide on the space $\mathcal{E}(\mathbb{R})$. Indeed for $f := \sum_{k=1}^n \alpha_k \mathbb{1}_{[0,t_k]}$ in $\mathcal{E}(\mathbb{R})$, remark 4.3 and equality (5.12) entail that $\mathcal{J}^h(f) = \sum_{k=1}^n \alpha_k B^{(h)}_{t_k}$ almost surely. According to (i) of proposition 5.14, we have the equality $\int_{\mathbb{R}} f(s)dB^{(h)}(s)ds = \sum_{k=1}^{n} \alpha_k \int_{\mathbb{R}} \mathbb{1}_{[0,t_k]}(s)W^{(h)}(s)ds = \sum_{k=1}^{n} \alpha_k B_{t_k}^{(h)}$ almost surely. This implies in particular that $|| \int_{\mathbb{R}} f(s)dB^{(h)}(s)ds||_{(L^2)} = ||f||_h$ for all f in $\mathcal{E}(\mathbb{R})$ since we have $||\mathcal{J}^h(f)||_{(L^2)} = ||f||_h$ for such f. Since Wiener integrals with respect to standard Brownian motion are the elements of the set $\{\int_{\mathbb{R}} f(s)\, dB(s),\ f \in L^2(\mathbb{R})\} = \overline{\{\int_{\mathbb{R}} f(s)\, dB(s),\ f \in \mathcal{E}(\mathbb{R})\}}^{(L^2)}$, it seems natural to give the following definition.

Definition 5.2. *(Wiener integral with respect to mBm)*

For an mBm $B^{(h)}$, define the Gaussian space $\Theta_h := \overline{\{\int_{\mathbb{R}} f(s)\, dB^{(h)}(s),\ f \in \mathcal{E}(\mathbb{R})\}}^{(L^2)}$. We call Wiener integral with respect to $B^{(h)}$ the elements of Θ_h.

Remark 5.26. *(i) Obviously, $\Theta_h = \overline{\{\sum_{k=0}^{+\infty} \left(\int_{\mathbb{R}} f(s) \frac{d}{ds}[g_{e_k}(s, h(s))]ds\right) < ., e_k > \ :\ f \in \mathcal{E}(\mathbb{R})\}}^{(L^2)}$. Thanks to definition 4.1, theorem 5.25 and the fact that $\mathcal{J}^h(f) = \int_{\mathbb{R}} f(s)\, dB^{(h)}(s)$ on $\mathcal{E}(\mathbb{R})$, we have*

$$\Theta_h = \{\mathcal{J}^h(u)\ :\ u \in \overline{\mathcal{E}(\mathbb{R})}^{<,>_h}\} \tag{5.44}$$

In other words, the set of Wiener integrals in the sense of definition 4.1 and 5.2 coincide.
(ii) When h is a constant function equal to H we find that $\Theta_h = \Theta_H = \{< ., M_H(f) > \ :\ f \in L^2_H(\mathbb{R})\}$ since $\overline{\mathcal{E}(\mathbb{R})}^{<,>_h} = \overline{\mathcal{E}(\mathbb{R})}^{<,>_H} = L^2_H(\mathbb{R})$. This is exactly what is expected in view of (4.1).

In fact we can be a little more precise in the case of fBm. Let $supp(\mathcal{K})$ denote the set of measurable functions $f : \mathbb{R} \to \mathbb{R}$ with compact support.

Proposition 5.27. *Let $H \in (0, 1)$. Then:*
(i) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be in $L^1_{loc}(\mathbb{R}) \cap L^2_H(\mathbb{R})$. Then $\int_{\mathbb{R}} f(s)\, dB^{(H)}(s)$ belongs to (L^2) if and only if $\int_{\mathbb{R}} f(s)\, dB^{(H)}(s) = \mathcal{J}^H(f)$.
(ii) $L^1_{loc}(\mathbb{R}) \cap L^2_H(\mathbb{R}) \cap supp(\mathcal{K}) \subset \{f : \mathbb{R} \to \mathbb{R} :\ \int_{\mathbb{R}} f(s)\, dB^{(H)}(s) \in (L^2)\}$.
(iii) For μ-almost every f in $L^1_{loc}(\mathbb{R}) \cap supp(\mathcal{K}) \cap \{f : \mathbb{R} \to \mathbb{R} :\ \int_{\mathbb{R}} f(s)\, dB^{(H)}(s) \in (L^2)\}$, f is in $L^2_H(\mathbb{R})$ and verifies $\int_{\mathbb{R}} f(s)\, dB^{(H)}(s) = \mathcal{J}^H(f)$.

Proof. (i) Let $f \in L^1_{loc}(\mathbb{R}) \cap L^2_H(\mathbb{R})$ and define $\Phi_f := \int_{\mathbb{R}} f(s)\, dB^{(H)}(s)$. By theorem 5.25, $\Phi_f = \sum_{k=0}^{+\infty} < f, M_H(e_k) >< ., e_k >$ where the equality holds in $(\mathcal{S})^*$. If we assume that Φ_f belongs to (L^2), then the equality is valid in (L^2). Besides, since f belongs to $L^1_{loc}(\mathbb{R}) \cap L^2_H(\mathbb{R})$ we have, according to theorem 3.7, $\mathcal{J}^H(f) =< ., M_H(f) >= \sum_{k=0}^{+\infty} < M_H(f), e_k >_{L^2(\mathbb{R})} < ., e_k >= \sum_{k=0}^{+\infty} < f, M_H(e_k) >< ., e_k >$ in (L^2). The converse part is obvious since $\int_{\mathbb{R}} f(s)\, dB^{(H)}(s) = \mathcal{J}^H(f)$ entails that $\int_{\mathbb{R}} f(s)\, dB^{(H)}(s) \in (L^2)$.
(ii) Since f is in $L^1_{loc}(\mathbb{R}) \cap supp(\mathcal{K})$ theorem 5.25 (by replacing $\mathbb{1}_{[0,t]}$ by $\mathbb{1}_{supp(f)}$, where $supp(f)$ denotes the support of f) entails that $\int_{\mathbb{R}} f(s)dB^{(H)}(s) = \sum_{k=0}^{+\infty} \left(\int_{\mathbb{R}} f(s)M_H(e_k)(s)ds\right) < ., e_k > = \sum_{k=0}^{+\infty} < f, M_H(e_k) >< ., e_k >$ in $(\mathcal{S})^*$. Besides, since f belongs to $L^2_H(\mathbb{R})$, $\mathcal{J}^H(f)$ exists and is equal to $\sum_{k=0}^{+\infty} < f, M_H(e_k) >< ., e_k >$ in (L^2).
(iii) Fix f in $L^1_{loc}(\mathbb{R}) \cap supp(\mathcal{K}) \cap \{f : \mathbb{R} \to \mathbb{R} :\ \int_{\mathbb{R}} f(s)\, dB^{(H)}(s) \in (L^2)\}$ and define $\widetilde{\Omega}$ as subset of ω in Ω such that (5.7) is true for all g in $\{e_k :\ k \in \mathbb{N}\}$. Clearly $\widetilde{\Omega}$ belongs to $\mathcal{G}$. As soon as f is in $\widetilde{\Omega}$, we can write $\int_{\mathbb{R}} f(s)\, dB^{(H)}(s) = \sum_{k=0}^{+\infty} < M_H(f), e_k >< ., e_k >$ in (L^2). This entails that $M_H(f)$ belongs to $L^2(\mathbb{R})$ and then, by bijectivity of M_H, that f belongs to $L^2_H(\mathbb{R})$. □

Remark 5.28. *This proposition shows in particular, that for $\mu-$almost every g in $supp(\mathcal{K})$:*

$$g \in L^2_H(\mathbb{R}) \Leftrightarrow \int_{\mathbb{R}} g(s)\, dB^{(H)}(s) \in (L^2).$$

Moreover, in this case, $\mathcal{J}^H(g) \overset{(L^2)}{=} \int_{\mathbb{R}} g(s)\, dB^{(H)}(s)$.

6 Itô Formulas

6.1 Itô Formula for generalized functionals of mBm on an interval $[a, b]$ with $0 < a < b$

Let us fix some notations. For a tempered distribution G and a positive integer n, let $G^{(n)}$ denote the n^{th} distributional derivative of G. We also write $G' := G^{(1)}$. Hence, by definition, the equality $< G', \varphi >= - < G, \varphi' >$ holds for all φ in $\mathscr{S}(\mathbb{R})$. For a map $t \mapsto F(t)$ from $[a, b]$ to $\mathscr{S}_{-p}(\mathbb{R})$ we will note $\frac{\partial^n F}{\partial x^n}(t)$ the quantity $(F(t))^{(n)}$, that is the n^{th} derivative in $\mathscr{S}'(\mathbb{R})$, of the tempered distribution $F(t)$. Hence we may consider the map $t \mapsto \frac{\partial^n F}{\partial x^n}(t)$ from $[a, b]$ to $\mathscr{S}'(\mathbb{R})$. Moreover for any t_0 in $[a, b]$, we will note $\frac{\partial F}{\partial t}(t_0)$ the quantity $\lim_{r \to 0} \frac{F(t_0+r)-F(t_0)}{r}$ when it exists in $\mathscr{S}_{-p}(\mathbb{R})$, for a certain integer p. When it exists, $\frac{\partial F}{\partial t}(t_0)$ is a tempered distribution, which is said to be the derivative of the distribution $F(t)$ with respect to t at point $t = t_0$. In line with section 5.3, we then define, for t_0 in $[a, b]$ and a positive integer n, the following quantities:

$$\frac{\partial^n F}{\partial x^n}(t_0, B^h(t_0)) := (F(t_0))^{(n)}(B^h(t_0)) \quad \text{and} \quad \frac{\partial F}{\partial t}(t_0, B^h(t_0)) := \Big(\frac{\partial F}{\partial t}(t_0)\Big)(B^h(t_0)).$$

Theorem 6.1. *Let $p \in \mathbb{N}$, a and b two reals with $0 < a < b$, and let F be an element of $C^1([a, b], \mathscr{S}_{-p}(\mathbb{R}))$ such that both maps $\frac{\partial F}{\partial x}$ and $\frac{\partial^2 F}{\partial x^2}$, from $[a, b]$ into $\mathscr{S}_{-p}(\mathbb{R})$, are continuous. Then the following equality holds in $(\mathcal{S})^*$:*

$$F(b, B^{(h)}(b)) - F(a, B^{(h)}(a)) = \int_a^b \frac{\partial F}{\partial t}(s, B^{(h)}(s))\, ds + \int_a^b \frac{\partial F}{\partial x}(s, B^{(h)}(s))\, dB^{(h)}(s) + \frac{1}{2} \int_a^b \Big(\frac{d}{ds}[R_h(s, s)]\Big) \frac{\partial^2 F}{\partial x^2}(s, B^{(h)}(s))\, ds. \quad (6.1)$$

Remark 6.2. *Recall that for all t in $[a, b]$, $\frac{d}{dt}[R_h(t, t)] = 2\, t^{2h(t)-1}\, (h'(t)\, t \ln t + h(t))$.*

Proof. We follow closely [8] p.97-98 for this proof. First notice that the three integrals on the right side of (6.1) exist since all integrands verify the assumptions of theorem 5.18. According to lemma 2.7 it is then sufficient to show equality of the S-transforms of both sides of (6.1). It is easy to see that, for every $\eta \in \mathscr{S}(\mathbb{R})$, the function $t \mapsto \gamma\Big(t^{2h(t)}, . - \int_0^t M_{h(t)}(\eta)\,(u)\,du\Big)$ is differentiable from $(0, b]$ into $\mathscr{S}(\mathbb{R})$. Using theorem 5.12 and theorem 5.17 we may write, for t in $[0, b]$:

$$\begin{aligned}
\frac{d}{dt} S(F(t, B^{(h)}(t))(\eta) &= \frac{d}{dt}\Big\langle F(t), \gamma\Big(t^{2h(t)}, . - \int_0^t M_{h(t)}(\eta)\,(u)\,du\Big)\Big\rangle \\
&= \Big\langle \frac{\partial F}{\partial t}(t), \gamma\Big(t^{2h(t)}, . - \int_0^t M_{h(t)}(\eta)\,(u)\,du\Big)\Big\rangle \\
&\quad + 2\, t^{2h(t)-1}\, (h'(t)\, t \ln t + h(t))\, \Big\langle F(t), \frac{\partial \gamma}{\partial t}\Big(t^{2h(t)}, . - \int_0^t M_{h(t)}(\eta)\,(u)\,du\Big)\Big\rangle \\
&\quad - \frac{d}{dt}[g_\eta(t, h(t))]\, \Big\langle F(t), \frac{\partial \gamma}{\partial x}\Big(t^{2h(t)}, . - \int_0^t M_{h(t)}(\eta)\,(u)\,du\Big)\Big\rangle =: I_1 + I_2 + I_3.
\end{aligned}$$

Now, $I_1 = S\Big(\frac{\partial F}{\partial t}(t)(B^{(h)}(t))\Big)(\eta) = S\Big(\frac{\partial F}{\partial t}(t, B^{(h)}(t))\Big)(\eta)$ using theorem 5.12 (iii). Besides, since γ fulfills the equality $\frac{\partial \gamma}{\partial t} = \frac{1}{2}\frac{\partial^2 \gamma}{\partial x^2}$, it is clear that

$$\begin{aligned}
I_2 &= t^{2h(t)-1}\, (h'(t)\, t \ln t + h(t))\, \Big\langle F(t), \frac{\partial^2 \gamma}{\partial x^2}\Big(t^{2h(t)}, . - \int_0^t M_{h(t)}(\eta)\,(u)\,du\Big)\Big\rangle \\
&= \frac{1}{2}\, \frac{d}{dt}[R_h(t, t)]\, S\Big(\frac{\partial^2 F}{\partial x^2}(t, B^{(h)}(t))\Big)(\eta).
\end{aligned}$$

Using (ii) of theorem 5.12, we get $I_3 = S(W^{(h)}(t))(\eta)\ \left\langle \frac{\partial F}{\partial x}(t), \gamma\left(t^{2h(t)}, . - \int_0^t M_{h(t)}(\eta)\ (u)\ du\right)\right\rangle$. Finally, we obtain

$$\frac{d}{dt}S(F(t,B^{(h)}(t))(\eta) = S\Big(\frac{\partial F}{\partial t}(t,B^{(h)}(t))\Big)(\eta) + S\left(W^{(h)}(t) \diamond \frac{\partial F}{\partial x}(t,B^{(h)}(t))\right)(\eta)$$
$$+ \frac{1}{2}\ \frac{d}{dt}[R_h(t,t)]\ S\Big(\frac{\partial^2 F}{\partial x^2}(t,B^{(h)}(t))\Big)(\eta). \qquad \square$$

In the proof of theorem 6.6 we will need the particular case where the function $F(.)$ is constant, equal to a tempered distribution that we denote F. In this case we have the following

Corollary 6.3. *Let $0 < a < b$ and F be a tempered distribution. Then the following equality holds in $(\mathcal{S})^*$:*

$$F(B^{(h)}(b)) - F(B^{(h)}(a)) = \int_a^b F'(B^{(h)}(s))\ dB^{(h)}(s) + \frac{1}{2}\int_a^b \left(\frac{d}{ds}[R_h(s,s)]\right)\ F''(B^{(h)}(s))\ ds.$$

Remark 6.4. *Of course when the function h is constant on $\mathbb{R}$, we get the Itô formula for fractional Brownian motion given in [8].*

6.2 Itô Formula in (L^2)

In this subsection, we give two further versions of Itô formula. The first one holds for functions with polynomial growth but weak differentiability assumptions, whereas the second one deals with $C^{1,2}$ functions with sub-exponential growth.

Itô Formula for certain generalized functionals of mBm on an interval

Theorem 6.1 does not extend immediately to the case $a = 0$ because the generalized functional is not defined in this situation, since $M_h(\mathbb{1}_{[0,t]})$ converges to 0 a.s and in $L^2(\mathbb{R})$ when t tends to 0 (see theorem-definition 5.2). As in [8], we now extend the formula to deal with this difficulty. We will need the following lemma which is a particular case of lemma 6.8 below.

Lemma 6.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists a couple (C, λ) in $\mathbb{R} \times \mathbb{R}_+$ with $|f(y)| \leqslant Ce^{\lambda y^2}$, for all real y. Let $g : \mathbb{R} \to \mathbb{R}_+^*$ be a measurable function such that $\lim_{t\to 0} g(t) = 0$ and define L_f on $\mathbb{R}_+^* \times \mathbb{R}$ by $L_f(u,x) := \int_{\mathbb{R}} f(y)\gamma(u, x-y)\ dy$. Then $\lim_{(t,x)\to(0^+,x_0)} L_f(g(t),x) = f(x_0)$, for all real x_0.*

Theorem 6.6. *Let $F : \mathbb{R} \to \mathbb{R}$ be continuous at 0 and of polynomial growth. Assume that the first distributional derivative of F is of function type (defined at the beginning of section 3.1). Then the following equality holds in (L^2):*

$$F(B^{(h)}(b)) - F(0) = \int_0^b F'(B^{(h)}(s))\ dB^{(h)}(s) + \frac{1}{2}\int_0^b \left(\frac{d}{ds}[R_h(s,s)]\right)\ F''(B^{(h)}(s))\ ds. \tag{6.2}$$

Proof. We follow again closely [8].

Step 1: $\lim_{t\to 0_+} F(B^{(h)}(t)) = F(0)$ in $(\mathcal{S}^*)$. In order to establish this fact, let us use theorem 8.6 of [52]. Since F is of polynomial growth, we may write, thanks to formula (29) of [8], that there exist two reals C and M and a positive integer m such that $\mathbb{E}[F(B^{(h)}(t))^2] \leqslant C^2(1 + \frac{(2m)!}{2^m m!}|t|^{2mh(t)}) \leqslant M^2$, for all t in $[0,b]$. Since $||:e^{<.,\eta>}:||_0 = e^{\frac{1}{2}|\eta|_0^2}$, Cauchy-Schwarz inequality yields, for all t in $[0,b]$ and η in $\mathscr{S}(\mathbb{R})$

$$|S[F(B^{(h)}(t))](\eta)| = |\mathbb{E}[F(B^{(h)}(t))\ :e^{<.,\eta>}:]| \leqslant ||F(B^{(h)}(t))||_0\ ||:e^{<.,\eta>}:||_0 \leqslant (1+M)\ e^{\frac{1}{2}|\eta|_0^2}. \tag{6.3}$$

It then just remains to show that $\lim_{t\to 0+} S[F(B^{(h)}(t))](\eta) = F(0)$. Thanks to theorem 5.12 and lemma 6.5, we get

$$\lim_{t\to 0+} S[F(B^{(h)}(t))](\eta) = \lim_{t\to 0+} \int_{\mathbb{R}} F(y)\frac{1}{\sqrt{2\pi t^{2h(t)}}} \exp\left\{-\frac{1}{2t^{2h(t)}}\left(\int_0^t M_{h(t)}(\eta)(u)\ du - y\right)^2\right\} dy$$
$$= \lim_{t\to 0+} \int_{\mathbb{R}} F(y)\ \gamma\left(t^{2h(t)}, \int_0^t M_{h(t)}(\eta)(u)\ du - y\right)\ dy$$
$$= \lim_{t\to 0+} L_F\left(t^{2h(t)}, \int_0^t M_{h(t)}(\eta)(u)\ du\right) = F(0). \tag{6.4}$$

Step 2: $\lim_{a\to 0+} \int_a^b F'(B^{(h)}(t))\ dB^{(h)}(t) = \int_0^b F'(B^{(h)}(t))\ dB^{(h)}(t)$ in $(S)^*$.
Define $[H_1, H_2] := [\min_{t\in[0,b]} h(t), \max_{t\in[0,b]} h(t)]$ and let us prove the two following facts

(i) There exists a constant D_1 which depends only of F such that $||F'(B^{(h)})(t)||_{-1} \leqslant D_1\ \max\left\{\frac{1}{t^{H_1}}, \frac{1}{t^{H_2}}\right\}$ for all t in $(0, b]$.

Let us first notice that, for all (x, b, t) in $\mathbb{R}\times\mathbb{R}_+^*\times(0, b]$, we have $\exp\{-x^2/4t^{2h(t)}\} \leqslant e^{-x^2/4}+\varepsilon(b)\ \exp\{-x^2/4b^{2H_2}\}$ where $\varepsilon(b) = 1$ if $b \geqslant 1$ and $\varepsilon(b) = 0$ if $b < 1$.
Note moreover that the function $x \mapsto F'(x)\ (e^{-x^2/4} + \varepsilon(b)\ \exp\{-x^2/4b^{2H_2}\})$ belongs to $L^1(\mathbb{R})$ since F' is of function type and belongs to $\mathscr{S}'(\mathbb{R})$. Since the operator A^{-1} has a norm operator equal to 1/2 (see [52] p.17) and using the equality $|M_{h(t)}(\mathbb{1}_{[0,t]})|_0^2 = t^{2h(t)}$, we get the following upper bound, valid for all k in $\mathbb{N}$,

$$|< F', \xi_{(t,h(t),k)} >|^2 = \left|\int_{\mathbb{R}} F'(x)\ \pi^{1/4}(k!)^{1/2}t^{kh(t)}\exp\{-\tfrac{x^2}{4t^{2h(t)}}\}e_k(x/(\sqrt{2}t^{h(t)}))\ dx\right|^2$$
$$\leqslant \pi^{1/2}\ \sup_{u\in\mathbb{R}}|e_k^2(u)|\ \left(\int_{\mathbb{R}} |F'(x)|\ \exp\{-\tfrac{x^2}{4t^{2h(t)}}\}\ dx\right)^2\ t^{2kh(t)}\ k!$$
$$\leqslant \underbrace{\pi^{1/2} \sup\left\{\sup_{u\in\mathbb{R}}|e_k^2(u)| : k \in \mathbb{N}\right\}\ \left(\int_{\mathbb{R}} |F'(x)|\ (e^{-x^2/4} + \varepsilon(b)\ \exp\{-x^2/4b^{2H_2}\})\ dx\right)^2}_{=:D_0}\ t^{2kh(t)}\ k!.$$

Using (ii) of remark 2.3 and again the fact that the operator A^{-1} has a norm operator equal to 1/2 (see (2) p.17 of [52]) we can write, for all real t in $(0, b]$, that

$$||F'(B^{(h)}(t))||_{-1}^2 = \left|\left|\frac{1}{\sqrt{2\pi}t^{h(t)}}\sum_{k=0}^{+\infty}(k!)^{-1}\ t^{-2kh(t)}\ < F', \xi_{(t,h(t),k)} > I_k\left((\ M_{h(t)}(\mathbb{1}_{[0,t]})\ )^{\otimes k}\right)\right|\right|_{-1}^2$$
$$= \frac{1}{2\pi t^{2h(t)}}\sum_{k=0}^{+\infty}(k!)^{-1}\ t^{-4kh(t)}\ |< F', \xi_{(t,h(t),k)} >|^2\ \underbrace{|(A^{-1})^{\otimes k}\left((\ M_{h(t)}(\mathbb{1}_{[0,t]})\ )^{\otimes k}\right)|_0^2}_{=|A^{-1}(M_{h(t)}(\mathbb{1}_{[0,t]}))|_0^{2k}}$$
$$\leqslant \frac{D_0}{2\pi t^{2h(t)}}\sum_{k=0}^{+\infty}(k!)^{-1}\ t^{-4kh(t)}\ t^{2kh(t)}\ k!\left(\frac{1}{2}\right)^k t^{2kh(t)} \leqslant \frac{D_0}{\pi}\ \max\left\{\frac{1}{t^{2H_1}}, \frac{1}{t^{2H_2}}\right\}. \tag{6.5}$$

(ii) $\int_0^b F'(B^{(h)}(t))\ dB^{(h)}(t)$ exists in $(\mathcal{S})^*$ and is equal to $\lim_{a\to 0+} \int_a^b F'(B^{(h)}(t))\ dB^{(h)}(t)$ in the sense of $(\mathcal{S})^*$.

In order to establish the existence of $\int_0^b F'(B^{(h)}(t))\ dB^{(h)}(t)$ in $(\mathcal{S})^*$, let us use theorem 2.8. From theorem-definition 2.2, we know that $F'(B^{(h)}(t)) \diamond W^{(h)}(t))$ belongs to $(\mathcal{S})^*$ for every t in $(0, b]$. Moreover using lemma 2.5 we get, for $\eta \in \mathscr{S}(\mathbb{R})$ and $t \in (0, b]$,

$$\left|S(F'(B^{(h)}(t)) \diamond W^{(h)}(t))(\eta)\right| \leqslant ||F'(B^{(h)}(t))||_{-1}\ ||W^{(h)}(t)||_{-2}\ e^{|\eta|_2^2} \leqslant \underbrace{\widehat{K}\ \max\{\tfrac{1}{t^{H_1}}, \tfrac{1}{t^{H_2}}\}}_{=:\mathcal{L}(t)}\ e^{|\eta|_2^2}, \tag{6.6}$$

where we have defined $\widehat{K} := \frac{D_0}{\pi} \sup_{t\in[0,b]} ||W^{(h)}(t)||_{-2}$. The function $t \mapsto S(F'(B^{(h)}(t))\ \diamond\ W^{(h)}(t))(\eta)$ is measurable on $[0, b]$ since $S(F'(B^{(h)}(t))\ \diamond\ W^{(h)}(t))(\eta) = S(F'(B^{(h)}(t)))(\eta)S(W^{(h)}(t))(\eta)$ using theorems 5.12

and 5.17. Moreover, since $\mathcal{L}$ belongs to $L^1([0,b])$, theorem 2.8 applies and shows that $\int_0^b F'(B^{(h)}(t))\, dB^{(h)}(t)$ is in $(\mathcal{S})^*$. It then just remains to use theorem 8.6 in [52] to show the convergence, in the sense of $(\mathcal{S})^*$, of $\int_a^b F'(B^{(h)}(t))\, dB^{(h)}(t)$ to $\int_0^b F'(B^{(h)}(t))\, dB^{(h)}(t)$ as a tends to 0_+. Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers which tends to 0 when n tends to $+\infty$ and $\Psi_n := \int_0^b F'(B^{(h)}(t))\, dB^{(h)}(t) - \int_{a_n}^b F'(B^{(h)}(t))\, dB^{(h)}(t)$. For every $\eta \in \mathscr{S}(\mathbb{R})$ and every $n \in \mathbb{N}$, $S(\Psi_n)(\eta) = \int_0^b \mathbb{1}_{[0,a_n]}(t)\, S(F'(B^{(h)}(t)) \diamond W^{(h)}(t))(\eta)\, dt$. Using (6.6) and the dominated convergence theorem, it is easy to show that $\lim_{n\to+\infty} S(\Psi_n)(\eta) = 0$. Hence theorem 8.6 of [52] applies and shows that $\lim_{a\to 0+} \int_a^b F'(B^{(h)}(t))\, dB^{(h)}(t) = \int_0^b F'(B^{(h)}(t))\, dB^{(h)}(t)$ in $(S)^*$.

Step 3: Proof of (6.2)

For any real a such that $0 < a < b$, we have, thanks to corollary 6.3,

$$F(B^{(h)}(b)) - F(B^{(h)}(a)) - \int_a^b F'(B^{(h)}(s))\, dB^{(h)}(s) = \frac{1}{2}\int_a^b \left(\frac{d}{ds}[R_h(s,s)]\right) F''(B^{(h)}(s))\, ds.$$

Steps 1 and 2 ensure that the left hand side has a limit in $(\mathcal{S})^*$ when a tends to 0. Using theorem 2.8, it is easy to see that $\int_0^b \left(\frac{d}{ds}[R_h(s,s)]\right) F''(B^{(h)}(s))\, ds$ belongs to $(\mathcal{S})^*$. Hence using the dominated convergence theorem and lemma 2.7 we deduce that $\lim_{a\to 0+} \int_a^b \left(\frac{d}{ds}[R_h(s,s)]\right) F''(B^{(h)}(s))\, ds$ is equal to $\int_0^b \left(\frac{d}{ds}[R_h(s,s)]\right) F''(B^{(h)}(s))\, ds$ in $(\mathcal{S})^*$. Since we have proved that, for all t in $[0,b]$, $F(B^{(h)}(t))$ belongs to (L^2), the same holds for the right hand side of (6.2) and then this equality holds also in (L^2). □

Remark 6.7. *As in the case of fBm (see [8]), the fact that both sides of the equality* (6.2) *are in* (L^2) *does not imply that every single element of the right hand side is in* (L^2). *This will be true if, for instance,* $F''(B^{(h)}(t))$ *belongs to* (L^2) *and* $\int_0^b \left|\frac{d}{ds}[R_h(s,s)]\right| \, ||F''(B^{(h)}(s))||_0\, ds < +\infty$.

Itô Formula in (L^2) for $C^{1,2}$ functions with sub-exponential growth

Let us begin with the following lemma:

Lemma 6.8. *Let* $T > 0$ *and* $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ *be a continuous function such that there exists a couple* (C_T, λ_T) *of* $\mathbb{R}\times\mathbb{R}_+^*$ *such that* $\max_{t\in[0,T]} |f(t,y)| \leqslant C_T e^{\lambda_T y^2}$ *for all real* y. *Define* $a > \lambda_T$, $I_a := (0, \frac{1}{4a})$ *and* $J_f : \mathbb{R}\times\mathbb{R}_+\times I_a \to \mathbb{R}$ *by* $J_f(x,t,u) := \int_{\mathbb{R}} f(t,y)\gamma(u, x-y)\, dy$. *Then* J_f *is well defined and moreover* $\lim_{(x,t,u)\to(x_0,0^+,0^+)} J(x,t,u) = f(0,x_0)$.

Proof. This is an immediate consequence of theorems 1 p.88 and 2 p.89 of [78]. □

Let us now give an Itô formula for functions with subexponential growth.

Theorem 6.9. *Let* $T > 0$ *and* $h : \mathbb{R} \to (0,1)$ *be a* C^1 *function such that* h' *is bounded on* $\mathbb{R}$. *Let* f *be a* $C^{1,2}([0,T]\times\mathbb{R},\mathbb{R})$ *function. Furthermore, assume that there are constants* $C \geqslant 0$ *and* $\lambda < \frac{1}{4\max_{t\in[0,T]} t^{2h(t)}}$ *such that for all* (t,x) *in* $[0,T]\times\mathbb{R}$,

$$\max_{t\in[0,T]} \left\{ |f(t,x)|, \left|\frac{\partial f}{\partial t}(t,x)\right|, \left|\frac{\partial f}{\partial x}(t,x)\right|, \left|\frac{\partial^2 f}{\partial x^2}(t,x)\right| \right\} \leqslant C e^{\lambda x^2}. \tag{6.7}$$

Then, for all t *in* $[0,T]$, *the following equality holds in* (L^2):

$$f(T, B^{(h)}(T)) = f(0,0) + \int_0^T \frac{\partial f}{\partial t}(t, B^{(h)}(t))\, dt + \int_0^T \frac{\partial f}{\partial x}(t, B^{(h)}(t))\, dB^{(h)}(t) + \frac{1}{2}\int_0^T \left(\frac{d}{dt}[R_h(t,t)]\right) \frac{\partial^2 f}{\partial x^2}(t, B^{(h)}(t))\, dt. \tag{6.8}$$

Proof. Our proof is similar to the one of theorem 5.3 in [9]. Let $T>0$ and $t\in[0,T]$. Formula (6.8) may be rewritten as

$$\int_0^T \frac{\partial f}{\partial x}(t,B^{(h)}(t))\,dB^{(h)}(t) = f(T,B^{(h)}(T)) - f(0,0) - \int_0^T \frac{\partial f}{\partial t}(t,B^{(h)}(t))\,dt - \frac{1}{2}\int_0^T \left(\frac{d}{dt}[R_h(t,t)]\right)\frac{\partial^2 f}{\partial x^2}(t,B^{(h)}(t))\,dt. \quad (6.9)$$

In order to show that all members on the right hand side of (6.9) are in (L^2), let us use theorem 2.9. If G belongs to $\left\{f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}\right\}$ we may write, thanks to (6.7),

$$\mathbb{E}\left[G(t,B^{(h)}(t))^2\right] \leqslant C^2\int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left(\frac{1}{t^{2h(t)}}-4\lambda\right)v^2\right\}\frac{dv}{\sqrt{2\pi t^{2h(t)}}} \leqslant \overbrace{C^2\,(1-4\lambda\max_{t\in[0,T]} t^{2h(t)})^{-1/2}}^{=:M^2}. \quad (6.10)$$

Since $B^{(h)}(t) = B^H(t)_{|H=h(t)}$ *a.s*, it is easy to see (in view of [9] p.978) that $B^{(h)}(t)$ is a Gaussian variable with mean equal to $\int_0^t M_{h(t)}(\eta)(u)\,du = g_\eta(t,h(t))$ and variance equal to $t^{2h(t)}$ under the probability $\mathbb{Q}_\eta$ which has been defined in (2.11). Hence, for every t in $(0,T]$ and η in $\mathscr{S}(\mathbb{R})$,

$$S(G(t,B^{(h)}(t))(\eta) = \mathbb{E}_{\mathbb{Q}_\eta}[G(t,B^{(h)}(t))] = \int_{\mathbb{R}} G(\,t,u+g_\eta(t,h(t))\,)\;\gamma(t^{2h(t)},u)\,du$$
$$= \int_{\mathbb{R}} G\left(t,u\,t^{h(t)} + \int_0^t M_{h(t)}(\eta)(x)\,dx\right)\frac{1}{\sqrt{2\pi}}\,e^{-u^2/2}\,du. \quad (6.11)$$

Using the theorem of continuity under the integral sign, we see that the functions $t\mapsto S[G(t,B^{(h)}(t))](\eta)$ and $t\mapsto S[G(t,B^{(h)}(t))\diamond W^{(h)}(t)](\eta)$ are continuous on $[0,T]$. Moreover, in view of (6.10), $t\mapsto ||G(t,B^{(h)}(t))||_0$ belongs to $L^1([0,T])$. Furthermore, note that

$$\int_0^T \left|\left|\left(\frac{d}{dt}[R_h(t,t)]\right)\frac{\partial^2 f}{\partial x^2}(t,B^{(h)}(t))\right|\right|_0 dt \leqslant 2M\int_0^T t^{2h(t)-1}\;|h'(t)\,t\ln t + h(t)|\;dt < +\infty.$$

Thus, theorem 2.9 applies and shows that all members on the right side of (6.9) are in (L^2).
Let us now show that $t\mapsto \frac{\partial f}{\partial x}(t,B^{(h)}(t))\,\diamond W^{(h)}(t)$ is $(\mathcal{S})^*$-integrable over $[0,T]$.
Reasoning as in the estimate (6.6), we note that there exists an integer $q\geqslant 2$ such that $\frac{\partial f}{\partial x}(t,B^{(h)}(t))\diamond W^{(h)}(t)$ belongs to $(\mathcal{S}_{-q})$ for every t in $[0,T]$. Moreover, for every η in $\mathscr{S}(\mathbb{R})$ and every t in $(0,T]$ we have, using lemma 2.5,

$$\left|S\left(\frac{\partial f}{\partial x}(t,B^{(h)}(t))\,\diamond W^{(h)}(t)\right)(\eta)\right| \leqslant \left|\left|\frac{\partial f}{\partial x}(t,B^{(h)}(t))\right|\right|_0 ||W^{(h)}(t)||_{-2}\,\exp\{|\eta|_2^2\}$$
$$\leqslant \left(\sup_{t\in[0,T]}||W^{(h)}(t)||_{-2}\right)\left|\left|\frac{\partial f}{\partial x}(t,B^{(h)}(t))\right|\right|_0 \exp\{|\eta|_2^2\},$$

Since $t\mapsto \left|\left|\frac{\partial f}{\partial x}(t,B^{(h)}(t))\right|\right|_0$ belongs to $L^1([0,T])$, theorem 2.8 applies on $[0,T]$ to the effect that $\int_0^T \frac{\partial f}{\partial x}(t,B^{(h)}(t))\,dB^{(h)}(t)$ belongs to $(\mathcal{S})^*$. It then just remains to show the following equality for all t in $[0,T]$ and η in $\mathscr{S}(\mathbb{R})$:

$$S\left(\int_0^T \frac{\partial f}{\partial x}(t,B^{(h)}(t))\,dB^{(h)}(t)\right)(\eta) = S\left(f(T,B^{(h)}(T)) - f(0,0) - \int_0^T \frac{\partial f}{\partial t}(t,B^{(h)}(t))\,dt\right)(\eta) - S\left(\frac{1}{2}\int_0^T \left(\frac{d}{dt}[R_h(t,t)]\right)\frac{\partial^2 f}{\partial x^2}(t,B^{(h)}(t))\,dt\right)(\eta). \quad (6.12)$$

Using (6.11) with $G=f$ and applying the theorem of differentiation under the integral sign, we get

$$\frac{d}{dt}\left[S(f(t,B^{(h)}(t))(\eta)\right] = \int_{\mathbb{R}}\frac{d}{dt}\left[f(\,t,u+g_\eta(t,h(t))\,)\;\gamma(t^{2h(t)},u)\right]du$$
$$= \underbrace{\int_{\mathbb{R}}\gamma(t^{2h(t)},u)\,\frac{d}{dt}\,[f(\,t,u+g_\eta(t,h(t))\,)]\,du}_{=:U_1(t)} + \underbrace{\int_{\mathbb{R}} f(\,t,u+g_\eta(t,h(t))\,)\;\frac{d}{dt}\left[\gamma(t^{2h(t)},u)\right]du}_{=:U_2(t)}.$$

Now, $U_1(t) = \int_{\mathbb{R}} \gamma(t^{2h(t)}, u)\, \frac{\partial f}{\partial t}(t, u + g_\eta(t, h(t)))\, du + \int_{\mathbb{R}} \gamma(t^{2h(t)}, u)\, \frac{\partial f}{\partial x}(t, u + g_\eta(t, h(t)))\, \frac{d}{dt} g_\eta(t, h(t)) du$

$$= S\left(\frac{\partial f}{\partial t}\big(t, B^{(h)}(t)\big)\right)(\eta) + S\left(\frac{\partial f}{\partial x}\big(t, B^{(h)}(t)\big)\right)(\eta)\;\; S\left(W^{(h)}(t)\right)(\eta).$$

Besides, using the equality $\frac{\partial \gamma}{\partial t} = \frac{1}{2}\frac{\partial^2 \gamma}{\partial x^2}$ and an integration by parts, we get

$$U_2(t) = \frac{1}{2}\frac{d}{dt}[t^{2h(t)}] \int_{\mathbb{R}} \frac{\partial^2 f}{\partial x^2}(t, u + g_\eta(t, h(t)))\, \gamma(t^{2h(t)}, u)\, du = \frac{1}{2}\frac{d}{dt}[t^{2h(t)}]\; S\left(\frac{\partial^2 f}{\partial x^2}\big(t, B^{(h)}(t)\big)\right)(\eta).$$

Hence we obtain, for any $\varepsilon > 0$, upon integrating $t \mapsto U_1(t) + U_2(t)$ between ε and T,

$$S(f(t, B^{(h)}(t))(\eta) - S(f(\varepsilon, B^{(h)}(\varepsilon))(\eta) = \int_\varepsilon^T S\left(\frac{\partial f}{\partial t}\big(t, B^{(h)}(t)\big)\right)(\eta)\, dt$$
$$+ \int_\varepsilon^T S\left(\frac{\partial f}{\partial x}\big(t, B^{(h)}(t)\big) \diamond W^{(h)}(t)\right)(\eta)\, dt + \frac{1}{2}\int_\varepsilon^T \frac{d}{dt}[t^{2h(t)}]\; S\left(\frac{\partial^2 f}{\partial x^2}\big(t, B^{(h)}(t)\big)\right)(\eta)\, dt. \quad (6.13)$$

Let us now show that $\lim_{\varepsilon\to 0+} S\left(f(\varepsilon, B^{(h)}(\varepsilon)\right)(\eta) = f(0,0) = S\left(f(0, B^{(h)}(0)\right)(\eta)$. For every $\varepsilon > 0$, (6.11) can be rewritten as $S(f(\varepsilon, B^{(h)}(\varepsilon))(\eta) = \int_{\mathbb{R}} f(\varepsilon, y)\, \gamma(\varepsilon^{2h(\varepsilon)}, g_\eta(\varepsilon, h(\varepsilon)) - y)\, dy$.
For a fixed $T > 0$, let λ_T, C_T be such that (6.7) is fulfilled. There exists $b > 0$ such that $\varepsilon^{2h(\varepsilon)}$ belongs to I_a (defined in lemma 6.8) as soon as $0 < \varepsilon < b$. Hence we may write, for any ε in $(0, b)$, $S(f(\varepsilon, B^{(h)}(\varepsilon))(\eta) = J_f(g_\eta(\varepsilon, h(\varepsilon)), \varepsilon, \varepsilon^{2h(\varepsilon)})$. Since $\lim_{\varepsilon\to 0+} \varepsilon^{2h(\varepsilon)}$ and $\lim_{\varepsilon\to 0+} g_\eta(\varepsilon, h(\varepsilon))$ are equal to 0, lemma 6.8 applies with $x_0 = 0$, and yields $\lim_{\varepsilon\to 0+} S\left(f(\varepsilon, B^{(h)}(\varepsilon)\right)(\eta) = f(0, 0)$.
Let us now establish (6.12). Thanks to the fact that both the functions $t \mapsto S[G(t, B^{(h)}(t))](\eta)$ and $t \mapsto S[G(t, B^{(h)}(t)) \diamond W^{(h)}(t)](\eta)$ are continuous on $[0, T]$ and using the dominated convergence we can take the limit when ε tends to 0 on the right hand side of (6.13) and finally get

$$S[f(T, B^{(h)}(T)) - f(0,0)](\eta) = S\left(\int_0^T \frac{\partial f}{\partial t}\big(t, B^{(h)}(t)\big)\, dt\right)(\eta) + S\left(\int_0^T \frac{\partial f}{\partial x}\big(t, B^{(h)}(t)\big)\, dB^{(h)}(t)\right)(\eta)$$
$$+ S\left(\frac{1}{2}\int_0^T \frac{d}{dt}[t^{2h(t)}]\, \frac{\partial^2 f}{\partial x^2}\big(t, B^{(h)}(t)\big)\right)(\eta)\, dt.$$

□

Remark 6.10. *We observe that if we take expectations on both sides of Itô's formula (6.7), we get exactly formula (1) of theorem 2.1 of [45], which is a general weak Itô formula for Gaussian processes, in the particular case where the Gaussian process is chosen to be an mBm.*

7 Tanaka formula and examples

In this section we first give a Tanaka formula as a corollary to theorem 6.6 with $F : x \mapsto |x - a|$. We then consider the case of two particular h functions that give noteworthy results.

7.1 Tanaka formula

Theorem 7.1 (Tanaka formula for mBm)**.** *Let $h : \mathbb{R} \to (0, 1)$ be of class C^1, $a \in \mathbb{R}$ and $T > 0$. The following equality holds in (L^2):*

$$|B^{(h)}(T) - a| = |a| + \int_0^T sign\left(B^{(h)}(t) - a\right)\, dB^{(h)}(t) + \int_0^T \frac{d}{dt}[R_h(t, t)]\, \delta_{\{a\}}(B^{(h)}(t))\, dt, \quad (7.1)$$

where the function sign is defined on $\mathbb{R}$ by $sign(x) := \mathbb{1}_{\mathbb{R}_+^*}(x) - \mathbb{1}_{\mathbb{R}_-}(x)$.

Proof. This is a direct application of theorem 6.6 with $F : x \mapsto |x-a|$. □

Remark 7.2. *That the previous equality holds true in (L^2) does not imply of course that both integrals above are in (L^2). This last result will be established in a forthcoming paper.*

7.2 Itô formula for functions h such that $\frac{d}{dt}[R_h(t,t)] = 0$

If h verifies $\frac{d}{dt}[R_h(t,t)] = 0$, then the second order term $\frac{\partial^2 f}{\partial x^2}(t, B^{(h)}(t))$ disappears in Itô formula. The formula is then formally the same one as in ordinary calculus. In this case, (7.1) reads:

$$|B^{(h)}(T) - a| = |a| + \int_0^T \operatorname{sign}(B^{(h)}(t) - a)\, dB^{(h)}(t).$$

Note that the “local time" part disappears in this equality.
Solving the differential equation $\frac{d}{dt}[R_h(t,t)] = 0$ yields two families of functions, denoted $(h_{1,\lambda})_{\lambda\in\mathbb{R}_+^*}$ and $(h_{2,\lambda})_{\lambda\in\mathbb{R}_-^*}$:

$$\forall\lambda > 0, \qquad h_{1,\lambda} : \begin{array}{ccc} (-\infty,-e^{\lambda})\cup(e^{\lambda},+\infty) & \to & (0,1) \\ t & \mapsto & \frac{\lambda}{\ln|t|}, \end{array}$$

and

$$\forall\lambda < 0, \qquad h_{2,\lambda} : \begin{array}{ccc} (-e^{\lambda},0) \;\cup\; (0,e^{\lambda}) & \to & (0,1) \\ t & \mapsto & \frac{\lambda}{\ln|t|}. \end{array}$$

In order to obtain an mBm defined on a compact interval, we may choose a compact subset of $(-\infty,-e^{\lambda})\cup(e^{\lambda},+\infty)$ when $\lambda > 0$ and a compact subset of $(-e^{\lambda},0) \;\cup\; (0,e^{\lambda})$ when $\lambda < 0$.
Figures 1 and 2 display examples of mBm with functions $h_1(t) := \frac{1}{\ln t}$ defined on $[e+10^{-3},100]$ and $h_2(t) := \frac{-1}{\ln t}$ defined on $[10^{-3}, 1/e - 10^{-2}]$.

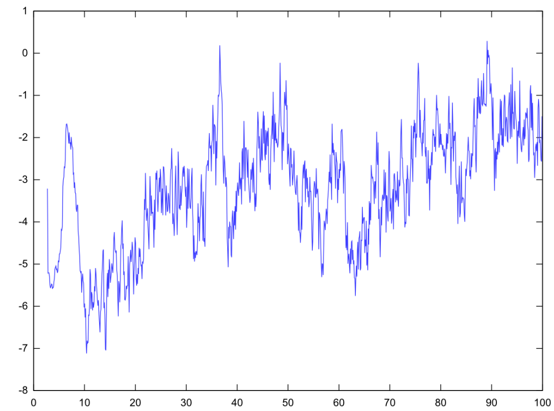

Figure 2.1: $t \mapsto B^{(h_1)}(t)$ with $h_1(t) := \frac{1}{\ln t}$ on $[e+10^{-3},100]$.

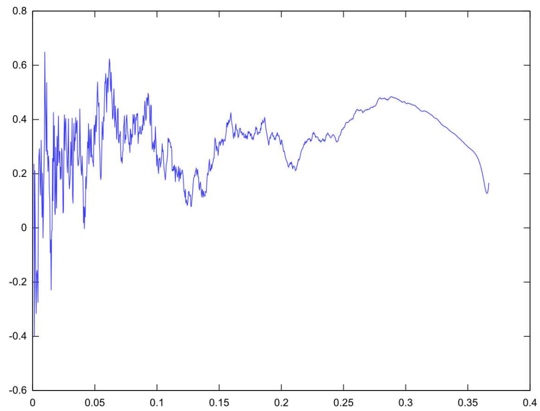

Figure 2.2: $t \mapsto B^{(h_2)}(t)$ with $h_2(t) := \frac{-1}{\ln t}$ on $[1.10^{-3}, 1/e - 10^{-3}]$.

Note moreover that $\lim_{t\to+\infty} h_i(t) = \lim_{t\to 0} h_i(t) = 0$ for $i = 1,2$.

7.3 Itô formula for functions h such that $\frac{d}{dt}[R_h(t,t)] = 1$

The situation where $\frac{d}{dt}[R_h(t,t)] = 1$ is interesting since then Itô formula is formally the same as in the case of standard Brownian motion. As a consequence, Tanaka formula takes the familiar form :

$$|B^{(h)}(T) - a| = |a| + \int_0^T \operatorname{sign}(B^{(h)}(t) - a)\, dB^{(h)}(t) + \int_0^T \delta_{\{a\}}(B^{(h)}(t))\, dt.$$

Thus, instead of a "weighted" local time as in (7.1), we get here an explicit expression for the local time of mBm for a family of h functions that we describe now.
The solutions $(h_c)_{c\in\mathbb{R}}$ of the differential equation are given by $h_c : (c,+\infty)\backslash\{-1,0,1\} \to \mathbb{R}$ and $h_c(t) := \frac{1}{2}\frac{\ln(t-c)}{\ln|t|}$. Recall that h_c is required to range in $(0,1)$. Denote, for $c \in \mathbb{R}$, $I_c := \{\, t \in (c,+\infty)\backslash\{-1,0,1\} \ : \ 0 < h_c(t) < 1\}$. For c in $(-\infty,1/4)$, let $t_1 := t_1(c) := \frac{1-\sqrt{1-4c}}{2}$ and $t_2 := t_2(c) := \frac{1+\sqrt{1-4c}}{2}$. Then I_c is explicitly given as follows:

$$
\begin{aligned}
&\forall c \in (-\infty,-2], && I_c = (1+c,t_1) \cup (t_2,+\infty),\\
&\forall c \in (-2,-1], && I_c = (t_1,1+c) \cup (t_2,+\infty),\\
&\forall c \in (-1,0), && I_c = (t_1,0) \cup (0,1+c) \cup (t_2,+\infty),\\
&\forall c \in [0,1/4), && I_c = (t_1,t_2) \cup (1+c,+\infty),\\
&\forall c \geqslant 1/4, && I_c = (1+c,+\infty).
\end{aligned}
$$

Figures 3 and 4 display examples of multifractional Brownian motion with regularity functions $h_3(t) := \frac{1}{2}\frac{\ln(t-1)}{\ln t}$ defined on $[2+10^{-3},5]$ and $h_4(t) := \frac{1}{2}\frac{\ln(t+1)}{\ln|t|}$ defined on $[\frac{1-\sqrt{5}}{2}+10^{-3},-10^{-3}]$.

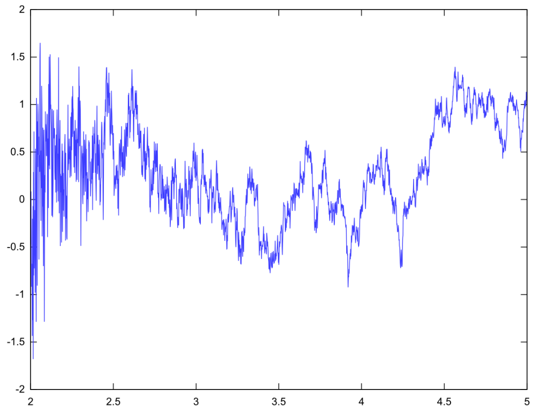

Figure 2.3: $t \mapsto B^{(h_3)}(t)$ with $h_3(t) := \frac{1}{2}\frac{\ln(t-1)}{\ln t}$ on $[2+10^{-3},5]$.

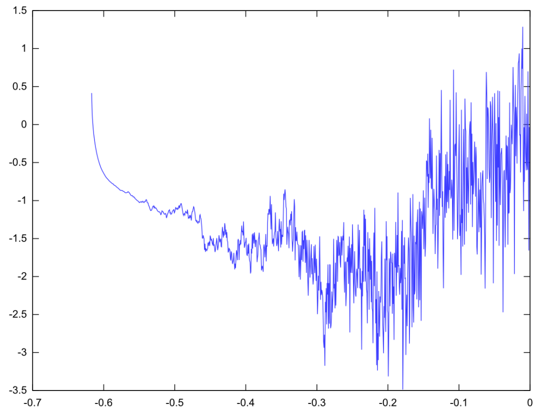

Figure 2.4: $t \mapsto B^{(h_4)}(t)$ with $h_4(t) := \frac{1}{2}\frac{\ln(t+1)}{\ln|t|}$ on $[\frac{1-\sqrt{5}}{2}+10^{-3},-10^{-3}]$.

Note that the case $c=0$ yields the constant function $h_c \equiv 1/2$, *i.e.* standard Brownian motion. Moreover, since $\lim_{t\to+\infty} h_c(t) = 1/2$ for every c, we see that the family of functions h_c behaves, asymptotically, like the constant function equal to $1/2$. However this does not mean that there is convergence in law of $B^{(h_c)}$ to Brownian motion; in fact one needs to scale $B^{(h_c)}$. For every t in $\mathbb{R}^*$, define the process X_t on $\mathbb{R}_+$ by $X_t(u) := \frac{B^{(h_c)}(tu)}{\sqrt{t}}$. Then $\{X_t(u);\ u \in \mathbb{R}_+\} \xrightarrow[t\to+\infty]{\mathcal{L}} \{B(u);\ u \in \mathbb{R}_+\}$ where B still denotes a Brownian motion and $\mathcal{L}$ denotes convergence in law.

8 Conclusion and future work

In this paper we have used a white noise approach to define a stochastic integral with respect to multifractional Brownian motion which generalizes the one for fBm based on the same approach. This stochastic calculus allows to solve some particular stochastic differential equations. We are currently investigating several extensions of this work. In order to apply this calculus to financial mathematics or to physics, it is necessary to study further the theory of stochastic differential equations driven by a mBm. This is the topic of a forthcoming paper.
The Tanaka formula we have obtained suggests that one can get several integral representations of local time with respect to mBm. Finally, since mBm is a Gaussian process, it seems also natural to investigate the links

between the construction of stochastic integral with respect to mBm that we gave and the one provided by Malliavin calculus. An extension in higher dimension is also desirable.

9 Acknowledgments

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X Appendix

X.A Bochner integral

All the following notions about the integral in the Bochner sense come from [44] p.72, 80 and 82 and from [52] p.247.

Definition X.1 (Bochner integral [52] p.247). *Let I be a subset of $\mathbb{R}$ endowed with the Lebesgue measure. One says that $\Phi : I \to (\mathcal{S})^*$ is Bochner integrable on I if it satisfies the two following conditions:*

1. *Φ is weakly measurable on I i.e $u \mapsto \ll \Phi(u), \varphi \gg$ is measurable on I for every φ in $(\mathcal{S})$.*
2. *$\exists p \in \mathbb{N}$ such that $\Phi(u) \in (\mathcal{S}_{-p})$ for almost every $u \in I$ and $u \mapsto ||\Phi(u)||_{-p}$ belongs to $L^1(I)$.*

The Bochner-integral of Φ on I is denoted $\int_I \Phi(s)\, ds$.

Properties X.1. *If $\Phi : I \to (\mathcal{S})^*$ is Bochner-integrable on I then*

1. *there exists an integer p such that $\big|\big|\int_I \Phi(s)\, ds\big|\big|_{-p} \leqslant \int_I ||\Phi(s)||_{-p}\, ds$.*
2. *Φ is also Pettis-integrable on I and both integrals coincide on I.*

Remark X.1. *The previous property shows that there is no risk of confusion by using the same notation for both the Bochner integral and the Pettis integral.*

Theorem X.2. *Let $p \in \mathbb{N}$ and $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of processes from I to $(\mathcal{S})^*$ such that $\Phi_n(u) \in (\mathcal{S}_{-p})$ for almost every $u \in I$ and for every n. Assume moreover that Φ_n is Bochner-integrable on I for every n and that*

$$\lim_{(n,m)\to(+\infty,+\infty)} \int_I ||\Phi_m(s) - \Phi_n(s)||_{-p}\, ds = 0.$$

Then there exists an $(\mathcal{S})^*$-process (almost surely $(\mathcal{S}_{-p})$-valued), denoted Φ, defined and Bochner-integrable on I such that

$$\lim_{n\to+\infty} \int_I ||\Phi(s) - \Phi_n(s)||_{-p}\, ds = 0. \tag{X.1}$$

Furthermore, if there exists an $(\mathcal{S})^*$-process, denoted Ψ, which verifies (X.1), then $\Psi(s) = \Phi(s)$ for almost every s in I. Finally the following equality holds:

$$\lim_{n\to+\infty} \int_I \Phi_n(s)\, ds = \int_I \Phi(s)\, ds \quad \text{in } (\mathcal{S}^*).$$

X.B Proof of proposition 4.1

Let $B^{(h)}$ be a normalized mBm on $\mathbb{R}$ with covariance function noted R_h. It is well known that one can define on the linear space $\text{span}_{\mathbb{R}}\{R_h(t,.) : t \in \mathbb{R}\}$ an inner product, denoted $<,>_{R_h}$, by $< R_h(t,.), R_h(s,.) >_{R_h} := R_h(t,s)$ (see [47] p.120 ff.). Define Ξ_h the closure of $\text{span}_{\mathbb{R}}\{R_h(t,.) : t \in \mathbb{R}\}$ for the norm $||\ ||_{R_h}$. The space $(\Xi_h, ||\ ||_{R_h})$ is called the Cameron-Martin space (or Reproducing Kernel Hilbert Space (R.K.H.S.)) associated to the the Gaussian process $B^{(h)}$. Let $\widetilde{\mathcal{E}}(\mathbb{R})$ denote the quotient space obtained by identifying all functions of $\mathcal{E}(\mathbb{R})$ which are equal almost everywhere. On $\widetilde{\mathcal{E}}(\mathbb{R}) \times \widetilde{\mathcal{E}}(\mathbb{R})$ define a bilinear form, noted

$<,>_h$, by $<\mathbb{1}_{[0,t]},\mathbb{1}_{[0,s]}>_h := R_h(t,s)$. Then $<,>_h$ is an inner product provided the linear map κ_h : $\widetilde{\mathcal{E}}(\mathbb{R}) \to \Xi_h$ defined by $\kappa_h(\mathbb{1}_{[0,t]}) := R_h(t,.)$, $t \in \mathbb{R}$, is injective. Define $\mathcal{I}_h := \overline{\text{vect}_{\mathbb{R}}\{B^{(h)}(t) : t \in \mathbb{R}\}}^{(L^2)}$ the first Wiener chaos of $B^{(h)}$. It is a well-known property of R.K.H.S. that the map $\tau_h : (\Xi_h, || \, ||_{R_h}) \to (\overline{\text{span}_{\mathbb{R}}\{B^{(h)}(t) : t \in \mathbb{R}\}}^{(L^2)}, || \, ||_{(L^2)})$, defined for all real t by $\tau_h(R_h(t,.)) = B^{(h)}(t)$ is an isometry. As a result, κ_h is injective if and only if $\tau_h \circ \kappa_h$ is injective. The next proposition states that this is indeed the case for any continuous function h:

Proposition X.3. *Let h be a continuous function defined on $\mathbb{R}$ and ranging in $(0,1)$. The family $(B^{(h)}(t))_{t\in\mathbb{R}^*}$ is linearly independent on $\mathbb{R}$, i.e for every positive integer n, $(\beta_1, \beta_2 \cdots, \beta_n)$ in $\mathbb{R}^n$ and $(t_1, t_2 \cdots, t_n)$ in $(\mathbb{R}^*)^n$, such that $t_i \neq t_j$ for $i \neq j$, the equality*

$$\sum_{j=1}^{n} \beta_j \; B^{(h)}(t_j) = 0 \quad a.s, \tag{X.2}$$

implies $\beta_1 = \beta_2 = \cdots = \beta_n = 0$.

The proof of this proposition requires the following lemma, the proof of which is easy and left to the reader.

Lemma X.4. *Define, for $t \in \mathbb{R}$, the function $A_t : \mathbb{R} \to \mathbb{C}$ by $A_t(\xi) := \frac{e^{it\xi}-1}{i\xi}$ if $\xi \neq 0$ and $A_t(0) := t$. Then, for all t, A_t is C^∞ on $\mathbb{R}$ and verifies, for every $n \in \mathbb{N}$, $A_t^{(n)}(0) = \frac{t\,(it)^n}{n+1}$, where $A_t^{(n)}$ denotes the n^{th} derivative of A_t.*

Proof. of proposition X.3. Let us use a proof by contradiction. By decreasing n if necessary we may always assume that $(\beta_1, \beta_2 \cdots, \beta_n)$ belongs to $(\mathbb{R}^*)^n$. Besides, since $B^{(h)}(t) = B^{(H)}(t)_{|_{H=h(t)}}$ and thanks to lemma 4.2 (i), equality (X.2) also reads $< ., \sum_{j=1}^n \beta_j \; M_{h(t_j)}(\mathbb{1}_{[0,t_j]}) >= 0 \quad a.s.$ By taking Fourier transforms, we get $\sum_{j=1}^n \beta_j \; \widehat{M_{h(t_j)}(\mathbb{1}_{[0,t_j]})} = 0 \quad a.e.$. Using (3.1) this yields

$$\sum_{j=1}^{n} \alpha_j \; |\xi|^{1/2-h(t_j)} \; \widehat{\mathbb{1}_{[0,t_j]}}(\xi) = 0, \quad \forall \xi \in \mathbb{R}^*, \tag{X.3}$$

where we have defined, for j in $\{1; 2; \cdots; n\}$, $\alpha_j := \beta_j \; (c_{h(t_j)})^{-1}$. By re-arranging if necessary the $(t_i)_i$, we may assume without loss of generality that $h(t_1) \geqslant h(t_2) \geqslant \cdots \geqslant h(t_n)$. Let $card(A)$ denote the cardinal of the set A. We distinguish three cases.

First case: $\text{card}(\{h(t_1); h(t_2); \cdots; h(t_n)\}) = 1$.
Since $h(t_1) = h(t_2) = \cdots = h(t_n) =: H$, we get, by multiplying equality (X.3) by $|\xi|^{H-1/2}$ and taking inverse Fourier transform, $\sum_{j=1}^n \alpha_j \; \mathbb{1}_{[0,t_j]} = 0$ almost everywhere on $\mathbb{R}$. This entails that $\{\alpha_1; \alpha_2; \cdots; \alpha_n\}$ and then $\{\beta_1; \beta_2; \cdots; \beta_n\}$ is equal to $\{0\}$.

Second case: $h(t_1) > h(t_2)$. Using that $\widehat{\mathbb{1}_{[0,t]}}(\xi) = A_t(\xi)$, (X.3) reads:

$$\alpha_1 \; (\tfrac{e^{it_1\xi}-1}{i\xi}) = -\sum_{j=2}^{n} \alpha_j \; |\xi|^{h(t_1)-h(t_j)} \; (\tfrac{e^{it_j\xi}-1}{i\xi}), \quad \forall \xi \in \mathbb{R}^*. \tag{X.4}$$

By lemma X.4 and taking the limit when ξ tends to 0 in (X.4), we get $\alpha_1 = 0$ which constitutes a contradiction.

Third case: $h(t_1) = h(t_2)$.
There exists an integer r in $\{2; 3; \cdots; n-1\}$, $(k_1, k_2, \cdots, k_r)$ in $(\mathbb{N}^*)^r$ with $2 \leqslant k_1 < k_2 < \cdots < k_r = n$, such that

$$\begin{array}{lcl} h(t_1) = h(t_2) = \cdots = h(t_{k_1}) & =: & H_1 \\ h(t_{k_1+1}) = h(t_{k_1+2}) = \cdots = h(t_{k_2}) & =: & H_2 \\ \vdots & & \vdots \\ h(t_{k_{r-1}+1}) = h(t_{k_{r-1}+2}) = \cdots = h(t_{k_r}) & =: & H_r, \end{array} \tag{X.5}$$

where $1 > H_1 > H_2 > \cdots > H_r > 0$. Note that when $r = 1$ we have $card(\{h(t_1); \cdots; h(t_n)\}) = 1$ (treated in the first case) and when $r = n$ we have $card(\{h(t_1); \cdots; h(t_n)\}) = n$ and then $h(t_1) > h(t_2)$ (treated in the second case). We hence assume from now that $2 \leqslant k_1 \leqslant n-1$ and $2 \leqslant r \leqslant n-1$.
Define the sets $I_1, I_2, \cdots, I_r$ by $I_1 := \{1; 2; \cdots; k_1\}$, $I_2 := \{k_1+1; k_1+2; \cdots; k_2\}, \cdots,\ I_r := \{k_{r-1}+1; k_{r-1}+2; \cdots; k_r\}$. Using lemma X.4, equality (X.3) can be rewritten as

$$\sum_{l=1}^{r} \sum_{j\in I_l} \alpha_j\ |\xi|^{1/2-H_l}\ A_{t_j}(\xi) = 0, \qquad \forall \xi \in \mathbb{R}^*. \tag{X.6}$$

Lemma X.5. *For every p in $\mathbb{N}^*$,*

$$\sum_{j\in I_1} \alpha_j\ t_j^p = 0. \tag{L_p}$$

Let us admit this lemma for the moment. The equalities (L_p) for p in $\{1; 2; \cdots; k_1\}$ yield the following linear system:

$$\underbrace{\begin{pmatrix} t_1 & t_2 & \cdots & t_{k_1-1} & t_{k_1} \\ t_1^2 & t_2^2 & \cdots & t_{k_1-1}^2 & t_{k_1}^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ t_1^{k_1} & t_2^{k_1} & \cdots & t_{k_1-1}^{k_1} & t_{k_1}^{k_1} \end{pmatrix}}_{=:D} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{k_1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{X.7}$$

The determinant of this system is a Vandermonde determinant which is non zero since all the t_i are distinct from each other. As the result, the only solution is $\alpha_1 = \alpha_2 = \cdots \alpha_{k_1} = 0$ which constitutes a contradiction and proves the proposition. □

We now present a sketch of proof of lemma X.5.
Proof of lemma X.5. By multiplying both sides of equality (X.6) by $|\xi|^{H_1-1/2}$ and then taking the limit when ξ tends to 0, we get, using lemma X.4, $\sum_{j\in I_1} \alpha_j\ t_j = 0$, which is equality (L_1). Now, fix p in $\mathbb{N}^*$. Starting from equality (X.6) we first
- multiply both sides of equality (X.6) by $|\xi|^{H_1-1/2}$ and call (X.6 bis) the resulting equality. For any ξ in $\mathbb{R}^*$, we then take the p^{th} derivative of both sides of equality (X.6 bis) at point ξ. We call (E_1) the equality thus obtained. It reads:

$$\sum_{l=1}^{r} \sum_{j\in I_l} \alpha_j\ \left[|\xi|^{H_1-H_l}\ A_{t_j}(\xi)\right]^{(p)} = 0, \qquad \forall \xi \in \mathbb{R}^*, \tag{E_1}$$

where $[g(\xi)]^{(p)}$ denotes the p^{th} derivative of the p-times differentiable map $\xi \mapsto g(\xi)$.
Now, starting from (E_1), we recursively perform the following operations successively for $l = 2, \ldots, r$:
- multiply both sides of equality (E_{l-1}) by $|\xi|^{H_l-H_{l-1}+p}$ and call (E_{l-1} bis) the resulting equality.
- take the p^{th} derivative of both sides of equality (E_{l-1} bis) at every point ξ in $\mathbb{R}^*$ and call (E_l) the resulting equality.
Equality (E_r) then reads:

$$\sum_{l=1}^{r} \sum_{j\in I_l} \alpha_j \times$$

$$\underbrace{\left[\left[\cdots\left[\left[|\xi|^{H_1-H_l}.\ A_{t_j}(\xi)\right]^{(p)} .|\xi|^{H_2-H_1+p}\right]^{(p)} .|\xi|^{H_3-H_2+p}\right]^{(p)} \cdots |\xi|^{H_{r-1}-H_{r-2}+p}\right]^{(p)} .|\xi|^{H_r-H_{r-1}+p}\right]^{(p)}}_{=:[\cdots]_{l,j}^{(p)}(\xi)}$$

$$= 0, \qquad \forall \xi \in \mathbb{R}^*, \tag{E_r}$$

Lemma X.4 yields that $\lim_{\xi\to 0} \alpha_j \ A^{(p)}_{t_j}(\xi) = \frac{i^p}{p+1} \ \alpha_j \ t_j^{p+1}$. We want to let ξ tend to 0 in the previous equality. However, for (l,j) in $\{1;2;\cdots;r\} \times I_l$, $\lim_{\xi\to 0} [\cdots]^{(p)}_{l,j}(\xi) = +\infty$. Nevertheless, it is easy to show that, for every (l,j) in $\{1;2;\cdots;r\} \times I_l$, $\lim_{\xi\to 0} \frac{[\cdots]^{(p)}_{l,j}(\xi)}{A^{(p)}_{t_j}(\xi) \ |\xi|^{H_r - H_l}}$ exists in $\mathbb{C}^*$ and is independent of j. Define $c_l := \lim_{\xi\to 0} \frac{[\cdots]^{(p)}_{l,j}(\xi)}{A^{(p)}_{t_{k_l}}(\xi) \ |\xi|^{H_r - H_l}}$, $l = 1, \ldots, r$. Denote, for (l,j) in $\{1;2;\cdots;r\} \times I_l$, $U_{l,j} : \mathbb{R} \to \mathbb{C}$, the continuous map on $\mathbb{R}$ such that $[\cdots]^{(p)}_{l,j}(\xi) = c_l \ |\xi|^{H_r - H_l} \ A^{(p)}_{t_j}(\xi) \ (1 + \ U_{l,j}(\xi))$. Equality (E_r) then reads

$$\sum_{l=1}^{r} c_l \sum_{j\in I_l} \alpha_j \ |\xi|^{H_r - H_l} \ A^{(p)}_{t_j}(\xi) \ (1 + \ U_{l,j}(\xi)) = 0, \qquad \forall \xi \in \mathbb{R}^*. \tag{X.8}$$

Upon multiplying both sides of the previous equality by $|\xi|^{H_1 - H_r}$ we get, for ξ in $\mathbb{R}^*$

$$c_1 \sum_{j\in I_1} \alpha_j \ A^{(p)}_{t_j}(\xi) \ (1 + \ U_{1,j}(\xi)) = -\sum_{l=2}^{r} c_l \sum_{j\in I_l} \alpha_j \ |\xi|^{H_1 - H_l} \ A^{(p)}_{t_j}(\xi) \ (1 + \ U_{l,j}(\xi)). \tag{X.9}$$

Since $H_1 > H_l$ for l in $\{2;\cdots;r\}$, taking the limit when ξ tends to 0 in (X.9) and using lemma X.4 yields $c_1 \ i^p \ (p+1)^{-1} \ \sum_{j\in I_1} \ \alpha_j \ t_j^{p+1} = 0$, which is nothing but (L_{p+1}). This ends the proof. □

Remark X.6. *Another way to establish that $R_H(.,.)$ defines an inner product on $\widetilde{\mathcal{E}}(\mathbb{R})$ for H in $(0,1)$ is to use* (3.3) *and* (3.9).

Chapter 3

Stochastic integration with respect to multifractional Brownian motion *via* tangent fractional Brownian motions

Joint work with E.Herbin and J. Lévy-Véhel

Abstract

Stochastic integration w.r.t. fractional Brownian motion (fBm) has raised huge interest in recent years, motivated in particular by applications in finance and Internet traffic modeling. Since fBm is not a semi-martingale, stochastic integration required specific developments. Multifractional Brownian motion (mBm) generalizes fBm by letting the Hölder exponent vary in time. The aim of this work is to explore some avenues in order to give firstly a new definition of mBm that generalizes the already existing one and secondly to define a stochastic calculus w.r.t. mBm from stochastic integrals w.r.t. fBm.

keywords: Brownian motion, multifractional Brownian motion, Gaussian processes, Skorohod integral white noise theory, S-transform, Wick-Itô integral, stochastic differential equations.

Contents

1 Introduction and Background

Fractional Brownian motion (fBm) is a centered Gaussian process with features that make it a useful model in various applications such as financial and Internet traffic modeling, image analysis and synthesis, physics, geophysics and more. These features include self-similarity, long range dependence and the ability to match any prescribed constant local regularity. Fractional Brownian motion is a centred Gaussian process indexed by a parameter, usually denoted H, that belongs to $(0,1)$, and is called the Hurst exponent. Its covariance function R_H reads:

$$R_H(t,s) := \frac{\gamma_H}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

where γ_H is a positive constant. An fBm with $\gamma_H = 1$ is termed normalized. Obviously, when $H = \frac{1}{2}$, fBm reduces to standard Brownian motion. Various integral representations of fBm are known, including the harmonizable and moving average ones [73], as well as representations by integrals over a finite domain [3, 18].

The fact that most of the properties of fBm are governed by the single real H restricts its application in some situations. In particular, its Hölder exponent remains the same all along its trajectory. This does not seem to be adapted to describe adequately natural terrains, for instance. In addition, long range dependence requires $H > 1/2$, and thus imposes paths smoother than the ones of Brownian motion. Multifractional Brownian motion was introduced to overcome these limitations. The basic idea is to replace the real H by a function $t \mapsto h(t)$ ranging in $(0,1)$.

Several definitions of multifractional Brownian motion exist. The first ones were proposed in [68] and in [7]. A more general one was introduced in [75]. In this work, we shall use a new definition that includes all previously known ones and which, in our opinion, is both more flexible and retains the essence of this class of processes. We first need to define a fractional Brownian field:

Definition 1.1 (Fractional Brownian field)**.** *Let* $(\Omega, \mathcal{F}, P)$ *be a fixed probability space. A fractional Brownian field on* $\mathbb{R}\times(0,1)$ *is a Gaussian field, noted* $(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$*, such that, for every* H *in* $(0,1)$*, the process* $(B_t^H)_{t\in\mathbb{R}}$ *defined by* $B_t^H := \mathbf{B}(t,H)$ *is a fractional Brownian motion*[1] *with Hurst parameter* H*.*

Note that this definition puts no constraint on the "inter-line" behaviour of the field, *i.e.* the relation between $(B_t^H)_{t\in\mathbb{R}}$ and $(B_t^{H'})_{t\in\mathbb{R}}$ for $H \neq H'$. In order to obtain a useful model, we need to control to some extent

1. One might also want to start from a family of fBms $(B^H)_{H\in(0,1)}$ ($i.e$ $B^H := (B_t^H)_{t\in\mathbb{R}}$ is an fBm for every H in $(0,1)$) and define from it the field $(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$ by $\mathbf{B}(t,H) := B_t^H$. However it is not true, in general, that the field $(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$ obtained in this way is Gaussian.

these relations. It turns out that the following condition is sufficient to prove all the results we will need in this section. ($\mathsf{E}[Y]$ denotes the expectation of a random variable Y in $L^1(\Omega,\mathcal{F},P)$):

$$(\mathcal{H}_1) : \forall [a,b] \subset \mathbb{R}, \forall [c,d] \subset (0,1), \exists (\Lambda,\delta) \in (\mathbb{R}_+^*)^2, \text{ such that } \mathsf{E}[(\mathbf{B}(t,H) - \mathbf{B}(t,H'))^2] \leqslant \Lambda\, |H-H'|^{\delta}, \quad \text{for every } (t,H,H') \text{ in } [a,b]\times[c,d]^2.$$

Note that assumption $(\mathcal{H}_1)$ is equivalent to the following one:

$$(\mathcal{H}) : \forall [a,b]\times[c,d] \subset \mathbb{R}\times(0,1), \exists (\Lambda,\delta) \in (\mathbb{R}_+^*)^2, s.t.\ \mathsf{E}[(\mathbf{B}(t,H) - \mathbf{B}(s,H'))^2] \leqslant \Lambda\, \left(|t-s|^{2c} + |H-H'|^{\delta}\right), \quad \text{for every } (t,s,H,H') \in [a,b]^2\times[c,d]^2.$$

Indeed, since $\mathbf{B}$ is a fractional field, for every (t,s,H) in $[a,b]^2\times[c,d]$, $\mathsf{E}[(\mathbf{B}(t,H) - \mathbf{B}(s,H))^2] = |t-s|^{2H}$. Then if $(\mathcal{H}_1)$ is true we have, for all (t,s,H,H') $[a,b]^2\times[c,d]^2$,

$$\mathsf{E}[(\mathbf{B}(t,H) - \mathbf{B}(s,H'))^2] \leqslant 2\Lambda\, (1+|b-a|^{2(d-c)})\, \left(|t-s|^{2c} + |H-H'|^{\delta}\right).$$

Thus, we will refer either to assumption $(\mathcal{H}_1)$ or $(\mathcal{H})$ in the sequel.

Remark 1.1. *(i) Assumption $(\mathcal{H})$ entails that the map $C : (t,s,H,H') \mapsto \mathsf{E}[\mathbf{B}(t,H)\,\mathbf{B}(s,H')]$ is continuous on $\mathbb{R}^2\times(0,1)^2$.*
(ii) Let p be an integer such that $p(2c\wedge\delta) > 2$. Let κ_p denote the constant such that $\mathbb{E}[Y^{2p}] = \kappa_p\, (\mathsf{E}[Y^2])^p$ where Y is a centered Gaussian random variable. Hypothesis $(\mathcal{H})$ entails that for all $(t,s,H,H') \in [a,b]^2\times[c,d]^2$,

$$\begin{aligned}\mathsf{E}[(\mathbf{B}(t,H) - \mathbf{B}(s,H'))^{2p}] = \kappa_p\Big(\mathsf{E}[(\mathbf{B}(t,H) - \mathbf{B}(s,H'))^2]\Big)^p &\leqslant \kappa_p\, \Lambda^p\, \left(|t-s|^{2c} + |H-H'|^{\delta}\right)^p \\ &\leqslant \kappa_p\, (2\widetilde{\Lambda})^p\, \max\{|t-s|; |H-H'|\}^{p(\delta\wedge 2c)},\end{aligned}$$

where $\widetilde{\Lambda} := \Lambda\,(1+b-a)^{\delta\vee 2c-\delta\wedge 2c}$, $x\wedge y := \min\{x;y\}$ and $x\vee y := \max\{x;y\}$. Kolmogorov's criterion implies that the field $\mathbf{B}$ has a d-Hölder continuous version for any d in $(0, \frac{2p}{p(\delta\wedge 2c)-2})$. In the sequel we will always work with this version.

An mBm is defined as follows:

Definition 1.2 (Multifractional Brownian motion)**.** *Let $h : \mathbb{R}\to(0,1)$ be a deterministic continuous function. A multifractional Brownian motion with functional parameter h is the Gaussian process $B^h := (B_t^h)_{t\in\mathbb{R}}$ defined by $B_t^h := \mathbf{B}(t,h(t))$ for every t in $\mathbb{R}$.*

A word on notation: $B^H_{\cdot}$ or $B^{h(t)}_{\cdot}$ will always denote an fBm with Hurst index H or $h(t)$, while $B^h_{\cdot}$ will stand for an mBm. Note that $B_t^h := \mathbf{B}(t,h(t)) = B_t^{h(t)}$, for every real t.
Furthermore, we say that h is the *regularity function* of the mBm. We will say that the fractional field $(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$ is normalized when, for all H in $(0,1)$, the fractional Brownian motion $(B_t^H)_{t\in\mathbb{R}}$, resulting from the fractional field $(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$, is a normalized fBm. In this case we will also say that the mBm $B^h := (B_t^h)_{t\in\mathbb{R}}$ is normalized.
It is straightforward to check that an mBm in the sense of [75, def.1.1] is also an mBm with our definition. [75] provides the correlation structure of various mBms. In general, it is clear that the correlation structures of two mBms derived from different fractional Brownian fields will differ whenever the "inter-line" correlations of the fields differ.
Fractional fields $(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$ leading to previously considered mBms include:

$$\mathbf{B}_1(t,H) := \frac{1}{c_H}\int_{\mathbb{R}} \frac{e^{itu}-1}{|u|^{H+1/2}}\widetilde{\mathbb{W}_1}(du), \qquad \mathbf{B}_2(t,H) := \int_{\mathbb{R}} \left(|t-u|^{H-1/2} - |u|^{H-1/2}\right) \mathbb{W}_2(du),$$

$$\mathbf{B}_3(t,H) := \int_{\mathbb{R}} \left((t-u)_+^{H-1/2} - (-u)_+^{H-1/2}\right) \mathbb{W}_3(du), \qquad \mathbf{B}_4(t,H) := \int_0^T \mathbb{1}_{\{0\leqslant u<t\leqslant T\}}(t,u)\, K_H(t,u)\, \mathbb{W}_4(du),$$

where $c_H := \left(\frac{2\cos(\pi H)\Gamma(2-2H)}{H(1-2H)}\right)^{\frac{1}{2}}$, $\alpha_H := \left(\frac{2H}{(1-2H)\beta(1-2H,H+1/2)}\right)^{1/2}$ and

$$K_H(t,u) := \begin{cases} \alpha_H \left[\left(\frac{t}{u}\right)^{H-1/2}(t-u)^{H-1/2} - (H-1/2)u^{1/2-H}\int_u^t (v-u)^{H-1/2}v^{H-1/2}dv\right] & \text{if } 0 < H < 1/2, \\ \left(\frac{H(2H-1)}{\beta(2-2H,H-1/2)}\right)^{1/2} u^{1/2-H}\int_u^t (v-u)^{H-3/2}v^{H-1/2}\, dv & \text{if } 1/2 < H < 1, \end{cases}$$

and where $\mathbb{W}_i$ denotes an independently scattered standard Gaussian measure on $\mathbb{R}$, $i \in \{2;3;4\}$, and $\widetilde{\mathbb{W}_1}$ denotes the complex-valued Gaussian measure which can be associated in a unique way to any independently scattered standard Gaussian measure $\mathbb{W}_1$ on $\mathbb{R}$ (see [75] p.203-204 and [73] p.325-326 for more details on the meaning of $\int_{\mathbb{R}} f(u)\, \mathbb{W}(du)$ and $\int_{\mathbb{R}} f(u)\, \widetilde{\mathbb{W}}(du)$ for a real or complex-valued function f in $L^2(\mathbb{R}, du)$). $\mathbf{B}_1(t,H)$ and $\mathbf{B}_2(t,H)$ lead to the so-called harmonisable mBm, first considered in [7]. $\mathbf{B}_2(t,H)$ yields the moving average mBm defined in [68]. All these are particular cases of mBms in the sense of [75]. $\mathbf{B}_4(t,H)$ corresponds to the Volterra multifractional Gaussian process studied in [18].

This paper is organized as follows. The section 2 is devoted to a new definition of mBm from fractional field and then to a result of approximation of a mBm by a sum of sequence of fBm. After giving an idea of what should be a reasonable integral w.r.t. mBm we give, in section 3, a criterion that allows to know when a sequence of integrals with respect to several fBms converge. We then give a definition of integral with respect to mBm as a sum of two terms, depending both on, the method of integration and the fractional field from which the mBm is defined.In section 4 we apply the definition given in section 3 to the Skorohod integral w.r.t. fBm obtained, using Malliavin calculus, in [3] and show that our integral with respect to mBm is the one obtained by [18]. In section 5 we first define an integral w.r.t. mBm (from fractional field $\mathbf{B}_1$). We then compare it to the one which has been defined in [53] and show that, when they exist, they both coincide.

2 Approximation of multifractional Brownian motion

Since an mBm is just a continuous path traced on a fractional Brownian field, a natural question is to enquire whether it may be approximated by patching adequately chosen fBms, and in which sense. Heuristically, for $a < b$, we divide $[a,b)$ into "small" intervals $[t_i, t_{i+1})$, and replace on each of these B^h by the fBm B^{H_i} where $H_i = h(t_i)$. It seems reasonable to expect that the resulting process $\sum_i B_t^{H_i}\mathbb{1}_{[t_i,t_{i+1})}(t)$ will "converge", in a sense to be made precise, to B^h when the size of the intervals $[t_i, t_{i+1})$ goes to 0. This idea is further supported by the fact that the moving average, harmonisable and Volterra mBms are all "tangents" to fBms in the following sense. This idea is further supported by the fact that the moving average, harmonisable and Volterra mBms are all "tangents" to fBms in the following sense: for every real u,

$$\left\{\frac{B^h_{u+rt} - B^h_u}{r^{h(u)}};\ t \in [a,b]\right\} \xrightarrow[r\to 0^+]{\text{law}} \{B_t^{h(u)};\ t \in [a,b]\}$$

Our aim in this section is to make this line of thought rigorous.

2.1 Approximation of mBm by piecewise fBms

In the sequel, we fix a fractional Brownian field $\mathbf{B}$ and a continuous function h, thus an mBm. We aim to prove that this mBm can be approximated on every compact interval $[a,b]$ by patching together fractional Brownian motions defined on a sequence of partitions of $[a,b]$. In that view, define an increasing sequence $(q_n)_{n\in\mathbb{N}}$ of integers such that $q_0 := 1$, $2^n \leqslant q_n \leqslant 2^{2^n}$ for every n in $\mathbb{N}^*$. For a compact interval $[a,b]$ of $\mathbb{R}$ and n in $\mathbb{N}$, let $x^{(n)} := \{x_k^{(n)}; k \in [\![0, q_n]\!]\}$ where $x_k^{(n)} := a + k\frac{(b-a)}{q_n}$ for every k in $[\![0,q_n]\!]$ (for integers p and q with $p < q$, $[\![p,q]\!]$ denotes the set $\{p; p+1; \cdots; q\}$). Now, if we define, for every n in $\mathbb{N}$, the partition $\mathcal{A}_n := \{[x_k^{(n)}, x_{k+1}^{(n)}); k \in [\![0, q_n - 1]\!]\} \cup \{x_{q_n}^{(n)}\}$, it is clear that $\mathcal{A} := (\mathcal{A}_n)_{n\in\mathbb{N}}$ is a decreasing nested sequence of subdivisions of $[a,b]$ (*i.e* $\mathcal{A}_{n+1} \subset \mathcal{A}_n$, *for every* n *in* $\mathbb{N}$).

For every t in $[a,b]$ and n in $\mathbb{N}$ there exists a unique integer p in $[\![0, q_n - 1]\!]$ such that $x_p^{(n)} \leqslant t < x_{p+1}^{(n)}$. We will note $x_t^{(n)}$ the real $x_p^{(n)}$ in the sequel. The sequence $(x_t^{(n)})_{n\in\mathbb{N}}$ is increasing and converges to t as n tends

to $+\infty$. Besides, define for every n in $\mathbb{N}$, the function $h_n : [a,b] \to (0,1)$ by setting $h_n(b) = h(b)$ and, for any t in $[a,b)$, $h_n(t) := h(x_t^{(n)})$. The sequence of functions $(h_n)_{n\in\mathbb{N}}$ converges pointwise to h on $[a,b]$.
Define, for t in $[a,b]$ and n in $\mathbb{N}$, the process

$$B_t^{h_n} := \mathbf{B}(t, h_n(t)) = \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t)\ \mathbf{B}(t, h(x_k^{(n)})) + \mathbb{1}_{\{b\}}(t)\ \mathbf{B}(b, h(b)). \tag{2.1}$$

Note that, despite the notation, the process B^{h_n} is not an mBm, as h_n is not continuous in general. We believe however there is no risk of confusion in using this notation. B^{h_n} is almost surely *càdlàg* and discontinuous at times $x_k^{(n)}$, k in $[\![0, q_n]\!]$.

Theorem 2.1 (Approximation theorem)**.** *Let* $\mathbf{B}$ *be a fractional Brownian field,* $h : \mathbb{R} \to (0,1)$ *be a continuous deterministic function and* B^h *be the associated mBm. Let* $[a,b]$ *be a compact interval of* $\mathbb{R}$*,* $\mathcal{A}$ *be a sequence of partitions as defined above, and consider the sequence of processes defined in* (2.1)*. Then:*

1. *If* $\mathbf{B}$ *is such that the map* C *(defined in* (i) *of remark 1.1) is continuous on* $([a,b])^2 \times h([a,b])^2$ *then the sequence of processes* $(B^{h_n})_{n\in\mathbb{N}}$ *converges in* $L^2(\Omega)$ *to* B^h*,i.e*

$$\forall t \in [a,b], \quad \lim_{n\to+\infty} E\left[\left(B_t^{h_n} - B_t^h\right)^2\right] = 0.$$

2. *If* $\mathbf{B}$ *satisfies assumption* $(\mathcal{H})$ *and if* h *is* β*-Hölder continuous for some positive real* β*, then the sequence of processes* $(B^{h_n})_{n\in\mathbb{N}^*}$ *converges*

   (**i**) *in law to* B^h*, i.e* $\quad \{B_t^{h_n};\ t \in [a,b]\} \xrightarrow[n\to+\infty]{law} \{B_t^h;\ t \in [a,b]\}.$

   (**ii**) *almost surely to* B^h*, i.e* $\quad P\left(\{\forall t \in [a,b], \ \lim_{n\to+\infty} B_t^{h_n} = B_t^h\}\right) = 1.$

Proof. **1**. Let $t \in [a,b]$. For any n in $\mathbb{N}$, and using the notations of (i) in remark 1.1,

$$E\left[\left(B_t^{h_n} - B_t^h\right)^2\right] = C(t,t,h(x_t^{(n)}), h(x_t^{(n)})) - 2\,C(t,t,h(x_t^{(n)}), h(t)) + C(t,t,h(t),h(t)).$$

The continuity of the maps h, $(t,H,H') \mapsto C(t,t,H,H')$ and the fact that $\lim\limits_{n\to+\infty} x_t^{(n)} = t$ entail that $\lim_{n\to\infty} E\left[(B_t^{h_n} - B_t^h)^2\right] = 0$.
2. By assumption, there exists η in $\mathbb{R}_+^*$ such that for all (s,t) in $[a,b]$,

$$|h(s) - h(t)| \leqslant \eta\ |s-t|^\beta. \tag{2.2}$$

(**i**) We proceed as usual in two steps (see for examples [16, 71]), **a)**: finite-dimensional convergence and **b)**: tightness of the sequence of probability measures $(P \circ B^{h_n})_{n\in\mathbb{N}}$.

a) Finite dimensional convergence
Since the process B^h and, for every n in $\mathbb{N}$, the process B^{h_n} defined by (2.1) are centered and Gaussian it is sufficient to prove that $\lim_{n\to\infty} E\left[B_t^{h_n} B_s^{h_n}\right] = E\left[B_t^h B_s^h\right]$ for every (s,t) in $[a,b]^2$.
The cases where $t = b$ or $s = t$ are consequences of point 1. above. We now assume that $a \leqslant s < t < b$.
For every integer n in $\mathbb{N}$, we get

$$E\left[B_t^{h_n} B_s^{h_n}\right] = \sum_{(k,j)\in[\![0,q_n-1]\!]^2} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t)\ \mathbb{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(s)\ E\left[\mathbf{B}(t, h_n(t))\mathbf{B}(s, h_n(s))\right].$$

We hence get $E\left[B_t^{h_n} B_s^{h_n}\right] = E\left[\mathbf{B}(t, h(x_t^{(n)}))\mathbf{B}(s, h(x_s^{(n)}))\right]$ for all large enough integers n, ($i.e$ such that $x_s^{(n)} \leqslant s < x_t^{(n)} \leqslant t$).
The continuity of h, (i) of remark 1.1, and the fact that $\lim_{n\to\infty}(x_t^{(n)}, x_s^{(n)}) = (t,s)$, entail that

$$\lim_{n\to\infty} E\left[B_t^{h_n} B_s^{h_n}\right] = E\left[B_t^h B_s^h\right].$$

b) Tightness of the sequence of probability measures $(P \circ B^{h_n})_{n\in\mathbb{N}}$.
We are in the particular case where a sequence of càdlàg processes converges to a continuous one. Theorem page 92 of [69] applies to this situation: it is sufficient to show that, for every positive reals ε and τ, there exist an integer m and a grid $\{t_i\}_{i\in[\![0,m]\!]}$ such that $a = t_0 < t_1 < \cdots < t_m = b$ that verify

$$\limsup_{n\to+\infty} P\left(\left\{\max_{0\leqslant i\leqslant m}\ \sup_{t\in[t_i,t_{i+1})} |B_t^{h_n} - B_{t_i}^{h_n}| > \tau\right\}\right) < \varepsilon. \tag{2.3}$$

Let us then fix (ε, τ) in $(\mathbb{R}_+^*)^2$. Define $[H_1, H_2] := [\inf_{u\in[a,b]} h(u), \sup_{u\in[a,b]} h(u)]$ and set, until the end of this proof, $q_n := 2^{2^n}, n \in \mathbb{N}$. Define $F := [a, b] \times [H_1, H_2]$. The process $(\mathbf{B}(t, H))_{(t,H)\in F}$ is Gaussian and the space of continuous real-valued functions defined on F endowed with the sup-norm is a separable Banach space. Fernique theorem (see [30, theorem 2.6 p.37]) applies to the effect that there exists a positive real α such that $A_\alpha := E\left[\exp\left\{\alpha \sup_{(t,H)\in F} \mathbf{B}(t, H)^2\right\}\right] < +\infty$. Set $G := L\ \sum_{p=0}^{+\infty} \frac{2^{p/2}}{q_p{}^{\delta/2}}$, where L is the universal constant in [76, Theorem 2.1.1 p. 33] . Define also $D := \Lambda\ (\eta\ (b-a)^\beta)^\delta$ where Λ and δ are the constants appearing in $(\mathcal{H})$ and η in (2.2). Let N be the smallest integer n such that

$$\max\left\{A_\alpha\ (1+q_n)\ \exp\left\{\frac{-\alpha\ \tau^2}{4|b-a|^{2H_2}}\ q_n^{2H_2}\right\}; 4\ (1+q_n)\ \exp\left\{\frac{-\ \tau^2}{2^7\ D}\ q_n^{\delta\beta}\right\}; \frac{b-a}{q_n}; \frac{G\ D^{1/2}}{q_n{}^{\frac{\delta\beta}{2}}}\right\} < 1 \wedge \frac{\tau}{8} \wedge \frac{\varepsilon}{2}. \tag{2.4}$$

Set $m := m(\tau, \varepsilon) = q_N$ and $t_i := x_i^{(m)}$ for i in $[\![0, m]\!]$. Note that this entails that, for all n larger than N, $h_n(t_i) = h(t_i)$. Besides, also as soon as $n \geqslant N$, $h_n(t)$ belongs to the set $h([t_i, t_{i+1}])$ when $t \in [t_i, t_{i+1}]$ (draw a picture). Let $J_n^{\tau,m} := P\big(\big\{\max_{0\leqslant i\leqslant m}\ \sup_{t\in[t_i,t_{i+1})} |B_t^{h_n} - B_{t_i}^{h_n}| > \tau\big\}\big)$. Then

$$J_n^{\tau,m} \leqslant (1+m) \max_{0\leqslant i\leqslant m} P\Big(\Big\{\sup_{t\in[t_i,t_{i+1})} |\mathbf{B}(t, h_n(t)) - \mathbf{B}(t_i, h_n(t_i))| > \tau\Big\}\Big) \leqslant (1+m)\ \Big(\max_{0\leqslant i\leqslant m} L_n^{\tau,i} + \max_{0\leqslant i\leqslant m} Q_n^{\tau,i}\Big), \tag{2.5}$$

where

$$L_n^{\tau,i} := P\Big(\Big\{\sup_{t\in[t_i,t_{i+1})} |\mathbf{B}(t, h_n(t)) - \mathbf{B}(t_i, h_n(t))| > \tau/2\Big\}\Big)$$

and

$$Q_n^{\tau,i} := P\left(\Big\{\sup_{t\in[t_i,t_{i+1})} |\mathbf{B}(t_i, h_n(t)) - \mathbf{B}(t_i, h_n(t_i))| > \tau/2\Big\}\right).$$

– Upper bound for $(1+m) \max_{0\leqslant i\leqslant m} L_n^{\tau,i}$:
The couple (i, n) being fixed in $[\![0, m]\!] \times \mathbb{N}$, the process $(\mathbf{B}(s, h_n(t)))_{s\in[t_i,t_{i+1}]}$ is a fractional Brownian motion of Hurst index $h_n(t)$. Using increment-stationarity and self-similarity of fractional Brownian motion yields:

$$\begin{aligned} L_n^{\tau,i} &= P\left(\sup_{t\in[t_i,t_{i+1})} |\mathbf{B}(t - t_i, h_n(t))| > \tau/2\right) \\ &= P\left(\sup_{u\in[0,t_{i+1}-t_i)} |t_{i+1} - t_i|^{h_n(u+t_i)} \left|\mathbf{B}\left(\frac{u}{t_{i+1}-t_i}, h_n(u+t_i)\right)\right| > \tau/2\right) \\ &\leqslant P\left(\sup_{v\in[0,1]} |\mathbf{B}(v, h_n(t_i + v(t_{i+1} - t_i)))| > \frac{\tau}{2|t_{i+1} - t_i|^{H_2}}\right). \end{aligned} \tag{2.6}$$

Using Markov identity and Fernique's theorem,

$$L_n^{\tau,i} \leqslant P\left(\sup_{(v,H)\in F}|\mathbf{B}(v,H)| > \frac{\tau}{2|t_{i+1}-t_i|^{H_2}}\right) = P\left(\exp\left\{\alpha \sup_{(v,H)\in F}|\mathbf{B}(v,H)|^2\right\} > \exp\left\{\frac{\alpha\,\tau^2}{4|t_{i+1}-t_i|^{2H_2}}\right\}\right)$$
$$\leqslant E\left[\exp\left\{\alpha \sup_{(t,H)\in F}\mathbf{B}(t,H)^2\right\}\right]\ \exp\left\{\frac{-\alpha\tau^2}{4|t_{i+1}-t_i|^{2H_2}}\right\} = A_\alpha\ \exp\left\{\frac{-\alpha\tau^2\ q_N^{2H_2}}{4|b-a|^{2H_2}}\right\}$$
$$< \frac{\varepsilon}{2(1+q_N)} = \frac{\varepsilon}{2(1+m)}.$$

We have shown that

$$\forall i\in[\![0,q_N]\!],\ \forall n\geqslant N, \qquad (1+m)\max_{0\leqslant i\leqslant m} L_n^{\tau,i} < \frac{\varepsilon}{2}. \tag{2.7}$$

– Upper bound for $(1+m)\ \max_{0\leqslant i\leqslant m} Q_n^{\tau,i}$:

Fix a couple (i,n) in $[\![0,m]\!]\times\mathbb{N}$. Recall that $h_n(t)$ belongs to $h([t_i,t_{i+1}]) =: \mathcal{K}^{(i)}$ for every t in $[t_i,t_{i+1})$. We hence have,

$$Q_n^{\tau,i} \leqslant P\Big(\Big\{\sup_{H\in\mathcal{K}^{(i)}}|\mathbf{B}(t_i,H)-\mathbf{B}(t_i,h(t_i))| > \tau/2\Big\}\Big) \leqslant 2\ P\Big(\Big\{\sup_{H\in\mathcal{K}^{(i)}}\overbrace{\mathbf{B}(t_i,H)-\mathbf{B}(t_i,h(t_i))}^{=:X_i(H)} > \tau/4\Big\}\Big). \tag{2.8}$$

Observe that the right hand side of the previous inequality does not depend on n any more.
Our aim is to apply [2, (2.6) p.43]. In that view, we first prove the following estimate:

Lemma 2.2. *For all i in $[\![0,q_N]\!]$,* $\qquad \mu_i := E\Big[\sup_{H\in\mathcal{K}^{(i)}} X_i(H)\Big] < \frac{G\ D^{1/2}}{q_N^{\delta\beta/2}} < \frac{\tau}{8}.$

Proof of lemma 2.2: Fix i in $[\![0,q_N]\!]$. We recall some notions from [76]. A sequence $C^{(i)} := (C_n^{(i)})_{n\in\mathbb{N}}$ of partitions of $\mathcal{K}^{(i)}$ is called admissible if it is increasing and such that $\mathrm{card}(C_n^{(i)}) \leqslant 2^{2^n}$, for every n in $\mathbb{N}$. Let d_i denote the pseudo-distance associated to the Gaussian process $(X_i(H))_{H\in\mathcal{K}^{(i)}}$, ($i.e$ $d_i(H,H') := (\mathbb{E}[(X_i(H)-X_i(H'))^2])^{1/2}$, for every (H,H') in $\mathcal{K}^{(i)}\times\mathcal{K}^{(i)}$).
For (H,p) in $\mathcal{K}^{(i)}\times\mathbb{N}$, let $C_p^{(i)}(H)$ be the unique element of the partition $C_p^{(i)}$ which contains H. The diameter of $C_p^{(i)}(H)$ is by definition $\Delta_i(C_p^{(i)}(H)) := \sup_{(H,H')\in C_p(H)^2} d_i(H,H')$. [76, Theorem 2.1.1] entails that:

$$\mu_i \leqslant L\ \gamma_2(\mathcal{K}^{(i)},d_i), \tag{2.9}$$

where $\gamma_2(\mathcal{K}^{(i)},d_i) := \inf\ \sup_{H\in\mathcal{K}^{(i)}}\sum_{p\geqslant 0} 2^{p/2}\ \Delta_i(C_p^{(i)}(H))$ and where the infimum is taken over all admissible sequences of partitions of $\mathcal{K}^{(i)}$. Let $H_1^{(i)}$ and $H_2^{(i)}$ be such that $\mathcal{K}^{(i)} =: [H_1^{(i)},H_2^{(i)}]$. Consider the sequence of partitions $C^{(i)} := (C_n^{(i)})_{n\in\mathbb{N}}$ of $\mathcal{K}^{(i)}$ defined, for every integer n, by setting $C_n^{(i)} := \{[y_k^{(n)},y_{k+1}^{(n)});k\in[\![0,q_n-1]\!]\}\cup\{y_{q_n}^{(n)}\}$, where $y_0^{(n)} = H_1^{(i)}$ and $y_{k+1}^{(n)}-y_k^{(n)} = \frac{H_2^{(i)}-H_1^{(i)}}{q_n}$, for (n,k) in $\mathbb{N}\times[\![0,q_n-1]\!]$.
It is clear that $C^{(i)}$ is a decreasing nested sequence of partitions of $\mathcal{K}^{(i)}$ and is hence admissible. For (H_0,p) in $\mathcal{K}^{(i)}\times\mathbb{N}$, denote $[y_{k_0}^{(p)},y_{k_0+1}^{(p)})$ the unique element of $C_p^{(i)}$ which contains H_0. Then, using $(\mathcal{H})$,

$$\mu_i \leqslant L\sum_{p\geqslant 0} 2^{p/2}\sup_{H_0\in\mathcal{K}^{(i)}}\ \sup_{(H,H')\in[y_{k_0}^{(p)},y_{k_0+1}^{(p)})^2}(\mathbb{E}[(X_i(H)-X_i(H'))^2])^{1/2}$$
$$\leqslant L\sum_{p\geqslant 0} 2^{p/2}\sup_{k\in[\![0,q_p-1]\!]}\ \sup_{(H,H')\in[y_k^{(p)},y_{k+1}^{(p)})^2}(\mathbb{E}[(X_i(H)-X_i(H'))^2])^{1/2}$$
$$\leqslant L\ \Lambda^{1/2}\sum_{p\geqslant 0} 2^{p/2}\sup_{k\in[\![0,q_p-1]\!]}|y_{k+1}^{(p)}-y_k^{(p)}|^{\delta/2} = G\ \Lambda^{1/2}\ (H_2^{(i)}-H_1^{(i)})^{\delta/2}.$$

By Hölder continuity of h, $H_2^{(i)} - H_1^{(i)} \leqslant \eta\ |t_{i+1} - t_i|^\beta = \eta\ \frac{|b-a|^\beta}{q_N^\beta}$. Using (2.4), we finally get $\mu_i < \frac{GD^{1/2}}{q_N^{\delta\beta/2}} < \tau/8$. □

Note that lemma 2.2 implies that $(\tau/8)^2 < (\tau/4 - \mu_i)^2$. Let's go back to the proof of tightness. [2, (2.6) p.43] yields, for all $n \geqslant N$,

$$Q_n^{\tau,i} \leqslant 4 \exp\left\{ \frac{-(\tau/4-\mu_i)^2}{2\sigma_i^2} \right\},$$

where $\sigma_i^2 := \sup_{H\in\mathcal{K}^{(i)}} \mathbb{E}\Big[X_i(H)^2\Big]$. By definition of $(X_i(H))_{H\in\mathcal{K}^{(i)}}$ and assumption $(\mathcal{H})$:

$$\sigma_i^2 \leqslant \Lambda \sup_{H\in\mathcal{K}^{(i)}} |H - h(t_i)|^\delta \leqslant \Lambda\ \eta^\delta\ |t_{i+1}-t_i|^{\beta\delta} = \Lambda\ \eta^\delta\ \frac{|b-a|^{\beta\delta}}{q_N^{\beta\delta}} = \frac{D}{q_N^{\beta\delta}}.$$

This yields that $Q_n^{\tau,i} \leqslant\ 4\ \exp\left\{\frac{-\tau^2\ q_N^{\delta\beta}}{2^7\ D}\right\}$ and finally:

$$\forall i \in [\![0, q_N]\!],\ \forall n \geqslant N, \qquad (1+m) \max_{0\leqslant i\leqslant m} Q_n^{\tau,i} < \frac{\varepsilon}{2}. \tag{2.10}$$

Using (2.7) and (2.10), inequality (2.5) then becomes $J_n^{\tau,m} < \varepsilon$ for every $n \geqslant N$, which ends the proof. □

(ii) Almost sure convergence

Denote $\widetilde{\Omega}$ the measurable subset of Ω, verifying $P(\widetilde{\Omega}) = 1$, such that for every ω in $\widetilde{\Omega}$, $(t,H) \mapsto B(t,H)(\omega)$ is continuous on $[a,b]\times[H_1,H_2]$. Then, for every ω in Ω, we get:

$$|B_t^{h_n}(\omega') - B_t^h(\omega')| = |B(t,h_n(t))(\omega') - B(t,h(t))(\omega')| = \left|B(t,h\big(x_t^{(n)}\big))(\omega') - B(t,h(t))(\omega')\right| \underset{n\to+\infty}{\longrightarrow} 0.$$

This ends the proof. □

The four fractional fields $(\mathbf{B}_i(,))_{i\in[\![1,4]\!]}$ defined at the end of section 1 verify assumption $(\mathcal{H})$, as shown by the next proposition:

Proposition 2.3. *The fractional Brownian fields* $\mathbf{B}_i := (\mathbf{B}_i(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$, $i \in [\![1,4]\!]$, *fulfill Assumption* $(\mathcal{H}_1)$.

Proof: The case of $\mathbf{B}_1$ and $\mathbf{B}_2$:
(3.10) of [75] yields that $\mathbf{B}_2(t,H) \overset{(law)}{=} \frac{2\cos((H+1/2)\frac{\pi}{2})}{\sqrt{2\pi}}\Gamma(H+1/2)\ c_H\ \mathbf{B}_1(t,H)$. It is thus sufficient to establish $(\mathcal{H}_1)$ for $\mathbf{B}_1$. Let $[a,b]\times[c,d]$ be fixed in $\mathbb{R}\times(0,1)$. For all (t,H,H') in $[a,b]\times[c,d]^2$,

$$\begin{aligned} I_t^{H,H'} := \mathsf{E}\left[(B(t,H)-B(t,H'))^2\right] &= \int_{\mathbb{R}} \left| \frac{e^{it\xi}-1}{c_H|\xi|^{H+1/2}} - \frac{e^{it\xi}-1}{c_{H'}|\xi|^{H'+1/2}} \right|^2 d\xi \\ &= \int_{\mathbb{R}} \left|\frac{e^{it\xi}-1}{\xi}\right|^2 \left| \tfrac{1}{c_H}\ |\xi|^{1/2-H} - \tfrac{1}{c_{H'}}\ |\xi|^{1/2-H'} \right|^2 d\xi. \end{aligned} \tag{2.11}$$

For ξ in $\mathbb{R}^*$, the map $f_\xi : [c,d] \to \mathbb{R}_+$, defined by $f_\xi(H) := \frac{1}{c_H}\ |\xi|^{1/2-H}$ is C^1 since $H \mapsto c_H$ is C^∞ on $(0,1)$. Thus there exists a positive real D such that

$$\forall(\xi,H)\in\mathbb{R}^*\times[c,d],\ |f'_\xi(H)| \leqslant D\ |\xi|^{1/2-H}\ (1+|\ln(|\xi|)|) \leqslant D\ \left(|\xi|^{1/2-c}+|\xi|^{1/2-d}\right)\ (1+|\ln(|\xi|)|).$$

Thanks to the mean-value theorem, (2.5) then reads

$$\begin{aligned} I_t^{H,H'} &\leqslant\ D^2\ |H-H'|^2 \int_{\mathbb{R}} \frac{|e^{it\xi}-1|^2}{|\xi|^2}\ \left(|\xi|^{1/2-c}+|\xi|^{1/2-d}\right)^2\ (1+|\ln(|\xi|)|)^2\ d\xi \\ &\leqslant\ D^2\ |H-H'|^2 \left(2^3 \int_{|\xi|>1} \frac{(1+\ln|\xi|)^2}{|\xi|^{1+2c}}\ d\xi + (2t)^2 \int_{|\xi|\leqslant 1} |\xi|^{1-2d}\ (1+|\ln(|\xi|)|)^2\ d\xi\right) \\ &\leqslant (2^3+T^2)\ D^2\ \left(\int_{|\xi|>1} \frac{(1+\ln|\xi|)^2}{|\xi|^{1+2c}}\ d\xi + \int_{|\xi|\leqslant 1} |\xi|^{1-2d}\ (1+|\ln(|\xi|)|)^2\ d\xi\right)\ |H-H'|^2. \end{aligned}$$

Since the two integrals in the last line are finite, assumption $(\mathcal{H}_1)$ is verified with $\delta = 2$.
The case of $\mathbf{B}_3$ is settled in [68, proof of theorem 4, (2.10)] and the one of $\mathbf{B}_4$ in [18, Proposition 3, (5)]. □

2.2 Particular cases of Kernel integral representations of the fractional field B

An integral representation of fBm is used, mostly, for analytical computations. It is thus relevant to recast assumption $(\mathcal{H})$ in terms of the kernel used in these representations. Two situations are present in the literature: an integral over a compact interval, and an integral over $\mathbb{R}$. We deal with both in the next subsections.

On a compact set $[0,T]$

In this case (see, *e.g* [3]), the fractional field $(\mathbf{B}(t,H))_{(t,H)\in[0,T]\times(0,1)}$ is defined by

$$\mathbf{B}(t,H) := \int_0^T \; K(t,u,H)\; \mathbb{W}(du),$$

where $\mathbb{W}$ is a Gaussian measure, K is defined on $[0,T]^2 \times (0,1)$ and verifies $u \mapsto K(t,u,H)$ belongs to $L^2([0,T],du)$, for all (t,H) in $[0,T]\times(0,1)$. This case is for instance the one of $\mathbf{B}_4$. As one can easily see, the following condition (C_K) entails $(\mathcal{H})$.

$$(C_K) : \forall (c,d) \text{ with } 0<c<d<1, H\mapsto K(t,u,H) \text{ is Hölder continuous on } [c,d], \text{uniformly in } (t,u) \text{ in } [0,T]^2,$$
$$i.e\ \exists (M,\delta)\in(\mathbb{R}_+^*)^2,\ \forall (t,u) \text{ in } [0,T]^2,\ |K(t,u,H)-K(t,u,H')| \leqslant M\ |H-H'|^{\delta}.$$

Condition (C_K) is fulfilled by the kernel K defining $\mathbf{B}_4$ (see [18, Proposition 3, (5)]).

On $\mathbb{R}$

A representation with an integral over $\mathbb{R}$ is used for instance in [14, 34]. The fractional field $(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$ is then defined by $\mathbf{B}(t,H) := \int_{\mathbb{R}} \; M(t,u,H)\; \mathbb{W}(du)$ where M is defined on $\mathbb{R}^2\times(0,1)$, and verifies $u \mapsto K(t,u,H)$ belongs to $L^2(\mathbb{R},du)$, for every (t,H) in $\mathbb{R}\times(0,1)$. This is the case for the fields $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$. Condition (C_M) entails $(\mathcal{H})$:

$$(C_M) : \forall [a,b]\subset\mathbb{R}, \forall [c,d]\subset(0,1), \exists \delta\in\mathbb{R}_+^*, \forall t\in[a,b], \exists \Phi_t\in L^2(\mathbb{R},du),\ \text{verifying } \sup_{t\in[a,b]}\int_{\mathbb{R}} |\Phi_t(u)|^2\ du < +\infty,$$
$$s.t.\ \forall (u,H,H')\in\mathbb{R}\times[c,d]^2,\ |M(t,u,H)-M(t,u,H')| \leqslant \Phi_t(u)\ |H-H'|^{\delta}.$$

Condition (C_M) is fulfilled by the kernel M defining $\mathbf{B}_1$ and $\mathbf{B}_2$ (see [18, Proposition 3, (5)]).

3 Stochastic integrals w.r.t. mBm as limits of integrals w.r.t. fBm

Our aim in this section is to show that one may define an integral with respect to mBm by using integrals with respect to fBms as approximations.
We consider as above a fractional field $(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$, but we assume in addition that the field is C^1 in H on $(0,1)$ in the $L^2(\Omega)$ sense, *i.e* we assume that the map $H\mapsto\mathbf{B}(t,H)$, from $(0,1)$ to $L^2(\Omega)$, is C^1 for every real t. We will denote $\frac{\partial\mathbf{B}}{\partial H}(t,H')$ the $L^2(\Omega)$-derivative at point H' of the map $H\mapsto\mathbf{B}(t,H)$. The field $(\frac{\partial\mathbf{B}(t,H)}{\partial H})_{(t,H)\in\mathbb{R}\times(0,1)}$ is of course Gaussian. We will need that the derivative field satisfies the same assumption $(\mathcal{H}_1)$ as $\mathbf{B}(t,H)$. More precisely, from now on, we assume that $\mathbf{B}(t,H)$ satisfies Assumption $(\mathcal{H}_2)$ we define now,

$(\mathcal{H}_2)$: $\forall [a,b]\times[c,d]\subset\mathbb{R}\times(0,1), H\mapsto\mathbf{B}(t,H)$ is C^1 in the $L^2(\Omega)$ sense from $(0,1)$ to $L^2(\Omega)$ for every t in $[a,b]$ and $\exists(\Delta,\alpha,\lambda)\in(\mathbb{R}_+^*)^3$ s.t. $\mathbb{E}\left[\left(\frac{\partial\mathbf{B}}{\partial H}(t,H)-\frac{\partial\mathbf{B}}{\partial H}(s,H')\right)^2\right] \leqslant \Delta\ \left(|t-s|^{\alpha}+|H-H'|^{\lambda}\right)$, for every (t,s,H,H') in $[a,b]^2\times[c,d]^2$.

Proposition 3.1. *The fractional Brownian fields* $\mathbf{B}_i := (\mathbf{B}_i(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$, $i \in [\![1,4]\!]$, *verify Assumption* $(\mathcal{H}_2)$.

Proof: The proof of this proposition in the case of $\mathbf{B}_1$ and $\mathbf{B}_2$ may be found in Appendix VI. The ones for $\mathbf{B}_3$ and $\mathbf{B}_4$ are easily obtained using results from [68] and [18] and are left to the reader. □

Let us consider from now on a C^1 deterministic function $h : \mathbb{R} \to (0,1)$, a fractional field $\mathbf{B}$ which fulfills assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, and the associated mBm $B_t^h := \mathbf{B}(t,h(t))$.

We now explain in a heuristic way how to define an integral with respect to mBm using approximating fBms. Write the total differential of $\mathbf{B}(t,H)$:

$$d\mathbf{B}(t,H) = \frac{\partial \mathbf{B}}{\partial t}(t,H)\ dt + \frac{\partial \mathbf{B}}{\partial H}(t,H)\ dH.$$

Of course, this is only formal as $t \mapsto \mathbf{B}(t,H)$ is not differentiable in L^2-sense nor almost surely with respect to t. It is, however, in the white noise sense, but we are not interested in these aspects at this stage. With a differentiable function h in place of H, this (again formally) yields

$$d\mathbf{B}(t,h(t)) = \frac{\partial \mathbf{B}}{\partial t}(t,h(t))\ dt + h'(t)\ \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\ dt. \tag{3.1}$$

The second term on the right-hand side of (3.1) is defined for almost every ω and every real t by assumption. Moreover, it is almost surely continuous as fonction of t and thus Riemann integrable on compact intervals. On the other hand, the first term of (3.1) has no meaning *a priori* since mBm is not differentiable with respect to t. However, since stochastic integrals with respect to fBm do exist, we are able to give a sense to $t \mapsto \frac{\partial \mathbf{B}}{\partial t}(t,H)$ for every fixed H in $(0,1)$. Continuing with our heuristic reasoning, we then approximate $\frac{\partial \mathbf{B}}{\partial t}(t,h(t))$ by $\lim\limits_{n\to+\infty} \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)},x_{k+1}^{(n)})}(t)\ \frac{\partial \mathbf{B}}{\partial t}(t,h_n(t))$. This formally yields:

$$d\mathbf{B}(t,h(t)) \approx \lim_{n\to+\infty} \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)},x_{k+1}^{(n)})}(t)\ \frac{\partial \mathbf{B}}{\partial t}(t,h_n(t))\ dt + h'(t)\ \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\ dt. \tag{3.2}$$

Assuming we may exchange integrals and limits, we would thus like to define, for suitable processes Y,

$$\int_0^1 Y_t\ d\mathbf{B}(t,h(t)) = \lim_{n\to+\infty} \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} Y_t\ dB_t^{h(x_k^{(n)})} + \int_0^1 Y_t\ h'(t)\ \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\ dt, \tag{3.3}$$

where the first term of (3.3) is a sum of integrals with respect to several fBms and the second term is, for almost every ω, a Riemann integral.

The interest of this approach is that one may use any of the numerous definitions of stochastic integration with respect to fBm, and automatically obtain a corresponding integral with respect to mBm. It is worthwhile to note that, with this approach, an integral with respect to mBm is a sum of two terms: the first one seems to depend only on the chosen method for integrating with respect to fBm, (*e.g.* a white noise or pathwise integral), while the second is, in most cases, a Riemann integral which appears to depend only on the field used to define the chosen mBm, *i.e* essentially on its correlation structure. This second term will imply that the integral with respect to the moving average mBm, for instance, is different from the one with respect to the harmonisable mBm. However, as the example of simple processes in the next subsection will show, the second term does also depend on the integration method with respect to fBm.

Let us now make the above idea precise. We first fix some notations. $(\mathcal{M})$ denotes a fixed method of integration with respect to fBm (*e.g* Skohorod, white noise, pathwise, $\cdots$). For the sake of notational simplicity, we will consider integrals over the interval $[0,1]$. For H in $(0,1)$, denote $\int_0^1 Y_t\ d^{(\mathcal{M})}B_t^H$ the integral of $Y := (Y_t)_{t\in[0,1]}$ on $[0,1]$ with respect to the fBm B^H, in sense of method $(\mathcal{M})$, assuming it does exist. With the same notations as in subsection 2.1, we consider a sequence $(q_n)_{n\in\mathbb{N}}$, a family $x^{(n)} := \{x_k^{(n)}; k \in [\![0,q_n]\!]\}$ (which is defined on $[0,1]$ by $x_k^{(n)} := \frac{k}{q_n}$ for k in $[\![0,q_n]\!]$) and the family of partition $\mathcal{A}_n := \{[x_k^{(n)},x_{k+1}^{(n)}); k \in [\![0,q_n-1]\!]\} \cup \{x_{q_n}^{(n)}\}$ of $[0,1]$. The following notations will be useful:

Definition 3.1 (integral with respect to lumped fBms). *Let $Y := (Y_t)_{t\in[0,1]}$ be a process on $[0,1]$ which is integrable with respect to all fBms of index H in $h([0,1])$ in the sense of method $(\mathcal{M})$. We denote the integral with respect to lumped fBms in sense of method $(\mathcal{M})$ by:*

$$\int_0^1 Y_t \ d^{(\mathcal{M})} B_t^{h_n} := \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \ \ Y_t \ d^{(\mathcal{M})} B_t^{h(x_k^{(n)})}, \quad n \in \mathbb{N}. \tag{3.4}$$

Note that the nature of $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h_n}$ depends on the method $(\mathcal{M})$ of integration. For example, $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{H}$ and hence $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h_n}$ will belong to $L^2(\Omega)$ if $(\mathcal{M})$ denotes the Skohorod integral, whereas $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{H}$ and hence $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h_n}$ belong to the space $(S)^*$ of stochastic distributions when $(\mathcal{M})$ denotes the integral in the sense of white noise theory.

We will write $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h}$ for the integral of Y on $[0,1]$ with respect to mBm in the sense of method $(\mathcal{M})$ (which is yet to be defined). When we do not want to specify a particular method $(\mathcal{M})$ but instead wish to refer to all methods at the same time, we will write $\int_0^1 Y_t \, dB_t^{h_n}$ and $\int_0^1 Y_t \, dB_t^{h}$ instead of $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h_n}$ and $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h}$.
In order to gain a better understanding of our approach, we treat in the following subsection the cases of simple deterministic and then random integrands.

3.1 Example: simple integrands

Deterministic simple integrands

Any reasonable definition of an integral with respect to mBm must be linear. Thus it suffices to consider the case of $Y = 1$. Obviously, we should find that $\int_0^1 1 \, dB_t^h = B_1^h$. Let us then compute the limit of the sequence $(\int_0^1 1 \, dB_t^{h_n})_{n\in\mathbb{N}}$.
In that view, it is convenient to use the S-transform. For a function η in the Schwartz space $\mathscr{S}(\mathbb{R})$ we will denote $S(\Phi)(\eta)$ the S-transform of the L^2-random variable Φ at point η (the reader who is not familiar with the S-transform may refer to section 5 of this paper where this notion is recalled in the more general setting of Hida spaces). For a field $\mathbf{B}$ and (t, H, η) in $\mathbb{R} \times (0,1) \times \mathscr{S}(\mathbb{R})$, we shall write $g_\eta(t,H)$ for the S-transform at point η of the random variable $\mathbf{B}(t,H)$. In other words, $g_\eta(t,H) := S(\mathbf{B}(t,H))(\eta)$.

Proposition 3.2. *Assume that the map h is C^2, that the map $H \mapsto g_\eta(t,H)$ is C^2 for every (t,η) in $\mathbb{R}\times\mathscr{S}(\mathbb{R})$ and that $(t,H) \mapsto \frac{\partial^i g_\eta}{\partial H^i}(t,H)$ is continuous on $\mathbb{R} \times (0,1)$ for i in $\{1;2\}$. The sequence $(\int_0^1 1 \, dB_t^{h_n})_{n\in\mathbb{N}}$ then converges in $L^2(\Omega)$ to $\mathbf{B}(1,h(1)) - \int_0^1 h'(t) \, \frac{\partial \mathbf{B}}{\partial H}(t,h(t)) \, dt$, where the second term is a pathwise integral.*

Proposition 3.2 implies that, for regular enough fields $\mathbf{B}$ and h functions, formula (3.3) does indeed yield $\int_0^1 1 \, dB_t^h = B_1^h$.
Proof:
Let us first show that $\int_0^1 h'(t) \, \frac{\partial \mathbf{B}}{\partial H}(t,h(t)) \, dt$ belongs to $L^2(\Omega)$. Cauchy-Schwarz inequality and assumption $(\mathcal{H}_2)$ entail:

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^1 h'(t) \, \frac{\partial \mathbf{B}}{\partial H}(t,h(t) \, dt\right)^2\right] &\leqslant \underbrace{\left(\int_0^1 |h'(t)|^2 \, dt\right)}_{:=\mu} \int_0^1 \mathbb{E}\left[\left(\frac{\partial \mathbf{B}}{\partial H}(t,h(t))\right)^2\right] dt \\ &\leqslant \mu\Delta \left(\int_0^1 (|t|^\alpha + |h(t)|^\lambda) \, dt\right) < +\infty. \end{aligned}$$

Let $I_n := \int_0^1 1 \, dB_t^{h_n}$. Thanks to (3.4) we have:

$$I_n = \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t)\ 1\ dB_t^{h(x_k^{(n)})} = \sum_{k=0}^{q_n-1} \left(B_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} - B_{x_k^{(n)}}^{h(x_k^{(n)})} \right)$$
$$= B_1^{h\left(x_{q_n-1}^{(n)}\right)} - \underbrace{\sum_{k=1}^{q_n-1} \left(B_{x_k^{(n)}}^{h(x_k^{(n)})} - B_{x_k^{(n)}}^{h(x_{k-1}^{(n)})} \right)}_{=:J_n} \tag{3.5}$$

It is thus sufficient to show that the sequence $(J_n)_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$ to $\int_0^1\ h'(t)\ \frac{\partial \mathbf{B}}{\partial H}(t, h(t)\ dt$. Thanks to [9, theorem 2.3], this is equivalent to showing that the sequence $((S(J_n)(\eta))_{n\in\mathbb{N}}$ (respectively $(\mathbb{E}\left[J_n^2\right])_{n\in\mathbb{N}}$) converges to $S(\int_0^1\ h'(t)\ \frac{\partial \mathbf{B}}{\partial H}(t, h(t)\ dt)(\eta)$ (respectively to $\mathbb{E}\left[\left(\int_0^1\ h'(t)\ \frac{\partial \mathbf{B}}{\partial H}(t, h(t)\ dt\right)^2\right]$) for all η in $\mathscr{S}(\mathbb{R})$. Now, for every integer n and η in $\mathscr{S}(\mathbb{R})$, taking S-transform of J_n at point η and using two times the finite increment theorem, there exist $(r_k^{(n)}, w_k^{(n)})$ in $[x_{k-1}^{(n)}, x_k^{(n)}]^2$ such that

$$S(J_n)(\eta) = \sum_{k=1}^{q_n-1}\ S\left(B_{x_k^{(n)}}^{h(x_k^{(n)})}\right)(\eta) - S\left(B_{x_k^{(n)}}^{h(x_{k-1}^{(n)})}\right)(\eta) = \sum_{k=1}^{q_n-1}\ g_\eta(x_k^{(n)}, h(x_k^{(n)})) - g_\eta(x_k^{(n)}, h(x_{k-1}^{(n)}))$$
$$= \sum_{k=1}^{q_n-1} \frac{\partial g_\eta}{\partial H}\left(x_k^{(n)}, h(r_k^{(n)})\right)\left(h(x_k^{(n)}) - h(x_{k-1}^{(n)})\right) = \sum_{k=1}^{q_n-1} \frac{\partial g_\eta}{\partial H}\left(x_k^{(n)}, h(r_k^{(n)})\right)\ h'(w_k^{(n)})(x_k^{(n)} - x_{k-1}^{(n)})$$
$$= \underbrace{\frac{1}{q_n}\sum_{k=1}^{q_n-1}\left(\frac{\partial g_\eta}{\partial H}\left(x_k^{(n)}, h(r_k^{(n)})\right)\ h'(w_k^{(n)}) - \frac{\partial g_\eta}{\partial H}\left(x_k^{(n)}, h(x_k^{(n)})\right)\ h'(x_k^{(n)})\right)}_{=:K_n} + \underbrace{\frac{1}{q_n}\sum_{k=1}^{q_n-1}\frac{\partial g_\eta}{\partial H}\left(x_k^{(n)}, h(x_k^{(n)})\right) h'(x_k^{(n)})}_{=:L_n}. \tag{3.6}$$

By definition of Riemann integral,

$$\lim_{n\to+\infty} L_n = \int_0^1\ h'(t)\ \frac{\partial g_\eta}{\partial H}(t, h(t))\ dt = \int_0^1\ h'(t)\ S\left(\frac{\partial \mathbf{B}}{\partial H}(t, h(t))\right)(\eta)\ dt = S\left(\int_0^1\ h'(t)\ \frac{\partial \mathbf{B}}{\partial H}(t, h(t))\ dt\right)(\eta). \tag{3.7}$$

Besides, the assumptions made on the map g_η and the mean value theorem entail that there exists a couple $(t_k^{(n)}, v_k^{(n)})$ in $[x_k^{(n)}, x_{k+1}^{(n)}]^2$ such that

$$|K_n| \leqslant \frac{1}{q_n}\sum_{k=1}^{q_n-1}\ \Bigg(|h(r_k^{(n)}) - h(x_k^{(n)})|\ |h'(w_k^{(n)})|\ \left|\frac{\partial^2 g_\eta}{\partial H^2}(x_k^{(n)}, h(t_k^{(n)})\right|$$
$$+ |h''(v_k^{(n)})|\ |w_k^{(n)} - x_k^{(n)}|\ \left|\frac{\partial g_\eta}{\partial H}\left(x_k^{(n)}, h(x_k^{(n)})\right)\right|\Bigg).$$

We hence have,

$$|K_n| \leqslant A_1\ \frac{1}{q_n}\sum_{k=1}^{q_n-1}\ \left(\eta\ |r_k^{(n)} - x_k^{(n)}|^\beta\ + |w_k^{(n)} - x_k^{(n)}|\right) \leqslant A_1\ (1+\eta)\ \left(\frac{1}{q_n^\beta} + \frac{1}{q_n}\right) \underset{n\to+\infty}{\longrightarrow} 0, \tag{3.8}$$

where $A_1 := \sup_{u\in[0,1]} |h'(u)| \sup_{(u,v)\in[0,1]\times h([0,1])} \left|\frac{\partial^2 g_\eta}{\partial H^2}(u,v)\right| + \sup_{u\in[0,1]} |h''(u)| \sup_{(u,v)\in[0,1]\times h([0,1])} \left|\frac{\partial g_\eta}{\partial H}(u,v)\right|$.
Thanks to (3.7) and (3.8) we get $\lim_{n\to+\infty} S(J_n)(\eta) = S\left(\int_0^1\ h'(t)\ \frac{\partial \mathbf{B}}{\partial H}(t, h(t))\ dt\right)(\eta)$ for every η in $\mathscr{S}(\mathbb{R})$. It then remains to show that $\lim_{n\to+\infty} \mathbb{E}[J_n^2] = \mathbb{E}[\left(\int_0^1\ h'(t)\ \frac{\partial \mathbf{B}}{\partial H}(t, h(t))\ dt\right)^2]$. Using Fubini's theorem,

$$Q := \mathbb{E}\left[\left(\int_0^1 h'(t)\, \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\, dt\right)^2\right] = \int_0^1\int_0^1 h'(t)\, h'(s)\, \overbrace{\mathbb{E}\left[\frac{\partial \mathbf{B}}{\partial H}(t,h(t))\, \frac{\partial \mathbf{B}}{\partial H}(s,h(s))\right]}^{=:\Theta((t,s,h(t)),h(s)))}\, ds\, dt.$$

For the same reason as in (i) of remark 1.1, assumption $(\mathcal{H}_2)$ entails that the function Θ is continuous on $\mathbb{R}^2 \times (0,1)^2$. Thus,

$$Q := \lim_{n\to+\infty} \frac{1}{q_n^2} \sum_{1\leqslant k,j\leqslant q_n-1} h'(x_j^{(n)})\, h'(x_k^{(n)})\, \mathbb{E}\left[\tfrac{\partial \mathbf{B}}{\partial H}(x_j^{(n)}, h(x_j^{(n)}))\tfrac{\partial \mathbf{B}}{\partial H}(x_k^{(n)}, h(x_k^{(n)}))\right]. \tag{3.9}$$

Besides for any n, thanks to (3.6),

$$J_n = \underbrace{\frac{1}{q_n}\sum_{k=1}^{q_n-1}\left(\frac{\partial \mathbf{B}}{\partial H}\left(x_k^{(n)}, h(r_k^{(n)})\right)\, h'(w_k^{(n)}) - \frac{\partial \mathbf{B}}{\partial H}\left(x_k^{(n)}, h(x_k^{(n)})\right)\, h'(x_k^{(n)})\right)}_{=:J_n^{(1)}}$$
$$+ \underbrace{\frac{1}{q_n}\sum_{k=1}^{q_n-1}\frac{\partial \mathbf{B}}{\partial H}\left(x_k^{(n)}, h(x_k^{(n)})\right)\, h'(x_k^{(n)})}_{=:J_n^{(2)}}. \tag{3.10}$$

In view of (3.9), it is clear that $(J_n^{(2)})_{n\in\mathbb{N}}$ tends to Q in $L^2(\Omega)$. Let us now show that $(J_n^{(1)})_{n\in\mathbb{N}}$ tends to 0 in $L^2(\Omega)$. In a way similar to the computations leading to (3.8), we get

$$J_n^{(1)} = \frac{1}{q_n}\sum_{k=1}^{q_n-1}\left(h''(v_k^{(n)})\,(w_k^{(n)} - x_k^{(n)})\,\frac{\partial \mathbf{B}}{\partial H}\left(x_k^{(n)}, h(x_k^{(n)})\right)\right)$$
$$+ \frac{1}{q_n}\sum_{k=1}^{q_n-1} h'(w_k^{(n)})\left(\frac{\partial \mathbf{B}}{\partial H}\left(x_k^{(n)}, h(r_k^{(n)})\right) - \frac{\partial \mathbf{B}}{\partial H}\left(x_k^{(n)}, h(x_k^{(n)})\right)\right) =: M_n + N_n. \tag{3.11}$$

Using (3.8), we get on the one hand

$$\mathbb{E}\left[M_n^2\right] = \frac{1}{q_n^2}\sum_{1\leqslant k,j\leqslant q_n-1} h''(v_k^{(n)})\, h''(v_j^{(n)})\,(w_k^{(n)} - x_k^{(n)})\,(w_j^{(n)} - x_j^{(n)})\,\Theta\left(x_k^{(n)}, x_j^{(n)}, h(x_k^{(n)}), h(x_j^{(n)})\right)$$
$$\leqslant \frac{A_2}{q_n^2}\sum_{1\leqslant k,j\leqslant q_n-1}(w_k^{(n)} - x_k^{(n)})\,(w_j^{(n)} - x_j^{(n)}) \leqslant \frac{A_2}{q_n^2} \underset{n\to+\infty}{\longrightarrow} 0, \tag{3.12}$$

where we set $A_2 := \sup_{u\in[0,1]} |h''(u)|^2 \sup_{(u,v)\in[0,1]^2} |\Theta(u,v,h(u),h(v))|$. On the other hand, setting $A_3 := \sup_{u\in[0,1]} |h'(u)|$ and using Cauchy-Schwarz's inequality and assumption $(\mathcal{H}_2)$, we get

$$\mathbb{E}\left[N_n^2\right] \leqslant$$

$$\frac{A_3}{q_n^2}\sum_{1\leqslant k,j\leqslant q_n-1}\left|\left|\tfrac{\partial \mathbf{B}}{\partial H}\left(x_k^{(n)},h(r_k^{(n)})\right)-\tfrac{\partial \mathbf{B}}{\partial H}\left(x_k^{(n)},h(x_k^{(n)})\right)\right|\right|_{L^2(\Omega)}\left|\left|\tfrac{\partial \mathbf{B}}{\partial H}\left(x_j^{(n)},h(r_j^{(n)})\right)-\tfrac{\partial \mathbf{B}}{\partial H}\left(x_j^{(n)},h(x_j^{(n)})\right)\right|\right|_{L^2(\Omega)}$$

$$\leqslant \frac{A_3\ \Delta}{q_n^2}\sum_{1\leqslant k,j\leqslant q_n-1}\left(|h(r_j^{(n)})-h(x_j^{(n)})|\ |h(r_k^{(n)})-h(x_k^{(n)})|\right)^{\lambda/2}$$

$$\leqslant \frac{A_3\ \Delta(1+\eta^\lambda)}{q_n^2}\left(\sum_{1\leqslant k\leqslant q_n-1}|r_k^{(n)}-x_k^{(n)}|^{\beta\lambda/2}\right)^2 \leqslant \frac{A_3\ \Delta(1+\eta^\lambda)}{q_n^2}\left(\sum_{1\leqslant k\leqslant q_n-1}\frac{1}{q_n^{\beta\lambda/2}}\right)^2$$

$$\leqslant \frac{A_3\ \Delta(1+\eta^\lambda)}{q_n^{\beta\lambda}}\underset{n\to+\infty}{\longrightarrow}0. \tag{3.13}$$

Inequalities (3.12) and (3.13) show that $(J_n^{(1)})_{n\in\mathbb{N}}$ tends to 0 in $L^2(\Omega)$ which ends the proof. □

Remark 3.3. (*i*) *One can also show that the sequence* $(\int_0^1 1\ dB_t^{h_n}\ dt)_{n\in\mathbb{N}}$ *converges almost surely.*
(*ii*) *Proposition* (3.2) *applies to the four fields considered in the introduction, since they all satisfy the required assumptions: in the case of* $\mathbf{B}_1$, *it has been proved in [53, lemma 5.5], that* $(t,H)\mapsto g_\eta(t,H)$ *belongs to* $C^\infty(\mathbb{R}\times(0,1))$ *for every* η *in* $\mathscr{S}(\mathbb{R})$. *Moreover, the result is similar when* $\mathbf{B}=\mathbf{B}_3$, *since [75] example 2 shows that there exist two* C^∞ *maps* $H\mapsto\alpha_H$ *and* $H\mapsto\upsilon_H$ *from* $(0,1)$ *to* $\mathbb{R}$ *such that* $\mathbf{B}_3(t,H)=\alpha_H\ \mathbf{B}_1(t,H)+\upsilon_H\underbrace{\int_{\mathbb{R}}\frac{e^{it\xi}-1}{i\xi|\xi|^{H-1/2}}\ \widetilde{\mathbb{W}_1}(d\xi)}_{=:Z_H(t)}$. *Since* $S(Z_H(t))(\eta)=\int_{\mathbb{R}}\eta(u)\ \frac{e^{itu}-1}{iu|u|^{H-1/2}}\ du$ *for every* η *in* $\mathscr{S}(\mathbb{R})$ *the conclusion follows. The case of* $\mathbf{B}_4$ *is similar and left to the reader.*

Simple processes

Given the result of Proposition 3.2 it seems natural to set

$$\int_0^1 Y_t\ d^{(\mathcal{M})}B_t^h := \lim_{n\to\infty}\int_0^1 Y_t\ d^{(\mathcal{M})}B_t^{h_n}+\Phi_h^{(\mathcal{M})}(Y), \tag{3.14}$$

where the process Y is integrable with respect to B^{h_n} in sense of method $(\mathcal{M})$, for every n in $\mathbb{N}$, and where we assume moreover that:
-$\lim_{n\to\infty}\int_0^1 Y_t\ d^{(\mathcal{M})}B_t^{h_n}$ exist in a sense to be made precise
-the functional Φ_h is defined on the set containing process Y with value in $L^2(\Omega)$ and fulfills $\Phi_h^{(\mathcal{M})}(Y)=\int_0^1 h'(t)\ Y_t\ \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\ dt$ for every constant process $(Y_t)_{t\in[0,1]}$ and $\Phi_h^{(\mathcal{M})}(Y)=0$ as soon as h is constant.
The following examples gives indication of what sould be the functional Φ_h in general cases.

Examples:
1. Let $(\mathcal{M}_1)$ denote the White Noise Theory

The reader which is not familiar with integral with respect to fBm in the sense of White noise Theory (also called fractional Wick-Itô integral) may refer to [34] and [9].
Let m be a fixed integer in $\mathbb{N}$. Define, for every i in $[\![0,q_m]\!]$, $t_i:=x_i^{(m)}$ and let Y be the process defined by $Y_t:=\sum_{i=0}^{q_m-1}\ Y_i\ \mathbb{1}_{(t_i,t_{i+1}]}(t)$ where, for every i, Y_i is a centred Gaussian random variable which is $\mathcal{F}(B^h)$-mesurable where $\mathcal{F}(B^h)$ denotes the σ-field generated by the random variables in the first Wiener chaos of the mBm B^h.
Since the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of partitions of $[0,1]$ is nested, we know that, for every integer $n\geqslant m$ and every ineteger i in $[\![0,q_m]\!]$, there exists an unique integer $k_i^{(n)}$ in $[\![0,q_n]\!]$ such that $t_i=x_{k_i^{(n)}}^{(n)}$. We hence have

$[t_i, t_{i+1}] = \bigcup_{k_i^{(n)} \leqslant k \leqslant k_{i+1}^{(n)}-1} [x_k^{(n)}, x_{k+1}^{(n)}]$ for every $n \geqslant m$. Define now $S_n := \int_0^1 Y_t \ d^{(\mathcal{M}_1)} B_t^{h_n}$ for every ineteger n in $\mathbb{N}$. Using (3.5) we get, for every $n \geqslant m$,

$$\begin{aligned}
S_n &= \sum_{i=0}^{q_m-1} \int_{t_i}^{t_{i+1}} Y_i \ d^{(\mathcal{M}_1)} B_t^{h(x_k^{(n)})} = \sum_{i=0}^{q_m-1} \sum_{k=k_i^{(n)}}^{k_{i+1}^{(n)}-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} Y_i \ d^{(\mathcal{M}_1)} B_t^{h(x_k^{(n)})} \\
&= \sum_{i=0}^{q_m-1} \sum_{k=k_i^{(n)}}^{k_{i+1}^{(n)}-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} Y_i \diamond W_t^{h(x_k^{(n)})} \ dt = \sum_{i=0}^{q_m-1} \sum_{k=k_i^{(n)}}^{k_{i+1}^{(n)}-1} Y_i \diamond \Big(\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} W_t^{h(x_k^{(n)})} \ dt \Big) \\
&= \sum_{i=0}^{q_m-1} Y_i \diamond \left(\sum_{k=k_i^{(n)}}^{k_{i+1}^{(n)}-1} \Big(B_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} - B_{x_k^{(n)}}^{h(x_k^{(n)})} \Big) \right). \qquad (3.15)
\end{aligned}$$

Using exactly the same method we used to show that $(J_n)_{n\in\mathbb{N}}$ converged in $L^2(\Omega)$ to $\int_0^1 h'(t) \ \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt$ in the proof of Proposition 3.2 we get the convergence in $L^2(\Omega)$ of the sequence $(\sum_{k=k_i^{(n)}}^{k_{i+1}^{(n)}-1} \Big(B_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} - B_{x_k^{(n)}}^{h(x_k^{(n)})} \Big))_{n\in\mathbb{N}}$ to $B_{t_{i+1}}^h - B_{t_i}^h - \int_{t_i}^{t_{i+1}} h'(t) \ \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt$, for every i in $[\![0, q_m - 1]\!]$. The continuity of Wick product then allow us to write

$$\lim_{n\to\infty} S_n = \sum_{i=0}^{q_m-1} Y_i \diamond (B_{t_{i+1}}^h - B_{t_i}^h) - \sum_{i=0}^{q_m-1} Y_i \diamond \int_{t_i}^{t_{i+1}} h'(t) \ \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt, \qquad (3.16)$$

where the previous limit holds in $L^2(\Omega)$. Since we have, for every i, $Y_i \diamond \int_{t_i}^{t_{i+1}} h'(t) \ \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt = \int_{t_i}^{t_{i+1}} h'(t) \ Y_i \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt = \int_{t_i}^{t_{i+1}} h'(t) \ Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt$ and $\int_{t_i}^{t_{i+1}} 1 \ d^{(\mathcal{M}_1)} B_t^h = B_{t_{i+1}}^h - B_{t_i}^h$, equality (3.16) reads,

$$\sum_{i=0}^{q_m-1} Y_i \diamond \int_{t_i}^{t_{i+1}} 1 \ d^{(\mathcal{M}_1)} B_t^h = \lim_{n\to\infty} \int_0^1 Y_t \ d^{(\mathcal{M}_1)} B_t^{h_n} + \int_0^1 h'(t) \ Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt, \qquad (3.17)$$

where the previous equality and limit hold in $L^2(\Omega)$. Remembering that, in the case of integral with respect to fBm in the White Noise sense we get the equality

$$Y_i \diamond \int_{t_i}^{t_{i+1}} 1 \ d^{\diamond} B_t^H = \int_{t_i}^{t_{i+1}} Y_i \ d^{\diamond} B_t^H = \int_{t_i}^{t_{i+1}} Y_t \ d^{\diamond} B_t^H,$$

for every H in $(0, 1)$, and assuming we have an analogous equality, in the case of the integral w.r.t. mBm, we would then get

$$Y_i \diamond \int_{t_i}^{t_{i+1}} 1 \ d^{\diamond} B_t^h = \int_{t_i}^{t_{i+1}} Y_i \ d^{\diamond} B_t^h = \int_{t_i}^{t_{i+1}} Y_t \ d^{\diamond} B_t^h,$$

for the White noise integral we want to define.
Thus we will hence have $\sum_{i=0}^{q_m-1} Y_i \diamond \int_{t_i}^{t_{i+1}} 1 \ d^{(\mathcal{M}_1)} B_t^h = \int_{t_i}^{t_{i+1}} Y_i \ d^{(\mathcal{M}_1)} B_t^h = \int_0^1 Y_t \ d^{(\mathcal{M}_1)} B_t^h$. Equality (3.17) will hence read

$$\int_0^1 Y_t \ d^{(\mathcal{M}_1)} B_t^h = \lim_{n\to\infty} \int_0^1 Y_t \ d^{(\mathcal{M}_1)} B_t^{h_n} + \int_0^1 h'(t) \ Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt,$$

where the limit and the equality hold in $L^2(\Omega)$. The previous equality and the fact that natural spaces of white noise theory are the spaces $(\mathcal{S}_{-p})$, for every p in $\mathbb{N}$ (see section 5 below), suggest to define the integral of any $(\mathcal{S}_{-p})$-valued process $Y := (Y_t)_{t\in[0,1]}$ with respect to mBm, in sense of $(\mathcal{M}_1)$, by setting,

$$\int_0^1 Y_t \ d^{(\mathcal{M}_1)} B_t^h := \lim_{n\to\infty} \int_0^1 Y_t \ d^{(\mathcal{M}_1)} B_t^{h_n} + \int_0^1 h'(t) \ Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt, \tag{3.18}$$

where the limit, the last integral, assuming they both exist, and the equality hold in $(\mathcal{S}_{-q})$ for some integer q in $\mathbb{N}$. This entails in particular that the functional $\Phi_h^{(\mathcal{M}_2)}$ is $(\mathcal{S}_{-q})$-valued for some q in $\mathbb{N}$ and is defined, assuming it exists, by $\Phi_h^{(\mathcal{M}_1)}(Y) := \int_0^1 h'(t) \ Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt$, where the last inetgral is taken in sense of Bochner. In fact it will be shown in section 5 that this definition of integral w.r.t. mBm in sense of white Noise theory is relevant since it coincide with the definition of the multifractional Wick-Itô integral given in [53] (see also definition 5.3 below).

2. Let $(\mathcal{M}_2)$ denote the Malliavin Calculus

Thanks to [64, proposition 8] and [9, corollary 3.5] we know that $\int_0^1 Y_t \ d^{(\mathcal{M}_1)} B_t^H = \int_0^1 Y_t \ d^{(\mathcal{M}_2)} B_t^H$ as soon as Y is integrable w.r.t. fBm of Hurst index H in sense of $\mathcal{M}_2$. Thus if we define $T_n := \int_0^1 Y_t \ d^{(\mathcal{M}_2)} B_t^{h_n}$ for every n in $\mathbb{N}$, we have $T_n := \int_0^1 Y_t \ d^{(\mathcal{M}_2)} B_t^{h_n} = S_n$. Keeping the same notaions as in the previous example, in particular for the simple process $Y := (Y_t)_{t\in[0,1]}$ defined by $Y_t := \sum_{i=0}^{q_m-1} Y_i \ \mathbb{1}_{(t_i,t_{i+1}]}(t)$, we get:

$$\lim_{n\to\infty} T_n = \sum_{i=0}^{q_m-1} Y_i \diamond (B_{t_{i+1}}^h - B_{t_i}^h) - \sum_{i=0}^{q_m-1} Y_i \diamond \int_{t_i}^{t_{i+1}} h'(t) \ \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt, \tag{3.19}$$

where the previous limit holds in $L^2(\Omega)$.
Now, in the case of Skorohod integral with respect to any Gaussian process $X := (X_t)_{t\in[0,1]}$, as it has been defined in [47, section 7], we get $Y_i \diamond \int_{t_i}^{t_{i+1}} 1 \ \delta X_t = \int_{t_i}^{t_{i+1}} Y_i \ \delta X_t$ thanks to [47, theorem (7.40)] and where $\int_a^b Y_i \ \delta X_t$ denotes the Skorohod integral of Y_i on $[a,b]$ w.r.t to X. Assuming this relation remains true for our Skorohod integral with respect to mBm B^h, we then would have by linearity,

$$\begin{aligned}\sum_{i=0}^{q_m-1} Y_i \diamond (B_{t_{i+1}}^h - B_{t_i}^h) &= \sum_{i=0}^{q_m-1} Y_i \diamond \int_{t_i}^{t_{i+1}} 1 \ d^{(\mathcal{M}_2)} B_t^h = \sum_{i=0}^{q_m-1} \int_{t_i}^{t_{i+1}} Y_i \ d^{(\mathcal{M}_2)} B_t^h \\ &= \sum_{i=0}^{q_m-1} \int_{t_i}^{t_{i+1}} Y_t \ d^{(\mathcal{M}_2)} B_t^h = \int_0^1 Y_t \ d^{(\mathcal{M}_2)} B_t^h.\end{aligned}$$

The previous equality and (3.19) then suggest to define the integral of Y on $[0,1]$ with respect to $(\mathcal{M}_2)$ by setting:

$$\int_0^1 Y_t \ d^{(\mathcal{M}_2)} B_t^h := \lim_{n\to\infty} \int_0^1 Y_t \ d^{(\mathcal{M}_2)} B_t^{h_n} + \int_0^1 h'(t) \ Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt, \tag{3.20}$$

where the previous equality and limit hold in $L^2(\Omega)$. This entails in particular that the functional $\Phi_h^{(\mathcal{M}_2)}$ is $L^2(\Omega)$-valued and is defined, assuming it exists, by $\Phi_h^{(\mathcal{M}_2)}(Y) := \int_0^1 h'(t) \ Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt$, where the last integral is pathwise (i.e $\Phi_h^{(\mathcal{M}_2)}(Y)(\omega)$ is defined, for almost every ω in Ω, as a Riemann integral).

Remark 3.4. *A notable advantage of these two definitions is to provide, by construction, the equality* $\int_0^1 Y_t \ d^{(\mathcal{M}_1)} B_t^h = \int_0^1 Y_t \ d^{(\mathcal{M}_2)} B_t^h$ *as soon as* Y *is integrable w.r.t. mBm in sense of* $\mathcal{M}_2$. *This result generalizes the case of integral w.r.t. the fractional Brownian motion.*

3. We can also see that, when $(\mathcal{M}_3)$ denotes the Zähle integrale or the Rough path theory (see [29, 79] for more details), we will want to set

$$\int_0^1 Y_t \ d^{(\mathcal{M}_2)} B_t^h := \lim_{n\to\infty} \int_0^1 Y_t \ d^{(\mathcal{M}_2)} B_t^{h_n} + \int_0^1 h'(t) \ Y_t \ \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \ dt. \tag{3.21}$$

3.2 Integral with respect to mBm in sense of method $(\mathcal{M})$

In regard of the previous examples it is now possible to give a rigorous definition of integral with respect to mBm in sense of method $(\mathcal{M})$. More precisely, let $(E, || \ ||_E)$ and $(F, || \ ||_F)$ are two normed linear spaces, endowed with their Borelian σ-field noted $\mathcal{B}(E)$ and $\mathcal{B}(F)$, and $Y := (Y_t)_{t\in[0,1]}$ be an E-valued process (i.e Y_t belongs to E for every real t in $[0,1]$ and $t \mapsto Y_t$ is measurable from $(0,1)$ to $(E, \mathcal{B}(E))$). The underlying method $(\mathcal{M})$ and the space F being fixed once and for all, define for every $\alpha \in (0,1)$,

$$\mathcal{H}_E^\alpha := \left\{ Y \in E^{[0,1]} : \int_{[0,1]} Y_t \ d^{(\mathcal{M})} B_t^\alpha \text{ exists and belongs to } F \right\},$$

where B^α denotes the fractional Brownian motion of Hurst index α. Denote also $\mathcal{H}_E = \bigcap_{\alpha\in h([0,1])} \mathcal{H}_E^\alpha$.

It is of practical interest to get a criterion which allow us to know when a process Y of $\mathcal{H}_E^\alpha$ is such that its integral w.r.t mixed fBm (definition 3.1) converges in F. This the purpose of the next subsection.

A general criterion for the convergence of the sequence $(\int_{[0,1]} Y_t \ dB_t^{h_n})_{n\in\mathbb{N}}$

We will always assume that there exists a subset Λ_E of $\mathcal{H}_E$ (maybe equal to $\mathcal{H}_E$) which may be endowed with a norm $\| \ \|_{\Lambda_E}$ such that $(\Lambda_E, \| \ \|_{\Lambda_E})$ is complete and which satisfies the following property: there exists $M > 0$ and a real χ such that for all partitions of $[0,1]$ in intervals $A_1, \ldots, A_n$ of equal size $\frac{1}{n}$,

$$\|Y.\mathbb{1}_{A_1}\|_{\Lambda_E} + \cdots + \|Y.\mathbb{1}_{A_n}\|_{\Lambda_E} \leqslant M \ n^\chi \|Y\|_{\Lambda_E}. \tag{3.22}$$

It turns out that there exists a simple suuficient condition that guarantees the existence of the limit $\lim_{n\to\infty} \int_0^1 \ Y_t \ \ d^{(\mathcal{M})} B_t^{h_n}$, when Y belongs to Λ_E. Define, for $n \in \mathbb{N}$, the map

$$\begin{aligned} L_n : \Lambda_E &\to F \\ Y &\mapsto \int_{[0,1]} Y_t \ d^{(\mathcal{M})} B_t^{h_n}. \end{aligned} \tag{3.23}$$

The following theorem provides a necessary condition under which $(L_n(Y))_{n\in\mathbb{N}}$ converges in F.

Theorem 3.5. *Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive integers such that $2^n \leqslant \prod_{0\leqslant k\leqslant n-1} a_k \leqslant 2^{2^n}$ for every n in $\mathbb{N}$ and such that $\lim_{n\to+\infty} (n(a_n - 1)) \ (\prod_{0\leqslant k\leqslant n-1} a_k)^{-1} = 0$. Choose the sequence $(q_n)_{n\in\mathbb{N}}$ used in (3.4) such that $q_0 = 1$ and $q_{n+1} = a_n \ q_n$ for all n in $\mathbb{N}$. Assume that the function $\mathcal{I} : \Lambda_E \times (0,1) \to F$ defined by*

$$\forall \ Y \in \Lambda_E, \ \forall \alpha \in (0,1), \quad \mathcal{I}(Y, \alpha) := \int_{[0,1]} Y_t \ d^{(\mathcal{M})} B_t^\alpha,$$

is θ-Hölder continuous with respect to α uniformly in Y for some positive real $\theta > \chi$, i.e. there exists $K > 0$ such that

$$\forall Y \in \Lambda_E, \forall (\alpha, \alpha') \in (0,1)^2, \qquad \sup_{\|Y\|_{\Lambda_E} \leqslant 1} \|\mathcal{I}(Y,\alpha) - \mathcal{I}(Y,\alpha')\|_F \leqslant K \ |\alpha - \alpha'|^\theta. \tag{3.24}$$

Then the sequence of functions $(L_n)_{n\in\mathbb{N}}$ defined in (3.23) converges pointwise to a function $L : \Lambda_E \to F$.

Remark 3.6. *As the following proof will show, it is of crucial importance that the sequence $(q_n)_{n\in\mathbb{N}}$ fulfills the assumption $q_{n+1} = a_n \ q_n$ for every integer n. The condition, $\prod_{0\leqslant k\leqslant n-1} a_k \leqslant 2^{2^n}$ for every n in $\mathbb{N}$, we asked results from section 2 where we have had to assume that $2^n \leqslant q_n \leqslant 2^{2^n}$ for every integer n. Since we want to use in the same time the result of section 2 and this theorem it is normal to have this supplementary condition.*

Proof: The general case being similar we will establish, for sake of simplicity, the proof in the case of the constant sequence $(a_n)_{n\in\mathbb{N}}$ equal to 2 which obviously fulfills the growing condition. For all $n \in \mathbb{N}$ and all $Y \in \Lambda_E$ we remark, thanks to (3.1), that $L_n(Y)$ can be decomposed as

$$L_n(Y) = \sum_{k=0}^{2^n-1} \Big(\int_{[\frac{2k}{2^{n+1}},\frac{2k+1}{2^{n+1}})} Y_t \; d^{(\mathcal{M})} B_t^{h(\frac{k}{2^n})} + \int_{[\frac{2k+1}{2^{n+1}},\frac{2k+2}{2^{n+1}})} Y_t \; d^{(\mathcal{M})} B_t^{h(\frac{k}{2^n})} \Big).$$

$$\text{Now, } L_{n+1}(Y) = \sum_{k=0}^{2^n-1} \Big(\int_{[\frac{2k}{2^{n+1}},\frac{2k+1}{2^{n+1}})} Y_t \; d^{(\mathcal{M})} B_t^{h(\frac{2k}{2^{n+1}})} + \int_{[\frac{2k+1}{2^{n+1}},\frac{2k+2}{2^{n+1}})} Y_t \; d^{(\mathcal{M})} B_t^{h(\frac{2k+1}{2^{n+1}})} \Big).$$

Using assumptions (3.22) and (3.24), one obtains

$$\begin{aligned}
\|L_n(Y) - L_{n+1}(Y)\|_F &= \Big\| \sum_{k=0}^{2^n-1} \Big(\mathcal{I}\big(Y.\mathbb{1}_{\left[\frac{2k+1}{2^{n+1}},\frac{k+1}{2^n}\right)}, h(\tfrac{k}{2^n})\big) - \mathcal{I}\big(Y.\mathbb{1}_{\left[\frac{2k+1}{2^{n+1}},\frac{k+1}{2^n}\right)}, h(\tfrac{2k+1}{2^{n+1}})\big) \Big) \Big\|_F \\
&\leqslant \sum_{k=0}^{2^n-1} \Big\| \Big(\mathcal{I}\big(Y.\mathbb{1}_{\left[\frac{2k+1}{2^{n+1}},\frac{k+1}{2^n}\right)}, h(\tfrac{k}{2^n})\big) - \mathcal{I}\big(Y.\mathbb{1}_{\left[\frac{2k+1}{2^{n+1}},\frac{k+1}{2^n}\right)}, h(\tfrac{2k+1}{2^{n+1}})\big) \Big) \Big\|_F \\
&\leqslant K \sum_{k=0}^{2^n-1} \big\| Y.\mathbb{1}_{[(2k+1).2^{-(n+1)},(k+1).2^{-n})} \big\|_{\Lambda_E} \; \Big| h((2k+1).2^{-(n+1)}) - h(k.2^{-n}) \Big|^\theta \\
&\leqslant K \; 2^{-\theta(n+1)} \sup_{t\in[0,1]} |h'(t)|^\theta \sum_{k=0}^{2^n-1} \big\| Y.\mathbb{1}_{[(2k+1).2^{-(n+1)},(k+1).2^{-n})} \big\|_{\Lambda_E} \\
&\leqslant K \; M \; 2^{-\theta(n+1)} \sup_{t\in[0,1]} |h'(t)|^\theta 2^{n\chi} \; \|Y\|_{\Lambda_E}.
\end{aligned}$$

It follows that the series $\sum_{n\in\mathbb{N}} (L_{n+1}(Y) - L_n(Y))$ converges absolutely for any fixed $Y \in \Lambda_E$, and consequently $(L_n(Y))_{n\in\mathbb{N}}$ converges to a limit $L(Y)$ as n goes to infinity. □

Definition of integral with respect to mBm in sense of $(\mathcal{M})$

The previous criterion allow us to define now rigorously the integral with respect to mBm in sense of method $(\mathcal{M})$. Let us first precise a notation. We will say that an E-valued process $(Z_t)_{t\in[0,1]}$ is integrable on $I \subset [0,1]$, in sense of F, if $\int_I \; Z_t \; dt$ exists:
-for almost every ω in sense of Riemann if $F \subset L^2(\Omega)$,
-in sense of Bochner if $L^2(\Omega) \subset F$,
and belongs to F (the reader who is no familiar wiht the integral in Bochner sense may refer to section 5 below and references therein).
Now, let $\varphi^{(\mathcal{M})} : E\times E \to F$ be the map defined by $\varphi^{(\mathcal{M})}(U,V) := U \diamond V$ if $(\mathcal{M})$ denotes the White noise theory or the Mallivin calculus and by $\varphi^{(\mathcal{M})}(U,V) := U \; V$ otherwise. For every process $Y := (Y_t)_{t\in[0,1]} \in E^{[0,1]}$, such that the map $t \mapsto h'(t) \; \varphi^{(\mathcal{M})}(Y_t, \frac{\partial \mathbf{B}}{\partial H}(t,h(t))$ is integrable in sense of F, define

$$\Phi_h^{(\mathcal{M})}(Y) := \int_0^1 \; h'(t) \; \varphi^{(\mathcal{M})}(Y_t, \frac{\partial \mathbf{B}}{\partial H}(t,h(t)) \; dt.$$

We get the following definition.

Definition 3.2 (Integral with respect to mBm in sense of $(\mathcal{M})$)**.** *Assume $(\mathcal{M})$ fulfills condition (3.24) and let $Y := (Y_t)_{t\in[0,1]}$ be an element of Λ_E such that the map $t \mapsto h'(t) \; \varphi^{(\mathcal{M})}(Y_t, \frac{\partial \mathbf{B}}{\partial H}(t,h(t))$ is integrable in sense of F. We call integral of Y with respect to mBm in sense of $(\mathcal{M})$, and note $\int_0^1 \; Y_t \; d^{(\mathcal{M})} B_t^h$, the quantity*

$$\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^h := \lim_{n\to\infty} \int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h_n} + \Phi_h^{(\mathcal{M})}(Y), \tag{3.25}$$

where the limit and the equality hold in F.

Remark 3.7. (*i*) *Contrary to what we might expect in regard of proposition 3.2, the second term on the right-hand side of* (3.25) *does not only depend on the choice of the fractional field* **B** *but also on the method* ($\mathcal{M}$) *of integration with respect to fBm that has been chosen. Note that the same is also true of the first term on the right-hand side of* (3.25).
(*ii*) *The main advantage of expression* (3.25) *is that any known construction of a stochastic integral with respect to fBm (e.g. pathwise, Malliavin calculus, white noise, rough path, ...) gives rise to a corresponding stochastic integral with respect to mBm.*
(*iii*) *Once again, note that E is not necessary a space of random variables (e.g $E := (\mathcal{S}_{-p})$ for some positive integer p; see section 5 below) and that E could be different of F as it will be the case in section 5*

4 Skohorod integral with respect to multifractional Brownian motion

In this section, we apply Theorem 3.5 to define a Skohorod-type integral with respect to mBm. The reference method of integration with respect to fBm here is the one based on Malliavin calculus, as exposed in [3]. We assume throughout this section that $H > 1/2$ and that h ranges in $(1/2, 1)$. For definiteness, we also set $\mathbf{B} = \mathbf{B}_4$ in this section.
Our notations are as follows (for a presentation of Malliavin calculus, see *e.g.* [6, 63]). Let:

$$\mathcal{S} = \left\{ R := f\left(W(h_1), W(h_2), \ldots, W(h_n)\right), f \in C_b^\infty(\mathbb{R}^n), h_i \in L^2([0,T]), i = 1, \ldots, n \right\}$$

where $W(h_i) := \int_{[0,T]} h_i(s) \, dW_s$ with $W := (W_s)_{s\in[0,T]}$ a Brownian motion, $C_b^\infty(\mathbb{R}^n)$ is the set of functions which are bounded as well as all their derivatives. For an element of $\mathcal{S}$, one defines the derivative operator D as:

$$DR = \sum_{i=1}^n \partial_i f\left(W(h_1), W(h_2), \ldots, W(h_n)\right) h_i.$$

D extends to the domain $\mathbb{D}$ which is the completion of $\mathcal{S}$ with respect to the norm:

$$\|R\|_{1,2} = \left(\mathsf{E}(R^2) + \mathsf{E}(\|DR\|^2_{L^2([0,T])})\right)^{\frac{1}{2}}.$$

We denote by δ the adjoint of D, and by $\mathrm{Dom}(\delta)$ its domain. More precisely, $\mathrm{Dom}(\delta)$ is the set of $u \in L^2(\Omega, [0,T])$ such that:

$$|\mathsf{E}(< DR, u >)| \leqslant c_u \, \mathsf{E}(R^2)$$

for all $R \in \mathcal{S}$ (we use $< .,. >$ to denote the scalar product on $L^2([0,T])$, and δ is defined on $\mathrm{Dom}(\delta)$ by the relation:

$$\mathsf{E}(R\delta(u)) = \mathsf{E}(< DR, u >).$$

The operator δ is a closed linear operator on $\mathrm{Dom}(\delta)$. It coincides with the Skohorod integral.
Let us now recall briefly the approach of [3] for the construction of a stochastic integral w.r.t. a class of Gaussian processes. Assume the continuous Gaussian process X may be written:

$$X_t = \int_0^t K(t,s) \, dW_s, \tag{4.1}$$

where the kernel $K(t,s)$ is defined for $0 < s < t < T$ and verifies

$$\sup_{t\in[0,T]} \int_0^t K(t,s)^2 ds < \infty. \tag{4.2}$$

Define the operator K^* on the set of step functions on $[0,T]$:

$$(K^*\varphi)(s) := \varphi(s)K(s^+,s) + \int_s^T \varphi(t)K(dt,s) \tag{4.3}$$

where $K(s^+,s) = K(T,s) - K((s,T],s)$. Then the stochastic integral w.r.t. X is defined for processes in $\mathrm{Dom}(\delta_X)$ ([3], formula (12)):

$$\mathrm{Dom}(\delta_X) := (K^*)^{-1}(\mathrm{Dom}(\delta)).$$

For a process v in $\mathrm{Dom}(\delta_X)$, one sets:

$$\delta_X(v) := \int_0^T v(s)\delta X(s) := \int_0^T (K^*v)(s)\delta W(s).$$

In the case of fBm, one has:

$$B_t^H = \int_0^t K_H(t,s)\, dW_s,$$

where

$$K_H(t,s) = d_H\,(t-s)^{H-\frac{1}{2}} + c_H\left(\frac{1}{2}-H\right)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1-\left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right)du \tag{4.4}$$

and

$$d_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(\frac{1}{2}+H)\Gamma(2-2H)}\right)^{\frac{1}{2}}.$$

We will index our operators and sets with H, *i.e.* we will write:

$$\mathrm{Dom}(\delta_H) = (K_H^*)^{-1}(\mathrm{Dom}(\delta))$$

for the domain of the Skohorod integral with respect to B^H and

$$\delta_H(v) := \int_0^T v(s)\delta B^H(s) := \int_0^T (K_H^*v)(s)\delta W(s)$$

for the integral. In other words, $\delta_H(v) = \delta(K_H^*v)$.
As the methodology of [3] works in a general framework, one may wonder whether it is possible to apply it directly to mBm. The prerequisite is to exhibit a kernel K such that (4.1) holds when X is an mBm. This is indeed the case for $\mathbf{B}_4$, and the work [18] develops this approach. In contrast, it does not seem to be an easy task to find such a kernel for the moving average and harmonizable mBms, as they have very different correlation structures (see [75]). As a matter of fact, we conjecture that such a kernel does not exist for the harmonizable mBm, based on the following fact: for all t_1,t_2 in $\mathbb{R}$ and H_1,H_2 in $(0,1)$,

$$\mathsf{E}\left[\mathbf{B}_1(t_1,H_1)\mathbf{B}_1(t_2,H_2)\right] = \mathsf{E}\left[\mathbf{B}_1(t_1,H_2)\mathbf{B}_1(t_2,H_1)\right].$$

This will investigated in a forthcoming work.
We now seek to apply Theorem 3.5 in order to define a Skohorod integral w.r.t. mBm through approximating Skohorod integrals w.r.t. fBms. In that view, we set $F = L^2(\Omega)$ and $\mathcal{I}(Y,\alpha) = \delta_\alpha(Y)$.
It is straightforward to check that, for every (t,s) in $[0,T]^2$, the function $H \mapsto K_H(t,s)$ is C^1. We denote its derivative by G_H, *i.e.* $G_{H_1}(t,s)$ is the derivative of the function $H \mapsto K_H(t,s)$ evaluated at H_1. We associate to G_H an operator G_H^* in a way similar to (4.3). Note that G_H^* is the derivative of the function $H \mapsto K_H^*$. One easily verifies that G_H fulfils (4.2), so that one may define as above $\delta_G(.) := \delta(G_H^*.)$ for a suitable class of processes. Let:

$$\mathcal{D} := \bigcap_{H\in h([0,1])} \mathrm{Dom}(\delta_H),$$

and

$$\mathcal{F} := \bigcap_{H\in h([0,1])} \mathcal{F}_H,$$

where

$$\mathcal{F}_H := (G_H^*)^{-1}(\mathrm{Dom}(\delta)).$$

Set

$$\Lambda := \mathcal{D} \cap \mathcal{F},$$

equipped with the norm $\|v\|_\Lambda = \sup_{H\in h([0,1])} \mathbb{E}\left(\int_0^T (K_H^* v)(s)^2 ds\right) + \sup_{H\in h([0,1])} \mathbb{E}\left(\int_0^T (G_H^* v)(s)^2 ds\right)$, which satisfies condition (3.22) with $\chi = 0$. By definition, $\delta_H(v)$ and $\int_0^T (G_H^* v)(s)\delta W(s)$ both exist for all H in $h([0,1])$ and all v in Λ. Fix (v, s, H, H') in $\Lambda \times [0,T] \times h([0,1])^2$ with $H < H'$. Consider the function $\varphi : \Omega \times [H, H'] \to \mathbb{R}$ defined by:

$$\varphi(\omega, H_1) := (K_{H_1}^* v(\omega))(s) - (K_H^* v(\omega))(s) - (H_1 - H)\frac{(K_{H'}^* v(\omega))(s) - (K_H^* v(\omega))(s)}{H' - H}.$$

For every ω in Ω, $\varphi(\omega, .)$ is C^1 and $\varphi(\omega, H) = \varphi(\omega, H') = 0$. As a consequence, there exists H''_ω in $[H, H']$ such that $\frac{\partial\varphi}{\partial H}(\omega, H''_\omega) = 0$. Thus, the set $A_\omega := \{H_1 \in [H, H'] : \frac{\partial\varphi}{\partial H}(\omega, H_1) = 0\}$ is a non-empty closed subset of $[H, H']$. It has a minimum, that we denote $H_0(\omega)$. The map $\omega \mapsto H_0(\omega)$ is measurable (*i.e.* H_0 is a random variable), and so is the map $(v, s, H, H', \omega) \mapsto H_0(v, s, H, H', \omega)$.
We wish to estimate $\|\mathcal{I}(v, H) - \mathcal{I}(v, H')\|_F$ for v in Λ. As we have just seen, there exists a measurable function $H_0 = H_0(H, H', v, s, \omega)$ such that:

$$\begin{aligned} u(s) &:= (K_{H'}^* v)(s) - (K_H^* v)(s) \\ &= (H' - H)(G_{H_0}^* v)(s). \end{aligned}$$

Thus:

$$\begin{aligned} \mathcal{I}(v, H') - \mathcal{I}(v, H) &= \delta(u) \\ &= (H' - H)\int_0^T (G_{H_0}^* v)(s)\delta W(s), \end{aligned}$$

and

$$\|\mathcal{I}(v, H') - \mathcal{I}(v, H)\|_{L^2(\Omega)} \leqslant |H - H'|\, \|v\|_\Lambda$$

i.e. (3.24) holds with $\theta = 1$, $E = F := L^2(\Omega)$ and $\Lambda_E := \Lambda$.
In order to define our integral with (3.25), we need to check that $h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t))$ is integrable. Define

$$Z_t := \int_0^t \frac{\partial \mathbf{B}}{\partial H}(s, h(s))\, ds = \int_0^t L(t, u)\, dB_u$$

with $L(t, u) := \int_u^t G_{h(s)}(s, u)\, ds$. Thus Z is a Volterra process. It follows from Proposition 7 in [64] [2] that any process in $L^2(\Omega \times [0,1])$ is Wick integrable w.r.t. Z. This implies in particular that $\int_0^1 h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t))\, dt$ exists for Y in Λ. In addition, adapting the arguments in Proposition 8 of [64], one may show that $\int_0^1 h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t))\, dt = \int_0^1 h'(t)Y(t)\, \delta Z_t$ is a Skohorod integral and thus belongs to $L^2(\Omega)$. We are then able to set the following definition and theorem:

Theorem-Definition 4.1. *Let $Y \in \Lambda$. Then the Skohorod integral of Y with respect to mBm is well-defined and given by:*

$$\int_0^1 Y_t\, d^{(\mathcal{M}_2)} B_t^h := \int_0^1 Y_t\, \delta B_t^h := \lim_{n\to\infty} \int_0^1 Y_t\, \delta B_t^{h_n} + \int_0^1 h'(t)\, Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t))\, dt. \tag{4.5}$$

where the equality holds in $L^2(\Omega)$.

Remark 4.1. *One may verify that the integral defined above coincides with the one studied in [18] when they are both defined. Comparing their domains would be an interesting task.*

2. it is a straightforward computation to check that the conditions of this proposition are verified by L

5 White noise approach to stochastic integration w.r.t. mBm

One can easily see that the function R_h, defined on $\mathbb{R}^2$, by

$$R_h(t,s) = \frac{c^2_{h_{t,s}}}{c_{h(t)}c_{h(s)}} \left[\frac{1}{2}\Big(|t|^{2h_{t,s}} + |s|^{2h_{t,s}} - |t-s|^{2h_{t,s}}\Big)\right], \tag{5.1}$$

is the covariance function of mBm build from the fractional field $\mathbf{B}_1$, which has been given at the beginning of section 1.

In [53] an integral with respect to mBm (built from fractional field $\mathbf{B}_1$) has been developed and studied. Our aim in this section is firstly to define an integral with respect to mBm using definition 3.2 (for $\mathcal{M} = \mathcal{M}_1$, where $(\mathcal{M}_1)$ still denote the white noise theory) and, secondly, to compare the integral w.r.t. mBm hence defined with the one defined in [53]. For this purpose, we have to choose and fix $\mathbf{B} = \mathbf{B}_1$ what we do from now and untill the end of this section [3]. Before comparing these two integrals w.r.t. mBm, we first need to particularize the construction of the stochastic integral given in definition 3.2 to $\mathbf{B}_1$.

Moreover, since this section is devoted to integral w.r.t mBm in sense of White noise theory, we recall, in the next subsection, the basic ideas and tools of white noise theory.

5.1 Reminders about White Noise theory and Bochner integral

White Noise theory

Define the measurable space $(\Omega, \mathcal{F})$ by setting $\Omega := \mathscr{S}'(\mathbb{R})$ and $\mathcal{F} := \mathcal{B}(\mathscr{S}'(\mathbb{R}))$, where be $\mathcal{B}$ denotes the σ-algebra of Borel sets. There exists an unique probability measure μ on $(\Omega, \mathcal{F})$ such that, for every f in $L^2(\mathbb{R})$, the map $< ., f >: \Omega \to \mathbb{R}$ defined by $< ., f > (\omega) =< \omega, f >$ (where $< \omega, f >$ is by definition $\omega(f)$, $i.e$ the action of the distribution ω on the function f) is a centered Gaussian random variable with variance equal to $||f||^2_{L^2(\mathbb{R})}$ under μ. For every n in $\mathbb{N}$, define $e_n(x) := (-1)^n\ \pi^{-1/4}(2^n n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2})$ the nth Hermite function. Let $(|\ \ |_p)_{p\in\mathbb{Z}}$ be a family norms defined by $|f|^2_p := \sum_{k=0}^{+\infty} (2k+2)^{2p} < f, e_k >^2_{L^2(\mathbb{R})}$, forall (p, f) in $\mathbb{Z} \times L^2(\mathbb{R})$. The operator A defined on $\mathscr{S}(\mathbb{R})$ by $A := -\frac{d^2}{dx^2} + x^2 + 1$ admits the sequence $(e_n)_{n\in\mathbb{N}}$ as eigenfunctions and the sequence $(2n+2)_{n\in\mathbb{N}}$ as eigenvalues.

From now on we will denote as is customary (L^2) the space $L^2(\Omega, \mathcal{G}, \mu)$ where $\mathcal{G}$ is the σ-field generated by $(< ., f >)_{f\in L^2(\mathbb{R})}$. For every random variable Φ of (L^2) there exists, according to the Wiener-Itô theorem, a unique sequence $(f_n)_{n\in\mathbb{N}}$ of functions f_n in $\widehat{L}^2(\mathbb{R}^n)$ such that Φ can be decomposed as $\Phi = \sum_{n=0}^{+\infty} I_n(f_n)$, where $\widehat{L}^2(\mathbb{R}^n)$ denotes the set of all symmetric functions f in $L^2(\mathbb{R}^n)$ and $I_n(f)$ denotes the nth multiple Wiener-Itô integral of f with the convention that $I_0(f_0) = f_0$ for constants f_0. Moreover we have the equality $\mathsf{E}[\Phi^2] = \sum_{n=0}^{+\infty} n!\, ||f_n||^2_{L^2(\mathbb{R}^n)}$ where E denotes the expectation with respect to μ. For any $\Phi := \sum_{n=0}^{+\infty} I_n(f_n)$ satisfying the condition $\sum_{n=0}^{+\infty} n!\ |A^{\otimes n} f_n|^2_0 < +\infty$, define the element $\Gamma(A)(\Phi)$ of (L^2) by $\Gamma(A)(\Phi) := \sum_{n=0}^{+\infty} I_n(A^{\otimes n} f_n)$, where $A^{\otimes n}$ denotes the nth tensor power of the operator A (see [47] appendix E for more details about tensor products of operators). The operator $\Gamma(A)$ is densely defined on (L^2) is called the second quantization operator of A. It is invertible and its inverse $\Gamma(A)^{-1}$ is bounded. Let us denote $||\varphi||^2_0 := ||\varphi||^2_{(L^2)}$ for any random variable φ in (L^2) and, for n in $\mathbb{N}$, let $\mathbb{D}\mathrm{om}(\Gamma(A)^n)$ be the domain of the nth iteration of $\Gamma(A)$. Define the family of norms $(||\ \ ||_p)_{p\in\mathbb{Z}}$ by:

$$||\Phi||_p := ||\Gamma(A)^p\Phi||_0 = ||\Gamma(A)^p\Phi||_{(L^2)}, \qquad \forall p \in \mathbb{Z}, \quad \forall \Phi \in (L^2) \cap \mathbb{D}\mathrm{om}(\Gamma(A)^p).$$

For any p in $\mathbb{N}$, define $(\mathcal{S}_p) := \{\Phi \in (L^2) :\ \Gamma(A)^p\Phi$ exists and belongs to $(L^2)\}$ and define $(\mathcal{S}_{-p})$ as being the completion of the space (L^2) with respect to the norm $||\ \ ||_{-p}$. As in [52], we let $(\mathcal{S})$ denote the projective limit of the sequence $((\mathcal{S}_p))_{p\in\mathbb{N}}$ and $(\mathcal{S})^*$ the inductive limit of the sequence $((\mathcal{S}_{-p}))_{p\in\mathbb{N}}$. The space $(\mathcal{S})$ is called the space of stochastic test functions and $(\mathcal{S})^*$ the space of Hida distributions. One can show that, for any p in $\mathbb{N}$, the dual space $(\mathcal{S}_p)^*$ of $\mathcal{S}_p$ is $(\mathcal{S}_{-p})$. Thus we will write $(\mathcal{S}_{-p})$, in the sequel, to denote the space $(\mathcal{S}_p)^*$. Note also that $(\mathcal{S})^*$ is the dual space of $(\mathcal{S})$. We will note $\ll\ ,\ \gg$ the duality bracket between

3. Of course one can define an integral w.r.t. mBm in sense of definition 3.2, with $(\mathcal{M}) := (\mathcal{M}_1)$, for other fractional fields $\mathbf{B}$ and not only for $\mathbf{B} = \mathbf{B}_1$.

$(\mathcal{S})^*$ and $(\mathcal{S})$. If Φ belongs to (L^2) then we have the equality $\ll\Phi,\varphi\gg \;= <\Phi,\varphi>_{(L^2)} = \mathbb{E}[\Phi\,\varphi]$. Since we have defined a topology given by a family of norms on the space $(\mathcal{S})^*$ it is possible to define a derivative and an integral in $(\mathcal{S})^*$. A function $\Phi : \mathbb{R} \to (\mathcal{S})^*$ is called a stochastic distribution process, or an $(\mathcal{S})^*$–process, or a Hida process for every real t_0. For every real t_0, the Hida process Φ is said to be differentiable at t_0 if $\lim_{r\to 0} \; r^{-1}\,(\Phi(t_0+r) \;-\; \Phi(t_0))$ exists in $(\mathcal{S})^*$.

For every f in $L^2(\mathbb{R})$ we define the *Wick exponential* of $< .,f >$, noted $: e^{<.,f>} :$, as being the (L^2) random variable equal to $e^{<.,f>-\frac{1}{2}|f|_0^2}$. The S-transform of an element Φ of $(\mathcal{S}^*)$, noted $S(\Phi)$, is defined as being the function from $\mathscr{S}(\mathbb{R})$ to $\mathbb{R}$ given by $S(\Phi)(\eta) := \;\ll\Phi, : e^{<.,\eta>} :\gg$ for every η in $\mathscr{S}(\mathbb{R})$. Finally for every $(\Phi,\Psi) \in (\mathcal{S})^* \times (\mathcal{S})^*$, there exists a unique element of $(\mathcal{S})^*$, called the Wick product of Φ and Ψ and noted $\Phi \diamond \Psi$, such that $S(\Phi \diamond \Psi)(\eta) = S(\Phi)(\eta)\; S(\Psi)(\eta)$; for every η in $\mathscr{S}(\mathbb{R})$.

Fractional and multifractional White noise

As we will see now, the operators M_H and $\frac{\partial M_H}{\partial H}$ are crucial in the study of an integral with respect to mBm in the framework of the white noise theory.

Operators M_H and $\frac{\partial M_H}{\partial H}$

Let H be a fixed real in $(0,1)$. Following [34] and references therein, define the operator M_H, specified in the Fourier domain, by $\widehat{M_H(u)}(y) := \frac{\sqrt{2\pi}}{c_H}\,|y|^{1/2-H}\widehat{u}(y)$ for every y in $\mathbb{R}^*$. This operator is well defined on the homogeneous Sobolev space of order $1/2-H$, denoted $L^2_H(\mathbb{R})$ and defined by $L^2_H(\mathbb{R}) := \{u \in \mathscr{S}'(\mathbb{R}) \;:\; \widehat{u} = T_f;\; f \in L^1_{loc}(\mathbb{R}) \text{ and } ||u||_H < +\infty\}$, where the norm $||\;||_H$ derives from the inner product $<,>_H$ defined on $L^2_H(\mathbb{R})$ by $< u,v >_H := \frac{1}{c_H^2}\int_{\mathbb{R}} |\xi|^{1-2H}\widehat{u}\,(\xi)\overline{\widehat{v}\,(\xi)}d\xi$ and where c_H has been given in the definition of the fractional field $\mathbf{B}_1$.

The definition of the operator $\frac{\partial M_H}{\partial H}$ is quite similar. More precisely, define for every H in $(0,1)$, the space $\Gamma_H(\mathbb{R}) := \{u \in \mathscr{S}'(\mathbb{R}) \;:\; \widehat{u} = T_f;\;\; f \in L^1_{loc}(\mathbb{R}) \;\text{ and }\; ||u||_{\delta_H(\mathbb{R})} < +\infty\}$, where the norm $||\;||_{\delta_H(\mathbb{R})}$ derives from the inner product on $\Gamma_H(\mathbb{R})$ defined by $< u,v >_{\delta_H} := \frac{1}{c_H^2}\int_{\mathbb{R}} (\beta_H + \ln|\xi|)^2\; |\xi|^{1-2H}\; \widehat{u}\,(\xi)\; \overline{\widehat{v}\,(\xi)}\; d\xi$. Following [53], define the operator $\frac{\partial M_H}{\partial H}$ from $(\Gamma_H(\mathbb{R}), <,>_{\delta_H(\mathbb{R})})$ to $(L^2(\mathbb{R}), <,>_{L^2(\mathbb{R})})$, in the Fourier domain, by: $\widehat{\frac{\partial M_H}{\partial H}(u)}(y) := -(\beta_H + \ln|y|)\,\frac{\sqrt{2\pi}}{c_H}\;|y|^{1/2-H}\;\widehat{u}(y)$, for every y in $\mathbb{R}^*$. The reader intersted in the properties of M_H and $\frac{\partial M_H}{\partial H}$ can refer to [53, section 3.2].

Fractional and multifractional White noise

Since we only consider mBm B^h with covariance function R_h, we can write, thanks to [53, (5.10)] and using the two last subsections,

$$\text{Almost surely, for every real } t, \qquad B^h_t = \mathbf{B}_1(t,h(t)) =< ., M_{h(t)}(\mathbb{1}_{[0,t]}) > . \tag{5.2}$$

However and for sake of simplicity we will only restrict ourselves to the case where t belongs to $[0,1]$. We can also write (5.2) under the form

$$\text{Almost surely, for every real } t, \qquad B^h_t = \sum_{k=0}^{+\infty}\left(\int_0^t M_{h(t)}(e_k)(s)ds\right) < ., e_k > . \tag{5.3}$$

We are now able to define the derivative in sense of $(\mathcal{S})^*$ of the mBm. Define the $(\mathcal{S}^*)$-valued process $W^h := (W^h_t)_{t\in[0,1]}$ by

$$W^h_t := \sum_{k=0}^{+\infty}\left[\frac{d}{dt}\Big(\int_0^t \; M_{h(t)}(e_k)(s)\; ds\Big)\right] \;< ., e_k > . \tag{5.4}$$

The following theorem states that, for all real t, the right hand side of (5.4) does indeed belong to $(\mathcal{S})^*$ and is exactly the $(\mathcal{S})^*$-derivative of B^h at point t of $[0,1]$.

Theorem-Definition 5.1. *[53, Theorem-definition 5.1] The process W^h defined by (5.4) is an $(\mathcal{S})^*$-process which verifies,in $(\mathcal{S})^*$, the following equality:*

$$W_t^h = \sum_{k=0}^{+\infty} M_{h(t)}(e_k)(t) < ., e_k > + h'(t) \sum_{k=0}^{+\infty} \Big(\int_0^t \frac{\partial M_H}{\partial H}(e_k)(s)\big|_{H=h(t)} ds \Big) < ., e_k >. \tag{5.5}$$

Moreover the process B^h is $(\mathcal{S})^*$-differentiable on $[0,1]$ and verifies $\frac{dB^h}{dt}(t) = W_t^h$ in $(\mathcal{S})^*$.

Once again if the function h_0 is constant, identically equal to H, we will note $W^H := (W_t^H)_{t\in[0,1]}$ and call fractional white noise the $(\mathcal{S})^*$-process W^{h_0}. One can rewrite (5.5), for every t in $[0,1]$, under the form

$$W_t^h = W_t^{h(t)} + h'(t)\, \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)), \tag{5.6}$$

where $W_t^{h(t)}$ is nothing but $W_t^H|_{H=h(t)}$ and where the equality holds in $(\mathcal{S})^*$.

Bochner integral

Since the objects we are now handling with are no longer, in general, random variables, the Riemann or Lebesgue integrals are not relevant here. However, taking advantage on the fact that we are working with vector linear spaces, we can use Pettis or Bochner integrals. We know that the space E, defined at the beginning of section 3.2, will be a space $(\mathcal{S}_{-p})$ for some integer p. The fact that we need to find a norm on Λ_E suggests the use of Bochner integral. A very good survey of this topic can be found in [52, p.247]. We here just recall the definition.

Definition 5.1 (Bochner integral [52],p.247)**.** *Let I be a subset of $[0,1]$ endowed with the Lebesgue measure. One says that $\Phi : I \to (\mathcal{S})^*$ is Bochner integrable on I if it satisfies the two following conditions:*

1. *Φ is weakly measurable on I i.e $t \mapsto \ll \Phi_t, \varphi \gg$ is measurable on I, for every φ in $(\mathcal{S})$.*
2. *There exists p in $\mathbb{N}$ such that $\Phi_t \in (\mathcal{S}_{-p})$ for almost every t in I and $t \mapsto ||\Phi_t||_{-p}$ belongs to $L^1(I, dt)$.*

The Bochner-integral of Φ on I is denoted $\int_I \Phi_t\, dt$. Moreover we will say that Φ is Bochner-integrable of index p.

Propertie 1. *If $\Phi : I \to (\mathcal{S})^*$ is Bochner-integrable on I with index p then we have the following inequality $\big|\big|\int_I \Phi_t\, dt\big|\big|_{-p} \leqslant \int_I ||\Phi_t||_{-p}\, dt$.*

Theorem 5.1 ([52], theorem 13.5)**.** *Let $\Phi := (\Phi_t)_{t\in[0,1]}$ be a $(\mathcal{S})^*$-valued process such that:*
(i) $t \mapsto S(\Phi_t)(\eta)$ is measurable for every η in $\mathscr{S}(\mathbb{R})$.
(ii) There exist p in $\mathbb{N}$, b in $\mathbb{R}^+$ and a function L in $L^1([0,1], dt)$ such that, for a.e. t in $[0,1]$, such that $|S(\Phi_t)(\eta)| \leqslant L(t)\, e^{b|\eta|_p^2}$, for every η in $\mathscr{S}(\mathbb{R})$.
Then Φ is Bochner integrable on $[0,1]$ and $\int_0^1 \Phi(s)\, ds \in (\mathcal{S}_{-q})$ for any $q > p$ such that $2be^2 D(q-p) < 1$ where e denotes the base of the natural logarithm and where $D(r)$ is defined by setting $D(r) := \frac{1}{2^{2r}} \sum_{n=1}^{+\infty} \frac{1}{n^{2r}}$ for every r of $(1/2, +\infty)$.

5.2 Wick-Itô integral with respect to fBm

The definition of the fractional Wick-Itô integral with respect to fBm or integral w.r.t fBm in the white noise sense has been firstly given in [34] and then extend in [8] using the Pettis integral. However, in order to use Theorem 3.5, we need that Y_s belongs to $(\mathcal{S}_{-p})$ for almost every real s. It then seems reasonable to assume that $(Y_s)_{s\in[0,1]}$ is Bochner integrable on $[0,1]$. For this reason, we now particularize the fractional Wick-Itô integral with respect to fBm of [34] and [8] into the framework of the Bochner integral.

Definition 5.2 (Wick-Itô integral w.r.t fBm in Bochner sense)**.** *Let H be a fixed real in $(0,1)$, I be a Borel subset of $[0,1]$, $B^H := (B_t^H)_{t\in I}$ be a fractional Brownian motion of hurst index H, and $Y := (Y_t)_{t\in I}$ be a $(\mathcal{S})^*$-valued process such that*

(i) There exists $p \in \mathbb{N}$ such that $Y_t \in (\mathcal{S}_{-p})$ for almost every $t \in I$,
(ii) the process $t \mapsto Y_t \diamond W_t^H$ is Bochner integrable on I
We then say that the process Y is Bochner-integrable with respect to fBm, or integrable on I with respect to fBm, in the Bochner sense and define this integral by setting:

$$\int_I Y_s \, d^\diamond B_s^H := \int_I Y_s \diamond W_s^H ds. \tag{5.7}$$

The following lemma, states that every Bochner integrable process is integrable with respect to fBm of any Hurst index H, in the Bochner sense.

Lemma 5.2. *Let $Y := (Y_t)_{t\in[0,1]}$ be a $(\mathcal{S})^*$-valued process, Bochner integrable of index $p_0 \in \mathbb{N}$. Then Y is integrable on $[0,1]$, with respect to fBm of any Hurst index H, in the Bochner sense. Moreover, for any H in $(0,1)$, $\int_{[0,1]} Y_s \, d^\diamond B_s^H$ belongs to $(\mathcal{S}_{-r_0})$ for every $r_0 \geqslant p_0+1$ if $p_0 \geqslant 2$ and for every $r_0 \geqslant p_0+2$ if $p_0 \in \{0;1\}$.*

Proof: Let $H \in (0,1)$, $p_0 \geqslant 2$ and $r_0 \geqslant p_0 + 1$ are fixed. For every t in $[0,1]$, it is clear that $t \mapsto Y_t \diamond W_t^H$ is weakly measurable since $t \mapsto S(Y_t \diamond W_t^H)(\eta)$ is measurable for every $\eta \in \mathscr{S}(\mathbb{R})$. Using [52, remark 2 p.92], we get, for almost every t in $[0,1]$, $||Y_t \diamond W_t^H||_{-r_0} \leqslant ||Y_t||_{-p_0} \, ||W_t^H||_{-p_0} < +\infty$ and hence $Y_t \diamond W_t^H$ belongs to $(\mathcal{S}_{-r_0})$. Since the map $t \mapsto ||W_t^H||_{-r}$ is continuous for every integer $r \geqslant 2$ (see [53, proposition 5.9]), we also get:

$$\int_0^1 ||Y_t \diamond W_t^H||_{-r_0} \, dt \leqslant \left(\sup_{t\in[0,1]} ||W_t^H||_{-p_0} \right) \int_0^1 ||Y_t||_{-p_0} dt < +\infty.$$

This shows that $t \mapsto Y_t \diamond W_t^H$ is Bochner-integrable of index r_0.
Let us now assume that $p_0 \in \{0;1\}$. We just have to verifiy that theorem 5.1 applies. The condition (i) is obviously fulffiled. Moreover, using, [52, p.79] we have for every (t,η) in $[0,1] \times \mathscr{S}(\mathbb{R})$,

$$|S(Y_t \diamond W_t^H)(\eta)| \leqslant \|Y_t\|_{-p_0} \, e^{\frac{1}{2}|\eta|_2^2} \sup_{t\in[0,1]} ||W_t^H||_{-2} =: L(t) \, e^{\frac{1}{2}|\eta|_2^2}.$$

Since Y is Bochner integrable of index p_0, it is clear that L belongs to $L^1([0,1],dt)$. Moreover, $e^2 D(r_0 - p_0) < 1$, for every $r_0 \geqslant p_0 + 2$. Theorem 5.1 then allows to conclude that $t \mapsto Y_t \diamond W_t^H$ is Bochner integrable of index r_0. □

Remark 5.3. *Of course if we make more assumptions on the $(\mathcal{S})^*$-process Y they may lead to $\int_{[0,1]} Y_s \, d^\diamond B_s^H$ belongs to (L^2). See [53, sections 6 and 7] for some examples.*

The following lemma, the proof of which is obvious in regard of hypothesis $(\mathcal{H}_2)$, will be useful in the proof of the proposition 5.5 below.

Lemma 5.4. *For every integer p in $\mathbb{N}$ and every real t in $[0,1]$, the map $(t,H) \mapsto \frac{\partial \mathbf{B}_1}{\partial H}(t,H)$ is continuous from $[0,t]$ into $((\mathcal{S}_{-p}), || \, ||_p)$. In particular, for every subset $[a,b]$ of $(0,1)$ there exists a positive real κ such that:*

$$\forall p \in \mathbb{N}, \qquad \sup_{(s,H)\in[0,t]\times[a,b]} \left|\left| \frac{\partial \mathbf{B}_1}{\partial H}(s,H) \right|\right|_{-p} \leqslant \kappa. \tag{5.8}$$

5.3 Stochastic integral with respect to mBm

The purpose of the next proposition is to show that we can apply Theorem 3.5 to the Wick-Itô integral w.r.t fBm and hence define an integral w.r.t to mBm. Let p_0 be a fixed integer in $\mathbb{N}$ and $s_0 \geqslant \max\{p_0+1;3\}$ be fixed in $\mathbb{N}$. Define $E := (\mathcal{S}_{-p_0})$ and $F := (\mathcal{S}_{-s_0})$. By definition we have the equality

$$\mathcal{H}_E = \left\{ (Y_t)_{t\in[0,1]} \in (\mathcal{S}_{-p_0})^{\mathbb{R}} : \int_{[0,1]} Y_t \, d^\diamond B_t^\alpha \in (\mathcal{S}_{-s_0}), \ \forall \alpha \in h([0,1]) \right\}.$$

Define also the set

$$\Lambda_E := \Big\{(Y_t)_{t\in[0,1]} \in (\mathcal{S}_{-p_0})^{\mathbb{R}} : Y \text{ is Bochner integrable of index } p_0 \text{ on } [0,1]\Big\}$$

and the norm $\| \ \|_{\Lambda_E}$ on Λ_E by $\|\Phi\|_{\Lambda_E} := \int_0^1 \ \|\Phi_t\|_{-p_0}\ dt$. The inclusion $\Lambda_E \subset \mathcal{H}_E$ results from lemma 5.2. Moreover the fact that $(\Lambda_E, \| \ \|_{\Lambda_E})$ is complete is a straightforward consequence of [44, Theorem 3.7.7 p.82]. The notations being fixed above we get the following

Proposition 5.5. *The white noise theory method (i.e* $(\mathcal{M}_1)$*) fulfills conditions* (3.22) *and* (3.24)*. Moreover, for every process* $Y := (Y_t)_{t\in[0,1]}$ *Bochner-integrable on* $[0,1]$*, the map* $t \mapsto h'(t)\ \varphi^{(\mathcal{M}_1)}(Y_t, \frac{\partial \mathbf{B}_1}{\partial H}(t,h(t))$ *is integrable in sense of* F*.*

The proof of this proposition stands on the two following lemmas.

Lemma 5.6. $(\mathcal{M}_1)$ *fulfills conditions* (3.22) *and* (3.24)*.*

Lemma 5.7. *for every process* $Y := (Y_t)_{t\in[0,1]}$ *Bochner-integrable on* $[0,1]$*, the map* $t \mapsto h'(t)\ Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t,h(t))$ *is integrable in sense of* F*.*

Proof of lemma 5.6: Since $(\mathcal{S}_{-p_0}) \subset (\mathcal{S}_{-2})$ if p_0 belongs to $\{0;1\}$, we assumme from now that $p_0 \geqslant 2$ and that $s_0 \geqslant p_0+1$. Let $A_1, A_2, \cdots, A_n$ are some fixed disjoint borelians of $[0,1]$. We get

$$\begin{aligned}\sum_{k=1}^{n} ||Y\mathbb{1}_{A_k}||_{\Lambda_E} &= \sum_{k=1}^{n} \int_0^1 ||Y\mathbb{1}_{A_k}(s)||_{-p_0} ds \leqslant \sum_{k=1}^{n} \int_0^1 \mathbb{1}_{A_k}(s)||Y_s||_{-p_0} ds \\ &\leqslant \int_0^1 \left(\sum_{k=1}^{n}\mathbb{1}_{A_k}(s)\right) ||Y_s||_{-p_0} ds \leqslant ||Y||_{\Lambda_E}. \end{aligned} \tag{5.9}$$

Besides we know, thanks to lemma 5.2, that Y is integrable with respect to fBm, in the Bochner sense, for any Hurst index α in $(0,1)$ and that $\mathcal{I}(Y,\alpha) = \int_{[0,1]} Y_t\ dB_t^{\diamond\alpha}$ belongs to $(\mathcal{S}_{-s_0})$. Now for any (α,α') in $(0,1)^2$, we get, using the same arguments we used in the proof of lemma 5.2,

$$\begin{aligned}||\mathcal{I}(Y,\alpha) - \mathcal{I}(Y,\alpha')||_{-s_0} &= \left|\left|\int_{[0,1]} Y_t \diamond (W_t^{\alpha} - W_t^{\alpha'})\ dt\right|\right|_{-s_0} \leqslant \int_0^1 ||Y_t||_{-p_0}\ ||W_t^{\alpha} - W_t^{\alpha'}||_{-p_0}\ dt \\ &\leqslant \left(\sup_{t\in[0,1]} ||W_t^{\alpha} - W_t^{\alpha'}||_{-p_0}\right) ||Y||_{\Lambda_E}, \text{ and then}\end{aligned}$$

$$\sup_{||Y||_{\Lambda_E}\leqslant 1} ||\mathcal{I}(Y,\alpha) - \mathcal{I}(Y,\alpha')||_{-s_0} \leqslant \sup_{t\in[0,1]} ||W_t^{\alpha} - W_t^{\alpha'}||_{-p_0}. \tag{5.10}$$

Furthermore, we have $||W_t^{\alpha} - W_t^{\alpha'}||_{-p_0}^2 = \sum_{k=0}^{+\infty} \frac{(M_\alpha(e_k)(t) - M_{\alpha'}(e_k)(t))^2}{(2k+2)^{2p_0}}$. For every (t,k) in $[0,1]\times\mathbb{N}$, define $f_{t,k} : (0,1) \to \mathbb{R}$ by $f_{t,k}(\alpha) := M_\alpha(e_k)(t)$. Thanks to [53, lemma 5.5] we know that $f_{t,k}$ is differeientable on $(0,1)$, for every (t,k) in $[0,1]\times\mathbb{N}$, and that its derivative, noted $f'_{t,k}$, satisfies $f'_{t,k}(\alpha) = \frac{\partial}{\partial\alpha}[M_\alpha(e_k)(t)] = \frac{\partial M_\alpha}{\partial\alpha}(e_k)(t)$ where the operator $\frac{\partial M_\alpha}{\partial\alpha}$ has been defined in [53, section 3.2]. Using point 1 of [53, lemma 5.6] and then the meanvalue theorem we get for every $[a,b] \subset (0,1)$, the existence of a positive real ρ such that for all $(t,\alpha,\alpha',k) \in [0,1]\times[a,b]^2\times\mathbb{N}$, $|M_\alpha(e_k)(t) - M_{\alpha'}(e_k)(t)| \leqslant \rho\ (k+1)^{2/3}\ \ln(k+1)\ |\alpha-\alpha'|$.
We hence have $||W_t^{\alpha} - W_t^{\alpha'}||_{-p_0}^2 \leqslant \rho^2\ |\alpha-\alpha'|^2\ \sum_{k=0}^{+\infty} \frac{(k+1)^{4/3}\ \ln^2(k+1)}{2^{2p_0}(k+1)^{2p_0}}$ and finally, using (5.10), we get

$$\sup_{||Y||_{\Lambda_E}\leqslant 1} ||\mathcal{I}(Y,\alpha) - \mathcal{I}(Y,\alpha')||_{-s_0} \leqslant |\alpha-\alpha'|\ \gamma_{p_0} \tag{5.11}$$

where $\gamma_{p_0} := \rho\left(\sum_{k=1}^{+\infty} \frac{\ln^2 k}{k^{2(p_0-2/3)}}\right)^{1/2}$ which is finite since $p_0 \geqslant 2$. Since γ_{p_0} is independent of α and α', Theorem 3.5 applies and gives us the existence of $\lim_{n\to+\infty} \int_{[0,1]} Y_t\ d^{\diamond}B_t^{h_n}$ in $(\mathcal{S}_{-s_0})$. □

Proof of lemma 5.7: Let $p_0 \geqslant 2$ and $s_0 \geqslant p_0 + 1$ are fixed. Using the same arguments as in the proof of lemma 5.2 we easily get the weak measurability of $t \mapsto h'(t)\ Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t))$ on $[0,1]$ and, using lemma 5.4, $\sup_{(s,H)\in[0,1]\times h([0,1])} \big|\big|\frac{\partial \mathbf{B}_1}{\partial H}(s,H)\big|\big|_{-p_0} \leqslant \kappa$, for every p_0. We hence get

$$\bigg|\bigg|h'(t)\ Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t,h(t))\bigg|\bigg|_{-s_0} \leqslant ||Y_s||_{-p_0} \left(\sup_{s\in[0,1]} |h'(s)|\right) \sup_{s\in[0,1]} \bigg|\bigg|\frac{\partial \mathbf{B}_1}{\partial H}(s,h(s))\bigg|\bigg|_{-p_0} < +\infty.$$

Thus there exists $\delta \in \mathbb{R}_+^*$, such that $\int_0^1 \big|\big|h'(s)\ Y_s \diamond \frac{\partial \mathbf{B}_1}{\partial H}(s,h(s))\big|\big|_{-s_0} ds \leqslant \delta \int_0^1 ||Y_s||_{-p_0}\ ds < +\infty$ which shows that $\int_0^1\ h'(t)\ Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t,h(t))dt$ is well defined in sense of Bochner. □

Corollary 5.8. *Let $(a_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are two sequences of integers that fulfill the growth condition of Theorem 3.5, then the element $\int_0^1\ Y_t\ d^{(\mathcal{M}_1)} B_t^h$ defined by*

$$\int_0^1\ Y_t\ d^{(\mathcal{M}_1)} B_t^h := \lim_{n\to\infty} \int_0^1\ Y_t\ d^{(\mathcal{M}_1)} B_t^{h_n} + \int_0^1\ h'(t)\ Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t,h(t))\ dt, \tag{5.12}$$

where the limit and the equality hold in $(\mathcal{S}_{-s_0})$, is well-defined and belongs to $(\mathcal{S}_{-s_0})$, where s_0 has been defined at the beginning of this subsection.

Proof: Obvious in regard of Proposition 5.5, Theorem 3.5 and Definition 3.2. □

5.4 Comparison between integrals in sense of $(\mathcal{M}_1)$ and Wick-Itô integral w.r.t. mBm

The definition of the multifractional Wick-Itô integral with respect to mBm or integral w.r.t mBm in the white noise sense has been given in [53]. In order to compare it to the integral w.r.t mBm defined by (5.5), we now particularize the multifractional Wick-Itô integral w.r.t mBm, defined in [53] in the Pettis integral framework, into the Bochner integral framework.

Definition 5.3 (Wick-Itô integral w.r.t mBm in Bochner sense)**.** *Let I be a Borelian connected subset of $[0,1]$, $B^h := (B_t^h)_{t\in I}$ be a multifractional Brownian motion and $Y := (Y_t)_{t\in I}$ be a $(\mathcal{S})^*$-valued process such that:*
(i) There exists $p \in \mathbb{N}$ such that $Y_t \in (\mathcal{S}_{-p})$ for almost every $t \in I$,
(ii) the process $t \mapsto Y(t) \diamond W_t^h$ is Bochner integrable on I.
We then say that the process Y is integrable on I, with respect to mBm, in the Bochner sense and define this integral by:

$$\int_I Y_s\ d^{\diamond} B_s^h := \int_I Y_s \diamond W_s^h ds. \tag{5.13}$$

Remark 5.9. (i) *Of course this definition generalizes the defintion 5.2 since we recover definition 5.2 when the function h is identically constant on I, equal to H.*
(ii) *Using the definition of $\left(W_t^h\right)_{t\in[0,1]}$, [53, proposition 5.9], and the proof of lemma 5.2, it is clear that every $(\mathcal{S})^*$-valued process $Y := (Y_t)_{t\in I}$, Bochner integrable on I of index p_0, is integrable on I with respect to mBm, in the Bochner sense. Moreover $\int_{[0,1]} Y_t\ dB_t^{\diamond h}$ belongs to $(\mathcal{S}_{-r_0})$, where r_0 has been defined in lemma 5.2.*

We now have two integrals with respect to mBm using white noise theory. The first one, which is noted $\int_0^1\ Y_t\ d^{(\mathcal{M}_1)} B_t^h$ has been defined in (5.12) and the second one, which is noted $\int_I Y_s\ d^{\diamond} B_s^h$, has been defined in (5.13).

In order to compare these two integrals with respect to mBm when they both exist, it seems natural, in view of lemma 5.2, proposition 5.5 and of the previous remark, to assume that $Y = (Y_t)_{t\in[0,1]}$ is a Bochner integrable process of index $p_0 \in \mathbb{N}$. The space E and the norm $||\ ||_{\Lambda_E}$ will then be taken as stated in Proposition 5.5. The following theorem shows that the integrals $\int_{[0,1]} Y_t\ d^{(\mathcal{M}_1)} B_t^h$ and $\int_{[0,1]} Y_t\ d^{\diamond} B_t^h$ are equal, assuming they both exist.

Theorem 5.10. *Let $Y = (Y_t)_{t\in[0,1]}$ be a Bochner integrable process of index $p_0 \in \mathbb{N}$. Then Y is integrable with respect to mBm in both sense* (5.12) *and* (5.13). *Moreover $\int_{[0,1]} Y_t \, d^{(\mathcal{M}_1)} B_t^h$ and $\int_{[0,1]} Y_t \, d^{\diamond} B_t^h$ are equal in $(\mathcal{S}^*)$.*

Proof: Since Y is a Bochner integrable process of index $p_0 \in \mathbb{N}$, we know, thanks to proposition 5.5 and to (ii) of the remark 5.9, that $\int_{[0,1]} Y_s \, d^{(\mathcal{M}_1)} B_s^h$ exists in $(\mathcal{S}_{-s_0})$ and $\int_{[0,1]} Y_s \, d^{\diamond} B_s^h$ exist in $(\mathcal{S}_{-r_0})$, where s_0 has been defined just above proposition 5.5 and r_0 has been defined in lemma 5.2. Moreover, thanks to (5.12) and using (3.4) and (5.7), we can write, in $(\mathcal{S}_{-s_0})$,

$$\begin{aligned}\int_0^1 Y_t \, d^{(\mathcal{M}_1)} B_t^h &= \lim_{n\to\infty} \int_0^1 Y_t \, d^{(\mathcal{M}_1)} B_t^{h_n} + \int_0^1 h'(t)\, Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) \, dt \\ &= \lim_{n\to\infty} \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \;\; Y_t \, d^{(\mathcal{M}_1)} B_t^{h(x_k^{(n)})} + \int_0^1 h'(t)\, Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) \, dt \\ &= \lim_{n\to\infty} \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \;\; Y_t \diamond W_t^{h(x_k^{(n)})} \, dt + \int_0^1 h'(t)\, Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) \, dt. \end{aligned} \tag{5.14}$$

Besides, thanks to equality (5.13) and using (5.6), we have, in $(\mathcal{S}_{-r_0})$,

$$\int_{[0,1]} Y_s \, d^{\diamond} B_s^h = \int_0^1 Y_t \diamond W_t^{h(t)} \, dt + \int_0^1 h'(t)\, Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) \, dt. \tag{5.15}$$

Since $s_0 \geqslant r_0$ we have $(\mathcal{S}_{-r_0}) \subset (\mathcal{S}_{-s_0})$. Thus we hence just have to show that, in $(\mathcal{S}_{-s_0})$,

$$L(Y) := \lim_{n\to\infty} \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \;\; Y_t \diamond W_t^{h(x_k^{(n)})} \, dt \text{ is equal to } \; M(Y) := \int_0^1 Y_t \diamond W_t^{h(t)} \, dt.$$

Since $L(Y)$ and $M(Y)$ both belong to $(\mathcal{S}_{-s_0})$, it is sufficient to show that they have the same S-transform. Let η be fixed in $\mathscr{S}(\mathbb{R})$, using [52, theorem 8.6] we get

$$S(L(Y))(\eta) = \lim_{n\to\infty} S\left(\sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \;\; Y_t \diamond W_t^{h(x_k^{(n)})} \, dt \right)(\eta).$$

Using now (ii) of [53, theorem 5.12] we get,

$$\begin{aligned} S(L(Y))(\eta) &= \lim_{n\to+\infty} \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} S(Y_t)(\eta)\; S(W_t^{h(x_k^{(n)})})(\eta) \, dt \\ &= \lim_{n\to+\infty} \int_{[0,1]} S(Y_t)(\eta) \left(\sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t)\; M_{h\left(x_k^{(n)}\right)}(\eta)(t) \right) dt. \end{aligned} \tag{5.16}$$

Furthermore, thanks to [53, lemma 5.5], we know that the map $(t, H) \mapsto M_H(\eta)(t)$ is continuous on every compact set of $\mathbb{R} \times (0,1)$. Define $K_\eta := \sup_{(t,H)\in[0,1]\times h([0,1])} M_H(\eta)(t)$. For every n in $\mathbb{N}$ and t in $[0,1]$ we have,

$$\begin{aligned} \left| S(Y_t)(\eta) \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t)\; M_{h\left(x_k^{(n)}\right)}(\eta)(t) \right| &\leqslant \sup_{(t,H)\in[0,1]\times h([0,1])} M_H(\eta)(t)\; e^{\frac{1}{2}|\eta|_{p_0}^2}\; ||Y_t||_{-p_0} \\ &\leqslant K_\eta \; e^{\frac{1}{2}|\eta|_{p_0}^2}\; ||Y_t||_{-p_0}. \end{aligned} \tag{5.17}$$

Since the map $t \mapsto K_\eta \, e^{|\eta|_{p_0}^2} \, ||Y_t||_{-p_0}$ belongs to $L^1(\mathbb{R}, dt)$ and since it is clear that, for almost every t in $[0,1]$,

$$\lim_{n\to+\infty} \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t)\; M_{h\left(x_k^{(n)}\right)}(\eta)(t) = M_{h(t)}(\eta)(t),$$

the dominated convergence theorem applies and gives us, using (5.17),

$$S(L(Y))(\eta) = \int_{[0,1]} S(Y_t)(\eta)\ M_{h(t)}(\eta)(t)\ dt = \int_{[0,1]} S(Y_t)(\eta)\ S(W_t^{h(t)})(\eta)\ dt = S\left(\int_{[0,1]} Y_t \diamond W_t^{h(t)}\ dt\right)(\eta),$$

The injectivity of the map $S : \Phi \mapsto S(\Phi)$ from $(\mathcal{S})^*$ into itself provides the equality $L(Y) = \int_{[0,1]} Y_t \diamond W_t^{h(t)}\ dt$, which ends the proof. □

Remark 5.11. *Thanks to the previous theorem we can use the Itô formulas and the Tanaka formula we obtained in chapter 2 (sections* 6 *and* 7*) for multifractional Wick-Itô integral.*

VI Appendix

Proof of proposition 3.1 in the case of $\mathbf{B}_1$ and $\mathbf{B}_2$

Once again it is sufficient to establish the proof for $\mathbf{B}_1$. Let $[a, b] \times [c, d]$ be fixed in $\mathbb{R} \times (0, 1)$ and (t, s, H, H') be fixed in $[a, b]^2 \times [c, d]^2$. We know, (see [34, section 2] for example) that, *a.s* $\mathbf{B}_1(t, H) :=< ., M_H(\mathbb{1}_{[0,t]}) >$, for every (t, H) in $\mathbb{R} \times (0, 1)$. Thanks to [53, Lemma 5.5, Proposition 3.8 and remark 3.9] we know that t the map $H \mapsto M_H(\mathbb{1}_{[0,t]})$ is C^1, for every real t, from $(0, 1)$ to $L^2(\mathbb{R})$. It is then clear that the map is $H \mapsto \mathbf{B}_1(t, H)$ is C^1 for every real t and that its derivative, noted $\frac{\partial \mathbf{B}_1}{\partial H}$ is such that: almost surely, $\frac{\partial \mathbf{B}_1}{\partial H}(t, H) =< ., \frac{\partial M_H}{\partial H}(\mathbb{1}_{[0,t]}) >$ for every (t, H) in $(0, 1)$. Note moreover that the previous equality also holds in $L^2(\Omega)$ and that the process $(\frac{\partial \mathbf{B}_1}{\partial H}(t, H))_{(t,H)\in\mathbb{R}\times(0,1)}$ is Gaussian and centred. Now and still using notations of [53], we get $\mathbb{E}[\frac{\partial \mathbf{B}_1}{\partial H}(t, H)\frac{\partial \mathbf{B}_1}{\partial H}(s, H')] = < \frac{\partial M_H}{\partial H}(\mathbb{1}_{[0,t]}), \frac{\partial M_H}{\partial H}(\mathbb{1}_{[0,s]}) >_{L^2(\mathbb{R})}$, for every $(t, H) \in \mathbb{R} \times (0, 1)$. We hence have

$$\begin{aligned} J := \mathbb{E}\left[\left(\tfrac{\partial \mathbf{B}_1}{\partial H}(t, H) - \tfrac{\partial \mathbf{B}_1}{\partial H}(s, H)\right)^2\right] &= \left|\left|\tfrac{\partial M_H}{\partial H}(\mathbb{1}_{[0,t]} - \mathbb{1}_{[0,s]})\right|\right|^2_{L^2(\mathbb{R})} \\ &= \tfrac{1}{c_H^2} \int_{\mathbb{R}} (\beta_H + \ln|y|)^2\ |y|^{1-2H} \left|\tfrac{1-e^{iy(t-s)}}{y^2}\right|^2 dy. \end{aligned}$$

where $\beta_H := \frac{c'_H}{c_H}$. Let τ be fixed in $(0, c)$ and $M := e^{\frac{\ln 2}{\tau}}$. Note that $M > 1$ and that $|y|^\tau \geqslant 2$ for every y such that $|y| \geqslant M$. We hence have:

$$\begin{aligned} J :&= \tfrac{1}{c_H^2} \int_{|y|>M} (\beta_H + \ln|y|)^2\ |y|^{1-2H} \left|\frac{1 - e^{iy(t-s)}}{y^2}\right|^2 dy + \tfrac{1}{c_H^2} \int_{|y|\leqslant M} (\beta_H + \ln|y|)^2\ |y|^{1-2H} \left|\frac{1 - e^{iy(t-s)}}{y^2}\right|^2 dy \\ &\leqslant \tfrac{1}{c_H^2} \int_{|y|>M} (\beta_H + \ln|y|)^2\ |y|^{1-2H}\ |t-s|^2 \frac{|y|^{2\tau}}{|y|^2}\ dy + |t-s|^2\ \tfrac{2^2}{c_H^2} \int_{|y|\leqslant M} \frac{(\beta_H + \ln|y|)^2}{|y|^{2H-1}}\ dy \\ &\leqslant |t-s|^2\ \tfrac{4}{c_H^2} \left(\int_{|y|>M} \frac{(\beta_H + \ln|y|)^2}{|y|^{1+2(H-\tau)}}\ dy + \int_{|y|\leqslant M} \frac{(\beta_H + \ln|y|)^2}{|y|^{2H-1}}\ dy\right) =: |t-s|^2\ Q(H). \end{aligned}$$

Since $\Delta_1 := \sup_{H\in[c,d]} Q(H) < +\infty$, we get

$$\mathbb{E}\left[\left(\tfrac{\partial \mathbf{B}_1}{\partial H}(t, H) - \tfrac{\partial \mathbf{B}_1}{\partial H}(s, H)\right)^2\right] \leqslant \Delta_1\ |t-s|^2. \tag{VI.1}$$

Besides, we get $\mathbb{E}\left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t, H) - \frac{\partial \mathbf{B}_1}{\partial H}(t, H')\right)^2\right] = \int_{\mathbb{R}} |1 - e^{ity}|^2\ (g_y(H) - g_y(H'))^2\ dy$, where the map $g_y : (0, 1) \to \mathbb{R}$ is defined by $g_y(H) := \frac{(\beta_H + \ln|y|)}{c_H}\ |y|^{1/2-H}$ for every y in $\mathbb{R}^*$, on $(0, 1)$. It is clear that g_y is C^1 on $[c, d]$, for every y in $\mathbb{R}^*$. The mean value theorem then applies and we know that there exists a positive constant K, which only depends on $[c, d]$, such that

$$\mathbb{E}\left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t,H) - \frac{\partial \mathbf{B}_1}{\partial H}(t,H')\right)^2\right] \leqslant |H-H'|^2 \ K \int_{\mathbb{R}} |1-e^{ity}|^2 \ |\Phi(y)|^2 \ dy,$$

where $\Phi(y) := 1 + \left(|y|^{1/2-c} + |y|^{1/2-d}\right)\left(1 + (1+\ln|y|)\ln|y|\right)$ for every y in $\mathbb{R}^*$. Since $\Delta_2 := K \int_{\mathbb{R}} |1 - e^{ity}|^2 \ |\Phi(y)|^2 \ dy < +\infty$, we have proven that

$$\mathbb{E}\left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t,H) - \frac{\partial \mathbf{B}_1}{\partial H}(t,H')\right)^2\right] \leqslant \Delta_2 \ |H-H'|^2. \tag{VI.2}$$

Setting $\Delta := 2(\Delta_1 + \Delta_2)$ and using (VI.1) and (VI.2) we finally get:

$$\begin{aligned}\mathbb{E}\left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t,H) - \frac{\partial \mathbf{B}_1}{\partial H}(s,H')\right)^2\right] &\leqslant 2\mathbb{E}\left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t,H) - \frac{\partial \mathbf{B}_1}{\partial H}(t,H')\right)^2\right] + 2\mathbb{E}\left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t,H') - \frac{\partial \mathbf{B}_1}{\partial H}(s,H')\right)^2\right] \\ &\leqslant \Delta \left(|t-s|^2 + |H-H'|^2\right).\end{aligned}$$

□

Chapter 4

Multifractional stochastic volatility models

Joint work with S.Corlay and J. Lévy-Véhel

Abstract
The aim of this work is to advocate the use of multifractional Brownian motion (mBm) as a relevant model in financial mathematics. Multifractional Brownian motion is an extension of fractional Brownian motion where the Hurst parameter is allowed to vary in time. This enables the possibility to accommodate for varying local regularity, and to decouple it from long-range dependence properties. While we believe that mBm is potentially useful in a variety of applications in finance, we focus here on a multifractional stochastic volatility Hull & White model that is an extension of the model studied in [24]. Using the stochastic calculus with respect to mBm developed in [53], we solve the corresponding stochastic differential equations. Since the solutions are of course not explicit, we take advantage of recently developed numerical techniques, namely functional quantization-based cubature methods, to get accurate approximations. This allows us to test the behaviour of our model (as well as the one in [24]) with respect to its parameters, and in particular its ability to explain the smile effect of implied volatility. An advantage of our model is that it is able both to fit smiles at different maturities, and to take into account volatility persistence in a more precise way than in [24].

keywords: Hull & White model, functional quantization, vector quantization, Karhunen-Loève, Gaussian process, fractional Brownian motion, multifractional Brownian motion, white noise theory, S-transform, Wick-Itô integral, stochastic differential equations.

Contents

Introduction

Volatility in financial markets is both of crucial importance and hard to model in an accurate way. It has been long known that a constant volatility as in the Black & Scholes model is not consistent with empirical findings, such as the smile effect. More basically, there is no reason to expect that instantaneous volatility should be constant. Popular models allowing for a varying volatility include ARCH models and their generalizations, stochastic volatility models and local volatility models. Local volatility models, in particular, enable the possibility to mimic in an exact way implied volatility surfaces. However, such models do not take into account another well documented fact: while stocks do not typically exhibit correlations, volatility does display long-range correlations (see, *e.g.* [4]). Stochastic volatility models, in contrast, are able to incorporate this feature, provided an adequate driving noise is used. In [24, 23], this is performed by using fractional integration. More precisely, the model considered in [24] for the dynamic of the price of a risky asset reads as follows:

$$\begin{cases} dS_t = \mu(t, S_t)dt + S_t\sigma_t dW_t, \\ d\ln(\sigma_t) = \theta\left(\mu - \ln(\sigma_t)\right)dt + \gamma dB_t^H, \quad \sigma > 0, \end{cases} \tag{0.1}$$

where W is a Brownian motion and B_t^H is an independent fractional Brownian motion (fBm) under the historical probability. Fractional Brownian motion is an extension of Brownian motion, parametrized by a real H in $(0,1)$, which has constant local Hölder regularity equal to H and whose increments display long-range dependence for $H > 1/2$. Such a model is consistent with the slow decay in the correlations of volatility observed in practice. It also accounts for two features related to the measured smile effect: the volatility process is less persistent in the short term than a standard diffusion, while it is more persistent in the long run ([23, p. 3]). We verify this fact in the case of Model (0.1) numerically in Section 6. However, by the very nature of this model, the evolution in time of the smile is governed by the single parameter H. In this work, we replace fBm appearing in (0.1) with a more general process called multifractional Brownian motion (mBm). Multifractional Brownian motion is an extension of fBm where the Hurst parameter H is replaced by a function h. This enables the possibility to accommodate for non-stationary local regularity, and to decouple it from long-range dependence properties. Indeed, there is no reason to believe that the regularity of the volatility should be constant. In addition, graphs of estimated historical volatility (see, *e.g.* [4]) seem to indicate that this process is highly irregular. Modelling this evolution with the help of an fBm would thus require to choose a "small" H, *i.e.* $H < 1/2$, which is not compatible with long-range dependence properties. In contrast, mBm has at each time t local regularity $h(t)$, and, no matter the value of h in $(0,1)$, always display long-range dependence as long h is not constant. In addition, as we will show from numerical experiments, the model (written in a risk-neutral setting):

$$\begin{cases} dF_t = F_t \sigma_t dW_t, \\ d\ln(\sigma_t) = \theta\left(\mu - \ln(\sigma_t)\right) dt + \gamma_h d^{\diamond} B_t^h + \gamma_\sigma dW_t^\sigma, \quad \sigma_0 > 0, \\ d\langle W, W^\sigma \rangle_t = \rho dt, \end{cases} \tag{0.2}$$

where B_t^h is an mBm, yields shapes of the smile at maturity T that are governed by a weighted average of the values of the function h up to time T: thus, by adequately choosing h, one may mimic a given implied volatility surface more faithfully than with a Hull & White model driven by fBm (the calibration of h for this purpose will be addressed in a forthcoming work).

In order to give a rigorous meaning to the model above, a stochastic integral with respect to mBm must be defined. Multifractional and fractional Brownian motion are not semimartingales, thus classical Itô theory does not apply to them. At the time [24] was written, no theory for integration with respect to fBm was available yet. Various approaches have been developed since. Among these, the one based on white noise theory is well fitted for an extension to mBm. In particular, it allows to deal with any $H \in (0,1)$ and to obtain Itô formulas. This integral was developed in [34, **?**, 13], and applied to option pricing in a fractional Black and Scholes model in [34] [1]. The white noise based stochastic integral was extended to mBm in [53]. This is the theory we will use in order to study precisely our stochastic volatility models.

While we focus here on the multifractional stochastic volatility model (0.2) (we also briefly consider a multifractional SABR model with $\beta = 1$ in Section 5), we would like to mention that mBm is useful in a variety of applications in finance (see [1] for a partial list of articles dealing with mBm in this field).

In order to assess the relevance of our model, we compute numerically the smiles at different maturities. Since the solution cannot be written in an explicit form, we need to resort to approximations. In our case, this is made possible by recent advances in the theory of functional quantization of Gaussian processes.

Functional quantization of Gaussian processes has become an active field of research in recent years since the seminal article [56]. As far as applications are concerned, cubature methods [66, 27] and variance reduction methods [28, 54] based on functional quantization have been proposed. However, as the numerical use of functional quantizers requires the evaluation of the Karhunen-Loève eigenfunctions, this method was restricted to processes for which a closed-form expression for this expansion is known, such as Brownian motion. In [26], a numerical method was proposed to perform numerical quadratic functional quantization of more general Gaussian processes, which will be applied here to multifractional Brownian motion.

We show that we can handle a fast and accurate forward start option pricing in this model thanks to a functional quantization-based cubature method similar to the one proposed in [66] and in [27]. This allows us to study the dependency of the smile dynamics on the functional parameter of the considered mBm.

The remaining of this paper is organized as follows. We recall in Section 1 basic facts about mBm. In Section 2, we explain how to perform functional quantization of mBm and investigate the rate of decay of the corresponding quantization error. Quantization-based cubature is also addressed in this section. Section 3 is devoted to recalls on the white noise based stochastic integral with respect to mBm. It also shows how to solve some stochastic differential equations (S.D.E.) in this frame and presents general remarks on the quantization of solutions of S.D.E. A detailed treatment of the Hull & White and SABR models are proposed in Sections 4 and 5. Numerical experiments and conclusions are gathered in Section 6.

1 Recalls on multifractional Brownian motion

Fractional Brownian motion (fBm) [51, 60] is a centred Gaussian process with features that make it a useful model in various applications such as financial and teletraffic modelling, image analysis and synthesis, geophysics and more. These features include self-similarity, long-range dependence and the ability to match any prescribed constant local regularity. Fractional Brownian motion depends on a parameter, usually

1. Such a fractional Black and Scholes model raises some financial and economical issues, see [10, 17].

denoted by H and called the Hurst exponent, that belongs to $(0,1)$. Its covariance function R_H reads:

$$R_H(t,s) := \frac{\gamma_H}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

where γ_H is a positive constant. A normalized fBm is one for which $\gamma_H = 1$. Obviously, when $H = \frac{1}{2}$, fBm reduces to standard Brownian motion. While fBm is a useful model, the fact that most of its properties are governed by the single number H restricts its application in some situations. In particular, its Hölder exponent remains the same all along its trajectory. Thus, for instance, long-range dependent fBm, which require $H > \frac{1}{2}$, must have smoother paths than Brownian motion. Multifractional Brownian motion [68, 7] was introduced to overcome these limitations. The basic idea is to replace the real H by a function $t \mapsto h(t)$ ranging in $(0,1)$.

The construction of mBm is best understood through the use of a fractional Brownian field. Fix a probability space $(\Omega, \mathcal{F}, P)$ and a positive real T. A fractional Brownian field on $[0,T]\times(0,1)$ is a Gaussian field, denoted $(\mathbf{B}(t,H))_{(t,H)\in[0,T]\times(0,1)}$, such that for every H in $(0,1)$ the process $(B_t^H)_{t\in[0,T]}$, where $B_t^H := \mathbf{B}(t,H)$, is a fractional Brownian motion with Hurst parameter H [2]. For a deterministic continuous function $h : [0,T] \to (0,1)$, we call multifractional Brownian motion with functional parameter h the Gaussian process $B^h := (B_t^h)_{t\in[0,T]}$ defined by $B_t^h := \mathbf{B}(t,h(t))$. We say that h is the *regularity function* of the mBm. The fractional field $(\mathbf{B}(t,H))_{(t,H)\in[0,T]\times(0,1)}$ is termed normalized when, for all H in $(0,1)$, $(B_t^H)_{t\in[0,T]}$ is a normalized fBm. In this case we will also say that B^h is normalized.

In order for mBm to posses interesting properties, we need some regularity of $\mathbf{B}(t,H)$ with respect to H. More precisely, we will always assume that $\mathbf{B}(t,H)$ satisfies the following condition:

$$\begin{array}{c}\forall T\in\mathbb{R}_+^*, \forall[c,d]\subset(0,1), \exists(\Lambda,\delta)\in(\mathbb{R}_+^*)^2 \text{ such that} \\ \mathsf{E}[(\mathbf{B}(t,H)-\mathbf{B}(t,H'))^2] \leqslant \Lambda\,|H-H'|^{\delta} \text{ for every } (t,H,H') \text{ in } [0,T]\times[c,d]^2.\end{array} \tag{$\mathcal{H}$}$$

Under this assumption, and if the functional parameter h is continuous, then the associated mBm has a continuous modification.

The class of mBm is rather large, since there is some freedom in choosing the correlations between the fBms composing the fractional field $\mathbf{B}(t,H)$. For definiteness, we will often consider in this work the so-called "well-balanced" version of multifractional Brownian motion. Essentially the same analysis could be conducted with other versions. More precisely, a well-balanced mBm is obtained from the field $\mathbf{B}(t,H) := \frac{1}{c_H}\int_{\mathbb{R}} \frac{e^{itu}-1}{|u|^{H+1/2}}\widetilde{W}(du)$ where $\widetilde{W}$ denotes a complex-valued Gaussian measure (*cf.* [75] for more details).

We show in Proposition 2.8 that assumption $\mathcal{H}$ is satisfied by the well-balanced fractional Brownian field (in fact, it is verified by all mBms considered so far in the literature). The proof of the following proposition can be found in [5]:

Proposition 1.1 (Covariance function of well-balanced mBm)**.** *The covariance function R_h of well-balanced mBm is given by*

$$R_h(t,s) = \frac{c_{h_{t,s}}^2}{c_{h(t)}c_{h(s)}}\left(\frac{1}{2}\left(|t|^{2h_{t,s}} + |s|^{2h_{t,s}} - |t-s|^{2h_{t,s}}\right)\right), \tag{1.1}$$

where $h_{t,s} := \frac{h(t)+h(s)}{2}$ *and* $c_x := \left(\frac{2\pi}{\Gamma(2x+1)\sin(\pi x)}\right)^{\frac{1}{2}}$.

The other main properties of mBm are as follows: the pointwise Hölder exponent at any point t of $B^{(h)}$ is almost surely equal to $h(t)\wedge\beta_h(t)$, where $\beta_h(t)$ is the pointwise Hölder exponent of h at t [39, 41]. For a smooth h, one thus may control the local regularity of the paths by the value of h. In addition, the increments of mBm display long range dependence for all non-constant $h(t)$ [5]. Finally, when h is C^1, mBm is tangent to fBm with exponent $h(u)$ in the neighbourhood of any u in the following sense [35]:

2. An alternative definition would be to start from a family of fBms $(B^H)_{H\in(0,1)}$ (*i.e.* $B^H := (B_t^H)_{t\in\mathbb{R}}$ is an fBm for every H in $(0,1)$) and define from it the field $(\mathbf{B}(t,H))_{(t,H)\in[0,T]\times(0,1)}$ by $\mathbf{B}(t,H) := B_t^H$. However it is not true, in general, that the field $(\mathbf{B}(t,H))_{(t,H)\in[0,T]\times(0,1)}$ obtained in this way is Gaussian.

$$\left\{\frac{B^h_{u+rt} - B^h_u}{r^{h(u)}};\ t \in [a,b]\right\} \xrightarrow[r\to 0^+]{\text{law}} \{B^{h(u)}_t;\ t \in [a,b]\}.$$

These properties show that mBm is a more versatile model that fBm: in particular, it is able to mimic in a more faithful way local properties of financial records, Internet traffic and natural landscapes [15, 55, 33] by matching their local regularity. This is important *e.g.* for purposes of detection or real-time control. The price to pay is of course that one has to deal with the added complexity brought by having a functional parameter instead of a single number.
In general, the increments of multifractional Brownian motion are neither independent nor stationary. Since an mBm B^h is an fBm of Hurst index H when h is constant and equal to H, there is no risk of confusion by denoting B^H the fractional Brownian motion with Hurst index H.

2 Functional quantization of multifractional Brownian motion

2.1 Computation of the quantization

The quantization of a random variable X valued in a reflexive separable Banach space $(E, |\cdot|)$ consists in its approximation by a random variable Y that is measurable with respect to X and that takes finitely many values in E. The resulting error of the discretization is usually measured by the L^p norm of the difference $|X-Y|$. If we settle on a fixed maximum cardinal N for $Y(\Omega)$, the minimization of the error reduces to the following optimization problem:

$$\min\left\{\|\,|X-Y|\,\|_p,\ Y:\Omega\to E \text{ measurable with respect to } X,\ \text{card}(Y(\Omega)) \leqslant N\right\}. \tag{2.1}$$

As Y is supposed to be measurable with respect to X, there exists a Borel map $\text{Proj} : E \to E$ valued in a finite subset Γ of E such that $Y = \text{Proj}(X)$. The finite subset Γ is called the codebook. Hence if $\Gamma = \{\gamma_1, \cdots, \gamma_N\}$, there exists a Borel partition $C = \{C_1, \cdots, C_N\}$ of E such that $\text{Proj} = \sum_{i=1}^{N} \gamma_i \mathbf{1}_{C_i}$. In other words, Proj performs the process of mapping the continuous set $X(\Omega)$ to the finite set Γ. Let Proj_Γ denote a nearest neighbour projection on Γ. Clearly,

$$|X - \text{Proj}_\Gamma(X)| \leqslant |X - \text{Proj}(X)| \quad \text{so that} \quad \big\||X - \text{Proj}_\Gamma(X)|\big\|_p \leqslant \big\||X - \text{Proj}(X)|\big\|_p.$$

Hence, in order to minimize the quantization error, it is optimal to use a nearest neighbour projection on the codebook Γ. A solution of (2.1) is called an L^p-optimal quantizer of X. An elementary property of an L^2-optimal quantizer is stationarity: $\mathsf{E}[X|Y] = Y$. We denote by $\mathcal{E}_{N,p}(X, |\cdot|)$ the minimal L^p quantization error for the random variable X and the norm $|\cdot|$:

$$\mathcal{E}_{N,p}(X, |\cdot|) = \min\left\{\big\||X-Y|\big\|_p,\ Y \text{ measurable with respect to } X \text{ and } |Y(\Omega)| \leqslant N\right\}$$

We now assume that X is a bi-measurable stochastic process on $[0,T]$ verifying $\int_0^T \mathsf{E}\left[|X_t|^2\right] dt < \infty$, that we see as a random variable valued in the Hilbert space $H = L^2([0,T])$. Suppose that its covariance function Γ^X is continuous. In [56], it is shown that, in the centred Gaussian case, linear subspaces U of H spanned by N-stationary quantizers correspond to principal components of X, in other words, are spanned by eigenvectors of the covariance operator of X. Thus, the quadratic optimal quantization of Gaussian processes consists in using its Karhunen-Loève decomposition $\left(e^X_n, \lambda^X_n\right)_{n\geqslant 1}$.
To perform optimal quantization, the Karhunen-Loève expansion is first truncated at a fixed order m and then the $\mathbb{R}^m$-valued Gaussian vector constituted of the m first coordinates of the process on its Karhunen-Loève decomposition is quantized. To reach optimal quantization, we have to determine both the optimal rank of truncation $d^X(N)$ (the quantization dimension) and the optimal $d^X(N)$-dimensional Gaussian quantizer corresponding to the first coordinates, $\bigotimes_{j=1}^{d^X(N)} \mathcal{N}\left(0, \lambda^X_j\right)$. We have the following representation of the quadratic

distortion $\mathcal{E}_N(X) := \mathcal{E}_{N,2}\left(X, |\cdot|_{L^2([0,T])}\right)$:

$$\mathcal{E}_N(X)^2 = \sum_{j\geqslant m+1} \lambda_j^X + \mathcal{E}_N\left(\bigotimes_{j=1}^m \mathcal{N}\left(0,\lambda_j^X\right)\right)^2.$$

From a numerical viewpoint, we are thus confronted on the one hand with the finite-dimensional quantization of the Gaussian distribution $\bigotimes_{j=1}^m \mathcal{N}\left(0,\lambda_j^X\right)$ and on the other hand with the numerical evaluation of the first Karhunen-Loève eigenfunctions $\left(e_n^X\right)_{1\leqslant n\leqslant d^X(N)}$. Various numerical algorithms have been developed to deal with the first point. Let us mention Lloyd's algorithm and the Competitive Learning Vector Quantization (CLVQ). A review of these methods is available in [65]. As far as the evaluation of the first Karhunen-Loève eigenfunctions is concerned, closed-expressions are available for standard Brownian motion, standard Brownian bridge and Ornstein-Uhlenbeck process. Other examples of explicit Karhunen-Loève expansions may be found in [32] and [70]. In the general case, the so-called Nyström method for approximating the solution of the associated integral equation may be used. It reads

$$\int_0^T \Gamma^X(\cdot,s)e_k^X(s)ds = \lambda_k^X e_k^X, \qquad k\geqslant 1, \tag{2.2}$$

where both the eigenvalues and the eigenvectors have to be determined. The Nyström method relies on the use of a quadrature scheme to approximate the integral, so that it turns into a matrix eigensystem. When dealing with the midpoint quadrature rule, and for sufficiently regular kernels Γ^X, the error admits an asymptotic expansion in the form of the sum of even powers of the step size, for both the eigenvalues and the eigenfunctions. We take advantage of this asymptotic expansion by using Richardson-Romberg extrapolation methods. This method has been benchmarked against the Karhunen-Loève eigensystems of standard Brownian motion, Brownian bridge and Ornstein-Uhlenbeck process in [26].

Instead of using an optimal quantization for the distribution $\bigotimes_{j=1}^{d^X(N)} \mathcal{N}\left(0,\lambda_j^X\right)$, another possibility is to use a product quantization, that is to use the Cartesian product of the optimal quadratic quantizers of the standard one-dimensional Gaussian distributions $\mathcal{N}\left(0,\lambda_j^X\right)_{1\leqslant j\leqslant d^X(N)}$. In the case of independent marginals, this yields a stationary quantizer, *i.e.* a quantizer Y of X which satisfies $\mathsf{E}[X|Y]=Y$. This property, shared with optimal quantizers, results in a convergence rate of a higher order for the quantization-based cubature scheme, as explained in [66]. An advantage of this approach is that one-dimensional Gaussian quantization is a fast procedure.

In [65], deterministic optimization methods (*e.g.* Newton-Raphson) are shown to converge rapidly to the unique optimal quantizer of the one-dimensional Gaussian distribution. Moreover, a sharply optimized database of quantizers of standard univariate and multivariate Gaussian distributions is available at `www.quantize.maths-fi.com` [67] for download. One sill has to determine the quantization level for each dimension to obtain optimal product quantization. In this case, the minimization of the distortion becomes:

$$\left(\mathcal{E}_N^{\text{prod}}(X)\right)^2 := \min\left\{\sum_{j=1}^d \mathcal{E}_{N_j}^2\left(\mathcal{N}\left(0,\lambda_j^X\right)\right) + \sum_{j\geqslant d+1}\lambda_j^X,\ N_1\times\cdots\times N_d\leqslant N,\ d\geqslant 1\right\}. \tag{2.3}$$

A solution of (2.3) is called an optimal K-L product quantizer. This problem can be solved by the "blind optimization procedure", which consists in computing the criterion for every possible decomposition $N_1\times\cdots\times N_d$ with $N_1\geqslant\cdots\geqslant N_d$. The result of this procedure can be stored for future use. Optimal decompositions for a wide range of values of N for both Brownian motion and Brownian bridge are available on the web site `www.quantize.maths-fi.com` [67]. Another fact on quadratic functional product quantization is that it is shown to be rate-optimal under certain assumptions on the K-L eigenvalues (see Theorem 2.1).

Quadratic product quantizers of fBms and well-balanced mBms for different H and h are displayed on Figures 4.1 and 4.2. A fixed product decomposition is used for simplicity.

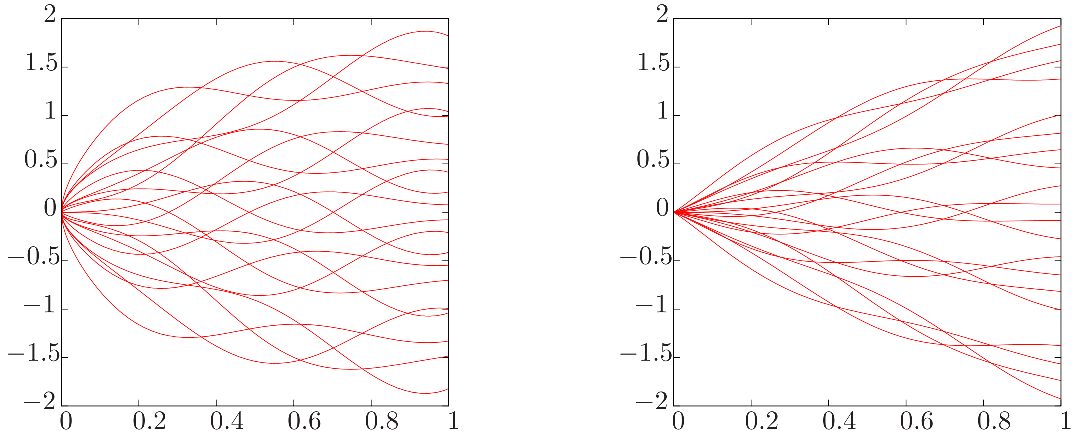

Figure 4.1: Quadratic $5 \times 2 \times 2$-product quantizer of fBm on $[0, 1]$ with $H = 0.25$ (left) and $H = 0.75$ (right).

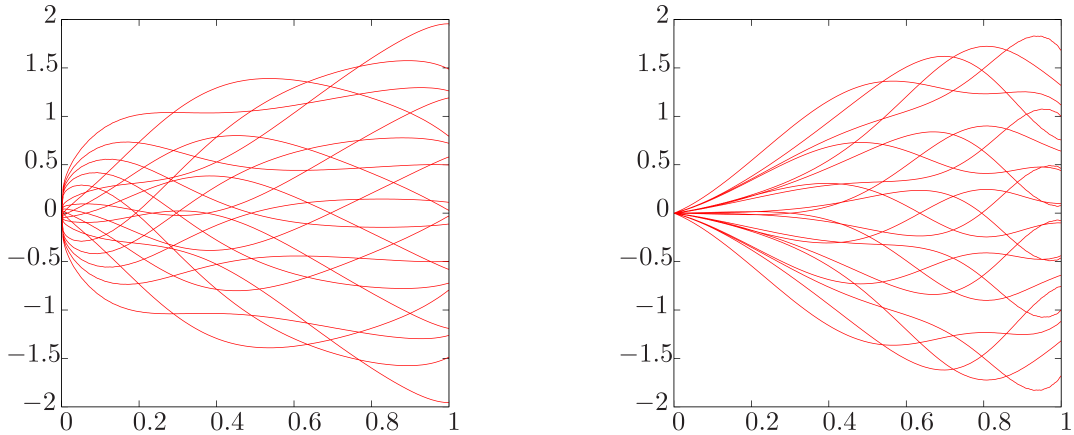

Figure 4.2: Quadratic $5 \times 2 \times 2$-product quantizer of mBm on $[0, 1]$ with $h(t) := 0.1 + 0.8t$ (left) and $h(t) := 0.9 - 0.8t$ (right).

These graphs reflect, to some extent, the features of the quantized process, in particular its correlation and regularity properties.

In the case of fBm (Figure 4.1), when H increases, the rate of decay of the Karhunen-Loève eigenvalues also increases (and so does the pathwise Hölder regularity of the underlying process), so that even though we do not change the quantization dimension in this example, the contribution of higher-order Karhunen-Loève eigenvalues decreases. In Figure 4.1, one can see that the curves of the functional quantizer localize around the quantization of the first Karhunen-Loève coordinate when $H = 0.75$, while this is not the case when $H = 0.25$.

In addition, the distribution of the curves on the plane is related to the fact that the almost sure Hausdorf dimension of the paths of fBm is 2-H: for small H, we expect the set of curves to be more space-filling than for large H, a feature that can be indeed be verified on the figure. In addition, the long-term correlation of fBm for $H > 1/2$, which results in paths typically having strong trends, translates here into curves which are roughly monotonous. Conversely, the negative correlations which characterizes the case $H < 1/2$ is reflected in the more oscillatory behaviour of the curves in the left pane.

The case of mBm (Figure 4.2) makes even clearer the relation between the properties of the process and the optimal quantizer. In the right pane of Figure 4.2, the function h decreases linearly from 0.9 to 0.1. One can see that, for small t, both the distribution of the curves and their trend look like the ones of fBm with large H. As t increases, the curves become more space-filling and oscillatory, in agreement with the fact that, for

t close to 1, the corresponding mBm has larger local Hausdorff dimension. Similar remarks hold for the case where h is an increasing function (right pane of Figure 4.2).
Another way of interpreting these figures is to recall that mBm is tangent, at each t, to fBm with exponent $H = h(t)$. The behaviour of the curves on Figure 4.2 is a translation of this fact in the quantization domain. Finally, note that the shape of the convex envelopes in each of the four figures roughly matches the time evolution of the variances of the corresponding processes, *i.e.* $t^{0.25}, t^{0.75}, t^{0.1+0.8t}$ and $t^{0.9-0.8t}$.

2.2 Rate of decay of the quantization error for mBm

The rate of decay of the quadratic functional quantization error was first investigated in [56]. More precise results were then established in [57]. These results rely on assumptions on the asymptotic behaviour of the Karhunen-Loève eigenvalues of the considered process. In Subsection 2.2, we recall the main result involving the rate of decay of these eigenvalues, leading to sharp rates of convergence for the quantization of fBm.
Unfortunately, such asymptotics for the Karhunen-Loève eigenvalues are not known at this time in the case of mBm. However, since the regularity of mBm is known, one may use another, less precise, type of results: these yield an upper estimate on the rate of decay of the quantization error [59]. This is explained in Subsection 2.2.

In the following, for two positive sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, we write $x_n \underset{n\to\infty}{\sim} y_n$ if $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$. The symbol $x_n \underset{n\to\infty}{\lesssim} y_n$ means that $\lim_{n\to\infty} \frac{x_n}{y_n} \leqslant 1$. Finally, $x_n \underset{n\to\infty}{\asymp} y_n$ means that $x_n = O(y_n)$ and $y_n = O(x_n)$ as $n \to \infty$.

Sharp rates based on asymptotics of Karhunen-Loève eigenvalues

Recall the following well-known definition:

Definition 2.1 (Regularly varying function at infinity). *A measurable function $\phi : (s,\infty) \to (0,\infty)$, ($s > 0$) is regularly varying at infinity with index $b \in \mathbb{R}$ if for every $t > 0$, $\lim_{x\to\infty} \frac{\phi(tx)}{\phi(x)} = t^b$.*

Let X be a bi-measurable centred Gaussian process on $[0,T]$ with a continuous covariance function Γ^X and such that $\int_0^T \mathsf{E}[X_s^2]ds < \infty$. Denote by $\left(e_n^X, \lambda_n^X\right)_{n\geqslant 1}$ its Karhunen-Loève eigensystem.

Theorem 2.1 (Quadratic quantization error asymptotics [57]). *Assume that $\lambda_n^X \sim \phi(n)$ as $n \to \infty$, where $\phi : (s,\infty) \to (0,\infty)$ is a decreasing regularly varying function of index $-b < -1$ and $s > 0$. Set $\psi(x) := \frac{1}{x\phi(x)}$. Then*

$$\mathcal{E}_N(X) \sim \left(\left(\frac{b}{2}\right)^{b-1} \frac{b}{b-1}\right)^{1/2} \psi(\log(N))^{-1/2} \quad \text{as } N \to \infty.$$

Moreover, the optimal product quantization dimension $m^X(N)$ verifies $m^X(N) \sim \frac{2}{b}\log(N)$ as $N \to \infty$, and the optimal product quantization error $\mathcal{E}_N^{\text{prod}}(X)$ of level N satisfies

$$\mathcal{E}_N^{\text{prod}}(X) \lesssim \left(\left(\frac{b}{2}\right)^{b-1} \frac{b}{b-1} + C(1)\right)^{1/2} \psi(\log(N))^{-1/2} \quad \text{as } N \to \infty,$$

where $C(1)$ is a universal positive constant.

Though the optimal product quantization is not asymptotically optimal, it still provides a rate-optimal sequence of quantizers. In the case where $b = 1$, a similar result is true, with the additional property that the optimal product quantization does yield an asymptotically optimal quadratic quantization error.

The case of fractional Brownian motion

In [56, 19], it is shown that the Karhunen-Loève eigenvalues of fBm on $[0,T]$ verify

$$\lambda_n^{B^H} \sim \frac{\nu_H}{n^{2H+1}} \quad \text{as } n \to \infty,$$

where ν_H is a positive constant. Thus, fBm satisfies the hypotheses of Theorem 2.1 and

$$\mathcal{E}_N\left(B^H\right) \sim \frac{K_H}{\log(N)^H} \text{ as } N \to \infty \text{ for some } K_H > 0, \quad \text{and} \quad \mathcal{E}_N^{\text{prod}}\left(B^H\right) \asymp \frac{1}{\log(N)^H} \text{ as } N \to \infty.$$

Mean regularity and domination of the functional quantization rate

We recall the definition of regular variation at 0:

Definition 2.2 (Regularly varying function at zero)**.** *A measurable function $\phi : (0, s) \to (0, \infty)$, $(s > 0)$ is regularly varying at 0 with index $b \geqslant 0$ if for every $t > 0$, $\lim_{x\to 0} \frac{\phi(tx)}{\phi(x)} = t^b$.*

Definition 2.3 (The ϕ-Lipschitz assumption)**.** *Let X be a bi-measurable process on $[0, T]$. We say that X satisfies the ϕ-Lipschitz assumption for $\rho > 0$, which we denote by $(L_{\phi,\rho})$, if there is a non-decreasing function $\phi : \mathbb{R}_+ \to [0, \infty]$, continuous at 0 with $\phi(0) = 0$, such that*

$$(L_{\phi,\rho}) \equiv \begin{cases} \forall (s,t) \in [0,T]^2, \mathsf{E}\left[|X_t - X_s|^\rho\right] \leqslant (\phi(|t-s|))^\rho, & \text{if } \rho \geqslant 1 \\ \forall t \in [0,T], \forall h \in (0,T], \mathsf{E}\left[\sup_{t\leqslant s\leqslant (t+h)\wedge T} |X_s - X_t|^\rho\right] \leqslant (\phi(h))^\rho & \text{if } 0 < \rho < 1. \end{cases}$$

Remark 2.2. *The ϕ-Lipschitz assumption implies that $\mathsf{E}\left[|X|^\rho_{L^\rho([0,T])}\right] < \infty$ so that $\mathbb{P}$-almost surely, $t \mapsto X_t$ lies in $L^\rho([0, T])$.*

Theorem 2.3 (Mean regularity and quantization rate)**.** *Let X be a bi-measurable process on $[0, T]$ such that $X_t \in L^\rho$ for every $t \in [0, T]$, $\rho > 0$. Assume that X satisfies $(L_{\phi,\rho})$ where ϕ is regularly varying at 0 with index b. Then*

$$\forall (r,p) \in [0,\rho]^2, \quad \mathcal{E}_{N,r}\left(X, |\cdot|_{L^p([0,T])}\right) \leqslant C_{r,p} \begin{cases} \phi(1/\log(N)), & \text{if } b > 0, \\ \psi(1/\log(N)), & \text{if } b = 0, \end{cases}$$

with $\psi(x) := \left(\int_0^x \frac{(\phi(\xi))^{(r\wedge 1)}}{\xi} d\xi\right)^{\frac{1}{r\wedge 1}}$, assuming in addition that $\int_0^x \frac{(\phi(\xi))^{(r\wedge 1)}}{\xi} d\xi < \infty$ if $b = 0$.

In particular, if $\phi(u) = cu^b$, $b > 0$, then

$$\mathcal{E}_{N,r}\left(X, |\cdot|_{L^p([0,T])}\right) = O(\log(N)^{-b}).$$

The case of multifractional Brownian motion

Theorem 2.4 (L^2-mean regularity of the multifractional Brownian motion)**.** *Let B^h be an mBm with functional parameter h satisfying assumption $(\mathcal{H})$. Assume that h is β-Hölder continuous, $\beta > 0$* [3]. *Then there exists a positive constant M such that*

$$\forall (s,t) \in [0,T]^2, \quad \mathsf{E}\left[\left(B^h_t - B^h_s\right)^2\right] \leqslant M\ |t-s|^{\left(2 \inf_{u\in[0,T]} h(u) \wedge \beta\delta\right)}, \tag{2.4}$$

where δ is given in assumption $(\mathcal{H})$.

Proof: We may assume that the fractional field $(\mathbf{B}(t, H))_{(t,H)\in[0,T]\times[c,d]}$ is normalized. For (t, s) in $[0, T]^2$:

$$\begin{aligned} \mathsf{E}\left[\left(B^h_t - B^h_s\right)^2\right] &\leqslant 2\ \mathsf{E}\left[(\mathbf{B}(t,h(t)) - \mathbf{B}(s,h(t)))^2\right] + 2\ \mathsf{E}\left[(\mathbf{B}(s,h(t)) - \mathbf{B}(s,h(s)))^2\right] \\ &\leqslant 2\left(|t-s|^{2h(t)} + \Lambda\ |h(t)-h(s)|^\delta\right) \leqslant 2\left(|t-s|^{2H_1}\ (1+T^{2(H_2-H_1)}) + \Lambda\ \eta^\beta |t-s|^{\beta\delta}\right) \\ &\leqslant 2\ \ (1+T^{2(H_2-H_1)})\ (1+\Lambda\ \eta^\beta)\ \left(|t-s|^{2H_1} + |t-s|^{\beta\delta}\right) \leqslant M\ |t-s|^{2H_1\wedge\beta\delta}, \end{aligned}$$

where $H_1 := \inf_{u\in[0,T]} h(u)$, $H_2 := \sup_{u\in[0,T]} h(u)$ and $M := 2(1 + T^{2(H_2-H_1)})\ (1 + T^{2H_1\vee\beta\delta-2H_1\wedge\beta\delta})\ (1 + \Lambda\ \eta^\beta)$. □

3. *i.e.* $\exists \eta \in \mathbb{R}^*_+,\ \forall (s,t) \in [0,T]^2,\ |h(s) - h(t)| \leqslant \eta\ |s-t|^\beta$.

Corollary 2.5 (Upper bound on the quantization error for multifractional Brownian motion)**.** *With the same notations and assumptions as in theorem 2.4:*

$$\mathcal{E}_{N,r}\left(B^h, |\cdot|_{L^p([0,T])}\right) = O\left(\log(N)^{-(H_1\wedge\frac{\beta\delta}{2})}\right),$$

for every (r,p) *in* $(\mathbb{R}_+)^2$*.*

Proof: Since B^h is a Gaussian process, Theorem 2.4 shows that B^h fulfils the ϕ-Lipschitz assumption for every integer ρ of the form $\rho := 2p$ where p is a positive integer and for the continuous function ϕ_ρ defined on $\mathbb{R}_+$ by $\phi_\rho(0) := 0$ and $\phi_\rho(x) := (\kappa_{\rho/2})^{1/2\rho}\ \sqrt{M}\ x^{H_1\wedge\frac{\beta\delta}{2}}$. We have denoted, for n in $\mathbb{N}$, κ_n the number such that $\mathsf{E}\left[Y^{2n}\right] = \kappa_n\ \mathsf{E}\left[Y^2\right]^n$ for the centred Gaussian random variable Y. It is clear that ϕ_ρ is regularly varying with index $H_1 \wedge \frac{\beta\delta}{2}$, which is positive. The result then follows from Theorem 2.3. □

Remark 2.6. *Corollary 2.5 extends to every process* $V^h := (V^h_t)_{t\in[0,T]}$*, of the form* $V^h_t := Z(t,h(t))$ *where* $Z := (Z(t,H))_{(t,H)\in\mathbb{R}\times[H_1,H_2]}$ *is a Gaussian field such that one can find* (Λ,γ,δ) *in* $(\mathbb{R}^*_+)^3$ *with*

$$\forall(s,t,H,H') \in [0,T]^2\times[H_1,H_2]^2, \qquad \mathsf{E}\left[\left(Z(t,H)-Z(s,H')\right)^2\right] \leqslant \Lambda\ (|t-s|^\gamma + |H-H'|^\delta),$$

In this case, for every (r,p) *in* $(\mathbb{R}^*_+)^2$*, we get* $\mathcal{E}_{N,r}\left(V^h, |\cdot|_{L^p([0,T])}\right) = O\left(\log(N)^{-(\frac{\gamma}{2}\wedge\frac{\beta\delta}{2})}\right)$*.*

Remark 2.7 (Sharp rate)**.** *We conjecture that the domination of the rate of decay of the quantization error for the mBm established in Corollary 2.5 is in fact a sharp rate and that we have, for every* $(r,p)\in(\mathbb{R}^*_+)^2$*,*

$$\mathcal{E}_{N,r}\left(B^h, |\cdot|_{L^p([0,T])}\right) \sim \frac{K_{h,r,p}}{\log(N)^{H_1\wedge\beta}} \text{ as } N\to\infty \textit{ for some } K_{h,r,p}>0,$$

and that the quadratic optimal product quantization error $\mathcal{E}^{\text{prod}}_N\left(B^h\right)$ *for the mBm satisfies*

$$\mathcal{E}^{\text{prod}}_N\left(B^h\right) \asymp \frac{1}{\log(N)^{H_1\wedge\beta}} \text{ as } N\to\infty,$$

which is consistent with the results of previous section on the fractional Brownian motion.

Proposition 2.8. *The well-balanced mBm* $B^h_t := \frac{1}{c_{h(t)}} \int_{\mathbb{R}} \frac{e^{it\xi}-1}{|u|^{h(t)+1/2}}\ \widetilde{W}(d\xi)$ *satisfies assumption* $(\mathcal{H})$*.*

Proof: One computes:

$$\begin{aligned} I^{H,H'}_t := \mathsf{E}\left[(\mathbf{B}(t,H)-\mathbf{B}(t,H'))^2\right] &= \int_{\mathbb{R}} \left|\frac{e^{it\xi}-1}{c_H|\xi|^{H+1/2}} - \frac{e^{it\xi}-1}{c_{H'}|\xi|^{H'+1/2}}\right|^2 du \\ &= \int_{\mathbb{R}} \left|\frac{e^{it\xi}-1}{\xi}\right|^2 \left|\tfrac{1}{c_H}\ |\xi|^{1/2-H} - \tfrac{1}{c_{H'}}\ |\xi|^{1/2-H'}\right|^2 d\xi. \end{aligned} \tag{2.5}$$

For every ξ in $\mathbb{R}^*$, the map $f_\xi : [c,d]\to\mathbb{R}_+$, defined by $f_\xi(H) := \frac{1}{c_H}\ |\xi|^{1/2-H}$ is C^1 since $H\mapsto c_H$ is C^1 on $(0,1)$. Thus there exists a positive real D such that

$$\forall(\xi,H)\in\mathbb{R}^*\times[c,d],\ |f'_\xi(H)| \leqslant D\ |\xi|^{1/2-H}\ (1+|\ln(|\xi|)|) \leqslant D\ \left(|\xi|^{1/2-c}+|\xi|^{1/2-d}\right)\ (1+|\ln(|\xi|)|).$$

Thanks to the mean-value theorem, (2.5) yields

$$\begin{aligned} I^{H,H'}_t &\leqslant\ D^2\ |H-H'|^2 \int_{\mathbb{R}} \frac{|e^{it\xi}-1|^2}{|\xi|^2}\ \left(|\xi|^{1/2-c}+|\xi|^{1/2-d}\right)^2\ (1+|\ln(|\xi|)|)^2\ d\xi \\ &\leqslant\ D^2\ |H-H'|^2 \left(2^3 \int_{|\xi|>1} \frac{(1+|\ln(|\xi|)|)^2}{|\xi|^{1+2c}}\ d\xi + (2t)^2 \int_{|\xi|\leqslant 1} |\xi|^{1-2d}\ (1+|\ln(|\xi|)|)^2\ d\xi\right) \\ &\leqslant (2^3+T^2)\ D^2\ \left(\int_{|\xi|>1} \frac{(1+|\ln(|\xi|)|)^2}{|\xi|^{1+2c}}\ d\xi + \int_{|\xi|\leqslant 1} |\xi|^{1-2d}\ (1+|\ln(|\xi|)|)^2\ d\xi\right)\ |H-H'|^2. \end{aligned}$$

Since the two integrals in the last line are finite, $(\mathcal{H})$ is verified with $\delta = 2$. □

2.3 Quantization-based cubature

Basic formula and related inequalities in the case of Lipschitz continuous functionals

The idea of quantization-based cubature methods is to approximate the distribution of the random variable X by the distribution of a quantizer Y of X. As Y is a discrete random variable, we have $\mathbb{P}_Y = \sum_{i=1}^{N} p_i \delta_{y_i}$. Therefore, if $F : E \to \mathbb{R}$ is a Borel functional,

$$\mathsf{E}[F(Y)] = \sum_{i=1}^{N} p_i F(y_i). \tag{2.6}$$

Hence, the weighted discrete distribution $(y_i, p_i)_{1 \leqslant i \leqslant N}$ of Y allows us to compute the sum (2.6). We review some error bounds that can be derived when approximating $\mathsf{E}[F(X))]$ by (2.6). See [66] for more details.

1. *If $X \in L^2$, Y a quantizer of X of size N and F is Lipschitz continuous, then*

$$|\mathsf{E}[F(X)] - \mathsf{E}[F(Y)]| \leqslant [F]_{\mathrm{Lip}} \|X - Y\|_2. \tag{2.7}$$

where $[F]_{\mathrm{Lip}}$ is the Lipschitz constant of F. In particular, if $(Y_N)_{N \geqslant 1}$ is a sequence of quantizers such that $\lim_{N \to \infty} \|X - Y_N\|_2 = 0$, then the distribution $\sum_{i=1}^{N} p_i^N \delta_{x_i^N}$ of Y_N weakly converges to the distribution $\mathbb{P}_X$ of X as $N \to \infty$.

2. *If Y is a stationary quantizer of X, i.e. $Y = \mathsf{E}[X|Y]$, and F is differentiable with a Lipschitz continuous derivative DF, then*

$$|\mathsf{E}[F(X)] - \mathsf{E}[F(Y)]| \leqslant [DF]_{\mathrm{Lip}} \|X - Y\|_2^2, \tag{2.8}$$

where $[DF]_{\mathrm{Lip}}$ is the Lipschitz constant of DF. If F is twice differentiable and D^2F is bounded, then we can replace $[DF]_{\mathrm{Lip}}$ by $\frac{1}{2}\|D^2F\|_\infty$.

3. *If F is a semi-continuous convex functional*[4] *and Y is a stationary quantizer of X,*

$$\mathsf{E}[F(Y)] \leqslant \mathsf{E}[F(X)]. \tag{2.9}$$

This is a simple consequence of Jensen's inequality. Indeed,

$$\mathsf{E}[F(Y)] \overset{\text{Stationarity}}{=} \mathsf{E}[F(\mathsf{E}[X|Y])] \overset{\text{Jensen}}{\leqslant} \mathsf{E}[\mathsf{E}[F(X)|Y])] \;=\; \mathsf{E}[F(X)].$$

The case of exponentials of continuous centred Gaussian processes

Let $(X_s)_{s \in [0,T]}$ be a continuous centred Gaussian process on $[0, T]$. Then the covariance function of X is also continuous.
In addition, Fernique's theorem entails that $\mathsf{E}\left[\int_0^T X_s^2 ds\right]$ is finite. We view X as a random variable valued in the separable Banach space $(C([0, T], \mathbb{R}), \|\cdot\|_\infty)$. Let $\widehat{X}$ be a stationary quantizer of X.

By the mean-value theorem, for all $(x, y) \in \mathbb{R}^2$, $|e^x - e^y| \leqslant e^{|x|+|y|}|x - y|$. Consequently, for $p \geqslant 1$, using Hölder's inequality:

$$\begin{aligned}\mathsf{E}\left[\int_0^T \left|e^{X_s} - e^{\widehat{X}_s}\right|^p ds\right]^{1/p} &\leqslant \mathsf{E}\left[\int_0^T e^{p|X_s| + p\left|\widehat{X}_s\right|} \left|X_s - \widehat{X}_s\right|^p ds\right]^{1/p} \\ &\leqslant \mathsf{E}\left[\int_0^T e^{p\tilde{p}|X_s|} e^{p\tilde{p}\left|\widehat{X}_s\right|} ds\right]^{\frac{1}{2p\tilde{p}}} \mathsf{E}\left[\int_0^T \left|X_s - \widehat{X}_s\right|^{p\tilde{q}} ds\right]^{\frac{1}{2p\tilde{q}}},\end{aligned}$$

where $(\tilde{p}, \tilde{q}) \in (1, \infty)^2$ are conjugate exponents. For $\epsilon > 0$, we choose $(\tilde{p}, \tilde{q})$ such that $p\tilde{q} = p + \epsilon$. This gives

$$\tilde{q} = 1 + \epsilon/p \qquad \text{and} \qquad \tilde{p} = 1 + p/\epsilon.$$

4. In the infinite-dimensional case, convexity does not imply continuity. In infinite-dimensional Banach spaces, a semi-continuity hypothesis is necessary for Jensen's inequality. See [80] for more details.

By Schwarz's inequality:

$$\mathsf{E}\left[\int_0^T \left|e^{X_s} - e^{\widehat{X}_s}\right|^p ds\right]^{1/p} \leqslant \mathsf{E}\left[\int_0^T e^{2p\tilde{p}|X_s|} ds\right]^{\frac{1}{2p\tilde{p}}} \mathsf{E}\left[\int_0^T e^{2p\tilde{p}\left|\widehat{X}_s\right|} ds\right]^{\frac{1}{2p\tilde{p}}} \left\|X - \widehat{X}\right\|_{p+\epsilon}.$$

Define the map $\phi : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ by $\phi(f) := \int_0^T e^{2p\tilde{p}|f(s)|} ds$. It is easily shown that ϕ is convex and continuous on $(C([0,T],\mathbb{R}), \|\cdot\|_\infty)$. Hence, Inequality (2.9) yields

$$\mathsf{E}\left[\int_0^T e^{2p\tilde{p}\left|\widehat{X}_s\right|} ds\right] \leqslant \mathsf{E}\left[\int_0^T e^{2p\tilde{p}|X_s|} ds\right].$$

Finally

$$\mathsf{E}\left[\int_0^T \left|e^{X_s} - e^{\widehat{X}_s}\right|^p ds\right]^{1/p} \leqslant \underbrace{\mathsf{E}\left[\int_0^T e^{2p\tilde{p}|X_s|} ds\right]^{\frac{1}{p\tilde{p}}}}_{<\infty} \left\|X - \widehat{X}\right\|_{p+\epsilon}. \tag{2.10}$$

We shall apply (2.10) with $p = 2 - \epsilon$ in section 4: this will allow us to control the $L^{2-\epsilon}$ quantization error of the exponential of a centred continuous Gaussian process X by the L^2 quantization of X.

Richardson-Romberg extrapolation

With respect to the quantization error

In the general setting of a non-uniform random variable X, a quadratic optimal N-quantizer Y_N of X and a C^1 functional with Lipschitz continuous derivative, Equation (2.8) does not provide a true asymptotic expansion which would allow one to use a Richardson-Romberg expansion, but it suggests the use a higher-order Taylor expansion of $F(X) - F(Y_N)$ to get one.
It follows from Taylor's formula that there exists a vector $\zeta \in [X, Y_N]$ such that

$$\begin{aligned}\mathsf{E}[F(X)] &= \mathsf{E}\left[F(Y_N)\right] + \underbrace{\mathsf{E}\left[\langle DF(Y_N), X - Y_N\rangle\right]}_{=\mathsf{E}\left[DF(Y_N).\mathsf{E}[X-Y_N|Y_N]\right]=0 \text{ by stationarity.}} + \tfrac{1}{2}\mathsf{E}\left[D^2F(Y_N)(X-Y_N)^{\otimes 2}\right] \\ &\quad + \tfrac{1}{6}\mathsf{E}\left[\zeta(X-Y_N)^{\otimes 3}\right] + o\left(\mathsf{E}\left[|X-Y_N|^3\right]\right) \\ &= \mathsf{E}\left[F(Y_N)\right] + \tfrac{1}{2}\mathsf{E}\left[D^2F(Y_N)(X-Y_N)^{\otimes 2}\right] + O\left(\mathsf{E}\left[|X-Y_N|^3\right]\right).\end{aligned} \tag{2.11}$$

In [37], it is proved that the asymptotics of the L^s quantization error induced by a sequence of L^r-optimal quantizers remains rate-optimal in the case of probability distributions on $\mathbb{R}^d$, with $s < r + d$ for a class of distributions including the Gaussian distribution. This leads to $\mathsf{E}\left[|X - Y_N|^3\right] = O\left(\mathsf{E}\left[|X - Y_N|^2\right]^{\frac{3}{2}}\right)$. This holds *e.g.* for Brownian motion.
Unfortunately, no sharp equivalence between $\|X - Y_N\|_2^2$ and $\mathsf{E}\left[D^2F(Y_N)(X - Y_N)^{\otimes 2}\right]$ has been established yet. Still, Equation (2.11) suggests to use a Richardson-Romberg extrapolation with respect to the quantization error $\mathsf{E}_N^2 := \|X - Y_N\|_2$. The two-steps extrapolation between $N = k$ and $N = l$ leads to approximate $\mathsf{E}[F(X)]$ by the quantity

$$\frac{\mathsf{E}[F(Y_l)]\mathsf{E}_k^2 - \mathsf{E}[F(Y_k)]\mathsf{E}_l^2}{\mathsf{E}_k^2 - \mathsf{E}_l^2}. \tag{2.12}$$

Although this kind of Richardson-Romberg extrapolation has not received a full theoretical justification yet, it does dramatically increase the efficiency of quantization-based cubature formulas.

With respect to the quantization level

When the value of E_k^2 is not known, one may rely on an asymptotic expansion with respect to the quantization level.

Remark 2.9 (Romberg extrapolation with respect to the quantization level). *In Section 2.2, we have seen that under some assumptions on the eigenvalues of the convergence operator, the rate of convergence of optimal quantizers and K-L optimal product quantizers is* $(\ln(N)^{-\alpha})$ *for some* $\alpha \in (0,1)$. *Replacing the distortion* E_N *by its asymptotics* $\frac{1}{\ln(N)^\alpha}$ *as* $N \to \infty$ *in Equation (2.12) suggests to approximate* $\mathsf{E}[F(X)]$ *by the quantity*

$$\frac{\mathsf{E}[F(Y_l)](\ln l)^{2\alpha} - \mathsf{E}[F(Y_k)](\ln k)^{2\alpha}}{(\ln l)^{2\alpha} - (\ln k)^{2\alpha}}. \tag{2.13}$$

Multidimensional Richardson-Romberg extrapolation

Let X^1 and X^2 be two independent random variables. We wish to estimate the expectation $\mathsf{E}\left[F\left(X^1, X^2\right)\right]$ for some regular functional F. In that view, one may use a cubature based on a product quantization $\left(\widehat{X}^1, \widehat{X}^2\right)$ of (X^1, X^2), and perform a multidimensional Richardson-Romberg extrapolation. This amounts to performing two Richardson-Romberg extrapolations as described already, one related to the quantization error of X^1 between quantization levels N_1 and M_1, and one related to the quantization error of X^2 between quantization levels N_2 and M_2. This leads to approximating $\mathsf{E}[F(X^1, X^2)]$ by the quantity

$$\frac{\mathsf{E}^2_{M_1}\mathsf{E}^2_{M_2}F^{N_1,N_2} - \mathsf{E}^2_{N_1}\mathsf{E}^2_{M_2}F^{M_1,N_2} - \mathsf{E}^2_{M_1}\mathsf{E}^2_{N_2}F^{N_1,M_2} + \mathsf{E}^2_{N_1}\mathsf{E}^2_{N_2}F^{M_1,M_2}}{\left(\mathsf{E}^2_{M_1} - \mathsf{E}^2_{N_1}\right)\left(\mathsf{E}^2_{M_2} - \mathsf{E}^2_{N_2}\right)}, \tag{2.14}$$

where $F^{p,q}$ denotes the estimated expectation obtained with the quantization-based cubature and quantization levels of p and q for X^1 and X^2 respectively. In other words, $F^{p,q}$ is defined by

$$F^{p,q} := \mathsf{E}\left[F\left(\widehat{X^1}^p, \widehat{X^2}^q\right)\right]$$

where $\widehat{X^1}^p$, $\widehat{X^2}^q$ are quantizers of levels p and q for X^1 and X^2 respectively. In Equation (2.14), E_{M_i} and E_{N_i} denote the quadratic quantization error of level M_i and N_i for X_i.

3 Stochastic calculus with respect to mBm

From now on and until the end of the work, we fix our mBm to be the well-balanced multifractional Brownian motion defined in Section 1. In addition, we will always assume that h is a C^1 function with derivative bounded on $\mathbb{R}$.

3.1 Some backgrounds on white noise theory

Define the probability space as $\Omega := \mathscr{S}'(\mathbb{R})$ and let $\mathcal{F} := \mathbb{B}(\mathscr{S}'(\mathbb{R}))$ be the σ-algebra of Borel sets.
There exists a probability measure μ such that, for every f in $L^2(\mathbb{R})$, the map $\langle \cdot, f\rangle : \Omega \to \mathbb{R}$ defined by $\langle \cdot, f\rangle(\omega) = \langle \omega, f\rangle$ (where $\langle \omega, f\rangle$ is by definition $\omega(f)$, *i.e* the action of the distribution ω on the function f) is a centred Gaussian random variable with variance equal to $\|f\|^2_{L^2(\mathbb{R})}$ under μ. For every n in $\mathbb{N}$, denote $e_n(x) := (-1)^n \pi^{-1/4}(2^n n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2})$ the nth Hermite function. Let $(|\cdot|_p)_{p\in\mathbb{Z}}$ be the family of norms defined by $|f|^2_p := \sum_{k=0}^{+\infty}(2k+2)^{2p}\langle f, e_k\rangle^2_{L^2(\mathbb{R})}$, for all (p, f) in $\mathbb{Z} \times L^2(\mathbb{R})$. The operator A defined on $\mathscr{S}(\mathbb{R})$ by $A := -\frac{d^2}{dx^2} + x^2 + 1$ admits the sequence $(e_n)_{n\in\mathbb{N}}$ as eigenfunctions and the sequence $(2n+2)_{n\in\mathbb{N}}$ as eigenvalues.
As is customary, we denote (L^2) the space $L^2(\Omega, \mathcal{G}, \mu)$ where $\mathcal{G}$ is the σ-field generated by $(\langle \cdot, f\rangle)_{f\in L^2(\mathbb{R})}$. For every random variable Φ of (L^2) there exists, according to the Wiener-Itô theorem, a unique sequence $(f_n)_{n\in\mathbb{N}}$ of functions f_n in $\widehat{L}^2(\mathbb{R}^n)$ such that Φ can be decomposed as $\Phi = \sum_{n=0}^{+\infty} I_n(f_n)$, where $\widehat{L}^2(\mathbb{R}^n)$ denotes the set of all symmetric functions f in $L^2(\mathbb{R}^n)$ and $I_n(f)$ denotes the nth multiple Wiener-Itô integral of f with the convention that $I_0(f_0) = f_0$ for constants f_0. Moreover we have the equality $\mathsf{E}[\Phi^2] = \sum_{n=0}^{+\infty} n!\,\|f_n\|^2_{L^2(\mathbb{R}^n)}$

where E denotes the expectation with respect to μ. For any $\Phi := \sum_{n=0}^{+\infty} I_n(f_n)$ satisfying the condition $\sum_{n=0}^{+\infty} n!\ |A^{\otimes n} f_n|_0^2 < +\infty$, define the element $\Gamma(A)(\Phi)$ of (L^2) by $\Gamma(A)(\Phi) := \sum_{n=0}^{+\infty} I_n(A^{\otimes n} f_n)$, where $A^{\otimes n}$ denotes the nth tensor power of the operator A (see [47, Appendix E] for more details about tensor products of operators). The operator $\Gamma(A)$ is densely defined on (L^2). It is invertible and its inverse $\Gamma(A)^{-1}$ is bounded. Let us denote $\|\varphi\|_0^2 := \|\varphi\|_{(L^2)}^2$ for φ in (L^2) and let $\mathbb{D}\mathrm{om}(\Gamma(A)^n)$ be the domain of the nth iteration of $\Gamma(A)$. Define the family of norms $(\|\cdot\|_p)_{p\in\mathbb{Z}}$ by:

$$\|\Phi\|_p := \|\Gamma(A)^p\Phi\|_0 = \|\Gamma(A)^p\Phi\|_{(L^2)}, \qquad \forall p \in \mathbb{Z}, \quad \forall \Phi \in (L^2) \cap \mathbb{D}\mathrm{om}(\Gamma(A)^p).$$

For any p in $\mathbb{N}$, let $(\mathcal{S}_p) := \{\Phi \in (L^2) : \Gamma(A)^p\Phi \text{ exists and belongs to } (L^2)\}$ and define $(\mathcal{S}_{-p})$ as the completion of the space (L^2) with respect to the norm $\|\cdot\|_{-p}$. As in [52], we let $(\mathcal{S})$ denote the projective limit of the sequence $((\mathcal{S}_p))_{p\in\mathbb{N}}$ and $(\mathcal{S})^*$ the inductive limit of the sequence $((\mathcal{S}_{-p}))_{p\in\mathbb{N}}$. The space $(\mathcal{S})$ is called the space of stochastic test functions and $(\mathcal{S})^*$ the space of Hida distributions. One can show that, for any p in $\mathbb{N}$, the dual space $(\mathcal{S}_p)^*$ of $\mathcal{S}_p$ is $(\mathcal{S}_{-p})$. Thus we will write $(\mathcal{S}_{-p})$, in the sequel, to denote the space $(\mathcal{S}_p)^*$. Note also that $(\mathcal{S})^*$ is the dual space of $(\mathcal{S})$. We will note $\langle\langle\cdot,\cdot\rangle\rangle$ the duality bracket between $(\mathcal{S})^*$ and $(\mathcal{S})$. If Φ belongs to (L^2) then we have the equality $\langle\langle\Phi,\varphi\rangle\rangle = \langle\Phi,\varphi\rangle_{(L^2)} = \mathsf{E}[\Phi\ \varphi]$. Since we have defined a topology given by a family of norms on the space $(\mathcal{S})^*$ it is possible to define a derivative and an integral in $(\mathcal{S})^*$. A function $\Phi : \mathbb{R} \to (\mathcal{S})^*$ is called a stochastic distribution process, or an $(\mathcal{S})^*$-process, or a Hida process.

The Hida process Φ is said to be differentiable at t_0 if $\lim_{r\to 0} r^{-1}\ (\Phi(t_0+r) - \Phi(t_0))$ exists in $(\mathcal{S})^*$. Moreover we may also define an integral of an Hida process:

Theorem 3.1 (Integral in $(\mathcal{S})^*$)**.** *Assume that $\Phi : \mathbb{R} \to (\mathcal{S})^*$ is weakly in $L^1(\mathbb{R}, dt)$, i.e. assume that for all φ in $(\mathcal{S})$, the mapping $u \mapsto \langle\langle\Phi(u),\varphi\rangle\rangle$ from $\mathbb{R}$ to $\mathbb{R}$ belongs to $L^1(\mathbb{R}, dt)$. Then, there exists a unique element in $(\mathcal{S})^*$, denoted by $\int_{\mathbb{R}} \Phi(u)du$, such that*

$$\left\langle\!\!\left\langle \int_{\mathbb{R}} \Phi(u)du, \varphi \right\rangle\!\!\right\rangle = \int_{\mathbb{R}} \langle\langle\Phi(u),\varphi\rangle\rangle\, du \quad \text{for all } \varphi \text{ in } (\mathcal{S}). \tag{3.1}$$

One says that Φ is $(\mathcal{S})^*$-integrable on $\mathbb{R}$ or integrable on $\mathbb{R}$ in the *Pettis* sense.

For every f in $L^2(\mathbb{R})$, define the *Wick exponential* of $\langle\cdot,f\rangle$, noted $: e^{\langle\cdot,f\rangle} :$, as the (L^2) random variable equal to $e^{\langle\cdot,f\rangle - \frac{1}{2}|f|_0^2}$. The S-transform of an element Φ of $(\mathcal{S}^*)$, noted $S(\Phi)$, is defined to be the function from $\mathscr{S}(\mathbb{R})$ to $\mathbb{R}$ given by $S(\Phi)(\eta) := \left\langle\!\left\langle \Phi, : e^{\langle\cdot,\eta\rangle} \right\rangle\!\right\rangle$ for every η in $\mathscr{S}(\mathbb{R})$. Finally for every $(\Phi,\Psi) \in (\mathcal{S})^* \times (\mathcal{S})^*$, there exists a unique element of $(\mathcal{S})^*$, called the Wick product of Φ and Ψ and noted $\Phi \diamond \Psi$, such that $S(\Phi \diamond \Psi)(\eta) = S(\Phi)(\eta)\ S(\Psi)(\eta)$; for every η in $\mathscr{S}(\mathbb{R})$.

The map $S : \Phi \mapsto S(\Phi)$, from $(\mathcal{S})^*$ to $(\mathcal{S})^*$, is injective. Furthermore, let $\Phi : \mathbb{R} \to (\mathcal{S})^*$ be a fixed $(\mathcal{S})^*$ process. If Φ is $(\mathcal{S})^*$-integrable over $\mathbb{R}$ then for all η in $\mathscr{S}(\mathbb{R})$, $S(\int_{\mathbb{R}} \Phi(u)\ du)(\eta) = \int_{\mathbb{R}} S(\Phi(u))(\eta)\ du$. If Φ is $(\mathcal{S})^*$-differentiable over $\mathbb{R}$ then for all η in $\mathscr{S}(\mathbb{R})$, $S[\frac{d\Phi}{dt}(t)](\eta) = \frac{d}{dt}\Big[[S\Phi(t)](\eta)\Big]$.

For any Φ in $(\mathcal{S})^*$ and k in $\mathbb{N}^*$, let $\Phi^{\diamond k}$ denote the element $\overbrace{\Phi \diamond \cdots \diamond \Phi}^{k \text{ times}}$ of $(\mathcal{S})^*$. One can generalize the definition of $\exp^{\diamond}$ to the case where Φ belongs to $(\mathcal{S})^*$. Indeed, for any Φ in $(\mathcal{S})^*$ such that the sum $\sum_{k=0}^{+\infty} \frac{\Phi^{\diamond k}}{k!}$ converges in $(\mathcal{S})^*$, define the element $\exp^{\diamond}\Phi$ of $(\mathcal{S})^*$ by setting $\exp^{\diamond}\Phi := \sum_{k=0}^{+\infty} \frac{\Phi^{\diamond k}}{k!}$. It is called Wick exponential of Φ.

For f in $L^2(\mathbb{R})$ and $\Phi := \langle\cdot,f\rangle$, it is easy to verify that $\exp^{\diamond}\Phi$ exists and coincides with $: e^{\langle\cdot,f\rangle} :$ defined at the beginning of this section.

Fractional and multifractional White noise

Operators M_H and $\frac{\partial M_H}{\partial H}$.

Let H belong to $(0,1)$. Following [34], the operator M_H is defined in the Fourier domain by

$$\widehat{M_H(u)}(y) := \frac{\sqrt{2\pi}}{c_H}\,|y|^{1/2-H}\widehat{u}(y), \qquad \forall y \in \mathbb{R}^*.$$

This operator is well defined on the homogeneous Sobolev space of order $1/2-H$ noted $L^2_H(\mathbb{R})$ and defined by $L^2_H(\mathbb{R}) := \{u \in \mathcal{S}'(\mathbb{R}) \ : \ \widehat{u} = T_f;\ f \in L^1_{loc}(\mathbb{R}) \text{ and } \|u\|_H < +\infty\}$. The norm $\|\cdot\|_H$ derives from the inner product $\langle \cdot,\cdot\rangle_H$ defined on $L^2_H(\mathbb{R})$ by: $\langle u, v\rangle_H := \frac{1}{c_H^2}\int_{\mathbb{R}} |\xi|^{1-2H}\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi$ where c_H is defined right after Definition 1.1.
The definition of the operator $\frac{\partial M_H}{\partial H}$ is quite similar [53]. Precisely, define, for H in $(0,1)$, the space $\Gamma_H(\mathbb{R}) := \{u \in \mathcal{S}'(\mathbb{R}) \ : \ \widehat{u} = T_f;\ \ f \in L^1_{loc}(\mathbb{R}) \ \text{ and } \|u\|_{\delta_H(\mathbb{R})} < +\infty\}$, where the norm $\|\cdot\|_{\delta_H(\mathbb{R})}$ derives from the inner product $\langle\cdot,\cdot\rangle_{\delta_H}$ defined on $\Gamma_H(\mathbb{R})$ by $\langle u, v\rangle_{\delta_H} := \frac{1}{c_H^2}\int_{\mathbb{R}}(\beta_H + \ln|\xi|)^2\ |\xi|^{1-2H}\ \widehat{u}(\xi)\ d\xi$.
The operator $\frac{\partial M_H}{\partial H}$, from $\left(\Gamma_H(\mathbb{R}), \langle\cdot,\cdot\rangle_{\delta_H(\mathbb{R})}\right)$ to $\left(L^2(\mathbb{R}), \langle\cdot,\cdot\rangle_{L^2(\mathbb{R})}\right)$, is defined in the Fourier domain by:

$$\widehat{\tfrac{\partial M_H}{\partial H}(u)}(y) := -(\beta_H + \ln|y|)\ \frac{\sqrt{2\pi}}{c_H}\ |y|^{1/2-H}\ \widehat{u}(y), \qquad \forall y \in \mathbb{R}^*.$$

Fractional and multifractional White noise.

For any measurable function $h : \mathbb{R} \to (0,1)$, it is easily seen that the process $B^h := \left(B^h_t\right)_{t\in\mathbb{R}}$ defined by

$$\forall(\omega,t) \in \Omega\times\mathbb{R}, \quad B^h_t := \sum_{k=0}^{+\infty}\left(\int_0^t M_{h(t)}(e_k)(s)ds\right)\langle\cdot, e_k\rangle$$

is an mBm. Assuming that h is differentiable, we define the $(\mathcal{S})^*$-valued function $W^h := (W^h_t)_{t\in\mathbb{R}}$ by

$$W^h_t := \sum_{k=0}^{+\infty}\left[\frac{d}{dt}\Big(\int_0^t \ M_{h(t)}(e_k)(s)\ ds\Big)\right]\ \langle\cdot, e_k\rangle. \tag{3.2}$$

The following theorem states that, for all real t, the right-hand side of (3.2) does indeed belong to $(\mathcal{S})^*$ and is exactly the $(\mathcal{S})^*$-derivative of B^h at point t.

Theorem-Definition 3.1 ([53, Theorem-definition 5.1])**.** *Let $h : \mathbb{R} \to (0,1)$ be a C^1 deterministic function such that its derivative function h' is bounded. The process W^h defined by (3.2) is an $(\mathcal{S})^*$-process which verifies the following equality (in $(\mathcal{S})^*$):*

$$W^h_t = \sum_{k=0}^{+\infty} M_{h(t)}(e_k)(t)\langle\cdot, e_k\rangle + \ h'(t)\ \sum_{k=0}^{+\infty}\Big(\int_0^t \frac{\partial M_H}{\partial H}(e_k)(s)\big|_{H=h(t)}ds\Big)\langle\cdot, e_k\rangle. \tag{3.3}$$

Moreover the process B^h is $(\mathcal{S})^*$-differentiable on $\mathbb{R}$ and verifies $\frac{dB^h}{dt}(t) = W^h_t$ in $(\mathcal{S})^*$.

When the function h is constant, identically equal to H, we will denote $W^H := \left(W^H_t\right)_{t\in\mathbb{R}}$ and call the process W^h fractional white noise. This process was defined and studied in [34, 13].

3.2 Stochastic integral with respect to mBm

We recall the definition of the Wick-Itô stochastic integral with respect to mBm from [53]:

Definition 3.1 (Multifractional Wick-Itô integral)**.** *Let B^h be a normalized multifractional Brownian motion and $Y : \mathbb{R} \to (\mathcal{S})^*$ be a process such that the process $t \mapsto Y_t \diamond W^h_t$ is $(\mathcal{S})^*$-integrable on $\mathbb{R}$. The process Y is* *said to be $d^\diamond B^h$-integrable on $\mathbb{R}$ or integrable on $\mathbb{R}$ with respect to mBm B^h. Moreover, the integral on $\mathbb{R}$ of Y with respect to B^h is defined by*

$$\int_{\mathbb{R}} Y_s \ d^\diamond B_s^h := \int_{\mathbb{R}} Y_s \diamond W_s^h \ ds. \tag{3.4}$$

For an interval I of $\mathbb{R}$, $\int_I Y_s \ d^\diamond B_s^h := \int_{\mathbb{R}} \ \mathbf{1}_I(s) \ Y_s \ d^\diamond B_s^h$.

When the function h is constant over $\mathbb{R}$, equal to H, the multifractional Wick-Itô integral coincides with the fractional Wick-Itô integral defined in [34], [13], [?] and [?]. In particular, when Y is adapted and when the function h is identically equal to $1/2$, (3.4) is nothing but the classical Itô integral with respect to Brownian motion. The multifractional Wick-Itô integral verifies the following properties:

Proposition 3.2. *Let B^h be an mBm and I be an interval of $\mathbb{R}$.*
- *For all (a,b) in $\mathbb{R}^2$ such that $a < b$, $\int_a^b \ 1 \ d^\diamond B_u^h = B_b^h - B_a^h$ almost surely.*
- *Let $X : I \to (\mathcal{S})^*$ be a $d^\diamond B^h$-integrable process over I. If $\int_I X_s \ d^\diamond B_s^h$ belongs to (L^2), then $\mathsf{E}[\int_I X_s \ d^\diamond B_s^h] = 0$.*

Multifractional Wick-Itô integral of deterministic elements

In order to solve differential equations driven by an mBm that will be encountered below, it is necessary to know the exact nature of multifractional Wick-Itô integrals of deterministic elements.
For H in $(0,1)$ and f in $\mathscr{S}(\mathbb{R})$, define the function $g_f : \mathbb{R} \times (0,1) \to \mathbb{R}$ by $g_f(t,H) := \int_0^t M_H(f)(x)dx$ where M_H is the operator defined in at the beginning of Section 3.1. It has been shown that g_f belongs to $C^\infty(\mathbb{R} \times (0,1), \mathbb{R})$ (*cf.* [53, Lemma 5.5]). The main result on the multifractional Wick-Itô integral of deterministic elements is the following:

Theorem 3.3. *([53, Theorem 5.25]) Let $h : \mathbb{R} \to (0,1)$ be a C^1 deterministic function and let $f : \mathbb{R} \to \mathbb{R}$ be a measurable deterministic function which belongs to $L^1_{\text{loc}}(\mathbb{R})$. Let $Z := (Z_t)_{t \in \mathbb{R}}$ be the process defined by $Z_t := \int_0^t f(s) \ d^\diamond B_s^h$. Then Z is an $(\mathcal{S})^*$-process which verifies the following equality in $(\mathcal{S})^*$*

$$\int_0^t f(s) \ d^\diamond B_s^h = \sum_{k=0}^{+\infty} \left(\int_0^t f(s) \ \tfrac{d}{ds}[g_{e_k}(s, h(s))] \ ds \right) \langle \cdot, e_k \rangle. \tag{3.5}$$

Moreover Z is a (centred) Gaussian process if and only if $\sum_{k=0}^{+\infty} \left(\int_0^t f(s) \tfrac{d}{ds}[g_{e_k}(s,h(s))] \ ds \right)^2 < +\infty$, for all t in $\mathbb{R}$. In this case we have, for every t in $\mathbb{R}$,

$$Z_t = \int_0^t f(s) \ d^\diamond B_s^h \overset{\mathcal{L}}{\sim} \mathcal{N}\left(0, \sum_{k=0}^{+\infty} \left(\int_0^t f(s) \ \tfrac{d}{ds}[g_{e_k}(s,h(s))] \ ds \right)^2 \right). \tag{3.6}$$

In particular, Z is a Gaussian process when f belongs to $C^1(\mathbb{R};\mathbb{R})$.

Deriving the quantity $\mathsf{E}\left[Z_t^2\right]$ in the previous theorem might be complicated using Equation (3.6). However, when f is a C^1 function, thanks to the Itô formula with respect to mBm [53, Theorem 6.9], we obtain the following integration-by-parts formula

$$\int_0^t \ f(s) \ d^\diamond B_s^h \overset{(L^2)}{=} f(t) \ B_t^h - \int_0^t \ f'(s) \ B_s^h \ ds, \tag{3.7}$$

which leads to

$$\mathsf{E}\left[Z_t^2\right] = f(t)^2 \ t^{2h(t)} \ + \int_0^t \int_0^t \ f'(s) \ f'(u) \ R_h(s,u) \ ds \ du - 2f(t) \int_0^t \ f'(s) \ R_h(t,s) \ ds. \tag{3.8}$$

Remark 3.4. *The integration-by-parts formula (3.7) allows to identify almost surely $\int_0^t \ f(s) \ d^\diamond B_s^h$ with the quantity $I_t^f(B^h)$ where the map $I_t^f : C^0([0,t];\mathbb{R}) \to \mathbb{R}$ is defined by*

$$I_t^f : g \mapsto \left(f(t)g(t) - \int_0^t \ f'(s)g(s) \ ds \right). \tag{3.9}$$

3.3 Stochastic differential equations

Mixed multifractional Brownian S.D.E.

Let us consider the following mixed multifractional stochastic differential equation, where γ_1 and γ_2 are positive constants and B_t is a Brownian motion:

$$\begin{cases} dX_t = X_t \left(\gamma_1 d^\diamond B_t + \gamma_2 d^\diamond B_t^h \right), \\ X_0 = x_0 \in \mathbb{R}. \end{cases} \tag{3.10}$$

Of course (3.10) is a shorthand notation for the equation

$$X_t = x_0 + \gamma_1 \int_0^t X_s d^\diamond B_s + \gamma_1 \int_0^t X_s d^\diamond B_s^h, \qquad X_0 = x_0 \in \mathbb{R},$$

where the previous equality holds in $(\mathcal{S})^*$. A solution of this equation will be called *geometric mixed multifractional Brownian motion*. Rewriting the previous equation in terms of derivatives in $(\mathcal{S})^*$, we get:

$$\frac{dX_t}{dt} = X_t \diamond \left(\gamma_1 W_t^{1/2} + \gamma_2 W_t^h \right), \qquad x_0 \in \mathbb{R}. \tag{3.11}$$

Theorem 3.5 (Geometric mixed multifractional Brownian motion)**.** *The $(\mathcal{S})^*$-process $(X_t)_{t\in[0,T]}$ defined by*

$$X_t := x_0 \exp^\diamond \left(\gamma_1 B_t + \gamma_2 B_t^h \right), \tag{3.12}$$

is the unique solution of (3.11) *in* $(\mathcal{S})^*$.

Proof: Applying the S-transform to both sides of Equation (3.11) and denoting by y_η the map $t \mapsto S(X_t)(\eta)$, for every η in $\mathscr{S}(\mathbb{R})$, we get:

$$y_\eta'(t) = y_\eta(t) \left(\gamma_1 M_{1/2}(\eta)(t) + \gamma_2 \frac{d}{dt}[g_\eta(t, h(t))] \right), \qquad y_\eta(0) = x_0.$$

This equation admits a unique solution which verifies $y_\eta(t) = x_0 \exp\{\gamma_1 \int_0^t M_{1/2}(\eta)(u)du + \gamma_2 \int_0^t \frac{d}{du}[g_\eta(u, h(u))]du\}$. Using (i) and (ii) of [53, Theorem 5.12] we hence get, for every η in $\mathscr{S}(\mathbb{R})$,

$$y_\eta(t) = x_0 \exp\{\gamma_1 S(B_t)(\eta) + \gamma_2 S(B_t^h)(\eta)\} = S\left(x_0 \exp^\diamond \{\gamma_1 B_t + \gamma_2 B_t^h\} \right)(\eta).$$

The injectivity of the S-transform allows us to conclude that $X_t = x_0 \exp^\diamond \left\{ \gamma_1 B_t + \gamma_2 B_t^h \right\}$ for every t in $[0, T]$. □

Remark 3.6. *(i) Using [47, Equality* (3.16)*], one sees that X is an (L^2)-valued process that may be represented as:*

$$X_t = x_0 \exp\left\{ \gamma_1 B_t + \gamma_2 B_t^h - \tfrac{1}{2}\left(\gamma_1^2 t + \gamma_2^2 t^{2h(t)} \right) \right\}.$$

(ii) The theorem is also a consequence of [46, Theorem 3.1.2*].*

Mixed multifractional Ornstein-Uhlenbeck S.D.E.

Let us now consider the following mixed stochastic differential equation:

$$\begin{cases} dU_t = \theta(\mu - U_t)dt + (\alpha_1 \ d^\diamond B_t + \alpha_2 \ d^\diamond B_t^h) \\ U_0 = u_0 \in \mathbb{R}, \end{cases} \tag{3.13}$$

where $(B_t)_{t\in\mathbb{R}}$ and $\left(B_t^h\right)_{t\in\mathbb{R}}$ are independent, $\theta \geqslant 0$ and μ, α_1, α_2 belong to $\mathbb{R}$. A solution of this equation will be called a *mixed multifractional Ornstein-Uhlenbeck process.*

Theorem 3.7 (Mixed multifractional Ornstein-Uhlenbeck process). *The $L^2(\Omega)$-valued process $(U_t)_{t\in\mathbb{R}}$ defined by*

$$U_t := u_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \alpha_1 \int_0^t e^{\theta(s-t)} d^\diamond B_s + \alpha_2 \int_0^t e^{\theta(s-t)} d^\diamond B_s^h, \tag{3.14}$$

is the unique solution of the stochastic differential equation (3.13).

Proof: The proof that the process U defined by (3.14) is the unique solution of (3.13) is very similar to the one of Theorem 3.5. Indeed, setting $y_\eta(t) := S(U_t)(\eta)$ for every (t, η) in $\mathbb{R}\times\mathbb{R}$ and applying the S-transform to both sides of (3.13) we get, for every η in $\mathscr{S}(\mathbb{R})$, the ordinary differential equation

$$y'_\eta(t) = \theta(\mu - y_\eta(t)) + \alpha_1 M_{1/2}(\eta)(t) + \alpha_2 \tfrac{d}{dt}[g_\eta(t, h(t))], \qquad y_\eta(0) = u_0. \tag{3.15}$$

Its unique solution is

$$y_\eta(t) = u_0 e^{-\theta t} + e^{-\theta t} \int_0^t e^{\theta s}\left(\theta\mu + \alpha_1 M_{1/2}(\eta)(s) + \alpha_2 \tfrac{d}{ds}[g_\eta(s, h(s))]\right) ds, \qquad y_\eta(0) = u_0.$$

Again, one concludes using the injectivity of the S-transform. □

3.4 Quantization of solutions of S.D.E. driven by mBm

Quantizing a Gaussian process X often yields as well a quantization of the solutions of stochastic differential equations driven by X: indeed, in many cases, these solutions may be expressed as functionals of X. A quantizer of the solution can then be obtained by simply plugging the quantizer of X into the functional.

In the one-dimensional setting, under rather general conditions on the diffusion coefficients and if X is a continuous semimartingale, this functional is easily determined using the Lamperti transform (see [58]). In this case, the corresponding quantizer of the stochastic differential equation is obtained by plugging the Gaussian quantizer in the S.D.E. written in the Stratonovich sense, leading to a finite set of ordinary differential equations. This leads to a simple and general constructive method to build a functional quantization of the solution of an S.D.E.

Unfortunately, no such result is available in the case of an S.D.E. driven by multifractional Brownian motion (or even by fractional Brownian motion). However, in some situations, and in particular when an explicit solution is known, one may sometimes still use the procedure just described: if the functional giving the solution is regular enough, quantization-based cubatures can then be used. This is for instance the case of geometric mixed multifractional Brownian motion defined in Section 3.3, which is a simple functional of a Brownian motion and a multifractional Brownian motion (see Remark 3.6 and section 2.3). We describe two other favourable situations in the next subsections.

The case of a Wiener integral

An easy case is the one of a Wiener integral $\int_0^t f(s) d^\diamond B_s^h$ where f is a C^1 deterministic function. The integration-by-parts formula for mBm (3.7) reads $\int_0^t f(s) d^\diamond B_s^h \overset{a.s.}{=} f(t) B_t^h - \int_0^t f'(s) B_s^h ds$. Thus, for $p \geqslant 1$, the stochastic process $t \mapsto \int_0^t f(s) d^\diamond B_s^h$, seen as a random variable valued in $L^p(0,T)$, is the image of B^h by the map

$$\begin{array}{rccc} J^f: & L^p([0,T]) & \to & L^p([0,T]) \\ & g & \mapsto & f(\cdot)g(\cdot) - \int_0^\cdot f'(s)g(s)ds. \end{array}$$

In other words we have (dt-almost everywhere) $J^f(g)(t) = I_t^f(g)$ where I_t^f was defined in Remark 3.4.

Proposition 3.8 (L^p-regularity of the Wiener map). *For every $p \geqslant 1$, the map J^f is Lipschitz continuous on $L^p([0,T])$.*

Proof: It is straightforward that for $(g_1, g_2) \in L^p([0,T])^2$

$$\left\|J^f(g_1) - J^f(g_2)\right\|_p \leqslant \|f(g_1 - g_2)\|_p + \left\|\int_0^\cdot f'(s)(g_1(s) - g_2(s))ds\right\|_p \leqslant (\|f\|_\infty + \|f'\|_\infty T)\,\|g_1 - g_2\|_p\,.$$

□

In Appendix VIII, we prove that if h is C^1, the Karhunen-Loève eigenfunctions of a well-balanced mBm B^h have bounded variations, and thus stationary quantizers of B^h have bounded variations as well (because they lie on a subspace of $L^2([0,T])$ spanned by a finite number of Karhunen-Loève eigenfunctions, as already mentioned). In this setting, another integration by parts gives $I_t^f\left(\widehat{B}^h\right) = \int_0^t f(s)d\widehat{B}_s^h$ where $d\widehat{B}_s^h(\omega)$ stands for the signed measure associated with the function of bounded variation $s \mapsto \widehat{B}_s^h(\omega)$.

The case of certain simple diffusions

Another easy case is the one of an S.D.E. of the form

$$Y_t = y_0 + \int_0^t \beta(s, Y_s)ds + X_t, \tag{3.16}$$

where $\beta(s,y)$ is assumed to be Lipschitz continuous in y uniformly in s. This setting is addressed in [58, p. 20-21] [5], where the authors consider the associated integral equation

$$y(t) = y_0 + \int_0^t \beta(s, y(s))ds + g(t), \tag{3.17}$$

where $g \in L^p([0,T])$ is fixed. The existence and uniqueness in $L^p([0,T])$ of a solution for the integral equation (3.17) follows from the same approach used for ordinary differential equations. Then the solution of the associated S.D.E. (3.16) simply reads $U_t = \Psi_p^\beta(X)_t$, where $\Psi_p^\beta : L^p([0,T]) \to L^p([0,T])$ is the functional that maps $g \in L^p([0,T])$ to the unique solution in $L^p([0,T])$ of Equation (3.17). In [58], the map Ψ_p^β is showed to be Lipschitz continuous in $L^p([0,T])$. More precisely, one has

$$c([\beta]_{\mathrm{Lip}}, T)\, \|g_1 - g_2\|_p^p \leqslant \left\|\Psi_p^\beta(g_1) - \Psi_p^\beta(g_2)\right\|_p^p \leqslant C([\beta]_{\mathrm{Lip}}, T)\, \|g_1 - g_2\|_p^p\,,$$

with $c([\beta]_{\mathrm{Lip}}, T) = \frac{1}{2^{p-1}\left(1 - [\beta]_{\mathrm{Lip}}^p T^p\right)}$ and $C([\beta]_{\mathrm{Lip}}, T) = e^{2^{p-1}[\beta]_{\mathrm{Lip}} T^{p-1}}$.
Mixed multifractional Ornstein-Uhlenbeck process, defined in Section 3.3, is of the form (3.17), with $\beta(s,u) = \theta(\mu - u)$ and $X = \alpha_1 B + \alpha_2 B^h$.

4 Multifractional Hull & White stochastic volatility model

We assume that, under the risk-neutral measure, the forward price of a risky asset is the solution of the S.D.E.

$$\begin{cases} dF_t = F_t \sigma_t dW_t, \\ d\ln(\sigma_t) = \theta\left(\mu - \ln(\sigma_t)\right) dt + \gamma_h d^\diamond B_t^h + \gamma_\sigma dW_t^\sigma, \quad \sigma_0 > 0, \end{cases} \tag{4.1}$$

where $\theta \geqslant 0$ and where W and W^σ are two standard Brownian motions and B^h is a well-balanced multifractional Brownian motion independent of W and W^σ with functional parameter h assumed to be continuously differentiable. We assume that W is decomposed into $\rho dW_t^\sigma + \sqrt{1-\rho^2} dW_t^F$, where W^F is a Brownian motion independent of W^σ. Hence, (4.1) writes

$$\begin{cases} dF_t = F_t \sigma_t \left(\rho dW_t^\sigma + \sqrt{1-\rho^2} dW_t^F\right) \\ d\ln(\sigma_t) = \theta\left(\mu - \ln(\sigma_t)\right) dt + \gamma_h d^\diamond B_t^h + \gamma_\sigma dW_t^\sigma, \quad \sigma_0 > 0. \end{cases} \tag{4.2}$$

We denote respectively by $\mathcal{F}^\sigma$, $\mathcal{F}^F$ and $\mathcal{F}^h$ the natural filtrations of W^σ, W^F and B^h. We define the filtration $\mathcal{F}^{\sigma,h}$ by $\mathcal{F}_t^{\sigma,h} = \sigma\left(\mathcal{F}_t^\sigma, \mathcal{F}_t^h\right)$ and $\mathcal{F}^{F,\sigma,h}$ by $\mathcal{F}_t^{F,\sigma,h} = \sigma\left(\mathcal{F}_t^F, \mathcal{F}_t^\sigma, \mathcal{F}_t^h\right)$.
The unique solution of (4.1) reads

$$\begin{cases} F_t = F_0 \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right) \\ \sigma_s = \exp\left(\ln(\sigma_0)e^{-\theta s} + \mu\left(1 - e^{-\theta s}\right) + \gamma_\sigma \int_0^s e^{\theta(u-s)} dW_u^\sigma + \gamma_h \int_0^s e^{\theta(u-s)} d^\diamond B_u^h\right). \end{cases} \tag{4.3}$$

5. Actually, in [58], the Lamperti transform is used to reduce a general Brownian diffusion to this case.

In other words, $\ln(\sigma_t)$ is a mixed multifractional Ornstein-Uhlenbeck process. Note that, although the volatility process is not a semimartingale, the process $(F_t)_{t\in[0,T]}$ remains a (positive) $\mathcal{F}^{F,\sigma,h}$-local martingale, and thus a super-martingale. The same proof as in [48] shows that, if $\rho = 0$, this local martingale is indeed a martingale. Numerical experiments seem to indicate that this property still holds for $\rho < 0$, a fact that remains to be proved.

We now consider the problem of pricing a forward start call option (the put case is handled similarly). The payoff of this option writes $\left(\frac{F_T}{F_\tau} - K\right)_+$ for some fixed maturity $\tau \in [0,T]$. We need to compute the risk-neutral expectation $\mathsf{E}\left[\left(\frac{F_T}{F_\tau} - K\right)_+\right]$.

The following decomposition holds:

$$F_t = F_0 \underbrace{\exp\left(\rho\int_0^t \sigma_s dW_s^\sigma - \frac{\rho^2}{2}\int_0^t \sigma_s^2 ds\right)}_{\text{Measurable with respect to } \mathcal{F}_t^{\sigma,h}} \exp\left(\sqrt{1-\rho^2}\int_0^t \sigma_s dW_s^F - \frac{1-\rho^2}{2}\int_0^t \sigma_s^2 ds\right).$$

Conditioning by $\mathcal{F}_T^{\sigma,h}$ yields

$$\begin{aligned}
\mathsf{E}\left[\left(\tfrac{F_T}{F_\tau} - K\right)_+\right] &= \mathsf{E}\left[\mathsf{E}\left[\left(\tfrac{F_T}{F_\tau} - K\right)_+ \Big| \mathcal{F}_T^{\sigma,h}\right]\right] \\
&= \mathsf{E}\left[\mathsf{E}\left[\left(F_{\tau,T}\exp\left(\sqrt{1-\rho^2}\textstyle\int_0^t \sigma_t dW_t^F - \frac{1-\rho^2}{2}\int_0^t \sigma_s^2 ds\right) - K\right)_+ \Big| \mathcal{F}_T^{\sigma,h}\right]\right] \\
&= \mathsf{E}\Big[\mathrm{PrimeBS}\Big(F_{\tau,T}, \Big((1-\rho^2)\tfrac{1}{T-\tau}\underbrace{\int_\tau^T \sigma_s^2 ds}_{=:I_{\tau,T}^\sigma}\Big)^{\frac{1}{2}}, T-\tau, K\Big)\Big],
\end{aligned} \tag{4.4}$$

where $F_{\tau,T} := \exp\left(\rho\int_\tau^T \sigma_s dW_s^\sigma - \frac{\rho^2}{2}\int_\tau^T \sigma_s^2 ds\right)$ and PrimeBS is the closed-form expression for the price of a Call option in the Black & Scholes model, detailed in Appendix VII. The aim is to estimate the expectation (4.4) by a quantization-based cubature associated with the functional quantization of B^h and W^σ. We thus need to write the terms $F_{\tau,T}$ and $\int_\tau^T \sigma_s^2 ds$ as explicit functionals of the paths of W^σ and B^h in $L^2([0,T])$.

Recall that σ is the exponential of a mixed multifractional Ornstein-Uhlenbeck process:

$$\sigma_t = \exp\left(\ln(\sigma_0)e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \gamma_\sigma e^{-\theta t} I_t^{e^{\theta\cdot}}(W^\sigma) + \gamma_h e^{-\theta t} I_t^{e^{\theta\cdot}}(B^h)\right). \tag{4.5}$$

This yields an explicit functional form for $\int_\tau^T \sigma_s^2 ds$ as a function of the paths of W^σ and B^h. Denote $(p_j^h)_{1\leqslant j\leqslant N_1}$ and $(\chi_j^h)_{1\leqslant j\leqslant N_1}$ the weights and the paths of the quantizer $\widehat{B}^h$ of B^h, and $(p_j^\sigma)_{1\leqslant j\leqslant N_2}$ and $(\chi_j^\sigma)_{1\leqslant j\leqslant N_2}$ the weights and the paths of the quantizer $\widehat{W}^\sigma$ of W^σ. Conditionally on $B^h = \chi_i^h$, one has $I_{\tau,T}^\sigma = I_{\tau,T}^{\sigma^i}$, where

$$I_{\tau,T}^{\sigma^i} := \int_\tau^T \sigma_s^i dW_s^\sigma$$

and

$$\sigma_t^i = \exp\left(\ln(\sigma_0)e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \gamma_\sigma\int_0^t e^{\theta(s-t)}dW_s^\sigma + \gamma_h e^{-\theta t}\, I_t^g\left(\chi_i^h\right)\right).$$

Appendix B shows that χ_i^h has bounded variations. This entails that σ^i is a semimartingale. Define $\langle\sigma^i, W^\sigma\rangle_{\tau,T} := \langle\sigma^i, W^\sigma\rangle_T - \langle\sigma^i, W^\sigma\rangle_\tau$, where $\langle\cdot,\cdot\rangle$ denotes the semimartingale bracket and let us denote by $\int_\tau^T \sigma_s^i \circ dW_s^\sigma$ the Stratonovich integral of σ^i. Then, $I_{\tau,T}^{\sigma^i}$ reads

$$I_{\tau,T}^{\sigma^i} = \int_\tau^T \sigma_s^i \circ dW_s^\sigma - \frac{1}{2}\langle\sigma^i, W^\sigma\rangle_{\tau,T}.$$

Itô's formula yields

$$\int_\tau^T \sigma_t^i\, dW_t^\sigma = \underbrace{\frac{\sigma_T^i - \sigma_\tau}{\gamma_\sigma} - \frac{1}{\gamma_\sigma}\int_\tau^T \sigma_t^i\theta\left(\mu - \ln\left(\sigma_t^i\right)\right)dt - \frac{\gamma_h}{\gamma_\sigma}\int_\tau^T \sigma_t^i d\chi_i^h(t)}_{=\int_\tau^T \sigma_t^i\, \circ\, dW_t^\sigma} - \underbrace{\frac{\gamma_\sigma}{2}\int_\tau^T \sigma_t^i dt}_{=\frac{1}{2}\langle\sigma^i, W^\sigma\rangle_{\tau,T}}.$$

Moreover,

$$\int_\tau^T \widehat{\sigma^i}_t \, d\widehat{W}_t^\sigma = \frac{\widehat{\sigma^i}_T - \widehat{\sigma}_\tau}{\gamma_\sigma} - \frac{1}{\gamma_\sigma}\int_\tau^T \widehat{\sigma^i}_t \theta\left(\mu - \ln\left(\widehat{\sigma^i}_t\right)\right) dt - \frac{\gamma_h}{\gamma_\sigma}\int_\tau^T \widehat{\sigma^i}_t d\chi_i^h(t).$$

This shows that $\int_\tau^T \sigma_t^i \circ dW_t^\sigma$ may be approximated by $\int_\tau^T \widehat{\sigma^i}_t \, d\widehat{W}_t^\sigma$ and $\int_\tau^T \sigma_t^i \, dt$ by $\int_\tau^T \widehat{\sigma_t^i} \, dt$. Thus we approximate $I_{\tau,T}^{\sigma^i}$ by $\widehat{I}_{\tau,T}^{\sigma^i} := \int_\tau^T \widehat{\sigma_s^i} d\widehat{W}_s^\sigma - \frac{\gamma_\sigma}{2}\int_\tau^T \widehat{\sigma_s^i} ds$.
The cubature formula is then fully explicit and one finally obtains the following approximation:

$$\mathsf{E}\left[\left(\frac{F_T}{F_\tau} - K\right)_+\right] \approx \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} p_i^h p_j^\sigma \text{PrimeBS}\left(F_{\tau,T}^{i,j}, \left(\left(1-\rho^2\right)\frac{1}{T-\tau}\int_\tau^T \left(\sigma^{i,j}(s)\right)^2 ds\right)^{\frac{1}{2}}, T-\tau, K\right),$$

where

$$F_{\tau,T}^{i,j} = \exp\left(\rho\int_\tau^T \sigma^{i,j}(s) d\chi_j^\sigma(s) - \rho\gamma^\sigma \frac{1}{2}\int_\tau^T \sigma^{i,j}(s) ds - \frac{\rho^2}{2}\int_\tau^T \left(\sigma^{i,j}(s)\right)^2 ds\right),$$

and

$$\sigma^{i,j}(t) := \exp\left(\ln(\sigma_0)e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \gamma_\sigma e^{-\theta t} I_t^{e^{\theta \cdot}}(\chi_j^\sigma) + \gamma_h e^{-\theta t} I_t^{e^{\theta \cdot}}(\chi_j^h)\right).$$

$\left(\ln(\sigma^{i,j})\right)_{1\leqslant i\leqslant N_1, 1\leqslant j\leqslant N_2}$ and $(p_i^h p_j^\sigma)_{1\leqslant i\leqslant N_1, 1\leqslant j\leqslant N_2}$ are the paths and weights of a stationary quantizer of the mixed multifractional Ornstein-Uhlenbeck process $\ln(\sigma)$. The results stated in Section 3.4 allow us to control its quadratic quantization error with the quantization error of $\widehat{W}^\sigma$ and $\widehat{B}^h$. We then apply (2.10) to get an upper bound for the $L^{2-\epsilon}$ quantization error of the process σ on $[0,T]$, for any $\epsilon > 0$.

5 Multifractional SABR model

We now assume that, under the risk-neutral measure, the forward price of a risky asset is the solution of the S.D.E.

$$\begin{cases} dF_t = F_t\sigma_t dW_t, \\ d\sigma_t = \sigma_t\left(\gamma_h d^\diamond B_t^h + \gamma_\sigma dW_t^\sigma\right), \quad \sigma_0 > 0, \end{cases} \tag{5.1}$$

where W and W^σ are two standard Brownian motions and B^h is a well-balanced multifractional Brownian motion independent of W and W^σ with functional C^1 parameter h. We assume that W is decomposed into $\rho dW_t^\sigma + \sqrt{1-\rho^2} dW_t^F$, where W^F is a Brownian motion independent of W^σ. We use the same notations as in the previous section for $\mathcal{F}^\sigma$, $\mathcal{F}^F$, $\mathcal{F}^h$, $\mathcal{F}^{\sigma,h}$ and $\mathcal{F}^{F,\sigma,h}$. Hence, (5.1) writes

$$\begin{cases} dF_t = F_t\sigma_t\left(\rho dW_t^\sigma + \sqrt{1-\rho^2} dW_t^F\right) \\ d\sigma_t = \sigma_t\left(\gamma_h d^\diamond B_t^h + \gamma_\sigma dW_t^\sigma\right), \quad \sigma_0 > 0. \end{cases} \tag{5.2}$$

The solution of the stochastic differential equation verified by σ, established in Theorem 3.5, is

$$\sigma_t = \sigma_0 \exp^\diamond\left(\gamma_\sigma W_t^\sigma + \gamma_h B_t^h\right) = \sigma_0 \exp\left(\gamma_\sigma W_t^\sigma + \gamma_h B_t^h - \frac{1}{2}\left(\gamma_\sigma^2 t + \gamma_h^2 t^{2h(t)}\right)\right). \tag{5.3}$$

Reasoning as in the case of the Hull & White model presented in Section 4, it can be shown that F is an $\mathcal{F}^{F,\sigma,h}$-martingale for $\rho = 0$. In addition, the same numerical procedures as above may be used.

6 Numerical experiments

6.1 Variance reduction method for the quantization-based cubature

Numerical experiments carried out in [27] showed that, in the case of vanilla options, computing the implied volatility using the *estimated forward* instead of the *theoretical forward* in the Black & Scholes formula

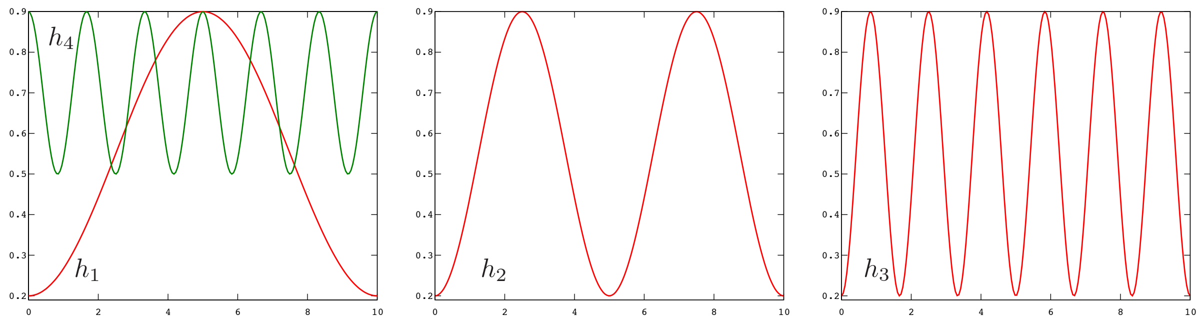

Figure 4.3: left: functions h_1 and h_4; middle: function h_2; right: function h_3.

improves the accuracy. The counterpart of this method in the frame of forward start options is to replace the "1" appearing in Formula (VII.3) by the quantity

$$\mathfrak{I}_{N_1,N_2} := \sum_{1\leqslant i\leqslant N_1,\ 1\leqslant j\leqslant N_2} p_i^h p_j^\sigma F_{\tau,T}^{i,j} \tag{6.1}$$

This also holds when using Richardson-Romberg extrapolation: in this case, one uses the extrapolated value of $\mathfrak{I}_{N_1,N_2}$ instead of 1 in Formula (VII.3).
These methods were used to generate the numerical results presented below.

6.2 Numerical results

We present results on the multifractional Hull & White model. We have computed the price as a function of strike for different maturities: 1, 2.5, 5 and 10 years. Driving noises were chosen in the class of fBms and mBms. More precisely, we display results of our experiments with:

1. An fBm with $H = 0.2$.
2. An fBm with $H = 0.5$.
3. An fBm with $H = 0.75$.
4. An fBm with $H = 0.9$.
5. An mBm with $h = h_1 = 0.35\sin\left(\frac{2\pi}{10}\left(t+\frac{15}{2}\right)\right) + 0.55$.
6. An mBm with $h = h_2 = 0.35\sin\left(\frac{2\pi}{5}\left(t+\frac{15}{4}\right)\right) + 0.55$.
7. An mBm with $h = h_3 = 0.35\sin\left(\frac{6\pi}{5}\left(t+\frac{5}{4}\right)\right) + 0.55$.
8. An mBm with $h = h_4 = -0.2\sin\left(\frac{6\pi}{5}\left(t+\frac{5}{4}\right)\right) + 0.7$.

The four functions are plotted on Figure 4.3. The values of the other parameters are $\gamma_h = 0.3$, $\gamma_\sigma = \rho = 0$ (except for the experiments displayed on Figure 4.6), $\theta = 0.3$, $\mu = \ln(0.2)$, $\sigma_0 = 0.2$ and $F_0 = 100$.
The results displayed below provide an experimental justification to the claims made in the introduction. Indeed, one sees that, for the short maturity $T = 1$ year, in the fractional Hull & White model (*i.e.* with h constant), the smiles are more pronounced for small H and decrease as H increase, while the reverse is true for all maturities larger than one year (Figure 4.4). Thus, stronger correlations in the driving noise do translate in this model into a slower decrease of the smile as maturities increase, as noted in [23]. However, with such an fBm-based model, an H larger than 1/2 is needed to ensure long-range dependence and thus a more realistic evolution of the smile as compared to the Brownian case. As mentioned above, this is not compatible with empirical graphs of the volatility which show a very irregular behaviour, and would require a small H. In addition, the local regularity of the volatility evolves in time, calling for a varying H, *i.e.* an mBm.
Another aspect is that a fixed H, as in a modelling with fBm, does not allow to control independently the shape of the smiles at different maturities. This is possible with mBm, where the smile at maturity T

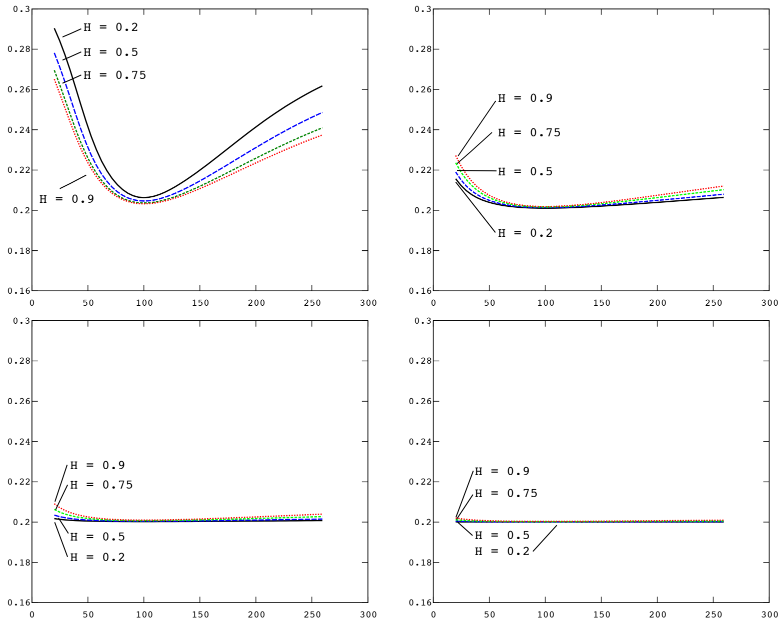

Figure 4.4: Comparisons of vanilla option volatility smiles for fBm with $H = 0.2, H = 0.5, H = 0.7$ and $H = 0.9$ at different maturities. Top left: $T = 1$. Top right: $T = 2.5$. Bottom left: $T = 5$. Bottom right: $T = 10$.

depends on a weighted average of the values of h up to time T, as can be inferred from equalities (3.7) and (4.3). This is apparent on Figure 4.5. We have compared fBms and mBms at various maturities T, where H and h are chosen such that $h(t) = H$, or, for the bottom right plot, $h_1(t) = h_4(t)$. One sees that the shape of the smile depends on a weighted average of past values of h. For instance, in the bottom left plot, the values of h before $T = 2.5$ are in average smaller than 0.9, resulting in a flatter smile. The fact that a *weighted* average must be considered is apparent on the bottom right plot: indeed, the smile is more pronounced for h_1, although the average from 0 to 5 of this function is smaller than the one of h_4. In contrast, the values in the immediate past of $t = 5$ are larger for h_1 than for h_4, as may be checked on Figure 4.3. An adequate choice of h may thus allow one to better approximate a whole implied volatility surface. This topic will be addressed in a future work.
Finally, we display on Figure 4.6 an example with $\rho \neq 0$ for illustration purposes.

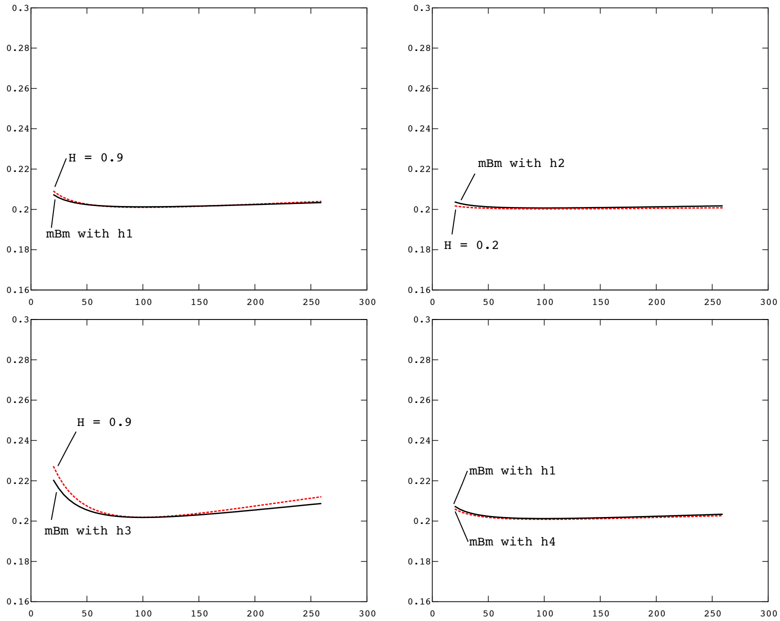

Figure 4.5: Comparisons of vanilla option volatility smiles for various fBm and mBm at several maturities. Top left: fBm with $H = 0.9$ and mBm with function h_1 at $T = 5$ ($h_1(5) = 0.9$). Top right: fBm with $H = 0.2$ and mBm with function h_2 at $T = 5$ ($h_2(5) = 0.2$). Bottom left: fBm with $H = 0.9$ and mBm with function h_3 at $T = 2.5$ ($h_3(2.5) = 0.9$). Bottom right: mBm with function h_1 and mBm with function h_4 at $T = 5$ ($h_1(5) = h_4(5) = 0.9$).

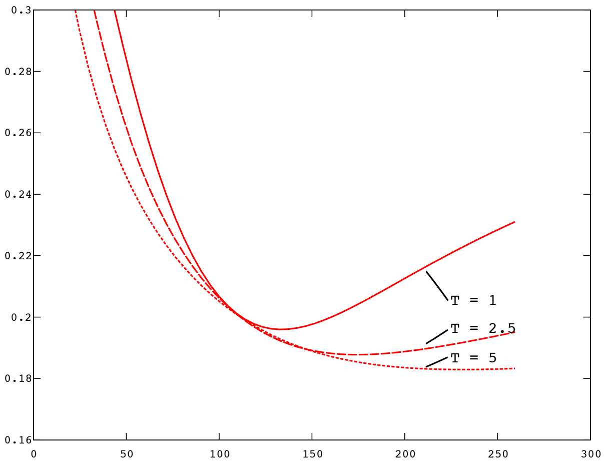

Figure 4.6: Vanilla option volatility smiles in the multifractional Hull & White model, with $\gamma_h = 0.3$, $\gamma_\sigma = 0.3$ $\rho = -0.5$, $\theta = 0.3$, $\mu = \ln(0.2)$, $\sigma_0 = 0.2$ and $F_0 = 100$, and $h = h_2$ for maturities $T = 1$, $T = 2.5$ and $T = 5$.

Acknowledgments

The authors are very thankful to Bruno Dupire, Gilles Pagès and Marc Yor for helpful remarks and comments.

VII Appendix

The implied forward start volatility

The forward start option price.
Let W be a standard Brownian motion on $[0,T]$ and $\tau \in (0,T)$. Let us consider the stochastic differential equation $dS_t = S_t\sigma_t dW_t$ (with $(\sigma_t)_{t\in[0,T]}$ a deterministic process) whose solution is a geometric Brownian motion $S_t = S_0 \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2\, ds\right)$. The forward start Call option price $\text{FSPrimeBS}(\sigma,\tau,T,K)$ is given by

$$\text{FSPrimeBS}(\sigma,\tau,T,K) = \mathsf{E}\left[\left(\frac{S_T}{S_\tau} - K\right)_+\right] = \mathcal{N}(d_1) - K\mathcal{N}(d_2),$$

where $d_1 := \frac{\overline{\sigma}\sqrt{T-\tau}}{2} + \frac{\ln(K)}{\overline{\sigma}\sqrt{T-\tau}}$, $d_2 := d_1 - \frac{\overline{\sigma}\sqrt{T-\tau}}{2}$ and $\overline{\sigma}^2 := \frac{1}{T-\tau}\int_\tau^T \sigma_s^2\, ds$. In other words, we have

$$\text{FSPrimeBS}(\sigma,\tau,T,K) = \text{PrimeBS}(1,\sigma,T-\tau,K), \tag{VII.1}$$

where $(S_0, \text{Vol}, \text{Mat}, \text{Strike}) \mapsto \text{PrimeBS}(S_0, \text{Vol}, \text{Mat}, \text{Strike})$ is the closed-form expression for the vanilla Call option price in the Black & Scholes model.

The implied forward start volatility.
In the Black & Scholes model, where the asset price follows a geometric Brownian motion with a constant volatility, the forward start Call (or Put) option price is an increasing function of the volatility (if the strike is not zero). Conversely, for a given forward start Call (or Put) option price, the Black & Scholes implied volatility is the unique value of the volatility for which the Black & Scholes formula recovers the price; in other words, the implied forward start volatility associated with a given forward F_0, a forward start date τ, a maturity $T > \tau$, a strike K, and an option price P is defined by

$$P = \textbf{FSPrimeBS}\left(\textbf{ImpliedFSVolBS}\left(\tau,T,K,P\right),\tau,T,K\right). \tag{VII.2}$$

Using Equation (VII.1), this yields

$$\textbf{ImpliedFSVolBS}\left(\tau,T,K,P\right) = \textbf{ImpliedVolBS}\left(1,T-\tau,K,P\right), \tag{VII.3}$$

where $\textbf{ImpliedVolBS}\,(\text{Fwd}, \text{maturity}, \text{Strike}, \text{Price})$ is the Black & Scholes implied volatility a certain forward, maturity, strike and option price.

VIII Variations of the Karhunen-Loève eigenfunctions of mBm

Let R_h denote the covariance function of a normalized mBm B^h with functional C^1 parameter h and e_k^h be the kth Karhunen-Loève eigenfunction of B^h. For k in $\mathbb{N}$, define the map $I_k : [0,T] \to \mathbb{R}$ by $I_k(t) := \int_0^T R_h(t,s)\, e_k^h(s)\, ds = \lambda_k^h e_k^h$, where λ_k^h is the eigenvalue associated with e_k^h.

Theorem VIII.1. *For every integer k, the map e_k^h has bounded variations on $[0,T]$.*

Proof: For every fixed (k,t) in $\mathbb{N}\times[0,T]$,

$$\begin{aligned} I_k(t) &= \int_0^T \frac{c_{h_{t,s}}^2}{c_{h(t)}c_{h(s)}}\, t^{2h_{t,s}}\, e_k^h(s)\, ds + \int_0^T \frac{c_{h_{t,s}}^2}{c_{h(t)}c_{h(s)}}\, s^{2h_{t,s}}\, e_k^h(s)\, ds - \int_0^T \frac{c_{h_{t,s}}^2}{c_{h(t)}c_{h(s)}}\, |t-s|^{2h_{t,s}}\, e_k^h(s)\, ds \\ &=: F_1(t) + F_2(t) - F_3(t). \end{aligned} \tag{VIII.1}$$

We show that F_i has bounded variations for every i in $\{1,2,3\}$. The cases of F_1, F_2 and F_3 are similar, and we only treat here F_1. Let $(t_i)_{0\leqslant i\leqslant N}$ be a sequence of elements of $[0,T]$ such that $0 = t_0 < t_1 < \cdots < t_N = T$. For any i in $\{1,\cdots,N\}$ we get,

$$|F_1(t_i)-F_1(t_{i-1})| \leqslant \overbrace{\sup_{s\in[0,T]}\left|\frac{e_k(s)}{c_{h(s)}}\right|}^{=:K_1} \int_0^T \left|\frac{c^2_{h_{t_i,s}}}{c_{h(t_i)}}\, t_i^{2h_{t_i,s}} - \frac{c^2_{h_{t_{i-1},s}}}{c_{h(t_{i-1})}}\, t_{i-1}^{2h_{t_{i-1},s}}\right| ds$$
$$\leqslant K_1 \Bigg(\underbrace{\int_0^T \frac{c^2_{h_{t_i,s}}}{c_{h(t_i)}} \left|t_i^{2h_{t_i,s}} - t_{i-1}^{2h_{t_{i-1},s}}\right| ds}_{=:G_i} + \underbrace{\int_0^T \left|\frac{c^2_{h_{t_i,s}}}{c_{h(t_i)}} - \frac{c^2_{h_{t_{i-1},s}}}{c_{h(t_{i-1})}}\right| t_{i-1}^{2h_{t_{i-1},s}}\, ds}_{=:L_i} \Bigg). \tag{VIII.2}$$

Since the map $(s,t)\mapsto \frac{c^2_{h_{t,s}}}{c_{h(t)}}$ is C^1 as soon as h is C^1, the mean-value theorem yields

$$\left|\frac{c^2_{h_{t_i,s}}}{c_{h(t_i)}} - \frac{c^2_{h_{t_{i-1},s}}}{c_{h(t_{i-1})}}\right| \leqslant \sup_{s\in[0,T]} |f_s'(t)|\ |t_i-t_{i-1}| =: K_2\ |t_i-t_{i-1}|,$$

where $f_s'(t)$ denotes, for every s in $[0,T]$, the derivative, at point t, of the map $t\mapsto \frac{c^2_{h_{t,s}}}{c_{h(t)}}$. Setting $[H_1,H_2] := \left[\inf_{u\in[0,T]} h(u),\ \sup_{u\in[0,T]} h(u)\right]$, one gets:

$$L_i \leqslant K_2\ |t_i-t_{i-1}| \int_0^T t_{i-1}^{2h_{t_{i-1},s}}\ ds \leqslant T\ (1+K_2)\ |t_i-t_{i-1}|\ \left(e^{2H_1T}+e^{2H_2T}\right) =: K_3\ \ |t_i-t_{i-1}|. \tag{VIII.3}$$

Besides, $G_i \leqslant \sup_{(t,s)\in[0,T]^2}\left|\frac{c^2_{h_{t,s}}}{c_{h(t)}}\right|\ \int_0^T \left|t_i^{2h_{t_i,s}} - t_{i-1}^{2h_{t_{i-1},s}}\right|\ ds =: K_4\ \int_0^T \left|t_i^{2h_{t_i,s}} - t_{i-1}^{2h_{t_{i-1},s}}\right|\ ds.$
Now, writing

$$t_i^{2h_{t_i,s}} - t_{i-1}^{2h_{t_{i-1},s}} = \underbrace{t_i^{2h_{t_i,s}} - t_{i-1}^{2h_{t_i,s}}}_{:=C_i(s)} - \underbrace{t_{i-1}^{2h_{t_i,s}} - t_{i-1}^{2h_{t_{i-1},s}}}_{:=D_i(s)},$$

we easily get that

$$\forall s\in[0,T],\ \ |C_i(s)| \leqslant 2H_2 \int_{t_{i-1}}^{t_i} \left(x^{2H_2-1}-x^{2H_1-1}\right)\ dx.$$

Define the family of maps $(g_\alpha)_{\alpha\in\mathbb{R}_+^*}$ from $\mathbb{R}_+$ to $\mathbb{R}_+$, by $g_\alpha(x) := \alpha^x$ if $x>0$ and $g_\alpha(x) := 1$ if $x=0$. Let $K_5 := \sup_{\alpha\in[0,T]} |\ln(\alpha)|\ (e^{2H_1\ln(\alpha)}+e^{2H_2\ln(\alpha)})$. The mean-value theorem applied to g_α yields

$$\forall s\in[0,T],\ \ |D_i(s)| \leqslant 2^{-1}\ K_5\ |2h_{t_i,s}-2h_{t_{i-1},s}| \leqslant K_5 \sup_{u\in[0,T]} |h'(u)|\ |t_i-t_{i-1}| =: K_6\ |t_i-t_{i-1}|.$$

We hence have shown that

$$\forall i\in\{1;\cdots;N\},\ \ G_i \leqslant \overbrace{(1+T)\ (1+2H_2)\ (1+K_4)\ (1+K_6)}^{=:K_7} \left(|t_i-t_{i-1}| + \int_{t_{i-1}}^{t_i} \left(x^{2H_2-1}-x^{2H_1-1}\right)\ dx\right). \tag{VIII.4}$$

Using (VIII.3) and (VIII.4) we finally obtain

$$\sum_{i=1}^N |F_1(t_i)-F_1(t_{i-1})| \leqslant 2K_7\ \left(1+\frac{1}{2H_1}\right)\left(T+T^{2H_1}+T^{2H_2}\right) < +\infty,$$

which ends the proof. □

Appendix A

Real and complex Gaussian measures

Contents

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $L^0(\Omega)$ be the set of all real random variables on Ω. Let $(E, \mathcal{E}, m)$ be a measured space, and $\mathcal{E}_0^{(m)} := \{A \in \mathcal{E};\ m(A) < +\infty\}$. Let us moreover note, $\widehat{L^2_{\mathbb{C}}}(d\xi) := \{h : \mathbb{R} \to \mathbb{C};\ \int_{\mathbb{R}} |h(\xi)|^2\, d\xi < +\infty \ \&\ \overline{h(\xi)} = h(-\xi)\ \forall \xi \in \mathbb{R}\}$ and for, $i \in \{1; 2\}$, $L^i_{\mathbb{R}}(du) := \{f : \mathbb{R} \to \mathbb{R};\ \int_{\mathbb{R}} |f(u)|^i\, du < +\infty\}$. Let us finally note $\mathcal{B}$ the Borel σ-field of $\mathbb{R}$, $\mathcal{B}_0^{(m)} := \{A \in \mathcal{B}(\mathbb{R});\ m(A) < +\infty\}$ and λ the Lebesgue measure on $\mathbb{R}$. Moreover, for every $\sigma \neq 0$ we will write $X \rightsquigarrow \mathcal{N}(0, \sigma^2)$ to mean that the probability measure of X is centred and Gaussian with variance equal to σ^2.

Definition 0.1 (Gaussian measure on $(\mathbb{R}, \mathcal{B})$)**.** *For any positive real b, define the measure $\lambda_b := b\lambda$ (when $b = 1$ we note λ instead of λ_1). An independently scattered σ-additive set function $M : \mathcal{B}_0^{(\lambda_b)} \to L^0(\Omega)$ such that $M(A) \rightsquigarrow \mathcal{N}(0; 2\lambda_b(A))$ for any A in $\mathcal{B}_0^{(\lambda_b)}$ is called a Gaussian random measure on $\mathbb{R}$ with control (or intensity) measure λ_b.*

(i) *Independently scattered* means that if $A_1, A_2, \cdots, A_k$ belongs to $\mathcal{B}_0^{(\lambda_b)}$ and are disjoint, then the random variables $M(A_1), M(A_2), \cdots, M(A_k)$ are independent.

(ii) *σ-additive* means that if $A_1, A_2, \cdots, A_k$ belong to $\mathcal{B}_0^{(\lambda_b)}$ are disjoint and such that $\overset{+\infty}{\underset{j=1}{\cup}} A_j \in \mathcal{B}_0^{(\lambda_b)}$, then $M(\overset{+\infty}{\underset{j=1}{\cup}} A_j) = \sum_{j=1}^{+\infty} M(A_j)$ a.s.

Definition 0.2 (Gaussian measure on $(\mathbb{R}, \mathcal{B})$, independently scattered on $\mathbb{R}_+$)**.** *A set function $M : \mathcal{B}_0^{(\lambda_b)} \to L^0(\Omega)$, independently scattered and σ-additive on $\mathbb{R}_+$ such that $M(A) \rightsquigarrow \mathcal{N}(0; 2\lambda_b(A))$ for any A in $\mathcal{B}_0^{(\lambda_b)}$ is called a Gaussian random measure on $\mathbb{R}$, independently scattered on $\mathbb{R}_+$, with control measure λ_b.*

Remark 0.2. *Note that when M is a Gaussian measure on $\mathbb{R}$, independently scattered on $\mathbb{R}_+$, $M(A)$ may not be independent of $M(B)$ even for disjoint sets A and B of $\mathcal{B}_0^{(\lambda_b)}$.*

There are two equivalent ways to understand integral with respect to Gaussian measure. We recall in two following sections these two ways and explain in the third section the links between them.

1 First method: [73] and references therein

Define firstly $\widetilde{\mathbb{W}} := \mathbb{W}_1 + i\mathbb{W}_2$ where $\mathbb{W}_1$ and $\mathbb{W}_2$ are independent Gaussian measures, independently scattered on $\mathbb{R}_+$, with control measure $\lambda_{1/4}$ satisfying $\mathbb{W}_1(A) = \mathbb{W}_1(-A)$ and $\mathbb{W}_2(A) = -\mathbb{W}_2(-A)$ for any A in $\mathcal{B}_0^{(\lambda_{1/4})}$. Then $\widetilde{\mathbb{W}}(A) = \overline{\widetilde{\mathbb{W}}(-A)}$ for any A in $\mathcal{B}_0^{(\lambda_{1/4})}$. Moreover, for any Borel set A in $\mathbb{R}_+$,

$$\mathbb{E}[|\widetilde{\mathbb{W}}(A)|^2] = \mathbb{E}[\mathbb{W}_1(A)^2] + \mathbb{E}[\mathbb{W}_2(A)^2] = \lambda(A) \text{ and } \mathbb{E}[\widetilde{\mathbb{W}}(A)^2] = \mathbb{E}[\mathbb{W}_1(A)^2] - \mathbb{E}[\mathbb{W}_2(A)^2] = 0.$$

Consider the set of complex-valued functions $h := h_1 + ih_2$ defined on $\mathbb{R}$ such that h_1 and h_2 are real-valued functions which satisfy, for every real ξ , $h_1(\xi) = h_1(-\xi)$, $h_2(\xi) = -h_2(-\xi)$ and $\int_{\mathbb{R}} h_i(\xi)^2 d\xi < +\infty$ for $i \in \{1;2\}$. Note that h belongs to $\widehat{L^2_{\mathbb{C}}}(d\xi)$. We are now able to give the following

Definition 1.1. *For any h in $\widehat{L^2_{\mathbb{C}}}(d\xi)$, we define the stochastic Wiener integral $I(h) = \int_{\mathbb{R}} h(\xi)\widetilde{\mathbb{W}}(d\xi)$ of h by*

$$I(h) := \int_{\mathbb{R}} h_1(\xi)\mathbb{W}_1(d\xi) - \int_{\mathbb{R}} h_2(\xi)\mathbb{W}_2(d\xi).$$

The process $\{I(h);\ h \in \widehat{L^2_{\mathbb{C}}}(d\xi)\}$ is a centred Gaussian process whose covariance function Σ is given, for all (h,g) in $\widehat{L^2_{\mathbb{C}}}(d\xi) \times \widehat{L^2_{\mathbb{C}}}(d\xi)$, by $\Sigma(h,g) =< h,g >_{L^2_{\mathbb{C}}(\mathbb{R})}$. In particular $I(h) \rightsquigarrow \mathcal{N}(0; ||h||^2_{L^2_{\mathbb{C}}(\mathbb{R})})$ for every h in $\widehat{L^2_{\mathbb{C}}}(d\xi)$.

Let us note $\widehat{f}$ the Fourier transform of any element f in $L^1_{\mathbb{R}}(du) \cup L^2_{\mathbb{R}}(du)$ defined, for every $\xi \in \mathbb{R}$, by $\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi u}\ f(u)\ du$, if f belongs to $L^1(\mathbb{R})$ and extended by continuity for abitrary f in $L^2_{\mathbb{R}}(du)$.
It is then clear that, for every Gaussian random measure $\mathbb{W}_0$ on $\mathbb{R}$ with control measure $\lambda_{1/2}$, we have the equality in law: $\int_{\mathbb{R}} f(u)\mathbb{W}(du) \overset{(d)}{=} \int_{\mathbb{R}}\ \mathcal{F}(f)(\xi)\ \widetilde{\mathbb{W}}(d\xi)$ for every f in $L^2_{\mathbb{R}}(du)$.

Remark 1.1. *Intuitively it seems natural to set $I(h) := \int_{\mathbb{R}}(h_1(\xi)+ih_2(\xi))(\mathbb{W}_1(d\xi)+i\mathbb{W}_2(d\xi)) := \int_{\mathbb{R}}(h_1(\xi)+ih_2(\xi))\mathbb{W}_1(d\xi) + \int_{\mathbb{R}}(ih_1(\xi) - h_2(\xi))\mathbb{W}_2(d\xi)$. We hence would have*

$$I(h) = \int_{\mathbb{R}} h_1(\xi)\mathbb{W}_1(d\xi) - \int_{\mathbb{R}} h_2(\xi)\mathbb{W}_2(d\xi) + i\Big(\underbrace{\int_{\mathbb{R}} h_2(\xi)\mathbb{W}_1(d\xi)}_{=:R_{2,1}} - \underbrace{\int_{\mathbb{R}} h_1(\xi)\mathbb{W}_2(d\xi)}_{=:R_{1,2}}\Big).$$

However, since $\int_{\mathbb{R}_-} h_2(\xi)\mathbb{W}_1(d\xi) = \int_{\mathbb{R}_+} h_2(-u)\mathbb{W}_1(-du) = -\int_{\mathbb{R}_+} h_2(u)\mathbb{W}_1(du)$ and $\int_{\mathbb{R}_-} h_1(\xi)\mathbb{W}_2(d\xi) = \int_{\mathbb{R}_+} h_1(-u)\mathbb{W}_2(-du) = -\int_{\mathbb{R}_+} h_1(u)\mathbb{W}_2(du)$ almost surely, we get that $R_{2,1} = R_{1,2} = 0$ almost surely and thus, finally, $I(h) = \int_{\mathbb{R}} h_1(\xi)\mathbb{W}_1(d\xi) - \int_{\mathbb{R}} h_2(\xi)\mathbb{W}_2(d\xi)$, for every ω as stated the definition.

2 Second method: [75] and references therein

Let $\Psi := (\psi_\lambda)_{\lambda\in\Lambda}$ be an orthonormal basis of the Hilbert space $L^2_{\mathbb{R}}(du)$ with inner product $<,>$ defined by $< f,g >:= \int_{\mathbb{R}} f(u)\ g(u)\ du$ for all (f,g) in $L^2_{\mathbb{R}}(du) \times L^2_{\mathbb{R}}(du)$. For example we can take Ψ to be a wavelet basis, namely $\psi_\lambda(u) := 2^{-j/2}\psi(2^{-j}u - k)$, where $\lambda = (j,k)$, j,k belongs to $\mathbb{Z}$ and where the "mother wavelet "ψ is such that Ψ is an orthonormal basis.
The image of the space $L^2_{\mathbb{R}}(du)$ with respect to the Fourier transform is nothing but the space $\widehat{L^2_{\mathbb{C}}}(d\xi)$ defined at the begining of section 1. Moreover, for any h and l in $\widehat{L^2_{\mathbb{C}}}(d\xi)$, denote $< h,l >:= \int_{\mathbb{R}} h(\xi)\overline{l(\xi)}\ d\xi$ the corresponding inner product.

Now, for any f in $L^2_{\mathbb{R}}(du)$ and h in $\widehat{L^2_{\mathbb{C}}}(d\xi)$, define the stochastic integrals

$$\mathscr{I}(f) = \int_{\mathbb{R}} f(u)\ W(du) := \sum_{\lambda\in\Lambda} < f,\psi_\lambda >\ Z_\lambda, \tag{2.1}$$

$$\mathscr{J}(h) = \int_{\mathbb{R}} h(\xi)\ \widehat{W}(d\xi) := \sum_{\lambda\in\Lambda} < h,\widehat{\psi_\lambda} >\ Z_\lambda, \tag{2.2}$$

where $Z := (Z_\lambda)_{\lambda\in\Lambda}$ is some fixed collection of independent and identically distributed standard Gaussian random variables on $(\Omega,\mathcal{F},\mathbb{P})$. These series converge in the L^2-sense, for all f in $L^2_{\mathbb{R}}(du)$ and h in $\widehat{L^2_{\mathbb{C}}}(d\xi)$. Moreover one has, for all f and g in $L^2_{\mathbb{R}}(du)$,

$$\mathbb{E}[\mathscr{I}(f)\mathscr{I}(g)] \;=\; <f,g> \;=\; <\widehat{f},\widehat{g}> \;=\; \mathbb{E}[\mathscr{J}(\widehat{f})\mathscr{J}(\widehat{g})], \tag{2.3}$$

There is a "stochastic Parseval identity "relating the stochastic integrals $\int_{\mathbb{R}} f(u)\, W(du)$ and $\int_{\mathbb{R}} h(\xi)\, \widehat{W}(d\xi)$ in (2.1) and (2.2), namely:

Proposition 2.1. *For all f in $L^2_{\mathbb{R}}(du)$ and h in $\widehat{L^2_{\mathbb{C}}}(d\xi)$, we have that:*

$$\int_{\mathbb{R}} f(u)\, W(du) \overset{a.s}{=} \int_{\mathbb{R}} h(\xi)\, \widehat{W}(d\xi), \quad \textit{if and only if} \quad \widehat{f}(\xi) = h(\xi), \quad a.e. \tag{2.4}$$

Proof. The equality beetwen the right-hand sides of (2.1) and (2.2) results simply from the Parseval identity. □

Remark 2.2. *We now have $\int_{\mathbb{R}} f(u)\, W(du) \overset{a.s}{=} \int_{\mathbb{R}} \widehat{f}(\xi)\, \widehat{W}(d\xi)$ for every f in $L^2_{\mathbb{R}}(du)$. Moreover $\{\mathscr{J}(h);\ h \in \widehat{L^2_{\mathbb{C}}}(d\xi)\}$ is a centred Gaussian process, and once again, $\mathscr{J}(h) \rightsquigarrow \mathcal{N}(0; ||h||^2_{L^2_{\mathbb{C}}(\mathbb{R})})$ for every h in $\widehat{L^2_{\mathbb{C}}}(d\xi)$.*

3 Links between the two methods

From second to first method

Starting from the second method framework, it is easy to build gaussian measures $\mathbb{W}_1, \mathbb{W}_2$ and $\widetilde{\mathbb{W}}$ on $\mathbb{R}$, satisfying the assumptions of the first method and hence recover the integral I. More precisely, we consider $\mathscr{I}_1$ and $\mathscr{I}_2$ as in (2.1) as well as $\mathscr{J}_1$ and $\mathscr{J}_2$ as in (2.2) such that, for every i in $\{1;2\}$

$$\mathscr{I}_i(f) = \int_{\mathbb{R}} f(u)\, W_i(du) := \sum_{\lambda\in\Lambda} <f,\psi_\lambda>\; Z^{(i)}_\lambda,$$

$$\mathscr{J}_i(h) = \int_{\mathbb{R}} h(\xi)\, \widehat{W_i}(d\xi) := \sum_{\lambda\in\Lambda} <h,\widehat{\psi_\lambda}>\; Z^{(i)}_\lambda,$$

where $Z^{(1)} := (Z^{(1)}_\lambda)_{\lambda\in\Lambda}$ and $Z^{(2)} := (Z^{(2)}_\lambda)_{\lambda\in\Lambda}$ are two fixed independent collections of independent and identically distributed standard Gaussian random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.
We then define the three maps $\mathbb{W}_1$, $\mathbb{W}_2$ and $\widetilde{\mathbb{W}}$ from $\mathcal{B}_0^{(\lambda_{1/4})}$ to $L^2(\Omega,\mathcal{F},\mathbb{P})$ by $\mathbb{W}_1(A) := \frac{1}{2}\mathscr{J}_1(\mathbb{1}_A + \mathbb{1}_{-A} + (\sqrt{2}-2)\mathbb{1}_{A\cap(-A)})$, $\mathbb{W}_2(A) := -\frac{1}{2}\mathscr{J}_2(i\mathbb{1}_A - i\mathbb{1}_{-A} + \sqrt{2}\mathbb{1}_{A\cap(-A)})$ and $\widetilde{\mathbb{W}} := \mathbb{W}_1 + \mathbb{W}_2$. We hence define, for every h in $\widehat{L^2_{\mathbb{C}}}(d\xi)$, $I(h) := \int_{\mathbb{R}} h_1(\xi)\mathbb{W}_1(d\xi) - \int_{\mathbb{R}} h_2(\xi)\mathbb{W}_2(d\xi)$. We have the following results:

Properties 3.1. *(i) $\mathbb{W}_1$ and $\mathbb{W}_2$ are independent Gaussian random measure on $\mathbb{R}$, independently scattered on $\mathbb{R}_+$, with control measure $\lambda_{1/4}$ which satisfy $\mathbb{W}_1(A) = \mathbb{W}_1(-A)$ and $\mathbb{W}_2(A) = -\mathbb{W}_2(-A)$ for any A in $\mathcal{B}_0^{(\lambda)}$.*

(ii) For any $h := h_1 + ih_2$ in $\widehat{L^2_{\mathbb{C}}}(d\xi)$ and i in $\{1;2\}$, the two Gaussian processes $(\mathscr{J}_i(h))_{h\in L^2_{\mathbb{C}}(d\xi)}$ and $(I(h))_{h\in\widehat{L^2_{\mathbb{C}}}(d\xi)}$ have the same distribution. In particular, $\int_{\mathbb{R}} h(\xi)\, \widehat{W}(d\xi) \overset{(d)}{=} \int_{\mathbb{R}} h(\xi)\, \widetilde{\mathbb{W}}(d\xi)$.

Proof. It is clear that the process $\mathbb{W}_1 := \{\mathbb{W}_1(A); A \in \mathcal{B}_0^{(\lambda_{1/4})}\}$ is gaussian. Moreover, for every A in $\mathcal{B}^{(\lambda_{1/4})}{}_0$, we have $\mathbb{W}_1(A) \rightsquigarrow \mathcal{N}(0; 1/4||\mathbb{1}_A + \mathbb{1}_{-A} + (\sqrt{2}-2)\mathbb{1}_{A\cap(-A)}||^2_{L^2(\mathbb{R})})$ and hence $\mathbb{W}_1(A) \rightsquigarrow \mathcal{N}(0; \frac{\lambda(A)}{2})$ since $||\mathbb{1}_A + \mathbb{1}_{-A} + (\sqrt{2}-2)\mathbb{1}_{A\cap(-A)}||^2_{L^2(\mathbb{R})} = 2\lambda(A)$.

Moreover, for every $A_1, A_2, \cdots, A_k$ in $(\mathcal{B}_0^{(\lambda_{1/4})})^k$ such that the A_i are disjoints and all in $\mathbb{R}_+$, the idependence of $\mathbb{W}_1(A_1), \mathbb{W}_1(A_2), \cdots, \mathbb{W}_1(A_k)$ results from the fact that $\mathbb{E}[\mathbb{W}_1(A_i)\mathbb{W}_1(A_j)] = \frac{1}{4} < \mathbb{1}_{A_i} + \mathbb{1}_{-A_i}, \mathbb{1}_{A_j} + \mathbb{1}_{-A_j} >= 0$. Finally, the equality, in $L^2(\Omega,\mathcal{F},\mathbb{P})$, between $\mathbb{W}(\underset{i\in\mathbb{N}}{\cup} A_i)$ and $\sum_{i\in\mathbb{N}} \mathbb{W}(A_i)$ is clear since $\mathbb{E}\left[\left(\mathbb{W}(\underset{i\in\mathbb{N}}{\cup} A_i) - \sum_{i=0}^N \mathbb{W}(A_i)\right)^2\right] = \sum_{i=N+1}^{+\infty} \lambda(A_i) \underset{N\to+\infty}{\longrightarrow} 0$. The series of independent random variables $\sum_{i\in\mathbb{N}} \mathbb{W}(A_i)$ converges in $L^2(\Omega,\mathcal{F},\mathbb{P})$ sense as well as almost surely. □

From first to second method

Conversly, let us start from the first method framework. We define for every function h in $\widehat{L^2_{\mathbb{C}}}(d\xi)$, $\mathscr{J}(h) := I(h) = \int_{\mathbb{R}} h_1(\xi)\mathbb{W}_1(d\xi) - \int_{\mathbb{R}} h_2(\xi)\mathbb{W}_2(d\xi)$, and for every f in $L^2_{\mathbb{R}}(du)$, $\int_{\mathbb{R}} f(u)W(du) := \mathscr{J}(\widehat{f})$. It is clear that (2.4) is verified.

Remark 3.1. *Since we have shown that the two methods of [75] and [73] are equivalent, we use, in the sequel, notations of both methods.*

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