Quasi Riesz transforms, Hardy spaces and generalized sub-Gaussian heat kernel estimates

Li Chen

UNIVERSITÉ PARIS-SUD
École Doctorale de Mathématiques de la région Paris-Sud
Laboratoire de Mathématiques de la Faculté des Sciences d'Orsay

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Quasi transformées de Riesz, espaces de Hardy et estimations sous-gaussiennes du noyau de la chaleur

Directeurs de thèse :	M.	Pascal	AUSCHER
	M.	Thierry	COULHON
Rapporteurs :	M.	Frédéric	BERNICOT
	M.	Peer	KUNSTMANN

Soutenue le 24 Avril 2014 devant le jury composé de :

M.	Pascal	AUSCHER	: Directeur de thèse
M.	Thierry	COULHON	: Directeur de thèse
M.	Andrew	HASSELL	: Président du jury
M.	Alan	MCINTOSH	: Examinateur
M.	Pierre	PORTAL	: Examinateur
M.	Adam	SIKORA	: Examinateur

li.chen@math.u-psud.fr
li.chen@anu.edu.au

Thèse préparée au

Département de Mathématiques d'Orsay

Laboratoire de Mathématiques d'Orsay (UMR 8628), Bât. 425

Université Paris-Sud 11

91405 Orsay Cedex

France

Declaration

Except where otherwise indicated, this thesis is my own original work.

The material in Section 1.1-Section 1.4 and Section 2.1-Section 2.3 is included in the preprint [Che14b] titled “Sub-gaussian heat kernel estimates and quasi Riesz transforms for $1 \leq p \leq 2$”.

Chapter 3 is included in the preprint [Che14a] titled “Hardy spaces on metric measure spaces with generalized heat kernel estimates”.

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Résumé

Dans cette thèse nous étudions les transformées de Riesz et les espaces de Hardy associés à un opérateur sur un espace métrique mesuré. Ces deux sujets sont en lien avec des estimations du noyau de la chaleur associé à cet opérateur.

Dans les Chapitres 1, 2 et 4, on étudie les transformées quasi de Riesz sur les variétés riemannienne et sur les graphes. Dans le Chapitre 1, on prouve que les quasi transformées de Riesz sont bornées dans L^p pour $1 < p \leq 2$. Dans le Chapitre 2, on montre que les quasi transformées de Riesz est aussi de type faible $(1,1)$ si la variété satisfait la propriété de doublement du volume et l'estimation sous-gaussienne du noyau de la chaleur. On obtient des résultats analogues sur les graphes dans le Chapitre 4.

Dans le Chapitre 3, on développe la théorie des espaces de Hardy sur les espaces métriques mesurés avec des estimations différentes localement et globalement du noyau de la chaleur. On définit les espaces de Hardy par les molécules et par les fonctions quadratiques. On montre tout d'abord que ces deux espaces H^1 sont les mêmes. Puis, on compare l'espace H^p défini par par les fonctions quadratiques et L^p. On montre qu'ils sont équivalents. Mais on trouve des exemples tels que l'équivalence entre L^p et H^p défini par les fonctions quadratiques avec l'homogénéité t^2 n'est pas vraie. Finalement, comme application, on montre que les quasi transformées de Riesz sont bornées de H^1 dans L^1 sur les variétés fractales.

Dans le Chapitre 5, on prouve des inégalités généralisées de Poincaré et de Sobolev sur les graphes de Vicsek. On aussi montre qu'elles sont optimales.
Mot-clés: Transformées de Riesz, espaces de Hardy, espaces métriques mesurés, graphes, l'estimations du noyau de la chaleur.

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Abstract

In this thesis, we mainly study Riesz transforms and Hardy spaces associated to operators. The two subjects are closely related to volume growth and heat kernel estimates.

In Chapter 1, 2 and 4, we study Riesz transforms on Riemannian manifold and on graphs. In Chapter 1, we prove that on a complete Riemannian manifold, the quasi Riesz transform is always L^p bounded for $1 < p \leq 2$. In Chapter 2, we prove that the quasi Riesz transform is also of weak

type $(1,1)$ if the manifold satisfies the doubling volume property and the sub-Gaussian heat kernel estimate. Similarly, we show in Chapter 4 the same results on graphs.

In Chapter 3, we develop a Hardy space theory on metric measure spaces satisfying the doubling volume property and different local and global heat kernel estimates. Firstly we define Hardy spaces via molecules and square functions which are adapted to the heat kernel estimates. Then we show that the two H^1 spaces via molecules and viasquare functions are the same. Also, we compare the H^p space defined via square functions with L^p. The corresponding H^p $(p > 1)$ space defined via square functions is equivalent to the Lebesgue space L^p. However, it is shown that in this situation, the H^p space corresponding to Gaussian estimates does not coincide with L^p any more. Finally, as an application of this Hardy space theory, we proved the $H^1 - L^1$ boundedness of quasi Riesz transforms on fractal manifolds.

In Chapter 5, we consider Vicsek graphs. We prove generalised Poincaré inequalities and Sobolev inequalities on Vicsek graphs and we show that they are optimal.

Keywords: Riesz transforms, Hardy spaces, metric measure spaces, graphs, heat kernel estimates.

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Contents

Introduction en français

Dans cette thèse nous étudions les transformées de Riesz et les espaces de Hardy associés à un opérateur sur un espace métrique mesuré. Ces deux sujets sont en lien avec des estimations du noyau de la chaleur associé à cet opérateur. Nos buts majeurs sont: le comportement des transformées de Riesz sur $L^p(M)$ où M est une variété satisfaisant la propriété de doublement du volume et l'estimation sous-gaussienne du noyau de la chaleur; la théorie des espaces de Hardy sur les espaces métriques mesurés avec des estimations différentes localement et globalement du noyau de la chaleur.

Nous commencerons par introduire nos différents cadres de travail: variétés riemanniennes, graphes, espaces métriques mesurés de Dirichlet. Puis nous décrirons des estimations du noyau de la chaleur dans tous les cadres que nous considérons. Et après, nous introduirons les problèmes de transformée de Riesz et la théorie des espaces de Hardy. Finalement, nous présenterons nos résultats principaux.

0.1 Cadre

Soit (M,d,μ) un espace métrique mesuré. Pour tout $x\in M$ et tout $r>0$, on désigne par $B(x,r)$ la boule de centre x et de rayon r, et par $V(x,r)$ sa mesure.

Notation 0.1. On note $x\simeq y$ pour signifier qu'il existe deux constantes $c,C>0$ avec $c<C$ telles que $cx\leq y\leq Cx$ (uniformément en x et y).

Soit B la boule $B(x,r)$. Nous notons λB la boule de centre x et de rayon λr. Nous notons $C_1(B)=4B$ et $C_j(B)=2^{j+1}B/2^jB$ pour $j=2,3,\cdots$.

Définition 0.2. On dit que (M,d,μ) satisfait la propriété de doublement du volume s'il existe $C>0$ tel que

$$V(x,2r)\leq CV(x,r),\ \forall x\in M,\forall r>0. \tag{D}$$

Variétés riemanniennes Soit M une variété riemannienne complète non-compacte. On note d la distance géodésique, μ la mesure riemannienne, ∇ le gradient riemannien et Δ l'opérateur de

Laplace-Beltrami positif. Par définition et par le théorème spectral, on a

$$\int_M \nabla f \cdot \nabla g d\mu = (\Delta f, g) = \int_M (\Delta^{1/2} f)(\Delta^{1/2} g) d\mu,\ \forall f, g \in \mathscr{C}_0^\infty(M).$$

Soit $e^{-t\Delta}$ le semi-groupe de la chaleur, et soit $p_t(x,y)$ le noyau de la chaleur associé. Alors $(e^{-t\Delta})_{t>0}$ est une famille de contractions linéaires sur $L^2(M)$. En fait, on a le théorème suivant

Théorème 0.3 ([Gri09, Str83])**.** Le semi-groupe de la chaleur $(e^{-t\Delta})_{t>0}$ sur $L^2(M)$ admet un noyau unique $p_t(x,y)$ satisfaisant

1. $p_t(x,y) > 0$ est une fonction C^∞ sur $\mathbb{R}^+ \times M \times M$.

2. $p_t(x,y) = p_t(y,x)$ pour tous $x, y \in M$ et $t > 0$.

3. Pour toute $f \in L^2(M)$, $e^{-t\Delta}$ s'écrit sous la forme de l'intégrale suivante:

$$e^{-t\Delta} f(x) = \int_M p_t(x,y) f(y) d\mu(y).$$

4. Le semi-groupe est sous-markovien. Si $f \in L^2(M)$, alors $0 \le f \le 1 \Rightarrow 0 \le e^{-t\Delta} f \le 1$.

5. $\|e^{-t\Delta} f\|_p \le \|f\|_p$ pour tout $t > 0$ et $f \in L^2 \bigcap L^p$, $1 \le p \le \infty$, avec

$$\|e^{-t\Delta} f - f\|_p \to 0$$

   pour $1 \le p < \infty$.

6. Pour tout $f \in L^2(M)$, on a $\frac{d}{dt} e^{-t\Delta} f = -\Delta e^{-t\Delta} f$. Cette égalité a lieu aussi pour $f \in L^p$, $1 \le p \le \infty$, si on définit $e^{-t\Delta} f$ comme dans 3.

Graphes Soit Γ un graphe infini connexe localement uniformément fini. On suppose Γ muni de sa distance naturelle d et muni d'un poids symétrique μ sur $\Gamma \times \Gamma$. Deux points x et y de Γ sont voisins si et seulement si $\mu_{xy} > 0$. On note $x \sim y$. On désigne $\mu(x) = \sum_{y \sim x} \mu_{xy}$. Alors pour $\Omega \subset \Gamma$, on a une mesure définie par

$$\mu(\Omega) = \sum_{x \in \Gamma} \mu(x).$$

Soit p un noyau de Markov réversible par rapport à μ. C'est-à-dire,

$$p(x,y)\mu(x) = \mu_{xy} = p(y,x)\mu(y),\ \ \forall x, y \in \Gamma;$$

$$\sum_{y \in \Gamma} p(x,y) = 1,\ \ \forall x \in \Gamma.$$

On suppose que Γ satisfait la condition $\Delta(\alpha)$. C'est-à-dire, il existe $c>0$ tel que pour tous $x,y\in\Gamma$

$$x\sim y \text{ implique } p(x,y)\geq c, \text{ et } x\sim x.$$

On définit les noyaux itérés de p par

$$p_0(x,y):=\delta(x,y)=\begin{cases}1, & x=y,\\ 0, & x\neq y;\end{cases}$$
$$p_k(x,y)=\sum_{z\in\Gamma}p_{k-1}(x,z)p(z,y),\ k\geq 1.$$

L'opérateur associé à p est défini par

$$Pf(x)=\sum_{y\in\Gamma}p(x,y)f(y).$$

Alors pour tout $k\in\mathbb{N}$, on a

$$P^kf(x)=\sum_{y\in\Gamma}p_k(x,y)f(y).$$

On dit que $I-P$ est le laplacien sur Γ.

On définit la longueur du gradient de f en $x\in\Gamma$ par

$$|\nabla f(x)|=\left[\frac{1}{2}\sum_y p(x,y)|f(x)-f(y)|^2\right]^{1/2}.$$

et on considère $p_{k+1}(y,x)-p_k(x,y)$ comme la dérivée discrète en temps de $p_k(y,x)$.

Par un simple calcul, on obtient

$$\|\nabla f\|_2^2=<(I-P)f,f>=\|(I-P)^{\frac{1}{2}}f\|_2^2,$$

où $(I-P)^{\frac{1}{2}}$ est défini par la théorie spectrale. L'hypothèse $\triangle(\alpha)$ implique $-1\notin \mathrm{Spec}\,(P)$ (voir [Rus00]). Par conséquent,

$$P=\int_a^1\lambda dE_\lambda,\text{ où } a>-1.$$

On a aussi l'analyticité P sur L^2 (voir [CSC90, Proposition 3]). C'est-à-dire, il existe $C>0$ tel que pour tout $n\in\mathbb{N}$,

$$\left\|P^n-P^{n+1}\right\|_{2\to 2}\leq Cn^{-1}.$$

Pour tout $y\in\Gamma$, $p_\cdot(\cdot,y)$ satisfait l'équation de la chaleur

$$\mu(x)(u(n+1,x)-u(n,x))=\sum_y\mu_{xy}(u(n,y)-u(n,x)),$$

ou

$$\mu(x)u(n+1,x)=\sum_y \mu_{xy}u(n,y).$$

Espaces métriques mesurés Soit (M,μ) un espace localement compact avec μ une mesure de Radon positive. Soit $\mathscr{E}$ une forme de Dirichlet régulière et fortement locale sur $\mathscr{D}\subset L^2(M,d\mu)$. C'est-à-dire,

- Soit $\mathscr{C}_0(M)$ l'ensemble des fonctions continues à support compact sur M. Alors $\mathscr{D}\cap\mathscr{C}_0(M)$ est dense dans $\mathscr{C}_0(M)$ sous la norme uniforme et est dense dans $\mathscr{D}$ sous la norme $(\|f\|_2^2+\mathscr{E}(f,f))^{1/2}$.
- Soit f_1,f_2 deux fonctions à support compact. Si f_2 est constante sur un ensemble U_1 contenant supp f, alors $\mathscr{E}(f_1,f_2)=0$.

On désigne par L l'opérateur positif auto-adjoint associé à $\mathscr{E}$. Alors

$$\mathscr{E}(f,g)=<Lf,g>=<f,Lg>=\int_M d\Gamma(f,g).$$

Le semi-groupe engendré e^{-tL} est une contraction sur $L^2(M)$. De plus, e^{-tL} est sous-markovien et on peut le considérer comme un opérateur contractant sur $L^p(M)$, $1\le p\le\infty$.

$\mathscr{E}$ admet une "mesure d'énergie" Γ:

$$\mathscr{E}(f,g)=\int_M d\Gamma(f,g),\ \forall f,g\in\mathscr{D}.$$

Pour $x,y\in M$, la distance est définie par

$$d(x,y)=\sup\{f(x)-f(y): d\Gamma(f,f)\le d\mu, f\in\mathscr{C}\},$$

où $d\Gamma(f,f)\le d\mu$ signifie que $\Gamma(f,f)$ est absolument continue par rapport à μ avec la dérivée de Radon-Nikodym bornée par 1. On doit supposer que d est une distance qui redonne la topologie initiale et pour laquelle M est complet.

Nous dirons que $(M,d,\mu,\mathscr{E})$ est un espace métrique mesuré de Dirichlet.

Si $d\Gamma$ est absolument continue par rapport à μ, nous dirons que $\mathscr{E}$ admet un carré du champ. C'est-à-dire, il existe une unique forme bilinéaire symétrique positive continue de $\mathscr{D}\times\mathscr{D}$ dans L^1 (qu'on désigne encore par Γ) telle que

$$\mathscr{E}(fh,g)+\mathscr{E}(gh,f)-\mathscr{E}(h,fg)=\int_M h\Gamma(f,g)d\mu,\ \forall f,g,h\in\mathscr{D}\cap L^\infty.$$

Voici quelques exemples d'espaces métriques mesurés munis d'une forme de Dirichlet:

Exemple 0.4. Variétés riemanniennes.

Soit (M,d,μ) une variété riemannienne et Δ l'opérateur de Laplace-Beltrami. Alors

$$\mathscr{E}(f,f) := \int_M (\Delta f) f d\mu,\ \forall f \in \mathscr{C}_0^\infty(M).$$

Exemple 0.5. Espaces euclidiens avec un opérateur elliptique sous forme divergence.

Soit $a = (a_{ij}(x))_{n\times n}$ une matrice bornée et mesurée. On definit

$$\mathscr{E}(f,f) := \int_{\mathbb{R}^n} \nabla f \cdot a \nabla f,\ \forall f \in \mathscr{C}_0^\infty(\mathbb{R}^n).$$

Exemple 0.6. Fractals.

Par exemple, les tapis de Sierpiński, les triangles de Sierpiński, les fractals de Vicsek etc (voir [BB92, BB99a, BCG01, Bar13]).

Sur un tapis de Sierpiński F, soit F_n une série d'approximations de F et $\mathscr{E}_n$ la forme convenable de Dirichlet sur F_n. Alors on peut construire une forme de Dirichlet $\mathscr{E}$ sur F. Voir [Bar13] pour les détails.

Exemple 0.7. Systèmes de câbles (voir [Var85, BB04]).

Soit (G,d,ν) un graphe. On definit le système de câbles G_C en remplaçant chaque arête de G par une copie de $(0,1)$. La mesure μ sur G_C est définie par $d\mu(t) = \nu_{xy} dt$, où t parcourt le câble reliant deux sommets x et y. Notons $\mathscr{C}$ les fonctions à support compact sur $C(G_C)$ et qui sont C^1 sur chaque câble. On définit

$$\mathscr{E}(f,f) := \int_{G_C} \left|f'(t)\right|^2 d\mu(t),\ \forall f \in \mathscr{C}.$$

0.2 Estimations du noyau de la chaleur

Soit (M,d,μ) un espace métrique mesuré complet. Soit L un opérateur positif auto-adjoint sur $L^2(M,\mu)$ qui engendre un semi-groupe analytique $(e^{-tL})_{t>0}$. Notons que $(e^{-tL})_{t>0}$ n'est pas nécessairement uniformément borné sur $L^1(M,\mu)$. On connaît des exemples de semi-groupes qui ne sont pas bornés sur L^1, mais seulement sur L^p où $p \in [p_0, p_0']$ pour un $p_0 > 1$ (voir par exemple, [Aus07]).

Soit $1 < \beta_1 \leq \beta_2$. Dans cette thèse, nous considérons les estimations du noyau de la chaleur suivantes.

L'hypothèse (A1): L'estimation $L^2 - L^2$ non-classique de Davies-Gaffney: $\forall j \geq 1$, il existe $C, c > 0$

tels que pour tous $x, y \subset M$,

$$\left\| \mathbb{1}_{B(x,t)} e^{-\rho(t)L} \mathbb{1}_{B(y,t)} \right\|_{2\to 2} \leq \begin{cases} C \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_1}{\beta_1-1}}\right) & 0<t<1, \\ C \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_2}{\beta_2-1}}\right), & t\geq 1. \end{cases} \tag{DG_{β_1,β_2}}$$

où

$$\rho(t) = \begin{cases} t^{\beta_1}, & 0<t<1, \\ t^{\beta_2}, & t \geq 1; \end{cases}$$

L'hypothèse (A2): Soit $1 \leq p_0 < 2$ et ρ comme dans (**A1**). L'estimation $L^{p_0} - L^{p_0'}$ non-classique hors-diagonale: pour tout $x, y \in M$ and $t > 0$, et $\forall j \geq 1$,

$$\left\| \mathbb{1}_{B(x,t)} e^{-\rho(t)L} \mathbb{1}_{B(y,t)} \right\|_{p_0\to p_0'} \leq \begin{cases} \dfrac{C}{V^{\frac{1}{p_0}-\frac{1}{p_0'}}(x,t)} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_1}{\beta_1-1}}\right) & 0<t<1, \\ \dfrac{C}{V^{\frac{1}{p_0}-\frac{1}{p_0'}}(x,t)} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_2}{\beta_2-1}}\right), & t\geq 1. \end{cases} \tag{$DG^{p_0,p_0'}_{\beta_1,\beta_2}$}$$

Remark 0.8. Une estimation équivalente à (**A2**) est l'estimation $L^{p_0} - L^2$ hors-diagonale

$$\left\| \mathbb{1}_{B(x,t)} e^{-\rho(t)L} \mathbb{1}_{B(y,t)} \right\|_{p_0\to 2} \leq \begin{cases} \dfrac{C}{V^{\frac{1}{p_0}-\frac{1}{2}}(x,t)} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_1}{\beta_1-1}}\right) & 0<t<1, \\ \dfrac{C}{V^{\frac{1}{p_0}-\frac{1}{2}}(x,t)} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_2}{\beta_2-1}}\right), & t\geq 1. \end{cases}$$

On renvoie à [BK05, Blu07, CS08] pour les démonstrations.

L'hypothèse (A3): Soit ρ comme dans (**A1**). L'estimation ponctuelle généralisée du noyau de la chaleur: pour tout $x, y \in M$ and $t > 0$,

$$p_t(x,y) \leq \frac{C_2}{V(y,\rho^{-1}(t))} \exp\left(-c_2 G(t, d(x,y))\right), \tag{UE_{β_1,β_2}}$$

où

$$G(r,t) \simeq \begin{cases} \left(\frac{r^{\beta_1}}{t}\right)^{1/(\beta_1-1)}, & t \le r, \\ \left(\frac{r^{\beta_2}}{t}\right)^{1/(\beta_2-1)}, & t \ge r. \end{cases}$$

Remarque 0.9. Notons que $\beta_1 \le \beta_2$. Alors pour $d(x,y) \le t$, on a

$$\left(\frac{d^{\beta_1}(x,y)}{t}\right)^{1/(\beta_1-1)} \le \left(\frac{d^{\beta_2}(x,y)}{t}\right)^{1/(\beta_2-1)}.$$

Et pour $t \le d(x,y)$, on a $\left(\frac{d^{\beta_1}(x,y)}{t}\right)^{1/(\beta_1-1)} \ge \left(\frac{d^{\beta_2}(x,y)}{t}\right)^{1/(\beta_2-1)}$. Donc (UE_{β_1,β_2}) implique l'estimation suivante:

$$p_t(x,y) \le \begin{cases} \frac{C}{V(x,t^{1/\beta_1})} \exp\left(-c\left(\frac{d^{\beta_1}(x,y)}{t}\right)^{1/(\beta_1-1)}\right), & 0<t<1, \\ \frac{C}{V(x,t^{1/\beta_2})} \exp\left(-c\left(\frac{d^{\beta_2}(x,y)}{t}\right)^{1/(\beta_2-1)}\right), & t \ge 1. \end{cases}$$

Remarque 0.10. On a $(\mathbf{A3}) \Rightarrow (\mathbf{A2}) \Rightarrow (\mathbf{A1})$. En fait,

Lemme 0.11. Soit M vérifiant (D) et (UE_{β_1,β_2}), $\beta_1 \le \beta_2$. Alors, pour tous $x \in M$ et $j \ge 1$, on a $\forall K \in \mathbb{N}$ et $1 \le p \le 2$

$$\frac{\left\|(\rho(t)L)^K e^{-\rho(t)L} f\right\|_{L^2(C_j(B(x,t)))}}{V^{1/2}(2^{j+1}B(x,t))} \le \begin{cases} \frac{C\exp\left(-c2^{\frac{j\beta_1}{\beta_1-1}}\right)}{V^{1/p}(x,t)} \|f\|_{L^p(B)}, & 0<t<1, \\ \frac{C\exp\left(-c2^{\frac{j\beta_2}{\beta_2-1}}\right)}{V^{1/p}(x,t)} \|f\|_{L^p(B)}, & t \ge 1. \end{cases}$$

Comme dans [BK05], on obtient

Lemme 0.12. Soit M vérifiant (D) et (DG_{β_1,β_2}) avec $\beta_1 \le \beta_2$. Soit B une boule de centre $x \in M$ et de rayon $r > 0$. Alors pour tous $k,l \in \mathbb{N}$ avec $0 \le l \le k-2$ et pour tout $t \in (0, 2^{k+1}r)$, on a

$$\left\|\mathbb{1}_{C_k(B)}(\rho(t)L)^K e^{-\rho(t)L}\mathbb{1}_{C_l(B)}\right\|_{2\to 2} \le \begin{cases} C\exp\left(-c2^{\frac{k\beta_1}{\beta_1-1}}\right), & 0<t<1, \\ C\exp\left(-c2^{\frac{k\beta_2}{\beta_2-1}}\right), & t \ge 1. \end{cases}$$

Exemple 0.13. Toutes les variétés riemanniennes vérifient $(DG_{2,2})$, voir [Dav92].

Les variétés riemanniennes à courbure de Ricci positive vérifient ($UE_{2,2}$), voir [LY86].

Exemple 0.14. Certaines variétés fractales vérifient ($UE_{2,m}$), $m > 2$.

Soit (G, F, ν) un graphe infini connexe satisfaisant la croissance polynomiale du volume: $V(x,r) \simeq r^D$ et l'estimation sous-gaussienne du noyau de la chaleur:

$$p_k(x,y) \leq \frac{C\mu(y)}{k^{D/m}} \exp\left(-c\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right), \tag{1}$$

où $D \geq 1$ et $2 \leq m \leq D+1$. En fait, pour tout $D \geq 1$ et tous m satisfaisant $2 \leq m \leq D+1$, il existe un graphe qui satisfait (1). On construit une variété riemannienne M à partir de G en remplaçant toutes les arêtes par les tubes de longueur 1 et puis en les collant de façon lisse aux sommets (voir [BCG01] pour la variété de Vicsek). On note Δ l'opérateur Laplace-Beltrami sur M et $e^{-t\Delta}$ le noyau de la chaleur associé. Alors le noyau de la chaleur $p_t(x,y)$ satisfait (UE_{β_1,β_2}) avec $\beta_1 = 2$ and $\beta_2 = m > 2$.

Exemple 0.15. Espace euclidien avec des opérateurs elliptiques sous forme divergence.

Soit L un opérateur homogène elliptique sur $L^2(\mathbb{R}^n)$ d'order $2m$ ($m \geq 1$) sous forme divergence:

$$L := (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha\left(a_{\alpha,\beta}\partial^\beta\right),$$

où $a_{\alpha,\beta}$ est borné pour tous α, β. Alors ($UE_{2m,2m}$) a lieu si $n \leq 2m$. Si $n > 2m$, Alors ($DG^{p_0,p_0'}_{\beta_1,\beta_2}$) a lieu pour $p_0 = \frac{2n}{n+2m}$. Voir par exemple, [Aus07, Dav95, Dav97a] etc.

Exemple 0.16. Systèmes de câbles. Par exemple, pour la système de câbles associé au triangle de Sierpinski (sur $\mathbb{Z}^2$), $p_t(x,y)$ satisfait (UE_{β_1,β_2}) avec $\beta_1 = 2$ and $\beta_2 = \log 5/\log 2$ (voir [Jon96, BB04]).

Plus généralement, Hebisch and Saloff-Coste [HSC01] considèrent un espace métrique mesuré complet non-compact de Dirichlet, sur lequel le noyau de la chaleur vérifie l'estimation supérieure $\rho-$Gaussienne

$$p_t(x,y) \simeq \frac{C_2}{V(y,\rho^{-1}(t))} \exp\left(-c_2 G(t, d(x,y))\right),$$

où ρ^{-1} est la fonction inverse de ρ. Notons que ρ et G vérifient certaines relations qu'on ignore ici (voir [HSC01, Section 5] pour les détails). En fait, cette estimation non-classique est équivalente à une inégalité de Harnack parabolique, voir aussi [GT12].

0.3 Transformée de Riesz sur les variétés non-compactes

Strichartz a posé en 1983 la question de savoir pour quelle variétés riemanniennes non-compactes M et pour quels $p \in (1, +\infty)$ les semi-normes $\| |\nabla f| \|_p$ et $\left\|\Delta^{1/2} f\right\|_p$ sont équivalentes, lorsque f

parcourt les fonctions lisses à support compact sur M. C'est à dire, quand il existe des constantes $c, C > 0$ telles que

$$c\|\Delta^{1/2} f\|_p \leq \||\nabla f|\|_p \leq C\|\Delta^{1/2} f\|_p, \ \forall f \in \mathscr{C}_0^\infty(M)? \tag{E_p}$$

Plus précisément, on dit que M satisfait (R_p) si

$$\||\nabla f|\|_p \leq C \left\| \Delta^{1/2} f \right\|_p \tag{R_p}$$

et que M satisfait (RR_p) si

$$\left\| \left| \Delta^{1/2} f \right| \right\|_p \leq C \|\nabla f\|_p. \tag{RR_p}$$

La propriété (R_p) revient à dire que la transformée de Riesz est continue sur $L^p(M)$; elle se dualise en $(RR_{p'})$ (voir [CD03]). Si (R_p) et (RR_p) ont lieu, cela signifie que les deux définitions concurrentes des espaces de Sobolev homogène d'ordre un dans L^p coïncident sur M. Clairement, (R_2) et (RR_2) sont vraies, mais pour $p \neq 2$, il s'agit d'un problème d'intégrales singulières, où les méthodes classiques de Calderón-Zygmund amenaient à des hypothèses trop coûteuses. Beaucoup de travaux ont été faits pour résoudre ce problème, voir par exemple [Bak87, CD99, CD03, ACDH04, AC05, CCH06, Car07, CS10, Devar] et leurs références.

Des estimations du noyau de la chaleur jouent un rôle très important dans ces problèmes. Voici des estimations du noyau de la chaleur qu'on connaît bien.

L'estimation supérieure diagonale:

$$p_t(x,x) \leq \frac{c}{V(x,\sqrt{t})}, \ \forall x \in M, t > 0. \tag{DUE}$$

L'estimation supérieure gaussienne:

$$p_t(x,y) \leq \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{Ct}\right), \ \forall x,y \in M, t > 0. \tag{UE}$$

En fait, sous l'hypothèse (D), les estimations (DUE) et (UE) sont équivalentes (voir [CS08], [Gri09]).

L'estimation de Li-Yau: pour tous $x, y \in M$ et pour tout $t > 0$,

$$\frac{c}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right) \leq p_t(x,y) \leq \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{Ct}\right). \tag{LY}$$

L'estimation du gradient du noyau de la chaleur:

$$|\nabla p_t(x,y)| \leq \frac{C}{\sqrt{t}V(y,\sqrt{t})}, \ \forall x,y \in M, t > 0. \tag{G}$$

En 1999, Coulhon et Duong ont prouvé le résultat suivant:

Theorem 0.17. *Soit M une variété riemannienne complète satisfaisant la propriété de doublement (D). Supposons une estimation gaussienne supérieure du noyau de la chaleur (DUE). Alors la transformée de Riesz $\nabla\Delta^{-1/2}$ est bornée sur L^p pour $1 < p \leq 2$.*

Si M satisfait (D) et (DUE), on sait que (R_p) peut être en défaut si $p > 2$, mais on ne sait pas si (RR_p) peut être en défaut pour $1 < p < 2$. Si M satisfait seulement (D), alors (RR_p) peut être en défaut pour $1 < p < 2$. On a les deux exemples suivants:

Variété somme connexe de deux espaces euclidiens On note M_n une variété obtenue comme somme connexe de deux espaces euclidiens $\mathbb{R}^n$.

Soit $n \geq 2$. Coulhon et Duong ([CD99, Section 5]) ont démontré que la transformée de Riesz sur M_n n'est pas bornée dans L^p pour $p > n$. Notons que M_n satisfait (D) et (DUE), mais (LY) est fausse.

Soit $n \geq 3$. Le résultat est amélioré dans [CCH06]. En fait, (R_p) est vraie pour $1 < p < n$ et est fausse pour $p \geq n$, voir [CCH06, Car07].

Variété de Vicsek (voir [BCG01, Section 6]) Soit M une variété de Vicsek à croissance polynômiale du volume: $V(x,r) \simeq r^D$, $r \geq 1$. Alors M vérifie l'estimation du noyau de la chaleur

$$\sup_{x\in M} p_t(x,x) \simeq t^{-\frac{D}{D+1}},\ t \geq 1.$$

Proposition 0.18 ([CD03])**.** Soit M une variété de Vicsek comme ci-dessus. Alors (RR_p) est fausse pour $1 < p < \frac{2D}{D+1}$. Par conséquence, (R_p) est fausse pour $p > \frac{2D}{D-1}$.

Si en outre M satisfait l'estimation gaussienne inférieure du noyau de la chaleur, c'est-à-dire, l'estimation de Li-Yau, ou bien également, des inégalités de Poincaré à l'échelle, Auscher et Coulhon [AC05] ont montré que

Théorème 0.19 ([AC05])**.** Soit M une variété riemannienne complète non-compacte satisfaisant la propriété de doublement du volume et les inégalités de Poincaré à l'échelle, c'est-à-dire, il existe $C > 0$ tel que $\forall B, \forall f \in C_0^\infty(B)$

$$\int_B |f - f_B|^2 d\mu \leq C r_B^2 \int_B |\nabla f|^2 d\mu,$$

où r_B est le rayon de B. Alors il existe $\varepsilon > 0$ tel que (R_p) est vrai pour $2 < p < 2+\varepsilon$.

Enfin,

Théorème 0.20 ([ACDH04])**.** Soit M une variété riemanniene complète non-compacte vérifiant (LY). Soit $p_0 \in (2,\infty]$. Alors les affirmations suivantes sont équivalentes:

1. Pour tout $p \in [2, p_0)$, on a
$$\left\|\left|\nabla e^{-t\Delta}\right|\right\|_{p\to p} \leq \frac{C}{\sqrt{t}}, \ \forall t > 0.$$

2. Pour tout $p \in [2, p_0)$, la transformée de Riesz est bornée sur L^p.

Théorème 0.21 ([ACDH04, CS10])**.** Soit M une variété riemannienne complète non-compacte vérifiant (D) et (G). Alors la transformée de Riesz est bornée sur L^p pour tout $1 < p < \infty$.

0.4 Espaces de Hardy

L'étude des espaces de Hardy trouve son origine dans la théorie des séries de Fourier et de l'analyse complexe en une variable. Depuis les années 1960, elle a été transférée à l'analyse réelle en plusieurs variables, ou plus généralement sur des espaces de type homogène. Il y a beaucoup de caractérisations équivalentes pour les espaces de Hardy. Par exemple, par les fonctions maximales, par les décompositions atomiques ou moléculaires, par les intégrales singulières, etc.

Nous rappelons une description de H^1 en utilisant des atomes sur un espace de type homogène M (voir [CW77]). On dit qu'une fonction $a \in L^2(M)$ est un atome de H^1 s'il existe une boule $B \in M$ telle que

1. $\operatorname{supp} a \subset B$;

2. $\|a\|_2 \leq \mu^{-1/2}(B)$;

3. $\int_M a(x)dx = 0$.

Une fonction f sur M appartient à $H^1(M)$ si et seulement s'il existe une suite $(\lambda_n)_{n\in\mathbb{N}} \in l^1$ et une suite de H^1-atomes $(a_n)_{n\in\mathbb{N}}$ tels que
$$f = \sum_{j=0}^{\infty} \lambda_j a_j,$$
et la convergence a lieu dans $L^1(M)$.

On peut aussi définir des espaces H^1 moléculaires. Contrairement à un atome, la molécule n'est pas à support compact, mais elle décroît très vite.

Dans $\mathbb{R}^n$, Coifman, Meyer and Stein ont développé la théorie des espaces de tentes ([CMS85], et voir [Rus07] sur des espaces de type homogène) qui connecte les espaces de Hardy à des fonctions quadratiques. On dit qu'une fonction mesurable F sur $\mathbb{R}^n \times (0, \infty)$ appartient à l'espace de tentes $T_2^p(\mathbb{R}^n)$ si
$$\|F\|_{T_2^p} := \left\| \left(\iint_{|y-x|<t} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2} \right\|_{L^p} < \infty.$$

En fait, pour toute fonction convenable f sur $\mathbb{R}^n$, la fonction quadratique conique est définie sur $\mathbb{R}^n$ par

$$Sf(x) = \left(\iint_{|y-x|<t} \left|t\sqrt{\Delta}e^{-t\sqrt{\Delta}}f(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2},$$

où $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ et $e^{-t\sqrt{\Delta}}$ est le semi-groupe de Poisson.

On dit que $f \in H^p_S$, $p \geq 1$ si $Sf \in L^p$, ou également, $t\sqrt{\Delta}e^{-t\sqrt{\Delta}}f \in T^p_2$. En effet, H^1_S et H^1 moléculaire sont équivalents. D'un côté, pour $f \in H^1(\mathbb{R}^n)$, on a $\|Sf\|_{L^1} \leq C\|f\|_{H^1}$ (voir [FS72]).

De l'autre côté, par la formule reproduisante de Calderón, on a

$$f = \int_0^\infty t\sqrt{\Delta}e^{-t\sqrt{\Delta}}F_t \frac{dt}{t}, \tag{2}$$

où $F_t := t\sqrt{\Delta}e^{-t\sqrt{\Delta}}f$ est dans l'espace de tentes T^1_2. Il est été prouvé dans [CMS85] que F_t admet une décomposition atomique de T^1_2. On la substitue dans (2) et on obtient une décomposition moléculaire pour f. Pour $p \in (1,\infty)$, $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ a lieu.

Les espaces de tentes sont intensivement utilisés dans la théorie des espaces de Hardy associés aux opérateurs, voir [AMR08], [HLM$^+$11], [Uhl11, KU12] etc.

Dans [AMR08], Auscher, McIntosh et Russ ont introduit des espaces de Hardy sur les variétés. Soit M une variété vérifiant le doublement du volume. Ils ont défini des espaces de Hardy sur les formes différentielles de tous les degrés (voir [AMM13] pour les espaces de Hardy de fibrés vectoriels sur les espaces métriques mesurés satisfaisant (D)). Notons $D = d + d^*$ l'opérateur de Hodge-Dirac et $\Delta = D^2$ l'opérateur de Hodge-Laplace. Par définition, la transformée de Riesz $D\Delta^{-1/2}$ est bornée sur $L^2(\Lambda T^*M)$. Les espaces de Hardy H^p_D sont définis par les fonctions convenables quadratiques tels que la transformée de Riesz est bornée sur $H^p_D(\Lambda T^*M)$, pour tout $1 \leq p \leq 2$. Il y a aussi une caractérisation par molécules. On a les comparaisons suivantes entre espaces H^p et L^p:

Théorème 0.22 ([AMR08])**.** Soit M une variété riemannienne complète satisfaisant le doublement du volume (D). Alors,

- Pour tout $1 \leq p \leq 2$, $H^p(\Lambda T^*M) \subset \overline{\mathscr{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)}$.
- Pour tout $2 \leq p < \infty$, $\overline{\mathscr{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)} \subset H^p(\Lambda T^*M)$.

De plus, si le noyau de la chaleur associé à l'opérateur laplacien de Hodge vérifie l'estimation gaussienne supérieure, c'est-à-dire, pour tout $0 \leq k \leq n$,

$$|p^k_t(x,y)| \leq \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{Ct}\right), \forall x,y \in M, t > 0, \tag{G_k}$$

où p^k_t est le noyau de $e^{-t\Delta_k}$ et Δ_k est l'opérateur laplacien de Hodge sur les $k-$formes, alors $H^p(\Lambda T^*M) = \overline{\mathscr{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)}$ pour tout $1 < p < \infty$.

En conséquence du calcul fonctionnel sur $H^p(\Lambda T^*M)$, on a

Théorème 0.23 ([AMR08])**.** Soit M une variété riemannienne complète satisfaisant le doublement du volume. Alors, pour tout $1 \leq p \leq \infty$, la transformée de Riesz $D\Delta^{-1/2}$ est bornée sur $H^p(\Lambda T^*M)$. Donc, elle est bornée de $H^1(\Lambda T^*M)$ dans $L^1(\Lambda T^*M)$.

Plus précisément, si on considère les fonctions (0-formes), on a les affirmations suivantes:

- Soit (D) et (G_0). Alors pour tout $1 < p < \infty$,

$$H^p(\Lambda^0 T^*M) = \overline{\mathscr{R}(D) \cap L^p(\Lambda^0 T^*M)}^{L^p(\Lambda^0 T^*M)}.$$

- Soit (D), (G_0) et (G_1). Alors la transformée de Riesz $d\Delta^{-1/2}$ sur les fonctions est bornée de $L^p(\Lambda^0 T^*M)$ dans $L^p(\Lambda^1 T^*M)$ pour tout $1 < p < \infty$.

Dans [HLM+11], Hofmann, Lu, Mitrea, Mitrea and Yan ont considéré des espaces de Hardy définis par les fonctions quadratiques sur les espaces métriques mesurés de type homogène. Soit X un espace métrique mesuré avec le doublement du volume et soit L un opérateur positif auto-adjoint qui engendre un semi-groupe analytique $(e^{-tL})_{t>0}$ sur $L^2(X)$ vérifiant l'estimation de Davies-Gaffney. C'est-à-dire, il existe deux constantes $C, c > 0$ telles que pour tous ensembles ouverts $U_1, U_2 \subset X$,

$$|< e^{-tL} f_1, f_2 >| \leq C \exp\left(-\frac{\mathrm{dist}^2(U_1,U_2)}{ct}\right) \|f_1\|_2 \|f_2\|_2, \ \forall t > 0,$$

pour toutes $f_i \in L^2(X)$ à support compact et $\mathrm{supp}\, f_i \subset U_i$, $i = 1, 2$, où

$$\mathrm{dist}\,(U_1,U_2) := \inf_{x \in U_1, y \in U_2} d(x,y).$$

Ils ont étendu les résultats dans [AMR08] par obtenir une decomposition atomique de H^1_L et par développer la théorie des espaces H^1 et BMO adaptés à L.

Plus généralement, Kunstmann and Uhl [Uhl11, KU12] ont étudié des opérateurs positifs auto-adjoints sur L^2 vérifiant l'estimation d'ordre m ($m \geq 2$) de Davies-Gaffney (DG_m):

$$|< e^{-tL} f_1, f_2 >| \leq C \exp\left(-c\left(\frac{d(x,y)}{t}\right)^{\frac{m}{m-1}}\right) \|f_1\|_2 \|f_2\|_2, \ \forall t > 0,$$

pour tous $x, y \in M$ et $f_1, f_2 \in L^2(X)$ avec $\mathrm{supp}\, f_1 \subset B(y, t^{1/m})$ et $\mathrm{supp}\, f_2 \subset B(x, t^{1/m})$. Ils ont défini des espaces de Hardy par les fonctions quadratiques et par les molécules adaptés à (DG_m). Par ailleurs, si l'estimation gaussienne (p_0, p_0') généralisée d'ordre m a lieu: $\forall t > 0, \forall x, y \in M$,

$$\left\| \mathbb{1}_{B(x,t^{1/m})} e^{-tL} \mathbb{1}_{B(y,t^{1/m})} \right\|_{p_0 \to p_0'} \leq C V^{-\left(\frac{1}{p_0} - \frac{1}{p_0'}\right)}(x, t^{1/m}) \exp\left(-c \left(\frac{d(x,y)}{t} \right)^{\frac{m}{m-1}} \right),$$

alors pour tout $p \in (p_0, 2)$, ils ont montré que l'espace de Hardy H^p défini par les fonctions quadratiques coïncide avec L^p.

0.5 Résultats de la thèse

0.5.1 Les quasi transformées de Riesz

Sur les variétés riemanniennes complètes Notons que nous ne faisons aucune hypothèse sur la croissance du volume ni sur le noyau de la chaleur.

Proposition 0.24. Soit M une variété riemannienne complète. Alors, pour tout $1 < p \leq 2$, on a

$$\left\| \left| \nabla e^{-t\Delta} \right| \right\|_{p \to p} \leq \frac{C}{\sqrt{t}}, \tag{G_p}$$

et également

$$\| |\nabla f| \|_p^2 \leq C \|f\|_p \|\Delta f\|_p. \tag{MI_p}$$

Comme application, on obtient une version faible de (R_p) pour $1 < p \leq 2$.

Théorème 0.25. Soit M une variété riemannienne complète. Alors, pour $0 < \alpha < 1/2$, la quasi transformée de Riesz $\nabla e^{-\Delta} \Delta^{-\alpha}$ est borné sur L^p pour $1 < p \leq 2$.

Proposition 0.26. Soit M une variété riemannienne complète satisfaisant (D_{loc}) et (DUE_{loc}). Alors $\nabla(I + \Delta)^{-1/2} + \nabla e^{-\Delta} \Delta^{-\alpha}$ avec $\alpha \in (0, 1/2)$ est bornée sur L^p pour $1 < p \leq 2$.

Sur les variétés riemanniennes vérifiant (D) et ($UE_{2,m}$) On ne sait pas si (DUE) est nécessaire pour la bornitude de la transformée de Riesz sur L^p pour $1 < p < 2$. Une question naturelle est la suivante:

Question 0.27. Soit M une variété riemannienne vérifiant (D) et ($UE_{2,m}$), est-ce que la transformée de Riesz est bornée sur $L^p(M)$ pour $1 < p < 2$?

Si oui, on améliore le résultat dans Théorème 0.17. Sinon, on trouve des contre-exemples intéressants. Pour le moment, on ne sait pas résoudre ce problème. Cependant, sous l'hypothèse ($UE_{2,m}$) ou plus faible, on considère le cas $p = 1$ dans Théorème 0.25.

Théorème 0.28. Soit M une variété complète non-compacte vérifiant (D) et ($UE_{2,m}$). Alors pour tout $0 < \alpha < 1/2$, l'opérateur $\nabla e^{-\Delta} \Delta^{-\alpha}$ est de type $(1, 1)$.

En plus, sous des hypothèses plus fortes, on peut obtenir

Théorème 0.29. Soit M une variété complète non-compacte vérifiant (D) et $(HK_{2,m})$, $m \geq 2$, c'est à dire, $(UE_{2,m})$ et l'estimation inférieure correspondante. Supposons (G_{p_0}), $p_0 \in (2,\infty]$, et l'estimation L^2 Davies-Gaffney $(DG_{\frac{1}{2}})$:

$$\left\| |\nabla e^{-t\Delta} f| \right\|_{L^2(B)} \leq \begin{cases} \dfrac{C}{t^{1/2}} e^{-c\frac{d^2(B,C_i(B))}{t}} \|f\|_{L^2(C_i(B))}, & 0 < t < 1, \\ \dfrac{C}{t^{1/2}} e^{-c\left(\frac{d^m(B,C_i(B))}{t}\right)^{1/(m-1)}} \|f\|_{L^2(C_i(B))}, & 1 \leq t < \infty. \end{cases}$$

Alors la transformée de Riesz à l'infini $\nabla e^{-\Delta}\Delta^{-1/2}$ est bornée sur L^p pour $p \in (2, p_0)$.

Contre-exemples pour $p > 2$ On améliore le résultat de la Proposition 0.18. En fait, il existe des variétés telles que la transformée de Riesz inverse n'est pas bornée sur L^p pour $1 < p < 2$ et donc (R_p) est faux pour $2 < p < \infty$.

Théorème 0.30. Soit M une variété de Vicsek à croissance polynomiale du volume: $V(x,r) \simeq r^D$ pour $r \geq 1$. Alors l'inégalité

$$\|\Delta^\beta f\|_p \leq C_p \| |\nabla f| \|_p$$

est fausse pour tout $\beta < \beta(p) := \frac{1}{D+1}\left(\frac{D}{p} + \frac{1}{p'}\right)$, où $1 < p < \infty$.

En particulier, (RR_p) est faux pour tout $1 < p < 2$. Par conséquent, (R_p) est fausse pour tout $p > 2$.

Sur les graphes Considérons maintenant le cas discret (les graphes). On a les résultats suivants qui sont similaires à la Proposition 0.26 et au Théorème 0.28.

Proposition 0.31. Soit Γ un graphe infini connexe vérifiant le doublement du volume local, c'est-à-dire, il existe une constante $c > 1$ telle que

$$\mu(B(x,1) \leq c\mu(x), \; \forall x \in \Gamma,$$

Soit $\alpha \in (0, 1/2)$ fixé. Alors, pour $1 < p \leq 2$, il existe $C > 0$ tel que

$$\| |\nabla (I-P)^{-\alpha} f| \|_p \leq C \|f\|_p.$$

Théorème 0.32. Soit Γ un graphe infini connexe vérifiant le doublement du volume et la condition $\triangle(\alpha)$. Supposons aussi l'estimation sous-gaussienne supérieure du noyau de la chaleur: $\forall x, y \in M$, $\forall k \in \mathbb{N}$,

$$p_k(x,y) \leq \frac{C\mu(y)}{V(x, k^{1/m})} \exp\left(-c\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right).$$

Alors pour tout $\beta \in (0,1/2)$, l'opérateur $\nabla(I-P)^{-\beta}$ est de type faible $(1,1)$.

0.5.2 Espaces de Hardy

Soit (M,d,μ) un espace métrique mesuré de type homogène satisfaisant une estimation différente en petit temps et en grand temps du noyau de la chaleur. Comme dans [AMR08, HLM$^+$11], on définit deux classes d'espace de Hardy sur M. La première classe est l'espace H^1 défini par les molécules. On le désigne $H^1_{L,\rho,\mathrm{mol}}(M)$. La deuxième classe est $H^p_{L,S_h^\rho}(M)$, $1 \le p \le \infty$, défini par les fonctions quadratiques. On montre tout d'abord

Théorème 0.33. Soit M un espace métrique mesuré vérifiant (D) et (DG_{β_1,β_2}), $\beta_1 \le \beta_2$. Alors $H^1_{L,\rho,\mathrm{mol}}(M) = H^1_{L,S_h^\rho}(M)$. De plus, on a $\|f\|_{H^1_{L,\rho,\mathrm{mol}}(M)} \simeq \|f\|_{H^1_{L,S_h^\rho}(M)}$.

En comparant $H^p_{L,S_h^\rho}(M)$ et L^p for $1 < p < \infty$, on obtient

Théorème 0.34. Soit M un espace métrique mesuré non-compact vérifiant la propriété de doublement du volume (D) et l'estimation supérieure du noyau de la chaleur ($DG^{p_0,p_0'}_{\beta_1,\beta_2}$). Alors $H^p_{L,S_h^\rho}(M) = \overline{R(L)\cap L^p(M)}^{L^p(M)}$ a lieu pour tout $p_0 < p < p_0'$.

Comme une corollaire, l'estimation ponctuelle du noyau de la chaleur nous donne

Corollaire 0.35. Soit M un espace métrique mesuré non-compact vérifiant la propriété de doublement du volume (D) et l'estimation supérieure du noyau de la chaleur (UE_{β_1,β_2}). Alors $H^1_{L,\rho,mol}(M) = H^1_{L,S_h^\rho}(M)$, et $H^p_{L,S_h^\rho}(M) = L^p(M)$ pour tout $1 < p < \infty$.

Cependant, pour $1 < p < 2$, l'équivalence entre L^p et H^p défini par les fonctions quadratiques avec le scaling t^2 n'est pas nécessairement vraie. En effet, on a le résultat suivant.

Théorème 0.36. Soit M une variété riemannienne complète non-compacte. Supposons la croissance polynômiale du volume:

$$V(x,r) \simeq r^d,\ r \ge 1,$$

et l'estimation sous-gaussienne du noyau de la chaleur ($HK_{2,m}$), où $2 < m < d/2$. Alors l'inclusion $L^p(M) \subset H^p_{\Delta,S_h}(M)$ est fausse pour $p \in \left(\frac{d}{d-m},2\right)$.

Comme application des développements précédents sur les espaces de Hardy, on montre

Théorème 0.37. Soit M une variété satisfaisant (D) et ($UE_{2,m}$), $m > 2$. Alors, pour $\alpha \in (0,1/2)$ fixé, l'opérator $\nabla e^{-\Delta}\Delta^{-\alpha}$ est borné de $H^1_{\Delta,m}$ dans L^1.

0.5.3 Les autres résultats

Sur les graphes de Vicsek, on obtient une inégalité généralisée de Poincaré.

Théorème 0.38. Soit Γ un graphe de Vicsek à croissance polynomiale du volume: $V(x,n) \simeq n^D$. Alors, pour $p \geq 1$,

$$\|f - f_n(x)\|_{L^p(B(x,n))} \leq CV(x,n)^{\frac{1}{D(p)}} \||\nabla f|\|_{L^p(B(x,2n))},$$

où $f_n(x) = \frac{1}{V(x,n)} \sum_{z \in B(x,n)} f(z)\mu(z)$ et $\frac{1}{D(p)} = \frac{1}{p} + \frac{1}{p'D}$.

On a aussi une inégalité généralisée de Sobolev.

Théorème 0.39. Soit Γ un graphe de Vicsek à croissance polynomiale du volume: $V(x,n) \simeq n^D$. Alors, pour $p \geq 1$,

$$\|f\|_p \lesssim \mu(\Omega)^{\frac{1}{D(p)}} \||\nabla f|\|_p, \forall \Omega \subset \Gamma, \forall f \in c_0(\Omega), \qquad (S^p_{D(p)})$$

où $D(p)$ est le même comme dans le Théorème 0.38.

Notons que ces deux inégalités sont optimales.

Introduction

In this thesis, we study Riesz transforms and Hardy spaces associated to operators. The two subjects are closely related to volume growth and heat kernel estimates. We mainly have two aims here: the L^p boundedness of the Riesz transform on any Riemannian manifold (or any graph) satisfying the doubling volume property and the sub-Gaussian heat kernel estimate; the Hardy space theory on some metric measure Dirichlet space where the local and global behaviour of the heat kernel are different.

In the following, we first introduce different settings: Riemannian manifolds, graphs and metric measure Dirichlet spaces. Then we enumerate several heat kernel estimates which we consider later. After that, we briefly recall the background and previous work about Riesz transforms and Hardy spaces. Finally we present our results and the structure of this thesis.

0.6 Setting

Let (M,d,μ) be a metric measure space. We write $B(x,r)=\{y:d(y,x)<r\}$ for the ball of centre $x\in M$ and radius $r>0$. Denote $V(x,r)=\mu(B(x,r))$.

Notation 0.40. In the sequel, the letters c,C denote positive constants, which always change in different places. For two positive quantities x and y, we say $x\lesssim y$ if there exists a constant $C>0$ such that $x\leq Cy$ and $x\simeq y$ if there exist two constants c,C with $0<c\leq C$ such that $cx\leq y\leq Cx$ (uniformly in the parameters on which x,y depend).

If B is a ball, we shall denote by x_B its centre and by r_B its radius. For any given $\lambda>0$, we will write λB for the λ dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B}=\lambda r_B$. We denote $C_1(B)=4B$, and $C_j(B)=2^{j+1}B\backslash 2^jB$ for $j=2,3,\cdots$.

Definition 0.41. We say that (M,d,μ) satisfies the doubling volume property if there exists a constant $C>0$ such that for any $x\in M$ and $r>0$,

$$V(x,2r)\leq CV(x,r). \tag{D}$$

A simple consequence of (D) is that there exist $\nu > 0$ and $C > 0$ such that

$$\frac{V(x,r)}{V(x,s)} \leq C\left(\frac{r}{s}\right)^{\nu}, \ \forall x \in M, \, r \geq s > 0. \tag{3}$$

It follows that, for any $x, y \in M$ and $r > 0$, we have

$$V(x,r) \leq C\left(1 + \frac{d(x,y)}{r}\right)^{\nu} V(y,r).$$

Therefore,

$$\int_{d(x,y)<r} \frac{1}{V(x,r)} d\mu(x) \simeq C, \ \forall y \in M, \, r > 0. \tag{4}$$

If M is non-compact and connected, we also have a reverse inequality of (3), that is, there exists $\nu' > 0$ such that

$$\frac{V(x,r)}{V(x,s)} \geq C\left(\frac{r}{s}\right)^{\nu'}, \ \forall x \in M, \, r \geq s > 0. \tag{5}$$

0.6.1 Riemannian manifolds

Let (M, d, μ) be a complete non-compact weighted Riemannian manifold, where d is the geodesic distance and μ is a measure with a smooth positive density with respect to the Riemannian measure.

Let ∇ be the Riemannian gradient. The Laplace-Beltrami operator Δ_μ on (M, g, μ) is defined by

$$(\Delta_\mu f, g) = \int_M \nabla f \cdot \nabla g \, d\mu, \ \forall f, g \in \mathscr{C}_0^\infty(M).$$

A nontrivial theorem says that, since M is complete, one can extend Δ_μ to a self-adjoint operator on $L^2(M, d\mu)$ (see [Dav90, Gri09]). In the sequel, we denote Δ_μ by Δ. The spectral theory yields for all $f \in \mathscr{C}_0^\infty(M)$

$$(\Delta f, f) = \int_M |\Delta^{1/2} f|^2 d\mu.$$

Moreover, the associated heat semigroup $\left(e^{-t\Delta}\right)_{t>0}$ is a family of linear contractions on $L^2(M)$. Indeed, the following theorem holds:

Theorem 0.42 ([Gri09, Str83])**.** *The heat semigroup $e^{-t\Delta}$ on $L^2(M)$ admits a unique heat kernel $p_t(x,y)$ satisfying*

1. *Smoothness: $p_t(x,y)$ is a C^∞ function on $\mathbb{R}^+ \times M \times M$.*
2. *Positivity: $p_t(x,y) > 0$ for all $x, y \in M$ and $t > 0$.*
3. *Symmetry: $p_t(x,y) = p_t(y,x)$ for all $x, y \in M$ and $t > 0$.*

4. *For any $f \in L^2(M)$, and for all $x \in M$ and $t > 0$,*

$$e^{-t\Delta}f(x) = \int_M p_t(x,y)f(y)d\mu(y). \tag{6}$$

5. *Sub-Markovian property: For $f \in L^2(M)$, $0 \le f \le 1 \Rightarrow 0 \le e^{-t\Delta}f \le 1$. In other words, $\int_M p_t(x,y)d\mu(y) \le 1$.*

6. *L^p boundedness and strongly continuity: $\|e^{-t\Delta}f\|_p \le \|f\|_p$ for all $t > 0$ and $f \in L^2 \bigcap L^p$, $1 \le p \le \infty$, with $\|e^{-t\Delta}f - f\|_p \to 0$ if $1 \le p < \infty$.*

7. *For all $f \in L^2(M)$, we have $\frac{d}{dt}e^{-t\Delta}f = -\Delta e^{-t\Delta}f$. This still holds for all $f \in L^p$, $1 \le p \le \infty$, if we define $e^{-t\Delta}f$ as in (6).*

0.6.2 Graphs

Let Γ be an infinite graph endowed with a symmetric weight μ on $\Gamma \times \Gamma$. For any $x, y \in \Gamma$, then $\mu_{xy} = \mu_{yx} \ge 0$. We say that x and y are neighbors (they are connected by an edge), denoted by $x \sim y$, if and only if $\mu_{xy} > 0$.

Define $\mu(x) = \sum_{y \sim x} \mu_{xy}$, then it extends a measure on Γ by

$$\mu(\Omega) = \sum_{x \in \Omega} \mu(x),$$

where Ω is a finite subset in Γ.

For $x, y \in \Gamma$, a path of length n between x and y in Γ is a sequence $x_i, 0 = 1, \cdots, n$ such that $x_0 = x, x_n = y$ and $x_i \sim x_{i+1}, i = 0, \cdots, n-1$. Assume that Γ is connected, that is, there exists a path between any two vertices of Γ. The induced graph distance $d(x,y)$ is the minimal number of edges in any path connecting x and y. Let $B(x,r) = \{y \in \Gamma : d(x,y) \le r\}$ be the closed ball with center $x \in \Gamma$ and radius $r > 0$, while $V(x,r) = \mu(B(x,r))$ denotes its mesure.

Define $p(x,y) = \frac{\mu_{xy}}{\mu(x)}$, for any $x, y \in \Gamma$. Then p is a reversible Markov kernel satisfying

$$\mu(x)p(x,y) = \mu_{xy} = \mu(y)p(y,x); \tag{7}$$

$$\sum_{y \in \Gamma} p(x,y) = 1; \tag{8}$$

$$p(x,y) = 0, \text{ if } d(x,y) \ge 2. \tag{9}$$

The third property is the so-called finite range property of Γ.

We always assume

- Γ is locally uniformly finite, which means that for any $x \in \Gamma$, there exists $N \in \mathbb{N}$ such that x has at most N neighbours;

- Γ satisfies the condition $\triangle(\alpha)$, that is, there exists $c > 0$ such that for all $x, y \in \Gamma$

$$x \sim y \text{ implies } p(x,y) \geq c, \text{ and } x \sim x. \tag{10}$$

Denote by p_k the iterated kernel given by

$$p_0(x,y) := \delta(x,y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y; \end{cases}$$
$$p_k(x,y) = \sum_{z \in \Gamma} p_{k-1}(x,z) p(z,y), \; k \geq 1.$$

The L^p norm of a function f on Γ is given by

$$\|f\|_p = \left[\sum_x |f(x)|^p \mu(x) \right]^{1/p},$$

and one denotes by $c_0(\Gamma)$ the set of functions with finite support on Γ.

The linear operator associated with the kernel p is defined by

$$Pf(x) = \sum_y p(x,y) f(y),$$

then P is self-adjoint on $L^2(\Gamma)$ and we have

$$P^k f(x) = \sum_y p_k(x,y) f(y).$$

We call the operator $L = I - P$ the discrete Laplacian on Γ.

The value of discrete gradient on Γ is defined by

$$|\nabla f(x)| = \left[\frac{1}{2} \sum_y p(x,y) |f(x) - f(y)|^2 \right]^{1/2},$$

while $p_{k+1}(y,x) - p_k(y,x)$ is regarded as the discrete time derivative of $p_k(y,x)$.

A simple calculation shows that

$$\|\nabla f\|_2^2 = <(I-P)f, f> = \|(I-P)^{\frac{1}{2}} f\|_2^2,$$

where $(I-P)^{1/2}$ is defined by spectral theory. As is pointed out in [Rus00], the assumption $\triangle(\alpha)$ in (10) implies $-1 \notin \operatorname{Spec}(P)$, and as a consequence

$$P = \int_a^1 \lambda \, dE_\lambda, \text{ where } a > -1. \tag{11}$$

This implies the analyticity of P on L^2, see [CSC90, Proposition 3]. That is, there exists $C > 0$such that for all $n \in \mathbb{N}$,

$$\left\|P^n - P^{n+1}\right\|_{2\to 2} \leq Cn^{-1}.$$

For any $y \in \Gamma$, $p_\cdot(\cdot, y)$ satisfies the discrete-time heat equation

$$\mu(x)(u(n+1,x) - u(n,x)) = \sum_y \mu_{xy}(u(n,y) - u(n,x)),$$

or equivalently

$$\mu(x)u(n+1,x) = \sum_y \mu_{xy} u(n,y).$$

0.6.3 Metric measure Dirichlet spaces

Some of the following presentation is taken from [HSC01, BBK06, Bar13]. Let (M,μ) be a locally compact separable space equipped with a positive Radon measure μ which is strictly positive on any non-empty open set. Denote by $\mathscr{C}_0(M)$ the set of all continuous functions with compact support. Let $\mathscr{E}$ be a regular Dirichlet form on M with domain $\mathscr{D} \in L^2(M, d\mu)$ (see [Stu94, Stu95, GSC11]). That is, there exists a core $\mathscr{C} \in \mathscr{D} \cap \mathscr{C}_0(M)$ which is dense in $\mathscr{C}_0(M)$ under the uniform norm and dense in $\mathscr{D}$ under the norm $(\|f\|_2^2 + \mathscr{E}(f,f))^{1/2}$. We also assume $\mathscr{E}$ is strongly local: f_1, f_2 with compact support and f_1 is constant on an open set U_1 containing $\sup f_2$, then $\mathscr{E}(f_1, f_2) = 0$. $\mathscr{E}$ admit an "energy measure" Γ:

$$\mathscr{E}(f,g) = \int_M d\Gamma(f,g),$$

for $f, g \in \mathscr{D}$.

For any $x, y \in M$, define

$$d(x,y) = \sup\{f(x) - f(y) : d\Gamma(f,f) \leq d\mu, f \in \mathscr{C}\}.$$

Here $d\Gamma(f,f) \leq d\mu$ means that $\Gamma(f,f)$ is absolutely continuous with respect to μ with Radon-Nikodym derivative bounded by 1. Assume that d is actually a true metric that generates the original topology on M and that (M,d) is complete. Then we say that $(M,d,\mu,\mathscr{E})$ is a metric measure Dirichlet space.

We say that $\mathscr{E}$ admits a "carré du champ" if $d\Gamma$ is absolutely continuous with respect to μ. That is, there exists a unique positive symmetric and continuous bilinear form from $\mathscr{D} \times \mathscr{D}$ into L^1, still denoted by Γ, such that

$$\mathscr{E}(fh,g) + \mathscr{E}(gh,f) - \mathscr{E}(h,fg) = \int_M h\Gamma(f,g)d\mu,\ \forall f,g,h \in \mathscr{D} \cap L^\infty.$$

Note that in general $d\Gamma$ needs not to be absolutely continuous with respect to μ. For more information, we refer to [BÉ85, BH91].

Denote by L the non-negative self-adjoint operator associated with $\mathscr{E}$. Then

$$\mathscr{E}(f,g) = < Lf, g > = < f, Lg > = \int_M d\Gamma(f,g).$$

The generated semigroup $\left(e^{-tL}\right)_{t>0}$ is a contraction on $L^2(M,\mu)$. Moreover, e^{-tL} is Markovian and can be extended as a contraction operator on $L^p(M,\mu)$, $1 \le p \le \infty$.

The following are some examples:

Example 0.43. Manifolds. Let (M,d,μ) be a Riemannian manifold and Δ be the Laplace-Beltrami operator. We take the core to be the C^∞ functions on M with compact support, and set

$$\mathscr{E}(f,f) := \int_M (\Delta f) g d\mu,\ \forall f \in \mathscr{C}_0^\infty(M).$$

Example 0.44. Euclidean spaces with divergence form operators.

Let $(a_{ij}(x))_{n\times n}$ be a bounded and measurable matrix, and set

$$\mathscr{E}(f,f) := \int_{\mathbb{R}^n} \nabla f \cdot a \nabla g,\ \forall f \in \mathscr{C}_0^\infty(\mathbb{R}^n).$$

Example 0.45. Fractals.

For example, Sierpinski carpets, Sierpinski gaskets, Vicsek sets, etc. See for references [BB92, BB99a, BCG01, Bar13].

On a Sierpinski carpet F, let F_n be a sequence of approximations to F and $\mathscr{E}_n$ be an appropriate Dirichlet form on F_n. Then one can construct a limiting Dirichlet form $\mathscr{E}$ on F. See [Bar13] for the construction.

Example 0.46. Cable systems (Quantum graphs) (see [Var85], [BB04, Section 2]).

Given a weighted graph (G,d,ν) (see Section 0.6.2), we define the cable system G_C by replacing each edge of G by a copy of $(0,1)$ joined together at the vertices. The measure μ on G_C is given by $d\mu(t) = \nu_{xy} dt$ for t in the cable connecting x and y; and μ assigns no mass to any vertex. The distance between two points x and y is given as follows: if x and y are on the same cable, the length is just the usual Euclidean distance $|x-y|$. If they are on different cables, then the distance is $\min\{|x-z_x| + d(z_x,z_y) + |z_y - y|\}$ (d is the usual graph distance), where the minimum is taken over all vertices z_x and z_y such that x is on a cable with one end at z_x and y is on a cable with one end at z_y. One takes as the core $\mathscr{C}$ the functions in $C(G_C)$ which have compact support and are C^1 on each cable, and sets

$$\mathscr{E}(f,f) := \int_{G_C} \left|f'(t)\right|^2 d\mu(t).$$

0.7 Heat kernel estimates

Let (M,d,μ) be a complete metric measure space. Let L be a non-negative self-adjoint operator on $L^2(M,\mu)$, which generates an analytic semigroup $(e^{-tL})_{t>0}$. Note that $(e^{-tL})_{t>0}$ is not necessarily uniformly bounded on $L^1(M,\mu)$. There are interesting semigroups that do not act on L^1, but only on a range of L^p spaces, where $p\in[p_0,p_0']$ for some $p_0>1$. See for example, [Aus07].

Let $1<\beta_1\le\beta_2$. In this paper, we will assume following heat kernel estimate:

Definition 0.47. Let $1\le p_0<2$. We say that M satisfies the generalised $L^{p_0}-L^{p_0'}$ off-diagonal estimate if for $x,y\in M$ and $t>0$,

$$\left\|\mathbb{1}_{B(x,t)}e^{-\rho(t)L}\mathbb{1}_{B(y,t)}\right\|_{p_0\to p_0'}\le\begin{cases}\dfrac{C}{V^{\frac{1}{p_0}-\frac{1}{p_0'}}(x,t)}\exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_1}{\beta_1-1}}\right) & 0<t<1,\\ \dfrac{C}{V^{\frac{1}{p_0}-\frac{1}{p_0'}}(x,t)}\exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_2}{\beta_2-1}}\right), & t\ge1,\end{cases}\qquad (DG^{p_0,p_0'}_{\beta_1,\beta_2})$$

where

$$\rho(t)=\begin{cases}t^{\beta_1}, & 0<t<1,\\ t^{\beta_2}, & t\ge1;\end{cases}\qquad(12)$$

Remark 0.48. In particular, if $p_0=2$, we say that M satisfies the generalised L^2-L^2 Davies-Gaffney estimate and denote it by (DG_{β_1,β_2}). That is,

$$\left\|\mathbb{1}_{B(x,t)}e^{-\rho(t)L}\mathbb{1}_{B(y,t)}\right\|_{2\to2}\le\begin{cases}C\exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_1}{\beta_1-1}}\right) & 0<t<1,\\ C\exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_2}{\beta_2-1}}\right), & t\ge1.\end{cases}\qquad (DG_{\beta_1,\beta_2})$$

If $p_0=1$, then $(DG^{p_0,p_0'}_{\beta_1,\beta_2})$ is equivalent to the following pointwise estimate of the heat kernel (see [BK05, Proposition 3.6]):

$$p_{\rho(t)}(x,y)\le\begin{cases}\dfrac{C}{V(x,t)}\exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_1}{\beta_1-1}}\right) & 0<t<1,\\ \dfrac{C}{V(x,t)}\exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_2}{\beta_2-1}}\right), & t\ge1.\end{cases}\qquad (UE_{\beta_1,\beta_2})$$

In the following, we will given some examples that satisfy ($DG^{p_0,p_0'}_{\beta_1,\beta_2}$).

Example 0.49. Euclidean spaces with higher order divergence form operators.

Let L be a homogeneous elliptic operator on $L^2(\mathbb{R}^n)$ of order $2m$ ($m \geq 1$) in divergence form:

$$L := (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha \left(a_{\alpha,\beta} \partial^\beta \right),$$

where we assume $a_{\alpha,\beta}$ is bounded for all α, β. Then $(UE_{2m,2m})$ holds if $n \leq 2m$. If $n > 2m$, then ($DG^{p_0,p_0'}_{\beta_1,\beta_2}$) holds for $p_0 = \frac{2n}{n+2m}$. See for example [Aus07, Dav95, Dav97a] etc.

Example 0.50. Riemannian manifolds.

Any Riemannian manifold with non-negative Ricci curvature satisfies $(UE_{2,2})$, which is the most familiar Gaussian heat kernel upper estimates. See [LY86, Gri09] for more information.

In general, any complete Riemannian manifold satisfies the classical Davies-Gaffney estimate $(DG_{2,2})$ (see [Dav92]).

Example 0.51. Fractals.

For example, Sierpinski carpets, Sierpinski gaskets, Vicsek sets, etc, satisfy $(UE_{m,m})$ with $m > 2$. We refer to [BB92, BB99a, BCG01, Bar13]. On a Sierpinski carpet F, let X_t be a Brownian motion on F which is constructed as the limit of time-changed reflecting Brownian motions on approximations to F. Let L be the infinitesimal generator of the limiting process X_t. The transition density of X_t is the heat kernel.

Example 0.52. Some fractal manifolds.

Let (G, E, ν) be a infinite connected graph satisfying polynomial volume growth $V(x,r) \simeq r^D$ and sub-Gaussian heat kernel estimate

$$p_k(x,y) \leq \frac{C\mu(y)}{k^{D/m}} \exp\left(-c \left(\frac{d^m(x,y)}{k} \right)^{1/(m-1)} \right),$$

where $D \geq 1$ and $2 \leq m \leq D+1$. In fact, one can choose any $D \geq 1$ and any $2 \leq m \leq D+1$ such that the above conditions hold (see [Bar04]). Let M be a manifold constructed from G by replacing the edges of the graph by tubes of length 1 and then gluing the tubes together smoothly at the vertices (see [BCG01] for a concrete example). Let d be the geodesic distance, μ be the Riemannian measure and Δ be the non-negative Laplace-Beltrami operator on M. Denote by $e^{-t\Delta}$ the heat semigroup associated with Δ. Then the heat kernel $p_t(x,y)$ satisfies (UE_{β_1,β_2}) with $\beta_1 = 2$ and $\beta_2 = m > 2$. That is, the small time behaviour of the heat kernel is Gaussian as in Euclidean spaces while the heat kernel has a sub-Gaussian decay for large time.

Example 0.53. Some cable systems (Quantum graphs).

Let G_C be the cable system as in Example 0.53. Let L be the associated non-negative self-adjoint operator associated with $\mathscr{E}$ and e^{-tL} be the generated semigroup. Then the associated kernel may satisfies (UE_{β_1,β_2}). For example, the cable graph associated with the Sierpinski gasket graph (in $\mathbb{Z}^2$) satisfies $(UE_{2,\log 5/\log 2})$ (see [Jon96, BB04]).

More generally, Hebisch and Saloff-Coste [HSC01] consider a non-compact complete metric space with regular and strongly local Dirichlet form, on which the related heat kernel satisfies the $\rho-$Gaussian heat kernel upper estimate

$$p_t(x,y) \leq \frac{C}{V(y,\rho^{-1}(t))} \exp\left(-cG(t,d(x,y))\right), \tag{13}$$

where ρ^{-1} is the inverse function of ρ, ρ and G satisfy certain conditions which we omit here. If the matching lower bound also holds, we say that M satisfies two-sided $\rho-$Gaussian heat kernel estimate. We refer to [HSC01] (Section 5) for a detailed introduction of this kind of non-classical heat kernel estimates and related parabolic Harnack inequalities. See also [GT12].

Indeed, Examples 0.52 and 0.53 satisfy two-sided $\rho-$Gaussian heat kernel estimate with

$$\rho(t) = \begin{cases} t^{\beta_1}, & 0 < t < 1, \\ t^{\beta_2}, & t \geq 1, \end{cases}$$

and

$$G(R,t) = \begin{cases} \left(\dfrac{R^{\beta_1}}{t}\right)^{1/(\beta_1-1)}, & t \leq R, \\ \left(\dfrac{R^{\beta_2}}{t}\right)^{1/(\beta_2-1)}, & t \geq R. \end{cases}$$

Note that in this case the upper bound here implies (UE_{β_1,β_2}). In fact, since $\beta_1 \leq \beta_2$. Then for $d(x,y) \leq t$, one has $\left(\frac{d^{\beta_1}(x,y)}{t}\right)^{1/(\beta_1-1)} \leq \left(\frac{d^{\beta_2}(x,y)}{t}\right)^{1/(\beta_2-1)}$. And for $t \leq d(x,y)$, one has

$$\left(\frac{d^{\beta_1}(x,y)}{t}\right)^{1/(\beta_1-1)} \geq \left(\frac{d^{\beta_2}(x,y)}{t}\right)^{1/(\beta_2-1)}.$$

Thus the upper bound of (13) implies the following estimate:

$$p_t(x,y) \leq \begin{cases} \dfrac{C}{V(x,t^{1/\beta_1})} \exp\left(-c\left(\dfrac{d^{\beta_1}(x,y)}{t}\right)^{1/(\beta_1-1)}\right), & 0 < t < 1, \\ \dfrac{C}{V(x,t^{1/\beta_2})} \exp\left(-c\left(\dfrac{d^{\beta_2}(x,y)}{t}\right)^{1/(\beta_2-1)}\right), & t \geq 1, \end{cases}$$

which is exactly (UE_{β_1,β_2}). In the sequel, we also denote (13) by (UE_{β_1,β_2}).

0.8 Riesz transforms on Riemannian manifolds

It was asked by Strichartz [Str83] in 1983 on which non-compact Riemannian manifold M, and for which p, $1 < p < +\infty$, the two semi-norms $\||\nabla f|\|_p$ and $\|\Delta^{1/2} f\|_p$ were equivalent on $\mathscr{C}_0^\infty(M)$. That is, when do there exist two constants c_p, C_p such that

$$c_p\|\Delta^{1/2} f\|_p \le \||\nabla f|\|_p \le C_p\|\Delta^{1/2} f\|_p,\ \forall f \in \mathscr{C}_0^\infty(M)? \tag{E_p}$$

One says that the Riesz transform $\nabla\Delta^{-1/2}$ is L^p bounded on M, or that M satisfies (R_p), if

$$\||\nabla f|\|_p \le C\|\Delta^{1/2} f\|_p,\ \forall f \in \mathscr{C}_0^\infty(M), \tag{R_p}$$

and M satisfies (RR_p) if

$$\|\Delta^{1/2} f\|_p \le C\||\nabla f|\|_p,\ \forall f \in \mathscr{C}_0^\infty(M). \tag{RR_p}$$

It is well-known (see for example [CD03]) that by duality (R_p) implies $(RR_{p'})$, where p' is the conjugate of p. The converse is false. Obviously, (R_2) and (RR_2) are always true. The real issue is the case $p \neq 2$ and a lot of work has addressed this problem. For a complete introduction about the background of this problem, we refer to [ACDH04] and the references therein.

Estimates of the heat kernel and its derivative happen to be a key ingredient for the boundedness of the Riesz transform. We list some Gaussian heat kernel estimates as follows:

The on-diagonal upper estimate

$$p_t(x,x) \le \frac{c}{V(x,\sqrt{t})},\ \forall x \in M, t > 0. \tag{DUE}$$

The off-diagonal Gaussian upper estimate

$$p_t(x,y) \le \frac{C}{V(x,\sqrt{t})}\exp\left(-\frac{d^2(x,y)}{Ct}\right),\ \forall x,y \in M, t > 0. \tag{UE}$$

In fact, assuming the doubling volume property, (DUE) self-improves into the Gaussian heat kernel estimate (UE) (See for example [CS08], [Gri09]).

The Li-Yau type estimate, i. e., $\forall x,y \in M, t > 0$,

$$\frac{c}{V(x,\sqrt{t})}\exp\left(-\frac{d^2(x,y)}{ct}\right) \le p_t(x,y) \le \frac{C}{V(x,\sqrt{t})}\exp\left(-\frac{d^2(x,y)}{Ct}\right). \tag{LY}$$

The gradient upper estimate

$$|\nabla p_t(x,y)| \leq \frac{C}{\sqrt{t}V(y,\sqrt{t})},\ \forall x,y \in M, t > 0. \tag{G}$$

Let us first recall a result of Coulhon and Duong in [CD99].

Theorem 0.54. *Let M be a complete non-compact manifold satisfying the doubling volume property (D) and the heat kernel estimate (DUE). Then (R_p) holds for $1 < p < 2$.*

Note that (R_p) may be false under (D) and (DUE) for $p > 2$, even (RR_p) for $1 < p < 2$. The following are two counterexamples.

Connected sum of $\mathbb{R}^n$ Let M_n be a manifold consisting in two copies of $\mathbb{R}^n \backslash B(0,1)$, with the Euclidean metric, glued smoothly along the unit circles.

For $n \geq 2$, Coulhon and Duong ([CD99, Section 5]) showed that the Riesz transform on M_n is not L^p bounded for $p > n$. Note that M_n satisfies (D) and (UE) (the assumptions in Theorem 0.54), but (LY) fails.

For $n \geq 3$, the result can be improved to the point that the Riesz transform on M_n is not L^p bounded for $p \geq n$ (see [CCH06]). While for $1 < p < n$, the Riesz transform is L^p bounded (see [Car07, CCH06]).

Vicsek manifold (see [BCG01, Section 6]) Let M be a Vicsek manifold with polynomial growth of exponent $D > 1$ (D can be arbitrarily large) for large radius, that is,

$$V(x,r) \simeq r^D,\ r \geq 1.$$

It satisfies the heat kernel estimate

$$\sup_{x \in M} p_t(x,x) \simeq t^{-\frac{D}{D+1}},\ t \geq 1.$$

Proposition 0.55 ([CD03])**.** *Let M be a Vicsek manifold with polynomial growth of exponent $D > 1$. Then (RR_p) is false for $1 < p < \frac{2D}{D+1}$. Consequently, (R_p) is false for $p > \frac{2D}{D-1}$.*

If we further assume the lower Gaussian bound on the heat kernel, that is the Li-Yau type estimates, or equivalently (D) and the Poincaré inequality (P) below (see [SC02]), then Theorem 0.54 can be improved.

Theorem 0.56 ([AC05])**.** *Let M be a complete non-compact Riemannian manifold satisfying (D) and the Poincaré inequality, that is, there exists $C > 0$ such that $\forall B$, $\forall f \in C_0^\infty(B)$*

$$\int_B |f - f_B|^2 d\mu \leq C r_B^2 \int_B |\nabla f|^2 d\mu, \tag{P}$$

where r_B is the radius of B. Then there exists $\varepsilon > 0$ such that (R_p) holds for $2 < p < 2+\varepsilon$.

Results for p not necessarily close to 2 are obtained under stronger conditions:

Theorem 0.57 ([ACDH04]). *Let M be a complete non-compact Riemannian manifold satisfying (LY). If for some $p_0 \in (2,\infty]$ there exists $C > 0$ such that for all $t > 0$,*

$$\left\| \left| \nabla e^{-t\Delta} \right| \right\|_{p_0 \to p_0} \leq \frac{C}{\sqrt{t}}, \tag{G_{p_0}}$$

then the Riesz transform is bounded on L^p for $2 < p < p_0$.

Theorem 0.58 ([ACDH04], [CS08]). *Let M be a complete non-compact Riemannian manifold satisfying the doubling volume property (D). If (G) holds, then the Riesz transform is bounded on L^p and (E_p) holds for $1 < p < \infty$.*

0.9 Hardy spaces associated with operators

The study of Hardy spaces originated in the 1910's and at the very beginning was confined to Fourier series and complex analysis in one variable. Since 1960's, it has been transferred to real analysis in several variables, or more generally to analysis on spaces of homogeneous type (see [CW77]). Hardy spaces H^p are defined for $p > 0$ and for $p > 1$ they usually coincide with L^p spaces. Here we mainly focus on the case $p \geq 1$, and in particular $p = 1$. There are many different equivalent definitions of Hardy spaces. They involve suitable maximal functions, the atomic decomposition, the molecule decomposition, singular integrals, etc (see [FS72, CW77], ...).

Let us first recall the H^1 space defined on a general homogeneous space M in terms of atoms (see [CW77]). A function $a \in L^2(M)$ is an H^1-atom if there exists a ball $B \in M$ such that

1. $\operatorname{supp} a \subset B$,
2. $\|a\|_2 \leq V(B)^{-1/2}$,
3. $\int_M a(x)dx = 0$.

A complex-valued function f defined on M belongs to $H^1(M)$ if and only if f has a decomposition

$$f = \sum_{j=0}^{\infty} \lambda_j a_j,$$

where $\sum_{j=0}^{\infty} |\lambda_j| < +\infty$ and a_j's are H^1-atoms. A similar definition can be devised for all $p > 0$.

The molecular Hardy spaces are defined similarly. Compared with atoms, molecules don't have compact supports but decay sufficiently fast. They can be used as an alternative to atoms for many purposes.

In the Euclidean space, Coifman, Meyer and Stein later introduced tent spaces in [CMS85] (see [Rus07] for homogeneous spaces), which connect atomic Hardy spaces and square functions. A measurable function F on $\mathbb{R}^n \times (0,\infty)$ belongs to the tent space $T_2^p(\mathbb{R}^n)$ if

$$\|F\|_{T_2^p} := \left\| \left(\iint_{|y-x|<t} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2} \right\|_{L^p} < \infty.$$

For more details about the tent spaces, see Section 4.2.3 below. Let us clarify the connection among tent spaces, square functions and Hardy spaces (see [FS72, AMR08]). Let f be a suitable function on $\mathbb{R}^n$. For all $x \in \mathbb{R}^n$, we define the so-called conical square function as follows

$$Sf(x) = \left(\iint_{|y-x|<t} \left| t\sqrt{\Delta} e^{-t\sqrt{\Delta}} f(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2},$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $e^{-t\sqrt{\Delta}}$ is the Poisson semigroup.

One can define Hardy spaces H^p via this square function. That is, $f \in H^p$, $p \geq 1$, if $Sf \in L^p$, i.e. $t\sqrt{\Delta}e^{-t\sqrt{\Delta}} f \in T_2^p$. Indeed, the H^1 space defined in this way is equivalent to the definition via molecules. On the one hand, it was shown in [FS72] that if $f \in H^1(\mathbb{R}^n)$, then

$$\|Sf\|_{L^1} \leq C\|f\|_{H^1}.$$

On the other hand, by the Calderón reproducing formula, f can be written as

$$f = \int_0^\infty t\sqrt{\Delta} e^{-t\sqrt{\Delta}} F_t \frac{dt}{t}, \tag{14}$$

where $F_t := t\sqrt{\Delta}e^{-t\sqrt{\Delta}} f$ belongs to the tent space T_2^1. It is proved in [CMS85] that F_t admits a T_2^1-atomic decomposition. One can plug this into (14) and obtain a molecular decomposition for f. For $p \in (1,\infty)$, it holds that $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

In [AMR08], Auscher, McIntosh and Russ went beyond the Euclidean setting and considered a complete Riemannian manifold M satisfying the doubling volume property and defined Hardy spaces of differential forms of all degrees (see [AMM13] for Hardy spaces of vector bundles on doubling metric measure spaces). Let $L^2(\Lambda T^*M)$ be the Hilbert space of square-integrable differential forms of every order on M. Denote by d and d^*, the exterior derivative and its adjoint on $L^2(\Lambda T^*M)$. The Hodge-Dirac operator is $D = d + d^*$ and the Hodge-Laplacian is $\Delta = D^2$. The Riesz transform $D\Delta^{-1/2}$ is bounded on $L^2(\Lambda T^*M)$ by definition. The authors then construct Hardy spaces H_D^p of differential forms by means of suitable square functions such that the Riesz transform $D\Delta^{-1/2}$ is bounded on $H_D^p(\Lambda T^*M)$ for $1 \leq p \leq 2$. A molecular characterisation of $H_D^1(\Lambda T^*M)$ is also given.

Theorem 0.59 ([AMR08])**.** *Let M be a complete Riemannian manifold satisfying the doubling* *volume property. Then,*

- *For all* $1 \leq p \leq 2$, $H^p(\Lambda T^*M) \subset \overline{\mathscr{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)}$.
- *For all* $2 \leq p < \infty$, $\overline{\mathscr{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)} \subset H^p(\Lambda T^*M)$.

Furthermore, if we assume the Gaussian upper bounds for the heat kernel of the Hodge Laplacian, that is, for all $0 \leq k \leq n$,

$$|p_t^k(x,y)| \leq \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{Ct}\right), \forall x,y \in M, t > 0, \qquad (G_k)$$

where p_t^k *is the kernel of* $e^{-t\Delta_k}$ *and* Δ_k *is the Hodge Laplacian restricted to* $k-$*forms, then* $H^p(\Lambda T^*M) = \overline{\mathscr{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)}$ *for* $1 < p < \infty$.

As a result of functional calculus on $H^p(\Lambda T^*M)$, we obtain

Theorem 0.60 ([AMR08]). *Let M be a complete Riemannian manifold satisfying the doubling volume property. Then, for all* $1 \leq p \leq \infty$, *the Riesz transform* $D\Delta^{-1/2}$ *is* $H^p(\Lambda T^*M)$ *bounded. Consequently, it is* $H^1(\Lambda T^*M) - L^1(\Lambda T^*M)$ *bounded.*

More specifically, focusing on the case of functions (0-forms), it holds

- Assume (D) and (G_0). Then it holds for $1 < p < \infty$,

$$H^p(\Lambda^0 T^*M) = \overline{\mathscr{R}(D) \cap L^p(\Lambda^0 T^*M)}^{L^p(\Lambda^0 T^*M)}.$$

- Assume (D), (G_0) and (G_1). Then the Riesz transform on functions $d\Delta^{-1/2}$ is $L^p(\Lambda^0 T^*M) - L^p(\Lambda^1 T^*M)$ bounded for $1 < p < \infty$.

In [HLM+11], Hofmann, Lu, Mitrea, Mitrea and Yan developed the theory of H^1 and BMO spaces on a space of homogeneous type M with a non-negative self-adjoint operator L, which generates an analytic semigroup $(e^{-tL})_{t>0}$ satisfying the following Davies-Gaffney estimate. That is, there exist $C, c > 0$ such that for any open sets $U_1, U_2 \subset M$,

$$| < e^{-tL} f_1, f_2 > | \leq C \exp\left(-\frac{\text{dist}^2(U_1,U_2)}{ct}\right) \|f_1\|_2 \|f_2\|_2, \ \forall t > 0, \qquad (15)$$

for every $f_i \in L^2(M)$ with $\text{supp}\, f_i \subset U_i$, $i = 1,2$, where $\text{dist}(U_1,U_2) := \inf_{x \in U_1, y \in U_2} d(x,y)$. The authors extended results of [AMR08] by obtaining an atomic decomposition of the H^1 space.

More generally, Kunstmann and Uhl [Uhl11, KU12] considered a non-negative self-adjoint operator L on L^2 which satisfies the Davies-Gaffney estimate (DG_m) (of order m), $m \geq 2$: $\forall t > 0, \forall x, y \in M$,

$$\left\| \mathbb{1}_{B(x,t^{1/m})} e^{-tL} \mathbb{1}_{B(y,t^{1/m})} \right\|_{2 \to 2} \leq C \exp\left(-c \left(\frac{d(x,y)}{t^{1/m}}\right)^{\frac{m}{m-1}}\right). \qquad (16)$$

They defined Hardy spaces via square functions and via molecules which are adapted to (DG_m). They proved that the two H^1 spaces are equivalent. Moreover, if the so-called generalised Gaussian $(p_0, p_0')-$estimates (of order m) holds: $\forall t > 0, \forall x, y \in M$,

$$\left\| \mathbb{1}_{B(x,t^{1/m})} e^{-tL} \mathbb{1}_{B(y,t^{1/m})} \right\|_{p_0 \to p_0'} \leq \frac{C}{V^{\frac{1}{p_0} - \frac{1}{p_0'}}(x, t^{1/m})} \exp\left(-c \left(\frac{d(x,y)}{t^{1/m}} \right)^{\frac{m}{m-1}} \right), \tag{17}$$

(which implies that $(e^{-tL})_{t>0}$ acts on L^p for $p \in (p_0, p_0')$), they prove that the Hardy space H^p defined via square functions coincides with L^p for $p \in (p_0, 2)$.

0.10 Presentation of results

In this thesis, we mainly consider two subjects: the L^p boundedness of (quasi) Riesz transforms and the Hardy space theory. Besides, we also consider some inequalities on Vicsek graphs. In the following, we first present our results on different subjects respectively, and then give the plan of this thesis.

0.10.1 About (quasi) Riesz transforms

We study the (quasi) Riesz transform in different settings: the general complete Riemannian manifold, the Riemannian manifold satisfying (D) and ($UE_{2,m}$), the graph satisfying (D) and (UE_m). See Chapter 2, 3, 5 and Section 4.4.

On complete Riemannian manifolds Consider an arbitrary complete Riemannian manifold without any volume and heat kernel assumptions.

Proposition 0.61. *Let M be a complete Riemannian manifold. Then for $1 < p \leq 2$, we have*

$$\left\| |\nabla e^{-t\Delta}| \right\|_{p \to p} \leq \frac{C}{\sqrt{t}},$$

and equivalently

$$\| |\nabla f| \|_p^2 \leq C \|f\|_p \|\Delta f\|_p.$$

As an application, one can obtain a weak version of (R_p) for $1 < p \leq 2$.

Theorem 0.62. *Let M be a complete manifold. Then, for any fixed $\alpha \in (0, 1/2)$, the operator $\nabla e^{-\Delta} \Delta^{-\alpha}$ is bounded on L^p for all $1 < p \leq 2$.*

Indeed, we shall see in Section 2.1 below that $\nabla e^{-\Delta} \Delta^{-1/2}$ is to be seen as Riesz transform at infinity. The above property is weaker than the L^p boundedness of $\nabla e^{-\Delta} \Delta^{-1/2}$.

Together with local results, this yields:

Proposition 0.63. *Let M be a complete Riemannian manifold satisfying (D_{loc}) and (DUE_{loc}), then the quasi Riesz transform* $\nabla(I+\Delta)^{-1/2}+\nabla e^{-\Delta}\Delta^{-\alpha}$ *with* $\alpha\in(0,1/2)$ *is* L^p *bounded for* $1<p\leq 2$.

Then one can try to replace (DUE) with other natural heat kernel upper bounds. For instance, the sub-Gaussian heat kernel upper bound ($UE_{2,m}$), $m\geq 2$ (see Section 1.2).

On Riemannian manifolds satisfying (D) and ($UE_{2,m}$) It is not known whether (DUE) is necessary for the L^p boundedness of Riesz transform for $1<p<2$. A natural question would be to ask whether (R_p) holds if we replace (DUE) with some other natural heat kernel estimates, for instance, ($UE_{2,m}$). That is,

Question 0.64. Are the Riesz transforms L^p bounded for $1<p<2$ on manifolds with the doubling volume property as well as ($UE_{2,m}$)?

If the Riesz transform is not bounded on such kind of manifolds, then they are interesting counter-examples. If the Riesz transform is bounded, then the result in Theorem 0.54 is improved.

So far, we are not able to answer this question. However, under ($UE_{2,m}$) or a weaker condition to be introduced below (see Remark 3.12), we are able to treat the endpoint case $p=1$ of Theorem 1.8. Indeed, we have

Theorem 0.65. *Let M be a complete Riemannian manifold satisfying (D) and ($UE_{2,m}$). Then for any* $0<\alpha<1/2$, *the quasi Riesz transform* $\nabla e^{-\Delta}\Delta^{-\alpha}$ *is of weak type* $(1,1)$.

If we have some additional assumptions, we can get the following positive result for $p>2$.

Theorem 0.66. *Let M be a complete non-compact Riemannian manifold satisfying (D) and the two-sided sub-Gaussian heat kernel bound ($HK_{2,m}$) with* $m\geq 2$, *i.e. ($UE_{2,m}$) and a similar lower bound. Assume that (G_{p_0}) holds for some* $p_0\in(2,\infty]$ *and also the* L^2 *Davies-Gaffney estimate ($DG_{\frac{1}{2}}$):*

$$\left\|\left|\nabla e^{-t\Delta}f\right|\right\|_{L^2(B)}\leq\begin{cases}\dfrac{C}{t^{1/2}}e^{-c\frac{d^2(B,C_i(B))}{t}}\|f\|_{L^2(C_i(B))}, & 0<t<1,\\ \dfrac{C}{t^{1/2}}e^{-c\left(\frac{d^m(B,C_i(B))}{t}\right)^{1/(m-1)}}\|f\|_{L^2(C_i(B))}, & 1\leq t<\infty.\end{cases}$$

Then the Riesz transform at infinity $\nabla e^{-\Delta}\Delta^{-1/2}$ *is* L^p *bounded for* $p\in(2,p_0)$.

Counterexamples for $p>2$ We improve the negative result of Proposition 0.55. That is, there exist manifolds such that the reverse Riesz transform is not L^p bounded for $1<p<2$ and thus (R_p) doesn't hold for $2<p<\infty$.

Theorem 0.67. *Consider the Vicsek manifold M with the polynomial volume growth* $V(x,r)\simeq r^D$ *for* $r\geq 1$. *Then the inequality*

$$\|\Delta^\beta f\|_p\leq C_p\||\nabla f|\|_p$$

is false for any $\beta < \beta(p) := \frac{1}{D+1}\left(\frac{D}{p} + \frac{1}{p'}\right)$*, where* $1 < p < \infty$*.*

In particular, (RR_p*) is false for any* $1 < p < 2$*. Consequently, (*R_p*) doesn't hold for all* $p > 2$*.*

On graphs Consider also the discrete case (graphs). We have the following results, which are similar to Proposition 0.63 and Theorem 0.65.

Proposition 0.68. *Let* Γ *be a infinite connected graph satisfying the local doubling property, that is, for some constant* $c > 1$*,*

$$\mu(B(x,1) \leq c\mu(x),\ \forall x \in \Gamma.$$

Then for any fixed $\beta \in (0,1/2)$ *and* $1 < p \leq 2$*, there exists* $C > 0$ *such that*

$$\| |\nabla (I-P)^{-\beta} f| \|_p \leq C\|f\|_p,$$

for all $f \in L^p$*.*

Theorem 0.69. *Let* Γ *be a infinite connected graph as above. Suppose that* Γ *satisfies the condition* $\triangle(\alpha)$ *and the sub-Gaussian upper heat kernel estimate (*UE_m*). Then for any* $\beta \in (0,1/2)$*, the quasi Riesz transform* $\nabla(I-P)^{-\beta}$ *is weak type* $(1,1)$*.*

0.10.2 About Hardy spaces

Similarly as in [AMR08, HLM$^+$11], we define two classes of Hardy spaces on metric measure space (M,d,μ) with different local and global heat kernel estimates for e^{-tL} where L is non-negative self-adjoint operator. The first class is the H^1 space defined via molecules, denoted by $H^1_{L,\rho,\mathrm{mol}}(M)$. The second class is $H^p_{L,S_h^\rho}(M)$, $p \geq 1$, defined via conical square function (or tent spaces). Note that here ρ is related to the heat kernel estimates. As in [HLM$^+$11], we will show that the two definitions of H^1 space are equivalent. That is

Theorem 0.70. *Let* M *be a metric measure space satisfying the doubling volume property (D) and the heat kernel estimate (*DG_{β_1,β_2}*),* $\beta_1 \leq \beta_2$*. Then* $H^1_{L,\rho,\mathrm{mol}}(M) = H^1_{L,S_h^\rho}(M)$*. Moreover,* $\|f\|_{H^1_{L,\rho,\mathrm{mol}}(M)} \simeq \|f\|_{H^1_{L,S_h^\rho}(M)}$*.*

Comparing $H^p_{L,S_h^\rho}(M)$ and L^p for $1 < p < \infty$, we get the following generalisation of [KU12, Uhl11].

Theorem 0.71. *Let* M *be a non-compact metric measure space satisfying the doubling volume property (D) and the heat kernel estimate (*$DG^{p_0,p_0'}_{\beta_1,\beta_2}$*). Then* $H^p_{L,S_h^\rho}(M) = \overline{R(L)\cap L^p(M)}^{L^p(M)}$ *for* $p_0 < p < p_0'$*.*

Remark 0.72. Compared with (16) and (17), the two assumptions ($DG^{p_0,p_0'}_{\beta_1,\beta_2}$) and ($DG_{\beta_1,\beta_2}$) admit different local and global behaviours.

As a corollary, the pointwise heat kernel estimate gives

Corollary 0.73. *Let M be a non-compact metric measure space satisfying the doubling volume property (D) and the following pointwise heat kernel estimate (UE_{β_1,β_2}). Then $H^1_{L,\rho,\mathrm{mol}}(M) = H^1_{L,S_h^\rho}(M)$, and $H^p_{L,S_h^\rho}(M) = L^p(M)$ for $1 < p < \infty$.*

In particular, we are interested in the case $\beta_1 = 2$. Indeed, fractal manifolds do satisfy (UE_{2,β_2}) with $\beta_2 > 2$ because locally they are like Euclidean spaces and globally they are like fractals (see Example 3). For convenience, we shall call $\beta_2 = m$ from now on. In that case, let Δ be the Laplace-Beltrami operator, we denote for simplicity by $H^1_{\Delta,m,mol}$ the H^1 space defined via molecules $H^1_{L,\rho,mol}$, H^p_{Δ,S_h^m} the H^p space defined via square functions H^p_{L,S_h^ρ}. In this notation, Theorem 1.1 states that we denote by $H^1_{\Delta,m}$ the H^1 space.

In the following theorem, we show that for $1 < p < 2$, the equivalence may not hold between L^p and H^p defined via conical square function S_h with scaling t^2. That is,

Theorem 0.74. *Let M be a Riemannian manifold with polynomial volume growth*

$$V(x,r) \simeq r^d,\ r \geq 1, \tag{18}$$

as well as two-sided sub-Gaussian heat kernel estimate ($HK_{2,m}$) with $2 < m < d/2$, that is, ($UE_{2,m}$) and the matching lower bound. Then $L^p(M) \subset H^p_{\Delta,S_h}(M)$ doesn't hold for $p \in \left(\frac{d}{d-m}, 2\right)$.

As an application of this Hardy space theory, we have

Theorem 0.75. *Let M be a manifold satisfying the doubling volume property (D) and the heat kernel estimate ($UE_{2,m}$), $m > 2$. Then for any fixed $\alpha \in (0,1/2)$, the operator $\nabla e^{-\Delta}\Delta^{-\alpha}$ is $H^1_{\Delta,m} - L^1$ bounded.*

Of course, this gives back Theorem 0.62 by interpolation in this particular case.

0.10.3 Other results

Consider the Vicsek graph, we have the following generalised Poincaré inequalities and Sobolev inequalities, which are all optimal.

Theorem 0.76. *Let Γ be a Vicsek graph with polynomial volume growth $V(x,n) \simeq n^D$. Then for $p \geq 1$, it holds*

$$\|f - f_n(x)\|_{L^p(B(x,n))} \leq CV(x,n)^{\frac{1}{D(p)}} \||\nabla f|\|_{L^p(B(x,2n))},$$

where $f_n(x) = \frac{1}{V(x,n)} \sum_{z \in B(x,n)} f(z)\mu(z)$ and $\frac{1}{D(p)} = \frac{1}{p} + \frac{1}{p'D}$.

Theorem 0.77. *Let Γ be a Vicsek graph with polynomial volume growth $V(x,n) \simeq n^D$. Then for $p \geq 1$, it holds*

$$\|f\|_p \lesssim \mu(\Omega)^{\frac{1}{D(p)}} \||\nabla f|\|_p, \forall \Omega \subset \Gamma, \forall f \in c_0(\Omega), \qquad (S^p_{D(p)})$$

where $D(p)$ is the same as in Theorem 0.76.

The plan of the thesis In Chapter 1, we explain the relation of Riesz transform, local Riesz transform, Riesz transform at infinity and the quasi Riesz transform. Our main result in this section is Theorem 0.62. We also get a negative result Theorem 1.12.

In Chapter 2, we concentrate on Riemannian manifolds satisfying (D) and ($UE_{2,m}$). We consider the endpoint case $p = 1$ for the quasi Riesz transform in two different ways. To be precise, we show that quasi Riesz transforms are weak $(1,1)$ bounded (Theorem 0.65). Moreover, we consider the case $p > 2$. Indeed, with additional assumptions on the gradient of the heat semigroup, the Riesz transform at infinity is L^p bounded.

In Chapter 3, we develop a Hardy space theory on metric measure spaces satisfying (D) and (DG_{β_1,β_2}). Firstly we define Hardy spaces via molecules $H^1_{L,\rho,mol}$ and via square functions H^p_{L,S^ρ_h} ($p \geq 1$), which are adapted to the heat kernel estimates. The two H^1 spaces are shown to be equivalent. Then we compare the H^p space defined via square functions with L^p. Assuming (D) and ($DG^{p_0,p_0'}_{\beta_1,\beta_2}$), we show the equivalence of $H^p_{S^\rho_h}$ and L^p for $p_0 < p < p_0'$. We also find some counterexamples such that $H^p_{S_h} \neq L^p$, where $H^p_{S_h}$ is the Hardy space defined via the square function with the scaling t^2. Finally, as an application of this Hardy space theory, we proved the $H^1 - L^1$ boundedness of quasi Riesz transforms on some fractal manifolds.

In Chapter 4, we work on the discrete case (graphs). In Section 4.1, we obtain the L^p boundedness of the discrete quasi Riesz transforms for $1 < p \leq 2$. In Section 4.2, we prove the weak $(1,1)$ boundedness of the quasi Riesz transform.

In Chapter 5, we consider Vicsek graphs. We prove the generalised Poincaré inequalities and Sobolev inequalities on Vicsek graphs which are all optimal.

Chapter 1

L^p boundedness of quasi Riesz transforms on Riemannian manifolds

In this chapter, we study the (quasi) Riesz transform on an arbitrary complete Riemannian manifold M without any other assumptions.

We first describe the relations between Riesz transform, local Riesz transform, Riesz transform at infinity and quasi Riesz transform. Then we prove the L^p boundedness of the gradient of the heat semigroup (G_p) for $1 < p \leq 2$, which implies the L^p boundedness of the quasi Riesz transform (Proposition 1.9). At last we give a counterexample on which (R_p) doesn't hold for $p > 2$.

We could as well consider a metric measure space setting associated with a regular and strongly local Dirichlet form, which admits a "carré du champ" (see Section 0.6.3, or see [BÉ85, GSC11] for more information).

1.1 Localisation of Riesz transforms

Write the Riesz transform

$$\nabla\Delta^{-1/2} = \int_0^\infty \nabla e^{-t\Delta} \frac{dt}{t^{1/2}}.$$

Alexpoulos [Ale02] separated the integral into local and global parts and considered the integrals respectively to show the L^p boundedness of Riesz transform.

An alternative and equivalent method given in [DtER03] is to consider the following two local Riesz transform and Riesz transform at infinity:

For $1 < p < \infty$, we say that the local Riesz transform is L^p bounded if

$$\||\nabla f|\|_p \leq C\|(I+\Delta)^{1/2} f\|_p, \ \forall f \in \mathscr{C}_0^\infty(M). \tag{R_p^{loc}}$$

and the Riesz transform at infinity is L^p bounded if

$$\||\nabla e^{-\Delta} f|\|_p \leq C\|\Delta^{1/2} f\|_p, \ \forall f \in \mathscr{C}_0^\infty(M). \tag{R_p^∞}$$

Remark 1.1. Note that at high frequencies, $(I+\Delta)^{-1/2} \simeq \Delta^{-1/2}$. Thus locally $\nabla(I+\Delta)^{-1/2}$ is the Riesz transform. Similarly, since $e^{-\Delta}\Delta^{-1/2} \simeq \Delta^{-1/2}$ when $\Delta \ll \varepsilon$ (i.e. at low frequencies), we can regard the operator $\nabla e^{-\Delta}\Delta^{-1/2}$ as the localisation of Riesz transform at infinity.

Let recall several results related to the local Riesz transform.

Theorem 1.2 ([Bak87]). *Let M be a Riemannian manifold. If the Ricci curvature is bounded from below. Then (R_p^{loc}) holds for $1 < p < \infty$.*

In the case $1 < p < 2$, a local version of Theorem 0.54 says:

Theorem 1.3 ([CD99]). *Let M be a complete Riemannian manifold satisfying the local doubling volume property (D_{loc})*

$$\forall r_0 > 0, \exists C_{r_0} \text{ such that } V(x,2r) \leq C_{r_0} V(x,r), \ \forall x \in M, r \in (0,r_0),$$

and whose volume growth at infinity is at most exponential in the sense that

$$V(x,\lambda r) \leq Ce^{c\lambda} V(x,r), \ \forall x \in M, \lambda > 1, r \leq 1. \tag{E}$$

Suppose (DUE_{loc})

$$p_t(x,x) \leq \frac{C'}{V(x,\sqrt{t})}, \ \forall x \in M, t \in (0,1].$$

Then (R_p^{loc}) holds for $1 < p \leq 2$.

Examples that satisfy the above assumptions include Riemannian manifolds with Ricci curvature bounded from below, fractal manifolds, etc.

Similarly, for $p > 2$, a local version of Theorem 0.57 is as follows:

Theorem 1.4 ([ACDH04]). *Let M be a complete Riemannian manifold satisfying the local doubling volume property (E) and (P_{loc}), i. e., for all $r_0 > 0$ there exists a constant C_{r_0} such that for any ball B with radius $r \leq r_0$,*

$$\int_B |f - f_B|^2 d\mu \leq C_{r_0} r^2 \int_B |\nabla f|^2 d\mu.$$

If for some $p_0 \in (2,\infty]$, $a_0 \geq 0$ and for all $t > 0$,

$$\left\| \left|\nabla e^{-t\Delta}\right| \right\|_{p_0 \to p_0} \leq \frac{Ce^{a_0 t}}{\sqrt{t}}. \tag{$G_{p_0}^{loc}$}$$

Then the local Riesz transform $\nabla(\Delta + a)^{-1/2}$ is L^p bounded for $2 < p < p_0$ and $a > a_0$.

We can characterise the L^p boundedness of Riesz transform by the combination of (R_p^{loc}) and (R_p^∞). That is,

Theorem 1.5. *Let M be a complete Riemannian manifold. Then, for $1 < p < \infty$, the Riesz transform $\nabla\Delta^{-1/2}$ is L^p bounded on M if and only if (R_p^{loc}) and (R_p^∞) hold.*

The proof relies on the following multiplier theorem due to Cowling:

Theorem 1.6 ([Cow83])**.** *Let (M,ρ,μ) be a metric measure space. Let L be the generator of a bounded analytic semigroup on $L^p(M)$ for $1 < p < \infty$ such that e^{-tL} is positive, contractive and sub-Markovian for $t > 0$. Suppose that F is a bounded holomorphic function in the sectorial $\Sigma_{\pi/2} = \{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| < \pi/2\}$. Then*

$$\|F(L)f\|_p \le C\|f\|_p,\ \forall f \in L^p(M),$$

where C depends on p, σ and F.

Proof of Theorem 1.5: First assume (R_p). For any $f \in C_0^\infty(M)$, on the one hand, we have

$$\begin{aligned}\left\|\left|\nabla(I+\Delta)^{-1/2}f\right|\right\|_p &= \left\|\left|\nabla\Delta^{-1/2}\Delta^{1/2}(I+\Delta)^{-1/2}f\right|\right\|_p \\ &\le C\left\|\Delta^{1/2}(I+\Delta)^{-1/2}f\right\|_p \le C\|f\|_p.\end{aligned}$$

Here the last inequality follows from the above theorem. This proves (R_p^{loc}).

On the other hand, (R_p^∞) holds obviously due to the L^p boundedness of heat semigroup. In fact,

$$\left\|\left|\nabla e^{-\Delta}\Delta^{-1/2}f\right|\right\|_p \le C\left\|e^{-\Delta}f\right\|_p \le C\|f\|_p.$$

Conversely, assume (R_p^{loc}) and (R_p^∞), then

$$\begin{aligned}\||\nabla f|\|_p &\le \left\|\left|\nabla e^{-\Delta}f\right|\right\|_p + \left\|\left|\nabla(I-e^{-\Delta})f\right|\right\|_p \\ &\lesssim \left\|\Delta^{1/2}f\right\|_p + \left\|\left|\nabla(I+\Delta)^{-1/2}(I+\Delta)^{1/2}(I-e^{-\Delta})\Delta^{-1/2}\Delta^{1/2}f\right|\right\|_p \\ &\lesssim \left\|\Delta^{1/2}f\right\|_p + \left\|(I+\Delta)^{1/2}(I-e^{-\Delta})\Delta^{-1/2}\Delta^{1/2}f\right\|_p \\ &\lesssim \left\|\Delta^{1/2}f\right\|_p.\end{aligned}$$

Here the last inequality is due to Theorem 1.6. This proves (R_p). □

We shall now introduce a variation of the Riesz transform at infinity. Let $0 < \alpha < 1/2$. We say that M satisfies ($R_p^{\infty,\alpha}$) if

$$\left\|\left|\nabla e^{-\Delta}f\right|\right\|_p \le C\|\Delta^\alpha f\|_p,\ \forall f \in \mathscr{C}_0^\infty(M). \tag{$R_p^{\infty,\alpha}$}$$

Together with the local Riesz transform, it will give us a notion of quasi Riesz transform.

Note that (R_p^∞) implies ($R_p^{\infty,\alpha}$) for all $\alpha \in (0,1/2)$. Indeed,

$$\begin{aligned} \left\| \left| \nabla e^{-\Delta}\Delta^{-\alpha} f \right| \right\|_p &\leq \left\| \left| \nabla e^{-\Delta/2}\Delta^{-1/2} e^{-\Delta/2}\Delta^{1/2-\alpha} f \right| \right\|_p \\ &\leq C \left\| e^{-\Delta/2}\Delta^{1/2-\alpha} f \right\|_p \\ &\leq C\|f\|_p. \end{aligned}$$

1.2 Equivalence of (G_p) and (MI_p)

For any $1 < p < \infty$, we say that M satisfies the multiplicative inequality if $\forall f \in \mathscr{D}(\Delta)$,

$$\||\nabla f|\|_p^2 \leq C\|\Delta f\|_p\|f\|_p, \tag{MI_p}$$

and M satisfies the gradient of the heat semigroup on M is L^p bounded if

$$\left\| \left| \nabla e^{-t\Delta} \right| \right\|_{p\to p} \leq \frac{C_p}{\sqrt{t}}, \ \forall t > 0. \tag{G_p}$$

Recall that (R_p) implies (G_p) and (MI_p). In fact, (G_p) and (MI_p) are equivalent for any $1 < p < \infty$, which was shown in [CS10].

Proposition 1.7 ([CS10])**.** *Let M be a complete Riemannian manifold. Then, for any $1 < p < \infty$, (G_p) is equivalent to (MI_p).*

See also [Dun08] for the proof and [CS10] for more information about the relations of (MI_p), the Riesz transforms, and estimates of the derivative of heat kernel. For the sake of completeness, we give a proof here.

Proof. First assume (MI_p). We substitute f by $e^{-t\Delta}f$ in (MI_p), then

$$\left\| \left| \nabla e^{-t\Delta} f \right| \right\|_p^2 \leq C \left\| \Delta e^{-t\Delta} f \right\|_p \left\| e^{-t\Delta} f \right\|_p.$$

Recall that the heat semigroup is analytic on $L^p(M)$ for $1 < p < \infty$, that is, there exist $C > 0$ such that for all $t > 0$,

$$\left\| t\Delta e^{-t\Delta} \right\|_{p\to p} \leq C.$$

Then we obtain

$$\left\| \left| \nabla e^{-t\Delta} f \right| \right\|_p \leq C t^{-1/2} \|f\|_p.$$

Conversely assume (G_p). For any $f \in C_0^\infty(M)$, write the identity

$$f = e^{-t\Delta} f + \int_0^t \Delta e^{-s\Delta} f ds, \forall t > 0.$$

Then (G_p) yields

$$\begin{aligned} \||\nabla f|\|_p &\leq C \left\| \left|\nabla e^{-t\Delta} f\right| \right\|_p + \left\| \int_0^t \left|\nabla \Delta e^{-s\Delta} f\right| ds \right\|_p \\ &\leq Ct^{-1/2} \|f\|_p + \int_0^t \left\| \left|\nabla e^{-s\Delta} \Delta f\right| \right\|_p ds \\ &\leq Ct^{-1/2} \|f\|_p + Ct^{1/2} \|\Delta f\|_p. \end{aligned}$$

Take $t = \|f\|_p \|\Delta f\|_p^{-1}$, we get (MI_p). □

Under (G_p) (or (MI_p)), the quasi Riesz transform at infinity is L^p bounded:

Proposition 1.8. *Let M be a complete Riemannian manifold satisfying (G_p) for some $p \in (1, \infty)$. Then for any $\alpha \in (0, 1/2)$, $(R_p^{\infty,\alpha})$ holds.*

Proof. For any $f \in \mathscr{C}_0^\infty(M)$, write

$$\nabla e^{-\Delta} \Delta^{-\alpha} f = \int_0^\infty \nabla e^{-(1+t)\Delta} f \frac{dt}{t^{1-\alpha}}.$$

Since (G_p) holds, then we have

$$\left\| \left| \nabla e^{-\Delta} \Delta^{-\alpha} f \right| \right\|_p \leq \int_0^\infty \left\| \left| \nabla e^{-(1+t)\Delta} f \right| \right\|_p \frac{dt}{t^{1-\alpha}} \leq C_p \|f\|_p \int_0^\infty \frac{dt}{(t+1)^{1/2} t^{1-\alpha}},$$

which obviously converges for $\alpha \in (0, 1/2)$. Therefore we have $(R_p^{\infty,\alpha})$ for any $\alpha \in (0, 1/2)$. □

1.3 L^p boundedness of quasi Riesz transform for $1 < p \leq 2$

This part is inspired by [CD03] and [Dun08], where (MI_p) and (G_p) for $1 < p \leq 2$ were shown on manifolds and graphs respectively.

In the following, we will give a different proof of (MI_p) and (G_p) on Riemannian manifolds. More precisely, we directly show (MI_p) and (G_p) by using the method to prove the L^p boundedness of the Littlewood-Paley-Stein function in [Ste70b] (see also [CDL03, Theorem 1.2]). While in [Dun08, Theorem 1.3], an analogue proof was given in the discrete case.

Proposition 1.9. *Let M be a complete Riemannian manifold. Then (MI_p) and (G_p) hold for $1 < p \leq 2$.*

Proof. Assume that $f \in \mathscr{C}_0^\infty(M)$ is non-negative and not identically zero. Set $u(x,t) = e^{-t\Delta} f(x)$. Then u is smooth and positive everywhere. For any $1 < p \leq 2$, we have

$$\begin{aligned} J(x,t) &:= \left(\frac{\partial}{\partial t} + \Delta\right) u^p(x,t) \\ &= p u^{p-1}(x,t) \left(\frac{\partial}{\partial t} + \Delta\right) u(x,t) - p(p-1) u^{p-2}(x,t) |\nabla u(x,t)|^2 \\ &= -p(p-1) u^{p-2}(x,t) |\nabla u(x,t)|^2, \end{aligned}$$

which yields

$$|\nabla u(x,t)|^2 = -\frac{1}{p(p-1)} u^{2-p}(x,t) J(x,t).$$

Also note that $J(x,t)$ is non-positive and $\int_M \Delta u^p(x,t) d\mu(x) = 0$.

With these preparations, we get by Hölder inequality that

$$\begin{aligned} \|\nabla u(\cdot,t)\|_p^p &\leq C_p \int_M u^{p(2-p)/2} (-J(x,t))^{p/2} d\mu(x) \\ &\leq C_p \left(\int_M u^p d\mu(x)\right)^{(2-p)/2} \left(-\int_M J(x,t) d\mu(x)\right)^{p/2}. \end{aligned} \tag{1.1}$$

Note that (1.1) uses the fact that $1 < p \leq 2$.

Next, since $\int_M \Delta u^p(x,t) d\mu(x) = 0$,

$$\begin{aligned} -\int_M J(x,t) d\mu(x) &= -\int_M \frac{\partial}{\partial t} u^p(x,t) d\mu(x) = \int_M p u^{p-1}(x,t) \Delta u(x,t) d\mu(x) \\ &\leq C_p \left(\int_M u^p(x,t) d\mu(x)\right)^{1/p'} \left(\int_M (\Delta u(x,t))^p d\mu(x)\right)^{1/p}, \end{aligned}$$

where the inequality is again due to Hölder inequality and p' is the conjugate of p.

Combining the above two estimates, then

$$\| |\nabla u(\cdot,t)| \|_p^p \leq C \|u(\cdot,t)\|_p^{p/2} \|\Delta u(\cdot,t)\|_p^{p/2}. \tag{1.2}$$

On the one hand, as t goes to zero, we get the multiplicative inequality from (1.2)

$$\| |\nabla f| \|_p^p \leq C \|f\|_p^{p/2} \|\Delta f\|_p^{p/2}.$$

On the other hand, by the analyticity of the heat semigroup, (1.2) yields

$$\| |\nabla u(\cdot,t)| \|_p^p \leq C t^{-p/2} \|f\|_p^p,$$

which is exactly (G_p). □

Remark 1.10. Note that the above argument only works for $1 < p \leq 2$. Indeed (G_p) for $p > 2$ has consequences that are not always true, see [ACDH04].

Combining Proposition 1.8 and Proposition 1.9, we get

Corollary 1.11. *Let M be a complete Riemannian manifold. Then for any fixed $\alpha \in (0,1/2)$, the operator $\nabla e^{-\Delta}\Delta^{-\alpha}$ is L^p bounded for $1 < p \leq 2$.*

This corollary, together with the local result Theorem 1.3, gives us

Proposition 1.12. *Let M be a complete Riemannian manifold satisfying (D_{loc}) and (DUE_{loc}), then the quasi Riesz transform $\nabla(I+\Delta)^{-1/2} + \nabla e^{-\Delta}\Delta^{-\alpha}$ with $\alpha \in (0,1/2)$ is L^p bounded for $1 < p \leq 2$.*

1.4 Conclusion

To summarise, we have proved until now that:

- For an arbitrary complete Riemannian manifold M and an arbitrary number $p \in (1,\infty)$, (R_p) is equivalent to (R_p^{loc}) and (R_p^{∞}).
- For an arbitrary complete Riemannian manifold M and an arbitrary number $p \in (1,\infty)$, (MI_p) is equivalent to (G_p). Furthermore, if M satisfies (G_p) (or (MI_p)), then the quasi Riesz transform $\nabla e^{-\Delta}\Delta^{-\alpha}$ is L^p bounded with $\alpha \in (0,1/2)$.
- For an arbitrary complete Riemannian manifold M and an arbitrary number $p \in (1,2]$, (MI_p) and (G_p) are true. Consequently, the quasi Riesz transform $\nabla e^{-\Delta}\Delta^{-\alpha}$ is L^p bounded with $\alpha \in (0,1/2)$

In particular, this yields Proposition 0.63.

1.5 A counterexample for $p > 2$

A negative result for the L^p boundedness of (reverse) Riesz transform with $p > 2$:

Theorem 1.13. *Consider the Vicsek manifold M with the polynomial volume growth $V(x,r) \simeq r^D$ for $r \geq 1$. Then the inequality*

$$\|\Delta^{\beta} f\|_p \leq C_p \||\nabla f|\|_p \tag{1.3}$$

is false for any $\beta < \beta(p) := \frac{1}{D+1}\left(\frac{D}{p} + \frac{1}{p'}\right)$, where $1 < p < \infty$.

In particular, the reverse Riesz transform (RR_p)

$$\|\Delta^{1/2} f\|_p \leq C_p \||\nabla f|\|_p$$

is false for any $p \in (1,2)$. Consequently, the L^p boundedness of the Riesz transform doesn't hold for any $p > 2$.

Remark 1.14. The argument of the proof is taken from [CD03] and we give another example to improve the result therein. We rewrite it for the sake of completeness.

Proof. It is well known [BCG01] that M satisfies the heat kernel estimate

$$\sup_{x\in M} p_t(x,x) \simeq t^{-\frac{D}{D+1}}, t \geq 1.$$

Let $D' = \frac{2D}{D+1}$. Therefore, we have for $p > 1$

$$\|e^{-t\Delta}\|_{1\to p} \leq t^{-\frac{D'}{2}(1-1/p)}, t \geq 1.$$

This implies the following Nash inequality (see [Cou92])

$$\|f\|_p^{1+\frac{2\beta p}{(p-1)D'}} \leq C_p \|f\|_1^{2\beta \frac{p}{(p-1)D'}} \|\Delta^\beta f\|_p, \forall f \in \mathscr{C}_0^\infty(M) \text{ such that } \frac{\|f\|_p}{\|f\|_1} \leq 1.$$

Assume that (1.3) holds. It follows

$$\|f\|_p^{1+\frac{2\beta p}{(p-1)D'}} \leq C_p \|f\|_1^{\frac{2\beta p}{(p-1)D'}} \||\nabla f|\|_p, \forall f \in \mathscr{C}_0^\infty(M) \text{ such that } \frac{\|f\|_p}{\|f\|_1} \leq 1. \tag{1.4}$$

Consider the Vicsek graph Γ, out of which M is constructed by replacing the edges with tubes. We have the discrete version of (1.4) on Γ (see [CSC95]):

$$\|f\|_p^{1+\frac{2\beta p}{(p-1)D'}} \leq C_p \|f\|_1^{\frac{2\beta p}{(p-1)D'}} \||\nabla_\Gamma f|\|_p. \tag{1.5}$$

Note that $\||\nabla_\Gamma f|\|_p^p \simeq \sum_{x,y\in\Gamma} |f(x)-f(y)|^p \mu_{xy}$.

We will show that there exists a function such that (1.5) doesn't hold.

In fact, take the same Ω_n and f as in [BCG01], Section 4 (see Figure 5.2 in Chapter 5). That is: $\Omega_n = \Gamma \bigcap [0,3^n]^N$, where $q = 2^N + 1 = 3^D$. Hence $|\Omega_n| \simeq q^n$. Denote by z_0 the centre of Ω_n and by $z_i, i \leq 1$ its corners. Define f as follows: $f(z_0) = 1, f(z_i) = 0, i \leq 1$, and extend f as a harmonic function in the rest of Ω_n. Then f is linear on each of the paths of length 3^n, which connects z_0 with the corners z_i, and is constant elsewhere. More exactly, if z belongs to some γ_{z_0,z_i}, then $f(z) = 3^{-n} d(z_i,z)$. If not, then $f(z) = f(z')$, where z' is the nearest vertex in certain line of z_0 and z_i.

On the one hand, we have

$$\sum_{x\in\Gamma} |f(x)|^p \mu(x) \leq |\Omega_n|.$$

On the other hand, for any x in the $n-1$ block with centre z_0, we have $f(x) \geq \frac{2}{3}$. Therefore

$$\sum_{x\in\Gamma} |f(x)|^p \mu(x) \geq (2/3)^p |\Omega_{n-1}| \simeq q^n \simeq |\Omega_n|.$$

Also, since $|f(x) - f(y)| = 3^{-n}$ for any two neighbours x, y on each of the diagonals connecting z_0 and z_i, and otherwise $f(x) - f(y) = 0$, we obtain

$$\|\nabla f\|_{L^p}^p \simeq \sum_{x,y} |f(x) - f(y)|^p \mu_{xy} \leq \sum_{i=1}^{2^N} 3^{-np} d(z_0, z_i) = 2^N 3^{-n(p-1)} \simeq |\Omega_n|^{-\frac{p-1}{D}}.$$

Thus (1.5) yields

$$|\Omega_n|^{\frac{1}{p}\left(1+\frac{2\beta p}{(p-1)D'}\right)} \lesssim |\Omega_n|^{\frac{2\beta p}{(p-1)D'} - \frac{1}{p'D}}. \tag{1.6}$$

Now we compare the exponents of $|\Omega_n|$ on both sides and get

$$\begin{aligned} \frac{2\beta p}{(p-1)D'} - \frac{1}{p'D} - \frac{1}{p}\left(1 + \frac{2\beta p}{(p-1)D'}\right) &= \left(1 - \frac{1}{p}\right)\frac{2\beta p}{(p-1)D'} - \frac{1}{p} - \frac{1}{p'D} \\ &= \beta\frac{D+1}{D} - \frac{1}{p} - \frac{1}{p'D} < 0, \end{aligned}$$

if $\beta < \beta(p) := \frac{1}{D+1}\left(\frac{D}{p} + \frac{1}{p'}\right)$.

Let $n \to \infty$, then $|\Omega_n|$ tends to infinity. Obviously, (1.6) is false for $\beta < \beta(p)$, $1 \leq p < \infty$, which contradicts our assumption (1.3).

In addition, it is easy to see that $\beta(p) > 1/2$ for any $1 < p < 2$. Thus (RR_p) is false for $1 < p < 2$, which also means that the Riesz transform isn't bounded on L^p for $2 < p < \infty$. Indeed, by duality (see for example [CD03]), the L^p boundedness of Riesz transform indicates (RR_p) for p', where p and p' are conjugates and $p \in (1, \infty)$.

□

Remark 1.15. Recall that Theorem 1.5 ([AC05, Theorem 0.4]) says for a complete non-compact manifold satisfying the doubling volume property and the L^2 Poincaré inequality, or equivalently, the two-sided Gaussian estimate of the heat kernel, there exists a constant $\varepsilon > 0$ such that the Riesz transform is L^p bounded for $p \in (2, 2+\varepsilon)$. While Theorem 0.67 shows that the result may be false if the L^2 Poincaré inequality doesn't hold.

Indeed, on a Vicsek graph (also Vicsek manifold) with volume growth $V(x,n) \simeq n^D$, we don't have the L^2 Poincaré inequality. Moreover, we have no L^p Poincaré inequality. A weaker inequality holds (we will prove it later):

$$\|f - f_n(x)\|_{L^2(B(x,n))} \leq C n^{\frac{D+1}{2}} \||\nabla f|\|_{L^2(B(x,2n))},$$

which is optimal as shown below in Chapter 5.

Chapter 2

Sub-Gaussian heat kernel estimates and quasi Riesz transforms on Riemannian manifolds

Remember that assuming only local doubling volume property and local Gaussian heat kernel upper bound, we get the L^p $(1 < p \leq 2)$ boundedness of quasi Riesz transforms $\nabla(I+\Delta)^{-1/2} + \nabla e^{-\Delta}\Delta^{-\alpha}$, where $0 < \alpha < 1/2$. If we assume in addition a global Gaussian heat kernel upper bound, the Riesz transform is L^p bounded for $1 < p \leq 2$ and weak $(1,1)$ bounded. What happens if we suppose globally another heat kernel upper bound, the so-called sub-Gaussian upper bound ($UE_{2,m}$), $m > 2$?

In this chapter, we study the (quasi) Riesz transform on a Riemannian manifold M satisfying (D) and ($UE_{2,m}$).

In Section 2.3, we prove that the quasi Riesz transform is also of weak type $(1,1)$, which gives back Proposition 0.63. Yet we don't know whether the Riesz transform, which corresponds to $\alpha = 1/2$, is L^p bounded or not for $1 < p \leq 2$.

Moreover, we consider the L^p boundedness of the Riesz transform for $p > 2$ in Section 2.4. If, in addition, we have the matching lower estimate for the heat kernel and further assumptions for the gradient of the heat kernel, we obtain the L^p boundedness of the Riesz transform for $p > 2$.

2.1 More about sub-Gaussian heat kernel estimates

Let M be a complete non-compact Riemannian manifold. Recall the sub-Gaussian heat kernel estimate ($UE_{2,m}$), $m > 2$, on M:

$$p_t(x,y) \leq \frac{C}{V(x,\rho^{-1}(t))} \exp\left(-cG(d(x,y),t)\right), \tag{$UE_{2,m}$}$$

where

$$\rho(t) = \begin{cases} t^2, & 0 < t < 1, \\ t^m, & t \geq 1; \end{cases}$$

and

$$G(r,t) = \begin{cases} \dfrac{r^2}{t}, & t \leq r, \\ \left(\dfrac{r^m}{t}\right)^{1/(m-1)}, & t \geq r. \end{cases}$$

Note that for $d(x,y) \leq t$, one has

$$\frac{d^2(x,y)}{t} \leq \left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}.$$

And for $t \leq d(x,y)$, one has

$$\frac{d^2(x,y)}{t} \geq \left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}.$$

Thus ($UE_{2,m}$) implies the following estimate:

$$p_t(x,y) \leq \begin{cases} \dfrac{C}{V(x,t^{1/2})} \exp\left(-c\dfrac{d^2(x,y)}{t}\right), & 0 < t < 1, \\ \dfrac{C}{V(x,t^{1/m})} \exp\left(-c\left(\dfrac{d^m(x,y)}{t}\right)^{1/(m-1)}\right), & t \geq 1. \end{cases} \tag{2.1}$$

That is, the small time behaviour of the heat kernel is Gaussian as in Euclidean spaces while the heat kernel has a sub-Gaussian decay for large time. There exist such manifolds for all $m \geq 2$. One can choose any $D \geq 1$ and any $2 \leq m \leq D+1$ such that $V(x,r) \simeq r^D$ for $r \geq 1$ and ($UE_{2,m}$) holds, see Example 0.52. The matching lower bound can also be considered, that is

$$p_t(x,y) \geq \frac{C}{V(x,\rho^{-1}(t))} \exp\left(-cG(d(x,y),t)\right). \tag{$LE_{2,m}$}$$

If M satisfies both ($UE_{2,m}$) and ($LE_{2,m}$), we say that M has two-sided sub-Gaussian heat kernel estimate, denoted by ($HK_{2,m}$).

Comparing ($UE_{2,m}$) with the Gaussian heat kernel estimate (UE):

$$p_t(x,y) \leq \frac{C}{V(x,\sqrt{t})} \exp\left(-c\frac{d^2(x,y)}{t}\right), \forall x,y \in M, t > 0.$$

For $t > 1$, we have $V(x,t^{1/2}) > V(x,t^{1/m})$. That means $p_t(x,x)$ decays with t more slowly in the sub-Gaussian case than in the Gaussian case. Also for $t \geq \max\{1, d(x,y)\}$, clearly

$$\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)} \geq \frac{d^2(x,y)}{t},$$

then $p_t(x,y)$ decays with $d(x,y)$ faster in the sub-Gaussian case than in the Gaussian case. But on the whole, the two kinds of pointwise estimates are incomparable.

As a simple consequence of $(UE_{2,m})$, for any $f \in L^1(M)$ with compact support and any x such that $\operatorname{dist}(x, \operatorname{supp} f) \geq r$ $(r \geq 0)$, we have

$$\begin{aligned} |e^{-\rho(t)\Delta} f(x)| &= \left| \int_M p_{\rho(t)}(x,y) f(y) d\mu(y) \right| \\ &\leq \frac{C}{V(x,t)} \int_M \exp\left(-c\left(\frac{d(x,y)}{t}\right)^{\gamma(t)}\right) |f(y)| d\mu(y) \\ &\leq \frac{C}{V(x,t)} \exp\left(-c\left(\frac{r}{t}\right)^{\tau(t)}\right) \|f\|_{L^1}, \end{aligned} \tag{2.2}$$

where

$$\tau(t) = \begin{cases} 2, & 0 < t < 1, \\ m/(m-1), & t \geq 1. \end{cases} \tag{2.3}$$

2.2 Weighted estimates of the heat kernel

Let (M, d, μ) be a non-compact complete manifold satisfying the doubling volume property (D) and the sub-Gaussian estimate $(UE_{2,m})$. In the following, we aim to get the integral estimates for the heat kernel and its derivative, which furthermore imply the $L^1 - L^2$ off-diagonal estimates. The method we use here is similar to the one in [CD99, Section 2.3].

First, we have the pointwise estimate of the time derivative of heat kernel:

Lemma 2.1. *Let M be as above, then we have for any $k \geq 1$ and for any $x, y \in M$,*

$$\left| \frac{\partial^k}{\partial t^k} p_t(x,y) \right| \leq \begin{cases} \dfrac{C}{t^k V\left(y, t^{1/2}\right)} \exp\left(-c\dfrac{d^2(x,y)}{t}\right), & t < 1, \\ \dfrac{C}{t^k V\left(y, t^{1/m}\right)} \exp\left(-c\left(\dfrac{d^m(x,y)}{t}\right)^{1/(m-1)}\right), & t \geq 1. \end{cases} \tag{2.4}$$

Proof. It is enough to show the case $k = 1$. For $k > 1$, the proof is similar.

Since $(UE_{2,m})$ implies (2.1), it follows from [Dav97b] (Thm. 4 and Cor. 5) that there exists an

$a \in (0,1)$ such that for $0 < t < a$

$$\left|\frac{\partial}{\partial t} p_t(x,y)\right| \leq \frac{C}{tV\left(y,t^{1/2}\right)} \exp\left(-c\frac{d^2(x,y)}{t}\right);$$

and for $t > a^{-1}$,

$$\left|\frac{\partial}{\partial t} p_t(x,y)\right| \leq \frac{C}{tV\left(y,t^{1/m}\right)} \exp\left(-c\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right).$$

(Observe that $d(x,y) > t \geq 1$ implies $\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)} < \frac{d^2(x,y)}{t}$ with $m > 2$.)

For $t \in (a,1)$, according to [Dav97b], Cor.5, it suffices to show that there exists a constant $\delta \in (0,1)$ such that for all $s \in [(1-\delta)t,(1+\delta)t]$

$$p_s(x,y) \leq \frac{C}{V\left(y,s^{1/2}\right)} \exp\left(-c\frac{d^2(x,y)}{s}\right).$$

This is obvious since $V(x,s^{1/2}) \simeq V(x,s^{1/m})$.

The case $t \in [1,a^{-1})$ is a little complicated. It is enough to show that there exists a constant $\delta \in (0,1)$ such that for all $s \in [(1-\delta)t,(1+\delta)t]$,

$$p_s(x,y) \leq \frac{C}{V\left(y,s^{1/m}\right)} \exp\left(-c\left(\frac{d^m(x,y)}{s}\right)^{1/(m-1)}\right).$$

As it is pointed out, this is true when $d(x,y) > s \geq 1$.

For $s < 1$, if $d(x,y)$ is small, then

$$\begin{aligned}\frac{1}{V\left(y,s^{1/2}\right)} \exp\left(-c\frac{d^2(x,y)}{s}\right) &\leq \frac{C'}{V\left(y,s^{1/m}\right)}\\ &\leq \frac{C''}{V\left(y,s^{1/m}\right)} \exp\left(-c\left(\frac{d^m(x,y)}{s}\right)^{1/(m-1)}\right)\end{aligned}$$

If $d(x,y)$ is large, the control is easy to see. □

Remark 2.2. Since

$$\left|\Delta^k e^{-t\Delta} f(x)\right| = \left|\frac{\partial^k}{\partial t^k} e^{-t\Delta} f(x)\right| = \left|\int_M \frac{\partial^k}{\partial t^k} p_t(x,y) f(y) d\mu(y)\right|,$$

it follows from (2.4) that for any $f \in L^1(M)$ with compact support and any x such that

$\operatorname{dist}(x, \operatorname{supp} f) \geq r \geq 0$,

$$|(\rho(t)\Delta)^k e^{-\rho(t)\Delta} f(x)| \leq \frac{C}{V(x,t)} \exp\left(-c\left(\frac{r}{t}\right)^{\tau(t)}\right) \|f\|_{L^1}, \tag{2.5}$$

here τ is as in (2.3).

Now we intend to estimate $\int_{B(x,r)^c} |\nabla p_t(x,y)| d\mu(x)$ for any $t > 0$ and $r \geq 0$.

Lemma 2.3. *For any $\alpha \in (1/m, 1/2)$, we have for any $y \in M$ and $r \geq 0$,*

$$\int_{M \setminus B(y,r)} |\nabla p_t(x,y)| d\mu(x) \leq \begin{cases} Ct^{-\frac{1}{2}} e^{-c\frac{r^2}{t}}, & 0 < t < 1, \\ Ct^{-\alpha} e^{-c\left(\frac{r^m}{t}\right)^{\frac{1}{m-1}}}, & t \geq 1. \end{cases} \tag{2.6}$$

Remark 2.4. Note that the estimate (2.6) holds for any $0 < \alpha < 1/2$. But in the proof below, α can not achieve $1/2$ unless $m = 2$. This allows us to obtain the weak $(1,1)$ boundedness of $\nabla e^{-\Delta}\Delta^{-\alpha}$, not the Riesz transform. If one could get (2.6) with $\alpha = 1/2$, the proof in Section 2.4 below would yield the boundedness of the Riesz transform.

Proof. For $0 < t < 1$, the above estimate is proved in [CD99].

Now for $t \geq 1$. Comparing with the proof of the Gaussian case in [CD99], we need to replace the weight $\exp\left(-c\frac{d^2(x,y)}{t}\right)$ by $\exp\left(-c\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right)$ (c is chosen appropriately).

Step 1: For any $c > 0$,

$$\int_{M \setminus B(y,r)} \exp\left(-c\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right) d\mu(x) \lesssim e^{-\frac{c}{2}\left(\frac{r^m}{t}\right)^{1/(m-1)}} V\left(y, t^{1/m}\right). \tag{2.7}$$

Indeed,

$$\begin{aligned} & \int_{M \setminus B(y,r)} \exp\left(-c\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right) d\mu(x) \\ \leq\ & e^{-\frac{c}{2}\left(\frac{r^m}{t}\right)^{1/(m-1)}} \int_M \exp\left(-\frac{c}{2}\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right) d\mu(x) \\ \leq\ & e^{-\frac{c}{2}\left(\frac{r^m}{t}\right)^{1/(m-1)}} \sum_{i=0}^{\infty} \int_{B(y,(i+1)t^{1/m})/B(y,it^{1/m})} \exp\left(-\frac{c}{2}\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right) d\mu(x) \\ \leq\ & e^{-\frac{c}{2}\left(\frac{r^m}{t}\right)^{1/(m-1)}} V\left(y, t^{1/m}\right) \sum_{i=0}^{\infty} (i+1)^{\nu} e^{-\frac{c}{2} i^{1/(m-1)}} \\ \leq\ & Ce^{-\frac{c}{2}\left(\frac{r^m}{t}\right)^{1/(m-1)}} V\left(y, t^{1/m}\right). \end{aligned}$$

Step 2: For $0<\gamma<2c$ (c is the constant in $(UE_{2,m})$), we have

$$\begin{aligned}&\int_M p_t(x,y)^2 \exp\left(\gamma\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right) d\mu(x)\\ \leq\ & \frac{C}{V^2\left(y,t^{1/m}\right)}\int_M \exp\left((\gamma-2c)\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right) d\mu(x) \leq \frac{C_\gamma}{V\left(y,t^{1/m}\right)}.\end{aligned}$$

This is a consequence of $(UE_{2,m})$ and Step 1 with $r=0$.

Step 3: Denote

$$I(t,y)=\int_M |\nabla_x p_t(x,y)|^2 \exp\left(\gamma\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right) d\mu(x),$$

with γ small enough. Using integration by parts,

$$\begin{aligned} I(t,y) &= \int_M p_t(x,y)\Delta p_t(x,y) \exp\left(\gamma\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right) d\mu(x)\\ &\quad -\int_M p_t(x,y)\nabla_x p_t(x,y)\cdot\nabla_x \exp\left(\gamma\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right) d\mu(x)\\ &= -\int_M p_t(x,y)\frac{\partial}{\partial t}p_t(x,y) \exp\left(\gamma\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right) d\mu(x)\\ &\quad -\frac{\gamma m}{m-1}\int_M p_t(x,y)\nabla_x p_t(x,y)\left(\frac{d(x,y)}{t}\right)^{1/(m-1)}\\ &\quad \cdot\nabla_x d(x,y) \exp\left(\gamma\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right) d\mu(x)\\ &= I_1(t,y)+I_2(t,y).\end{aligned}$$

According to Lemma 2.1 and Step 1,

$$|I_1(t,y)| \leq \frac{C'_\gamma}{tV\left(y,t^{1/m}\right)}.$$

For I_2, since $|\nabla_x d(x,y)|\leq 1$ and $\left(\frac{d(x,y)}{t}\right)^{1/(m-1)} = \left(\frac{d^m(x,y)}{t}\right)^{1/m(m-1)} t^{-1/m}$, then from Step 2 and Cauchy-Schwarz inequality,

$$|I_2(t,y)| \leq C''_\gamma t^{-1/m}(I(t,y))^{1/2}\left(\frac{C_\gamma}{V\left(y,t^{1/m}\right)}\right)^{1/2}.$$

We get

$$\begin{aligned} I(t,y) &\leq \frac{C'_\gamma}{tV\left(y,t^{1/m}\right)} + t^{-1/m}(I(t,y))^{\frac{1}{2}}\left(\frac{C''_\gamma}{V\left(y,t^{1/m}\right)}\right)^{1/2} \\ &\leq \frac{C'_\gamma}{t^{2/m}V\left(y,t^{1/m}\right)} + (I(t,y))^{1/2}\left(\frac{C''_\gamma}{t^{2/m}V\left(y,t^{1/m}\right)}\right)^{1/2}. \end{aligned}$$

Therefore

$$I(t,y) \leq \frac{C}{t^{2/m}V\left(y,t^{1/m}\right)}. \tag{2.8}$$

Step 4: We divide the integral $\int_{M\setminus B(y,r)} |\nabla p_t(x,y)| d\mu(x)$ as follows

$$\begin{aligned} \int_{M\setminus B(y,r)} |\nabla p_t(x,y)| d\mu(x) &= \sum_{i=0}^{\infty} \int_{2^i r < d(x,y) \leq 2^{i+1} r} |\nabla p_t(x,y)| d\mu(x) \\ &\leq \sum_{i=0}^{\infty} V^{1/2}\left(y, 2^{i+1} r\right) \left(\int_{2^i r < d(x,y) \leq 2^{i+1} r} |\nabla p_t(x,y)|^2 d\mu(x) \right)^{1/2}. \end{aligned}$$

For each $i \geq 0$, it follows from (2.8) that

$$\begin{aligned} &\left(\int_{2^i r < d(x,y) \leq 2^{i+1} r} |\nabla p_t(x,y)|^2 d\mu(x) \right)^{1/2} \\ \leq &\left(\int_{2^i r < d(x,y) \leq 2^{i+1} r} |\nabla p_t(x,y)|^2 \exp\left(c \left(\frac{d^m(x,y)}{t} \right)^{1/(m-1)} \right) d\mu(x) \right)^{1/2} \cdot e^{-c\left(\frac{2^{im} r^m}{t}\right)^{1/(m-1)}} \\ \leq &\frac{C}{t^{1/m} V^{1/2}(y, t^{1/m})} e^{-c\left(\frac{2^{im} r^m}{t}\right)^{1/(m-1)}}. \end{aligned} \tag{2.9}$$

On the other hand, applying (2.7) with $r = 0$ (as well as the corresponding estimate for $t/2 < 1$),

$$\begin{aligned} &\left(\int_{2^i r < d(x,y) \leq 2^{i+1} r} |\nabla p_t(x,y)|^2 d\mu(x) \right)^{1/2} \\ \leq &\left\| \left| \nabla e^{-\frac{t}{2}\Delta} \right| \right\|_{2\to 2} \left\| p_{\frac{t}{2}}(\cdot, y) \right\|_2 \\ \leq &\frac{C}{t^{1/2} V(y, \rho^{-1}(t/2))} \left(\int_M \exp\left(-cG(t/2, d(y,z))\right) d\mu(z) \right)^{1/2} \\ \leq &\frac{C}{t^{1/2} V^{1/2}\left(y, t^{1/m}\right)}. \end{aligned} \tag{2.10}$$

Thus taking $\theta = \frac{\frac{1}{2}-\alpha}{\frac{1}{2}-\frac{1}{m}}$, we get from (2.9) and (2.10) that

$$\begin{aligned}&\left(\int_{2^i r<d(x,y)\leq 2^{i+1}r} |\nabla p_t(x,y)|^2 d\mu(x)\right)^{1/2}\\ \leq&\left(\frac{C}{t^{1/m}V^{1/2}(y,t^{1/m})}e^{-c\left(\frac{2^{im}r^m}{t}\right)^{1/(m-1)}}\right)^{\theta}\left(\frac{C}{t^{1/2}V^{1/2}(y,t^{1/m})}\right)^{1-\theta}\\ \leq&\frac{C}{t^{\alpha}V^{1/2}(y,t^{1/m})}e^{-c\left(\frac{2^{im}r^m}{t}\right)^{1/(m-1)}},\end{aligned} \tag{2.11}$$

where c depends on α.

Finally (3) and (2.11) yield

$$\begin{aligned}\int_{M\setminus B(y,r)}|\nabla p_t(x,y)|d\mu(x) &\leq \sum_{i=0}^{\infty} V^{1/2}(y,2^{i+1}r)\frac{C}{t^{\alpha}V^{1/2}(y,t^{1/m})}e^{-c\left(\frac{2^{im}r^m}{t}\right)^{1/(m-1)}}\\ &\leq Ct^{-\alpha}e^{-c\left(\frac{r^m}{t}\right)^{1/(m-1)}}.\end{aligned}$$

□

Notice the pointwise estimate of the time derivative of the heat kernel in Lemma 2.1. Then we can use the same method as in Lemma 2.3 and get

Lemma 2.5. *For any $0<\alpha<1/2$ and $k\geq 1$, there exist two constants $c,C>0$ such that for any $y\in M$ and $r\geq 0$,*

$$\int_{B(y,r)^c}\left|\nabla\frac{\partial^k}{\partial t^k}p_t(x,y)\right|d\mu(x)\leq\begin{cases}C\dfrac{1}{t^{k+\frac{1}{2}}}e^{-c\frac{r^2}{t}}, & t<1,\\ C\dfrac{1}{t^{k+\alpha}}e^{-c\left(\frac{r^m}{t}\right)^{\frac{1}{m-1}}}, & t\geq 1.\end{cases} \tag{2.12}$$

We also have the following Davies-Gaffney estimates for the heat kernel.

Corollary 2.6. *Let M satisfy (D) and ($UE_{2,m}$). Then for any ball B with radius $r_B>0$, we have for $1\leq p\leq 2$ and $j\geq 2$,*

$$\frac{1}{\mu^{1/2}(2^{j+1}B)}\|e^{-t\Delta}f\|_{L^2(C_j(B))}\leq\begin{cases}\dfrac{Ce^{-c\frac{4^j r_B^2}{t}}}{\mu(B)}\|f\|_{L^1(B)}, & 0<t<1,\\ \dfrac{Ce^{-c\left(\frac{2^{jm}r_B^m}{t}\right)^{1/(m-1)}}}{\mu(B)}\|f\|_{L^1(B)}, & t\geq 1.\end{cases} \tag{2.13}$$

The proof depends on the estimate in Step 2 of the previous lemma, which we omit here.

Now we give the Davies-Gaffney estimates for the derivatives of the heat semigroup. In fact, we adopt a similar proof as in [AM08].

Corollary 2.7. *Let M be a complete non-compact Riemannian manifold satisfying (D) and ($UE_{2,m}$). Fix any $\alpha \in (1/m, 1/2)$. Then for any ball B with radius r_B and centre x_B, it holds for $t > 1$ and $j \geq 2$ that*

$$\left\| \left| \nabla e^{-t\Delta} f \right| \right\|_{L^2(C_j(B))} \leq \begin{cases} \dfrac{C_\alpha e^{-c\frac{4^j r_B^2}{t}}}{t^{1/2} V^{1/2}(x_B, t^{1/2})} \|f\|_{L^1(B)}, & 0 < t < 1, \\ \dfrac{C_\alpha e^{-c\left(\frac{2^{jm} r_B^m}{t}\right)^{1/(m-1)}}}{t^{\alpha} V^{1/2}(x_B, t^{1/m})} \|f\|_{L^1(B)}, & t \geq 1; \end{cases} \tag{2.14}$$

and

$$\left\| \left| \nabla \Delta^k e^{-t\Delta} f \right| \right\|_{L^2(C_j(B))} \leq \begin{cases} \dfrac{C_\alpha e^{-c\frac{4^j r_B^2}{t}}}{t^{k+1/2} V^{1/2}(x_B, t^{1/2})} \|f\|_{L^1(B)}, & 0 < t < 1, \\ \dfrac{C_\alpha e^{-c\left(\frac{2^{jm} r_B^m}{t}\right)^{1/(m-1)}}}{t^{k+\alpha} V^{1/2}(x_B, t^{1/m})} \|f\|_{L^1(B)}, & t \geq 1. \end{cases} \tag{2.15}$$

Proof. We focus on proving the case $t \geq 1$ in (2.14).

On the one hand, we claim

$$\left\| \left| \nabla e^{-t\Delta} f \right| \right\|_{L^2(C_j(B))} \leq \frac{C}{t^{1/m} V^{1/2}(x_B, t^{1/m})} e^{-c\left(\frac{2^{jm} r_B^m}{t}\right)^{1/(m-1)}} \|f\|_{L^1(B)}. \tag{2.16}$$

Indeed, it has been shown in Step 3 of Lemma 2.3 that for $t \geq 1$,

$$\left(\int_M |\nabla p_t(x,y)|^2 \exp\left(c \left(\frac{d^m(x,y)}{t} \right)^{1/(m-1)} \right) d\mu(x) \right)^{1/2} \leq \frac{C}{t^{1/m} V^{1/2}(y, t^{1/m})}.$$

Thus for all $y \in B$, $j \geq 2$ and $t \geq 1$, it holds that

$$\begin{aligned} &\left(\int_{C_j(B)} |\nabla p_t(x,y)|^2 d\mu(x) \right)^{1/2} \\ \leq &\left(\int_M |\nabla p_t(x,y)|^2 \exp\left(c \left(\frac{d^m(x,y)}{t} \right)^{1/(m-1)} \right) d\mu(x) \right)^{1/2} \cdot e^{-c\left(\frac{2^{jm} r_B^m}{t}\right)^{1/(m-1)}} \\ \leq &\frac{C}{t^{1/m} V^{1/2}(y, t^{1/m})} e^{-c\left(\frac{2^{jm} r_B^m}{t}\right)^{1/(m-1)}}. \end{aligned} \tag{2.17}$$

Now for $f \in C_0^\infty(B)$, by using Minkowski inequality and (D), we get

$$\left\| \left| \nabla e^{-t\Delta} f \right| \right\|_{L^2(C_j(B))}$$

$$
\begin{aligned}
&\leq \int_B |f(y)| \left(\int_{C_j(B)} |\nabla p_t(x,y)|^2 d\mu(x) \right)^{1/2} d\mu(y) \\
&\leq \frac{C}{t^{1/m}} e^{-c\left(\frac{2^{jm} r_B^m}{t}\right)^{1/(m-1)}} \int_B \frac{|f(y)|}{V^{1/2}(y,t^{1/m})} d\mu(y) \\
&\leq \frac{C}{t^{1/m} V^{1/2}(y,t^{1/m})} e^{-c\left(\frac{2^{jm} r_B^m}{t}\right)^{1/(m-1)}} \int_B |f(y)| \frac{V^{1/2}\left(y, r_B + t^{1/m}\right)}{V^{1/2}(y,t^{1/m})} d\mu(y) \\
&\leq \frac{C}{t^{1/m} V^{1/2}(x_B,t^{1/m})} e^{-c\left(\frac{2^{jm} r_B^m}{t}\right)^{1/(m-1)}} \|f\|_{L^1(B)}.
\end{aligned}
$$

On the other hand, since $\||\nabla f|\|_2 = \|\Delta^{1/2} f\|_2$ always holds, it follows from spectral theory that

$$
\left\| \left| \nabla e^{-t\Delta} \right| \right\|_{2\to 2} \leq C t^{-1/2}.
$$

For any function $f \in \mathscr{C}_0^\infty(B)$, we have for $t \geq 1$

$$
\begin{aligned}
\left\| \left| \nabla e^{-t\Delta} f \right| \right\|_{L^2} &\leq \left\| \left| \nabla e^{-\frac{t}{2}\Delta} \right| \right\|_{2\to 2} \|e^{-\frac{t}{2}\Delta} f\|_{L^2(M)} \leq \frac{C\left(1 + r_B/\sigma\left(\frac{t}{2}\right)\right)^{\nu/2}}{t^{1/2} V^{1/2}(x_B, \sigma\left(\frac{t}{2}\right))} \|f\|_{L^1(B)} \\
&\leq \frac{C(1 + r_B/t^{1/m})^{\nu/2}}{t^{1/2} V^{1/2}(x_B, t^{1/m})} \|f\|_{L^1(B)},
\end{aligned}
\tag{2.18}
$$

where the last inequality is due to the doubling volume property and the pointwise heat kernel estimate.

Combining the two estimates above (2.16) and (2.18), for any fixed $\theta \in (0,1)$, we obtain

$$
\begin{aligned}
&\left\| \left| \nabla e^{-t\Delta} f \right| \right\|_{L^2(C_j(B))} \\
\leq\ & \left(\frac{C(1 + r_B/t^{1/m})^{\nu/2}}{t^{1/2} V^{1/2}(x_B, t^{1/m})} \|f\|_{L^1(B)} \right)^{\theta} \left(\frac{C e^{-c\left(\frac{2^{jm} r_B^m}{t}\right)^{1/(m-1)}}}{t^{1/m} V^{1/2}(x_B, t^{1/m})} \|f\|_{L^1(B)} \right)^{1-\theta} \\
\leq\ & \frac{C e^{-c\left(\frac{2^{jm} r_B^m}{t}\right)^{1/(m-1)}}}{t^{\frac{\theta}{2} + \frac{1-\theta}{m}} V^{1/2}(x_B, t^{1/m})} \|f\|_{L^1(B)},
\end{aligned}
$$

that is, (2.14) holds.

In a similar way, we can get the other estimates. □

Remark 2.8. Note that the estimate (2.14) holds for any $\alpha < 1/2$, but in the above proof, α can not achieve $1/2$ unless $m = 2$. See Remark 2.4.

We will see that (2.14) is the crucial ingredient in the later proofs. While (2.15) is only used in

Section 3.5.

Besides, (2.14) is stronger than the L^2 Davies-Gaffney estimate for the gradient of the heat semigroup (DG_α) (see Section 2.4 below) because of the L^1 norm instead of the L^2 norm in the right hand side. We do need to use the stronger version (2.14) in Section 2.3 and Section 3.5.

2.3 Weak $(1,1)$ boundedness of quasi Riesz transforms

In order to show the weak $(1,1)$ boundedness of quasi Riesz transform, we will use the Calderon-Zygmund decomposition, which we now recall:

Theorem 2.9. *Let (M,d,μ) be a metric measured space satisfying the doubling volume property. Let $1 \le q \le \infty$ and $f \in L^q$. Let $\lambda > 0$. Then there exists a decomposition of f, $f = g + b = g + \sum_i b_i$ so that*

1. *$|g(x)| \le C\lambda$ for almost all $x \in M$;*
2. *There exists a sequence of balls $B_i = B(x_i, r_i)$ so that each b_i is supported in B_i,*
$$\int |b_i(x)|^p d\mu(x) \le C\lambda^p \mu(B_i)$$
3. *$\sum_i \mu(B_i) \le \frac{C}{\lambda^q} \int |f(x)|^q d\mu(x)$;*
4. *$\|b\|_q \le C\|f\|_q$ and $\|g\|_q \le C\|f\|_q$;*
5. *There exists $k \in \mathbb{N}^*$ such that each $x \in M$ is contained in at most k balls B_i.*

We refer to [CW71] and [Ste70a] for the proof.

Remark 2.10. In this section, we use Theorem 2.9 for $q = 1$. In the proof of Theorem 3.16, we use the L^q version for $q \in (1,2)$ to prove the equivalence of Hardy spaces and Lebesgue spaces.

Our result is

Theorem 2.11. *Let M be a complete Riemannian manifold satisfying (D) and ($UE_{2,m}$). Then for any $0 < \alpha < 1/2$, the quasi Riesz transform $\nabla e^{-\Delta}\Delta^{-\alpha}$ is of weak type $(1,1)$ and thus L^p bound for $1 < p \le 2$.*

Remark 2.12. If we replace the assumption ($UE_{2,m}$) by the weaker assumptions (2.13) for $j \ge 1$ and (2.14) for $j \ge 2$, the above theorem also holds. Indeed, we can prove this by using functional calculus, in the same way as the proof of Theorem 3.16 (or Theorem 2.15). The above statement is enough to beat the main example we have in mind, namely the Vicsek manifold.

Remark 2.13. The following proof is essentially taken from [CD99]. More precisely, we use the singular integral technique which was first developed [DM99].

Proof. Denote $T = \nabla e^{-\Delta}\Delta^{-\alpha}$, it suffices to show that

$$\mu(\{x : |Tf(x)| > \lambda\}) \le C\frac{\|f\|_1}{\lambda}.$$

Fix $f \in L^1(M) \cap L^2(M)$, we take the Calderón-Zygmund decomposition of f at the level of λ for $q = 1$, i. e., $f = g + b = g + \sum_i b_i$, then

$$\mu(\{x : |Tf(x)| > \lambda\}) \le \mu(\{x : |Tg(x)| > \lambda/2\}) + \mu(\{x : |Tb(x)| > \lambda/2\}).$$

Since T is L^2 bounded, by using the property of the Calderón-Zygmund decomposition we get

$$\mu(\{x : |Tg(x)| > \lambda/2\}) \le C\lambda^{-2}\|g\|_2^2 \le C\lambda^{-1}\|g\|_1 \le C\lambda^{-1}\|f\|_1.$$

As for the second term, we divide $\{B_i\}$ into two classes: the one in which the balls have radius no less than 1 and the one in which the balls have radius smaller than 1. Denote by

$$\mathscr{C}_1 = \{i : B_i = B(x_i, r_i) \text{ with } r_i \ge 1\};$$
$$\mathscr{C}_2 = \{i : B_i = B(x_i, r_i) \text{ with } r_i < 1\}.$$

Then we have

$$\begin{aligned}\mu(\{x : |T\sum_i b_i(x)| > \lambda/2\}) \le\ & \mu(\{x : |T\sum_{i\in\mathscr{C}_1} b_i(x)| > \lambda/4\}) \\ & + \mu(\{x : |T\sum_{i\in\mathscr{C}_2} b_i(x)| > \lambda/4\}).\end{aligned}$$

Write

$$Tb_i = Te^{-t_i\Delta}b_i + T\left(I - e^{-t_i\Delta}\right)b_i, \tag{2.19}$$

where $t_i = \rho(r_i)$ with ρ defined in (12). In the following, we will consider the two cases of balls separately.

Case 1: For balls with radius no less than 1, our aim here is to prove

$$\mu(\{x : |T\sum_{i\in\mathscr{C}_1} b_i(x)| > \lambda/4\}) \le C\lambda^{-1}\|f\|_1.$$

Then from (2.19), we have

$$\mu(\{x : |T\sum_{i\in\mathscr{C}_1} b_i(x)| > \lambda/4\}) \le \mu(\{x : |T\sum_{i\in\mathscr{C}_1} e^{-t_i\Delta}b_i(x)| > \lambda/8\})$$

$$+\mu(\{x:|T\sum_{i\in\mathscr{C}_1}(I-e^{-t_i\Delta})b_i(x)|>\lambda/8\}).$$

We begin to estimate the first term. Since T is L^2 bounded, then

$$\mu(\{x:|T\sum_{i\in\mathscr{C}_1}e^{-t_i\Delta}b_i(x)|>\lambda/8\})\leq\frac{C}{\lambda^2}\left\|\sum_{i\in\mathscr{C}_1}e^{-t_i\Delta}b_i\right\|_2^2.$$

Note that for any $j\geq1$, it holds

$$\begin{aligned}\left\|e^{-t_i\Delta}b_i\right\|_{L^2(C_j(B_i))}&\leq C\frac{\mu^{1/2}(2^{j+1}B_i)}{\mu(B_i)}e^{-c2^{jm/(m-1)}}\|b_i\|_1\\&\leq C\lambda\mu^{1/2}(2^{j+1}B_i)e^{-c2^{jm/(m-1)}}.\end{aligned}\tag{2.20}$$

Indeed, for $j\geq2$, the first inequality follows from Corollary 2.6. For $j=1$, we have

$$\begin{aligned}\left\|e^{-t_i\Delta}b_i\right\|_{L^2(4B_i)}&=\left(\int_{4B_i}\left|\int_{B_i}p_{t_i}(x,y)b_i(y)d\mu(y)\right|^2d\mu(x)\right)^{1/2}\\&\leq C\|b_i\|_1\left(\int_{4B_i}V^{-2}(x,r_i)d\mu(x)\right)^{1/2}\\&\leq C\frac{\mu^{1/2}(4B_i)}{\mu(B_i)}\|b_i\|_1^2\leq C\lambda\mu^{1/2}(4B_i).\end{aligned}$$

By a duality argument,

$$\begin{aligned}\left\|\sum_{i\in\mathscr{C}_1}e^{-t_i\Delta}b_i\right\|_2&=\sup_{\|\phi\|_2=1}\int_M\left|\sum_{i\in\mathscr{C}_1}e^{-t_i\Delta}b_i\right||\phi|d\mu\\&\leq\sup_{\|\phi\|_2=1}\sum_{i\in\mathscr{C}_1}\sum_{j=1}^{\infty}\int_{C_j(B_i)}\left|e^{-t_i\Delta}b_i\right||\phi|d\mu\\&:=\sup_{\|\phi\|_2=1}\sum_{i\in\mathscr{C}_1}\sum_{j=1}^{\infty}A_{ij}.\end{aligned}$$

Applying Cauchy-Schwarz inequality and (2.20), we get

$$\begin{aligned}A_{ij}&\leq\left\|e^{-t_i\Delta}b_i\right\|_{L^2(C_j(B_i))}\|\phi\|_{L^2(C_j(B_i))}\\&\leq C\lambda\mu(2^{j+1}B_i)e^{-c2^{jm/(m-1)}}\operatorname*{ess\,inf}_{y\in B_i}\left(\mathscr{M}\left(|\phi|^2\right)(y)\right)^{1/2}\\&\leq C\lambda2^{jD}e^{-c2^{jm/(m-1)}}\mu(B_i)\operatorname*{ess\,inf}_{y\in B_i}\left(\mathscr{M}\left(|\phi|^2\right)(y)\right)^{1/2}.\end{aligned}$$

Here $\mathscr{M}$ denotes the Littlewood-Paley maximal operator:

$$\mathscr{M}f(x) = \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y). \tag{2.21}$$

Then

$$\begin{aligned}
\Big\| \sum_{i\in\mathscr{C}_1} e^{-t_i\Delta} b_i \Big\|_2 &\leq C\lambda \sup_{\|\phi\|_2=1} \sum_{i\in\mathscr{C}_1} \sum_{j=1}^{\infty} 2^{jD} e^{-c2^{jm/(m-1)}} \mu(B_i) \operatorname*{ess\,inf}_{y\in B_i} \big(\mathscr{M}\big(|\phi|^2\big)(y)\big)^{1/2} \\
&\leq C\lambda \sup_{\|\phi\|_2=1} \int \sum_{i\in\mathscr{C}_1} \chi_{B_i}(y) \big(\mathscr{M}\big(|\phi|^2\big)(y)\big)^{1/2} d\mu(y) \\
&\leq C\lambda \sup_{\|\phi\|_2=1} \int_{\cup_{i\in\mathscr{C}_1} B_i} \big(\mathscr{M}\big(|\phi|^2\big)(y)\big)^{1/2} d\mu(y) \\
&\leq C\lambda \mu^{1/2}\big(\cup_{i\in\mathscr{C}_1} B_i\big) \leq C\lambda^{1/2} \|f\|_1^{1/2}.
\end{aligned}$$

Here the third inequality is due to the finite overlapping of the Calderón-Zygmund decomposition. In the fourth inequality, we use Kolmogorov's inequality (see for example [Gra08, page 91]).

Therefore, we obtain

$$\mu(\{x : |T \sum_{i\in\mathscr{C}_1} e^{-t_i\Delta} b_i(x)| > \lambda/8\}) \leq C\lambda^{-1} \|f\|_1.$$

It remains to show $\mu(\{x : |T \sum_{i\in\mathscr{C}_1} (I - e^{-t_i\Delta}) b_i(x)| > \lambda/8\}) \leq C\lambda \|f\|_1$. We have

$$\begin{aligned}
&\mu(\{x : |T \sum_{i\in\mathscr{C}_1} (I - e^{-t_i\Delta}) b_i(x)| > \lambda/8\}) \\
\leq\ &\mu(\{x \in \bigcup_{i\in\mathscr{C}_1} 2B_i : |T \sum_{i\in\mathscr{C}_1} (I - e^{-t_i\Delta}) b_i(x)| > \lambda/8\}) \\
&+\mu(\{x \in M\setminus \bigcup_{i\in\mathscr{C}_1} 2B_i : |T \sum_{i\in\mathscr{C}_1} (I - e^{-t_i\Delta}) b_i(x)| > \lambda/8\}) \\
\leq\ &\sum_{i\in\mathscr{C}_1} \mu(2B_i) + \lambda/8 \sum_{i\in\mathscr{C}_1} \int_{M\setminus 2B_i} |T(I - e^{-t_i\Delta}) b_i(x)| d\mu(x).
\end{aligned}$$

We claim: $\forall t \geq 1$, $\forall b$ with support in B, then

$$\int_{M\setminus 2B} |T(I - e^{-t\Delta}) b(x)| d\mu(x) \leq C\|b\|_1.$$

Denote by $k_t(x,y)$ the kernel of the operator $T(I - e^{-t_i\Delta})$, then

$$\int_{M\setminus 2B} |T(I - e^{-t\Delta}) b(x)| d\mu(x) \leq \int_{M\setminus 2B} \int_B |k_t(x,y)||b(y)| d\mu(y) d\mu(x)$$

$$\leq \int_M |b(y)| \int_{d(x,y)\geq t^{1/m}} |k_t(x,y)| d\mu(x) d\mu(y).$$

It is enough to show that $\int_{d(x,y)\geq t^{1/m}} |k_t(x,y)| d\mu(x)$ is uniformly bounded for $t \geq 1$. The identity $\Delta^{-\alpha} = \int_0^\infty e^{-s\Delta} \frac{ds}{s^{1-\alpha}}$ (we ignore the constant here) gives us

$$T(I-e^{-t\Delta}) = \int_0^\infty \nabla e^{-(s+1)\Delta}(I-e^{-t\Delta}) \frac{ds}{s^{1-\alpha}},$$

that is,

$$\begin{aligned} k_t(x,y) &= \int_0^\infty (\nabla p_{s+1}(x,y) - \nabla p_{s+t+1}(x,y)) \frac{ds}{s^{1-\alpha}} \\ &= \int_0^\infty \left(\frac{1}{s^{1-\alpha}} - \frac{1_{\{s>t\}}}{(s-t)^{1-\alpha}} \right) \nabla p_{s+1}(x,y) ds. \end{aligned}$$

Thus from Lemma 2.3, we have

$$\begin{aligned} & \int_{d(x,y)\geq t^{1/m}} |k_t(x,y)| \, d\mu(x) \\ = & \int_{d(x,y)\geq t^{1/m}} \left| \int_0^\infty \left(\frac{1}{s^{1-\alpha}} - \frac{1_{\{s>t\}}}{(s-t)^{1-\alpha}} \right) \nabla p_{s+1}(x,y) ds \right| d\mu(x) \\ \leq & \int_0^\infty \left| \frac{1}{s^{1-\alpha}} - \frac{1_{\{s>t\}}}{(s-t)^{1-\alpha}} \right| \cdot \int_{d(x,y)\geq t^{1/m}} |\nabla p_{s+1}(x,y)| \, d\mu(x) ds \\ \lesssim & \int_0^\infty \left| \frac{1}{s^{1-\alpha}} - \frac{1_{\{s>t\}}}{(s-t)^{1-\alpha}} \right| (s+1)^{-\alpha} e^{-c\left(\frac{t}{s+1}\right)^{1/(m-1)}} ds \\ = & \left(\int_0^1 + \int_1^t \right) \frac{1}{s^{1-\alpha}(s+1)^\alpha} e^{-c\left(\frac{t}{s+1}\right)^{1/(m-1)}} ds \\ & + \int_t^\infty \left| \frac{1}{s^{1-\alpha}} - \frac{1}{(s-t)^{1-\alpha}} \right| (s+1)^{-\alpha} e^{-c\left(\frac{t}{s+1}\right)^{1/(m-1)}} ds \\ = & K_1 + K_2 + K_3. \end{aligned}$$

In fact, K_1, K_2, K_3 are uniformly bounded:

$$K_1 \leq \int_0^1 s^{\alpha-1} ds < \infty;$$

Since $s+1 \simeq s$ for $s>1$ and we can dominate the e^{-x} by Cx^{-c} for any fixed $c>0$, we have

$$K_2 \leq \int_1^t e^{-c'\left(\frac{t}{s}\right)^{1/(m-1)}} \frac{ds}{s} \leq C \int_1^t \left(\frac{s}{t}\right)^c \frac{ds}{s} < \infty;$$

For K_3,

$$K_3 \leq \int_t^\infty \left| \frac{1}{s^{1-\alpha}} - \frac{1}{(s-t)^{1-\alpha}} \right| s^{-\alpha} ds$$

$$
\begin{aligned}
&= \int_0^\infty \left| \frac{1}{(u+1)^{1-\alpha}} - \frac{1}{u^{1-\alpha}} \right| (u+1)^{-\alpha} du \\
&\leq \int_0^1 \left(\frac{1}{(u+1)} + \frac{1}{u^{1-\alpha}} \right) (u+1)^{-\alpha} du + \int_1^\infty \frac{1}{(u+1)u^{1-\alpha}} du \\
&\leq \int_0^1 \frac{2}{u^{1-\alpha}} du + \int_1^\infty \frac{1}{u^{2-\alpha}} du < \infty.
\end{aligned}
$$

Note that we get the second line by changing variable with $u = \frac{s}{t} - 1$.

Case 2: It remains to show

$$
\mu\left(\left\{x : |T \sum_{i \in \mathscr{C}_2} b_i(x)| > \lambda/4\right\}\right) \leq C\lambda^{-1} \|f\|_1.
$$

We repeat the arguement as Case 1. Still from (2.19), we have

$$
\begin{aligned}
\mu(\{x : |T \sum_{i \in \mathscr{C}_2} b_i(x)| > \lambda/4\}) \leq\ & \mu(\{x : |T \sum_{i \in \mathscr{C}_2} e^{-t_i \Delta} b_i(x)| > \lambda/8\}) \\
& + \mu(\{x : |T \sum_{i \in \mathscr{C}_2} (I - e^{-t_i \Delta}) b_i(x)| > \lambda/8\}).
\end{aligned}
$$

By using the L^2 boundedness of T and Corollary 2.6, the same duality argument in Case 1 yields

$$
\mu(\{x : |T \sum_{i \in \mathscr{C}_2} e^{-t_i \Delta} b_i(x)| > \lambda/8\}) \leq C\lambda^{-1} \|f\|_1.
$$

For the estimate of $\mu(\{x : |T \sum_{i \in \mathscr{C}_2} (I - e^{-t_i \Delta}) b_i(x)| > \lambda/8\})$, it suffices to show that

$$
\int_{d(x,y) \geq t^{1/2}} |k_t(x,y)| \, d\mu(x)
$$

is finite and does not depend on $t < 1$. In fact, Lemma 2.3 yields

$$
\begin{aligned}
& \int_{d(x,y) \geq t^{1/2}} |k_t(x,y)| \, d\mu(x) \\
\leq\ & \int_0^\infty \left| \frac{1}{s^{1-\alpha}} - \frac{1_{\{s>t\}}}{(s-t)^{1-\alpha}} \right| \cdot \int_{d(x,y) \geq t^{1/2}} |\nabla p_{s+1}(x,y)| \, d\mu(x) ds \\
\lesssim\ & \int_0^\infty \left| \frac{1}{s^{1-\alpha}} - \frac{1_{\{s>t\}}}{(s-t)^{1-\alpha}} \right| (s+1)^{-\alpha} e^{-c\left(\frac{t^{m/2}}{s+1}\right)^{1/(m-1)}} ds \\
=\ & \int_0^t \frac{1}{s^{1-\alpha}(s+1)^\alpha} e^{-c\left(\frac{t^{m/2}}{s+1}\right)^{1/(m-1)}} ds \\
& + \int_t^\infty \left| \frac{1}{s^{1-\alpha}} - \frac{1}{(s-t)^{1-\alpha}} \right| (s+1)^{-\alpha} e^{-c\left(\frac{t^{m/2}}{s+1}\right)^{1/(m-1)}} ds
\end{aligned}
$$

$$:= \quad K_1' + K_2'.$$

Because $t<1$, thus $K_1' < K_1$ converges.

We can estimate K_2' in the same way as for K_3 and get an bound that does not depend on t. □

2.4 Remarks on the L^p boundedness of Riesz transform for $p > 2$

Under assumptions of (D) and $(UE_{2,m})$, we have already seen that there exist Riemnnian manifolds such that the Riesz transform is not L^p bounded for $p>2$ (see Section 2.5). However, we will show in the following that under some additional assumptions on M, the Riesz transform (at infinity) is L^p bounded for $p>2$.

Definition 2.14. Let $0<\alpha<1$. We say that M satisfies the Davies-Gaffney estimate (DG_α) if for all $i\geq 1$,

$$\left\|\left|\nabla e^{-t\Delta} f\right|\right\|_{L^2(B)} \leq \begin{cases} \frac{C}{t^{1/2}} e^{-c\frac{d^2(B,C_i(B))}{t}} \|f\|_{L^2(C_i(B))}, & 0<t<1, \\ \frac{C}{t^{\alpha}} e^{-c\left(\frac{d^m(B,C_i(B))}{t}\right)^{1/(m-1)}} \|f\|_{L^2(C_i(B))}, & 1\leq t<\infty. \end{cases}$$

Theorem 2.15. *Let M be a complete non-compact Riemannian manifold satisfying (D) and the two-sided sub-Gaussian heat kernel bound ($HK_{2,m}$). Let $p_0\in(2,\infty]$, assume that*

$$\left\|\left|\nabla e^{-t\Delta}\right|\right\|_{p_0\to p_0} \leq \frac{C}{t^{1/2}}, \forall t>0. \tag{G_{p_0}}$$

and also the L^2 Davies-Gaffney estimate ($DG_{\frac{1}{2}}$). Then the Riesz transform at infinity $\nabla e^{-\Delta}\Delta^{-1/2}$ is L^p bounded for $p\in(2,p_0)$.

We have two observations here:

1. (G_{p_0}) is almost necessary in the sense that (R_p^∞) implies (G_p) for all $2<p<p_0$ and $t\geq 1$.
2. If in addition, M has Ricci curvature bounded from below, then the Riesz transform itself is L^p bounded under the above assumptions. In fact, this follows from the above theorem and Bakry's result about the local Riesz transform (see [Bak87] and more generally [ACDH04]). That is, for a complete noncompact Riemannian manifold with Ricci curvature bounded from below, the local Riesz transform is L^p bounded for $1<p<\infty$.
3. We don't know so far how to obtain $(DG_{\frac{1}{2}})$.

Our main tool is:

Theorem 2.16 ([ACDH04]). *Let (M,d,μ) satisfy (D) and let T be a sublinear operator which is bounded on $L^2(M,\mu)$. Let $p_0 \in (2,\infty]$. Let $A_r, r>0$, be a family of linear operators acting on $L^2(M,\mu)$. Assume*

$$\left(\frac{1}{\mu(B)}\int_B \left|T(I-A_{r(B)})f\right|^2 d\mu\right)^{1/2} \leq C\left(\mathscr{M}\left(|f|^2\right)\right)^{1/2}(x), \tag{2.22}$$

and

$$\left(\frac{1}{\mu(B)}\int_B \left|TA_{r(B)}f\right|^{p_0} d\mu\right)^{1/p_0} \leq C\left(\mathscr{M}\left(|Tf|^2\right)\right)^{1/2}(x), \tag{2.23}$$

for all $f \in L^2(M,\mu)$, all $x \in M$ and all balls $B \ni x$, $r(B)$ being the radius of B. Here $\mathscr{M}$ denotes the Hardy-Littlewood operator as in (2.21). If $2<p<p_0$ and $Tf \in L^p(M,\mu)$ when $f \in L^p(M,\mu)$, then T is of strong type (p,p) and its operator norm is bounded by a constant depending only on its $(2,2)$ norm, on the constant in (D), on p and p_0, and on the constants in (2.22) and (2.23).

Consider more generally $T_\alpha = \nabla e^{-\Delta}\Delta^{-\alpha}$, $0<\alpha<1/2$. We define the regularising operator $A_r, r>0$ by

$$I - A_r = \left(I - e^{-\rho(r)\Delta}\right)^N,$$

where the integer $N>0$ will be chosen later.

Lemma 2.17. *Let M be a complete non-compact Riemannian manifold satisfying the doubling volume property and (DG_α), where $\alpha \in (0,1/2]$ is fixed. Then for some N large enough (depending on the doubling volume property), for every ball B and all $x \in B$,*

$$\left(\frac{1}{\mu(B)}\int_B \left|T_\alpha\left(I-e^{-\rho(r)\Delta}\right)^N f\right|^2 d\mu\right)^{1/2} \leq C\left(\mathscr{M}\left(|f|^2\right)\right)^{1/2}(x). \tag{2.24}$$

Remark 2.18. If we assume that M satisfies $(UE_{2,m})$ instead of (DG_α), then (2.24) holds for T_α with any $0<\alpha<1/2$. Indeed, Corollary 2.7 shows that (D) and $(UE_{2,m})$ imply (DG_α) for any $0<\alpha<1/2$.

Proof. We decompose f as $f=\sum_{i=1}^\infty f_i = \sum_{i=1}^\infty f\chi_{C_i}$. By Minkowski inequality we have

$$\left\|\left|T_\alpha\left(I-e^{-\rho(r)\Delta}\right)^N f\right|\right\|_{L^2(B)} \leq \sum_{i=1}^\infty \left\|\left|T_\alpha\left(I-e^{-\rho(r)\Delta}\right)^N f_i\right|\right\|_{L^2(B)}.$$

In order to obtain (2.24), it suffices to show for every $i \geq 1$, it holds

$$\left\|\left|T_\alpha\left(I-e^{-\rho(r)\Delta}\right)^N f_i\right|\right\|_{L^2(B)} \leq C2^{-ci}\mu^{1/2}(2^{i+1}B)\left(\mathscr{M}\left(|f|^2\right)\right)^{1/2}(x), \tag{2.25}$$

where $c>D$.

For $i=1$, using the L^2 boundedness of $T_\alpha\left(I-e^{-\rho(r)\Delta}\right)^N$, we get

$$\left\|\left|T_\alpha\left(I-e^{-\rho(r)\Delta}\right)^N f_1\right|\right\|_{L^2(B)} \leq \|f\|_{L^2(4B)} \leq \mu^{1/2}(4B)\left(\mathscr{M}\left(|f|^2\right)\right)^{1/2}(x).$$

For $i\geq 2$, write

$$T_\alpha\left(I-e^{-\rho(r)\Delta}\right)^N = \int_0^\infty t^\alpha \nabla e^{-(t+1)\Delta}\left(I-e^{-\rho(r)\Delta}\right)^N \frac{dt}{t}.$$

Let $\Phi_{t,r}(\zeta)=e^{-(t+1)\zeta}(1-e^{-\rho(r)\zeta})^N$. Then

$$\left\|\left|T_\alpha\left(I-e^{-\rho(r)\Delta}\right)^N f_i\right|\right\|_{L^2(B)} \leq \int_0^\infty t^\alpha \||\nabla\Phi_{t,r}(\Delta)f_i|\|_{L^2(B)}\frac{dt}{t}. \tag{2.26}$$

We will estimate $\|\Phi_{t,r}(\Delta)f_i\|_{L^2(B)}$ by functional calculus. The notation mainly comes from [Aus07, Section 2.2].

For any fixed t and r, then $\Phi_{t,r}$ is a holomorphic function satisfying

$$|\Phi_{t,r}(\zeta)| \leq C|\zeta|^N(1+|\zeta|)^{-2N},$$

for all $\zeta\in\Sigma=\{z\in\mathbb{C}^*: |\arg z|<\xi\}$ with any $\xi\in(0,\pi/2)$ and for some constant C.

Since Δ is a nonnegative self-adjoint operator, or equivalently Δ is a sectorial operator of type 0, we can express $\Phi_{t,r}(\Delta)$ by functional calculus. Let $0<\theta<\omega<\xi<\pi/2$, we have

$$\Phi_{t,r}(\Delta) = \int_{\Gamma_+} e^{-z\Delta}\eta_+(z)dz + \int_{\Gamma_-} e^{-z\Delta}\eta_-(z)dz,$$

and

$$\eta_\pm(z) = \int_{\gamma_\pm} e^{\zeta z}\Phi_{t,r}(\zeta)d\zeta.$$

Here $\Gamma_\pm$ is the half-ray $\mathbb{R}^+e^{\pm i(\pi/2-\theta)}$ and $\gamma_\pm$ is the half-ray $\mathbb{R}^+e^{\pm i\omega}$.

Then for any $z\in\Gamma_\pm$,

$$\begin{aligned}
|\eta_\pm(z)| &= \left|\int_{\gamma_\pm} e^{\zeta z}e^{-(t+1)\zeta}(1-e^{-\rho(r)\zeta})^N d\zeta\right| \\
&\leq \int_{\gamma_\pm} |e^{\zeta z-(t+1)\zeta}||1-e^{-\rho(r)\zeta}|^N|d\zeta| \\
&\leq \int_{\gamma_\pm} e^{-c|\zeta|(|z|+t+1)}|1-e^{-\rho(r)\zeta}|^N|d\zeta| \\
&\leq C\int_0^\infty e^{-cs(|z|+t+1)}\rho^N(r)s^N ds \leq \frac{C\rho^N(r)}{(|z|+t+1)^{N+1}}.
\end{aligned}$$

In the second inequality, the constant $c > 0$ depends on θ and ω. Indeed, $\Re(\zeta z) = |\zeta||z|\Re e^{\pm i(\pi/2-\theta+\omega)}$. Since $\theta < \omega$, then $\pi/2 < \pi/2-\theta+\omega < \pi$ and $|e^{\zeta z}| = e^{-c_1|\zeta||z|}$ with $c_1 = -\cos(\pi/2-\theta+\omega)$. Also it is obvious to see that $|e^{\rho(t)\zeta}| = e^{-c_2\rho(t)|\zeta|}$. Thus the second inequality follows. In the third inequality, let $\zeta = se^{\pm i\omega}$, we have $|d\zeta| = ds$. In addition, we dominate $|1-e^{-\rho(r)\zeta}|^N$ by $(\rho(r)\zeta)^N$.

We choose θ appropriately such that $|z| \sim \Re z$ for $z \in \Gamma_\pm$, then for any $j \geq 2$ fixed,

$$\begin{aligned}
&\||\nabla\Phi_{t,r}(\Delta)f_i|\|_{L^2(B)} \\
\leq\ & C\left(\int_{\Gamma_+}+\int_{\Gamma_-}\right)\left\|\left|\nabla e^{-\Re z\Delta}f_i\right|\right\|_{L^2(B)}\frac{\rho^N(r_i)}{(|z|+t+1)^{N+1}}|dz| \\
\leq\ & C\int_0^\infty \left\|\left|\nabla e^{-s\Delta}f_i\right|\right\|_{L^2(B)}\frac{\rho^N(r_i)}{(s+t+1)^{N+1}}ds.
\end{aligned}$$

In the following, we estimate $\||\nabla\Phi_{t,r}(\Delta)f_i|\|_{L^2(B)}$ for t small and large separately by applying (2.14) with $p = 2$.

For $0 < t < 1$, we get

$$\begin{aligned}
&\||\nabla\Phi_{t,r}(\Delta)f_i|\|_{L^2(B)} \\
&\leq C\left(\int_0^1 \frac{e^{-c\frac{4^ir^2}{s}}}{s^{1/2}}\frac{\rho^N(r)}{(s+t+1)^{N+1}}ds + \int_1^\infty \frac{e^{-c\left(\frac{2^{im}r^m}{s}\right)^{1/(m-1)}}}{s^\alpha}\frac{\rho^N(r)}{(s+t+1)^{N+1}}ds\right)\|f_i\|_{L^2} \\
&\leq C\left(e^{-c4^ir^2}\rho^N(r)\int_0^1\frac{1}{s^{1/2}}ds + \int_1^\infty\left(\frac{s}{2^{im}r^m}\right)^c\frac{\rho^N(r)}{s^{N+\alpha+1}}ds\right)\mu^{1/2}(2^{i+1}B)\left(\mathscr{M}\left(|f|^2\right)\right)^{1/2}(x) \\
&\leq C\left(e^{-c4^ir^2}\rho^N(r)+4^{-iN}\right)\mu^{1/2}(2^{i+1}B)\left(\mathscr{M}\left(|f|^2\right)\right)^{1/2}(x) \\
&\leq C4^{-iN}\mu^{1/2}(2^{i+1}B)\left(\mathscr{M}\left(|f|^2\right)\right)^{1/2}(x).
\end{aligned}$$

For the second term in the third line, c equals to N if $r \geq 1$ and $2N/m$ if $0 < r < 1$. Indeeds, for any fixed $c > 0$, it holds $e^{-t} < Ct^{-c}$ for all $t > 0$.

For $t \geq 1$, then

$$\begin{aligned}
\||\nabla\Phi_{t,r}(\Delta)f_i|\|_{L^2(B)} \leq\ & \int_0^1\frac{\|f_i\|_{L^2}}{s^{1/2}}e^{-c\frac{4^ir^2}{s}}\frac{\rho^N(r)}{(s+t+1)^{N+1}}ds \\
&+\left(\int_1^t+\int_t^\infty\right)\frac{\|f_i\|_{L^2}}{s^\alpha}e^{-c\left(\frac{2^{im}r^m}{s}\right)^{1/(m-1)}}\frac{\rho^N(r)}{(s+t+1)^{N+1}}ds \\
:=\ & (I_1+I_2+I_3)\mu^{1/2}(2^{i+1}B)\left(\mathscr{M}\left(|f|^2\right)\right)^{1/2}(x).
\end{aligned}$$

For I_1, we have

$$I_1 \leq e^{-c4^ir^2}\rho^N(r)t^{-N-1}\int_0^1 s^{-1/2}ds \leq Ce^{-c4^ir^2}\rho^N(r)t^{-N-1}. \tag{2.27}$$

Estimating I_2 in a similar way, then

$$I_2 \le e^{-c\left(\frac{2^{im}r^m}{t}\right)^{1/(m-1)}}\rho^N(r)t^{-N-1}\int_1^t s^{-\alpha}ds \le Ce^{-c\left(\frac{2^{im}r^m}{t}\right)^{1/(m-1)}}\rho^N(r)t^{-N-\alpha}. \tag{2.28}$$

For I_3, we claim

$$I_3 \le C\rho^N(r)\inf\left\{(2^{im}r^m)^{-N-\alpha}, t^{-N+\alpha}\right\}. \tag{2.29}$$

In fact, we change the variable s by $u=\left(\frac{2^{im}r^m}{s}\right)^{1/(m-1)}$, then

$$\begin{aligned} I_3 &\le \rho^N(r)\int_t^\infty e^{-c\left(\frac{2^{im}r^m}{s}\right)^{1/(m-1)}}\frac{ds}{s^{N+\alpha+1}} \\ &= C\rho^N(r)\left(2^{im}r^m\right)^{-N-\alpha}\int_0^{\left(\frac{2^{im}r^m}{t}\right)^{1/(m-1)}} e^{-cu}u^{(N+\alpha)(m-1)-1}du. \end{aligned}$$

On the one hand, the integral in the second line above is bounded, thus

$$I_3 \le C\rho^N(r)\left(2^{im}r^m\right)^{-N-\alpha}. \tag{2.30}$$

On the other hand, the integral is also dominated by $\left(\frac{2^{im}r^m}{t}\right)^{N+\alpha}$. Therefore

$$I_3 \le C\rho^N(r)t^{-N-\alpha}. \tag{2.31}$$

Now we are ready to estimate $\left\|T_\alpha\left(I-e^{-\rho(r)\Delta}\right)^N f_i\right\|_{L^2(B)}$. The above estimates give us

$$\begin{aligned} &\left\|T_\alpha\left(I-e^{-\rho(r)\Delta}\right)^N f_i\right\|_{L^2(B)} \\ \le\ & \int_0^\infty t^\alpha \|\Phi_{t,r}(\Delta)f_i\|_{L^2(B)}\frac{dt}{t} \\ \le\ & C\left(2^{-2iN/m}\int_0^1 t^\alpha\frac{dt}{t}+\int_1^\infty t^\alpha(I_1+I_2+I_3)\frac{dt}{t}\right)\mu^{1/2}(2^{i+1}B)\left(\mathscr{M}\left(|f|^2\right)\right)^{1/2}(x) \\ \le\ & C\left(2^{-2iN/m}+\int_1^\infty(I_1+I_2+I_3)\frac{dt}{t^{1-\alpha}}\right)\mu^{1/2}(2^{i+1}B)\left(\mathscr{M}\left(|f|^2\right)\right)^{1/2}(x). \end{aligned}$$

We claim: $\int_1^\infty(I_1+I_2+I_3)\frac{dt}{t^{1-\alpha}} \le C4^{-Ni}$. Therefore, (2.25) holds as long as we choose N to be large enough and the lemma is proved.

Indeed, it is easy to see from (2.27) that

$$\int_1^\infty I_1\frac{dt}{t^{1-\alpha}} \le Ce^{-c4^ir^2}\rho^N(r)\int_1^\infty t^{-N-2+\alpha}dt \le 4^{-Ni}.$$

It follows from (2.28) that

$$\int_1^\infty I_2 \frac{dt}{t^{1-\alpha}} \le C\rho^N(r)\int_1^\infty e^{-c\left(\frac{2^{im}r^m}{t}\right)^{1/(m-1)}} t^{-N-1}dt \le C4^{-Ni}.$$

More specifically, if $r \ge 1$, then $\rho(r) = r^m$ and

$$\begin{aligned}\int_1^\infty I_2 \frac{dt}{t^{1-\alpha}} &\le Cr^{mN}\int_1^\infty e^{-c\left(\frac{2^{im}r^m}{t}\right)^{1/(m-1)}} t^{-N}\frac{dt}{t} \\ &\le C2^{-imN}\int_1^\infty e^{-cu}u^{N(m-1)-1}du \le C2^{-imN}.\end{aligned}$$

If $0 < r < 1$, then $\rho(r) = r^2$ and

$$\int_1^\infty t^\alpha I_2 \frac{dt}{t} \le Cr^{2N}\int_1^\infty \left(\frac{t}{2^{im}r^m}\right)^{2N/m} t^{-N}\frac{dt}{t} \le C4^{-Ni}.$$

Now for $\int_1^\infty I_3 \frac{dt}{t^{1-\alpha}}$. If $r \le 2^{-i}$, using the estimate (2.31) and we get

$$\int_1^\infty I_3 \frac{dt}{t^{1-\alpha}} \le r^{2N}\int_1^\infty t^{-N}\frac{dt}{t} \le C4^{-Ni}.$$

If $r > 2^{-i}$, we use (2.30) in the following way

$$\begin{aligned}\int_1^\infty I_3 \frac{dt}{t^{1-\alpha}} &\le \rho^N(r)(2^{im}r^m)^{-N}\int_1^{2^{mi}r^m} \left(\frac{t}{2^{mi}r^m}\right)^\alpha \frac{dt}{t} \\ &\quad +\rho^N(r)(2^{im}r^m)^{-N}\int_{2^{mi}r^m}^\infty \left(\frac{2^{im}r^m}{t}\right)^N \frac{dt}{t} \\ &\le \rho^N(r)(2^{im}r^m)^{-N} \le C4^{-Ni}.\end{aligned}$$

If we assume $(DG_{\frac{1}{2}})$, then the same proof gives us (2.24) for $T_{\frac{1}{2}}$. □

Lemma 2.19. *Let M satisfies the doubling volume property, $(HK_{2,m})$, (G_{p_0}) as well as $(DG_{\frac{1}{2}})$. Then, for every $p \in (2, p_0)$, every ball B with radius r and all $x \in B$, we have for any arbitrary $\alpha \in (0,1)$*

$$\left(\frac{1}{\mu(B)}\int_B \left|T_\alpha\left(I-(I-e^{-\rho(r)\Delta})^N\right)f\right|^p d\mu\right)^{1/p} \le C\big(\mathscr{M}\left(|T_\alpha f|^2\right)\big)^{1/2}(x). \tag{2.32}$$

Before verifying (2.32), we prove the following two lemmas as preparation.

Lemma 2.20. *Let M satisfies the doubling volume property, (G_{p_0}) as well as $(DG_{\frac{1}{2}})$. Then for every $p \in (2, p_0)$, every ball B with radius r and every function $f \in L^2(M)$ supported in $C_i(B)$ and every* *integer* $1\le k\le N$, *we have*

$$\frac{1}{\mu^{1/p}(B)}\left\|\left|\nabla e^{-k\rho(r)\Delta}f\right|\right\|_{L^p(B)}\le\begin{cases}\dfrac{Ce^{-c4^i}}{r\mu^{1/2}(2^{i+1}B)}\|f\|_{L^2(C_i(B))}, & 0<r<1,\\ \dfrac{Ce^{-c2^{im/(m-1)}}}{r^{m/2}\mu^{1/2}(2^{i+1}B)}\|f\|_{L^2(C_i(B))}, & r\ge 1.\end{cases}\tag{2.33}$$

Proof. Since (G_{p_0}) and $(DG_{\frac{1}{2}})$, it follows by interpolation that for $p\in(2,p_0)$ and $i\ge 2$,

$$\left\|\left|\nabla e^{-t\Delta}f\right|\right\|_{L^p(B)}\le\begin{cases}\dfrac{C}{t^{1/2}}e^{-c\frac{4^ir^2}{t}}\|f\|_{L^p(C_i(B))}, & 0<t<1,\\ \dfrac{C}{t^{1/2}}e^{-c\left(\frac{2^{im}r^m}{t}\right)^{1/(m-1)}}\|f\|_{L^p(C_i(B))}, & 1\le t<\infty.\end{cases}\tag{2.34}$$

First consider any ball B with radius $r\ge 2$.

If $\operatorname{supp} f\subset 4B$, we have

$$\left\|e^{-k\rho(r)\Delta}f\right\|_{L^p(B)}\le\left\|\left|\nabla e^{-\frac{k}{2}\rho(r)\Delta}\right|\right\|_{p\to p}\left\|e^{-\frac{k}{2}\rho(r)\Delta}f\right\|_{L^p}\le\frac{C}{\rho^{1/2}(r)}\left\|e^{-\frac{k}{2}\rho(r)\Delta}f\right\|_{L^p}.$$

Note that from the heat kernel upper bound and the doubling volume property, we have

$$\begin{aligned}\left|e^{-\frac{k}{2}\rho(r)\Delta}f(y)\right| &= \left|\int_{4B}p_{\frac{k}{2}\rho(r)}(y,z)f(z)d\mu(z)\right|\le C\int_{4B}\frac{|f(z)|}{V(z,k'r)}d\mu(z)\\ &\le \frac{C}{\mu(4B)}\int_{4B}\frac{\mu(4B)}{V(z,k'r)}|f(z)|d\mu(z)\\ &\le \frac{C}{\mu^{1/2}(4B)}\|f\|_{L^2(4B)}.\end{aligned}$$

Here $k'=\left(\frac{k}{2}\right)^{1/m}$.

Therefore

$$\begin{aligned}\left\|e^{-\frac{k}{2}\rho(r)\Delta}f\right\|_{L^p(B)} &\le C\left(\sup\left|e^{-\frac{k}{2}\rho(r)\Delta}f(y)\right|^{p-2}\right)^{1/p}\cdot\left(\int_M\left|e^{-\frac{k}{2}\rho(r)\Delta}f(y)\right|^2\right)^{1/p}\\ &\le \frac{C}{\mu^{\frac{1}{2}-\frac{1}{p}}(4B)}\|f\|_{L^2(4B)}.\end{aligned}$$

And

$$\left(\frac{1}{\mu(B)}\int_B\left|\nabla e^{-k\rho(r)\Delta}f\right|^p d\mu\right)^{1/p}\le\frac{C}{r^{m/2}}\left(\frac{1}{\mu(4B)}\int_{4B}|f|^2d\mu\right)^{1/2}.$$

If supp $f \subset C_i(B)$ with $i \geq 2$,

$$\nabla e^{-k\rho(r)\Delta} f = \sum_{l=1}^{\infty} h_l := \sum_{l=1}^{\infty} \nabla e^{-\frac{k}{2}\rho(r)\Delta} \chi_{C_l} e^{-\frac{k}{2}\rho(r)\Delta} f.$$

Then

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B |h_l|^p d\mu\right)^{1/p} \\
\leq\ & \left|\left|\left|\nabla e^{-\frac{k}{2}\rho(r)\Delta}\right|\right|\right|_{L^p(C_l(B))\to L^p(B)} \left(\frac{1}{\mu(B)}\int_{C_l(B)} |e^{-\frac{k}{2}\rho(r)\Delta} f|^p d\mu\right)^{1/p} \\
\leq\ & \left(\frac{\mu(2^{l+1}B)}{\mu(B)}\right)^{1/p} \frac{Ce^{-c2^{lm/(m-1)}}}{r^{m/2}} \left(\frac{1}{\mu(2^{l+1}B)}\int_{C_l} \left|e^{-\frac{k}{2}\rho(r)\Delta} f\right|^p d\mu\right)^{1/p}.
\end{aligned}$$

Similarly, for $y \in C_l(B)$, we have

$$\begin{aligned}
&\left|e^{-\frac{k}{2}\rho(r)\Delta} f(y)\right| \\
=\ & \left|\int_{4B} p_{\frac{k}{2}\rho(r)}(y,z) f(z) d\mu(z)\right| \leq C\int_{C_i(B)} \frac{|f(z)|}{V(z,k'r)} d\mu(z) \\
\leq\ & \frac{C}{\mu(2^{i+1}B)}\int_{C_i(B)} \frac{\mu(2^{i+1}B)}{V(z,k'r)} \exp\left(-c\left(\frac{d^m(C_i,C_l)}{r^m}\right)^{1/(m-1)}\right) |f(z)| d\mu(z) \\
\leq\ & \frac{C}{\mu^{1/2}(2^{i+1}B)} K_{il} \|f\|_{L^2(4B)},
\end{aligned}$$

where

$$K_{il} \leq \begin{cases} Ce^{-c2^{im/(m-1)}}, & l \leq i-2, \\ C2^{i\nu}, & i-2 < l < i+2, \\ Ce^{-c2^{lm/(m-1)}}, & l \geq i+2. \end{cases}$$

At the same time, we have

$$\left(\frac{1}{\mu(2^{l+1}(B))}\int_{C_l} \left|e^{-\frac{k}{2}\rho(r)\Delta} f\right|^2 d\mu\right)^{1/2} \leq CK_{il}\left(\frac{1}{\mu(2^{i+1}(B))}\int_{C_i(B)} |f|^2 d\mu\right)^{1/2}.$$

The two estimates gives us

$$\left(\frac{1}{\mu(2^{l+1}B)}\int_{C_l} \left|e^{-\frac{k}{2}\rho(r)\Delta} f\right|^p d\mu\right)^{1/p} \leq CK_{il}\left(\frac{1}{\mu(2^{i+1}B)}\int_{C_i(B)} |f|^2 d\mu\right)^{1/2}.$$

Combining the above estimate, we obtain

$$\left(\frac{1}{\mu(B)}\int_B \left|\nabla e^{-k\rho(r)\Delta} f\right|^p d\mu\right)^{1/p}$$

$$\begin{aligned}
&\leq C\sum_{l=1}^{\infty}\left\|\left|\nabla e^{-\frac{k}{2}\rho(r)\Delta}\right|\right\|_{L^p(C_l(B))\to L^p(B)}\left(\frac{1}{\mu(B)}\int_{C_i(B)}\left|e^{-\frac{k}{2}\rho(r)\Delta}f\right|^p d\mu\right)^{1/p}\\
&\leq C\sum_{l=1}^{\infty}2^{(l+1)\nu/p}\frac{e^{-c2^{lm/(m-1)}}}{r^{m/2}}K_{il}\left(\frac{1}{\mu(2^{i+1}B)}\int_{C_i(B)}|f|^2d\mu\right)^{1/2}\\
&\leq C\frac{e^{-c2^{im/(m-1)}}}{r^{m/2}}\left(\frac{1}{\mu(2^{i+1}B)}\int_{C_i(B)}|f|^2d\mu\right)^{1/2}
\end{aligned}$$

For $r < N^{-1/2}$, i.e. $kr^2 < 1$ for every $k = 1,2,\cdots,N$, repeating the above proof by using the estimate (2.34) for $0 < t < 1$, we obtain

$$\left(\frac{1}{\mu(B)}\int_B\left|\nabla e^{-k\rho(r)\Delta}f\right|^p d\mu\right)^{1/p}\leq C\frac{e^{-c4^i}}{r}\left(\frac{1}{\mu(2^{i+1}B)}\int_{C_i(B)}|f|^2d\mu\right)^{1/2} \tag{2.35}$$

It remains to consider the case $N^{-1/2} \leq r \leq 2$. In that case, $r \simeq 1$ and the estimate is easy to obtain.

□

Lemma 2.21. *Let M be a Riemannian manifold satisfying ($HK_{2,m}$). Then for any $i \geq 1$, we have*

$$\left(\frac{1}{\mu(2^{i+1}B)}\int_{C_i(B)}|f_i|^2d\mu\right)^{1/2}\leq C\sqrt{\rho(2^{i+1}r)}\left(\mathscr{M}\left(|\nabla f|^2\right)\right)^{1/2}(x),\ \forall B\in M,\ \forall x\in B. \tag{2.36}$$

Proof. It is well known that ($HK_{2,m}$) implies the following rescaled Poincaré inequality (see for example [BB04], [BBK06]) on M:

$$\int_{B(x,r)}(f-f_{B(x,r)})^2\leq C\rho(r)\int_{B(x,r)}|\nabla f|^2d\mu,\ \forall x\in M,\forall r>0, \tag{2.37}$$

which is a generalisation of the standard Poincaré inequality (P). However, unlike (P), there is no equivalence between ($HK_{2,m}$) and (D)+(2.37).

For $i = 1$, (2.36) follows directly from (2.37).

For $i \geq 2$, observe that

$$f - f_{4B} = f - f_{2^{i+1}B} + \sum_{l=2}^{i}(f_{2^lB} - f_{2^{l+1}B}).$$

and Poincaré inequality gives us

$$|f_{2^lB}-f_{2^{l+1}B}|^2\leq\frac{1}{\mu(2^{l+1}B)}\int_{2^{l+1}B}|f-f_{2^{l+1}B}|^2d\mu\leq C\rho(2^{l+1}r)\mathscr{M}\left(|\nabla f|^2\right)(x).$$

Thus

$$\begin{aligned}\left(\frac{1}{\mu(2^{i+1}B)}\int_{C_i(B)}|f_i|^2d\mu\right)^{1/2} &\leq \left(\frac{1}{\mu(2^{i+1}B)}\int_{2^{i+1}B}|f-f_{4B}|^2d\mu\right)^{1/2}\\ &\leq \left(\frac{1}{\mu(2^{i+1}B)}\int_{2^{i+1}B}|f-f_{2^{i+1}B}|^2d\mu\right)^{1/2}\\ &\quad +\sum_{l=2}^{i}\left(\frac{1}{\mu(2^{i+1}B)}\int_{2^{i+1}B}\left|f_{2^l(B)}-f_{2^{l+1}B}\right|^2d\mu\right)^{1/2}\\ &\leq C\sqrt{\rho(2^{i+1}r)}\big(\mathscr{M}\left(|\nabla f|^2\right)\big)^{1/2}(x).\end{aligned}$$

□

Proof of Lemma 2.19: Note that

$$A_r = I-(I-e^{-\rho(r)\Delta})^N = \sum_{k=1}^{\infty}(-1)^{k+1}\binom{N}{k}e^{-k\rho(r)\Delta}.$$

In order to show (2.32), it is enough to prove that for $k=1,2,\cdots,N$,

$$\left(\frac{1}{\mu(B)}\int_B\left|\nabla e^{-k\rho(r)\Delta}f\right|^pd\mu\right)^{1/p}\leq C\big(\mathscr{M}\left(|\nabla f|^2\right)\big)^{1/2}(x). \tag{2.38}$$

We consider any ball B with radius $r\geq 1$($r<1$ follows in the same way). Combining the estimates in Lemma 2.19 and Lemma 2.21, we obtain for $i=1$,

$$\begin{aligned}\left(\frac{1}{\mu(B)}\int_B\left|\nabla e^{-k\rho(r)\Delta}f_1\right|^pd\mu\right)^{1/p} &\leq \frac{C}{r^{m/2}}\left(\frac{1}{\mu(B)}\int_{4B}|f-f_{4B}|^2d\mu\right)^{1/2}\\ &\leq C\left(\frac{1}{\mu(B)}\int_{4B}|\nabla f|^2d\mu\right)^{1/2}\\ &\leq C\big(\mathscr{M}\left(|\nabla f|^2\right)\big)^{1/2}(x),\end{aligned}$$

and for $i\geq 2$,

$$\begin{aligned}&\left(\frac{1}{\mu(B)}\int_B\left|\nabla e^{-k\rho(r)\Delta}f_i\right|^pd\mu\right)^{1/p}\\ \leq\ & \frac{Ce^{-c2^{im/(m-1)}}}{r^{m/2}}\left(\frac{1}{\mu(2^{i+1}B)}\int_{C_i}|f_i|^2d\mu\right)^{1/2}\\ \leq\ & C2^{im/2}e^{-c2^{im/(m-1)}}\left(\frac{1}{\mu(2^{i+1}B)}\int_{2^{i+1}B}|\nabla f|^2d\mu\right)^{1/2}\\ \leq\ & C2^{im/2}e^{-c2^{im/(m-1)}}\big(\mathscr{M}\left(|\nabla f|^2\right)\big)^{1/2}(x).\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B \left|\nabla e^{-k\rho(r)\Delta} f\right|^p d\mu\right)^{1/p} \\
\leq\ & \sum_{i=1}^{\infty}\left(\frac{1}{\mu(B)}\int_B \left|\nabla e^{-k\rho(r)\Delta} f_i\right|^p d\mu\right)^{1/p} \\
\leq\ & C\left(1+\sum_{i=2}^{\infty} 2^{im/2} e^{-c2^{im/(m-1)}}\right)\left(\mathscr{M}\left(|\nabla f|^2\right)\right)^{1/2}(x) \\
\leq\ & \left(\mathscr{M}\left(|\nabla f|^2\right)\right)^{1/2}(x).
\end{aligned}$$

□

To summarise, let M be a complete non compact Riemannian manifold satisfying (D), we have shown that (DG_α) implies (2.22) with T_α and $(DG_{\frac{1}{2}})$ implies (2.23) for any T_α. Thus Theorem 2.15 holds. Notice also that (DG_α) follows from $(HK_{2,m})$, but unfortunately we can't obtain $(DG_{\frac{1}{2}})$ from $(HK_{2,m})$.

Chapter 3

Hardy spaces on metric measure spaces with generalised heat kernel estimates

In this Chapter, we develop a Hardy space theory on metric measure space (M,d,μ) with different local and global heat kernel estimates for e^{-tL} where L is non-negative self-adjoint operator. This extends the work of Kunstmann and Uhl in [Uhl11, KU12].

In Section 3.2, we define two classes of Hardy spaces associated with operators on M satisfying (D) and (DG_{β_1,β_2}). One is the H^1 space defined via molecules, denoted by $H^1_{L,\rho,\mathrm{mol}}(M)$. The other is $H^p_{L,S_h^\rho}(M)$, $p\geq 1$, defined via conical square function. Note that here ρ is related to the heat kernel estimates.

In Section 3.3, we prove that the two definitions of H^1 space are equivalent.

In Section 3.4, we compare the H^p space defined via square functions, denoted by $H^p_{S_h^\rho}$, with L^p. Assuming (D) and ($DG^{p_0,p_0'}_{\beta_1,\beta_2}$), we show the equivalence of $H^p_{S_h^\rho}$ and L^p for $p_0<p<p_0'$ (Theorem 3.16). However, for the Hardy space $H^p_{S_h}$ defined via the square function with the scaling t^2 under sub-Gaussian heat kernel estimates, the equivalence may not be true. Indeed, we find some counterexamples (Theorem 3.19).

In the last section, we apply this Hardy space theory to show the H^1-L^1 boundedness of quasi Riesz transforms on Riemannian manifolds satisfying (D) and ($UE_{2,m}$).

Throughout this chapter, unless otherwise stated, we always consider a metric measure spaces (M,d,μ) satisfying (D) and (DG_{β_1,β_2}), where $\beta_1\leq\beta_2$. If necessary we may smoothen $\rho(t)$ as

$$\rho(t)=\begin{cases} t^{\beta_1}, & \text{if } 0<t\leq 1/2,\\ \text{smooth part}, & \text{if } 1/2<t<2,\\ t^{\beta_2}, & \text{if } t\geq 2;\end{cases}$$

with $\rho'(t)\simeq 1$ for $1/2<t<2$, which we still denote by $\rho(t)$. Since $\frac{\rho'(t)}{\rho(t)}=\frac{\beta_1}{t}$ for $0<t\leq 1/2$ and

$\frac{\rho'(t)}{\rho(t)} = \frac{\beta_2}{t}$ for $t \geq 2$, we have in a uniform way

$$\frac{\rho'(t)}{\rho(t)} \simeq \frac{1}{t}. \tag{3.1}$$

3.1 More about assumptions on the heat kernel

Recall the heat kernel estimates on (M,d,μ) in Section 0.7:

The generalised $L^{p_0} - L^{p_0'}$ off-diagonal estimate for $p \in (1,2)$: for $x,y \in M$ and $t > 0$,

$$\left\| \mathbb{1}_{B(x,t)} e^{-\rho(t)L} \mathbb{1}_{B(y,t)} \right\|_{p_0 \to p_0'} \leq \begin{cases} \dfrac{C}{V^{\frac{1}{p_0}-\frac{1}{p_0'}}(x,t)} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_1}{\beta_1-1}}\right) & 0<t<1, \\ \dfrac{C}{V^{\frac{1}{p_0}-\frac{1}{p_0'}}(x,t)} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_2}{\beta_2-1}}\right), & t \geq 1. \end{cases} \tag{$DG^{p_0,p_0'}_{\beta_1,\beta_2}$}$$

The generalised $L^2 - L^2$ Davies-Gaffney estimate: $x,y \in M$ and $t > 0$

$$\left\| \mathbb{1}_{B(x,t)} e^{-\rho(t)L} \mathbb{1}_{B(y,t)} \right\|_{2 \to 2} \leq \begin{cases} C\exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_1}{\beta_1-1}}\right) & 0<t<1, \\ C\exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_2}{\beta_2-1}}\right), & t \geq 1. \end{cases} \tag{DG_{β_1,β_2}}$$

The pointwise estimate of the heat kernel: $x,y \in M$ and $t > 0$

$$p_{\rho(t)}(x,y) \leq \begin{cases} \dfrac{C}{V(x,t)} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_1}{\beta_1-1}}\right) & 0<t<1, \\ \dfrac{C}{V(x,t)} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_2}{\beta_2-1}}\right), & t \geq 1. \end{cases} \tag{UE_{β_1,β_2}}$$

In the following, we will give some lemmas from the heat kernel estimates. We first observe that (UE_{β_1,β_2}) $\Rightarrow$ ($DG^{p_0,p_0'}_{\beta_1,\beta_2}$) $\Rightarrow$ (DG_{β_1,β_2}). Indeed,

Lemma 3.1 ([BK05]). *Let (M,d,μ) be a space of homogeneous type. Let L be a non-negative self-adjoint operator on $L^2(M,\mu)$. Assume that ($DG^{p_0,p_0'}_{\beta_1,\beta_2}$) holds. Then for all $p_0 \leq u \leq v \leq p_0'$, we* *have*

$$\left\|\mathbb{1}_{B(x,t)}e^{-\rho(t)L}\mathbb{1}_{B(y,t)}\right\|_{u\to v}\leq\begin{cases}\dfrac{C}{V^{\frac{1}{u}-\frac{1}{v}}(x,t)}\exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_1}{\beta_1-1}}\right) & 0<t<1,\\ \dfrac{C}{V^{\frac{1}{u}-\frac{1}{v}}(x,t)}\exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_2}{\beta_2-1}}\right), & t\geq 1.\end{cases} \tag{3.2}$$

Remark 3.2. The estimate $(DG^{p_0,p_0'}_{\beta_1,\beta_2})$ is equivalent to the $L^{p_0}-L^2$ off-diagonal estimate

$$\left\|\mathbb{1}_{B(x,t)}e^{-\rho(t)L}\mathbb{1}_{B(y,t)}\right\|_{p_0\to 2}\leq\begin{cases}\dfrac{C}{V^{\frac{1}{p_0}-\frac{1}{2}}(x,t)}\exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_1}{\beta_1-1}}\right) & 0<t<1,\\ \dfrac{C}{V^{\frac{1}{p_0}-\frac{1}{2}}(x,t)}\exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_2}{\beta_2-1}}\right), & t\geq 1.\end{cases}$$

We refer to [BK05, CS08] for the proof.

In fact, we also have

Lemma 3.3 ([BK05, Uhl11])**.** *Let (M,d,μ) be a space of homogeneous type. Let L be a non-negative self-adjoint operator on $L^2(M,\mu)$. Assume that $(DG^{p_0,p_0'}_{\beta_1,\beta_2})$ holds. Then for all $p_0\leq u\leq v\leq p_0'$ and $k\in\mathbb{N}$, we have*

1. *For any ball $B\in M$ with radius $r>0$, and any $i\geq 2$,*

$$\left\|\mathbb{1}_B(tL)^k e^{-tL}\mathbb{1}_{C_i(B)}\right\|_{u\to v}\leq\begin{cases}\dfrac{C2^{i\nu}}{\mu^{\frac{1}{u}-\frac{1}{v}}(B)}e^{-c\left(\frac{2^{i\beta_1}r^{\beta_1}}{t}\right)^{1/(\beta_1-1)}} & 0<t<1,\\ \dfrac{C2^{i\nu}}{\mu^{\frac{1}{u}-\frac{1}{v}}(B)}e^{-c\left(\frac{2^{i\beta_1}r^{\beta_2}}{t}\right)^{1/(\beta_2-1)}}, & t\geq 1.\end{cases} \tag{3.3}$$

2. *For all $\alpha,\beta\geq 0$ such that $\alpha+\beta=\frac{1}{u}-\frac{1}{v}$,*

$$\left\|V^{\alpha}(\cdot,t)(\rho(t)L)^k e^{-\rho(t)L}V^{\beta}(\cdot,t)\right\|_{u\to v}\leq C.$$

More generally, similarly as in [BK05, Uhl11], we can obtain

Lemma 3.4. *Let M satisfy (D) and (DG_{β_1,β_2}) with $\beta_1\leq\beta_2$. Let B be a ball with centre $x\in M$ and* *radius* $r > 0$. *Then for all* $k, l \in \mathbb{N}$ *with* $0 \leq l \leq k-2$ *and* $t \in (0, 2^{k+1}r)$, *it holds*

$$\left\| \mathbb{1}_{C_k(B)} (\rho(t)L)^K e^{-\rho(t)L} \mathbb{1}_{C_l(B)} \right\|_{2\to 2} \leq \begin{cases} C\exp\left(-c2^{\frac{k\beta_1}{\beta_1-1}}\right), & 0 < t < 1, \\ C\exp\left(-c2^{\frac{k\beta_2}{\beta_2-1}}\right), & t \geq 1. \end{cases} \tag{3.4}$$

Note that here we can change the position of k and l.

3.2 Definitions of Hardy spaces

We present two definitions of Hardy space associated with L, that is, H^1 via molecules and $H^p, 1 \leq p < \infty$, via square function. In fact, the Hardy spaces we define here are modifications of these in [HLM+11, Uhl11], which are adapted to the heat kernel estimates (DG_{β_1,β_2}). Despite the fact that L is self-adjoint, we cannot use atoms as in [HLM+11]. Indeed, atoms are related to a Gaussian behavior, not a sub-Gaussian one.

3.2.1 Hardy space via molecules

Definition 3.5. Let $\varepsilon > 0$ and $K > \frac{\nu}{2\beta_1}$. A function $a \in L^2(M)$ is called a $(1,2,\varepsilon)-$molecule associated to L if there exist a function $b \in \mathscr{D}(L)$ and a ball B with radius r_B such that

1. $a = L^K b$;

2. It holds that for every $k = 0, 1, \cdots, K$ and $i = 0, 1, 2, \cdots$, we have

$$\|(\rho(r_B)L)^k b\|_{L^2(C_i(B))} \leq \rho^K(r_B) 2^{-i\varepsilon} V(2^i B)^{-1/2}.$$

Definition 3.6. We say that $f = \sum_{n=0}^{\infty} \lambda_n a_n$ is a molecular $(1,2,\varepsilon)-$representation of f if $(\lambda_n)_{n\in\mathbb{N}} \in l^1$, each a_n is a molecule as above, and the sum converges in the L^2 sense. We denote the collection of all the functions with a molecular representation by $\mathbb{H}^1_{\Delta,\rho,mol}$, where the norm of $f \in \mathbb{H}^1_{\Delta,\rho,mol}$ is given by

$$\|f\|_{\mathbb{H}^1_{\Delta,\rho,mol}(M)} = \inf\left\{ \sum_{n=0}^{\infty} |\lambda_n| : f = \sum_{n=0}^{\infty} \lambda_n a_n \text{ is a molecular } (1,2,\varepsilon)-\text{representation} \right\}.$$

The Hardy space $H^1_{L,\rho,\mathrm{mol}}(M)$ is defined as the completion of $\mathbb{H}^1_{L,\rho,mol}(M)$ with respect to this norm.

3.2.2 Hardy spaces via square function

Consider the quadratic operator associated with the heat kernel defined by the following conical square function

$$S_h^\rho f(x) = \left(\iint_{\Gamma(x)} |\rho(t) L e^{-\rho(t)L} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}, \tag{3.5}$$

where the cone $\Gamma(x) = \{(y,t) \in M \times (0,\infty) : d(y,x) < t\}$.

We define the $L^2(M)$ adapted Hardy space $H^2(M)$ as the closure of the range of L in $L^2(M)$ norm, i.e., $H^2(M) := \overline{R(L)}$. Due to the self-adjointness of L in $L^2(M)$, we get $L^2(M) = \overline{R(L)} \bigoplus N(L)$, where the sum is orthogonal. Under the assumptions (D) and (UE_{β_1,β_2}), we have $N(L) = 0$ and thus $H^2(M) = L^2(M)$. Indeed, since for any $f \in L^2(M)$,

$$e^{-tL} f - f = \int_0^t \frac{\partial}{\partial s} e^{-sL} f ds = -\int_0^t L e^{-sL} f ds,$$

then $f \in N(L)$ implies $f = e^{-tL} f$, $\forall t \geq 0$. Also, as a consequence of (UE_{β_1,β_2}) and Hölder inequality, it holds $\forall x \in M$, $\forall t \geq 0$,

$$\begin{aligned}
|f(x)| &= |e^{-\rho(t)L} f(x)| = \left| \int_M p_{\rho(t)}(x,y) f(y) d\mu(y) \right| \\
&\leq \frac{C}{V(x,t)} \int_M \exp\left(-c \left(\frac{d(x,y)}{t} \right)^{\tau(t)} \right) |f(y)| d\mu(y) \\
&\leq \frac{C}{V(x,t)} \left(\int_M \exp\left(-2c \left(\frac{d(x,y)}{t} \right)^{\tau(t)} \right) d\mu(y) \right)^{1/2} \|f\|_2 \leq \frac{C}{V(x,t)^{1/2}} \|f\|_2.
\end{aligned}$$

Here $\tau(t) = \frac{\beta_1}{\beta_1 - 1}$ for $0 < t < 1$, and otherwise $\tau(t) = \frac{\beta_2}{\beta_2 - 1}$. Now let $t \to \infty$, we obtain that $f = 0$.

Definition 3.7. The Hardy space $H^p_{L,S_h^\rho}(M)$, $p \geq 1$ is defined as the completion of the set $\{f \in H^2(M) : \|S_h^\rho f\|_{L^p} < \infty\}$ with respect to the norm $\|S_h^\rho f\|_{L^p}$. The $H^p_{L,S_h^\rho}(M)$ norm is defined by $\|f\|_{H^p_{L,S_h^\rho}(M)} := \|S_h^\rho f\|_{L^p(M)}$.

For $p = 2$, the operator S_h^ρ is bounded on $L^2(M)$. Indeed, for every $f \in L^2(M)$,

$$\begin{aligned}
\|S_h^\rho f\|^2_{L^2(M)} &= \int_M \iint_{\Gamma(x)} \left| \rho(t) L e^{-\rho(t)L} f(y) \right|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} d\mu(x) \\
&\simeq \iint_{M \times (0,\infty)} \left| \rho(t) L e^{-\rho(t)L} f(y) \right|^2 d\mu(y) \frac{dt}{t} \\
&\simeq \iint_{M \times (0,\infty)} \left| \rho(t) L e^{-\rho(t)L} f(y) \right|^2 d\mu(y) \frac{\rho'(t) dt}{\rho(t)} \\
&= \int_0^\infty < (\rho(t)L)^2 e^{-2\rho(t)L} f, f > \frac{\rho'(t) dt}{\rho(t)} = C \|f\|^2_{L^2(M)}.
\end{aligned} \tag{3.6}$$

Note that the second step follows from Fubini theorem and (4). The third step is obtained by using the fact (3.1): $\rho'(t)/\rho(t) \simeq 1/t$. The last one is a consequence of spectral theory.

3.2.3 Tent spaces

We introduce the definition and properties of tent spaces over the homogeneous spaces M (satisfying the doubling volume property), following [CMS85], [Rus07] (see also [HLM$^+$11]).

For any $x \in M$ and for any closed subset $F \subset M$, a saw-tooth region is defined as $\mathscr{R}(F) := \bigcup_{x \in F} \Gamma(x)$. If O is an open subset of X, then the "tent" over O, denoted by $\widehat{O}$, is defined as

$$\widehat{O} := [\mathscr{R}(O^c)]^c = \{(x,t) \in M \times (0,\infty) : d(x,O^c) \geq t\}.$$

For a measurable function F on $M \times (0,\infty)$, consider

$$\mathscr{A}F(x) = \left(\iint_{\Gamma(x)} |F(y,t)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}.$$

Given $0 < p < \infty$, say that a measurable function $F \in T_2^p(M \times (0,\infty))$ if

$$\|F\|_{T_2^p(M)} := \|\mathscr{A}F\|_{L^p(M)} < \infty.$$

For simplicity, we denote $T_2^p(M \times (0,\infty))$ by $T_2^p(M)$ from now on.

Therefore, for $f \in H^p_{\Delta,S_h^m}(M)$ and $0 < p < \infty$, write $F(y,t) = \rho(t)\Delta e^{-\rho(t)\Delta} f(y)$, we have

$$\|f\|_{H^p_{\Delta,S_h^m}(M)} = \|F\|_{T_2^p(M)}.$$

Consider another functional

$$\mathscr{C}F(x) = \sup_{x \in B} \left(\iint_{\widehat{B}} |F(y,t)|^2 \frac{d\mu(y)dt}{t} \right)^{1/2},$$

we say that a measurable function $F \in T_2^\infty(M)$ if $\mathscr{C}F \in L^\infty(M)$.

Proposition 3.8. *Suppose* $1 < p < \infty$*, let* p' *be the conjugate of* p*. Then the pairing* $< F, G > \longrightarrow \int_{M \times (0,\infty)} F(x,t)G(x,t)d\mu(x)dt/t$ *realizes* $T_2^{p'}(M)$ *as the dual of* $T_2^p(M)$*.*

Denote by $[\,,]_\theta$ the complex method of interpolation described in [BL76]. Then we have the following result of interpolation of tent spaces.

Proposition 3.9. *Suppose* $1 \leq p_0 < p < p_1 \leq \infty$*, with* $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $0 < \theta < 1$*. Then*

$$[T_2^{p_0}(M), T_2^{p_1}(M)]_\theta = T_2^p(M).$$

Next we review the atomic theory for tent spaces as originally developed in [CMS85], and extended to the setting of spaces of homogeneous type in [Rus07].

Definition 3.10. A measurable function A on $M \times (0,\infty)$ is said to be a T_2^1-atom if there exists a ball $B \in M$ such that A is supported in $\widehat{B}$ and

$$\int_{M\times(0,\infty)} |A(x,t)|^2 d\mu(x)\frac{dt}{t} \leq \mu^{-1}(B).$$

Proposition 3.11 ([HLM$^+$11],[Rus07])**.** *For every element $F \in T_2^1(M)$ there exist a sequence of numbers $\{\lambda_j\}_{j=0}^\infty \in l^1$ and a sequence of T_2^1-atoms $\{A_j\}_{j=0}^\infty$ such that*

$$F = \sum_{j=0}^\infty \lambda_j A_j \text{ in } T_2^1(M) \text{ and a.e. in } M \times (0,\infty). \tag{3.7}$$

Moreover, $\sum_{j=0}^\infty \lambda_j \approx \|F\|_{T_2^1(M)}$, where the implicit constants depend only on the homogeneous space properties of M.

Finally, if $F \in T_2^1(M) \cap T_2^2(M)$, then the decomposition (3.7) also converges in $T_2^2(M)$.

3.3 $H^1_{L,\rho,mol}(M) = H^1_{L,S_h^\rho}(M)$

In this section, we will show that, under the assumptions of (D) and the Davies-Gaffney estimate (DG_{β_1,β_2}), the two H^1 spaces: $H^1_{L,\rho,\mathrm{mol}}(M)$ and $H^1_{L,S_h^\rho}(M)$, are equivalent. We denote

$$H^1_{L,\rho}(M) := H^1_{L,S_h^\rho}(M) = H^1_{L,\rho,\mathrm{mol}}(M).$$

Theorem 3.12. *Let M be a metric measure space satisfying the doubling volume property (D) and the heat kernel estimate (DG_{β_1,β_2}). Then $H^1_{L,\rho,\mathrm{mol}}(M) = H^1_{L,S_h^\rho}(M)$. Moreover,*

$$\|f\|_{H^1_{L,\rho,\mathrm{mol}}(M)} \simeq \|f\|_{H^1_{L,S_h^\rho}(M)}.$$

The outline of the proof is the same as in [HLM$^+$11]. Since $H^1_{L,\rho,\mathrm{mol}}(M)$ and $H^1_{L,S_h^\rho}(M)$ are completions of $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)$ and $H^1_{L,S_h^\rho}(M) \cap H^2(M)$, it is enough to show $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M) = H^1_{L,S_h^\rho}(M) \cap H^2(M)$ with equivalent norms. In the following, we will prove the two-sided inclusions separately. Before proceeding to the proof, we first note the following criterion to prove $H^1_{L,\rho,\mathrm{mol}}(M) - L^1(M)$ boundedness of an operator, which is an analogue of Lemma 4.3 in [HLM$^+$11].

Lemma 3.13. *Assume that T is a linear operator, or a nonnegative sublinear operator, satisfying* *the weak-type* $(2,2)$ *bound*

$$\mu\left(\{x\in M : |Tf(x)|>\eta\}\right)\le C\eta^{-2}\|f\|_2^2,\ \ \forall\eta>0 \tag{3.8}$$

and that for every $(1,2,\varepsilon)-$*molecule a, we have*

$$\|Ta\|_{L^1}\le C, \tag{3.9}$$

with constant C independent of a. Then T is bounded from $\mathbb{H}^1_{\Delta,\rho,mol}(M)$ *to* $L^1(M)$ *with*

$$\|Tf\|_{L^1}\le C\|f\|_{\mathbb{H}^1_{\Delta,\rho,mol}(M)}.$$

Consequently, by density, T extends to be a bounded operator from $H^1_{\Delta,\rho,mol}(M)$ *to* $L^1(M)$.

The proof in [HLM$^+$11] is applicable in our case. For the sake of completeness, we rewrite it here.

Proof. Let $f\in\mathbb{H}^1_{\Delta,\rho,mol}(M)$, where $f=\sum_n\lambda_n a_n$ is a molecular $(1,2,\varepsilon)-$representation such that

$$\|f\|_{\mathbb{H}^1_{\Delta,\rho,mol}(M)}\approx\sum_n|\lambda_n|.$$

Since the sum converges in L^2 from Definition 3.6 and T is of weak type $(2,2)$, we have at almost every point

$$|Tf|\le\sum_n|\lambda_n||Ta_n|. \tag{3.10}$$

Indeed, denote $f^N=\sum_{n>N}\lambda_n a_n$. Because T is linear or non-negative sub linear,we have

$$|Tf|\le\left|T\left(f^N\right)\right|+\left|T\left(\sum_{n=0}^{N}\lambda_n a_n\right)\right|\le\left|T\left(f^N\right)\right|+\sum_{n=0}^{\infty}|\lambda_n||T(a_n)|.$$

Thus for every $\eta>0$,

$$\begin{aligned}\mu\left(\left\{|Tf|-\sum_{n=0}^{\infty}|\lambda_n||T(a_n)|>\eta\right\}\right) &\le \limsup_{N\to\infty}\mu\left(\left\{\left|T\left(f^N\right)\right|>\eta\right\}\right)\\ &\le C\eta^{-2}\limsup_{N\to\infty}\|f^N\|_2^2=0.\end{aligned}$$

As a result, (3.10) follows. Combining (3.10) and (3.9), we get that Tf is L^1 bounded. Finally, by a standard density argument, we conclude that T is bounded from $H^1_{\Delta,\rho,mol}(M)$ to $L^1(M)$. □

The inclusion $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)\subseteq H^1_{L,S_h^\rho}(M)\cap H^2(M)$**.**

Theorem 3.14. *Let M be a metric measure space satisfying the doubling volume property (D) and the heat kernel estimate (DG_{β_1,β_2}). Then* $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M) \subseteq H^1_{L,S_h^\rho}(M) \cap H^2(M)$ *and*

$$\|f\|_{H^1_{L,S_h^\rho}(M)} \le C\|f\|_{\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)}.$$

Proof. First observe $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M) \subseteq H^2(M)$. Indeed, by Definition 3.5, any $(1,2,\varepsilon)-$molecule belongs to $R(L)$. Thus any finite linear combination of molecules belongs in $R(L)$. Since $f \in \mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)$ is the $L^2(M)$ limit of finite linear combination of molecules, we get $f \in \overline{R(L)} = H^2(M)$.

It remains to show $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M) \subseteq H^1_{L,S_h^\rho}(M)$, that is, S_h^ρ is bounded from $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)$ to $L^1(M)$. Since S_h^ρ is L^2 bounded (see (3.6)) and thus of weak type $(2,2)$, Lemma 3.13 tells us that it suffices to prove, for any $(1,2,\varepsilon)-$molecule a, there exists a constant C such that $\|S_h^\rho a\|_{L^1(M)} \le C$. In other words, one needs to prove $\|A\|_{T_2^1(M)} \le C$ with

$$A(y,t) = \rho(t)Le^{-\rho(t)L}a(y).$$

Assume that a is related to a function b and a ball B with radius r, that is, $a = L^K b$ and for every $k = 0,1,\cdots,K$ and $i = 0,1,2,\cdots$, it holds that

$$\|(\rho(r)L)^k b\|_{L^2(C_i(B))} \le \rho(r)2^{-i\varepsilon}\mu(2^iB)^{-1/2}.$$

Similarly as in [AMR08], we divide A into four parts:

$$\begin{aligned} A &= \mathbb{1}_{2B\times(0,2r)}A + \sum_{i\ge1}\mathbb{1}_{C_i(B)\times(0,r)}A + \sum_{i\ge1}\mathbb{1}_{C_i(B)\times(r,2^{i+1}r)}A \\ &\quad + \sum_{i\ge1}\mathbb{1}_{2^iB\times(2^ir,2^{i+1}r)}A = A_0 + A_1 + A_2 + A_3. \end{aligned}$$

Here $\mathbb{1}$ denotes the characteristic function and $C_i(B) = 2^{i+1}B\backslash 2^iB$, $i \ge 1$. It suffices to show that for every $j = 0,1,2,3$, we have $\|A_j\|_{T_2^1} \le C$.

Firstly consider A_0. Observe that

$$\mathscr{A}(A_0)(x) = \left(\iint_{\Gamma(x)} \left|\mathbb{1}_{2B\times(0,2r)}(y,t)A(y,t)\right|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\right)^{1/2}$$

is supported on $4B$. Indeed, let x_B be the center of B, then

$$d(x,x_B) \le d(x,y) + d(y,x_B) \le 4r.$$

Therefore, we have

$$\begin{aligned}
\|A_0\|^2_{T_2^2(M)} &= \|\mathscr{A}(A_0)\|_2^2 \le \int_{4B}\iint_{\Gamma(x)} \left|\mathbb{1}_{2B\times(0,2r)}(y,t)A(y,t)\right|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}d\mu(x)\\
&\le \int_0^\infty\int_M \left|\rho(t)Le^{-\rho(t)L}a(y)\right|^2 d\mu(y)\frac{dt}{t} \le C\|a\|^2_{L^2(M)}\\
&= C\sum_{i=0}^\infty \|a\|^2_{L^2(C_i(B))} \le C\sum_{i=0}^\infty 2^{-2i\varepsilon}\mu^{-1}(2^iB)\\
&\le C\sum_{i=0}^\infty 2^{-2i\varepsilon}\mu^{-1}(B) \le C\mu^{-1}(B).
\end{aligned}$$

Similarly as in (3.6), applying the Fubini theorem, (4) and spectral theory, we obtain the second inequality and the third one. Combing this with Cauchy-Schwarz inequality, then

$$\|A_0\|_{T_2^1(M)} \le \|A\|_{T_2^2(M)}\mu(4B)^{1/2} \le C.$$

Now for A_1. For each $i\ge 1$, we have $\operatorname{supp}\mathscr{A}(\mathbb{1}_{C_i(B)\times(0,r)}A)\subset 2^{i+2}B$. In fact, $d(x,x_B)\le d(x,y)+d(y,x_B)\le t+2^{i+1}r<2^{i+2}r$. Then

$$\begin{aligned}
&\left\|\mathbb{1}_{C_i(B)\times(0,r)}A\right\|_{T_2^2} = \left\|\mathscr{A}(\mathbb{1}_{C_i(B)\times(0,r)}A)\right\|_2\\
\le\ &\left(\int_{2^{i+2}B}\iint_{\Gamma(x)}\left|\mathbb{1}_{C_i(B)\times(0,r_B)}(y,t)\rho(t)Le^{-\rho(t)L}a(y)\right|^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}d\mu(x)\right)^{1/2}\\
\le\ &\left(\int_0^r\int_{C_i(B)}\left|\rho(t)Le^{-\rho(t)L}a(y)\right|^2 d\mu(y)\frac{dt}{t}\right)^{1/2}\\
\le\ &\sum_{l=0}^\infty\left(\int_0^r\int_{C_i(B)}\left|\rho(t)Le^{-\rho(t)L}\mathbb{1}_{C_l(B)}a(y)\right|^2 d\mu(y)\frac{dt}{t}\right)^{1/2}\\
:=\ &\sum_{l=0}^\infty I_l.
\end{aligned}$$

We estimate I_l with $|i-l|>3$ and $|i-l|\le 3$ respectively. Firstly assume that $|i-l|\le 3$, then by spectral theorem (see (3.6)), we have

$$\begin{aligned}
I_l^2 &\le \int_0^\infty\int_M\left|\rho(t)Le^{-\rho(t)L}\mathbb{1}_{C_l(B)}a(y)\right|^2 d\mu(y)\frac{dt}{t}\le C\|a\|^2_{L^2(C_l(B))}\\
&\le C2^{-2i\varepsilon}\mu^{-1}(2^iB).
\end{aligned}$$

Assume now $|i-l|>3$. Note that $\operatorname{dist}(C_l(B),C_i(B))\ge c2^{\max\{l,i\}}r_B$. First if $i>l+3$, it follows from Lemma 3.4

$$\begin{aligned} I_l^2 &\le \int_0^r \exp\left(-c\left(\frac{2^i r}{t}\right)^{\tau(t)}\right)\|a\|^2_{L^2(C_l(B))}d\mu(y)\frac{dt}{t} \\ &\le C2^{-2l\varepsilon}\mu^{-1}(2^lB)\int_0^r \left(\frac{t}{2^ir}\right)^{c+\nu}\frac{dt}{t} \le C2^{-ci}2^{-2l\varepsilon}\mu^{-1}(2^iB). \end{aligned} \tag{3.11}$$

Here $\tau(t) = \frac{\beta_1}{\beta_1-1}$ if $0<t<1$, otherwise $\tau(t) = \frac{\beta_2}{\beta_2-1}$. In the second inequality, we dominate $\exp\left(-c\left(\frac{2^ir}{t}\right)^{\tau(t)}\right)$ by $\left(\frac{t}{2^ir}\right)^{c+\nu}$. The last inequality comes from (3).

If $i<l-3$, then Lemma 3.4 yields

$$\begin{aligned} I_l^2 &\le \int_0^r \exp\left(-c\left(\frac{2^l r}{t}\right)^{\tau(t)}\right)\|a\|^2_{L^2(C_l(B))}d\mu(y)\frac{dt}{t} \\ &\le C2^{-2l\varepsilon}\mu^{-1}(2^lB)\int_0^r \left(\frac{t}{2^ir}\right)^{c}\frac{dt}{t} \le C2^{-ci}2^{-2l\varepsilon}\mu^{-1}(2^iB). \end{aligned} \tag{3.12}$$

It follows from above that

$$\begin{aligned} \left\|\mathbb{1}_{C_i(B)\times(0,r)}A\right\|_{T_2^2} &\le C\sum_{l:|l-i|\le 3} 2^{-i\varepsilon}\mu^{-1/2}(2^iB) + C\sum_{l:|l-i|>3} 2^{-kc}2^{-l\varepsilon}\mu^{-1/2}(2^iB) \\ &\le C2^{-ic}\mu^{-1/2}(2^iB), \end{aligned}$$

where c depends on ε, M. Therefore

$$\|A_1\|_{T_2^1} \le \sum_{i\ge 1}\left\|\mathbb{1}_{C_i(B)\times(0,r)}A\right\|_{T_2^2}\mu^{1/2}(2^{i+2}B) \le C\sum_{i\ge 1}2^{-ic} \le C.$$

We estimate A_2 in a similar way as before except that we replace a by L^Kb. Note that for each $i\le 1$, we have supp $\mathscr{A}(\mathbb{1}_{C_i(B)\times(r,2^{i+1}r)}A) \subset 2^{i+2}B$. Indeed,

$$d(x,x_B) \le d(x,y)+d(y,x_B) \le t+2^{i+1}r_B \le 2^{i+2}r_B.$$

Then

$$\begin{aligned} &\left\|\mathbb{1}_{C_i(B)\times(r_B,2^{i+1}r)}A\right\|_{T_2^2} = \left\|\mathscr{A}(\mathbb{1}_{C_i(B)\times(r,2^{i+1}r_B)}A)\right\|_2 \\ \le\ & \left(\int_{2^{i+2}B}\iint_{\Gamma(x)}\left|\mathbb{1}_{C_i(B)\times(r_B,2^ir_B)}(y,t)A(y,t)\right|^2\frac{d\mu(y)dt}{V(x,t)t}d\mu(x)\right)^{1/2} \\ \le\ & \left(\int_{r_B}^{2^{i+1}r}\int_{C_i(B)}\left|(\rho(t)L)^{K+1}e^{-\rho(t)L}b(y)\right|^2 d\mu(y)\frac{dt}{t\rho^{2K}(t)}\right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\leq \left(\sum_{l=0}^{\infty}\int_{r_B}^{2^{i+1}r}\int_{C_i(B)}\left|(\rho(t)L)^{K+1}e^{-\rho(t)L}\mathbb{1}_{C_l(B)}b(y)\right|^2 d\mu(y)\frac{dt}{t\rho^{2K}(t)}\right)^{1/2}\\
&:= \sum_{l=0}^{\infty} J_l
\end{aligned}$$

When $|i-l|\leq 3$, by spectral theorem we get $J_l^2\leq C2^{-2i\varepsilon}V^{-1}(2^iB)$. And when $|i-l|>3$, it holds $\operatorname{dist}(C_l(B),C_i(B))\geq c2^{\max\{l,i\}}r_B$. If $i>l+3$, by using Lemma 3.4, we get

$$\begin{aligned}
J_l^2 &\leq \int_r^{2^{i+1}r}\exp\left(-c\left(\frac{2^ir}{t}\right)^{\tau(t)}\right)\|b\|^2_{L^2(C_l(B))}d\mu(y)\frac{dt}{t\rho^2(t)}\\
&\leq \rho^{2K}(r)2^{-2l\varepsilon}\mu^{-1}(2^{l+1}B)\int_r^{2^{i+1}r}\left(\frac{t}{2^ir}\right)^c\frac{dt}{t\rho^{2K}(t)}\\
&\leq C2^{-i(c-\nu)}2^{-l(2\varepsilon+\nu)}\mu^{-1}(2^iB).
\end{aligned}$$

Here we take $\nu<c<2\beta_1K$. The same estimate also holds for $l>i+3$, in the same way as (3.12).

Hence

$$\left\|\mathbb{1}_{C_i(B)\times(r,2^ir)}A\right\|^2_{T_2^2}\leq C2^{-ic}\mu^{-1}(2^iB),$$

and

$$\|A_2\|_{T_2^1}\leq\sum_{i\geq 1}\left\|\mathbb{1}_{C_i(B)\times(r,2^ir_B)}A\right\|_{T_2^2}\mu^{1/2}(2^{i+2}B)\leq C\sum_{i\geq 1}2^{-ic/2}\leq C.$$

It remains to estimate the last term A_3. For each $i\geq 1$, we still have

$$\operatorname{supp}\mathscr{A}(\mathbb{1}_{2^iB\times(2^ir,2^{i+1}r_B)}A)\subset 2^{i+2}B.$$

Then we obtain as before that

$$\begin{aligned}
&\left\|\mathbb{1}_{2^iB\times(2^ir,2^{i+1}r)}A\right\|_{T_2^2} = \left\|\mathscr{A}(\mathbb{1}_{2^iB\times(2^ir,2^{i+1}r_B)}A)\right\|_2\\
&\leq \left(\int_{2^{i+2}B}\iint_{\Gamma(x)}\left|\mathbb{1}_{2^iB\times(2^ir_B,2^{i+1}r)}(y,t)A(y,t)\right|^2\frac{d\mu(y)dt}{V(x,t)t}d\mu(x)\right)^{1/2}\\
&\leq \left(\int_{2^ir}^{2^{i+1}r}\int_{2^iB}\left|(\rho(t)L)^{K+1}e^{-\rho(t)L}b(y)\right|^2\frac{d\mu(y)dt}{t\rho^{2K}(t)}\right)^{1/2}\\
&\leq \sum_{l=0}^{\infty}\left(\int_{2^ir}^{2^{i+1}r}\int_{2^iB}\left|(\rho(t)L)^{K+1}e^{-\rho(t)L}\mathbb{1}_{C_l(B)}b(y)\right|^2\frac{d\mu(y)dt}{t\rho^{2K}(t)}\right)^{1/2}\\
&:= K_l.
\end{aligned}$$

In fact, due to the doubling volume property, the pointwise heat kernel estimate as well as the definition of molecules, we get

$$\begin{aligned} K_l^2 &\leq \int_{2^i r}^{2^{i+1} r} \|\mathbb{1}_{C_l(B)} b\|_{L^2}^2 \frac{dt}{t\rho^{2K}(t)} \leq C\rho^{2K}(r) 2^{-2l\varepsilon} \mu^{-1}(2^l B) \int_{2^i r}^{2^{i+1} r} \frac{dt}{t\rho^{2K}(t)} \\ &\leq C 2^{-2l\varepsilon} 2^{-ic} \mu^{-1}(2^i B). \end{aligned}$$

Hence

$$\|A_3\|_{T_2^1} \leq \sum_{i\geq 1} \|\mathbb{1}_{2^i B\times(2^i r_B, 2^{i+1} r_B)} A\|_{T_2^2} \mu^{1/2}(2^{i+2} B) \leq C \sum_{i\geq 1} 2^{-2i} \leq C.$$

This finishes the proof. □

The inclusion $H^1_{L,S_h^\rho} \cap H^2(M) \subseteq \mathbb{H}^1_{L,\rho,\mathrm{mol}}$. We closely follow the proof of Theorem 4.13 in [HLM+11] and get

Theorem 3.15. *Let M be a metric measure space satisfying the doubling volume property (D) and the heat kernel estimate (DG_{β_1,β_2}). If $f \in H^1_{L,S_h^\rho}(M) \cap H^2(M)$, then there exist a sequence of numbers $\{\lambda_j\}_{j=0}^\infty \subset l^1$ and a sequence of $(1,2,\varepsilon)-$molecules $\{a_j\}_{j=0}^\infty$ such that f can be represented in the form $f = \sum_{j=0}^\infty \lambda_j a_j$, with the sum converging in $L^2(M)$, and*

$$\|f\|_{\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)} \leq C \sum_{j=0}^\infty \lambda_j \leq C \|f\|_{H^1_{L,S_h^\rho}(M)},$$

where C is independent of f. In particular, $H^1_{L,S_h^\rho}(M) \cap H^2(M) \subseteq \mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)$.

Proof. For $f \in H^1_{L,S_h^\rho}(M) \cap H^2(M)$, denote $F(x,t) = \rho(t) L e^{-\rho(t)L} f(x)$. Then by the definition of $H^1_{L,S_h^\rho}(M)$, we have $F \in T_2^1(M) \cap T_2^2(M)$.

From Theorem 3.11, we decompose F as $F = \sum_{j=0}^\infty \lambda_j A_j$, where $\{\lambda_j\}_{j=0}^\infty \in l^1$, $\{A_j\}_{j=0}^\infty$ is a sequence of T_2^1-atoms supported in a sequence of sets $\{\widehat{B}_j\}_{j=0}^\infty$, and the sum converges in both $T_2^1(M)$ and $T_2^2(M)$. Also

$$\sum_{j=0}^\infty \lambda_j \leq C\|F\|_{T_2^1(X)} = C\|f\|_{H^1_{L,S_h^\rho}(M)}.$$

For $f \in H^2(M)$, by functional calculus, we have the following "Calderón reproducing formula"

$$f = C\int_0^\infty (\rho(t)L)^{K+1} e^{-2\rho(t)L} f \frac{\rho'(t)dt}{\rho(t)} = C\int_0^\infty (\rho(t)L)^K e^{-\rho(t)L} F(\cdot,t) \frac{\rho'(t)dt}{\rho(t)} := C\pi_{h,L}(F).$$

Denote $a_j = C\pi_{h,L}(A_j)$, then $f = \sum_{j=0}^\infty \lambda_j a_j$. Since for $F \in T_2^2(M)$, we have $\|\pi_{h,L}(F)\|_{L^2(M)} \leq C\|F\|_{T_2^2(M)}$. Thus we learn from Lemma 4.12 in [HLM+11] that the sum also converges in $L^2(M)$.

We claim that $a_j, j = 0,1,...,$ are $(1,2,\varepsilon)-$molecules up to multiplication to some uniform constant, which implies our conclusion. Indeed, $a_j = L^K b_j$ for $b_j = C\int_0^\infty \rho^K(t) e^{-\rho(t)L} A_j(\cdot,t) \frac{\rho'(t)dt}{\rho(t)}$.

Now we estimate the norm $\|(\rho(r_{B_j})L)^k b_j\|_{L^2(C_i(B))}$. For simplicity we ignore the index j. Consider any function $g \in L^2(C_i(B))$ with $\|g\|_{L^2(C_i(B))} = 1$, then for $k = 0, 1, \cdots, K$,

$$\begin{aligned}
&\left|\int_M (\rho(r_B)L)^k b(x)g(x)d\mu(x)\right| \\
\leq\ & C\left|\int_M \left(\int_0^\infty (\rho(r_B)L)^k \rho^K(t)e^{-\rho(t)L}(A_j(\cdot,t))(x)\frac{\rho'(t)dt}{\rho(t)}\right)g(x)d\mu(x)\right| \\
=\ & C\left|\int_{\widehat{B}} \left(\frac{\rho(r_B)}{\rho(t)}\right)^k \rho^K(t)A_j(x,t)(\rho(t)L)^k e^{-\rho(t)L}g(x)d\mu(x)\frac{\rho'(t)dt}{\rho(t)}\right| \\
\leq\ & C\left(\int_{\widehat{B}} |A_j(x,t)|^2 d\mu(x)\frac{dt}{t}\right)^{1/2} \\
& \cdot\left(\int_{\widehat{B}} \left|\left(\frac{\rho(r_B)}{\rho(t)}\right)^k \rho^K(t)(\rho(t)L)^k e^{-\rho(t)L}g(x)\right|^2 d\mu(x)\frac{dt}{t}\right)^{1/2}.
\end{aligned}$$

In the last inequality, we apply Hölder inequality as well as (3.1).

We continue to estimate by using the definition of T_2^1-atoms and the off-diagonal estimates of heat kernel.

For $i = 0, 1$, the above quantity is dominated by

$$C\mu^{-1/2}(B)\rho(r_B)\left(\int_{\widehat{B}} \left|(\rho(t)L)^k e^{-\rho(t)L}g(x)\right|^2 d\mu(x)\frac{dt}{t}\right)^{1/2} \leq C\mu^{-1/2}(B)\rho(r_B).$$

Next for $i \geq 2$, by using Lemma 3.3 ,the above quantity is controlled

$$\begin{aligned}
& C\mu^{-1/2}(B)\left(\int_0^{r_B} \left(\frac{\rho(r_B)}{\rho(t)}\right)^{2k} \rho^{2K}(t)\left\|(\rho(t)L)^k e^{-\rho(t)L}g\right\|^2_{L^2(B)}\frac{dt}{t}\right)^{1/2} \\
\leq\ & C\mu^{-1/2}(B)\left(\int_0^{r_B} \left(\frac{\rho(r_B)}{\rho(t)}\right)^{2k} \rho^{2K}(t)\exp\left(-c\left(\frac{2^i r_B}{t}\right)^\tau\right)\frac{dt}{t}\right)^{1/2} \\
\leq\ & C\mu^{-1/2}(B)\left(\int_0^{r_B} \left(\frac{\rho(r_B)}{\rho(t)}\right)^{2k} \rho^{2K}(t)\left(\frac{t}{2^i r_B}\right)^{\varepsilon+\nu}\frac{dt}{t}\right)^{1/2} \\
\leq\ & C\mu^{-1/2}(2^i B)\rho^K(r_B)2^{-i\varepsilon}.
\end{aligned}$$

Since $k = 0, 1, \cdots, K$, the last inequality always holds for any $\varepsilon > 0$.

Therefore,

$$\begin{aligned}
\|(\rho(r_B)L)^k b\|_{L^2(C_i(B))} &= \sup_{\|g\|_{L^2(C_i(B))}=1} \left|\int_M (\rho(r_B)L)^k b(x)g(x)d\mu(x)\right| \\
&\leq C\mu^{-1/2}(2^i B)\rho^K(r_B)2^{-i\varepsilon}.
\end{aligned}$$

□

3.4 Comparison of Hardy spaces and Lebesgue spaces

For this topic, we refer to [AMR08, HLM$^+$11, Uhl11, Dev13].

Let (M,d,μ) be a metric measure space of homogeneous type and L be a non-negative self-adjoint operator satisfying the Davies-Gaffney estimate (3.2). We first recall the definition of the Hardy space $H^p_{L,S_h}(M)$, $p \geq 1$, in [HLM$^+$11]. Consider the conical square function

$$S_h f(x) = \left(\iint_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}. \tag{3.13}$$

The Hardy space $H^p_{L,S_h}(M)$, $p \geq 1$ is defined as the completion of the set $\{f \in H^2(M) : \|S_h f\|_{L^p} < \infty\}$ with respect to the norm $\|f\|_{H^p_{L,S_h}} = \|S_h f\|_{L^p}$, for $f \in H^2(M)$. Here $H^2(M) = \overline{R(L)}^{L^2}$, see Section 3.1.2.

It always holds that $H^p_{L,S_h} \subset L^p$, $1 < p < 2$. Indeed, since the L^p, $2 < p < \infty$, norm of the conical square function is controlled by its vertical analogue (see [AHM12]). By an argument of duality (see [AMR08]), we obtain $H^p_{L,S_h} \subset L^p$, $1 < p < 2$. We also refer to [AMM13] for the embedding $H^p_{L,S_h} \subset L^p$, $1 \leq p < 2$.

If one assumes in addition the Gaussian upper bound for the heat kernel of the operator L, we have $H^p_{L,S_h}(M) = L^p(M)$, $1 < p < \infty$ (see for example [AMR08, Theorem 8.5] for Hardy spaces of $0-$forms on Riemannian manifold). We are going to show that this equivalence does not hold in general.

Coming back to the setting of a metric measure space (M,d,μ) satisfying (D) and $(DG^{p_0,p_0'}_{\beta_1,\beta_2})$, a natural question to ask is whether $H^p_{L,S^\rho_h}(M)$ and $L^p(M)$ are equivalent.

In this section, we will study the relations between $L^p(M)$, $H^p_{L,S^\rho_h}(M)$ and $H^p_{L,S_h}(M)$ under the assumptions of (D) and $(DG^{p_0,p_0'}_{\beta_1,\beta_2})$. We first show in the following that $L^p(M)$ and $H^p_{L,S^\rho_h}(M)$ are equivalent. Next we give some examples such that $L^p(M)$ and $H^p_{L,S_h}(M)$ are not equivalent. More precisely, the inclusion $L^p \subset H^p_{L,S_h}$ may be false for $1 < p < 2$.

3.4.1 Equivalence of $L^p(M)$ and $H^p_{L,S^\rho_h}(M)$ for $1 < p < \infty$

Theorem 3.16. *Let M be a non-compact metric measure space satisfying the doubling volume property (D) and $(DG^{p_0,p_0'}_{\beta_1,\beta_2})$. Then $H^p_{L,S^\rho_h}(M) = \overline{R(L) \cap L^p(M)}^{L^p(M)}$ for $p_0 < p < p_0'$.*

As explained in Section 3.1.2, if we assume (D) and (UE_{β_1,β_2}), then it holds that $\overline{R(L)} = L^2$. Therefore we have

Corollary 3.17. *Let M be a metric measure space satisfying the doubling volume property (D) and (UE_{β_1,β_2}). Then $H^p_{L,S_h^\rho}(M) = L^p(M)$ for $1 < p < \infty$.*

Proof. It suffices to prove that for any $f \in R(L) \cap L^p(M)$ with $p_0 < p < p_0'$,

$$\|S_h^\rho f\|_{L^p} \le C\|f\|_{L^p}. \tag{3.14}$$

With this fact at hand, we can obtain by duality that $\|f\|_{L^p} \le C\|S_h^\rho f\|_{L^p}$ for $p_0 < p < p_0'$.

Indeed, for $f \in R(L)$, write the identity

$$f = C\int_0^\infty (\rho(t)L)^2 e^{-2\rho(t)L} f \frac{\rho'(t)dt}{\rho(t)},$$

where the integral $C\int_\varepsilon^{1/\varepsilon} (\rho(t)L)^2 e^{-2\rho(t)L} f \frac{\rho'(t)dt}{\rho(t)}$ converges to f in $L^2(M)$ as $\varepsilon \to 0$.

Then for $f \in R(L) \cap L^p(M)$, we have

$$\begin{aligned}
\|f\|_{L^p} &= \sup_{\|g\|_{L^{p'}} \le 1} |< f, g >| \approx \sup_{\|g\|_{L^{p'}} \le 1} \left| \iint_{M\times(0,\infty)} F(y,t)G(y,t)d\mu(y)\frac{\rho'(t)dt}{\rho(t)} \right| \\
&\approx \sup_{\|g\|_{L^{p'}} \le 1} \left| \int_M \iint_{\Gamma(x)} F(y,t)G(y,t)\frac{d\mu(y)}{V(x,t)}\frac{\rho'(t)dt}{\rho(t)} d\mu(x) \right| \\
&\le C \sup_{\|g\|_{L^{p'}} \le 1} \|F\|_{T_2^p}\|G\|_{T_2^{p'}} = C \sup_{\|g\|_{L^{p'}} \le 1} \|S_h f\|_{L^p}\|S_h g\|_{L^{p'}} \\
&\le C \sup_{\|g\|_{L^{p'}} \le 1} \|S_h f\|_{L^p}\|g\|_{L^{p'}} = \|S_h f\|_{L^p}.
\end{aligned}$$

Here $F(y,t) = \rho(t)Le^{-\rho(t)L}f(y)$ and $G(y,t) = \rho(t)Le^{-\rho(t)L}g(y)$. The second line's equivalence is due to the doubling volume property.

By an approximating process, the above argument holds for $f \in L^p(M)$.

For $p > 2$, the L^p norm of the conical square function is controlled by its vertical analogue (for a reference, see [AHM12], where the proof can be adapted to the homogenous setting), which is always L^p bounded for $p_0 < p < p_0'$ by adapting the proofs in [Blu07] and [CDMY96] (if $(e^{-tL})_{t>0}$ is a symmetric Markov semigroup, then it is L^p bounded for $1 < p < \infty$, according to [Ste70b]). Hence (3.14) holds.

It remains to show (3.14) for $p_0 < p < 2$.

In the following, we will prove the weak (p_0, p_0) boundedness of S_h^ρ by using the L^{p_0} Calderón-Zygmund decomposition (see Theorem 2.9 in Chapter 2). Since S_h^ρ is also L^2 bounded as shown in (3.6), then by interpolation, (3.14) holds for every $p_0 < p < 2$. The proof is similar to [Aus07, Proposition 6.8] and [AHM12, Theorem 3.1].

We take the Calderón-Zygmund decomposition of f at the height λ, that is, $f = g + \sum b_i$ with supp $b_i \subset B_i$. Since S_h^ρ is a sublinear operator, write

$$\begin{aligned} S_h^\rho\left(\sum_i b_i\right) &= S_h^\rho\left(\sum_i \left(I-\left(I-e^{-\rho(r_i)}\right)^N+\left(I-e^{-\rho(r_i)L}\right)^N\right)b_i\right) \\ &\leq S_h^\rho\left(\sum_i \left(I-\left(I-e^{-\rho(r_i)}\right)^N\right)b_i\right)+S_h^\rho\left(\sum_i \left(I-e^{-\rho(r_i)}\right)^N b_i\right). \end{aligned}$$

Here $N\in\mathbb{N}$ is chosen to be larger than $2\nu/\beta_1$ where ν is as in (3).

Then it is enough to prove that

$$\begin{aligned} &\mu\left(\{x\in M: S_h^\rho(f)(x)>\lambda\}\right)\leq \mu\left(\left\{x\in M: S_h^\rho(g)(x)>\frac{\lambda}{3}\right\}\right) \\ &+\mu\left(\left\{x\in M: S_h^\rho\left(\sum_i\left(I-\left(I-e^{-\rho(r_i)}\right)^N\right)b_i\right)(x)>\frac{\lambda}{3}\right\}\right) \\ &+\mu\left(\left\{x\in M: S_h^\rho\left(\sum_i\left(I-e^{-\rho(r_i)L}\right)^N b_i\right)(x)>\frac{\lambda}{3}\right\}\right)\leq \frac{C}{\lambda^{p_0}}\int |f(x)|^{p_0}d\mu(x). \end{aligned}$$

We treat g in a routine way. Since S_h^ρ is L^2 bounded as shown in (3.6), then

$$\mu\left(\left\{x\in M: S_h^\rho(g)(x)>\frac{\lambda}{3}\right\}\right)\leq C\lambda^{-2}\|g\|_2^2\leq C\lambda^{-p_0}\|g\|_{p_0}\leq C\lambda^{-p_0}\|f\|_{p_0}.$$

Now for the second term. Note that $I-\left(I-e^{-\rho(r_i)L}\right)^N=\sum_{k=1}^N(-1)^{k+1}\binom{N}{k}e^{-k\rho(r_i)L}$. Since the sum has N terms, it is enough to show that for every $1\leq k\leq N$,

$$\mu\left(\left\{x\in M: S_h^\rho\left(\sum_i e^{-k\rho(r_i)L}b_i\right)(x)>\frac{\lambda}{3N}\right\}\right)\leq \frac{C_k}{\lambda^{p_0}}\int |f(x)|^{p_0}d\mu(x). \tag{3.15}$$

Note the following slight improvement of Assumption ($DG^{p_0,p_0'}_{\beta_1,\beta_2}$) (or (3.3) for $u=p_0$ and $v=2$): for every $1\leq k\leq N$ and for every $j\geq 1$, we have

$$\left\|e^{-k\rho(r_i)L}b_i\right\|_{L^2(C_j(B_i))}\leq C_k\frac{\mu^{1/2}(2^{j+1}B_i)}{\mu^{1/p_0}(B_i)}e^{-c_k 2^{j\tau(k\rho(r_i))}}\|b_i\|_{L^{p_0}(B_i)}. \tag{3.16}$$

Here $\tau(r)=\beta_1/(\beta_1-1)$ if $0<r<1$, otherwise $\tau(r)=\beta_2/(\beta_2-1)$.

Indeed, if $r_i\geq 1$, it holds that

$$\begin{aligned} \left\|e^{-k\rho(r_i)L}b_i\right\|_{L^2(C_j(B_i))} &\leq \left\|e^{-k\rho(r_i)L}b_i\right\|_{L^2(C_j(k^{1/\beta_2}B))} \\ &\leq C_k\frac{\mu^{1/2}(2^{j+1}k^{1/\beta_2}B)}{\mu^{1/p_0}(k^{1/\beta_2}B)}e^{-c_k 2^{j\beta_2/(\beta_2-1)}}\|b_i\|_{L^{p_0}(B_i)} \end{aligned}$$

$$\leq C_k \frac{\mu^{1/2}(2^j B_i)}{\mu^{1/p_0}(B_i)} e^{-c_k 2^{j\beta_2/(\beta_2-1)}} \|b_i\|_{L^{p_0}(B_i)}.$$

The last inequality follows from (D).

Now for $0 < r_i < 1$. Fixed k, if $0 < r_i < k^{-1/\beta_1}$, then

$$\begin{aligned}
\left\| e^{-k\rho(r_i)L} b_i \right\|_{L^2(C_j(B_i))} &\leq \left\| e^{-k\rho(r_i)L} b_i \right\|_{L^2(C_j(k^{1/\beta_1}B))} \\
&\leq C_k \frac{\mu^{1/2}(2^{j+1}k^{1/\beta_1}B)}{\mu^{1/p_0}(k^{1/\beta_1}B)} e^{-c_k 2^{j\beta_1/(\beta_1-1)}} \|b_i\|_{L^{p_0}(B_i)} \\
&\leq C_k \frac{\mu^{1/2}(2^j B_i)}{\mu^{1/p_0}(B_i)} e^{-c_k 2^{j\beta_1/(\beta_1-1)}} \|b_i\|_{L^{p_0}(B_i)}.
\end{aligned}$$

Otherwise, note that $k^{-1/\beta_1} \leq r_i < 1$, that is, $r_i \simeq 1$, then

$$\begin{aligned}
\left\| e^{-k\rho(r_i)L} b_i \right\|_{L^2(C_j(B_i))} &\leq \left\| e^{-k\rho(r_i)L} b_i \right\|_{L^2(C_j(k^{1/\beta_2}B(x_B,k^{1/\beta_2}r^{\beta_1/\beta_2})))} \\
&\leq C_k \frac{V^{1/2}(2^{j+1}B(x_B,k^{1/\beta_2}r^{\beta_1/\beta_2})))}{V^{1/p_0}(x_B,k^{1/\beta_2}r^{\beta_1/\beta_2}))} e^{-c_k 2^{j\beta_2/(\beta_2-1)}} \|b_i\|_{L^{p_0}(B_i)} \\
&\leq C_k \frac{\mu^{1/2}(2^j B_i)}{\mu^{1/p_0}(B_i)} e^{-c_k 2^{j\beta_2/(\beta_2-1)}} \|b_i\|_{L^{p_0}(B_i)}.
\end{aligned}$$

With the above preparations, we can show (3.15) now. Write

$$\mu\left(\left\{x : \left| S_h^\rho \left(\sum_i e^{-k\rho(r_i)L} b_i \right)(x) \right| > \frac{\lambda}{3N} \right\}\right) \leq \frac{C}{\lambda^2} \left\| \sum_i e^{-k\rho(r_i)L} b_i \right\|_2^2$$

By a duality argument,

$$\begin{aligned}
\left\| \sum_i e^{-k\rho(r_i)L} b_i \right\|_2 &= \sup_{\|\phi\|_2=1} \int_M \left| \sum_i e^{-k\rho(r_i)L} b_i \right| |\phi| d\mu \\
&\leq \sup_{\|\phi\|_2=1} \sum_i \sum_{j=1}^{\infty} \int_{C_j(B_i)} \left| e^{-k\rho(r_i)L} b_i \right| |\phi| d\mu \\
&:= \sup_{\|\phi\|_2=1} \sum_i \sum_{j=1}^{\infty} A_{ij}.
\end{aligned}$$

Applying Cauchy-Schwarz inequality, (3.16) and (3), we get

$$\begin{aligned}
A_{ij} &\leq \left\| e^{-k\rho(r_i)L} b_i \right\|_{L^2(C_j(B_i))} \|\phi\|_{L^2(C_j(B_i))} \\
&\leq C\lambda \mu(2^{j+1}B_i) e^{-c 2^{j\tau(k\rho(r_i))}} \operatorname*{ess\,inf}_{y\in B_i} \left(\mathscr{M}\left(|\phi|^2\right)(y) \right)^{1/2}
\end{aligned}$$

$$\leq \quad C\lambda 2^{jD} e^{-c2^{j\tau(k\rho(r_i))}} \mu(B_i) \operatorname*{ess\,inf}_{y\in B_i} \left(\mathscr{M}\left(|\phi|^2\right)(y)\right)^{1/2}.$$

Here $\mathscr{M}$ denotes the Littlewood-Paley maximal operator.

Then

$$\begin{aligned}
\left\|\sum_i e^{-k\rho(r_i)L} b_i\right\|_2 &\leq C\lambda \sup_{\|\phi\|_2=1} \sum_i \sum_{j=1}^{\infty} 2^{jD} e^{-c2^{j\tau(k\rho(r_i))}} \mu(B_i) \operatorname*{ess\,inf}_{y\in B_i} \left(\mathscr{M}\left(|\phi|^2\right)(y)\right)^{1/2} \\
&\leq C\lambda \sup_{\|\phi\|_2=1} \int \sum_i \mathbb{1}_{B_i}(y) \left(\mathscr{M}\left(|\phi|^2\right)(y)\right)^{1/2} d\mu(y) \\
&\leq C\lambda \sup_{\|\phi\|_2=1} \int_{\cup_i B_i} \left(\mathscr{M}\left(|\phi|^2\right)(y)\right)^{1/2} d\mu(y) \\
&\leq C\lambda \mu^{1/2}(\cup_i B_i) \leq C\lambda^{1-p_0/2} \left(\int |f|^{p_0} d\mu\right)^{1/2}.
\end{aligned}$$

The third inequality is due to the finite overlapping of the Calderón-Zygmund decomposition. In the last line, for the first inequality, we use Kolmogorov's inequality (see for example [Gra08, page 91]).

Therefore, we obtain

$$\mu\left(\left\{x : \left|S_h^{\rho}\left(\sum_i e^{-k\rho(r_i)L} b_i\right)(x)\right| > \frac{\lambda}{3N}\right\}\right) \leq C\lambda^{-p_0} \int |f|^{p_0} d\mu. \tag{3.17}$$

For the third term, we have

$$\begin{aligned}
&\mu\left(\left\{x \in M : S_h^{\rho}\left(\sum_i \left(I - e^{-\rho(r_i)L}\right)^N b_i\right)(x) > \lambda/3\right\}\right) \\
\leq \quad &\mu\left(\cup_j 4B_j\right) + \mu\left(\left\{x \in M \setminus \cup_j 4B_j : S_h^{\rho}\left(\sum_i \left(I - e^{-\rho(r_i)L}\right)^N b_i\right)(x) > \lambda/3\right\}\right).
\end{aligned}$$

From the Calderón-Zygmund decomposition and doubling volume property, we get

$$\mu\left(\cup_j 4B_j\right) \leq \sum_j \mu(4B_j) \leq C\sum_j \mu(B_j) \leq \frac{C}{\lambda^{p_0}} \|f\|_{p_0}.$$

It remains to show that

$$\begin{aligned}
\Lambda \quad &:= \quad \mu\left(\left\{x \in M \setminus \cup_j 4B_j : S_h^{\rho}\left(\sum_i \left(I - e^{-\rho(r_i)L}\right)^N b_i\right)(x) > \frac{\lambda}{3}\right\}\right) \\
&\leq \quad \frac{C}{\lambda^{p_0}} \int |f(x)|^{p_0} d\mu(x).
\end{aligned}$$

As a consequence of the Chebichev inequality, Λ is dominated by

$$
\begin{aligned}
& \frac{9}{\lambda^2}\int_{M\setminus\cup_j 4B_j}\left(S_h^\rho\left(\sum_i\left(I-e^{-\rho(r_i)L}\right)^N b_i\right)(x)\right)^2 d\mu(x)\\
\leq\ & \frac{9}{\lambda^2}\int_{M\setminus\bigcup_j 4B_j}\iint_{\Gamma(x)}\left(\sum_i \rho(t)Le^{-\rho(t)L}\left(I-e^{-\rho(r_i)L}\right)^N b_i(y)\right)^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}d\mu(x)\\
\leq\ & \frac{18}{\lambda^2}\int_{M\setminus\cup_j 4B_j}\iint_{\Gamma(x)}\left(\sum_i \mathbb{1}_{2B_i}(y)\rho(t)Le^{-\rho(t)L}\left(I-e^{-\rho(r_i)L}\right)^N b_i(y)\right)^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}d\mu(x)\\
& +\frac{18}{\lambda^2}\int_{M\setminus\cup_j 4B_j}\iint_{\Gamma(x)}\left(\sum_i \mathbb{1}_{M\setminus 2B_i}(y)\rho(t)Le^{-\rho(t)L}\left(I-e^{-\rho(r_i)L}\right)^N b_i(y)\right)^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}d\mu(x)\\
:=\ & \frac{18}{\lambda^2}(\Lambda_{loc}+\Lambda_{glob}).
\end{aligned}
$$

For the estimate of Λ_{loc}. Thanks to the bounded overlapping of $2B_i$, we can put the sum of i out of the square up to a multiplicative constant. That is,

$$
\begin{aligned}
& \Lambda_{loc}\\
\leq\ & C\sum_i\int_{M\setminus\cup_j 4B_j}\int_0^\infty\int_{B(x,t)}\left(\mathbb{1}_{2B_i}(y)\rho(t)Le^{-\rho(t)L}\left(I-e^{-\rho(r_i)L}\right)^N b_i(y)\right)^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}d\mu(x)\\
\leq\ & C\sum_i\int_{M\setminus\cup_j 4B_j}\int_{2r_i}^\infty\int_{B(x,t)}\left(\mathbb{1}_{2B_i}(y)\rho(t)Le^{-\rho(t)L}\left(I-e^{-\rho(r_i)L}\right)^N b_i(y)\right)^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}d\mu(x)\\
\leq\ & C\sum_i\int_{2r_i}^\infty\int_M\left(\int_{B(y,t)}\frac{d\mu(x)}{V(x,t)}\right)\left(\mathbb{1}_{2B_i}(y)\rho(t)Le^{-\rho(t)L}\left(I-e^{-\rho(r_i)L}\right)^N b_i(y)\right)^2 d\mu(y)\frac{dt}{t}\\
\leq\ & C\sum_i\int_{2r_i}^\infty\int_{2B_i}\left(\rho(t)Le^{-\rho(t)L}\left(I-e^{-\rho(r_i)L}\right)^N b_i(y)\right)^2 d\mu(y)\frac{dt}{t}.
\end{aligned}
$$

For the second inequality, note that for every i, $x\in M\setminus\cup_j 4B_j$ means $x\notin 4B_i$. Then $y\in 2B_i$ and $d(x,y)<t$ imply that $t\geq 2r_i$. Thus the integral is zero for every i. We obtain the third inequality by using the Fubini theorem and (4).

Then by using (3.16), it follows

$$
\begin{aligned}
& \Lambda_{loc}\\
\leq\ & C\sum_i\int_{2r_i}^\infty\int_{2B_i}\left(\frac{\mu^{\frac{1}{p_0}-\frac12}(B_i)}{V^{\frac{1}{p_0}-\frac12}(y,t)}\frac{V^{\frac{1}{p_0}-\frac12}(y,t)}{\mu^{\frac{1}{p_0}-\frac12}(B_i)}\rho(t)Le^{-\rho(t)L}\left(I-e^{-\rho(r_i)L}\right)^N b_i(y)\right)^2 d\mu(y)\frac{dt}{t}\\
\leq\ & C\sum_i\int_{2r_i}^\infty\int_{2B_i}\left(\frac{V^{\frac{1}{p_0}-\frac12}(y,4r_i)}{V^{\frac{1}{p_0}-\frac12}(y,t)}\frac{V^{\frac{1}{p_0}-\frac12}(y,t)}{\mu^{\frac{1}{p_0}-\frac12}(B_i)}\rho(t)Le^{-\rho(t)L}\left(I-e^{-\rho(r_i)L}\right)^N b_i(y)\right)^2 d\mu(y)\frac{dt}{t}
\end{aligned}
$$

$$\begin{aligned}
&\leq C\mu^{1-\frac{2}{p_0}}(B_i)\sum_i\int_{2r_i}^{\infty}\left(\frac{4r_i}{t}\right)^{\nu'\left(\frac{2}{p_0}-1\right)}\left\|V^{\frac{1}{p_0}-\frac{1}{2}}(\cdot,t)\rho(t)Le^{-\rho(t)L}\left(I-e^{-\rho(r_i)L}\right)^N b_i\right\|_2^2\frac{dt}{t}\\
&\leq C\mu^{1-\frac{2}{p_0}}(B_i)\sum_i\left\|\left(I-e^{-\rho(r_i)L}\right)^N b_i\right\|_{p_0}^2\\
&\leq C\mu^{1-\frac{2}{p_0}}(B_i)\sum_i\|b_i\|_{p_0}^2\leq C\lambda^2\sum_i\mu(B_i)\leq C\lambda^{2-p_0}\int|f|^{p_0}d\mu.
\end{aligned}$$

For the second inequality, we use the reverse doubling property (5). The third inequality follows from the $L^{p_0}-L^2$ boundedness of the operator $V^{\frac{1}{p_0}-\frac{1}{2}}(\cdot,t)\rho(t)Le^{-\rho(t)L}$ (see Lemma 3.3). Then by using the L^{p_0} boundedness of the heat semigroup, we get the fourth inequality.

Now for the global part. We split the integral into annuli, that is,

$$\begin{aligned}
&\Lambda_{glob}\\
&\leq \int_M\iint_{\Gamma(x)}\left(\sum_i \mathbb{1}_{M\setminus 2B_i}(y)\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\right)^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}d\mu(x)\\
&\leq \int_0^\infty\int_M\int_{B(y,t)}\left(\sum_i \mathbb{1}_{M\setminus 2B_i}(y)\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\right)^2\frac{d\mu(x)}{V(x,t)}d\mu(y)\frac{dt}{t}\\
&\leq \int_0^\infty\int_M\left(\sum_i \mathbb{1}_{M\setminus 2B_i}(y)\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\right)^2 d\mu(y)\frac{dt}{t}.
\end{aligned}$$

In order to estimate the above L^2 norm, we use an argument of dualization. Take the supreme of all functions $h(y,t)\in L^2(M\times(0,\infty),\frac{d\mu(y)dt}{t})$ with norm 1, then

$$\begin{aligned}
&\Lambda_{glob}^{1/2}\\
&\leq \left(\int_0^\infty\int_M\left(\sum_i \mathbb{1}_{M\setminus 2B_i}(y)\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\right)^2 d\mu(y)\frac{dt}{t}\right)^{1/2}\\
&= \sup_h\iint_{M\times(0,\infty)}\left|\sum_i \mathbb{1}_{M\setminus 2B_i}(y)\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\right||h(y,t)|\frac{d\mu(y)dt}{t}\\
&\leq \sup_h\sum_i\sum_{j\geq 2}\int_0^\infty\int_{C_j(B_i)}\left|\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\right||h(y,t)|\frac{d\mu(y)dt}{t}\\
&\leq \sup_h\sum_i\sum_{j\geq 2}\left(\int_0^\infty\int_{C_j(B_i)}\left|\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\right|^2\frac{d\mu(y)dt}{t}\right)^{1/2}\\
&\quad\times\left(\int_0^\infty\int_{C_j(B_i)}|h(y,t)|^2\frac{d\mu(y)dt}{t}\right)^{1/2}.
\end{aligned}$$

Denote $I_{ij}=\left(\int_0^\infty\int_{C_j(B_i)}\left|\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\right|^2\frac{d\mu(y)dt}{t}\right)^{1/2}$.

Let $H_{t,r}(\zeta)=\rho(t)\zeta e^{-\rho(t)\zeta}(1-e^{-\rho(r)\zeta})^N$. Then

$$I_{ij}=\left(\int_0^\infty \|H_{t,r_i}(L)b_i\|^2_{L^2(C_j(B_i))}\frac{dt}{t}\right)^{1/2}. \tag{3.18}$$

We will estimate $\|H_{t,r_i}(L)b_i\|_{L^2(C_j(B_i))}$ by functional calculus. The notation mainly comes from [Aus07, Section 2.2].

For any fixed t and r, then $H_{t,r}$ is a holomorphic function satisfying

$$|H_{t,r}(\zeta)|\le C|\zeta|^{N+1}(1+|\zeta|)^{-2(N+1)},$$

for all $\zeta\in\Sigma=\{z\in\mathbb{C}^*:|\arg z|<\xi\}$ with any $\xi\in(0,\pi/2)$ and for some constant C.

Since L is a nonnegative self-adjoint operator, or equivalently L is a bisectorial operator of type 0, we can express $H_{t,r}(L)$ by functional calculus. Let $0<\theta<\omega<\xi<\pi/2$, we have

$$H_{t,r}(L)=\int_{\Gamma_+}e^{-zL}\eta_+(z)dz+\int_{\Gamma_-}e^{-zL}\eta_-(z)dz,$$

and

$$\eta_\pm(z)=\int_{\gamma_\pm}e^{\zeta z}H_{t,r}(\zeta)d\zeta.$$

Here $\Gamma_\pm$ is the half-ray $\mathbb{R}^+e^{\pm i(\pi/2-\theta)}$ and $\gamma_\pm$ is the half-ray $\mathbb{R}^\pm e^{\pm i\omega}$.

Then for any $z\in\Gamma_\pm$,

$$\begin{aligned}
|\eta_\pm(z)| &= \left|\int_{\gamma_\pm}e^{\zeta z}\rho(t)\zeta e^{-\rho(t)\zeta}(1-e^{-\rho(r)\zeta})^N d\zeta\right| \\
&\le \int_{\gamma_\pm}|e^{\zeta z-\rho(t)\zeta}|\rho(t)|\zeta||1-e^{-\rho(r)\zeta}|^N|d\zeta| \\
&\le \int_{\gamma_\pm}e^{-c|\zeta|(|z|+\rho(t))}\rho(t)|\zeta||1-e^{-\rho(r)\zeta}|^N|d\zeta| \\
&\le C\int_0^\infty e^{-cs(|z|+\rho(t))}\rho(t)\rho^N(r)s^{N+1}ds\le\frac{C\rho(t)\rho^N(r)}{(|z|+\rho(t))^{N+2}}.
\end{aligned}$$

In the second inequality, the constant $c>0$ depends on θ and ω. Indeed,

$$\Re(\zeta z)=|\zeta||z|\Re e^{\pm i(\pi/2-\theta+\omega)}.$$

Since $\theta<\omega$, then $\pi/2<\pi/2-\theta+\omega<\pi$ and $|e^{\zeta z}|=e^{-c_1|\zeta||z|}$ with $c_1=-\cos(\pi/2-\theta+\omega)$. Also it is obvious to see that $|e^{\rho(t)\zeta}|=e^{-c_2\rho(t)|\zeta|}$. Thus the second inequality follows. In the third inequality, let $\zeta=se^{\pm i\omega}$, we have $|d\zeta|=ds$. In addition, we dominate $|1-e^{-\rho(r)\zeta}|^N$ by $(\rho(r)\zeta)^N$.

We choose θ appropriately such that $|z| \sim \Re z$ for $z \in \Gamma_{\pm}$, then for any $j \geq 2$ fixed,

$$
\begin{aligned}
& \|H_{t,r_i}(L)b_i\|_{L^2(C_j(B_i))} \\
\leq\ & C\left(\int_{\Gamma_+} + \int_{\Gamma_-}\right)\left\|e^{-\Re z L} b_i\right\|_{L^2(C_j(B_i))} \frac{\rho(t)}{(|z|+\rho(t))^2} \frac{\rho^N(r_i)}{(|z|+\rho(t))^N}|dz| \\
\leq\ & C\int_0^\infty \left\|e^{-sL} b_i\right\|_{L^2(C_j(B_i))} \frac{\rho(t)\rho^N(r_i)}{(s+\rho(t))^{N+2}} ds.
\end{aligned}
$$

Applying (3.3) with $u = p_0$ and $v = 2$, then

$$
\begin{aligned}
& \|H_{t,r_i}(L)b_i\|_{L^2(C_j(B_i))} \\
& \leq \frac{C2^{j\nu}\|b_i\|_{p_0}}{\mu^{\frac{1}{p_0}-\frac{1}{2}}(B_i)} \int_0^\infty e^{-c\left(\frac{2^j r_i}{\rho^{-1}(s)}\right)^{\tau(s)}} \frac{\rho(t)\rho^N(r_i)}{(s+\rho(t))^{N+2}} ds \\
& \leq \frac{C2^{j\nu}\|b_i\|_{p_0}}{\mu^{\frac{1}{p_0}-\frac{1}{2}}(B_i)} \left(\int_0^{\rho(t)} + \int_{\rho(t)}^\infty\right) e^{-c\left(\frac{2^j r_i}{\rho^{-1}(s)}\right)^{\tau(s)}} \frac{\rho(t)\rho^N(r_i)}{(s+\rho(t))^{N+2}} ds \\
& := \frac{C2^{j\nu}\|b_i\|_{p_0}}{\mu^{\frac{1}{p_0}-\frac{1}{2}}(B_i)} (H_1(t,r_i,j) + H_2(t,r_i,j)).
\end{aligned} \tag{3.19}
$$

In the second and the third lines, $\tau(s)$ is originally defined in (3.16). In fact, it should be $\tau(\rho^{-1}(s))$. Since $\rho^{-1}(s)$ and s are unanimously larger or smaller than one, we always have $\tau(s) = \tau(\rho^{-1}(s))$.

Hence, by Minkowski inequality, we get from (3.18) and (3.19)

$$
I_{ij} \leq \frac{C2^{j\nu}\|b_i\|_{p_0}}{\mu^{\frac{1}{p_0}-\frac{1}{2}}(B_i)} \left(\left(\int_0^\infty H_1^2(t,r_i,j)\frac{dt}{t}\right)^{1/2} + \left(\int_0^\infty H_2^2(t,r_i,j)\frac{dt}{t}\right)^{1/2}\right). \tag{3.20}
$$

It remains to estimate the two integrals $\int_0^\infty H_1^2(t,r_i,j)\frac{dt}{t}$ and $\int_0^\infty H_2^2(t,r_i,j)\frac{dt}{t}$ separately.

Claim 1:

$$
\int_0^\infty H_1(t,r_i,j)^2\frac{dt}{t} \leq C2^{-2\beta_1 N j}. \tag{3.21}
$$

Estimate first $H_1(t,r_i,j)$. We control $\frac{\rho(t)\rho^N(r_i)}{(s+\rho(t))^{N+2}}$ by $\frac{\rho^N(r_i)}{\rho(t)^{N+1}}$. Thus for $0 < t < 1$, we have

$$
H_1(t,r_i,j) \leq \int_0^{t^{\beta_1}} e^{-c\left(\frac{2^{j\beta_1} r_i^{\beta_1}}{s}\right)^{1/(\beta_1-1)}} \frac{\rho^N(r_i)}{t^{\beta_1(N+1)}} ds \leq Ce^{-c\left(\frac{2^j r_i}{t}\right)^{\beta_1/(\beta_1-1)}} \frac{\rho^N(r_i)}{t^{\beta_1 N}}, \tag{3.22}
$$

For $t \geq 1$,

$$
H_1(t,r_i,j) \quad \leq \quad \int_0^1 e^{-c\left(\frac{2^{j\beta_1} r_i^{\beta_1}}{s}\right)^{1/(\beta_1-1)}} \frac{\rho^N(r_i)}{t^{\beta_1(N+1)}} ds + \int_1^{t^{\beta_2}} e^{-c\left(\frac{2^{j\beta_2} r_i^{\beta_2}}{s}\right)^{1/(\beta_2-1)}} \frac{\rho^N(r_i)}{t^{m(N+1)}} ds
$$

$$
\begin{aligned}
&\leq C\left(e^{-c2^{j\beta_1}r_i^{\beta_1}}+e^{-c\left(\frac{2^{j\beta_2}r_i^{\beta_2}}{t^{\beta_2}}\right)^{1/(\beta_2-1)}}\right)\frac{\rho^N(r_i)}{t^{\beta_2 N}}.\\
&\leq Ce^{-c\left(\frac{2^{j\beta_2}r_i^{\beta_2}}{t_2^{\beta}}\right)^{1/(\beta_2-1)}}\frac{\rho^N(r_i)}{t^{\beta_2 N}}.
\end{aligned}
$$

To summarise, we get

$$
H_1(t,r_i,j)\leq\begin{cases}Ce^{-c\left(\frac{2^j r_i}{t}\right)^{\beta_1/(\beta_1-1)}}\dfrac{\rho^N(r_i)}{t^{\beta_1 N}}, & \text{for } t\in(0,1),\\ Ce^{-c\left(\frac{2^{j\beta_2}r_i^{\beta_2}}{t_2^{\beta}}\right)^{1/(\beta_2-1)}}\dfrac{\rho^N(r_i)}{t^{\beta_2 N}}, & \text{for } t\geq 1.\end{cases} \tag{3.23}
$$

In order to get (3.21), we consider three cases separately. That is, $r_i<2^{-j}$, $2^{-j}\leq r_i<1$ and $r_i\geq 1$.

Firstly for $r_i<2^{-j}$, then $\rho(r_i)=r_i^{\beta_1}$. We divide the integral from 0 to infinity into three parts as follows

$$
\begin{aligned}
&\int_0^\infty H_1^2(t,r_i,j)\frac{dt}{t}\\
=\;&\int_0^{2^j r_i}H_1^2(t,r_i,j)\frac{dt}{t}+\int_{2^j r_i}^1 H_1^2(t,r_i,j)\frac{dt}{t}+\int_1^\infty H_1^2(t,r_i,j)\frac{dt}{t}\\
\leq\;& C\int_0^{2^j r_i}\left(\frac{t}{2^j r_i}\right)^c\frac{r_i^{2\beta_1 N}}{t^{2\beta_1 N}}\frac{dt}{t}+C\int_{2^j r_i}^1\frac{r_i^{2\beta_1 N}}{t^{2\beta_1 N}}\frac{dt}{t}+C\int_1^\infty\frac{r_i^{2\beta_1 N}}{t^{2\beta_2 N}}\frac{dt}{t}\\
\leq\;& C2^{-cj}r_i^{2\beta_1 N-c}\int_0^{2^j r_i}t^{c-2\beta_1 N}\frac{dt}{t}+Cr_i^{2\beta_1 N}\int_{2^j r_i}^\infty t^{-2\beta_1 N}\frac{dt}{t}+Cr_i^{2\beta_1 N}\int_1^\infty t^{-2\beta_2 N}\frac{dt}{t}\\
\leq\;& C2^{-2\beta_1 j}+C2^{-2\beta_1 j}+Cr_i^{2\beta_1 N}\leq C2^{-2\beta_1 j}.
\end{aligned}
$$

In the first integral, we use the first estimate of $H_1(t,r_i,j)$ in (3.23). We also dominate the exponential term by a polynomial one, where c in the second line is chosen to be larger than $2mN$. In fact, for any fixed $L>0$, $x^L e^{-cx}$ is uniformly bounded for $x>0$. In the second and the third integrals, we use the third bound in (3.23).

Secondly for r_i satisfying $2^{-j}\leq r_i<1$. Then $\rho(r_i)=r_i^{\beta_1}$ and $2^j r_i\geq 1$. We divide the integral into three different parts compared with the first case. That is,

$$
\begin{aligned}
&\int_0^\infty H_1^2(t,r_i,j)\frac{dt}{t}\\
=\;&\int_0^1 H_1^2(t,r_i,j)\frac{dt}{t}+\int_1^{2^j r_i}H_1^2(t,r_i,j)\frac{dt}{t}+\int_{2^j r_i}^\infty H_1^2(t,r_i,j)\frac{dt}{t}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \; C\int_0^1\left(\frac{t}{2^jr_i}\right)^c\frac{r_i^{2\beta_1N}}{t^{2\beta_1N}}\frac{dt}{t}+C\int_1^{2^jr_i}\left(\frac{t}{2^jr_i}\right)^c\frac{r_i^{2\beta_1N}}{t^{2\beta_2N}}\frac{dt}{t}+C\int_{2^jr_i}^\infty\frac{r_i^{2\beta_1N}}{t^{2\beta_2N}}\frac{dt}{t}\\
&\leq \; C2^{-cj}r_i^{2\beta_1N-c}\int_0^1 t^{c-2\beta_1N}\frac{dt}{t}+C2^{-cj}r_i^{2\beta_1N-c}\int_0^{2^jr_i}t^{c-2\beta_2N}\frac{dt}{t}+Cr_i^{2\beta_1N}\int_{2^jr_i}^\infty t^{-2\beta_2N}\frac{dt}{t}\\
&\leq \; C2^{-cj}r_i^{2\beta_1N-c}+C2^{-2\beta_2Nj}r_i^{2\beta_1N-2\beta_2N}+C2^{-2\beta_2Nj}r_i^{2\beta_1N-2\beta_2N}\\
&\leq \; C2^{-cj}2^{-(2\beta_1N-c)j}+C2^{-2\beta_2Nj}2^{-(2\beta_1N-2\beta_2N)j}\leq C2^{-2\beta_1Nj}.
\end{aligned}
$$

In the first and the second integrals, we take the first and the seconds estimates of $H_1(t,r_i,j)$ in (3.23) respectively. Then we use the same trick to control exponential terms. The constant c is the same as before. In the last integrals, we use the third bound in (3.23).

Finally for $r_i\geq 1$. Obviously $2^jr_i\geq 1$. The estimate is similar to the second case. We make the same division of the integral and the same estimate for the same part. Then

$$
\begin{aligned}
&\int_0^\infty H_1^2(t,r_i,j)\frac{dt}{t}\\
&\leq \; C\int_0^1\left(\frac{t}{2^jr_i}\right)^c\frac{r_i^{2\beta_2N}}{t^{2\beta_1N}}\frac{dt}{t}+C\int_1^{2^jr_i}\left(\frac{t}{2^jr_i}\right)^c\frac{r_i^{2\beta_2N}}{t^{2\beta_2N}}\frac{dt}{t}+C\int_{2^jr_i}^\infty\frac{r_i^{2\beta_2N}}{t^{2\beta_2N}}\frac{dt}{t}\\
&\leq \; C2^{-cj}r_i^{2\beta_2N-c}\int_0^1 t^{c-2\beta_1N}\frac{dt}{t}+C2^{-cj}r_i^{2\beta_2N-c}\int_0^{2^jr_i}t^{c-2\beta_2N}\frac{dt}{t}+Cr_i^{2\beta_2N}\int_{2^jr_i}^\infty t^{-2\beta_2N}\frac{dt}{t}\\
&\leq \; C2^{-cj}r_i^{2\beta_2N-c}+C2^{-2\beta_2Nj}+C2^{-2\beta_2Nj}\leq C2^{-2\beta_1Nj}.
\end{aligned}
$$

Here c is chosen to be larger than $2\beta_2N$.

So we get (3.21) for all cases.

Claim 2:

$$
\int_0^\infty H_2^2(t,r_i,j)\frac{dt}{t}\leq C2^{-2\beta_1Nj}. \tag{3.24}
$$

Now estimate $H_2(t,r_i,j)$. Write $\frac{\rho(t)\rho^N(r_i)}{(s+\rho(t))^{N+2}}\leq\frac{\rho(t)\rho^N(r_i)}{s^{N+2}}$. On the one hand,

$$
H_2(t,r_i,j)=\int_{\rho(t)}^\infty e^{-c\left(\frac{2^jr_i}{\sigma(s)}\right)^{\tau(s)}}\frac{\rho(t)\rho^N(r_i)}{(s+\rho(t))^{N+2}}ds\leq\int_{\rho(t)}^\infty\frac{\rho(t)\rho^N(r_i)}{s^{N+2}}ds=C\frac{\rho^N(r_i)}{\rho^N(t)}. \tag{3.25}
$$

On the other hand, we also have

$$
H_2(t,r_i,j)\leq\begin{cases} C2^{-\beta(r_i)Nj}\left(\frac{t}{2^jr_i}\right)^{\beta_1} & \text{for}\quad 0<t<1,\\ C2^{-\beta(r_i)Nj}\left(\frac{t}{2^jr_i}\right)^{\beta_2} & \text{for}\quad t\geq 1.\end{cases} \tag{3.26}
$$

where β is defined in (3.37). That is, $\beta(t)=\beta_1$ if $0<t<1$, or else $\beta(t)=\beta_2$. Note that $\rho(r_i)=r_i^{\beta(r_i)}$.

In fact, for $t \geq 1$, note that $s^{-1-N} \leq s^{-(1+\beta(r_i)N/\beta_2)}$ for $s \geq 1$, then

$$\begin{aligned} H_2(t,r_i,j) &\leq \int_{t^{\beta_2}}^{\infty} e^{-c\left(\frac{2^{j\beta_2}r_i^{\beta_2}}{s}\right)^{1/(\beta_2-1)}} \frac{t^{\beta_2}\rho^N(r_i)}{s^{N+1}}\frac{ds}{s} \\ &\leq \int_{t^{\beta_2}}^{\infty} e^{-c\left(\frac{2^{j\beta_2}r_i^{\beta_2}}{s}\right)^{1/(\beta_2-1)}} \frac{t^{\beta_2}\rho^N(r_i)}{s^{1+\beta(r_i)N/\beta-2}}\frac{ds}{s} \\ &\leq C\frac{t^{\beta_2}\rho^N(r_i)}{(2^{j\beta_2}r_i^{\beta_2})^{1+\beta(r_i)N/\beta_2}}\int_0^{\infty} e^{-cu}u^{(\beta_2-1)(1+\beta(r_i)N/\beta_2)-1}du \\ &\leq C2^{-\beta(r_i)Nj}\left(\frac{t}{2^jr_i}\right)^{\beta_2}. \end{aligned}$$

For $t<1$, since $s^{-1-N} \leq s^{-(1+\beta(r_i)N/\beta_1)}$ for $s<1$ and $s^{-1-N} \leq s^{-(\beta_1+\beta(r_i)N)/\beta_2}$ for $s \geq 1$, then

$$\begin{aligned} &H_2(t,r_i,j) \\ \leq\ & \int_{t^{\beta_1}}^{\infty} e^{-c\left(\frac{2^jr_i}{\sigma(s)}\right)^{\tau(s)}} \frac{t^{\beta_1}\rho^N(r_i)}{s^{N+1}}\frac{ds}{s} \\ \leq\ & \int_{t^{\beta_1}}^{1} e^{-c\left(\frac{2^{j\beta_1}r_i^{\beta_1}}{s}\right)^{1/(\beta_1-1)}} \frac{t^{\beta_1}\rho^N(r_i)}{s^{1+\beta(r_i)N/\beta_1}}\frac{ds}{s} + \int_1^{\infty} e^{-c\left(\frac{2^{j\beta_2}r_i^{\beta_2}}{s}\right)^{1/(\beta_2-1)}} \frac{t^{\beta_1}\rho^N(r_i)}{s^{(\beta_1+\beta(r_i)N)/m}}\frac{ds}{s} \\ \leq\ & C2^{-\beta(r_i)Nj}\left(\frac{t}{2^jr_i}\right)^{\beta_1}\left(\int_0^{\infty} e^{-cu}u^{\beta(r_i)N/\beta_1}du + \int_0^{\infty} e^{-cv}v^{(\beta_2-1)(\beta_1+\beta(r_i)N)/\beta_2-1}dv\right) \\ \leq\ & C2^{-\beta(r_i)Nj}\left(\frac{t}{2^jr_i}\right)^{\beta_1}. \end{aligned}$$

Combining (3.25) and (3.26), we get

$$H_2(t,r_i,j) \leq C\inf\left\{\frac{\rho^N(r_i)}{\rho^N(t)}, 2^{-\beta(r_i)Nj}\left(\frac{t}{2^jr_i}\right)^{\beta(t)}\right\}. \tag{3.27}$$

Let us show the estimate (3.24). Similarly as the estimate for $\int_0^\infty H_1^2(t,r_i,j)\frac{dt}{t}$, we also consider three cases: $r_i<2^{-j}$, $2^{-j} \leq r_i < 1$ and $r_i \geq 1$. Moreover, we split the integral in the same way as for $H_1(t,r_i,j)$ to treat each case.

If $r_i<2^{-j}$, then $\rho(r_i)=r_i^{\beta_1}$ and $\beta(r_i)=\beta_1$. We get

$$\begin{aligned} \int_0^{\infty} H_2^2(t,r_i,j)\frac{dt}{t} &= \int_0^{2^jr_i} H_2^2(t,r_i,j)\frac{dt}{t} + \int_{2^jr_i}^{1} H_2^2(t,r_i,j)\frac{dt}{t} + \int_1^{\infty} H_2^2(t,r_i,j)\frac{dt}{t} \\ &\leq C2^{-2\beta_1Nj}\int_0^{2^jr_i}\left(\frac{t}{2^jr_i}\right)^{\beta_1}\frac{dt}{t} + C\int_{2^jr_i}^{1}\frac{r_i^{2\beta_1N}}{t^{2\beta_1N}}\frac{dt}{t} + C\int_1^{\infty}\frac{r_i^{2\beta_1N}}{t^{2\beta_2N}}\frac{dt}{t} \end{aligned}$$

$$\leq \quad C2^{-2\beta_1 Nj} + C2^{-2\beta_1 Nj} + Cr_i^{42\beta_1 N} \leq C2^{-2\beta_1 Nj}.$$

If $2^{-j} \leq r_i < 1$, we have $\rho(r_i) = r_i^{\beta_1}$, $\beta(r_i) = \beta_1$ and $2^j r_i \geq 1$. Then

$$\begin{aligned}
&\int_0^\infty H_2^2(t, r_i, j)\frac{dt}{t} \\
= \;& \int_0^1 H_2^2(t, r_i, j)\frac{dt}{t} + \int_1^{2^j r_i} H_2^2(t, r_i, j)\frac{dt}{t} + \int_{2^j r_i}^\infty H_2^2(t, r_i, j)\frac{dt}{t} \\
\leq \;& C2^{-2\beta_1 Nj}\int_0^1 \left(\frac{t}{2^j r_i}\right)^{\beta_1}\frac{dt}{t} + C2^{-2\beta_1 Nj}\int_1^{2^j r_i}\left(\frac{t}{2^j r_i}\right)^{\beta_2}\frac{dt}{t} + C\int_{2^j r_i}^\infty \frac{r_i^{2\beta_1 N}}{t^{2\beta_2 N}}\frac{dt}{t} \\
\leq \;& C2^{-2\beta_1 Nj} + C2^{-2\beta_1 Nj} + C2^{-2\beta_2 Nj} r_i^{2\beta_1 N - 2\beta_2 N} \leq C2^{-2\beta_1 Nj}.
\end{aligned}$$

Here the last inequality holds since $r_i \geq 2^{-j}$.

If $r_i \geq 1$, we have $\rho(r_i) = r_i^m$, $\beta(r_i) = m$ and $2^j r_i \geq 1$. Then

$$\begin{aligned}
&\int_0^\infty H_2^2(t, r_i, j)\frac{dt}{t} \\
= \;& \int_0^1 H_2^2(t, r_i, j)\frac{dt}{t} + \int_1^{2^j r_i} H_2^2(t, r_i, j)\frac{dt}{t} + \int_{2^j r_i}^\infty H_2^2(t, r_i, j)\frac{dt}{t} \\
\leq \;& C2^{-2\beta_2 Nj}\int_0^1 \left(\frac{t}{2^j r_i}\right)^{\beta_1}\frac{dt}{t} + C2^{-2\beta_2 Nj}\int_1^{2^j r_i}\left(\frac{t}{2^j r_i}\right)^{\beta_2}\frac{dt}{t} + C\int_{2^j r_i}^\infty \frac{r_i^{2\beta_2 N}}{t^{2\beta_2 N}}\frac{dt}{t} \\
\leq \;& C2^{-2\beta_2 Nj} + C2^{-2\beta_2 Nj} + C2^{-2\beta_2 Nj} \leq C2^{-2\beta_1 Nj}.
\end{aligned}$$

Thus (3.24) is true for every case.

Therefore, it follows from(3.20), (3.21) and (3.24)

$$I_{ij} \leq \frac{C2^{j\nu}\|b_i\|_{p_0}}{\mu^{\frac{1}{p_0}-\frac{1}{2}}(B_i)}2^{-\beta_1 Nj}. \tag{3.28}$$

Now for the integral $\left(\int_0^\infty \int_{C_j(B_i)} |h(y,t)|^2 \frac{d\mu(y)dt}{t}\right)^{1/2}$. Take $\tilde{h}(y) = \int_0^\infty |h(y,t)|^2 \frac{dt}{t}$, then

$$\left(\int_0^\infty \int_{C_j(B_i)} |h(y,t)|^2 \frac{d\mu(y)dt}{t}\right)^{1/2} \leq \mu^{1/2}(2^{j+1}B_i) \operatorname*{ess\,inf}_{z\in B_i} \mathscr{M}^{1/2}\tilde{h}(z), \tag{3.29}$$

where $\mathscr{M}$ is the Hardy-Littlewood maximal function in (2.21).

Using the same strategy as the proof of (3.15), we get from (3.28) and (3.29) that

$$\begin{aligned}
\Lambda_{glob}^{1/2} \;\; & \leq \;\; \sup_h \sum_i \sum_{j\geq 2} \frac{C2^{j\nu}\mu^{1/2}(2^j B_i)\|b_i\|_{p_0}}{\mu^{\frac{1}{p_0}-\frac{1}{2}}(B_i)} 2^{-\beta_1 Nj} \operatorname*{ess\,inf}_{z\in B_i} \mathscr{M}^{1/2}\tilde{h}(z) \\
& \leq \;\; C\lambda \sup_h \int_M \sum_i \mathbb{1}_{B_i}(y)\mathscr{M}^{1/2}\tilde{h}(y)d\mu(y)
\end{aligned}$$

$$\leq C\lambda \sup_h \int_{\cup_i B_i} \mathscr{M}^{1/2}\tilde{h}(y)d\mu(y)$$
$$\leq C\lambda\mu(\cup_i B_i)^{1/2} \leq C\lambda^{1-p_0/2}\left(\int |f|^{p_0}d\mu\right)^{1/2}.$$

Here the supreme is taken over all the functions h with $\|h\|_{L^2\left(\frac{d\mu(y)dt}{t}\right)} = 1$. Since $N > 2\nu/\beta_1$, the sum $\sum_{j\geq 2} 2^{-\beta_1 Nj+3\nu j/2}$ converges and we get the second inequality. The fourth one is a result of Kolmogorov's inequality.

Thus we have shown $\Lambda_{glob} \leq C\lambda^{2-p_0}\int |f|^{p_0}d\mu$.

□

Remark 3.18. Note that in [Uhl11], Uhl considered the Littlewood-Paley-Stein $g^*_{\lambda,\psi}-$function adapted to the operator instead of the square function S_ψ. Indeed, the two functionals behave similarly due to the doubling volume property.

3.4.2 $H^p_{\Delta,S_h}(M) \neq L^p(M)$

Theorem 3.19. *Let M be a Riemannian manifold with polynomial volume growth*

$$V(x,r) \simeq r^d,\ r\geq 1, \tag{3.30}$$

as well as two-sided sub-Gaussian heat kernel estimate ($UE_{2,m}$) and ($LE_{2,m}$) (see Section 2.1), where $2<m<d/2$. Then $L^p(M)\subset H^p_{\Delta,S_h}(M)$ doesn't hold for $p\in\left(\frac{d}{d-m},2\right)$.

More generally, a slight adaption of Theorem 3.19 plus Theorem 3.16 yields the following result.

Theorem 3.20. *Let M be be a Riemannian manifold as above. Then for any $0<m'\leq m$ and for any $p\in\left(\frac{d}{d-m},2\right)$, $L^p(M)=H^p_{S_h^{m'}}(M)$ if and only if $m'=m$.*

Before moving forward to the proof of Theorem 3.19, let us recall the following two theorems about the Sobolev inequality and the Green operator.

Theorem 3.21 ([Cou90])**.** *Let (M,μ) be a $\sigma-$finite measure space. Let T_t be a semigroup on L^s, $1\leq s\leq\infty$, with infinitesimal generator $-L$. Assume that T_t is equicontinuous on L^1 and L^∞. Then the following two conditions are equivalent:*

1. *There exists $C>0$ such that $\|T_t\|_{1\to\infty}\leq Ct^{-D/2}$, $\forall t\geq 1$.*
2. *T_1 is bounded from L^1 to L^∞ and for $q>1$, $\exists C$ such that*

$$\|f\|_p \leq C\left(\left\|L^{\alpha/2}f\right\|_q + \left\|L^{\alpha/2}f\right\|_p\right),\ f\in\mathscr{D}(L^{\alpha/2}), \tag{3.31}$$

where $0<\alpha q<D$ and $\frac{1}{p}=\frac{1}{q}-\frac{\alpha}{D}$.

Theorem 3.22 ([Li12]). *Let M be a complete manifold without boundary. There exists a Green's function $G(x,y)$ which is smooth on $(M\times M)\backslash D$ satisfying*

$$\Delta_x \int_M G(x,y)f(y)d\mu(y) = f(x),\ \forall f\in \mathscr{C}_0^\infty(M). \tag{3.32}$$

We also observe that

Lemma 3.23. *Let M be as above. Let B be an arbitrary ball with radius $r\geq 4$. Then there exists a constant $c>0$ depending on d and m such that for all t with $r^m/2\leq t\leq r^m$,*

$$\int_B p_t(x,y)d\mu(y)\geq c,\ \forall x\in B.$$

Proof. Note that for any $x,y\in B$, we have $t\geq r^m/2\geq 2r\geq d(x,y)$. Then we have

$$\begin{aligned}\int_B p_t(x,y)d\mu(y) &\geq \int_B \frac{c}{t^{d/m}}\exp\left(-C\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}\right)d\mu(y)\\ &\geq \frac{cV(B)}{t^{d/m}}\exp\left(-C\left(\frac{r^m}{t}\right)^{1/(m-1)}\right)\geq c.\end{aligned}$$

□

Proof of Theorem 3.19

Let $\phi_n\in\mathscr{C}_0^\infty(M)$ be a cut-off function as follows: $0\leq\phi_n\leq 1$ and for some $x_0\in M$,

$$\phi_n(x)=\begin{cases}1, & x\in B(x_0,n),\\ 0, & x\in M\backslash B(x_0,2n).\end{cases}$$

For simplicity, we denote $B(x_0,n)$ by B_n.

Taking $f_n=G\phi_n$, Theorem 3.22 says that $\Delta f_n=\phi_n$.

On the one hand, we apply Theorem 3.21 by choosing $T_t=e^{-t\Delta}$. Indeed, $e^{-t\Delta}$ is Markov hence bounded on L^p, equicontinuous on L^1,L^∞ and satisfies

$$\left\|e^{-t\Delta}\right\|_{1\to\infty}=\sup_{x,y\in M} p_t(x,y)\leq Ct^{-D/2},$$

where $D=2d/m>2$. Then taking $\alpha=2$ and $p>\frac{D}{D-2}$, it follows that

$$\|f_n\|_p\leq C\Big(\|\Delta f_n\|_q+\|\Delta f_n\|_p\Big),$$

where $\frac{1}{p}=\frac{1}{q}-\frac{\alpha}{D}$, that is, $q=\frac{Dp}{D+2p}=\frac{dp}{d+mp}$.

Using the fact that $\Delta f_n = \phi_n$ and $\phi_n \le \mathbb{1}_{B(x_0,2n)}$, we get

$$\begin{aligned}\|f_n\|_p &\le C\Big(\|\phi_n\|_{\frac{dp}{d+mp}} + \|\phi_n\|_p\Big)\\ &\le C\Big(V^{\frac{d+mp}{dp}}(x_0,2n) + V^{\frac{1}{p}}(x_0,2n)\Big)\\ &\le C\Big(n^{m+d/p} + n^{d/p}\Big)\\ &\le Cn^{m+d/p}.\end{aligned} \tag{3.33}$$

In particular, $\|f_n\|_2 \le Cn^{m+d/2}$.

On the other hand,

$$\begin{aligned}\|S_h f_n\|_p^p &= \int_M \left(\iint_{\Gamma(x)} \left|t^2 \Delta e^{-t^2\Delta} f_n(y)\right|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\right)^{p/2} d\mu(x)\\ &= \int_M \left(\iint_{\Gamma(x)} \left(t^2 e^{-t^2\Delta} \phi_n(y)\right)^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\right)^{p/2} d\mu(x).\end{aligned}$$

Since $\phi_n \ge \mathbb{1}_{B_n} \ge 0$, it follows from the Markovian property of the heat semigroup that

$$\|S_h f_n\|_p^p \ge \int_M \left(\iint_{\Gamma(x)} \left(t^2 e^{-t^2\Delta} \mathbb{1}_{B_n}(y)\right)^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\right)^{p/2} d\mu(x).$$

By using Lemma 3.23, it holds that $e^{-t^2\Delta}\mathbb{1}_{B_{n/2}} \ge c$ if $\frac{n^{m/2}}{2} \le t \le n^{m/2}$. Then we get

$$\|S_h f_n\|_p^p \ge \int_{B\left(x_0,\frac{n^{m/2}}{4}\right)} \left(\int_{\frac{n^{m/2}}{2}}^{n^{m/2}} \int_{B(x,t)\cap B_{n/2}} \frac{Ct^3}{V(x,t)} d\mu(y)dt\right)^{p/2} d\mu(x).$$

Observe also that, for $t > \frac{n^{m/2}}{2}$ and $x \in B\left(x_0, \frac{n^{m/2}}{4}\right)$, we have $B_n \subset B(x,t)$ as long as n is large enough. Then the volume growth (3.30) gives us a lower bound in terms of n. That is,

$$\|S_h f_n\|_p^p \ge \int_{B\left(x_0,\frac{n^{m/2}}{4}\right)} \left(\int_{\frac{n^{m/2}}{2}}^{n^{m/2}} \frac{C\mu(B_n)t^3}{V(x,n^{m/2})} dt\right)^{p/2} d\mu(x) \ge Cn^{\frac{md}{2}(1-\frac{p}{2})} n^{mp+dp/2}.$$

Comparing the upper bound of $\|f_n\|_p$ in (3.33) for $p > \frac{D}{D-2}$, we obtain

$$\|S_h f_n\|_p \ge Cn^{\frac{md}{2}(\frac{1}{p}-\frac{1}{2})+m+\frac{d}{2}} \ge Cn^{d\left(\frac{m}{2}-1\right)\left(\frac{1}{p}-\frac{1}{2}\right)}\|f_n\|_p, \tag{3.34}$$

where $p > \frac{D}{D-2}$.

Assume $D > 4$, i.e. $m < d/2$, we have $\frac{D}{D-2} < 2$. Then for $\frac{D}{D-2} < p < 2$, since $m > 2$,

$$n^{d\left(\frac{m}{2}-1\right)\left(\frac{1}{p}-\frac{1}{2}\right)} \to \infty \text{ as } n \to \infty.$$

Thus (3.34) implies that $L^p \subset H^p_{S_h}$ is not true for $p \in \left(\frac{D}{D-2}, 2\right)$, i.e. $p \in \left(\frac{d}{d-m}, 2\right)$, where $2 < m < d/2$.

To conclude, for any fixed $p \in \left(\frac{d}{d-m}, 2\right)$, according to (3.33) and (3.34), there exists a family of functions $\left\{g_n = \frac{f_n}{n^{m+d/p}}\right\}_{n\geq 1}$ such that

$$\|g_n\|_p \leq C, \; \|g_n\|_2 \leq n^{\frac{d}{2}-\frac{d}{p}} \to 0$$

and

$$\|S_h g_n\|_p \geq n^{d(\frac{m}{2}-1)(\frac{1}{p}-\frac{1}{2})} \to +\infty$$

as n goes to infinity.

Therefore S_h is not L^p bounded for $p \in \left(\frac{d}{d-m}, 2\right)$ and the inclusion $L^p \subset H^p_{S_h^{m'}}$ doesn't hold for $p \in \left(\frac{d}{d-m}, 2\right)$. □

Proof of Theorem 3.20 If $m' = m$, Theorem 3.16 says that $L^p \subset H^p_{S_h^m}$.

Conversely, by doing a slight adjustment for the above proof, we can prove that $L^p \subset H^p_{S_h^{m'}}$ is false for $p \in \left(\frac{d}{d-m}, 2\right)$, where $2 < m < d/2$ and $m' < m$. □

3.5 Application: the $H^1_{\Delta,m}(M) - L^1(M)$ boundedness of quasi Riesz transforms on Riemannian manifolds

This section is devoted to an application of the Hardy space theory we introduced above.

Let (M, d, μ) be a Riemannian manifold satisfying the doubling volume property (D) and the sub-Gaussian estimate ($UE_{2,m}$). Note that we could as well consider a metric measure space Dirichlet space which admits a "carré du champ".

It is known that the quasi Riesz transform $\nabla e^{-\Delta}\Delta^{-\alpha}$ with $0 < \alpha < 1/2$ is of weak type $(1,1)$ (see [Che14b]). Our aim here is to prove that the quasi Riesz transform is $H^1_{\Delta,m,mol}(M) - L^1(M)$ bounded. Due to Theorem 3.12, it is $H^1_{\Delta,m}(M) - L^1(M)$ bounded.

Theorem 3.24. *Let M be a manifold satisfying the doubling volume property (D) and the heat kernel estimate ($UE_{2,m}$), $m \geq 2$. Then, for any $0 < \alpha < 1/2$, the operator $T = \nabla e^{-\Delta}\Delta^{-\alpha}$ is bounded from $H^1_{\Delta,m,\mathrm{mol}}(M)$ into $L^1(M)$, i. e., there exists a constant C such that*

$$\|Tf\|_{L^1(M)} \leq C\|f\|_{H^1_{\Delta,m,\mathrm{mol}}(M)}.$$

Proof. It suffices to show that, for any $(1,2,\varepsilon)-$molecule a associated to a function b and a ball B

with radius r_B, there exists a constant C such that $\|Ta\|_{L^1(M)} \leq C$.

Write

$$\begin{aligned} Ta &= \int_0^\infty \nabla e^{-(t+1)\Delta} a \frac{dt}{t^{1-\alpha}} = \int_0^{\rho(r_B)} \nabla e^{-(t+1)\Delta} a \frac{dt}{t^{1-\alpha}} + \int_{\rho(r_B)}^\infty \nabla e^{-(t+1)\Delta} \Delta^K b \frac{dt}{t^{1-\alpha}} \\ &:= D_1 + D_2. \end{aligned}$$

We will estimate the L^1 norm of D_1 and D_2 separately. First for D_1, we split the integral into annuli and use the Minkowski inequality as well as the Hölder inequality, then

$$\begin{aligned} \|D_1\|_{L^1(M)} &\leq \sum_{i=0}^\infty \left\| \int_0^{\rho(r_B)} \nabla e^{-(t+1)\Delta} \mathbb{1}_{C_i(B)} a \frac{dt}{t^{1-\alpha}} \right\|_{L^1(C_j(2^iB))} \\ &\leq \sum_{i=0}^\infty \sum_{j=0}^\infty \left\| \int_0^{\rho(r_B)} \nabla e^{-(t+1)\Delta} \mathbb{1}_{C_i(B)} a \frac{dt}{t^{1-\alpha}} \right\|_{L^1(M)} \\ &\leq \sum_{i=0}^\infty \sum_{j=0}^\infty \int_0^{\rho(r_B)} \left\| \nabla e^{-(t+1)\Delta} \mathbb{1}_{C_i(B)} a \right\|_{L^1(C_j(2^iB))} \frac{dt}{t^{1-\alpha}} \\ &\leq \sum_{i=0}^\infty \sum_{j=0}^\infty \int_0^{\rho(r_B)} \left\| \nabla e^{-(t+1)\Delta} \mathbb{1}_{C_i(B)} a \right\|_{L^2(C_j(2^iB))} \mu^{1/2}(2^{i+j+1}B) \frac{dt}{t^{1-\alpha}}. \end{aligned}$$

Fix $i \in \mathbb{N}$. For j small, i.e., $j = 0,1,2$, note the fact $\left\|\nabla e^{-t\Delta}\right\|_{2\to 2} \leq Ct^{-1/2}$, then

$$\begin{aligned} &\int_0^{\rho(r_B)} \left\| \nabla e^{-(t+1)\Delta} \mathbb{1}_{C_i(B)} a \right\|_{L^2(C_j(2^iB))} \mu^{1/2}(2^{i+j}B) \frac{dt}{t^{1-\alpha}} \\ &\leq C \int_0^{\rho(r_B)} (t+1)^{-1/2} \|a\|_{L^2(C_i(B))} \mu^{1/2}(2^{i+j}B) \frac{dt}{t^{1-\alpha}} \\ &\leq C 2^{-\varepsilon i} \int_0^{\rho(r_B)} (t+1)^{-1/2} \frac{dt}{t^{1-\alpha}} \leq C 2^{-\varepsilon i}. \end{aligned} \tag{3.35}$$

Note that the last inequality holds for all r_B. In fact, if $r_B \geq 1$, then

$$\int_0^{\rho(r_B)} (t+1)^{-1/2} \frac{dt}{t^{1-\alpha}} \leq \int_0^1 \frac{dt}{t^{1-\alpha}} + \int_1^{r_B^m} \frac{dt}{t^{3/2-\alpha}} \leq C.$$

While if $r_B < 1$, we have

$$\int_0^{\rho(r_B)} (t+1)^{-1/2} \frac{dt}{t^{1-\alpha}} \leq \int_0^{r_B^2} \frac{dt}{t^{1-\alpha}} \leq C r_B^{2\alpha} \leq C.$$

For $j \geq 3$, we will use the $L^1 - L^2$ off-diagonal estimate for the gradient of the heat kernel, say

Corollary 2.7. Denote $D_1^{ij}=\left\|\nabla e^{-(t+1)\Delta}\mathbb{1}_{L^2(C_i(B))}a\right\|_{C_j(2^iB)}$, we get

$$\begin{aligned}D_1^{ij} &\leq \frac{C_\alpha}{(t+1)^\alpha\mu^{1/2}(2^{i+j}B)}e^{-c\left(\frac{2^{(i+j)m}r^m}{t+1}\right)^{1/(m-1)}}\|a\|_{L^1(C_i(B))}\\ &\leq 2^{-\varepsilon i}\frac{C}{(t+1)^\alpha\mu^{1/2}(2^{i+j}B)}e^{-c\left(\frac{2^{(i+j)m}r^m}{t+1}\right)^{1/(m-1)}}.\end{aligned}$$

Then we have

$$\begin{aligned}&\int_0^{\rho(r_B)}D_1^{ij}\frac{dt}{t^{1-\alpha}}\cdot\mu^{1/2}(2^{i+j}B)\\ &=2^{-\varepsilon i}\int_0^{\rho(r_B)}(t+1)^{-\alpha}\exp\left(-c\left(\frac{2^{(i+j)m}r_B^m}{t+1}\right)^{1/(m-1)}\right)\frac{dt}{t^{1-\alpha}}\\ &\leq C2^{-\varepsilon i}\int_0^{\rho(r_B)}(t+1)^{-\alpha}\left(\frac{t+1}{2^{(i+j)m}r_B^m}\right)^{\alpha\beta(r_B)/m}\frac{dt}{t^{1-\alpha}}\\ &\leq C2^{-\varepsilon i}(2^{(i+j)}r_B)^{-\alpha\beta(r_B)}\int_0^{\rho(r_B)}\frac{dt}{t^{1-\alpha}}\leq C2^{-\varepsilon i}2^{-\alpha\beta(i+j)}.\end{aligned} \tag{3.36}$$

Here β is defined by

$$\beta(r)=\begin{cases}2, & 0<r<1,\\ m, & r\geq 1;\end{cases} \tag{3.37}$$

And the last inequality holds since $\alpha\beta(r_B)/m\leq\alpha$, which indicates $(t+1)^{-\alpha+\alpha\beta(r_B)/m}\leq 1$.

Hence, from (3.35) and (3.36) we get

$$\|D_1\|_{L^1(M)}\leq C\sum_{i=0}^\infty\sum_{j=0}^2 2^{-\varepsilon i}+C\sum_{i=0}^\infty\sum_{j=3}^\infty 2^{-\varepsilon i}2^{-\alpha\beta(r_B)(i+j)}\leq C.$$

Now for D_2, we use the same trick as before. we get at first

$$\begin{aligned}\|D_2\|_{L^1(M)} &\leq \sum_{i=0}^\infty\left\|\int_{\rho(r_B)}^\infty\nabla e^{-(t+1)\Delta}\Delta^K\mathbb{1}_{C_i(B)}b\frac{dt}{t^{1-\alpha}}\right\|_{L^1(M)}\\ &\leq \sum_{i=0}^\infty\sum_{j=0}^\infty\left\|\int_{\rho(r_B)}^\infty\nabla e^{-(t+1)\Delta}\Delta^K\mathbb{1}_{C_i(B)}b\frac{dt}{t^{1-\alpha}}\right\|_{L^1(C_j(2^iB))}\\ &\leq \sum_{i=0}^\infty\sum_{j=0}^\infty\int_{\rho(r_B)}^\infty\left\|\nabla\Delta^K e^{-(t+1)\Delta}\mathbb{1}_{C_i(B)}b\right\|_{L^2(C_j(2^iB))}\mu^{1/2}(2^{i+j}B)\frac{dt}{t^{1-\alpha}}.\end{aligned}$$

Still for $j=0,1,2$. Since $\left\|\nabla\Delta^K e^{-t\Delta}\right\|_{2\to 2}\le Ct^{-K-1/2}$. Then we have

$$\begin{aligned}
&\int_{\rho(r_B)}^{\infty}\left\|\nabla\Delta e^{-(t+1)\Delta}\mathbb{1}_{C_i(B)}b\right\|_{L^2(C_j(2^iB))}\mu^{1/2}(2^{i+j}B)\frac{dt}{t^{1-\alpha}}\\
&\le C\int_{\rho(r_B)}^{\infty}(t+1)^{-K-1/2}\|b\|_{L^2(C_i(B))}\mu^{1/2}(2^{i+j}B)\frac{dt}{t^{1-\alpha}}\\
&\le C2^{-\varepsilon i}\rho(r_B)\int_{\rho(r_B)}^{\infty}(t+1)^{-K-1/2}\frac{dt}{t^{1-\alpha}}\le C2^{-\varepsilon i}.
\end{aligned}\tag{3.38}$$

As in the estimate of D_1, the last inequality always holds. Similarly, if $r_B<1$, then

$$\int_{\rho(r_B)}^{\infty}(t+1)^{-K-1/2}\frac{dt}{t^{1-\alpha}}\le\int_{r_B^2}^{1}\frac{dt}{t^{1-\alpha}}+\int_1^{\infty}\frac{dt}{t^{K+3/2-\alpha}}\le C.$$

If $r_B>1$, then

$$\int_{\rho(r_B)}^{\infty}(t+1)^{-K-1/2}\frac{dt}{t^{1-\alpha}}\le\int_{r_B^m}^{\infty}\frac{dt}{t^{K+3/2-\alpha}}\le C.$$

While for $j\ge 3$, denote $D_2^{ij}=\|\nabla\Delta^K e^{-(t+1)\Delta}\mathbb{1}_{C_i(B)}b\|_{L^2(C_j(2^iB))}$, we get from (2.15) in Corollary 2.7 that

$$\begin{aligned}
D_2^{ij}&\le\left\|\nabla\Delta^K e^{-(t+1)\Delta}\mathbb{1}_{C_i(B)}b\right\|_{L^2(C_j(2^iB))}\\
&\le\frac{C_\alpha}{(t+1)^{K+\alpha}\mu^{1/2}(2^{i+j}B)}e^{-c\left(\frac{2^{(i+j)m}r^m}{t+1}\right)^{1/(m-1)}}\|b\|_{L^1(C_i(B))}\\
&\le 2^{-\varepsilon i}\rho^K(r_B)\frac{C}{(t+1)^{K+\alpha}\mu^{1/2}(2^{i+j}B)}e^{-c\left(\frac{2^{(i+j)m}r^m}{t+1}\right)^{1/(m-1)}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{\rho(r_B)}^{\infty}D_2^{ij}\frac{dt}{t^{1-\alpha}}\cdot\mu^{1/2}(2^{i+j}B)\\
&=2^{-\varepsilon i}\rho^K(r_B)\int_{\rho(r_B)}^{\infty}(t+1)^{-K-\alpha}\exp\left(-c\left(\frac{2^{(i+j)m}r_B^m}{t+1}\right)^{1/(m-1)}\right)\frac{dt}{t^{1-\alpha}}\\
&\le C2^{-\varepsilon i}\rho^K(r_B)\int_{\rho(r_B)}^{\infty}t^{-K}(t+1)^{-\alpha}\left(\frac{t+1}{2^{(i+j)m}r_B^m}\right)^{\alpha\beta(r_B)/m}\frac{dt}{t^{1-\alpha}}\\
&\le C2^{-\varepsilon i}\rho^K(r_B)(2^{(i+j)}r_B)^{\alpha\beta(r_B)}\int_{\rho(r_B)}^{\infty}\frac{dt}{t^{K+1-\alpha}}\\
&\le C2^{-\varepsilon i}\rho^K(r_B)(2^{(i+j)}r_B)^{-\alpha\beta(r_B)}\rho(r_B)^{(\alpha-K)}\le C2^{-\varepsilon i}2^{-\alpha\beta(r_B)(i+j)}.
\end{aligned}\tag{3.39}$$

The second inequality is also due to the fact $\alpha\beta(r_B)/m\le\alpha$, where β is defined in (3.37).

As a result of (3.38) and (3.39),

$$\|D_2\|_{L^1(M)} \leq C \sum_{i=0}^{\infty} \sum_{j=0}^{2} 2^{-\varepsilon i} + C \sum_{i=0}^{\infty} \sum_{j=3}^{\infty} 2^{-\varepsilon i} 2^{-\alpha\beta(i+j)} \leq C.$$

Finally we get $\|Ta\|_{L^1(M)} \leq C$ and our proof is finished. □

Remark 3.25. We can recover again the L^p boundedness of quasi Riesz transforms from the complex interpolation theorem over Hardy spaces $H^p_{\Delta,S_h^m}(M)$.

In fact, consider the operator $T_z = \nabla e^{-\Delta} \Delta^{-\alpha+(\alpha-1/2)z}$, $z \in \mathbb{C}$. Then for $y \in \mathbb{R}$,

$$T_{iy} f = \nabla e^{-\Delta} \Delta^{-\alpha+(\alpha-1/2)iy} f,$$
$$T_{1+iy} f = \nabla e^{-\Delta} \Delta^{-1/2+(\alpha-1/2)iy} f.$$

We have already shown in Theorem 3.24 that $T_0 = \nabla e^{-\Delta} \Delta^{-\alpha}$ is $H^1_{\Delta,m,mol}(M) - L^1(M)$ bounded, or equivalently $H^1_{\Delta,S_h^m}(M) - L^1(M)$ bounded by Theorem 3.12. It is also true that T_{iy} is $H^1_{\Delta,S_h^m}(M) - L^1(M)$ bounded. In fact,

$$T_{iy} = \int_0^\infty \nabla e^{-(t+1)\Delta} \frac{dt}{t^{1-\alpha+(\alpha-1/2)iy}}.$$

Besides, T_{1+iy} is bounded on $L^2(M)$. By Theorem 3.9, we conclude that $T_{\alpha(p)} = \nabla e^{-\Delta} \Delta^{-\alpha(p)}$ is $H^p_{\Delta,S_h^m}(M) - L^p(M)$ bounded, with $p \in (1,2)$ and $\alpha(p) = \alpha - (\alpha - 1/2)\frac{2}{p'}$. Since $H^p_{\Delta,S_h^m}(M) = L^p(M)$ for $1 < p < \infty$ from Corollary 3.17, it comes out that $\nabla e^{-\Delta} \Delta^{-\alpha(p)}$ is strong type (p,p) bounded.

Chapter 4

Quasi Riesz transforms on graphs

In this chapter, we consider the Riesz transform on graphs. Note that unlike fractal manifolds, we do not need to deal with local problems.

In the first section, we prove that on a general graph, the quasi Riesz transform is L^p bounded for $1 < p \leq 2$ (Proposition 4.2). In the second section, we consider the endpoint case $p = 1$ for quasi Riesz transforms on graphs satisfying (D) and (UE_m). We show that quasi Riesz transforms are of weak type $(1,1)$ (Theorem 4.3).

4.1 L^p boundedness of quasi Riesz transforms on graphs

First recall the multiplicative inequality and the derivative estimate of the heat semigroup given in [Dun08, Corollary 1.2]:

Proposition 4.1. *Let (Γ, d, μ) satisfy the local doubling property, that is, for some constant $c > 1$,*

$$\mu(B(x,1) \leq c\mu(x), \ \forall x \in \Gamma,$$

and let $p \in (1,2]$. There exists $c_p > 0$ such that

$$|||\nabla f|||_p^2 \leq c_p \|(I-P)f\|_p \|f\|_p, \ \forall f \in L^p(\Gamma). \tag{4.1}$$

If, in addition, $-1 \notin \sigma_{L^2}(P)$, then there exists $c'_p > 0$ such that

$$|||\nabla P^k|||_{p \to p} \leq c'_p k^{-1/2}, \tag{4.2}$$

for $k \in \mathbb{N}$.

Therefore, we have the analogue of Proposition 1.9 on graphs.

Proposition 4.2. *Let (Γ, d, μ) as before. Then for any fixed $\beta \in (0,1/2)$ and $1 < p \leq 2$, there exists $C > 0$ such that*

$$\||\nabla(I-P)^{-\beta} f|\|_p \leq C\|f\|_p,$$

for all $f \in L^p$.

Proof. For any $f \in L^p$, the Taylor expansion of $\nabla(I-P)^{-\beta}$ yields

$$\nabla(I-P)^{-\beta} f = \sum_{n=0}^{\infty} a_n \nabla P^n f,$$

where $a_n \leq C_\alpha n^{\beta-1}$.

From Proposition 4.1, we have (4.2) for any $p \in (1,2]$. Thus

$$\begin{aligned}
\||\nabla(I-P)^{-\beta} f|\|_p &= \||\sum_{n=0}^{\infty} a_n \nabla P^n f|\|_p \leq \||\nabla f|\|_p + \sum_{n=1}^{\infty} a_n \||\nabla P^n f|\|_p \\
&\leq \||\nabla f|\|_p + C\sum_{n=1}^{\infty} a_n n^{-1/2}\|f\|_p \\
&\leq c_p \||(I-P)f|\|_p^{1/2} \|f\|_p^{1/2} + C\sum_{n=1}^{\infty} a_n n^{-1/2}\|f\|_p \leq C\|f\|_p.
\end{aligned}$$

□

4.2 Weak $(1,1)$ boundedness of quasi Riesz transforms on graphs with sub-Gaussian heat kernel estimates

Let (Γ, d, μ) be a graph as in Section 0.6.2 satisfying the doubling volume property (D). Assume that p_k satisfies the sub-Gaussian heat kernel upper estimate

$$p_k(x,y) \leq \frac{C\mu(y)}{V(x,k^{1/m})} \exp\left(-c\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right). \tag{UE_m}$$

There are a lot of graphs that satisfy (UE_m). For example, Vicsek graphs (see for details in [BCG01]), graphical Sierpinski carpets ([BB99b]), graphical Sierpinski gaskets ([Jon96]), etc. For more examples, we refer to [Bar04] (see Remark 0.52).

The aim here is to show the analogue of Theorem 2.11 on graphs. Our result is

Theorem 4.3. *Let Γ be a infinite connected graph satisfying the doubling volume property (D) and the condition $\triangle(\alpha)$. Assume also the sub-Gaussian upper heat kernel estimate (UE_m). Then for any $\beta \in (0,1/2)$, the quasi Riesz transform $\nabla(I-P)^{-\beta}$ is weak type $(1,1)$.*

Remark 4.4. It is also interesting to develop the Hardy space theory in Chapter 3 on graphs. For recent work of this subject, we refer to [BDar], where the authors assume the doubling volume property and the Gaussian heat kernel estimate. However, so far we don't know how to generalise in the sub-Gaussian setting. One difficulty we meet is how to get a proper Calderón reproducing formula.

4.2.1 Sub-Gaussian heat kernel estimates on graphs

We will establish the $L^1 - L^2$ off-diagonal estimate for the gradient of the heat kernel, which is crucial tool to our proof. The following results are analogues of Lemma 2.1, Lemma 2.3 and Corollary 2.7 of the continuous case in Chapter 2.

Let's recall the discrete "time derivative" estimate of p_k which was initially given by M. Christ [Chr95], while in [Blu00], S. Blunck gave a much simpler proof. See also N. Dungey [Dun06].

Lemma 4.5. *There exist $C,c>0$ such that for any $k\in\mathbb{N}^*$ and any $x,y\in\Gamma$,*

$$|p_{k+1}(x,y)-p_k(x,y)|\leq \frac{C\mu(y)}{kV\left(x,k^{1/m}\right)}\exp\left(-c\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right).$$

Lemma 4.6. *For any $x\in\Gamma,k\in\mathbb{N}^*$ and $l\in\mathbb{N}$, there exist $\alpha>0$ such that*

$$\sum_{y\in B(x,l^{1/m})^c}|\nabla_y p_k(y,x)|\mu(y)\leq C_l\mu(x)e^{-\alpha\left(\frac{l}{k}\right)^{1/(m-1)}}k^{-1/m}.$$

Proof. Similarly as in [CD99], [Rus00], the proof is decomposed into several steps.
Step 1: For any $l\in\mathbb{N}$,

$$\sum_{y\in B(x,l^{1/m})^c}\exp\left(-2\beta\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\mu(y)\lesssim e^{-\beta\left(\frac{l}{k}\right)^{1/(m-1)}}V(x,k^{1/m}). \tag{4.3}$$

Step 2: For $0<\gamma<2c$ (c is the constant in (UE_m)), we have

$$\sum_y[p_k(y,x)]^2\exp\left(\gamma\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\mu(y)\leq\frac{C_\gamma\mu^2(x)}{V(x,k^{1/m})}.$$

This is a consequence of (UE_m) and Step 1 with $r=0$.
Step 3:

$$\sum_y|\nabla_y p_k(y,x)|^2\exp\left(\gamma\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\mu(y)\leq\frac{C'_\gamma\mu^2(x)}{k^{2/m}V(x,k^{1/m})}. \tag{4.4}$$

Denote $2J(k,x)=\sum_y|\nabla_y p_k(y,x)|^2\exp\left(\gamma\left(\frac{d^m(x,y)}{k}\right)^{\frac{1}{m-1}}\right)\mu(y)$, with γ small enough. Then we have

$$
\begin{aligned}
&2J(k,x)\\
=&2\sum_{y,z} p_k(y,x)[p_k(y,x)-p_k(z,x)]p(y,z)\exp\left(\gamma\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\mu(y)\\
&+\sum_{y,z} p_k(y,x)[p_k(y,x)-p_k(z,x)]p(y,z)\\
&\cdot\left[\exp\left(\gamma\left(\frac{d^m(x,z)}{k}\right)^{1/(m-1)}\right)-\exp\left(\gamma\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\right]\mu(y)\\
=&2J_1(t,x)+J_2(t,x).
\end{aligned}
$$

In fact,

$$
\begin{aligned}
&J_1(k,x)\\
=&\sum_{y,z} p_k(y,x)[p_k(y,x)-p_k(z,x)]p(y,z)\exp\left(\gamma\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\mu(y)\\
=&\sum_{y} p_k(y,x)\exp\left(\gamma\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\mu(y)\sum_z[p_k(y,x)-p_k(z,x)]p(y,z)\\
=&\sum_{y} p_k(y,x)\exp\left(\gamma\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\mu(y)[p_k(y,x)-p_{k+1}(y,x)].
\end{aligned}
$$

Now using the discrete "time derivative" estimate in Lemma 2.1, then

$$
\begin{aligned}
J_1(k,x) &\leq \frac{C\mu(x)}{kV^{1/2}\left(x,k^{1/m}\right)}\sum_y \frac{p_k(y,x)\mu(y)}{V^{1/2}\left(y,k^{1/m}\right)}\exp\left(\gamma_1\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\\
&\leq \frac{C\mu^2(x)}{kV^{1/2}\left(x,k^{1/m}\right)}\sum_y \frac{p_k(x,y)}{V^{1/2}\left(y,k^{1/m}\right)}\exp\left(\gamma_1\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\\
&\leq \frac{C\mu^2(x)}{kV^2\left(x,k^{1/m}\right)}\sum_y \mu(y)\exp\left((\gamma_1-c)\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\\
&\leq \frac{C_{\gamma_1}\mu^2(x)}{k^{2/m}V\left(x,k^{1/m}\right)}.
\end{aligned}
$$

The second inequality is due to (7) and the third one uses (D) and (UE_m).

For $J_2(k,x)$, by the mean value theorem,

$$\begin{aligned}&\left|\exp\left(\gamma\left(\frac{d^m(x,z)}{k}\right)^{1/(m-1)}\right)-\exp\left(\gamma\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\right|\\ \leq&\frac{m\gamma}{(m-1)k^{1/(m-1)}}|d(x,z)-d(x,y)|\max(d(x,y),d(x,z))^{1/(m-1)}\\ &\times\exp\left(\gamma\left(\frac{\max(d(x,y),d(x,z))^m}{k}\right)^{1/(m-1)}\right).\end{aligned}\tag{4.5}$$

Since $p(y,z)>0$ implies $y\sim z$. In this case, we have $d(x,z)\leq d(x,y)+1$, thus

$$d(x,z)^{m/(m-1)}\leq(d(x,y)+1)^{m/(m-1)}\leq C_m\left(d(x,y)^{m/(m-1)}+1\right).$$

Besides,

$$\left(\frac{d(x,y)+1}{k}\right)^{1/(m-1)}=\left(\frac{d(x,y)+1}{k^{1/m}}\right)^{1/(m-1)}\frac{1}{k^{1/m}}$$

Therefore by Cauchy-Schwartz inequality,

$$\begin{aligned}&J_2(k,x)\\ \leq\ &\frac{m\gamma}{m-1}\sum_{y\in\Gamma}\sum_{z\sim y}p_k(y,x)[p_k(y,x)-p_k(z,x)]p(y,z)\left(\frac{d(x,y)+r_0}{k}\right)^{1/(m-1)}\\ &\cdot\exp\left(\gamma\left(\frac{(d(x,y)+1)^m}{k}\right)^{1/(m-1)}\right)\mu(y)\\ \leq\ &\frac{C_{m,\gamma}}{k^{1/m}}\sum_{y\in\Gamma}\sum_{z\sim y}p_k(y,x)[p_k(y,x)-p_k(z,x)]p(y,z)\exp\left(\gamma'\left(\frac{(d(x,y)+1)^m}{k}\right)^{1/(m-1)}\right)\mu(y)\\ \leq\ &\frac{C_{m,\gamma}}{k^{1/m}}\left(\sum_{y\in\Gamma}\sum_{z\sim y}p_k^2(y,x)p(y,z)\exp\left(\gamma_2\left(\frac{(d(x,y)+1)^m}{k}\right)^{1/(m-1)}\right)\mu(y)\right)^{1/2}\\ &\cdot\left(\sum_{y\in\Gamma}\sum_{z\sim y}[p_k(y,x)-p_k(z,x)]^2p(y,z)\exp\left(\gamma\left(\frac{(d(x,y)+1)^m}{k}\right)^{1/(m-1)}\right)\mu(y)\right)^{1/2}\\ \leq\ &\frac{C_{\gamma_2}\mu(x)}{k^{1/m}V^{1/2}\left(x,k^{1/m}\right)}J(k,x)^{1/2}.\end{aligned}$$

The last inequality is obtained by (8) and the estimate in Step 2.

Now we get

$$J(k,x)\leq\frac{C_{\gamma_1}\mu^2(x)}{k^{2/m}V\left(x,k^{1/m}\right)}+\frac{C_{\gamma_2}\mu(x)}{k^{1/m}V^{1/2}\left(x,k^{1/m}\right)}I(k,x)^{1/2},$$

which gives us

$$J(k,x) \leq \frac{C\mu^2(x)}{k^{2/m}V\left(x,k^{1/m}\right)}.$$

Step 4: By Cauchy-Schwartz inequality, we have

$$\begin{aligned}
& \sum_{x\in B(y,l)^c} |\nabla_x p_k(x,y)|\mu(x) \\
\leq & \left(\sum_x |\nabla_x p_k(x,y)|^2 \exp\left(2\alpha\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\mu(x)\right)^{1/2} \\
& \cdot\left(\sum_x \exp\left(-2\alpha\left(\frac{d^m(x,y)}{k}\right)^{1/(m-1)}\right)\mu(x)\right)^{1/2} \\
\leq & \frac{C_\gamma\mu(y)}{V(y,k^{1/m})}\cdot e^{-\alpha\left(\frac{l^m}{k}\right)^{m/(m-1)}} V(y,k^{1/m}) = C\mu(y)k^{-1/m}e^{-\alpha\left(\frac{l^m}{k}\right)^{1/(m-1)}}.
\end{aligned}$$

□

Corollary 4.7. *Fix $p\in[1,2]$ and $j\geq 2$. For any ball B with radius r, we have*

$$\frac{1}{V^{1/2}(2^{j+1}B)}\|P^k f\|_{L^2(C_j(B))} \leq \frac{e^{-c\left(\frac{2^{jm}r^m}{k}\right)^{1/m-1}}}{V^{1/p}(B)}\|f\|_{L^p(B)}, \tag{4.6}$$

$$\frac{1}{V^{1/2}(2^{j+1}B)}\||\nabla P^k f|\|_{L^2(C_j(B))} \leq \frac{Ce^{-c\left(\frac{2^{jm}r^m}{k}\right)^{1/(m-1)}}}{k^{1/m}V^{1/p}(B)}\|f\|_{L^p(B)}. \tag{4.7}$$

Proof. Let us prove (4.7) by using the derivative estimate of the heat kernel (4.4). For any $z\in B$, we have

$$\begin{aligned}
& \||\nabla p_k(\cdot,z)|\|_{L^2(C_j(B))} \\
= & \left(\sum_{x\in C_j(B)} |\nabla p_k(x,z)|^2\mu(x)\right)^{1/2} \\
\leq & C\left(\sum_{x\in C_j(B)} |\nabla p_k(x,z)|^2 e^{c\left(\frac{d^m(x,z)}{k}\right)^{1/(m-1)}}\mu(x)\right)^{1/2}\cdot e^{-c\left(\frac{2^{jm}r^m}{k}\right)^{1/(m-1)}} \\
\leq & \frac{\mu(z)}{k^{1/m}V^{1/2}(z,k^{1/m})}\cdot e^{-c\left(\frac{2^{jm}r^m}{k}\right)^{1/(m-1)}}.
\end{aligned}$$

Therefore, similarly as in the continuous case, we get

$$\frac{1}{V^{1/2}(2^{j+1}B)}\||\nabla P^k f|\|_{L^2(C_j(B))}$$

$$
\begin{aligned}
&= \frac{1}{V^{1/2}(2^{j+1}B)}\left(\frac{1}{2}\sum_{x\in C_j(B)}\sum_{y\sim x}p(x,y)|P^kf(x)-P^kf(y)|^2\mu(x)\right)^{1/2}\\
&\leq \frac{C}{V^{1/2}(2^{j+1}B)}\sum_{z\in B}|f(z)|\left(\sum_{x\in C_j(B)}\sum_{y\sim x}p(x,y)|p_k(x,z)-p_k(y,z)|^2\mu(x)\right)^{1/2}\\
&\leq \frac{C}{V^{1/2}(2^{j+1}B)}\sum_{z\in B}|f(z)|\left(\sum_{x\in C_j(B)}|\nabla p_k(x,z)|^2\mu(x)\right)^{1/2}\\
&\leq \frac{Ce^{-c\left(\frac{2^{jm}r^m}{k}\right)^{1/(m-1)}}}{k^{1/m}V(2^jB)}\sum_{z\in B}|f(z)|\frac{V(z,2^{jr})}{V(z,k^{1/m})}\leq \frac{Ce^{-c\left(\frac{2^{jm}r^m}{k}\right)^{1/(m-1)}}}{k^{1/m}V(B)}\sum_{z\in B}|f(z)|\\
&\leq \frac{Ce^{-c\left(\frac{2^{jm}r^m}{k}\right)^{1/(m-1)}}}{k^{1/m}V^{1/p}(B)}\|f\|_{L^p(B)}.
\end{aligned}
$$

In the same way, the first estimate (4.6) follows from the estimate in Step 2. □

Corollary 4.8. *Assume that Γ satisfies $\Delta(\alpha)$, (D) and (UE_m). Fix $\beta\in(1/m,1/2)$. Then for any ball B with radius r, we have*

$$
\frac{1}{V^{1/2}(2^{j+1}B)}\||\nabla P^kf|\|_{L^2(C_j(B))}\leq \frac{C}{k^\beta V^{1/p}(B)}\|f\|_{L^1(B)}. \tag{4.8}
$$

Proof. For $k=1,2$, it is obvious to see that (4.7) implies (4.8).

Now consider $k\geq 3$. For any f supported in $B=B(x_B,r_B)$, since

$$
\left\|\left|\nabla P^k\right|\right\|_{L^2\to L^2}\leq Ck^{-1/2},
$$

we get by using the (UE_m)

$$
\left\|\left|\nabla P^kf\right|\right\|_{L^2}\leq \left\|\left|\nabla P^{[k/2]+1}\right|\right\|_{L^2\to L^2}\left\|P^{k-1-[k/2]}f\right\|_{L^2}\leq \frac{C(1+r_B/k^{1/m})^{\nu/2}}{k^{1/2}V^{1/2}(x_B,k^{1/m})}\|f\|_{L^1(B)}.
$$

Combining this with (4.7), we obtain (4.8). □

4.2.2 Weak $(1,1)$ boundedness of quasi Riesz transforms

In order to show the weak $(1,1)$ boundedness of quasi Riesz transform, we will use the following criterion. Blunck and Kunstmann [BK03] proved the original version on metric measure space. See also [Aus07] for a nice presentation in the Euclidean case.

Theorem 4.9. *Let $p_0\in[1,2)$. Assume that Γ satisfies the doubling property (D) and let T be a sublinear operator of strong type $(2,2)$. For any ball B, Let A_B be a linear operator acting on*

$L^2(\Gamma)$. Assume that, for all $j \geq 1$, there exists $g(j) > 0$ such that, for all ball $B \subset \Gamma$ and all function f supported in B,

$$\frac{1}{V^{1/2}(2^{j+1}B)} \|T(I-A_B)f\|_{L^2(C_j(B))} \leq g(j)\frac{1}{V^{1/p_0}(B)} \|f\|_{L^{p_0}} \tag{4.9}$$

for all $j \geq 2$ and

$$\frac{1}{V^{1/2}(2^{j+1}B)} \|A_B f\|_{L^2(C_j(B))} \leq g(j)\frac{1}{V^{1/p_0}(B)} \|f\|_{L^{p_0}} \tag{4.10}$$

for all $j \geq 1$. If $\sum_{j\geq 1} g(j)2^{\nu j} < +\infty$ where ν is given by (3), then T is of weak type (p_0, p_0), and is therefore of strong type (p,p) for all $p_0 < p < 2$.

Proof of Theorem 4.3 We apply Theorem 4.9 with $T = \nabla(I-P)^{-\beta}$ and $p_0 = 1$. We first show that T is L^2 bounded. In fact, for any $f \in \mathscr{C}_0(\Gamma)$, we have already seen that $\||\nabla f|\|_2 = \left\|(I-P)^{1/2}f\right\|_2$ always holds, then

$$\begin{aligned}
\|Tf\|_2 &= \left\|\left|\nabla(I-P)^{-\beta}f\right|\right\|_2 \\
&\leq \left\|\left|\nabla(I-P)^{-1/2}f\right|\right\|_{2\to 2}\left\|(I-P)^{1/2-\beta}f\right\|_2 \\
&\leq \left\|(I-P)^{1/2-\beta}f\right\|_2.
\end{aligned}$$

Then by spectral theory we have

$$\|Tf\|_2^2 \leq \left\|(I-P)^{1/2-\beta}f\right\|_2^2 = \int_a^1 (1-\lambda)^{\frac{1}{2}-\frac{1}{m}} d < E_\lambda f, f > \leq C\|f\|_2.$$

Now for any ball B with radius k, take

$$A_B = I - (I-P^{k^m})^N = \sum_{l=1}^{N} (-1)^{l+1}\binom{N}{l}P^{lk^m},$$

where $N \in \mathbb{N}$ will be chosen later. Note that $I - A_B = \sum_{l=0}^{N}(-1)^l\binom{N}{l}P^{lk^m}$.

Write the Taylor expansion of $(1-x)^{-\beta}$, i. e.

$$(1-x)^{-\beta} = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{\beta(\beta+1)\cdots(\beta+n-1)}{n!}x^n. \tag{4.11}$$

where the classical estimate says $a_n \leq Cn^{\beta-1}$.

Therefore

$$\nabla(I-P)^{-\beta}(I-A_B) = \sum_{n=0}^{\infty}\sum_{l=0}^{N}(-1)^l a_n\binom{N}{l}\nabla P^{n+lk^m}$$

$$= \sum_{n=0}^{\infty}\sum_{l=0}^{N} g_k(n)\nabla P^n,$$

where

$$g_k(n) = \sum_{0\le l\le N, lk^m\le n} (-1)^l a_{n-lk^m}\binom{N}{l}.$$

Let us first show (4.9). For any $j\ge 2$ and any function f supported in B, the off-diagonal estimate in Lemma 4.8 as well as the Minkowshi inequality give us

$$\begin{aligned}
&\frac{1}{V^{1/2}(2^{j+1}B)}\left\|\left|\nabla(I-P)^{-\beta}(I-A_B)f\right|\right\|_{L^2(C_j(B))}\\
\le\ &\frac{1}{V^{1/2}(2^{j+1}B)}\left\|\left|\sum_{n=0}^{\infty} g_k(n)\nabla P^n f\right|\right\|_{L^2(C_j(B))}\\
\le\ &\sum_{n=1}^{\infty}|g_k(n)|\frac{1}{V^{1/2}(2^{j+1}B)}\||\nabla P^n f|\|_{L^2(C_j(B))}\\
=\ &\frac{C\|f\|_{L^1(B)}}{V(B)}\sum_{n=1}^{\infty}|g_k(n)|n^{-\beta}e^{-c\left(\frac{2^{jm}k^m}{n}\right)^{1/(m-1)}}.
\end{aligned}$$

Note that in the third line, the term $n=0$ vanishes. Indeed, since f is supported in B, we have $|\nabla P^0 f(x)| = |\nabla f(x)| = 0$, for any $x\in C_j(B)$, $j\ge 2$.

It remains to estimate the sum $g(j) := \sum_{n=1}^{\infty}|g_k(n)|n^{-\beta}e^{-c\left(\frac{2^{jm}k^m}{n}\right)^{1/(m-1)}}$ in the last line. In fact, we have the following estimate for which we will prove later in Lemma 4.10:

$$|g_k(n)| \le \begin{cases} C(n-lk^m)^{\beta-1}, & 0\le lk^m < n \le (l+1)k^m \le (N+1)k^m,\\ Ck^{mN}n^{-N-1+\beta}, & n > (N+1)k^m. \end{cases} \tag{4.12}$$

Write $g(j)$ into two parts

$$\begin{aligned}
g(j) = &\sum_{l=0}^{N}\sum_{n=lk^m+1}^{(l+1)k^m}|g_k(n)|n^{-\beta}e^{-c\left(\frac{2^{jm}k^m}{n}\right)^{1/(m-1)}}\\
&+\sum_{n=(N+1)k^m+1}^{\infty}|g_k(n)|n^{-\beta}e^{-c\left(\frac{2^{jm}k^m}{n}\right)^{1/(m-1)}} := G_1+G_2.
\end{aligned}$$

For G_1, using the estimate in (4.12) for $n \leq (N+1)k^m$, we get

$$\begin{aligned} G_1 &\leq C \sum_{l=0}^{N} \sum_{n=lk^m+1}^{(l+1)k^m} (n-lk^m)^{\beta-1} n^{-\beta} e^{-c\left(\frac{2^{jm}k^m}{n}\right)^{1/(m-1)}} \\ &= C \sum_{l=0}^{N} \sum_{n=1}^{k^m} (n+lk^m)^{\beta-1} n^{-\beta} e^{-c\left(\frac{2^{jm}k^m}{n+lk^m}\right)^{1/(m-1)}} \\ &\leq C \sum_{n=1}^{k^m} n^{-1} e^{-c\left(\frac{2^{jm}k^m}{n+lk^m}\right)^{1/(m-1)}} \\ &\leq Ce^{-c2^{jm/(m-1)}}. \end{aligned} \tag{4.13}$$

Next for G_2, this time applying the estimate in (4.12) for $n > (N+1)k^m$,

$$\begin{aligned} G_2 &= \sum_{n=(N+1)k^m+1}^{\infty} k^{mN} n^{-N-1+\beta} n^{-\beta} e^{-c\left(\frac{2^{jm}k^m}{n}\right)^{1/(m-1)}} \\ &\leq Ck^{mN} \int_0^{\infty} t^{-N-1} e^{-c\left(\frac{2^{jm}k^m}{t}\right)^{1/(m-1)}} dt \\ &\leq C2^{-jmN} \int_0^{\infty} u^{N(m-1)-1} e^{-cu} du \\ &\leq C2^{-jmN}. \end{aligned} \tag{4.14}$$

Choosing any integer $N > \nu/m$, we get from (4.13) and (4.14) that $\sum_{j\geq 1} g(j)2^{\nu j} < \infty$. Therefore (4.9) holds.

Now let us verify (4.10). For any $1 \leq l \leq N+1$, if $j \geq 2$, we apply (4.6) and get

$$\frac{1}{V^{1/2}(2^{j+1}B)} \left\| P^{lk^m} f \right\|_{l^2(C_j(B))} \leq \frac{Ce^{-c2^{jm/(m-1)}}}{V(B)} \|f\|_{L^1(B)}.$$

If $j = 1$, then

$$\begin{aligned} \left\| P^{lk^m} f \right\|_{L^2(4B)} &= \left(\sum_{x\in 4B} \left| \sum_{y\in B} p_{lk^m}(x,y) f(y) \right|^2 \mu(x) \right)^{1/2} \\ &\leq \sum_{y\in B} |f(y)| \left(\sum_{x\in 4B} |p_{lk^m}(x,y)|^2 \mu(x) \right)^{1/2} \\ &\leq \sum_{y\in B} |f(y)| \mu(y) \left(\sum_{x\in 4B} \frac{C}{V(x,l^{1/m}k)^2} \mu(x) \right)^{1/2} \\ &\leq \frac{C}{V(B)} \|f\|_{L^1(B)}. \end{aligned}$$

Thus (4.10) holds. □

Lemma 4.10. *For* $g_k(n) = \sum_{0\le l\le N, lk^m\le n} (-1)^l a_{n-lk^m}\binom{N}{l}$, *we have the following estimates:*

$$|g_k(n)| \le \begin{cases} C(n-lk^m)^{\beta-1}, & 0\le lk^m < n \le (l+1)k^m \le (N+1)k^m, \\ Ck^{mN}n^{-N-1+\beta}, & n > (N+1)k^m. \end{cases} \tag{4.15}$$

We will use similarly method as in [BR09, Lemma 4.1].

Proof. First for $0 \le lk^m < n \le (l+1)k^m \le (N+1)k^m$, the estimate follows from the fact that $a_n \le Cn^{\beta-1}$.

Now for $n > (N+1)k^m$, we will use the following inequality, as in [ACDH04, BR09]: for any positive integer N, any C^N function φ on $(0,\infty)$, any positive integer k and any $t > (N+1)k^m$

$$\left|\sum_{0\le l\le N, lk^m\le n} (-1)^l \binom{N}{l}\varphi(t-lk^m)\right| \le C \sup_{u\ge\frac{t}{n+1}} \left|\varphi^{(N)}(u)\right| k^m, \tag{4.16}$$

where $C>0$ depends on N.

Note that

$$a_n = \frac{\beta(\beta+1)\cdots(\beta+n-1)}{n!} = \frac{1}{\Gamma(\beta)\Gamma(1-\beta)}B(n+\beta,1-\beta),$$

where B denotes the Beta function.

Take $\varphi(x) = \frac{1}{\Gamma(\beta)\Gamma(1-\beta)}B(x+\beta,1-\beta)$, then $a_n = \varphi(n)$. We claim: for all $x>1$,

$$\left|\frac{\partial^N}{\partial x^N}\varphi(x)\right| \le Cx^{-N-1+\beta}. \tag{4.17}$$

The desired result then follows from (4.16).

It is enough to consider $B(x+\beta,1-\beta) = \int_0^1 t^{x+\beta-1}(1-t)^{-\beta}dt$. Indeed, for any $\delta\in(0,1)$ and any $x>1$,

$$\begin{aligned} \left|\frac{\partial^N}{\partial x^N}B(x+\beta,1-\beta)\right| &= \left|\int_0^1 (\ln t)^N t^{x+\beta-1}(1-t)^{-\beta}dt\right| \\ &= \int_0^\delta |\ln t|^N t^{x+\beta-1}(1-t)^{-\beta}dt + \int_\delta^1 |\ln t|^N t^{x+\beta-1}(1-t)^{-\beta}dt \\ &:= I_N(x)+J_N(x). \end{aligned} \tag{4.18}$$

Observe that

$$I_N(x) \le \delta^{x-1}\int_0^\delta |\ln t|^N t^\beta (1-t)^{-\beta}dt \le C\delta^{x-1}. \tag{4.19}$$

For $J_N(x)$. Since $|\ln t| \simeq 1-t$ when $t \to 1^-$, we fix $\delta \in (0,1)$ such that $|\ln t| \leq \frac{1}{4}(1-t)$. Then

$$J_N(t) \leq C\int_{\delta}^{1} t^{x+\beta-1}(1-t)^{-\beta+N}dt \leq CB(x+\beta, N+1-\beta) \leq Cx^{-N-1+\beta}. \tag{4.20}$$

In the last line, C depends on β and N.

Combining (4.18), (4.19) and (4.20), we get (4.16). □

Chapter 5

Poincaré inequalities and Sobolev inequalities on Vicsek graphs

In this chapter, we consider Poincaré inequalities and Sobolev inequalities. We prove in the first first section the generalised L^p Poincaré inequality and in the second section the generalised L^p Sobolev inequality. Finally we show the optimality of these inequalities, which connects to Section 2.5.

Recall the construction of a Vicsek graph Γ on $\mathbb{R}^n$. The following in taken from [CG03].

Let Q_r denote the cube in $\mathbb{R}^n$

$$Q_r = \{x \in \mathbb{R}^n : 0 \leq x_i \leq r,\ i = 1, 2, \cdots, n\}.$$

Construct an increasing sequence $\{\Gamma_k\}$ of finite graphs as subsets of Q_{3^k}. Let Γ_1 be the set of $2^n + 1$ points containing all vertices of Q_1 and the center of Q_1. Define 2^n edges in Γ_1 as segments connecting the center with the corners. Assuming that Γ_k is already constructed, define Γ_{k+1} as follows. The cube $Q_{3^{k+1}}$ is naturally divided into 3^n congruent copies of Q_{3^k}; select $2^n + 1$ of the copies of Q_{3^k} by taking the corner cubes and the center one. In each of the selected copies of Q_{3^k} construct a congruent copy of graph Γ_k, and define Γ_{k+1} as the union of all $2^n + 1$ copies of Γ_k (merged at the corners). Then the Vicsek tree Γ is the union of all Γ_k, $k \geq 1$ (see Figure 5.1). Then for all $x \in \Gamma$, $r \geq 1$ and $k > 0$, Γ satisfies

$$V(x, r) \simeq r^D$$

and

$$p_k(x, x) \leq Ck^{-\frac{D}{D+1}}.$$

where $D = \log_3(2^n + 1)$.

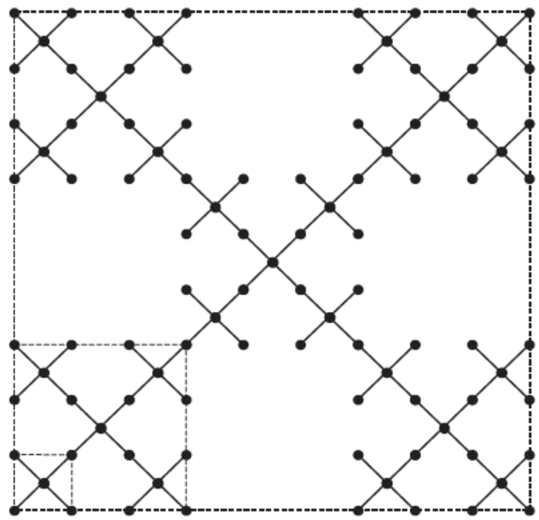

Figure 5.1: A fragment of the Vicsek graph on the plane

5.1 Poincaré inequalities on Vicsek graphs

In the following, we will prove a L^p Poincaré inequality by using the idea in [DS91], see also [CSC93], where the L^2 case is considered.

Theorem 5.1. *Let Γ be a Vicsek graph with polynomial volume growth $V(x,n) \simeq n^D$. Then for $p \geq 1$, it holds on Γ that*

$$\|f - f_n(x)\|_{L^p(B(x,n))} \leq CV(x,n)^{\frac{1}{D(p)}} \||\nabla f|\|_{L^p(B(x,2n))},$$

where $f_n(x) = \frac{1}{V(x,n)} \sum_{z \in B(x,n)} f(z)\mu(z)$ and $\frac{1}{D(p)} = \frac{1}{p} + \frac{1}{p'D}$.

Proof. For any $y,z \in B(x,n)$, we choose $\gamma_{y,z}$ as one of the shortest paths from y to z. Define $\Gamma_{x,n} := \{\gamma_{y,z} : y,z \in B(x,n)\}$. Let e be an oriented edge in a path from e_- to e_+ and $\mu(e) = \mu_{e_-e_+}$.

Since

$$|f(y) - f(z)| \leq \sum_{e \in \gamma_{y,z}} |f(e_+) - f(e_-)|,$$

for $p = 1$, we get

$$\begin{aligned}
& V(x,n) \sum_{y \in B(x,n)} |f(y) - f_n(x)|\mu(y) \\
\leq \quad & \sum_{y,z \in B(x,n)} |f(y) - f(z)|\mu(y)\mu(z) \\
\leq \quad & \sum_{y,z \in B(x,n)} \sum_{e \in \gamma_{y,z}} |f(e_+) - f(e_-)| \frac{\mu(e)}{\mu(e)} \mu(y)\mu(z)
\end{aligned}$$

$$\leq \; c_\alpha \sum_{y,z\in B(x,n)} \left[\sum_{e\in\gamma_{y,z}} |f(e_+)-f(e_-)|\mu(e) \right] \mu(y)\mu(z).$$

In the last inequality, c_α depends on $(\Delta(\alpha))$ in (10).

For $p>1$, Hölder inequality leads to

$$\begin{aligned}
& V(x,n)\sum_{y\in B(x,n)} |f(y)-f_n(x)|^p\mu(y) \\
\leq\; & V(x,n)^{1-p}\sum_{y\in B(x,n)} \left| \sum_{z\in B(x,n)} [f(y)-f(z)]\mu(z)^{\frac{1}{p}+\frac{1}{p'}} \right|^p \mu(y) \\
\leq\; & V(x,n)^{1-p}\sum_{y\in B(x,n)} \left[\sum_{z\in B(x,n)} |f(y)-f(z)|^p\mu(z) \right]\cdot \left[\sum_{z\in B(x,n)} \mu(z) \right]^{p-1} \mu(y) \\
=\; & \sum_{y,z\in B(x,n)} |f(y)-f(z)|^p\mu(y)\mu(z) \\
\leq\; & \sum_{y,z\in B(x,n)} \left[\sum_{e\in\gamma_{y,z}} |f(e_+)-f(e_-)| \frac{\mu^{1/p}(e)}{\mu^{1/p}(e)} \right]^p \mu(y)\mu(z) \\
\leq\; & \sum_{y,z\in B(x,n)} \left[\sum_{e\in\gamma_{y,z}} |f(e_+)-f(e_-)|^p\mu(e) \right]\cdot \left[\sum_{e\in\gamma_{y,z}} \mu(e)^{-1/(p-1)} \right]^{p-1} \mu(y)\mu(z).
\end{aligned}$$

Due to the property $(\Delta(\alpha))$ in (10), the last term is bounded from above by

$$C\sum_{e\in B(x,2n)} |f(e_+)-f(e_-)|^p\mu(e)\cdot \sum_{y,z\in B(x,n)} |\gamma_{y,z}|^{p-1}\mu(y)\mu(z).$$

Take $K(x,n)=\sum_{y,z\in B(x,n)}|\gamma_{y,z}|^{p-1}\mu(y)\mu(z)$ for $p\geq 1$, then

$$\|f-f_n(x)\|^p_{L^p(B(x,n))} \leq CK(x,n)\||\nabla f|\|^p_{L^p(B(x,2n))},\ \forall p\geq 1.$$

It is easy to see that $K(x,n)\leq (2n)^{p-1}V(x,n)^2$. In particular, for a Vicsek graph with the volume $V(x,n)\simeq n^D$, we have

$$\|f-f_n(x)\|^p_{L^p(B(x,n))} \leq CV(x,n)^{1+\frac{p-1}{D}}\||\nabla f|\|^p_{L^p(B(x,2n))} \simeq n^{D+p-1}\||\nabla f|\|^p_{L^p(B(x,2n))}.$$

□

It is well-known that under the assumption of Poincaré inequality and the doubling volume property, one can deduce the pseudo-Poincaré inequality (see for example [SC92], [SC02]). The same proof is also applicable here. Consequently,

$$\|f-f_n\|_p \leq Cn^{\frac{D}{p}+\frac{1}{p'}}\||\nabla f|\|_p. \qquad (PP^p_{D(p)})$$

Remark 5.2. The exponent $\frac{D}{p}+\frac{1}{p'}$ in ($PP^p_{D(p)}$) is optimal. Indeed, we will show in the sequel that ($PP^p_{D(p)}$) implies ($S^p_{D(p)}$), which is optimal. If there existed a better exponent, it would follow from the proof below that ($S^p_{D(p)}$) could also be improved, which contradicts its optimality.

5.2 General Sobolev inequalities

In this section, we consider a kind of Sobolev inequalities in the following form: for $p \geq 1$ and a non-decreasing positive function ϕ

$$\|f\|_p \leq \phi(|\Omega|)\||\nabla f|\|_p, \forall \Omega \subset \Gamma, \forall f \in c_0(\Omega).$$

For a detailed exploration of this kind of inequalities on weighted manifolds, see the survey [Cou03].

By using the same method as in [Cou03, Proposition 2.6], we will prove

Theorem 5.3. *Let Γ be a Vicsek graph with polynomial volume growth $V(x,n) \simeq n^D$. Then for $p \geq 1$, it holds*

$$\|f\|_p \lesssim \mu(\Omega)^{\frac{1}{D(p)}}\||\nabla f|\|_p, \forall \Omega \subset \Gamma, \forall f \in c_0(\Omega), \tag{$S^p_{D(p)}$}$$

where $D(p)$ is the same as in Theorem 5.1.

Remark 5.4. On Vicsek graphs, we already know the inequalities ($S^p_{D(p)}$) for $p = 1, 2, \infty$. That is

1. If $p = 1$, then $D(1) = 1$. Indeed, (S^1_1) is equivalent to the isoperimetric inequality with the isoperimetric dimension 1, i.e. $|\partial \Omega| \geq C$, which can be seen from the structure of Vicsek graph.

2. If $p = 2$, then $D(2) = \frac{2D}{D+1}$. ($S^2_{\frac{2D}{D+1}}$) is equivalent to the Nash inequality or the heat kernel upper estimates as follows:

$$\|f\|_2^{1+\frac{D+1}{D}} \leq C\|f\|_1^{\frac{D+1}{D}}\||\nabla f|\|_2;$$
$$p_k(x,x) \leq Ck^{-\frac{D}{D+1}}, \forall k \in \mathbb{N}.$$

3. If $p = \infty$, then $D(\infty) = D$. (S^∞_D) is equivalent to the lower volume bound: $V(x,n) \geq n^D$.

Besides, in the three cases, ($S^p_{D(p)}$) is also optimal. We refer to [Cou03], [BCG01] for more information.

Proof. We will show the result for $p > 1$. Note that for any $f \in L^1(\Gamma)$,

$$|f_n(x)| = \left|\frac{1}{V(x,n)}\sum_{y \in B(x,n)} f(y)\mu(y)\right| \leq C_D n^{-D}\|f\|_{L^1}. \tag{5.1}$$

Take a finite set $\Omega \subset \Gamma$ and a non-negative function $f \in c_0(\Omega)$. Write

$$\|f\|_p^p = (f - f_n, f^{p-1}) + (f_n, f^{p-1}).$$

It follows from Hölder inequalities, ($PP^p_{D(p)}$), and (5.1) that

$$\begin{aligned} \|f\|_p^p &\leq \|f - f_n\|_p \cdot \|f\|_p^{p-1} + \|f_n\|_\infty \cdot \|f^{p-1}\|_1 \\ &\leq CV(n)^{\frac{1}{p}+\frac{1}{p'D}} \||\nabla f|\|_p \cdot \|f\|_p^{p-1} + \frac{C_D \mu(\Omega)}{n^D} \|f\|_p^p. \end{aligned}$$

Choosing n such that $n^D = 2C_D\mu(\Omega)$, then

$$\|f\|_p \leq C'\mu(\Omega)^{\frac{1}{p}+\frac{1}{p'D}} \||\nabla f|\|_p.$$

The inequality also holds for any $f \in \mathscr{C}_0(\Omega)$. □

Here is another proof without using the Poincaré inequality, which is inspired from [BCG01].

For any function $f \in c_0(\Omega)$, assume $\max_{x\in\Omega} |f(x)| = 1$ (otherwise we can normalise f). Then we have

$$\sum_{x\in\Omega} |f(x)|^p \mu(x) \leq \mu(\Omega).$$

Consider a point x_0 such that $|f(x_0)| = 1$ and the largest integer n such that the ball $B(x_0, n) \subset \Omega$. Define

$$r(\Omega) = \max\{r \in \mathbb{N} : \exists x \in \Omega \text{ such that } B(x, r) \subset \Omega\}.$$

Obviously $n \leq r(\Omega)$. Also, there exists a sequence of points

$$x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_n \sim x_{n+1}$$

starting from x_0 and terminating at a point $x_{n+1} \notin \Omega$.

Therefore, by Hölder inequality

$$\sum_{x\in\Omega} |\nabla f(x)|^p \mu(x) \simeq \sum_{x,y\in\Omega} |f(x) - f(y)|^p \mu_{xy} \geq \frac{c}{n^{p-1}} \left(\sum_{i=0}^{n} |f(x_i) - f(x_{i+1})| \right)^p \geq \frac{c}{n^{p-1}}.$$

The last inequality is due to the fact

$$\sum_{i=0}^{n} |f(x_i) - f(x_{i+1})| \geq |f(x_0) - f(x_{n+1})| = 1.$$

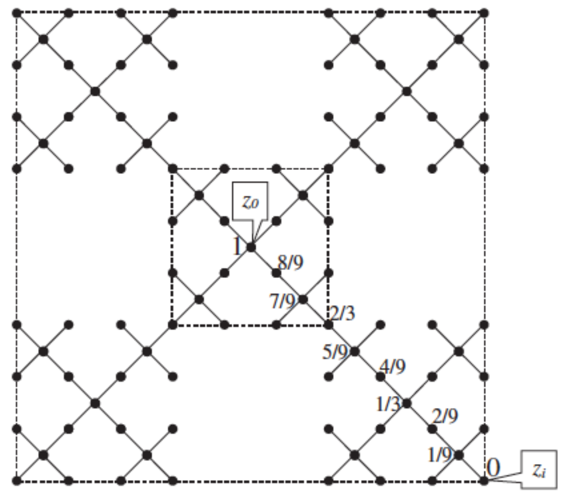

Figure 5.2: the function f on the diagonal $z_0 z_i$

Eventually, we obtain

$$\frac{\sum_{x\in\Omega} |\nabla f(x)|^p \mu(x)}{\sum_{x\in\Omega} |f(x)|^p \mu(x)} \geq \frac{c}{n^{p-1}\mu(\Omega)} \geq \frac{c}{\mu(\Omega)^{1+(p-1)/D}}.$$

5.3 Optimality of $(S^p_{D(p)})$ for $p > 2$ on Vicsek graphs

In [BCG01, Section 4], the authors established the sharpness of the Faber-Krahn inequality $(S^2_{D(2)})$ by constructing a family of sets Ω_n and functions on them. In fact, with the same functions and subsets, we can also show the optimality of $(S^p_{D(p)})$ for $p > 2$.

The weight μ here is the standard weight. Let $\Omega_n = \Gamma \bigcap [0, 3^n]^N$ be the same subset as in [BCG01], where $q = 2^N + 1 = 3^D$. Hence $\mu(\Omega_n) \simeq q^n$. Denote by z_0 the centre of Ω_n and by $z_i, i \leq 1$ its corners. Define f as follows: $f(z_0) = 1, f(z_i) = 0, i \leq 1$, and extend f as a harmonic function in the rest of Ω_n. Then f is linear on each of the paths of length 3^n, which connects z_0 with the corners z_i, and is constant elsewhere. More exactly, if z belongs to some γ_{z_0,z_i}, then $f(z) = 3^{-n} d(z_i, z)$. If not, then $f(z) = f(z')$, where z' is the nearest vertex in certain line of z_0 and z_i. See Figure 5.2 (from [BCG01]).

For any x in the $n-1$ block with centre z_0, we have $f(x) \geq \frac{2}{3}$. Therefore

$$\sum_{x\in\Omega_n} |f(x)|^p \mu(x) \geq (2/3)^p \mu(\Omega_{n-1}) \simeq q^n.$$

Also, since $|f(x) - f(y)| = 3^{-n}$ for any two neighbours x, y on each of the diagonals connecting

z_0 and z_i, and otherwise $f(x) - f(y) = 0$, we obtain

$$\|\nabla f\|_p^p \simeq \sum_{x,y\in\Omega_n} |f(x) - f(y)|^p \mu_{xy} \le \sum_{i=1}^{2^N} 3^{-np} d(z_0, z_i) = 2^N 3^{-n(p-1)}.$$

Finally combining the above two estimates, we have

$$\frac{\|\nabla f\|_p}{\|f\|_p} \le C_p 3^{-\frac{n}{p'}} q^{-\frac{n}{p}} = C_p q^{-\frac{n}{p'D} - \frac{n}{p}} \simeq \mu(\Omega_n)^{-\frac{1}{p'D} - \frac{1}{p}}.$$

This finishes the proof.

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