Limites diffusives pour des équations cinétiques stochastiques

Sylvain de Moor

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présentée par

Sylvain De Moor

Préparée à l'unité mixte de recherche 6625

Institut de recherche mathématique de Rennes

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devant le jury composé de :

Thierry Goudon / *rapporteur*
Directeur de recherche à l'INRIA Sophia Antipolis

Mauro Mariani / *rapporteur*
Chercheur à l'Université de Rome La Sapienza

Erwan Faou / *examinateur*
Directeur de recherche à l'INRIA Rennes

Josselin Garnier / *examinateur*
Professeur à l'Université Paris 7

Arnaud Debussche / *directeur de thèse*
Professeur à l'ENS Rennes

Julien Vovelle / *directeur de thèse*
Chargé de recherche CNRS à l'Université Lyon 1

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Thèse encadrée par Arnaud Debussche et Julien Vovelle

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Introduction

Cette thèse présente des travaux de recherche sur les équations aux dérivées partielles stochastiques (EDPS), c'est-à-dire sur des équations aux dérivées partielles (EDP) que l'on perturbe de façon aléatoire. La théorie des probabilités sera omniprésente dans ces travaux afin de gérer mathématiquement la perturbation stochastique. D'un point de vue déterministe, les équations aux dérivées partielles permettent de décrire ou de modéliser un large panel de phénomènes, qu'ils soient par exemple de nature physique, chimique ou encore biologique. Leur étude mathématique présente ainsi un grand intérêt et donne naissance à un riche champ d'applications. Il apparaît souvent qu'il est pertinent de vouloir introduire un objet aléatoire afin de perturber une EDP, que ce soit par exemple pour modéliser une composante trop imprévisible du phénomène observé ou encore pour rendre compte d'une incertitude sur des données observées. On parle alors de bruitage de l'équation. La théorie des équations aux dérivées partielles stochastiques apparaît donc comme une extension naturelle de celle portant sur les EDP. Cette thèse porte principalement sur l'étude du procédé d'approximation-diffusion sur des équations cinétiques stochastiques. Elle contient également un résultat de régularité pour des EDPS quasi-linéaires de type parabolique ainsi qu'une étude des mesures invariantes d'une équation de Fokker-Planck stochastique.

Dans la suite de cette introduction, on se propose dans un premier temps d'exposer brièvement le procédé de l'approximation-diffusion, en s'intéressant successivement au cas déterministe puis stochastique. On présente ensuite une introduction à la méthode des fonctions-test perturbées qui est un outil efficace et élégant pour obtenir des résultats d'approximation-diffusion dans un contexte d'EDP stochastiques. Cette méthode sera, en particulier, utilisée dans plusieurs travaux présentés. Enfin, on propose un résumé des articles qui composent cette thèse : on présente une description des problèmes étudiés, les résultats obtenus, des idées de preuve ainsi que les techniques utilisées.

Limites diffusives

Dans cette première section, nous proposons une brève présentation du procédé d'approximation-diffusion pour des équations cinétiques collisionnelles. On s'intéresse dans un premier temps au cas déterministe puis, dans un second temps, au cas stochastique, où l'équation cinétique collisionnelle est bruitée.

Le cas déterministe

Nous nous intéressons à des équations cinétiques collisionnelles. De façon générale, ce type d'équation permet de modéliser la dynamique mésoscopique d'un nuage de particules (photons, neutrons, molécules gazeuses par exemple) dans un milieu extérieur (plasma, semi-conducteur par exemple). L'inconnue est la fonction de distribution f du nuage de particules : $f(t, x, v)$ représente la proportion de particules qui, au temps t, sont à la position spatiale x et ont la vitesse v. L'équation satisfaite par la fonction de distribution f peut se mettre sous la forme

$$\partial_t f \; + \; a(v) \cdot \nabla_x f \; = \; Cf,$$

où C est un opérateur qui modélise l'interaction des particules. Pour cette introduction, nous supposerons que la variable temporelle varie dans l'intervalle de temps $[0, \infty)$, la variable d'espace dans le tore $\mathbb{T}^N$ et la variable de vitesse dans un espace mesuré (V, μ), ce qui recouvre la majorité des cas présentés dans cette thèse. L'opérateur C n'agit que sur la variable de vitesse $v \in V$. La partie transport libre de l'équation est gouvernée par un champ $a : V \to \mathbb{R}^N$.

Le principe de l'approximation-diffusion est d'approcher la solution f qui rend compte du modèle à l'échelle mésoscopique par la solution d'une équation qui décrit l'évolution du système cette fois à l'échelle macroscopique. Pour cela, on modifie l'échelle à laquelle on observe la solution mésoscopique f. Précisément, donnons-nous $\varepsilon > 0$. On observe maintenant la solution f à l'échelle macroscopique en posant

$$f^\varepsilon(\theta(\varepsilon)t, \varepsilon x, v) := f(t, x, v), \quad \varepsilon > 0,$$

où θ est une fonction qui caractérise l'échelle de la variable temporelle et qui satisfait $\theta(\varepsilon) \to 0$ lorsque $\varepsilon \to 0$. L'équation vérifiée par f^ε est

$$\theta(\varepsilon)\partial_t f^\varepsilon \; + \; \varepsilon a(v) \cdot \nabla_x f^\varepsilon \; = \; Cf^\varepsilon. \tag{0.1}$$

On cherche maintenant à étudier le comportement de la solution f^ε quand le paramètre ε tend vers 0. Ce comportement dépend fortement de la forme de l'opérateur de collision C. Dans de nombreux cas de choix de l'opérateur de collision, on observe que, pour une échelle temporelle θ appropriée, la solution f^ε converge, en un certain sens, vers la solution d'une équation aux dérivées partielles diffusive, appelée limite diffusive ou encore limite hydrodynamique. Ainsi, on approche la solution f^ε par la solution d'une équation de diffusion : c'est le principe de l'approximation-diffusion.

À titre d'exemple, étudions le cas où l'opérateur de collision est donné par un opérateur linéaire L de relaxation de la forme

$$Cf = Lf = \int_V f(v) \, \mathrm{d}\mu(v) \, F - f,$$

où $v \mapsto F(v)$ est une fonction d'équilibre des vitesses qui vérifie $F > 0$ p.p. et $\int_V F(v) \, \mathrm{d}\mu(v) = 1$. On peut par exemple penser à une distribution Maxwellienne. On remarque que F est dans le noyau de L puisque $LF = 0$, et réciproquement que le noyau de l'opérateur L est engendré par la fonction d'équilibre F, à savoir

$$\ker(L) = \{\rho F, \, \rho \in \mathbb{R}\}.$$

De façon générale, la détermination du noyau de l'opérateur de collision C est primordiale puisque, à la limite $\varepsilon \to 0$ dans (0.1), on obtient que l'éventuelle limite formelle f de f^ε vérifie $Cf = 0$. Ainsi, dans notre cas particulier d'une relaxation linéaire, la limite f de f^ε s'écrira $f(t, x, v) = \rho(t, x)F(v)$. On observe une convergence vers l'équilibre F en la variable de vitesse $v \in V$. Il reste alors à caractériser le coefficient ρ. C'est cette dernière quantité qui va satisfaire une équation aux dérivées partielles diffusive. Il faut remarquer que l'on obtient donc une équation satisfaite par une variable macroscopique, à savoir par $\rho = \int_V f \, \mathrm{d}\mu$, la moyenne sur les vitesses de la limite formelle f.

Nous allons maintenant, toujours dans le cas d'une relaxation linéaire, expliquer formellement comment on obtient l'équation diffusive satisfaite par ρ. On utilise la méthode du développement de Hilbert, c'est-à-dire que l'on développe selon les puissances de ε la solution f^ε de (0.1). Formellement, on écrit

$$f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + ...,$$

et on introduit ce développement de la solution f^ε dans l'équation (0.1). On supposera que θ est de la forme $\theta(\varepsilon) = \varepsilon^\alpha$ pour un certain $\alpha > 0$ à déterminer de sorte que l'échelle soit la bonne pour obtenir une limite non triviale. On obtient donc

$$\begin{aligned} &\varepsilon^\alpha \partial_t f_0 + \varepsilon^{1+\alpha}\partial_t f_1 + \varepsilon^{2+\alpha}\partial_t f_2 + \varepsilon a(v)\cdot\nabla_x f_0 + \varepsilon^2 a(v)\cdot\nabla_x f_1 + \varepsilon^3 a(v)\cdot\nabla_x f_2 \\ &\quad = Lf_0 + \varepsilon L f_1 + \varepsilon^2 L f_2 + ... \end{aligned} \tag{0.2}$$

À la limite $\varepsilon \to 0$, on obtient $Lf_0 = 0$ de sorte que f_0 s'écrit $f_0 = \int_V f_0 \,\mathrm{d}\mu\, F =: \rho F$. On retrouve bien entendu le fait que f^ε tend vers une quantité de la forme ρF et nous cherchons maintenant à expliciter l'équation satisfaite par ρ. En identifiant les termes ayant une puissance ε^α, l'équation (0.2) montre que $\alpha \in \mathbb{N}^*$ sans quoi on obtient l'équation triviale $\partial_t \rho = 0$. Si $\alpha = 1$, l'identification des termes ayant une puissance ε dans l'équation (0.2) donne

$$\partial_t \rho F + a(v)\cdot\nabla_x \rho F = Lf_1. \tag{0.3}$$

Afin que f_1 soit correctement défini, il faut vérifier que cette équation d'inconnue f_1 admet des solutions. Si $g \in L^2(V)$, on peut facilement montrer que l'équation $Lf = g$ d'inconnue $f \in L^2(V)$ admet une solution si et seulement si

$$\int_V g \,\mathrm{d}\mu = 0,$$

auquel cas les solutions sont données par $f = -g + qF$, $q \in \mathbb{R}$. Ainsi, afin que le problème (0.3) soit bien posé, il faut que l'intégrale en vitesse du terme de gauche s'annule, c'est-à-dire que

$$\partial_t \rho + \operatorname{div}\left(\rho \int_V a(v)F(v)\,\mathrm{d}\mu(v)\right) = 0,$$

ce qui nous donne une équation aux dérivées partielles satisfaite par ρ. Il s'agit cependant d'un cas que nous allons écarter : cette équation est encore de type cinétique, nous souhaitons approcher f^ε par la solution d'une EDP diffusive. Ceci suggère donc d'étudier maintenant le cas $\alpha = 2$. On rappelle que l'équation (0.2) avec $\alpha = 2$ dans laquelle nous avions imposé $L(f_0) = 0$ est donnée par

$$\begin{aligned} &\varepsilon^2 \partial_t f_0 + \varepsilon^3 \partial_t f_1 + \varepsilon^4 \partial_t f_2 + \varepsilon a(v)\cdot\nabla_x f_0 + \varepsilon^2 a(v)\cdot\nabla_x f_1 + \varepsilon^3 a(v)\cdot\nabla_x f_2 \\ &\quad = \varepsilon L f_1 + \varepsilon^2 L f_2 + ... \end{aligned} \tag{0.4}$$

À l'ordre 1 en ε, on doit s'assurer que

$$a(v)\cdot\nabla_x \rho F = Lf_1, \tag{0.5}$$

ce qui nous permet de définir f_1 à condition que l'intégrale en vitesse du terme de gauche s'annule. Pour cela, nous allons imposer la condition suivante que le flux de a contre F est nul, c'est-à-dire que

$$\int_V a(v)F(v)\,\mathrm{d}\mu(v) = 0. \tag{0.6}$$

Ainsi, l'équation (0.5) est bien posée et on a

$$f_1 = -a(v)\cdot\nabla_x \rho F + q(x)F.$$

Ensuite, à l'ordre 2 en ε dans l'équation (0.4) on a

$$\partial_t \rho F + a(v) \cdot \nabla_x f_1 = L f_2. \tag{0.7}$$

De façon similaire à ce qui précède, cela nous permettra de définir f_2 à condition que l'intégrale en vitesse du terme de gauche s'annule, c'est-à-dire si

$$\partial_t \rho + \operatorname{div}\left(\int_V a(v) f_1 \, \mathrm{d}\mu(v) \right) = 0.$$

En injectant l'expression de f_1 trouvée plus haut et en remarquant que $\int_V a(v) q(x) F(v) \, \mathrm{d}\mu = 0$ grâce à (0.6), cette dernière équation se réécrit

$$\partial_t \rho - \operatorname{div}\left(K \nabla_x \rho \right) = 0, \tag{0.8}$$

où K est la matrice

$$K := \int_V a(v) \otimes a(v) F(v) \, \mathrm{d}\mu(v).$$

Nous supposerons alors que cette matrice K est bien définie, i.e. que $K < \infty$. Ainsi, si ρ satisfait l'équation aux dérivées partielles diffusive (0.8), on peut bien définir f_2. En résumé, nous avons donc montré formellement que si $\theta(\varepsilon) = \varepsilon^2$ et si (0.6) est vérifiée, alors $f^\varepsilon = \rho F + O(\varepsilon)$ où ρ satisfait une EDP diffusive, ce que l'on voulait. Souvent, afin que l'EDP diffusive (0.8) obtenue soit non-dégénérée, nous supposerons de plus que la matrice de diffusion K est définie positive.

Une preuve rigoureuse du résultat de limite de diffusion que nous venons d'étudier formellement peut être trouvé dans l'article de P. Degond, T. Goudon, F. Poupaud [DGP00]. De façon générale, dans le cas déterministe, les problèmes d'approximation-diffusion ont suscité beaucoup d'intérêt et de nombreux travaux à ce sujet ont été menés ; à commencer par les articles de E. W.Larsen, J. B. Keller [LK74] et A. Bensoussan, J.-L. Lions, G. Papanicolaou [BLP79]. Des opérateurs de collision non-linéaires ont été étudiés, notamment dans les articles [BGP87, BGPS88] en ce qui concerne les équations de transfert radiatif et dans les articles [Mel02] et [GM03] dans le cas de l'opérateur de collision de Boltzmann-Pauli. Nous mentionnons également une revue des résultats et des méthodes employées dans ce domaine de F. Golse [Gol98]. Enfin, de nombreuses références à propos de problèmes variés de limite diffusive pourront être trouvées dans [DGP00].

La méthode du développement de Hilbert fonctionne dans de nombreux cas et permet d'obtenir facilement de façon formelle la limite hydrodynamique du système cinétique collisionnel étudié. Noter qu'en toute généralité, en ce qui concerne l'opérateur de collision C, le développement de la solution f^ε fait apparaître la différentielle de C à savoir $f \mapsto DC(f)$.

Le cas stochastique

Dans un contexte stochastique, le principe de l'approximation-diffusion reste le même excepté le fait que l'on étudie des équations cinétiques collisionnelles *bruitées* et que l'on cherche à les approcher par des équations aux dérivées partielles *stochastiques* diffusives. Nous commençons par exposer la façon dont on bruite les équations cinétiques collisionelles. Nous rappelons le problème déterministe dans le cas où l'échelle temporelle θ est donnée par $\theta(\varepsilon) = \varepsilon^2$ (c'est précisément l'échelle appropriée dans la grande majorité des résultats qui composent cette thèse) :

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} C f^\varepsilon. \tag{0.9}$$

On s'intéresse dans cette thèse à un bruit de type multiplicatif. Précisément, on étudie l'équation bruitée

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} C f^\varepsilon + \frac{1}{\varepsilon^\beta} m^\varepsilon f^\varepsilon, \tag{0.10}$$

où le processus aléatoire m^ε est défini par

$$m^\varepsilon(t,x) := m\left(\frac{t}{\varepsilon^\gamma}, x\right),$$

le processus m étant un processus de Markov stationnaire sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ adapté à une filtration $(\mathcal{F}_t)_{t\geq 0}$ et centré sous sa mesure invariante. Le processus m est à valeurs dans un espace *ad hoc* de fonctions dépendant de la variable d'espace. Les paramètres d'échelle β et γ sont à déterminer de façon adéquate. On souligne que le processus m ne dépend pas de la vitesse $v \in V$. On cherche alors à étudier le comportement de la solution f^ε lorsque ε tend vers 0. On aimerait, à l'instar du cas déterministe, montrer que f^ε converge, dans un sens à préciser, vers la solution d'une EDPS diffusive.

Nous discutons maintenant du choix des paramètres d'échelle β et γ. Dans la suite, nous imposons $\gamma = 2\beta$. On peut comprendre cette hypothèse en observant que le choix $\gamma = 2\beta$ est précisément celui sous lequel le processus

$$M_t^\varepsilon := \frac{1}{\varepsilon^\beta}\int_0^t m^\varepsilon(s)\,\mathrm{d}s = \frac{1}{\varepsilon^\beta}\int_0^t m(\varepsilon^{-\gamma}s)\,\mathrm{d}s, \quad t \geq 0,$$

converge en loi lorsque ε tend vers 0, en l'occurrence, vers $(Q^{\frac{1}{2}}W_t, t \geq 0)$ où Q un opérateur de covariance et W un processus de Wiener cylindrique sur $L^2(\mathbb{T}^N)$. Prenons f et g dans $L^2(\mathbb{T}^N)$ et notons $(.,.)$ le produit scalaire de $L^2(\mathbb{T}^N)$. Formellement, si $\gamma = 2\beta$, on a, en utilisant la stationnarité du processus m,

$$\begin{aligned}
\mathbb{E}(f, M_t^\varepsilon)(g, M_s^\varepsilon) &= \varepsilon^{-2\beta}\,\mathbb{E}\iint_{\mathbb{T}^{2N}}\int_0^t\int_0^s f(x)g(y)\,m(\varepsilon^{-2\beta}u, x)\,m(\varepsilon^{-2\beta}r, y)\,\mathrm{d}u\,\mathrm{d}r\,\mathrm{d}(x,y)\\
&= \varepsilon^{-2\beta}\,\mathbb{E}\iint_{\mathbb{T}^{2N}}\int_0^t\int_0^s f(x)g(y)\,m(\varepsilon^{-2\beta}(u-r), x)\,m(0, y)\,\mathrm{d}u\,\mathrm{d}r\,\mathrm{d}(x,y)\\
&= \mathbb{E}\iint_{\mathbb{T}^{2N}}\int_0^t\int_{-\varepsilon^{-2\beta}r}^{\varepsilon^{-2\beta}(s-r)} f(x)g(y)\,m(u, x)\,m(0, y)\,\mathrm{d}u\,\mathrm{d}r\,\mathrm{d}(x,y).
\end{aligned}$$

Si l'on suppose $s > t$, on a $s - r > 0$ pour tout $r \in [0,t]$ de sorte que, lorsque ε tend vers 0,

$$\begin{aligned}
\mathbb{E}(f, M_t^\varepsilon)(g, M_s^\varepsilon) &\longrightarrow \mathbb{E}\iint_{\mathbb{T}^{2N}}\int_0^t\int_{\mathbb{R}} f(x)g(y)\,m(u, x)\,m(0, y)\,\mathrm{d}u\,\mathrm{d}r\,\mathrm{d}(x,y)\\
&= t\iint_{\mathbb{T}^{2N}} f(x)g(y)\,k(x,y)\,\mathrm{d}(x,y) = t\,(f, Qg),
\end{aligned}$$

où l'on a posé

$$k(x,y) := \mathbb{E}\int_{\mathbb{R}} m(u,x)m(0,y)\,\mathrm{d}u, \quad x, y \in \mathbb{T}^N, \tag{0.11}$$

et où Q est l'opérateur intégral de noyau k sur $L^2(\mathbb{T}^N)$

$$Qf(x) := \int_{\mathbb{T}^N} k(x,y)f(y)\,\mathrm{d}y, \quad f \in L^2(\mathbb{T}^N). \tag{0.12}$$

En envisageant le cas $s \leq t$, on obtient finalement, à la limite $\varepsilon \to 0$,

$$\mathbb{E}(f, M_t^\varepsilon)(g, M_s^\varepsilon) \longrightarrow (t \wedge s)\,(f, Qg).$$

Ceci suggère bien, comme annoncé, la convergence en loi du processus $(M_t^\varepsilon, t \geq 0)$ vers le bruit blanc $(Q^{\frac{1}{2}}W_t, t \geq 0)$. Ces calculs sont bien entendus formels et il faut s'assurer de l'existence de tous les objets introduits ci-dessus. Ceci est fait en détails dans les chapitres de la thèse concernés.

Sous la condition $\gamma = 2\beta$, nous étudions dans cette thèse le cas $\beta = 1$ de sorte que l'on s'intéresse à l'équation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} C f^\varepsilon + \frac{1}{\varepsilon} m^\varepsilon f^\varepsilon, \quad m^\varepsilon(t,x) := m\left(\frac{t}{\varepsilon^2}, x\right). \tag{0.13}$$

Nous terminons avec une remarque sur l'équation (0.13). Une idée naturelle afin d'éliminer le terme de bruit dans cette équation est de faire le changement de variable suivant : on introduit $g^\varepsilon = f^\varepsilon \exp(-M^\varepsilon)$ où $M_t^\varepsilon = \frac{1}{\varepsilon} \int_0^t m_s^\varepsilon \,\mathrm{d}s$. La nouvelle variable g^ε vérifie alors l'équation

$$\partial_t g^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x g^\varepsilon = \frac{1}{\varepsilon^2} C g^\varepsilon - \frac{1}{\varepsilon} a(v) \cdot \nabla_x M^\varepsilon g^\varepsilon.$$

En étudiant cette nouvelle équation, on peut se convaincre que ce changement de variable n'apporte rien qui pourrait améliorer de façon significative le résultat d'approximation-diffusion que nous prouvons en travaillant directement sur l'équation (0.13). De plus, la preuve que nous utilisons en travaillant sur l'équation (0.13) est robuste dans le sens où nous pourrions par exemple traiter sans problèmes le cas où le terme de bruit $\frac{1}{\varepsilon} m^\varepsilon f^\varepsilon$ est remplacé par $\frac{1}{\varepsilon} m^\varepsilon K f^\varepsilon$ où K est un opérateur borné, cas où le changement de variable exhibé ci-dessus tombe immédiatement en défaut. C'est pourquoi dans la suite nous n'utilisons jamais ce changement de variable bien qu'il semble pourtant naturel au premier abord.

Les premiers travaux sur des problèmes d'approximation-diffusion dans le cas stochastique sont dus à R. Z. Hasminskii [Has66a, Has66b]. L'article pionnier de G. C. Papanicolaou, D. Stroock, S. R. S. Varadhan [PSV77] traite un cas de limite fluide stochastique en utilisant une approche martingale et la méthode des fonctions-test perturbées. Nous mentionnons également l'ouvrage de J.-P. Fouque, J. Garnier, G. C. Papanicolaou, K. Solna [FGPS10]. Enfin une généralisation de la méthode des fonctions-test perturbées permettant de s'attaquer à des problèmes infini-dimensionnels est présentée dans les travaux de A. Debussche, J. Vovelle [DV12] et A. de Bouard, M. Gazeau [dBG12].

La méthode des fonctions-test perturbées

Dans cette section, nous présentons la méthode des fonctions-test perturbées qui est un outil très efficace dans la mise en place des preuves de résultats d'approximation-diffusion dans un contexte stochastique. Elle a été introduite par G. C. Papanicolaou, D. Stroock, S. R. S. Varadhan [PSV77]. Nous présentons en détail la méthode dans le cas d'un bruit multiplicatif et d'un opérateur de collision C linéaire de relaxation de la forme

$$Cf = Lf = \int_V f(v) \,\mathrm{d}\mu(v) \, F - f,$$

où $v \mapsto F(v)$ est une fonction d'équilibre des vitesses qui vérifie $F > 0$ presque partout et $\int_V F(v) \,\mathrm{d}\mu(v) = 1$. Ce cadre est celui de l'article [DV12]. On rappelle que dans ce cas l'équation cinétique collisionnelle bruitée qui nous intéresse s'écrit, en notant $A = v \cdot \nabla_x$ l'opérateur de transport,

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} A f^\varepsilon = \frac{1}{\varepsilon^2} L f^\varepsilon + \frac{1}{\varepsilon} m^\varepsilon f^\varepsilon.$$

Nous soulignons [1] que le processus f^ε prend ses valeurs dans l'espace de Lebesgue à poids $L^2_{F^{-1}} := L^2(\mathbb{T}^N \times V, \mathrm{d}x \, F^{-1} \mathrm{d}\mu(v))$ et que m^ε est à valeurs dans une boule E de l'espace

1. Voir les chapitres de thèse concernés pour plus de détails.

$W^{1,\infty}(\mathbb{T}^N)$. Le processus f^ε n'est pas Markovien, le processus m^ε devant être lui aussi connu pour connaître exactement f^ε. En revanche, le processus $(f^\varepsilon, m^\varepsilon)$ est Markovien. Écrivons son générateur infinitésimal, noté $\mathscr{L}^\varepsilon$. Si $\varphi : L^2_{F^{-1}} \times E \to \mathbb{R}$ est une fonction-test assez régulière, en notant $D\varphi(f)$ la différentielle de φ par rapport à f et en l'identifiant au gradient, on a

$$\mathscr{L}^\varepsilon\varphi(f,n) = -\frac{1}{\varepsilon}(Af, D\varphi(f)) + \frac{1}{\varepsilon^2}(Lf, D\varphi(f)) + \frac{1}{\varepsilon}(fn, D\varphi(f)) + \frac{1}{\varepsilon^2}M\varphi(f,n),$$

où M désigne le générateur infinitésimal du processus m et $(.,.)$ le produit scalaire dans $L^2_{F^{-1}}$. Afin de comprendre le comportement de la solution f^ε lorsque ε tend vers 0, il est naturel de vouloir étudier l'asymptotique du générateur $\mathscr{L}^\varepsilon$ lorsque ε tend vers 0, c'est-à-dire d'identifier les limites $\varepsilon \to 0$ des quantités $\mathscr{L}^\varepsilon\varphi$ pour toute fonction-test φ convenable. Cependant, en l'état, la quantité $\mathscr{L}^\varepsilon\varphi$ dégénère lorsque ε tend vers 0. Pour contourner ce problème, nous allons corriger la fonction-test φ. En s'inspirant de la méthode du développement de Hilbert qui est basée sur le développement de la solution f^ε elle-même, on réalise ici un développement de la fonction-test. On considère donc la *fonction-test perturbée* φ^ε définie par

$$\varphi^\varepsilon := \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2.$$

Les fonctions φ_1 et φ_2 (et suivantes, si nécessaire) sont appelées des correcteurs : elles permettent de corriger le point du vue que l'on a de la fonction-test φ visée en la perturbant. Le but est alors de choisir les correcteurs φ_1 et φ_2 convenablement de sorte que l'on ait

$$\mathscr{L}^\varepsilon\varphi^\varepsilon = \mathscr{L}\varphi + O(\varepsilon),$$

où $\mathscr{L}$ sera interprété comme le générateur de la diffusion stochastique limite. Considérons une fonction-test φ assez régulière et qui ne dépend que de la variable f (on ne cherche à caractériser le générateur limite que sur la première composante du couple $(f^\varepsilon, m^\varepsilon)$). Dès lors, puisque φ ne dépend pas de $n \in E$, on obtient

$$\begin{aligned}\mathscr{L}^\varepsilon\varphi^\varepsilon(f,n) = &-\frac{1}{\varepsilon}(Af, D\varphi(f)) + \frac{1}{\varepsilon^2}(Lf, D\varphi(f)) + \frac{1}{\varepsilon}(fn, D\varphi(f)) \\ &- (Af, D\varphi_1(f)) + \frac{1}{\varepsilon}(Lf, D\varphi_1(f)) + (fn, D\varphi_1(f)) + \frac{1}{\varepsilon}M\varphi_1 \\ &- \varepsilon(Af, D\varphi_2(f)) + (Lf, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)) + M\varphi_2.\end{aligned} \tag{0.14}$$

On identifie alors successivement les termes ayant une même puissance de ε.

Ordre ε^{-2}

À l'ordre ε^{-2} on obtient que

$$(Lf, D\varphi(f)) = 0 \tag{0.15}$$

pour tout $f \in L^2_{F^{-1}}$. Introduisons ici le semi-groupe $g(t,f)$ de l'opérateur L sur $L^2_{F^{-1}}$, c'est-à-dire que $g(t,f)$ satisfait l'équation

$$\begin{cases} \dfrac{d.}{dt}g(t,f) = Lg(t,f), \\ g(0,f) = f. \end{cases}$$

L'opérateur de relaxation linéaire L étant simple, on peut facilement obtenir une expression du semi-groupe g, à savoir

$$g(t,f) = \rho F + (f - \rho F)e^{-t}, \quad t \geq 0, \qquad \rho = \int_V f \,\mathrm{d}\mu.$$

Noter que l'on retrouve l'effet de relaxation vers l'équilibre en vitesse de l'opérateur L puisque $g(t,f) \to \rho F$ lorsque $t \to \infty$. L'équation (0.15) implique que $\varphi(g(t,f))$ est constant au cours du temps. Ainsi, $\varphi(f) = \varphi(\lim_{t\to\infty} g(t,f)) = \varphi(\rho F)$, c'est-à-dire que φ ne dépend de f qu'à travers ρF. C'est ce que nous supposerons dorénavant. En particulier, cela implique que pour tout $f, h \in L^2_{F^{-1}}$,

$$(h, D\varphi(f)) = (\overline{h}F, D\varphi(\overline{f}F)), \tag{0.16}$$

où l'on a introduit la notation $\overline{f}$ qui désigne la moyenne en vitesse de la fonction f,

$$\overline{f} = \int_V f(v)\, \mathrm{d}\mu(v).$$

Ordre ε^{-1}

L'identification des termes d'ordre ε^{-1} dans l'équation (0.14) donne

$$(Af, D\varphi(f)) + (fn, D\varphi(f)) + (Lf, D\varphi_1(f)) + M\varphi_1 = 0. \tag{0.17}$$

On considère le processus de Markov $((g(t,f), m(t,n)), t \geq 0)$ dont le générateur infinitésimal, que l'on notera B, est clairement donné par

$$B\varphi(f,n) = (Lf, D\varphi(f)) + M\varphi.$$

Ainsi, l'équation (0.17) se réécrit sous la forme de l'équation de Poisson suivante

$$B\varphi_1 = -(Af, D\varphi(f)) - (fn, D\varphi(f)). \tag{0.18}$$

Avant de continuer notre analyse, nous rappelons quelques faits sur les équations de Poisson. Soit G le générateur infinitésimal d'un processus de Markov stationnaire de mesure invariante λ à valeurs dans un espace X. On note e^{tG} le semi-groupe associé à G. Pour $\psi, \theta : X \to \mathbb{R}$ fixées, on cherche à résoudre l'équation dite de Poisson

$$G\psi = \theta,$$

i.e. à inverser G. On peut montrer que si $\int_X \theta\, \mathrm{d}\lambda = 0$ alors

$$\psi(x) = -\int_0^\infty e^{tG}\theta(x)\, \mathrm{d}t, \quad x \in X,$$

est, sous réserve d'existence de l'intégrale, une solution de l'équation de Poisson. On notera $\psi = G^{-1}\theta$. Revenons à l'équation (0.18). Le correcteur φ_1 sera parfaitement défini si cette équation de Poisson admet une solution. Étudions la condition d'annulation. La mesure stationnaire du processus $((g(t,f), m(t,n)), t \geq 0)$ est $\delta_{\rho F} \times \nu$ où ρ désigne la moyenne de f en vitesse, $\rho = \int_V f\, \mathrm{d}\mu$, et ν la mesure stationnaire de m. On doit donc vérifier que

$$\int_E (A\rho F, D\varphi(\rho F)) + (\rho F n, D\varphi(\rho F))\, \mathrm{d}\nu(n) = 0.$$

Le premier terme ci-dessus est nul puisque l'on peut écrire, grâce à (0.16),

$$(A\rho F, D\varphi(\rho F)) = (\overline{A\rho F}F, D\varphi(\rho F))$$

et que l'on a supposé en (0.6) que

$$\int_V a(v)F(v)\, \mathrm{d}\mu(v) = 0$$

de sorte que $\overline{A\rho F} = 0$. Le second terme est également nul, puisque, le processus m ayant été supposé centré sous la mesure invariante ν, on a $\int_V n \, \mathrm{d}\nu(n) = 0$. De plus, on peut calculer de façon formelle le premier correcteur grâce à la formule

$$\begin{aligned}\varphi_1(f,n) &= \int_0^\infty e^{tB}[(Af, D\varphi(f)) + (fn, D\varphi(f))] \, \mathrm{d}t \\ &= \int_0^\infty \mathbb{E}[(Ag(t,f), D\varphi(g(t,f))) + (g(t,f)m(t,n), D\varphi(g(t,f)))] \, \mathrm{d}t,\end{aligned}$$

et l'on trouve, après calculs, l'expression

$$\varphi_1(f,n) = (Af, D\varphi(f)) - (fM^{-1}I(n), D\varphi(f)). \tag{0.19}$$

Ordre 1

Le correcteur φ ne dépendant que de ρF et le correcteur φ_1 étant défini par (0.19), l'expression de $\mathscr{L}^\varepsilon \varphi^\varepsilon$ se résume à

$$\begin{aligned}\mathscr{L}^\varepsilon \varphi^\varepsilon(f,n) = -(Af, D\varphi_1(f)) + (fn, D\varphi_1(f)) - \varepsilon(Af, D\varphi_2(f)) \\ + (Lf, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)) + M\varphi_2,\end{aligned}$$

que l'on réécrit sous la forme

$$\begin{aligned}\mathscr{L}^\varepsilon \varphi^\varepsilon(f,n) &= B\varphi_2 - (Af, D\varphi_1(f)) + (fn, D\varphi_1(f)) + O(\varepsilon) \\ &= B\varphi_2 + q(f,n) + O(\varepsilon),\end{aligned}$$

où nous avons défini $q(f,n) = -(Af, D\varphi_1(f)) + (fn, D\varphi_1(f))$. À ce stade, il faut remarquer que le terme d'ordre 1 du terme de droite ne peut être le générateur infinitésimal limite que l'on cherche. En effet, ce dernier dépend encore de f (et non de la variable ρ) et également de la variable n. Le correcteur φ_2 est là pour corriger ce défaut. Étant donné qu'on ne peut uniquement corriger des quantités dont la moyenne sous la mesure invariante du processus $((g(t,f), m(t,n)), t \geq 0)$ est nulle, on ajoute artificiellement cette moyenne, c'est-à-dire que l'on écrit

$$\begin{aligned}\mathscr{L}^\varepsilon \varphi^\varepsilon(f,n) = B\varphi_2 + \left[q(f,n) - \int_E q(\rho F, n) \, \mathrm{d}\nu(n)\right] \\ + \int_E q(\rho F, n) \, \mathrm{d}\nu(n) + O(\varepsilon).\end{aligned}$$

On choisit alors φ_2 solution de l'équation de Poisson

$$B\varphi_2 = -q(f,n) + \int_E q(\rho F, n) \, \mathrm{d}\nu(n),$$

cette dernière équation étant bien posée puisque le terme de droite est bien de moyenne nulle sous la mesure invariante du processus $((g(t,f), m(t,n)), t \geq 0)$. On obtient ainsi

$$\mathscr{L}^\varepsilon \varphi^\varepsilon(f,n) = \mathscr{L}\varphi(\rho F) + O(\varepsilon),$$

où l'opérateur $\mathscr{L}$ est défini par

$$\mathscr{L}\varphi(\rho F) = \int_E q(\rho F, n) \, \mathrm{d}\nu(n) = -\int_E [(A\rho F, D\varphi_1(\rho F)) - (\rho F n, D\varphi_1(\rho F))] \, \mathrm{d}\nu(n).$$

De plus, grâce à l'expression (0.19) de φ_1, le générateur limite $\mathscr{L}$ peut entièrement s'expliciter. On vérifie alors, conformément au résultat de [DV12], que c'est le générateur d'un processus de diffusion qui vérifie l'équation

$$\mathrm{d}\rho = \mathrm{div}(K\nabla_x \rho) \, \mathrm{d}t + \rho \circ Q^{1/2} \mathrm{d}W_t,$$

où Q est un opérateur positif à trace qui s'exprime en fonction de m, sa définition étant celle donnée ci-dessus en (0.11) et (0.12). À titre de comparaison, on rappelle quand dans le cas déterministe ($m^\varepsilon \equiv 0$), la limite de diffusion correspondante s'écrivait $\mathrm{d}\rho = \mathrm{div}(K\nabla_x \rho)\,\mathrm{d}t$.

Une limite de diffusion fractionnaire stochastique

Le premier chapitre de cette thèse présente un résultat d'approximation-diffusion fractionnaire dans un contexte stochastique. Nous commençons par rappeler le cadre du problème dans le cas déterministe. On étudie une équation cinétique collisionnelle remise à l'échelle dont l'opérateur de collision L est une relaxation linéaire. Précisément, l'équation s'écrit

$$\theta(\varepsilon)\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = L(f^\varepsilon). \tag{0.20}$$

L'inconnue f^ε dépend du temps $t \in [0,\infty)$, de l'espace $x \in \mathbb{R}^N$ et de la vitesse $v \in V$ où $(V,\mu) = (\mathbb{R}^N, \mathrm{d}v)$. Nous continuerons à noter l'espace des vitesses V et non $\mathbb{R}^N$ pour le différencier de celui d'espace. Le champ de vitesse a de la partie transport est ici donné par $a(v) = v$. Enfin, l'opérateur de collision est donné par

$$L(f) = \int_V f(v)\,\mathrm{d}v\, F - f,$$

où $v \mapsto F(v)$ est une fonction d'équilibre des vitesses, bornée, paire, strictement positive presque partout et d'intégrale $\int_V F(v)\,\mathrm{d}v = 1$. Noter que la parité de la fonction F garantit que la condition (0.6) est vérifiée, à savoir $\int_V vF(v)\,\mathrm{d}v = 0$.

Lorsque la matrice $K = \int_V v \otimes vF(v)\,\mathrm{d}v$ est finie, on rappelle que, sous l'échelle $\theta(\varepsilon) = \varepsilon^2$, la solution f^ε de (0.20) converge vers ρF où ρ est solution de l'équation de diffusion

$$\partial_t \rho - \mathrm{div}(K\nabla_x \rho) = 0.$$

Le cas où la matrice K n'est plus finie a été étudié par A. Mellet dans l'article [Mel10]. Plus précisément, on suppose que la fonction d'équilibre F a un comportement de type puissance à l'infini, c'est-à-dire que

$$F(v) \underset{|v|\to\infty}{\sim} \frac{\kappa_0}{|v|^{N+\alpha}} \tag{0.21}$$

avec $\alpha \in (0,2)$ et $\kappa_0 > 0$. Noter que l'on a bien $K = \infty$ sous cette hypothèse. La question qui se pose est maintenant la suivante : sous cette nouvelle condition, peut-on, éventuellement sous une échelle θ différente, obtenir tout de même une limite diffusive pour l'équation cinétique (0.20) ? Sous réserve de choisir une échelle θ appropriée par rapport au comportement à l'infini de la fonction d'équilibre F, la réponse est positive : A. Mellet prouve que dans le cas $\theta(\varepsilon) = \varepsilon^\alpha$ la solution f^ε de (0.20) converge vers ρF où ρ est une solution de l'équation diffusive fractionnaire

$$\partial_t \rho + \kappa(-\Delta)^{\alpha/2}\rho = 0,$$

pour un certain $\kappa > 0$ et où $(-\Delta)^s$ désigne l'opérateur Laplacien fractionnaire, qui peut-être défini, par exemple, par transformation de Fourier sur $L^2(\mathbb{R}^N)$ en posant

$$\mathcal{F}[(-\Delta)^s f](\xi) = |\xi|^{2s}\mathcal{F}[f](\xi), \quad \xi \in \mathbb{R}^N.$$

Le premier chapitre de cette thèse s'intéresse à la problématique correspondante dans le cas stochastique où l'équation cinétique est perturbée par un bruit de type multiplicatif. Pour $0 < \alpha < 2$, on s'intéresse à l'équation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon^{\alpha-1}} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^\alpha} L f^\varepsilon + \frac{1}{\varepsilon^{\frac{\alpha}{2}}} m^\varepsilon f^\varepsilon, \tag{0.22}$$

avec condition initiale $f^\varepsilon(0) = f_0^\varepsilon$, où l'on a défini $m^\varepsilon(t,x) := m\left(t/\varepsilon^\alpha, x\right)$, le processus m étant un processus de Markov stationnaire centré sous la mesure invariante sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ et adapté à une filtration $(\mathcal{F}_t)_{t\geq 0}$; son générateur infinitésimal sera noté M. Noter que l'échelle du bruit, qui dépend de α, est elle-aussi en corrélation avec le comportement de la fonction d'équilibre des vitesses F à l'infini. On étudie la convergence en loi du processus f^ε dans un espace adéquat. Introduisons sur $\mathbb{R}^N$ l'opérateur

$$J := -\Delta_x + |x|^2,$$

et définissons, pour $\gamma \in \mathbb{R}$,

$$S^\gamma(\mathbb{R}^N) := \{u \in \mathcal{S}'(\mathbb{R}^N),\ J^{\frac{\gamma}{2}} u \in L^2(\mathbb{R}^N)\},$$

où $\mathcal{S}'$ désigne l'ensemble des distributions tempérées. On étudie alors la convergence en loi du processus f^ε dans l'espace $C([0,T]; S^{-\eta}(\mathbb{R}^N))$ pour tout $\eta > 0$. On notera que dans [DV12], il était possible de travailler avec les espaces de Sobolev $H^\eta(\mathbb{T}^N)$ car le contexte était périodique en espace. Dans notre cas, sur l'espace $\mathbb{R}^N$, il nous faut récupérer de la compacité. C'est le rôle de la multiplication par $|x|^2$ dans la définition de l'opérateur J puisqu'alors $S^\eta(\mathbb{R}^N)$ s'injecte de façon compacte dans $L^2(\mathbb{R}^N)$ si $\eta > 0$. Le résultat prouvé est le suivant.

Théorème 0.0.1. *On suppose que* $(f_0^\varepsilon)_{\varepsilon>0}$ *est borné dans* $L^2_{F^{-1}} = L^2(\mathbb{R}^N \times V, dx\, F^{-1} dv)$ *et que l'on a la convergence*

$$\rho_0^\varepsilon := \int_V f_0^\varepsilon \, dv \underset{\varepsilon\to 0}{\longrightarrow} \rho_0 \ \text{dans } L^2(\mathbb{R}^N).$$

Alors, pour tout $\eta > 0$ *et* $T > 0$, $\rho^\varepsilon := \int_V f^\varepsilon \, dv$ *converge en loi dans* $C([0,T], S^{-\eta}(\mathbb{R}^N))$ *vers la solution* ρ *de l'équation diffusive fractionnaire stochastique*

$$d\rho = -\kappa(-\Delta)^{\frac{\alpha}{2}} \rho dt + \frac{1}{2} H\rho + \rho Q^{\frac{1}{2}} dW_t, \ \text{dans } [0,T] \times \mathbb{R}^N,$$

avec condition initiale $\rho(0) = \rho_0$ *dans* $L^2(\mathbb{R}^N)$, *où* W *est un processus de Wiener cylindrique sur* $L^2(\mathbb{R}^N)$,

$$\kappa := \frac{\kappa_0}{c_{d,\alpha}} \int_0^\infty |t|^\alpha e^{-t} \, dt,$$

$$H := \int_E n M^{-1} I(n) \, d\nu(n) \in W^{1,\infty},$$

et où Q *est un opérateur positif à trace qui s'exprime*[2] *en fonction du processus* m.

L'équation limite (0.29) peut également être écrite sous sa forme Stratonovich, à savoir

$$\mathrm{d}\rho = -\kappa(-\Delta)^{\frac{\alpha}{2}} \rho \, \mathrm{d}t + \rho \circ Q^{\frac{1}{2}} \mathrm{d}W_t.$$

Ainsi, l'équation limite dans le cas stochastique est l'équation limite du cas déterministe bruitée par un bruit cylindrique multiplicatif.

Dans le cas déterministe ($m^\varepsilon \equiv 0$), la preuve d'A. Mellet dans l'article [Mel10] s'articule comme suit. Après avoir prouvé quelques estimées *a priori* sur la solution f^ε, on veut passer à la limite dans la formulation faible de l'équation (0.22), que l'on obtient en la multipliant par une fonction-test ξ dépendant de (t,x) et en intégrant sur $\mathbb{R}^+ \times \mathbb{R}^N \times V$. Plus précisément, on ne passe pas à la limite dans la formulation faible proprement dite, on utilise la méthode des moments qui consiste à ne pas multiplier directement l'équation (0.22) par une fonction-test ξ

2. voir le chapitre 1 pour plus de détails.

mais par une modification *ad hoc* de ξ. En effet, on considère ξ une fonction-test cible, et on introduit χ^ε solution de l'équation auxiliaire

$$\chi^\varepsilon - \varepsilon v \cdot \nabla_x \chi^\varepsilon = \xi.$$

La fonction χ^ε est une fonction-test qui approche ξ dans le sens où elle est régulière et satisfait $\chi^\varepsilon \to \xi$ lorsque $\varepsilon \to 0$. On multiplie alors l'équation (0.22) par χ^ε, on intègre sur $\mathbb{R}^+ \times \mathbb{R}^N \times V$ et on est en mesure de passer à la limite lorsque $\varepsilon \to 0$.

Dans notre contexte stochastique, on ne peut pas adapter facilement la preuve décrite dans [DV12], le fait de considérer des vitesses non bornées $a(v) = v$ sur $V = \mathbb{R}^N$ engendrant de sérieuses difficultés pour contrôler la partie transport de l'équation cinétique. Nous utilisons donc la méthode des moments, que l'on couple à la méthode des fonctions-test perturbées. Pour ce faire, nous remarquons dans un premier temps qu'il nous sera suffisant d'étudier les limites de $\mathscr{L}^\varepsilon \varphi$ [3] où $\varphi(f) = (f, \xi F)$ et $\varphi(f) = (f, \xi F)^2$ avec $\xi \in L^2(\mathbb{R}^N)$ une fonction-test régulière. Ensuite, dans le cas où $\varphi(f) = (f, \xi F)$ par exemple, en s'inspirant de la méthode des fonctions-test perturbées, adaptée à l'échelle α de l'équation (0.22), et de la méthode des moments, on définit la *fonction-test perturbée*

$$\varphi^\varepsilon = (f, \chi^\varepsilon F) + \varepsilon^{\frac{\alpha}{2}}(f, \delta^\varepsilon F) + \varepsilon^\alpha (f, \theta^\varepsilon F),$$

où, comme dans le cas déterministe, χ^ε satisfait l'équation

$$\chi^\varepsilon - \varepsilon v \cdot \nabla_x \chi^\varepsilon = \xi,$$

et où δ^ε et θ^ε satisfont des équations de la forme [4]

$$(L + \varepsilon v \cdot \nabla_x + M)(\delta^\varepsilon F) = \psi_1, \quad (L + \varepsilon v \cdot \nabla_x + M)(\theta^\varepsilon F) = \psi_2.$$

Ce faisant, on est bien en mesure de prouver que $\mathscr{L}^\varepsilon \varphi^\varepsilon = \mathscr{L}\varphi + O(\varepsilon)$ où l'on montre que $\mathscr{L}$ est le générateur infinitésimal de la diffusion stochastique fractionnaire annoncée ci-dessus. Le reste de la preuve est standard et suit les mêmes lignes que l'article [DV12].

L'équation de transfert radiatif perturbée par un processus de Wiener

Dans le deuxième chapitre, nous présentons un résultat de limite hydrodynamique stochastique sur une équation cinétique avec un opérateur de collision non-linéaire bruitée par un processus de Wiener cylindrique.

Au vu du résultat d'A. Debussche et J. Vovelle [DV12] et de celui du chapitre 1 de cette thèse, il est naturel de vouloir s'intéresser à la limite hydrodynamique d'un problème cinétique collisionnel bruité plus complexe, en particulier d'un cas présentant un opérateur de collision non-linéaire. Une idée en guise de première étape avant de considérer des opérateurs non-linéaires plus compliqués est de perturber légèrement l'opérateur de collision linéaire de relaxation $L(f) = \int_V f \, d\mu - f$ pour le rendre non-linéaire. Nous considérons alors l'opérateur de collision

$$C(f) = \sigma(\overline{f})(\overline{f} - f), \qquad \overline{f} = \int_V f \, d\mu,$$

3. Nous reprenons ici les notations de la Section : $\mathscr{L}^\varepsilon$ est le générateur infinitésimal du processus Markovien $(f^\varepsilon, m^\varepsilon)$, $(.,.)$ désigne le produit scalaire de $L^2_{F^{-1}}$ et M le générateur infinitésimal de m.

4. Voir le chapitre 1 pour plus de détails.

où $\sigma : \mathbb{R} \to \mathbb{R}$ est une fonction sur laquelle nous porterons des hypothèses par la suite. Dans le cas déterministe et sous l'échelle habituelle $\theta(\varepsilon) = \varepsilon^2$, on étudie donc l'équation cinétique suivante

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} C(f^\varepsilon). \tag{0.23}$$

Avec l'opérateur C défini ci-dessus, cette équation cinétique collisionnelle est appelée équation de transfert radiatif. Elle modélise l'interaction entre un milieu environnant continu et un flux de photons rayonnant au travers. La fonction $\sigma : \mathbb{R} \to \mathbb{R}$ qui perturbe l'opérateur linéaire de relaxation est appelée l'opacité. Noter que la perturbation non-linéaire n'opère que sur la moyenne de f en vitesse, ce qui est un fait important. Toujours dans le cas déterministe, des résultats d'approximation-diffusion ont déjà été étudiés par C. Bardos, F. Golse, B. Perthame, R. Sentis dans les articles [BGP87] et [BGPS88] où il est prouvé que la solution f^ε de (0.23) converge vers ρ solution de l'équation diffusive non-linéaire, dite de Rosseland,

$$\partial_t \rho - \mathrm{div}_x(\sigma(\rho)^{-1} K \nabla_x \rho) \ = \ 0,$$

où la matrice de diffusion K est donnée par $K = \int_V a(v) \otimes a(v)\, \mathrm{d}\mu(v)$. Dans ce contexte, on parle alors d'approximation de Rosseland.

Le résultat exposé dans le deuxième chapitre présente un équivalent stochastique de ce résultat dans le cas où l'équation cinétique collisionnelle est bruitée par un processus de Wiener cylindrique. Présentons le cadre de l'étude. On considère l'équation

$$\begin{cases} \mathrm{d}f^\varepsilon + \dfrac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon \mathrm{d}t = \dfrac{1}{\varepsilon^2} \sigma(\overline{f^\varepsilon})(\overline{f^\varepsilon} - f^\varepsilon)\mathrm{d}t + f^\varepsilon \circ Q \mathrm{d}W_t, \\ f^\varepsilon(0) = \rho_{\mathrm{in}}. \end{cases} \tag{0.24}$$

L'inconnue f^ε dépend du temps $t \in [0, \infty)$, de l'espace $x \in \mathbb{T}^N$ et de la vitesse $v \in V$ où $(V, \mu) = (\mathbb{T}^N, \mathrm{d}v)$. Le champ de vitesse $a : V \to \mathbb{R}^N$ de la partie transport vérifie les conditions habituelles $\int_V a(v)\, \mathrm{d}v = 0$ et le fait que la matrice $K = \int_V a(v) \otimes a(v)\, \mathrm{d}v$ est finie et définie positive. On formule également des hypothèses sur l'opacité $\sigma : \mathbb{R} \to \mathbb{R}$ que nous ne détaillerons pas ici [5]. Le bruit W est un processus de Wiener cylindrique sur l'espace de Hilbert $L^2(\mathbb{T}^N)$ que l'on peut définir en posant

$$\mathrm{d}W_t \ = \ \sum_{k \geq 0} e_k \, \mathrm{d}\beta_k(t), \tag{0.25}$$

où $(\beta_k)_{k \geq 0}$ sont des mouvements Browniens réels sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ filtré et $(e_k)_{k \geq 0}$ une base orthonormale de l'espace de Hilbert $L^2(\mathbb{T}^N)$. L'opérateur de covariance Q est un opérateur linéaire auto-adjoint sur $L^2(\mathbb{T}^N)$ sur lequel nous faisons une hypothèse de type « propriété de régularisation » ; nous reviendrons sur cette hypothèse dans un moment.

Il est à noter que la nature du bruit est différente des précédents cas étudiés : d'une part il ne dépend pas de l'échelle ε et d'autre part il est déjà sous la forme limite d'un processus de Wiener cylindrique. En particulier, l'équation (0.24) est une équation de type Itô pour laquelle on peut utiliser la formule d'Itô. Ce contexte plus favorable nous permet d'obtenir un résultat bien plus fort que les résultats d'approximation-diffusion stochastiques présentés jusqu'ici. En effet, le processus f^ε ne converge plus vers ρ en loi mais fortement dans un espace *ad hoc*. Le résultat obtenu est le suivant.

Théorème 0.0.2. *Soit f^ε la solution du problème cinétique (0.24) et ρ la solution de l'équation diffusive stochastique non-linéaire*

$$\begin{cases} d\rho - \mathrm{div}_x \left(\sigma(\rho)^{-1} K \, \nabla_x \rho\right) dt = \rho \circ Q dW_t, \\ \rho(0) = \rho_{in}. \end{cases} \tag{0.26}$$

5. Voir le chapitre 2 pour plus de détails.

Alors la solution f^ε converge quand ε tend vers 0 vers la limite fluide ρ dans le sens où

$$\sup_{t\in[0,T]} \mathbb{E}\|f_t^\varepsilon - \rho_t\|_{L^1_{x,v}} \leq C\varepsilon.$$

En ce qui concerne la preuve, l'équation étudiée (0.24) étant de type Itô, nous pouvons utiliser la méthode du développement de Hilbert, qui est souvent utilisée dans le cadre déterministe. On développe donc f^ε selon les puissances de ε. Contrairement à de nombreux cas déterministes où un développement jusqu'à l'ordre 2 suffit, nous sommes contraints ici, à cause de la présence du bruit, de pousser le développement jusqu'à l'ordre 3. On écrit

$$f^\varepsilon = \rho + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3^\varepsilon + r^\varepsilon, \tag{0.27}$$

où ρ est la solution du problème limite (0.26), f_1, f_2 et f_3^ε sont des correcteurs à définir de façon appropriée et r^ε le reste du développement. Notons que le troisième correcteur f_3^ε dépend de ε. En introduisant ce développement dans l'équation (0.24), il est facile de déduire formellement les équations qui doivent être satisfaites par les correcteurs f_1, f_2 et f_3^ε, ces derniers pouvant alors être totalement explicités en fonction de ρ. Afin de conclure la preuve il reste alors d'une part à estimer le reste r^ε du développement et d'autre part à étudier le comportement des correcteurs f_1, f_2 et f_3^ε.

En ce qui concerne l'estimation du reste r^ε, on peut facilement montrer que r^ε satisfait une équation de type Itô que l'on peut expliciter. On observe alors qu'en appliquant la formule d'Itô on peut estimer r^ε dans l'espace $X := L^\infty(0,T;L^1(\Omega;L^1(\mathbb{T}^N\times V)))$ et montrer qu'il est de l'ordre de ε :

$$\|r^\varepsilon\|_X = O(\varepsilon).$$

Ainsi, puisque le résultat que l'on cherche à démontrer s'écrit $\|f^\varepsilon - \rho\|_X = O(\varepsilon)$, il reste, en vertu du développement (0.27), à prouver que $\|\varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3^\varepsilon\|_X = O(\varepsilon)$. Ceci revient à contrôler les correcteurs f_1, f_2 et f_3^ε dans X. Ces derniers s'expriment en fonction de ρ et de ses dérivées. En effet, on a $f_1 = -\sigma(\rho)^{-1}a(v)\cdot\nabla_x\rho$ et le correcteur f_2 (resp. f_3^ε) fait intervenir les dérivées jusqu'à l'ordre 2 (resp. 3) de ρ. Ainsi, afin de contrôler les correcteurs, il est nécessaire d'avoir de la régularité sur la solution ρ du problème limite (0.26). Pour ce faire, on utilise le résultat de régularité pour les équations aux dérivées partielles stochastiques quasi-linéaires de type parabolique prouvé dans le chapitre 4. Ce dernier s'applique bien ici, sous réserve que le bruit soit assez régulier. C'est à ce moment qu'intervient l'hypothèse de type « propriété de régularisation » sur l'opérateur de covariance Q que nous avions évoqué plus haut. Précisément, on impose la condition $\sum_{k\geq 0}\|Qe_k\|^2_{W^{4,\infty}_x} < \infty$ où $(e_k)_{k\in\mathbb{N}}$ désigne la base Hilbertienne de $L^2(\mathbb{T}^N)$ selon laquelle nous avions développé le processus de Wiener cylindrique W.

L'équation de transfert radiatif perturbée par un bruit Markovien

Dans le troisième chapitre de cette thèse, on s'intéresse de nouveau au cas de l'équation de transfert radiatif, qui présente un opérateur de collision non-linéaire, mais que l'on bruite cette fois-ci de façon multiplicative avec un processus Markovien qui dépend de l'échelle ε. On étudie donc l'équation non-linéaire suivante

$$\begin{cases} \partial_t f^\varepsilon + \dfrac{1}{\varepsilon}a(v)\cdot\nabla_x f^\varepsilon = \dfrac{1}{\varepsilon^2}\sigma(\overline{f^\varepsilon})(\overline{f^\varepsilon}F - f^\varepsilon) + \dfrac{1}{\varepsilon}f^\varepsilon m^\varepsilon, \\ f^\varepsilon(0) = f_0^\varepsilon, \end{cases} \tag{0.28}$$

Précisons le contexte. L'inconnue f^ε dépend du temps $t \in [0, \infty)$, de l'espace $x \in \mathbb{T}^N$ et de la vitesse $v \in V$ où (V, μ) est un espace mesuré. On rappelle que la notation $\overline{f}$ désigne la moyenne de f en vitesse : $\overline{f} = \int_V f(v)\,\mathrm{d}\mu$. Le champ de vitesse $a : V \to \mathbb{R}^N$ de la partie transport est supposé borné et vérifie les conditions habituelles $\int_V a(v)F(v)\,\mathrm{d}\mu = 0$ et le fait que la matrice $K = \int_V a(v) \otimes a(v)F(v)\,\mathrm{d}\mu$ est finie et définie positive. Nous imposons également une autre contrainte sur a que nous préciserons dans un moment. L'équilibre en vitesses $v \mapsto F(v)$ est tel que

$$F > 0 \text{ p.p.}, \quad \int_V F(v)\,\mathrm{d}\mu = 1, \quad \sup_{v \in V} F(v) < \infty.$$

L'opacité $\sigma : \mathbb{R} \to \mathbb{R}$ est supposé Lipschitz continue et bornée inférieurement et supérieurement par des constantes strictement positives. Enfin, le terme stochastique m^ε est donné par

$$m^\varepsilon(t, x) := m\left(\frac{t}{\varepsilon^2}, x\right),$$

le processus m étant un processus de Markov stationnaire centré sous la mesure invariante sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ et adapté à une filtration $(\mathcal{F}_t)_{t\geq 0}$; son générateur infinitésimal sera noté M.

Nous soulignons que les hypothèses faites sur l'opacité σ sont bien plus faibles que celles requises pour le résultat du chapitre précédent. Nous énonçons maintenant le résultat obtenu dans ce chapitre.

Théorème 0.0.3. *On suppose que* $(f_0^\varepsilon)_{\varepsilon>0}$ *est bornée dans* $L^2_{F^{-1}} = L^2(\mathbb{R}^N \times V, dx\, F^{-1}dv)$ *et que l'on a la convergence*

$$\rho_0^\varepsilon := \int_V f_0^\varepsilon\, dv \underset{\varepsilon \to 0}{\longrightarrow} \rho_0 \text{ dans } L^2(\mathbb{T}^N).$$

Alors, pour tous $\eta > 0$ *et* $T > 0$, $\rho^\varepsilon := \overline{f^\varepsilon}$ *converge en loi dans* $C([0,T], H^{-\eta}(\mathbb{T}^N))$ *et* $L^2(0,T; L^2(\mathbb{T}^N))$ *vers la solution* ρ *de l'équation stochastique de diffusion non-linéaire*

$$d\rho - \mathrm{div}_x(\sigma(\rho)^{-1} K \nabla_x \rho)\, dt = H\rho dt + \rho\, Q^{\frac{1}{2}}\, dW_t, \text{ dans } [0,T] \times \mathbb{T}^N, \tag{0.29}$$

avec condition initiale $\rho(0) = \rho_0$ *dans* $L^2(\mathbb{T}^N)$*, et où* W *est un processus de Wiener cylindrique sur* $L^2(\mathbb{T}^N)$,

$$K := \int_V a(v) \otimes a(v) F(v)\, d\mu(v)$$

et

$$H := \int_E n M^{-1} I(n)\, d\nu(n) \in W^{1,\infty}.$$

L'équation limite obtenue dans le cas stochastique est donc l'équation limite dérivée dans le cas déterministe bruitée de façon multiplicative par un processus de Wiener cylindrique. Il est à noter que l'on obtient la convergence en loi du processus ρ^ε non seulement dans l'espace $C([0,T], H^{-\eta}(\mathbb{T}^N))$ mais également dans $L^2(0,T; L^2(\mathbb{T}^N))$.

La structure de la preuve est identique au cas présenté dans l'article [DV12]. C'est-à-dire que l'on utilise la méthode des fonctions-test perturbées pour identifier le générateur limite, on montre ensuite la tension de la famille de processus $(\rho^\varepsilon)_{\varepsilon>0}$ dans un bon espace et on passe enfin à la limite dans la formulation martingale du problème. Le problème majeur est que le fait de prouver, comme dans [DV12], la tension de $(\rho^\varepsilon)_{\varepsilon>0}$ dans l'espace $C([0,T]; H^{-\eta}(\mathbb{T}^N))$ pour $\eta > 0$ n'est désormais plus suffisant pour passer à la limite dans le terme non-linéaire $\sigma(\rho^\varepsilon)(\rho^\varepsilon F - f^\varepsilon)$. Nous devons donc obtenir la tension de $(\rho^\varepsilon)_{\varepsilon>0}$ dans un meilleur espace, en l'occurrence nous la prouvons dans l'espace $L^2(0,T; L^2(\mathbb{T}^N))$, ce qui sera suffisant pour passer à la limite dans le terme non-linéaire. Ce faisant, nous récupérons également à la fin la convergence en loi du processus ρ^ε dans $L^2(0,T;L^2(\mathbb{T}^N))$. Afin de prouver la tension de la famille $(\rho^\varepsilon)_{\varepsilon>0}$ dans $L^2(0,T;L^2(\mathbb{T}^N))$, nous utilisons un lemme de moyenne.

D'un point de vue déterministe, considérons pour tout $\varepsilon > 0$ l'équation cinétique

$$\varepsilon\partial_t g^\varepsilon + v\cdot\nabla_x g^\varepsilon = h^\varepsilon, \quad (t,x,v)\in[0,T]\times\mathbb{T}^N\times V,$$

où $g^\varepsilon, h^\varepsilon$ sont uniformément bornées par rapport à ε dans $L^2(0,T;L^2(\mathbb{T}^N\times V))$. Les lemmes de moyenne permettent de montrer que la moyenne en vitesse $\int_V g^\varepsilon\,\mathrm{d}\mu(v)$ de la solution g^ε est plus régulière uniformément en ε dans le sens où

$$\|g^\varepsilon\|_{H^{1/2}((0,T)\times\mathbb{T}^N)} \le C,$$

la constante C étant indépendante de ε. Les travaux sur ces lemmes de moyenne sont nombreux mais nous citons par exemple l'article de F. Golse, P.-L. Lions, B. Perthame, R. Sentis [GLPS88]. Dans la preuve de notre résultat, nous utilisons un lemme de moyenne prouvé dans [Jab09]. L'équation (0.28) se réécrit en effet

$$\varepsilon\partial_t f^\varepsilon + a(v)\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon}\sigma(\overline{f^\varepsilon})(\overline{f^\varepsilon}F - f^\varepsilon) + f^\varepsilon m^\varepsilon,$$

et il est possible de montrer que le terme de droite est bien uniformément borné en ε dans $L^2(0,T;L^2(\mathbb{T}^N\times V))$. Le lemme de moyenne nous permet alors de prouver que pour un $s>0$,

$$\mathbb{E}\int_0^T \|\rho_s^\varepsilon\|^2_{H^s(\mathbb{T}^N)}\,\mathrm{d}s \le C,$$

la constante C étant indépendante de ε. Ceci donne de la compacité en espace pour le processus ρ^ε et permet ensuite d'obtenir aisément la tension annoncée de $(\rho^\varepsilon)_{\varepsilon>0}$ dans l'espace $L^2(0,T;L^2(\mathbb{T}^N))$. Nous soulignons que pour pouvoir appliquer le lemme de moyenne présenté dans [Jab09], nous devons supposer que le champ a de la partie transport libre vérifie la condition suivante, usuelle dans le cadre des lemmes de moyenne,

$$\forall\varepsilon>0,\ \forall(\xi,\alpha)\in S^{N-1}\times\mathbb{R},\ \ \mu\left(\{v\in V, |a(v)\cdot\xi+\alpha|<\varepsilon\}\right)\le\varepsilon^\theta,$$

pour un certain $\theta\in(0,1]$. Ceci est précisément l'hypothèse supplémentaire sur le champ a que nous avions évoqué plus haut mais pas encore explicité.

Enfin, remarquons que le fait de pouvoir obtenir la tension de la famille $(\rho^\varepsilon)_{\varepsilon>0} = (\overline{f^\varepsilon})_{\varepsilon>0}$ dans $L^2(0,T;L^2(\mathbb{T}^N))$ permet également de traiter aisément le cas légèrement plus général

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}a(v)\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon^2}\sigma(\overline{f^\varepsilon})(\overline{f^\varepsilon}F - f^\varepsilon) + \frac{1}{\varepsilon}\lambda(\overline{f^\varepsilon})f^\varepsilon m^\varepsilon,$$

où $\lambda:\mathbb{R}\to\mathbb{R}$ est continue bornée. Cette remarque s'applique en particulier dans le cas linéaire $\sigma\equiv 1$, ce qui permet donc de renforcer le résultat traité dans [DV12], d'une part en donnant la convergence en loi de ρ^ε dans l'espace $L^2(0,T;L^2(\mathbb{T}^N))$ et d'autre part en autorisant un bruit multiplicatif de la forme $\frac{1}{\varepsilon}\lambda(\overline{f^\varepsilon})f^\varepsilon m^\varepsilon$.

Un résultat de régularité pour les EDPS quasi-linéaires paraboliques

Nous laissons maintenant de côté les problèmes d'approximation-diffusion dans le cas stochastique pour présenter le résultat du quatrième chapitre de cette thèse, à savoir un résultat de régularité pour des équations aux dérivées partielles stochastiques quasi-linéaires de type parabolique. Ce résultat est utilisé dans la preuve du résultat du chapitre 2. Le bruit considéré sera multiplicatif. Commençons par présenter le contexte. On se donne $D \subset \mathbb{R}^N$ un domaine borné à frontière lisse, et pour $T > 0$ on définit $D_T = (0,T) \times D$, $S_T = (0,T] \times \partial D$. On étudie le problème suivant

$$\begin{cases} \mathrm{d}u = \mathrm{div}(B(u))\,\mathrm{d}t + \mathrm{div}\,(A(u)\nabla u)\,\mathrm{d}t + F(u)\,\mathrm{d}t + H(u)\,\mathrm{d}W & \text{dans } D_T, \\ u = 0 & \text{dans } S_T, \\ u(0) = u_0 & \text{dans } D, \end{cases} \tag{0.30}$$

où W est un processus de Wiener cylindrique sur un certain espace de Hilbert K et où H est une fonction à valeurs dans l'espace $\gamma(K,X)$ des opérateurs γ-radonifiants de K dans divers espaces de Sobolev X. On souhaite s'intéresser à la régularité de la solution u du problème (0.30).

Un tel résultat de régularité est important et intéressant en soi. En effet, l'équation (0.30) est relativement générale et permet de modéliser de nombreux phénomènes dans divers domaines. La régularité des solutions de cette équation permet d'obtenir de nombreuses informations qualitatives sur ces solutions ; c'est également une première étape nécessaire lors de l'étude de schémas d'approximation numérique. D'autre part, ce résultat est intéressant d'un point de vue de l'étude de l'approximation-diffusion stochastique. En effet, la classe de modèles du type (0.30) est riche et contient de nombreuses équations qui sont des limites diffusives de systèmes cinétiques bruités, par exemple l'équation limite diffusive parabolique de l'article [DV12] ainsi que les limites diffusives exhibées dans cette thèse. On s'attend également, sur des modèles cinétiques plus complexes, à des limites de diffusion de la forme (0.30).

En ce qui concerne les travaux passés en relation avec ce type de problème, nous commençons par mentionner, dans le cas déterministe, la théorie classique de Schauder pour les équations paraboliques linéaires, voir [Lie96], qui permet d'estimer les normes de Hölder de la solution en fonction des paramètres connus de l'équation. Toujours dans le cas déterministe, le cas quasi-linéaire parabolique, c'est-à-dire le cas d'équations du type (0.30) sans le bruit, est traité en détails dans le livre classique [LSU68]. Dans le contexte stochastique, nous citons le travail de L. Denis et A. Matoussi [DM13] où un principe du maximum est prouvé pour une EDPS de type similaire à (0.30), mais avec un coefficient H plus général qui peut dépendre du gradient de la solution. Un résultat de régularité Höldérienne des solutions de systèmes paraboliques non-linéaires est prouvée dans l'article de L. Beck et F. Flandoli [BF13] en utilisant des méthodes d'énergie.

En ce qui concerne notre résultat, nous prouvons que la solution u est Hölderienne en la variable de temps d'exposant $1/2 - \varepsilon$, la régularité temporelle étant évidemment limitée par le terme stochastique, et aussi régulière que souhaité en la variable d'espace tant que les coefficients de l'équation le sont. Avant d'énoncer le résultat, nous présentons l'hypothèse faite sur le facteur H du bruit. On introduit les espaces de Bessel qui sont une certaine échelle d'espaces de régularité. Précisément, on définit l'espace de Bessel $H^{a,r}(D)$, $a \geq 0$, $r \in (1,\infty)$, par interpolation complexe : si $a \geq 0$ et $m \in \mathbb{N}$ est tel que $a \leq m < a+1$ alors on définit

$$H^{a,r}(D) := [W^{m,r}(D), L^r(D)]_{(m-a)/m},$$

où $L^r(D)$ et $W^{m,r}(D)$ désignent respectivement l'espace de Lebesgue et l'espace classique de Sobolev-Slobodeckij [6]. Dès lors, on définit les espaces $H_0^{a,r}(D)$ comme étant l'adhérence de $C_c^\infty(D)$ dans $H^{a,r}(D)$. Ces espaces sont liés à l'opérateur de Laplace avec conditions de Dirichlet sur D que l'on notera Δ_D. En tant qu'opérateur sur L^r, son domaine est $H_0^{2,r}$. De plus, les domaines de ses puissances sont les espaces de Bessel, c'est-à-dire que l'on a $D((-\Delta_D)^\alpha) = H_0^{2\alpha,r}$. Concernant le coefficient H, on notera, pour $a \geq 0$ et $r \in [2,\infty)$, ($\mathrm{H}_{a,r}$) l'hypothèse suivante

6. Voir le chapitre 4 pour plus de détails sur les espaces de régularité en jeu.

$$\|H(u)\|_{\gamma(K,H_0^{a,r})} \leq \begin{cases} C\big(1+\|u\|_{H_0^{a,r}}\big), & a\in[0,1], \\ C\big(1+\|u\|_{H_0^{a,r}}+\|u\|^a_{H_0^{1,ar}}\big), & a>1, \end{cases} \tag{H$_{a,r}$}$$

c'est-à-dire que l'on suppose que H envoie $H_0^{a,r}$ dans $\gamma(K,H_0^{a,r})$ lorsque $a\in[0,1]$ et qu'il envoie $H_0^{a,r}\cap H_0^{1,ar}$ dans $\gamma(K,H_0^{a,r})$ si $a>1$. Le résultat est alors le suivant.

Théorème 0.0.4. *Soit $k\in\mathbb{N}$. Soit u une solution faible du problème (0.30) telle que, pour tout $p\in[2,\infty)$,*

$$u\in L^2(\Omega;C([0,T];L^2))\cap L^p(\Omega;L^\infty(0,T;L^p))\cap L^2(\Omega;L^2(0,T;W_0^{1,2})).$$

On suppose que

(i) $u_0\in L^m(\Omega;C^{k+\iota}(\overline{D}))$ pour un certain $\iota>0$ et tout $m\in[2,\infty)$,

(ii) A, $B\in C_b^k$ et $F\in C_b^{k-1}$,

(iii) l'hypothèse (H$_{a,r}$) est satisfaite pour tous $a<k+1$ et $r\in[2,\infty)$.

Alors pour tout $\lambda\in(0,1/2)$ il existe $\beta>0$ tel que, pour tout $m\in[2,\infty)$, la solution faible u appartient à $L^m(\Omega;C^{\lambda,k+\beta}(\overline{D_T}))$.

Pour la preuve du résultat, une adaptation au cas stochastique de la méthode déterministe présentée dans [LSU68] semble difficile. On introduit une nouvelle méthode basée sur une idée très simple qui consiste à séparer le déterministe de l'aléatoire : la solution faible u de (0.30) est décomposée en deux parties $u=y+z$ où y est solution d'une équation aux dérivées partielles linéaire de type parabolique à coefficients aléatoires et z est solution d'une équation aux dérivées partielles stochastique linéaire de type parabolique avec le même bruit que dans (0.30). Ce faisant, la régularité de u est réduite à l'étude de la régularité de y et z qui s'obtient par des techniques bien connues d'équations aux dérivées partielles déterministes pour y et de régularisation par convolution stochastique pour z.

Nous présentons maintenant un peu plus en détails cette méthode. Les principales difficultés étant dues à l'opérateur différentiel de second ordre et au terme stochastique, nous supposons que $B=F=0$ et considérons des conditions aux bords périodiques, c'est-à-dire que l'on se place sur $D=\mathbb{T}^N$ le tore de dimension N. On considère u une solution faible de

$$\begin{cases} \mathrm{d}u=\operatorname{div}(A(u)\nabla u)\,\mathrm{d}t+H(u)\,\mathrm{d}W, \\ u(0)=u_0, \end{cases} \tag{0.31}$$

et z une solution de

$$\begin{cases} \mathrm{d}z=\Delta z\,\mathrm{d}t+H(u)\,\mathrm{d}W, \\ z(0)=0. \end{cases}$$

La solution z peut être exprimée par la convolution stochastique avec le semi-groupe généré par l'opérateur Laplacien avec conditions de Dirichlet sur D, que l'on notera $(S(t))_{t\geq0}$, à savoir

$$z(t)=\int_0^t S(t-s)H(u)\,\mathrm{d}W(s) \tag{0.32}$$

pour laquelle des propriétés de régularisation sont bien connues. En définissant ensuite $y=u-z$, on obtient facilement que y est solution de

$$\begin{cases} \partial_t y=\operatorname{div}(A(u)\nabla y)+\operatorname{div}((A(u)-\mathrm{I})\nabla z), \\ y(0)=u_0, \end{cases} \tag{0.33}$$

qui est une équation aux dérivées partielles linéaire de type parabolique à coefficients aléatoires. Ceci étant, la structure de la preuve est la suivante. Grâce à des estimées *a priori* pour l'équation (0.31), nous avons

$$u \in L^p(\Omega; L^\infty(0,T; L^p(\mathbb{T}^N))) \cap L^2(\Omega; L^2(0,T; W_0^{1,2}(\mathbb{T}^N))), \quad \forall p \in [2,\infty).$$

En utilisant alors la méthode de factorisation, qui permet d'étudier la régularité d'une convolution stochastique, il est possible de montrer que z défini par (0.32) possède assez de régularité pour que ∇z soit une fonction avec de bonnes propriétés d'intégrabilité. Grâce à cette régularité accrue de z, il est maintenant possible d'utiliser un résultat déterministe classique concernant les EDP linéaires paraboliques à coefficients discontinus, voir [LSU68], pour l'équation (0.33) de sorte que y est Hölder continue en temps et espace. Et cette régularité est propagée sur $u = y + z$. La solution u étant maintenant plus régulière, z l'est également, à un niveau où la théorie de Schauder pour les EDP linéaires de type parabolique avec coefficients Hölder continus s'applique pour l'équation (0.33), voir [Lie96]. Ainsi y est-elle plus régulière, et $u = y + z$ également. Tant que les coefficients et la donnée initiale de l'équation (0.30) sont suffisamment réguliers, il est possible d'itérer ce raisonnement afin de prouver la régularité annoncée pour la solution u.

Mesures invariantes pour une équation de Fokker-Planck stochastique

Enfin, nous présentons ici le dernier chapitre de cette thèse, qui présente un résultat d'existence et unicité d'une mesure invariante pour une équation de Fokker-Planck stochastique, où l'équation standard de Fokker-Planck est bruitée par un terme de force aléatoire. On dérive en particulier dans ce chapitre une estimation hypocoercitive sur la solution de cette équation. Plus précisément, présentons le cadre d'étude. On s'intéresse à l'équation suivante

$$\mathrm{d}f \;+\; v \cdot \nabla_x f \;\mathrm{d}t \;+\; \lambda \nabla_v f \odot \mathrm{d}W_t \;=\; \mathcal{Q}(f)\;\mathrm{d}t \tag{0.34}$$

où l'inconnue f dépend des variables $t \in [0,\infty)$, $x \in \mathbb{T}^N$ et $v \in \mathbb{R}^N$. L'opérateur $\mathcal{Q}$ est l'opérateur de Fokker-Planck, dont l'expression est donnée par

$$\mathcal{Q}(f) \;=\; \Delta_v f \;+\; \mathrm{div}_v(vf).$$

Nous définissons maintenant le bruit $\mathrm{d}W_t$. Pour cela, on introduit tout d'abord un opérateur Γ auto-adjoint positif sur $L^2(\mathbb{T}^N;\mathbb{R}^N)$ vérifiant $\mathrm{Tr}(\Gamma) < \infty$ ainsi qu'un système orthonormé complet $(H_j)_{j\in\mathbb{N}}$ dans $L^2(\mathbb{T}^N;\mathbb{R}^N)$ constitué de vecteurs propres de Γ dont les valeurs propres positives associées sont notées $(\gamma_j)_{j\in\mathbb{N}}$:

$$\Gamma H_j = \gamma_j H_j, \quad j \in \mathbb{N}.$$

La perturbation aléatoire $\mathrm{d}W_t$ est un Γ–processus de Wiener sur $L^2(\mathbb{T}^N;\mathbb{R}^N)$, que l'on peut par exemple définir par la série

$$\mathrm{d}W_t(x) \;=\; \sum_j \Gamma^{\frac{1}{2}} H_j(x)\;\mathrm{d}\beta_j(t) = \sum_j \gamma_j^{\frac{1}{2}} H_j(x)\;\mathrm{d}\beta_j(t)$$

où les $(\beta_j)_{j\in\mathbb{N}}$ sont des mouvements browniens réels indépendants. On introduit alors la notation $F_j := \Gamma^{\frac{1}{2}} H_j$ de sorte que l'on écrira désormais le bruit sous la forme

$$\mathrm{d}W_t(x) \;=\; \sum_j F_j(x)\;\mathrm{d}\beta_j(t).$$

Concernant ces coefficients $(F_j)_{j\in\mathbb{N}}$, nous supposerons qu'ils vérifient la condition suivante

$$\sum_j \|F_j\|_\infty^2 + \|\nabla_x F_j\|_\infty^2 \leq 1.$$

La notation $\odot$ tient compte du produit scalaire sur $\mathbb{R}^N$ et du fait que le terme stochastique est considéré sous forme Stratonovich. La paramètre λ s'apparente à la taille du bruit. On introduit la distribution de Maxwell $\mathcal{M}$ sur $\mathbb{R}^N$ dont l'expression est donnée par

$$\mathcal{M}(v) = (2\pi)^{-N/2} e^{-|v|^2/2}, \quad v \in \mathbb{R}^N.$$

Il faut remarquer que l'opérateur $\mathcal{Q}$ est agréable à manipuler dans l'espace à poids $L^2(\mathbb{R}^N, \mathcal{M}^{-1}\mathrm{d}v)$ dans lequel il est auto-adjoint. Ainsi, dans ce qui suit, on ne travaillera pas exactement sur la variable f mais plutôt sur la variable g définie par $f = \mathcal{M}^{\frac{1}{2}} g$. Ce faisant, on peut facilement vérifier que g est solution du problème

$$\begin{cases} \mathrm{d}g \;+\; v\cdot\nabla_x g \;\mathrm{d}t \;+\; \lambda\left(\nabla_v - \dfrac{v}{2}\right) g \;\odot \mathrm{d}W_t \;=\; Lg\;\mathrm{d}t, \\ g(0) = g_{\mathrm{in}}, \end{cases} \tag{0.35}$$

et où l'opérateur L, qui est cette fois auto-adjoint sur $L^2(\mathbb{R}^N)$, est donné par

$$Lg \;=\; \Delta_v g \;+\; \left(\frac{N}{2} - \frac{|v|^2}{4}\right) g.$$

Bien entendu, les résultats obtenus sur l'équation satisfaite par g se transmettent facilement à l'équation satisfaite par f.

Le but du chapitre est d'étudier l'existence et l'unicité d'une mesure invariante pour le problème (0.35). Nous obtenons ceci en prouvant une estimation hypocoercitive sur la solution g. Commençons par dire quelques mots sur la théorie de l'hypocoercivité qui a été introduite par C. Villani dans son mémoire [Vil09]. Le principe est de s'intéresser à la convergence de solutions à des modèles cinétiques collisionnels vers leurs équilibres. Pour fixer les idées, considérons une équation de la forme

$$\partial_t f + v\cdot\nabla_x f = Qf, \tag{0.36}$$

où Q est un opérateur auto-adjoint qui agit uniquement sur la variable de vitesse. Nous supposons également que le noyau de l'opérateur Q est de dimension finie et, en introduisant Π_ℓ la projection orthogonale sur $\ker(Q)$ dans l'espace $L^2(\mathbb{R}^N, \mathrm{d}v)$, que l'on a une propriété de coercivité locale (en espace) dans $L^2(\mathbb{R}^N, \mathrm{d}v)$: pour tout $h \in L^2(\mathbb{R}^N, \mathrm{d}v)$,

$$\langle Qh, h\rangle \leq -c\|h - \Pi_\ell h\|,$$

pour un certain $c > 0$. Cela implique en particulier que l'opérateur Q a un trou spectral. La classe d'équations (0.36) est relativement générale et comprend entre autres les équations linéarisées de Boltzmann, les équations de relaxation, de Landau ou encore de Fokker-Planck. Les équilibres globaux de ces modèles appartiennent à $\ker(Q)$. Enfin, nous introduisons la projection globale $\bar{\Pi}$ sur $\ker(Q)$ dans l'espace $L^2(\mathbb{T}^N \times \mathbb{R}^N)$ qui est définie par

$$\bar{\Pi}h = \int_{\mathbb{T}^N} \Pi_\ell h(x, v)dx, \qquad h \in L^2(\mathbb{T}^N \times \mathbb{R}^N).$$

Si f est une solution du modèle cinétique (0.36), il faut remarquer que la fonction $t \mapsto \bar{\Pi}f(t)$ est constante au cours du temps égale à $\bar{\Pi}f(0)$. En effet, en multipliant l'équation (0.36) par $\bar{\Pi}f(t)$, en intégrant en espace et vitesse, et en utilisant que $Q^*\bar{\Pi}f(t) = Q\bar{\Pi}f(t) = 0$ on obtient

$$\frac{d}{dt}\bar{\Pi}f(t) = 0.$$

Dès lors, la théorie de l'hypocoercivité nous donne la convergence exponentielle de la solution f vers son équilibre global $\bar{\Pi} f(0)$:

$$\|f(t) - \bar{\Pi} f(0)\|_{\mathcal{H}} \leq K e^{-\tau t}, \quad t \geq 0,$$

dans un certain espace de Sobolev $\mathcal{H}$. Le taux de convergence τ est explicite, il dépend des paramètres du problème et en particulier du trou spectral de Q que l'on peut quantifier avec la constante c introduite ci-dessus. Pour plus de détails sur l'hypocoercivité, nous renvoyons au mémoire de C. Villani [Vil09] et aux références de ce mémoire ainsi qu'à l'article de C. Mouhot et L. Neumann [MN06] où l'hypocoercivité est utilisée pour étudier la convergence à l'équilibre de nombreux modèles cinétiques incluant le modèle de Fokker-Planck.

Revenons à l'étude du problème (0.35) où nous regardons pour le moment le cas déterministe $\lambda = 0$. Le noyau de L est porté par la fonction $\mathcal{M}^{\frac{1}{2}}$ et l'on peut exprimer la projection globale sur $\ker(L)$ comme

$$\bar{\Pi} h = \rho_\infty(h) \mathcal{M}^{\frac{1}{2}}, \qquad h \in L^2(\mathbb{T}^N \times \mathbb{R}^N)$$

où l'on a défini $\rho_\infty(h) := \iint h(t) \mathcal{M}^{\frac{1}{2}} \, dx \, dv = \iint h(0) \mathcal{M}^{\frac{1}{2}} \, dx \, dv$ (cette quantité est constante au cours du temps). L'hypocoercivité nous donne alors, voir par exemple [MN06, Section 5.3], une convergence exponentielle vers 0 pour la quantité $g(t) - \rho_\infty(g) \mathcal{M}^{\frac{1}{2}}$ dans l'espace H^1.

Dans ce chapitre, on souhaite obtenir de l'hypocoercivité sur le modèle de Fokker-Planck (0.35) qui est perturbé par une force aléatoire. Le résultat prouvé, qui contient également le résultat d'existence et d'unicité des solutions du problème (0.35) est énoncé ci-après. On introduit juste avant quelques notations : $\langle .,. \rangle$ et $\|\cdot\|$ désigneront respectivement le produit scalaire et la norme de l'espace $L^2_{x,v} := L^2(\mathbb{T}^N \times \mathbb{R}^N)$ et on définit l'opérateur $D = \nabla v + v/2$ et les espaces [7]

$$L^2_D = \{f \in L^2(\mathbb{R}^N); Df \in L^2(\mathbb{R}^N)\}, \quad L^2_{x,D} = L^2(\mathbb{T}^N; L^2_D), \quad L^2_{\nabla,D} = \{f \in L^2_{x,D}; \nabla_x f \in L^2_{x,v}\}.$$

Théorème 0.0.5. *Soit $g_{\text{in}} \in L^2(\Omega; L^2_{x,v})$. Pour tout $\lambda < 1$, il existe un unique processus adapté $\{g(t),\, t \geq 0\}$ qui satisfait :*

(i) pour tout $T > 0$, $g \in C_w([0,T]; L^2(\Omega; L^2_{x,v}))$ et $Dg \in L^2(\Omega \times (0,T); L^2_{x,v})$;
(ii) $g(0) = g_{\text{in}}$;
(iii) pour tout φ dans $C_c^\infty(\mathbb{T}^N \times \mathbb{R}^N)$ et tout $t \geq 0$,

$$\begin{aligned} \langle g(t), \varphi \rangle &= \langle g_{\text{in}}, \varphi \rangle + \int_0^t \langle g(s), v \cdot \nabla_x \varphi \rangle ds + \lambda \sum_{j \geq 0} \int_0^t \langle g(s), F_j \cdot D\varphi \rangle \, d\beta_j(s) \\ &+ \int_0^t \langle g(s), L^* \varphi \rangle ds + \frac{\lambda^2}{2} \sum_{j \geq 0} \int_0^t \left\langle g(s), (F_j \cdot D)^2 \varphi \right\rangle ds, \quad p.s. \end{aligned} \tag{0.37}$$

La quantité $\rho_\infty(g) := \iint g \mathcal{M}^{\frac{1}{2}}$ est constante au cours du temps. De plus, il existe $\lambda_0(N) > 0$ tel que, pour tout $\lambda < \lambda_0$, g a les propriétés suivantes. La solution g devient plus régulière dès que $t > 0$: pour tout $t_0 > 0$, il existe une constante $C(N, t_0) > 0$ telle que

$$\mathbb{E}\|g(t_0)\|^2_{L^2_{\nabla,D}} \leq C \mathbb{E}\|g_{in}\|^2. \tag{0.38}$$

De plus, si $t_0 > 0$, il existe des constantes c, C et K dépendant seulement de N telles que g satisfait, pour $t \geq t_0$, l'estimation

$$\mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}} + c\, \mathbb{E} \int_{t_0}^t \|g(s)\|^2_{L^2_{\nabla,D}} + \|D\nabla_x g(s)\|^2 + \|D^2 g(s)\|^2 ds \;\leq\; C\mathbb{E}\|g(t_0)\|^2_{L^2_{\nabla,D}} + C\mathbb{E}|\rho_\infty|^2 (t - t_0), \tag{0.39}$$

7. Pour plus de détails, notamment sur les normes considérées sur ces espaces, voir le chapitre 5.

et, pour $t \geq t_0$, l'estimation hypocoercitive

$$\mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}} \leq \; Ce^{-c(t-t_0)}\mathbb{E}\|g(t_0)\|^2_{L^2_{\nabla,D}} + K\mathbb{E}|\rho_\infty(g)|^2. \tag{0.40}$$

Concernant la preuve du résultat, l'existence du processus g est obtenue par un schéma de Galerkin sur lequel on passe à la limite grâce aux estimations d'énergie uniformes qui découlent de l'estimation (plus forte) hypocoercitive. Noter que l'on doit imposer une condition de petitesse sur λ, c'est-à-dire que le bruit n'est pas trop important, ceci afin d'empêcher que le bruit vienne altérer les propriétés dissipatives de l'opérateur L, voir également [MN06, Section 3.2] à ce sujet. Enfin, la preuve de l'estimation hypocoercitive (0.40) est prouvée de la même façon que dans [MN06] et en utilisant la formule d'Itô afin de traiter le terme stochastique.

Nous énonçons maintenant le résultat relatif à l'existence et l'unicité des mesures invariantes du problème (0.35).

Théorème 0.0.6. *Soit $g_{\text{in}} \in L^2_{x,v}$. On suppose que $\lambda < \lambda_0$ où λ_0 a été introduit dans le Théorème 0.0.5. Pour $w \in \mathbb{R}$, on introduit l'espace*

$$X_w := \left\{ g \in L^2_{x,v},\ \iint g\mathcal{M}^{\frac{1}{2}} = w \right\}.$$

Alors, pour tout $w \in \mathbb{R}$, le problème

$$\begin{cases} dg \;+\; v\cdot\nabla_x g\ dt \;+\; \lambda\left(\nabla_v - \dfrac{v}{2}\right) g \;\odot dW_t \;=\; Lg\ dt, \\ g(0) = g_{in}, \\ \iint g_{\text{in}}\mathcal{M}^{\frac{1}{2}} = w, \end{cases} \tag{P_w}$$

admet une unique mesure invariante sur l'espace X_w.

La preuve du résultat repose essentiellement sur l'estimation hypocoercitive (0.40). En effet, on peut facilement déduire de cette dernière que si g_1 et g_2 sont deux solutions du problème (0.35) de conditions initiales respectives $g_{\text{in},1}$ et $g_{\text{in},2}$ telles que $\iint g_{\text{in},1}\mathcal{M}^{\frac{1}{2}} = \iint g_{\text{in},2}\mathcal{M}^{\frac{1}{2}}$, alors les solutions se rejoignent exponentiellement vite. Ceci nous permet d'établir l'existence ainsi que l'unicité d'une mesure invariante pour le problème (0.35).

Chapter 1

A fractional diffusion limit for a stochastic kinetic equation

Abstract: We study the stochastic fractional diffusive limit of a kinetic equation involving a small parameter and perturbed by a smooth random term. We show, under an appropriate scaling for the small parameter, the convergence in law to a stochastic fluid limit involving a fractional Laplacian. The proof relies on a generalization in the infinite dimensional case of the perturbed test-functions method and on the moments method used in the proof of the corresponding result in the deterministic setting.

Keywords: Kinetic equations, diffusion limit, stochastic partial differential equations, perturbed test functions, fractional diffusion, moments method.

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1.1 Introduction

In this chapter, we consider the following equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon^{\alpha-1}} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^\alpha} L f^\varepsilon + \frac{1}{\varepsilon^{\frac{\alpha}{2}}} m^\varepsilon f^\varepsilon \ \text{ in } \mathbb{R}_t^+ \times \mathbb{R}_x^d \times \mathbb{R}_v^d, \tag{1.1}$$

with initial condition

$$f^\varepsilon(0) = f_0^\varepsilon \ \text{ in } \ \mathbb{R}_x^d \times \mathbb{R}_v^d, \tag{1.2}$$

where $0 < \alpha < 2$, L is a linear operator (see (1.3) below) and m^ε a random process depending on $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ (see Section 1.2.2). We will study the behaviour in the limit $\varepsilon \to 0$ of its solution f^ε.

The solution $f^\varepsilon(t,x,v)$ to this kinetic equation can be interpreted as a distribution function of particles having position x and degrees of freedom v at time t. The variable v belongs to the velocity space $\mathbb{R}^d$ that we denote by V. The collision operator L models diffusive and mass-preserving interactions of the particles with the surrounding medium; it is given by

$$Lf = \int_V f \,\mathrm{d}v\, F - f, \tag{1.3}$$

where F is a velocity equilibrium function such that $F \in L^\infty$, $F(-v) = F(v)$, $F > 0$ a.e., $\int_V F(v)\mathrm{d}v = 1$ and which is a power tail distribution

$$F(v) \underset{|v|\to\infty}{\sim} \frac{\kappa_0}{|v|^{d+\alpha}}. \tag{1.4}$$

Note that $F \in \ker(L)$. Power tail distribution functions arise in various contexts, such as astrophysical plasmas or in the study of granular media. For more details on the subject, we refer to [MMM11].

In this chapter, we derive a stochastic diffusive limit to the random kinetic model (1.1) using the method of perturbed test functions. This method provides an elegant way of deriving stochastic diffusive limit from random kinetic systems; it was first introduced by Papanicolaou, Stroock and Varadhan [PSV77]. The book of Fouque, Garnier, Papanicolaou and Solna [FGPS10] presents many applications to this method. A generalisation in infinite dimension of the perturbed test functions method arose in recent papers of Debussche and Vovelle [DV12] and de Bouard and Gazeau [dBG12].

We make some remarks about the deterministic case where $m^\varepsilon \equiv 0$. First of all, we consider the following collisional kinetic equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} L f^\varepsilon,$$

obtained from (1.1) by taking $\alpha = 2$ and discarding the noise term. As ε goes to 0, under the condition that the matrix $K = \int_V v \otimes v\, F(v)\,\mathrm{d}v$ is finite and definite positive, one can show that the solution f^ε converges to the solution of the following diffusive equation

$$\partial_t \rho = \mathrm{div}_x(K \nabla_x \rho).$$

The proof of this approximation-diffusion result can be found in [DGP00] for instance. A natural idea is now to investigate the case where the matrix K is not finite and to wonder whether we are still able to derive a diffusive limit or not. In the articles [Mel10, MMM11], Mellet studied the case where F is a power tail distribution as defined in (1.4) (note that the matrix K is infinite in this case) and succeed in proving an approximation-diffusion result under a different scaling, precisely on the equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon^{\alpha-1}} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^\alpha} L f^\varepsilon.$$

In this case, the hydrodynamic limit involves a fractional Laplacian operator:

$$\partial_t \rho = -\kappa(-\Delta)^{\alpha/2}\rho.$$

Thus, in this chapter, we are interested in the corresponding stochastic result where the previous kinetic equation has been perturbed multiplicatively by a noise term as described by Equation (1.1).

For the random kinetic model (1.1), the case $\alpha = 2$ with K finite and v replaced by $a(v)$ where a is bounded is derived in the paper of Debussche and Vovelle [DV12]. Here we study a behaviour for the velocity equilibrium function F parametrized by $0 < \alpha < 2$ under which the classical diffusive matrix K is infinite and we relax the boundedness hypothesis on a since we study the case $a(v) = v$. Note that, in our case, in order to get a non-trivial limiting equation as ε goes to 0, we exactly must have $a(v)$ unbounded; furthermore, we can easily extend the result to velocities of the form $a(v)$ where a is a C^1-diffeomorphism from V onto V. We expect a limiting stochastic equation with a fractional Laplacian.

To derive a stochastic diffusive limit to the random kinetic model (1.1), we use a generalization in infinite dimension of the perturbed test functions method. Nevertheless, the fact that the velocities are not bounded gives rise to non-trivial difficulties to control the transport term $v \cdot \nabla_x$. As a result, we also use the moments method applied in [Mel10] in the deterministic case. The moments method consists in working on weak formulations and in introducing new auxiliary problems, namely in the deterministic case

$$\chi^\varepsilon - \varepsilon v \cdot \nabla_x \chi^\varepsilon = \varphi,$$

where φ is some smooth function; thus we introduce in the sequel several additional auxiliary problems to deal with the stochastic part of the kinetic equation. Solving these problems is based on the inversion of the operator $L - \varepsilon A + M$ where M is the infinitesimal generator of the driving process m. Finally, we have to combine appropriately the moments and the perturbed test functions methods.

We also point out similar works using a more probabilistic approach of Basile and Bovier [BB10] and Jara, Komorowski and Olla [JKO09].

1.2 Preliminaries and main result

1.2.1 Notations

In the sequel, $L^2_{F^{-1}}$ denotes the F^{-1} weighted $L^2(\mathbb{R}^d \times V)$ space equipped with the norm

$$\|f\|^2 := \int_{\mathbb{R}^d}\int_V \frac{|f(x,v)|^2}{F(v)}\,\mathrm{d}v\,\mathrm{d}x.$$

We denote its scalar product by $(.,.)$. We also need to work in the space $L^2(\mathbb{R}^d)$, or L^2_x for short. The scalar product in L^2_x will be denoted by $(.,.)_x$. When $f \in L^2_{F^{-1}}$, we denote by ρ the first moment of f over V i.e. $\rho = \int_V f\,\mathrm{d}v$. We often use the following inequality

$$\|\rho\|_{L^2_x} \leq \|f\|,$$

which is just Cauchy-Schwarz inequality and the fact that $\int_V F(v)\,\mathrm{d}v = 1$. Finally, $\mathcal{S}(\mathbb{R}^d)$ stands for the Schwartz space on $\mathbb{R}^d$, and $\mathcal{S}'(\mathbb{R}^d)$ for the space of tempered distributions on $\mathbb{R}^d$.

We recall that the operator L is defined by (1.3). It can easily be seen that L is a bounded operator from $L^2_{F^{-1}}$ to $L^2_{F^{-1}}$. Note also that L is dissipative since, for $f \in L^2_{F^{-1}}$,

$$(Lf, f) = -\|Lf\|^2. \tag{1.5}$$

In the sequel, we denote by $g(t,\cdot)$ the semi-group generated by the operator L on $L^2_{F^{-1}}$. It satisfies, for $f \in L^2_{F^{-1}}$,

$$\begin{cases} \dfrac{d}{\mathrm{d}t} g(t,f) = Lg(t,f), \\ \quad g(0,f) = f, \end{cases}$$

and it is given by

$$g(t,f) = \int_V f\,\mathrm{d}v\,F(1-e^{-t}) + fe^{-t}, \quad t \geq 0,\ f \in L^2_{F^{-1}},$$

so that $g(t,\cdot)$ is a contraction, that is, for $f \in L^2_{F^{-1}}$,

$$\|g(t,f)\| \leq \|f\|, \quad t \geq 0. \tag{1.6}$$

We now introduce the following spaces S^γ for $\gamma \in \mathbb{R}$. First, we define the following operator on $L^2(\mathbb{R}^d)$

$$J := -\Delta_x + |x|^2,$$

with domain

$$\mathrm{D}(J) := \left\{ f \in L^2(\mathbb{R}^d),\ \Delta_x f,\ |x|^2 f \in L^2(\mathbb{R}^d) \right\}.$$

Let $(p_j)_{j\in\mathbb{N}^d}$ be the Hermite functions, defined as

$$p_j(x_1, ..., x_d) := H_{j_1}(x_1) \cdots H_{j_d}(x_d) e^{-\frac{|x|^2}{2}},$$

where $j = (j_1, ..., j_d) \in \mathbb{N}^d$ and H_i stands for the $i-$th Hermite's polynomial on $\mathbb{R}$. The functions $(p_j)_{j\in\mathbb{N}^d}$ are the eigenvectors of J with associated eigenvalues $(\mu_j)_{j\in\mathbb{N}^d} := (2|j|+1)_{j\in\mathbb{N}^d}$ where $|j| := |j_1| + \cdots + |j_d|$. Furthermore, one can check that J is invertible from $\mathrm{D}(J)$ to $L^2(\mathbb{R}^d)$, and that it is self-adjoint. As a result, we can define J^γ for any $\gamma \in \mathbb{R}$.
Then, for $\gamma \in \mathbb{R}$, we can also view J^γ as an operator on $\mathcal{S}'(\mathbb{R}^d)$. Let $u \in \mathcal{S}'(\mathbb{R}^d)$, we define $J^\gamma u \in \mathcal{S}'(\mathbb{R}^d)$ by setting, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle J^\gamma u, \varphi \rangle := \langle u, J^\gamma \varphi \rangle.$$

Finally, we introduce, for $\gamma \in \mathbb{R}$,

$$S^\gamma(\mathbb{R}^d) := \{u \in \mathcal{S}'(\mathbb{R}^d),\ J^{\frac{\gamma}{2}} u \in L^2(\mathbb{R}^d)\},$$

equipped with the norm

$$\|u\|_{S^\gamma(\mathbb{R}^d)} = \|J^{\frac{\gamma}{2}} u\|_{L^2(\mathbb{R}^d)}.$$

In the sequel, we need to know the asymptotic behaviour of the quantities $\|p_j\|_{L^2_x}$, $\|\nabla_x p_j\|_{L^2_x}$, $\|D^2 p_j\|_{L^2_x}$ and $\|(-\Delta)^{\frac{\alpha}{2}} p_j\|_{L^2_x}$ as $|j| \to \infty$. In fact, classical properties of the Hermite functions give the following bounds

$$\begin{aligned} &\|p_j\|_{L^2_x} = 1, && \|\nabla_x p_j\|_{L^2_x} \leq {\mu_j}^{\frac{1}{2}}, \\ &\|D^2 p_j\|_{L^2_x} \leq \mu_j, && \|(-\Delta)^{\frac{\alpha}{2}} p_j\|_{L^2_x} \leq 1 + \mu_j. \end{aligned} \tag{1.7}$$

We finally recall the definition of the fractional power of the Laplacian. It can be introduced using the Fourier transform in $\mathcal{S}'(\mathbb{R}^d)$ by setting, for $u \in \mathcal{S}'(\mathbb{R}^d)$,

$$\mathcal{F}((-\Delta)^{\frac{\alpha}{2}} u)(\xi) = |\xi|^\alpha \mathcal{F}(u)(\xi).$$

Alternatively, we have the following singular integral representation, see [Val09],

$$-(-\Delta)^{\frac{\alpha}{2}} u(x) = c_{d,\alpha} \mathrm{PV} \int_{\mathbb{R}^d} [u(x+h) - u(x)] \frac{\mathrm{d}h}{|h|^{d+\alpha}},$$

for some constant $c_{d,\alpha}$ which only depends on d and α.

1.2.2 The random perturbation

The random term m^ε is defined by

$$m^\varepsilon(t,x) := m\left(\frac{t}{\varepsilon^\alpha}, x\right),$$

where m is a stationary process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and is adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$. Note that m^ε is adapted to the filtration $(\mathcal{F}^\varepsilon_t)_{t\geq 0} = (\mathcal{F}_{\varepsilon^{-\alpha}t})_{t\geq 0}$. We assume that, considered as a random process with values in a space of spatially dependent functions, m is a stationary homogeneous Markov process taking values in a subset E of $L^2(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$. In the sequel, E will be endowed with the norm $\|\cdot\|_\infty$ of $L^\infty(\mathbb{R}^d)$. Besides, we denote by $\mathcal{B}(E,X)$ (or $\mathcal{B}(E)$ when $X = \mathbb{R}$) the set of bounded functions from E to X endowed with the norm $\|g\|_\infty := \sup_{n\in E} \|g(n)\|_X$ for $g \in \mathcal{B}(E,X)$.

We assume that m is stochastically continuous. Note that m is supposed not to depend on the variable v. For all $t \geq 0$, the law ν of m_t is supposed to be centered

$$\mathbb{E}[m_t] = \int_E n \,\mathrm{d}\nu(n) = 0.$$

The subset E has the following properties. We fix a family $(\eta_i)_{i\in\mathbb{N}}$ of functions in $W^{1,\infty}(\mathbb{R}^d)$ such that

$$S := \sum_{i\in\mathbb{N}} \|\eta_i\|_{W^{1,\infty}} < \infty,$$

and we assume that every $n \in E$ can be uniquely written as

$$n = \sum_{i\in\mathbb{N}} n_i(n)\eta_i, \tag{1.8}$$

with $|n_i(n)| \leq K$ for all $i \in \mathbb{N}$ and all $n \in E$. Note that the preceding series converges absolutely and that E is included in the ball $\mathrm{B}(0, KS)$ of $W^{1,\infty}(\mathbb{R}^d)$. Finally, since m is centered, we also suppose that for all $i \in \mathbb{N}$,

$$\int_E n_i(n)\mathrm{d}\nu(n) = 0. \tag{1.9}$$

We denote by e^{tM} a transition semi-group on E associated to m. We suppose that the transition semi-group is Feller i.e. e^{tM} maps continuous functions of n on continuous functions of n for all $t \geq 0$. In the sequel we also need to consider e^{tM} as a transition semi-group on the space $\mathcal{B}(E, L^2_{F^{-1}})$ and not only on $\mathcal{B}(E)$. Thus, if $g \in \mathcal{B}(E, L^2_{F^{-1}})$, e^{tM} acts on g pointwise, that is,

$$[\widetilde{e^{tM}} g](x,v) = e^{tM}[g(x,v)], \quad (x,v) \in \mathbb{R}^d \times V.$$

In both cases, we denote by M the infinitesimal generator associated to the transition semi-group. Note that we do not distinguish on which space $\mathcal{B}(E,X)$, $X=\mathbb{R}$ or $L^2_{F^{-1}}$, the operators are acting since it will always be clear from the context. Then, for $X=\mathbb{R}$ or $X=L^2_{F^{-1}}$, $\mathrm{D}(M)$ stands for the domain of M; it is defined as follows:

$$\mathrm{D}(M) := \left\{ u \in \mathcal{B}(E,X),\ \lim_{h\to 0} \frac{e^{hM}-I}{h} u \text{ exists in } \mathcal{B}(E,X) \right\},$$

and if $u \in \mathrm{D}(M)$, we set

$$Mu := \lim_{h\to 0} \frac{e^{hM}-I}{h} u \text{ in } \mathcal{B}(E,X).$$

We suppose that there exists $\mu > 0$ such that for all $g \in \mathcal{B}(E)$ verifying the condition $\int_E g(n)\mathrm{d}\nu(n) = 0$,

$$\|e^{tM} g\|_\infty \le e^{-\mu t}\|g\|_\infty, \quad t \ge 0. \tag{1.10}$$

Moreover, we suppose that m is ergodic and satisfies some mixing properties in the sense that there exists a subspace $\mathscr{P}_M$ of $\mathcal{B}(E)$ such that for any $g \in \mathscr{P}_M$, the Poisson equation

$$M\psi = g - \int_E g(n)\,\mathrm{d}\nu(n) =: \widehat{g},$$

has a unique solution $\psi \in \mathrm{D}(M)$ satisfying $\int_E \psi(n)\,\mathrm{d}\nu(n) = 0$. We denote by $M^{-1}\widehat{g}$ this unique solution, and assume that it is given by

$$M^{-1}\widehat{g}(n) = -\int_0^\infty e^{tM}\widehat{g}(n)\mathrm{d}t, \quad n \in E. \tag{1.11}$$

Thanks to (1.10), the above integral is well defined. In particular, it implies that for all $n \in E$,

$$\lim_{t\to\infty} e^{tM}\widehat{g}(n) = 0.$$

We assume that for all $i \in \mathbb{N}$, $n \mapsto n_i(n)$ is in $\mathscr{P}_M$ and that for all $n \in E$, $|M^{-1}n_i(n)| \le K$. As a consequence, we simply define $M^{-1}I$ by

$$M^{-1}I(n) := \sum_{i\in\mathbb{N}} M^{-1}n_i(n)\eta_i, \quad n \in E.$$

We also suppose that for all $f \in L^2(\mathbb{R}^d)$, the functions $g_f : n \in E \mapsto (f,n)_x$ and $n \in E \mapsto M^{-1}g_f(n)$ are in $\mathscr{P}_M$.

We will suppose that for all $t \ge 0$,

$$\mathbb{E}\|m_t\|^2_{L^2_x} < \infty, \quad \mathbb{E}\|M^{-1}I(m_t)\|^2_{L^2_x} < \infty. \tag{1.12}$$

To describe the limiting stochastic partial differential equation, we then set

$$k(x,y) = \mathbb{E}\int_{\mathbb{R}} m_0(y)m_t(x)\,\mathrm{d}t, \quad x,y \in \mathbb{R}^d.$$

The kernel k is, thanks to (1.12), the fact that m is stationary and Cauchy-Schwarz inequality, in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$ and such that

$$\int_{\mathbb{R}^d} k(x,x)\,\mathrm{d}x < \infty.$$

Furthermore, we can check (see [DV12]), since m is stationary, that k is symmetric. As a result, we introduce the operator Q on $L^2(\mathbb{R}^d)$ associated to the kernel k

$$Qf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d,$$

which is self-adjoint and trace class. Furthermore, since we assumed that the functions $g_f : n \in E \mapsto (f,n)_x$ and $n \in E \mapsto M^{-1}g_f(n)$ are in $\mathscr{P}_M$, we can show, see [DV12, Lemma 1], that Q is non-negative, that is $(Qf,f)_x \geq 0$ for all $f \in L^2(\mathbb{R}^d)$. As a result, we can define the square root $Q^{\frac{1}{2}}$ which is Hilbert-Schmidt on $L^2(\mathbb{R}^d)$.

It remains to make some hypothesis on M. We set, for all $n \in E$,

$$\theta(n) = \int_E nM^{-1}I(n)\mathrm{d}\nu(n) - nM^{-1}I(n), \tag{1.13}$$

and, for $i,j \in \mathbb{N}$, $\theta_{i,j} = \int_E n_i M^{-1} n_j \mathrm{d}\nu - n_i M^{-1} n_j$, so that

$$\theta = \sum_{i,j\in\mathbb{N}} \theta_{i,j}\eta_i\eta_j.$$

We suppose that for all $i,j,k,l \in \mathbb{N}$ and $s,t \geq 0$,

$$\begin{cases} n \mapsto n_i(n), \\ n \mapsto \theta_i(n), \\ n \mapsto e^{tM}n_i(n)e^{sM}n_j(n), \\ n \mapsto e^{tM}\theta_{i,j}(n)e^{sM}n_k(n), \\ n \mapsto e^{tM}\theta_{i,j}(n)e^{sM}\theta_{k,l}(n), \end{cases} \tag{1.14}$$

are in $\mathrm{D}(M)$, with

$$\|n_i\|_\infty + \|\theta_{i,j}\|_\infty + \|Mn_i\|_\infty + \|M\theta_{i,j}\|_\infty \leq K, \tag{1.15}$$

$$\|M[e^{tM}n_ie^{sM}n_j]\|_\infty + \|M[e^{tM}\theta_{i,j}e^{sM}n_j]\|_\infty + \|M[e^{tM}\theta_{i,j}e^{sM}\theta_{k,l}]\|_\infty \leq Ke^{-\mu(s+t)}. \tag{1.16}$$

Remark The above assumptions (1.10) – (1.16) on the process m are verified, for instance, when m is a Poisson process taking values in E.

We now state two lemmas which will be very useful in the following.

Lemma 1.2.1. *Let $p \in \mathcal{B}(E)$ be a function in $\mathrm{D}(M)$ such that $\|Mp\|_\infty \leq K$. Then we have, for all $h > 0$,*

$$\left\| \frac{e^{hM} - I}{h}p - Mp \right\|_\infty \leq 2K.$$

Proof. We just write, for all $n \in E$,

$$\begin{aligned} \left| \frac{e^{hM} - I}{h}p(n) - Mp(n) \right| &= \left| \frac{1}{h}\int_0^h Me^{sM}p(n)\,\mathrm{d}s - Mp(n) \right| \\ &= \left| \frac{1}{h}\int_0^h e^{sM}Mp(n)\,\mathrm{d}s - Mp(n) \right| \leq 2K, \end{aligned}$$

where we used the contraction property of the semigroup e^{tM}. This concludes the proof. □

Remark The proof is still valid if $p \in \mathcal{B}(E, L^2_{F^{-1}})$; we just have to replace the absolute values by the $L^2_{F^{-1}}$-norm.

Lemma 1.2.2. *For all $i, j, k, l \in \mathbb{N}$ and $s, t \geq 0$, the functions*

$$\begin{cases} n \mapsto n_i(n), \\ n \mapsto \theta_{i,j}(n), \\ n \mapsto e^{tM} n_i e^{sM} n_j(n), \\ n \mapsto e^{tM} \theta_{i,j} e^{sM} n_k(n), \\ n \mapsto e^{tM} \theta_{i,j} e^{sM} \theta_{k,l}(n), \end{cases} \qquad \begin{cases} n \mapsto M n_i(n), \\ n \mapsto M\theta_{i,j}(n), \\ n \mapsto M[e^{tM} n_i e^{sM} n_j](n), \\ n \mapsto M[e^{tM} \theta_{i,j} e^{sM} n_k](n), \\ n \mapsto M[e^{tM} \theta_{i,j} e^{sM} \theta_{k,l}](n), \end{cases} \tag{1.17}$$

are continuous.

Proof. We fix $i, j, k, l \in \mathbb{N}$ and $s, t \geq 0$. First of all, $n \mapsto n_i(n)$ is obviously continuous since it is linear. We recall that $\theta_{i,j} = \int_E n_i M^{-1} n_j \mathrm{d}\nu - n_i M^{-1} n_j$. With (1.9) and (1.11), we have

$$M^{-1} n_j = \int_0^\infty e^{tM} n_j \, \mathrm{d}t,$$

which is continuous with respect to $n \in E$ by (1.9), (1.10), (1.15) and the dominated convergence Theorem. As a result, $n \mapsto n_i(n) M^{-1} n_j(n)$ is continuous; and also the map $n \mapsto \int_E n_i(n) M^{-1} n_j(n) \mathrm{d}\nu(n)$ by the dominated convergence Theorem. Hence $n \mapsto \theta_{i,j}(n)$ is continuous. The continuity of n_i and $\theta_{i,j}$ now immediately gives the continuity of the three last functions of the left group of the lemma by the Feller property of the semigroup e^{tM}.
For the remaining functions, just remark that if $p \in \mathcal{B}(E)$ is in $\mathrm{D}(M)$ and continuous, then Mp is the uniform limit on E when $h \to 0$ of the functions

$$\frac{e^{hM} - I}{h} p,$$

which are continuous by the Feller property of the semigroup. Hence Mp is continuous. This ends the proof. □

1.2.3 Resolution of the kinetic equation

In this section, we solve the linear evolution problem (1.1) – (1.2) thanks to a semigroup approach. We thus introduce the linear operator $A := -v \cdot \nabla_x$ on $L^2_{F^{-1}}$ with domain

$$\mathrm{D}(A) := \{ f \in L^2_{F^{-1}}, \; v \cdot \nabla_x f \in L^2_{F^{-1}} \}.$$

The operator A has dense domain and, since it is skew-adjoint, it is m-dissipative. Consequently A generates a contraction semigroup $(\mathcal{T}(t))_{t \geq 0}$, see [CH98]. We recall that $\mathrm{D}(A)$ is endowed with the norm $\| \cdot \|_{\mathrm{D}(A)} := \| \cdot \| + \|A \cdot \|$, and that it is a Banach space.

Proposition 1.2.3. *Let $T > 0$ and $f_0^\varepsilon \in L^2_{F^{-1}}$. Then there exists a unique mild solution of (1.1) – (1.2) on $[0, T]$ in $L^\infty(\Omega)$, that is there exists a unique $f^\varepsilon \in L^\infty(\Omega, \mathcal{C}([0,T], L^2_{F^{-1}}))$ such that $\mathbb{P}-a.s.$*

$$f_t^\varepsilon = \mathcal{T}\left(\frac{t}{\varepsilon^{\alpha-1}}\right) f_0^\varepsilon + \int_0^t \mathcal{T}\left(\frac{t-s}{\varepsilon^{\alpha-1}}\right) \left(\frac{1}{\varepsilon^\alpha} L f_s^\varepsilon + \frac{1}{\varepsilon^{\frac{\alpha}{2}}} m_s^\varepsilon f_s^\varepsilon\right) ds, \quad t \in [0, T].$$

Assume further that $f_0^\varepsilon \in \mathrm{D}(A)$, then there exists a unique strong solution f^ε in the space $L^\infty(\Omega, C^1([0,T], L^2_{F^{-1}})) \cap L^\infty(\Omega, C([0,T], \mathrm{D}(A)))$ to (1.1) – (1.2).

Proof. Subsections 4.3.1 and 4.3.3 in [CH98] gives that $\mathbb{P}-$a.s. there exists a unique mild solution $f^\varepsilon \in \mathcal{C}([0,T], L^2_{F^{-1}})$ and it is not difficult to slightly modify the proof to obtain that in fact $f^\varepsilon \in L^\infty(\Omega, \mathcal{C}([0,T], L^2_{F^{-1}}))$ (we intensively use that for all $t \geq 0$ and $\varepsilon > 0$, $\|m_t^\varepsilon\|_\infty \leq K$).
Similarly, subsections 4.3.1 and 4.3.3 in [CH98] gives us $\mathbb{P}-$a.s. a strong solution f^ε in the space $C^1([0,T], L^2_{F^{-1}}) \cap C([0,T], \mathrm{D}(A))$ to (1.1) – (1.2) and once again one can easily get that in fact $f^\varepsilon \in L^\infty(\Omega, C^1([0,T], L^2_{F^{-1}})) \cap L^\infty(\Omega, C([0,T], \mathrm{D}(A)))$. □

1.2.4 Main result

We are now ready to state our main result:

Theorem 1.2.4. *Assume that $(f_0^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2_{F^{-1}}$ and that*

$$\rho_0^\varepsilon := \int_V f_0^\varepsilon \, dv \underset{\varepsilon \to 0}{\longrightarrow} \rho_0 \ \text{in } L^2(\mathbb{R}^d).$$

Then, for all $\eta > 0$ and $T > 0$, $\rho^\varepsilon := \int_V f^\varepsilon \, dv$ converges in law in $C([0,T], S^{-\eta})$ to the solution ρ to the stochastic diffusion equation

$$d\rho = -\kappa(-\Delta)^{\frac{\alpha}{2}} \rho dt + \frac{1}{2} H\rho + \rho Q^{\frac{1}{2}} dW_t, \ \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x^d, \tag{1.18}$$

with initial condition $\rho(0) = \rho_0$ in $L^2(\mathbb{R}^d)$, and where W is a cylindrical Wiener process on $L^2(\mathbb{R}^d)$,

$$\kappa := \frac{\kappa_0}{c_{d,\alpha}} \int_0^\infty |t|^\alpha e^{-t} \, dt, \tag{1.19}$$

and

$$H := \int_E n M^{-1} I(n) \, d\nu(n) \in W^{1,\infty}. \tag{1.20}$$

Remark The limiting equation (1.18) can also be written in Stratonovich form

$$d\rho = -\kappa(-\Delta)^{\frac{\alpha}{2}} \rho \mathrm{d}t + \rho \circ Q^{\frac{1}{2}} dW_t.$$

Notation In the sequel, we will note $\lesssim$ the inequalities which are valid up to constants of the problem, namely K, S, μ, d, α, $\|L\|$, $\sup_{\varepsilon>0} \|f_0^\varepsilon\|$ and real constants. Nevertheless, when we need to emphasize the dependence of a constant on a parameter, we index the constant C by the parameter; for instance the constant C_φ depends on φ.

1.3 The generator

The process f^ε is not Markov (indeed, by (1.1), we need m^ε to know the increments of f^ε) but the couple $(f^\varepsilon, m^\varepsilon)$ is. From now on, we denote by $\mathscr{L}^\varepsilon$ its infinitesimal generator, it is given by

$$\mathscr{L}^\varepsilon \Psi(f,n) = \frac{1}{\varepsilon^\alpha}(Lf + \varepsilon A f, D\Psi(f,n)) + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(fn, D\Psi(f,n)) + \frac{1}{\varepsilon^\alpha} M\Psi(f,n),$$

provided $\Psi : L^2_{F^{-1}} \times E \to \mathbb{R}$ is enough regular to be in the domain of $\mathscr{L}^\varepsilon$. Thus we begin this section by introducing a special set of functions which lie in the domain of $\mathscr{L}^\varepsilon$ and satisfy the associated martingale problem. In the following, if $\Psi : L^2_{F^{-1}} \to \mathbb{R}$ is differentiable with respect to $f \in L^2_{F^{-1}}$, we denote by $D\Psi(f)$ its differential at a point f and we identify the differential with the gradient.

Definition 1.3.1. *We say that $\Psi : L^2_{F^{-1}} \times E \to \mathbb{R}$ is a good test function if*

(i) $(f, n) \mapsto \Psi(f, n)$ *is differentiable with respect to* f*;*

(ii) $(f, n) \mapsto D\Psi(f, n)$ *is continuous from* $L^2_{F^{-1}} \times E$ *to* $L^2_{F^{-1}}$ *and maps bounded sets onto bounded sets;*

(iii) for any $f \in L^2_{F^{-1}}$, $\Psi(f, \cdot) \in D_M$*;*

(iv) $(f, n) \mapsto M\Psi(f, n)$ *is continuous from* $L^2_{F^{-1}} \times E$ *to* $\mathbb{R}$ *and maps bounded sets onto bounded sets.*

Proposition 1.3.1. *Let* Ψ *be a good test function. If* $f_0^\varepsilon \in \mathrm{D}(A)$*,*

$$M_\Psi^\varepsilon(t) := \Psi(f_t^\varepsilon, m_t^\varepsilon) - \Psi(f_0^\varepsilon, m_0^\varepsilon) - \int_0^t \mathscr{L}^\varepsilon \Psi(f_s^\varepsilon, m_s^\varepsilon)\, ds$$

is a continuous and integrable $(\mathcal{F}_t^\varepsilon)_{t \geq 0}$ *martingale, and if* $|\Psi|^2$ *is a good test function, its quadratic variation is given by*

$$\langle M_\Psi^\varepsilon \rangle_t = \int_0^t (\mathscr{L}^\varepsilon |\Psi|^2 - 2\Psi \mathscr{L}^\varepsilon \Psi)(f_s^\varepsilon, m_s^\varepsilon)\, ds.$$

Proof. This is classical, we use the same kind of ideas and follow the proof of [DV12, Proposition 6] and [FGPS10, Appendix 6.9]. □

1.4 The limit generator

In this section, we study the limit of the generator $\mathscr{L}^\varepsilon$ when $\varepsilon \to 0$. The limit generator $\mathscr{L}$ will characterize the limit stochastic fluid equation.

1.4.1 Formal derivation of the corrections

To derive the diffusive limiting equation, one has to study the limit as ε goes to 0 of quantities of the form $\mathscr{L}^\varepsilon \Psi$ where Ψ is a good test function. From now on, we choose a specific form for the test functions that we keep thorough the chapter. We take φ in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and we set

$$\Psi(f, n) := (f, \varphi F) \tag{1.21}$$

It is clear that Ψ is a good test function. Remember that, when $\varepsilon \to 0$, we will obtain a fluid limit equation verified by the macroscopic quantity ρF; the test function Ψ takes this point in consideration since $\Psi(f, n) = \Psi(f) = \Psi(\rho F)$. In the sequel, we will show that the knowledge of the limits $\mathscr{L}^\varepsilon \Psi$ and $\mathscr{L}^\varepsilon |\Psi|^2$ as ε goes to 0 where Ψ is defined as (1.21) is sufficient to obtain our result. Nevertheless, we now have to correct Ψ and $|\Psi|^2$ so as to obtain non-singular limits. Here, we show formally how we correct Ψ (the formal work on $|\Psi|^2$ is similar).
We search the correction Ψ^ε of Ψ. First of all, to correct the deterministic part, we follow the moments method presented in [Mel10] so we set

$$\Psi^\varepsilon(f, n) = (f, \chi^\varepsilon F)$$

where χ^ε solves the auxiliary problem

$$\chi^\varepsilon - \varepsilon v \cdot \nabla_x \chi^\varepsilon = \varphi.$$

Now, to correct the stochastic part, we try an Hilbert expansion method (adapted to our scaling) coupled with the idea of auxiliary equation brought in the moments method so that we complete the definition of Ψ^ε as

$$\Psi^\varepsilon(f, n) = (f, \chi^\varepsilon F) + \varepsilon^{\frac{\alpha}{2}}(f, \delta^\varepsilon F) + \varepsilon^\alpha (f, \theta^\varepsilon F),$$

where δ^ε and θ^ε are to be defined. We then compute, since the first term in the expansion of Ψ^ε does not depend on $n \in E$,

$$\mathscr{L}^\varepsilon \Psi^\varepsilon(f,n) = \frac{1}{\varepsilon^\alpha}(Lf + \varepsilon Af, \chi^\varepsilon F) \tag{1.22}$$
$$+ \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(fn, \chi^\varepsilon F) + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(Lf + \varepsilon Af, \delta^\varepsilon F) + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(f, M\delta^\varepsilon F) \tag{1.23}$$
$$+ (fn, \delta^\varepsilon F) + (Lf + \varepsilon Af, \theta^\varepsilon F) + (f, M\theta^\varepsilon F) + \varepsilon^{\frac{\alpha}{2}}(fn, \theta^\varepsilon F). \tag{1.24}$$

The first term (1.22) above converges as ε goes to 0 to $(-\kappa(-\Delta)^{\frac{\alpha}{2}} f, \varphi F)$, see [Mel10], that is to the infinitesimal generator of the fractional Laplacian applied to Ψ: we get the deterministic term of the limiting equation.
Since L is auto-adjoint and A is skew-adjoint, the three following terms (1.23) can be rewritten as

$$\frac{1}{\varepsilon^{\frac{\alpha}{2}}}(f, n\chi^\varepsilon F) + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(f, (L - \varepsilon A + M)(\delta^\varepsilon F)).$$

Then we cancel these singular term by choosing δ^ε such that

$$(L - \varepsilon A + M)(\delta^\varepsilon F) = -n\chi^\varepsilon F.$$

Formally, this equation can be solved with the resolvent operator of $L - \varepsilon A + M$ so that we have

$$\delta^\varepsilon(x, v, n)F(v) = \int_0^{+\infty} e^{tM} g(t, n\chi^\varepsilon F)(x + \varepsilon vt, v)\,\mathrm{d}t.$$

With this expression of $\delta^\varepsilon F$ and since $\chi^\varepsilon \to \varphi$ as $\varepsilon \to 0$, see [Mel10], we have that $\delta^\varepsilon F$ converges to $-M^{-1}I(n)\varphi F$ when $\varepsilon \to 0$. So, neglecting an error term, we can suppose that (1.24) writes

$$(f, -nM^{-1}I(n)\varphi F) + (Lf + \varepsilon Af, \theta^\varepsilon F) + (f, M\theta^\varepsilon F) + \varepsilon^{\frac{\alpha}{2}}(fn, \theta^\varepsilon F).$$

Note that, for now, the limit of $\mathscr{L}^\varepsilon \Psi^\varepsilon$ as ε goes to 0 does depend on $n \in E$. Since the expected limit is $\mathscr{L}\Psi$ where Ψ does not depend on n, we have to correct once again the remaining terms to break the dependence with respect to n of the limit. The right way to do so, given the mixing properties of the operator M, is to subtract the mean value: we write (1.24) as

$$(f, -H\varphi F) + (f, \theta(n)\varphi F) + (Lf + \varepsilon Af, \theta^\varepsilon F) + (f, M\theta^\varepsilon F) + \varepsilon^{\frac{\alpha}{2}}(fn, \theta^\varepsilon F),$$

where H and θ are respectively defined in (1.20) and (1.13). Now, we choose θ^ε so that

$$(L - \varepsilon A + M)(\theta^\varepsilon F) = -\theta(n)\varphi F,$$

so that (1.24) becomes

$$(f, -H\varphi F) + \varepsilon^{\frac{\alpha}{2}}(fn, \theta^\varepsilon F);$$

it allows us to conclude that $\mathscr{L}^\varepsilon \Psi^\varepsilon$ converges to $\mathscr{L}\Psi$ as $\varepsilon \to 0$ where $\mathscr{L}$ is the infinitesimal generator of the equation (1.18) (note that $D^2\Psi \equiv 0$ so that no stochastic appears here).
As we said previously, the same kind of work can be done to correct $|\Psi|^2$. In the following subsections, we define rigorously the corrections of Ψ and $|\Psi|^2$.

1.4.2 Preliminaries to the deterministic correction

As it is said above, we use the moments method presented in [Mel10] to correct the deterministic part of the equation (1.1). Let χ^ε be the solution of the auxiliary problem

$$\chi^\varepsilon - \varepsilon v \cdot \nabla_x \chi^\varepsilon = \varphi. \tag{1.25}$$

We recall, see [Mel10], that the solution of (1.25) is given by

$$\chi^\varepsilon(x,v) = \int_0^{+\infty} e^{-t}\varphi(x+\varepsilon vt)\,\mathrm{d}t, \quad x\in\mathbb{R}^d,\ v\in V. \tag{1.26}$$

We now detail few results on χ^ε.

Proposition 1.4.1. *The function $\chi^\varepsilon F$ is in $L^2_{F^{-1}}$ with*

$$\|\chi^\varepsilon F\| \leq \|\varphi\|_{L^2_x}. \tag{1.27}$$

Furthermore, for any $\lambda>0$, we have the following estimate:

$$\|(\chi^\varepsilon-\varphi)F\|^2 \lesssim C_\lambda^2\varepsilon^2\|\nabla_x\varphi\|^2_{L^2_x} + \|\varphi\|^2_{L^2_x}\lambda^2. \tag{1.28}$$

Proof. The estimate (1.27) is proved in [Mel10, Lemma 3.1] and the estimate (1.28) is a slight refinement of what is addressed in [Mel10, Lemma 3.1]. Nevertheless, for a precise proof of the proposition, we refer the reader to Appendix A. □

In the two following lemmas, we study in detail the convergence to the fractional Laplace operator. We recall that κ has been defined by (1.19).

Lemma 1.4.2. *For any $\lambda>0$, we have the following estimate:*

$$\left\|\varepsilon^{-\alpha}\int_V[\chi^\varepsilon(\cdot,v)-\varphi(\cdot)]F(v)dv + \kappa(-\Delta)^{\frac{\alpha}{2}}\varphi\right\|_{L^2_x} \lesssim (\Lambda(\varepsilon)+\lambda)(\|\varphi\|_{L^2_x}+\|D^2\varphi\|_{L^2_x}), \tag{1.29}$$

for a certain function Λ, which only depends on ε, such that $\Lambda(\varepsilon)\to 0$ when $\varepsilon\to 0$.

Proof. Once again, the bound (1.29) is a refinement of what is proved in [Mel10, Proposition 3.2]. We refer the reader to Appendix A for a precise proof. □

Lemma 1.4.3. *For any $\lambda>0$, we have the following estimate:*

$$\left|\varepsilon^{-\alpha}(\varepsilon Af + Lf, \chi^\varepsilon F) + (\kappa(-\Delta)^{\frac{\alpha}{2}}f,\varphi F)\right| \lesssim (\Lambda(\varepsilon)+\lambda)\|f\|(\|\varphi\|_{L^2_x}+\|D^2\varphi\|_{L^2_x}), \tag{1.30}$$

for a certain function Λ, which only depends on ε, such that $\Lambda(\varepsilon)\to 0$ when $\varepsilon\to 0$.

Proof. For the reader's convenience, the proof is deferred to Appendix A. □

1.4.3 Preliminaries to the stochastic corrections

The corrector δ^ε

We recall that $g(t,\cdot)$ denotes the semi-group generated by the operator L on $L^2_{F^{-1}}$ and that the function χ^ε has been defined in (1.25). Then, we define the function $\delta^\varepsilon:\mathbb{R}^d\times V\times E\to\mathbb{R}$ by

$$\delta^\varepsilon(x,v,n)F(v) := \int_0^{+\infty} e^{tM}g(t,n\chi^\varepsilon F)(x+\varepsilon vt,v)\,\mathrm{d}t,$$

and we give here some properties of δ^ε. We recall that the test function φ has been fixed in Section 1.4.1.

Proposition 1.4.4. *The function $\delta^\varepsilon F$ belongs to $\mathcal{B}(E, L^2_{F^{-1}})$ with*

$$\|\delta^\varepsilon F\|_{\mathcal{B}(E,L^2_{F^{-1}})} \lesssim \|\varphi\|_{L^2_x}. \tag{1.31}$$

It satisfies

$$(L - \varepsilon A + M)(\delta^\varepsilon F) = -n\chi^\varepsilon F, \tag{1.32}$$

with

$$\|M\delta^\varepsilon F\|_{\mathcal{B}(E,L^2_{F^{-1}})} \lesssim \|\varphi\|_{L^2_x}. \tag{1.33}$$

Furthermore, for any $\lambda > 0$, we have the two following estimates:

$$\|\delta^\varepsilon F + M^{-1}I(n)\varphi F\|_{\mathcal{B}(E,L^2_{F^{-1}})} \lesssim C_\lambda \|\nabla_x\varphi\|_{L^2_x}\varepsilon + \|\varphi\|_{L^2_x}\lambda, \tag{1.34}$$

$$\|M\delta^\varepsilon F + n\chi^\varepsilon F\|_{\mathcal{B}(E,L^2_{F^{-1}})} \lesssim C_\lambda \|\nabla_x\varphi\|_{L^2_x}\varepsilon + \|\varphi\|_{L^2_x}\lambda. \tag{1.35}$$

Proof. Proof of (1.31). The definition of $\delta^\varepsilon F$ can be rewritten, thanks to (1.8), as

$$\delta^\varepsilon(x,v,n)F(v) = \sum_{i=0}^{+\infty}\int_0^{+\infty} e^{tM} n_i(n) g(t, \eta_i\chi^\varepsilon F)(x+\varepsilon vt, v)\,\mathrm{d}t =: \sum_{i=0}^{+\infty}\alpha_i(x,v,n).$$

Then we fix $i \in \mathbb{N}$ and $n \in E$. We have

$$\begin{aligned}
\|\alpha_i(\cdot,\cdot,n)\|^2 &= \int_{\mathbb{R}^d}\int_V \left(\int_0^{+\infty} e^{tM} n_i(n) g(t,\eta_i\chi^\varepsilon F)(x+\varepsilon vt, v)\,\mathrm{d}t\right)^2 \frac{\mathrm{d}v}{F(v)}\mathrm{d}x \\
&\le \int_{\mathbb{R}^d}\int_V \left(\int_0^{+\infty} Ke^{-\mu t}|g(t,\eta_i\chi^\varepsilon F)|(x+\varepsilon vt, v)\,\mathrm{d}t\right)^2 \frac{\mathrm{d}v}{F(v)}\mathrm{d}x \\
&\le \frac{K^2}{\mu}\int_{\mathbb{R}^d}\int_V\int_0^{+\infty} e^{-\mu t}|g(t,\eta_i\chi^\varepsilon F)|^2(x+\varepsilon vt, v)\,\mathrm{d}t\frac{\mathrm{d}v}{F(v)}\mathrm{d}x \\
&= \frac{K^2}{\mu}\int_0^\infty e^{-\mu t}\|g(t,\eta_i\chi^\varepsilon F)\|^2\,\mathrm{d}t \le \frac{K^2}{\mu^2}\|\eta_i\chi^\varepsilon F\|^2 \le \frac{K^2}{\mu^2}\|\eta_i\|^2_{W^{1,\infty}}\|\varphi F\|^2,
\end{aligned}$$

where we used (1.9), (1.10), (1.15), Cauchy-Schwarz inequality, the contraction property of the semigroup $g(t,\cdot)$ (1.6) and finally (1.27). We thus get

$$\|\alpha_i\|_{\mathcal{B}(E,L^2_{F^{-1}})} \le \frac{K}{\mu}\|\eta_i\|_{W^{1,\infty}}\|\varphi F\|.$$

Since $S = \sum_{i\in\mathbb{N}}\|\eta_i\|_{W^{1,\infty}} < \infty$, we finally deduce that the series defining $\delta^\varepsilon F$ converges absolutely in $\mathcal{B}(E, L^2_{F^{-1}})$ and that

$$\|\delta^\varepsilon F\|_{\mathcal{B}(E,L^2_{F^{-1}})} \lesssim \|\varphi F\| = \|\varphi\|_{L^2_x}.$$

Proof of (1.32). Fix $i \in \mathbb{N}$, α_i maps E into $L^2_{F^{-1}}$. We claim that $\alpha_i \in \mathrm{D}(M)$ with, for all $n \in E$,

$$M\alpha_i(x,v,n) = \int_0^{+\infty} Me^{tM} n_i(n) g(t,\eta_i\chi^\varepsilon F)(x+\varepsilon vt, v)\,\mathrm{d}t =: \beta_i(x,v,n)$$

in $L^2_{F^{-1}}$. Indeed, for $n \in E$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_V \left(\frac{e^{hM}\alpha_i(x,v,n) - \alpha_i(x,v,n)}{h} - \beta_i(x,v,n)\right)^2 \frac{\mathrm{d}v}{F(v)}\mathrm{d}x \\
&= \int_{\mathbb{R}^d}\int_V \left(\int_0^\infty \left[\frac{e^{(t+h)M} - e^{tM}}{h} - Me^{tM}\right] n_i(n) g(t,\eta_i\chi^\varepsilon F)(x+\varepsilon vt, v)\,\mathrm{d}t\right)^2 \frac{\mathrm{d}v}{F(v)}\mathrm{d}x \\
&\leq \int_{\mathbb{R}^d}\int_V \left(\int_0^\infty e^{-\mu t}\left\|\left[\frac{e^{hM}-I}{h} - M\right] n_i\right\|_\infty |g(t,\eta_i\chi^\varepsilon F)|(x+\varepsilon vt, v)\,\mathrm{d}t\right)^2 \frac{\mathrm{d}v}{F(v)}\mathrm{d}x \\
&\leq \frac{1}{\mu^2}\left\|\left[\frac{e^{hM}-I}{h} - M\right] n_i\right\|_\infty^2 \|\eta_i\|^2_{W^{1,\infty}}\|\varphi F\|^2.
\end{aligned}$$

Since by (1.14), $n \mapsto n_i(n) \in \mathrm{D}(M)$ we deduce that

$$\left\|\frac{e^{hM}\alpha_i - \alpha_i}{h} - \beta_i\right\|_{\mathcal{B}(E,L^2_{F^{-1}})} \leq \frac{1}{\mu}\left\|\left[\frac{e^{hM}-I}{h} - M\right] n_i\right\|_\infty \|\eta_i\|_{W^{1,\infty}}\|\varphi F\| \underset{h\to 0}{\longrightarrow} 0,$$

which is just what we needed. Now, with (1.15), we apply Lemma 1.2.1 so that we deduce, with the fact that $\sum_{i\in\mathbb{N}}\|\eta_i\|_{W^{1,\infty}} < \infty$ and the dominated convergence Theorem, that $\delta^\varepsilon F \in \mathrm{D}(M)$ with

$$M[\delta^\varepsilon F](x,v,n) = \sum_{i=0}^\infty \beta_i(x,v,n),$$

where the series converges absolutely in $\mathcal{B}(E, L^2_{F^{-1}})$. We fix $i \in \mathbb{N}$, $n \in E$ and $v \in V$. We recall that η_i is in $W^{1,\infty}(\mathbb{R}^d)$ and that χ^ε is defined by (1.26) where φ is in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Then it is easily seen that $\eta_i\chi^\varepsilon F$ and $\overline{\eta_i\chi^\varepsilon F}$ are in $W^{1,2}(\mathbb{R}^d)$ with respect to x. Therefore, since $g(t,\eta_i\chi^\varepsilon F) = \overline{\eta_i\chi^\varepsilon F}F(1-e^{-t}) + \eta_i\chi^\varepsilon F e^{-t}$, we obtain that $h_1 := t \in (0,\infty) \mapsto g(t,\eta_i\chi^\varepsilon F)(x+\varepsilon vt, v)$ is in $W^{1,\infty}((0,\infty), L^2_x)$ with

$$h_1'(t)(x,v) = Lg(t,\eta_i\chi^\varepsilon F)(x+\varepsilon vt,v) + \varepsilon v\cdot\nabla_x g(t,\eta_i\chi^\varepsilon F)(x+\varepsilon vt, v),$$

in L^2_x. Furthermore, with (1.10), $h_2 := t \in (0,\infty) \mapsto e^{tM}n_i(n)$ is clearly in $W^{1,1}((0,\infty),\mathbb{R})$ with $h_2'(t) = Me^{tM}n_i(n)$. We now get by integration by parts

$$\begin{aligned}
\beta_i(x,v,n) &= \int_0^{+\infty} Me^{tM}n_i(n)g(t,\eta_i\chi^\varepsilon F)(x+\varepsilon vt,v)\,\mathrm{d}t \\
&= \left[e^{tM}n_i(n)g(t,\eta_i\chi^\varepsilon F)(x+\varepsilon vt,v)\right]_0^\infty - \int_0^{+\infty} e^{tM}n_i(n)\frac{d.}{\mathrm{d}t}g(t,\eta_i\chi^\varepsilon F)(x+\varepsilon vt,v)\,\mathrm{d}t \\
&= -n_i(n)\eta_i\chi^\varepsilon F(x,v) - \int_0^{+\infty} e^{tM}n_i(n)Lg(t,\eta_i\chi^\varepsilon F)(x+\varepsilon vt,v)\,\mathrm{d}t \\
&\quad - \varepsilon v\cdot\int_0^{+\infty} e^{tM}n_i(n)\nabla_x g(t,\eta_i\chi^\varepsilon F)(x+\varepsilon vt,v)\,\mathrm{d}t,
\end{aligned}$$

where all the equalities have to be understood in L^2_x. We easily see that the last two terms of the preceding equality are respectively equal in L^2_x to $-L\alpha_i(x,v,n)$ and $\varepsilon A\alpha_i(x,v,n)$. As a result, we just proved that for all $i \in \mathbb{N}$ and $n \in E$, we have the following equality for almost all $x \in \mathbb{R}^d$ and $v \in V$:

$$(L - \varepsilon A + M)\alpha_i(x,v,n) = -n_i(n)\eta_i\chi^\varepsilon F(x,v). \tag{1.36}$$

Now, the right hand term of the last equality is clearly in $L^2_{F^{-1}}$. Since α_i is in $L^2_{F^{-1}}$, $L\alpha_i \in L^2_{F^{-1}}$; and we proved above that $M\alpha_i \in L^2_{F^{-1}}$. As a consequence $A\alpha_i$ is in $L^2_{F^{-1}}$ and the preceding equality is valid in $L^2_{F^{-1}}$. We want to sum over $i \in \mathbb{N}$. We previously proved that we have, in $\mathcal{B}(E, L^2_{F^{-1}})$, $\sum_{i=0}^{+\infty} M\alpha_i = \sum_{i=0}^{+\infty} \beta_i = M[\delta^\varepsilon F]$. Since the series $\sum_{i\in\mathbb{N}} \alpha_i$ converges absolutely in $\mathcal{B}(E, L^2_{F^{-1}})$ and since L is a bounded operator on $L^2_{F^{-1}}$, we also deduce that we have, in $\mathcal{B}(E, L^2_{F^{-1}})$, $\sum_{i=0}^{+\infty} L\alpha_i = L[\delta^\varepsilon F]$. Since $\sum_{i\in\mathbb{N}} n_i\eta_i$ converges absolutely in $W^{1,\infty}(\mathbb{R}^d)$ to n, we obtain that $\sum_{i\in\mathbb{N}} n_i\eta_i\chi^\varepsilon F$ converges absolutely in $\mathcal{B}(E, L^2_{F^{-1}})$ to $n\chi^\varepsilon F$. Finally, with (1.36) and the fact that A is a closed operator, we also have, in $\mathcal{B}(E, L^2_{F^{-1}})$, $\sum_{i=0}^{+\infty} A\alpha_i = A[\delta^\varepsilon F]$. Summing (1.36) over $i \in \mathbb{N}$ now gives $(L - \varepsilon A + M)(\delta^\varepsilon F) = -n\chi^\varepsilon F$.

Proof of (1.33) We just proved that $M\delta^\varepsilon F = \sum_{i=0}^{+\infty} \beta_i$, with

$$\begin{aligned}\beta_i(x, v, n) &= \int_0^{+\infty} Me^{tM} n_i(n) g(t, \eta_i\chi^\varepsilon F)(x + \varepsilon vt, v)\,\mathrm{d}t\\ &= \int_0^{+\infty} e^{tM} M n_i(n) g(t, \eta_i\chi^\varepsilon F)(x + \varepsilon vt, v)\,\mathrm{d}t,\end{aligned}$$

so that we immediately deduce (1.33) thanks to (1.15).

Proof of (1.34). Let $\lambda > 0$. First of all, we point out that $g(t, \eta_i\varphi F) = \eta_i\varphi F$ so that

$$-M^{-1}n_i(n)\eta_i\varphi F(x, v) = \int_0^{\infty} e^{tM} n_i(n) g(t, \eta_i\varphi F)(x, v)\mathrm{d}t.$$

We can then write, for $i \in \mathbb{N}$ and $n \in E$,

$$\begin{aligned}&\|\alpha_i(\cdot, \cdot, n) + M^{-1}n_i(n)\eta_i\varphi F\|^2\\ &\le \int_{\mathbb{R}^d}\int_V \left(\int_0^{+\infty} e^{tM} n_i(n) g(t, \eta_i(\chi^\varepsilon - \varphi)F)(x + \varepsilon vt, v)\,\mathrm{d}t\right)^2 \frac{\mathrm{d}v}{F(v)}\mathrm{d}x\\ &+ \int_{\mathbb{R}^d}\int_V \left(\int_0^{+\infty} e^{tM} n_i(n)\left[g(t, \eta_i\varphi F)(x + \varepsilon vt, v) - g(t, \eta_i\varphi F)(x, v)\right]\mathrm{d}t\right)^2 \frac{\mathrm{d}v}{F(v)}\mathrm{d}x.\end{aligned}$$

Similarly as the very beginning of the proof, we can bound the first term by

$$\frac{K^2}{\mu^2}\|\eta_i\|^2_{W^{1,\infty}}\|(\chi^\varepsilon - \varphi)F\|^2,$$

and we recall that we have, with (1.28),

$$\|(\chi^\varepsilon - \varphi)F\|^2 \le 2C_\lambda^2\varepsilon^2\|\nabla_x\varphi\|^2_{L^2_x} + 4\|\varphi\|^2_{L^2_x}\lambda^2.$$

For the second term, B say, we write

$$\begin{aligned}B &= \int_{\mathbb{R}^d}\int_V \left(\int_0^{+\infty} e^{tM} n_i(n)\left[\eta_i\varphi F(x + \varepsilon vt, v) - \eta_i\varphi F(x, v)\right]\mathrm{d}t\right)^2 \frac{\mathrm{d}v}{F(v)}\mathrm{d}x.\\ &\le \frac{K^2}{\mu}\|\eta_i\|^2_{W^{1,\infty}}\int_{\mathbb{R}^d}\int_V\int_0^{+\infty} e^{-\mu t}\left[\varphi(x + \varepsilon vt) - \varphi(x)\right]^2 \mathrm{d}tF(v)\mathrm{d}v\mathrm{d}x.\end{aligned}$$

We can then mimic the proof of Proposition 1.4.1 to get the following bound

$$\int_{\mathbb{R}^d}\int_V\int_0^{+\infty} e^{-\mu t}\left[\varphi(x + \varepsilon vt) - \varphi(x)\right]^2 \mathrm{d}tF(v)\mathrm{d}v\mathrm{d}x \le \frac{2C_\lambda^2}{\mu^3}\varepsilon^2\|\nabla_x\varphi\|^2_{L^2_x} + \frac{4}{\mu}\|\varphi\|^2_{L^2_x}\lambda^2.$$

To sum up, we just obtained, for $i \in \mathbb{N}$ and $n \in E$,

$$\begin{aligned}\|\alpha_i(\cdot, \cdot, n) + M^{-1}n_i(n)\eta_i\varphi F\| &\lesssim \|\eta_i\|_{W^{1,\infty}}\left(C_\lambda^2\|\nabla_x\varphi\|^2_{L^2_x}\varepsilon^2 + \|\varphi\|^2_{L^2_x}\lambda^2\right)^{\frac{1}{2}}\\ &\lesssim \|\eta_i\|_{W^{1,\infty}}\left(C_\lambda\|\nabla_x\varphi\|_{L^2_x}\varepsilon + \|\varphi\|_{L^2_x}\lambda\right).\end{aligned}$$

We can now sum over $i \in \mathbb{N}$ to obtain,

$$\|\delta^\varepsilon F + M^{-1}I(n)\varphi F\|_{\mathcal{B}(E,L^2_{F^{-1}})} \lesssim C_\lambda \|\nabla_x \varphi\|_{L^2_x}\varepsilon + \|\varphi\|_{L^2_x}\lambda,$$

which is the bound expected.
Proof of (1.35). We recall that $M\delta^\varepsilon F = \sum_{i=0}^{+\infty} \beta_i$, with β_i defined above. Note that

$$n\chi^\varepsilon F(x,v) = \sum_{i=0}^{+\infty} \int_0^{+\infty} e^{tM} M n_i(n) \eta_i \chi^\varepsilon F(x,v)\, \mathrm{d}t,$$

so that we decompose $M\delta^\varepsilon(x,v,n)F(v) + n\chi^\varepsilon F(x,v)$ into two terms

$$\sum_{i=0}^{+\infty} \int_0^{+\infty} e^{tM} M n_i(n) \left[g(t,\eta_i \chi^\varepsilon F)(x+\varepsilon vt, v) - g(t,\eta_i\varphi F)(x,v)\right] \mathrm{d}t$$
$$+ \sum_{i=0}^{+\infty} \int_0^{+\infty} e^{tM} M n_i(n) \left[\eta_i \varphi F)(x,v) - \eta_i \chi^\varepsilon F(x,v)\right] \mathrm{d}t.$$

As we have done previously, we can show that the first term is, in $\mathcal{B}(E, L^2_{F^{-1}})$, $\lesssim \left(C_\lambda \|\nabla_x\varphi\|_{L^2_x}\varepsilon + \|\varphi\|_{L^2_x}\lambda\right)$. We bound the second term in $\mathcal{B}(E, L^2_{F^{-1}})$ as $\lesssim \|(\chi^\varepsilon - \varphi)F\|$, that is, thanks to (1.28), $\lesssim \left(C_\lambda \|\nabla_x\varphi\|_{L^2_x}\varepsilon + \|\varphi\|_{L^2_x}\lambda\right)$. It finally gives the bound expected. This concludes the proof. □

The corrector θ^ε

We recall that, for all $n \in E$,

$$\theta(n) = \int_E nM^{-1}I(n)\mathrm{d}\nu(n) - nM^{-1}I(n),$$

and that, for $i,j \in \mathbb{N}$, $\theta_{i,j} = \int_E n_i M^{-1} n_j \mathrm{d}\nu - n_i M^{-1} n_j$. Then we define the function $\theta^\varepsilon : \mathbb{R}^d \times V \times E \to \mathbb{R}$ by

$$\theta^\varepsilon(x,v,n)F(v) := \int_0^{+\infty} e^{tM} g(t, \theta(n)\varphi F)(x+\varepsilon vt, v)\, \mathrm{d}t,$$

that is,

$$\theta^\varepsilon(x,v,n)F(v) := \sum_{i,j=0}^{+\infty} \int_0^{+\infty} e^{tM} \theta_{i,j}(n) g(t, \eta_i\eta_j\varphi F)(x+\varepsilon vt, v)\, \mathrm{d}t,$$

and, similarly as Proposition 1.4.4, we obtain the

Proposition 1.4.5. *The function $\theta^\varepsilon F$ belongs to $\mathcal{B}(E, L^2_{F^{-1}})$ with*

$$\|\theta^\varepsilon F\|_{\mathcal{B}(E,L^2_{F^{-1}})} \lesssim \|\varphi\|_{L^2_x}. \tag{1.37}$$

It satisfies

$$(L - \varepsilon A + M)(\theta^\varepsilon F) = -\theta(n)\varphi F, \tag{1.38}$$

with

$$\|M\theta^\varepsilon F\|_{\mathcal{B}(E,L^2_{F^{-1}})} \lesssim \|\varphi\|_{L^2_x}. \tag{1.39}$$

The corrector ζ^ε

We set, for all $(f,n) \in L^2_{F^{-1}} \times E$,

$$\xi^\varepsilon(f,n) = (f, \delta^\varepsilon F)n - \int_E (f, \delta^\varepsilon F) n \mathrm{d}\nu(n),$$

and, for $i \in \mathbb{N}$, $\xi_i^\varepsilon = (f, \delta^\varepsilon F)n_i$. We then define the function $\zeta^\varepsilon : \mathbb{R}^d \times V \times L^2_{F^{-1}} \times E \to \mathbb{R}$ by

$$\zeta^\varepsilon(x,v,f,n)F(v) := \int_0^{+\infty} e^{tM} g(t, \xi^\varepsilon(f,n)\varphi F)(x + \varepsilon vt, v)\, \mathrm{d}t.$$

Similarly as Proposition 1.4.4, we have the

Proposition 1.4.6. *Let $f \in L^2_{F^{-1}}$ be fixed. The function $\zeta^\varepsilon F(f)$ belongs to $\mathcal{B}(E, L^2_{F^{-1}})$ with*

$$\|\zeta^\varepsilon F(f)\|_{\mathcal{B}(E,L^2_{F^{-1}})} \lesssim \|f\| \|\varphi\|^2_{L^2_x}. \tag{1.40}$$

It satisfies

$$(L - \varepsilon A + M)(\zeta^\varepsilon F(f)) = -\xi^\varepsilon(f,n)\varphi F, \tag{1.41}$$

with

$$\|M\zeta^\varepsilon F(f)\|_{\mathcal{B}(E,L^2_{F^{-1}})} \lesssim \|f\| \|\varphi\|^2_{L^2_x}. \tag{1.42}$$

Note that $f \mapsto \zeta^\varepsilon F(f)$ is linear. Furthermore, we have for all $f \in \mathrm{D}(A)$,

$$\|\zeta^\varepsilon(Lf + \varepsilon Af, \cdot)F\|_{\mathcal{B}(E,L^2_{F^{-1}})} \lesssim \|f\| \|\varphi\|_{L^2_x} \left(C_\lambda \|\nabla_x \varphi\|_{L^2_x} \varepsilon + \|\varphi\|_{L^2_x} \lambda\right). \tag{1.43}$$

Proof. We will only prove (1.42) and (1.43). For the former, we write for $i \in \mathbb{N}$ and $(f,n) \in L^2_{F^{-1}} \times E$,

$$\begin{aligned}
M\xi_i^\varepsilon(f,n) &= M(f, \delta^\varepsilon(n)F)n_i(n) - \int_E M(f, \delta^\varepsilon(n)F)n_i(n)\mathrm{d}\nu(n) \\
&= \sum_{j=0}^{+\infty} \int_0^{+\infty} M n_i(n) e^{tM} n_j(n) (f, g(t, \eta_j \chi^\varepsilon F)F)\mathrm{d}t \\
&\quad - \int_E \sum_{j=0}^{+\infty} \int_0^{+\infty} M n_i(n) e^{tM} n_j(n)(f, g(t, \eta_j \chi^\varepsilon F)F)\mathrm{d}t\mathrm{d}\nu(n),
\end{aligned}$$

so that, with (1.16), we have $|M\xi_i^\varepsilon(f,n)| \lesssim \|f\| \|\varphi\|_{L^2_x}$. With the definition of ζ^ε, it is now easy to obtain (1.42).
For (1.43), we fix $i \in \mathbb{N}$ and focus on $\xi_i^\varepsilon(f,n)$. We have for all $(f,n) \in \mathrm{D}(A) \times E$,

$$\begin{aligned}
\xi_i^\varepsilon(Lf + \varepsilon Af, n) &= (Lf + \varepsilon Af, \delta^\varepsilon(n)F)n_i - \int_E (Lf + \varepsilon Af, \delta^\varepsilon(n)F)n_i \mathrm{d}\nu(n) \\
&= (f, (L - \varepsilon A)[\delta^\varepsilon(n)F])n_i - \int_E (f, (L - \varepsilon A)[\delta^\varepsilon(n)F])n_i \mathrm{d}\nu(n) \\
&= -(f, M\delta^\varepsilon(n)F + n\chi^\varepsilon F)n_i + \int_E (f, M\delta^\varepsilon(n)F + n\chi^\varepsilon F)n_i \mathrm{d}\nu(n),
\end{aligned}$$

where we used (1.32). Thanks to (1.35), we thus obtain that, for all $(f,n) \in \mathrm{D}(A) \times E$,

$$|\xi_i^\varepsilon(Lf + \varepsilon Af, n)| \lesssim \|f\| \left(C_\lambda \|\nabla_x \varphi\|_{L^2_x} \varepsilon + \|\varphi\|_{L^2_x} \lambda\right).$$

With the expression of ζ^ε, it is now easy to get the required estimate. This concludes the proof. □

1.4.4 Definition of the corrections

In this section, we precisely define the corrections of the two test functions Ψ and $|\Psi|^2$ that we derived in a formal way in Subsection 1.4.1.

First, we define a deterministic correction by

$$\Psi_*^\varepsilon(f,n) := (f,\chi^\varepsilon F), \qquad f \in L^2_{F^{-1}},\ n \in E.$$

Then, the stochastic corrections for Ψ are defined by, for $(f,n) \in L^2_{F^{-1}} \times E$,

$$\begin{cases} \varphi_1^\varepsilon(f,n) := (f,\delta^\varepsilon(n)F), \\ \varphi_2^\varepsilon(f,n) := (f,\theta^\varepsilon(n)F). \end{cases}$$

The stochastic corrections for $|\Psi|^2$ are defined by, for $(f,n) \in L^2_{F^{-1}} \times E$,

$$\begin{cases} \phi_1^\varepsilon(f,n) := 2(f,\chi^\varepsilon F)(f,\delta^\varepsilon(n)F), \\ \phi_2^\varepsilon(f,n) := 2(f,\zeta^\varepsilon(f,n)F) + 2(f,\chi^\varepsilon F)(f,\theta^\varepsilon(n)F). \end{cases}$$

Finally, the corrections $\Psi^{\varepsilon,1}$ and $\Psi^{\varepsilon,2}$ of Ψ and $|\Psi|^2$ are defined by

$$\begin{cases} \Psi^{\varepsilon,1}(f,n) := \Psi_*^\varepsilon + \varepsilon^{\frac{\alpha}{2}}\varphi_1^\varepsilon + \varepsilon^\alpha \varphi_2^\varepsilon, \\ \Psi^{\varepsilon,2}(f,n) := |\Psi_*^\varepsilon|^2 + \varepsilon^{\frac{\alpha}{2}}\phi_1^\varepsilon + \varepsilon^\alpha \phi_2^\varepsilon. \end{cases}$$

Proposition 1.4.7. *For $i = 1,2$ and $(f,n) \in L^2_{F^{-1}} \times E$, we have the following estimates:*

$$\varphi_i^\varepsilon(f,n) \lesssim \|f\|\|\varphi\|_{L^2_x}, \qquad \phi_i^\varepsilon(f,n) \lesssim \|f\|^2\|\varphi\|^2_{L^2_x}, \tag{1.44}$$

$$M\varphi_i^\varepsilon(f,n) \lesssim \|f\|\|\varphi\|_{L^2_x}, \quad M\phi_i^\varepsilon(f,n) \lesssim \|f\|^2\|\varphi\|^2_{L^2_x}. \tag{1.45}$$

Furthermore, the functions Ψ_*^ε, $|\Psi_*^\varepsilon|^2$, φ_1^ε, φ_2^ε, ϕ_1^ε and ϕ_2^ε are good test functions. Besides, for $(f,n) \in L^2_{F^{-1}} \times E$,

$$|(f, D\phi_2^\varepsilon(f,n))| \lesssim \|f\|^2\|\varphi\|^2_{L^2_x}. \tag{1.46}$$

Proof. Estimates (1.44) and (1.45) are justified by Cauchy Schwarz inequality and (1.31), (1.33), (1.37), (1.39), (1.40) and (1.42).
Concerning the fact that all the functions cited above are good test functions, we first note that the case of Ψ_*^ε and $|\Psi_*^\varepsilon|^2$ is easy to prove.
Let us deal with the case of φ_1^ε. Conditions (i) and (iii) of Definition 1.3.1 are obviously verified. For condition (ii), we have to prove that $D\varphi_1^\varepsilon(f,n) \equiv \delta^\varepsilon(n)F$ is continuous with respect to $(f,n) \in L^2_{F^{-1}} \times E$, i.e. that $n \mapsto \delta^\varepsilon(n)F$ is continuous. We recall that $\delta^\varepsilon(x,v,n)F(v) = \sum_{i=0}^{+\infty} \alpha_i(x,v,n)$ in $\mathcal{B}(E, L^2_{F^{-1}})$ where

$$\alpha_i(x,v,n) := \int_0^{+\infty} e^{tM} n_i(n) g(t,\eta_i\chi^\varepsilon F)(x+\varepsilon vt, v)\,\mathrm{d}t.$$

Now, $n \mapsto n_i(n)$ is continuous with Lemma 1.2.2, and we thus have thanks to (1.9), (1.10), (1.15) and the dominated convergence Theorem that $n \mapsto \alpha_i(n)$ is continuous. Since the series of the α_i defining $\delta^\varepsilon F$ converges in $\mathcal{B}(E, L^2_{F^{-1}})$, we obtain the continuity of $n \mapsto \delta^\varepsilon(n)F$. Furthermore, we can show that $(f,n) \mapsto D\varphi_1^\varepsilon(f,n)$ maps bounded sets onto bounded sets thanks to (1.31). So condition (ii) is verified. Similarly, by the continuity of $n \mapsto Mn_i(n)$ (Lemma 1.2.2) and by (1.45), we prove that condition (iv) is verified.
Similarly, we can prove that φ_2^ε, ϕ_1^ε and ϕ_2^ε are good test functions.
Finally, since $\zeta^\varepsilon(f,n)$ is linear in f, for $(f,n) \in L^2_{F^{-1}} \times E$,

$$D\phi_2^\varepsilon(f,n)(f) = 4(f,\zeta^\varepsilon(f,n)F) + 4(f,\chi^\varepsilon F)(f,\theta^\varepsilon(n)F),$$

so that (1.27), (1.31) and (1.40) gives (1.46). □

Proposition 1.4.8. *The function $(f,n) \mapsto |\Psi^{\varepsilon,1}|^2(f,n)$ is a good test function. Furthermore, we have, for all $(f,n) \in L^2_{F^{-1}} \times E$, the following bounds:*

$$\begin{cases} |M|\varphi_1^\varepsilon|^2(f,n)| \lesssim \|f\|^2\|\varphi\|^2_{L^2_x}, \\ |M[\varphi_1^\varepsilon\varphi_2^\varepsilon](f,n)| \lesssim \|f\|^2\|\varphi\|^2_{L^2_x}, \\ |M|\varphi_2^\varepsilon|^2(f,n)| \lesssim \|f\|^2\|\varphi\|^2_{L^2_x}, \end{cases} \tag{1.47}$$

and

$$\varepsilon^{-\alpha}|M|\Psi^{\varepsilon,1}|^2 - 2\Psi^{\varepsilon,1}M\Psi^{\varepsilon,1}| \lesssim \|f\|^2\|\varphi\|^2_{L^2_x}. \tag{1.48}$$

Proof. In the expression of $|\Psi^{\varepsilon,1}|^2$, since Ψ_*^ε, φ_1^ε and φ_2^ε are good test functions by Proposition 1.4.7, it is easy to prove that $|\Psi_*^\varepsilon|^2$, $\Psi_*^\varepsilon\varphi_1^\varepsilon$ and $\Psi_*^\varepsilon\varphi_2^\varepsilon$ are also good test functions. It remains to focus on the cases of $|\varphi_1^\varepsilon|^2$, $\varphi_1^\varepsilon\varphi_2^\varepsilon$ and $|\varphi_2^\varepsilon|^2$. We only show the case of $|\varphi_1^\varepsilon|^2$ since the others are proved similarly.
First, note that point (i) of Definition 1.3.1 is clearly verified by $|\varphi_1^\varepsilon|^2$ with $D|\varphi_1^\varepsilon|^2(f,n)(h) = 2(f,\delta^\varepsilon(n)F)(h,\delta^\varepsilon(n)F)$ and this function of (f,n) maps bounded sets onto bounded sets (thanks to (1.31)) and is continuous (is it linear in f and continuous in n since $n \mapsto \delta^\varepsilon(n)F$ is continuous, see the proof of Proposition 1.4.7). Then we write

$$\begin{aligned} |\varphi_1^\varepsilon|^2(f,n) = (f,\delta^\varepsilon(n)F)^2 &= \left(\sum_{i=0}^{+\infty}\int_0^{+\infty} e^{tM}n_i(n)(f,g(t,\eta_i\chi^\varepsilon F)F)\mathrm{d}t\right)^2 \\ &= \sum_{i,j}\int_0^\infty\int_0^\infty e^{tM}n_i(n)e^{sM}n_j(n)(f,g(t,\eta_i\chi^\varepsilon F)F)(f,g(s,\eta_j\chi^\varepsilon F)F)\mathrm{d}t\mathrm{d}s, \end{aligned}$$

so that, with (1.14), (1.16) and Lemma 1.2.1, we can mimic the proof of Proposition 1.4.4 to show that $|\varphi_1^\varepsilon|^2 \in \mathrm{D}(M)$ with

$$M|\varphi_1^\varepsilon|^2(f,n) = \sum_{i,j}\int_0^\infty\int_0^\infty M[e^{tM}n_ie^{sM}n_j](n)(f,g(t,\eta_i\chi^\varepsilon F)F)(f,g(s,\eta_j\chi^\varepsilon F)F)\mathrm{d}t\mathrm{d}s.$$

Furthermore, with (1.16), $(f,n) \mapsto M|\varphi_1^\varepsilon|^2(f,n)$ maps bounded sets onto bounded sets (it gives the first bound of (1.47)); with (1.2.2), (1.16) and the dominated convergence Theorem, it is continuous with respect to n. Since it is linear in f and maps bounded sets onto bounded sets, it is continuous with respect to (f,n).
To sum up, we proved that $|\varphi_1^\varepsilon|^2(f,n)$ verifies points (ii), (iii) and (iv) of Definition 1.3.1. Finally, we obtain (1.48) thanks to (1.44), (1.45) and (1.47). □

1.4.5 Convergence to the limit generator

We first define the limit generator $\mathscr{L}$. For $\psi = \Psi$ or $\psi = |\Psi|^2$, and all $\rho \in L^2(\mathbb{R}^d)$, we set

$$\begin{aligned} \mathscr{L}\psi(\rho) := (\rho F, -\kappa(-\Delta)^{\frac{\alpha}{2}}D\psi(\rho F)) &- \int_E (\rho F n M^{-1}I(n), D\psi(\rho F))\mathrm{d}\nu(n) \\ &- \int_E D^2\psi(\rho F)(\rho F n, \rho F M^{-1}I(n))\mathrm{d}\nu(n), \end{aligned}$$

and one can easily verify that it is well defined. Then, we state the two results of convergence.

Proposition 1.4.9. *If $(f,n) \in \mathrm{D}(A) \times E$, for any $\lambda > 0$, we have the following estimate:*

$$\begin{aligned} \left|\mathscr{L}^\varepsilon\Psi^{\varepsilon,1}(f,n) - \mathscr{L}\Psi(\rho)\right| \lesssim \|f\| \big[&\Lambda(\varepsilon)(\|\varphi\|_{L^2_x} + \|D^2\varphi\|_{L^2_x}) + C_\lambda\|\nabla_x\varphi\|_{L^2_x}\varepsilon \\ &+\|\varphi\|_{L^2_x}\varepsilon^{\frac{\alpha}{2}} + (\|\varphi\|_{L^2_x} + \|D^2\varphi\|_{L^2_x})\lambda\big]. \end{aligned} \tag{1.49}$$

We can also write the right-hand side of the previous bound as

$$\|f\|(\Lambda(\varepsilon)C_{\varphi,\lambda} + C_\varphi \lambda), \tag{1.50}$$

where in each case Λ stands for a function which only depends on ε such that $\Lambda(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Proof. We recall that, thanks to Proposition 1.4.7, Ψ_*^ε, φ_1^ε and φ_2^ε are good test functions. Then, we compute:

$$\mathscr{L}^\varepsilon \Psi_*^\varepsilon(f,n) = \frac{1}{\varepsilon^\alpha}(Lf + \varepsilon Af, \chi^\varepsilon F) + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(fn, \chi^\varepsilon F),$$

where we used the fact that $M\Psi_*^\varepsilon(f,n) = 0$ since Ψ_*^ε does not depend on n. We also have

$$\begin{aligned}\varepsilon^{\frac{\alpha}{2}} \mathscr{L}^\varepsilon \varphi_1^\varepsilon(f,n) &= \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(Lf + \varepsilon Af, \delta^\varepsilon(n)F) + (fn, \delta^\varepsilon(n)F) + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(f, M\delta^\varepsilon(n)F) \\ &= \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(f, (L - \varepsilon A + M)[\delta^\varepsilon(n)F]) + (fn, \delta^\varepsilon(n)F),\end{aligned}$$

where we used the fact that L (resp. A) is auto-adjoint (resp. skew-adjoint) and due to the equation verified by $\delta^\varepsilon F$ (1.32), we are led to

$$\varepsilon^{\frac{\alpha}{2}} \mathscr{L}^\varepsilon \varphi_1^\varepsilon(f,n) = -\frac{1}{\varepsilon^{\frac{\alpha}{2}}}(fn, \chi^\varepsilon F) + (fn, \delta^\varepsilon(n)F).$$

Furthermore, we have

$$\varepsilon^\alpha \mathscr{L}^\varepsilon \varphi_2^\varepsilon(f,n) = (f, (L - \varepsilon A + M)[\theta^\varepsilon(n)F]) + \varepsilon^{\frac{\alpha}{2}}(fn, \theta^\varepsilon(n)F),$$

that we rewrite, thanks to the equation verified by $\theta^\varepsilon F$ (1.38), as

$$\varepsilon^\alpha \mathscr{L}^\varepsilon \varphi_2^\varepsilon(f,n) = -(f, \theta(n)\varphi F) + \varepsilon^{\frac{\alpha}{2}}(fn, \theta^\varepsilon(n)F).$$

To sum up, $\mathscr{L}^\varepsilon \Psi^{\varepsilon,1}(f,n) = \mathscr{L}^\varepsilon \Psi_*^\varepsilon(f,n) + \varepsilon^{\frac{\alpha}{2}} \mathscr{L}^\varepsilon \varphi_1^\varepsilon(f,n) + \varepsilon^\alpha \mathscr{L}^\varepsilon \varphi_2^\varepsilon(f,n)$, hence

$$\begin{aligned}\mathscr{L}^\varepsilon \Psi^{\varepsilon,1}(f,n) &= \frac{1}{\varepsilon^\alpha}(Lf + \varepsilon Af, \chi^\varepsilon F) + (fn, \delta^\varepsilon(n)F) - (f, \theta(n)\varphi F) + \varepsilon^{\frac{\alpha}{2}}(fn, \theta^\varepsilon(n)F) \\ &= \frac{1}{\varepsilon^\alpha}(\varepsilon Af + Lf, \chi^\varepsilon F) - \int_E (fnM^{-1}I(n), \varphi F)\mathrm{d}\nu(n) \\ &\qquad + (fn, (\delta^\varepsilon(n)F + M^{-1}I(n)\varphi F)) + \varepsilon^{\frac{\alpha}{2}}(fn, \theta^\varepsilon(n)F).\end{aligned}$$

We point out that $D^2\Psi(f) \equiv 0$ and $(f\psi_1, \psi_2 F) = (\rho F\psi_1, \psi_2 F)$ if ψ_1 and ψ_2 do not depend on $v \in V$ so that we have

$$\begin{aligned}|\mathscr{L}^\varepsilon \Psi^\varepsilon(f,n) - \mathscr{L}\Psi(\rho)| &\le |\varepsilon^{-\alpha}(\varepsilon Af + Lf, \chi^\varepsilon F) + (\kappa(-\Delta)^{\frac{\alpha}{2}} f, \varphi F)| \\ &\qquad + |(fn, (\delta^\varepsilon(n)F + M^{-1}I(n)\varphi F))| + \varepsilon^{\frac{\alpha}{2}}|(fn, \theta^\varepsilon(n)F)|.\end{aligned}$$

We recall that, for all $n \in E$, $\|n\|_{W^{1,\infty}} \lesssim 1$ so that

$$\begin{cases} |(fn, (\delta^\varepsilon(n)F + M^{-1}I(n)\varphi F))| \lesssim \|f\|\|\delta^\varepsilon F + M^{-1}I\varphi F\|_{\mathcal{B}(E, L^2_{F^{-1}})}, \\ |(fn, \theta^\varepsilon(n)F)| \lesssim \|f\|\|\theta^\varepsilon F\|_{\mathcal{B}(E, L^2_{F^{-1}})}. \end{cases}$$

Then the bounds (1.30), (1.34) and (1.37) immediately give the result; this concludes the proof. □

Proposition 1.4.10. *If $(f,n) \in \mathrm{D}(A) \times E$, for any $\lambda > 0$, we have the following estimate:*

$$|\mathscr{L}^\varepsilon \Psi^{\varepsilon,2}(f,n) - \mathscr{L}|\Psi|^2(\rho)| \lesssim \Lambda(\varepsilon) C_{\varphi,\lambda} \|f\|^2 + C_\varphi \|f\|^2 \lambda,$$

for a certain function Λ, which only depends on ε, such that $\Lambda(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Proof. We recall that, thanks to Proposition 1.4.7, $|\Psi_*^\varepsilon|^2$, ϕ_1^ε and ϕ_2^ε are good test functions. Then, we compute:

$$\mathscr{L}^\varepsilon |\Psi_*^\varepsilon|^2(f,n) = \frac{2}{\varepsilon^\alpha}(L + \varepsilon A f, \chi^\varepsilon F)(f, \chi^\varepsilon F) + \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(fn, \chi^\varepsilon F)(f, \chi^\varepsilon F),$$

where we used the fact that $M|\Psi_*^\varepsilon|^2(f,n) = 0$ since Ψ_*^ε does not depend on n. We also have, with the fact that $D\varphi_1(f)(h) = 2(h, \chi^\varepsilon F)(f, \delta^\varepsilon(n)F) + 2(h, \delta^\varepsilon(n)F)(f, \chi^\varepsilon F)$,

$$\begin{aligned}
\varepsilon^{\frac{\alpha}{2}} \mathscr{L}^\varepsilon \phi_1^\varepsilon(f,n) &= \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(L + \varepsilon A f, \chi^\varepsilon F)(f, \delta^\varepsilon(n)F) + \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(L + \varepsilon A f, \delta^\varepsilon(n)F)(f, \chi^\varepsilon F) \\
&\quad + 2(fn, \chi^\varepsilon F)(f, \delta^\varepsilon(n)F) + 2(fn, \delta^\varepsilon(n)F)(f, \chi^\varepsilon F) + \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(f, M\delta^\varepsilon(n)F)(f, \chi^\varepsilon F) \\
&= \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(L + \varepsilon A f, \chi^\varepsilon F)(f, \delta^\varepsilon(n)F) + \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(f, (L - \varepsilon A + M)[\delta^\varepsilon(n)F])(f, \chi^\varepsilon F) \\
&\quad + 2(fn, \chi^\varepsilon F)(f, \delta^\varepsilon(n)F) + 2(fn, \delta^\varepsilon(n)F)(f, \chi^\varepsilon F).
\end{aligned}$$

Thanks to the equation satisfied by $\delta^\varepsilon F$ (1.32), we finally get

$$\begin{aligned}
\varepsilon^{\frac{\alpha}{2}} \mathscr{L}^\varepsilon \phi_1^\varepsilon(f,n) &= \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(L + \varepsilon A f, \chi^\varepsilon F)(f, \delta^\varepsilon(n)F) - \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(fn, \chi^\varepsilon F)(f, \chi^\varepsilon F) \\
&\quad + 2(fn, \chi^\varepsilon F)(f, \delta^\varepsilon(n)F) + 2(fn, \delta^\varepsilon(n)F)(f, \chi^\varepsilon F).
\end{aligned}$$

Besides, we have

$$\begin{aligned}
\varepsilon^\alpha \mathscr{L}^\varepsilon \phi_2^\varepsilon(f,n) &= 2(f, (L - \varepsilon A + M)[\zeta^\varepsilon(f,n)F]) + 2(f, (L - \varepsilon A + M)[\theta^\varepsilon(n)F])(f, \chi^\varepsilon F) \\
&\quad + 2(Lf + \varepsilon A f, \chi^\varepsilon F)(f, \theta^\varepsilon(n)F) + 2(f, \zeta^\varepsilon(Lf + \varepsilon A f, n)F) + \varepsilon^{\frac{\alpha}{2}}(fn, D\phi_2^\varepsilon(f,n)),
\end{aligned}$$

that is, due to equations verified by $\theta^\varepsilon F$ and $\zeta^\varepsilon F$ (1.38) and (1.41),

$$\begin{aligned}
\varepsilon^\alpha \mathscr{L}^\varepsilon \phi_2^\varepsilon(f,n) &= -2(f, \xi^\varepsilon \varphi F) - 2(f, \theta(n)\varphi F)(f, \chi^\varepsilon F) \\
&\quad + 2(Lf + \varepsilon A f, \chi^\varepsilon F)(f, \theta^\varepsilon(n)F) + 2(f, \zeta^\varepsilon(Lf + \varepsilon A f, n)F) + \varepsilon^{\frac{\alpha}{2}}(fn, D\phi_2^\varepsilon(f,n)).
\end{aligned}$$

To sum up, $\mathscr{L}^\varepsilon \Psi^{\varepsilon,2}(f,n) = \mathscr{L}^\varepsilon |\Psi_*^\varepsilon|^2(f,n) + \varepsilon^{\frac{\alpha}{2}} \mathscr{L}^\varepsilon \phi_1^\varepsilon(f,n) + \varepsilon^\alpha \mathscr{L}^\varepsilon \phi_2^\varepsilon(f,n)$, hence

$$\begin{aligned}
\mathscr{L}^\varepsilon \Psi^{\varepsilon,2}(f,n) &= \frac{2}{\varepsilon^\alpha}(L + \varepsilon A f, \chi^\varepsilon F)(f, \chi^\varepsilon F) + \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(L + \varepsilon A f, \chi^\varepsilon F)(f, \delta^\varepsilon(n)F) \\
&\quad + 2(fn, \chi^\varepsilon F)(f, \delta^\varepsilon(n)F) + 2(fn, \delta^\varepsilon(n)F)(f, \chi^\varepsilon F) - 2(f, \xi^\varepsilon \varphi F) \\
&\quad - 2(f, \theta(n)\varphi F)(f, \chi^\varepsilon F) + 2(Lf + \varepsilon A f, \chi^\varepsilon F)(f, \theta^\varepsilon(n)F) + 2(f, \zeta^\varepsilon(Lf + \varepsilon A f, n)F) \\
&\quad + \varepsilon^{\frac{\alpha}{2}}(fn, D\phi_2^\varepsilon(f,n)).
\end{aligned}$$

Now, with the definitions of θ, ξ and the limit generator $\mathscr{L}$, we write the following decomposition $\mathscr{L}^\varepsilon \Psi^{\varepsilon,2}(f,n) - \mathscr{L}|\Psi|^2(\rho) = \sum_{i=1}^9 \tau_i(f,n)$, where

$$\begin{aligned}
\tau_1 &:= \frac{2}{\varepsilon^\alpha}(L + \varepsilon A f, \chi^\varepsilon F)(f, \chi^\varepsilon F) - 2(-\kappa(-\Delta)^{\frac{\alpha}{2}} f, \varphi F)(f, \varphi F), \\
\tau_2 &:= -2 \int_E (f, nM^{-1}I(n)\varphi F)(f, (\chi^\varepsilon - \varphi)F) \mathrm{d}\nu(n), \\
\tau_3 &:= 2 \int_E (f, (\delta^\varepsilon(n)F + M^{-1}I(n)\varphi F))(fn, \varphi F) \mathrm{d}\nu(n),
\end{aligned}$$

$$\tau_4 := 2(fn,(\delta^\varepsilon(n)F+M^{-1}I(n)\varphi F))(f,\chi^\varepsilon F),\quad \tau_5 := 2(f,\delta^\varepsilon(n)F)(f,(\chi^\varepsilon-\varphi)F),$$
$$\tau_6 := \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(Lf+\varepsilon Af,\chi^\varepsilon F)(f,\delta^\varepsilon(n)F),\qquad \tau_7 := 2(Lf+\varepsilon Af,\chi^\varepsilon F)(f,\theta^\varepsilon(n)F),$$
$$\tau_8 := 2(f,\zeta^\varepsilon(Lf+\varepsilon Af,n)F),\qquad \tau_9 := \varepsilon^{\frac{\alpha}{2}}(fn,D\phi_2^\varepsilon(f,n)).$$

To conclude the proof, we are now about to bound every τ_i. For τ_1, we write

$$\tau_1 = \frac{2}{\varepsilon^\alpha}(L+\varepsilon Af,\chi^\varepsilon F)(f,\chi^\varepsilon F) - 2(-\kappa(-\Delta)^{\frac{\alpha}{2}}f,\varphi F)(f,\chi^\varepsilon F) + 2(f,-\kappa(-\Delta)^{\frac{\alpha}{2}}\varphi F)(f,(\chi^\varepsilon-\varphi)F),$$

so that, with (1.27),

$$|\tau_1| \lesssim \|f\|\|\varphi\|_{L^2_x}\left|\frac{1}{\varepsilon^\alpha}(L+\varepsilon Af,\chi^\varepsilon F)+(\kappa(-\Delta)^{\frac{\alpha}{2}}f,\varphi F)\right| + 2\|f\|^2\|\kappa(-\Delta)^{\frac{\alpha}{2}}\varphi\|_{L^2_x}\|(\chi^\varepsilon-\varphi)F\|,$$

and we use (1.30) and (1.28). Similarly, we bound τ_2 thanks to (1.28), τ_3 thanks to (1.34), τ_4 thanks to (1.27) and (1.34), τ_5 thanks to (1.31) and (1.28). For τ_6, we write

$$\tau_6 = 2\varepsilon^{\frac{\alpha}{2}}\left(\frac{1}{\varepsilon^\alpha}(Lf+\varepsilon Af,\chi^\varepsilon F) - (-\kappa(-\Delta)^{\frac{\alpha}{2}}f,\varphi F)\right)(f,\delta^\varepsilon(n)F) + 2\varepsilon^{\frac{\alpha}{2}}(f,-\kappa(-\Delta)^{\frac{\alpha}{2}}\varphi F)(f,\delta^\varepsilon(n)F),$$

so that, with (1.31),

$$|\tau_6| \lesssim \varepsilon^{\frac{\alpha}{2}}\|f\|\|\varphi\|_{L^2_x}\left|\frac{1}{\varepsilon^\alpha}(L+\varepsilon Af,\chi^\varepsilon F)+(\kappa(-\Delta)^{\frac{\alpha}{2}}f,\varphi F)\right| + \varepsilon^{\frac{\alpha}{2}}\|f\|^2\|\kappa(-\Delta)^{\frac{\alpha}{2}}\varphi\|_{L^2_x}\|\varphi\|_{L^2_x},$$

and we use (1.30). We handle the case of τ_7 similarly. We bound τ_8 thanks to (1.43), and τ_9 thanks to (1.46).
Finally, the combination of the bounds on the τ_i exactly yields the required result. This concludes the proof. □

1.5 Uniform bound in $L^2_{F^{-1}}$

In this section, we prove a uniform estimate of the $L^2_{F^{-1}}$ norm of the solution f^ε with respect to ε. To do so, we will again use the perturbed test functions method. Thus, let us begin by defining a correction function. Namely, we introduce the function $\iota^\varepsilon : \mathbb{R}^d \times V \times E \to \mathbb{R}$ with

$$\iota^\varepsilon(x,v,n) := \sum_{i=0}^{+\infty}\int_0^{+\infty} e^{tM}n_i(n)\eta_i(x+\varepsilon vt)\,dt.$$

Similarly as Proposition 1.4.4, we can prove the

Proposition 1.5.1. *The function ι^ε is in $L^\infty(\mathbb{R}^d\times V\times E)$ with*

$$\|\iota^\varepsilon\|_{L^\infty(\mathbb{R}^d\times V\times E)} \lesssim 1. \tag{1.51}$$

It satisfies

$$(M-\varepsilon A)(\iota^\varepsilon) = -n. \tag{1.52}$$

Proposition 1.5.2. *For all $p \geq 1$ and $f_0 \in \mathrm{D}(A)$, we have the following bound*

$$\mathbb{E} \sup_{t\in[0,T]} \|f_t^\varepsilon\|^p \lesssim 1. \tag{1.53}$$

Proof. We set, for all $f \in L^2_{F^{-1}}$, $\Theta(f) := \frac{1}{2}\|f\|^2$, which is easily seen to be a good test function. Then,, with the fact that A is skew-adjoint, (1.5), and the fact that Θ does not depend on $n \in E$, we get for $f \in \mathrm{D}(A)$ and $n \in E$,

$$\begin{aligned} \mathscr{L}^\varepsilon \Theta(f,n) &= \frac{1}{\varepsilon^\alpha}(Lf + \varepsilon Af, f) + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(fn, f) + \frac{1}{\varepsilon^\alpha} M\Theta(f,n) \\ &= -\frac{1}{\varepsilon^\alpha}\|Lf\|^2 + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(fn,f). \end{aligned}$$

The first term has a favourable sign to obtain our bound. The second term is more difficult to control, and we correct Θ as follows. We set $\phi^\varepsilon(f,n) = (f, \iota^\varepsilon(n)f)$ and $\Theta^\varepsilon(f,n) := \Theta(f,n) + \varepsilon^{\frac{\alpha}{2}}\phi^\varepsilon(f,n)$. We can show, with the same method as in the proof of Proposition 1.4.7, that ϕ^ε is a good test function. We then use integrations by parts and (1.52) to discover

$$\begin{aligned} \varepsilon^{\frac{\alpha}{2}}\mathscr{L}^\varepsilon \phi^\varepsilon(f,n) &= \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(Lf, \iota^\varepsilon(n)f) + \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(\varepsilon Af, \iota^\varepsilon(n)f) + 2(fn, \iota^\varepsilon(n)f) + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(f, M\iota^\varepsilon(n)f) \\ &= \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(Lf, \iota^\varepsilon(n)f) + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(f, (M - \varepsilon A)[\iota^\varepsilon(n)]f) + 2(fn, \iota^\varepsilon(n)f) \\ &= \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(Lf, \iota^\varepsilon(n)f) - \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(fn, f) + 2(fn, \iota^\varepsilon(n)f). \end{aligned}$$

To sum up, since $\mathscr{L}^\varepsilon\Theta^\varepsilon(f,n) = \mathscr{L}^\varepsilon\Theta(f,n) + \varepsilon^{\frac{\alpha}{2}}\mathscr{L}^\varepsilon\phi^\varepsilon(f,n)$, we have

$$\mathscr{L}^\varepsilon\Theta^\varepsilon(f,n) = -\frac{1}{\varepsilon^\alpha}\|Lf\|^2 + \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(Lf, \iota^\varepsilon(n)f) + 2(fn, \iota^\varepsilon(n)f).$$

We use (1.51) to bound the second term:

$$\begin{aligned} \frac{2}{\varepsilon^{\frac{\alpha}{2}}}(Lf, \iota^\varepsilon(n)f) &\lesssim \frac{1}{\varepsilon^{\frac{\alpha}{2}}}\|Lf\|\|f\| \\ &\leq \frac{\|Lf\|^2}{2\varepsilon^\alpha} + \frac{1}{2}\|f\|^2 \lesssim \frac{\|Lf\|^2}{2\varepsilon^\alpha} + \|f\|^2. \end{aligned}$$

Besides, note that with (1.51) the third term is $\lesssim \|f\|^2$. Finally we just proved that

$$|\mathscr{L}^\varepsilon\Theta^\varepsilon(f,n)| \lesssim \|f\|^2. \tag{1.54}$$

As in Proposition 1.3.1, since Θ^ε is a good test function, we now set,

$$M^\varepsilon_{\Theta^\varepsilon}(t) := \Theta^\varepsilon(f^\varepsilon_t, m^\varepsilon_t) - \Theta^\varepsilon(f^\varepsilon_0, m^\varepsilon_0) - \int_0^t \mathscr{L}^\varepsilon\Theta^\varepsilon(f^\varepsilon_s, m^\varepsilon_s)\,\mathrm{d}s,$$

which is a continuous and integrable $(\mathcal{F}^\varepsilon_t)_{t\geq 0}$ martingale. By definition of Θ, Θ^ε and M^ε,

$$\frac{1}{2}\|f^\varepsilon_t\|^2 = \frac{1}{2}\|f^\varepsilon_0\|^2 - \varepsilon^{\frac{\alpha}{2}}(\phi^\varepsilon(f^\varepsilon_t, m^\varepsilon_t) - \phi^\varepsilon(f^\varepsilon_0, m^\varepsilon_0)) + \int_0^t \mathscr{L}^\varepsilon\Theta^\varepsilon(f^\varepsilon_s, m^\varepsilon_s)\,\mathrm{d}s + M^\varepsilon_{\Theta^\varepsilon}(t).$$

Since with (1.51) we have $|\phi^\varepsilon(f,n)| \lesssim \|f\|^2$, we can write, with (1.54),

$$\|f^\varepsilon_t\|^2 \lesssim \|f^\varepsilon_0\|^2 + \varepsilon^{\frac{\alpha}{2}}\|f^\varepsilon_t\| + \int_0^t \|f^\varepsilon_s\|^2\,\mathrm{d}s + \sup_{t\in[0,T]}|M^\varepsilon_{\Theta^\varepsilon}(t)|,$$

that is, for ε sufficiently small and by the Gronwall Lemma,

$$\|f_t^\varepsilon\|^2 \lesssim \|f_0^\varepsilon\|^2 + \sup_{t\in[0,T]} |M^\varepsilon_{\Theta^\varepsilon}(t)|. \tag{1.55}$$

Furthermore, similarly as Proposition 1.4.8, we can show that $|\Theta^\varepsilon|^2$ is a good test function, and that

$$|\mathscr{L}^\varepsilon|\Theta^\varepsilon|^2 - 2\Theta^\varepsilon \mathscr{L}^\varepsilon\Theta^\varepsilon| = \varepsilon^{-\alpha}|M|\Theta^\varepsilon|^2 - 2\Theta^\varepsilon M\Theta^\varepsilon| \lesssim \|f\|^4(1+\Lambda(\varepsilon)),$$

for some function Λ which only depends on ε and such that $\Lambda(\varepsilon)\to 0$ as $\varepsilon\to 0$. In particular, for ε small enough,

$$|\mathscr{L}^\varepsilon|\Theta^\varepsilon|^2 - 2\Theta^\varepsilon \mathscr{L}^\varepsilon\Theta^\varepsilon| \lesssim \|f\|^4.$$

Besides, with Proposition 1.3.1, the quadratic variation of $M^\varepsilon_{\Theta^\varepsilon}(t)$ is given by

$$\langle M^\varepsilon_{\Theta^\varepsilon}\rangle_t = \int_0^t (\mathscr{L}^\varepsilon|\Theta^\varepsilon|^2 - 2\Theta^\varepsilon\mathscr{L}^\varepsilon\Theta^\varepsilon)(f_s^\varepsilon, m_s^\varepsilon)\,\mathrm{d}s.$$

As a result, with Burkholder-Davis-Gundy and Hölder inequalities, we get

$$\mathbb{E}[\sup_{t\in[0,T]} |M^\varepsilon_{\Theta^\varepsilon}|^p] \lesssim \mathbb{E}[|\langle M^\varepsilon_{\Theta^\varepsilon}\rangle_T|^{\frac{p}{2}}] \lesssim \int_0^T \mathbb{E}[\|f_s^\varepsilon\|^{2p}]\,\mathrm{d}s. \tag{1.56}$$

By (1.55), we have

$$\mathbb{E}[\|f_t^\varepsilon\|^{2p}] \lesssim \mathbb{E}[\|f_0^\varepsilon\|^{2p}] + \mathbb{E}[\sup_{t\in[0,T]} |M^\varepsilon_{\Theta^\varepsilon}(t)|^p],$$

so that we get

$$\mathbb{E}[\|f_T^\varepsilon\|^{2p}] \lesssim \mathbb{E}[\|f_0^\varepsilon\|^{2p}] + \int_0^T \mathbb{E}[\|f_s^\varepsilon\|^{2p}]\,\mathrm{d}s,$$

that is, by the Gronwall lemma,

$$\mathbb{E}[\|f_T^\varepsilon\|^{2p}] \lesssim \mathbb{E}[\|f_0^\varepsilon\|^{2p}].$$

This actually holds true for any $t\in[0,T]$. Thus, using (1.56) and then (1.55) gives finally the result. □

1.6 Summary of the results

In this section we state the following proposition which sums up all the results obtained above. This will be convenient to handle the tightness and convergence steps. We recall that the corrections $\Psi^{\varepsilon,i}$, $i=1,2$ are defined in Section 1.4.4.

Proposition 1.6.1. *Let $f_0^\varepsilon \in \mathrm{D}(A)$. For $i=1,2$,*

$$M^{\varepsilon,i}(t) := \Psi^{\varepsilon,i}(f_t^\varepsilon, m_t^\varepsilon) - \Psi^{\varepsilon,i}(f_0^\varepsilon, m_0^\varepsilon) - \int_0^t \mathscr{L}^\varepsilon\Psi^{\varepsilon,i}(f_s^\varepsilon, m_s^\varepsilon)\,ds, \quad t\in[0,T],$$

is a continuous and integrable martingale for the filtration $(\mathcal{F}_t^\varepsilon)_{t\geq 0}$ generated by $(m_t^\varepsilon, t\geq 0)$. The quadratic variation of M_1^ε is given by

$$\langle M^{\varepsilon,i}\rangle_t = \int_0^t (\mathscr{L}^\varepsilon|\Psi^{\varepsilon,1}|^2 - 2\Psi^{\varepsilon,1}\mathscr{L}^\varepsilon\Psi^{\varepsilon,1})(f_s^\varepsilon, m_s^\varepsilon)\,ds, \quad t\in[0,T]$$

and we have, for all $t \in [0,T]$,

$$|\mathscr{L}^\varepsilon|\Psi^{\varepsilon,1}|^2 - 2\Psi^{\varepsilon,1}\mathscr{L}^\varepsilon\Psi^{\varepsilon,1}|(f_t^\varepsilon, m_t^\varepsilon) \lesssim \sup_{t\in[0,T]} \|f_t^\varepsilon\|^2\|\varphi\|^2_{L^2_x}. \tag{1.57}$$

Furthermore, for any $\lambda > 0$, $0 \le s_1 \le \dots \le s_n \le s \le t$ and $G \in C_b((L^2(\mathbb{R}^d))^n)$,

$$\left|\mathbb{E}\left[\left(\Psi(\rho_t^\varepsilon F) - \Psi(\rho_s^\varepsilon F) - \int_s^t \mathscr{L}\Psi(\rho_\sigma^\varepsilon)\, d\sigma\right) G(\rho^\varepsilon_{s_1}, ..., \rho^\varepsilon_{s_n})\right]\right| \lesssim \Lambda(\varepsilon)C_{\varphi,\lambda} + C_\varphi\lambda, \tag{1.58}$$

$$\left|\mathbb{E}\left[\left(|\Psi|^2(\rho_t^\varepsilon F) - |\Psi|^2(\rho_s^\varepsilon F) - \int_s^t \mathscr{L}|\Psi|^2(\rho_\sigma^\varepsilon)\, d\sigma\right) G(\rho^\varepsilon_{s_1}, ..., \rho^\varepsilon_{s_n})\right]\right| \lesssim \Lambda(\varepsilon)C_{\varphi,\lambda} + C_\varphi\lambda, \tag{1.59}$$

for some function Λ, which only depends on ε, such that $\Lambda(\varepsilon) \to 0$ when $\varepsilon \to 0$. Finally, for all $t \in [0,T]$, we have the following estimate:

$$|\mathscr{L}^\varepsilon\Psi^{\varepsilon,1}|(f_t^\varepsilon, m_t^\varepsilon) \lesssim \sup_{t\in[0,T]} \|f_t^\varepsilon\|(\|\varphi\|_{L^2_x} + \|\nabla_x\varphi\|_{L^2_x} + \|D^2\varphi\|_{L^2_x} + \|(-\Delta)^{\frac{\alpha}{2}}\varphi\|_{L^2_x}). \tag{1.60}$$

Proof. For $i = 1, 2$, Proposition 1.4.7 gives that $\Psi^{\varepsilon,i}$ is a good test function, and it implies, with Proposition 1.3.1, that $M^{\varepsilon,i}$ is a continuous and integrable martingale. Besides, with Proposition 1.4.8, $|\Psi^{\varepsilon,1}|^2$ is a good test function, hence the formula for the quadratic variation of $M^{\varepsilon,1}$.
Note that $\mathscr{L}^\varepsilon|\Psi^{\varepsilon,1}|^2 - 2\Psi^{\varepsilon,1}\mathscr{L}^\varepsilon\Psi^{\varepsilon,1} = \varepsilon^{-\alpha}(M|\Psi^{\varepsilon,1}|^2 - 2\Psi^{\varepsilon,1}M\Psi^{\varepsilon,1})$ from which we deduce (1.57) due to (1.48).
We continue with the proof of (1.58). Observe that $\Psi = \Psi^{\varepsilon,1} + (\Psi - \Psi^\varepsilon_*) - \varepsilon^{\frac{\alpha}{2}}\varphi_1^\varepsilon - \varepsilon^\alpha\varphi_2^\varepsilon$ so that we can write

$$\begin{aligned}\Psi(f_t^\varepsilon) - \Psi(f_s^\varepsilon) - \int_s^t \mathscr{L}\Psi(\rho_\sigma^\varepsilon)\,\mathrm{d}\sigma &= M^{\varepsilon,1}(t) - M^{\varepsilon,1}(s)\\ &+ (\Psi - \Psi^\varepsilon_*)(f_t^\varepsilon) - (\Psi - \Psi^\varepsilon_*)(f_s^\varepsilon) - \varepsilon^{\frac{\alpha}{2}}\varphi_1^\varepsilon(f_t^\varepsilon) - \varepsilon^\alpha\varphi_2^\varepsilon(f_t^\varepsilon)\\ &\quad + \varepsilon^{\frac{\alpha}{2}}\varphi_1^\varepsilon(f_s^\varepsilon) + \varepsilon^\alpha\varphi_2^\varepsilon(f_s^\varepsilon) + \int_s^t \mathscr{L}^\varepsilon\Psi^{\varepsilon,1}(f_\sigma^\varepsilon, m_\sigma^\varepsilon) - \mathscr{L}\Psi(\rho_\sigma^\varepsilon)\,\mathrm{d}\sigma.\end{aligned}$$

Then, we multiply by $G(\rho^\varepsilon_{s_1}, ..., \rho^\varepsilon_{s_n})$ and take the expectation. Note that, since $M^{\varepsilon,1}$ is a martingale for the filtration $(\mathcal{F}_t^\varepsilon)_{t\ge 0}$ generated by $(m_t^\varepsilon, t \ge 0)$, we have

$$\mathbb{E}[(M^{\varepsilon,1}(t) - M^{\varepsilon,1}(s))G(J_r^{-\eta}\rho^\varepsilon_{s_1}, ..., J_r^{-\eta}\rho^\varepsilon_{s_n})] = 0.$$

Then, it suffices to use (1.28), (1.44), (1.50), the uniform $L^2_{F^{-1}}$ bound (1.53) and $\Psi(f) = \Psi(\rho F)$ to obtain (1.58). A similar work can be done to obtain (1.59).

It remains to prove (1.60). We simply write, for $(f, n) \in \mathrm{D}(A) \times E$,

$$|\mathscr{L}^\varepsilon\Psi^{\varepsilon,1}(f,n)| \le |\mathscr{L}^\varepsilon\Psi^{\varepsilon,1}(f,n) - \mathscr{L}\Psi(f,n)| + |\mathscr{L}\Psi(f,n)|.$$

We apply (1.49) with $\varepsilon \le 1$ and $\lambda = 1$ so that

$$|\mathscr{L}^\varepsilon\Psi^{\varepsilon,1}(f,n) - \mathscr{L}\Psi(f,n)| \lesssim \|f\|(\|\varphi\|_{L^2_x} + \|\nabla_x\varphi\|_{L^2_x} + \|D^2\varphi\|_{L^2_x}).$$

Since, clearly,

$$|\mathscr{L}\Psi(f,n)| \lesssim \|f\|(\|\kappa(-\Delta)^{\frac{\alpha}{2}}\varphi\|_{L^2_x} + \|\varphi\|_{L^2_x}),$$

the proof is complete. □

1.7 Tightness

In this section, in order to be able to take the limit $\varepsilon \to 0$ in law of the family of processes $(\rho^\varepsilon)_{\varepsilon>0}$, we prove its tightness in an appropriate space, namely $C([0,T], S^{-\eta}(\mathbb{R}^d))$. Precisely, the result is the following.

Proposition 1.7.1. *Let $\eta > 0$. Then the family $(\rho^\varepsilon)_{\varepsilon>0}$ is tight in $C([0,T], S^{-\eta}(\mathbb{R}^d))$.*

Proof. We will here specialize the test function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ into the functions $(p_j)_{j\in\mathbb{N}^d}$, which are defined in Section 1.2.1. So we set, for $j \in \mathbb{N}^d$ and $f \in L^2_{F^{-1}}$,

$$\Psi_j(f) := (f, p_j F),$$

and we index by $j \in \mathbb{N}^d$ all the corrections defined in Section 1.4.4. Thanks to Proposition 1.6.1, we consider the continuous martingales

$$M_j^{\varepsilon,1}(t) := \Psi_j^{\varepsilon,1}(f_t^\varepsilon, m_t^\varepsilon) - \Psi_j^{\varepsilon,1}(f_0^\varepsilon, m_0^\varepsilon) - \int_0^t \mathscr{L}^\varepsilon \Psi_j^{\varepsilon,1}(f_s^\varepsilon, m_s^\varepsilon)\,\mathrm{d}s.$$

We also define, for $j \in \mathbb{N}^d$ and $t \in [0,T]$,

$$\theta_j^\varepsilon(t) := \Psi_j(f_0^\varepsilon) + \int_0^t \mathscr{L}^\varepsilon \Psi_j^{\varepsilon,1}(f_s^\varepsilon, m_s^\varepsilon)\,\mathrm{d}s + M_j^{\varepsilon,1}(t).$$

Note that

$$\theta_j^\varepsilon(t) = \Psi_j(f_0^\varepsilon) + \Psi_j^{\varepsilon,1}(f_t^\varepsilon, m_t^\varepsilon) - \Psi_j^{\varepsilon,1}(f_0^\varepsilon, m_0^\varepsilon),$$

so that, with Cauchy-Schwarz inequality and (1.44),

$$|\theta_j^\varepsilon(t)| \lesssim \sup_{t\in[0,T]} \|f^\varepsilon(t)\| \|p_j\|_{L^2_x} = \sup_{t\in[0,T]} \|f^\varepsilon(t)\|.$$

Hence, by the uniform $L^2_{F^{-1}}$ bound (1.53),

$$\mathbb{E} \sup_{t\in[0,T]} \left|\theta_j^\varepsilon(t)\right| \lesssim 1. \tag{1.61}$$

We now observe that, for $t \in [0,T]$,

$$\begin{aligned}\Psi_j(f_t^\varepsilon) - \theta_j^\varepsilon(t) = &\left[(\Psi_j - \Psi_{*,j}^\varepsilon) - \varepsilon^{\frac{\alpha}{2}}\varphi_{1,j}^\varepsilon - \varepsilon^\alpha \varphi_{2,j}^\varepsilon\right](f_t^\varepsilon, m_t^\varepsilon) \\ &- \left[(\Psi_j - \Psi_{*,j}^\varepsilon) - \varepsilon^{\frac{\alpha}{2}}\varphi_{1,j}^\varepsilon - \varepsilon^\alpha \varphi_{2,j}^\varepsilon\right](f_0^\varepsilon, m_0^\varepsilon),\end{aligned}$$

and it gives, with Cauchy-Schwarz inequality, (1.28), (1.44), and (1.7),

$$\begin{aligned}\left|\Psi_j(f_t^\varepsilon) - \theta_j^\varepsilon(t)\right| &\lesssim \sup_{t\in[0,T]} \|f_t^\varepsilon\| \|(\chi_j^\varepsilon - p_j)F\| + (\varepsilon^{\frac{\alpha}{2}} + \varepsilon^\alpha)\|f_t^\varepsilon\|\|p_j\|_{L^2_x} \\ &\leq \sup_{t\in[0,T]} \|f_t^\varepsilon\|(C_\lambda \varepsilon \|\nabla_x p_j\|_{L^2_x} + \|p_j\|_{L^2_x}\lambda + (\varepsilon^{\frac{\alpha}{2}} + \varepsilon^\alpha)\|p_j\|_{L^2_x}) \\ &\leq \sup_{t\in[0,T]} \|f_t^\varepsilon\|(C_\lambda \varepsilon \mu_j^{\frac{1}{2}} + \lambda + \varepsilon^{\frac{\alpha}{2}} + \varepsilon^\alpha).\end{aligned} \tag{1.62}$$

From now on, we fix $\gamma > d/2 + 1$. Observe that, by (1.61), a.s. and for all $t \in [0,T]$, the series defined by $u_t^\varepsilon := \sum_{j\in\mathbb{N}^d} \theta_j^\varepsilon(t) J^{-\gamma} p_j$ converges in $L^2(\mathbb{R}^d)$, which is embedded in $\mathcal{S}'(\mathbb{R}^d)$. We then set

$$\theta_t^\varepsilon := J^\gamma \sum_{j\in\mathbb{N}^d} \theta_j^\varepsilon(t) J^{-\gamma} p_j,$$

which exists a.s. and for all $t \in [0,T]$ in $\mathcal{S}'(\mathbb{R}^d)$. In fact, we see that a.s. and for all $t \in [0,T]$, θ_t^ε is in $S^{-\gamma}(\mathbb{R}^d)$. Indeed,

$$\|\theta_t^\varepsilon\|^2_{S^{-\gamma}(\mathbb{R}^d)} = \|J^\gamma u_t^\varepsilon\|^2_{S^{-\gamma}(\mathbb{R}^d)} = \|u_t^\varepsilon\|^2_{L^2_x} < \infty.$$

We point out that $\Psi_j(f_t^\varepsilon) = (\rho_t^\varepsilon F, p_j F) = (\rho_t^\varepsilon, p_j)_x$ so that

$$\begin{aligned}\langle \rho^\varepsilon(t) - \theta^\varepsilon(t), p_j\rangle &= \Psi_j(f_t^\varepsilon) - \langle J^\gamma u_t^\varepsilon, p_j\rangle = \Psi_j(f_t^\varepsilon) - \langle u_t^\varepsilon, J^\gamma p_j\rangle \\ &= \Psi_j(f_t^\varepsilon) - \langle u_t^\varepsilon, p_j\rangle \mu_j^\gamma = \Psi_j(f_t^\varepsilon) - \theta_j^\varepsilon(t).\end{aligned}$$

By (1.62), it permits to write, for $t \in [0,T]$,

$$\begin{aligned}\|\rho^\varepsilon(t) - \theta^\varepsilon(t)\|^2_{S^{-\gamma}(\mathbb{R}^d)} &\lesssim \sum_{j\in\mathbb{N}^d} \mu_j^{-2\gamma} \sup_{t\in[0,T]} \|f_t^\varepsilon\|^2 (C_\lambda \varepsilon^2 \mu_j + \lambda^2 + \varepsilon^\alpha + \varepsilon^{2\alpha}) \\ &\lesssim \sup_{t\in[0,T]} \|f_t^\varepsilon\|^2 (C_\lambda \varepsilon^2 + \varepsilon^\alpha + \varepsilon^{2\alpha} + \lambda^2)\end{aligned}$$

where the second bound comes from our choice $\gamma > d/2 + 1$ (we recall, see Section 1.2.1, that $\mu_j = 2|j| + 1$). Thanks to the uniform $L^2_{F^{-1}}$ bound (1.53), it finally leads to the following estimate:

$$\mathbb{E} \sup_{t\in[0,T]} \|\rho^\varepsilon(t) - \theta^\varepsilon(t)\|_{S^{-\gamma}(\mathbb{R}^d)} \lesssim C_\lambda \varepsilon + \varepsilon^{\frac{\alpha}{2}} + \varepsilon^\alpha + \lambda. \tag{1.63}$$

We now fix $\eta > 0$. For any $\delta > 0$, let

$$w(\rho, \delta) := \sup_{|t-s|<\delta} \|\rho(t) - \rho(s)\|_{S^{-\eta}(\mathbb{R}^d)}$$

denote the modulus of continuity of a function $\rho \in C([0,T], S^{-\eta}(\mathbb{R}^d))$. Since the embedding $L^2(\mathbb{R}^d) \subset S^{-\eta}(\mathbb{R}^d)$ is compact, and by Ascoli's Theorem, the set

$$K_R := \left\{ \rho \in C([0,T], S^{-\eta}(\mathbb{R}^d)),\ \sup_{t\in[0,T]} \|\rho\|_{L^2(\mathbb{R}^d)} \le R,\ w(\rho,\delta) < \varepsilon(\delta) \right\},$$

where $R > 0$ and $\varepsilon(\delta) \to 0$ when $\delta \to 0$, is compact in $C([0,T], S^{-\eta}(\mathbb{R}^d))$. To prove the tightness of $(\rho^\varepsilon)_{\varepsilon>0}$ in $C([0,T], S^{-\eta}(\mathbb{R}^d))$, it thus suffices, see [Bil09], to prove that for all $\sigma > 0$, there exists $R > 0$ such that

$$\mathbb{P}(\sup_{t\in[0,T]} \|\rho^\varepsilon\|_{L^2(\mathbb{R}^d)} > R) < \sigma, \tag{1.64}$$

and

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \mathbb{P}(w(\rho^\varepsilon, \delta) > \sigma) = 0. \tag{1.65}$$

By the continuous embedding $L^2(\mathbb{R}^d) \subset S^{-\eta}(\mathbb{R}^d)$ and Markov's inequality, we have

$$\mathbb{P}(\sup_{t\in[0,T]} \|\rho^\varepsilon\|_{L^2(\mathbb{R}^d)} > R) \le \mathbb{P}(\sup_{t\in[0,T]} \|f^\varepsilon\|_{L^2_{F^{-1}}} > R) \le \frac{1}{R}\mathbb{E}[\sup_{t\in[0,T]} \|f^\varepsilon\|_{L^2_{F^{-1}}}],$$

and it gives (1.64) thanks to the uniform $L^2_{F^{-1}}$ bound (1.53).
Similarly, we will deduce (1.65) by Markov's inequality and a bound on $\mathbb{E}[w(\rho^\varepsilon, \delta)]$ for $\delta > 0$. Actually, by interpolation, the continuous embedding $L^2(\mathbb{R}^d) \subset S^{-\eta}(\mathbb{R}^d)$ and the uniform $L^2_{F^{-1}}$ bound (1.53), we have

$$\mathbb{E} \sup_{|t-s|<\delta} \|\rho(t) - \rho(s)\|_{S^{-\eta^\flat}} \le \mathbb{E} \sup_{|t-s|<\delta} \|\rho(t) - \rho(s)\|^v_{S^{-\eta^\sharp}}$$

for a certain $\upsilon > 0$ if $\eta^\sharp > \eta^\flat > 0$. As a result, it is indeed sufficient to work with $\eta = \gamma$. In view of (1.63), we first want to obtain an estimate of the increments of θ^ε. We have, for $j \in \mathbb{N}^d$ and $0 \le s \le t \le T$,

$$\theta_j^\varepsilon(t) - \theta_j^\varepsilon(s) = \int_s^t \mathscr{L}^\varepsilon \Psi_j^{\varepsilon,1}(f_\sigma^\varepsilon, m_\sigma^\varepsilon)\, \mathrm{d}\sigma + M_j^{\varepsilon,1}(t) - M_j^{\varepsilon,1}(s).$$

By (1.60) and the uniform $L^2_{F^{-1}}$ bound (1.53), we have

$$\mathbb{E}\left|\int_s^t \mathscr{L}^\varepsilon \Psi_j^{\varepsilon,1}(f_\sigma^\varepsilon, m_\sigma^\varepsilon)\, \mathrm{d}\sigma\right|^4 \lesssim C_j |t-s|^4,$$

where

$$C_j := (\|p_j\|_{L^2_x} + \|\nabla_x p_j\|_{L^2_x} + \|D^2 p_j\|_{L^2_x} + \|(-\Delta)^{\frac{\alpha}{2}} p_j\|_{L^2_x}).$$

Furthermore, using Burkholder-Davis-Gundy inequality,

$$\mathbb{E}|M_j^{\varepsilon,1}(t) - M_j^{\varepsilon,1}(s)|^4 \lesssim \mathbb{E}|\langle M_j^{\varepsilon,1}\rangle_t - \langle M_j^{\varepsilon,1}\rangle_s|^2,$$

and thanks to (1.57), the uniform $L^2_{F^{-1}}$ bound (1.53) and the fact that $\|p_j\|_{L^2_x} = 1$, we are led to

$$\mathbb{E}|M_j^{\varepsilon,1}(t) - M_j^{\varepsilon,1}(s)|^4 \lesssim |t-s|^2.$$

Finally we have $\mathbb{E}|\theta_j^\varepsilon(t) - \theta_j^\varepsilon(s)|^4 \lesssim (1+C_j)|t-s|^2$. Now, note that with (1.7), $C_j \lesssim 1 + \sqrt{\mu_j} + \mu_j$. Since we took $\gamma > d/2 + 1$, we can conclude that

$$\mathbb{E}\|\theta_t^\varepsilon - \theta_s^\varepsilon\|^4_{S^{-\gamma}(\mathbb{R}^d)} \lesssim |t-s|^2.$$

It easily follows that, for $\upsilon < 1/2$, $\mathbb{E}\|\theta^\varepsilon\|^4_{W^{\upsilon,4}(0,T,S^{-\gamma}(\mathbb{R}^d))} \lesssim 1$ so that by the Sobolev embedding $W^{\upsilon,4}(0,T,S^{-\gamma}(\mathbb{R}^d)) \subset C^{0,\tau}(0,T,S^{-\gamma}(\mathbb{R}^d))$ which holds true whenever $\tau < \upsilon - 1/4$, we obtain that $\mathbb{E}w(\theta^\varepsilon, \delta) \lesssim \delta^\tau$ for a certain positive τ.
Thus, we deduce, with (1.63),

$$\begin{aligned}\mathbb{E}w(\rho^\varepsilon, \delta) &\le 2\mathbb{E} \sup_{t\in[0,T]} \|\rho_t^\varepsilon - \theta_t^\varepsilon\|_{S^{-\gamma}(\mathbb{R}^d)} + \mathbb{E}w(\theta^\varepsilon, \delta)\\ &\lesssim C_\lambda \varepsilon + \varepsilon^{\frac{\alpha}{2}} + \varepsilon^\alpha + \lambda + \delta^\tau.\end{aligned}$$

To conclude, we then have

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \mathbb{P}(w(\rho^\varepsilon, \delta) > \sigma) \le \lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \sigma^{-1}\mathbb{E}w(\rho^\varepsilon, \delta) \lesssim \sigma^{-1}\lambda,$$

and since $\lambda > 0$ was arbitrary, we just proved (1.65). This concludes the proof. □

1.8 Convergence

In this section, we conclude the proof of Theorem 1.2.4. The idea is now, by the tightness result proved above and Prokhorov's Theorem, to take a subsequence of $(\rho^\varepsilon)_{\varepsilon>0}$ that converges in law to some probability measure. Then we show that this limit probability is actually uniquely determined thanks to the convergences to the limit generator $\mathscr{L}$ proved above.

Let us fix $\eta > 0$. By Proposition 1.7 and Prokhorov's Theorem, there exist a subsequence of $(\rho^\varepsilon)_{\varepsilon>0}$, still denoted $(\rho^\varepsilon)_{\varepsilon>0}$, and a probability measure P on $C([0,T], S^{-\eta}(\mathbb{R}^d))$ such that

$$P^\varepsilon \to P \text{ weakly on } C([0,T], S^{-\eta}(\mathbb{R}^d)),$$

where P^ε stands for the law of ρ^ε. We will now identify the probability measure P. Since $C([0,T],S^{-\eta}(\mathbb{R}^d))$ is separable, we can apply Skohorod representation Theorem [Bil09], so that there exist a new probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ and random variables

$$\widetilde{\rho^\varepsilon},\widetilde{\rho}:\widetilde{\Omega}\to C([0,T],S^{-\eta}(\mathbb{R}^d)),$$

with respective law P^ε and P such that $\widetilde{\rho^\varepsilon}\to\widetilde{\rho}$ in $C([0,T],S^{-\eta}(\mathbb{R}^d))$, $\widetilde{\mathbb{P}}-$a.s. In the sequel, for the sake of clarity, we do not write any more the tildes.

Let us pass to the limit $\varepsilon\to 0$ in the left-hand side of (1.58), namely in the quantity

$$\mathbb{E}\left[\left(\Psi(\rho^\varepsilon_t F)-\Psi(\rho^\varepsilon_s F)-\int_s^t \mathscr{L}\Psi(\rho^\varepsilon_\sigma)\,\mathrm{d}\sigma\right)G(\rho^\varepsilon_{s_1},...,\rho^\varepsilon_{s_n})\right]=:\mathbb{E}[\mathcal{A}(\rho^\varepsilon)].$$

Without loss of any generality, we may assume that the function $G\in C_b((L^2(\mathbb{R}^d))^n)$ is also continuous on the space $H^{-\eta}$; this is always possible with an approximation argument: it suffices to consider $G_r:=G((\mathrm{I}+rJ)^{-\frac{\eta}{2}}\cdot,...,(\mathrm{I}+rJ)^{-\frac{\eta}{2}}\cdot)$ as $r\to 0$. Then, with the $\mathbb{P}-$a.s. convergence of ρ^ε to ρ in the space $C([0,T],S^{-\eta}(\mathbb{R}^d))$, we have that

$$\mathcal{A}(\rho^\varepsilon)\to\mathcal{A}(\rho),\quad \text{a.s.}$$

Furthermore, thanks to the uniform $L^2_{F^{-1}}$ bound (1.53) and the boundedness of G, $(\mathcal{A}(\rho^\varepsilon))_{\varepsilon>0}$ is uniformly integrable since it is bounded in $L^2(\Omega)$, hence

$$\mathbb{E}\mathcal{A}(\rho^\varepsilon)\to\mathbb{E}\mathcal{A}(\rho).$$

As a consequence, we can now pass to the limit $\varepsilon\to 0$ in (1.58) to discover

$$\left|\mathbb{E}\left[\left(\Psi(\rho_t F)-\Psi(\rho_s F)-\int_s^t \mathscr{L}\Psi(\rho_\sigma)\,\mathrm{d}\sigma\right)G(\rho_{s_1},...,\rho_{s_n})\right]\right|\lesssim C_\varphi\lambda.$$

Since this holds true for arbitrary $\lambda>0$, it yields

$$\mathbb{E}\left[\left(\Psi(\rho_t F)-\Psi(\rho_s F)-\int_s^t \mathscr{L}\Psi(\rho_\sigma)\,\mathrm{d}\sigma\right)G(\rho_{s_1},...,\rho_{s_n})\right]=0. \tag{1.66}$$

Similarly, we can pass to the limit $\varepsilon\to 0$ in (1.59); it gives

$$\mathbb{E}\left[\left(|\Psi|^2(\rho_t F)-|\Psi|^2(\rho_s F)-\int_s^t \mathscr{L}|\Psi|^2(\rho_\sigma)\,\mathrm{d}\sigma\right)G(\rho_{s_1},...,\rho_{s_n})\right]=0. \tag{1.67}$$

Since (1.66) and (1.67) are valid for all $n\in\mathbb{N}$, $s_1\le...\le s_n\le s\le t\in[0,T]$ and all $G\in C_b((L^2(\mathbb{R}^d))^n)$, we deduce that

$$N(t):=\Psi(\rho_t F)-\Psi(\rho_0 F)-\int_0^t \mathscr{L}\Psi(\rho_\sigma)\,\mathrm{d}\sigma,\quad t\in[0,T],$$

and

$$S(t):=|\Psi|^2(\rho_t F)-|\Psi|^2(\rho_0 F)-\int_0^t \mathscr{L}|\Psi|^2(\rho_\sigma)\,\mathrm{d}\sigma,\quad t\in[0,T],$$

are martingales with respect to the filtration generated by $(\rho_s)_{s\in[0,T]}$. It implies that, see [FGPS10, Appendix 6.9], the quadratic variation of N is given by

$$\langle N\rangle_t=\int_0^t\left[\mathscr{L}|\Psi|^2(\rho_\sigma)-2\Psi(\rho_\sigma)\mathscr{L}\Psi(\rho_\sigma)\right]\,\mathrm{d}\sigma,\quad t\in[0,T].$$

Furthermore, we have

$$\begin{aligned}\mathscr{L}|\Psi|^2(\rho_\sigma) - 2\Psi(\rho_\sigma)\mathscr{L}\Psi(\rho_\sigma) &= -2\int_E (\rho_\sigma n, \varphi)_x (\rho_\sigma M^{-1} I(n), \varphi)_x \,\mathrm{d}\nu(n) \\ &= 2\mathbb{E}[\int_0^\infty (\rho_\sigma m_0, \varphi)_x (\rho_\sigma m_t, \varphi)_x \,\mathrm{d}t] \\ &= \mathbb{E}[\int_{\mathbb{R}} (\rho_\sigma m_0, \varphi)_x (\rho_\sigma m_t, \varphi)_x \,\mathrm{d}t] \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho_\sigma(x)\varphi(x)\rho_\sigma(y)\varphi(y)k(x,y)\,\mathrm{d}x\mathrm{d}y \\ &= \|\rho_\sigma Q^{\frac{1}{2}}\varphi\|^2_{L^2_x}.\end{aligned}$$

Here, we recall that $\Psi(\rho F) = (\rho F, \varphi F) = (\rho, \varphi)_x$ and that the results above are valid for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. As a consequence, the martingale N gives us that

$$M(t) := \rho_t - \rho_0 - \int_0^t [-\kappa(-\Delta)^{\frac{\alpha}{2}}\rho_\sigma - \frac{1}{2}\rho_\sigma H]\,\mathrm{d}\sigma, \quad t \in [0,T],$$

is a continuous martingale in $L^2(\mathbb{R}^d)$ with respect to the filtration generated by $(\rho_s)_{s\in[0,T]}$ with quadratic variation

$$\langle M\rangle_t = \int_0^t (\rho_\sigma Q^{\frac{1}{2}})(\rho_\sigma Q^{\frac{1}{2}})^*\,\mathrm{d}\sigma, \quad t \in [0,T].$$

Thanks to martingale representation Theorem, see [DPZ08, Theorem 8.2], up to a change of probability space, there exists a cylindrical Wiener process W in $L^2(\mathbb{R}^d)$ such that

$$\rho_t - \rho_0 - \int_0^t [-\kappa(-\Delta)^{\frac{\alpha}{2}}\rho_\sigma - \frac{1}{2}\rho_\sigma H]\,\mathrm{d}\sigma = \int_0^t \rho_\sigma Q^{\frac{1}{2}}\,\mathrm{d}W_\sigma, \quad t \in [0,T].$$

This equality gives that ρ has the law of a weak solution to the equation (1.18) with paths in $C([0,T], S^{-\eta}(\mathbb{R}^d))$. Since this equation has a unique solution with paths in the space $C([0,T], S^{-\eta}(\mathbb{R}^d)) \cap L^\infty([0,T], L^2(\mathbb{R}^d))$, and since pathwise uniqueness implies uniqueness in law, we deduce that P is the law of this solution and is uniquely determined. Finally, by the uniqueness of the limit, the whole sequence $(P^\varepsilon)_{\varepsilon>0}$ converges to P weakly in the space of probability measures on $C([0,T], S^{-\eta}(\mathbb{R}^d))$.

Appendix A

Proof of Lemma 1.4.1.

Proof. For the first bound, we write, thanks to Cauchy-Schwarz inequality,

$$\begin{aligned}\|\chi^\varepsilon F\|^2 &= \int_{\mathbb{R}^d}\int_V \left(\int_0^{+\infty} e^{-t}\varphi(x+\varepsilon vt)\,\mathrm{d}t\right)^2 F(v)\mathrm{d}v\mathrm{d}x \\ &\le \int_{\mathbb{R}^d}\int_V\int_0^{+\infty} e^{-t}\varphi^2(x+\varepsilon vt)F(v)\,\mathrm{d}t\mathrm{d}v\mathrm{d}x \\ &= \|\varphi\|^2_{L^2_x}\int_V\int_0^{+\infty} e^{-t}F(v)\,\mathrm{d}t\mathrm{d}v = \|\varphi\|^2_{L^2_x}.\end{aligned}$$

To prove the second estimate, we fix $\lambda > 0$. Since F is integrable with respect to v, we take $C_\lambda > 0$ such that $\int_{\{|v|\geq C_\lambda\}} F(v)\,\mathrm{d}v < \lambda^2$. We have

$$\|(\chi^\varepsilon - \varphi)F\|^2 = \int_{\mathbb{R}^d}\int_V \left(\int_0^{+\infty} e^{-t}[\varphi(x+\varepsilon vt) - \varphi(x)]\,\mathrm{d}t\right)^2 F(v)\mathrm{d}v\mathrm{d}x.$$

Then we split the v-integral into two terms τ_1 and τ_2:

$$\begin{aligned}
\tau_1 &:= \int_{\mathbb{R}^d}\int_{|v|\geq C_\lambda}\int_0^{+\infty} e^{-z}\left[\varphi(x+\varepsilon vz) - \varphi(x)\right]^2 F(v)\,\mathrm{d}z\mathrm{d}v\mathrm{d}x \\
&\leq 2\int_{\mathbb{R}^d}\int_{|v|\geq C_\lambda}\int_0^{+\infty} e^{-z}\left(|\varphi(x+\varepsilon vz)|^2 + |\varphi(x)|^2\right) F(v)\,\mathrm{d}z\mathrm{d}v\mathrm{d}x \\
&= 4\|\varphi\|^2_{L^2_x}\int_{|v|\geq C_\lambda}\int_0^{+\infty} e^{-z}F(v)\,\mathrm{d}z\mathrm{d}v < 4\|\varphi\|^2_{L^2_x}\lambda^2\ ;
\end{aligned}$$

$$\begin{aligned}
\tau_2 &:= \int_{\mathbb{R}^d}\int_{|v|\leq C_\lambda}\int_0^{+\infty} e^{-z}\left[\varphi(x+\varepsilon vz) - \varphi(x)\right]^2 F(v)\,\mathrm{d}z\mathrm{d}v\mathrm{d}x \\
&= \int_{\mathbb{R}^d}\int_{|v|\leq C_\lambda}\int_0^{+\infty} e^{-z}\left(\int_0^1 \varepsilon zv\cdot\nabla_x\varphi(x+t\varepsilon zv)\,\mathrm{d}t\right)^2 F(v)\,\mathrm{d}z\mathrm{d}v\mathrm{d}x \\
&\leq \varepsilon^2\int_{\mathbb{R}^d}\int_{|v|\leq C_\lambda}\int_0^{+\infty}\int_0^1 e^{-z}z^2|v|^2|\nabla_x\varphi(x+t\varepsilon zv)|^2F(v)\,\mathrm{d}t\mathrm{d}z\mathrm{d}v\mathrm{d}x \\
&\leq 2\varepsilon^2C_\lambda^2\|\nabla_x\varphi\|^2_{L^2_x},
\end{aligned}$$

and this is the result. □

Proof of Lemma 1.4.2.

Proof. We fix $\lambda > 0$. Then we choose C such that, for all $|v| \geq C$,

$$\left|F(v) - \frac{\kappa_0}{|v|^{d+\alpha}}\right| \leq \frac{\lambda\kappa_0}{|v|^{d+\alpha}}. \tag{1.68}$$

Now, we write, for $x \in \mathbb{R}^d$,

$$\begin{aligned}
&\varepsilon^{-\alpha}\int_V\int_0^{+\infty} e^{-t}\left[\varphi(x+\varepsilon vt) - \varphi(x)\right]F(v)\,\mathrm{d}t\mathrm{d}v \\
&= \varepsilon^{-\alpha}\int_{|v|\leq C}\int_0^{+\infty} e^{-t}\left[\varphi(x+\varepsilon vt) - \varphi(x)\right]F(v)\,\mathrm{d}t\mathrm{d}v \\
&+ \varepsilon^{-\alpha}\int_{|v|\geq C}\int_0^{+\infty} e^{-t}\left[\varphi(x+\varepsilon vt) - \varphi(x)\right]\frac{\kappa_0}{|v|^{d+\alpha}}\,\mathrm{d}t\mathrm{d}v \\
&+ \varepsilon^{-\alpha}\int_{|v|\geq C}\int_0^{+\infty} e^{-t}\left[\varphi(x+\varepsilon vt) - \varphi(x)\right]\left[F(v) - \frac{\kappa_0}{|v|^{d+\alpha}}\right]\mathrm{d}t\mathrm{d}v \\
&=: I_1(x) + I_2(x) + I_3(x).
\end{aligned}$$

We begin by bounding $\|I_1\|^2_{L^2_x}$. Since $F(v) = F(-v)$, we rewrite $I_1(x)$ as follows

$$\begin{aligned}
I_1(x) &= \varepsilon^{-\alpha}\int_{|v|\leq C}\int_0^{+\infty} e^{-t}\left[\varphi(x+\varepsilon vt) - \varphi(x) - \varepsilon vt\cdot\nabla_x\varphi(x)\right]F(v)\,\mathrm{d}t\mathrm{d}v \\
&= \varepsilon^{-\alpha}\int_{|v|\leq C}\int_0^{+\infty}\int_0^1 e^{-t}\left[D^2\varphi(x+\varepsilon vts)(\varepsilon vt,\varepsilon vt)\right]F(v)\,\mathrm{d}s\mathrm{d}t\mathrm{d}v.
\end{aligned}$$

Then, with Cauchy-Schwarz inequality, we can write

$$\begin{aligned}
\|I_1\|_{L_x^2}^2 &= \varepsilon^{-2\alpha}\int_{\mathbb{R}^d}\left(\int_{|v|\le C}\int_0^{+\infty}\int_0^1 e^{-t}\left[D^2\varphi(x+\varepsilon vts)(\varepsilon vt,\varepsilon vt)\right]F(v)\,\mathrm{d}s\mathrm{d}t\mathrm{d}v\right)^2\mathrm{d}x\\
&\le \varepsilon^{-2\alpha}\int_{\mathbb{R}^d}\int_{|v|\le C}\int_0^{+\infty}\int_0^1 e^{-t}\varepsilon^4|v|^4t^4|D^2\varphi(x+\varepsilon vts)|^2F(v)\,\mathrm{d}s\mathrm{d}t\mathrm{d}v\,\mathrm{d}x\\
&= \varepsilon^{4-2\alpha}\|D^2\varphi\|_{L_x^2}^2\int_{|v|\le C}\int_0^{+\infty}\int_0^1 e^{-t}t^4|v|^4F(v)\,\mathrm{d}s\mathrm{d}t\mathrm{d}v\\
&\le 24C^4\varepsilon^{4-2\alpha}\|D^2\varphi\|_{L_x^2}^2.
\end{aligned}$$

We are now interested in I_2. We first rewrite I_2 thanks to the change of variables $w := \varepsilon vt$

$$\begin{aligned}
I_2(x) &= \varepsilon^{-\alpha}\int_0^{+\infty}\int_{|w|\ge C\varepsilon t} e^{-t}\left[\varphi(x+w)-\varphi(x)\right]\frac{\kappa_0|\varepsilon t|^{d+\alpha}}{|w|^{d+\alpha}}\frac{dw}{\varepsilon^{\mathrm{d}}t^d}\mathrm{d}t\\
&= \kappa_0\int_0^{+\infty}\int_{|w|\ge C\varepsilon t} e^{-t}|t|^\alpha\left[\varphi(x+w)-\varphi(x)\right]\frac{dw}{|w|^{d+\alpha}}\mathrm{d}t.
\end{aligned}$$

Here we recall that the fractional laplacian can be written as

$$\begin{aligned}
-(-\Delta)^{\frac{\alpha}{2}}\varphi(x) &= c_{d,\alpha}\mathrm{PV}\int_V\left[\varphi(x+w)-\varphi(x)\right]\frac{dw}{|w|^{d+\alpha}}\\
&= c_{d,\alpha}\int_{|w|\ge 1}\left[\varphi(x+w)-\varphi(x)\right]\frac{dw}{|w|^{d+\alpha}}\\
&\quad + c_{d,\alpha}\int_{|w|\le 1}\left[\varphi(x+w)-\varphi(x)-w\cdot\nabla_x\varphi(x)\right]\frac{dw}{|w|^{d+\alpha}}\\
&= L_1(x)+L_2(x).
\end{aligned}$$

It prompts us to use a similar decomposition of $I_2(x)$; we thus write

$$\begin{aligned}
I_2(x) &= \kappa_0\int_0^{1/(C\varepsilon)} e^{-t}|t|^\alpha\int_{|w|\ge 1}\left[\varphi(x+w)-\varphi(x)\right]\frac{dw}{|w|^{d+\alpha}}\mathrm{d}t\\
&+ \kappa_0\int_0^{1/(C\varepsilon)} e^{-t}|t|^\alpha\int_{C\varepsilon t\le|w|\le 1}\left[\varphi(x+w)-\varphi(x)-w\cdot\nabla_x\varphi(x)\right]\frac{dw}{|w|^{d+\alpha}}\mathrm{d}t\\
&+ \kappa_0\int_{1/(C\varepsilon)}^{+\infty} e^{-t}|t|^\alpha\int_{|w|\ge C\varepsilon t}\left[\varphi(x+w)-\varphi(x)\right]\frac{dw}{|w|^{d+\alpha}}\mathrm{d}t\\
&= J_1(x)+J_2(x)+J_3(x).
\end{aligned}$$

We recall the definition (1.19) of κ :

$$\kappa = \frac{\kappa_0}{c_{d,\alpha}}\int_0^{+\infty} e^{-t}|t|^\alpha\,\mathrm{d}t.$$

Then, with Cauchy-Schwarz inequality,

$$\begin{aligned}\|J_1-\kappa L_1\|_{L^2_x}^2 &= \int_{\mathbb{R}^d}\left(\kappa_0\int_{1/(C\varepsilon)}^{+\infty} e^{-t}|t|^\alpha \int_{|w|\geq 1}[\varphi(x+w)-\varphi(x)]\,\frac{dw}{|w|^{d+\alpha}}\mathrm{d}t\right)^2 \mathrm{d}x\\ &\leq \kappa_0^2\left(\int_{|w|\geq 1}\frac{dw}{|w|^{d+\alpha}}\right)\int_{\mathbb{R}^d}\int_{1/(C\varepsilon)}^{+\infty} e^{-t}|t|^{2\alpha}\int_{|w|\geq 1}[\varphi(x+w)-\varphi(x)]^2\,\frac{dw}{|w|^{d+\alpha}}\mathrm{d}t\,\mathrm{d}x\\ &\leq 4\kappa_0^2\left(\int_{|w|\geq 1}\frac{dw}{|w|^{d+\alpha}}\right)^2\|\varphi\|_{L^2_x}^2\int_{1/(C\varepsilon)}^{+\infty} e^{-t}|t|^{2\alpha}\,\mathrm{d}t.\end{aligned}$$

To continue, we decompose $J_2(x)-\kappa L_2(x)$ into two terms $\tau_1(x)+\tau_2(x)$

$$\begin{aligned}&-\kappa_0\int_0^{1/(C\varepsilon)} e^{-t}|t|^\alpha\int_{0\leq|w|\leq C\varepsilon t}\int_0^1 D^2\varphi(x+ws)(w,w)\mathrm{d}s\frac{dw}{|w|^{d+\alpha}}\mathrm{d}t\\ &\qquad -\kappa_0\int_{1/(C\varepsilon)}^{+\infty} e^{-t}|t|^\alpha\int_{|w|\leq 1}\int_0^1 D^2\varphi(x+ws)(w,w)\mathrm{d}s\frac{dw}{|w|^{d+\alpha}}\mathrm{d}t.\end{aligned}$$

We work on $\|\tau_1\|_{L^2_x}^2$, using Cauchy-Schwarz inequality, and the change of variables $v=w/(\varepsilon t)$:

$$\begin{aligned}\|\tau_1\|_{L^2_x}^2 &= \int_{\mathbb{R}^d}\left(\kappa_0\int_0^{1/(C\varepsilon)} e^{-t}|t|^\alpha\int_{0\leq|w|\leq C\varepsilon t}\int_0^1 D^2\varphi(x+ws)(w,w)\mathrm{d}s\frac{dw}{|w|^{d+\alpha}}\mathrm{d}t\right)^2\mathrm{d}x\\ &\leq \int_{\mathbb{R}^d}\left(\kappa_0\int_0^{1/(C\varepsilon)} e^{-t}|t|^\alpha\int_{0\leq|w|\leq C\varepsilon t}\int_0^1 |D^2\varphi(x+ws)|\mathrm{d}s\frac{dw}{|w|^{d+\alpha-2}}\mathrm{d}t\right)^2\mathrm{d}x\\ &\leq \kappa_0^2\int_{|w|\leq 1}\frac{dw}{|w|^{d+\alpha-2}}\int_{\mathbb{R}^d}\int_0^{1/(C\varepsilon)} e^{-t}|t|^{2\alpha}\int_{0\leq|w|\leq C\varepsilon t}\int_0^1 |D^2\varphi(x+ws)|^2\mathrm{d}s\frac{dw}{|w|^{d+\alpha-2}}\mathrm{d}t\,\mathrm{d}x\\ &\leq \kappa_0^2\int_{|w|\leq 1}\frac{dw}{|w|^{d+\alpha-2}}\|D^2\varphi\|_{L^2_x}^2\int_0^{+\infty} e^{-t}|t|^{2\alpha}\int_{0\leq|w|\leq C\varepsilon t}\frac{dw}{|w|^{d+\alpha-2}}\mathrm{d}t\\ &= \kappa_0^2\int_{|w|\leq 1}\frac{dw}{|w|^{d+\alpha-2}}\int_0^{+\infty} e^{-t}|t|^{\alpha+2}\mathrm{d}t\int_{|v|\leq C}\frac{\mathrm{d}v}{|v|^{d+\alpha-2}}\varepsilon^{2-\alpha}\|D^2\varphi\|_{L^2_x}^2.\end{aligned}$$

With the same kind of computations, we are led to

$$\|\tau_2\|_{L^2_x}^2 \leq \kappa_0^2\left(\int_{|w|\leq 1}\frac{dw}{|w|^{d+\alpha-2}}\right)^2\|D^2\varphi\|_{L^2_x}^2\int_{1/(C\varepsilon)}^{+\infty} e^{-t}|t|^{2\alpha}\,\mathrm{d}t,$$

and

$$\|J_3\|_{L^2_x}^2 \leq 4\kappa_0^2\left(\int_{|w|\geq 1}\frac{dw}{|w|^{d+\alpha}}\right)^2\|\varphi\|_{L^2_x}^2\int_{1/(C\varepsilon)}^{+\infty} e^{-t}|t|^{2\alpha}\,\mathrm{d}t.$$

Finally, about the case of I_3, thanks to (1.68), we can do the same work as for I_2; then we just have to put together all the bounds obtained to get the result. This concludes the proof. □

Proof of Lemma 1.4.3.

Proof. First, we write

$$
\begin{aligned}
\varepsilon^{-\alpha}(Lf+\varepsilon Af,\chi^\varepsilon F) &= \varepsilon^{-\alpha}\int_{\mathbb{R}^d}\int_V \rho F\chi^\varepsilon - f\chi^\varepsilon - \varepsilon v\cdot\nabla_x f\chi^\varepsilon \,\mathrm{d}v\mathrm{d}x \\
&= \varepsilon^{-\alpha}\int_{\mathbb{R}^d}\int_V \rho F\chi^\varepsilon - f(\chi^\varepsilon - \varepsilon v\cdot\nabla_x\chi^\varepsilon)\,\mathrm{d}v\mathrm{d}x \\
&= \varepsilon^{-\alpha}\int_{\mathbb{R}^d}\int_V \rho F\chi^\varepsilon - f\varphi\,\mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^d}\rho\int_V \varepsilon^{-\alpha}\left[\chi^\varepsilon-\varphi\right]F\,\mathrm{d}v\mathrm{d}x,
\end{aligned}
$$

where we used an integration by part and (1.25). Furthermore, we have

$$
\begin{aligned}
(-\kappa(-\Delta)^{\frac{\alpha}{2}}f,\varphi F) &= (f,-\kappa(-\Delta)^{\frac{\alpha}{2}}\varphi F) \\
&= -\kappa\int_{\mathbb{R}^d}\int_V f(-\Delta)^{\frac{\alpha}{2}}\varphi\,\mathrm{d}v\mathrm{d}x \\
&= -\kappa\int_{\mathbb{R}^d}\rho(-\Delta)^{\frac{\alpha}{2}}\varphi\,\mathrm{d}v\mathrm{d}x.
\end{aligned}
$$

As a consequence, with Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
\varepsilon^{-\alpha}(\varepsilon Af+Lf,\chi^\varepsilon F)+(\kappa(-\Delta)^{\frac{\alpha}{2}}f,\varphi F) &= \int_{\mathbb{R}^d}\rho\left[\int_V \varepsilon^{-\alpha}\left[\chi^\varepsilon-\varphi\right]F\mathrm{d}v+\kappa(-\Delta)^{\frac{\alpha}{2}}\varphi\right]\mathrm{d}x \\
&\le \|\rho\|_{L^2_x}\left\|\int_V \varepsilon^{-\alpha}\left[\chi^\varepsilon-\varphi\right]F\mathrm{d}v+\kappa(-\Delta)^{\frac{\alpha}{2}}\varphi\right\|_{L^2_x} \\
&\le \|f\|\left\|\int_V \varepsilon^{-\alpha}\left[\chi^\varepsilon-\varphi\right]F\mathrm{d}v+\kappa(-\Delta)^{\frac{\alpha}{2}}\varphi\right\|_{L^2_x},
\end{aligned}
$$

and an application of Lemma 1.4.2 then concludes the proof. □

Chapter 2

The radiative transfer equation perturbed by a Wiener process

Abstract: We provide in this chapter the rigorous derivation of a stochastic non-linear diffusion equation from a radiative transfer equation perturbed with a Wiener process. The proof of the convergence relies on a formal Hilbert expansion and the estimation of the remainder. The Hilbert expansion has to be done up to order 3 to overcome some difficulties caused by the random noise.

Keywords: Kinetic equations, diffusion limit, stochastic partial differential equations, Hilbert expansion, radiative transfer, averaging lemma.

The results of this chapter are available as a preprint:

[DDV14b] A. Debussche, S. De Moor, and J. Vovelle. Diffusion limit for the radiative transfer equation perturbed by a Wiener process. *ArXiv e-prints*, May 2014.

2.1 Introduction

In this chapter, we are interested in the following non-linear equation

$$\begin{aligned} &\mathrm{d}f^\varepsilon \ + \ \frac{1}{\varepsilon}a(v)\cdot\nabla_x f^\varepsilon \ \mathrm{d}t \ = \ \frac{1}{\varepsilon^2}\sigma(\overline{f^\varepsilon})L(f^\varepsilon)\mathrm{d}t \ + \ f^\varepsilon\circ Q\mathrm{d}W_t, \\ &f^\varepsilon(0)=\rho_{\mathrm{in}}, \qquad t\in[0,T],\, x\in\mathbb{T}^N,\, v\in V. \end{aligned} \tag{2.1}$$

where V is an N-dimensional torus, $a : V \to V$ and $\sigma : \mathbb{R} \to \mathbb{R}$. The notation $\overline{f}$ stands for the average over the velocity space V of the function f, that is

$$\overline{f} = \int_V f\,\mathrm{d}v.$$

The operator L is a linear operator of relaxation which acts on the velocity variable $v \in V$ only. It is given by

$$L(f) := \overline{f} - f. \tag{2.2}$$

The random noise term W is a cylindrical Wiener process on the Hilbert space $L^2(\mathbb{T}^N)$. The covariance operator Q is a linear self-adjoint operator on $L^2(\mathbb{T}^N)$. The precise description of the problem setting will be given in the next section. In this chapter, we investigate the behaviour in the limit $\varepsilon \to 0$ of the solution f^ε of (2.1).

Concerning the physical background in the deterministic case ($Q \equiv 0$), the equation (2.1) describes the interaction between a surrounding continuous medium and a flux of photons radiating through it in the absence of hydrodynamical motion. The unknown $f^\varepsilon(t, x, v)$ then stands for a distribution function of photons having position x and velocity v at time t. The function σ is the opacity of the matter. When the surrounding medium becomes very large compared to the mean free paths ε of photons, the solution f^ε to (2.1) is known to behave like ρ where ρ is the solution of the Rosseland equation

$$\partial_t\rho - \mathrm{div}_x(\sigma(\rho)^{-1}K\nabla_x\rho) \ = \ 0, \qquad (t,x)\in[0,T]\times\mathbb{T}^N.$$

with $K := \int_V a(v) \otimes a(v)\,\mathrm{d}v$. This is what is called the Rosseland approximation. In this chapter, we investigate such an approximation where we have perturbed the deterministic equation by a smooth multiplicative random noise of the form $f^\varepsilon \circ Q\mathrm{d}W$. Note in particular that the noise is independent of the scaling ε of the equation. In the deterministic case, the Rosseland approximation has been widely studied. In the paper of Bardos, Golse and Perthame [BGP87], they derive the Rosseland approximation on a slightly more general equation of radiative transfer type than (2.1) where the solution also depends on the frequency variable ν. Using the so-called Hilbert's expansion method, they prove a strong convergence of the solution to the radiative transfer equation to the solution to the Rosseland equation. In [BGPS88], the stationary and evolution Rosseland approximation are proved in a weaker sense with weakened hypothesis on the various parameters of the radiative transfer equation, in particular on the opacity function σ.

In the stochastic setting, the paper of Debussche and Vovelle [DV12] deals with the problem of the radiative transfer equation where the opacity function is constant ($\sigma \equiv 1$) and with a multiplicative noise of the form $\frac{1}{\varepsilon}f^\varepsilon m^\varepsilon$ where $m^\varepsilon(t, x) = m(t/\varepsilon^2, x)$ with m a stationary Markov process. Note that in this setting, the noise also depends on the scaling ε of the equation and that formally $\frac{1}{\varepsilon}m^\varepsilon\mathrm{d}t$ converges in law to some Wiener process $Q\mathrm{d}W_t$ where Q is a covariance operator which can be expressed in terms of the driving process m. In the paper [DV12], the authors prove the convergence in law of the solution to (2.1) to a limit stochastic fluid equation by mean of a generalization of the perturbed test-functions method.

In this present work, we consider a non-linear operator $\sigma(\overline{f})Lf$, which can be seen as a simple non-linear perturbation of the classical linear relaxation operator L considered in [DV12]. Nevertheless, we consider that the noise is already in its limit form QdW. In particular, we point out that the fact that the noise is already in an Itô form permits the application of the Itô formula. As a consequence, we are able to prove in this chapter a stronger result of convergence of f^ε to ρ, namely a strong convergence in the space $X := L^\infty(0,T; L^1(\Omega; L^1_{x,v}))$ with rate ε. The proof relies on the so-called Hilbert expansion method: we expand the solution f^ε to (2.1) as $f^\varepsilon = \rho + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + r^\varepsilon$ where ρ is the solution to the limit problem, f_1, f_2, f_3 are three correctors to be defined appropriately and where r^ε denotes the remainder of the expansion. First, we prove that the correctors $(f_i)_{1\le i\le 3}$ behave correctly in the space X so that $\varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 = O(\varepsilon)$ in X. This step requires some regularity on the limit solution ρ and we make use of the regularity result of Chapter 4. Then, to conclude the proof, we estimate the remainder by mean of an Itô formula to show that r^ε is of order ε in X. Note that an Hilbert expansion up to order 2 is usually sufficient in many well-known deterministic cases; here we need to push the expansion up to order 3 to overcome some difficulties caused by the noise term.

We point out that, in the sequel, when proving existence and uniqueness for the problem (2.1), we use a stochastic averaging lemma which can be interesting by itself. It provides a better regularity for the average over the velocity space of solutions to kinetic stochastic equations, see Lemma 2.4.3. The proof of this lemma is detailed in Appendix B; it is mainly based on an adaptation to a stochastic setting of the paper of Bouchut and Desvillettes [BD99].

The chapter is organized as follows. In Section 2.2, we introduce the setting and the notations and give the main result to be proved, Theorem 2.2.2. In Section 2.3, we derive formally the limit equation. Finally, in Section 2.4, we provide the proof of the main result, which is divided in three main steps. First, we study the existence, uniqueness and regularity of the solutions to the radiative transfer equation (2.1) and to the stochastic Rosseland problem. Then we define and study the correctors of the Hilbert expansion. Finally, we estimate the remainder to conclude the proof.

2.2 Preliminaries and main result

2.2.1 Notations and hypothesis

Let us now introduce the precise setting of equation (2.1). We work on a finite-time interval $[0,T]$, $T>0$, and consider periodic boundary conditions for the space variable: $x \in \mathbb{T}^N$ where $\mathbb{T}^N$ is the N-dimensional torus. Regarding the velocity space V, we also consider periodic boundary conditions, that is $V = \mathbb{T}^N$, but we keep the notation V to distinguish the velocity space from the space one.

For $p \in [1,\infty]$, the Lebesgue spaces $L^p(\mathbb{T}^N \times V)$ will be denoted by $L^p_{x,v}$ for short. The associated norm will be written $\|\cdot\|_{L^p_{x,v}}$. Similarly, we define the Lebesgue spaces L^p_x, L^p_v and, if $k \in \mathbb{Z}$, the Sobolev spaces $W^{k,p}_{x,v}$ and $W^{k,p}_x$ or $H^k_{x,v}$ and H^k_x when $p=2$. The scalar product of $L^2_{x,v}$ will be denoted by $(\cdot,\cdot)$. We finally introduce, for $k \in \mathbb{N}$, the space $C^{0,k}([0,T]\times\mathbb{T}^N)$ constituted by the functions of the variables $(t,x) \in [0,T]\times\mathbb{T}^N$ which are continuous in time and k-times continuously differentiable in space.

Concerning the velocity mapping $a : V \to V$, we shall assume that it is C^1_b. Furthermore, we suppose that the following null flux hypothesis holds

$$\int_V a(v)\,\mathrm{d}v = 0. \tag{2.3}$$

We also define the following matrix

$$K := a(v) \otimes a(v) \tag{2.4}$$

and assume that $\overline{K}$ is definite positive. Furthermore, we use a stochastic version of averaging lemmas to prove the existence of the solution f^ε to (2.1). To do so, we need to assume the following standard condition:

$$\forall \varepsilon > 0,\ \forall (\xi, \sigma) \in S^{N-1} \times \mathbb{R},\ \ \mathrm{Leb}\left(\{v \in V, |a(v)\cdot \xi + \sigma| < \varepsilon\}\right) \le \varepsilon^\alpha, \tag{2.5}$$

for some $\alpha \in (0, 1]$ and where Leb denotes the normalized Lebesgue measure on $V = \mathbb{T}^N$.

Regarding the opacity function $\sigma : \mathbb{R} \to \mathbb{R}$, we assume that

(H1) There exist two positive constants σ_*, $\sigma^* > 0$ such that for almost all $x \in \mathbb{R}$, we have

$$\sigma_* \le \sigma(x) \le \sigma^*;$$

(H2) the function σ is $\mathcal{C}_b^3$, in particular σ is Lipschitz continuous;

(H3) the mappings $x \mapsto \sigma(x)$ and $x \mapsto \sigma(x)x$ are respectively non-increasing and non-decreasing.

Finally, the initial condition ρ_{in} is supposed to be a smooth non-negative function which does not depend on the variable $v \in V$.

2.2.2 The random noise

Regarding the stochastic term, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. The random noise $\mathrm{d}W_t$ is a cylindrical Wiener process on the Hilbert space $L^2(\mathbb{T}^N)$. We can define it by setting

$$\mathrm{d}W_t \ = \sum_{k\ge 0} e_k \ \mathrm{d}\beta_k(t), \tag{2.6}$$

where the $(\beta_k)_{k\ge 0}$ are independent Brownian motions on the real line and $(e_k)_{k\ge 0}$ a complete orthonormal system in the Hilbert space $L^2(\mathbb{T}^N)$. The covariance operator Q is a linear self-adjoint operator on $L^2(\mathbb{T}^N)$. We assume the following regularity property

$$\sum_{k\ge 0} \|Qe_k\|^2_{W_x^{4,\infty}} < \infty. \tag{2.7}$$

In particular, we define

$$\kappa_{0,\infty} := \sum_{k\ge 0} \|Qe_k\|^2_{L_x^\infty} < \infty, \qquad \kappa_{1,\infty} := \sum_{k\ge 0,\, 1\le i\le N} \|\partial_{x_i} Qe_k\|^2_{L_x^\infty} < \infty. \tag{2.8}$$

As a consequence, we can introduce

$$G := \frac{1}{2}\sum_{k\ge 0} (Qe_k)^2,$$

which will be useful when switching Stratonovich integrals into Itô form. Precisely, we point out that for Equation (2.1) we can write $f^\varepsilon \circ Q\mathrm{d}W_t \ = \ f^\varepsilon Q\mathrm{d}W_t + Gf^\varepsilon \mathrm{d}t$ where

$$Q\mathrm{d}W_t \ = \sum_{k\ge 0} Qe_k \ \mathrm{d}\beta_k(t).$$

In the sequel, we will have to consider stochastic integrals of the form $hQ\mathrm{d}W_t$ where $h \in L^p_{x,v}$, $p \geq 2$, and we should ensure the existence of the stochastic integrals as $L^p_{x,v}$-valued processes. We recall that the Lebesgue spaces $L^p_{x,v}$ with $p \geq 2$ belong to a class of the so-called 2-smooth Banach spaces, which are well suited for stochastic Itô integration (see [Brz97], [BP99] for a precise construction). So, let us denote by $\gamma(L^2(\mathbb{T}^N), X)$ the space of the γ-radonifying operators from $L^2(\mathbb{T}^N)$ to a 2-smooth Banach space X. We recall that $\Psi \in \gamma(L^2(\mathbb{T}^N), X)$ if the series

$$\sum_{k\geq 0} \gamma_k \Psi(e_k)$$

converges in $L^2(\widetilde{\Omega}, X)$, for any sequence $(\gamma_k)_{k\geq 0}$ of independent normal real valued random variables on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. Then, the space $\gamma(L^2(\mathbb{T}^N), X)$ is endowed with the norm

$$\|\Psi\|_{\gamma(K,X)} := \left(\widetilde{\mathbb{E}} \left| \sum_{k\geq 0} \gamma_k \Psi(e_k) \right|_X^2 \right)^{1/2}$$

(which does not depend on $(\gamma_k)_{k\geq 0}$) and is a Banach space. Now, if $h \in L^p_{x,v}$, $p \geq 2$, $hQ\mathrm{d}W$ can be interpreted as $\Psi \mathrm{d}W$ where Ψ is the following γ-radonifying operator from $L^2(\mathbb{T}^N)$ to $L^p_{x,v}$:

$$\Psi(e_k) := hQe_k.$$

Let us compute the γ-radonifying norm of Ψ. We fix $(\gamma_j)_{j\in\mathbb{N}}$ a sequence of independent $\mathcal{N}(0,1)$-random variables.

$$\begin{aligned} \|\Psi\|^2_{\gamma(L^2(\mathbb{T}^N), L^p_{x,v})} &= \widetilde{\mathbb{E}} \Big\| \sum_k \gamma_k h(e_k) \Big\|^2_{L^p_{x,v}} = \widetilde{\mathbb{E}} \Big\| \sum_k \gamma_k hQe_k \Big\|^2_{L^p_{x,v}} \\ &\leq \left(\widetilde{\mathbb{E}} \Big\| \sum_k \gamma_k hQe_k \Big\|^p_{L^p_{x,v}} \right)^{2/p} = \left(\widetilde{\mathbb{E}} \int_{\mathbb{T}^N\times V} \Big| \sum_k \gamma_k hQe_k \Big|^p \right)^{2/p}. \end{aligned}$$

Observe that, almost everywhere in $\mathbb{T}^N \times V$, $\sum_k \gamma_k hQe_k$ is a real centered Gaussian with covariance $\sum_k |h\,Qe_k|^2$. As a consequence, there exists a constant $C_p \in (0,\infty)$ such that

$$\widetilde{\mathbb{E}} \Big| \sum_k \gamma_k hQe_k \Big|^p = C_p \Big(\sum_k |hQe_k|^2 \Big)^{p/2}.$$

We use this equality in the computations of the γ-radonifying norm to obtain, thanks to (2.8),

$$\begin{aligned} \|\Psi\|^2_{\gamma(L^2(\mathbb{T}^N), L^p_{x,v})} &\leq C_p^{2/p} \left(\int_{\mathbb{T}^N\times V} \Big(\sum_k (Qe_k)^2 \Big)^{p/2} |h|^p \right)^{2/p} \\ &\leq C_p^{2/p} \kappa_{0,\infty} \|h\|^2_{L^p_{x,v}}. \end{aligned} \tag{2.9}$$

2.2.3 Properties of the operator $\sigma(\bar{\cdot})L(\cdot)$.

Similarly as in the deterministic case, we expect with (2.1) that $\sigma(\overline{f^\varepsilon})L(f^\varepsilon)$ tends to zero with ε, so that we should determine the equilibrium of the operator $\sigma(\bar{\cdot})L(\cdot)$. In this case, since $\sigma > 0$, they are clearly constituted by the functions independent of $v \in V$.
In the space $L^2_{x,v}$, the operator $\sigma(\bar{\cdot})L(\cdot)$ is dissipative. Namely, we have, for $f \in L^2_{x,v}$,

$$(\sigma(\overline{f})Lf, f) = -\|\sigma(\overline{f})^{1/2} Lf\|^2_{L^2_{x,v}} \leq 0. \tag{2.10}$$

In the space L^1_v we have some accretivity properties for the operator $\sigma(\bar{\cdot})L(\cdot)$. Namely, (see [BGP87]), if f, $g \in L^1_v$ with $f \geq 0$, we have

$$\int_V \mathrm{sgn}^+(f-g)\left[\sigma(\overline{f})L(f) - \sigma(\overline{g})L(g)\right] \mathrm{d}v \leq 0, \tag{2.11}$$

where $\mathrm{sgn}^+(x) := \mathbf{1}_{x \geq 0}$. In the deterministic setting, the quantity above is involved when deriving the equation satisfied by $(f-g)^+$ where f and g are solutions to the equation (2.1) without noise and where $x^+ := \max(0,x)$ stands for the positive part of x. This is the main argument that permits to prove uniqueness for equation (2.1) without noise. In our stochastic setting, this procedure will be replaced by the application of Itô formula with the function $x \mapsto x^+$ to the process $f-g$. To make this plainly rigorous, we have to approximate the map $x \mapsto x^+$ by regular (at least C^2) functions. Therefore, we have to investigate what we have lost in the bound (2.11) above when replacing sgn^+ by some smooth approximation. To this end, take ψ a smooth (at least C^2) non-decreasing function such that

$$\begin{cases} \psi(x) = 0, \quad x \in (-\infty, 0], \\ \psi(x) = 1, \quad x \in [1, +\infty), \\ 0 < \psi(x) < 1, \quad x \in (0,1). \end{cases}$$

and define

$$\varphi_\delta(x) := \int_0^x \psi\left(\frac{y}{\delta}\right) \mathrm{d}y, \quad x \in \mathbb{R}. \tag{2.12}$$

Then, we have the following lemma.

Lemma 2.2.1. *Let $\delta > 0$. Suppose that f, $g \in L^1_v$ with $f \geq 0$. We have the two following estimates*

$$\int_V \varphi'_\delta(f-g)\left[\sigma(\overline{f})L(f) - \sigma(\overline{g})L(g)\right] dv \leq C\left(1 + \|f\|_{L^1_v}\right)\delta, \tag{2.13}$$

$$\int_V \varphi'_\delta(g-f)\left[\sigma(\overline{g})L(g) - \sigma(\overline{f})L(f)\right] dv \leq C\left(1 + \|f\|_{L^1_v}\right)\delta. \tag{2.14}$$

Proof. The proof is given in Appendix A. □

2.2.4 Main result

We may now state our main result, the proof of which will be given throughout this chapter.

Theorem 2.2.2. *Let f^ε denote the solution of the kinetic problem (2.1) in the sense of Proposition 2.4.1 and ρ the solution of the non-linear stochastic partial differential equation*

$$\begin{cases} d\rho - \mathrm{div}_x\left(\sigma(\rho)^{-1}\overline{K}\,\nabla_x\rho\right) dt = \rho \circ Q dW_t, \\ \rho(0) = \rho_{in}, \end{cases} \tag{2.15}$$

in the sense of Proposition 2.4.5 and where K denotes the matrix (2.4). Then, the solution f^ε converges as ε tends to 0 to the fluid limit ρ and we have the estimate

$$\sup_{t\in[0,T]} \mathbb{E}\|f^\varepsilon_t - \rho_t\|_{L^1_{x,v}} \leq C\varepsilon. \tag{2.16}$$

2.3 Formal Hilbert expansion

In this section, we derive formally the limit equation satisfied by f^ε as ε goes to 0. To do so, we classically introduce the following Hilbert expansion of the solution f^ε:

$$f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots$$

Then, discarding the terms with positive power of ε, equation (2.1) reads

$$\begin{aligned}\mathrm{d}f_0 = & -\frac{1}{\varepsilon}a(v)\cdot\nabla_x f_0 \,\mathrm{d}t - a(v)\cdot\nabla_x f_1 \,\mathrm{d}t + \frac{1}{\varepsilon^2}\sigma(\overline{f_0} + \varepsilon\overline{f_1} + \varepsilon^2\overline{f_2})L(f_0 + \varepsilon f_1 + \varepsilon^2 f_2)\,\mathrm{d}t \\ & + f_0 \circ Q\mathrm{d}W_t + O(\varepsilon).\end{aligned}$$

Putting the terms with the same power of ε together and omitting once again those with positive power of ε, we have

$$\begin{aligned}\mathrm{d}f_0 = & \frac{1}{\varepsilon^2}\sigma(\overline{f_0})L(f_0)\,\mathrm{d}t + \left(-\frac{1}{\varepsilon}a(v)\cdot\nabla_x f_0 + \frac{1}{\varepsilon}\sigma(\overline{f_0})L(f_1)\right. \\ & \left. + \frac{1}{\varepsilon^2}\left[\sigma(\overline{f_0} + \varepsilon\overline{f_1}) - \sigma(\overline{f_0})\right] L(f_0)\right)\mathrm{d}t \\ & + \left(-a(v)\cdot\nabla_x f_1 + \sigma(\overline{f_0})L(f_2) + \frac{1}{\varepsilon^2}\left[\sigma(\overline{f_0} + \varepsilon\overline{f_1} + \varepsilon^2\overline{f_2}) - \sigma(\overline{f_0} + \varepsilon\overline{f_1})\right] L(f_0)\right. \\ & \left. + \frac{1}{\varepsilon}\left[\sigma(\overline{f_0} + \varepsilon\overline{f_1}) - \sigma(\overline{f_0})\right] L(f_1)\right)\mathrm{d}t + f_0 \circ Q\mathrm{d}W_t + O(\varepsilon).\end{aligned}$$

Next, we identify the terms having the same power of ε. At the order ε^{-2}, we find $\sigma(\overline{f_0})L(f_0) = 0$, which implies $L(f_0) = 0$; thus we have $f_0 = \overline{f_0} =: \rho$. Then, at the order ε^{-1}, with the fact that $L(f_0) = 0$, we find

$$L(f_1) = \sigma(\rho)^{-1}\, a(v)\cdot\nabla_x\rho.$$

Since the integral with respect to $v \in V$ of the right-hand side vanishes thanks to (2.3), this equation can be solved by

$$f_1 := -\sigma(\rho)^{-1}\, a(v)\cdot\nabla_x\rho, \tag{2.17}$$

and we point out that $\overline{f_1} = 0$. Finally, at the order ε^0, we get

$$\mathrm{d}\rho = -a(v)\cdot\nabla_x f_1 \,\mathrm{d}t + \sigma(\rho)L(f_2)\,\mathrm{d}t + \rho\circ Q\mathrm{d}W_t. \tag{2.18}$$

By integration with respect to $v \in V$ and with $\int_V L(f_2)\mathrm{d}v = 0$, we discover

$$\mathrm{d}\rho = -\mathrm{div}_x(\overline{a(v)f_1})\,\mathrm{d}t + \rho\circ\mathrm{d}W_t,$$

that is, thanks to the expression of f_1 given by (2.17),

$$\mathrm{d}\rho - \mathrm{div}_x\left(\sigma(\rho)^{-1}\overline{K}\,\nabla_x\rho\right)\mathrm{d}t = \rho\circ Q\mathrm{d}W_t, \tag{2.19}$$

where $K = a(v)\otimes a(v)$. Furthermore, if ρ satisfies equation (2.19), equation (2.18) now reads

$$\sigma(\rho)L(f_2) = \mathrm{div}_x\left(\sigma(\rho)^{-1}(\overline{K} - K)\nabla_x\rho\right),$$

and since the integral with respect to $v \in V$ of the right-hand side vanishes, this can indeed be solved by setting

$$f_2 := -\sigma(\rho)^{-1}\mathrm{div}_x\left(\sigma(\rho)^{-1}(\overline{K} - K)\nabla_x\rho\right). \tag{2.20}$$

To conclude, the solution f^ε of the kinetic problem (2.1) formally converges to an equilibrium state ρ which satisfies the *non-linear* stochastic partial differential equation (2.19) given above.

2.4 Convergence of f^ε

In this section, we now give a rigorous proof of the convergence of f^ε. The main difficulty is that the remainder $r^\varepsilon := f^\varepsilon - \rho - \varepsilon f_1 - \varepsilon^2 f_2$ can only be appropriately estimated in $L^1_{x,v}$. As a result, in our stochastic case, we will need to apply Itô formula in $L^1_{x,v}$. This gives rise to some difficulties. So, in the sequel, we will need to push the Hilbert expansion of f^ε up to order 3 to overcome these problems. To begin with, we solve the kinetic problem (2.1) and the limiting equation (2.15) and investigate the regularity and properties of the solutions.

2.4.1 Resolution of the kinetic problem

Let us study the kinetic problem (2.1). We solve it using a standard semigroup approach combined with a regularization of the random noise term. Let $p \in [1, \infty]$. We introduce the contraction semigroup $(\mathcal{U}(t))_{t\geq 0}$ generated by the linear operator $-a(v) \cdot \nabla_x$ on the space $L^p_{x,v}$.

Proposition 2.4.1. *Let ρ_{in} be a smooth non-negative function which does not depend on $v \in V$. Then there exists a unique non-negative strong Itô solution f^ε to the kinetic problem (2.1) which belongs to $L^2(\Omega; L^2(0,T; L^2_{x,v}))$ with $\nabla_x f^\varepsilon \in L^2(0,t; L^2_{x,v})$ a.s. for all $t < T$, that is, $\mathbb{P}-$a.s. for all $t \in [0,T]$,*

$$f^\varepsilon(t) = \rho_{in} - \frac{1}{\varepsilon}\int_0^t a(v)\cdot\nabla_x f^\varepsilon_s \, ds + \frac{1}{\varepsilon}\int_0^t \sigma(\overline{f^\varepsilon_s})L(f^\varepsilon_s)\, ds + \int_0^t G f^\varepsilon_s \, dt + \int_0^t f^\varepsilon_s \, Q dW_s.$$

Furthermore, we have the following uniform bound

$$\sup_{t\in[0,T]} \mathbb{E}\|f^\varepsilon(t)\|^2_{L^2_{x,v}} \leq C. \tag{2.21}$$

Before giving the proof of the proposition, we recall a classical result about the regularization of the stochastic convolution.

Lemma 2.4.2. *Let $p \in (2, \infty)$. Let $\Psi \in L^p(\Omega; L^p(0,T; L^p_{x,v}))$. We define*

$$z(t) := \int_0^t \mathcal{U}(t-s)\Psi(s)\, Q dW_s, \quad t \in [0,T].$$

Then $z \in L^p(\Omega; C([0,T]; L^p_{x,v}))$ and

$$\mathbb{E}\sup_{t\in[0,T]} \|z(t)\|^p_{L^p_{x,v}} \leq C\,\mathbb{E}\|\Psi\|^p_{L^p(0,T;L^p_{x,v})},$$

for some constant C which depends on p and $\kappa_{0,\infty}$.

The proof relies on the so-called factorization method (see [PZ07b, Section 11]) combined with the application of the Burkholder-David-Gundy inequality for martingales with values in a 2-smooth Banach space (see [Brz97] and [BP99]) and the bound (2.9).

Proof. Existence and uniqueness part. In this part of the proof, for the sake of convenience, we set $\varepsilon = 1$.

Step 1: Uniqueness. We first begin with the proof of uniqueness for equation (2.1). So let f and g be two non-negative solutions of (2.1) with the same initial condition ρ_{in} and which at least belong to $L^1(\Omega; L^1(0,T; L^1_{x,v}))$. We set $r := f - g$ and estimate r in $L^1_{x,v}$ by applying the Itô formula with the C^2 function φ_δ defined by (2.12) which approximates $x \mapsto x^+$. This gives (note that the term relative to $a(v)\cdot\nabla_x r^\varepsilon$ cancels)

$$\begin{aligned}\mathbb{E}\int_{\mathbb{T}^N\times V}\varphi_\delta(r_t) \;=\; &\mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V}\varphi_\delta'(f_s-g_s)\left[\sigma(\overline{f_s})L(f_s)-\sigma(\overline{g_s})L(g_s)\right]\,\mathrm{d}s\\ &+\;\mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V}\varphi_\delta'(r_s)Gr_s\,\mathrm{d}s+\;\mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V}\varphi_\delta''(r_s)G|r_s|^2\,\mathrm{d}s.\end{aligned}$$

Since $x^+\le\varphi_\delta(x)+\delta$, we have

$$\mathbb{E}\|(r_t)^+\|_{L^1_{x,v}}\;\le\;\mathbb{E}\int_{\mathbb{T}^N\times V}\varphi_\delta(r_t)+\delta.$$

Then, for the next term, we use the accretivity property of the operator $\sigma(\bar{\cdot})L(\cdot)$. Namely, with Lemma 2.2.1, we get

$$\mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V}\varphi_\delta'(f_s-g_s)\left[\sigma(\overline{f_s})L(f_s)-\sigma(\overline{g_s})L(g_s)\right]\le\;C\delta\left(1+\mathbb{E}\int_0^T\|f_s\|_{L^1_{x,v}}\mathrm{d}s\right)\le C\delta.$$

For the following term, we just observe that $|\varphi_\delta'|\le 1$ and that $\|G\|_{L^\infty_x}<\infty$ with (2.8) so that

$$\mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V}\varphi_\delta'(r_s)Gr_s\,\mathrm{d}s\;\le\;C\,\mathbb{E}\int_0^t\|r_s\|_{L^1_{x,v}}\,\mathrm{d}s.$$

For the last term of the Itô formula, we point out that φ_δ'' is zero on $[0,\delta]^c$ and that $|\varphi_\delta''|\le 1/\delta$ on $[0,\delta]$. Thus, we obtain

$$\mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V}\varphi_\delta''(r_s)G|r_s|^2\,\mathrm{d}s\;\le\;C\delta.$$

Summing up all the previous bounds now yields

$$\mathbb{E}\|(r_t)^+\|_{L^1_{x,v}}\;\le\;C\delta+C\,\mathbb{E}\int_0^t\|r_s\|_{L^1_{x,v}}\,\mathrm{d}s.$$

A similar work can be done for $(r)^-=(-r)^+$. As a result we obtain the estimate

$$\mathbb{E}\|r_t\|_{L^1_{x,v}}\;\le\;C\delta+C\,\mathbb{E}\int_0^t\|r_s\|_{L^1_{x,v}}\,\mathrm{d}s.$$

Since this inequality holds true for all $\delta>0$, an application of the Gronwall lemma yields $f=g$ in $L^1(\Omega;L^1(0,T;L^1_{x,v}))$.

Step 2: Resolution of a regularized equation. For $\delta>0$, we will denote by ξ_δ a mollifier on $\mathbb{T}^N\times V$ as $\delta\to0$. This step is devoted to the proof of existence of a solution f^δ to the regularized equation

$$\mathrm{d}f\;+\;a(v)\cdot\nabla_x f\;\mathrm{d}t\;=\;\sigma(\overline{f})L(f)\;\mathrm{d}t\;+\;Gf\;\mathrm{d}t\;+\;f*\xi_\delta\;Q\mathrm{d}W_t,\tag{2.22}$$

with $\delta>0$ being fixed. Let us fix $p>N$. We will apply a fixed point argument in the space $L^p(\Omega;C([0,T_0];L^\infty_{x,v}))$ with T_0 sufficiently small. Before doing this, we first need to truncate the equation to overcome with the non-linear term $f\mapsto\sigma(\overline{f})Lf$ which is not Lipschitz. Following for example [dBD99] or [Gyö98], we introduce $\theta\in C_0^\infty(\mathbb{R})$ whose compact support is embedded in $(-2,2)$ and such that $\theta(x)=1$ for $x\in[-1,1]$ and $0\le\theta\le 1$ on $\mathbb{R}$. Then, for $R>0$, we set $\theta_R(x)=\theta(x/R)$. We are now considering the following equation:

$$\mathrm{d}f \;+\; a(v)\cdot\nabla_x f\;\mathrm{d}t \;=\; \theta_R(\|f\|_{L^\infty_{x,v}})\sigma(\overline{f})L(f)\;\mathrm{d}t \;+\; Gf\;\mathrm{d}t \;+\; f*\xi_\delta\;Q\mathrm{d}W_t, \tag{2.23}$$

and we are looking for a mild solution $f^{R,\delta}$, that is,

$$\begin{aligned} f(t) \;&= \mathcal{U}(t)\rho_{\text{in}} + \int_0^t \mathcal{U}(t-s)\theta_R(\|f_s\|_{L^\infty_{x,v}})\sigma(\overline{f_s})L(f_s)\,\mathrm{d}s + \int_0^t \mathcal{U}(t-s)Gf_s\,\mathrm{d}t \\ &\quad + \int_0^t \mathcal{U}(t-s)f_s*\xi_\delta\,Q\mathrm{d}W_s. \end{aligned} \tag{2.24}$$

Here, as usual, if $f\in L^p(\Omega;C([0,T_0];L^\infty_{x,v}))$, we denote by $\mathcal{T}f$ the right-hand side of the previous equation and we shall verify that the Banach fixed-point Theorem applies. We refer the reader to [dBD99, Proof of Proposition 3.1] for a precise proof in a similar setting. Here, we just prove the contraction property of the stochastic integral. Thanks to Lemma 2.4.2 and with Young's inequality, we easily obtain

$$\mathbb{E}\sup_{t\in[0,T_0]}\left\|\int_0^t \mathcal{U}(t-s)(f_s-g_s)*\xi_\delta\,Q\mathrm{d}W_s\right\|^p_{L^p_{x,v}} \le C\,T_0\,\mathbb{E}\sup_{s\in[0,T_0]}\|f_s-g_s\|^p_{L^\infty_{x,v}},$$

where the constant C depends on p and $\kappa_{0,\infty}$. Now, since $\nabla_x\mathcal{U}(t)g=\mathcal{U}(t)\nabla_x g$, we can similarly obtain

$$\mathbb{E}\sup_{t\in[0,T_0]}\left\|\nabla_x\int_0^t \mathcal{U}(t-s)(f_s-g_s)*\xi_\delta\,Q\mathrm{d}W_s\right\|^p_{L^p_{x,v}} \le C\,T_0\,\mathbb{E}\sup_{s\in[0,T_0]}\|f_s-g_s\|^p_{L^\infty_{x,v}},$$

where the constant C now depends on p, $\kappa_{0,\infty}$, $\kappa_{1,\infty}$ and $\|\nabla_x\xi_\delta\|_{L^1_{x,v}}$. Furthermore, with the identity $\nabla_v\mathcal{U}(t)g=-ta'(v)\mathcal{U}(t)\nabla_x g+\mathcal{U}(t)\nabla_v g$, a similar bound can be proved for the derivatives of the stochastic integral with respect to $v\in V$. To sum up, we are led to

$$\mathbb{E}\sup_{t\in[0,T_0]}\left\|\int_0^t \mathcal{U}(t-s)(f_s-g_s)*\xi_\delta\,Q\mathrm{d}W_s\right\|^p_{W^{1,p}_{x,v}} \le C\,(T_0+T_0^2)\,\mathbb{E}\sup_{s\in[0,T_0]}\|f_s-g_s\|^p_{L^\infty_{x,v}},$$

for some constant C which depends on p, $\kappa_{0,\infty}$, $\kappa_{1,\infty}$, $\|\nabla_x\xi_\delta\|_{L^1_{x,v}}$ and $\|\nabla_v\xi_\delta\|_{L^1_{x,v}}$. Finally, with the Sobolev embedding $W^{1,p}_{x,v}\subset L^\infty_{x,v}$ which holds true since $p>N$, we can conclude that the contraction property of the stochastic term is satisfied in $L^p(\Omega;C([0,T_0];L^\infty_{x,v}))$ provided T_0 is sufficiently small. The Banach fixed-point Theorem then applies and gives us a mild solution $f^{R,\delta}$ of (2.24) in $L^p(\Omega;C([0,T_0];L^\infty_{x,v}))$. Iterating this argument yields a solution in the space $L^p(\Omega;C([0,T];L^\infty_{x,v}))$. Let us introduce, for $R>0$ and $\delta>0$, the following stopping times

$$\tau_{R,\delta}:=\inf\{t\in[0,T],\ \|f^{R,\delta}_t\|_{L^\infty_{x,v}}>R\}.$$

We can show, with a similar method as in [dBD99, Lemma 4.1], that $\tau_{R,\delta}$ is nondecreasing with R so that we can define $\tau^*_\delta:=\lim_{R\to\infty}\tau_{R,\delta}$. The next step is devoted to the proof of some estimates on the solution $f^{R,\delta}$.

Step 3: Estimates on the solution $f^{R,\delta}$. In this step, we emphasize the dependence through the parameters R and δ of the constants C appearing in the estimates. For instance C_δ depends on δ but not on R. With the mild formulation (2.24), using the boundedness of θ_R, σ and G, the contraction property of the semigroup $\mathcal{U}$ in $L^\infty_{x,v}$ and evaluating the stochastic integral in $L^\infty_{x,v}$ similarly as above, we can obtain the following bound

$$\mathbb{E}\sup_{t\in[0,T]}\|f^{R,\delta}_t\|^p_{L^\infty_{x,v}}\le C_\delta. \tag{2.25}$$

Note that the dependence with respect to δ of this bound is due to the evaluation of the stochastic integral in $L^\infty_{x,v}$ by estimating its $W^{1,p}_{x,v}$-norm: this gives rise to the terms $\|\nabla_x \xi_\delta\|_{L^1_{x,v}}$ and $\|\nabla_v \xi_\delta\|_{L^1_{x,v}}$ which depend on δ. Nevertheless, estimating the solution $f^{R,\delta}$ in $L^p_{x,v}$ with $p>2$ gives a uniform bound with respect to R and δ. Precisely, with the mild formulation (2.24), using the boundedness of θ_R, σ and G, the contraction property of the semigroup $\mathcal{U}$ in $L^p_{x,v}$ and evaluating the stochastic integral in $L^p_{x,v}$, $p>2$, thanks to Lemma 2.4.2, we can obtain the following bound

$$\mathbb{E}\sup_{t\in[0,T]}\|f^{R,\delta}_t\|^p_{L^p_{x,v}} \leq C. \tag{2.26}$$

Finally, we point out that we can also estimate $\nabla_x f^{R,\delta}$ in $L^p_{x,v}$, $p>2$, by differentiating equation (2.24). We obtain the bound

$$\mathbb{E}\sup_{t\in[0,T]}\|\nabla_x f^{R,\delta}\|^p_{L^p_{x,v}} \leq C_R. \tag{2.27}$$

Step 4: Definition of f^δ. From (2.25) we easily deduce that for all $\delta>0$, $\tau^*_\delta = T$ a.s. Thus, we define f^δ on $[0,T]=\cup_{R>0}[0,\tau_{R,\delta}]$ by $f^\delta = f^{R,\delta}$ on $[0,\tau_{R,\delta}]$. Note that this definition makes sense since we have proved uniqueness for the equation (2.24) satisfied by $f^{R,\delta}$. Since $f^{R,\delta}$ is a mild solution of (2.23) and since for all $t\in[0,T)$ we have that $\nabla_x f^\delta$ exists a.s. in $L^p(0,t;L^p_{x,v}))$, $p>2$, thanks to (2.27), we get that f^δ is a strong solution of (2.22), that is, $\mathbb{P}-$a.s. for all $t\in[0,T]$,

$$f^\delta(t) = \rho_{\text{in}} - \int_0^t a(v)\cdot\nabla_x f^\delta_s \,\mathrm{d}s + \int_0^t \sigma(\overline{f^\delta_s})L(f^\delta_s)\,\mathrm{d}s + \int_0^t G f^\delta_s\,\mathrm{d}s + \int_0^t f^\delta_s * \xi_\delta\, Q\mathrm{d}W_s. \tag{2.28}$$

Furthermore, with (2.26) and the fact that $\tau^*_\delta = T$ a.s., we deduce that for $p>2$,

$$\mathbb{E}\sup_{t\in[0,T]}\|f^\delta_t\|^p_{L^p_{x,v}} \leq C.$$

Thanks to the Hölder inequality, the previous bound holds true when $p=2$, that is

$$\mathbb{E}\sup_{t\in[0,T]}\|f^\delta_t\|^2_{L^2_{x,v}} \leq C. \tag{2.29}$$

Finally, note that, thanks to the equation (2.28), we can show that $f^\delta\geq 0$. Indeed, it suffices to apply the Itô formula with the function φ_δ defined by (2.12) to the process $-f^\delta$. Similarly as in *Step 1*, since $\rho_{\text{in}}\geq 0$, this yields $(f^\delta)^- = 0$, hence the result.

Step 5: Convergence $\delta\to 0$. Thanks to (2.29), up to a subsequence, the sequence $(f^\delta)_{\delta>0}$ converges weakly in $L^2(\Omega;L^2(0,T;L^2_{x,v}))$ to some f. This is not sufficient to pass to the limit in (2.28) due to the non-linear term. Thus we use the following stochastic averaging lemma, the proof of which in given in Appendix B.

Lemma 2.4.3. *Let $\alpha\in(0,1]$. We assume that hypothesis (2.5) is satisfied. Let f be bounded in $L^2(\Omega;L^2(0,T;L^2_{x,v}))$ such that*

$$df + a(v)\cdot\nabla_x f\,dt = h\,dt + g\,Q\,dW_t, \tag{2.30}$$

with g and h bounded in $L^2(\Omega;L^2(0,T;L^2_{x,v}))$. Then the quantity $\rho=\overline{f}$ verifies

$$\mathbb{E}\int_0^T \|\rho_s\|^2_{H^{\alpha/2}_x}\,ds \leq C.$$

With (2.28) and (2.29), we apply this lemma to the process $\rho^\delta := \overline{f^\delta}$ to obtain

$$\mathbb{E}\int_0^T \|\rho_s^\delta\|^2_{H_x^{\alpha/2}} \mathrm{d}s \;\leq\; C. \tag{2.31}$$

Furthermore, thanks to (2.28) and (2.29), we get that

$$\mathbb{E}\int_0^{T-h} \|f_{s+h}^\delta - f_s^\delta\|^2_{H^{-1}_{x,v}} \mathrm{d}s \;\leq\; Ch, \tag{2.32}$$

which also implies

$$\mathbb{E}\int_0^{T-h} \|\rho_{s+h}^\delta - \rho_s^\delta\|^2_{H^{-1}_x} \mathrm{d}s \;\leq\; Ch. \tag{2.33}$$

Then, with the bounds (2.29) and (2.32) and [Sim87, Theorem 1] we obtain that the sequence of the laws of the processes $(f^\delta)_{\delta>0}$ is tight in $L^2(0,T;H^{-1}_{x,v})$. With the bounds (2.31) and (2.33) and [Sim87, Theorem 4] we also get that the sequence of the laws of the processes $(\rho^\delta)_{\delta>0}$ is tight in $L^2(0,T;L^2_{x,v})$. As a consequence, with Prokhorov's Theorem, we can assume that, up to a subsequence, the laws of the processes $(\rho^\delta)_{\delta>0}$ converges weakly to the law of some process ρ in the space of probability measures on $L^2(0,T;L^2_{x,v})$. Then, using then the Skorohod representation Theorem, there exist a new probability space $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{P}})$ where lives a cylindrical Wiener process $\widehat{W}$ on the Hilbert space $L^2(\mathbb{T}^N)$ and some random variables $\widehat{f^\delta}$, $\widehat{f}$ with respective laws $\mathbb{P}(f^\delta \in \cdot)$ and $\mathbb{P}(f \in \cdot)$ such that $\int_V \widehat{f^\delta}\,\mathrm{d}v$ converges $\widehat{\mathbb{P}}-$a.s. in $L^2(0,T;L^2_{x,v})$ to $\int_V \widehat{f}\,\mathrm{d}v$. Furthermore, we recall that we have the weak convergence of $\widehat{f^\delta}$ to $\widehat{f}$ in $L^2(\widehat{\Omega};L^2(0,T;L^2_{x,v}))$. We also point out that, with (2.27), we can suppose that $\nabla_x\widehat{f}$ exists a.s. in $L^2(0,t;L^2_{x,v})$ for all $t\in[0,T)$. We now have all in hands to pass to the limit $\delta\to 0$ in (2.28) to discover that $\widehat{\mathbb{P}}-$a.s. for all $t\in[0,T]$,

$$\widehat{f}(t) = \rho_{\mathrm{in}} - \int_0^t a(v)\cdot\nabla_x\widehat{f}_s\,\mathrm{d}s + \int_0^t \sigma(\overline{\widehat{f}_s})L(\widehat{f}_s)\,\mathrm{d}s + \int_0^t G\widehat{f}_s\,\mathrm{d}t + \int_0^t \widehat{f}_s\,Q\mathrm{d}\widehat{W}_s. \tag{2.34}$$

Step 6: Conclusion. In this final step, we want to get rid of the change of probability space. To this purpose, we recall that we proved pathwise uniqueness for positive solutions to the equation (2.34) above in *Step 1*. As a consequence, we will make use of the Gyöngy-Krylov characterization of convergence in probability introduced in [GK96]. We recall here the precise result

Lemma 2.4.4. *Let X be a Polish space equipped with the Borel σ-algebra. A sequence of X-valued random variables $\{Y_n,\, n\in\mathbb{N}\}$ converges in probability if and only if for every subsequence of joint laws $\{\mu_{n_k,m_k},\, k\in\mathbb{N}\}$, there exists a further subsequence which converges weakly to a probability measure μ such that*

$$\mu\left((x,y)\in X\times X,\, x=y\right) = 1.$$

Thanks to the pathwise uniqueness of equation (2.34), we can make use of this characterization of convergence in probability here (see for instance [Gyö98, Proof of Theorem 2.1] for more details about the arguments) to deduce that, up to a subsequence, the sequence $(f^\delta)_{\delta>0}$ defined on the initial probability space $(\Omega,\mathcal{F},\mathbb{P})$ converges in probability in $L^2(0,T;L^2_{x,v})$ to a process f. Without loss of generality, we can assume that the convergence is almost sure. Then, using again the method used above in *Step 5*, we deduce that $\mathbb{P}-$a.s. for all $t\in[0,T]$,

$$f(t) = \rho_{\mathrm{in}} - \int_0^t a(v)\cdot\nabla_x f_s\,\mathrm{d}s + \int_0^t \sigma(\overline{f_s})L(f_s)\,\mathrm{d}s + \int_0^t Gf_s\,\mathrm{d}t + \int_0^t f_s\,Q\mathrm{d}W_s. \tag{2.35}$$

Thus f is a non-negative strong solution of the kinetic problem (2.1) and belongs to the expected spaces.

Uniform bound part. The bound (2.21) is easily obtained with an application of the Itô formula with the C^2 function $f \mapsto \|f\|^2_{L^2_{x,v}}$ to the process f^ε. We then make use of the dissipation property (2.10) of the operator $\sigma(\bar{\cdot})L(\cdot)$ in $L^2_{x,v}$ and of the Gronwall lemma. □

2.4.2 Existence and regularity for the limiting equation

Let us now study the limiting stochastic fluid equation (2.15) and the regularity of its solution. Precisely, we have the following result.

Proposition 2.4.5. *Let $p \geq 1$. There exists a strong solution ρ in $L^p(\Omega; C^{0,3}([0,T]\times\mathbb{T}^N))$ to the limit equation* (2.15)

$$\begin{cases} d\rho - \mathrm{div}_x\left(\sigma(\rho)^{-1}\overline{K}\,\nabla_x\rho\right)dt = \rho\circ Q\,dW_t, \\ \rho(0) = \rho_{in}, \end{cases}$$

that is, $\mathbb{P}-$*a.s. for all* $t\in[0,T]$,

$$\rho(t) = \rho_{in} + \int_0^t \mathrm{div}_x\left(\sigma(\rho)^{-1}\overline{K}\,\nabla_x\rho\right)ds + \int_0^t \rho\circ Q\,dW_s.$$

Proof. Note that the Stratonovich integral $\rho\circ Q\mathrm{d}W_t$ rewrites in Itô form $G\rho\,\mathrm{d}t + \rho\,Q\mathrm{d}W_t$. As a consequence, with the hypothesis made on σ (H1)−(H3), a, and the noise (2.7), we can easily show that Theorem 4.2.3 of Chapter 4 applies so that the proof is complete. □

2.4.3 Definition of the two first correctors

Following the computations done in a formal way in section 2.3, we define:

$$\begin{aligned} f_1 &:= -\sigma(\rho)^{-1}\,a(v)\cdot\nabla_x\rho, \\ f_2 &:= -\sigma(\rho)^{-1}\mathrm{div}_x\left(\sigma(\rho)^{-1}(\overline{K}-K)\nabla_x\rho\right). \end{aligned} \tag{2.36}$$

We state two propositions giving the properties of the processes f_1 and f_2.

Proposition 2.4.6. *Let $p \geq 1$. The first corrector f^1, defined by* (2.36), *satisfies*

$$\sigma(\rho)L(f_1) = a(v)\cdot\nabla_x\rho \tag{2.37}$$

with the estimate

$$\mathbb{E}\sup_{t\in[0,T]}\|f_1(t)\|^p_{L^\infty_{x,v}} < \infty, \quad \mathbb{E}\sup_{t\in[0,T]}\|a(v)\cdot\nabla_x f_1(t)\|^p_{L^\infty_{x,v}} < \infty. \tag{2.38}$$

Furthermore, we have

$$df_1 = f_{1,d}\;dt + f_1\left(1-\sigma(\rho)^{-1}\sigma'(\rho)\rho\right)Q\,dW_t, \tag{2.39}$$

where $f_{1,d}$ satisfies

$$\mathbb{E}\sup_{t\in[0,T]}\|f_{1,d}(t)\|^p_{L^\infty_{x,v}} < \infty. \tag{2.40}$$

Proof. The equation (2.37) is a straightforward consequence of the definition of L and f_1 and of (2.3). The estimate (2.38) is a consequence of the regularity of ρ given in Proposition 2.4.5, the bounds (H1) on σ and the boundedness of a. One can easily verify that equation (2.39) holds true; then the bound (2.40) comes once again from the regularity of ρ, the bounds (H1) on σ, the regularity (H2) of σ and the boundedness of a. □

Similarly, we can prove the following properties of the second corrector f_2.

Proposition 2.4.7. *Let $p \geq 1$. The second corrector f^2, defined by (2.36), satisfies*

$$\sigma(\rho)L(f_2) = \mathrm{div}_x\left(\sigma(\rho)^{-1}(\overline{K} - K)\nabla_x\rho\right) = \mathrm{div}_x\left(\sigma(\rho)^{-1}\overline{K}\,\nabla_x\rho\right) + a(v)\cdot\nabla_x f_1 \tag{2.41}$$

with the estimates

$$\mathbb{E}\sup_{t\in[0,T]}\|f_2(t)\|^p_{L^\infty_{x,v}} < \infty, \quad \mathbb{E}\sup_{t\in[0,T]}\|a(v)\cdot\nabla_x f_2(t)\|^p_{L^\infty_{x,v}} < \infty. \tag{2.42}$$

Furthermore, we have

$$df_2 = f_{2,d}\ dt + f_{2,s}\ QdW_t, \tag{2.43}$$

where $f_{2,d}$ and $f_{2,s}$ satisfy

$$\mathbb{E}\sup_{t\in[0,T]}\|f_{2,d}(t)\|^p_{L^\infty_{x,v}} < \infty, \quad \mathbb{E}\sup_{t\in[0,T]}\|f_{2,s}(t)\|^p_{L^\infty_{x,v}} < \infty. \tag{2.44}$$

2.4.4 Equation satisfied by the remainder

From now on, f^ε denotes the solution to problem (2.1) and ρ the solution of the limiting equation (2.15). We define the remainder r^ε by

$$r^\varepsilon := f^\varepsilon - \rho - \varepsilon f_1 - \varepsilon^2 f_2 - \varepsilon^3 f_3^\varepsilon,$$

where the correctors f_1, f_2 have been defined above. The third corrector f_3^ε will be defined below; its aim will be to cancel all the noise terms of order $O(\varepsilon)$ so that the remainder has a noise term of order $O(\varepsilon^2)$. Let us write the equation satisfied by r^ε. We have

$$\mathrm{d}r^\varepsilon = -\frac{1}{\varepsilon}a(v)\cdot\nabla_x f^\varepsilon\ \mathrm{d}t + \frac{1}{\varepsilon^2}\sigma(\overline{f^\varepsilon})L(f^\varepsilon)\ \mathrm{d}t + f^\varepsilon\ Q\mathrm{d}W_t + Gf^\varepsilon\ \mathrm{d}t - \mathrm{d}\rho - \varepsilon\mathrm{d}f_1 - \varepsilon^2\mathrm{d}f_2 - \varepsilon^3\mathrm{d}f_3^\varepsilon.$$

We recall that $L(\rho) = 0$ so that we have

$$\begin{aligned}\mathrm{d}r^\varepsilon = &-\frac{1}{\varepsilon}a(v)\cdot\nabla_x f^\varepsilon\ \mathrm{d}t + \frac{1}{\varepsilon}\sigma(\rho)L(f_1)\ \mathrm{d}t + \sigma(\rho)L(f_2)\ \mathrm{d}t + \varepsilon\sigma(\rho)L(f_3^\varepsilon)\ \mathrm{d}t\\ &+ \frac{1}{\varepsilon^2}\left[\sigma(\overline{f^\varepsilon})L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]\mathrm{d}t\\ &+ f^\varepsilon\ Q\mathrm{d}W_t + Gf^\varepsilon\ \mathrm{d}t - \mathrm{d}\rho - \varepsilon\mathrm{d}f_1 - \varepsilon^2\mathrm{d}f_2 - \varepsilon^3\mathrm{d}f_3^\varepsilon.\end{aligned}$$

Using the equations satisfied by f_1, f_2 and ρ, that is (2.37), (2.41) and (2.15), we obtain

$$\begin{aligned}\mathrm{d}r^\varepsilon = &-\frac{1}{\varepsilon}a(v)\cdot\nabla_x f^\varepsilon\ \mathrm{d}t + \frac{1}{\varepsilon}a(v)\cdot\nabla_x\rho\ \mathrm{d}t + a(v)\cdot\nabla_x f_1\ \mathrm{d}t + \mathrm{div}_x\left(\overline{K}\sigma(\rho)^{-1}\nabla_x\rho\right)\mathrm{d}t\\ &+ \varepsilon\sigma(\rho)L(f_3^\varepsilon)\ \mathrm{d}t + \frac{1}{\varepsilon^2}\left[\sigma(\overline{f^\varepsilon})L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]\mathrm{d}t\\ &+ f^\varepsilon\ Q\mathrm{d}W_t + Gf^\varepsilon\ \mathrm{d}t - \mathrm{div}_x\left(\overline{K}\sigma(\rho)^{-1}\nabla_x\rho\right)\mathrm{d}t - \rho\ Q\mathrm{d}W_t - G\rho\ \mathrm{d}t\\ &- \varepsilon\mathrm{d}f_1 - \varepsilon^2\mathrm{d}f_2 - \varepsilon^3\mathrm{d}f_3^\varepsilon.\end{aligned}$$

After simplification, we have,

$$\begin{aligned}\mathrm{d}r^\varepsilon + \frac{1}{\varepsilon}a(v)\cdot\nabla_x r^\varepsilon\ \mathrm{d}t = &-\varepsilon a(v)\cdot\nabla_x f_2\ \mathrm{d}t - \varepsilon^2 a(v)\cdot\nabla_x f_3^\varepsilon\ \mathrm{d}t\\ &+ \frac{1}{\varepsilon^2}\left[\sigma(\overline{f^\varepsilon})L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]\mathrm{d}t + (f^\varepsilon - \rho)\,Q\mathrm{d}W_t\\ &+ G(f^\varepsilon - \rho)\,\mathrm{d}t - \varepsilon\mathrm{d}f_1 - \varepsilon^2\mathrm{d}f_2 - \varepsilon^3\mathrm{d}f_3^\varepsilon + \varepsilon\sigma(\rho)L(f_3^\varepsilon)\ \mathrm{d}t.\end{aligned}$$

Using the expression (2.39) of $\mathrm{d}f_1$, we discover

$$\begin{aligned}
\mathrm{d}r^\varepsilon + \frac{1}{\varepsilon}a(v)\cdot\nabla_x r^\varepsilon \mathrm{d}t \;=\; & -\varepsilon a(v)\cdot\nabla_x f_2 \,\mathrm{d}t - \varepsilon^2 a(v)\cdot\nabla_x f_3^\varepsilon \,\mathrm{d}t \\
& + \frac{1}{\varepsilon^2}\left[\sigma(\overline{f^\varepsilon})L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]\mathrm{d}t + (f^\varepsilon - \rho)\,Q\mathrm{d}W_t \\
& + G(f^\varepsilon - \rho)\,\mathrm{d}t - \varepsilon f_{1,d}\,\mathrm{d}t - \varepsilon f_1\left(1 - \sigma(\rho)^{-1}\sigma'(\rho)\rho\right)Q\mathrm{d}W_t \\
& - \;\varepsilon^2\mathrm{d}f_2 - \varepsilon^3\mathrm{d}f_3^\varepsilon + \varepsilon\sigma(\rho)L(f_3^\varepsilon)\,\mathrm{d}t.
\end{aligned}$$

In the sequel, when estimating the remainder, we need the noise term to be of order $O(\varepsilon^2)$, see Section 2.4.6. As a consequence, we would like to choose f_3^ε to delete the terms of order $O(\varepsilon)$ in front of the noise. Namely, we would like to impose

$$\varepsilon^2\mathrm{d}f_3^\varepsilon - \sigma(\rho)L(f_3^\varepsilon)\,\mathrm{d}t = f_1\sigma(\rho)^{-1}\sigma'(\rho)\rho\,Q\mathrm{d}W_t, \tag{2.45}$$

so that the equation satisfied by the remainder r^ε is finally given by

$$\begin{aligned}
\mathrm{d}r^\varepsilon + \frac{1}{\varepsilon}a(v)\cdot\nabla_x r^\varepsilon \mathrm{d}t = \; & -\varepsilon a(v)\cdot\nabla_x f_2 \,\mathrm{d}t - \varepsilon^2 a(v)\cdot\nabla_x f_3^\varepsilon \,\mathrm{d}t \\
& + \frac{1}{\varepsilon^2}\left[\sigma(\overline{f^\varepsilon})L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]\mathrm{d}t \\
& + \;(f^\varepsilon - \rho - \varepsilon f_1)\,Q\mathrm{d}W_t + G(f^\varepsilon - \rho)\,\mathrm{d}t - \varepsilon f_{1,d}\,\mathrm{d}t - \varepsilon^2\mathrm{d}f_2.
\end{aligned}$$

Note that f_1 and f_2 do not depend on ε. In the following, we shall prove that f_3^ε is of order $O(\varepsilon^{-1})$ with respect to ε. As a consequence, the drift term (excepted the singular one) is of order $O(\varepsilon)$. We also recall that we precisely added f_3^ε in the development of f^ε to get a term of order $O(\varepsilon^2)$ in front of the noise; this will be necessary further in the estimate of the remainder. We point out that $L^1_{x,v}$ is indeed the appropriate space in which the estimate of the remainder will give a favourable sign to the singular term $\varepsilon^{-2}\left[\sigma(\overline{f^\varepsilon})L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]$ thanks to the accretivity property of the operator $\sigma(\bar{\cdot})L(\cdot)$, see section 2.2.3. The next section is devoted to the definition of the third corrector by solving the equation (2.45).

2.4.5 Definition of the third corrector.

In this part, we study the following equation for the third corrector which was suggested in a formal way in the computations done just above:

$$\varepsilon^2\mathrm{d}f_3^\varepsilon - \sigma(\rho)L(f_3^\varepsilon)\,\mathrm{d}t = f_1\sigma(\rho)^{-1}\sigma'(\rho)\rho\,Q\mathrm{d}W_t. \tag{2.46}$$

We solve this equation thanks to a stochastic convolution with the semigroup generated by the *non-autonomous* operator $\sigma(\rho)L$ on $L^p_{x,v}$ where $p \geq 1$. Let us begin with the study of this semigroup. We point out that we only need to know its behaviour on the subspace $\{g \in L^p_{x,v},\, \overline{g} = 0\}$.

Proposition 2.4.8. *Let $p \geq 1$ and $g \in L^p_{x,v}$ such that $\overline{g} = 0$. For $s \in [0,T]$, the problem*

$$\left\{\begin{array}{l} \varepsilon^2 u'(t) - \sigma(\rho(t))L(u(t)) = 0, \quad t \in [s,T], \\ u(s) = g, \end{array}\right. \tag{2.47}$$

admits a.s. a unique classical solution in $\mathcal{C}^1([s,T];L^p_{x,v})$ that we write $u(t) = U^\varepsilon(t,s)g$. It is given by

$$U^\varepsilon(t,s)g = g\exp\left(-\varepsilon^{-2}\int_s^t \sigma(\rho)(r)\,dr\right), \quad t \in [s,T]. \tag{2.48}$$

Furthermore, we have the bound

$$\|U^\varepsilon(t,s)g\|_{L^p_{x,v}} \leq \|g\|_{L^p_{x,v}}\exp\left(-\varepsilon^2\sigma_*(t-s)\right). \tag{2.49}$$

Proof. Note that with (2.47) and $\int_V \sigma(\rho)L(u)\,\mathrm{d}v = 0$, we immediately have that $\overline{u}' = 0$ so that $\overline{u}$ is constant and equals $\overline{g}$, which is zero. Then equation (2.47), with the definition of L, rewrites

$$\varepsilon^2 u'(t) = -\sigma(\rho(t))u(t),$$

which gives easily (2.48). This proves existence and uniqueness in $\mathcal{C}^1([0,T];L^p_{x,v})$ for the problem (2.47). The bound $\sigma \geq \sigma_*$ (H1) immediately yields (2.49). This concludes the proof. □

Before stating the main result about the third corrector, we need the following lemma about the regularity of the stochastic convolution.

Lemma 2.4.9. *Let $p \geq 2$. Suppose that $\varphi \in L^p(\Omega; L^\infty(0,T;L^p_{x,v}))$ satisfies $\overline{\varphi} = 0$. We define*

$$z(t) := \varepsilon^{-2}\int_0^t U^\varepsilon(t,s)\varphi(s)\,Q\,dW_s, \quad t \in [0,T].$$

Then, we have the bound

$$\sup_{t\in[0,T]} \mathbb{E}\|z(t)\|^p_{L^p_{x,v}} \leq C\varepsilon^{-p}\,\mathbb{E}\|\varphi\|^p_{L^\infty(0,T;L^p_{x,v})}.$$

Proof. Here, we recall that a.s. and for s, $t \in [0,T]$, $U^\varepsilon(t,s)\varphi(s)$ is an element of $L^p_{x,v}$. The stochastic integral $U^\varepsilon(t,s)\varphi(s)Q\mathrm{d}W_s$ can be interpreted as $\Psi^\varepsilon(t,s)\mathrm{d}W_s$ where $\Psi^\varepsilon(t,s)$ is the following γ-radonifying operator from $L^2(\mathbb{T}^N)$ to $L^p_{x,v}$ (see Subsection 2.2.2)

$$\Psi^\varepsilon(t,s)(e_k) := U^\varepsilon(t,s)\varphi(s)Qe_k.$$

Then, we use the Burkholder-Davis-Gundy's inequality for martingales with values in $L^p_{x,v}$ (see [Brz97] and [BP99]) and the bound (2.9) to obtain

$$\begin{aligned}\mathbb{E}\|z(t)\|^p_{L^p_{x,v}} &\leq C\varepsilon^{-2p}\,\mathbb{E}\left(\int_0^t \|\Psi^\varepsilon(t,s)\|^2_{\gamma(L^2_x,L^p_{x,v})}\,\mathrm{d}s\right)^{p/2}\\ &\leq C\varepsilon^{-2p}\,\mathbb{E}\left(\int_0^t \|U^\varepsilon(t,s)\varphi(s)\|^2_{L^p_{x,v}}\,\mathrm{d}s\right)^{p/2}.\end{aligned}$$

Next, thanks to (2.49) with the hypothesis $\overline{\varphi} = 0$, we have

$$\begin{aligned}\mathbb{E}\|z(t)\|^p_{L^p_{x,v}} &\leq C\varepsilon^{-2p}\,\mathbb{E}\|\varphi\|^p_{L^\infty(0,T;L^p_{x,v})}\left(\int_0^t \exp\left(-\varepsilon^{-2}\sigma_*(t-s)\right)\,\mathrm{d}s\right)^{p/2}\\ &\leq C\varepsilon^{-p}\,\mathbb{E}\|\varphi\|^p_{L^\infty(0,T;L^p_{x,v})},\end{aligned}$$

which concludes the proof. □

The existence and the properties of the third corrector f_3^ε are collected in the following proposition.

Proposition 2.4.10. *Let $p \geq 1$. There exists a process f_3^ε with values in $L^\infty(0,T;L^p(\Omega;L^p_{x,v}))$ which satisfies $\overline{f_3^\varepsilon} = 0$ and*

$$\varepsilon^2 df_3^\varepsilon - \sigma(\rho)L(f_3^\varepsilon)\;dt = f_1\sigma(\rho)^{-1}\sigma'(\rho)\rho\;Q\,dW_t, \tag{2.50}$$

that is, $\mathbb{P}-$a.s. for all $t \in [0,T]$,

$$f_3^\varepsilon(t) = \varepsilon^{-2}\int_0^t \sigma(\rho(s))L(f_3^\varepsilon(s))\,ds + \varepsilon^{-2}\int_0^t f_1(s)\sigma(\rho(s))^{-1}\sigma'(\rho(s))\rho(s)\,Q\,dW_s.$$

Furthermore, we have the estimates

$$\sup_{t\in[0,T]} \mathbb{E}\|f_3(t)\|^p_{L^p_{x,v}} \le C\varepsilon^{-p}, \qquad \sup_{t\in[0,T]} \mathbb{E}\|a(v)\cdot\nabla_x f_3(t)\|^p_{L^p_{x,v}} \le C\varepsilon^{-p}. \tag{2.51}$$

Proof. We fix $p \ge 2$. We set $\varphi := f_1\sigma(\rho)^{-1}\sigma'(\rho)\rho$ and we define

$$f_3^\varepsilon(t) := \varepsilon^{-2}\int_0^t U^\varepsilon(t,s)\varphi(s)\,Q\mathrm{d}W_s, \quad t\in[0,T].$$

Observe that with the definition (2.36) of f_1 we have $\varphi = -\sigma(\rho)^{-2}\,\sigma'(\rho)\rho\, a(v)\cdot\nabla_x\rho$. Thanks to the regularity of ρ, σ and a, we obviously have that φ belongs to $L^p(\Omega; L^\infty(0,T;L^\infty_{x,v}))$ which is embedded in $L^p(\Omega;L^\infty(0,T;L^p_{x,v}))$. As a consequence, since $\overline{\varphi}=0$, we can apply Lemma 2.4.9 to find that

$$\sup_{t\in[0,T]} \mathbb{E}\|f_3^\varepsilon(t)\|^p_{L^p_{x,v}} \le C\varepsilon^{-p}. \tag{2.52}$$

This proves in particular the existence of the stochastic integral which defines f_3^ε. Next, for $t\in[0,T]$, we can easily compute the quantity

$$\int_0^t \sigma(\rho(s))L(f_3^\varepsilon(s))\,\mathrm{d}s,$$

by using the stochastic version of Fubini's Theorem and the fact that, when $g \in L^p_{x,v}$, $\partial_s U^\varepsilon(s,r) = \varepsilon^{-2}\sigma(\rho(s))L(U^\varepsilon(s,r)g)$ by Proposition 2.47; we obtain that f_3^ε is a strong solution of (2.50), that is, $\mathbb{P}-$a.s.,

$$f_3^\varepsilon(t) = \varepsilon^{-2}\int_0^t \sigma(\rho(s))L(f_3^\varepsilon(s))\,\mathrm{d}s + \varepsilon^{-2}\int_0^t \varphi(s)\,Q\mathrm{d}W_s. \tag{2.53}$$

With this expression of f_3^ε, it is clear that $\overline{f_3^\varepsilon}=0$. To conclude the proof, it remains to bound $a(v)\cdot\nabla_x f_3^\varepsilon$ in $L^\infty(0,T;L^p(\Omega;L^p_{x,v}))$. Let $i\in\{1,...,N\}$, we differentiate equation (2.53) with respect to the space variable x_i to discover

$$\begin{aligned}\partial_{x_i} f_3^\varepsilon(t) = &\ \varepsilon^{-2}\int_0^t \partial_{x_i}\rho_s\sigma'(\rho(s))L(f_3^\varepsilon(s))\,\mathrm{d}s + \varepsilon^{-2}\int_0^t \sigma(\rho(s))L(\partial_{x_i} f_3^\varepsilon(s))\,\mathrm{d}s\\ &+ \ \varepsilon^{-2}\int_0^t \partial_{x_i}\varphi(s)\,Q\mathrm{d}W_s + \varepsilon^{-2}\int_0^t \varphi(s)\,Q\mathrm{d}\partial_{x_i}W_s.\end{aligned}$$

As a consequence, we see that we can write $\partial_{x_i} f_3^\varepsilon$ into the following mild form

$$\begin{aligned}\partial_{x_i} f_3^\varepsilon(t) = &\ \varepsilon^{-2}\int_0^t U^\varepsilon(t,s)\partial_{x_i}\rho_s\sigma'(\rho(s))L(f_3^\varepsilon(s))\,\mathrm{d}s + \varepsilon^{-2}\int_0^t U^\varepsilon(t,s)\partial_{x_i}\varphi(s)\,Q\mathrm{d}W_s\\ &+ \ \varepsilon^{-2}\int_0^t U^\varepsilon(t,s)\varphi(s)\,Q\mathrm{d}\partial_{x_i}W_s.\end{aligned}$$

Let us deal with the first term of the last equality. We set $\phi = \partial_{x_i}\rho_s\sigma'(\rho(s))L(f_3^\varepsilon(s))$. Thanks to the regularity of ρ, σ and the bound (2.52), it clearly belongs to the space $L^p(\Omega;L^p(0,T;L^p_{x,v}))$ with

$$\mathbb{E}\|\phi\|^p_{L^p(0,T;L^p_{x,v})} \le C\varepsilon^{-p}.$$

Therefore, since $\overline{\phi}=0$, we can use (2.49) to write, with the Young and Hölder inequalities,

$$\begin{aligned}\mathbb{E}\left\|\varepsilon^{-2}\int_0^t U^\varepsilon(t,s)\phi(s)\mathrm{d}s\right\|^p_{L^p_{x,v}} &\le \mathbb{E}\left(\int_0^t \varepsilon^{-2}\|U^\varepsilon(t,s)\phi(s)\|_{L^p_{x,v}}\mathrm{d}s\right)^p\\ &\le C\mathbb{E}\|\phi\|^p_{L^p(0,T;L^p_{x,v})} \le C\varepsilon^{-p}.\end{aligned}$$

For the two remaining terms, we can easily verify that Lemma 2.4.9 applies (even with the noise $\mathrm{d}\partial_{x_i} W$ thanks to the hypothesis (2.8) $\sum_k \|\partial_{x_i} Q e_k\|_\infty^2 < \infty$). Finally, we combine the two applications of Lemma 2.4.9 with the previous bound to obtain

$$\sup_{t\in[0,T]} \mathbb{E}\|\nabla_x f_3^\varepsilon(t)\|_{L^p_{x,v}}^p \leq C\varepsilon^{-p},$$

which concludes the proof of the second estimate of (2.51) due to the boundedness of a. It remains to prove the proposition when $p \in [1,2)$ but it is a straightforward consequence of the Hölder's inequality. This concludes the proof. □

2.4.6 Estimate of the remainder

Finally, we estimate the remainder r^ε in the space $L^1_{x,v}$; this will conclude the proof of Theorem 2.2.2. We point out that the correctors f_1, f_2 and f_3^ε are now properly defined in the previous sections. We recall that we set:

$$r^\varepsilon := f^\varepsilon - \rho - \varepsilon f^1 - \varepsilon^2 f_2 - \varepsilon^3 f_3^\varepsilon.$$

Thanks to the calculations made in Subsection 2.4.4, r^ε now satisfies:

$$\begin{aligned} \mathrm{d}r^\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_x r^\varepsilon \mathrm{d}t = & -\varepsilon a(v)\cdot\nabla_x f_2 \,\mathrm{d}t - \varepsilon^2 a(v)\cdot\nabla_x f_3^\varepsilon \,\mathrm{d}t \\ & + \frac{1}{\varepsilon^2}\left[\sigma(\overline{f^\varepsilon})L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]\mathrm{d}t \\ & + \ (f^\varepsilon - \rho - \varepsilon f_1)\, Q\mathrm{d}W_t + G(f^\varepsilon-\rho)\,\mathrm{d}t - \varepsilon f_{1,d}\,\mathrm{d}t - \varepsilon^2 \mathrm{d}f_2. \end{aligned}$$

We will estimate r^ε in $L^1_{x,v}$ by estimating $(r^\varepsilon)^+$ and $(r^\varepsilon)^-$ in $L^1_{x,v}$ using the Itô formula, where $x^+ = \max(0,x)$ and $x^- = (-x)^+$. We write the equation verified by r^ε as follows:

$$\mathrm{d}r^\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_x r^\varepsilon = \ D_t\,\mathrm{d}t + \frac{1}{\varepsilon^2} D_t^*\,\mathrm{d}t + H_t\, Q\mathrm{d}W_t,$$

where

$$\begin{cases} D := -\varepsilon a(v)\cdot\nabla_x f_2 - \varepsilon^2 a(v)\cdot\nabla_x f_3^\varepsilon + G(f^\varepsilon-\rho) - \varepsilon f_{1,d} - \varepsilon^2 f_{2,d}, \\ D^* := \sigma(\overline{f^\varepsilon})L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon), \\ H := (f^\varepsilon - \rho - \varepsilon f_1) - \varepsilon^2 f_{2,s}. \end{cases}$$

Since $f^\varepsilon - \rho = \varepsilon f^1 + \varepsilon^2 f_2 + \varepsilon^3 f_3^\varepsilon + r^\varepsilon$, thanks to (2.38), (2.40), (2.42), (2.44), (2.51) with $p=1$ and with $\|G\|_{L^\infty} < \infty$, we have the bound

$$\mathbb{E}\int_0^t \int_{\mathbb{T}^N\times V} |D_s|\,\mathrm{d}s \ \leq \ C\varepsilon + \mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V} |r_s^\varepsilon|\,\mathrm{d}s. \tag{2.54}$$

Similarly, for any $\delta > 0$, with $f^\varepsilon - \rho - \varepsilon f^1 = \varepsilon^2 f_2 + \varepsilon^3 f_3^\varepsilon + r^\varepsilon$ and thanks to (2.42), (2.44), (2.51) with $p = 2$ and with $\|G\|_{L^\infty} < \infty$, we have the bound

$$\mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V} G|H_s|^2 \mathbf{1}_{|r_s^\varepsilon|\leq\delta}\,\mathrm{d}s \ \leq \ C(\varepsilon^4 + \delta^2). \tag{2.55}$$

Now, $\delta > 0$ being fixed, we apply the Itô formula with the C^2 approximation φ_δ of the function $x \mapsto x^+$ defined by (2.12) to the process r^ε to obtain (note that the term relative to $\varepsilon^{-1} a(v)\cdot\nabla_x r^\varepsilon$ cancels)

$$\begin{aligned} \mathbb{E}\int_{\mathbb{T}^N\times V} \varphi_\delta(r_t^\varepsilon) \ = \ & \mathbb{E}\int_{\mathbb{T}^N\times V} \varphi_\delta(r_{\mathrm{in}}^\varepsilon) \ + \ \mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V} \varphi_\delta'(r_s^\varepsilon) D_s\,\mathrm{d}s \\ & + \ \frac{1}{\varepsilon^2}\mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V} \varphi_\delta'(r_s^\varepsilon) D_s^*\,\mathrm{d}s \ + \ \mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V} \varphi_\delta''(r_s^\varepsilon) G|H_s|^2\,\mathrm{d}s. \end{aligned}$$

Since $x^+ \leq \varphi_\delta(x) + \delta$, we have

$$\mathbb{E}\|(r^\varepsilon_t)^+\|_{L^1_{x,v}} \;\leq\; \mathbb{E}\int_{\mathbb{T}^N\times V} \varphi_\delta(r^\varepsilon_t) + \delta$$

and thanks to $\varphi_\delta(x) \leq x^+$, we get

$$\mathbb{E}\int_{\mathbb{T}^N\times V} \varphi_\delta(r^\varepsilon_{\rm in}) \;\leq\; \mathbb{E}\|(r^\varepsilon_{\rm in})^+\|_{L^1_{x,v}}.$$

With $|\varphi'_\delta| \leq 1$ and (2.54), we have

$$\mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V} \varphi'_\delta(r^\varepsilon_s)D_s\,{\rm d}s \;\leq\; C\varepsilon + \mathbb{E}\int_0^t \|r^\varepsilon_s\|_{L^1_{x,v}}{\rm d}s.$$

Next, we study the term

$$\int_V \varphi'_\delta(r^\varepsilon_s)D^*_s\,{\rm d}v = \int_V \varphi'_\delta(r^\varepsilon_s)\left[\sigma(\overline{f^\varepsilon_s})L(f^\varepsilon_s) - \sigma(\rho_s)L(f^\varepsilon_s - r^\varepsilon_s)\right]{\rm d}v.$$

To this end, we define $g^\varepsilon := f^\varepsilon - r^\varepsilon$; note that $\overline{g^\varepsilon} = \rho$. The term we are interested in thus rewrites

$$J := \int_V \varphi'_\delta(f^\varepsilon - g^\varepsilon)\left[\sigma(\overline{f^\varepsilon})L(f^\varepsilon) - \sigma(\overline{g^\varepsilon})L(g^\varepsilon)\right]{\rm d}v,$$

so that, with the positivity of f^ε, we can apply the accretivity bound (2.13) of Lemma 2.2.1 to find

$$J \leq C(1 + \|f^\varepsilon\|_{L^1_v})\delta.$$

We immediately deduce, using Cauchy-Schwarz's inequality and the uniform bound (2.21) of f^ε in $L^2(\Omega; L^2(0,T; L^2_{x,v}))$, that we have

$$\begin{aligned}\frac{1}{\varepsilon^2}\mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V} \varphi'_\delta(r^\varepsilon_s)D^*_s\,{\rm d}s \;&=\; \frac{1}{\varepsilon^2}\mathbb{E}\int_0^t\int_{\mathbb{T}^N} J_s\,{\rm d}x{\rm d}s \;\leq\; \frac{C\delta}{\varepsilon^2}\left(1 + \mathbb{E}\int_0^t \|f^\varepsilon_s\|_{L^1_{x,v}}{\rm d}s\right)\\ &\leq\; \frac{C\delta}{\varepsilon^2}.\end{aligned}$$

Let us now study the last term of the Itô formula. We point out that φ''_δ is zero on $[0,\delta]^c$ and that $|\varphi''_\delta| \leq 1/\delta$ on $[0,\delta]$. Thus, with (2.55), we may write

$$\mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V} \varphi''_\delta(r^\varepsilon_s)G|H_s|^2\,{\rm d}s \;\leq\; \frac{1}{\delta}\mathbb{E}\int_0^t\int_{\mathbb{T}^N\times V} G|H_s|^2\mathbf{1}_{|r^\varepsilon_s|\leq\delta}\,{\rm d}s \;\leq\; \frac{C}{\delta}(\varepsilon^4 + \delta^2).$$

Summing up all the previous bounds now yields

$$\mathbb{E}\|(r^\varepsilon_t)^+\|_{L^1_{x,v}} \;\leq\; \mathbb{E}\|(r^\varepsilon_{\rm in})^+\|_{L^1_{x,v}} + \delta + C\varepsilon + \mathbb{E}\int_0^t \|r^\varepsilon_s\|_{L^1_{x,v}}{\rm d}s + \frac{C\delta}{\varepsilon^2} + \frac{C}{\delta}(\varepsilon^4 + \delta^2).$$

Now observe that $(r^\varepsilon)^- = (-r^\varepsilon)^+ = (g^\varepsilon - f^\varepsilon)^+$ to obtain similarly (making use of the bound (2.14) instead of (2.13) when applying Lemma 2.2.1)

$$\mathbb{E}\|(r^\varepsilon_t)^-\|_{L^1_{x,v}} \leq\; \mathbb{E}\|(r^\varepsilon_{\rm in})^-\|_{L^1_{x,v}} + \delta + C\varepsilon + \mathbb{E}\int_0^t \|r^\varepsilon_s\|_{L^1_{x,v}}{\rm d}s + \frac{C\delta}{\varepsilon^2} + \frac{C}{\delta}(\varepsilon^4 + \delta^2).$$

Summing the two previous bounds and applying the Gronwall's lemma, we get

$$\mathbb{E}\|r^\varepsilon_t\|_{L^1_{x,v}} \leq C\left(\mathbb{E}\|r^\varepsilon_{\rm in}\|_{L^1_{x,v}} + \delta + \varepsilon + \frac{\delta}{\varepsilon^2} + \frac{\varepsilon^4}{\delta} + \delta\right).$$

Since this bound is valid for all $\delta > 0$, we choose $\delta = \varepsilon^3$ to discover

$$\mathbb{E}\|r_t^\varepsilon\|_{L^1_{x,v}} \leq C\left(\mathbb{E}\|r_{\text{in}}^\varepsilon\|_{L^1_{x,v}} + \varepsilon\right).$$

We point out that $r_{\text{in}}^\varepsilon = -\varepsilon f^1(0) - \varepsilon^2 f^2(0)$, so that

$$\mathbb{E}\|r_t^\varepsilon\|_{L^1_{x,v}} \leq C\varepsilon.$$

Finally, thanks to (2.38), (2.42) and (2.51) with $p = 1$, we have

$$\sup_{t\in[0,T]} \mathbb{E}\|\varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3^\varepsilon\|_{L^1_{x,v}} \leq C\varepsilon$$

so that we obtain the estimate

$$\sup_{t\in[0,T]} \mathbb{E}\|f_t^\varepsilon - \rho_t\|_{L^1_{x,v}} \leq C\varepsilon,$$

which concludes the proof of Theorem 2.2.2.

Appendix A

Proof of Lemma 2.2.1.

Proof. Let us prove the first estimate; the second one is proved similarly. We are interested in the term

$$J := \int_V \varphi_\delta'(f-g)\left[\sigma(\overline{f})L(f) - \sigma(\overline{g})L(g)\right] \mathrm{d}v.$$

Here, we observe that

$$\begin{aligned} 0 &= \varphi_\delta'(\overline{f}-\overline{g})\left[\sigma(\overline{g})(\overline{g}-\overline{g}) - \sigma(\overline{f})(\overline{f}-\overline{f})\right] \\ &= \int_V \varphi_\delta'(\overline{f}-\overline{g})\left[\sigma(\overline{g})(\overline{g}-g) - \sigma(\overline{f})(\overline{f}-f)\right] \mathrm{d}v. \end{aligned}$$

As a consequence, we can write

$$\begin{aligned} J &= \int_V \varphi_\delta'(f-g)\left[\sigma(\overline{f})\overline{f} - \sigma(\overline{f})f - \sigma(\overline{g})\overline{g} + \sigma(\overline{g})g\right] \mathrm{d}v \\ &+ \int_V \varphi_\delta'(\overline{f}-\overline{g})\left[\sigma(\overline{g})(\overline{g}-g) - \sigma(\overline{f})(\overline{f}-f)\right] \mathrm{d}v \\ &= \int_V \left[\sigma(\overline{f})\overline{f} - \sigma(\overline{g})\overline{g}\right]\left[\varphi_\delta'(f-g) - \varphi_\delta'(\overline{f}-\overline{g})\right] \mathrm{d}v \\ &+ \int_V \left[\sigma(\overline{f})f - \sigma(\overline{g})g\right]\left[\varphi_\delta'(\overline{f}-\overline{g}) - \varphi_\delta'(f-g)\right] \mathrm{d}v \\ &=: J_1 + J_2. \end{aligned}$$

We will now bound J_1 and J_2 separately. Let us begin with the case of J_1. We decompose J_1

as:

$$\begin{aligned}
J_1 = & \int_V \left[\sigma(\overline{f})\overline{f} - \sigma(\overline{g})\overline{g}\right] \left[\varphi'_\delta(f-g) - \varphi'_\delta(\overline{f}-\overline{g})\right] \mathbf{1}_{f-g\leq 0} \mathrm{d}v \\
& + \int_V \left[\sigma(\overline{f})\overline{f} - \sigma(\overline{g})\overline{g}\right] \left[\varphi'_\delta(f-g) - \varphi'_\delta(\overline{f}-\overline{g})\right] \mathbf{1}_{\overline{f}-\overline{g}\leq 0} \mathrm{d}v \\
& + \int_V \left[\sigma(\overline{f})\overline{f} - \sigma(\overline{g})\overline{g}\right] \left[\varphi'_\delta(f-g) - \varphi'_\delta(\overline{f}-\overline{g})\right] \mathbf{1}_{f-g\in[0,\delta], \overline{f}-\overline{g}\in[0,\delta]} \mathrm{d}v \\
& + \int_V \left[\sigma(\overline{f})\overline{f} - \sigma(\overline{g})\overline{g}\right] \left[\varphi'_\delta(f-g) - \varphi'_\delta(\overline{f}-\overline{g})\right] \mathbf{1}_{f-g\in[0,\delta], \overline{f}-\overline{g}\geq\delta} \mathrm{d}v \\
& + \int_V \left[\sigma(\overline{f})\overline{f} - \sigma(\overline{g})\overline{g}\right] \left[\varphi'_\delta(f-g) - \varphi'_\delta(\overline{f}-\overline{g})\right] \mathbf{1}_{f-g\geq\delta, \overline{f}-\overline{g}\in[0,\delta]} \mathrm{d}v \\
& + \int_V \left[\sigma(\overline{f})\overline{f} - \sigma(\overline{g})\overline{g}\right] \left[\varphi'_\delta(f-g) - \varphi'_\delta(\overline{f}-\overline{g})\right] \mathbf{1}_{f-g\geq\delta, \overline{f}-\overline{g}\geq\delta} \mathrm{d}v \\
& =: J_1^{(1)} + J_1^{(2)} + J_1^{(3)} + J_1^{(4)} + J_1^{(5)} + J_1^{(6)}.
\end{aligned}$$

Study of $J_1^{(1)}$: Note that when $f-g\leq 0$, we have $\varphi'_\delta(f-g) = 0$. If $\overline{f}\leq\overline{g}$, we also have $\varphi'_\delta(\overline{f}-\overline{g}) = 0$, and if $\overline{f}\geq\overline{g}$, we have $\sigma(\overline{f})\overline{f} - \sigma(\overline{g})\overline{g}\geq 0$ thanks to the monotonicity of $x\mapsto\sigma(x)x$ (see (H3)) and $\varphi'_\delta(\overline{f}-\overline{g})\in[0,1]$. As a result, we conclude

$$J_1^{(1)} \leq 0.$$

Study of $J_1^{(2)}$: Note that when $\overline{f}-\overline{g}\leq 0$, we have $\varphi'_\delta(\overline{f}-\overline{g}) = 0$, $\sigma(\overline{f})\overline{f} - \sigma(\overline{g})\overline{g}\leq 0$ thanks to the monotonicity of $x\mapsto\sigma(x)x$ and $\varphi'_\delta(f-g)\in[0,1]$ so that we obtain

$$J_1^{(2)} \leq 0.$$

Study of $J_1^{(3)}$: First, we write

$$J_1^{(3)} = \int_V \left[(\sigma(\overline{f}) - \sigma(\overline{g}))\overline{f} + \sigma(\overline{g})(\overline{f}-\overline{g})\right] \left[\varphi'_\delta(f-g) - \varphi'_\delta(\overline{f}-\overline{g})\right] \mathbf{1}_{f-g\in[0,\delta], \overline{f}-\overline{g}\in[0,\delta]} \mathrm{d}v.$$

Since $\varphi'_\delta(f-g) - \varphi'_\delta(\overline{f}-\overline{g})\in[-1,1]$, we obtain with (H1) and the Lipschitz continuity of σ (see (H2)) that

$$\begin{aligned}
J_1^{(3)} & \leq \int_V \left(|\overline{f}|\|\sigma\|_{\mathrm{Lip}}\delta + \sigma^*\delta\right) \mathbf{1}_{f-g\in[0,\delta], \overline{f}-\overline{g}\in[0,\delta]} \mathrm{d}v \\
& \leq C(1+|\overline{f}|)\delta.
\end{aligned}$$

Study of $J_1^{(4)}$: Note that when $\overline{f}-\overline{g}\geq\delta$ we have $\varphi'_\delta(\overline{f}-\overline{g}) = 1$ and $\sigma(\overline{f})\overline{f} - \sigma(\overline{g})\overline{g}\geq 0$ thanks to the monotonicity of $x\mapsto\sigma(x)x$. Since $\varphi'_\delta(f-g)\in[0,1]$, we thus get

$$J_1^{(4)} \leq 0.$$

Study of $J_1^{(5)}$: Exactly as in the case of $J_1^{(3)}$, we get

$$\begin{aligned}
J_1^{(5)} & \leq \int_V \left(|\overline{f}|\|\sigma\|_{\mathrm{Lip}}\delta + \sigma^*\delta\right) \mathbf{1}_{f-g\geq\delta, \overline{f}-\overline{g}\in[0,\delta]} \mathrm{d}v \\
& \leq C(1+|\overline{f}|)\delta.
\end{aligned}$$

Study of $J_1^{(6)}$: When $f-g\geq\delta$ and $\overline{f}-\overline{g}\geq\delta$ we have $\varphi'_\delta(f-g) = \varphi'_\delta(\overline{f}-\overline{g}) = 1$ so that

$$J_1^{(6)} = 0.$$

Now, let us study the case of J_2. Similarly, we decompose J_2 as:

$$\begin{aligned}
J_2 = &\int_V \left[\sigma(\overline{f})f - \sigma(\overline{g})g\right]\left[\varphi'_\delta(\overline{f}-\overline{g}) - \varphi'_\delta(f-g)\right]\mathbf{1}_{f-g\le 0}\mathrm{d}v \\
&+ \int_V \left[\sigma(\overline{f})f - \sigma(\overline{g})g\right]\left[\varphi'_\delta(\overline{f}-\overline{g}) - \varphi'_\delta(f-g)\right]\mathbf{1}_{\overline{f}-\overline{g}\le 0}\mathrm{d}v \\
&+ \int_V \left[\sigma(\overline{f})f - \sigma(\overline{g})g\right]\left[\varphi'_\delta(\overline{f}-\overline{g}) - \varphi'_\delta(f-g)\right]\mathbf{1}_{f-g\in[0,\delta],\overline{f}-\overline{g}\in[0,\delta]}\mathrm{d}v \\
&+ \int_V \left[\sigma(\overline{f})f - \sigma(\overline{g})g\right]\left[\varphi'_\delta(\overline{f}-\overline{g}) - \varphi'_\delta(f-g)\right]\mathbf{1}_{f-g\in[0,\delta],\overline{f}-\overline{g}\ge\delta}\mathrm{d}v \\
&+ \int_V \left[\sigma(\overline{f})f - \sigma(\overline{g})g\right]\left[\varphi'_\delta(\overline{f}-\overline{g}) - \varphi'_\delta(f-g)\right]\mathbf{1}_{f-g\ge\delta,\overline{f}-\overline{g}\in[0,\delta]}\mathrm{d}v \\
&+ \int_V \left[\sigma(\overline{f})f - \sigma(\overline{g})g\right]\left[\varphi'_\delta(\overline{f}-\overline{g}) - \varphi'_\delta(f-g)\right]\mathbf{1}_{f-g\ge\delta,\overline{f}-\overline{g}\ge\delta}\mathrm{d}v \\
=: &\, J_2^{(1)} + J_2^{(2)} + J_2^{(3)} + J_2^{(4)} + J_2^{(5)} + J_2^{(6)}.
\end{aligned}$$

Study of $J_2^{(1)}$*:* When $f-g\le 0$, we have $\varphi'_\delta(f-g)=0$. If $\overline{f}\le\overline{g}$, we also have $\varphi'_\delta(\overline{f}-\overline{g})=0$; and if $\overline{f}\ge\overline{g}$, we have $\sigma(\overline{f})f-\sigma(\overline{g})g\le 0$ thanks to the monotonicity of σ (see (H3)) and the positivity of f. Since $\varphi'_\delta(\overline{f}-\overline{g})\in[0,1]$, we conclude

$$J_1^{(1)} \le 0.$$

Study of $J_2^{(2)}$*:* When $\overline{f}-\overline{g}\le 0$, we have $\varphi'_\delta(\overline{f}-\overline{g})=0$ and $\sigma(\overline{f})-\sigma(\overline{g})\ge 0$ thanks to the monotonicity of σ. If $f\le g$, we also have $\varphi'_\delta(f-g)=0$. If $f\ge g\ge 0$, we have $\sigma(\overline{f})f-\sigma(\overline{g})g\ge 0$. If $f\ge 0\ge g$, we still have $\sigma(\overline{f})f-\sigma(\overline{g})g\ge 0$ since $\sigma\ge 0$. Note that the case $0\ge f\ge g$ is impossible by positivity of f. Finally, since $\varphi'_\delta(f-g)\in[0,1]$, we conclude

$$J_2^{(2)} \le 0.$$

Study of $J_2^{(3)}$*:* First, we write

$$J_2^{(3)} = \int_V \left[(\sigma(\overline{f})-\sigma(\overline{g}))f + \sigma(\overline{g})(f-g)\right]\left[\varphi'_\delta(\overline{f}-\overline{g}) - \varphi'_\delta(f-g)\right]\mathbf{1}_{f-g\in[0,\delta],\overline{f}-\overline{g}\in[0,\delta]}\mathrm{d}v.$$

Since $\varphi'_\delta(f-g)-\varphi'_\delta(\overline{f}-\overline{g})\in[-1,1]$, we obtain with (H1) and the Lipschitz continuity of σ that

$$\begin{aligned}
J_2^{(3)} &\le \int_V \left(|f|\|\sigma\|_{\mathrm{Lip}}\delta + \sigma^*\delta\right)\mathbf{1}_{f-g\in[0,\delta],\overline{f}-\overline{g}\in[0,\delta]}\mathrm{d}v \\
&\le C(1+\overline{|f|})\delta.
\end{aligned}$$

Study of $J_2^{(4)}$*:* We write

$$J_2^{(4)} = \int_V \left[(\sigma(\overline{f})-\sigma(\overline{g}))f + \sigma(\overline{g})(f-g)\right]\left[\varphi'_\delta(\overline{f}-\overline{g}) - \varphi'_\delta(f-g)\right]\mathbf{1}_{f-g\in[0,\delta],\overline{f}-\overline{g}\ge\delta}\mathrm{d}v.$$

Note that when $\overline{f}-\overline{g}\ge\delta$ we have $\varphi'_\delta(\overline{f}-\overline{g})-\varphi'_\delta(f-g) = 1-\varphi'_\delta(f-g)\in[0,1]$ and $\sigma(\overline{f})-\sigma(\overline{g})\le 0$ thanks to the monotonicity of σ. With the positivity of f, we thus get

$$J_2^{(4)} \le \sigma^*\delta.$$

Study of $J_2^{(5)}$*:* We have

$$J_2^{(5)} = \int_V \left[(\sigma(\overline{f})-\sigma(\overline{g}))f + \sigma(\overline{g})(f-g)\right]\left[\varphi'_\delta(\overline{f}-\overline{g}) - \varphi'_\delta(f-g)\right]\mathbf{1}_{f-g\ge\delta,\overline{f}-\overline{g}\in[0,\delta]}\mathrm{d}v.$$

Note that when $f-g\geq\delta$ we have $\varphi_\delta'(\overline{f}-\overline{g})-\varphi_\delta'(f-g)=\varphi_\delta'(\overline{f}-\overline{g})-1\in[-1,0]$. We thus get

$$J_2^{(5)}\leq\|\sigma\|_{\mathrm{Lip}}\overline{|f|}\delta.$$

Study of $J_2^{(6)}$: When $f-g\geq\delta$ and $\overline{f}-\overline{g}\geq\delta$ we have $\varphi_\delta'(f-g)=\varphi_\delta'(\overline{f}-\overline{g})=1$ so that

$$J_2^{(6)}=0.$$

To sum up, we get the following bound on J

$$J\leq C(1+\|f\|_{L^1_v})\delta,$$

which concludes the proof. □

Appendix B

Proof of Lemma 2.4.3. We recall the Lemma to be proved.

Lemma. *Let $\alpha\in(0,1]$. We assume that hypothesis (2.5) is satisfied. Let f be bounded in $L^2(\Omega;L^2(0,T;L^2_{x,v}))$ such that*

$$df+a(v)\cdot\nabla_x f\,dt=h\,dt+g\,Q\,dW_t, \tag{2.56}$$

with g and h bounded in $L^2(\Omega;L^2(0,T;L^2_{x,v}))$. Then the quantity $\rho=\overline{f}$ verifies

$$\mathbb{E}\int_0^T\|\rho_s\|^2_{H_x^{\alpha/2}}ds\leq C.$$

Proof. We adapt in our stochastic context the proof of [BD99, Theorem 2.3]. We recall that $Q\mathrm{d}W_t=\sum_{k\geq0}Qe_k\mathrm{d}\beta_k(t)$ but, in order to simplify the notations, we assume in the proof that the noise is one-dimensional, namely of the form $Qe_l\mathrm{d}\beta_l(t)$, $l\geq0$, the generalization to an infinite dimensional noise being straightforward. We set $\theta_l=Qe_l$. Let $k\in\mathbb{Z}^N\mapsto\widehat{f}(k)$ denote the Fourier transform of f with respect to the space variable $x\in\mathbb{T}^N$. We take the spatial Fourier transform in Equation (2.56) and we add artificially on both sides of the equation a term $\lambda\widehat{f}$ for some constant $\lambda>0$ to be chosen later. We obtain, for $k\in\mathbb{Z}^N$,

$$d\widehat{f}(k)-ia(v)\cdot k\widehat{f}(k)\mathrm{d}t+\lambda\widehat{f}(k)=\widehat{h}\mathrm{d}t+\widehat{g\theta_l}\mathrm{d}\beta_l(t)+\lambda\widehat{f}(k).$$

Using Duhamel's formula, we have

$$\begin{aligned}\widehat{f}(t,k,v)&=e^{-(\lambda-ia(v)\cdot k)t}\widehat{f}(0,k,v)+\int_0^t e^{-(\lambda-ia(v)\cdot k)(t-s)}[\widehat{h}+\lambda\widehat{f}](s,k,v)\,\mathrm{d}s\\&\quad+\int_0^t e^{-(\lambda-ia(v)\cdot k)(t-s)}\widehat{g\theta_l}(s,k,v)\,\mathrm{d}\beta_l(s).\end{aligned}$$

Integrating in the velocity variable $v\in V$, we get

$$\begin{aligned}\widehat{\rho}(t,k)&=e^{-\lambda t}\int_V e^{ia(v)\cdot kt}\widehat{f}(0,k,v)\mathrm{d}v+\int_0^t e^{-\lambda(t-s)}\int_V e^{ia(v)\cdot k(t-s)}[\widehat{h}+\lambda\widehat{f}](s,k,v)\,\mathrm{d}v\,\mathrm{d}s\\&\quad+\int_0^t e^{-\lambda(t-s)}\int_V e^{ia(v)\cdot k(t-s)}\widehat{g\theta_l}(s,k,v)\,\mathrm{d}v\,\mathrm{d}\beta_l(s)\\&=T_d(t,k)+T_s(t,k),\end{aligned}$$

where

$$T_d(t,k) := e^{-\lambda t}\int_V e^{ia(v)\cdot kt}\widehat{f}(0,k,v)\mathrm{d}v + \int_0^t e^{-\lambda(t-s)}\int_V e^{ia(v)\cdot k(t-s)}[\widehat{h}+\lambda\widehat{f}](s,k,v)\,\mathrm{d}v\,\mathrm{d}s$$

and

$$T_s(t,k) := \int_0^t e^{-\lambda(t-s)}\int_V e^{ia(v)\cdot k(t-s)}\widehat{g\theta_l}(s,k,v)\,\mathrm{d}v\,\mathrm{d}\beta_l(s)$$

denote respectively the deterministic and stochastic part of $\widehat{\rho}(t,k)$. Let $k\in\mathbb{Z}^N$, $k\neq 0$. The deterministic term can be handled exactly as in the proof of [BD99, Theorem 2.3] and we obtain, up to a real multiplicative constant,

$$\mathbb{E}\int_0^T |T_d|^2(t,k)\,\mathrm{d}t \le \frac{1}{\lambda^{1-\alpha}|k|^\alpha}\mathbb{E}\int_V |\widehat{f}|^2(0,k,v)\,\mathrm{d}v + \frac{1}{\lambda^{2-\alpha}|k|^\alpha}\mathbb{E}\int_0^T\int_V |\widehat{h}+\lambda\widehat{f}|^2(s,k,v)\,\mathrm{d}v\,\mathrm{d}s.$$

So let us now focus on the stochastic term T_s. First, using the Itô isometry, we have

$$\begin{aligned}\mathbb{E}|T_s|^2(t,k) &= \mathbb{E}\int_0^t e^{-2\lambda(t-s)}\Big|\int_V e^{ia(v)\cdot k(t-s)}\widehat{g\theta_l}(s,k,v)\,\mathrm{d}v\Big|^2\,\mathrm{d}s\\ &= \mathbb{E}\int_0^t e^{-2\lambda s}\Big|\int_V e^{ia(v)\cdot ks}\widehat{g\theta_l}(t-s,k,v)\,\mathrm{d}v\Big|^2\,\mathrm{d}s,\end{aligned}$$

so that, by the Fubini Theorem and the change of variable $\tau := t-s$, we have

$$\begin{aligned}\mathbb{E}\int_0^T |T_s|^2(t,k)\,\mathrm{d}t &= \mathbb{E}\int_0^T\int_0^{T-\tau} e^{-2\lambda s}\Big|\int_V e^{ia(v)\cdot ks}\widehat{g\theta_l}(\tau,k,v)\,\mathrm{d}v\Big|^2\,\mathrm{d}s\,\mathrm{d}\tau\\ &\le \mathbb{E}\int_0^T\int_{\mathbb{R}} e^{-2\lambda s}\Big|\int_V e^{ia(v)\cdot ks}\widehat{g\theta_l}(\tau,k,v)\,\mathrm{d}v\Big|^2\,\mathrm{d}s\,\mathrm{d}\tau\\ &= \frac{1}{|k|}\mathbb{E}\int_0^T\int_{\mathbb{R}} e^{-\frac{2\lambda s}{|k|}}\Big|\int_V e^{ia(v)\cdot\frac{k}{|k|}s}\widehat{g\theta_l}(\tau,k,v)\,\mathrm{d}v\Big|^2\,\mathrm{d}s\,\mathrm{d}\tau.\end{aligned}$$

We use the bound

$$e^{-\frac{2\lambda s}{|k|}} \le \frac{1}{1+\frac{4\lambda}{|k|^2}s^2},\qquad s\ge 0,$$

and estimate the oscillatory integral thanks to [BD99, Lemma 2.4] and (2.5); we therefore get

$$\mathbb{E}\int_0^T |T_s|^2(t,k)\,\mathrm{d}t \le \frac{C}{\lambda^{1-\alpha}|k|^\alpha}\mathbb{E}\int_0^T\int_V |\widehat{g\theta_l}|^2(\tau,k,v)\,\mathrm{d}v\,\mathrm{d}\tau.$$

As a result, summing up the previous bounds, we have, up to a real multiplicative constant,

$$\begin{aligned}\mathbb{E}\int_0^T |\widehat{\rho}|^2(t,k)\,\mathrm{d}t \le \frac{1}{\lambda^{1-\alpha}|k|^\alpha}\mathbb{E}\int_0^T\int_V |\widehat{g\theta_l}|^2(\tau,k,v)\,\mathrm{d}v\,\mathrm{d}\tau + \frac{1}{\lambda^{1-\alpha}|k|^\alpha}\mathbb{E}\int_V |\widehat{f}|^2(0,k,v)\,\mathrm{d}v\\ + \frac{1}{\lambda^{2-\alpha}|k|^\alpha}\mathbb{E}\int_0^T\int_V |\widehat{h}+\lambda\widehat{f}|^2(s,k,v)\,\mathrm{d}v\,\mathrm{d}s.\end{aligned}$$

We choose $\lambda\equiv 1$, multiply the last equation by $|k|^\alpha$ and sum over $k\in\mathbb{Z}^N$ to find

$$\begin{aligned}\mathbb{E}\int_0^T \|\rho(t)\|^2_{H_x^{\alpha/2}}\mathrm{d}t &\le C\mathbb{E}\left[\|g\theta_l\|^2_{L^2(0,T;L^2_{x,v})} + \|h+f\|^2_{L^2(0,T;L^2_{x,v})} + \|f(0)\|^2_{L^2_{x,v}}\right]\\ &\le C\mathbb{E}\left[\|Qe_l\|^2_{L^\infty_x}\|g\|^2_{L^2(0,T;L^2_{x,v})} + \|h+f\|^2_{L^2(0,T;L^2_{x,v})} + \|f(0)\|^2_{L^2_{x,v}}\right].\end{aligned}$$

This concludes the proof when the noise is finite dimensional. For the infinite dimensional case, we recall that, thanks to (2.8), we have $\kappa_{0,\infty} = \sum_{l\ge 0}\|Qe_l\|^2_{L^\infty_x} < \infty$. □

Chapter 3

The radiative transfer equation perturbed by a Markovian process

Abstract: We study the stochastic diffusive limit of a kinetic radiative transfer equation involving a small parameter and perturbed by a smooth random term. Under an appropriate scaling for the small parameter, we prove the convergence in law to a stochastic non-linear fluid limit. The proof relies on a generalization in the infinite dimensional case of the perturbed test-functions method. Furthermore, in order to pass to the limit in the non-linear term, we use an averaging lemma to obtain the tightness of the process in a suitable space.

Keywords: Kinetic equations, non-linear diffusion limit, stochastic partial differential equations, perturbed test functions, Rosseland approximation, radiative transfer, averaging lemma.

The results of this chapter are available as a preprint:

[DDV14a] A. Debussche, S. De Moor, and J. Vovelle. Diffusion limit for the radiative transfer equation perturbed by a Markovian process. *ArXiv e-prints*, May 2014.

3.1 Introduction

In this chapter, we are interested in the following non-linear equation

$$\begin{cases} \partial_t f^\varepsilon \ + \ \dfrac{1}{\varepsilon} a(v)\cdot\nabla_x f^\varepsilon \ = \ \dfrac{1}{\varepsilon^2}\sigma(\overline{f^\varepsilon})L(f^\varepsilon) \ + \ \dfrac{1}{\varepsilon} f^\varepsilon m^\varepsilon, \\ f^\varepsilon(0) = f_0^\varepsilon, \qquad t\in[0,T],\, x\in\mathbb{T}^N,\, v\in V. \end{cases} \tag{3.1}$$

where (V,μ) is a measured space, $a : V \to \mathbb{R}^N$, $\sigma : \mathbb{R}\to\mathbb{R}$. The notation $\overline{f}$ stands for the average over the velocity space V of the function f, that is

$$\overline{f} = \int_V f \, \mathrm{d}\mu(v).$$

The operator L is a linear operator of relaxation which acts on the velocity variable $v \in V$ only. It is given by

$$L(f) := \overline{f}F - f, \tag{3.2}$$

where $v \mapsto F(v)$ is a velocity equilibrium function such that

$$F > 0 \text{ a.s.}, \quad \overline{F} = 1, \quad \sup_{v\in V} F(v) < \infty. \tag{3.3}$$

The term m^ε is a random process depending on $(t,x) \in \mathbb{R}^+\times\mathbb{R}^N$ (see section 3.2.2). The precise description of the problem setting will be given in the next section. In this chapter, we study the behaviour in the limit $\varepsilon\to 0$ of the solution f^ε of (3.1).

Concerning the physical background in the deterministic case ($m^\varepsilon \equiv 0$), equation (3.1) describes the interaction between a surrounding continuous medium and a flux of photons radiating through it in the absence of hydrodynamical motion. The unknown $f^\varepsilon(t,x,v)$ then stands for a distribution function of photons having position x and velocity v at time t. The function σ is the opacity of the matter. When the surrounding medium becomes very large compared to the mean free paths ε of photons, the solution f^ε to (3.1) is known to behave like ρF where ρ is the solution of the Rosseland equation

$$\partial_t \rho - \mathrm{div}_x(\sigma(\rho)^{-1}K\nabla_x\rho) \ = \ 0, \qquad (t,x)\in[0,T]\times\mathbb{T}^N,$$

and F is the velocity equilibrium defined above. This is what we call the Rosseland approximation. In this chapter, we investigate such an approximation where we have perturbed the deterministic equation by a smooth multiplicative random noise. To do so, we use the method of perturbed test-functions. This method provides an elegant way of deriving stochastic diffusive limit from random kinetic systems; it was first introduced by Papanicolaou, Stroock and Varadhan [PSV77]. The book of Fouque, Garnier, Papanicolaou and Solna [FGPS10] presents many applications to this method. A generalization in infinite dimension of the perturbed test-functions method arose in recent papers of Debussche and Vovelle [DV12] and de Bouard and Gazeau [dBG12].

In the deterministic case (that is when $m^\varepsilon\equiv 0$), the Rosseland approximation has been widely studied. In the paper of Bardos, Golse and Perthame [BGP87], they derive the Rosseland approximation on a slightly more general equation of radiative transfer type than (3.1) where the solution also depends on the frequency variable ν. Using the so-called Hilbert's expansion method, they prove a strong convergence of the solution of the radiative transfer equation to the solution of the Rosseland equation. In [BGPS88], the Rosseland approximation is proved in a weaker sense with weakened hypothesis on the various parameters of the radiative transfer equation, in particular on the opacity function σ.

In the stochastic setting, the case where $\sigma \equiv \sigma_0$ is constant has been studied in the paper of Debussche and Vovelle [DV12] where they prove the convergence in law of the solution of (3.1) to a limit stochastic fluid equation by mean of a generalization of the perturbed test-functions method. Thus the radiative transfer equation (3.1) is a first step in studying approximation diffusion on non-linear stochastic kinetic equations since the operator $\sigma(\overline{f})Lf$ stands for a simple non-linear perturbation of the classical linear relaxation operator L.

As expected, we have to handle some difficulties caused by this non-linearity. In the paper of Debussche and Vovelle [DV12] is proved the tightness of the family of processes $(r^\varepsilon)_{\varepsilon>0}$ in the space of time-continuous function with values in some negative Sobolev space $H^{-\eta}(\mathbb{T}^N)$. In our non-linear setting, this is not any more sufficient to succeed in passing to the limit as ε goes to 0. As a consequence, the main step to overcome this difficulty is to prove the tightness of the family of processes $(r^\varepsilon)_{\varepsilon>0}$ in the space $L^2(0,T;L^2(\mathbb{T}^N))$. This is made using averaging lemmas in the L^2 setting with a slight adaptation to our stochastic context. The main results about deterministic averaging lemmas that we will use in the sequel can be found in the paper of Jabin [Jab09]. We point out that, thanks to this additional tightness result, we could handle the case of a more general and non-linear noise term in (3.1) of the form $\frac{1}{\varepsilon}m^\varepsilon\lambda(\overline{f^\varepsilon})f^\varepsilon$ where $\lambda:\mathbb{R}\to\mathbb{R}$ is a bounded and continuous function. In particular, this remains valid in the linear case $\sigma \equiv 1$ studied in the paper [DV12] of Debussche and Vovelle so that this chapter can provide some improvements to their result.

3.2 Preliminaries and main result

3.2.1 Notations and hypothesis

Let us now introduce the precise setting of equation (3.1). We work on a finite-time interval $[0,T]$ where $T>0$ and consider periodic boundary conditions for the space variable: $x\in\mathbb{T}^N$ where $\mathbb{T}^N$ is the N-dimensional torus. Regarding the velocity space V, we assume that (V,μ) is a measured space.

In the sequel, $L^2_{F^{-1}}$ denotes the F^{-1} weighted $L^2(\mathbb{T}^N\times V)$ space equipped with the norm

$$\|f\|^2 := \int_{\mathbb{T}^N}\int_V \frac{|f(x,v)|^2}{F(v)}\,\mathrm{d}\mu(v)\mathrm{d}x.$$

We denote its scalar product by $(.,.)$. We also need to work in the space $L^2(\mathbb{T}^N)$, which will be often written L^2 for short when the context is clear. In what follows, we will often use the inequality

$$\|\overline{f}\|_{L^2_x} \le \|f\|,$$

which is just Cauchy-Schwarz inequality and the fact that $\overline{F}=1$. We also introduce the Sobolev spaces on the torus $H^\gamma(\mathbb{T}^N)$, or H^γ for short. For $\gamma\in\mathbb{N}$, they consist of periodic functions which are in $L^2(\mathbb{T}^N)$ as well as their derivatives up to order γ. For general $\gamma\ge 0$, they are easily defined by Fourier series. For $\gamma<0$, $H^\gamma(\mathbb{T}^N)$ is the dual of $H^{-\gamma}(\mathbb{T}^N)$.

Concerning the velocity mapping $a:V\to\mathbb{R}^N$, we shall assume that it is bounded, that is

$$\sup_{v\in V}|a(v)|<\infty. \tag{3.4}$$

Furthermore, we suppose that the following null flux hypothesis holds

$$\int_V a(v)F(v)\,\mathrm{d}\mu(v)=0, \tag{3.5}$$

and that the following matrix

$$K := \int_V a(v) \otimes a(v) F(v) \, \mathrm{d}\mu(v)$$

is definite positive. Finally, to obtain some compactness in the space variable by means of averaging lemmas, we also assume the following standard condition:

$$\forall \varepsilon > 0,\ \forall (\xi, \alpha) \in S^{N-1} \times \mathbb{R},\ \ \mu\left(\{v \in V, |a(v) \cdot \xi + \alpha| < \varepsilon\}\right) \leq \varepsilon^{\theta}, \tag{3.6}$$

for some $\theta \in (0, 1]$.

Let us now give several hypothesis on the opacity function $\sigma : \mathbb{R} \to \mathbb{R}$. We assume that

(H1) There exist two positive constants σ_*, $\sigma^* > 0$ such that for almost all $x \in \mathbb{R}$, we have

$$\sigma_* \leq \sigma(x) \leq \sigma^*;$$

(H2) the function σ is Lipschitz continuous.

Similarly as in the deterministic case, we expect with (3.1) that $\sigma(\overline{f^\varepsilon})L(f^\varepsilon)$ tends to zero with ε, so that we should determine the equilibrium of the operator $\sigma(\bar{\cdot})L(\cdot)$. In this case, since $\sigma > 0$, they are clearly constituted by the functions of the form ρF with ρ being independent of $v \in V$. Note that it can easily be seen that $\sigma(\bar{\cdot})L(\cdot)$ is a bounded operator from $L^2_{F^{-1}}$ to $L^2_{F^{-1}}$ and that it is dissipative; precisely, for $f \in L^2_{F^{-1}}$,

$$(\sigma(\overline{f})Lf, f) = -\|\sigma^{\frac{1}{2}}(\overline{f})Lf\|^2 \leq 0. \tag{3.7}$$

In the sequel, we denote by $g(t, \cdot)$ the semi-group generated by the operator $\sigma(\bar{\cdot})L(\cdot)$ on $L^2_{F^{-1}}$. It verifies, for $f \in L^2_{F^{-1}}$,

$$\begin{cases} \dfrac{d}{dt} g(t, f) = \sigma(\overline{g(t, f)}) L g(t, f), \\ g(0, f) = f, \end{cases}$$

and we can show that it is given by

$$g(t, f) = \overline{f} F + (f - \overline{f} F) e^{-t\sigma(\overline{f})}, \quad t \geq 0,\ f \in L^2_{F^{-1}}.$$

With the hypothesis (H1) made on σ, we deduce the following relaxation property of the operator $\sigma(\bar{\cdot})L(\cdot)$

$$g(t, f) \longrightarrow \overline{f} F, \quad t \to \infty, \qquad \text{in } L^2_{F^{-1}}. \tag{3.8}$$

3.2.2 The random perturbation

The random term m^ε is defined by

$$m^\varepsilon(t, x) := m\left(\frac{t}{\varepsilon^2}, x\right),$$

where m is a stationary process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and is adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$. Note that m^ε is adapted to the filtration $(\mathcal{F}^\varepsilon_t)_{t \geq 0} = (\mathcal{F}_{\varepsilon^{-2} t})_{t \geq 0}$.

We assume that, considered as a random process with values in a space of spatially dependent functions, m is a stationary homogeneous Markov process taking values in a subset E of $W^{1,\infty}(\mathbb{T}^N)$. In the sequel, E will be endowed with the norm $\|\cdot\|_\infty$ of $L^\infty(\mathbb{T}^N)$. Besides, we denote by $\mathcal{B}(E)$ the set of bounded functions from E to $\mathbb{R}$ endowed with the norm $\|g\|_\infty := \sup_{n \in E} |g(n)|$ for $g \in \mathcal{B}(E)$.

We assume that m is stochastically continuous. Note that m is supposed not to depend on the variable v. For all $t \geq 0$, the law ν of m_t is supposed to be centered

$$\mathbb{E}m_t = \int_E n \, \mathrm{d}\nu(n) = 0.$$

We denote by e^{tM} a transition semi-group on E associated to m and by M its infinitesimal generator. $\mathrm{D}(M)$ stands for the domain of M; it is defined as follows:

$$\mathrm{D}(M) := \left\{ u \in \mathcal{B}(E), \lim_{h \to 0} \frac{e^{hM} - I}{h} u \text{ exists in } \mathcal{B}(E) \right\},$$

and if $u \in \mathrm{D}(M)$, we have

$$Mu := \lim_{h \to 0} \frac{e^{hM} - I}{h} u \text{ in } \mathcal{B}(E).$$

Moreover, we suppose that m is ergodic and satisfies some mixing properties in the sense that there exists a subspace $\mathscr{P}_M$ of $\mathcal{B}(E)$ such that for any $g \in \mathscr{P}_M$, the Poisson equation

$$M\psi = g - \int_E g(n) \, \mathrm{d}\nu(n) =: \widehat{g},$$

has a unique solution $\psi \in \mathrm{D}(M)$ satisfying $\int_E \psi(n) \, d\nu(n) = 0$. We denote by $M^{-1}\widehat{g}$ this unique solution, and assume that it is given by

$$M^{-1}\widehat{g}(n) = -\int_0^\infty e^{tM}\widehat{g}(n)\mathrm{d}t, \quad n \in E. \tag{3.9}$$

In particular, we suppose that the above integral is well defined. We need that $\mathscr{P}_M$ contains sufficiently many functions. Thus we assume that for all $f, g \in L^2_{F^{-1}}$, we have

$$\psi^{(1)}_{f,g} : n \mapsto (fn, g) \in \mathscr{P}_M, \tag{3.10}$$

and we then define $M^{-1}I$ from E into $W^{1,\infty}(\mathbb{T}^N)$ by

$$(fM^{-1}I(n), g) := M^{-1}\psi^{(1)}_{f,g}(n), \quad \forall f, g \in L^2_{F^{-1}}. \tag{3.11}$$

Then, we also suppose that for all $f, g, h \in L^2_{F^{-1}}$ and all continuous operator B from $L^2_{F^{-1}}$ to the space of the continuous bilinear operators on $L^2_{F^{-1}} \times L^2_{F^{-1}}$,

$$\psi^{(2)}_{f,g} : n \mapsto (fnM^{-1}I(n), g), \quad \psi^{(3)}_{B,f,g,h} : n \mapsto B(f)(gn, hM^{-1}I(n)) \in \mathscr{P}_M. \tag{3.12}$$

We need a uniform bound in $W^{1,\infty}(\mathbb{T}^N)$ of all the functions of the variable $n \in E$ introduced above. Namely, we assume, for all $f, g \in L^2_{F^{-1}}$ and all continuous operator B on $L^2_{F^{-1}}$,

$$\begin{aligned} &\|n\|_{W^{1,\infty}(\mathbb{T}^N)} \leq C_*, && \|M^{-1}I(n)\|_{W^{1,\infty}(\mathbb{T}^N)} \leq C_*, \\ &|M^{-1}\psi^{(2)}_{f,g}| \leq C_*\|f\|\|g\|, && |M^{-1}\psi^{(3)}_{B,f,g}| \leq C_*\|B(f)\|\|f\|\|g\|. \end{aligned} \tag{3.13}$$

Finally, we suppose that for all $f, g \in L^2_{F^{-1}}$,

$$n \mapsto (fM^{-1}I(n), g)^2 \in \mathrm{D}(M) \text{ with } |M[(fM^{-1}I(n), g)^2]| \leq C_*\|f\|^2\|g\|^2. \tag{3.14}$$

To describe the limiting stochastic partial differential equation, we then set

$$k(x, y) = \mathbb{E}\int_{\mathbb{R}} m_0(y)m_t(x) \, dt, \quad x, y \in \mathbb{T}^N.$$

We can easily show that the kernel k belong to $L^\infty(\mathbb{T}^N \times \mathbb{T}^N)$ and, m being stationary, that it is symmetric (see [DV12]). As a result, we introduce the operator Q on $L^2(\mathbb{T}^N)$ associated to the kernel k

$$Qf(x) = \int_{\mathbb{T}^N} k(x,y)f(y)\,\mathrm{d}y,$$

which is self-adjoint, compact and non-negative (see [DV12]). As a consequence, we can define the square root $Q^{\frac{1}{2}}$ which is Hilbert-Schmidt on $L^2(\mathbb{T}^N)$.

Remark 3.2.1. *The above assumptions on the process m are verified, for instance, when m is a Poisson process taking values in a bounded subset E of $W^{1,\infty}(\mathbb{T}^N)$.*

3.2.3 Resolution of the kinetic equation

In this section, we solve the linear evolution problem (3.1) thanks to a semi-group approach. We thus introduce the linear operator $A := a(v)\cdot\nabla_x$ on $L^2_{F^{-1}}$ with domain

$$\mathrm{D}(A) := \{f \in L^2_{F^{-1}}, \nabla_x f \in L^2_{F^{-1}}\}.$$

The operator A has dense domain and, since it is skew-adjoint, it is m-dissipative. Consequently A generates a contraction semigroup $(\mathcal{T}(t))_{t\geq 0}$ (see [CH98]). We recall that $\mathrm{D}(A)$ is endowed with the norm $\|\cdot\|_{\mathrm{D}(A)} := \|\cdot\| + \|A\cdot\|$, and that it is a Banach space.

Proposition 3.2.1. *Let $T>0$ and $f_0^\varepsilon \in L^2_{F^{-1}}$. Then there exists a unique mild solution of (3.1) on $[0,T]$ in $L^\infty(\Omega)$, that is there exists a unique $f^\varepsilon \in L^\infty(\Omega, C([0,T], L^2_{F^{-1}}))$ such that $\mathbb{P}-$a.s.*

$$f_t^\varepsilon = \mathcal{T}\left(\frac{t}{\varepsilon}\right) f_0^\varepsilon + \int_0^t \mathcal{T}\left(\frac{t-s}{\varepsilon}\right)\left(\frac{1}{\varepsilon^2}\sigma(\overline{f_s^\varepsilon})Lf_s^\varepsilon + \frac{1}{\varepsilon}m_s^\varepsilon f_s^\varepsilon\right) ds, \quad t\in[0,T].$$

Assume further that $f_0^\varepsilon \in \mathrm{D}(A)$, then there exists a unique strong solution f^ε which belongs to the spaces $L^\infty(\Omega, C^1([0,T], L^2_{F^{-1}}))$ and $L^\infty(\Omega, C([0,T], \mathrm{D}(A)))$ of (3.1).

Proof. Subsections 4.3.1 and 4.3.3 in [CH98] gives that $\mathbb{P}-$a.s. there exists a unique mild solution $f^\varepsilon \in C([0,T], L^2_{F^{-1}})$ and it is not difficult to slightly modify the proof to obtain that in fact $f^\varepsilon \in L^\infty(\Omega, C([0,T], L^2_{F^{-1}}))$ (we intensively use that for all $t\geq 0$ and $\varepsilon>0$, $\|m_t^\varepsilon\|_{W^{1,\infty}(\mathbb{T}^N)} \leq C_*$).
Similarly, subsections 4.3.1 and 4.3.3 in [CH98] gives us $\mathbb{P}-$a.s. a strong solution f^ε in the spaces $C^1([0,T], L^2_{F^{-1}})$ and $C([0,T], \mathrm{D}(A))$ of (3.1) and once again one can easily get that in fact f^ε belongs to the spaces $L^\infty(\Omega, C^1([0,T], L^2_{F^{-1}}))$ and $L^\infty(\Omega, C([0,T], \mathrm{D}(A)))$. □

Remark 3.2.2. *If $f_0^\varepsilon \in \mathrm{D}(A)$, we thus have, for $\varepsilon>0$ fixed,*

$$\sup_{t\in[0,T]} \|f_t^\varepsilon\| + \sup_{t\in[0,T]} \|Af_t^\varepsilon\| \in L^\infty(\Omega). \tag{3.15}$$

3.2.4 Main result

We are now ready to state our main result.

Theorem 3.2.2. *Assume that $(f_0^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2_{F^{-1}}$ and that*

$$\rho_0^\varepsilon := \int_V f_0^\varepsilon \, d\mu(v) \underset{\varepsilon\to 0}{\longrightarrow} \rho_0 \text{ in } L^2(\mathbb{T}^N).$$

Then, for all $\eta > 0$ and $T > 0$, $r^\varepsilon := \overline{f^\varepsilon}$ converges in law in $C([0,T], H^{-\eta}(\mathbb{T}^N))$ and $L^2(0,T; L^2(\mathbb{T}^N))$ to the solution ρ to the non-linear stochastic diffusion equation

$$d\rho - \operatorname{div}_x(\sigma(\rho)^{-1}K\nabla_x\rho)\ dt \ = \ H\rho\ dt + \rho\, Q^{\frac{1}{2}}\, dW_t,\ \ in\ [0,T]\times\mathbb{T}^N, \tag{3.16}$$

with initial condition $\rho(0) = \rho_0$ in $L^2(\mathbb{T}^N)$, and where W is a cylindrical Wiener process on $L^2(\mathbb{T}^N)$,

$$K := \int_V a(v)\otimes a(v) F(v)\, d\mu(v) \tag{3.17}$$

and

$$H := \int_E nM^{-1}I(n)\, d\nu(n) \in W^{1,\infty}. \tag{3.18}$$

Remark 3.2.3. *The limit equation (3.16) can also be written in Stratonovich form*

$$d\rho - \operatorname{div}_x(\sigma(\rho)^{-1}K\nabla_x\rho)\ dt \ = \ \rho\circ Q^{\frac{1}{2}}\, dW_t.$$

Notation In the sequel, we denote by $\lesssim$ the inequalities which are valid up to constants of the problem, namely C_*, N, $\sup_{\varepsilon>0}\|f_0^\varepsilon\|$, $\sup_{v\in V}|a(v)|$, $\sup_{v\in V} F(v)$, σ_*, σ^*, $\|\sigma\|_{\mathrm{Lip}}$ and real constants.

3.3 The generator

The process f^ε is not Markov (indeed, by (3.1), we need m^ε to know the increments of f^ε) but the couple $(f^\varepsilon, m^\varepsilon)$ is. From now on, we denote by $\mathscr{L}^\varepsilon$ its infinitesimal generator, that is

$$\mathscr{L}^\varepsilon\varphi(f,n) := \lim_{h\to 0}\frac{1}{h}\mathbb{E}\left[\varphi(f_h^\varepsilon, m_h^\varepsilon) - \varphi(f,n)\middle|(f_0^\varepsilon, m_0^\varepsilon) = (f,n)\right],$$

where $\varphi : L^2_{F^{-1}}\times E\to\mathbb{R}$ belongs to the domain of $\mathscr{L}^\varepsilon$. Thus we begin this section by introducing a special set of functions which lie in the domain of $\mathscr{L}^\varepsilon$ and satisfy the associated martingale problem.

In the following, if $\varphi : L^2_{F^{-1}}\to\mathbb{R}$ is differentiable with respect to $f\in L^2_{F^{-1}}$, we denote by $D\varphi(f)$ its differential at a point f and we identify the differential with the gradient.

Definition 3.3.1. *We say that $\varphi : L^2_{F^{-1}}\times E\to\mathbb{R}$ is a good test function if*

(i) *$(f,n)\mapsto\varphi(f,n)$ is differentiable with respect to f;*

(ii) *$(f,n)\mapsto D\varphi(f,n)$ is continuous from $L^2_{F^{-1}}\times E$ to $L^2_{F^{-1}}$ and maps bounded sets onto bounded sets;*

(iii) *for any $f\in L^2_{F^{-1}}$, $\varphi(f,\cdot)\in D_M$;*

(iv) *$(f,n)\mapsto M\varphi(f,n)$ is continuous from $L^2_{F^{-1}}\times E$ to $\mathbb{R}$ and maps bounded sets onto bounded sets.*

Proposition 3.3.1. *Let φ be a good test function. Then, for all $(f,n)\in\mathrm{D}(A)\times E$,*

$$\mathscr{L}^\varepsilon\varphi(f,n) = -\frac{1}{\varepsilon}(Af, D\varphi(f)) + \frac{1}{\varepsilon^2}(\sigma(\overline{f})Lf, D\varphi(f)) + \frac{1}{\varepsilon}(fn, D\varphi(f)) + \frac{1}{\varepsilon^2}M\varphi(f,n).$$

Furthermore, if $f_0^\varepsilon\in\mathrm{D}(A)$,

$$M_\varphi^\varepsilon(t) := \varphi(f_t^\varepsilon, m_t^\varepsilon) - \varphi(f_0^\varepsilon, m_0^\varepsilon) - \int_0^t \mathscr{L}^\varepsilon\varphi(f_s^\varepsilon, m_s^\varepsilon)\, ds$$

is a continuous and integrable $(\mathcal{F}_t^\varepsilon)_{t\geq 0}$ martingale, and if $|\varphi|^2$ is a good test function, its quadratic variation is given by

$$\langle M_\varphi^\varepsilon\rangle_t = \int_0^t (\mathscr{L}^\varepsilon|\varphi|^2 - 2\varphi\mathscr{L}^\varepsilon\varphi)(f_s^\varepsilon, m_s^\varepsilon)\, ds.$$

Proof. We compute the expression of the infinitesimal generator as follows :

$$\begin{aligned}\mathscr{L}^\varepsilon\varphi(f,n) &= \lim_{h\to 0}\frac{1}{h}\mathbb{E}\left[\varphi(f_h^\varepsilon, m_h^\varepsilon) - \varphi(f,n)\big|(f_0^\varepsilon, m_0^\varepsilon) = (f,n)\right]\\ &= \lim_{h\to 0}\frac{1}{h}\mathbb{E}\left[\varphi(f_h^\varepsilon, m_h^\varepsilon) - \varphi(f,m_h^\varepsilon)\big|(f_0^\varepsilon, m_0^\varepsilon) = (f,n)\right]\\ &+ \lim_{h\to 0}\frac{1}{h}\mathbb{E}\left[\varphi(f, m_h^\varepsilon) - \varphi(f,n)\big|m_0^\varepsilon = n\right]\end{aligned}$$

Since φ verifies point (iii) of Definition 3.3.1, the second term of the last equality goes to $\varepsilon^{-2}M\varphi(f,n)$ when $h\to 0$. We now focus on the first term. With points $(i)-(ii)$ of Definition 3.3.1, we have that φ is continuously differentiable with respect to f. Thus

$$\varphi(f_h^\varepsilon, m_h^\varepsilon) - \varphi(f, m_h^\varepsilon) = \int_0^1 D\varphi(f + s(f_h^\varepsilon - f), m_h^\varepsilon)(f_h^\varepsilon - f)\,\mathrm{d}s.$$

Besides, since $f_0^\varepsilon = f \in \mathrm{D}(A)$, $f^\varepsilon \in C^1([0,T], L^2_{F^{-1}})$ and we have

$$f_h^\varepsilon - f = h\int_0^1 \partial_t f_{uh}^\varepsilon\,\mathrm{d}u.$$

Thus, we can rewrite the first term as

$$\begin{aligned}&= \lim_{h\to 0}\frac{1}{h}\mathbb{E}\left[\varphi(f_h^\varepsilon, m_h^\varepsilon) - \varphi(f,m_h^\varepsilon)\big|(f_0^\varepsilon, m_0^\varepsilon) = (f,n)\right]\\ &= \lim_{h\to 0}\mathbb{E}_{(f,n)}\left[\int_0^1\int_0^1 a_h(w,s,u)\,\mathrm{d}u\,\mathrm{d}s\right],\end{aligned}$$

with $a_h(w,s,u) := D\varphi(f + s(f_h^\varepsilon - f), m_h^\varepsilon)(\partial_t f_{uh}^\varepsilon)$ and where $\mathbb{E}_{(f,n)}$ denotes the expectation under the probability measure $\mathbb{P}_{(f,n)} := \mathbb{P}(\,\cdot\,|(f_0^\varepsilon, m_0^\varepsilon) = (f,n))$.

Recall that $D\varphi$ is continuous with respect to (f,n) thanks to point (ii) of Definition 3.3.1, that f^ε is $\mathbb{P}-$a.s. in $C^1([0,T], L^2_{F^{-1}})$ and that m^ε is stochastically continuous to conclude that a_h converges in probability as $h\to 0$ to $D\varphi(f,n)(\partial_t f^\varepsilon(0))$ in the probability space $\tilde{\Omega} := (\Omega\times[0,1]\times[0,1], \mathbb{P}_{(f,n)}\otimes\mathrm{d}x\otimes\mathrm{d}s)$. Furthermore, we prove that $(a_h)_{0\leq h\leq 1}$ is uniformly integrable in $\tilde{\Omega}$ since it is uniformly bounded with respect to $0\leq h\leq 1$ in $L^\infty(\tilde{\Omega})$. Indeed, with the fact that L is a bounded operator, with (H1) and the fact that $\|n\|_{L^\infty(\mathbb{T}^N)} \lesssim 1$ for all $n\in E$, we get

$$|a_h| \lesssim \|D\varphi(f + s(f_h^\varepsilon - f), m_h^\varepsilon)\|(\|f_{uh}^\varepsilon\| + \|Af_{uh}^\varepsilon\|).$$

With (3.15), we set

$$R := \sup_{t\in[0,T]}\|f_t^\varepsilon\| + \sup_{t\in[0,T]}\|Af_t^\varepsilon\| \in L^\infty(\Omega),$$

and define $r := \|R\|_{L^\infty(\Omega)}$. Then, since $D\varphi$ maps bounded sets on bounded sets, we can bound the term $||D\varphi(f + s(f_h^\varepsilon - f), m_h^\varepsilon)||$ by

$$C := \sup\left\{\|D\varphi(f,n)\|, f\in B_{L^2_{F^{-1}}}(0, \|f\| + r), n \in B_E(0, C_*)\right\}.$$

So we are led to

$$\|a_h\|_{L^\infty(\tilde{\Omega})} \lesssim C\cdot r,$$

which is what we announced. To prove the sequel of the proposition, we use the same kind of ideas and follow the proofs of [DV12, Proposition 6] and [FGPS10, Appendix 6.9]. □

3.4 The limit generator

In this section, we study the limit of the generator $\mathscr{L}^\varepsilon$ when $\varepsilon \to 0$. The limit generator $\mathscr{L}$ will characterize the limit stochastic fluid equation.

3.4.1 Formal derivation of the corrections

To derive the diffusive limiting equation, one has to study the limit as ε goes to 0 of quantities of the form $\mathscr{L}^\varepsilon\varphi$ where φ is a good test function. To do so, following the perturbed test-functions method, we have to correct φ so as to obtain a non-singular limit. We search the correction φ^ε of φ under the classical form:

$$\varphi^\varepsilon := \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2.$$

In this decomposition, φ_1 and φ_2 are respectively the first and second order corrections and are to be defined in the sequel so that

$$\mathscr{L}^\varepsilon\varphi^\varepsilon = \mathscr{L}\varphi + O(\varepsilon),$$

where $\mathscr{L}$ will be the limit generator. We restrict our study to smooth test-functions. Precisely, we introduce the set of spatial derivative operators up to order 3:

$$\mathcal{R} := \{\partial_{i_1}^{e_1}\partial_{i_2}^{e_2}\partial_{i_3}^{e_3},\ e \in \{0,1\}^3,\ i \in \{1, ..., N\}^3,\ |i| \leq 3\}$$

and we suppose that the test-function φ is a good test, that $\varphi \in C^3(L^2_{F^{-1}})$ and that there exists a constant $C_\varphi > 0$ such that

$$\begin{cases} |\varphi(f)| \leq C_\varphi(1 + \|f\|^2), \\ \|\Lambda D\varphi(f)\| \leq C_\varphi(1 + \|f\|), \\ |D^2\varphi(f)(\Lambda_1 h, \Lambda_2 k)| \leq C_\varphi\|h\|\|k\|, \\ |D^3\varphi(f)(\Lambda_1 h, \Lambda_2 k, \Lambda_3 l)| \leq C_\varphi\|h\|\|k\|\|l\|, \end{cases} \tag{3.19}$$

for any $f, h, k, l \in L^2_{F^{-1}}$ and $\Lambda, \Lambda_1, \Lambda_2, \Lambda_3 \in \mathcal{R}$. Thanks to Proposition 3.3.1, and since φ does not depend on $n \in E$, we can write

$$\mathscr{L}^\varepsilon\varphi^\varepsilon(f,n) = -\frac{1}{\varepsilon}(Af, D\varphi(f)) + \frac{1}{\varepsilon^2}(\sigma(\overline{f})Lf, D\varphi(f)) + \frac{1}{\varepsilon}(fn, D\varphi(f)) \tag{3.20}$$

$$- (Af, D\varphi_1(f)) + \frac{1}{\varepsilon}(\sigma(\overline{f})Lf, D\varphi_1(f)) + (fn, D\varphi_1(f)) + \frac{1}{\varepsilon}M\varphi_1 \tag{3.21}$$

$$- \varepsilon(Af, D\varphi_2(f)) + (\sigma(\overline{f})Lf, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)) + M\varphi_2. \tag{3.22}$$

In the sequel, we do not care about the terms relative to the transport part A of the equation since these terms will be handled as in the deterministic case (when $m^\varepsilon \equiv 0$). To be more precise, and as it will be shown in the sequel, the first term of (3.20) will give rise, as ε goes to 0, to the deterministic term in the limit generator $\mathscr{L}$ and the first terms of (3.21) and (3.22) are respectively of orders ε and ε^2. For the remaining terms, in a first step, we would like to cancel those who have a singular power of ε. Thus we should impose that the two following equations hold:

$$(\sigma(\overline{f})Lf, D\varphi(f)) = 0, \tag{3.23}$$

$$(\sigma(\overline{f})Lf, D\varphi_1(f)) + M\varphi_1 + (fn, D\varphi(f)) = 0. \tag{3.24}$$

Let us say a word about the fact that we chose to handle the terms relative to the transport part of the equation separately. When trying to correct these terms thanks to the correctors φ_1 and φ_2, the non-linearity σ implies that the second corrector φ_2, unless we can write it formally, does not behave properly any more.

Equation on φ

Let us solve (3.23). We recall that $(g(t,f))_{t\geq 0}$ denotes the semigroup of the operator $\sigma(\cdot)L$. Equation (3.23) gives immediately that the map $t\mapsto\varphi(g(t,f))$ is constant. As a result, with (3.8),

$$\varphi(f)=\varphi(g(0,f))=\varphi(\varphi(g(\infty,f))=\varphi(\overline{f}F),$$

so that φ only depends on $\overline{f}F$. This implies, for all $h\in L^2_{F^{-1}}$,

$$(h,D\varphi(f))=(\overline{h}F,D\varphi(\overline{f}F)). \tag{3.25}$$

Equation on φ_1

Next, we solve (3.24). We consider the Markov process $(g(t,f),m(t,n))_{t\geq 0}$. Its generator will be denoted by $\mathscr{M}$. We observe that equation (3.24) rewrites:

$$\mathscr{M}\varphi_1(f,n)=-(fn,D\varphi(f)).$$

This Poisson equation will have a solution if the integral of $(f,n)\mapsto(fn,D\varphi(f))$ over $L^2_{F^{-1}}\times E$ equipped with the invariant measure of the process $(g(t,f),m(t,n))_{t\geq 0}$ is zero. So, we must verify that

$$\int_E(\overline{f}Fn,D\varphi(\overline{f}F))\,\mathrm{d}\nu(n)=0,$$

and this relation does hold since m is centered. As a consequence, if we can prove the existence of the integral, we can write φ_1 as

$$\varphi_1(f,n)=\int_0^\infty\mathbb{E}(g(t,f)m(t,n),D\varphi(g(t,f)))\,\mathrm{d}t.$$

Then, we use (3.25), $\overline{g(t,f)}=\overline{f}$ and (3.10) and (3.11) to obtain

$$\begin{aligned}\varphi_1(f,n)&=\int_0^\infty\mathbb{E}(\overline{f}Fm(t,n),D\varphi(\overline{f}F))\,\mathrm{d}t=-(\overline{f}FM^{-1}I(n),D\varphi(\overline{f}F))\\&=-(fM^{-1}I(n),D\varphi(f)).\end{aligned}$$

We are now able to state the

Proposition 3.4.1 (First corrector)**.** *Let $\varphi\in C^3(L^2_{F^{-1}})$ be a good test-function satisfying (3.19) and depending only on $\overline{f}F$. For any $(f,n)\in L^2_{F^{-1}}\times E$, we define the first corrector φ_1 as*

$$\varphi_1(f,n):=-(fM^{-1}I(n),D\varphi(f)).$$

Furthermore, it satisfies the bounds

$$(i)\ \ |\varphi_1(f,n)|\lesssim C_\varphi(1+\|f\|)^2,\quad (ii)\ \ \|AD\varphi_1(f,n)\|\lesssim C_\varphi(1+\|f\|). \tag{3.26}$$

Note that the bounds (3.26) are consequences of (3.13) and (3.19).

Equation on φ_2

At this stage, we have

$$\begin{aligned}\mathscr{L}^\varepsilon\varphi^\varepsilon(f,n)=&-\frac{1}{\varepsilon}(Af,D\varphi(f))+\mathscr{M}\varphi_2+(fn,D\varphi_1(f))\\&-(Af,D\varphi_1(f))-\varepsilon(Af,D\varphi_2(f))+\varepsilon(fn,D\varphi_2(f)).\end{aligned} \tag{3.27}$$

Note that the limit of $\mathscr{L}^\varepsilon\varphi^\varepsilon$ as ε goes to 0 does depend on $n \in E$ with the term $(fn, D\varphi_1(f))$. Since the expected limit is $\mathscr{L}\varphi$ where φ does not depend on n, we have to correct this term to cancel the dependence with respect to n of the limit. This is the aim of the second order correction φ_2. The right way to do so, given the mixing properties of the operator $\mathscr{M}$, is to subtract the mean value of this term under the invariant measure of the Markov process $(g(t,f), m(t,n))_{t\geq 0}$ governed by $\mathscr{M}$. We write

$$\begin{aligned}\mathscr{L}^\varepsilon\varphi^\varepsilon(f,n) = &-\frac{1}{\varepsilon}(Af, D\varphi(f)) + \int_E (\overline{f}Fn, D\varphi_1(\overline{f}F))\,\mathrm{d}\nu(n)\\ &+ \mathscr{M}\varphi_2 + (fn, D\varphi_1(f)) - \int_E (\overline{f}Fn, D\varphi_1(\overline{f}F))\,\mathrm{d}\nu(n)\\ &- (Af, D\varphi_1(f)) - \varepsilon(Af, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)),\end{aligned}$$

and we can now define φ_2 as the solution of the well-posed Poisson equation

$$\mathscr{M}\varphi_2 = -(fn, D\varphi_1(f)) + \int_E (\overline{f}Fn, D\varphi_1(\overline{f}F))\,\mathrm{d}\nu(n).$$

Note that, thanks to the definition of φ_1 given above, we can compute

$$(\overline{f}Fn, D\varphi_1(\overline{f}F)) = -(fnM^{-1}I(n), D\varphi(f)) - D^2\varphi(f)(fM^{-1}I(n), fn) =: q(f,n)$$

As a result, we easily have the following proposition.

Proposition 3.4.2 (Second corrector)**.** *Let $\varphi \in C^3(L^2_{F^{-1}})$ be a good test-function satisfying* (3.19) *and depending only on $\overline{f}F$. For any $(f,n) \in L^2_{F^{-1}} \times E$, we define the second corrector φ_2 as*

$$\varphi_2(f,n) := \mathbb{E}\int_0^\infty \left(\int_E (q(\overline{f}F,n)\,d\nu(n) - q(g(t,f), m(t,n))\right)\,dt,$$

which is well defined and satisfies the bounds

$$(i)\ \ |\varphi_2(f,n)| \lesssim C_\varphi(1+\|f\|)^2, \quad (ii)\ \ \|AD\varphi_2(f,n)\| \lesssim C_\varphi(1+\|f\|). \tag{3.28}$$

The existence of φ_2 is based on (3.12) and the bounds (3.28) are proved using (3.13) and (3.19).

Summary

The correctors φ_1 and φ_2 being defined as above in Propositions 3.4.1 and 3.4.2, we are finally led to

$$\begin{aligned}\mathscr{L}^\varepsilon\varphi^\varepsilon(f,n) = &-\frac{1}{\varepsilon}(Af, D\varphi(f)) + \int_E (\overline{f}Fn, D\varphi_1(\overline{f}F))\,\mathrm{d}\nu(n)\\ &- (Af, D\varphi_1(f)) - \varepsilon(Af, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)).\end{aligned}$$

We are now able to define the limit generator $\mathscr{L}$ as, for all $\rho \in L^2(\mathbb{T}^N)$,

$$\begin{aligned}\mathscr{L}\varphi(\rho) := (\mathrm{div}_x(\sigma(\rho)^{-1}K\nabla_x\rho)F, D\varphi(\rho F)) &- \int_E (\rho FnM^{-1}I(n), D\varphi(\rho F))\,\mathrm{d}\nu(n)\\ &- \int_E D^2\varphi(\rho F)(\rho FM^{-1}I(n), \rho Fn)\,\mathrm{d}\nu(n),\end{aligned} \tag{3.29}$$

and we have shown the following equality

$$\begin{aligned}\mathscr{L}^\varepsilon\varphi^\varepsilon(f,n) = \mathscr{L}\varphi(\overline{f}) &- \frac{1}{\varepsilon}(Af, D\varphi(f)) - (\mathrm{div}_x(\sigma(\overline{f})^{-1}K\nabla_x\overline{f})F, D\varphi(\overline{f}F))\\ &- (Af, D\varphi_1(f)) - \varepsilon(Af, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)).\end{aligned} \tag{3.30}$$

3.5 Uniform bound in $L^2_{F^{-1}}$

In this section, we prove a uniform estimate of the $L^2_{F^{-1}}$ norm of the solution f^ε with respect to ε. To do so, we will again use the perturbed test functions method. The result is the following:

Proposition 3.5.1. *Let $p \geq 1$ and $f_0^\varepsilon \in \mathrm{D}(A)$. We have the two following bounds*

$$\mathbb{E} \sup_{t\in[0,T]} \|f_t^\varepsilon\|^p \lesssim 1, \tag{3.31}$$

$$\mathbb{E}\left(\int_0^T \|\sigma^{\frac{1}{2}}(\overline{f_s^\varepsilon})Lf_s^\varepsilon\|^2\,ds\right)^p \lesssim \varepsilon^{2p}. \tag{3.32}$$

Proof. We set, for all $f \in L^2_{F^{-1}}$, $\varphi(f) := \frac{1}{2}\|f\|^2$, which is easily seen to be a good test function. Then, with Proposition 3.3.1, the fact that A is skew-adjoint, (3.7), and the fact that φ does not depend on $n \in E$, we get for $f \in \mathrm{D}(A)$ and $n \in E$,

$$\begin{aligned}\mathscr{L}^\varepsilon\varphi(f,n) &= -\frac{1}{\varepsilon}(Af,f) + \frac{1}{\varepsilon^2}(\sigma(\overline{f})Lf,f) + \frac{1}{\varepsilon}(fn,f) + \frac{1}{\varepsilon^2}M\varphi(f,n)\\ &= -\frac{1}{\varepsilon^2}\|\sigma^{\frac{1}{2}}(\overline{f})Lf\|^2 + \frac{1}{\varepsilon}(fn,f).\end{aligned}$$

The first term has a favourable behaviour for our purpose. The second term is more difficult to control and we correct φ thanks to the perturbed test-functions method to get rid of it: we recall the formal computations done in Section 3.4.1 and we set $\varphi_1(f,n) = -(f, M^{-1}I(n)f)$ and $\varphi^\varepsilon := \varphi(f,n) + \varepsilon\varphi_1$. We can show that φ_1 is a good test function with, thanks to Proposition 3.3.1,

$$\begin{aligned}\varepsilon\mathscr{L}^\varepsilon\varphi_1(f,n) = &-\frac{2}{\varepsilon}(\sigma(\overline{f})Lf, M^{-1}I(n)f) - 2(Af, M^{-1}I(n)f)\\ &- 2(fn, M^{-1}I(n)f) - \frac{1}{\varepsilon}(fn,f).\end{aligned}$$

As a consequence, we are led to

$$\begin{aligned}\mathscr{L}^\varepsilon\varphi^\varepsilon(f,n) = -\frac{1}{\varepsilon^2}\|\sigma^{\frac{1}{2}}(\overline{f})Lf\|^2 - \frac{2}{\varepsilon}(\sigma(\overline{f})Lf, M^{-1}I(n)f) - 2(Af, M^{-1}I(n)f)\\ - 2(fn, M^{-1}I(n)f).\end{aligned}$$

We use (3.13) and the hypothesis (H1) made on σ to bound the second term:

$$\begin{aligned}\frac{2}{\varepsilon}(\sigma(\overline{f})Lf, M^{-1}I(n)f) &\leq 2C_*(\sigma^*)^{\frac{1}{2}}\varepsilon^{-1}\|\sigma^{\frac{1}{2}}(\overline{f})Lf\|\|f\|\\ &\leq \frac{1}{2\varepsilon^2}\|\sigma^{\frac{1}{2}}(\overline{f})Lf\|^2 + 2C_*^2\sigma^*\|f\|^2.\end{aligned}$$

Furthermore, for the last two terms, we write

$$\begin{aligned}-2(Af, M^{-1}I(n)f) - 2(fn, M^{-1}I(n)f) &= (f^2, AM^{-1}I(n)) - 2(fn, M^{-1}I(n)f)\\ &\leq \|f\|^2\|a\|_{L^\infty(V)}C_* + 2C_*^2\|f\|^2.\end{aligned}$$

To sum up, we have proved that

$$\mathscr{L}^\varepsilon\varphi^\varepsilon(f,n) \lesssim -\frac{1}{2\varepsilon^2}\|\sigma^{\frac{1}{2}}(\overline{f})Lf\|^2 + \|f\|^2. \tag{3.33}$$

As in Proposition 3.3.1, since φ^ε is a good test function, we now define

$$M^\varepsilon(t) := \varphi^\varepsilon(f^\varepsilon_t, m^\varepsilon_t) - \varphi^\varepsilon(f^\varepsilon_0, m^\varepsilon_0) - \int_0^t \mathscr{L}^\varepsilon \varphi^\varepsilon(f^\varepsilon_s, m^\varepsilon_s)\,\mathrm{d}s,$$

which is a continuous and integrable $(\mathcal{F}^\varepsilon_t)_{t\geq 0}$ martingale. By definition of φ, φ^ε and M^ε, we obtain

$$\frac{1}{2}\|f^\varepsilon_t\|^2 = \frac{1}{2}\|f^\varepsilon_0\|^2 - \varepsilon(\varphi_1(f^\varepsilon_t, m^\varepsilon_t) - \varphi_1(f^\varepsilon_0, m^\varepsilon_0)) + \int_0^t \mathscr{L}^\varepsilon \varphi^\varepsilon(f^\varepsilon_s, m^\varepsilon_s)\,\mathrm{d}s + M^\varepsilon(t).$$

Since we have obviously $|\varphi_1(f,n)| \lesssim \|f\|^2$, we can write, with (3.33),

$$\|f^\varepsilon_t\|^2 \ \lesssim\ \|f^\varepsilon_0\|^2 + \varepsilon\|f^\varepsilon_t\| + \int_0^t -\frac{1}{2\varepsilon^2}\|\sigma^{\frac{1}{2}}(\overline{f^\varepsilon_s})Lf^\varepsilon_s\|^2 + \|f^\varepsilon_s\|^2\,\mathrm{d}s + \sup_{t\in[0,T]}|M^\varepsilon(t)|,$$

i.e. for ε sufficiently small,

$$\int_0^t \frac{1}{2\varepsilon^2}\|\sigma^{\frac{1}{2}}(\overline{f^\varepsilon_s})Lf^\varepsilon_s\|^2\,\mathrm{d}s + \|f^\varepsilon_t\|^2 \ \lesssim\ \|f^\varepsilon_0\|^2 + \int_0^t \|f^\varepsilon_s\|^2\,ds + \sup_{t\in[0,T]}|M^\varepsilon(t)|,$$

and by Gronwall lemma,

$$\int_0^t \frac{1}{2\varepsilon^2}\|\sigma^{\frac{1}{2}}(\overline{f^\varepsilon_s})Lf^\varepsilon_s\|^2\,\mathrm{d}s + \|f^\varepsilon_t\|^2 \ \lesssim\ \|f^\varepsilon_0\|^2 + \sup_{t\in[0,T]}|M^\varepsilon(t)|. \tag{3.34}$$

Note that $|\varphi^\varepsilon|^2$ is a good test function with, thanks to (3.13) and (3.14),

$$|\mathscr{L}^\varepsilon|\varphi^\varepsilon|^2 - 2\varphi^\varepsilon\mathscr{L}^\varepsilon\varphi^\varepsilon| = |M|\varphi_1|^2 - 2\varphi_1 M\varphi_1| \lesssim \|f\|^4,$$

and that, with Proposition 3.3.1, the quadratic variation of $M^\varepsilon(t)$ is given by

$$\langle M^\varepsilon\rangle_t = \int_0^t (\mathscr{L}^\varepsilon|\varphi^\varepsilon|^2 - 2\varphi^\varepsilon\mathscr{L}^\varepsilon\varphi^\varepsilon)(f^\varepsilon_s, m^\varepsilon_s)\,\mathrm{d}s.$$

As a result, with Burkholder-Davis-Gundy and Hölder inequalities, we get

$$\mathbb{E}\sup_{t\in[0,T]}|M^\varepsilon(t)|^p \ \lesssim\ \mathbb{E}|\langle M^\varepsilon\rangle_T|^{\frac{p}{2}} \lesssim \int_0^T \mathbb{E}\|f^\varepsilon_s\|^{2p}\,\mathrm{d}s. \tag{3.35}$$

Neglecting the first (positive) term of the left-hand side in (3.34), we have

$$\mathbb{E}\|f^\varepsilon_t\|^{2p} \ \lesssim\ \mathbb{E}\|f^\varepsilon_0\|^{2p} + \mathbb{E}\sup_{t\in[0,T]}|M^\varepsilon(t)|^p,$$

so that we get

$$\mathbb{E}\|f^\varepsilon_T\|^{2p} \ \lesssim\ \mathbb{E}\|f^\varepsilon_0\|^{2p} + \int_0^T \mathbb{E}\|f^\varepsilon_s\|^{2p}\,\mathrm{d}s,$$

and, by Gronwall lemma,

$$\mathbb{E}\|f^\varepsilon_T\|^{2p} \ \lesssim\ \mathbb{E}\|f^\varepsilon_0\|^{2p}. \tag{3.36}$$

This actually holds true for any $t \in [0,T]$. Thus, using (3.35) and (3.36) in (3.34) finally gives the expected bounds. □

Remark 3.5.1. *We define $g^\varepsilon := f^\varepsilon - r^\varepsilon F = -Lf^\varepsilon$. Since we have $\sigma \geq \sigma_*$, the bound (3.32) gives that, for all $p \geq 1$,*

$$(\varepsilon^{-1} g^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^p(\Omega; L^2(0,T; L^2_{F^{-1}})). \tag{3.37}$$

In the sequel, we must deal with the non-linear term. To do so, we need some compactness in the space variable of the process $(r^\varepsilon)_{\varepsilon>0}$. The following proposition is a first step to this purpose.

Proposition 3.5.2. *We assume that hypothesis (3.6) is satisfied. Let $p \geq 1$ and $s \in (0, \theta/2)$. We have the bound*

$$\mathbb{E}\left(\int_0^T \|r_s^\varepsilon\|^2_{H^s(\mathbb{T}^N)}\, ds\right)^p \lesssim 1. \tag{3.38}$$

Proof. Note that with $\sigma \leq \sigma^*$, the remark (3.37) and equation (3.1), we observe that

$$(\varepsilon \partial_t f^\varepsilon + a(v)\cdot \nabla_x f^\varepsilon - f^\varepsilon m^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^p(\Omega; L^2(0,T; L^2_{F^{-1}})).$$

Furthermore, $(f^\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(\Omega; L^2(0,T; L^2_{F^{-1}}))$ with (3.31) and $|m^\varepsilon| \leq C_*$ so that

$$(\varepsilon \partial_t f^\varepsilon + a(v)\cdot \nabla_x f^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^p(\Omega; L^2(0,T; L^2_{F^{-1}})). \tag{3.39}$$

Then, thanks to (3.6), we apply an averaging lemma to conclude. Precisely, [Jab09, Theorem 3.1] in the unstationary case applies a.s. with $\beta = \gamma = 0$, $p_1 = q_1 = p_2 = q_2 = 2$, $a = 0$, $k = \theta$ and

$$f = f^\varepsilon, \qquad g = \varepsilon \partial_t f^\varepsilon + a(v)\cdot \nabla_x f^\varepsilon,$$

and gives the bound

$$\|r^\varepsilon\|_{B^{\frac{\theta}{2},2}_{\infty,\infty}} \leq C \|f^\varepsilon\|^{\frac12} \|\varepsilon \partial_t f^\varepsilon + a(v)\cdot \nabla_x f^\varepsilon\|^{\frac12} \quad \text{a.s.}$$

Since, for any $s < \theta/2$, $H^s \subset B^{\frac{\theta}{2}}_{\infty,\infty}$, it yields, for $p \geq 1$,

$$\mathbb{E}\left(\int_0^T \|r_s^\varepsilon\|^2_{H^s}\, \mathrm{d}s\right)^p \leq C\mathbb{E}\left(\int_0^T \|f_s^\varepsilon\| \|\varepsilon \partial_t f_s^\varepsilon + a(v)\cdot \nabla_x f_s^\varepsilon\|\, \mathrm{d}s\right)^p,$$

so that the result follows with Cauchy Schwarz inequality and (3.31) and (3.39). This concludes the proof. □

3.6 Tightness

We want to prove the convergence in law of the family $(r^\varepsilon)_{\varepsilon>0}$: in this section, we study the tightness of the processes $(r^\varepsilon)_{\varepsilon>0}$ in the space $C([0,T], H^{-\eta}(\mathbb{T}^N))$ where $\eta > 0$. In fact, this will not be sufficient to pass to the limit in the non-linear term. As a consequence, we also prove that $(r^\varepsilon)_{\varepsilon>0}$ is tight in the space $L^2(0,T; L^2(\mathbb{T}^N))$.

Proposition 3.6.1. *Let $\eta > 0$. The sequence $(\rho^\varepsilon)_{\varepsilon>0}$ is tight in the spaces $C([0,T], H^{-\eta}(\mathbb{T}^N))$ and $L^2(0,T; L^2(\mathbb{T}^N))$.*

Proof. Step 1: control of the modulus of continuity of r^ε in $H^{-\eta}(\mathbb{T}^N)$. Let $\eta > 0$ be fixed. For any $\delta > 0$, we define

$$w(\rho, \delta) := \sup_{|t-s|<\delta} \|\rho(t) - \rho(s)\|_{H^{-\eta}(\mathbb{T}^N)}$$

the modulus of continuity of a function $\rho \in C([0,T], H^{-\eta}(\mathbb{T}^N))$. In this first step of the proof, we want to obtain the following bound

$$\mathbb{E}w(r^\varepsilon, \delta) \lesssim \varepsilon + \delta^\tau, \tag{3.40}$$

for some positive τ. To do so, we use the perturbed test-functions method. Let $(p_j)_{j\in\mathbb{N}^N}$ the Fourier orthonormal basis of $L^2(\mathbb{T}^N)$ and J the operator

$$J := (\mathrm{I} - \Delta_x)^{-\frac{1}{2}}.$$

Let $j \in \mathbb{N}^N$. We set

$$\varphi_j(f) := (f, p_j F), \quad f \in L^2_{F^{-1}},$$

and we define the first order corrections by, see Section 3.4.1,

$$\varphi_{1,j}(f,n) := -(fM^{-1}I(n), p_j F), \quad (f,n) \in L^2_{F^{-1}} \times E.$$

We finally define $\varphi_j^\varepsilon := \varphi_j + \varepsilon\varphi_{1,j}$, which is easily seen to be a good test-function, so that, thanks to Proposition 3.3.1, we consider the continuous martingales

$$M_j^\varepsilon(t) := \varphi_j^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) - \varphi_j^\varepsilon(f_0^\varepsilon, m_0^\varepsilon) - \int_0^t \mathscr{L}^\varepsilon \varphi_j^\varepsilon(f_s^\varepsilon, m_s^\varepsilon)\,\mathrm{d}s.$$

We also define,

$$\theta_j^\varepsilon(t) := \varphi_j(f_0^\varepsilon) + \int_0^t \mathscr{L}^\varepsilon \varphi_j^\varepsilon(f_s^\varepsilon, m_s^\varepsilon)\,\mathrm{d}s + M_j^\varepsilon(t).$$

Note that

$$\theta_j^\varepsilon(t) = \varphi_j(f_t^\varepsilon) + \varepsilon(\varphi_{1,j}(f_t^\varepsilon, m_t^\varepsilon) - \varphi_{1,j}(f_0^\varepsilon, m_0^\varepsilon)), \tag{3.41}$$

so that, with the definitions of φ_j and $\varphi_{1,j}$, Cauchy-Schwarz inequality, we easily get

$$|\theta_j^\varepsilon(t)| \lesssim \sup_{t\in[0,T]} \|f^\varepsilon(t)\| \|p_j\|_{L^2_x} = \sup_{t\in[0,T]} \|f^\varepsilon(t)\|.$$

Hence, by the uniform $L^2_{F^{-1}}$ bound (3.31),

$$\mathbb{E} \sup_{t\in[0,T]} \left|\theta_j^\varepsilon(t)\right| \lesssim 1. \tag{3.42}$$

With (3.41) and the uniform $L^2_{F^{-1}}$ bound (3.31), we also deduce

$$\mathbb{E} \sup_{t\in[0,T]} \left|\varphi_j(r_t^\varepsilon) - \theta_j^\varepsilon(t)\right| \lesssim \varepsilon. \tag{3.43}$$

From now on, we fix $\gamma > N/2 + 2$ and we remark that, by (3.42), a.s. and for all $t \in [0,T]$, the series defined by $u_t^\varepsilon := \sum_{j\in\mathbb{N}^N} \theta_j^\varepsilon(t) J^\gamma p_j$ converges in $L^2(\mathbb{T}^N)$. We then set

$$\theta^\varepsilon(t) := J^{-\gamma} \sum_{j\in\mathbb{N}^N} \theta_j^\varepsilon(t) J^\gamma p_j,$$

which exists a.s. and for all $t \in [0,T]$ in $H^{-\gamma}(\mathbb{T}^N)$. And with (3.43), we obtain

$$\mathbb{E} \sup_{t\in[0,T]} \|r^\varepsilon(t) - \theta^\varepsilon(t)\|_{H^{-\gamma}(\mathbb{T}^N)} \lesssim \varepsilon. \tag{3.44}$$

Actually, by interpolation, the continuous embedding $L^2(\mathbb{T}^N) \subset H^{-\eta}(\mathbb{T}^N)$ and the uniform $L^2_{F^{-1}}$ bound (3.31), we have

$$\mathbb{E} \sup_{|t-s|<\delta} \|\rho(t) - \rho(s)\|_{H^{-\eta^\flat}} \leq \mathbb{E} \sup_{|t-s|<\delta} \|\rho(t) - \rho(s)\|_{H^{-\eta^\sharp}}^{\upsilon}$$

for a certain $\upsilon > 0$ if $\eta^\sharp > \eta^\flat > 0$. As a result, it is indeed sufficient to work with $\eta = \gamma$. In view of (3.44), we first want to obtain an estimate of the increments of θ^ε. We have, for $j \in \mathbb{N}^N$ and $0 \le s \le t \le T$,

$$\theta_j^\varepsilon(t) - \theta_j^\varepsilon(s) = \int_s^t \mathscr{L}^\varepsilon \varphi_j^\varepsilon(f_\sigma^\varepsilon, m_\sigma^\varepsilon)\, \mathrm{d}\sigma + M_j^\varepsilon(t) - M_j^\varepsilon(s). \tag{3.45}$$

We then control the two terms on the right-hand side of (3.45). Let us begin with the first one. Note that, since $D\varphi_j(f) \equiv p_j F$ and $D\varphi_{1,j}(f) \equiv -M^{-1}I(n)p_j F$, we obtain thanks to (3.27) with $\varphi_2 \equiv 0$,

$$\mathscr{L}^\varepsilon \varphi_j^\varepsilon(f_\sigma^\varepsilon, m_\sigma^\varepsilon) = -\frac{1}{\varepsilon}(Af_\sigma^\varepsilon, p_j F) + (Af_\sigma^\varepsilon, M^{-1}I(m_\sigma^\varepsilon)p_j F) - (f_\sigma^\varepsilon m_\sigma^\varepsilon, M^{-1}I(m_\sigma^\varepsilon)p_j F).$$

Since, with (3.5), we have $\overline{a(v)f_\sigma^\varepsilon} = \overline{a(v)g_\sigma^\varepsilon}$ where g^ε has been defined previously as $g^\varepsilon := f^\varepsilon - r^\varepsilon F$, we can write

$$(Af_\sigma^\varepsilon, p_j F) = \int_{\mathbb{T}^N} \mathrm{div}_x(\overline{a(v)f_\sigma^\varepsilon}) p_j \, \mathrm{d}x = \int_{\mathbb{T}^N} \mathrm{div}_x(\overline{a(v)g_\sigma^\varepsilon}) p_j \, \mathrm{d}x = (Ag_\sigma^\varepsilon, p_j F)$$

and, as a consequence, since a is bounded, we are led to

$$\frac{1}{\varepsilon}(Af_\sigma^\varepsilon, p_j F) \lesssim \|\varepsilon^{-1} g_\sigma^\varepsilon\| \|\nabla_x p_j\|_{L^2}.$$

Similarly, we can show that

$$(Af_\sigma^\varepsilon, M^{-1}I(m_\sigma^\varepsilon)p_j F) \lesssim \|g_\sigma^\varepsilon\|(1 + \|\nabla_x p_j\|_{L^2}).$$

Since we have obviously $(f_\sigma^\varepsilon m_\sigma^\varepsilon, M^{-1}I(m_\sigma^\varepsilon)p_j F) \lesssim \|f_\sigma^\varepsilon\|$, we can conclude that

$$|\mathscr{L}^\varepsilon \varphi_j^\varepsilon(f_\sigma^\varepsilon, m_\sigma^\varepsilon)| \lesssim C_j \left[\|\varepsilon^{-1} g_\sigma^\varepsilon\| + \|g_\sigma^\varepsilon\| + \|f_\sigma^\varepsilon\|\right], \tag{3.46}$$

where $C_j := 1 + \|\nabla_x p_j\|_{L^2} \le 1 + |j|$. Thanks to (3.31) and (3.37) with $p = 4$, we have that $(\varepsilon^{-1} g^\varepsilon)_{\varepsilon>0}$, $(g^\varepsilon)_{\varepsilon>0}$ and $(f^\varepsilon)_{\varepsilon>0}$ are bounded in $L^4(\Omega; L^2(0,T; L^2_{F^{-1}}))$. As a consequence, (3.46) and an application of Hölder's inequality gives

$$\mathbb{E}\left|\int_s^t \mathscr{L}^\varepsilon \varphi_j^\varepsilon(f_\sigma^\varepsilon, m_\sigma^\varepsilon)\, d\sigma\right|^4 \lesssim C_j^4 |t-s|^2.$$

Furthermore, using Burkholder-Davis-Gundy inequality, we can control the second term of the right-hand side of (3.45) as

$$\mathbb{E}|M_j^\varepsilon(t) - M_j^\varepsilon(s)|^4 \lesssim \mathbb{E}|\langle M_j^\varepsilon\rangle_t - \langle M_j^\varepsilon\rangle_s|^2,$$

where the quadratic variation $\langle M_j^\varepsilon\rangle$ is given by

$$\langle M_j^\varepsilon\rangle_t = \int_0^t (M|\varphi_{1,j}|^2 - 2\varphi_{1,j} M\varphi_{1,j})(f_s^\varepsilon, m_s^\varepsilon)\, \mathrm{d}s.$$

With the definition of $\varphi_{1,j}$, (3.13), (3.14) and the uniform $L^2_{F^{-1}}$ bound (3.31), it is now easy to get

$$\mathbb{E}|M_j^\varepsilon(t) - M_j^\varepsilon(s)|^4 \lesssim |t-s|^2.$$

Finally we have $\mathbb{E}|\theta_j^\varepsilon(t) - \theta_j^\varepsilon(s)|^4 \lesssim (1 + |j|^4)|t-s|^2$. Since we took $\gamma > N/2 + 2$, we can conclude that

$$\mathbb{E}\|\theta^\varepsilon(t) - \theta^\varepsilon(s)\|^4_{H^{-\gamma}(\mathbb{T}^N)} \lesssim |t-s|^2.$$

It easily follows that, for $\upsilon < 1/2$,

$$\mathbb{E}\|\theta^\varepsilon\|^4_{W^{\upsilon,4}(0,T,H^{-\gamma}(\mathbb{T}^N))} \lesssim 1$$

and by the embedding

$$W^{\upsilon,4}(0,T,H^{-\gamma}(\mathbb{T}^N)) \subset \mathcal{C}^\tau(0,T,H^{-\gamma}(\mathbb{T}^N)), \quad \tau < \upsilon - \frac{1}{4},$$

we obtain that $\mathbb{E}w(\theta^\varepsilon,\delta) \lesssim \delta^\tau$ for a certain positive τ. Finally, with (3.44), we can now conclude the first step of the proof since

$$\mathbb{E}w(\rho^\varepsilon,\delta) \le 2\mathbb{E}\sup_{t\in[0,T]}\|\rho^\varepsilon_t - \theta^\varepsilon_t\|_{H^{-\gamma}(\mathbb{T}^N)} + \mathbb{E}w(\theta^\varepsilon,\delta) \lesssim \varepsilon + \delta^\tau. \tag{3.47}$$

Step 2: tightness in $C([0,T];H^{-\eta}(\mathbb{T}^N))$. Since the embedding $L^2(\mathbb{T}^N) \subset H^{-\eta}(\mathbb{T}^N)$ is compact, and by Ascoli's Theorem, the set

$$K_R := \left\{\rho \in C([0,T],H^{-\eta}(\mathbb{T}^N)),\ \sup_{t\in[0,T]}\|\rho\|_{L^2(\mathbb{T}^N)} \le R,\ w(\rho,\delta) < \varepsilon(\delta)\right\},$$

where $R > 0$ and $\varepsilon(\delta) \to 0$ when $\delta \to 0$, is compact in $C([0,T],H^{-\eta}(\mathbb{T}^N))$. By Prokohrov's Theorem, the tightness of $(r^\varepsilon)_{\varepsilon>0}$ in $C([0,T],H^{-\eta}(\mathbb{T}^N))$ will follow if we prove that for all $\sigma > 0$, there exists $R > 0$ such that

$$\mathbb{P}(\sup_{t\in[0,T]}\|\rho^\varepsilon\|_{L^2(\mathbb{T}^N)} > R) < \sigma, \tag{3.48}$$

and

$$\lim_{\delta\to0}\limsup_{\varepsilon\to0}\mathbb{P}(w(\rho^\varepsilon,\delta) > \sigma) = 0. \tag{3.49}$$

With Markov's inequality and the uniform $L^2_{F^{-1}}$ bound (3.31), we have

$$\mathbb{P}(\sup_{t\in[0,T]}\|\rho^\varepsilon\|_{L^2(\mathbb{T}^N)} > R)\ \le\ \mathbb{P}(\sup_{t\in[0,T]}\|f^\varepsilon\| > R)\ \lesssim\ R^{-1},$$

which gives (3.48). And we deduce (3.49) by Markov's inequality and the bound (3.40) since

$$\begin{aligned}\lim_{\delta\to0}\limsup_{\varepsilon\to0}\mathbb{P}(w(\rho^\varepsilon,\delta) > \sigma)\ &\le\ \lim_{\delta\to0}\limsup_{\varepsilon\to0}\sigma^{-1}\mathbb{E}w(\rho^\varepsilon,\delta)\\ &\lesssim\ \lim_{\delta\to0}\limsup_{\varepsilon\to0}\sigma^{-1}(\varepsilon+\delta^\tau)\ =\ 0.\end{aligned}$$

Step 3: tightness in $L^2(0,T;L^2(\mathbb{T}^N))$. Similarly, due to [Sim87, Theorem 5], the set

$$K_R := \left\{\rho \in L^2(0,T;L^2(\mathbb{T}^N)),\ \int_0^T\|\rho_t\|^2_{H^s(\mathbb{T}^N)}\mathrm{d}t \le R,\ w(\rho,\delta) < \varepsilon(\delta)\right\},$$

where $R > 0$, $s > 0$ and $\varepsilon(\delta) \to 0$ when $\delta \to 0$, is compact in $L^2(0,T;L^2(\mathbb{T}^N))$. By Prokhorov's Theorem, the tightness of $(r^\varepsilon)_{\varepsilon>0}$ in $L^2(0,T;L^2(\mathbb{T}^N))$ will follow if we prove that for all $\sigma > 0$, there exists $R > 0$ such that

$$\mathbb{P}(\int_0^T\|\rho_t\|^2_{H^s(\mathbb{T}^N)}\mathrm{d}t > R) < \sigma, \tag{3.50}$$

and

$$\lim_{\delta\to0}\limsup_{\varepsilon\to0}\mathbb{P}(w(\rho^\varepsilon,\delta) > \sigma) = 0. \tag{3.51}$$

But (3.50) and (3.51) are consequences of Markov's inequality and the bounds (3.38) with $p = 1$ and (3.40) so that the proof is complete. □

3.7 Convergence

We conclude here the proof of Theorem 3.2.2. The idea is now, by the tightness result and Prokhorov Theorem, to take a subsequence of $(\rho^\varepsilon)_{\varepsilon>0}$ that converges in law to some probability measure. Then we show that this limiting probability is actually uniquely determined by the limit generator $\mathscr{L}$ defined above.

We fix $\eta > 0$. By Proposition 3.6.1 and Prokhorov's Theorem, there is a subsequence of $(\rho^\varepsilon)_{\varepsilon>0}$, still denoted $(\rho^\varepsilon)_{\varepsilon>0}$, and a probability measure P on the spaces $C([0,T],H^{-\eta})$ and $L^2(0,T;L^2)$ such that

$$P^\varepsilon \to P \text{ weakly in } C([0,T],H^{-\eta}) \text{ and } L^2(0,T;L^2),$$

where P^ε stands for the law of ρ^ε. We now identify the probability measure P.

Since the spaces $C([0,T],H^{-\eta})$ and $L^2(0,T;L^2)$ are separable, we can apply Skohorod representation Theorem [Bil09], so that there exists a new probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ and random variables

$$\widetilde{r^\varepsilon},\widetilde{\rho}:\widetilde{\Omega}\to C([0,T],H^{-\eta})\cap L^2(0,T;L^2),$$

with respective law P^ε and P such that $\widetilde{\rho^\varepsilon}\to\widetilde{\rho}$ in $C([0,T],H^{-\eta})$ and $L^2(0,T;L^2)$ $\widetilde{\mathbb{P}}-$a.s. In the sequel, for the sake of clarity, we do not write any more the tildes.

Note that, with (3.37), we can also suppose that $\varepsilon^{-1}g^\varepsilon$ converges to some g weakly in the space $L^2(\Omega;L^2(0,T;L^2_{F^{-1}}))$. Similarly, with (3.13), we assume that m^ε converges to m weakly in $L^2(\Omega;L^2(0,T;L^2_{F^{-1}}))$. Before going on the proof, we want to identify the weak limit g of $\varepsilon^{-1}g^\varepsilon$.

Lemma 3.7.1. *In $L^2(\Omega;L^2(0,T;L^2))$, we have the relation*

$$\overline{a(v)g} = -\sigma(\rho)^{-1}K\nabla_x\rho.$$

Proof. We define $D_T := (0,T)\times\mathbb{T}^N$. Since f^ε satisfies equation (3.1), we can write, for any $\psi\in C_c^\infty(D_T)$ and $\theta\in L^\infty(V\times\Omega;\mathbb{R}^N)$,

$$\begin{aligned}\mathbb{E}\int_{D_T\times V} f^\varepsilon F^{-1}\left(-\varepsilon\partial_t\psi - a\cdot\nabla_x\psi\right)\theta \;&=\; \mathbb{E}\int_{D_T\times V}\frac{1}{\varepsilon}\sigma(\overline{f^\varepsilon})Lf^\varepsilon F^{-1}\psi\theta\\ &\quad+\;\mathbb{E}\int_{D_T\times V} m^\varepsilon f^\varepsilon F^{-1}\psi\theta.\end{aligned}$$

We recall that we set $g^\varepsilon := f^\varepsilon - r^\varepsilon F$ and that $Lf^\varepsilon = Lg^\varepsilon$ so that we have

$$\begin{aligned}&\mathbb{E}\int_{D_T\times V} -\varepsilon f^\varepsilon F^{-1}\partial_t\psi\,\theta - r^\varepsilon a\cdot\nabla_x\psi\,\theta - g^\varepsilon F^{-1}a\cdot\nabla_x\psi\,\theta\\ &\qquad= \mathbb{E}\int_{D_T\times V}\sigma(r^\varepsilon)L(\varepsilon^{-1}g^\varepsilon)F^{-1}\psi\theta \;+\; \mathbb{E}\int_{D_T\times V} m^\varepsilon f^\varepsilon F^{-1}\psi\theta.\end{aligned}$$

Since $(f^\varepsilon)_{\varepsilon>0}$ and $(\varepsilon^{-1}g^\varepsilon)_{\varepsilon>0}$ are bounded in $L^2(\Omega;L^2(0,T;L^2_{F^{-1}}))$ by (3.31) and (3.37), and with the $\mathbb{P}-$a.s. convergence $r^\varepsilon\to\rho$ in $L^2(0,T;L^2_{F^{-1}})$ coupled with the uniform integrability of the family $(r^\varepsilon)_{\varepsilon>0}$ obtained with (3.31), we have that the left-hand side of the previous equality actually converges as $\varepsilon\to 0$ to

$$\mathbb{E}\int_{D_T\times V} -\rho a\cdot\nabla_x\psi\,\theta.$$

Note that, $\mathbb{P}-$a.s., we have the following convergences in $L^2(0,T;L^2_{F^{-1}})$

$$\sigma(r^\varepsilon) \to \sigma(\rho), \quad L(\varepsilon^{-1}g^\varepsilon) \rightharpoonup Lg, \quad f^\varepsilon \to \rho F, \quad m^\varepsilon \rightharpoonup \mathrm{m},$$

where the first convergence is justified by the Lipschitz continuity of σ. As a result, since all the quantities above are uniformly integrable with respect to ε thanks to (3.31), (3.37) and (3.13), the right-hand side of the previous equality converges as $\varepsilon \to 0$ to

$$\mathbb{E}\int_{D_T\times V} \sigma(\rho)L(g)F^{-1}\psi\theta \;+\; \mathbb{E}\int_{D_T\times V} \mathrm{m}\rho\psi\theta.$$

Thus, we have

$$\mathbb{E}\int_{D_T\times V} -\rho a\cdot\nabla_x\psi\,\theta \;=\; \mathbb{E}\int_{D_T\times V} \sigma(\rho)L(g)F^{-1}\psi\theta \;+\; \mathbb{E}\int_{D_T\times V} \mathrm{m}\rho\psi\theta.$$

Let ξ be an arbitrary bounded measurable function on Ω. We now set $\theta(v,\omega) = a(v)F(v)\xi(\omega)$; note that we do have $\theta \in L^\infty(V\times\Omega,\mathbb{R}^N)$. With (3.5) and the relation $Lg = \overline{g}F - g$, we obtain

$$-\mathbb{E}\int_{D_T\times V} \rho a\cdot\nabla_x\psi\, aF \;=\; -\mathbb{E}\int_{D_T\times V} \sigma(\rho)ga(v)\psi.$$

Since this relation holds for any $\xi \in L^\infty(\Omega)$ and $\psi \in C_c^\infty(D_T)$, we deduce that $\nabla_x\rho \in L^2(\Omega, L^2(D_T))$ and that

$$\overline{a(v)g} = -\sigma(\rho)^{-1}K\nabla_x\rho,$$

and this concludes the proof. □

Let $\varphi \in C^3(L^2_{F^{-1}})$ a good test-function satisfying (3.19). We define φ^ε as in Section 3.4.1. Since φ^ε is a good test-function, we have that

$$\varphi^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) - \varphi^\varepsilon(f_0^\varepsilon, m_0^\varepsilon) - \int_0^t \mathscr{L}^\varepsilon\varphi^\varepsilon(f_s^\varepsilon, m_s^\varepsilon)\,\mathrm{d}s, \quad t\in[0,T],$$

is a continuous martingale for the filtration generated by $(f_s^\varepsilon)_{s\in[0,T]}$. As a result, if Ψ denotes a continuous and bounded function from $L^2(\mathbb{T}^N)^n$ to $\mathbb{R}$, we have

$$\mathbb{E}\left[\left(\varphi^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) - \varphi^\varepsilon(f_s^\varepsilon, m_s^\varepsilon) - \int_s^t \mathscr{L}^\varepsilon\varphi^\varepsilon(f_u^\varepsilon, m_u^\varepsilon)\,\mathrm{d}u\right)\Psi(r_{s_1}^\varepsilon,...,r_{s_n}^\varepsilon)\right] = 0, \tag{3.52}$$

for any $0\le s_1\le ...\le s_n\le s\le t$. Our final purpose is to pass to the limit $\varepsilon\to 0$ in (3.52). In the sequel, we assume that the function φ and Ψ are also continuous on the space $H^{-\eta}$, which is always possible with an approximation argument: it suffices to consider $\varphi_r := \varphi((\mathrm{I}-r\Delta_x)^{-\frac{\eta}{2}}\cdot)$ and $\Psi_r := \Psi((\mathrm{I}-r\Delta_x)^{-\frac{\eta}{2}}\cdot,...,(\mathrm{I}-r\Delta_x)^{-\frac{\eta}{2}}\cdot)$ as $r\to 0$. With (3.30), we divide the left-hand side of (3.52) in four parts. Precisely, we define, for $i\in\{1,...,4\}$

$$\begin{aligned}
\tau_1^\varepsilon &:= \varphi^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) - \varphi^\varepsilon(f_s^\varepsilon, m_s^\varepsilon),\\
\tau_2^\varepsilon &:= \int_s^t \mathscr{L}\varphi(r_u^\varepsilon)\,\mathrm{d}u,\\
\tau_3^\varepsilon &:= \int_s^t -\frac{1}{\varepsilon}(Af_u^\varepsilon, D\varphi(f_u^\varepsilon)) - (\mathrm{div}_x(\sigma(r_u^\varepsilon)^{-1}K\nabla_x r_u^\varepsilon)F, D\varphi(r_u^\varepsilon F))\,\mathrm{d}u,\\
\tau_4^\varepsilon &:= \int_s^t -(Af_u^\varepsilon, D\varphi_1(f_u^\varepsilon)) - \varepsilon(Af_u^\varepsilon, D\varphi_2(f_u^\varepsilon)) + \varepsilon(f_u^\varepsilon m_u^\varepsilon, D\varphi_2(f_u^\varepsilon))\,\mathrm{d}u.
\end{aligned}$$

Study of τ_1^ε. We recall that $\varphi^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) = \varphi(r_t^\varepsilon F) + \varepsilon\varphi_1(f_t^\varepsilon, m_t^\varepsilon) + \varepsilon^2\varphi_2(f_t^\varepsilon, m_t^\varepsilon)$ so that, with the $\mathbb{P}-$a.s. convergence of r^ε to ρ in $C([0,T], H^{-\eta})$ and the bounds (i) of (3.26) and (3.28), we have that τ_1^ε converges $\mathbb{P}-$a.s. to $\varphi(\rho_t F) - \varphi(\rho_s F)$ as ε goes to 0. Furthermore, with the continuity of Ψ in $H^{-\eta}$, we also have that $\Psi(r_{s_1}^\varepsilon, ..., r_{s_n}^\varepsilon)$ converges $\mathbb{P}-$a.s. to $\Psi(\rho_{s_1}, ..., \rho_{s_n})$. Finally, since the family $\tau_1^\varepsilon \Psi(r_{s_1}^\varepsilon, ..., r_{s_n}^\varepsilon)$ is uniformly integrable with respect to ε thanks to (3.19), the bounds (i) of (3.26) and (3.28) and the uniform $L^2_{F^{-1}}$ bound (3.31), we have that

$$\mathbb{E}[\tau_1^\varepsilon \Psi(r_{s_1}^\varepsilon, ..., r_{s_n}^\varepsilon)] \to \mathbb{E}\left[(\varphi(\rho_t F) - \varphi(\rho_s F))\, \Psi(\rho_{s_1}, ..., \rho_{s_n})\right].$$

Study of τ_2^ε. We recall, with (3.29), that

$$\begin{aligned}\mathscr{L}\varphi(r_u^\varepsilon) = (\mathrm{div}_x(\sigma(r_u^\varepsilon)^{-1} K \nabla_x r_u^\varepsilon) F, D\varphi(r_u^\varepsilon F)) &- \int_E (r_u^\varepsilon F n M^{-1} I(n), D\varphi(r_u^\varepsilon F))\, d\nu(n) \\ &- \int_E D^2\varphi(r_u^\varepsilon F)(r_u^\varepsilon F M^{-1} I(n), r_u^\varepsilon F n)\, d\nu(n).\end{aligned}$$

Thanks to the $\mathbb{P}-$a.s. convergence of r^ε to ρ in $L^2(0,T;L^2)$ and with $\varphi \in C^3(L^2_{F^{-1}})$, we can pass to the limit $\varepsilon \to 0$ in the term

$$\int_s^t \int_E -(r_u^\varepsilon F n M^{-1} I(n), D\varphi(r_u^\varepsilon F)) - D^2\varphi(r_u^\varepsilon F)(r_u^\varepsilon F M^{-1} I(n), r_u^\varepsilon F n)\, d\nu(n)\, \mathrm{d}u.$$

Regarding the first term of $\mathscr{L}\varphi(r_u^\varepsilon)$, we introduce

$$G(\rho) := \int_0^\rho \frac{\mathrm{d}y}{\sigma(y)},$$

which is, thanks to the hypothesis (H1) made on σ, Lipschitz continuous on $L^2(\mathbb{T}^N)$. Now the first term of $\mathscr{L}\varphi(r_u^\varepsilon)$ writes

$$(\mathrm{div}_x(\sigma(r_u^\varepsilon)^{-1} K \nabla_x r_u^\varepsilon) F, D\varphi(r_u^\varepsilon F)) = (\mathrm{div}_x \nabla_x G(r_u^\varepsilon) F, D\varphi(r_u^\varepsilon F)).$$

Furthermore, with (3.19), the mapping $\rho \mapsto \partial^2_{x_i,x_j} D\varphi(\rho F)$ is continuous on $L^2(\mathbb{T}^N)$. As a result, we can now pass to the limit in the term

$$\int_s^t (\mathrm{div}_x(\sigma(r_u^\varepsilon)^{-1} K \nabla_x r_u^\varepsilon) F, D\varphi(r_u^\varepsilon F))\, \mathrm{d}u.$$

To sum up, we obtain that τ_2^ε converges $\mathbb{P}-$a.s. to $\int_s^t \mathscr{L}\varphi(\rho_u)\, \mathrm{d}u$ as ε goes to 0. Finally, since the family $\tau_2^\varepsilon \Psi(r_{s_1}^\varepsilon, ..., r_{s_n}^\varepsilon)$ is uniformly integrable with respect to ε thanks to (3.19) and the uniform $L^2_{F^{-1}}$ bound (3.31), we have that

$$\mathbb{E}[\tau_2^\varepsilon \Psi(r_{s_1}^\varepsilon, ..., r_{s_n}^\varepsilon)] \to \mathbb{E}\left[\left(\int_s^t \mathscr{L}\varphi(\rho_u)\, \mathrm{d}u\right) \Psi(\rho_{s_1}, ..., \rho_{s_n})\right].$$

Study of τ_3^ε. First of all, we observe that, with the decomposition $f^\varepsilon = r^\varepsilon F + g^\varepsilon$, (3.25) and (3.5),

$$-\varepsilon^{-1}(A f_u^\varepsilon, D\varphi(f_u^\varepsilon)) = -\varepsilon^{-1}(A g_u^\varepsilon, D\varphi(f_u^\varepsilon)),$$

so that, with the $\mathbb{P}-$a.s. convergences in $L^2(0,T;L^2)$

$$\varepsilon^{-1} g^\varepsilon \rightharpoonup g, \quad r^\varepsilon \to \rho,$$

and the continuity of the mapping $\rho \mapsto AD\varphi(\rho F)$ thanks to (3.19), we obtain that the first term of τ_3^ε converges $\mathbb{P}-$a.s. to

$$-\int_s^t (\overline{A g_u} F, D\varphi(\rho_u F))\, \mathrm{d}u.$$

And, with Lemma 3.7.1, this term writes

$$\int_s^t (\mathrm{div}_x(\sigma(\rho_u)^{-1}K\nabla_x\rho_u)F, D\varphi(\rho_u F))\,\mathrm{d}u. \tag{3.53}$$

Furthermore, similarly as the case of τ_2^ε, we have that the second term of τ_3^ε converges $\mathbb{P}-$a.s. to the opposite of (3.53). As a result, τ_3^ε converges $\mathbb{P}-$a.s. to 0. Finally, since the family $\tau_3^\varepsilon\Psi(r_{s_1}^\varepsilon,...,r_{s_n}^\varepsilon)$ is uniformly integrable with respect to ε thanks to (3.19), the uniform $L^2_{F^{-1}}$ bound (3.31) and the bound (3.37) on $(\varepsilon^{-1}g^\varepsilon)_{\varepsilon>0}$, we have that

$$\mathbb{E}[\tau_3^\varepsilon\Psi(r_{s_1}^\varepsilon,...,r_{s_n}^\varepsilon)] \to 0.$$

Study of τ_4^ε. If we transform the two first terms of τ_4^ε exactly as we do for the first term of τ_3^ε, it is then easy, using the uniform bounds (3.31) and (3.37) and the bounds (ii) of (3.26) and (3.28), to get

$$\mathbb{E}[\tau_4^\varepsilon\Psi(r_{s_1}^\varepsilon,...,r_{s_n}^\varepsilon)] = O(\varepsilon).$$

To sum up, we can pass to the limit $\varepsilon \to 0$ in (3.52) to obtain

$$\mathbb{E}\left[\left(\varphi(\rho_t F) - \varphi(\rho_s F) - \int_s^t \mathscr{L}\varphi(\rho_u)\,\mathrm{d}u\right)\Psi(\rho_{s_1},...,\rho_{s_n})\right] = 0. \tag{3.54}$$

We recall that this is valid for all $n\in\mathbb{N}$, $0\le s_1\le ...\le s_n\le s\le t\in[0,T]$ and all Ψ continuous and bounded function on $L^2(\mathbb{T}^N)^n$. Now, let ξ be a smooth function on $L^2(\mathbb{T}^N)$. We choose $\varphi(f) = (f,\xi F)$. We can easily verify that φ and $|\varphi|^2$ belong to $C^3(L^2_{F^{-1}})$ and that they are good test-function satisfying (3.19). Thus, we obtain that

$$N_t := \varphi(\rho_t F) - \varphi(\rho_0 F) - \int_0^t \mathscr{L}\varphi(\rho_u)\,\mathrm{d}u, \quad t\in[0,T],$$
$$|\varphi|^2(\rho_t F) - |\varphi|^2(\rho_0 F) - \int_0^t \mathscr{L}|\varphi|^2(\rho_u)\,\mathrm{d}u, \quad t\in[0,T],$$

are continuous martingales with respect to the filtration generated by $(\rho_s)_{s\in[0,T]}$. It implies (see appendix 6.9 in [FGPS10]) that the quadratic variation of N is given by

$$\langle N\rangle_t = \int_0^t \left[\mathscr{L}|\varphi|^2(\rho_u) - 2\varphi(\rho_u)\mathscr{L}\varphi(\rho_u)\right]\mathrm{d}u, \quad t\in[0,T].$$

Furthermore, we have

$$\begin{aligned}\mathscr{L}|\varphi|^2(\rho_u) - 2\varphi(\rho_u)\mathscr{L}\varphi(\rho_u) &= -2\int_E (\rho_u F n, \xi F)(\rho_u F M^{-1}I(n), \xi F)\,\mathrm{d}\nu(n)\\ &= 2\mathbb{E}\int_0^\infty (\rho_u F m_0, \xi F)(\rho_u F m_t, \xi F)\,\mathrm{d}t\\ &= \mathbb{E}\int_\mathbb{R} (\rho_u F m_0, \xi F)(\rho_u F m_t, \xi F)\,\mathrm{d}t\\ &= \int_{\mathbb{T}^N}\int_{\mathbb{T}^N} \rho_u(x)\xi(x)\rho_u(y)\xi(y)k(x,y)\,\mathrm{d}x\mathrm{d}y\\ &= \|\rho_u Q^{\frac{1}{2}}\xi\|^2_{L^2}.\end{aligned}$$

This is valid for all smooth function ξ of $L^2(\mathbb{T}^N)$ so we deduce that

$$M_t := \rho_t - \rho_0 - \int_0^t \mathrm{div}_x(\sigma(\rho_s)^{-1}K\nabla_x\rho_s)\,\mathrm{d}s - \int_0^t \rho_s H\,\mathrm{d}s, \quad t\in[0,T],$$

is a martingale with quadratic variation

$$\int_0^t \rho_s Q^{\frac{1}{2}} \left(\rho_s Q^{\frac{1}{2}}\right)^* \, \mathrm{d}s.$$

Thanks to martingale representation Theorem, see [DPZ08, Theorem 8.2], up to a change of probability space, there exists a cylindrical Wiener process W such that

$$\rho_t - \rho_0 - \int_0^t \mathrm{div}_x(\sigma(\rho_s)^{-1} K \nabla_x \rho_s) \, \mathrm{d}s - \int_0^t \rho_s H \, \mathrm{d}s = \int_0^t \rho_s Q^{\frac{1}{2}} \, \mathrm{d}W_s, \quad t \in [0, T].$$

This gives that ρ has the law of a weak solution to the equation (3.16) with paths in $C([0,T], H^{-\eta}) \cap L^2(0,T;L^2)$. Since this equation has a unique solution with paths in the space $C([0,T], H^{-\eta}) \cap L^2(0,T;L^2)$, and since pathwise uniqueness implies uniqueness in law, we deduce that P is the law of this solution and is uniquely determined. Finally, by the uniqueness of the limit, the whole sequence $(P^\varepsilon)_{\varepsilon>0}$ converges to P weakly in the spaces of probability measures on $C([0,T], H^{-\eta})$ and $L^2(0,T;L^2)$. This concludes the proof of Theorem 3.2.2.

Chapter 4

A regularity result for quasilinear stochastic partial differential equations of parabolic type

Abstract: We consider a quasilinear parabolic stochastic partial differential equation driven by a multiplicative noise and study regularity properties of its weak solution satisfying classical *a priori* estimates. In particular, we determine conditions on coefficients and initial data under which the weak solution is Hölder continuous in time and possesses spatial regularity that is only limited by the regularity of the given data.

Keywords: Stochastic partial differential equations, regularity, Hölder, stochastic convolution, Schauder theory, quasilinear parabolic partial differential equations.

The results of this chapter are a joint work with A. Debussche and M. Hofmanova and are available as a preprint:

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4.1 Introduction

In this chapter, we are interested in the regularity of weak solutions of quasilinear parabolic stochastic partial differential equation driven by a multiplicative noise. Let $D \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, let $T > 0$ and set $D_T = (0,T) \times D$, $S_T = (0,T] \times \partial D$. We study the following problem

$$\begin{cases} \mathrm{d}u = \mathrm{div}(B(u))\,\mathrm{d}t + \mathrm{div}\,(A(u)\nabla u)\,\mathrm{d}t + F(u)\,\mathrm{d}t + H(u)\,\mathrm{d}W & \text{in } D_T, \\ \quad u = 0 & \text{in } S_T, \\ u(0) = u_0 & \text{in } D. \end{cases} \tag{4.1}$$

where W a cylindrical Wiener process on some Hilbert space K and H a mapping with values in the space of the γ-radonifying operators from K to certain Sobolev spaces. The precise description of the problem setting will be given in the next section.

It is a well known fact in the field of PDEs and SPDEs that many equations do not, in general, have classical or strong solutions and can be solved only in some weaker sense. Unlike deterministic problems, in the case of stochastic equations we can only ask whether the solution is smooth in the space variable since the time regularity is limited by the regularity of the stochastic integral. Thus, the aim of the present work is to determine conditions on coefficients and initial data under which there exists a spatially smooth solution to (4.1).

Such a regularity result is fundamental and interesting by itself. Equations of the form (4.1) appear in many sciences. Regularity of solutions is an important property when one wants to study qualitative behaviour. It is also a preliminary step when studying numerical approximations. Our original motivation is that such models arise as limit of random kinetic equations. An example of such equations is treated in [DV12]. The problem is linear there and the limit is a limit stochastic parabolic equation. But we wish to treat more general kinetic equations and expect limit equations of the form (4.1). The rigorous justification of this limit requires the results obtained in this chapter.

The issue of existence of a classical solution to deterministic parabolic problems is well understood, among the main references stands the extensive book [LSU68] which is mainly concerned with the solvability of initial-boundary value problems and the Cauchy problem to the basic linear and quasilinear second order PDEs of parabolic type. A special attention is paid to the connection between the smoothness of solutions and the smoothness of known data entering into the problem (initial condition and coefficients), nevertheless, due to technical complexity of the proofs a direct generalization to the stochastic case is not obvious.

In the case of linear parabolic problems, let us mention the classical Schauder theory (see e.g. [Lie96]) that provides *a priori* estimates relating the norms of solutions of initial- boundary value problems, namely the parabolic Hölder norms, to the norms of the known quantities in the problems. These results are usually employed in order to deal with quasilinear equations: the application of the Schauder fixed point theorem leads easily to the existence of a smooth solution under very weak hypotheses on the coefficients. In our proof, we make use of the Schauder theory as well, yet in an entirely different approach.

Regularity of parabolic problems in the stochastic setting was also studied in several works. In the previous work of the third author [Hof13], semilinear parabolic SPDEs (i.e. the diffusion matrix A independent of the solution) were studied and a regularity result established by using semigroup arguments. In [DM13], a maximum principle is obtained for a SPDE similar to (4.1) but with a more general diffusion H, it may depend on the gradient of u. In [Ges12], existence and uniqueness of strong solutions to SPDEs with drift given by the subdifferential of a quasi-convex function is proved. Hölder continuity of solutions to nonlinear parabolic systems under suitable structure conditions was proved in [BF13] by energy methods. In comparison to this work, the quasilinear case considered in the present chapter is more delicate and different techniques need to be applied.

The transposition of the deterministic method exposed in [LSU68] seems to be quite difficult. Fortunately, we have found a trick to avoid this. We introduce a new method that is based on a very simple idea: a weak solution to (4.1) that satisfies *a priori* estimates is decomposed into two parts $u = y + z$ where z is a solution to a linear parabolic SPDE with the same noise term as (4.1) and y solves a linear parabolic PDE with random coefficients. As a consequence, the problem of regularity of u is reduced to showing regularity of z and regularity of y which can be handled by known techniques for stochastic convolutions and deterministic PDEs. It is rather surprising that this classical idea used to treat semilinear equations can be applied also for quasilinear problems.

Let us explain this method more precisely. As the main difficulties come from the second order and the stochastic term, for simplicity of the introduction we assume $B = F = 0$ and consider periodic boundary conditions, i.e. $D = \mathbb{T}^N$ is the N-dimensional torus. Let u be a weak solution to

$$\begin{cases} \mathrm{d}u = \operatorname{div}\left(A(u)\nabla u\right)\mathrm{d}t + H(u)\,\mathrm{d}W, \\ u(0) = u_0, \end{cases} \tag{4.2}$$

and let z be a solution to

$$\begin{cases} \mathrm{d}z = \Delta z\,\mathrm{d}t + H(u)\,\mathrm{d}W, \\ z(0) = 0. \end{cases}$$

Then z is given by the stochastic convolution with the semigroup generated by the Laplacian, denoted by $(S(t))_{t\geq 0}$, i.e.

$$z(t) = \int_0^t S(t-s)H(u)\,\mathrm{d}W(s)$$

and regularization properties are known. Setting $y = u - z$ it follows immediately that y solves

$$\begin{cases} \partial_t y = \operatorname{div}(A(u)\nabla y) + \operatorname{div}((A(u) - \mathrm{I})\nabla z), \\ y(0) = u_0, \end{cases} \tag{4.3}$$

which is a (pathwise) deterministic linear parabolic PDE. According to *a priori* estimates for (4.2), it holds

$$u \in L^p(\Omega; L^\infty(0,T; L^p(\mathbb{T}^N))) \cap L^2(\Omega; L^2(0,T; W^{1,2}(\mathbb{T}^N))), \quad \forall p \in [2,\infty),$$

and making use of the factorization method it is possible to show that z possesses enough regularity so that ∇z is a function with good integrability properties. Now, a classical result for deterministic linear parabolic PDEs with discontinuous coefficients (see [LSU68]) yields Hölder continuity of y (in time and space) and consequently also Hölder continuity of u itself. Having this in hand, the regularity of z can be increased to a level where the Schauder theory for linear parabolic PDEs with Hölder continuous coefficients applies to (4.3) (see [Lie96]) and higher regularity of y is obtained. Repeating this approach then allows us to conclude that u is λ-Hölder continuous in time with $\lambda < 1/2$ and possesses as much regularity in space as allowed by the regularity of the coefficients and the initial data.

The chapter is organized as follows. In Section 4.2, we introduce the basic setting and state our regularity results, Theorem 4.2.2, Theorem 4.2.3. Section 4.3 gives a preliminary result concerning the stochastic convolution. The remainder of the chapter is devoted to the proof of Theorem 4.2.2 and Theorem 4.2.3 that is divided into several parts. In Section 4.4, we establish our first regularity result, Theorem 4.2.2, that gives some Hölder continuity in time and space of a weak solution to (4.1). The regularity is then inductively improved in the final Section 4.5 and Theorem 4.2.3 is proved.

4.2 Notations, hypotheses and the main result

4.2.1 Notations

In this chapter, we adopt the following conventions. For $r \in [1,\infty]$, the Lebesgue spaces $L^r(D)$ are denoted by L^r and the corresponding norm by $\|\cdot\|_r$. In order to measure higher regularity of functions we make use of the Bessel potential spaces $H^{a,r}(D)$, $a \in \mathbb{R}$ and $r \in (1,\infty)$, we also shorten the notation to $H^{a,r}$ with the norm $\|\cdot\|_{a,r}$. The choice of this scale of function spaces is more natural for our method than the Sobolev-Slobodeckij spaces $W^{a,r}$, namely, the spaces $H_0^{a,r}$ coincide with the domains of fractional powers of the Laplace operator with null Dirichlet boundary conditions, which is an important ingredient for proving regularity of the stochastic convolution. For the reader's convenience we include a reminder of the basic properties of these spaces in Section 4.3.

Another important scale of function spaces which is used throughout the chapter are the Hölder spaces. In particular, if X and Y are two Banach spaces and $\alpha \in (0,1)$, $C^\alpha(X;Y)$ denotes the space of bounded Hölder continuous functions with values in Y equipped with the norm

$$\|f\|_{C^\alpha(X;Y)} \;=\; \sup_{x\in X}\|f(x)\|_Y \;+\; \sup_{x,x'\in X, x\neq x'}\frac{\|f(x)-f(x')\|_Y}{\|x-x'\|_X^\alpha}.$$

In the sequel, we consider the spaces $C^\alpha(\overline{D}) = C^\alpha(\overline{D};\mathbb{R})$, $C^\alpha([0,T];X)$ where $X = H^{a,r}$ or $X = C^\beta(\overline{D})$ and $C^\alpha([0,T]\times\overline{D}) = C^\alpha([0,T]\times\overline{D};\mathbb{R})$. Besides, we employ Hölder spaces with different regularity in time and space, i.e. $C^{\alpha,\beta}([0,T]\times\overline{D})$ equipped with the norm

$$\|f\|_{C^{\alpha,\beta}} = \sup_{(t,x)}|f(t,x)| + \sup_{(t,x)\neq(s,y)}\frac{|f(t,x)-f(s,y)|}{|t-s|^\alpha + |x-y|^\beta}.$$

With usual modifications we can also consider $\alpha,\beta \geq 1$. Note that it holds $C^\alpha([0,T];C^\beta(\overline{D})) \subsetneq C^{\alpha,\beta}([0,T]\times\overline{D})$ and therefore we have to distinguish these two spaces (see [Rab11]).

4.2.2 Hypotheses

Let us now introduce the precise setting of (4.1). We work on a finite-time interval $[0,T]$, $T>0$, and on a bounded domain D in $\mathbb{R}^N$ with smooth boundary. We denote by D_T the cylinder $(0,T)\times D$ and by S_T the lateral surface of D_T, that is $S_T = (0,T]\times\partial D$. Concerning the coefficients A, B, F, H, we only state here the basic assumptions that guarantee the existence of a weak solution and are valid throughout the chapter. Further regularity hypotheses are necessary in order to obtain better regularity of the weak solution and will be specified later. We assume that the flux function

$$B = (B_1,\dots,B_N) : \mathbb{R} \longrightarrow \mathbb{R}^N$$

is continuous with linear growth. The diffusion matrix

$$A = (A_{ij})_{i,j=1}^N : \mathbb{R} \longrightarrow \mathbb{R}^{N\times N}$$

is supposed to be continuous, symmetric, positive definite and bounded. In particular, there exist constants ν, $\mu > 0$ such that for all $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$,

$$\nu|\xi|^2 \leq A(u)\xi\cdot\xi \leq \mu|\xi|^2. \tag{4.4}$$

The drift coefficient $F : \mathbb{R} \to \mathbb{R}$ is continuous with linear growth.

Regarding the stochastic term, let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. The driving process W is a cylindrical Wiener process: $W(t) = \sum_{k\geq 1} \beta_k(t) e_k$ with $(\beta_k)_{k\geq 1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)_{t\geq 0}$ and $(e_k)_{k\geq 1}$ a complete orthonormal system in a separable Hilbert space K. For each $u \in L^2(D)$ we consider a mapping $H(u) : K \to L^2(D)$ defined by $H(u)\, e_k = H_k(\cdot, u(\cdot))$. In particular, we suppose that $H_k \in C(D \times \mathbb{R})$ and the following linear growth condition holds true

$$\sum_{k\geq 1} |H_k(x,\xi)|^2 \leq C\big(1 + |\xi|^2\big), \qquad \forall x \in D,\, \xi \in \mathbb{R}. \tag{4.5}$$

This assumption implies in particular that H maps $L^2(D)$ to $L_2(K;L^2(D))$ where $L_2(K;L^2(D))$ denotes the collection of Hilbert-Schmidt operators from K to $L^2(D)$. Thus, given a predictable process u that belongs to $L^2(\Omega; L^2(0,T; L^2(D)))$, the stochastic integral $t \mapsto \int_0^t H(u)\mathrm{d}W$ is a well defined process taking values in $L^2(D)$ (see [DPZ08] for a thorough exposition).

Later on we are going to estimate the weak solution of (4.1) in certain Bessel potential spaces $H^{a,r}$ with $a \geq 0$ and $r \in [2, \infty)$ and therefore we need to ensure the existence of the stochastic integral in (4.1) as an $H^{a,r}$-valued process. We recall that the Bessel potential spaces $H^{a,r}$ with $a \geq 0$ and $r \in [2,\infty)$ belong to the class of 2-smooth Banach spaces since they are isomorphic to $L^r(0,1)$ according to [Tri95, Theorem 4.9.3] and hence they are well suited for the stochastic Itô integration (see [Brz97], [BP99] for the precise construction of the stochastic integral). So, let us denote by $\gamma(K,X)$ the space of the γ-radonifying operators from K to a 2-smooth Banach space X. We recall that $\Psi \in \gamma(K,X)$ if the series

$$\sum_{k\geq 0} \gamma_k \Psi(e_k)$$

converges in $L^2(\widetilde{\Omega}, X)$, for any sequence $(\gamma_k)_{k\geq 0}$ of independent Gaussian real-valued random variables on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and any orthonormal basis $(e_k)_{k\geq 0}$ of K. Then, the space $\gamma(K,X)$ is endowed with the norm

$$\|\Psi\|_{\gamma(K,X)} := \left(\widetilde{\mathbb{E}} \left| \sum_{k\geq 0} \gamma_k \Psi(e_k) \right|_X^2 \right)^{1/2}$$

(which does not depend on $(\gamma_k)_{k\geq 0}$, nor on $(e_k)_{k\geq 0}$) and is a Banach space. Now, if $a \geq 0$ and $r \in [2, \infty)$ we denote by ($\mathrm{H}_{a,r}$) the following hypothesis

$$\|H(u)\|_{\gamma(K,H_0^{a,r})} \leq \begin{cases} C\big(1 + \|u\|_{H_0^{a,r}}\big), & a \in [0,1], \\ C\big(1 + \|u\|_{H_0^{a,r}} + \|u\|^a_{H_0^{1,ar}}\big), & a > 1, \end{cases} \tag{$\mathrm{H}_{a,r}$}$$

i.e. H maps $H_0^{a,r}$ to $\gamma(K, H_0^{a,r})$ provided $a \in [0,1]$ and it maps $H_0^{a,r} \cap H_0^{1,ar}$ to $\gamma(K, H_0^{a,r})$ provided $a > 1$. The precise values of parameters a and r will be given later in each of our regularity results.

Remark 4.2.1. *We point out that, thanks to the linear growth hypothesis* (4.5) *on the functions* $(H_k)_{k\geq 1}$*, one can easily verify that, for all* $r \in [2,\infty)$*, the bound* ($\mathrm{H}_{0,r}$) *holds true.*

In order to clarify the assumption ($\mathrm{H}_{a,r}$), let us present the main examples we have in mind.

Example Let W be a d-dimensional $(\mathscr{F}_t)$-Wiener process, that is $W(t) = \sum_{k=1}^d W_k(t)\, e_k$, where W_k, $k = 1, \ldots, d$, are independent standard $(\mathscr{F}_t)$-Wiener processes and $(e_k)_{k=1}^d$ is an orthonormal basis of $K = \mathbb{R}^d$. Then the hypothesis ($\mathrm{H}_{a,r}$) is satisfied for $a \geq 0$, $r \in [2,\infty)$ provided the functions $H_1, \dots, H_d$ are sufficiently smooth and respect the boundary conditions in the following sense:

$$H_k(x,0) = 0, \qquad x \in \partial D, \quad \forall k = 1,\dots,d,$$

(for more details we refer the reader to [RS96]). Note that in this example it is necessary to restrict ourselves to the subspace $H_0^{a,r} \cap H_0^{1,ar}$ of $H_0^{a,r}$ so that the corresponding Nemytskij operators $u \mapsto H_k(\cdot, u(\cdot))$ take values in $H_0^{a,r}$. In fact, if $1 + 1/r \leq a \leq N/r$, $r \in (1,\infty)$, then only linear operators map $H_0^{a,r}$ to itself (see [RS96]).

Example In the case of linear operator H we are able to deal with an infinite dimensional noise. Namely, let W be a $(\mathscr{F}_t)$-cylindrical Wiener process on $K = L^2(D)$, that is $W(t) = \sum_{k\geq 1} W_k(t)\, e_k$, where W_k, $k \geq 1$, are independent standard $(\mathscr{F}_t)$-Wiener processes and $(e_k)_{k\geq 1}$ an orthonormal basis of K. We assume that H is linear of the form $H(u)e_k := u\, Qe_k$, $k \geq 1$, where Q denotes a linear operator from K to K. Then, one can verify that the hypothesis ($\mathrm{H}_{a,r}$) is satisfied for $a \geq 0$, $r \in [2,\infty)$ provided we assume the following regularity property: $\sum_{k\geq 1} \|Qe_k\|^2_{a,\infty} < \infty$. We point out that, in this example, H maps $H_0^{a,r}$ to $\gamma(K, H_0^{a,r})$ for any $a \geq 0$ and $r \in [2,\infty)$.

As we are interested in proving the regularity up to the boundary for weak solutions of (4.1), it is necessary to impose certain compatibility conditions upon the initial data and the null Dirichlet boundary condition. To be more precise, since u_0 can be random in general, let us assume that $u_0 \in L^0(\Omega; C(\overline{D}))$ with $u_0 = 0$ on ∂D. Further integrability and regularity assumptions on u_0 will be specified later.
Note that other boundary conditions could be studied with similar arguments.

4.2.3 Existence of weak solutions

Let us only give a short comment here as the existence of weak solutions is not our main concern and we will only make use of *a priori* estimates for parabolic equations of the form (4.1). In the recent work [DHV13], the authors gave a well-posedness result for degenerate parabolic SPDEs (with periodic boundary conditions) of the form

$$\begin{cases} \mathrm{d}u = \operatorname{div}(B(u))\,\mathrm{d}t + \operatorname{div}(A(u)\nabla u)\,\mathrm{d}t + H(u)\,\mathrm{d}W, \\ u(0) = u_0, \end{cases}$$

where the diffusion matrix was supposed to be positive semidefinite. One can easily verify that the Dirichlet boundary conditions and the drift term $F(u)$ in (4.1) do not cause any additional difficulties in the existence part of the proofs and therefore the corresponding results in [DHV13], namely Section 4 (with the exception of Subsection 4.3) and Proposition 5.1, are still valid in the case of (4.1). In particular, we have the following.

Theorem 4.2.1. *There exists* $\big((\tilde\Omega, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \tilde W, \tilde u\big)$ *which is a weak martingale solution to* (4.1) *and, for all* $p \in [2,\infty)$,

$$\tilde u \in L^2(\tilde\Omega; C([0,T]; L^2)) \cap L^p(\tilde\Omega; L^\infty(0,T; L^p)) \cap L^2(\tilde\Omega; L^2(0,T; W^{1,2})).$$

In the sequel, we assume the existence of a weak solution on the original probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and show that it possesses as much regularity as we want provided the coefficients and initial data are sufficiently regular. We point out that this assumption is taken without loss of generality since pathwise uniqueness can be proved once we have sufficient regularity in hand and hence existence of a pathwise solution can be then obtained by usual methods (cf. [DHV13, Subsection 4.3]).
A similar result can be obtained in the case of null Dirichlet boundary conditions as well.

4.2.4 The main result

To conclude this section let us state our main results to be proved precisely.

Theorem 4.2.2. *Let u be a weak solution to (4.1) such that, for all $p \in [2,\infty)$,*

$$u \in L^2(\Omega; C([0,T]; L^2)) \cap L^p(\Omega; L^\infty(0,T; L^p)) \cap L^2(\Omega; L^2(0,T; W_0^{1,2})).$$

Assume that

(i) $u_0 \in L^m(\Omega; C^\iota(\overline{D}))$ *for some* $\iota > 0$ *and all* $m \in [2,\infty)$,

(ii) ($\mathrm{H}_{1,2}$) *is fulfilled.*

Then there exists $\eta > 0$ such that, for all $m \in [2,\infty)$, the weak solution u belongs to the space $L^m(\Omega; C^\eta(\overline{D_T}))$.

Theorem 4.2.3. *Let $k \in \mathbb{N}$. Let u be a weak solution to (4.1) such that, for all $p \in [2,\infty)$,*

$$u \in L^2(\Omega; C([0,T]; L^2)) \cap L^p(\Omega; L^\infty(0,T; L^p)) \cap L^2(\Omega; L^2(0,T; W_0^{1,2})).$$

Assume that

(i) $u_0 \in L^m(\Omega; C^{k+\iota}(\overline{D}))$ *for some* $\iota > 0$ *and all* $m \in [2,\infty)$,

(ii) $A,\, B \in C_b^k$ *and* $F \in C_b^{k-1}$,

(iii) ($\mathrm{H}_{a,r}$) *is fulfilled for all* $a < k+1$ *and* $r \in [2,\infty)$.

Then for all $\lambda \in (0,1/2)$ there exists $\beta > 0$ such that, for all $m \in [2,\infty)$, the weak solution u belongs to $L^m(\Omega; C^{\lambda,k+\beta}(\overline{D_T}))$.

4.3 Regularity of the stochastic convolution

Our proof of Theorem 4.2.2 and Theorem 4.2.3 is based on a regularity result that concerns mild solutions to linear SPDEs of the form

$$\begin{cases} \mathrm{d}Z = \Delta_D Z\,\mathrm{d}t + \Psi(t)\,\mathrm{d}W_t, \\ Z(0) = 0, \end{cases} \tag{4.6}$$

where Δ_D is the Laplacian on D with null Dirichlet boundary conditions acting on various Bessel potential spaces.

In order to motivate the use of these spaces let us recall their basic properties (for a thorough exposition we refer the reader to the books of Triebel [Tri95], [Tri92]). In the case of $\mathbb{R}^N$ (or $\mathbb{T}^N$) the Bessel potential spaces are defined in terms of Fourier transform of tempered distributions: let $a \in \mathbb{R}$, $r \in (1,\infty)$ then

$$H^{a,r}(\mathbb{R}^N) = \left\{ f \in \mathcal{S}'(\mathbb{R}^N);\ \|f\|_{H^{a,r}} := \left\| \mathcal{F}^{-1}(1+|\xi|^2)^{a/2}\mathcal{F}f \right\|_{L^r} < \infty \right\}$$

and they belong to the Triebel-Lizorkin scale $F^a_{r,s}(\mathbb{R}^N)$ in the sense that $H^{a,r}(\mathbb{R}^N) = F^a_{r,2}(\mathbb{R}^N)$. As a consequence, they are generally different from the Sobolev-Slobodeckij spaces $W^{a,r}(\mathbb{R}^N)$ which belong to the Besov scale $B^a_{r,s}(\mathbb{R}^N)$ in the sense that $W^{a,r}(\mathbb{R}^N) = B^a_{r,r}(\mathbb{R}^N)$ if $a > 0$, $a \notin \mathbb{N}$. Nevertheless, we have the following two relations which link the two scales of function spaces together

$$W^{a,r}(\mathbb{R}^N) = H^{a,r}(\mathbb{R}^N) \quad \text{if} \quad a \in \mathbb{N}_0,\ r \in (1,\infty) \quad \text{or} \quad a \geq 0,\ r = 2,$$

and

$$H^{a+\varepsilon,r}(\mathbb{R}^N) \hookrightarrow W^{a,r}(\mathbb{R}^N) \hookrightarrow H^{a-\varepsilon,r}(\mathbb{R}^N) \qquad a \in \mathbb{R},\ r \in (1,\infty),\ \varepsilon > 0.$$

The Bessel potential spaces $H^{a,r}(\mathbb{R}^N)$ behave well under the complex interpolation, i.e. for $a_0, a_1 \in \mathbb{R}$ and $r_0, r_1 \in (1, \infty)$ it holds that

$$[H^{a_0,r_0}(\mathbb{R}^N), H^{a_1,r_1}(\mathbb{R}^N)]_\theta = H^{a,r}(\mathbb{R}^N), \tag{4.7}$$

where $\theta \in (0,1)$ and $a = (1-\theta)a_0 + \theta a_1$, $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, which makes them more suitable for studying regularity for linear elliptic and parabolic problems. Indeed, under the assumption of bounded imaginary powers of a positive operator $\mathcal{A}$ on a Banach space X, the domains of fractional powers of $\mathcal{A}$ are given by the complex interpolation as well: let $0 \leq \alpha < \beta < \infty$, $\theta \in (0,1)$ then

$$[D(\mathcal{A}^\alpha), D(\mathcal{A}^\beta)]_\theta = D(\mathcal{A}^{(1-\theta)\alpha+\theta\beta}).$$

Furthermore, the expression (4.7) suggests how the spaces $H^{a,r}(D)$ may be defined for a general domain D: if $a \geq 0$ and $m \in \mathbb{N}$ such that $a \leq m < a+1$ then we define

$$H^{a,r}(D) := [W^{m,r}(D), L^r(D)]_{(m-a)/m}.$$

If D is sufficiently regular then $H^{a,r}(D)$ coincides with the space of restrictions to D of functions in $H^{a,r}(\mathbb{R}^N)$ and the Sobolev embedding theorem holds true. The spaces $H_0^{a,r}(D)$, $a \geq 0$, $r \in (1,\infty)$, are then defined as the closure of $C_c^\infty(D)$ in $H^{a,r}(D)$. Note, that $H_0^{a,r}(D) = H^{a,r}(D)$ if $a \leq 1/r$ and $H_0^{a,r}(D)$ is strictly contained in $H^{a,r}$ if $a > 1/r$. Besides, an interpolation result similar to (4.7) holds for these spaces as well

$$[H_0^{a_0,r_0}(D), H_0^{a_1,r_1}(D)]_\theta = H_0^{a,r}(D).$$

Let us now take a closer look at the Dirichlet Laplacian Δ_D. Considered as an operator on L^r, its domain is $H_0^{2,r}$ and it is the infinitesimal generator of an analytic semigroup denoted by $S = (S(t))_{t\geq 0}$. Moreover, it follows from the above considerations that the domains of its fractional powers coincide with the Bessel potential spaces, that is $D((-\Delta_D)^\alpha) = H_0^{2\alpha,r}$, $\alpha \geq 0$. Therefore, one can build a fractional power scale (or a Sobolev tower, see [Ama95], [EN06]) generated by $(L^r, -\Delta_D)$ to get

$$\big[\big(H_0^{2\alpha,r}, -\Delta_{D,2\alpha,r}\big);\ \alpha \geq 0\big], \tag{4.8}$$

where $-\Delta_{D,2\alpha,r}$ is the $H_0^{2\alpha,r}$-realization of $-\Delta_D$. Having this in hand, an important result [Ama95, Theorem V.2.1.3] describes the behaviour of the semigroup S in this scale. More precisely, the operator $\Delta_{D,2\alpha,r}$ generates an analytic semigroup $S_{2\alpha,r}$ on $H_0^{2\alpha,r}$ which is naturally obtained from S by restriction, i.e. $S_{2\alpha,r}(t)$ is the $H_0^{2\alpha,r}$-realization of $S(t)$, $t \geq 0$, and we have the following regularization property: for any $\delta > 0$ and $t > 0$, $S_{2\alpha,r}(t)$ maps $H_0^{2\alpha,r}$ into $H_0^{2\alpha+\delta,r}$ with

$$\big\|S_{2\alpha,r}(t)\big\|_{\mathcal{L}(H_0^{2\alpha,r}, H_0^{2\alpha+\delta,r})} \leq \frac{C}{t^{\delta/2}}. \tag{4.9}$$

For notational simplicity of the sequel we do not directly specify the spaces where the operators Δ_D and $S(t)$, $t \geq 0$, are acting since this is always clear from the context.

The solution to (4.6) is given by the stochastic convolution, that is

$$Z(t) = \int_0^t S(t-s)\Psi(s)\,\mathrm{d}W_s, \quad t \in [0,T].$$

In order to describe the connection between its regularity and the regularity of Ψ, we recall the following proposition.

Proposition 4.3.1. *Let $a \geq 0$ and $r \in [2,\infty)$ and let Ψ be a progressively measurable process in $L^p(\Omega; L^p(0,T; \gamma(K, H_0^{a,r})))$.*

(i) Let $p \in (2,\infty)$ and $\delta \in (0, 1-2/p)$. Then, for any $\gamma \in [0, 1/2 - 1/p - \delta/2)$, $Z \in L^p(\Omega; C^\gamma(0,T; H_0^{a+\delta,r}))$ and

$$\mathbb{E}\|Z\|^p_{C^\gamma(0,T;H_0^{a+\delta,r})} \leq C\,\mathbb{E}\|\Psi\|^p_{L^p(0,T;\gamma(K,H_0^{a,r}))}.$$

(ii) Let $p \in [2,\infty)$ and $\delta \in (0,1)$. Then $Z \in L^p(\Omega; L^p(0,T; H_0^{a+\delta,r}))$ and

$$\mathbb{E}\|Z\|^p_{L^p(0,T;H_0^{a+\delta,r})} \leq C\,\mathbb{E}\|\Psi\|^p_{L^p(0,T;\gamma(K,H_0^{a,r}))}.$$

Proof. Having established the behaviour of the Dirichlet Laplacian and the corresponding semigroup along the fractional power scale (4.8), the proof of (i) is an application of the factorization method and can be found in [Brz97, Corollary 3.5] whereas the point (ii) follows from the Burkholder-Davis-Gundy inequality and regularization properties (4.9) of the semigroup. □

4.4 First step in the regularity problem

In this section, we show the first step towards regularity of the weak solution u to (4.1). We consider the following auxiliary problem

$$\begin{cases} \mathrm{d}z = \Delta z \,\mathrm{d}t + H(u)\,\mathrm{d}W_t & \text{in } D_T,\\ z = 0 & \text{in } S_T,\\ z(0) = 0 & \text{in } D. \end{cases} \tag{4.10}$$

It can be rewritten in the abstract form

$$\begin{cases} \mathrm{d}z = \Delta_D z \,\mathrm{d}t + H(u)\,\mathrm{d}W_t,\\ z(0) = 0 \end{cases}$$

and hence its solution is given by the stochastic convolution

$$z(t) = \int_0^t S(t-s)H(u_s)\,\mathrm{d}W_s, \quad t \in [0,T]. \tag{4.11}$$

Next, we define the process $y := u - z$. It follows immediately that y solves the following linear parabolic PDE with random coefficients

$$\begin{cases} \partial_t y = \operatorname{div}(A(u)\nabla y) + \operatorname{div}(B(u)) + F(u) + \operatorname{div}((A(u)-\mathrm{I})\nabla z) & \text{in } D_T,\\ y = 0 & \text{in } S_T,\\ y(0) = u_0 & \text{in } D. \end{cases} \tag{4.12}$$

This way, we have split u into two parts, i.e. y and z, that are much more convenient in order to study regularity. Our first regularity result reads as follows.

Proposition 4.4.1. *Let $u_0 \in L^m(\Omega; C^\iota(\overline{D}))$ for some $\iota > 0$ and all $m \in [2,\infty)$. We assume that ($\mathrm{H}_{1,2}$) is fulfilled. Then, there exists $\eta > 0$ such that, for all $m \in [2,\infty)$, the weak solution u to (4.1) belongs to $L^m(\Omega; C^\eta(\overline{D_T}))$.*

Proof. Step 1: Regularity of z. According to the hypothesis, the weak solution u to (4.1) belongs to $L^2(\Omega; L^2(0,T; H_0^{1,2}))$ so that, thanks to the hypothesis ($\mathrm{H}_{1,2}$), we have that $H(u)$ belongs to $L^2(\Omega; L^2(0,T; \gamma(K, H_0^{1,2})))$. As a result, with Proposition 4.3.1 - (ii), the bound ($\mathrm{H}_{1,2}$) and the embedding $H_0^{a,r} \subset H^{a,r}$, we have that for any $a \in (0,2)$, $z \in L^2(\Omega; L^2(0,T; H^{a,2}))$ with

$$\mathbb{E}\|z\|^2_{L^2(0,T;H^{a,2})} \leq C\Big(1 + \mathbb{E}\|u\|^2_{L^2(0,T;H_0^{1,2})}\Big).$$

Besides, since for all $p \in [2,\infty)$, the weak solution u to (4.1) belongs to $L^p(\Omega; L^p(0,T;L^p))$, we obtain, with the hypothesis ($\mathrm{H}_{0,p}$) (see Remark 4.2.1), that $H(u)$ belongs to the space $L^p(\Omega; L^p(0,T;\gamma(K,L^p)))$. As a consequence, with Proposition 4.3.1 - (ii), the bound ($\mathrm{H}_{0,p}$) and the embedding $H_0^{a,r} \subset H^{a,r}$, we have that for any $b \in (0,1)$, $z \in L^p(\Omega; L^p(0,T;H^{b,p}))$ with

$$\mathbb{E}\|z\|^p_{L^p(0,T;H^{b,p})} \leq C\Big(1 + \mathbb{E}\|u\|^p_{L^p(0,T;L^p)}\Big).$$

We have proved that for any $a \in (0,2)$ and $b \in (0,1)$, we have $z \in L^2(\Omega; L^2(0,T;H^{a,2}))$ and $z \in L^p(\Omega; L^p(0,T;H^{b,p}))$. We can now interpolate to obtain that (see [Ama00])

$$z \in L^r(\Omega; L^r(0,T;H^{c,r})),$$

where, for $\theta \in (0,1)$,

$$\begin{cases} \dfrac{1}{r} = \dfrac{\theta}{2} + \dfrac{1-\theta}{p}, \\ c = a\theta + b(1-\theta), \end{cases}$$

with the bound

$$\mathbb{E}\|z\|^r_{L^r(0,T;H^{c,r})} \leq \Big(\mathbb{E}\|z\|^2_{L^2(0,T;H_0^{a,2})}\Big)^{r\theta/2}\Big(\mathbb{E}\|z\|^p_{L^p(0,T;H_0^{b,p})}\Big)^{r(1-\theta)/p} < \infty. \tag{4.13}$$

Note that by choosing $\theta \in (0,1)$ and $p \in [2,\infty)$ appropriately, r can be arbitrary in $[2,\infty)$. Furthermore, when $\theta \in (0,1)$ is fixed, it is always possible to take $(a,b) \in (0,2)\times(0,1)$ such that $c > 1$. As a result, for all $r \in [2,\infty)$, there exists $c_r > 1$ such that

$$z \in L^r(\Omega; L^r(0,T;H^{c_r,r})).$$

This gives, for all $r \in [2,\infty)$,

$$\nabla z \in L^r(\Omega; L^r(0,T;L^r)),$$

and, due to the boundedness of the mapping A,

$$(A(u) - \mathrm{I})\nabla z \in L^r(\Omega; L^r(0,T;L^r)),$$

with, thanks to (4.13),

$$\mathbb{E}\|(A(u)-\mathrm{I})\nabla z\|^r_{L^r(0,T;L^r)} \leq C\mathbb{E}\|z\|^r_{L^r(0,T;H^{c,r})} < \infty, \tag{4.14}$$

where $C > 0$ depends on $\|A\|_\infty$. Note that, thanks to the linear growth property of the coefficients B and F, we obviously have, for all $r \in [2,\infty)$,

$$\mathbb{E}\|B(u)\|^r_{L^r(0,T;L^r)} + \mathbb{E}\|F(u)\|^r_{L^r(0,T;L^r)} \leq C(1 + \mathbb{E}\|u\|^r_{L^r(0,T;L^r)}) < \infty. \tag{4.15}$$

Step 2: Regularity of y. From now on, we consider that $r \geq r_0$ where r_0 is fixed such that for all $r \geq r_0$,

$$\frac{2+N}{r} < \frac{1}{2}. \tag{4.16}$$

Concerning the regularity of y, we intend to apply the regularization result given in the second part of [LSU68, Theorem 10.1, Ch. III] to deduce that y has in fact α-Hölder continuous paths in $\overline{D_T}$ for some $\alpha > 0$. Precisely, we set $\Gamma' = S_T$ and

$$a_i = b_i = a = 0, \quad f_i = B_i(u) + ((A(u)-\mathrm{I})\nabla z)_i, \quad f = F(u),$$

and observe that the conditions (1.2), (7.1) and (7.2) in [LSU68, Ch. III] are satisfied thanks to (4.4) and the bounds (4.14)–(4.15) coupled with (4.16). Note also that [LSU68, Theorem

7.1, Ch. III] applies and gives $y \in L^\infty(D_T)$ a.s. Thus we can now employ the second part of [LSU68, Theorem 10.1, Ch. III] which yields $y \in C^{\alpha/2,\alpha}(\overline{D_T})$ where $\alpha \in (0,\iota]$ is determined by N, ν, μ and r_0. In particular, we point out that α is deterministic. Furthermore, studying the proofs of [LSU68, Theorem 7.1, Theorem 10.1, Ch. III] in detail, we have the following bound

$$\begin{aligned} \|y\|_{C^{\alpha/2,\alpha}(\overline{D_T})} \le C(1+\|u_0\|_{C^\iota(\overline{D})})\times \\ (1+\|B(u)+(A(u)-\mathrm{I})\nabla z\|^{2N+1}_{L^r(0,T;L^r)}+\|F(u)\|^{2N+1}_{L^r(0,T;L^r)}) \end{aligned} \tag{4.17}$$

for some deterministic constant $C>0$ depending on the constants of the problem and on r_0. Therefore, if $2(2N+1)m<r$, we obtain due to (4.14)–(4.15), the hypothesis made on u_0 and the Cauchy-Schwarz inequality

$$\begin{aligned} \mathbb{E}\|y\|^m_{C^{\alpha/2,\alpha}} \le C\,(1+\mathbb{E}\|u_0\|^{2m}_{C^\iota(\overline{D})})\times \\ (1+\mathbb{E}\|B(u)+(A(u)-\mathrm{I})\nabla z\|^r_{L^r(0,T;L^r)}+\mathbb{E}\|F(u)\|^r_{L^r(0,T;L^r)})<\infty. \end{aligned} \tag{4.18}$$

Since r is arbitrary in $[r_0,\infty)$, the result holds for all $m\in[2,\infty)$.

Step 3: Hölder regularity of z. In order to complete the proof it is necessary to improve the regularity of z. Recall that for all $m\in[2,\infty)$, the solution u to (4.1) belongs to $L^m(\Omega;L^m(0,T;L^m))$ and that $H(u)$ belongs to $L^m(\Omega;L^m(0,T;\gamma(K,L^m)))$. We now apply Proposition 4.3.1 - (i) and ($\mathrm{H}_{0,m}$) to obtain, since $H_0^{a,r}\subset H^{a,r}$, that for $m\in(2,\infty)$, $\delta\in(0,1-2/m)$ and $\gamma\in[0,1/2-1/m-\delta/2)$, $z\in L^m(\Omega;C^\gamma([0,T];H^{\delta,m}))$ with

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];H^{\delta,m})}\le C\left(1+\mathbb{E}\|u\|^m_{L^m(0,T;L^m)}\right).$$

Note that we can choose δ and γ to be independent of m. For instance, let us suppose in the sequel that $m\ge 3$; then $\delta=1/6$ and $\gamma=1/12$ satisfies the conditions above for any $m\ge 3$. Furthermore, from now on, we also suppose that $m\ge 7N:=m_0$. This implies that $m\ge 3$ and $\delta m>N$, so that the following Sobolev embedding holds true

$$H^{\delta,m}\hookrightarrow C^\lambda,\quad \lambda:=\delta-N/m_0.$$

We conclude that, for all $m\ge m_0$,

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];C^\lambda)}\le C\left(1+\mathbb{E}\|u\|^m_{L^m(0,T;L^m)}\right)<\infty. \tag{4.19}$$

Note that for $m\in[2,m_0)$, we can write with the Hölder inequality

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];C^\lambda)}\le\left(\mathbb{E}\|z\|^{m_0}_{C^\gamma([0,T];C^\lambda)}\right)^{m/m_0}<\infty. \tag{4.20}$$

Step 4: Conclusion. Finally, we set $\eta:=\min(\alpha/2,\gamma,\lambda)>0$ and we recall that $u=y+z$ so that the conclusion follows from (4.18), (4.19), (4.20) due to the fact that $C^\eta([0,T];C^\eta(\overline{D}))\subset C^\eta([0,T]\times\overline{D})$. □

4.5 Increasing the regularity

In this final section, we complete the proof of Theorem 4.2.3. Having Proposition 4.4.1 in hand, it is quite straightforward to significantly increase the regularity of u using the same auxiliary problems (4.10) and (4.12) together with the Schauder theory for deterministic parabolic PDEs with Hölder continuous coefficients.

Proposition 4.5.1. *Let $u_0 \in L^m(\Omega; C^{1+\iota}(\overline{D}))$ for some $\iota > 0$ and all $m \in [2,\infty)$. Suppose that A, $B \in C_b^1$. If ($H_{a,r}$) is fulfilled for all $a < 2$ and $r \in [2,\infty)$, then for all $\lambda \in (0,1/2)$ there exists $\beta > 0$ such that for all $m \in [2,\infty)$ the weak solution u to (4.1) belongs to $L^m(\Omega; C^{\lambda,1+\beta}(\overline{D_T}))$.*

Proof. The proof is divided in two parts: we first increase the regularity in space and then in time.

Spatial regularity. Step 1: Regularity of z. First, we improve the regularity of z that was defined in (4.11). According to Proposition 4.4.1, there exists $\eta > 0$ such that for all $m \in [2,\infty)$, $u \in L^m(\Omega; C^\eta(\overline{D_T}))$. In particular, since u satisfies Dirichlet boundary conditions, this implies that $u \in L^m(\Omega; L^m(0,T; H_0^{\kappa,m}))$ provided $\kappa < \eta$. With ($H_{\kappa,m}$), we deduce that $H(u) \in L^m(\Omega; L^m(0,T; \gamma(K, H_0^{\kappa,m})))$. An application of Proposition 4.3.1 and the embedding $H_0^{a,r} \subset H^{a,r}$ yields that $z \in L^m(\Omega; C^\gamma([0,T]; H^{\kappa+\delta,m}))$ for every $m \in (2,\infty)$ with

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];H^{\kappa+\delta,m})} \le C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;H_0^{\kappa,m})}\right),$$

where $\delta \in (0, 1-2/m)$ and $\gamma \in [0, 1/2 - 1/m - \delta/2)$. In the sequel, we assume that $m \ge (N+4)/\kappa := m_0$. Then $\delta = 1 - 3/m_0$ and $\gamma = 1/(4m_0)$ satisfies the conditions above uniformly in $m \ge m_0$. Furthermore, observe that $(\kappa+\delta)m > \kappa m \ge \kappa m_0 \ge N$ so that the following Sobolev embedding holds true

$$H^{\kappa+\delta,m} \hookrightarrow C^\sigma, \quad \sigma = \kappa + \delta - N/m_0.$$

Besides, by definition of m_0, $\sigma = \kappa + 1 - (N+3)/m_0 > 1$. Finally, we deduce that for all $m \ge m_0$, $z \in L^m(\Omega; C^\gamma([0,T]; C^\sigma(\overline{D})))$ with

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];C^\sigma)} \le C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;H_0^{\kappa,m})}\right). \tag{4.21}$$

Step 2: Regularity of y. Next, we improve the regularity of y that is given by (4.12). Namely, we intend to make use of the classical Schauder theory for deterministic parabolic PDEs, see e.g. [Lie96, Theorem 6.48]. As a consequence of Proposition 4.4.1 and (4.21), we obtain due to the assumptions upon A, B and F that, for all $m \in [2,\infty)$

$$\begin{aligned} A(u) &\in L^m(\Omega; C^{\alpha/2,\alpha}(\overline{D_T})), \\ B(u) + (A(u) - \mathrm{I})\nabla z &\in L^m(\Omega; C^{\alpha/2,\alpha}(\overline{D_T})), \\ F(u) &\in L^m(\Omega; L^m(0,T; L^m)), \\ u_0 &\in L^m(\Omega; C^{1+\alpha}(\overline{D})), \end{aligned}$$

where $\alpha := \min(\iota, \eta, \sigma - 1, \gamma) > 0$. Thus the hypotheses of [Lie96, Theorem 4.8, Theorem 6.48] are fulfilled and we obtain the following (pathwise) estimate

$$\|y\|_{C^{(1+\alpha)/2,1+\alpha}} \le C\Big(\|u_0\|_{C^{1+\alpha}} + \|B(u) + (A(u) - \mathrm{I})\nabla z\|_{C^{\alpha/2,\alpha}} + \|F(u)\|_{L^r(0,T;L^r)}\Big),$$

where $r \in [2,\infty)$ is large enough. We conclude that, for all $m \in [2,\infty)$,

$$y \in L^m(\Omega; C^{(1+\alpha)/2,1+\alpha}(\overline{D_T})) \tag{4.22}$$

which together with (4.21) yields $u \in L^m(\Omega; C^{\gamma,1+\alpha}(\overline{D_T}))$.

Time regularity. Having in hand the improved regularity of u, we consider again the stochastic convolution z, repeat the approach from the first step of this proof and obtain due to Proposition 4.4.1 (with $\delta = 0$) and ($H_{1+\kappa,m}$)

$$\mathbb{E}\|z\|^m_{C^\lambda([0,T];H^{1+\kappa,m})} \le C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;H_0^{1+\kappa,m})} + \mathbb{E}\|u\|^{(1+\kappa)m}_{L^{(1+\kappa)m}(0,T;H_0^{1,(1+\kappa)m})}\right) < \infty, \tag{4.23}$$

where $\kappa < \alpha$ and $\lambda \in (0, 1/2 - 1/m)$. Therefore for any $\lambda \in (0, 1/2)$ there exists m_0 large enough so that (4.23) holds true for any $m \geq m_0$ and the Sobolev embedding then implies that $z \in L^m(\Omega; C^\lambda([0,T]; C^{1+\beta}(\overline{D})))$ for $\beta < \kappa$. Since we already have (4.22) the proof is complete. □

Due to the properties of the stochastic convolution it is not possible to increase the time regularity of u. Nevertheless, it is possible to continue in the same manner as before and increase its space regularity.

Proposition 4.5.2. *Let $u_0 \in L^m(\Omega; C^{2+\iota}(\overline{D}))$ for some $\iota > 0$ and all $m \in [2, \infty)$. Suppose that A, $B \in C_b^2$ and that $F \in C_b^1$. If ($\mathrm{H}_{a,r}$) is fulfilled for all $a < 3$ and $r \in [2, \infty)$, then for all $\lambda \in (0, 1/2)$ there exists $\beta > 0$ such that for all $m \in [2, \infty)$ the weak solution u to (4.1) belongs to $L^m(\Omega; C^{\lambda, 2+\beta}(\overline{D_T}))$.*

First, we give the proof of Proposition 4.5.2 in the periodic case where $D = \mathbb{T}^N$. We point out that in this simpler setting the proof can exactly be reproduced in order to establish Proposition 4.5.3 below which achieves higher regularity of u. Then, we give the proof of Proposition 4.5.2 in the general case of a bounded domain D of $\mathbb{R}^N$ with smooth boundary. Unlike the periodic setting, this proof does not directly extend to the proof of Proposition 4.5.3. Thus, the technique requires some improvements which are detailed in the proof of Proposition 4.5.3.

Proof. The periodic case. From now on, let $D = \mathbb{T}^N$. The proof follows similar ideas as in Proposition 4.5.1 only with some modifications in the second step.

Spatial regularity. Step 1: Regularity of z. As in Proposition 4.5.1, we first increase the regularity of z. With Proposition 4.5.1, for any $\lambda \in (0, 1/2)$, there exists $\beta > 0$ such that for all $m \in [2, \infty)$, $u \in L^m(\Omega; C^{\lambda, 1+\beta}(\overline{D_T}))$. Then we deduce

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T]; H^{1+\kappa+\delta,m})} \leq C \left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;H^{1+\kappa,m})} + \mathbb{E}\|u\|^{(1+\kappa)m}_{L^{(1+\kappa)m}(0,T;H^{1,(1+\kappa)m})}\right) < \infty,$$

where $\kappa < \beta$, $\delta \in (0, 1 - 2/m)$ and $\gamma \in [0, 1/2 - 1/m - \delta/2)$. By a similar reasoning as above we obtain due to the Sobolev embedding that $z \in L^m(\Omega; C^\gamma([0,T]; C^\sigma(\overline{D})))$ where $m \in [2, \infty)$ and $\sigma > 2$.

Step 2: Regularity of y. In order to improve the space regularity of y we derive the equation that is satisfied by ∂y where the operator ∂ can stand for any partial derivative with respect to space variable x: $\partial = \partial_{x_i}$, $i = 1, \dots, N$. We obtain

$$\begin{cases} \partial_t(\partial y) = \operatorname{div}(A(u)\nabla(\partial y)) + \operatorname{div}(\partial A(u)\nabla u) + \operatorname{div}(\partial B(u)) \\ \qquad\qquad + \partial F(u) + \operatorname{div}((A(u) - \mathrm{I})\nabla(\partial z)) \qquad \text{in } D_T, \\ \partial y(0) = \partial u_0. \end{cases}$$

The above is again a (pathwise) linear parabolic PDE hence we need to show that its coefficients satisfy the hypotheses of [Lie96, Theorem 6.48]. In particular, according to what was already proved, we have

$$\begin{aligned} A(u) &\in L^m(\Omega; C^{\alpha/2,\alpha}(\overline{D_T})), \\ \partial A(u)\nabla u + \partial B(u) + (A(u) - \mathrm{I})\nabla(\partial z) &\in L^m(\Omega; C^{\alpha/2,\alpha}(\overline{D_T})), \\ \partial F(u) &\in L^m(\Omega; L^m(0,T; L^m)), \\ \partial u_0 &\in L^m(\Omega; C^{1+\alpha}(\overline{D})), \end{aligned}$$

for some $\alpha \in (0, \sigma - 2]$ and all $m \in [2, \infty)$ provided A, $B \in C_b^1$, $F \in C_b^1$. Therefore [Lie96, Theorem 6.48] applies and we deduce

$$\partial y \in L^m(\Omega; C^{(1+\alpha)/2, 1+\alpha}(\overline{D_T})).$$

As a consequence, we see that

$$y \in L^m(\Omega; C^{(1+\alpha)/2,2+\alpha}(\overline{D_T}))$$

hence

$$u \in L^m(\Omega; C^{\gamma,2+\alpha}(\overline{D_T})).$$

Time regularity. Finally, we improve the time regularity of u by considering the stochastic convolution again as in Proposition 4.5.1. We obtain that for any $\lambda \in (0,1/2)$ there exists m_0 large enough so that

$$\mathbb{E}\|z\|^m_{C^\lambda([0,T];H^{2+\kappa,m})} \leq C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;H_0^{2+\kappa,m})} + \mathbb{E}\|u\|^{(2+\kappa)m}_{L^{(2+\kappa)m}(0,T;H_0^{1,(2+\kappa)m})}\right),$$

holds true for any $m \geq m_0$ and the Sobolev embedding then implies that z belongs to $L^m(\Omega; C^\lambda([0,T]; C^{2+\beta}(\overline{D})))$ for $\beta < \kappa$ which completes the proof. □

Let us now prove Proposition 4.5.2 in the general case. In the sequel D is again a bounded domain in $\mathbb{R}^N$ with smooth boundary.

Proof. The general case. The proof follows the same scheme as in the periodic case except for the *Step 2: Regularity of y*. Let us now detail the proof of this step.

Step 2: Regularity of y. In order to improve the space regularity of y we make use of [LSU68, Theorem 5.2, Ch. IV]. In particular, we set

$$a_{ij} = A_{ij}(u), \quad a_j = \nabla u \cdot A'_{\cdot j}(u), \quad a = 0, \quad f = \operatorname{div}\big(B(u) + (A(u) - \mathrm{I})\nabla z\big) + F(u).$$

According to what was already proved, we have

$$\begin{aligned} a_{ij},\, a_j,\, a,\, f &\in L^m(\Omega; C^{\alpha/2,\alpha}(\overline{D_T})), \\ u_0 &\in L^m(\Omega; C^{2+\alpha}(\overline{D})), \end{aligned} \tag{4.24}$$

for some $\alpha \in (0, \sigma - 2]$ and all $m \in [2,\infty)$ provided A, $B \in C_b^2$, $F \in C_b^1$. Therefore [LSU68, Theorem 5.2, Ch. IV] applies and we deduce

$$y \in L^m(\Omega; C^{1+\alpha/2,2+\alpha}(\overline{D_T})),$$

hence

$$u \in L^m(\Omega; C^{\gamma,2+\alpha}(\overline{D_T})).$$

This completes the proof. □

Finally, we achieve even higher regularity of u provided the coefficients are smooth enough. We obtain the following result.

Proposition 4.5.3. *Let $k \in \{3,4,\dots\}$. Let $u_0 \in L^m(\Omega; C^{k+\iota}(\overline{D}))$ for some $\iota > 0$ and all $m \in [2,\infty)$. Suppose that A, $B \in C_b^k$ and $F \in C_b^{k-1}$. If ($\mathrm{H}_{a,r}$) is fulfilled for all $a < k+1$ and $r \in [2,\infty)$, then for all $\lambda \in (0,1/2)$ there exists $\beta > 0$ such that for all $m \in [2,\infty)$ the weak solution u to (4.1) belongs to $L^m(\Omega; C^{\lambda,k+\beta}(\overline{D_T}))$.*

As previously mentioned, the proof of Proposition 4.5.2 in the periodic case can exactly be reproduced here so that the result of Proposition 4.5.3 is proved in the setting of periodic boundary conditions.
Nevertheless, the proof of Proposition 4.5.2 made in the general case does not apply here any more. Indeed, the problem arises from the fact that the regularization result given by [LSU68, Theorem 5.2, Ch. IV] is stated under the condition that the regularity of the coefficients and

the source term is in the parabolic scaling, that is, the space regularity is exactly twice the time regularity. In our case, since the time regularity is limited to $\frac{1}{2}^-$, we are limited to 1^- for the space regularity of the coefficients and the source term if we want to fit in the setting of [LSU68, Theorem 5.2, Ch. IV]. As a consequence, we wouldn't obtain a better space regularity of our solution u than 3^-. To handle this issue, we prove an alternative version of the result [LSU68, Theorem 5.2, Ch. IV] where we avoid the hypothesis of the parabolic regularity of the coefficients and initial data. The result is the following.

Theorem 4.5.4. *Let $\mathscr{L}$ denote the linear parabolic differential operator given by [LSU68, (5.1), Ch. IV]*

$$\mathscr{L}u = \partial_t u - \sum_{i,j=1}^{N} a_{ij}\partial^2_{x_i,x_j} u + \sum_{i=1}^{N} a_i \partial_{x_i} u + au,$$

and u the solution to the null Dirichlet problem [LSU68, (5.3), Ch. IV]

$$\begin{cases} \mathscr{L}u = f & \text{in } D_T, \\ u = 0 & \text{in } S_T, \\ u(0) = u_0 & \text{in } D. \end{cases}$$

Let α, $\beta \geq 0$ such that $2\alpha \leq \beta$. Assume that the coefficients of $\mathscr{L}$ and the source f belong to $C^{\alpha,\beta}(\overline{D_T})$ and that u_0 belongs to $C^{\beta}(\overline{D})$. Then, for all $\varepsilon > 0$, u is $C^{\alpha+1-\varepsilon,\beta+2-\varepsilon}(\overline{D_T})$ with

$$\|u\|_{C^{\alpha+1-\varepsilon,\beta+2-\varepsilon}} \leq C(\|f\|_{C^{\alpha,\beta}} + \|u_0\|_{C^{\beta}}).$$

Proof of Proposition 4.5.3. For the time being, let us suppose that this result holds true. The proof of Proposition 4.5.3 is then exactly the same as in Proposition 4.5.2 in the general case except that (4.24) is replaced by

$$\begin{aligned} a_{ij},\, a_j,\, a,\, f &\in L^m(\Omega; C^{\gamma,(k-2)+\alpha}(\overline{D_T})), \\ u_0 &\in L^m(\Omega; C^{k+\alpha}(\overline{D})), \end{aligned} \tag{4.25}$$

for any $\gamma < 1/2$ and some $\alpha \in (0, \sigma - k]$ where $\sigma > k$ and that we then apply Theorem 4.5.4 instead of [LSU68, Theorem 5.2, Ch. IV]. □

Thus it only remains to prove Theorem 4.5.4.

Proof of Theorem 4.5.4. The proof of [LSU68, Theorem 5.2, Ch. IV] is divided into two steps. The first one is to prove the desired result on the whole space and on the half-space in the case where a_{ij} are constant coefficients and $a_i = a = 0$; the results are the bounds (6.4) and (6.5) in [LSU68, Theorem 6.1, Ch. IV] (the bound (6.6) deals with the case of Neumann boundary conditions). The second one is to freeze the coefficients, to use a localization technique and to handle the lower order terms of $\mathscr{L}$ by some compactness argument and finally to prove [LSU68, Theorem 5.2, Ch. IV] using (6.4) and (6.5) of [LSU68, Theorem 6.1, Ch. IV]; this second step is achieved in [LSU68, Section 7, Ch. IV]. As a result, we only need to prove that the bounds (6.4) and (6.5) of [LSU68, Theorem 6.1, Ch. IV] hold true whenever the regularity of the source term is not in the parabolic scaling. Furthermore, as explained in the proof of [LSU68, Theorem 6.1, Ch. IV], it is sufficient to deal with the case $a_{ij} = \delta_{ij}$.
To sum up, let $f \in C^{\alpha,\beta}([0,T] \times \mathbb{R}^N)$, $g \in C^{\alpha,\beta}([0,T] \times \overline{\mathbb{R}^N_+})$, and w, v the solutions of

$$\begin{cases} \partial_t w - \Delta w = f & \text{in } (0,T) \times \mathbb{R}^N, \\ w(0) = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} \partial_t v - \Delta v = g & \text{in } (0,T) \times \mathbb{R}^N_+, \\ v|_{x_N=0} = 0, \\ v(0) = 0, \end{cases}$$

where $\mathbb{R}^N_+$ denotes the half-space $\{(x_1,...,x_N)\in\mathbb{R}^N,\ x_N>0\}$, it remains to prove that, for all $\varepsilon>0$,

$$\|w\|_{C^{\alpha+1-\varepsilon,\beta+2-\varepsilon}([0,T]\times\mathbb{R}^N)} \leq C\|f\|_{C^{\alpha,\beta}([0,T]\times\mathbb{R}^N)}, \tag{4.26}$$

$$\|v\|_{C^{\alpha+1-\varepsilon,\beta+2-\varepsilon}([0,T]\times\overline{\mathbb{R}^N_+})} \leq C\|g\|_{C^{\alpha,\beta}([0,T]\times\overline{\mathbb{R}^N_+})}. \tag{4.27}$$

The bound (4.26) can be justified exactly as in the case of the parabolic scaling, see the proof of [LSU68, (2.1), Ch. IV]. It gives the bound (4.26) where we can take $\varepsilon=0$, that is

$$\|w\|_{C^{\alpha+1,\beta+2}([0,T]\times\mathbb{R}^N)} \leq C\|f\|_{C^{\alpha,\beta}([0,T]\times\mathbb{R}^N)}.$$

Unfortunately, the proof made in [LSU68] in the case of the half-space does not work any more when we are not in the parabolic scaling. So, let us define $(S(t))_{t\geq0}$ the semigroup of the Dirichlet Laplacian on the half-space $\mathbb{R}^N_+$. Precisely, $\psi=S(t)h$ satisfies

$$\begin{cases}\partial_t\psi-\Delta\psi=0 \ \text{ in } (0,\infty)\times\mathbb{R}^N_+,\\ \psi|_{x_N=0}=0,\\ \psi(0)=h.\end{cases} \tag{P$_h^+$}$$

It is classical that $S(1)$ maps $C^\gamma(\overline{\mathbb{R}^N_+})$ to $C^\infty(\overline{\mathbb{R}^N_+})$ so that we can deduce the following bound, for any $h\in C^\gamma(\overline{\mathbb{R}^N_+})$ and $\delta>0$,

$$\|S(1)h\|_{C^{\gamma+\delta}(\overline{\mathbb{R}^N_+})} \leq C\|h\|_{C^\gamma(\overline{\mathbb{R}^N_+})}. \tag{4.28}$$

Now, let $t>0$ and $h\in C^\gamma(\overline{\mathbb{R}^N_+})$. We define $\widetilde{h}(x):=h(xt^{\frac12})$ and consider the solution ψ to the problem (P$_{\widetilde{h}}^+$). Finally, we set $\varphi(s,x):=\psi(st^{-1},xt^{-\frac12})$ which is well defined in the half-space and satisfies (P$_h^+$). As a result, $\varphi(s,x)=S(s)h$. Thus observe that we have $S(t)h=\varphi(t,x)=\psi(1,xt^{-\frac12})=S(1)\widetilde{h}(xt^{-\frac12})$ so that we deduce, with (4.28),

$$\|S(t)h\|_{C^{\gamma+\delta}(\overline{\mathbb{R}^N_+})}=\|S(1)\widetilde{h}(\cdot\, t^{-\frac12})\|_{C^{\gamma+\delta}(\overline{\mathbb{R}^N_+})} \leq Ct^{-(\gamma+\delta)/2}\|\widetilde{h}\|_{C^\gamma(\overline{\mathbb{R}^N_+})}.$$

As a result, since $\|\widetilde{h}\|_{C^\gamma(\overline{\mathbb{R}^N_+})}\leq t^{\gamma/2}\|h\|_{C^\gamma(\overline{\mathbb{R}^N_+})}$, we are led to

$$\|S(t)h\|_{C^{\gamma+\delta}(\overline{\mathbb{R}^N_+})} \leq Ct^{-\delta/2}\|h\|_{C^\gamma(\overline{\mathbb{R}^N_+})}. \tag{4.29}$$

Finally, let us conclude the proof of the bound (4.27). The solution v is given by

$$v(t)=\int_0^t S(t-s)g(s)\,\mathrm{d}s,$$

so that with (4.29) we deduce

$$\|v\|_{C^{0,\gamma+\delta}([0,T]\times\overline{\mathbb{R}^N_+})} \leq C\|g\|_{C^{0,\gamma}([0,T]\times\overline{\mathbb{R}^N_+})}, \tag{4.30}$$

provided $\delta<2$. Besides, thanks to the result [LSU68, (6.5), Ch. IV] in the parabolic scaling, we have the bound

$$\|v\|_{C^{\sigma/2+1,\sigma+2}([0,T]\times\overline{\mathbb{R}^N_+})} \leq C\|g\|_{C^{\sigma/2,\sigma}([0,T]\times\overline{\mathbb{R}^N_+})}. \tag{4.31}$$

Since the bounds (4.30) and (4.31) holds true for any γ, $\sigma\geq0$ and $\delta<2$, we deduce, by interpolation, that for any $\varepsilon>0$,

$$\|v\|_{C^{\alpha+1-\varepsilon,\beta+2-\varepsilon}([0,T]\times\overline{\mathbb{R}^N_+})} \leq C\|g\|_{C^{\alpha,\beta}([0,T]\times\overline{\mathbb{R}^N_+})},$$

which concludes the proof. □

Chapter 5

Invariant measures for a stochastic Fokker-Planck equation

Abstract: We study a Fokker-Plank equation perturbed by a random force and prove, if the contribution of the noise is small enough, existence and uniqueness of the solutions to the problem. We also derive an hypocoercive estimate on the solutions. Finally, using the hypocoercivity, we can prove existence and uniqueness of the invariant measures of the problem.

Keywords: Stochastic partial differential equations, Fokker-Planck, cylindrical Wiener process, hypocoercivity, invariant measures.

The results of this chapter are a joint work with L. M. Rodrigues and J. Vovelle.

5.1 Introduction

In this chapter, we are interested in studying the invariant measures of the following stochastic Fokker-Planck equation

$$\mathrm{d}f \;+\; v\cdot\nabla_x f\;\mathrm{d}t \;+\; \lambda\nabla_v f\odot\mathrm{d}W_t \;=\; \mathcal{Q}(f)\;\mathrm{d}t. \tag{5.1}$$

The unknown f depends on the variables $t\in[0,\infty)$, $x\in\mathbb{T}^N$ and $v\in\mathbb{R}^N$. The operator $\mathcal{Q}$ is the Fokker-Planck operator whose expression is given by

$$\mathcal{Q}(f) \;=\; \Delta_v f \;+\; \mathrm{div}_v(vf).$$

Let us introduce the noise term. We take Γ a self-adjoint and non-negative operator on $L^2(\mathbb{T}^N;\mathbb{R}^N)$ with $\mathrm{Tr}(\Gamma)<\infty$. Let $(H_j)_{j\in\mathbb{N}}$ a complete orthonormal system in $L^2(\mathbb{T}^N;\mathbb{R}^N)$ of eigenvectors of Γ with associated non-negative eigenvalues $(\gamma_j)_{j\in\mathbb{N}}$:

$$\Gamma H_j=\gamma_j H_j,\quad j\in\mathbb{N}.$$

The random perturbation $\mathrm{d}W_t$ is a Γ–Wiener process on $L^2(\mathbb{T}^N;\mathbb{R}^N)$, see for instance [DPZ08, Section 4.1]. It can be written as

$$\mathrm{d}W_t(x) \;=\; \sum_j \Gamma^{\frac12}H_j(x)\;\mathrm{d}\beta_j(t)=\sum_j \gamma_j^{\frac12}H_j(x)\;\mathrm{d}\beta_j(t)$$

where the $(\beta_j)_{j\in\mathbb{N}}$ are real independent Brownian motions. In what follows, we set $F_j:=\Gamma^{\frac12}H_j$ and write the noise under the form

$$\mathrm{d}W_t(x) \;=\; \sum_j F_j(x)\;\mathrm{d}\beta_j(t).$$

The notation $\odot$ emphasizes the scalar product in $\mathbb{R}^N$ and the fact that we consider the stochastic term in the Stratonovich sense. The parameter $\lambda>0$ represents the size of the random perturbation. Concerning the coefficients $(F_j)_{j\in\mathbb{N}}$ of the noise, we suppose in the sequel that the following condition holds

$$\sum_j \|F_j\|_\infty^2+\|\nabla_x F_j\|_\infty^2\le 1. \tag{5.2}$$

Note that the operator $\mathcal{Q}$ is self-adjoint in the weighted space $L^2(\mathbb{R}^N,\mathcal{M}^{-1}\mathrm{d}v)$ where we have introduced the Maxwellian distribution $\mathcal{M}$ on $\mathbb{R}^N$, which is defined by

$$\mathcal{M}(v)=(2\pi)^{-N/2}e^{-|v|^2/2},\quad v\in\mathbb{R}^N.$$

For this reason, in what follows, we do not work exactly on the variable f and we set $f=\mathcal{M}^{\frac12}g$. By doing so, g satisfies the problem

$$\left\{\begin{array}{l} \mathrm{d}g \;+\; v\cdot\nabla_x g\;\mathrm{d}t \;+\; \lambda\Big(\nabla_v-\dfrac{v}{2}\Big)g\;\odot\mathrm{d}W_t \;=\; Lg\;\mathrm{d}t,\\ g(0)=g_{\mathrm{in}}. \end{array}\right. \tag{5.3}$$

with

$$Lg \;=\; \Delta_v g \;+\; \left(\frac{N}{2}-\frac{|v|^2}{4}\right)g$$

being a self-adjoint operator on $L^2(\mathbb{R}^N)$.

From a physical point of view, this kind of equation can describe the evolution of the distribution function $g(t,x,v)$ of a cloud of particles which, at a time t, are at position x and have velocity v. The transport term $v \cdot \nabla_x g$ corresponds to the free flow of particles while the Fokker-Planck operator L models the interactions between the particles and with the surrounding medium. The noisy term $\lambda\,(\nabla_v - v/2)\, g \odot \mathrm{d}W_t$ describes the effect of a random force acting on the particles.

The aim of this chapter is twofold. First of all, we want to study existence and uniqueness for the problem (5.3). Then, we investigate existence and uniqueness of an invariant measure for this problem.

We obtain the existence of the solutions to Equation (5.3) by a standard Galerkin scheme. Precisely, we project Equation (5.3) on some finite dimensional space. By doing so, we construct a sequence $(g_m)_m$ of approximate solutions to our problem. Then, one has to derive energy estimates on the sequence $(g_m)_m$ in order to pass to the limit in the approximate problem. Note that, to ensure existence, we need that the coefficient λ in front of the noise is small enough so that the random perturbation does not affect too much the dissipation of the operator L (see for instance [MN06, Section 3.2]). Uniqueness is proved with the bounds derived on the approximate solutions and that remain valid for the solution g by passing to the limit.

In the sequel, we derive hypocoercive estimates on the approximate solutions. Note that some uniform energy estimates would have been sufficient to prove existence and uniqueness of solutions to (5.3) but these hypocoercive estimates will be our main tool to prove existence and uniqueness of an invariant measure for the problem (5.3). Let us say a few words about the theory of hypocoercivity which has been introduced by Villani [Vil09]. It provides a method to study the rates of convergence to equilibrium of the solutions to kinetic collisional models. For instance, we consider the following class of kinetic models

$$\partial_t f + v \cdot \nabla_x f = Qf, \tag{5.4}$$

where Q is a linear self-adjoint collisional operator which acts on the velocity variable only. We also suppose that the kernel of the operator Q is finite dimensional and, denoting by Π_ℓ the orthogonal projection on $\ker(Q)$ in $L^2(\mathbb{R}^N, \mathrm{d}v)$, that the following local (in space) coercivity assumption holds in $L^2(\mathbb{R}^N, \mathrm{d}v)$:

$$\langle Qh, h\rangle \leq -c\|h - \Pi_\ell h\|,$$

for some $c > 0$. This implies that Q has a spectral gap. The class we have just introduced includes, among others, the cases of linearised Boltzmann, classical relaxation, Landau and Fokker-Planck equations. Note that the global steady states of these models belong to $\ker(Q)$. Finally, we introduce the global projection $\bar{\Pi}$ on $\ker(Q)$ in $L^2(\mathbb{T}^N \times \mathbb{R}^N)$ defined by

$$\bar{\Pi} h = \int_{\mathbb{T}^N} \Pi_\ell h(x,v)dx.$$

It can be easily seen that, if f is a solution to Equation (5.4), $\bar{\Pi} f(t) = \bar{\Pi} f(0)$ is independent of time. Then the hypocoercivity theory gives us the exponential damping of the solution f to equilibrium:

$$\|f(t) - \bar{\Pi} f(0)\|_{\mathcal{H}} \leq K e^{-\tau t}, \quad t \geq 0,$$

in some Sobolev space $\mathcal{H}$. We refer the reader to the memoir of Villani [Vil09] and references therein and also to the paper of Mouhot and Neumann [MN06] where the hypocoercivity is used to study the convergence to equilibrium of many kinetic models including Fokker-Planck equations.

In the case of the deterministic Fokker-Planck equation (5.3) where $\lambda = 0$, the kernel of L is spanned by the function $\mathcal{M}^{\frac{1}{2}}$ and we have

$$\bar{\Pi} h = \rho_\infty(h)\mathcal{M}^{\frac{1}{2}},$$

where we have defined $\rho_\infty(h) := \iint h(t)\mathcal{M}^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}v = \iint h(0)\mathcal{M}^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}v$ (this quantity is time independent). Thus we can prove (see [MN06, Section 5.3]) an exponential damping for the quantity $g(t) - \rho_\infty(g)\mathcal{M}^{\frac{1}{2}}$ in the $H^1(\mathbb{T}^N \times \mathbb{R}^N)$ norm.

In this chapter, we prove hypocoercive estimates on the Fokker-Planck model (5.3) which has been perturbed by a random force. We follow the proof in the paper of Mouhot and Neumann [MN06] and use the Itô formula to handle the stochastic term. We obtain the following hypocoercive estimate:

$$\mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}} \leq \; Ce^{-ct}\mathbb{E}\|g_{\mathrm{in}}\|^2_{L^2_{\nabla,D}} + K\mathbb{E}|\rho_\infty(g)|^2, \quad t \geq 0, \tag{5.5}$$

where $L^2_{\nabla,D}$ can be understood as an $H^1(\mathbb{T}^N \times \mathbb{R}^N)$ Sobolev space (see below for the precise definition). In fact, this hypocoercive estimate only holds true if the initial condition g_{in} is regular, that is in $L^2_{\nabla,D}$. In the sequel, we solve Problem (5.3) with initial condition g_{in} in $L^2_{x,v}$. To overcome this defect, we take advantage of the regularising effect of Equation (5.3): we prove that the solution g with initial condition in $L^2_{x,v}$ belongs to $L^2_{\nabla,D}$ as soon as $t > 0$. Then, once $g(t)$ is in $L^2_{\nabla,D}$, we are able to obtain hypocoercivity.

Concerning the proof of existence and uniqueness of an invariant measure for the problem (5.3), we mainly use the hypocoercive estimates. Indeed, we can deduce from the estimate (5.5) the following property of the solutions: let g_1 and g_2 be two solutions of the problem (5.3) with respective initial conditions $g_{\mathrm{in},1}$ and $g_{\mathrm{in},2}$ such that $\iint g_{\mathrm{in},1}\mathcal{M}^{\frac{1}{2}} = \iint g_{\mathrm{in},2}\mathcal{M}^{\frac{1}{2}}$, then the solutions meet exponentially fast. That is the main argument for proving the uniqueness of the invariant measure. We therefore obtain a family of unique invariant measures to the problem (5.3) indexed by the quantity $\iint g_{\mathrm{in}}\mathcal{M}^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}v$.

5.2 Existence of solutions

5.2.1 Preliminaries and main result

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which is supposed to be right continuous and such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-–null sets of $\mathcal{F}$. We study the following stochastic equation in $\mathbb{T}^N \times \mathbb{R}^N$:

$$\left\{\begin{array}{l} \mathrm{d}g \;+\; v\cdot\nabla_x g\;\mathrm{d}t \;+\; \lambda\left(\nabla_v - \dfrac{v}{2}\right)g \;\odot \mathrm{d}W_t \;=\; Lg\;\mathrm{d}t, \\ g(0) = g_{\mathrm{in}}. \end{array}\right. \tag{5.6}$$

Note that we can write the Stratonovich correction explicitly and that the first equation then reads in Itô form:

$$\begin{aligned} \mathrm{d}g + v\cdot\nabla_x g\;\mathrm{d}t + \lambda\left(\nabla_v - \frac{v}{2}\right)g\cdot \mathrm{d}W_t = Lg\;\mathrm{d}t \\ + \frac{\lambda^2}{2}\sum_j F_j\cdot\left(\nabla_v - \frac{v}{2}\right)\left(F_j\cdot\left(\nabla_v - \frac{v}{2}\right)h\right)\mathrm{d}t. \end{aligned} \tag{5.7}$$

In the following, we denote by $\langle .,. \rangle$ and $\|\cdot\|$ respectively the scalar product and the norm of $L^2_{x,v} := L^2(\mathbb{T}^N \times \mathbb{R}^N)$. Let H be an Hilbert space. For any $T > 0$, we denote by $C_w([0,T], H)$

the space of weakly continuous functions on $[0,T]$ with values in H. Let us introduce the differential operators

$$D = \nabla v + \frac{v}{2}, \quad D^* = -\nabla v + \frac{v}{2},$$

where D^* is the formal adjoint of D component-wise, *i.e.* $D_k^* = (D_k)^*$, $k = 1, ..., N$. Note that, for f smooth enough, we have the two following identities

$$\|Df\|^2 \;=\; \|\nabla_v f\|^2 + \frac{1}{4}\|vf\|^2 \;-\; \frac{N}{2}\|f\|^2 \tag{5.8}$$

and

$$\|D^*f\|^2 \;=\; \|\nabla_v f\|^2 + \frac{1}{4}\|vf\|^2 \;+\; \frac{N}{2}\|f\|^2. \tag{5.9}$$

We introduce the space

$$L^2_D = \{f \in L^2(\mathbb{R}^N); Df \in L^2(\mathbb{R}^N)\} = \{f \in L^2(\mathbb{R}^N); D^*f \in L^2(\mathbb{R}^N)\}$$

and then define the spaces

$$L^2_{x,D} = L^2(\mathbb{T}^N; L^2_D), \quad L^2_{\nabla,D} = \{f \in L^2_{x,D}; \nabla_x f \in L^2_{x,v}\},$$

equipped respectively with the norms

$$\|f\|^2_{L^2_{x,D}} = \|D^*f\|^2, \quad \|f\|^2_{L^2_{\nabla,D}} = \|D^*f\|^2 + \|\nabla_x f\|^2.$$

For the sake of convenience, we define the transport operator $A = v \cdot \nabla_x$ which is skew-symmetric, that is which satisfies $A^* = -A$. Concerning the Fokker-Planck operator L, let us present some of its properties. First, we recall its definition:

$$Lf \;=\; \Delta_v f \;+\; \left(\frac{N}{2} - \frac{|v|^2}{4}\right) f.$$

We have the following expressions:

$$Lf \;=\; -\sum_k D_k^* D_k f \;=\; Nf - \sum_k D_k D_k^* f,$$

which can be written $L = -D^*D = N\mathrm{Id} - DD^*$ for short. Note in particular that we have the following dissipative bound for the operator:

$$-\langle f, Lf\rangle \;=\; \|Df\|^2. \tag{5.10}$$

Furthermore, on the space $L^2_v(\mathbb{R}^N)$ the operator L possesses an Hilbertian basis of eigenfunctions $(q_j)_{j\in\mathbb{N}^N} = \left(c_j D^j \mathcal{M}^{\frac{1}{2}}\right)_{j\in\mathbb{N}^N}$ associated with eigenvalues $-|j|$. In particular, introducing the orthogonal projector Π_ℓ on $\langle q_0\rangle$ (which we extend trivially to $L^2_{x,v}$),

$$\Pi_\ell(f)(x,v) \;=\; \langle \mathcal{M}^{\frac{1}{2}}, f(x,\cdot)\rangle_{L^2_v(\mathbb{R}^N)}\, \mathcal{M}^{\frac{1}{2}}(v), \quad \Pi_\ell^\perp = I - \Pi_\ell,$$

we have

$$-\langle f, Lf\rangle \geq \|\Pi_\ell^\perp f\|^2. \tag{5.11}$$

Using (5.10), we then deduce from (5.11) that

$$\|f\|^2 \leq \|\Pi_\ell f\|^2 + \|Df\|^2. \tag{5.12}$$

Finally, in the sequel, we denote by $\{A,B\} := AB - BA$ the commutator of the operators A and B. We point out that one can easily show the following identities

$$\{D, A\} = \nabla_x, \qquad \{D, L\} = -ND.$$

We are now ready to state our main result concerning existence and uniqueness of solutions to the problem (5.6). The question of invariant measures will be studied further.

Theorem 5.2.1. *Suppose that hypothesis (5.2) holds and let $g_{\rm in} \in L^2(\Omega; L^2_{x,v})$. For any $\lambda < 1$, there exists a unique adapted process $\{g(t),\, t \geq 0\}$ which satisfies:*

(i) for all $T > 0$, $g \in C_w([0,T]; L^2(\Omega; L^2_{x,v}))$ and $Dg \in L^2(\Omega \times (0,T); L^2_{x,v})$;

(ii) $g(0) = g_{\rm in}$;

(iii) for all φ in $C_c^\infty(\mathbb{T}^N \times \mathbb{R}^N)$ and all $t \geq 0$,

$$\begin{aligned} \langle g(t), \varphi\rangle &= \langle g_{\rm in}, \varphi\rangle + \int_0^t \langle g(s), v\cdot\nabla_x\varphi\rangle\, ds + \lambda \sum_{j\geq 0}\int_0^t \langle g(s), F_j\cdot D\varphi\rangle\, d\beta_j(s) \\ &\quad + \int_0^t \langle g(s), L^*\varphi\rangle\, ds + \frac{\lambda^2}{2}\sum_{j\geq 0}\int_0^t \left\langle g(s), (F_j\cdot D)^2\varphi\right\rangle ds, \quad a.s. \end{aligned} \tag{5.13}$$

The quantity $\rho_\infty(g) := \iint g\mathcal{M}^{\frac{1}{2}}$ is constant in time. Furthermore, there exists $\lambda_0(N) > 0$ such that, for all $\lambda < \lambda_0$, g have the following properties. The solution g becomes more regular as soon as $t > 0$: for any $t_0 > 0$, there exists a constant $C(N, t_0) > 0$ such that

$$\mathbb{E}\|g(t_0)\|^2_{L^2_{\nabla,D}} \leq C\mathbb{E}\|g_{in}\|^2. \tag{5.14}$$

Besides, if $t_0 > 0$, there exist constants c, C and K depending on N only such that g satisfies, for $t \geq t_0$, the bound

$$\mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}} + c\,\mathbb{E}\int_{t_0}^t \|g(s)\|^2_{L^2_{\nabla,D}} + \|D\nabla_x g(s)\|^2 + \|D^2 g(s)\|^2\, ds \;\leq\; C\mathbb{E}\|g(t_0)\|^2_{L^2_{\nabla,D}} + C\mathbb{E}|\rho_\infty|^2(t-t_0), \tag{5.15}$$

and, for $t \geq t_0$, the hypocoercive estimate

$$\mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}} \leq\; Ce^{-c(t-t_0)}\mathbb{E}\|g(t_0)\|^2_{L^2_{\nabla,D}} + K\mathbb{E}|\rho_\infty(g)|^2. \tag{5.16}$$

To prove the existence part, we use a Galerkin projection method: one projects the equation (5.6) onto the finite dimensional space spanned by some vectors $\{e_0, ..., e_m\}$ where $(e_k)_{k\in\mathbb{N}}$ are smooth functions constituting an orthonormal basis of $L^2_{x,v}$.

5.2.2 The Galerkin scheme

Let $(p_i)_{i\in\mathbb{N}}$ be an orthonormal basis of normalized eigenfunctions for $-\Delta_x$ in $L^2(\mathbb{T}^N)$ and $(q_j)_{j\in\mathbb{N}}$ an orthonormal basis of eigenvectors for the Fokker-Planck operator L in $L^2(\mathbb{R}^N)$. We introduce the orthonormal basis $(e_k)_{k\in\mathbb{N}^2}$ of $L^2_{x,v}$ defined by

$$e_k(x,v) := p_i(x)q_j(v), \quad k = (i,j) \in \mathbb{N}^2,\, x \in \mathbb{T}^N,\, v \in \mathbb{R}^N.$$

Clearly, the functions $(e_k)_{k\in\mathbb{N}^2}$ are smooth with respect to (x, v). For the sake of convenience, we re-index this basis to write it $(e_k)_{k\in\mathbb{N}}$. We set $E_m := \mathrm{Span}\{e_0, ..., e_m\}$ and introduce Π_m the orthogonal projection on E_m in $L^2_{x,v}$. We are looking for an approximate solution $g_m : [0,T] \to E_m$ of (5.6) of the form

$$g_m(t) := \sum_{k=0}^m d_k(t)e_k. \tag{5.17}$$

We have the following result:

Proposition 5.2.2. *For all $m \geq 0$, there exists a unique adapted process $g_m \in C(0,T;E_m)$ a.s. of the form (5.17) and satisfying, for all $t \in [0,T]$ and $0 \leq k \leq m$,*

$$\begin{aligned}\langle g_m(t), e_k\rangle &= \langle g_{\rm in}, e_k\rangle + \int_0^t \langle g_m(s), v\cdot\nabla_x e_k\rangle ds + \lambda\sum_{j\geq 0}\int_0^t \langle g_m(s), F_j\cdot D e_k\rangle\, d\beta_j(s)\\ &\quad + \int_0^t \langle g_m(s), L^* e_k\rangle ds + \frac{\lambda^2}{2}\sum_{j\geq 0}\int_0^t \Big\langle g_m(s), (F_j\cdot D)^2 e_k\Big\rangle\, ds, \quad a.s.\end{aligned} \tag{5.18}$$

Proof. Suppose that g_m has the form (5.17). Then we can write equation (5.18) in terms of the coefficients $d_k = \langle g_m, e_k\rangle$ of g_m in the basis of E_m for k from 0 to m and we clearly obtain that $(d_k)_{0\leq k\leq m}$ satisfies a usual finite dimensional Itô system with globally Lipschitz coefficients. It is standard that we have existence and uniqueness of an adapted and continuous processes $(d_k)_{0\leq k\leq m}$, for which g_m of the form (5.17) is adapted, continuous with values in E_m and satisfying (5.18). □

Clearly, the process $(g_m(t))_{t\in[0,T]}$ with values in E_m satisfies

$$\mathrm{d}g_m + \Pi_m v\cdot\nabla_x g_m \mathrm{d}t - \lambda \Pi_m D^* g_m \odot \mathrm{d}W_t = L g_m \mathrm{d}t, \tag{5.19}$$

whith initial condition

$$g_m(0) = \Pi_m g_{\rm in}.$$

5.2.3 Estimates on the approximate solutions

In this section, we derive some estimates on the approximate solutions $(g_m)_m$. In the next sections, we deduce from these basic estimates a regularisation property (see Section 5.2.4) and hypocoercive estimates (see Section 5.2.5).

Formal computation

Our aim will be to evaluate $\mathbb{E}\Phi(g_m)$ where Φ is a quadratic functional of the form

$$\Phi(g) = \langle Sg, Tg\rangle,$$

where S and T are operators in the variables x or v of order at most one. In particular, S and T are *linear*. The procedure which we describe below is rigorous when applied to the finite-dimensional system satisfied by g_m but we will still use in what follows the equation (5.6) satisfied by g for simplicity. We apply S to (5.6) and then test against Tg, and do the same with the roles of S and T exchanged, to obtain

$$d\Phi(g) = -\langle SAg, Tg\rangle dt + \lambda\sum_j \langle S(F_j\cdot D^*)g, Tg\rangle \circ d\beta_j(t) + \langle SLg, Tg\rangle dt + \mathsf{sym}, \tag{5.20}$$

where by "$B(S,T)+\mathsf{sym}$" in the right-hand side of (5.20), we mean $B(S,T)+B(T,S)$. Switching to Itô form and taking expectation in (5.20) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\Phi(g) = -\mathbb{E}\langle SAg, Tg\rangle + \mathbb{E}\langle SLg, Tg\rangle + \frac{\lambda^2}{2}\mathbb{E}\mathcal{N}_{S,T}(g) + \mathsf{sym}. \tag{5.21}$$

where we have introduced the notation

$$\mathcal{N}_{S,T}(g) := \sum_j \langle S(F_j\cdot D^*)^2 g, Tg\rangle + \langle S(F_j\cdot D^*)g, T(F_j\cdot D^*)g\rangle.$$

Note also, in the case $S=T$, that, by (5.10),

$$\mathbb{E}\langle SLg, Sg\rangle = \mathbb{E}\langle LSg, Sg\rangle + \mathbb{E}\langle \{S,L\}g, Sg\rangle = -\mathbb{E}\|DSg\|^2 + \mathbb{E}\langle \{S,L\}g, Sg\rangle, \tag{5.22}$$

where $\{S,L\}=SL-LS$, which means that the term $\mathbb{E}\langle SLg, Sg\rangle$ in (5.21) will almost provide the part $-\mathbb{E}\|DSg\|^2$ needed for the exponential damping to obtain hypocoercivity.

First estimate: $\mathbb{E}\|g_m\|^2$

Taking $S=T=\mathrm{Id}$, we have by (5.21), (5.22) and the fact that A is skew-symmetric,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|g_m\|^2 + \mathbb{E}\|Dg_m\|^2 \leq \frac{\lambda^2}{2}\mathbb{E}\mathcal{N}_{\mathrm{Id},\mathrm{Id}}(g_m).$$

We recall

$$\mathcal{N}_{\mathrm{Id},\mathrm{Id}}(g_m) = \sum_j \langle (F_j\cdot D^*)^2 g_m, g_m\rangle + \langle (F_j\cdot D^*)g, (F_j\cdot D^*)g_m\rangle,$$

so that, using (5.2) and the bound $\|Df\|\leq\|D^*f\|$, we have $\mathcal{N}_{\mathrm{Id},\mathrm{Id}}(g_m)\leq 2\|D^*g_m\|^2$. As a result we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|g_m\|^2 + \mathbb{E}\|Dg_m\|^2 \leq \lambda^2\mathbb{E}\|D^*g_m\|^2. \tag{5.23}$$

Second estimate: $\mathbb{E}\|\nabla_x g_m\|^2$

We apply (5.21), (5.22) with $S=T=\nabla_x$. We obtain, due to the fact that A is skew-symmetric,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|\nabla_x g_m\|^2 + \mathbb{E}\|D\nabla_x g_m\|^2 \leq \frac{\lambda^2}{2}\mathbb{E}\mathcal{N}_{\nabla_x,\nabla_x}(g_m),$$

with

$$\mathcal{N}_{\nabla_x,\nabla_x}(g_m) = \sum_j \langle \nabla_x(F_j\cdot D^*)^2 g_m, \nabla_x g_m\rangle + \langle \nabla_x(F_j\cdot D^*)g_m, \nabla_x(F_j\cdot D^*)g_m\rangle.$$

By (5.2) and the bound $\|Df\|\leq\|D^*f\|$, we have

$$\mathcal{N}_{\nabla_x,\nabla_x}(g_m) \leq \|D^*g_m\|^2 + 4\|D^*g_m\|\|D^*\nabla_x g_m\| + 2\|D^*\nabla_x g_m\|^2.$$

As a result, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|\nabla_x g_m\|^2 + \mathbb{E}\|D\nabla_x g_m\|^2 \leq \frac{\lambda^2}{2}\mathbb{E}\big[\|D^*g_m\|^2 + 4\|D^*g_m\|\|D^*\nabla_x g_m\| + 2\|D^*\nabla_x g_m\|^2\big] \tag{5.24}$$

Third estimate: $\mathbb{E}\|Dg_m\|^2$

We apply (5.21) and (5.22) with $S=T=D$. It gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|Dg_m\|^2 = -\mathbb{E}\langle DAg_m, Dg_m\rangle - \mathbb{E}\|D^2g_m\|^2 + \mathbb{E}\langle \{D,L\}g_m, Dg_m\rangle + \frac{\lambda^2}{2}\mathbb{E}\mathcal{N}_{D,D}(g_m).$$

Note that $\{A,D\}=\nabla_x$ and $\{D,L\}=-ND$ so that, since A is skew-symmetric, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|Dg_m\|^2 = -\mathbb{E}\langle \nabla_x g_m, Dg_m\rangle - \mathbb{E}\|D^2g_m\|^2 - N\mathbb{E}\|Dg_m\|^2 + \frac{\lambda^2}{2}\mathbb{E}\mathcal{N}_{D,D}(g_m).$$

Furthermore, by (5.2) and the bound $\|Df\| \le \|D^* f\|$,

$$\begin{aligned}\mathcal{N}_{D,D}(g_m) &= \sum_j \langle D(F_j \cdot D^*)^2 g_m, Dg_m\rangle + \langle D(F_j \cdot D^*)g_m, D(F_j \cdot D^*)g_m\rangle \\ &\le \|(D^*)^2 g_m\|^2 + \|DD^* g_m\|^2.\end{aligned}$$

It follows then that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|Dg_m\|^2 + \mathbb{E}\|D^2 g_m\|^2 \le \mathbb{E}\|\nabla_x g_m\|\|Dg_m\| + \frac{\lambda^2}{2}\mathbb{E}\big[\|(D^*)^2 g_m\|^2 + \|DD^* g_m\|^2\big]. \quad (5.25)$$

Fourth estimate: $\mathbb{E}\langle \nabla_x g_m, Dg_m\rangle$

We apply (5.21) with $S = \nabla_x$ and $T = D$. It yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\langle \nabla_x g_m, Dg_m\rangle &= -\mathbb{E}\langle \nabla_x A g_m, Dg_m\rangle - \mathbb{E}\langle DAg_m, \nabla_x g_m\rangle \\ &\quad + \mathbb{E}\langle \nabla_x L g_m, Dg_m\rangle + \mathbb{E}\langle DLg_m, \nabla_x g_m\rangle \\ &\quad + \frac{\lambda^2}{2}\mathbb{E}[\mathcal{N}_{\nabla_x,D}(g_m) + \mathcal{N}_{D,\nabla_x}(g_m)].\end{aligned}$$

First of all, with the identities $\nabla_x A = A\nabla_x$, $A^* = -A$ and $\{A, D\} = \nabla_x$, we have

$$\begin{aligned}-\mathbb{E}\langle \nabla_x A g_m, Dg_m\rangle - \mathbb{E}\langle DAg_m, \nabla_x g_m\rangle &= -\mathbb{E}\langle A\nabla_x g_m, Dg_m\rangle - \mathbb{E}\langle DAg_m, \nabla_x g_m\rangle \\ &= -\mathbb{E}\langle (AD - DA)g_m, \nabla_x g_m\rangle \\ &= -\mathbb{E}\|\nabla_x g_m\|^2.\end{aligned}$$

Besides, with the identity $L = -D^*D = N\mathrm{Id} - DD^*$, we have

$$\begin{aligned}\mathbb{E}\langle \nabla_x L g_m, Dg_m\rangle + \mathbb{E}\langle DLg_m, \nabla_x g_m\rangle &= -\mathbb{E}\langle D^*D\nabla_x g_m, Dg_m\rangle - \mathbb{E}\langle DD^*Dg_m, \nabla_x g_m\rangle \\ &= -\mathbb{E}\langle D\nabla_x g_m, D^2 g_m\rangle - \mathbb{E}\langle D^*DDg_m, \nabla_x g_m\rangle - N\mathbb{E}\langle Dg_m, \nabla_x g_m\rangle \\ &= -2\mathbb{E}\langle D\nabla_x g_m, D^2 g_m\rangle - N\mathbb{E}\langle Dg_m, \nabla_x g_m\rangle.\end{aligned}$$

Concerning the terms $\mathcal{N}_{\nabla_x,D}(g_m)$ and $\mathcal{N}_{D,\nabla_x}(g_m)$, we have

$$\begin{aligned}\mathcal{N}_{\nabla_x,D}(g_m) + \mathcal{N}_{D,\nabla_x}(g_m) &= \sum_j \langle \nabla_x (F_j \cdot D^*)^2 g_m, Dg_m\rangle + \langle \nabla_x (F_j \cdot D^*)g_m, D(F_j \cdot D^*)g_m\rangle \\ &\quad + \sum_j \langle D(F_j \cdot D^*)^2 g_m, \nabla_x g_m\rangle + \langle D(F_j \cdot D^*)g_m, \nabla_x (F_j \cdot D^*)g_m\rangle\end{aligned}$$

which is bounded, thanks to (5.2) and $\|Df\| \le \|D^* f\|$, by

$$\begin{aligned}&\|D^*\nabla_x g_m\|\|D^2 g_m\| + 2\|D^* g_m\|\|D^2 g_m\| + \|(D^*)^2 g_m\|\|D^*\nabla_x g_m\| \\ &\quad + 2\|D^*\nabla_x g_m\|\|DD^* g_m\| + 2\|D^* g_m\|\|DD^* g_m\|.\end{aligned}$$

As a result, we finally obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\langle \nabla_x g_m, Dg_m\rangle + \mathbb{E}\|\nabla_x g_m\|^2 &\le 2\mathbb{E}\|D\nabla_x g_m\|\|D^2 g_m\| + N\mathbb{E}\|Dg_m\|\|\nabla_x g_m\| \\ &\quad + \frac{\lambda^2}{2}\mathbb{E}\big[\|D^*\nabla_x g_m\|\|D^2 g_m\| + 2\|D^* g_m\|\|D^2 g_m\| + \|(D^*)^2 g_m\|\|D^*\nabla_x g_m\| \\ &\quad + 2\|D^*\nabla_x g_m\|\|DD^* g_m\| + 2\|D^* g_m\|\|DD^* g_m\|\big]. \quad (5.26)\end{aligned}$$

Summary and rewriting of the estimates

In this section, we sum up the estimates (5.23), (5.24), (5.25) and (5.26) derived above and try to "close" them with respect to the variables g_m, $\nabla_x g_m$, Dg_m, $D\nabla_x g_m$ and $D^2 g_m$. Note in particular that the operator D^* appears in the right-hand sides of these estimates and we are about to correct this. To do so, we use the formula

$$\|D^* f\|^2 = \|Df\|^2 + N\|f\|^2 \tag{5.27}$$

proved by (5.8) and (5.9). Now we focus on each estimate (5.23), (5.24), (5.25) and (5.26).

First estimate. The first bound (5.23) can now be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\mathbb{E}\|g_m\|^2 + \mathbb{E}\|Dg_m\|^2 \leq \lambda^2 \mathbb{E}[\|Dg_m\|^2 + N\|g_m\|^2]. \tag{5.28}$$

Second estimate. The second one (5.24) becomes

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\mathbb{E}\|\nabla_x g_m\|^2 + \mathbb{E}\|D\nabla_x g_m\|^2 &\leq \frac{\lambda^2}{2}\mathbb{E}\big[\|Dg_m\|^2 + N\|g_m\|^2 \\ &+ 4(\|Dg_m\| + \sqrt{N}\|g_m\|)(\|D\nabla_x g_m\| + \sqrt{N}\|\nabla_x g_m\|) + 2\|D\nabla_x g_m\|^2 + 2N\|\nabla_x g_m\|^2\big].\end{aligned} \tag{5.29}$$

Third estimate. Concerning the third one (5.25), let us precise how to handle the term $\|DD^* g_m\|^2$. Using the identity $DD^* = N\mathrm{Id} + D^*D$, we can write

$$\|DD^* f\|^2 = N^2\|f\|^2 + 2N\|Dg\|^2 + \|D^*Df\|^2.$$

As a consequence, we can show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\mathbb{E}\|Dg_m\|^2 + \mathbb{E}\|D^2 g_m\|^2 \leq \mathbb{E}\|Dg_m\|\|\nabla_x g_m\| + \frac{\lambda^2}{2}\mathbb{E}\big[3N^2\|g_m\|^2 + 7N\|Dg_m\|^2 + 2\|D^2 g_m\|^2\big]. \tag{5.30}$$

Fourth estimate. Finally, with a similar work, the fourth bound (5.26) writes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\mathbb{E}\langle\nabla_x g_m, Dg_m\rangle + \mathbb{E}\|\nabla_x g_m\|^2 &\leq 2\mathbb{E}\|D\nabla_x g_m\|\|D^2 g_m\| + N\mathbb{E}\|Dg_m\|\|\nabla_x g_m\| \\ &+ \frac{\lambda^2}{2}\mathbb{E}\big[(\|D\nabla_x g_m\| + \sqrt{N}\|\nabla_x g_m\|)(\sqrt{2}N\|g_m\| + 2\sqrt{N}\|Dg_m\| + 2\|D^2 g_m\|) \\ &+ 2\|D^2 g_m\|(\|Dg_m\| + \sqrt{N}\|g_m\|) \\ &+ (N\|g_m\| + \sqrt{3N}\|Dg_m\| + \|D^2 g_m\|)(2\|D\nabla_x g_m\| + 2\sqrt{N}\|\nabla_x g_m\| + 2\|Dg_m\| + 2\sqrt{N}\|g_m\|)\big].\end{aligned} \tag{5.31}$$

5.2.4 Regularisation for $t > 0$

In this part, we show that the solution g to Equation (5.6) with initial condition g_{in} in $L^2_{x,v}$ gains regularity as soon as $t > 0$. Precisely, $g(t) \in L^2_{\nabla,D}$ if $t > 0$. In what follows, we work on the approximate solutions $(g_m)_m$ but the result remains valid on the solution g by passing to the limit, see Section 5.2.6. The result is the following.

Proposition 5.2.3. *Let $T > 0$, there exist constants $\lambda^*(N,T) > 0$ and $C(N,T) > 0$ such that for any $t \in (0,T]$ and $\lambda < \lambda^*$, we have the bounds*

$$\mathbb{E}\|g_m(t)\|^2 \leq C\mathbb{E}\|g_{in}\|^2, \quad \mathbb{E}\|Dg_m(t)\|^2 \leq \frac{C}{t}\mathbb{E}\|g_{in}\|^2, \quad \mathbb{E}\|\nabla_x g_m(t)\|^2 \leq \frac{C}{t^3}\mathbb{E}\|g_{in}\|^2. \tag{5.32}$$

We now start the proof of this result. To simplify the notations, we take $T = 1$, the adaptation of the proof being straightforward in the case $T > 0$. Let k, a, b and c some positive constants to be chosen later on. We introduce, for $t \in [0,1]$,

$$\mathcal{K}_t(g) := k\|g\|^2 + at^3\|\nabla_x g\|^2 + bt\|Dg\|^2 + 2ct^2\langle \nabla_x g, Dg\rangle.$$

We suppose in the sequel that the condition $c^2 < ab$ is satisfied. Note that, using the Young inequality $2cxy \le \nu c x^2 + \frac{c}{\nu}y^2$ with $\nu = \frac{1}{2}\left(\frac{c}{b} + \frac{a}{b}\right)$, we have

$$\mathcal{K}_t(g) \ge k\|g\|^2 + \frac{ab - c^2}{2b}t^3\|\nabla_x g\|^2 + bt\frac{ab-c^2}{ab+c^2}\|Dg\|^2. \tag{5.33}$$

The main step of the proof is to show that there exists a constant $C(N) > 0$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_t(g_m) \le C\|g_m\|^2,$$

from which one can easily deduce (5.32). The previous bound is obtained thanks to the estimates (5.28), (5.29), (5.30) and (5.31). Precisely, we state the following preliminary result.

Lemma 5.2.4. *We have the bounds*

(i) $\frac{1}{2}\frac{d}{dt}\mathbb{E}\|g_m\|^2 + \mathbb{E}\|Dg_m\|^2 \le \lambda^2\mathbb{E}Q_1(\|g_m\|, \|Dg_m\|, \|\nabla_x g_m\|, \|D\nabla_x g_m\|, \|D^2 g_m\|)$,

(ii) $\frac{1}{2}\frac{d}{dt}\mathbb{E}\|\nabla_x g_m\|^2 + \mathbb{E}\|D\nabla_x g_m\|^2 \le \lambda^2\mathbb{E}Q_2(\|g_m\|, \|Dg_m\|, \|\nabla_x g_m\|, \|D\nabla_x g_m\|, \|D^2 g_m\|)$,

(iii) $\frac{1}{2}\frac{d}{dt}\mathbb{E}\|Dg_m\|^2 + \mathbb{E}\|D^2 g_m\|^2 \le \mathbb{E}\|Dg_m\|\|\nabla_x g_m\| + \lambda^2\mathbb{E}Q_3(\|g_m\|, \|Dg_m\|, \|\nabla_x g_m\|, \|D\nabla_x g_m\|, \|D^2 g_m\|)$,

(iv) $\frac{d}{dt}\mathbb{E}\langle \nabla_x g_m, Dg_m\rangle + \mathbb{E}\|\nabla_x g_m\|^2 \le 2\mathbb{E}\|D\nabla_x g_m\|\|D^2 g_m\| + N\mathbb{E}\|Dg_m\|\|\nabla_x g_m\| + \lambda^2\mathbb{E}Q_4(\|g_m\|, \|Dg_m\|, \|\nabla_x g_m\|, \|D\nabla_x g_m\|, \|D^2 g_m\|)$,

where the Q_i, $i \in \{1, ..., 4\}$, are quadratic forms on $\mathbb{R}^5$ whose coefficients depend only on N. Furthermore, these quadratic forms satisfy the following property: if $\varepsilon > 0$, there exists $\lambda^*(N, \varepsilon) > 0$ and $C(N) > 0$ such that for all $t \in [0,1]$, $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ and $\lambda < \lambda^*$, each of the quantities

$$\lambda^2 Q_1(x_1,x_2,x_3,x_4,x_5), \quad \lambda^2 t^3 Q_2(x_1,x_2,x_3,x_4,x_5),$$
$$\lambda^2 t Q_3(x_1,x_2,x_3,x_4,x_5), \quad \lambda^2 t^2 Q_4(x_1,x_2,x_3,x_4,x_5),$$

are bounded by

$$Cx_1^2 + \varepsilon x_2^2 + \varepsilon t^2 x_3^2 + \varepsilon t^3 x_4^2 + \varepsilon t x_5^2.$$

Proof. Note that the bounds (i), (ii), (iii) and (iv) are respectively obtained from the estimates (5.28), (5.29), (5.30) and (5.31). It only remains to prove that the quadratic forms Q_i, $i \in \{1, ..., 4\}$ satisfy the announced property. We focus on the cases of Q_1 and Q_2; the others are justified by the same method.

Study of Q_1. Thanks to (5.28), the quadratic form Q_1 is given by

$$Q_1(x_1,x_2,x_3,x_4,x_5) = Nx_1^2 + x_2^2.$$

As a consequence, if $\varepsilon > 0$, we can choose λ small enough to ensure that $\lambda^2 Q_1(x_1,x_2,x_3,x_4,x_5)$ is bounded by $Cx_1^2 + \varepsilon x_2^2 + \varepsilon t^2 x_3^2 + \varepsilon t^3 x_4^2 + \varepsilon t x_5^2$.

Study of Q_2. With (5.29), the quadratic form Q_2 is given by

$$Q_2(x_1,x_2,x_3,x_4,x_5) = \frac{N}{2}x_1^2 + \frac{1}{2}x_2^2 + Nx_3^2 + x_4^2 + 2Nx_1x_3 + 2\sqrt{N}x_1x_4 + 2\sqrt{N}x_2x_3 + 2x_2x_4.$$

We now work on each term of the quantity t^3Q_2. We recall that $t \leq 1$. As a result, concerning the four first terms,

$$\frac{N}{2}t^3x_1^2 + \frac{1}{2}t^3x_2^2 + Nt^3x_3^2 + t^3x_4^2 \leq \frac{N}{2}x_1^2 + \frac{1}{2}x_2^2 + Nt^2x_3^2 + t^3x_4^2.$$

For the fifth term, we use $t \leq 1$ and the inequality $2xy \leq x^2 + y^2$ to obtain

$$2Nt^3x_1x_3 \leq 2Ntx_1x_3 \leq Nx_1^2 + Nt^2x_3^2.$$

Finally, we handle the three remaining terms similarly:

$$\begin{aligned}
&2\sqrt{N}t^3x_1x_4 \leq 2\sqrt{N}t^{3/2}x_1x_4 \leq \sqrt{N}x_1^2 + \sqrt{N}t^3x_4,\\
&2\sqrt{N}t^3x_2x_3 \leq 2\sqrt{N}tx_2x_3 \leq \sqrt{N}x_2^2 + \sqrt{N}t^2x_3^2,\\
&2t^3x_2x_4 \leq 2t^{3/2}x_2x_4 \leq x_2^2 + t^3x_4^2.
\end{aligned}$$

To conclude, if $\varepsilon > 0$, using the bounds above, one can choose λ small enough to ensure that $\lambda^2t^3Q_2(x_1,x_2,x_3,x_4,x_5)$ is bounded by $Cx_1^2 + \varepsilon x_2^2 + \varepsilon t^2x_3^2 + \varepsilon t^3x_4^2 + \varepsilon tx_5^2$. This concludes the proof. □

We now have all in hands to conclude the proof of Proposition 5.2.3. We compute, thanks to Lemma 5.2.4,

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\mathcal{K}_t(g_m) \;\leq\; &\mathbb{E}\Big[-(k-b)\|Dg_m\|^2 - at^3\|D\nabla_x g_m\|^2 - bt\|D^2g_m\|^2 - (c-3a)t^2\|\nabla_x g_m\|^2\\
&+\;(bt + Nct^2 + 4ct)\|\nabla_x g_m\|\|Dg_m\| + 2ct^2\|D\nabla_x g_m\|\|D^2g_m\|\Big]\\
+\;\lambda^2\mathbb{E}&\left[\{kQ_1 + at^3Q_2 + btQ_3 + ct^2Q_4\}(\|g_m\|,\|Dg_m\|,\|\nabla_x g_m\|,\|D\nabla_x g_m\|,\|D^2g_m\|)\right].
\end{aligned} \tag{5.34}$$

We first focus on the behaviour of the deterministic part of the right-hand side, obtained when $\lambda = 0$. We set

$$k = 2(10+N)^2 + 6 + \frac{1}{8}, \quad a = \frac{1}{4}, \quad b = 6, \quad c = 1, \tag{5.35}$$

and prove that for this choice of the constants k, a, b and c we have

$$\begin{aligned}
&-(k-b)\|Dg_m\|^2 - at^3\|D\nabla_x g_m\|^2 - bt\|D^2g_m\|^2 - (c-3a)t^2\|\nabla_x g_m\|^2\\
&\qquad + (bt + Nct^2 + 4ct)\|\nabla_x g_m\|\|Dg_m\| + 2ct^2\|D\nabla_x g_m\|\|D^2g_m\|\\
&\qquad\qquad \leq -\frac{1}{8}\|Dg_m\|^2 - \frac{t^3}{20}\|D\nabla_x g_m\|^2 - t\|D^2g_m\|^2 - \frac{t^2}{8}\|\nabla_x g_m\|^2.
\end{aligned} \tag{5.36}$$

Thanks to (5.35) and $Nct^2 \leq Nct$ for $t \leq 1$, the left-hand side of (5.36) is bounded by

$$\begin{aligned}
&-(2(10+N)^2 + 1/8)\|Dg_m\|^2 - \frac{t^3}{4}\|D\nabla_x g_m\|^2 - 6t\|D^2g_m\|^2 - \frac{t^2}{4}\|\nabla_x g_m\|^2\\
&\qquad + (10+N)t\|\nabla_x g_m\|\|Dg_m\| + 2t^2\|D\nabla_x g_m\|\|D^2g_m\|.
\end{aligned} \tag{5.37}$$

Furthermore, using the Young inequality $\ell xy \leq \frac{\ell x^2}{2\nu} + \frac{\ell \nu y^2}{2}$, we have the bounds

$$\begin{aligned}(10+N)t\|\nabla_x g_m\|\|Dg_m\| &\leq \frac{t^2}{8}\|\nabla_x g_m\|^2 + 2(10+N)^2\|Dg_m\|^2,\\ 2t^2\|D\nabla_x g_m\|\|D^2 g_m\| &\leq \frac{t^3}{5}\|D\nabla_x g_m\|^2 + 5t\|D^2 g_m\|^2,\end{aligned}$$

from which we immediately deduce that (5.37) is bounded by the right-hand side of (5.36), hence (5.36). Concerning the stochastic part in (5.34), we use Lemma 5.2.4 to obtain, for all $\varepsilon > 0$, some constants $\lambda^*(N, \varepsilon) > 0$ and $C(N) > 0$ such that for all $t \in [0,1]$ and $\lambda < \lambda^*$,

$$\begin{aligned}&\lambda^2\mathbb{E}\left[\{kQ_1 + at^3Q_2 + btQ_3 + ct^2Q_4\}(\|g_m\|, \|Dg_m\|, \|\nabla_x g_m\|, \|D\nabla_x g_m\|, \|D^2 g_m\|)\right.\\ &\leq\ (k+a+b+c)\mathbb{E}\Big[C\|g_m\|^2 + \varepsilon\|Dg_m\|^2 + \varepsilon t^2\|\nabla_x g_m\|^2 + \varepsilon t^3\|D\nabla_x g_m\|^2 + \varepsilon t\|D^2 g_m\|^2\Big].\end{aligned} \tag{5.38}$$

We recall with (5.35) that $k+a+b+c$ is a constant depending only on N. As a consequence, plugging the bounds (5.36) and (5.38) in (5.34) and choosing $\varepsilon > 0$ sufficiently small, we deduce that there exists $\lambda^*(N, \varepsilon) > 0$ and $C(N) > 0$ such that for all $t \in [0,1]$ and $\lambda < \lambda^*$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\mathcal{K}_t(g_m)\ \leq\ C\mathbb{E}\|g_m\|^2. \tag{5.39}$$

We point out that, assuming that $\lambda < \lambda^* < 1$, we deduce from (5.28) and Gronwall's lemma that

$$\sup_{t\in[0,1]} \mathbb{E}\|g_m(t)\|^2 \leq e^{2\lambda^2 N}\mathbb{E}\|g_{\mathrm{in}}\|^2.$$

As a consequence, we integrate (5.39) and use the previous bound to obtain, for $t \in [0,1]$,

$$\mathbb{E}\mathcal{K}_t(g_m) \leq C\mathbb{E}\|g_{\mathrm{in}}\|^2,$$

for some constant $C(N) > 0$. Finally, the bounds (5.32) are a consequence of the bound (5.33) and the fact that the condition $c^2 < ab$ is satisfied by (5.35). This concludes the proof of Proposition 5.2.3.

5.2.5 Hypocoercive estimates

In this section, we derive hypocoercive estimates on the approximate solutions $(g_m)_m$. In particular, it provides uniform energy estimates which are necessary to prove that a subsequence of $(g_m)_m$ indeed converges to a solution of our problem (5.6). Note that these hypocoercive estimates remain valid for g by passing to the limit, see Section 5.2.6.

Closure of the estimates for the exponential damping

The proof of these hypocoercive estimates relies on the bounds (5.28), (5.29), (5.30) and (5.31) of Section 5.2.3. In order to obtain an exponential damping, we need to estimate each terms in the right-hand side of those equations by terms figuring in the left-hand side of one of them. This is not the case for the time being since the quantity $\|g_m\|$ appears in some right-hand sides. Our first step is to correct this defect. To do so, we recall with (5.12) that

$$\|g_m\|^2 \leq \|\Pi_\ell g_m\|^2 + \|Dg_m\|^2.$$

As a consequence, it suffices to estimate the term $\|\Pi_\ell g_m\|$. Denote by

$$\bar{\Pi}f = \int_{\mathbb{T}^N} \Pi_\ell f(x,v)dx, \quad \bar{\Pi}^\perp = \Pi_\ell - \bar{\Pi},$$

the orthogonal projections on $\langle 1\otimes q_0\rangle$ and $\langle 1\otimes q_0\rangle^\perp$ respectively, where

$$1\otimes q_0(x,v) := q_0(v).$$

We decompose $\Pi_\ell g_m = \bar\Pi^\perp g_m + \bar\Pi g_m$ and use the Poincaré-Wirtinger inequality to get an estimate on the first part $\bar\Pi^\perp g_m$:

$$\|f\|^2_{L^2(\mathbb{T}^N)} \leq C_{\mathrm{pw}}\|\nabla_x f\|^2_{L^2(\mathbb{T}^N)}, \tag{5.40}$$

for all $f\in H^1(\mathbb{T}^N)$ satisfying

$$\int_{\mathbb{T}^N} f(x)dx = 0. \tag{5.41}$$

Since $\bar\Pi^\perp g_m$ satisfies the cancellation condition (5.41), we obtain indeed, by integrating (5.40) with respect to v,

$$\|\bar\Pi^\perp g_m\|^2 \leq C_{\mathrm{pw}}\|\nabla_x\bar\Pi^\perp g_m\|^2 = C_{\mathrm{pw}}\|\nabla_x g_m\|^2.$$

The remaining term $\bar\Pi g_m$ is constant in time: we have $\bar\Pi g_m = \rho^m_\infty\mathcal{M}^{\frac12}$ where $\rho^m_\infty := \iint g_m\mathcal{M}^{\frac12}$ is independent on t. To sum up,

$$\|g_m\|^2 \leq \|Dg_m\|^2 + C_{\mathrm{pw}}\|\nabla_x g_m\|^2 + |\rho^m_\infty|^2. \tag{5.42}$$

Using this bound, we write successively the estimates that we obtain from the bounds (5.28), (5.29), (5.30) and (5.31) of Section 5.2.3.

First estimate. There exists a constant $K_1(C_{\mathrm{pw}},N)>0$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|g_m\|^2 + \mathbb{E}\|Dg_m\|^2 \leq \lambda^2K_1\mathbb{E}\big[\|Dg_m\|^2 + \|\nabla_x g_m\|^2 + |\rho^m_\infty|^2\big]. \tag{5.43}$$

Second estimate. There exist a constant $K_2(C_{\mathrm{pw}},N)>0$ and some quadratic form B_2 on $\mathbb{R}^4$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|\nabla_x g_m\|^2 + \mathbb{E}\|D\nabla_x g_m\|^2 \leq K_2\lambda^2\mathbb{E}\,B_2\Big(\|Dg_m\|,\|\nabla_x g_m\|,\|D\nabla_x g_m\|,|\rho^m_\infty|\Big). \tag{5.44}$$

Third estimate. There exist a constant $K_3(C_{\mathrm{pw}},N)>0$ and some quadratic form B_3 on $\mathbb{R}^5$ such that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|Dg_m\|^2 + \mathbb{E}\|D^2g_m\|^2 &\leq \mathbb{E}\|\nabla_x g_m\|\|Dg_m\|\\ &+ K_3\lambda^2\mathbb{E}\,B_3\Big(\|Dg_m\|,\|\nabla_x g_m\|,\|D\nabla_x g_m\|,\|D^2g_m\|,|\rho^m_\infty|\Big).\end{aligned} \tag{5.45}$$

Fourth estimate. There exist a constant $K_4(C_{\mathrm{pw}},N)>0$ and some quadratic form B_4 on $\mathbb{R}^5$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\langle\nabla_x g_m,Dg_m\rangle + \mathbb{E}\|\nabla_x g_m\|^2 &\leq 2\mathbb{E}\|D\nabla_x g_m\|\|D^2g_m\| + N\mathbb{E}\|Dg_m\|\|\nabla_x g_m\|\\ &K_4\lambda^2\mathbb{E}\,B_4\Big(\|Dg_m\|,\|\nabla_x g_m\|,\|D\nabla_x g_m\|,\|D^2g_m\|,|\rho^m_\infty|\Big).\end{aligned} \tag{5.46}$$

Final estimate

Let α, β, γ be some positive coefficients that we will choose later and set

$$\mathcal{F}(g) = \|g\|^2 + \alpha\|\nabla_x g\|^2 + \beta\|Dg\|^2 + 2\gamma\langle\nabla_x g, Dg\rangle.$$

Note that, if $\gamma^2 < \alpha\beta$, then, using the Young inequality $2\gamma ab \le \mu\gamma a^2 + \frac{\gamma}{\mu}b^2$ with $\mu = \frac{1}{2}\left(\frac{\gamma}{\beta} + \frac{\alpha}{\beta}\right)$, we obtain

$$\mathcal{F}(g) \ge \|g\|^2 + \frac{\alpha\beta - \gamma^2}{2\beta}\|\nabla_x g\|^2 + \beta\frac{\alpha\beta - \gamma^2}{\alpha\beta + \gamma^2}\|Dg\|^2.$$

It follows, by (5.27), that for a good choice of the coefficients, we have the following equivalence:

$$C_1\|g\|^2_{L^2_{\nabla,D}} \le \mathcal{F}(g) \le C_2\|g\|^2_{L^2_{\nabla,D}}, \tag{5.47}$$

for some constants C_1, $C_2 > 0$. More precisely, we obtain (5.47) under the hypothesis

$$\gamma^2 < \alpha\beta, \quad \beta\frac{\alpha\beta - \gamma^2}{\alpha\beta + \gamma^2} < \frac{1}{N}. \tag{5.48}$$

Besides, adding (5.43), (5.44), (5.45), (5.46), we have the estimate

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\mathcal{F}(g_m) \ \le\ & \mathbb{E}\Big[-\|Dg_m\|^2 - \alpha\|D\nabla_x g_m\|^2 - \beta\|D^2 g_m\|^2 - \gamma\|\nabla_x g_m\|^2 \\ & + (\beta + N\gamma)\|\nabla_x g_m\|\|Dg_m\| + 2\gamma\|D\nabla_x g_m\|\|D^2 g_m\|\Big] \\ & + \ K\lambda^2\mathbb{E}\, B\Big(\|Dg_m\|, \|\nabla_x g_m\|, \|D\nabla_x g_m\|, \|D^2 g_m\|, |\rho^m_\infty|\Big),\end{aligned} \tag{5.49}$$

for some constant K which depends on K_i, $i \in \{1, ..., 4\}$, and some quadratic form B on $\mathbb{R}^5$ depending on the quadratic forms B_2, B_3 and B_4 and whose coefficients depend on α, β and γ. Now, we let $\alpha = \beta = 2\gamma$ and γ small enough such that

$$(i) \quad (2\gamma + N\gamma)^2 \le \gamma, \qquad (ii) \quad \gamma < 5/(6N). \tag{5.50}$$

Note that $\alpha = \beta = 2\gamma$ and (5.50) – *(ii)* ensures that (5.48) holds. Furthermore, $\alpha = \beta = 2\gamma$ also gives

$$\alpha\beta \ge 4\gamma^2. \tag{5.51}$$

As a result, (5.51) and (5.50) – *(i)* gives that for all p, q ,r and $s \in \mathbb{R}$,

$$\frac{1}{2}\left(p - (\beta + N\gamma)q\right)^2 + \frac{1}{2}\left(\gamma - (\beta + N\gamma)^2\right)q^2 + \frac{\alpha}{2}\left(r - \frac{2\gamma s}{\alpha}\right)^2 + \frac{1}{2\alpha}\left(\alpha\beta - 4\gamma^2\right)s^2 \ \ge\ 0.$$

Expanding this estimate exactly gives, for all p, q ,r and $s \in \mathbb{R}$, the following bound

$$(\beta + N\gamma)pq + 2\gamma rs \ \le\ \frac{1}{2}p^2 + \frac{\gamma}{2}q^2 + \frac{\alpha}{2}r^2 + \frac{\beta}{2}s^2.$$

We deduce from (5.49) and the previous bound applied to $p = \|Dg_m\|$, $q = \|\nabla_x g_m\|$, $r = \|D\nabla_x g_m\|$ and $s = \|D^2 g_m\|$ the estimate

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\mathcal{F}(g_m) \ \le\ & \frac{1}{2}\mathbb{E}\Big[-\|Dg_m\|^2 - \alpha\|D\nabla_x g_m\|^2 - \beta\|D^2 g_m\|^2 - \gamma\|\nabla_x g_m\|^2\Big] \\ & + \ K\lambda^2\mathbb{E}\, B\Big(\|Dg_m\|, \|\nabla_x g_m\|, \|D\nabla_x g_m\|, \|D^2 g_m\|, |\rho^m_\infty|\Big).\end{aligned} \tag{5.52}$$

Now, α, β and γ being fixed as above, we take λ small enough, say $\lambda < \lambda_0$, such that for all p, q, r, s and $\rho \in \mathbb{R}$,

$$K\lambda^2 B(p,q,r,s,\rho) \;\leq\; \frac{1}{4}p^2 + \frac{\gamma}{4}q^2 + \frac{\alpha}{4}r^2 + \frac{\beta}{4}s^2 + C\rho^2$$

for some constant $C > 0$. Using this estimate in (5.52) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\mathcal{F}(g_m) \leq C\mathbb{E}|\rho_\infty^m|^2 - c\left(\mathbb{E}\|\nabla_x g_m\|^2 + \mathbb{E}\|Dg_m\|^2 + \mathbb{E}\|D^2 g_m\|^2 + \mathbb{E}\|D\nabla_x g_m\|^2\right),$$

for some positive constants $C, c > 0$ depending on C_{pw} and N only. Note that λ_0 also depends only on C_{pw} and N. By the formula $\|D^* f\|^2 = \|Df\|^2 + N\|f\|^2$ and (5.42), this gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\mathcal{F}(g_m) \leq C\mathbb{E}|\rho_\infty^m|^2 - c\left(\mathbb{E}\|g_m\|^2_{L^2_{\nabla,D}} + \mathbb{E}\|D^2 g_m\|^2 + \mathbb{E}\|D\nabla_x g_m\|^2\right), \tag{5.53}$$

for some constants $C, c > 0$. By integrating in time from $t_0 > 0$ to $t \geq t_0$ the bound (5.53) and using (5.47), we deduce the first estimate (5.15) for $g = g_m$. Furthermore, (5.53) and (5.47) imply the bound

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\mathcal{F}(g_m) \leq C\mathbb{E}|\rho_\infty^m|^2 - c\mathbb{E}\mathcal{F}(g_m).$$

If $t_0 > 0$, it follows that the function $\varphi(t) := e^{2ct}\mathcal{F}(g_m) - C\mathbb{E}|\rho_\infty^m|^2 c^{-1} e^{2ct}$ defined on $[t_0, +\infty)$ satisfies $\varphi' \leq 0$ so that we deduce, for $t \geq t_0$,

$$\begin{aligned}\mathbb{E}\mathcal{F}(g_m) \leq&\; e^{-2c(t-t_0)}\left(\mathbb{E}\mathcal{F}(g_m(t_0)) - \frac{C\mathbb{E}|\rho_\infty|^2}{c}\right) + \frac{C\mathbb{E}|\rho_\infty|^2}{c}\\ \leq&\; e^{-2c(t-t_0)}\mathbb{E}\mathcal{F}(g_m(t_0)) + \frac{C\mathbb{E}|\rho_\infty|^2}{c}.\end{aligned}$$

Thanks to (5.47), this exactly yields the hypocoercive estimate (5.16) for $g = g_m$. To conclude, the estimates (5.15) and (5.16) remain valid for g by passing to the limit $m \to +\infty$, a procedure which we give in detail now.

5.2.6 Proof of Theorem 5.2.1

In this section, we conclude the proof of Theorem 5.2.1.

Existence. Let $T > 0$. We use the estimate (5.28) and assume that $\lambda < 1$ to obtain uniform estimates on g_m in $L^\infty(0,T;L^2(\Omega;L^2_{x,v}))$ and on Dg_m in $L^2(\Omega\times(0,T);L^2_{x,v})$ by some quantities depending on N, T, and the norm $\mathbb{E}\|g_{\mathrm{in}}\|^2$. As a consequence, $(g_m)_m$ admits a subsequence (still denoted $(g_m)_m$) such that

$$g_m \rightharpoonup g \text{ in } L^2(\Omega\times(0,T);L^2_{x,v})$$

where g, $Dg \in L^2(\Omega\times(0,T);L^2_{x,v})$. From (5.18) and the uniform estimates on the approximate solutions g_m in $L^\infty(0,T;L^2(\Omega;L^2_{x,v}))$, we can deduce (using Ascoli's Theorem and a diagonal argument) that there is a further subsequence of $(g_m)_m$ such that for all $t \in [0,T]$,

$$g_m(t) \rightharpoonup g(t) \text{ in } L^2(\Omega;L^2_{x,v}).$$

In particular, $g \in C_w([0,T];L^2(\Omega;L^2_{x,v}))$. We now have all in hands to pass to the limit $m \to \infty$ in (5.18). We deduce the existence of a solution g satisfying the points (i), (ii) and (iii) of Theorem 5.2.1.

Uniqueness. The uniqueness of the solution is a consequence of the fact that, if g is a solution of our problem in the sense of (i), (ii) and (iii) of Theorem 5.2.1, then g satisfies the following energy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|g\|^2 + \mathbb{E}\|Dg\|^2 \leq \lambda^2\mathbb{E}[\|Dg\|^2 + N\|g\|^2]. \tag{5.54}$$

Indeed, since $\lambda < 1$, (5.54) immediately gives, with Gronwall's lemma, that a solution with initial condition $g_{\mathrm{in}} \equiv 0$ is zero in $L^\infty(0,T;L^2(\Omega;L^2_{x,v}))$ for every $T > 0$. Hence the uniqueness by linearity of the problem. So let us explain why the estimate (5.54) holds true if g is a solution of our problem in the sense of (i), (ii) and (iii) of Theorem 5.2.1.
Assume for the time being that $v \cdot \nabla_x g \in L^2(\Omega \times (0,T);L^2_{x,v})$. Then take $\varphi = e_k$ in (iii) of Theorem 5.2.1, compute $\langle g(t), e_k\rangle^2$ thanks to the Itô formula, and sum over $k \in \mathbb{N}$ to obtain exactly (5.54). This computation makes sense thanks to the regularity of g given by (i) in Theorem 5.2.1: $g \in C_w([0,T];L^2(\Omega;L^2_{x,v}))$ and $Dg \in L^2(\Omega \times (0,T);L^2_{x,v})$, and thanks to the fact that the term $\int_0^t \langle v\cdot\nabla_x g(s), g(s)\rangle \,\mathrm{d}s$ is well defined and equals zero by the skew-symmetry of the transport operator.
Now, in the general case when $v \cdot \nabla_x g$ does not belong to $L^2(\Omega \times (0,T);L^2_{x,v})$, we regularise in space and truncate in velocity the solution g. Let us sketch the main ideas. Let $\rho \in C^\infty(\mathbb{T}^N)$ such that $\rho \geq 0$ and $\int_{\mathbb{T}^N} \rho = 1$ and $\Theta \in C_c^\infty(\mathbb{R}^N)$ such that $\mathrm{supp}(\Theta) \subset \{|v| \leq 2\}$ and $\Theta \equiv 1$ in $\{|v| \leq 1\}$. For any $\varepsilon > 0$, we introduce $\rho^\varepsilon(x) := \varepsilon^{-N}\rho(\varepsilon^{-1}x)$, $x \in \mathbb{T}^N$ and $\Theta^\varepsilon(v) := \Theta(\varepsilon v)$, $v \in \mathbb{R}^N$. Finally, we consider $g^\varepsilon := \Theta^\varepsilon\rho^\varepsilon * g$. We recall the equation satisfied by g

$$\mathrm{d}g + v\cdot\nabla_x g\;\mathrm{d}t - \lambda D^*g \odot \mathrm{d}W_t \;=\; -D^*Dg\;\mathrm{d}t.$$

As a consequence, g^ε satisfies

$$\mathrm{d}g^\varepsilon + v\cdot\nabla_x g^\varepsilon\;\mathrm{d}t - \Theta^\varepsilon\rho^\varepsilon * [\lambda D^*g \odot \mathrm{d}W_t] \;=\; -D^*[\Theta^\varepsilon\rho^\varepsilon * Dg]\;\mathrm{d}t + r^\varepsilon\,\mathrm{d}t \tag{5.55}$$

with

$$r^\varepsilon := v\cdot\nabla_x[\Theta^\varepsilon\rho^\varepsilon * g] - \Theta^\varepsilon\rho^\varepsilon * [v\cdot\nabla_x g] + D^*[\Theta^\varepsilon\rho^\varepsilon * Dg] - \Theta^\varepsilon\rho^\varepsilon * [D^*Dg].$$

We multiply Equation (5.55) by g^ε; the transport term disappears. Now we let $\varepsilon \to 0$. We use $g \in C_w([0,T];L^2(\Omega;L^2_{x,v}))$ and $Dg \in L^2(\Omega \times (0,T);L^2_{x,v})$ to justify the limit $\varepsilon \to 0$ of all the terms except the one with r^ε. The latter vanishes: we apply the commutation lemma of DiPerna, Lions, see [DL89, Lemma II.1.] Finally, at the limit, we recover (5.54) for the solution g.

Properties of the solution g. The fact that the quantity $\rho_\infty(g)$ is constant in time is straightforward. We now explain how to obtain the regularisation bound (5.14) and the estimates (5.15) and (5.16). They are proved using the corresponding bounds we derived on the approximate solutions g_m, that is (5.32) of Proposition 5.2.3 and the estimates obtained in Section 5.2.5. These estimates on g_m give in particular uniform energy bounds which, by considering a weakly converging subsequence in appropriate spaces and using the lower semi-continuity of the norm, allow us to pass to the limit $m \to \infty$ in the estimates on the approximate solutions g_m.

This ends the proof of Theorem 5.2.1.

5.3 Invariant measures

In this section, we prove the following result about existence and uniqueness of an invariant measure to the problem (5.6).

Theorem 5.3.1. *Suppose that hypothesis (5.2) is satisfied and let $g_{\mathrm{in}} \in L^2_{x,v}$. We assume $\lambda < \lambda_0$ where λ_0 is introduced in Theorem 5.2.1. For $w \in \mathbb{R}$, we introduce the space*

$$X_w := \left\{ g \in L^2_{x,v},\ \iint g\mathcal{M}^{\frac{1}{2}} = w \right\}.$$

Then, for any $w \in \mathbb{R}$, the problem

$$\begin{cases} dg \ + \ v\cdot\nabla_x g \ dt \ + \ \lambda\Big(\nabla_v - \dfrac{v}{2}\Big)\, g \ \odot dW_t \ = \ Lg \ dt, \\ g(0) = g_{in}, \\ \iint g_{\mathrm{in}}\mathcal{M}^{\frac{1}{2}} = w, \end{cases} \tag{P_w}$$

admits a unique invariant measure on X_w.

Proof. We fix $w \in \mathbb{R}$ and $t_0 > 0$. We suppose that $\lambda < \lambda_0$.

Proof of existence. Let $g_{\mathrm{in}} \in L^2_{x,v}$ such that $\iint g_{\mathrm{in}}\mathcal{M}^{\frac{1}{2}} = w$. We consider the unique solution g to the problem (P_w) given by Theorem 5.2.1. First of all, using the regularisation result (5.14) of Theorem 5.2.1, we deduce that there exists a constant $C(N, t_0) > 0$ such that

$$\mathbb{E}\|g(t_0)\|^2_{L^2_{\nabla,D}} \leq C\mathbb{E}\|g_{\mathrm{in}}\|^2. \tag{5.56}$$

We also recall the hypocoercive estimate (5.16) of Theorem 5.2.1: for $t \geq t_0$, we have

$$\mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}} \leq \ Ce^{-c(t-t_0)}\mathbb{E}\|g(t_0)\|^2_{L^2_{\nabla,D}} + K\mathbb{E}|\rho_\infty(g)|^2.$$

It implies, with (5.56),

$$\sup_{t\geq t_0} \mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}} \leq \ C\mathbb{E}\|g_{\mathrm{in}}\|^2 + Kw^2. \tag{5.57}$$

In the sequel, if X is a random variable, we denote by $\mathscr{L}(X)$ its law. We introduce, for any $T > 0$, the probability measures $(\mu_T)_{T>0}$ on $L^2_{x,v}$ defined by

$$\mu_T := \frac{1}{T}\int_{t_0}^{t_0+T} \mathscr{L}(g(t))\,\mathrm{d}t.$$

We show that the sequence $(\mu_T)_{T>0}$ is tight. Since the embedding $L^2_{\nabla,D} \subset L^2_{x,v}$ is compact, for any $R > 0$, the set

$$K_R := \{f \in L^2_{x,v},\ \|f\|_{L^2_{\nabla,D}} \leq R\}$$

is compact in $L^2_{x,v}$. Furthermore, we have, thanks to Markov's inequality and (5.57),

$$\begin{aligned} \mu_T(K_R^c) \ &= \ \frac{1}{T}\int_{t_0}^{t_0+T} \mathbb{P}(\|g(t)\|_{L^2_{\nabla,D}} > R)\,\mathrm{d}t \\ &\leq \ \frac{1}{TR^2}\int_{t_0}^{t_0+T} \mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}}\,\mathrm{d}t \\ &\leq \ \frac{1}{R^2}(C\mathbb{E}\|g_{\mathrm{in}}\|^2 + Kw^2). \end{aligned}$$

This easily implies that the sequence $(\mu_T)_{T>0}$ is tight. By Prohorov's Theorem, we obtain that $(\mu_T)_{T>0}$ admits a subsequence (still denoted $(\mu_T)_{T>0}$) such that μ_T converges to some probability measure μ on $L^2_{x,v}$ as $T \to \infty$. Furthermore, it is classical to show that this limit measure μ is indeed an invariant measure for the problem (P_w), see for instance [DPZ08, Proposition 11.3].

Proof of uniqueness. To conclude, we prove uniqueness of the invariant measure. Let μ_1 and μ_2 be two invariant measures of the problem (P_w). We choose $g_{\mathrm{in},1}$ and $g_{\mathrm{in},2}$ two random variables with respective laws μ_1 and μ_2 and denote by g_1 and g_2 the solutions to (P_w) with respective initial conditions $g_{\mathrm{in},1}$ and $g_{\mathrm{in},2}$. We introduce

$$r(t) := g_1(t) - g_2(t), \quad t \geq 0.$$

Thanks to the regularisation result (5.14) of Theorem 5.2.1, we deduce that there exists a constant $C(N, t_0) > 0$ such that

$$\mathbb{E}\|r(t_0)\|^2_{L^2_{\nabla,D}} \leq C\mathbb{E}\|g_{\mathrm{in},1}\|^2 + C\mathbb{E}\|g_{\mathrm{in},2}\|^2. \tag{5.58}$$

Using the hypocoercive estimate (5.16) of Theorem 5.2.1, we have, for $t \geq t_0$,

$$\mathbb{E}\|r(t)\|^2_{L^2_{\nabla,D}} \leq \; Ce^{-c(t-t_0)}\mathbb{E}\|r(t_0)\|^2_{L^2_{\nabla,D}} + K\mathbb{E}|\rho_\infty(r)|^2.$$

With (5.58) and the identity $\rho_\infty(r) = \rho_\infty(g_{\mathrm{in},1}) - \rho_\infty(g_{\mathrm{in},2}) = w - w = 0$, it implies, for $t \geq t_0$,

$$\mathbb{E}\|r(t)\|^2_{L^2_{\nabla,D}} \leq \; Ce^{-c(t-t_0)}(\mathbb{E}\|g_{\mathrm{in},1}\|^2 + \mathbb{E}\|g_{\mathrm{in},2}\|^2).$$

Note that, with (5.9), we have $\frac{N}{2}\|f\|^2 \leq \|f\|^2_{L^2_{\nabla,D}}$ so that we finally deduce

$$\mathbb{E}\|r(t)\|^2 \leq \; Ce^{-c(t-t_0)}(\mathbb{E}\|g_{\mathrm{in},1}\|^2 + \mathbb{E}\|g_{\mathrm{in},2}\|^2). \tag{5.59}$$

To conclude, we take a Lipschitz continuous function $\Psi : L^2_{x,v} \to \mathbb{R}$ and write, thanks to the Cauchy-Schwarz inequality, for any $t \geq t_0$,

$$\begin{aligned} |\langle \mu_1 - \mu_2, \Psi\rangle|^2 &= |\mathbb{E}\left[\Psi(g_1(t)) - \Psi(g_2(t))\right]|^2 \\ &\leq \|\Psi\|^2_{\mathrm{Lip}}\mathbb{E}\|r(t)\|^2, \end{aligned}$$

from which we deduce, using (5.59) with $t \to \infty$, that $\mu_1 = \mu_2$. □

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Résumé

Cette thèse présente quelques résultats dans le domaine des équations aux dérivées partielles stochastiques. Une majeure partie d'entre eux concerne l'étude de limites diffusives de modèles cinétiques perturbés par un terme aléatoire. On présente également un résultat de régularité pour une classe d'équations aux dérivées partielles stochastiques ainsi qu'un résultat d'existence et d'unicité de mesures invariantes pour une équation de Fokker-Planck stochastique.

Dans un premier temps, on présente trois travaux d'approximation-diffusion dans le contexte stochastique. Le premier s'intéresse au cas d'une équation cinétique avec opérateur de relaxation linéaire dont l'équilibre des vitesses a un comportement de type puissance à l'infini. L'équation est perturbée par un processus Markovien. Cela donne lieu à une limite fluide stochastique fractionnaire. Les deux autres résultats concernent l'étude de l'équation de transfert radiatif qui est un problème cinétique non linéaire. L'équation est bruitée dans un premier temps avec un processus de Wiener cylindrique et dans un second temps par un processus Markovien. Dans les deux cas, on obtient à la limite une équation de Rosseland stochastique.

Dans la suite, on présente un résultat de régularité pour les équations aux dérivées partielles quasi-linéaires de type parabolique dont la partie aléatoire est gouvernée par un processus de Wiener cylindrique. Enfin, on étudie une équation de Fokker-Planck qui présente un terme de forçage aléatoire régi par un processus de Wiener cylindrique. On prouve d'une part l'existence et l'unicité des solutions de ce problème et d'autre part l'existence et l'unicité de mesures invariantes pour la dynamique de cette équation.

Mots-clés
Équation aux dérivées partielles stochastiques, approximation diffusion, limite de diffusion, limite fluide, limite hydrodynamique, méthode des fonctions test perturbées, développement de Hilbert, lemme de moyenne stochastique, régularité d'équations aux dérivées partielles quasi linéaires de type parabolique, mesures invariantes, équation de Fokker-Planck.

Abstract

This thesis presents several results about stochastic partial differential equations. The main subject is the study of diffusive limits of kinetic models perturbed with a random term. We also present a result about the regularity of a class of stochastic partial differential equations and a result of existence and uniqueness of invariant measures for a stochastic Fokker-Planck equation.

First, we give three results of approximation-diffusion in a stochastic context. The first one deals with the case of a kinetic equation with a linear operator of relaxation whose velocity equilibrium has a power tail distribution at infinity. The equation is perturbed with a Markovian process. This gives rise to a stochastic fluid fractional limit. The two remaining results consider the case of the radiative transfer equation which is a non-linear kinetic equation. The equation is perturbed successively with a cylindrical Wiener process and with a Markovian process. In both cases, we are led to a stochastic Rosseland fluid limit.

Then, we introduce a result of regularity for a class of quasi-linear stochastic partial differential equations of parabolic type whose random term is driven by a cylindrical Wiener process. Finally, we study a Fokker-Planck equation with a noisy force governed by a cylindrical Wiener process. We prove existence and uniqueness of solutions to the problem and then existence and uniqueness of invariant measures to the equation.

Keywords
Stochastic partial differential equations, approximation diffusion, diffusive limit, fluid limit, hydrodynamic limit, perturbed test functions method, Hilbert expansion, stochastic averaging lemma, regularity of quasi linear stochastic partial differential equations of parabolic type, invariant measures, Fokker-Planck equation.

N° d'ordre : ....

École normale supérieure de Rennes
Campus de Ker Lann - Avenue Robert Schuman - 35170 BRUZ
Tél : +33(0)2 99 05 93 00 - Fax : +33(0)2 99 05 93 29 - **www.ens-rennes.fr**