Application of quantum chaos methods to the oscillations of rapidly rotating stars

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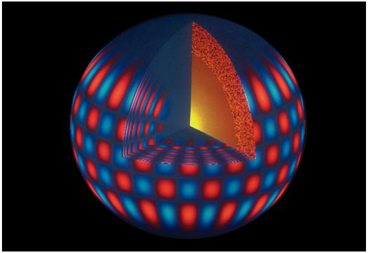
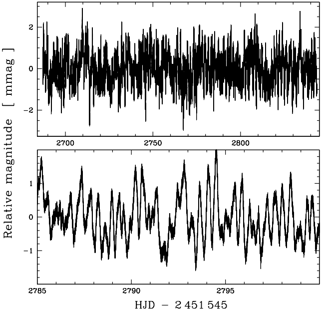
Asteroseismology

- Theory of stellar structure and evolution needs constraints.
- Stellar interiors are not accessible directly.
- Pulsating stars sustain global mode oscillations.
- Show periodic variations in their luminosity.
- Asteroseismology relates oscillation frequencies to internal properties of the star.

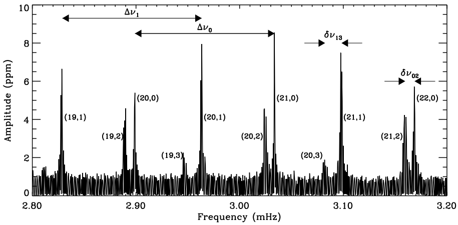




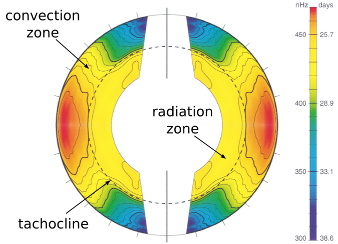
(Carrier et al., 2010)

Successes of asteroseismology: solar-like pulsators

- Seismology greatly improved knowledge of Sun's interior.
- Space missions CoRoT & Kepler for precision measurements of stellar brightness explore other solar-like pulsators.
- Oscillation spectrum of these stars is simple.
- Systematic mode identification, possibility of seismic inversion.



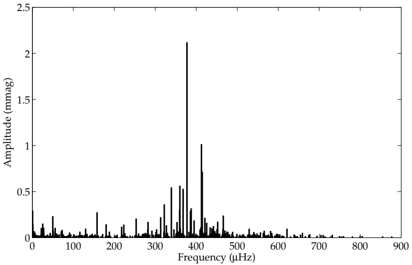
(Bedding and Kjeldsen, 2003).



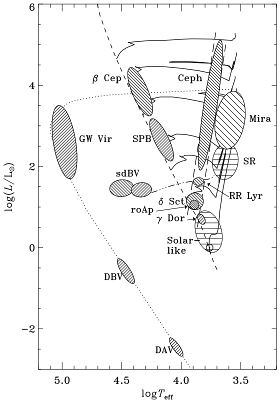
(Schou et al. 1998)

Trouble with non-evolved hot stars

- In the $M > 1.5\,M_{\odot}$ region of Hertzsprung-Russel diagram, no reliable mode identification.
- A large fraction of stars inaccessible to asteroseismic studies.
- Identified reason: rotational effects.



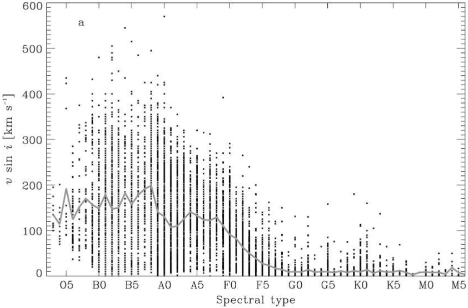
(Garciá Hernández et al. 2009)



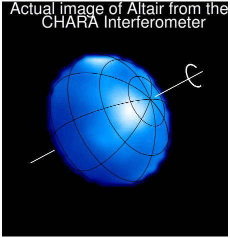
(Christensen-Dalsgaard).

Rapid rotation of non-evolved hot stars

- Non-evolved hot stars rotate rapidly on average.
- Centrifugal force causes spherical symmetry breaking.



(Royer 2009)



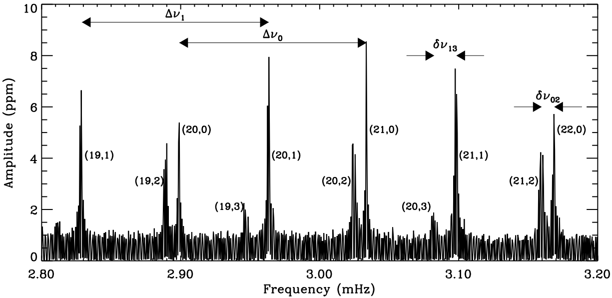
(Monnier et al 2007)

- Fundamental obstacle to mode identification.

Mode identification in the spherically symmetric case

- Very useful a priori information for mode identification: asymptotic structure of spectrum.
- For high-frequency pressure modes, 'Tassoul's asymptotic formula' (Vandakurov 1967; Tassoul 1980):

$$\nu_{n,\ell} \simeq \frac{1}{2\int_0^R \frac{dr}{c_s(r)}} \left(n + \frac{\ell}{2} + \frac{1}{4} + \alpha \right) .$$



- Formula not valid for rapidly rotating stars.

An asymptotic formula for rapidly rotating stars ?

- Accurate two-dimensional oscillation codes have been developed recently (Reese et al. 2006).
- Hints of regular asymptotic behaviour at high rotation rates.
- Empirical asymptotic formula for pressure mode frequencies (Lignières et al. 2006; Reese et al. 2008, 2009):

$$\omega_{n,\ell,m} \simeq \Delta_n n + \Delta_\ell \ell + \Delta_m |m| + \alpha.$$

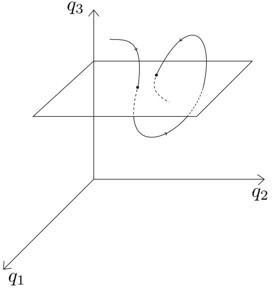
- Our goal: a theory to obtain analytical form of asymptotic formula for rapidly rotating stars.

Ray theory and Hamiltonian mechanics

- In their short-wavelength asymptotic limit, waves can be described by rays.
- Classical trajectories of Hamiltonian equations of motion:

$$\frac{d\vec{x}}{dt} = \frac{dH}{d\vec{k}}, \; \frac{d\vec{k}}{dt} = -\frac{dH}{d\vec{x}}.$$

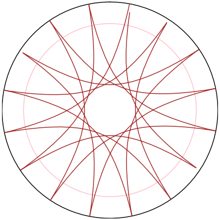
- For two degrees of freedom, phase space $(\vec{x}, \vec{k})$ is of dimension four.

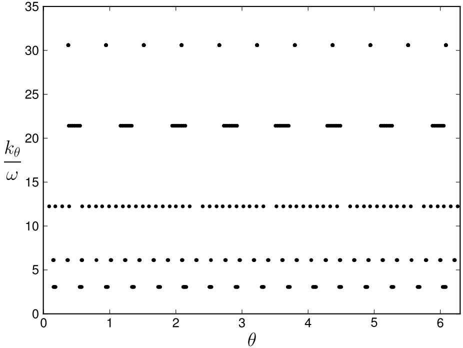


Poincaré Surface of Section

Acoustic rays in non-rotating stars

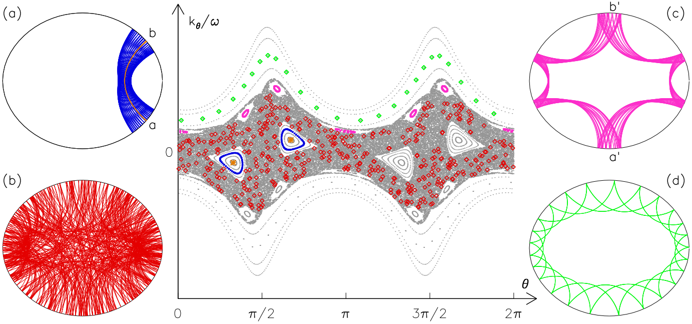
- For pressure waves, acoustic rays in the high-frequency limit.
- Non-rotating star: spherical symmetry $\Rightarrow$ ray dynamics is integrable.





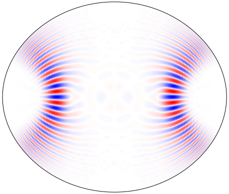
Acoustic rays in rapidly rotating stars

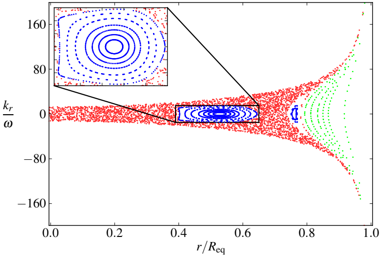
- Breaking of spherical symmetry.
- Generic transition to chaos for increasing rotation rates.
- Regular and chaotic zones in phase space ⇒ ray dynamics is mixed.
- Correspondence for waves in the short-wavelength limit ?



Quantum chaos and wave-ray correspondence

- Quantum chaos studies properties of wave systems whose ray limit is (partly or fully) chaotic.
- Modes in the short-wavelength limit are localised on phase space zones of ray dynamics.
- Spectrum is a superposition of independent sub-spectra.
- Sub-spectrum of modes in stable zones: $\omega = f(n, \ell, m)$.





Thesis work

- Analytical work to construct oscillation modes from main stable zone of phase space.
- Numerical implementation of the method with ray tracing code.
- Comparison with computations from complete numerical oscillation code.
- Consequences for observations of oscillations in rapidly rotating stars.

'Parabolic equation method' (Babich, 1968)

- Step 1: Wave equation expressed in ray-centered coordinates, ω-expansion yields Gaussian wavepacket solution.
- Step 2: Evolution of Gaussian width is linked to ray dynamics.
- Step 3: Quantization condition.

Step 0: Reduction to Helmholtz wave equation

- Adiabatic perturbations of uniformly rotating stellar model.
- Hypotheses: no perturbation of gravitational potential, no Coriolis force, no gravity waves.
- Cylindrical symmetry $\Rightarrow \Psi = \Psi_m \exp(im\phi)$.
- Pressure wave equation in the meridian plane of the star:

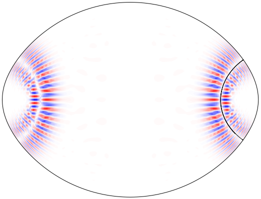
$$\vec{\nabla}^2 \Phi_m + \frac{\omega^2 - \omega_c^2 - \frac{c_s^2(m^2 - 1/4)}{d^2}}{c_s^2} \Phi_m = 0,$$

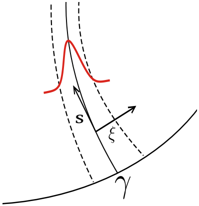
where $\Phi_m = \sqrt{d}.\Psi_m$, d distance to the rotation axis, c_s speed of sound, and ω_c cut-off frequency.

- $\tilde{c}_s = c_s / \sqrt{1 - \frac{1}{\omega^2}\left[\omega_c^2 + \frac{c_s^2(m^2-1/4)}{d^2}\right]}$.

Step 1: ω-expansion of wave equation

- Stable zone centered on stable periodic ray γ:
  ray-centered coordinates (s, ξ).





- WKB Ansatz: $\Phi_m(s,\xi) = \exp[i\omega\tau(s)]U_m(s,\xi)$.
- Assumption : localisation on transverse scale $\xi = O(1/\sqrt{\omega})$.
- Order ω^2: WKB phase $\tau = \int_0^s ds'/\tilde{c}_s(s')$.
- Order ω:

$$\frac{\partial^2 V_m}{\partial \nu^2} - K(s)\nu^2 V_m + \frac{2i}{\tilde{c}_s}\frac{\partial V_m}{\partial s} = 0,$$

  with $\nu = \sqrt{\omega}\xi$, $V_m = U_m/\sqrt{\tilde{c}_s}$ and $K(s) = \frac{1}{\tilde{c}_s(s)^3}\frac{\partial^2 \tilde{c}_s}{\partial \xi^2}\Big|_{\xi=0}$.
- Quantum harmonic oscillator in ν.

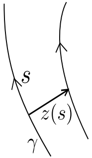
Step 2: Evolution of Gaussian width from ray dynamics

- Gaussian wavepacket solution: $V_m^0 = A(s)\exp\left(i\frac{\Gamma(s)}{2}\nu^2\right)$, where $\Gamma(s)$ is complex.
- $\frac{1}{\tilde{c}_s}\frac{d\Gamma(s)}{ds} + \Gamma(s)^2 + K(s) = 0$ and $\frac{1}{A(s)}\frac{dA(s)}{ds} = \frac{-\tilde{c}_s}{2}\Gamma(s)$.
- Change of variable $\Gamma(s) = p(s)/z(s)$ yields:

$$\frac{1}{\tilde{c}_s(s)}\frac{d}{ds}\begin{bmatrix} z(s) \\ p(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K(s) & 0 \end{bmatrix}\begin{bmatrix} z(s) \\ p(s) \end{bmatrix}.$$

$\Leftrightarrow$ Equations for linear deviations of rays from periodic ray γ:

$$\begin{pmatrix} z_f \\ p_f \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} z_i \\ p_i \end{pmatrix}$$



- Periodic ray γ is stable $\Rightarrow |\mathrm{Tr}(M)| < 2$ and $\Lambda^{\pm} = \exp(\pm i\alpha)$.
- $\forall s$, $\mathrm{Im}(\Gamma) < 0$ to have localisation of wavepacket on ray γ.

Step 3: Quantization condition

- Higher-order solutions of quantum harmonic oscillator are Hermite-Gauss polynomials:

$$V_m^\ell(s,\nu) = \left(\frac{i}{\sqrt{2}}\right)^\ell \left(\frac{\bar{z}}{z}\right)^{\ell/2} H_\ell(\sqrt{\mathrm{Im}(\Gamma)}\nu)\frac{\exp\left(i\frac{\Gamma}{2}\nu^2\right)}{\sqrt{z}} \ .$$

- Recall that $\Phi_m(s,\nu) = \exp\left(i\omega\int^s \frac{ds'}{c_s(s')}\right)\sqrt{\tilde{c}_s}V_m^\ell(s,\nu)$.
- Phase accumulated by Φ_m after one period of γ must be a multiple of 2π:

$$\omega_{n,\ell,m}\oint_\gamma \frac{ds}{\tilde{c}_s} - \frac{\alpha + 2\pi N_r}{2} - (\alpha + 2\pi N_r)\ell = 2\pi n + \pi \ .$$

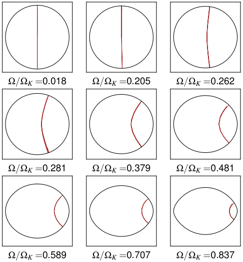
Numerical implementation of the method

- Formula for frequency spacings:

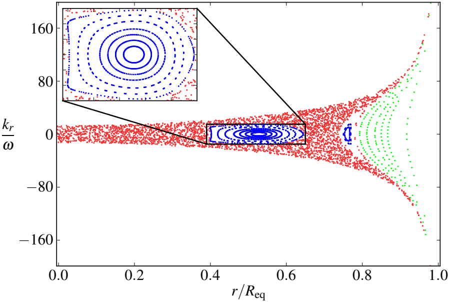
$$\omega_{n,l,m} = \delta_n(m)n + \delta_\ell(m)\ell + \beta(m) \ ,$$

with

$$\delta_n(m) = \frac{2\pi}{\oint_\gamma \frac{ds}{\tilde{c}_s}} \ , \quad \delta_\ell(m) = \frac{2\pi N_r + \alpha}{\oint_\gamma \frac{ds}{\tilde{c}_s}} \ \text{ and } \ \beta(m) = \frac{\delta_n + \delta_\ell}{2} \ .$$



with $\Omega_K = \sqrt{GM/R_{\mathrm{eq}}^3}$



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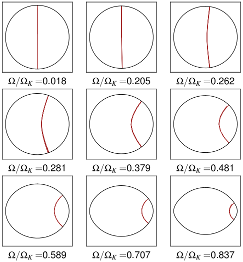
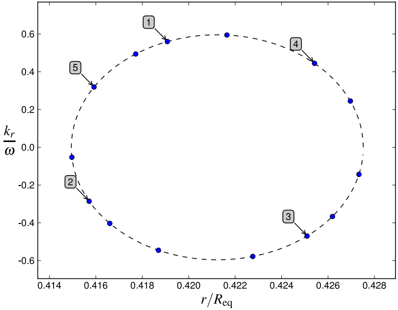




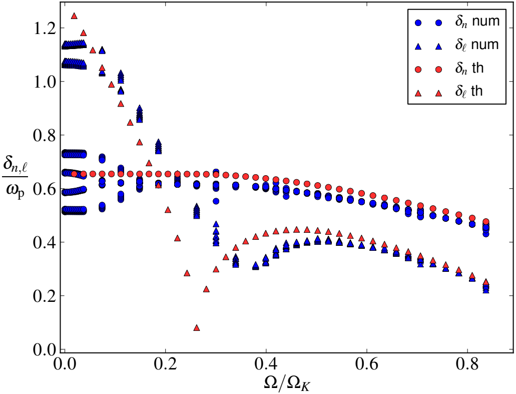
with $\Omega_K = \sqrt{GM/R_{\rm eq}^3}$

Comparisons with complete numerical computations: the TOP code

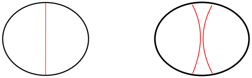
- High-frequency modes from accurate two-dimensional oscillation code using spectral method.
- Includes Coriolis force and gravitational potential perturbations.
- Polytropic stellar models ($p_0 = K\rho_0^{1+1/N}$) with $N = 3$, at different rotation rates.
- Mode following from $\Omega/\Omega_K = 0$ to 0.896, in the frequency range $n \in [42, 51]$.

Comparisons with complete numerical computations: frequency spacings at $m = 0$

- Good agreement outside bifurcation range (Pasek *et al.* 2011).

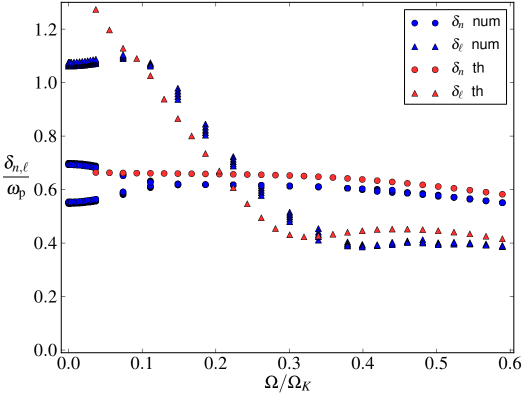


$m = 0$



Comparisons with complete numerical computations: frequency spacings at $m = \pm 1$

- Good agreement over a large range of rotation rates (Pasek *et al.* 2011).

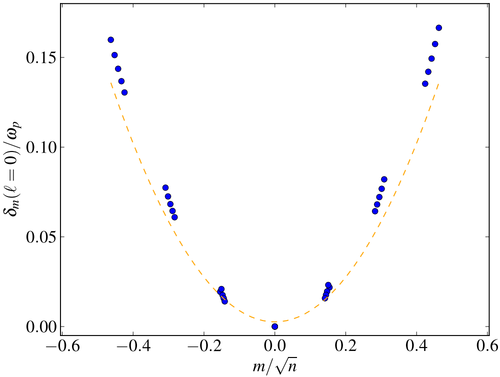


$|m| = 1$

Comparisons with complete numerical computations: rotational splitting δ_m

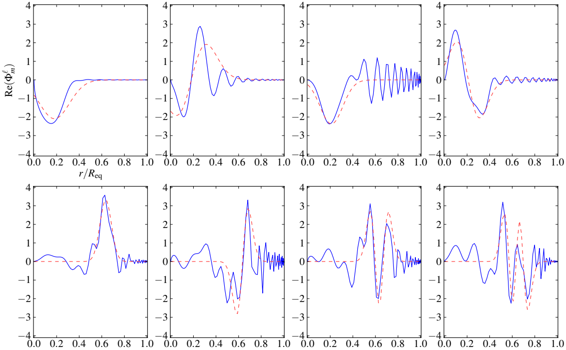
- Rotational splitting in co-rotating frame: $\delta_m = \omega_{n,\ell,m} - \omega_{n,\ell,0}$.
- From explicit dependence of δ_n on m, we derive

$$\delta_m(\ell = 0) \simeq \left(\frac{m}{\sqrt{n}}\right)^2 \frac{1}{4\pi} \oint \frac{c_s}{d^2} ds.$$



$\Omega/\Omega_K = 0.419$, good agreement for $\Omega/\Omega_K > 0.4$.

Comparisons with complete numerical computations: eigenfunctions



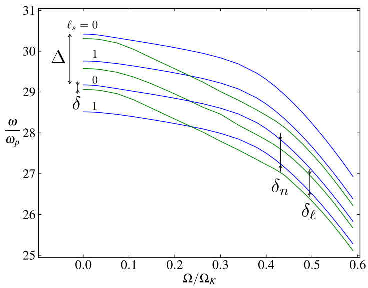
Equatorial cuts of eigenfunctions for different values of quantum numbers and Ω/Ω_K

Astrophysical applications of regular modes

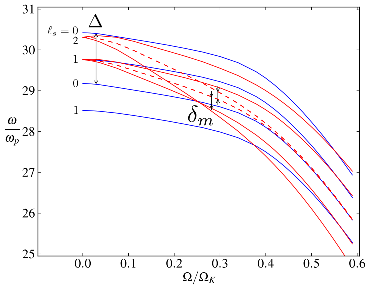
- For mode identification: a priori information on the structure of spectrum.
- For seismic diagnostic and inversion: seismic observables linked to internal properties of the star.
- For numerical codes: asymptotic regularities help search for patterns in spectrum.

Phenomenology of spectral observables

- Structure of regular spectrum given by δ_n, δ_ℓ, δ_m, and $m\Omega$ in the inertial frame.
- At small Ω, rapid evolution of δ_ℓ: reorganisation of spectrum.
- At high Ω, evolution dominated by advection term $m\Omega$ for frequencies in inertial frame.
- Frequency clustering for $\delta_n \sim \delta_\ell$, and $\delta_n \sim \Omega$ in inertial frame.



$m = 0$



$m = 0,\ |m| \in 1, 2$

Conclusion

- Application of wave chaos to a very large scale system: $\lambda \sim 10$ times the size of the Earth.
- Central result: asymptotic formulas for regular oscillation frequencies in rapidly rotating stars using ray theory.
- These formulas provide physical understanding of frequency spacings, which is useful for mode identification and inversion.

Perspectives

- Search for regular frequency spacings in observed spectra of rapidly rotating stars:
  - Hints of regularities recently found (García Hernández et al. 2009; Mantegazza et al. 2012).
  - Need to develop tools to search for regularities using theoretical calculations.
- Refinements of theory: bifurcation and near-integrable regime, discontinuities, etc.
- Asymptotic theory of chaotic modes.