Phénomènes de Stokes et approche galoisienne des problèmes de confluence

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Je rêve d'un jour où l'égoïsme ne régnera plus dans les sciences, où on s'associera pour étudier, au lieu d'envoyer aux académiciens des plis cachetés, on s'empressera de publier ses moindres observations pour peu qu'elles soient nouvelles, et on ajoutera « je ne sais pas le reste ».
Évariste Galois (1811-1832)

Résumé

Résumé

Cette thèse porte sur la théorie de Galois différentielle. Elle est divisée en deux parties. La première concerne la théorie de Galois différentielle paramétrée, et la seconde, les équations aux q-différences.

Dans le chapitre 2, nous exposons une généralisation de l'algorithme de Kovacic qui permet de calculer le groupe de Galois paramétré de certaines équations différentielles paramétrées d'ordre 2.

Dans le chapitre 3, nous présentons une généralisation du théorème de densité de Ramis qui donne un ensemble de générateurs topologiques du groupe de Galois pour les équations différentielles linéaires paramétrées à coefficients dans un anneau convenable. Nous obtenons une contribution au problème inverse dans cette théorie de Galois, donnons un critère d'isomonodromie, et répondons partiellement à une question posée par Sibuya.

Dans le chapitre 4, il est question de confluence et d'équations aux q-différences. Nous prouvons comment la transformée de Borel-Laplace d'une série formelle divergente solution d'une équation différentielle linéaire à coefficients dans $\mathbb{C}(z)$ peut être uniformément approchée par un q-analogue de la transformée de Borel-Laplace appliqué à une série formelle solution d'une famille d'équations aux q-différences linéaires qui discrétise l'équation différentielle. Nous faisons directement les calculs dans le cas des séries hypergéométriques basiques, et nous prouvons sous des hypothèses raisonnables, qu'une matrice fondamentale d'une équation différentielle linéaire à coefficients dans $\mathbb{C}(z)$ peut être uniformément approchée par une matrice fondamentale d'une famille d'équations aux q-différences linéaires correspondante.

Mots-clefs

Théorie de Galois différentielle. Équations différentielles linéaires. Théorie de Galois différentielle à paramètre. Équations différentielles linéaires paramétrées. Phénomène de Stokes. Transformées de Borel-Laplace. Algorithme de Kovacic. Systèmes intégrables. Isomonodromie. Théorème de densité. Groupes différentiels. Problème inverse. Équations aux q-différences. Confluence. Séries hypergéométriques basiques. Séries hypergéométriques.

Stokes phenomenoms and Galoisian approach to confluence problems

Abstract

This thesis deal with the differential Galois theory and more particularly the Borel-Laplace summation of the divergent power series. There are two parts. The first one (Chapters 2 and 3), is about parameterized differential Galois theory while the second one (Chapter 4), involves q-difference equations.

In Chapter 2, we present a generalization of Kovacic's algorithm that allow us to compute the parameterized Galois group of some parameterized differential equations of order two.

In Chapter 3, we prove a generalization of the density theorem of Ramis that gives a set of topological generator for the Galois group of parameterized differential equations having coefficients on a convenient ring. We obtain a contribution to the inverse problem in this Galois theory, give a criterion of isomonodromy, and provide a partial answer to a question of Sibuya.

Chapter 4 deal with confluence and q-difference equations. We show how the Borel-Laplace transformation of a divergent formal power series solution of a linear differential equation in coefficients in $\mathbb{C}(z)$ could be uniformly approximated by a q-analogue of the Borel-Laplace summation applied to a formal power series solution of a family of linear q-difference equations that discretize the differential equation. We make explicitly the computations for the basic hypergeometric series and prove under convenient assumptions, that a fundamental solution of a linear differential equation may be uniformly approximated by a fundamental solution of a corresponding family of linear q-difference equations.

Keywords

Differential Galois theory. Linear differential equations. Parameterized differential Galois theory. Parameterized linear differential equations. Stokes phenomenon. Borel-Laplace transformations. Kovacic's algorithm. Completely integrable system. Isomonodromy. Density theorem. Differential groups. Inverse problem. q-difference equations. Confluence. Basic hypergeometric series. Confluent hypergeometric series.

Table des matières

Chapitre 1

Introduction

Sommaire

Cette thèse compile trois articles qui correspondent respectivement aux chapitres 2 à 4. Ces derniers peuvent donc être lus indépendamment les uns des autres.

– [Dre12], *Computing the Galois group of some parameterized linear differential equation of order two,* est à paraître dans Proceedings of the American Mathematical Society. Il traite de la généralisation de l'algorithme de Kovacic dans le cadre de la théorie de Galois différentielle à paramètres.

– [Dre13], *A density theorem for parameterized differential Galois theory,* est à paraître dans Pacific Journal of Mathematics. Le résultat principal est la généralisation pour la théorie de Galois différentielle paramétrée du théorème de densité de Ramis.

– [Dre14], *Confluence of meromorphic solutions of q-difference equations,* est un preprint. Nous prouvons comment la sommation de Borel-Laplace d'une série divergente solution d'une équation différentielle peut être approchée par une fonction méromorphe, qui est solution d'une famille d'équations aux q-différences.

* * *

Le sujet principal de cette thèse est la théorie de Galois différentielle et son lien avec les problèmes de ressommation de séries divergentes. Nous allons commencer par un bref survol historique de ces questions et ne prétendons absolument pas à l'exhaustivité. En particulier, nous nous restreindrons à la théorie linéaire. Nous renvoyons à [Ram93, Bal00], pour plus de détails sur ce qui suit, et aux récents travaux de Malgrange [Mal02], pour la théorie non linéaire.

La veille de sa mort, en 1832, Galois mentionne dans une lettre à son ami Chevalier, une théorie de l'ambiguïté qui pourrait faire penser aux prémices de la théorie de Galois différentielle. Ce n'est qu'une décennie plus tard, en 1846, que Liouville publie les manuscrits de Galois qui rencontrent un succès important. Il travaille ensuite sur la résolubilité par quadratures des équations différentielles. Cela mènera entre autres aux extensions Liouvilliennes et aux systèmes Hamiltoniens Liouville intégrables, qui sont aujourd'hui un domaine de recherche actif. En 1883, Picard s'intéresse aux équations différentielles linéaires à coefficients dans $\mathbb{C}(z)$. Il définit deux groupes, que l'on appelle aujourd'hui groupe de Galois différentiel et groupe de monodromie. Il remarque que le groupe de Galois différentiel est en réalité un groupe algébrique mais ne développe aucune théorie. On attribue classiquement à Schlesinger le fait que pour les équations différentielles linéaires singulières régulières, le groupe de monodromie est dense pour la topologie de Zariski dans le groupe de Galois différentiel. Vessiot étend ensuite les travaux de Picard pour des systèmes à coefficients plus généraux. Il prouve aussi qu'une équation différentielle linéaire est résoluble par quadrature si et seulement si son groupe de Galois différentiel est résoluble. Il est à noter que c'est Kolchin, quelques décennies plus tard, qui énonce ce résultat sous la forme que nous connaissons aujourd'hui. Il en profite pour formaliser la théorie sous sa forme actuelle, en considérant des équations différentielles linéaires à coefficients dans un corps différentiel dont le corps des constantes est algébriquement clos et de caractéristique nulle. Il introduit la notion d'extension de Picard-Vessiot, prouve que le groupe de Galois est un groupe algébrique et démontre le théorème de correspondance galoisienne.

Le phénomène de Stokes va tenir un rôle important dans la théorie de Galois différentielle. Étant donnée une série formelle divergente solution d'une équation différentielle linéaire à coefficients méromorphes, on peut lui associer des fonctions analytiques au voisinage de 0 dans des secteurs, qui sont asymptotiques à la série formelle et solutions de la même équation. Le fait que deux solutions définies sur deux secteurs différents ne se recollent pas en une unique solution est appelé phénomène de Stokes. Ce physicien anglais a découvert ce phénomène dans les années 1850 en travaillant sur l'équation d'Airy, qui provient d'un problème d'optique, mais ne fit pas de théorie générale. Au début du XXe siècle, Birkhoff prouve sous des hypothèses restrictives que ce phénomène engendre des objets qui font partie d'un ensemble qui caractérise les classes d'équivalences méromorphes des systèmes différentiels. Voir §1.1 pour plus de détails. Wasow démontre bien plus tard que pour toute série formelle solution d'un système différentiel linéaire à coefficients méromorphes, pour tout secteur d'angle suffisamment petit, on peut trouver une solution du système analytique au voisinage de 0 sur le secteur et qui a pour développement asymptotique la série formelle initiale. Cependant, ce théorème ne garantit pas l'unicité de la solution asymptotique. Il faut attendre la fin des années 1970, pour que Ramis introduise la notion de séries k-sommables et obtienne un résultat de ce type. Ecalle rend ensuite le calcul de « la » solution asymptotique effectif, grâce à l'accélératrice d'Ecalle : il prouve qu'à une série formelle, solution d'une équation différentielle linéaire à coefficients méromorphes, on peut appliquer successivement un nombre fini de transformations de Borel et Laplace pour obtenir une solution méromorphe au voisinage de 0 sur un secteur donné. Il faut noter que grâce à Balser, on sait qu'une série multisommable peut s'écrire comme une somme finie de séries sommables sur un seul niveau. Si l'on est capable d'exhiber la décomposition, ce qui n'est pas toujours faisable, cela peut faciliter le calcul de la sommation de Borel-Laplace. C'est Ramis qui remarquera le caractère « galoisien » du phénomène de Stokes, c'est-à-dire que les relations algébriques et différentielles entre des séries formelles solutions d'équations différentielles sont préservées par la sommation de Borel-Laplace. Ceci mènera au théorème de densité de Ramis qui a été démontré dans les années 1980 : étant donnée une équation différentielle linéaire à coefficients germes de fonctions méromorphes, le groupe engendré par la monodromie, le tore exponentiel, et les matrices de Stokes, est dense pour la topologie de Zariski dans le groupe de Galois différentiel de l'équation. Voir §1.1 pour une définition précise de ces objets.

* * *

Les domaines tels que l'étude de déformations isomonodromiques ou les équations de Painlevé ont justifié l'introduction par Cassidy et Singer notamment, voir [CS07], d'une théorie de Galois pour les systèmes différentiels linéaires dépendant de paramètres. Le groupe de Galois mesure ici les relations algébriques et différentielles, par rapport aux paramètres, entre les solutions. Ce dernier n'est plus un groupe algébrique mais un groupe différentiel, c'est-à-dire qu'il annule un nombre fini d'équations différentielles polynomiales à coefficients dans un corps.

Dans le chapitre 2, nous généralisons l'algorithme de Kovacic pour cette théorie, ce qui permettra de calculer les groupes des Galois différentiels paramétrés de certaines équations différentielles paramétrées d'ordre 2. De plus, dans le cas où il n'y a pas de solution Liouvillienne, nous donnons une condition nécessaire et suffisante pour que l'équation considérée puisse être complétée en un système intégrable.

Dans le chapitre 3, nous prouvons un analogue du théorème de densité pour la théorie de Galois différentielle paramétrée. En particulier, nous sommes amenés à étudier comment s'insère le phénomène de Stokes dans la théorie de Galois différentielle à paramètres. Nous prouvons que si les coefficients de l'équation sont analytiques par rapport aux paramètres, alors les matrices de Stokes paramétrées ont des entrées analytiques en les paramètres sur un ouvert à priori plus petit. Comme applications de notre théorème de densité nous obtenons : une contribution au problème inverse dans la théorie de Galois différentielle paramétrée ; un critère d'isomonodromie ; ainsi qu'une réponse partielle à une question posée par Sibuya.

* * *

Lorsque q tend vers 1, l'opérateur aux q-différences $d_q := f \mapsto \frac{f(qz)-f(z)}{(q-1)z}$, « tend » vers l'opérateur de dérivation classique. Par conséquent, toute équation différentielle linéaire peut être discrétisée en une famille d'équations aux q-différences linéaires. Considérons une série formelle solution d'une équation différentielle linéaire. Dans le chapitre 4, nous prouvons que la sommation de Borel-Laplace de la série formelle peut être approchée par une fonction méromorphe, qui est solution d'une famille d'équations aux q-différences correspondante. Cette solution méromorphe s'obtient en appliquant successivement des q-analogues des transformées de Borel et Laplace à une série formelle qui est solution d'une famille d'équations aux q-différences correspondante. Nous explicitons les calculs dans le cas des séries hypergéométriques basiques, et nous prouvons sous des hypothèses raisonnables, qu'une matrice fondamentale d'une équation différentielle linéaire à coefficients dans $\mathbb{C}(z)$ peut être uniformément approchée par une matrice fondamentale d'une famille d'équations aux q-différences linéaires correspondante. Ces résultats sont analogues à ceux prouvés par Sauloy dans [Sau00], mais peuvent s'appliquer à des systèmes non fuchsiens.

* * *

La présente introduction est divisée en trois sections. La première traite de la théorie de Galois différentielle classique. La deuxième porte sur la théorie de Galois différentielle paramétrée. Nous en profitons pour présenter nos résultats de la partie 1. Enfin, la dernière section a pour sujet les équations aux q-différences. Nous y détaillons les principales contributions contenues dans la partie 2.

1.1 Théorie de Galois différentielle

Le but de cette section est de faire un survol rapide de la théorie de Galois pour les systèmes différentiels linéaires. Nous commencerons par rappeler succinctement les bases de cette théorie. Nous nous intéresserons ensuite aux systèmes différentiels fuchsiens. Nous en profiterons pour parler du problème de Riemann-Hilbert et du problème inverse. Nous traiterons les cas local et global dans toutes leurs généralités. Nous finirons par exposer brièvement l'algorithme de Kovacic. Nous référons à l'ouvrage [vdPS03] pour plus de détails.

1.1.1 Premières définitions, premières propriétés

Dans tout ce qui suit, les corps seront supposés commutatifs et de caractéristique nulle. Soit (K, ∂) un corps différentiel, c'est-à-dire un corps muni d'une dérivation ∂. Pour alléger les notations, nous écrirons K plutôt que (K, ∂) lorsqu'il n'y aura pas de confusions possibles. Notons par C le corps des constantes de K, à savoir le sous-corps de K des éléments de dérivées nulles. Nous ferons l'hypothèse que C est algébriquement clos.

Considérons un système différentiel linéaire de la forme $\partial Y = AY$, où $A \in \mathrm{M}_m(K)$, c'est-à-dire une matrice carrée de taille m à coefficients dans K. Il existe une extension de Picard-Vessiot $L|K$ associée à ce système, à savoir que $L|K$ est une extension de corps différentiel dont le corps des constantes est C, et qui est engendrée par K et les entrées d'une matrice fondamentale, à savoir une matrice inversible de solutions. De plus, deux extensions de Picard-Vessiot de $\partial Y = AY$ sont isomorphes en tant que corps différentiel.

Fixons $L|K$, extension de Picard-Vessiot de $\partial Y = AY$, et F matrice fondamentale à coefficients dans L. Le groupe de Galois différentiel $Gal_\partial\left(L\middle|K\right)$ est le groupe des automorphismes de corps de L, commutant avec la dérivation ∂ et laissant K invariant. L'application suivante est un morphisme de groupe injectif

$$\begin{array}{rccc} \rho_F : & Gal_\partial\left(L\middle|K\right) & \longrightarrow & \mathrm{GL}_m(C) \\ & \sigma & \mapsto & F^{-1}\sigma(F), \end{array}$$

où $\mathrm{GL}_m(C)$ désigne les matrices inversibles à coefficients dans C. L'image de l'application dépend de la matrice fondamentale à coefficients dans L. Un autre choix donnera lieu à un morphisme injectif conjugué au premier. Un fait très important est que $\rho_F\left(Gal_\partial\left(L\middle|K\right)\right)$ est un sous-groupe algébrique de $\mathrm{GL}_m(C)$. Nous identifierons désormais le groupe de Galois différentiel avec un sous-groupe algébrique de $\mathrm{GL}_m(C)$, pour une matrice fondamentale fixée. Le groupe de Galois différentiel mesure les relations algébriques entre les entrées de la matrice fondamentale. Grosso modo, plus celui ci est gros, moins il y aura de relations entre les solutions.

Il existe un équivalent de la correspondance galoisienne pour la théorie de Galois différentielle. Soit $\partial Y = AY$ un système différentiel linéaire à coefficients dans K, soit $L|K$ l'extension de Picard-Vessiot et soit $Gal_\partial\left(L\middle|K\right)$ le groupe de Galois différentiel. Définissons S, comme l'ensemble des sous-groupes algébriques de $Gal_\partial\left(L\middle|K\right)$, et T comme l'ensemble des sous-corps différentiels de L qui contiennent K. Les deux applications suivantes sont inverses l'une de l'autre :

$$\begin{array}{rccc} s: & S & \longrightarrow & T \\ & G & \mapsto & L^G := \{l \in L | \forall \sigma \in G, \sigma(l) = l\}, \end{array}$$

$$\begin{array}{rccc} t: & T & \longrightarrow & S \\ & M & \mapsto & Gal_\partial \left(L \middle| M\right) := \left\{\sigma \in Gal_\partial \left(L \middle| K\right) \middle| \forall m \in M, \sigma(m) = m\right\}. \end{array}$$

En particulier, pour prouver qu'un sous-groupe G est dense pour la topologie de Zariski dans $Gal_\partial \left(L \middle| K\right)$, il suffit de prouver que $L^G = K$. Ce sera l'ingrédient de base de la preuve des théorèmes de densité, cf. les théorèmes 1.1.3 et 1.1.4.

Les systèmes différentiels linéaires $\partial Y = AY$ et $\partial Y = BY$, avec $A, B \in \mathrm{M}_m\left(K\right)$, sont dit équivalents sur K, s'il existe une matrice inversible $H \in \mathrm{GL}_m(K)$, telle que

$$A = H\left[B\right]_\partial := HB\left(H\right)^{-1} + \partial\left(H\right)H^{-1}.$$

Un calcul rapide montre que :

$$\partial Y = BY \Longleftrightarrow \partial\left(HY\right) = AHY.$$

Notons $\mathbb{C}\{z\}$ l'anneau des germes de fonctions holomorphes en 0, et $\mathbb{C}(\{z\})$ son corps des fractions, c'est-à-dire le corps des germes de fonctions méromorphes en 0. Son corps des constantes est $\mathbb{C}$, qui est algébriquement clos et de caractéristique nulle, et la théorie de Picard-Vessiot peut s'appliquer pour les systèmes différentiels linéaires à coefficients dans $\mathbb{C}(\{z\})$, que nous munirons de la dérivation $\delta := z\frac{d}{dz}$. Soit $\mathbb{C}[[z]]$ l'anneau des séries formelles et soit $\mathbb{C}((z))$ son corps des fractions. Lorsque deux systèmes différentiels linéaires à coefficients dans $\mathbb{C}(\{z\})$ seront équivalents sur $\mathbb{C}(\{z\})$ (resp. sur $\mathbb{C}((z))$), nous dirons qu'ils sont méromorphiquement équivalents (resp. formellement équivalents). Les classes d'équivalences méromorphes (resp. formelles) des systèmes différentiels linéaires à coefficients dans $\mathbb{C}(\{z\})$, sont l'ensemble des systèmes différentiels linéaires à coefficients dans $\mathbb{C}(\{z\})$, quotienté par la relation d'équivalence

« $\delta Y = AY$ et $\delta Y = BY$ sont méromorphiquement (resp. formellement) équivalents » .

Classifier méromorphiquement et formellement les systèmes différentiels linéaires à coefficients dans $\mathbb{C}(\{z\})$, signifie chercher des objets qui caractérisent les classes d'équivalences méromorphes (resp. formelles) des systèmes différentiels linéaires à coefficients dans $\mathbb{C}(\{z\})$.

1.1.2 Systèmes différentiels linéaires fuchsiens

Nous allons maintenant considérer le système différentiel linéaire $\delta Y = AY$, où la matrice A appartient à $\mathrm{M}_m\left(\mathbb{C}\{z\}\right)$. Un tel système est dit localement fuchsien. Comme les coefficients sont définis localement, on appellera le groupe de Galois différentiel associé à une matrice fondamentale, le groupe de Galois différentiel local. Fixons une détermination du logarithme que nous noterons $\log(z)$, et soit $\widetilde{\mathbb{C}}$ la surface de Riemann du logarithme. L'algorithme de Frobenius, cf. [vdPS03], §3.1.1, pour un énoncé équivalent à ce qui suit, nous donne une matrice fondamentale de la forme

$$\hat{H}(z)e^{L\log(z)},$$

où $L \in \mathrm{M}_m(\mathbb{C})$ et $\hat{H}(z) \in \mathrm{GL}_m\Big(\mathbb{C}(\{z\})\Big)$. Les entrées de la matrice fondamentale sont des germes de fonctions analytiques sur $\widetilde{\mathbb{C}}$. Le prolongement analytique de $\hat{H}(z)e^{L\log(z)}$, le long d'un simple lacet γ orienté positivement autour de 0, transforme celle ci en $\hat{H}(z)e^{L\log(z)}e^{2i\pi L}$. La matrice de monodromie $M := e^{2i\pi L}$ ne dépend pas de γ. Par construction, cette dernière caractérise les classes d'équivalences méromorphes des systèmes localement fuchsiens. Grâce aux propriétés du prolongement analytique, on trouve qu'elle appartient au groupe de Galois différentiel du système. Le théorème de Schlesinger nous dit que le groupe engendré par M est dense pour la topologie de Zariski dans le groupe de Galois différentiel local, vu comme sous-groupe algébrique de $\mathrm{GL}_m(\mathbb{C})$.

Nous nous intéressons désormais au cas global. Considérons le système différentiel linéaire $\frac{d}{dz}Y = AY$ avec $A \in \mathrm{M}_m\Big(\mathbb{C}(z)\Big)$. On appellera le groupe de Galois différentiel associé à une matrice fondamentale, le groupe de Galois différentiel global. Si le système différentiel n'admet pas de singularité, alors le groupe de Galois différentiel global est l'identité. Supposons donc que ce ne soit pas le cas. Soit $S := \{\alpha_1, \dots, \alpha_{k+1}\} \subset \mathbb{P}_1(\mathbb{C})$ l'ensemble des singularités de $\frac{d}{dz}Y = AY$. Quitte à faire un changement de variable de la forme $z \mapsto (z-a)^{-1}$, nous pouvons nous ramener au cas où $\alpha_{k+1} = \infty$. Nous supposerons que le système est fuchsien, ce qui signifie que pour chacun des changements de variable $z \mapsto z - \alpha_i$, pour $i \leq k$ et $z \mapsto z^{-1}$, nous obtenons un système localement fuchsien en 0.

Fixons $x \in \mathbb{P}_1(\mathbb{C}) \setminus S$. Nous pouvons construire une matrice fondamentale F de $\frac{d}{dz}Y = AY$ ayant des entrées analytiques en x. Le groupe fondamental de

$$\pi_1\Big(\mathbb{P}_1(\mathbb{C}) \setminus S; x\Big),$$

est engendré par k générateurs $\gamma_1, \dots, \gamma_k$ qui correspondent aux lacets homotopes au lacet faisant un cercle orienté positivement de rayon suffisamment petit autour de la singularité α_i. Le prolongement analytique de F suivant un lacet homotope à γ_i transforme F en FM_i avec $M_i \in \mathrm{GL}_m(\mathbb{C})$, que nous appellerons matrice de monodromie de la singularité α_i. Nous construisons la matrice de monodromie de la singularité ∞, $M_{k+1} \in \mathrm{GL}_m(\mathbb{C})$, comme étant la matrice vérifiant $\prod_{i=1}^{k+1} M_i = \mathrm{Id}$. Notons que comme $\mathbb{P}_1(\mathbb{C}) \setminus S$ est connexe, les matrices de monodromie ne dépendent pas du point $x \in \mathbb{P}_1(\mathbb{C}) \setminus S$ choisi, mais uniquement de la matrice fondamentale choisie.

Les matrices de monodromie appartiennent au groupe de Galois différentiel global. Le théorème de Schlesinger nous dit que le groupe engendré par les M_i est dense pour la topologie de Zariski dans le groupe de Galois différentiel global, vu comme sous-groupe algébrique de $\mathrm{GL}_m(\mathbb{C})$.

Intéressons-nous maintenant à la version faible du problème de Riemann-Hilbert. Soit $S := \{\alpha_1, \dots, \alpha_{k+1}\}$, un ensemble fini de $\mathbb{P}_1(\mathbb{C})$, et soit $x \in \mathbb{P}_1(\mathbb{C}) \setminus S$. Étant donnée une représentation

$$\rho : \pi_1\Big(\mathbb{P}_1(\mathbb{C}) \setminus S; x\Big) \mapsto \mathrm{GL}_m(\mathbb{C}),$$

du groupe fondamental $\pi_1\Big(\mathbb{P}_1(\mathbb{C}) \setminus S; x\Big)$, peut-on trouver un système différentiel fuchsien, ayant toutes ses singularités dans S, tel que pour une matrice fondamentale analytique en x, le groupe engendré par les matrices de monodromie, par rapport à cette matrice fondamentale, soit l'image de ρ ? La réponse est oui, cf. [AB94, Bea93] et [vdPS03], §5.3. De ce résultat et du théorème de Schlesinger découle la résolution du problème inverse en théorie de Galois différentielle, qui a été originellement prouvée dans [TT79]. Voir aussi [MS96a, MS96b].

Théorème 1.1.1 (Carol et Marvin Tretkoff). *Pour tout sous-groupe algébrique G de $\mathrm{GL}_m(\mathbb{C})$, il existe $A \in \mathrm{M}_m\Big(\mathbb{C}(z)\Big)$, et F une matrice fondamentale de $\frac{d}{dz}Y = AY$, telles que l'on a un isomorphisme de groupe*

$$\begin{array}{rccc} \rho_F : & Gal_{\frac{d}{dz}}\left(L\middle|\mathbb{C}(z)\right) & \longrightarrow & G \\ & \sigma & \mapsto & F^{-1}\sigma(F), \end{array}$$

où L désigne l'extension de Picard-Vessiot associée à la matrice F.

1.1.3 Systèmes différentiels linéaires : cas général

Considérons maintenant le système différentiel linéaire $\delta Y = AY$, où A appartient à $\mathrm{M}_m\Big(\mathbb{C}(\{z\})\Big)$. La situation est plus compliquée que le cas localement fuchsien, puisque des séries formelles divergentes peuvent être solutions, et nous sommes obligés de raisonner dans un premier temps par classes d'équivalences formelles. Comme nous pouvons le voir dans [BJL80, LR01], il existe

- $\hat{H} \in \mathrm{M}_m\Big(\mathbb{C}[[z]]\Big)$,
- $L \in \mathrm{M}_m(\mathbb{C})$,
- $q_1, \dots, q_m \in z^{-1/\nu}\mathbb{C}[z^{-1/\nu}]$, avec $\nu \in \mathbb{N}^*$,

tels que

$$A = \hat{H}\Big[L + \mathrm{Diag}(\delta q_i)\Big]_{\delta}. \tag{1.1.1}$$

Autrement dit, le système $\delta Y = AY$ est formellement équivalent à $\delta Y = (L + \delta q_i)\, Y$. Grosso modo ce théorème nous dit qu'il existe une matrice fondamentale de la forme $\hat{H}(z)e^{L\log(z)}\mathrm{Diag}\left(e^{q_i(z)}\right)$. Cependant, cette dernière formulation est imprécise, car les différentes matrices ne peuvent à priori pas être multipliées entre elles. Notons que le premier résultat de ce type a été démontré conjointement par Hukuhara et Turrittin, cf. [vdPS03], Théorème 3.1, pour une formulation équivalente. Ils ont prouvé que $\delta Y = AY$ est équivalent sur $\mathbb{C}((z^{1/\nu}))$ à un système de la forme $\delta Y = (M + \mathrm{Diag}(\delta q_i))\, Y$, où les q_i sont les mêmes que précédemment, et $M \in \mathrm{M}_m(\mathbb{C})$ vérifie $M\mathrm{Diag}\,(q_i) = \mathrm{Diag}\,(q_i)\, M$.

Les coefficients de $\hat{H}$ sont à priori divergents, mais nous verrons plus loin qu'il existe des couples de réels (a, b) avec $a < b$, tels que nous pouvons construire des solutions de (1.1.1) dans $\mathcal{A}(a, b)$, le corps des germes en 0 de fonctions méromorphes sur le secteur

$$\overline{S}(a, b) := \left\{ z \in \widetilde{\mathbb{C}} \middle| \arg(z) \in]a, b[\right\},$$

en ressommant les séries formelles. Nous introduisons maintenant les transformées de Borel et Laplace qui vont nous permettre de faire cela.

Définition 1.1.2. (1) Soit $k \in \mathbb{Q}_{>0}$. La transformée de Borel formelle d'ordre k, $\hat{\mathcal{B}}_k$, est l'application qui transforme la série formelle $\sum a_n z^n$ en la série formelle

$$\hat{\mathcal{B}}_k\left(\sum a_n z^n\right) = \sum \frac{a_n}{\Gamma(1 + \frac{n}{k})}\zeta^n,$$

où Γ désigne la fonction Gamma.

(2) Soient $d \in \mathbb{R}$, $k \in \mathbb{Q}_{>0}$, $\varepsilon > 0$ et soit f analytique sur le secteur $\overline{S}(d-\varepsilon, d+\varepsilon)$. Nous supposerons que f est à croissance exponentielle d'ordre k à l'infini dans la direction d, ce qui signifie qu'il existe des constantes $A, B > 0$ telles que pour $\zeta \in \widetilde{\mathbb{C}}$ avec $\arg(\zeta) = d$,

$$|f(\zeta)| \leq Ae^{B|\zeta|^k}.$$

Alors, l'intégrale suivante définit un élément de $\mathcal{A}\left(d - \frac{\pi}{2k}, d + \frac{\pi}{2k}\right)$, cf. [Bal94], Page 13 pour la preuve, et est appelée transformée de Laplace d'ordre k de direction d de f :

$$\mathcal{L}_{k,d}(f)(z) = \int_0^{\infty e^{id}} f(\zeta) e^{-\left(\frac{\zeta}{z}\right)^k} d\left(\left(\frac{\zeta}{z}\right)^k\right).$$

Comme nous le voyons dans §3.1.3, les entrées de la matrice $\hat{H}(z)$ sont multisommables, ce qui signifie qu'il existe $\Sigma \subset \mathbb{R}$, fini modulo 2π, appelé ensemble des directions singulières, $0 < \mu_1 < \cdots < \mu_r$ des rationnels positifs, $\varepsilon > 0$, tels que si d n'est pas une direction singulière, pour $j = 0$, (resp. $j = 1, \ldots, j = r-1$), $H_0^d(\zeta) := \hat{\mathcal{B}}_{\mu_1} \circ \cdots \circ \hat{\mathcal{B}}_{\mu_r}\left(\hat{H}(z)\right)$ (resp. $H_j^d(\zeta) := \mathcal{L}_{\mu_j,d}\left(H_{j-1}^d(\zeta)\right)$) admet un prolongement analytique sur $\overline{S}(d-\varepsilon, d+\varepsilon)$, lequel est exponentiel d'ordre μ_{j+1} à l'infini, et que nous noterons toujours $H_j^d(\zeta)$. Notons $\widetilde{S}^d\left(\hat{H}\right) := \mathcal{L}_{\mu_r,d}\left(H_{r-1}^d(\zeta)\right)$. Les entrées de cette dernière matrice appartiennent à $\mathcal{A}\left(d - \frac{\pi}{2\mu_r}, d + \frac{\pi}{2\mu_r}\right)$ et sont Gevrey asymptotiques de niveau μ_r à celles de $\hat{H}(z)$. Voir §3.1.3 pour la définition de Gevrey asymptoticité. De plus, d'après [Bal94], §6.4, Théorème 2, l'application qui à $\hat{H}$ associe $\widetilde{S}^d\left(\hat{H}\right)$ induit un isomorphisme de corps différentiel du sous-corps différentiel de $\mathbb{C}((z))$ engendré par $\mathbb{C}(\{z\})$, et les entrées de $\hat{H}(z)$, vers le corps $\mathcal{A}\left(d - \frac{\pi}{2\mu_r}, d + \frac{\pi}{2\mu_r}\right)$. En particulier, si d n'est pas une direction singulière,

$$\widetilde{S}^d\left(\hat{H}\right) e^{L\log(z)} \mathrm{Diag}\left(e^{q_i(z)}\right)$$

est une matrice fondamentale de l'équation $\delta Y = AY$, dont les entrées appartiennent à $\mathcal{A}\left(d - \frac{\pi}{2\mu_r}, d + \frac{\pi}{2\mu_r}\right)$. Soit L_d le corps différentiel engendré par $\mathbb{C}(\{z\})$ et les entrées de $\widetilde{S}^d\left(\hat{H}\right) e^{L\log(z)} \mathrm{Diag}\left(e^{q_i(z)}\right)$. Puisque nous avons affaire à des fonctions méromorphes, il est clair que le corps des constantes de L_d est $\mathbb{C}$. Par conséquent, pour toute direction non singulière d, $L_d|\mathbb{C}(\{z\})$ est une extension de Picard-Vessiot.

Fixons une direction non singulière d. Contrairement au cas localement fuchsien, nous ne pouvons pas prolonger analytiquement $\widetilde{S}^d\left(\hat{H}\right) e^{L\log(z)} \mathrm{Diag}\left(e^{q_i(z)}\right)$ le long d'un simple lacet orienté positivement autour de 0. Cependant, nous allons « mimer » l'action de la monodromie. Définissons par $\hat{m}$, le morphisme de corps différentiel de L_d qui envoie $e^{\alpha\log(z)}$ pour $\alpha \in \mathbb{C}^*$ sur $e^{2i\pi\alpha} e^{\alpha\log(z)}$, e^{q_i} sur $e^{\hat{m}(q_i)}$, $\log(z)$ sur $\log(z) + 2i\pi$, et laisse $\mathbb{C}((z))$ invariant. Cf. [vdPS03], page 79, pour la justification que $\hat{m}$ est bien un morphisme de corps différentiel. Nous définissons la matrice de monodromie formelle, comme étant la matrice inversible $\hat{M} \in \mathrm{GL}_m(\mathbb{C})$ égale à

$$\widetilde{S}^d\left(\hat{H}\right) e^{L\log(z)} \mathrm{Diag}\left(e^{q_i(z)}\right) \hat{M} := \hat{m}\left(\widetilde{S}^d\left(\hat{H}\right) e^{L\log(z)} \mathrm{Diag}\left(e^{q_i(z)}\right)\right).$$

Par construction, cette matrice appartient au groupe de Galois différentiel local, vu comme sous-groupe de $\mathrm{GL}_m(\mathbb{C})$. Nous définissons le tore exponentiel, comme étant le sous-groupe du groupe de Galois différentiel local des éléments qui laissent invariants $\mathbb{C}((z))$, $\log(z)$ et les $\left(e^{a\log(z)}\right)_{a\in\mathbb{C}}$. Nous identifierons le tore exponentiel avec son image dans $\mathrm{GL}_m(\mathbb{C})$. La monodromie formelle et le tore exponentiel caractérisent les classes d'équivalences formelles des systèmes différentiels linéaires à coefficients dans $\mathbb{C}(\{z\})$. Cf. [vdPS03], Chapitre 3, Section 1. Ils demeurent en revanche insuffisants pour caractériser les classes d'équivalences méromorphes. Il faudra rajouter les matrices de Stokes que nous définissons maintenant.

Soit $d \in \mathbb{R}$, et soient $d^{\pm}$ tels que

$$d-\frac{\pi}{2\mu_r}<d^-<d<d^+<d+\frac{\pi}{2\mu_r},$$

et tels qu'il n'y ait pas de directions singulières dans $[d^-, d[\bigcup]d, d^+]$. Nous définissons la matrice de Stokes de direction d comme étant la matrice inversible $St_d \in \mathrm{GL}_m(\mathbb{C})$ telle que

$$\widetilde{S}^{d^+}\left(\hat{H}\right)e^{L\log(z)}\mathrm{Diag}\left(e^{q_i(z)}\right)=\widetilde{S}^{d^-}\left(\hat{H}\right)e^{L\log(z)}\mathrm{Diag}\left(e^{q_i(z)}\right)St_d.$$

Grâce au principe du prolongement analytique, nous déduisons que les matrices St_d sont indépendantes des choix de d^- et d^+. Comme l'extension de Picard-Vessiot est unique à isomorphisme différentiel près, les corps différentiels L^{d^+} et L^{d^-} sont isomorphes en tant que corps différentiel. En particulier, les matrices St_d appartiennent au groupe de Galois différentiel local. Elles sont l'identité si d n'est pas une direction singulière. De plus, la monodromie formelle, le tore exponentiel et les matrices de Stokes caractérisent les classes d'équivalences méromorphes des systèmes différentiels linéaires à coefficients dans $\mathbb{C}(\{z\})$. Cf. [vdPS03], Chapitre 9.

Nous pouvons enfin énoncer le théorème de densité de Jean-Pierre Ramis, cf. [vdPS03], Chapitre 8, dont la généralisation au cas paramétré est l'objet principal du chapitre 3.

Théorème 1.1.3 (Jean-Pierre Ramis)**.** *Le groupe engendré par le monodromie, le tore exponentiel et les matrices de Stokes est dense pour la topologie de Zariski dans le groupe de Galois différentiel local, vu comme sous-groupe algébrique de* $\mathrm{GL}_m(\mathbb{C})$*.*

Considérons maintenant $\frac{d}{dz}Y = AY$ avec A à coefficients dans $\mathbb{C}(z)$ quelconque. Nous exposons ici le théorème de densité dans sa version globale. Il est dû à Jean-Pierre Ramis et une preuve peut être trouvée dans [Mit96], Proposition 1.3. Soit $S := \{\alpha_1, \dots, \alpha_{k+1}\} \subset \mathbb{P}_1(\mathbb{C})$ l'ensemble des singularités de $\frac{d}{dz}Y = AY$ et supposons que $\alpha_{k+1} = \infty$. Par des transformations de type $z \mapsto z-\alpha$ et $z \mapsto 1/z$, nous pouvons définir les directions singulières pour chaque singularité de $\delta Y = AY$. Soient $d_1, \dots d_{k+1}$ des réels, tels que d_i n'est pas une direction singulière pour la singularité α_i. Soit $x \in \mathbb{P}_1(\mathbb{C})\backslash S$, et soit F une matrice fondamentale analytique en x. Par prolongement analytique, nous pouvons étendre cette solution en une solution, qui pour chaque singularité α_i, est définie sur un voisinage sectoriel de direction d_i. Ainsi, nous pouvons définir la monodromie, le tore exponentiel, les matrices de Stokes pour chaque singularité de $\delta Y = AY$. Ils appartiennent au groupe de Galois différentiel global.

Théorème 1.1.4 (Théorème de densité global)**.** *Le groupe engendré par la monodromie, le tore exponentiel, les matrices de Stokes pour chaque singularité est dense pour la topologie de Zariski, dans le groupe de Galois différentiel global, vu comme sous-groupe algébrique de* $\mathrm{GL}_m(\mathbb{C})$*.*

1.1.4 Algorithme de Kovacic

Le théorème de densité global nous donne une liste de générateurs topologiques du groupe de Galois différentiel global. Cependant, on ne sait pas toujours les calculer, sauf dans certains cas particuliers, cf. par exemple [Mit96]. En revanche, lorsque l'ordre du système est 2, cf. [Kov86], ou 3, cf. [SU93a, SU93b], nous disposons d'algorithmes pour déterminer le groupe de Galois différentiel global. Il est à noter que deux algorithmes à priori jamais implémentés, cf. [Hru02, vdH07], permettent en théorie de calculer n'importe quel groupe de Galois différentiel global. Intéressons-nous à l'ordre 2. Grâce au vecteur cyclique, nous pouvons nous placer dans le cas où le système différentiel linéaire est une équation différentielle linéaire :

$$\frac{d}{dz}\begin{pmatrix} Y \\ \frac{d}{dz}Y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r & s \end{pmatrix}\begin{pmatrix} Y \\ \frac{d}{dz}Y \end{pmatrix},$$

avec $r, s \in \mathbb{C}(z)$. Par le changement de variable $W := Ye^{\frac{1}{2}\int r}$, nous pouvons nous ramener au cas où $s = 0$. Notons H le groupe de Galois différentiel global. Dans ce cas précis, H est un sous-groupe algébrique de $\mathrm{SL}_2(\mathbb{C})$, cf. [Kov86], §1.3. L'algorithme de Kovacic repose sur la classification des groupes algébriques de $\mathrm{SL}_2(\mathbb{C})$. En particulier, cf. [Kov86] pour plus de détails, nous avons le théorème suivant

Théorème 1.1.5. *Il y a quatre possibilités.*

1. *Le groupe H est conjugué à un sous-groupe de*

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \text{ où } a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$$

   et il existe une solution de la forme $e^{\int_0^z f(u)du}$, avec $f \in \mathbb{C}(z)$.

2. *Le premier cas ne se produit pas, et H est conjugué à un sous-groupe de*

$$D_\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \bigcup \begin{pmatrix} 0 & b^{-1} \\ -b & 0 \end{pmatrix}, \text{ où } a, b \in \mathbb{C}^* \right\}$$

   et il existe une solution de la forme $e^{\int_0^z f(u)du}$, où f est algébrique sur $\mathbb{C}(z)$ de degré 2.

3. *Les deux premier cas ne se produisent pas et H est fini. Dans ce cas toutes les solutions sont algébriques sur $\mathbb{C}(z)$.*

4. *Si aucun des trois précédents cas ne se produit, alors $H = \mathrm{SL}_2(\mathbb{C})$.*

Kovacic a développé un algorithme permettant de déterminer lequel des quatre cas se produit, et lorsque le cas (4) ne se produit pas, de calculer une matrice fondamentale ainsi que le groupe de Galois différentiel global. Plus généralement, cet algorithme fonctionne toujours lorsque $\mathbb{C}$ est remplacé par un corps algébriquement clos et de caractéristique nulle. Ce sera le point de départ pour généraliser cet algorithme dans le cas des équations différentielles linéaires paramétrées. Cf. §1.2.4 et le chapitre 2.

1.2 Théorie de Galois différentielle paramétrée

Dans cette section, il est question de théorie de Galois des systèmes différentiels linéaires dépendant de paramètres. Nous en profiterons pour présenter nos résultats de la partie 1. Nous commençons par rappeler les bases de la théorie de Galois des systèmes différentiels linéaires paramétrés. Nous renvoyons aux chapitres 2 et 3 pour les exemples. Nous parlons ensuite d'isomonodromie, et plus généralement de systèmes complètement intégrables, ainsi que du lien de ces deux notions avec la théorie de Galois différentielle paramétrée. Nous énonçons un résultat de descente du groupe de Galois. Ce résultat va nous servir pour les deux théorèmes de densité, local et global, du chapitre 3. Nous donnons ensuite trois applications du théorème de densité, à savoir :
- une contribution au problème inverse dans cette théorie de Galois ;
- un critère d'isomonodromie ;
- une réponse partielle à une question posée par Sibuya.

Nous terminons par présenter brièvement la généralisation de l'algorithme de Kovacic qui est l'objet central du chapitre 2.

1.2.1 Préliminaires

Nous allons maintenant décrire brièvement la théorie de Galois des systèmes différentiels linéaires paramétrés, dont les bases ont été posées dans [CS07]. Voir aussi [Lan08, Rob59, Ume96b] et §3.2.1.

Soit $(K, \partial_0, \ldots, \partial_n)$ (nous le noterons simplement K lorsqu'il n'y aura aucune ambiguïté) un corps de caractéristique nulle, muni de $n+1$ dérivations qui commutent. Considérons $C := \{c \in K | \partial_0 c = 0\}$, le corps des constantes par rapport à la dérivation ∂_0. Le fait que les dérivations commutent entraîne que $(C, \partial_1, \ldots, \partial_n)$ est un corps muni de n dérivations $\partial_1, \ldots, \partial_n$. Nous considérons le système différentiel linéaire paramétré

$$\partial_0 Y = AY, \tag{1.2.1}$$

où $A \in \mathrm{M}_m(K)$. Les dérivations $\partial_1, \ldots, \partial_n$ peuvent être vues comme des dérivées par rapport aux paramètres.

Dans la théorie de Galois différentielle classique, nous faisions l'hypothèse que C est algébriquement clos. Dans le cas paramétré, nous avons besoin de la notion de corps différentiellement clos.

Définition 1.2.1 ([CS07], Définition 3.2)**.** Nous dirons que $(C, \partial_1, \ldots, \partial_n)$ est différentiellement clos s'il a la propriété suivante : pour tout entiers $k, l \in \mathbb{N}$ et pour tout $P_1, \ldots, P_k \in C\{X_1, \ldots, X_l\}_{\partial_1, \ldots, \partial_n}$, polynômes différentiels à l variables et à coefficients dans C, le système

$$\begin{cases} P_1(\alpha_1, \ldots, \alpha_l) & = & 0 \\ & \vdots & \\ P_{k-1}(\alpha_1, \ldots, \alpha_l) & = & 0 \\ P_k(\alpha_1, \ldots, \alpha_l) & \neq & 0, \end{cases}$$

a une solution dans C si et seulement si, il en a une dans un corps $(\partial_1, \ldots, \partial_n)$-différentiel contenant C.

Un corps différentiellement clos contient beaucoup d'éléments. Par exemple, $\overline{\mathbb{C}(t)}$, la clôture algébrique de $\mathbb{C}(t)$, n'est pas différentiellement clos puisque $e^t \notin \overline{\mathbb{C}(t)}$, alors qu'il est solution de $y' = y$. Cf. [Blu77, Kol74, Mar00, McG00, Rob59], pour plus de détails sur la définition 1.2.1. Citons quelques résultats qui peuvent être retrouvés dans les références ci-dessus.

– Étant donné un corps différentiel, il existe un corps différentiellement clos le contenant.
– Étant donné un corps différentiellement clos k_0, il existe un corps différentiellement clos $k_1 \supset k_0$, tel que pour tout corps différentiellement clos $K \supset k_0$, il existe un isomorphisme de corps différentiels de k_1 dans K.

Une extension de Picard-Vessiot paramétrée pour (1.2.1) est une extension de corps $(\partial_0, \dots, \partial_n)$-différentielle $\widetilde{K}\middle|K$ qui satisfait les propriétés suivantes :

– il existe une matrice fondamentale de $\partial_0 Y = AY$ à coefficients dans $\widetilde{K}$, c'est-à-dire une matrice inversible $F = (F_{i,j})$ à coefficients dans $\widetilde{K}$, telle que $\partial_0 F = AF$;
– le corps $(\partial_0, \dots, \partial_n)$-différentiel $\widetilde{K}$ est engendré par K est les $F_{i,j}$;
– le corps des constantes de $\widetilde{K}$ par rapport à la dérivation ∂_0 est C.

Un fait important est que si C est différentiellement clos, alors une extension de Picard-Vessiot paramétrée de (1.2.1) existe et est unique, à isomorphisme de corps $(\partial_0, \dots, \partial_n)$-différentiel près.

Nous ferons désormais l'hypothèse qu'il existe une extension de Picard-Vessiot paramétrée de (1.2.1), que nous noterons $\widetilde{K}|K$. Nous définissons $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$, comme étant le groupe des automorphismes de corps de $\widetilde{K}$, commutant avec les dérivations $\partial_0, \dots, \partial_n$ et laissant K invariant.

Soit $F \in \mathrm{GL}_m\left(\widetilde{K}\right)$ une matrice fondamentale pour (1.2.1) et considérons le morphisme de groupes injectif

$$\begin{array}{cccc} \rho_F : & Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right) & \longrightarrow & \mathrm{GL}_m(C) \\ & \sigma & \mapsto & F^{-1}\sigma(F). \end{array}$$

Contrairement au cas non paramétré, l'image de ρ_F n'est pas un sous-groupe algébrique de $\mathrm{GL}_m(C)$. Si C est différentiellement clos, alors l'image de ρ_F est un sous-groupe différentiel de $\mathrm{GL}_m(C)$, cf. définition ci-dessous. Nous renvoyons à [Kol73, Kol85], pour plus de détails sur les groupes différentiels.

Définition 1.2.2. On dira que le sous-groupe G de $\mathrm{GL}_m(C)$ est un sous-groupe différentiel s'il existe $P_1, \dots, P_k$, des polynômes $(\partial_1, \dots, \partial_n)$-différentiels algébriques à m^2 variables, tels que pour $A = (a_{i,j}) \in \mathrm{GL}_m(C)$,

$$A \in G \Longleftrightarrow P_1(a_{i,j}) = \dots = P_k(a_{i,j}) = 0.$$

Désormais, nous identifierons $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$, avec un sous-groupe différentiel de $\mathrm{GL}_m(C)$, pour une matrice fondamentale fixée. Le changement de matrice fondamentale donnera un groupe conjugué au premier.

Le groupe de Galois différentiel paramétré mesure les relations algébriques et différentielles (par rapports aux paramètres) entre les différentes solutions. Grosso modo, plus le groupe de Galois différentiel paramétré est gros, moins il y a de relations entre les solutions. Cf. [HS08], Page 374 pour une formulation exacte de ce fait.

Supposons que C est différentiellement clos. Il est en particulier algébriquement clos. Nous pouvons donc définir le groupe de Galois différentiel paramétré (resp. non paramétré) et pour une matrice fondamentale donnée, le voir comme un sous-groupe différentiel (resp. algébrique) de $\mathrm{GL}_m(C)$. Par construction, le groupe de Galois différentiel non paramétré contient le groupe de Galois différentiel paramétré. La proposition 6.21 de [HS08], nous indique que le groupe de Galois différentiel paramétré, est dense pour la topologie de Zariski dans le groupe de Galois différentiel non paramétré.

Nous allons maintenant énoncer le théorème de correspondance pour cette théorie de Galois. Munissons $\mathrm{GL}_m(C)$ de la topologie pour laquelle les fermés sont les lieux d'annulation des polynômes $(\partial_1, \dots, \partial_n)$-différentiels algébriques à m^2 variables. Cette topologie sera appelée topologie de Kolchin. Considérons (1.2.1) et supposons que C est différentiellement clos. Soit $\widetilde{K}|K$ l'extension de Picard-Vessiot et soit $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$ le groupe de Galois différentiel paramétré, que nous verrons comme un sous-groupe différentiel de $\mathrm{GL}_m(C)$. Soient S l'ensemble des sous-groupes de $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$ fermés pour la topologie de Kolchin, et T l'ensemble des sous-corps $(\partial_0, \dots, \partial_n)$-différentiels de $\widetilde{K}$ qui contiennent K. D'après [CS07], Proposition 3.5, les deux applications suivantes sont inverses l'une de l'autre :

$$\begin{array}{cccc} s: & S & \longrightarrow & T \\ & G & \mapsto & \widetilde{K}^G := \left\{k \in \widetilde{K}\middle|\forall \sigma \in G, \sigma(k) = k\right\}, \end{array}$$

$$\begin{array}{cccc} t: & T & \longrightarrow & S \\ & M & \mapsto & Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|M\right) := \left\{\sigma \in Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)\middle|\forall m \in M, \sigma(m) = m\right\}. \end{array}$$

En particulier, pour prouver qu'un sous-groupe G est dense pour la topologie de Kolchin dans $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$, il suffit de prouver que $\widetilde{K}^G = K$.

1.2.2 Autour des équations isomonodromiques

L'étude des systèmes différentiels linéaires dépendant de paramètres peut être motivée par l'étude des équations isomonodromiques. Soit $\mathcal{D}$ un ouvert connexe de $\mathbb{P}_1(\mathbb{C})$. Pour simplifier, nous supposerons que $\mathcal{D} \subset \mathbb{C}$. Soient $a := (a_1, \dots, a_n) \in \mathbb{C}^n$, $r := (r_1, \dots, r_n) \in (\mathbb{R}_{>0})^n$, et définissons le polydisque $D(a, r) := \prod_{i\leq n} D(a_i, r_i)$, où $D(a_i, r_i) \subset \mathbb{C}$ représente le disque ouvert de centre a_i et de rayon r_i. Considérons un système différentiel linéaire dépendant du paramètre $t := (t_1, \dots, t_n) \in D(a, r)$:

$$\frac{d}{dz}Y(z,t) = A(z,t)Y(z,t), \tag{1.2.2}$$

où $A \in \mathrm{M}_m\Big(\mathcal{A}(\mathcal{D} \times D(a, r))\Big)$ et $\mathcal{A}(\mathcal{D} \times D(a, r))$ désigne l'anneau des fonctions analytiques pour $(z, t) \in \mathcal{D} \times D(a, r)$. Nous supposerons en outre que à $t \in D(a, r)$ fixé, $z \mapsto A(z, t)$ est à coefficients dans $\mathbb{C}(z)$ et que chaque composante connexe de $\mathbb{P}_1(\mathbb{C}) \setminus \mathcal{D}$ contient exactement une singularité de (1.2.2). Il y a beaucoup de définitions de l'isomonodromie présentes dans la littérature. Nous en donnons une.

Définition 1.2.3 ([CS07], Définition 5.1)**.** Une matrice fondamentale de solutions de (1.2.2) est une famille de couples $\big(D(x_i, s_i), F_i(z, t)\big)_{i\in I}$, tels que :
– les $D(x_i, s_i)$ recouvrent $\mathcal{D}$;

– pour tout $i \in I$, la matrice inversible $F_i(z,t) \in \mathrm{GL}_m\Big(\mathcal{A}(D(x_i,s_i) \times D(a,r))\Big)$ est solution de (1.2.2).

Définissons $C_{i,j}(t) := F_i(z,t)^{-1}F_j(z,t)$ lorsque $D(x_i,s_i) \cap D(x_j,s_j) \neq \varnothing$. On dira que (1.2.2) est isomonodromique sur $\mathcal{D} \times D(a,r)$, s'il existe une matrice fondamentale $\big(D(x_i,s_i), F_i(z,t)\big)_{i\in I}$ de (1.2.2), telle que les $C_{i,j}(t)$ soient indépendants de t.

Si (1.2.2) est isomonodromique, alors par définition ses matrices de monodromie ne dépendent pas du paramètre t. La réciproque est vraie dans le cas où pour chaque valeur du paramètre fixé le système différentiel est fuchsien. Malheureusement, nous allons voir que la situation est plus compliquée dans le cas non fuchsien, cf. §1.2.3.3.

La définition d'isomonodromie n'est pas très commode d'utilisation. Nous verrons plus loin que (1.2.2) est isomonodromique si et seulement si, il peut être complété en un système complètement intégrable. Nous en donnons la définition maintenant.

Définition 1.2.4 ([vdPS03], Définition D.7)**.** Soit K un corps muni de $n+1$ dérivations $(\partial_0,\dots,\partial_n)$ qui commutent. Soient $A_0,\dots,A_n \in \mathrm{M}_m(K)$. Nous dirons que le système

$$[S]\left\{\begin{array}{ccc}\partial_0 Y & = & A_0 Y\\ & \vdots & \\ \partial_n Y & = & A_n Y,\end{array}\right.$$

est complètement intégrable, si et seulement si pour tout $0 \leq i,j \leq n$:

$$\partial_j A_i - \partial_i A_j = A_j A_i - A_i A_j.$$

Puisque les dérivations commutent, ceci est une condition nécessaire pour avoir l'existence d'une matrice fondamentale pour $[S]$, c'est-à-dire une matrice inversible F, telle que

$$\left\{\begin{array}{ccc}\partial_0 F & = & A_0 F\\ & \vdots & \\ \partial_n F & = & A_n F.\end{array}\right.$$

En réalité, la théorie de Galois différentiel classique, entraîne que c'est aussi une condition suffisante, à partir du moment où le corps des constantes par rapport aux $n+1$ dérivations est algébriquement clos. Cf. [vdPS03], Annexe D.

Revenons à ce qui précède. Notons $\partial_z, \partial_{t_1}, \dots, \partial_{t_n}$, les différentes dérivations par rapport aux variables $z, t_1, \dots, t_n$. Le résultat suivant est attribué à Schlesinger et une preuve peut en être trouvée dans [Sib90], Théorème A.5.2.3.

Proposition 1.2.5. *L'équation (1.2.2) est isomonodromique sur $\mathcal{D} \times D(a,r)$, si et seulement si, il existe n matrices $A_i(z,t) \in \mathrm{M}_m\Big(\mathcal{A}(\mathcal{D} \times D(a,r))\Big)$, telles que le système*

$$[S] : \left\{\begin{array}{ccc}\partial_z Y(z,t) & = & A(z,t)Y(z,t)\\ \partial_{t_1} Y(z,t) & = & A_1(z,t)Y(z,t)\\ & \vdots & \\ \partial_{t_n} Y(z,t) & = & A_n(z,t)Y(z,t),\end{array}\right.$$

soit complètement intégrable.

Étant donné que la définition de complète intégrabilité est plus simple à manipuler que la définition d'isomonodromie, cette proposition s'avère très utile. Cf. la proposition 1.2.17, pour une autre condition nécessaire et suffisante pour que (1.2.2) soit isomonodromique.

Nous terminons cette sous-section en étudiant le lien entre la théorie de Galois différentielle paramétrée et les systèmes qui peuvent être complétés en système complètement intégrables. Considérons (1.2.1) et supposons que C, le corps des constantes par rapport à la dérivation ∂_0, soit différentiellement clos. Soit C_0 le corps des constantes par rapport aux $n+1$ dérivations. Soit $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$, le groupe de Galois de (1.2.1), que nous voyons comme un sous-groupe différentiel de $\mathrm{GL}_m(C)$. Nous avons les résultats suivants :

Proposition 1.2.6 ([CS07], Proposition 3.9)**.** *Il existe une matrice $P \in \mathrm{GL}_m(C)$ telle que $PGal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)P^{-1} \subset \mathrm{GL}_m(C_0)$, si et seulement si $\partial_0 Y = A_0 Y$ peut être complété en un système complètement intégrable.*

Proposition 1.2.7 ([GO12], Théorème 6.3)**.** *Le système (1.2.1) peut être complété en un système complètement intégrable, si et seulement si pour tout $i \leq n$, il existe $A_i \in \mathrm{M}_m(K)$, telles que*

$$\partial_0 A_i - \partial_i A = AA_i - A_i A.$$

Cette dernière proposition a d'abord été démontrée dans le chapitre 2, pour le cas $m = 2$, en utilisant un argument présent dans la preuve de [Sit75], Théorème 1.2, Chapitre 2, avant d'être généralisée pour m quelconque par les auteurs de [GO12].

Lorsque le nombre de paramètres devient important, cette proposition permet de vérifier plus facilement si un système peut être complété en un système complètement intégrable : au lieu de vérifier $\binom{n}{2}$ relations, de type

$$\partial_j A_i - \partial_i A_j = A_j A_i - A_i A_j,$$

où $A_0 := A$, il suffit d'en vérifier n. En particulier, nous utiliserons cette proposition dans le chapitre 2 pour généraliser l'algorithme de Kovacic dans le cas paramétré. Cf. le cas (4) du théorème 1.2.18.

Démonstration de la proposition 1.2.7. Par définition, si $\partial_0 Y = AY$ peut être complété en un système complètement intégrable, alors pour tout $i \leq n$, il existe $A_i \in \mathrm{M}_m(K)$, telle que

$$\partial_0 A_i - \partial_i A = AA_i - A_i A.$$

Réciproquement, supposons que pour tout $i \leq n$, il existe $A_i \in \mathrm{M}_m(K)$, telle que

$$\partial_0 A_i - \partial_i A_0 = A_0 A_i - A_i A_0.$$

Fixons $i \leq n$. Par la proposition 1.2.6, $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$, vu comme sous-groupe différentiel de $\mathrm{GL}_m(C)$, est conjugué à un sous-groupe du groupe de matrices de dérivées nulles par rapport à ∂_i. Comme le corps des constantes par rapport à ∂_0 et ∂_i est algébriquement clos, et que deux matrices semblables ont les mêmes valeurs propres, les éléments de $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$ ont des valeurs propres qui appartiennent au corps des constantes par rapport aux dérivations ∂_0 et ∂_i. Le même raisonnement avec les autres valeurs possibles de i entraîne que les valeurs propres des éléments de $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$ appartiennent à C_0. Si $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$ n'était pas conjugué à un sous-groupe de $\mathrm{GL}_m(C_0)$, il contiendrait un élément ayant des valeurs propres qui n'appartiennent pas à C_0. D'après ce qui précède on obtient que $Gal^{\partial_1,\dots,\partial_n}_{\partial_0}\left(\widetilde{K}\middle|K\right)$ est conjugué à un sous-groupe de $\mathrm{GL}_m(C_0)$. En vertu de la proposition 1.2.6, le système (1.2.1) peut être complété en un système complètement intégrable. □

1.2.3 Théorèmes de densité pour les systèmes différentiels linéaires paramétrés (Chapitre 3)

Le but de cette sous-section est de résumer les résultats du chapitre 3. Considérons un système différentiel linéaire paramétré à coefficients dans un anneau de fonctions holomorphes, que nous définirons plus loin. Dans un premier temps, nous expliquons que l'on peut construire un groupe de Galois pour ce système, qui est de plus un groupe différentiel défini sur un corps de fonctions méromorphes sur un certain ouvert. Nous exposons deux théorèmes de densité, un local et un global, puis nous finissons par présenter des applications du théorème de densité global.

1.2.3.1 Descente du groupe de Galois différentiel paramétré

La plupart des résultats de [CS07] requièrent l'hypothèse que le corps des constantes soit différentiellement clos. Malheureusement, c'est une hypothèse qui peut se révéler contraignante lorsque l'on veut appliquer la théorie, puisque un corps différentiellement clos est un corps contenant beaucoup d'éléments, et que l'interprétation de ces éléments comme fonctions n'est pas aisée. Plusieurs résultats d'existence d'extensions de Picard-Vessiot paramétrées sans l'hypothèse que le corps des constantes est différentiellement clos peuvent être trouvés dans [GGO13, Wib12]. Lorsque les coefficients du système sont rationnels par rapport à la variable de dérivation, un résultat de Seidenberg nous permet de nous ramener dans le cas où le corps des constantes est un corps de fonctions méromorphes sur un ouvert. On utilisera cette stratégie dans le chapitre 2, pour généraliser l'algorithme de Kovacic dans le cas paramétré. Dans le chapitre 3, pour prouver l'analogue paramétré du théorème de densité de Ramis, cf. le théorème 3.2.20, nous utilisons une autre approche pour contourner l'hypothèse que le corps des constantes soit différentiellement clos. Nous la détaillons maintenant.

Soit K un corps différentiel équipé de $n+1$ dérivations qui commutent entre elles : $\partial_0,\dots,\partial_n$. Soit C, le corps des constantes par rapport à ∂_0, qui n'est pas supposé être différentiellement clos. Considérons $\partial_0 Y = AY$, avec $A \in \mathrm{M}_m(K)$, et faisons l'hypothèse qu'il existe $\widetilde{K}\middle|K$, une extension de Picard-Vessiot paramétrée pour $\partial_0 Y = AY$. Soit $F = (F_{i,j}) \in \mathrm{GL}_m\left(\widetilde{K}\right)$ une matrice fondamentale et notons $Gal^{\partial_1,\dots,\partial_n}_{\partial_0}\left(\widetilde{K}\middle|K\right)$, le groupe de Galois différentiel paramétré.

Proposition 1.2.8 (Proposition 3.2.9)**.** *(1) Considérons le morphisme de groupe injectif :*

$$\begin{array}{rccc}\rho_F: & Gal^{\partial_1,\dots,\partial_n}_{\partial_0}\left(\widetilde{K}\middle|K\right) & \longrightarrow & \mathrm{GL}_m(C)\\ & \varphi & \longmapsto & F^{-1}\varphi(F).\end{array}$$

Alors,

$$Im\ \rho_F = \left\{F^{-1}\varphi(F), \varphi \in Gal^{\partial_1,\dots,\partial_n}_{\partial_0}\left(\widetilde{K}\middle|K\right)\right\}$$

est un sous-groupe différentiel de $\mathrm{GL}_m(C)$.

(2) Soit G un sous-groupe de $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$*. Si* $\widetilde{K}^G = K$*, alors G est dense pour la topologie de Kolchin dans* $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$*, vu comme sous-groupe différentiel de* $\mathrm{GL}_m(C)$*.*

La correspondance galoisienne n'est plus vraie si C n'est pas différentiellement clos. Grosso modo, ceci est dû au fait que l'on n'a pas assez d'éléments dans C. En particulier, un élément dans $\widetilde{K}\backslash K$ peut être fixé par $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$, cf. [CS07], Exemple 3.1.

Nous introduisons maintenant certaines notations du chapitre 3. Soit U un polydisque non vide de $\mathbb{C}^n$. On notera par $\mathcal{M}_U$ le corps des fonctions méromorphes sur U. Les variables des éléments de $\mathcal{M}_U$ seront notées $t := (t_1,\dots,t_n)$. Le corps $\mathcal{M}_U$ est muni de n dérivations $\partial_{t_1},\dots,\partial_{t_n}$ qui commutent. Nous noterons $\Delta_t := \{\partial_{t_1},\dots,\partial_{t_n}\}$. Définissons le corps $\hat{K}_U := \mathcal{M}_U((z))$ et l'anneau

$$\mathcal{O}_U(\{z\}) := \left\{ f(z,t) = \sum f_i(t)z^i \in \hat{K}_U \middle| \forall t \in U, z \mapsto \sum f_i(t)z^i \in \mathbb{C}(\{z\}) \right\}.$$

Soit $f(z,t) \in \mathcal{O}_U(\{z\})$. Comme nous pouvons le voir dans la remarque 3.2.11, le rayon de convergence $R(t)$ de $z \mapsto f(z,t)$ est localement minoré.

Nous considérons maintenant le système différentiel linéaire paramétré

$$\partial_z Y(z,t) = A(z,t)Y(z,t), \tag{1.2.3}$$

avec $A(z,t) \in \mathrm{GL}_m\Big(\mathcal{O}_U(\{z\})\Big)$. Le corps des constantes de $\mathcal{O}_U(\{z\})$ par rapport à $\partial_z = \frac{d}{dz}$ est $\mathcal{M}_U$, qui n'est pas différentiellement clos. Le but principal du chapitre 3 est de prouver que (1.2.3) possède un groupe de Galois différentiel paramétré, qui peut être vu comme sous-groupe différentiel de $\mathrm{GL}_m(\mathcal{M}_U)$, et d'en exhiber un sous-groupe qui est dense pour la topologie de Kolchin. Cf. le théorème 3.2.20. Nous allons d'abord démontrer l'existence d'une extension de Picard-Vessiot paramétrée de (1.2.3), que nous allons obtenir grâce à la version paramétrée du théorème de Hukuhara-Turrittin que nous exposons ci-dessous.

Étant donné que l'anneau $\mathcal{O}_U(\{z\})$ est intègre, nous pouvons définir K_U comme étant son corps des fractions. De la proposition 3.1.3, nous tirons la proposition suivante :

Proposition 1.2.9 (Analogue paramétré du théorème de Hukuhara-Turrittin)**.** *Considérons (1.2.3). Il existe un polydisque non vide* $U' \subset U$*, ainsi que*

– $\hat{H}(z,t) \in \mathrm{GL}_m\Big(\hat{K}_{U'}\left[z^{1/\nu}\right]\Big)$*, pour un certain* $\nu \in \mathbb{N}^*$*, dont toutes les entrées ont des z-coefficients analytiques sur* U' *;*
– $L(t) \in \mathrm{M}_m(\mathcal{M}_{U'})$ *;*
– $q_1,\dots,q_m \in z^{-1/\nu}\mathcal{M}_{U'}\left[z^{-1/\nu}\right]$*, et* $\mathrm{Diag}\,(q_i)\,L(t) = L(t)\mathrm{Diag}\,(q_i)$ *;*

tels que, cf. §1.1.1 *pour la notation,*

$$A(z,t) = \hat{H}(z,t)\Big[L(t) + \mathrm{Diag}\,(\partial_z q_i)\Big]_{\partial_z}.$$

Cf. [Sch01], pour d'autres résultats de ce type, que nous comparons avec le nôtre dans la remarque 3.1.6. Dans l'appendice du chapitre 3, nous prouvons un résultat légèrement différent, cf. le théorème A.1.

Revenons maintenant à la proposition 1.2.9. Grosso modo, cette proposition nous dit qu'il existe une matrice fondamentale de (1.2.3), de la forme

$$\hat{H}(z,t)e^{L(t)\log(z)}\mathrm{Diag}\left(e^{q_i(z,t)}\right),$$

cette expression n'ayant pas plus de sens que dans le cas non paramétré, cf. §1.1.3. Pour pouvoir écrire une matrice fondamentale formelle, il faut introduire des symboles formels satisfaisant des relations algébriques et différentielles entre eux, qui jouerons le rôle de $\log(z)$, $\left(e^{a\log(z)}\right)_{a\in\mathcal{M}_{U'}}$, $\bigcup_{\nu\in\mathbb{N}^*}(e^q)_{q\in z^{-1/\nu}\mathcal{M}_{U'}[z^{-1/\nu}]}$. Cf. §3.1.1. Notons $\hat{F}(z,t)$ la matrice fondamentale formelle obtenue. Quitte à restreindre le polydisque U, nous pouvons supposer que $U=U'$. Nous prouvons ensuite dans §3.2.3, que le corps (∂_z,Δ_t)-différentiel $\widetilde{K_U}$ engendré par K_U et les entrées de $\hat{F}(z,t)$, a un corps des constantes par rapport à ∂_z égal à $\mathcal{M}_U$. Par conséquent, $\widetilde{K_U}\Big|K_U$, est une extension de Picard-Vessiot paramétrée de (1.2.3). En vertu de la proposition 1.2.8, le groupe de Galois différentiel paramétré $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\Big|K_U\right)$, peut être vu comme un sous-groupe différentiel de $\mathrm{GL}_m(\mathcal{M}_U)$.

Soit C un corps différentiellement clos qui contient $\mathcal{M}_U$. Munissons $C[[z]][z^{-1}]$ d'une structure de corps (∂_z,Δ_t)-différentiel : z est une (Δ_t)-constante avec $\partial_z z=1$, C est le corps des constantes par rapport à ∂_z, et ∂_z commute avec toutes les dérivations. On munit l'anneau $K_U\otimes_{\mathcal{M}_U}C$ de la structure d'anneau (∂_z,Δ_t)-différentiel donnée par :

$$\forall a\in K_U,\forall c\in C,\forall\partial\in\{\partial_z,\Delta_t\},\quad \partial(a\otimes_{\mathcal{M}_U}c)=\partial a\otimes_{\mathcal{M}_U}c+a\otimes_{\mathcal{M}_U}\partial c.$$

Cet anneau différentiel s'injecte dans $C[[z]][z^{-1}]$, ce qui implique qu'il est intègre. Par conséquent, nous pouvons définir son corps des fractions $\mathcal{K}_{C,U}$, que nous identifions comme un sous-corps de $C[[z]][z^{-1}]$.

Proposition 1.2.10 (Proposition 3.2.12). *Soit $\widetilde{\mathcal{K}_{C,U}}$, le corps (∂_z,Δ_t)-différentiel engendré par $\mathcal{K}_{C,U}$ et les entrées de $\hat{F}(z,t)$. Alors, l'extension de corps $\widetilde{\mathcal{K}_{C,U}}\Big|\mathcal{K}_{C,U}$ est une extension de Picard-Vessiot paramétrée de (1.2.3). De plus, il existe $P_1,\dots,P_k$, polynômes différentiels à coefficients dans $\mathcal{M}_U$, tels que*

$$\begin{array}{rl} & \left\{\hat{F}^{-1}\varphi\left(\hat{F}\right),\varphi\in Gal_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{K}_{C,U}}\Big|\mathcal{K}_{C,U}\right)\right\}\\ = & \left\{A=(a_{i,j})\in\mathrm{GL}_m(C)\Big|P_1(a_{i,j})=\dots=P_k(a_{i,j})=0\right\}\end{array}$$

$$\begin{array}{rl} & \left\{\hat{F}^{-1}\varphi\left(\hat{F}\right),\varphi\in Gal_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\Big|K_U\right)\right\}\\ = & \left\{A=(a_{i,j})\in\mathrm{GL}_m(\mathcal{M}_U)\Big|P_1(a_{i,j})=\dots=P_k(a_{i,j})=0\right\}.\end{array}$$

1.2.3.2 Énoncés des théorèmes de densité

Nous conserverons les notations de §1.2.3.1. Nous sommes maintenant prêts pour énoncer le résultat principal du chapitre 3 : l'analogue paramétré du théorème de densité de Ramis. Cf. le théorème 3.2.20. Considérons (1.2.3). Soient $U'\subset U$, $\nu\in\mathbb{N}^*$ et $\hat{H}(z,t)\in\mathrm{GL}_m\left(\hat{K}_{U'}[z^{1/\nu}]\right)$ donnés par la proposition 1.2.9.

Comme nous l'avons vu dans §1.1.3, pour toute valeur du paramètre $t\in U'$ fixé, il existe $\Sigma_t\subset\mathbb{R}$ fini modulo 2π, $0<\mu_1(t)<\dots<\mu_{r(t)}(t)$ des rationnels positifs dépendants de t, $\varepsilon(t)>0$, tels que si $d(t)\notin\Sigma_t$, pour $j=0$, (resp. $j=1,\dots,j=r(t)-1$), la matrice $H_j^{d(t)}(z,t):=\hat{\mathcal{B}}_{\mu_1(t)}\circ\dots\circ\hat{\mathcal{B}}_{\mu_{r(t)}(t)}\left(\hat{H}(z,t)\right)$ (resp. $H_j^{d(t)}(z,t):=\mathcal{L}_{\mu_j(t),d(t)}\left(H_{j-1}^{d(t)}(z,t)\right)$) admet un prolongement analytique sur $\overline{S}\left(d-\varepsilon(t),d+\varepsilon(t)\right)$ exponentiel d'ordre $\mu_{j+1}(t)$ à l'infini, que nous noterons toujours $H_j^{d(t)}(z,t)$. Notons enfin $\widetilde{S}^{d(t)}\left(\hat{H}\right)(z,t):=\mathcal{L}_{\mu_{r(t)}(t),d(t)}\left(H_{r(t)-1}^{d(t)}(z,t)\right)$.

Nous prouvons dans §3.1.4 que, quitte à restreindre U, nous pouvons supposer que :

1. $U' = U$;

2. il existe $(d_i(t))_{i\in\mathbb{N}}$ continues en t, telles que pour tout $t \in U$, $\bigcup\limits_{i\in\mathbb{N}} d_i(t) = \Sigma_t$, et pour tout $i \in \mathbb{N}$, $d_i(t) < d_{i+1}(t)$;

3. ni $r(t)$, ni les $\mu_i(t)$ ne dépendent de t. Il existe une constante $\varepsilon > 0$, telle que pour tout $t \in U$, $\varepsilon < \varepsilon(t)$.

Nous pouvons maintenant déterminer la dépendance des matrices asymptotiques en fonction des paramètres. Nous donnons une version affaiblie de la proposition 3.1.13.

Proposition 1.2.11. *Considérons $U \ni t \mapsto d(t) \in \mathbb{R}$, continue en t, telle que pour tout $t \in U$, $d(t) \notin \Sigma_t$. Alors, il existe une constante positive dépendante de t, $\varepsilon_1(t) > 0$, sur laquelle on ne fait pas d'hypothèse sur la manière dont elle dépend de t, telle que $\widetilde{S}^{d(t)}\left(\hat{H}\right)(z,t)$ est méromorphe pour :*

$$(z,t) \in \left\{ z \in \overline{S}\left(d(t) - \frac{\pi}{2\mu_r}, d(t) + \frac{\pi}{2\mu_r}\right) \middle| 0 < |z| < \varepsilon_1(t) \right\} \times U.$$

Conservons les mêmes notations que dans la proposition. Nous obtenons, cf. le lemme 3.1.14, que l'application qui à $\hat{H}(z,t)$ associe $\widetilde{S}^{d(t)}\left(\hat{H}\right)(z,t)$ induit un morphisme de corps (∂_z, Δ_t)-différentiel du sous-corps différentiel de $\hat{K}_U$ engendré par K_U et les entrées de $\hat{H}(z,t)$, dans le corps des fonctions $f(z,t)$ qui pour tout $t_0 \in U$ fixé, sont des germes de fonctions méromorphes sur le secteur $\overline{S}\left(d(t_0) - \frac{\pi}{2\mu_r}, d(t_0) + \frac{\pi}{2\mu_r}\right)$. Autrement dit, nous obtenons, cf. la proposition 1.2.9 pour les notations, une matrice fondamentale de (1.2.3) de la forme :

$$\widetilde{S}^{d(t)}\left(\hat{H}\right)(z,t)e^{\log(z)L(t)}\mathrm{Diag}\left(e^{q_i(z,t)}\right).$$

De plus, le corps (∂_z, Δ_t)-différentiel engendré par K_U et les entrées de la matrice fondamentale ci-dessus est isomorphe, en tant que corps (∂_z, Δ_t)-différentiel, au corps $\widetilde{K_U}$ que nous avons défini dans §1.2.3.1. Nous rappelons que $\widetilde{K_U}\middle|K_U$ est une extension de Picard-Vessiot paramétrée de (1.2.3) et que le groupe de Galois différentiel paramétré $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\middle|K_U\right)$ est un sous-groupe différentiel de $\mathrm{GL}_m(\mathcal{M}_U)$. Nous pouvons maintenant définir les générateurs topologiques de groupe de Galois différentiel paramétré. Cf. §3.2.4 pour les justifications.

On définit la matrice de monodromie formelle paramétrée comme étant la matrice qui, pour tout $t_0 \in U$, est égale à la matrice de monodromie formelle du système différentiel non paramétré spécialisé en t_0 :

$$\partial_z Y(z,t_0) = A(z,t_0)Y(z,t_0).$$

D'après l'hypothèse 2 ci-dessus, il existe $U \ni t \mapsto d(t) \in \mathbb{R}$ continue en t, telle que pour tout $t \in U$, $d(t)$ est une direction singulière paramétrée, c'est-à-dire une fonction continue telle que pour tout $t \in U$, $d(t) \in \Sigma_t$. On définit la matrice de Stokes paramétrée de direction $d(t)$ comme étant la matrice qui, pour tout $t_0 \in U$, est égale à la matrice de Stokes de direction $d(t_0)$ du système différentiel non paramétré spécialisé en t_0 :

$$\partial_z Y(z,t_0) = A(z,t_0)Y(z,t_0).$$

Ces matrices appartiennent à $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\middle|K_U\right)$, vu comme sous-groupe différentiel de $\mathrm{GL}_m(\mathcal{M}_U)$.

On définit le tore exponentiel paramétré comme étant le sous-groupe de $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\middle|K_U\right)$ des éléments laissant fixes les entrées de $\widetilde{S}^{d(t)}\left(\hat{H}\right)(z,t)e^{\log(z)L(t)}$. Si nous voyons le tore exponentiel paramétré comme sous-groupe de $\mathrm{GL}_m(\mathcal{M}_U)$, alors ses éléments appartiennent en réalité à $\mathrm{GL}_m(\mathbb{C})$. En particulier, pour t_0 fixé, le tore exponentiel paramétré évalué en t_0 n'est pas nécessairement égal au tore exponentiel du système différentiel non paramétré spécialisé en t_0. Cf. l'exemple 3.2.16.

Nous appellerons ces éléments monodromie formelle, tore exponentiel et matrice de Stokes de direction $d(t)$, lorsqu'il n'y aura pas de confusions possibles. Nous pouvons maintenant énoncer le résultat principal du chapitre 3, le théorème 3.2.20.

Théorème 1.2.12 (Analogue paramétré du théorème de densité de Ramis)**.** *Le groupe engendré par la monodromie formelle, le tore exponentiel et les matrices de Stokes est dense pour la topologie de Kolchin dans* $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\middle|K_U\right)$*, vu comme sous-groupe différentiel de* $\mathrm{GL}_m(\mathcal{M}_U)$*.*

Dans §3.2.5, nous nous intéressons au cas global paramétré. Nous renvoyons à cette section pour les détails. Nous considérons l'équation :

$$\partial_z Y(z,t) = A(z,t)Y(z,t), \tag{1.2.4}$$

avec $A(z,t) \in \mathrm{GL}_m\Big(\mathcal{M}_U(z)\Big)$. Les singularités paramétrées de (1.2.4) sont les pôles de $A(z,t)$, ce qui peut inclure ∞, vu comme fonction rationnelle en z. Elles appartiennent à la clôture algébrique de $\mathcal{M}_U$. Quitte à restreindre U, nous pouvons nous ramener au cas où :

- les singularités paramétrées $\alpha_1(t),\dots,\alpha_k(t)$ de (1.2.4) appartiennent à $\mathcal{M}_U$;
- les z-coefficients des entrées de la matrice A sont analytiques sur U ;
- il existe $\varepsilon > 0$ tel que pour tout $t \in U$, deux singularités paramétrées de (1.2.4) sont distantes d'au moins ε ;
- si $\alpha_i(t) \not\equiv \infty$ (resp. si $\alpha_i(t) \equiv \infty$), les hypothèses (1), (2) et (3) que nous avons énoncées ci-dessus sont satisfaites pour le système différentiel local paramétré

$$\partial_z Y(z-\alpha_i(t),t) = A(z-\alpha_i(t),t)Y(z-\alpha_i(t),t)$$

$$\text{resp. } \partial_z Y(z^{-1},t) = A(z^{-1},t)Y(z^{-1},t).$$

Soit $x(t) \in \mathbb{P}_1(\mathcal{M}_U(z))$, qui pour tout $t_0 \in U$, est différent des singularités paramétrées évaluées en t_0 de (1.2.4), et considérons une matrice fondamentale analytique en $(x(t),t)$. Soit $d(t)$ continue en t, telle que pour tout $t_0 \in U$, $d(t_0)$ n'est pas une direction singulière de $\partial_z Y(z-\alpha_i(t_0),t) = A(z-\alpha_i(t_0),t_0)Y(z-\alpha_i(t_0),t_0)$

(resp. $\partial_z Y(z^{-1},t_0) = A(z^{-1},t_0)Y(z^{-1},t_0)$). Par prolongement analytique, nous pouvons étendre la solution analytique en $(x(t),t)$ en une solution, qui pour la singularité $\alpha_i(t) \not\equiv \infty$ (resp. $\alpha_i(t) \equiv \infty$), est définie sur un voisinage sectoriel de direction $d(t)$. Ainsi, nous pouvons construire une extension de Picard-Vessiot paramétrée de (1.2.4) que l'on notera $\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)$. La proposition 1.2.8, entraîne que le groupe de Galois différentiel paramétré $Gal^{\Delta_t}_{\partial_z}\left(\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)\right)$ peut être vu comme un sous-groupe différentiel de $\mathrm{GL}_m(\mathcal{M}_U)$. De plus, nous avons un analogue de la proposition 1.2.10 dans le cas global. Voir la proposition 3.2.23. La construction de la matrice fondamentale ci-dessus nous permet de définir la monodromie, le tore exponentiel, les matrices de Stokes pour chaque singularité de (1.2.4). Ils appartiennent au groupe de Galois différentiel paramétré global.

Théorème 1.2.13 (Analogue paramétré du théorème de densité global, Théorème 3.2.24)**.** *Le sous-groupe de $Gal^{\Delta_t}_{\partial_z}\left(\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)\right)$ engendré par la matrice de monodromie formelle, le tore exponentiel et les matrices de Stokes, pour chaque singularité de (1.2.4) est dense pour la topologie de Kolchin dans $Gal^{\Delta_t}_{\partial_z}\left(\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)\right)$, vu comme sous-groupe différentiel de $\mathrm{GL}_m(\mathcal{M}_U)$.*

Ce théorème généralise [MS12], Théorème 4.2, qui présentait un théorème de densité global pour les systèmes différentiels paramétrés $\partial_z Y(z,t) = A(z,t)Y(z,t)$ à coefficients dans $\mathcal{M}_U(z)$, qui satisfont entre autres conditions que pour tout $t_0 \in U$, $\partial_z Y(z,t_0) = A(z,t_0)Y(z,t_0)$ est fuchsien. Nous appellerons systèmes différentiels paramétrés fuchsiens de tels systèmes. Dans ce cas, les matrices de monodromie engendrent un groupe dense pour la topologie de Kolchin dans le groupe de Galois différentiel paramétré global. C'est l'analogue paramétré du théorème de Schlesinger.

1.2.3.3 Applications du théorème 1.2.13

Nous présentons ici trois applications du théorème 1.2.13. La première application concerne le problème inverse. La deuxième est un critère d'isomonodromie faisant intervenir les générateurs topologiques du théorème 1.2.13. La dernière concerne enfin une réponse partielle à une question posée par Sibuya. Commençons par le problème inverse.

Nous aurons besoin de la définition suivante, cf. [Kol73], Chapitre 3, Section 7 pour plus de détails.

Définition 1.2.14. Soit C un corps de caractéristique nulle muni de n dérivations $\Delta := \{\partial_1, \ldots, \partial_n\}$ qui commutent. On dit que C est un corps (Δ)-universel, si pour tout corps (Δ)-différentiel $C_0 \subset C$, (Δ)-finiment engendré sur $\mathbb{Q}$, et pour toute extension de corps (Δ)-différentielle finie C_1 de C_0, il existe un C_0-isomorphisme de corps (Δ)-différentiel de C_1 dans C.

En particulier, un tel corps C est différentiellement clos. Cette notion a un avantage important. Étant donné un nombre fini d'éléments de C, corps (Δ)-universel, on en déduit grâce à un résultat de Seidenberg, cf. le théorème 3.3.10, qu'il existe un polydisque non vide U, tel que le corps (Δ)-différentiel engendré par $\mathbb{Q}$ et ces éléments soit (Δ)-isomorphe à $\mathcal{M}_U$. Nous pouvons donc identifier un nombre fini d'éléments du corps abstrait C comme des fonctions méromorphes sur un ouvert donné. Pour cette raison, les dérivations des corps universels que nous rencontrerons seront notées Δ_t.

Soit C un corps (Δ_t)-universel. Notons $C(z)$ le corps (∂_z, Δ_t)-différentiel des fractions d'indéterminée z, à coefficients dans C, où z est une (Δ_t)-constante avec $\partial_z z = 1$, C est le corps des constantes par rapport à ∂_z, et ∂_z commute avec toutes les dérivations.

Dans [MS12], les auteurs déduisent le corollaire suivant des équivalents paramétriques du problème de Riemann-Hilbert, cf. [MS12], Théorème 5.1, et du théorème de Schlesinger.

Corollaire 1.2.15 ([MS12], Corollaire 5.2)**.** *Soient C un corps (Δ_t)-universel, H un sous-groupe de $\mathrm{GL}_m(C)$ qui est finiment engendré et G la clôture de Kolchin de H. Alors, G est le groupe de Galois différentiel paramétré global d'un système différentiel linéaire paramétré à coefficients dans $C(z)$.*

Nous utilisons le théorème 1.2.13, pour prouver la réciproque du corollaire précédent. Finalement, nous obtenons :

Théorème 1.2.16 (Théorème 3.3.11)**.** *Soit G un sous-groupe différentiel de $\mathrm{GL}_m(C)$. Alors, G est le groupe de Galois différentiel paramétré global d'un système différentiel linéaire paramétré à coefficients dans $C(z)$ si et seulement si G contient un sous-groupe finiment engendré qui est dense pour la topologie de Kolchin.*

Nous allons maintenant donner un critère d'isomonodromie. Nous avons vu que si (1.2.4) est isomonodromique, voir §1.2.2, alors ses matrices de monodromie ne dépendent pas du paramètre t. La réciproque est malheureusement fausse. Comme nous allons le voir, ceci est dû au fait que dans le cas non fuchsien, les matrices de monodromie paramétrées ne suffisent plus à engendrer un sous-groupe dense pour la topologie de Kolchin dans le groupe de Galois. Nous rappelons que (1.2.4) est isomonodromique, si et seulement si (1.2.4) peut être complété en un système complètement intégrable, que $\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)$ désigne l'extension de Picard-Vessiot paramétrée de (1.2.4) construite dans §1.2.3.2, et que $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)\right)$ est le groupe de Galois. Du théorème 1.2.13 nous déduisons :

Proposition 1.2.17 (Proposition 3.3.2)**.** *Le système différentiel linéaire paramétré (1.2.4) peut être complété en un système complètement intégrable, si et seulement si, il existe une matrice fondamentale $F(z,t)$ à coefficients dans $\widetilde{\mathcal{M}_U(z)}$, telle que les images des générateurs topologiques de $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)\right)$ décrits dans le théorème 1.2.13 par rapport à la représentation associée à $F(z,t)$ appartiennent à $\mathrm{GL}_m(\mathbb{C})$.*

Grosso modo, cette proposition nous dit qu'un système différentiel linéaire paramétré peut être complété en un système complètement intégrable, si et seulement si, il existe une matrice fondamentale à coefficients dans $\widetilde{\mathcal{M}_U(z)}$, telle que les matrices de monodromie, le tore exponentiel et les matrices de Stokes de chaque singularité soient constants. De manière équivalente, l'équation (1.2.4) peut être complétée en un système complètement intégrable, si et seulement si les générateurs topologiques du théorème 1.2.13 sont conjugués sur $\mathrm{GL}_m(\mathcal{M}_U)$ à des matrices constantes.

Dans le cas des systèmes différentiels paramétrés fuchsiens, le tore exponentiel et les matrices de Stokes sont l'identité. Nous retrouvons donc le fait que le système peut être complété en un système complètement intégrable si et seulement si, il existe une matrice fondamentale telle que les matrices de monodromie soient constantes.

Nous présentons maintenant une dernière application. Nous renvoyons à §3.3.2 et [Sib75] pour plus de détails. Considérons l'équation

$$z^7\partial_z^2 W(z,t) = (1+tz^3)W(z,t),$$

dont l'unique singularité est 0. Les directions singulières sont les $\frac{2k\pi}{5}$ avec $k \in \mathbb{Z}$. La matrice de Stokes de direction $\frac{8\pi}{5}$ est de la forme :

$$\begin{pmatrix} 1 & -C_0(t)e^{\frac{3i\pi}{5}} \\ 0 & 1 \end{pmatrix},$$

où $C_0(t) \in \mathcal{M}_{\mathbb{C}}$ est définie dans [Sib75]. L'auteur se demandait si cette fonction satisfaisait ou non des équations différentielles polynomiales. Nous utilisons la théorie de Galois différentielle paramétrée et en particulier notre théorème 1.2.13, pour prouver que $C_0(t)$ ne satisfait pas d'équations différentielles linéaires à coefficients dans un certain corps.

1.2.4 Calcul du groupe de Galois différentiel paramétré pour certaines équations d'ordre deux (Chapitre 2)

Nous finissons cette section par la généralisation de l'algorithme de Kovacic pour la théorie de Galois différentielle paramétrée. C'est l'objet central du chapitre 2. Certains arguments du chapitre ont été utilisés dans [Arr12], pour traiter d'autres cas.

Soient C un corps (Δ_t)-universel et $C(z)$ le corps (∂_z, Δ_t)-différentiel que nous avons définis dans §1.2.3.3. Considérons

$$\partial_z \begin{pmatrix} Y \\ \partial_z Y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix} \begin{pmatrix} Y \\ \partial_z Y \end{pmatrix}, \tag{1.2.5}$$

avec $r \in C(z)$. Soient G le groupe de Galois paramétré et H le groupe de Galois non paramétré. Comme nous le voyons dans §2.2, en utilisant le théorème de Seidenberg, le corps (Δ_t)-différentiel engendré par les z-coefficients de r, vu comme quotient de deux polynômes, et les coefficients des polynômes qui définissent G et H, vus comme sous-groupes différentiels et algébriques de $\mathrm{GL}_m(C)$, est isomorphe au corps (Δ_t)-différentiel $\mathcal{M}_U$, où U, est un polydisque non vide de $\mathbb{C}^n$. Nous pouvons donc voir G (resp. H) comme un sous-groupe différentiel (resp. algébrique) de $\mathrm{GL}_m(\mathcal{M}_U)$, et $r(z)$ comme un élément de $\mathcal{M}_U(z)$ que nous noterons $r(z,t)$. De plus, quitte à restreindre U, nous pouvons supposer que si G et H sont conjugués sur $\mathrm{GL}_m(C)$ à d'autres groupes, alors la conjugaison à lieu sur $\mathrm{GL}_m(\mathcal{M}_U)$.

Les solutions particulières trouvées ne vont faire intervenir qu'un nombre fini d'éléments de C. Appliquant le même raisonnement que ci-dessus, quitte à restreindre U, nous pouvons supposer qu'elles font intervenir des éléments de $\mathcal{M}_U$. Puisque le corps C est algébriquement clos et de caractéristique nulle, l'algorithme de Kovacic fonctionne dans ce cas, ce qui nous permet de calculer H. Nous avons vu dans §1.2.1 que G est dense pour la topologie de Kolchin dans H. Il s'agit donc de calculer G, connaissant sa clôture de Zariski.

Avant d'énoncer le prochain théorème, nous introduisons quelques notations. Soit $\mathbb{D}$, un sous $\mathcal{M}_U$-espace vectoriel du $\mathcal{M}_U$-espace vectoriel des dérivations engendrées par Δ_t. On définit :

$$\mathcal{M}_U^{\mathbb{D}} := \{m \in \mathcal{M}_U | \forall \partial \in \mathbb{D}, \partial m = 0\}.$$

Si g est une fonction appartenant à une extension de corps (∂_z, Δ_t)-différentiel de $\mathcal{M}_U(z)$, on note

$$\mathcal{M}_U(z)\langle g \rangle_{\partial_z, \Delta_t},$$

le corps (∂_z, Δ_t)-différentiel engendré par $\mathcal{M}_U(z)$ et g. Nous trouvons (cf. §1.2.1 pour les autres notations et §1.1.4 pour le cas non paramétré) :

Théorème 1.2.18 (Théorème 2.2.10)**.** *Il y a quatre possibilités.*

1. *H est conjugué à un sous-groupe $B = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \text{ où } a \in \mathbb{C}^*, b \in \mathbb{C}\right\}$. Dans ce cas, il existe une solution de la forme $g(z,t) = e^{\int_0^z f(u,t)du}$, avec $f(z,t) \in \mathcal{M}_U(z)$. Il y a deux possibilités s'excluant mutuellement.*
   - *(a) Si $g(z,t) \in \mathcal{M}_U(z)$, alors nous pouvons calculer une autre solution $g(z,t)\displaystyle\int_{u=0}^{z} g(u,t)^{-2}du$ qui est linéairement indépendante de $g(z,t)$. Dans cette base, on peut calculer explicitement G grâce au lemme 2.2.2.*
   - *(b) Si $g(z,t) \notin \mathcal{M}_U(z)$, alors G est conjugué à*

   $$\left\{\begin{pmatrix} m(t) & a(t) \\ 0 & m(t)^{-1} \end{pmatrix}, \text{ où } m(t) \in M, a(t) \in A\right\},$$

   avec :

   M, que nous pouvons calculer explicitement grâce au lemme 2.2.2;

   $$A = \left\{ a(t) \in \mathcal{M}_U \;\middle|\; \begin{array}{l} \forall P \in \mathcal{M}_U\{y\}_{\Delta_t}, \\ P\left(\int_{u=0}^{z} g(u,t)^{-2}du\right) \in \mathcal{M}_U(z)\langle g\rangle_{\partial_z,\Delta_t} \iff P(a(t)) = 0 \end{array} \right\}.$$

2. *Il existe une solution de la forme $g(z,t) = e^{\int_0^z f(u,t)du}$, où $f(z,t)$ est algébrique sur $\mathcal{M}_U(z)$ de degré 2 et $f(z,t) \notin \mathcal{M}_U(z)$. Dans ce cas, G est conjugué à*

   $$\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \bigcup \begin{pmatrix} 0 & b^{-1} \\ -b & 0 \end{pmatrix}, \text{ où } a, b \in M\right\}, \text{ avec,}$$

   $$M = \left\{ a(t) \in \mathcal{M}_U \;\middle|\; \begin{array}{l} \forall P \in \mathcal{M}_U\{y_1 \dots, y_n\}_{\Delta_t}, \\ P\left(\partial_{t_i} \int_{u=0}^{z} f(u,t)du\right) \in \mathcal{M}_U(z) \Leftrightarrow P\left(\frac{\partial_{t_i} a(t)}{a(t)}\right) = 0 \end{array} \right\}.$$

3. *Les deux premiers cas ne se produisent pas et H est fini. Dans ce cas $G = H$.*
4. *Si aucun des trois précédents cas ne se produit, $H = \mathrm{SL}_2(\mathcal{M}_U)$ et il existe $\mathbb{D}$, un sous $\mathcal{M}_U$-espace vectoriel de l'espace vectoriel des dérivations engendrées par Δ_t, tel que G est conjugué à $\mathrm{SL}_2\left(\mathcal{M}_{\mathrm{U}}^{\mathbb{D}}\right)$. De plus, $\partial \in \mathbb{D}$ si et seulement si le système différentiel linéaire paramétré suivant a une solution dans $\mathcal{M}_U(z)$:*

   $$\frac{\partial_z^3 b(z,t)}{2} = 2\partial_z b(z,t) r(z,t) + b(z,t)\partial_z r(z,t) - \partial r(z,t).$$

1.3 Équations aux q-différences

Dans cette section, nous faisons un rapide survol de la classification méromorphe des systèmes au q-différences linéaires à coefficients dans $\mathbb{C}(z)$ et du phénomène de confluence. Pour éviter des énoncés trop techniques, la plupart des résultats seront écrits dans un cadre moins général que celui pour lesquels ils ont été originellement prouvés. Dans §1.3.1 nous résumons les travaux de [Sau00] qui traitent de la confluence dans le cas fuchsien. Dans §1.3.2 nous exposons les résultats de [RSZ13] sur la classification méromorphe des systèmes aux q-différences à coefficients dans $\mathbb{C}(\{z\})$. Au cours des deux dernières sous-sections, nous exposons les résultats du chapitre 4 qui portent sur la confluence des solutions asymptotiques des séries divergentes.

1.3.1 Cas fuchsien

Nous présenterons ici brièvement les résultats de [Sau00]. Dorénavant, q désignera un nombre réel strictement plus grand que 1. Définissons par σ_q l'opérateur de dilatation :

$$\sigma_q(f(z)) := f(qz).$$

Nous souhaitons résoudre les systèmes aux q-différences linéaires de la forme :

$$\sigma_q Y(z) = B(z)Y(z), \tag{1.3.1}$$

où $B(z) \in \mathrm{GL}_m\Big(\mathbb{C}(z)\Big)$. Pour cela, il faut choisir des solutions de $\sigma_q Y = zY$, $\sigma_q Y = aY$ avec $a \in \mathbb{C}^*$ et $\sigma_q Y = Y + 1$, qui joueront le rôle qu'opèrent e^z, z^a et $\log(z)$ dans le cas différentiel. Les « anciens », cf. [Ada31, Ada29, Bir30], utilisaient pour cela les fonctions multivaluées $z^{\log(z)/\log(q)}$, $\left(z^{\log(a)/\log(q)}\right)_{a\in\mathbb{C}^*}$ et $\log(z)/\log(q)$ respectivement. Cependant, un résultat de Praagman nous indique que tout système aux q-différences linéaires à coefficients dans $\mathbb{C}(z)$ admet une base de solutions méromorphes sur $\mathbb{C}^*$. Construire des solutions méromorphes sur $\mathbb{C}^*$ plutôt que des solutions multivaluées, va permettre aux auteurs de [Sau04, RS07, RS09] de définir un groupe de Galois sur $\mathbb{C}$, plutôt que sur le corps des fonctions invariantes sous l'action de σ_q. Nous préférerons donc l'approche de J. Sauloy en utilisant les fonctions :

- $\Theta_q(z) := \sum_{n\in\mathbb{Z}} q^{\frac{-n(n-1)}{2}} z^n$,
- $\Big(\Lambda_{q,a}(z)\Big)_{a\in\mathbb{C}^*} := \left(\frac{\Theta_q(z)}{\Theta_q(z/a)}\right)_{a\in\mathbb{C}^*}$,
- $l_q(z) := \dfrac{\delta\left(\Theta_q(z)\right)}{\Theta_q(z)}$ (rappelons que $\delta = z\frac{d}{dz}$).

La fonction Θ_q est méromorphe sur $\mathbb{C}^*$ et s'annule sur la q-spirale $-q^{\mathbb{Z}}$. Par conséquent, $\Big(\Lambda_{q,a}(z)\Big)_{a\in\mathbb{C}^*}$ et $l_q(z)$ sont aussi méromorphes sur $\mathbb{C}^*$. Nous reviendrons plus en détail sur ces fonctions au cours du chapitre 4. Soit B une matrice complexe inversible et considérons sa décomposition de Jordan $B = P(DN)P^{-1}$, où $D := \mathrm{Diag}(d_i)$ est diagonale, N est nilpotente avec $DN = ND$, et P est une matrice complexe inversible. Nous construisons la matrice

$$\Lambda_{q,B} := P\left(\mathrm{Diag}\left(\Lambda_{q,d_i}\right)e^{\log(N)l_q}\right)P^{-1} \in \mathrm{GL}_m\Big(\mathbb{C}\left[l_q, (\Lambda_{q,a})_{a\in\mathbb{C}^*}\right]\Big)$$

qui satisfait :

$$\sigma_q\Lambda_{q,B} = B\Lambda_{q,B} = \Lambda_{q,B}B.$$

Nous supposerons maintenant que le système (1.3.1) est fuchsien en 0 et que ses exposants en 0 sont non résonants, c'est-à-dire que $B(0) \in \mathrm{GL}_m(\mathbb{C})$ et que deux de ses valeurs distinctes ont nécessairement une image différente dans $\mathbb{C}^*/q^{\mathbb{Z}}$. Comme dans le cas différentiel, l'algorithme de Frobenius nous donne un système fondamental de solutions. Dans [Sau00], §1, il est prouvé :

Proposition 1.3.1. *Sous les hypothèses précédentes, il existe* $\hat{H}(z) \in \mathrm{Id} + z\mathrm{GL}_m\Big(\mathbb{C}\{z\}\Big)$, *telle que*

$$\hat{H}(z)\Lambda_{q,B(0)}$$

soit une matrice fondamentale de (1.3.1).

Considérons maintenant q comme un paramètre que nous allons faire tendre vers 1. Soit $\delta_q := \frac{\sigma_q - \mathrm{Id}}{q-1}$, qui converge formellement vers l'opérateur δ. Soit $B(z) \in \mathrm{GL}_m\Big(\mathbb{C}(z)\Big)$ telle que $B(0) \in \mathrm{GL}_m(\mathbb{C})$ a des valeurs propres distinctes non congrues modulo $\mathbb{Z}$. En particulier, on trouve, cf. [Sau00], que pour q proche de 1, $B(0)$ satisfait les hypothèses de la proposition 1.3.1. Nous obtenons donc une matrice fondamentale de la famille d'équations $\delta_q Y(z,q) = B(z)Y(z,q)$ de la forme

$$\hat{H}(z,q)\Lambda_{q,\mathrm{Id}+(q-1)B(0)},$$

avec $z \mapsto \hat{H}(z,q) \in \mathrm{Id} + z\mathrm{GL}_m\Big(\mathbb{C}\{z\}\Big)$. Il est naturel de se demander si la matrice ci-dessus converge vers une solution fondamentale de $\delta\widetilde{Y}(z) = B(z)\widetilde{Y}(z)$ calculée grâce à l'algorithme de Frobenius dans sa version différentielle. La réponse à cette question est donnée dans [Sau00], §3. Cf. le théorème ci-dessous.

Théorème 1.3.2 (J. Sauloy)**.** *Soit* $B(z) \in \mathrm{GL}_m\Big(\mathbb{C}(z)\Big)$ *telle que* $B(0) \in \mathrm{GL}_m(\mathbb{C})$ *a des valeurs propres distinctes non congrues modulo* $\mathbb{Z}$. *Soit* $\widetilde{H}(z) \in \mathrm{Id} + z\mathrm{GL}_m\Big(\mathbb{C}\{z\}\Big)$, *telle que* $\widetilde{H}(z)e^{B(0)\log(z)}$ *soit une matrice fondamentale de* $\delta\widetilde{Y}(z) = B(z)\widetilde{Y}(z)$.

– *On a la convergence uniforme*

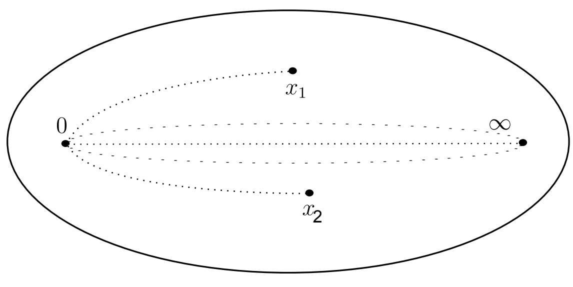
$$\lim_{q\to 1} \Lambda_{q,\mathrm{Id}+(q-1)B(0)} = e^{B(0)\log(z)},$$

sur les compacts de $\mathbb{C} \setminus \mathbb{R}_{<0}$.

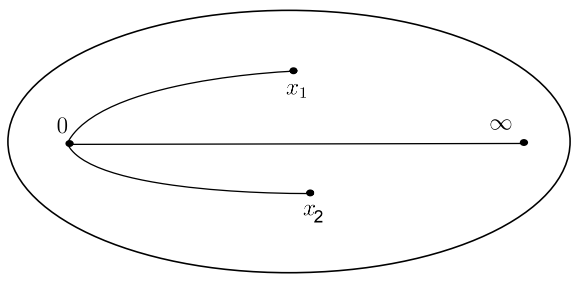
– *Soient* $x_1, \dots, x_r \in \mathbb{C}^*$ *les pôles de* $B(z)$. *On a la convergence uniforme*

$$\lim_{q\to 1} \hat{H}(z,q) = \widetilde{H}(z),$$

sur les compacts de $\mathbb{C} \setminus \{\mathbb{R}_{\geq 1}x_1, \dots, \mathbb{R}_{\geq 1}x_r\}$, *où* $\mathbb{R}_{\geq 1}x_i := \{\alpha x_i | \alpha \geq 1\}$. *L'égalité ici est à prendre au sens de « égale au prolongement analytique de ».*



Domaine de définition de $\hat{H}(z,q)\Lambda_{q,\mathrm{Id}+(q-1)B(0)}$



Domaine de définition de $\widetilde{H}(z)e^{B(0)\log(z)}$

Il est à noter que ce résultat est le point crucial de la démonstration du théorème 2.6 de [DVZ09] qui donne un résultat de confluence de ressommation des séries formelles de même nature que le théorème 1.3.7 ci-dessous. Voir la remarque 4.4.7 pour plus de détails.

Nous résumons ici les résultats de [Sau00], §4. Nous y renvoyons le lecteur pour plus de détails. Nous considérons toujours $\delta_q Y(z,q) = B(z)Y(z,q)$ où $B(z) \in \mathrm{GL}_m\Big(\mathbb{C}(z)\Big)$ satisfait les hypothèses du théorème ci-dessus. Nous faisons désormais les mêmes hypothèses à l'infini. De plus, nous supposerons que les arguments des pôles de $B(z)$ sont tous différents et que aucun n'est congru à π modulo 2π.

Nous obtenons grâce au théorème 1.3.2 des matrices fondamentales de la famille d'équations $\delta_q Y(z,q) = B(z)Y(z,q)$ en 0 et en l'infini que nous noterons par $\Phi_0(z,q)$ et $\Phi_\infty(z,q)$. Soient $\widetilde{\Phi}_0(z)$ et $\widetilde{\Phi}_\infty(z)$ les solutions de $\delta \widetilde{Y}(z) = B(z)\widetilde{Y}(z)$ que nous obtenons comme limite lorsque $q \to 1$. La matrice de Birkhoff $\Phi_0(z,q)\Big(\Phi_\infty(z,q)\Big)^{-1}$ est invariante sous l'action de σ_q et converge lorsque q tend vers 1 vers la matrice $\widetilde{\Phi}_0(z)\Big(\widetilde{\Phi}_\infty(z)\Big)^{-1}$ qui est localement constante. Soient $x_1, \dots, x_r$ les pôles de $B(z)$ que nous ordonnons tels que l'on ait $-\pi < \arg(x_1) < \dots < \arg(x_r) < \pi$. Le domaine de définition de $\widetilde{\Phi}_0(z)\Big(\widetilde{\Phi}_\infty(z)\Big)^{-1}$ est

$$\widetilde{\Omega} := \mathbb{C}^* \setminus \{\mathbb{R}_{<0}, \mathbb{R}_{\geq 0}x_1, \dots, \mathbb{R}_{\geq 0}x_r\} \text{ où } \mathbb{R}_{\geq 0}x_i := \{\alpha x_i | \alpha \geq 0\}.$$

Soient $a_0 := -\pi$, $a_{r+1} := \pi$ et pour $i \in \{1, \dots, r\}$, posons $a_i := \arg(x_i)$. Les composantes connexes de $\widetilde{\Omega}$ sont les $\widetilde{U}_i := \{z \in \mathbb{C}^* | \arg(z) \in]a_i, a_{i+1}[\}$ avec $i \in \{0, \dots, r\}$. Notons $\widetilde{P}_i$ la valeur de $\widetilde{\Phi}_0(z)\Big(\widetilde{\Phi}_\infty(z)\Big)^{-1}$ en $\widetilde{U}_i$. Fixons $i \in \{1, \dots, r\}$ et a, un élément de $\widetilde{U}_{i-1}$. Définissons le lacet parcourant le cercle γ orienté positivement autour de la singularité x_i qui passe par a. Nous pouvons choisir a de telle sorte que ce cercle soit inclus dans $\widetilde{U}_{i-1} \bigcup \mathbb{R}_{>0}x_i \bigcup \widetilde{U}_i$. Dans [Sau00], §4, il est prouvé :

Théorème 1.3.3 (J. Sauloy)**.** *Le prolongement analytique le long de γ change $\widetilde{\Phi}_0(z)$ en $\widetilde{\Phi}_0(z)\Big(\widetilde{P}_i\Big)^{-1}\widetilde{P}_{i-1}$. Autrement dit, la matrice de monodromie de $\delta\widetilde{Y}(z) = B(z)\widetilde{Y}(z)$ autour de la singularité x_i dans la base $\widetilde{\Phi}_0(z)$ est $\Big(\widetilde{P}_i\Big)^{-1}\widetilde{P}_{i-1}$.*

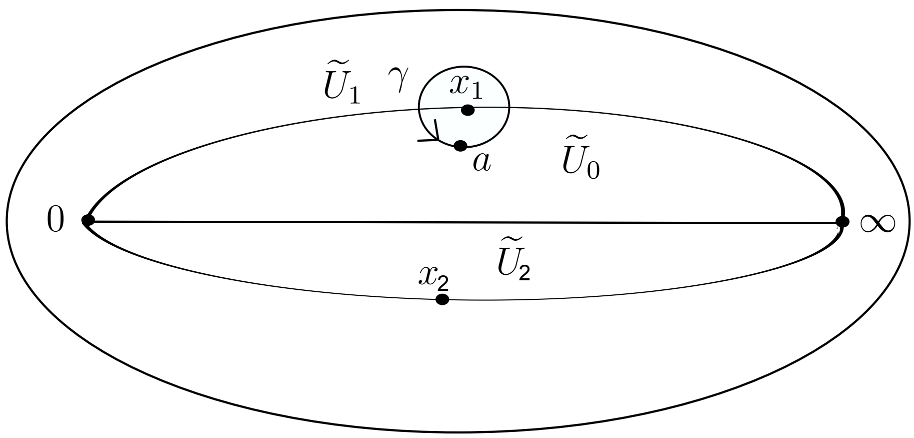


FIGURE 1.1 – Domaine de définition de $\widetilde{\Phi}_0(z)\Big(\widetilde{\Phi}_\infty(z)\Big)^{-1}$

1.3.2 Classification méromorphe des systèmes aux q-différences à coefficients rationnels

Nous nous intéressons maintenant aux systèmes aux q-différences linéaires à coefficients dans $\mathbb{C}(z)$, non nécessairement fuchsiens. Dans cette section, le réel $q > 1$ sera à nouveau fixé. Comme pour le cas différentiel, des séries formelles divergentes peuvent être solutions et nous sommes obligés de raisonner par classes d'équivalences formelles.

Nous étendons l'action de σ_q sur $\bigcup_{\nu\in\mathbb{N}^*} \mathbb{C}\left(\left(z^{1/\nu}\right)\right)$ par $\sigma_q z^{1/\nu} = e^{\log(q)/\nu} z^{1/\nu}$, pour $\nu \in \mathbb{N}^*$. Soit K une extension de corps intermédiaire $\bigcup_{\nu\in\mathbb{N}^*} \mathbb{C}\left(\left(z^{1/\nu}\right)\right) \Big| K \Big| \mathbb{C}(z)$, que nous supposerons stable par σ_q. Soient $A, B \in \mathrm{GL}_m(K)$. Les deux systèmes aux q-différences linéaires $\sigma_q Y = AY$ et $\sigma_q Y = BY$ sont dit équivalents sur K si il existe $P \in \mathrm{GL}_m(K)$, appelée transformation de jauge, telle que

$$A = P[B]_{\sigma_q} := (\sigma_q P) B P^{-1}.$$

En particulier,

$$\sigma_q Y = BY \Longleftrightarrow \sigma_q (PY) = APY.$$

Le théorème suivant reste vrai pour les systèmes à coefficients dans $\mathbb{C}((z))$.

Théorème 1.3.4 ([RSZ13], §2.2)**.** *Soit $B \in \mathrm{GL}_m\Big(\mathbb{C}(z)\Big)$ et considérons le système $\sigma_q Y(z) = B(z)Y(z)$. Il existe*

- $\nu \in \mathbb{N}^*$, $\mu_1 < \cdots < \mu_k \in \mathbb{Z}/\nu$,
- $m_1, \ldots, m_k \in \mathbb{N}$,
- $B_i \in \mathrm{GL}_{m_i}(\mathbb{C})$,
- $\hat{H} \in \mathrm{GL}_m\Big(\mathbb{C}\left(\left(z^{1/\nu}\right)\right)\Big)$,

tels que :

$$B = \hat{H}\left[A_f\right]_{\sigma_q} \text{ avec } A_f := \begin{pmatrix} z^{\mu_1}B_1 & & 0 \\ & \ddots & \\ 0 & & z^{\mu_k}B_k \end{pmatrix}.$$

La classification méromorphe et l'étude du phénomène de Stokes pour les équations aux q-différences ont récemment obtenues des contributions importantes, cf. [RSZ13]. Voir aussi [Bug11], pour un résultat de même nature valable dans un autre cadre. Tout ce qui suit reste vrai si l'on remplace $\mathbb{C}(z)$ par $\mathbb{C}(\{z\})$. Le théorème suivant nous dit qu'un système à coefficients dans $\mathbb{C}(z)$, peut se réduire via une transformation de jauge méromorphe à un système sous forme normale de Birkhoff-Guenther.

Théorème 1.3.5 ([RSZ13], §2.2)**.** *Soit $B \in \mathrm{GL}_m\Big(\mathbb{C}(z)\Big)$ et considérons le système $\sigma_q Y(z) = B(z)Y(z)$. Il existe*

- $\nu \in \mathbb{N}^*$, $\mu_1 < \cdots < \mu_k \in \mathbb{Z}/\nu$,
- $m_1 \ldots, m_k \in \mathbb{N}$,
- $B_i \in \mathrm{GL}_{m_i}(\mathbb{C})$,
- $U_{i,j}(z)$, $m_i \times m_j$ *matrices à coefficients dans* $\sum_{l=\mu_i\nu}^{\mu_j\nu-1} \mathbb{C} z^{l/\nu}$,
- $F \in \mathrm{GL}_m\Big(\mathbb{C}\left(\left\{z^{1/\nu}\right\}\right)\Big)$,

tels que :

$$B = F\left[A\right]_{\sigma_q}, \text{ avec } A := \begin{pmatrix} z^{\mu_1}B_1 & \dots & \dots & \dots & \dots \\ 0 & \ddots & \dots & U_{i,j}(z) & \dots \\ \vdots & \ddots & \ddots & \dots & \dots \\ \vdots & \dots & \ddots & \ddots & \dots \\ 0 & \dots & \dots & 0 & z^{\mu_k}B_k \end{pmatrix}.$$

Supposons que $\nu = 1$. Dans le chapitre 6 de [RSZ13], les auteurs prouvent qu'il existe un ensemble fini $\Sigma \subset \mathbb{C}^*/q^{\mathbb{Z}}$, tel que si $\lambda \in \mathbb{C}^*/q^{\mathbb{Z}}$ n'est pas dans Σ, alors il existe une unique solution fondamentale de $\sigma_q Y(z) = B(z)Y(z)$ de la forme $F(z)C(z)$ avec

$$C(z) := \begin{pmatrix} \mathrm{Id} & \dots & \dots & \dots & \dots \\ 0 & \ddots & \dots & U_{i,j}^{[\lambda]}(z) & \dots \\ \vdots & \ddots & \ddots & \dots & \dots \\ \vdots & \dots & \ddots & \ddots & \dots \\ 0 & \dots & \dots & 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \Theta_q(z)^{\mu_1}\Lambda_{q,B_1} & & \\ & \ddots & \\ & & \Theta_q(z)^{\mu_k}\Lambda_{q,B_k} \end{pmatrix},$$

telle que pour tout i, j, $U_{i,j}^{[\lambda]}(z)$ est méromorphe sur $\mathbb{C}^*$ et possède des pôles uniquement sur la q-spirale $q^{\mathbb{Z}}\lambda$ d'ordre au plus $\mu_j - \mu_i$.

Les solutions que nous venons d'écrire sont méromorphes sur $\mathbb{C}^*$ et peuvent se calculer algorithmiquement, cf. [RSZ13], §6.1. Ce sont les solutions de ce type qui permettent de faire la classification méromorphe des systèmes aux q-différences linéaires à coefficients dans $\mathbb{C}(\{z\})$: ils jouent donc un rôle analogue à la sommation de Borel-Laplace des systèmes différentiels. Cependant, le point de vue pour énoncer et démontrer le théorème de densité pour les systèmes aux q-différences linéaires à coefficients dans $\mathbb{C}(\{z\})$ est très différent de celui que nous avons exposé dans §1.1, cf. [RS07, RS09]. Il est à noter que ces résultats s'appuient sur la forme normale de Birkhoff-Guenther qui n'a pas d'équivalent différentiel. C'est pour cela que pour énoncer les résultats de confluence du chapitre 4, nous n'utiliserons pas ces solutions méromorphes.

1.3.3 Confluence des solutions asymptotiques (Chapitre 4)

Dans cette sous-section nous résumons dans un cas particulier les résultats des sections 4.4 et 4.7 du chapitre 4. Nous voyons à nouveau $q > 1$ comme un paramètre que nous allons faire tendre vers 1. L'expression « q proche de 1 », signifiera « au voisinage de 1 dans $]1, \infty[$ ».

Nous commençons par introduire quelques notations. Nous définissons la famille $(\rho_a)_{a\in\mathbb{C}}$ d'applications continues de $\widetilde{\mathbb{C}}$, la surface de Riemann du logarithme, dans lui même, qui envoie z sur $e^{a\log(z)}$. Pour $\widetilde{f} := \sum f_l z^l \in \bigcup_{\nu\in\mathbb{N}^*} \mathbb{C}\left(\left(z^{1/\nu}\right)\right)$ (resp. $f \in \mathcal{A}(a,b)$, cf. §1.1) et $c \in \mathbb{Q}_{>0}$, nous posons $\rho_c\left(\widetilde{f}\right) := \sum f_l z^{lc} \in \bigcup_{\nu\in\mathbb{N}^*} \mathbb{C}\left(\left(z^{1/\nu}\right)\right)$ (resp. $\rho_c(f) := f(z^c)$).

Pour tout $d \in \mathbb{R}$ nous définissons :

$$\int_{q^{\mathbb{Z}}e^{id}} f(\zeta)d_q\zeta := (q-1)\sum_{l\in\mathbb{Z}} f\left(q^l e^{id}\right) q^l e^{id},$$

dès lors que la série converge. Nous définissons par ailleurs la q-exponentielle

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_q^!} \in \mathbb{C}\{z\},$$

avec $[n]_q^! := \prod_{l=0}^{n} 1 + ... + q^l =: \prod_{l=0}^{n} [l]_q$. Elle est méromorphe sur $\mathbb{C}^*$ et s'annule sur la q-spirale $\frac{q^{\mathbb{N}^*}}{1-q}$.

Dans la littérature, nous pouvons trouver de nombreuses définitions des transformées de q-Borel et de q-Laplace, cf. [Zha00, Zha02, MZ00, DVZ09, RZ02]. Nous définissons maintenant celles qui nous seront utiles.

Définition 1.3.6. Soient $k \in \mathbb{Q}_{>0}$ et $d \in \mathbb{R}$.

(1) Soit $\nu \in \mathbb{N}^*$ minimal tel que $\nu k \in \mathbb{N}^*$. Nous définissons $\hat{\mathcal{B}}_{q,k}$ comme étant l'application de $\mathbb{C}\left[\left[z^{\nu k}\right]\right]$ dans $\mathbb{C}\left[\left[\zeta^{\nu k}\right]\right]$ satisfaisant :

$$\hat{\mathcal{B}}_{q,k}\left(\sum a_l z^l\right) := \sum \frac{a_l}{[l/k]_q^!} \zeta^l.$$

Remarquons que nous avons

$$\hat{\mathcal{B}}_{q,k} = \rho_k \circ \hat{\mathcal{B}}_{q,1} \circ \rho_{1/k}.$$

(2) Soit f une fonction telle qu'il existe un réel positif $\varepsilon > 0$, tel que pour q proche de 1, $z \mapsto f(z,q) \in \mathcal{A}(d-\varepsilon, d+\varepsilon)$. Nous dirons que f appartient à $\overline{\mathbb{H}}_k^d$, si pour q proche de 1, $z \mapsto f(z,q)$ admet un prolongement analytique sur $\overline{S}(d-\varepsilon, d+\varepsilon)$, que nous appellerons toujours f, tel qu'il existe des constantes, $J, L > 0$, telles que pour tout $z \in \mathbb{R}_{>0}$:

$$\left|f\left(e^{id}z, q\right)\right| < J e_q\left(L z^k\right).$$

(3) Comme nous le verrons dans §4.3, les applications suivantes sont bien définies et nous les appellerons transformées de q-Laplace d'ordre 1 et k respectivement :

$$\begin{array}{rcccl} \overline{\mathbb{H}}_1^d & \ni & f & \mapsto & \mathcal{L}_{q,1}^{[d]}(f)(z,q) & := & \displaystyle\int_{q^{\mathbb{Z}}e^{id}} \frac{f(\zeta,q)}{z e_q\left(\frac{q\zeta}{z}\right)} d_q\zeta, \\ \overline{\mathbb{H}}_k^d & \ni & g & \mapsto & \mathcal{L}_{q,k}^{[d]}(g)(z,q) & := & \rho_k \circ \mathcal{L}_{q,1}^{[d]} \circ \rho_{1/k}(g)(z,q). \end{array}$$

La fonction $z \mapsto \mathcal{L}_{q,1}^{[d]}(f)(z,q)$ est méromorphe sur un voisinage épointé de 0 dans $\mathbb{C}^*$. Pour $|z|$ petit, $\mathcal{L}_{q,1}^{[d]}(f)(z,q)$ a des pôles d'ordre au plus 1 contenus dans la q-spirale $q^{\mathbb{Z}}(1-q)e^{id}$.

Nous présentons maintenant une version légèrement affaiblie du théorème principal du chapitre 4, le théorème 4.4.5. Nous rappelons que $\delta_q = \frac{\sigma_q - \mathrm{Id}}{q-1}$. Considérons $z \mapsto \hat{h}(z,q) \in \mathbb{C}[[z]]$ dont les z-coefficients convergent lorsque q tend vers 1. Soit $\widetilde{h}(z) \in \mathbb{C}[[z]]$ la limite formelle. Nous supposons en outre que :

(A1) Il existe $m+1$ polynômes

$$z \mapsto b_0(z,q), \dots, b_m(z,q) \in \mathbb{C}[z],$$

dont les coefficients convergent lorsque q tend vers 1, tels que quel que soit q proche de 1, $\hat{h}(z,q)$ est solution de :

$$b_m(z,q)\delta_q^m\left(\hat{h}(z,q)\right) + \dots + b_0(z,q)\hat{h}(z,q) = 0.$$

Soient $\widetilde{b}_0(z), \dots, \widetilde{b}_m(z) \in \mathbb{C}[z]$, les limites des $b_0(z,q), \dots, b_m(z,q)$.

(A2) Pour q proche de 1, les valuations des $z \mapsto b_i(z,q) \in \mathbb{C}[z]$ sont indépendantes de q et sont égales aux valuations des $\widetilde{b}_i(z)$.

(A3) Il existe $c > 0$, tel que pour tout $i \leq m$ et q proche de 1 :

$$\left| b_i(z,q) - \widetilde{b}_i(z) \right| < (q-1)c \left(\left| \widetilde{b}_i(z) \right| + 1 \right).$$

Remarquons que la série $\widetilde{h}(z)$ est solution de :

$$\widetilde{b}_m(z)\delta^m\left(\widetilde{y}(z)\right) + \cdots + \widetilde{b}_0(z)\widetilde{y}(z) = 0. \tag{1.3.2}$$

Si tous les $\widetilde{b}_i(z)$ ont des valuations supérieures ou égales à $\widetilde{b}_m(z)$, alors pour q proche de 1, $z \mapsto \hat{h}(z,q), \widetilde{h}(z) \in \mathbb{C}\{z\}$ et on a d'après la remarque 4.4.4,

$$\lim_{q \to 1} \hat{h}(z,q) = \widetilde{h}(z),$$

uniformément sur un disque fermé centré en 0. Nous supposerons désormais que ce cas de figure ne se produit pas. Nous rappelons que nous avons vu dans §1.1.3, qu'il existe un ensemble fini modulo 2π de $\mathbb{R}$, tel que si d n'est pas dans cet ensemble, il existe $\varepsilon > 0$ et $\widetilde{S}^d\left(\widetilde{h}\right) \in \mathcal{A}\left(d-\varepsilon, d+\varepsilon\right)$, solution de (1.3.2) que l'on peut calculer en appliquant successivement à $\widetilde{h}(z)$ des transformées de Borel et Laplace. Nous pouvons maintenant énoncer le résultat principal du chapitre 4. Cf. l'appendice du chapitre 4 pour une variante de ce théorème avec un autre choix de transformée de q-Laplace.

Théorème 1.3.7 (Théorème 4.4.5)**.** *Notons* $\hat{h}(z,q) := \sum_{n=0}^{\infty} \hat{h}_n(q) z^n$. *Il existe*

- $\widetilde{\kappa}_1, \dots, \widetilde{\kappa}_s \in \mathbb{Q}_{>0}$,
- $\beta \in \mathbb{N}^*$,
- $\Sigma_{\widetilde{h}} \subset \mathbb{R}$ *fini modulo* $2\pi\mathbb{Z}$,

tels que si l'on se donne $d \in \mathbb{R} \setminus \Sigma_{\widetilde{h}}$ *et* $l \in \{0, \dots, \beta-1\}$, *alors la série suivante* $g_{1,l} := \hat{\mathcal{B}}_{q,\widetilde{\kappa}_1} \circ \cdots \circ \hat{\mathcal{B}}_{q,\widetilde{\kappa}_s}\left(\sum_{n=0}^{\infty} \hat{h}_{l+n\beta}(q) z^{n\beta}\right)$ *converge et appartient à* $\overline{\mathbb{H}}^d_{\widetilde{\kappa}_1}$.

De plus, pour $j=2$ *(resp.* $j=3,\dots,j=s$*),* $g_{j,l} := \mathcal{L}^{[d]}_{q,\widetilde{\kappa_{j-1}}}\left(g_{j-1,l}\right)$ *appartient à* $\overline{\mathbb{H}}^d_{\widetilde{\kappa}_j}$. *Soit* $S^{[d]}_{q,l}\left(\hat{h}\right) := \mathcal{L}^{[d]}_{q,\widetilde{\kappa}_s}\left(g_{r,l}\right)$. *La fonction*

$$S^{[d]}_q\left(\hat{h}\right) := \sum_{l=0}^{\beta-1} z^l S^{[d]}_{q,l}\left(\hat{h}\right) \in \mathcal{A}\left(d - \frac{\pi}{\widetilde{\kappa}_s}, d + \frac{\pi}{\widetilde{\kappa}_s}\right)$$

est solution de (1.2.1) et nous avons la convergence

$$\lim_{q \to 1} S^{[d]}_q\left(\hat{h}\right) = \widetilde{S}^d\left(\widetilde{h}\right),$$

uniformément sur les compacts de $\overline{S}\left(d - \frac{\pi}{2k_r}, d + \frac{\pi}{2k_r}\right) \setminus \bigcup \mathbb{R}_{\geq 1}\alpha_i$, *où* α_i *sont les racines de* $\widetilde{b}_m \in \mathbb{C}[z]$.

Exemple 1.3.8 (Confluence des séries hypergéométriques basiques). Soient $p = 1/q$, $r, s \in \mathbb{N}$ avec $r > s+1$, $a_1, \dots, a_r, b_1, \dots, b_s \in \mathbb{C} \setminus q^{\mathbb{N}}$, avec images différentes dans $\mathbb{C}^*/q^{\mathbb{Z}}$. Considérons la série formelle divergente :

$$ {}_r\varphi_s \left(\begin{matrix} a_1, \dots, a_r \\ \\ b_1, \dots, b_s \end{matrix} ; p, z \right) := \sum_{n=0}^{\infty} \frac{(a_1;p)_n \dots (a_r;p)_n}{(p;p)_n (b_1;p)_n \dots (b_s;p)_n} \left((-1)^n p^{n(n-1)/2} \right)^{1+s-r} z^n, $$

avec $(a;p)_{n+1} := (1 - ap^n)(a;p)_n$ et $(a;p)_0 := 1$, pour tout nombre complexe a. Soient $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s \in \mathbb{C} \setminus -\mathbb{N}$ avec images différentes dans $\mathbb{C}/\mathbb{Z}$, et définissons la série formelle divergente

$$ {}_rF_s \left(\begin{matrix} \alpha_1, \dots, \alpha_r \\ \\ \beta_1, \dots, \beta_s \end{matrix} ; z \right) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_r)_n}{n!(\beta_1)_n \dots (\beta_s)_n} z^n, $$

où $(\alpha)_0 := 1; (\alpha)_{n+1} := (\alpha + n)(\alpha)_n$ pour $\alpha \in \mathbb{C}$.

Si nous posons $\underline{p} := q^{-1/(r-s-1)}$, $\underline{q} := q^{1/(r-s-1)}$, $x := z\left(1 - \underline{p}\right)^{1+s-r}$, $a_i := \underline{p}^{\alpha_1}$ et $b_i = \underline{p}^{\beta_1}$, nous avons la convergence terme à terme de ${}_r\varphi_s$ vers ${}_rF_s$ lorsque $\underline{q} \to 1$. Dans la section 4.7, nous prouvons que si d n'est pas congru à $(r-s-1)\pi$ modulo 2π, alors on peut appliquer des q-analogues des transformées de Borel et Laplace à ${}_r\varphi_s$. De plus, nous démontrons, cf. le théorème 4.7.4, que la fonction obtenue converge uniformément sur tous les compacts de $\mathbb{C}_d := \{z \in \mathbb{C}^* | \arg(-z) \neq d\}$, lorsque $\underline{q}$ tend vers 1 vers la fonction,

$$ \sum_{j=1}^{r} \frac{\prod_{i=1}^{s} \Gamma(\beta_i) \prod_{\substack{i=1 \\ i \neq j}}^{r} \Gamma(\alpha_i - \alpha_j) \left((-1)^{s-r} z \right)^{-\alpha_j}}{\prod_{\substack{i=1 \\ i \neq j}}^{r} \Gamma(\alpha_i) \prod_{i=1}^{s} \Gamma(\beta_i - \alpha_j)} $$

$$ \times {}_{s+1}F_{r-1} \left(\begin{matrix} \alpha_j, \alpha_j - \beta_1 + 1, \dots, \alpha_j - \beta_s + 1 \\ \\ \alpha_j - \alpha_1 + 1, \dots, \widehat{\alpha_j - \alpha_j + 1}, \dots, \alpha_j - \alpha_r + 1 \end{matrix} ; \frac{(-1)^{s-r}}{z} \right), $$

où $(a_1, \dots, \widehat{a_j}, \dots, a_r)$ désigne la suite $(a_1, \dots, , a_r)$ dans laquelle le j-ème terme a été supprimé. L'égalité ici est à prendre au sens de « égale au prolongement analytique de ». Notons que le cas $r = 2$ et $s = 0$ avait déjà été traité dans [Zha02], §2.

1.3.4 Confluence des matrices fondamentales (Chapitre 4)

Dans cette sous-section, nous présentons une version affaiblie des résultats de §4.8. Nous voyons toujours $q > 1$ comme un paramètre réel que nous allons faire tendre vers 1. Nous considérons la famille d'équations linéaires

$$ \begin{cases} \Delta_q := b_m(z) \;\; \delta_q^m + b_{m-1}(z) \;\; \delta_q^{m-1} + \dots + b_0(z) \\ \\ \widetilde{\Delta} := b_m(z) \;\; \delta^m + b_{m-1}(z) \;\; \delta^{m-1} + \dots + b_0(z), \end{cases} $$

où $b_i \in \mathbb{C}[z]$. Considérons maintenant les systèmes correspondants

$$\begin{cases} \delta_q Y(z,q) &= B(z)Y(z,q) \\ \delta \widetilde{Y}(z) &= B(z)\widetilde{Y}(z). \end{cases} \tag{1.3.3}$$

Puisque les b_i sont indépendants de q, le polygone de Newton, cf. §4.2 pour une définition, de Δ_q, vu comme une σ_q-équation ne dépend pas de q. En particulier, les entiers et les rationnels ν, m_i et μ_i donnés par le théorème 1.3.4 qui ne dépendent en réalité que du polygone de Newton, ne dépendent pas de q. Ainsi, le théorème de Hukuhara-Turrittin et le théorème 1.3.4 nous assurent l'existence de

- $\nu \in \mathbb{N}^*$, $z \mapsto \hat{H}(z,q), \widetilde{H}(z) \in \mathrm{GL}_m\Big(\mathbb{C}\left(\left(z^{1/\nu}\right)\right)\Big)$;
- une matrice $A_f(z,q)$ diagonale par bloc ;
- des éléments $\widetilde{\lambda}_i(z) \in z^{-1/\nu}\mathbb{C}[z^{-1/\nu}]$, et une matrice $\widetilde{L} \in \mathrm{M}_m(\mathbb{C})$;

tels que

$$\begin{cases} \hat{H}(z,q)\Big[A_f(z,q)\Big]_{\sigma_q} = \mathrm{Id} + (q-1)B(z,q) \\ \widetilde{H}(z)\Big[\widetilde{L} + \mathrm{Diag}\left(\delta\widetilde{\lambda}_i(z)\right)\Big]_{\delta} = B(z). \end{cases} \tag{1.3.4}$$

Nous ferons en outre l'hypothèse que :

- les matrices $\widetilde{L}$ et $A_f(z,q)$ sont diagonales ;
- $\nu = 1$;
- le sous-espace vectoriel de $\mathrm{M}_m\Big(\mathbb{C}((z))\Big)$ des solutions du système $\widetilde{H}(z)\left[\widetilde{L} + \mathrm{Diag}\left(\delta\widetilde{\lambda}_i(z)\right)\right]_{\delta} = B(z)$ est de dimension 1.

Nous renvoyons à la remarque 4.8.1 pour plus d'explications sur ces hypothèses. Nous y voyons en particulier que sans perte de généralité, nous pouvons nous ramener au cas où :

- pour q proche de 1, les valuations des entrées de la première ligne de $\hat{H}(z,q)$ sont égales à 0 ;
- les valuations des entrées de la première ligne de $\widetilde{H}(z)$ sont nulles ;
- les termes constants des entrées de la première ligne de $\hat{H}(z,q)$ tendent vers ceux de $\widetilde{H}(z)$.

Nous définissons $\mathcal{O}_m^*$ comme étant l'anneau des matrices carrées inversibles de taille m, tel que $F(z,q) \in \mathcal{O}_m^*$, si pour q proche de 1, les entrées de $z \mapsto F(z,q)$ sont méromorphes sur $\mathbb{C}^*$, et $F(z,q)$ satisfait :

- on a la convergence uniforme sur les compacts de $\mathbb{C}^*$

$$\lim_{q\to 1}\Big(\delta_q F(z,q)\Big)F(z,q)^{-1} = 0;$$

- pour tout $z \in \mathbb{C}^*$, on a la simple convergence

$$\lim_{q\to 1} F(z,q) = \mathrm{Id}.$$

Sous ces hypothèses, nous prouvons :

Théorème 1.3.9 (Théorème 4.8.4)**.** *Écrivons* $\widetilde{\lambda}_i(z) = \sum_{j=1}^{k_i} \widetilde{\lambda}_{i,j} z^{-j}$.

(1) Il existe $z \mapsto F_1(z,q) \in \mathrm{GL}_m\Big(\mathbb{C}\{z\}\Big)$, $F_2(z,q) \in \mathcal{O}_m^*$, $z \mapsto N(z,q) \in \mathrm{M}_m\Big(\mathbb{C}(z)\Big)$ *tels que*

$$F_1(z,q)\Big[\mathrm{Id} + (q-1)N(z,q)\Big]_{\sigma_q} = \mathrm{Diag}\Big(B_i(q) z^{-\mu_i}\Big),$$

où $N(z,q)$ *satisfait*

$$\delta_q\left(F_2(z,q)\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}}\mathrm{Diag}\left(\prod_{j=1}^{k_i} e_{q^j}\left(\widetilde{\lambda}_{i,j} z^{-j}\right)\right)\right) =$$

$$N(z,q)F_2(z,q)\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}}\mathrm{Diag}\left(\prod_{j=1}^{k_i} e_{q^j}\left(\widetilde{\lambda}_{i,j} z^{-j}\right)\right).$$

(2) Les z-coefficients de $\hat{H}(z,q)F_1(z,q)$ *convergent simplement lorsque q tend vers* 1*, vers les z-coefficients de* $\widetilde{H}(z)$*. De plus, il existe* $N \in \mathbb{N}$*, tel que pour tout q proche de* 1*,* $z \mapsto z^N \hat{H}(z,q)F_1(z,q)$ *appartient à* $\mathrm{M}_m\Big(\mathbb{C}[[z]]\Big)$.

Le point (1) du théorème nous dit grosso modo qu'il existe une matrice fondamentale formelle de $\delta_q Y(z,q) = B(z,q)Y(z,q)$ de la forme

$$\hat{H}(z,q)F_1(z,q)F_2(z,q)\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}}\mathrm{Diag}\left(\prod_{j=1}^{k_i} e_{q^j}\left(\widetilde{\lambda}_{i,j} z^{-j}\right)\right),$$

qui converge formellement vers une matrice fondamentale de $\delta\widetilde{Y}(z) = \widetilde{B}(z)\widetilde{Y}(z)$ donnée par le théorème de Hukuhara-Turrittin

$$\widetilde{H}(z)e^{\log(z)\widetilde{L}}\mathrm{Diag}\left(e^{\widetilde{\lambda}_i(z)}\right).$$

Cf. la remarque 4.8.3. Le lemme 4.8.7 nous permet de combiner ce résultat avec le théorème 1.3.7. Cf. §4.8.3. Nous obtenons l'existence de $\Sigma \subset \mathbb{R}$ fini modulo 2π, tel que si $d \notin \Sigma$:

- nous pouvons appliquer lorsque q est proche de 1, des transformations de q-Borel et q-Laplace de direction d à chaque entrée de $z^N \hat{H}(z,q)F_1(z,q)$, pour obtenir une matrice $\mathcal{S}_q^{[d]}\left(z^N \hat{H}F_1\right)$ dont les entrées sont méromorphes sur $\mathbb{C}^*$;
- nous pouvons appliquer des transformations de Borel et Laplace de direction d à chaque entrée de $z^N\widetilde{H}(z)$, pour obtenir la matrice $\widetilde{\mathcal{S}}^d\left(z^N\widetilde{H}(z)\right)$ qui a été définie dans §1.1.3.

Pour tout $d \notin \Sigma$, nous obtenons une matrice fondamentale de $\delta_q Y(z,q) = B(z)Y(z,q)$ de la forme

$$\Phi_0^{[d]}(z,q) := z^{-N}\mathcal{S}_q^{[d]}\left(z^N\hat{H}F_1\right)F_2(z,q)\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}}\mathrm{Diag}\left(\prod_{j=1}^{k_i} e_{q^j}\left(\widetilde{\lambda}_{i,j} z^{-j}\right)\right),$$

qui converge uniformément vers

$$\widetilde{\Phi}_0^d(z) := z^{-N}\widetilde{\mathcal{S}}^d\left(z^N \widetilde{H}(z)\right)e^{\widetilde{L}\log(z)}\mathrm{Diag}\left(e^{\widetilde{\lambda}_i(z)}\right),$$

sur les compacts de $\overline{S}\left(d-\frac{\pi}{2k}, d+\frac{\pi}{2k}\right)\setminus\{\mathbb{R}_{\geq 1}x_1,\dots,\mathbb{R}_{\geq 1}x_r,\mathbb{R}_{<0}\}$, où k appartient à $\mathbb{Q}_{>0}$, et $x_1,\dots,x_r$ sont les pôles de $B(z)$ appartenant à $\overline{S}\left(d-\frac{\pi}{2k}, d+\frac{\pi}{2k}\right)$.

Nous nous intéressons désormais à la convergence des matrices de q-Stokes vers les matrices de Stokes. Soient $d^- < d^+$ tels que nous pouvons définir $\Phi_0^{[d^\pm]}(z,q)$. La q-matrice de Stokes $ST^{[d^-],[d^+]}(z,q)$ est la matrice inversible invariante sous l'action de σ_q définie par

$$\Phi_0^{[d^+]}(z,q) = \Phi_0^{[d^-]}(z,q)ST^{[d^-],[d^+]}(z,q).$$

Théorème 1.3.10 (Théorème 4.8.10)**.** *Soient $d-\frac{\pi}{2k} < d^- < d < d^+ < d+\frac{\pi}{2k}$ tels que*

$$\left([d^-,d[\bigcup]d,d^+]\right)\bigcap\Sigma = \varnothing.$$

Pour q proche de 1, nous pouvons définir $ST^{[d^-],[d^+]}(z,q)$ et nous avons

$$\lim_{q\to 1} ST^{[d^-],[d^+]}(z,q) = St_d,$$

uniformément sur les compacts de $\overline{S}\left(d-\frac{\pi}{2k}, d+\frac{\pi}{2k}\right)\setminus\{\mathbb{R}_{\geq 1}x_1,\dots,\mathbb{R}_{\geq 1}x_r,\mathbb{R}_{<0}\}$, où St_d désigne la matrice de Stokes définie en §1.1.3.

Nous faisons désormais les mêmes hypothèses à l'infini. De plus, nous supposerons que les arguments des pôles de $B(z)$ sont tous différents et que aucun n'est congru à π modulo 2π. Nous ordonnons les pôles x_i dans $\overline{S}\left(d-\frac{\pi}{2k}, d+\frac{\pi}{2k}\right)$ tels que l'on ait $d-\frac{\pi}{2k} < \arg(x_1) < \dots < \arg(x_r) < d+\frac{\pi}{2k}$. Quitte à prendre un plus gros ensemble Σ et à réduire k, nous pouvons supposer que si $d\notin\Sigma$, alors nous pouvons calculer les solutions à l'infini $\Phi_\infty^{[d]}(z,q), \widetilde{\Phi}_\infty^d(z)$ de la même manière que nous avons calculé $\Phi_0^{[d]}(z,q), \widetilde{\Phi}_0^d(z)$, et que nous avons la convergence uniforme :

$$\lim_{q\to 1}\Phi_\infty^{[d]}(z,q) = \widetilde{\Phi}_\infty^d(z),$$

sur les compacts de $\overline{S}\left(d-\frac{\pi}{2k}, d+\frac{\pi}{2k}\right)\setminus\{tx_1,\dots,tx_r,\mathbb{R}_{<0}, t\in]0,1[\}$.

Fixons $d\notin\Sigma$. La matrice de Birkhoff $\Phi_0^{[d]}(z,q)\left(\Phi_\infty^{[d]}(z,q)\right)^{-1}$ est invariante sous l'action de σ_q et nous avons la convergence uniforme vers la matrice localement constante

$$\lim_{q\to 1}\Phi_0^{[d]}(z,q)\left(\Phi_\infty^{[d]}(z,q)\right)^{-1} = \widetilde{\Phi}_0^d(z)\left(\widetilde{\Phi}_\infty^d(z)\right)^{-1},$$

sur les compacts de

$$\widetilde{\Omega} := \overline{S}\left(d-\frac{\pi}{2k}, d+\frac{\pi}{2k}\right)\setminus\left\{\mathbb{R}_{<0},\mathbb{R}_{>0}x_1,\dots,\mathbb{R}_{>0}x_r\right\},$$

où $x_1,\dots,x_r$ désignent toujours les pôles de $B(z)$ appartenant à $\overline{S}\left(d-\frac{\pi}{2k}, d+\frac{\pi}{2k}\right)$. Soient $a_0 := d-\frac{\pi}{2k}$, $a_{r+1} := d+\frac{\pi}{2k}$ et pour $i\in\{1,\dots,r\}$, posons $a_i := \arg(x_i)$. Les composantes connexes de $\widetilde{\Omega}$ sont les $\widetilde{U}_i := \left\{\overline{S}\left(d-\frac{\pi}{2k}, d+\frac{\pi}{2k}\right)\Big|\arg(z)\in]a_i,a_{i+1}[\right\}$ avec $i\in\{0,\dots,r\}$. Notons $\widetilde{P}_i$ la valeur de $\widetilde{\Phi}_0^d(z)\left(\widetilde{\Phi}_\infty^d(z)\right)^{-1}$ sur $\widetilde{U}_i$.

Fixons $i\in\{1,\dots,r\}$ et $a\in\widetilde{U}_{i-1}$. Définissons le lacet parcourant le cercle γ orienté positivement autour de la singularité x_i qui passe par a. Nous pouvons choisir a de telle sorte que ce cercle soit inclus dans $\widetilde{U}_{i-1}\bigcup\mathbb{R}_{>0}x_i\bigcup\widetilde{U}_i$.

Théorème 1.3.11 (Théorème 4.8.11)**.** *La matrice de monodromie de* $\delta\widetilde{Y}(z) = B(z)\widetilde{Y}(z)$ *autour de la singularité* x_i *dans la base* $\widetilde{\Phi}_0^d(z)$ *est* $\left(\widetilde{P}_i\right)^{-1}\widetilde{P}_{i-1}$.

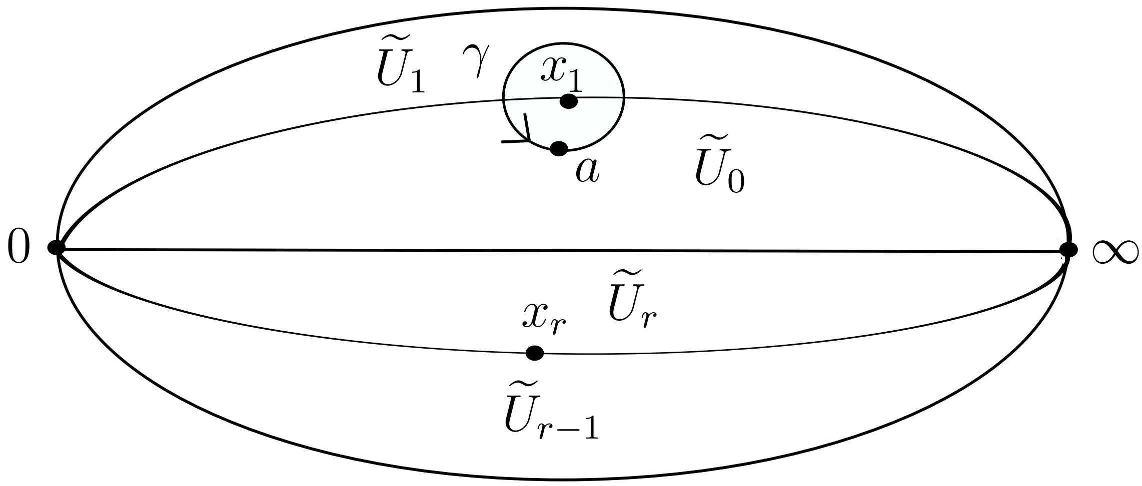


Figure 1.2 – Intersection de $\overline{S}\left(d - \frac{\pi}{2k}, d + \frac{\pi}{2k}\right)$ et du domaine de définition de $\widetilde{\Phi}_0(z)\left(\widetilde{\Phi}_\infty(z)\right)^{-1}$.

Première partie

Théorie de Galois différentielle paramétrée

Chapitre 2

Computing the Galois group of some parameterized linear differential equation of order two.

Sommaire

Résumé: We extend Kovacic's algorithm to compute the differential Galois group of some second order parameterized linear differential equation. In the case where no Liouvillian solutions could be found, we give a necessary and sufficient condition for the integrability of the system. We give various examples of computation.

Introduction

Let us consider the linear differential equation

$$\partial_z \begin{pmatrix} Y(z) \\ \partial_z Y(z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r(z) & 0 \end{pmatrix} \begin{pmatrix} Y(z) \\ \partial_z Y(z) \end{pmatrix},$$

where $r(z)$ is a rational function with coefficients in $\mathbb{C}$. We have a Galois theory for this type of equation; see [vdPS03]. In particular, we can associate to this equation a group H, which we call the differential Galois group, that measures the algebraic relations of the solutions. In this case, this group can be viewed as a linear algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})$. Kovacic in [Kov86] (see also [vdP99]) uses the classification of the linear algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})$ to obtain an algorithm that determines the Liouvillian solutions, which are the solutions that involve exponentials, indefinite integrals and solutions of polynomial equations. In particular, four cases happen:

1. H is conjugated to a subgroup of $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \text{ where } a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$ and there exists a Liouvillian solution of the form $e^{\int_0^z f(u)du}$, with $f(z) \in \mathbb{C}(z)$.

2. H is conjugated to a subgroup of $D_\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \bigcup \begin{pmatrix} 0 & b^{-1} \\ -b & 0 \end{pmatrix}, \text{ where } a, b \in \mathbb{C}^* \right\}$ and there exists a Liouvillian solution of the form $e^{\int_0^z f(u)du}$, where $f(z)$ is algebraic over $\mathbb{C}(z)$ of degree two and $f(z) \notin \mathbb{C}(z)$.

3. H is finite and all the solutions are algebraic over $\mathbb{C}(z)$.

4. $H = \mathrm{SL}_2(\mathbb{C})$ and there are no Liouvillian solutions.

Various improvements of this important algorithm have been made. See for example [DLR92, HvdP95, UW96, Zha95]. The case where H is finite has been totally solved in [SU93a, SU93b]; see also [vHW05].

Let $\{\partial_0, \partial_1, \dots, \partial_n\}$ be a set of $n+1$ commuting derivations. In this chapter, we are interested in the parameterized linear differential equation of the form

$$\partial_0 \begin{pmatrix} Y \\ \partial_0 Y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix} \begin{pmatrix} Y \\ \partial_0 Y \end{pmatrix},$$

where r belongs in a suitable $(\partial_0, \partial_1, \dots, \partial_n)$-differential field. The derivations $\partial_1, \dots, \partial_n$ should be thought of as derivations with respect to the parameters. We will denote by C its field of the ∂_0-constants. In [Lan08] and [CS07, HS08], the authors develop a Galois theory for the parameterized linear differential equations. They define a parameterized differential Galois group that measures the $(\partial_1, \dots, \partial_n)$-differential and algebraic relations between the solutions; see §2.1. This group can be seen as a differential group in the sense of Kolchin: this is a group of matrices whose entries lie in the differential field C and satisfy a set of polynomial differential equations with coefficients in C. In the case of the equation $\partial_0^2 Y = rY$, the Galois group will be a linear differential algebraic subgroup of $\mathrm{SL}_2(\mathrm{C})$. The goal of this chapter is to extend the algorithm from Kovacic and compute the parameterized differential Galois group of the equation $\partial_0^2 Y = rY$.

* * *

The chapter is presented as follows. In the first section, we recall some basic facts about parameterized differential Galois theory. This theory needs to use a field of the ∂_0-constants which is $(\partial_1, \dots, \partial_n)$-differentially closed (see [CS07], Definition 3.2). We will make a stronger assumption on the field of the ∂_0-constants C: we will assume that C is a universal $(\partial_1, \dots, \partial_n)$-field (see §2.1). We use this assumption on C because a field $(\partial_1, \dots, \partial_n)$-differentially closed is an abstract field which has no interpretation as a field of functions. We will see in §2 that a result of Seidenberg will allow us to identify the elements of the universal $(\partial_1, \dots, \partial_n)$-field C which we will consider as meromorphic functions on a polydisc D of $\mathbb{C}^n$.

In the second section, we recall the result of Seidenberg which implies that the parameterized differential Galois group can be seen as a linear differential algebraic subgroup defined over a field of meromorphic functions on a polydisc D of $\mathbb{C}^n$. Since the original algorithm from [Kov86] can be applied if we consider rational functions having coefficients in an algebraically closed field, we apply Kovacic's algorithm for the field of rational functions having coefficients in C. We obtain Liouvillian solutions that can be interpreted as meromorphic functions. Then we explain how to compute the Galois group in the four cases of Kovacic's algorithm. In the case number 4, the Galois group is Zariski dense in SL_2. We recall the definition of integrable systems and the link with integrable systems and equations with a Galois group that is Zariski dense in SL_2. We decrease the number of integrability conditions by showing that this is enough to check the integrability condition for the pairs of derivations (∂_z, ∂), where ∂ belongs in the vectorial space spanned by the derivations with respect to the parameters. Then, we obtain an effective way to compute the Galois group in the case number 4, see Proposition 2.2.8. We summarize the results of the section in Theorem 2.2.10.

In the last section we give various examples of computation.

* * *

After this chapter was written, the authors in [GO12] has generalized Proposition 2.2.8 for equations with order more than two. Moreover, Carlos E Arreche has proved some other results in touch with parameterized Kovacic's algorithm. See [Arr12].

2.1 Parameterized differential Galois theory

Let K be a differential field equipped with $n+1$ commuting derivations $\partial_0, \dots, \partial_n$ and let $\Delta = \{\partial_1, \dots, \partial_n\}$. We will assume that its field of the ∂_0-constants C is a universal (Δ)-field with characteristic 0; that is, a (Δ)-field such that for any (Δ)-field $C_0 \subset C$, (Δ)-finitely generated over $\mathbb{Q}$, and any (Δ)-finitely generated extension C_1 of C_0, there is a (Δ)-differential C_0-isomorphism of C_1 into C. See [Kol73], Chapter 3, Section 7, for more details. In particular, C is (Δ)-differentially closed. In this section, we will recall the result from [CS07] of Galois theory for the parameterized linear differential equation of the form

$$\partial_0 \begin{pmatrix} Y \\ \partial_0 Y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix} \begin{pmatrix} Y \\ \partial_0 Y \end{pmatrix}, \tag{2.1.1}$$

with $r \in K$. A parameterized Picard-Vessiot extension of the equation (2.1.1) on K is a $(\partial_0, \dots, \partial_n)$-differential field extension $\mathcal{K}|K$ generated over K by the entries of an invertible solution matrix (we will call it a fundamental solution) and such that the field of the ∂_0-constants of $\mathcal{K}$ is equal to C. We can apply [CS07], Theorem 9.5, for the equation (2.1.1), and deduce the existence and the uniqueness up to $(\partial_0, \dots, \partial_n)$-differential isomorphism of the parameterized Picard-Vessiot extension $\mathcal{K}|K$. If $\Delta = \varnothing$, we recover the usual unparameterized Picard-Vessiot extension.

The parameterized (resp. unparameterized) differential Galois group G (resp. H) is the group of field automorphisms of the parameterized Picard-Vessiot extension (resp. the unparameterized Picard-Vessiot extension) of the equation (2.1.1), which induces the identity on K and commutes with all the derivations (resp. with the derivation ∂_0). Let U be a fundamental solution. In the unparameterized case,

$$\{U^{-1}\varphi(U), \varphi \in H\}$$

is a linear algebraic subgroup of $\mathrm{GL}_2(\mathrm{C})$. In the parameterized case we find that

$$\{U^{-1}\varphi(U), \varphi \in G\}$$

is a linear differential algebraic subgroup, that is, a subgroup of $\mathrm{GL}_2(\mathrm{C})$ which is the zero of a set of (Δ)-differential polynomials in 4 variables. See [CS07], Theorem 9.10. Any other fundamental solution yields another differential algebraic subgroup of $\mathrm{GL}_2(\mathrm{C})$ which are all conjugated over $\mathrm{GL}_2(\mathrm{C})$. We will identify G (resp. H) with a linear differential algebraic subgroup of $\mathrm{GL}_2(\mathrm{C})$ (resp. with a linear algebraic subgroup of $\mathrm{GL}_2(\mathrm{C})$) for a chosen fundamental solution. The next lemma is a classical result.

Lemma 2.1.1 ([Kov86], Section 1.3)**.** $G \subset \mathrm{SL}_2(\mathrm{C})$.

Proof. Let b and c be two independents solutions. A simple computation shows that the Wronskian: $W = b\partial_0 c - (\partial_0 b)c$ satisfies $\partial_0 W = 0$, and hence $W \in C$. Therefore it is fixed by the elements of the parameterized differential Galois group G. Let us consider $\begin{pmatrix} b & c \\ \partial_0 b & \partial_0 c \end{pmatrix}$ a fundamental solution. Let $\sigma \in G$. We obtain the existence of $\begin{pmatrix} \alpha & \beta \\ \delta & \varepsilon \end{pmatrix} \in \mathrm{GL}_2(\mathrm{C})$, such that: $\sigma \begin{pmatrix} b & c \\ \partial_0 b & \partial_0 c \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \delta & \varepsilon \end{pmatrix} \begin{pmatrix} b & c \\ \partial_0 b & \partial_0 c \end{pmatrix}$.

A direct computation shows that: $\sigma W = W = (\alpha\varepsilon - \beta\delta)W$ and therefore:

$$G \subset \mathrm{SL}_2(\mathrm{C}).$$

□

2.2 Computation of the parameterized differential Galois group

Until the end of the chapter, C denotes a universal (Δ)-field equipped with n commuting derivations. Let $C(z)$ be the (∂_z, Δ)-differential field of rational functions in the indeterminate z, with coefficients in C, where z is a (Δ)-constant with $\partial_z z = 1$, C is the field of the ∂_z-constants and such that ∂_z commutes with all the derivations. Let us consider the parameterized linear differential equation

$$\partial_z \begin{pmatrix} Y(z) \\ \partial_z Y(z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r(z) & 0 \end{pmatrix} \begin{pmatrix} Y(z) \\ \partial_z Y(z) \end{pmatrix}, \tag{2.2.1}$$

with $r(z) \in C(z)$. We want to apply Kovacic's algorithm for the parameterized linear differential equation (2.2.1). Let $G \subset \mathrm{SL}_2(\mathrm{C})$ be the parameterized differential Galois group. The algorithm from [Kov86] can be applied if the field of the ∂_z-constants is algebraically closed, which is the case here. The problem is that C is an abstract field which is not very convenient for the computations. In fact we have an interpretation of the elements of C as meromorphic functions. Let C_1 be the (Δ)-differential field generated over $\mathbb{Q}$ by the z-coefficients of $r(z)$. Using the following result of Seidenberg (see [Sei58, Sei69]) with $K_0 = \mathbb{Q}$ and $K_1 = C_1$, we find the existence of a polydisc D of $\mathbb{C}^n$ such that the z-coefficients of $r(z)$ can be considered as meromorphic functions on D.

Theorem 2.2.1 (Seidenberg)**.** *Let $\mathbb{Q} \subset K_0 \subset K_1$ be finitely generated (Δ)-differential extensions of $\mathbb{Q}$ and assume that K_0 consists of meromorphic functions on some domain Ω of $\mathbb{C}^n$. Then K_1 is isomorphic to the field K_1^* of meromorphic functions on $\Omega_1 \subset \Omega$ such that $K_0|_{\Omega_1} \subset K_1^*$, and the derivations in Δ can be identified with the derivations with respect to the coordinates on Ω_1.*

Let $(\mathcal{M}_D, \partial_{t_1}, \dots, \partial_{t_n})$ denotes the $\Delta_t = \{\partial_{t_1}, \dots, \partial_{t_n}\}$-differential field of meromorphic functions on D, a polydisc of $\mathbb{C}^n$. Let $t = (t_1, \dots, t_n)$. The discussion above tell us that the $r(z)$ of the equation (2.2.1) can be identified with $r(z,t)$, an element of $\mathcal{M}_D(z)$ *, where D is a polydisc of $\mathbb{C}^n$. We will consider the parameterized linear differential equation

$$\partial_z \begin{pmatrix} Y(z,t) \\ \partial_z Y(z,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r(z,t) & 0 \end{pmatrix} \begin{pmatrix} Y(z,t) \\ \partial_z Y(z,t) \end{pmatrix},$$

with $r(z,t) \in \mathcal{M}_D(z)$. The group G is defined by a finite number of (Δ)-differential polynomials. Again, using the result of Seidenberg with the (Δ)-differential field generated over $\mathbb{Q}$ by the coefficients of the (Δ)-differential polynomials that define G and the z-coefficients of $r(z)$, we deduce that G can be seen as a linear differential algebraic subgroup of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$. Again using the result of Seidenberg, we remark that after shrinking D, we can assume that if G is conjugated over $\mathrm{SL}_2(\mathrm{C})$ to Q, we can identify Q and G as linear differential algebraic subgroups of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$, and they are conjugated over $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$. Furthermore, we obtain that the Liouvillian solutions found are defined over the algebraic closure of $\mathcal{M}_D(z)$. We will compute G as a linear differential algebraic subgroup of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$. We recall that we have four cases to consider:

1. There exists a Liouvillian solution that is of the following form $g(z,t) = e^{\int_0^z f(u,t)du}$, with $f(z,t) \in \mathcal{M}_D(z)$.

*. $\mathcal{M}_D(z)$ denotes the (∂_z, Δ_t)-differential field of rational functions with indeterminate z and with coefficients in $\mathcal{M}_D$ such that $\partial_z z = 1$, z is a (Δ_t)-constant and the field $\mathcal{M}_D$ is the field of the ∂_z-constants.

2. There exists a Liouvillian solution of the form $g(z,t) = e^{\int_0^z f(u,t)du}$, where $f(z,t)$ is algebraic over $\mathcal{M}_D(z)$ of degree two and $f(z,t) \notin \mathcal{M}_D(z)$.
3. All the solutions are algebraic over $\mathcal{M}_D(z)$.
4. There are no Liouvillian solutions.

These correspond to the four cases recalled in the introduction. Proposition 6.26 of [HS08] says that if we take the same fundamental solution, the Zariski closure of G is the unparameterized differential Galois group. This means that in each case we are looking at the Zariski dense subgroups of the group given by usual Kovacic's algorithm.

2.2.1 Case number 1

We start with the case number 1. There exists a Liouvillian solution of the form:

$$g(z,t) = e^{\int_0^z f(u,t)du},$$

with $f(z,t) \in \mathcal{M}_D(z)$. The action of G on the solution $g(z,t)$ can be computed with the following lemma:

Lemma 2.2.2. *Let $\sigma \in G$.*

(1) Let $\alpha(t) \in \mathcal{M}_D$ and $p,q \in \mathbb{N}$, such that $GCD(p,q) = 1$. Then there exists $k \in \mathbb{N}$ such that $\sigma\big((z-\alpha(t))^{p/q}\big) = e^{\frac{2ik\pi}{q}}(z-\alpha(t))^{p/q}$.

(2) Let $\alpha(t), \beta(t) \in \mathcal{M}_D$ and $\beta(t) \notin \mathbb{Q}$. Then there exists $a \in \mathbb{C}$ and $c \in \mathbb{C}^*$ such that $\sigma\big((z-\alpha(t))^{\beta(t)}\big) = ce^{a\beta(t)}(z-\alpha(t))^{\beta(t)}$.

(3) Let $Q(z,t) \in \mathcal{M}_D(z)$. Then there exists $a \in \mathbb{C}^*$ such that $\sigma\left(e^{Q(z,t)}\right) = ae^{Q(z,t)}$.

Proof. (1) We use the fact the elements of G are fields automorphisms that leave $\mathcal{M}_D$ invariant.

(2) A computation shows that

$$\partial_{t_i}(z-\alpha(t))^{\beta(t)} = \left[\log(z-\alpha(t))\partial_{t_i}\beta(t) - \frac{\partial_{t_i}\alpha(t)\beta(t)}{z-\alpha(t)}\right](z-\alpha(t))^{\beta(t)}.$$

The fact that σ commutes with all the derivations implies the existence of $a \in \mathbb{C}$ and $f(t) \in \mathcal{M}_D$ such that

$$\sigma\big(\log(z-\alpha(t))\big) = \log(z-\alpha(t)) + a$$

and

$$\sigma\big((z-\alpha(t))^{\beta(t)}\big) = f(t)(z-\alpha(t))^{\beta(t)}.$$

Since $\partial_{t_i}\sigma = \sigma\partial_{t_i}$, we obtain that

$$\begin{aligned}\left[\log(z-\alpha(t))\partial_{t_i}\beta(t) + a\partial_{t_i}\beta(t) - \tfrac{\partial_{t_i}\alpha(t)\beta(t)}{z-\alpha(t)}\right]f(t) &= \\ \partial_{t_i}f(t) + f(t)\left[\log(z-\alpha(t))\partial_{t_i}\beta(t) - \tfrac{\partial_{t_i}\alpha(t)\beta(t)}{z-\alpha(t)}\right].\end{aligned}$$

Finally, $f(t)$ satisfies the parameterized linear differential equation

$$\partial_{t_i}\left(\frac{\partial_{t_i}f(t)}{f(t)a\partial_{t_i}\beta(t)}\right) = 0.$$

This means that $\frac{\partial_{t_i}f(t)}{f(t)a\partial_{t_i}\beta(t)} = c \in \mathbb{C}^*$ and $\log f(t) = a\beta(t) + \log(c)$. Then we deduce that $f(t) = ce^{a\beta(t)}$.

(3) We use the fact that

$$\partial_{t_i}\sigma\left(e^{Q(z,t)}\right)=\sigma\left(\partial_{t_i}\left(e^{Q(z,t)}\right)\right)=\sigma\left(\partial_{t_i}\left(Q(z,t)\right)e^{Q(z,t)}\right)=\partial_{t_i}Q(z,t)\sigma\left(e^{Q(z,t)}\right).$$

The equation $\partial_{t_i}\sigma\left(e^{Q(z,t)}\right)=\partial_{t_i}Q(z,t)\sigma\left(e^{Q(z,t)}\right)$ admits $\sigma(e^{Q(z,t)})=ae^{Q(z,t)}$ with $a\in\mathbb{C}^*$ as a solution.

□

We deduce that the matrices of G are upper triangular. We will denote by $G_m \backsimeq \mathrm{GL}_1(\mathcal{M}_{\mathrm{D}})$ the multiplicative group. The proof of the following proposition is inspired by the proof of [Sit75], Theorem 1.4. Let $p: G\to G_m$ that sends $\begin{pmatrix} m(t) & a(t)\\ 0 & m^{-1}(t)\end{pmatrix}$ on $m(t)$. Let M be the image of p and $A\subset\mathcal{M}_D$ such that

$$\left\{\begin{pmatrix}1 & a(t)\\ 0 & 1\end{pmatrix}, \text{ where } a(t)\in A\right\}$$

is the kernel of p. We have already computed M with Lemma 2.2.2. For $m(t)\in M$, let $\Gamma_{m(t)}$ be the set of $\gamma_{m(t)}\in\mathcal{M}_D$ such that $\begin{pmatrix} m(t) & \gamma_{m(t)}\\ 0 & m(t)^{-1}\end{pmatrix}\in G$. We will identify $\mathbb{C}^*$ with the field of the constant elements of $\mathcal{M}_D$. If $\mathbb{C}^*\not\subset M$, because of Lemma 2.2.2, $g(z,t)\in\mathcal{M}_D(z)$, and we can compute explicitly $g(z,t)\int_{u=0}^{z} g(u,t)^{-2}du$, which is another solution. We obtain explicitly a fundamental solution and we can compute G. The next proposition explains how to compute G when $\mathbb{C}^*\subset M$.

Proposition 2.2.3. *Let us keep the same notations. Assume that $\mathbb{C}^*\subset M$. Then G is conjugated to*

$$\left\{\begin{pmatrix} m(t) & a(t)\\ 0 & m(t)^{-1}\end{pmatrix}, \text{ where } m(t)\in M, a(t)\in A\right\}.$$

For the proof of the proposition, we will need the following lemmas.

Lemma 2.2.4. *Assume that $\mathbb{C}^*\subset M$. Let $m(t)\in M$ and $a(t)\in A$. Then $m(t)a(t)\in A$.*

Proof. With Lemma 2.2.2, we obtain that for all $m(t)\in M$, there exists $b(t)\in M$ such that $b(t)^2=m(t)$. Let $m(t)\in M$, $b(t)^2=m(t)$, $\gamma_{b(t)}\in\Gamma_{b(t)}$, and $a(t)\in A$. The computation

$$\begin{pmatrix} b(t) & \gamma_{b(t)}\\ 0 & b(t)^{-1}\end{pmatrix}\begin{pmatrix}1 & a(t)\\ 0 & 1\end{pmatrix}\begin{pmatrix} b(t) & \gamma_{b(t)}\\ 0 & b(t)^{-1}\end{pmatrix}^{-1}=\begin{pmatrix}1 & m(t)a(t)\\ 0 & 1\end{pmatrix}$$

shows that if $m(t)\in M$ and $a(t)\in A$, then $m(t)a(t)\in A$. □

Lemma 2.2.5. *Assume that $\mathbb{C}^*\subset M$. Let $m(t)\in M$. Then $\gamma_{m(t)},\gamma'_{m(t)}\in\Gamma_{m(t)}$ if and only if $\left(\gamma_{m(t)}-\gamma'_{m(t)}\right)\in A$.*

Proof. Let $\gamma_{m(t)},\gamma'_{m(t)}\in\Gamma_{m(t)}$. The computation

$$\begin{pmatrix} m(t) & \gamma_{m(t)}\\ 0 & m(t)^{-1}\end{pmatrix}\begin{pmatrix} m(t) & \gamma'_{m(t)}\\ 0 & m(t)^{-1}\end{pmatrix}^{-1}=\begin{pmatrix}1 & m(t)(\gamma_{m(t)}-\gamma'_{m(t)})\\ 0 & 1\end{pmatrix}$$

shows that $m(t)(\gamma_{m(t)}-\gamma'_{m(t)})\in A$, and then $(\gamma_{m(t)}-\gamma'_{m(t)})\in A$, because of Lemma 2.2.4. Conversely, if $(\gamma_{m(t)}-\gamma'_{m(t)})\in A$ and $\gamma_{m(t)}\in\Gamma_{m(t)}$, then $m(t)(\gamma_{m(t)}-\gamma'_{m(t)})\in A$, because of Lemma 2.2.4. The same computation shows that $\gamma'_{m(t)}\in\Gamma_{m(t)}$. □

Lemma 2.2.6. *Assume that $\mathbb{C}^* \subset M$. Let $b \in \mathbb{C}^* \setminus \{\pm 1\}$ and $\gamma_b \in \Gamma_b$. Let*

$$\beta(t) = b(b^2-1)^{-1}\gamma_b.$$

Then, $\beta(t)(m(t) - m(t)^{-1}) \in \Gamma_{m(t)}$, for all $m(t) \in M$.

Proof. Let $m(t) \in M$ and $\gamma_{m(t)} \in \Gamma_{m(t)}$. The computation

$$\begin{pmatrix} b & \gamma_b \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} m(t) & \gamma_{m(t)} \\ 0 & m(t)^{-1} \end{pmatrix} \begin{pmatrix} b & \gamma_b \\ 0 & b^{-1} \end{pmatrix}^{-1} \begin{pmatrix} m(t) & \gamma_{m(t)} \\ 0 & m(t)^{-1} \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 1 & (1-m(t)^2)b\gamma_b - (1-b^2)m(t)\gamma_{m(t)} \\ 0 & 1 \end{pmatrix}$$

implies that $(1-m(t)^2)b\gamma_b - (1-b^2)m(t)\gamma_{m(t)} \in A$. Since $(1-b^2)m(t) \in M$, Lemma 2.2.4 implies that

$$(1-b^2)^{-1}m(t)^{-1}(1-m(t)^2)b\gamma_b - \gamma_{m(t)} = \beta(t)(m(t)-m(t)^{-1}) - \gamma_{m(t)} \in A.$$

Therefore $\beta(t)(m(t)-m(t)^{-1}) \in \Gamma_{m(t)}$, because of Lemma 2.2.5. □

Proof of Proposition 2.2.3. With Lemmas 2.2.5 and 2.2.6, we find that

$$G \simeq \left\{ \begin{pmatrix} m(t) & \beta(t)(m(t)-m(t)^{-1}) + a(t) \\ 0 & m(t)^{-1} \end{pmatrix}, \text{ where } m(t) \in M, a(t) \in A \right\}.$$

If we change the fundamental solution, i.e, if we conjugate G over $\mathrm{GL}_2(\mathcal{M}_\mathrm{D})$, we can simplify the expression of G. After conjugation by the element $P = \begin{pmatrix} 1 & \beta(t) \\ 0 & 1 \end{pmatrix}$, we obtain that

$$PGP^{-1} \simeq \left\{ \begin{pmatrix} m(t) & a(t) \\ 0 & m(t)^{-1} \end{pmatrix}, \text{ where } m(t) \in M, a(t) \in A \right\}.$$

□

We now want to compute G when $\mathbb{C}^* \subset M$. The computation of M has already been done in Lemma 2.2.2. We are now interested in the computation of A, which is a linear differential algebraic subgroup of $(\mathcal{M}_D, +)$. Cassidy classifies the linear differential algebraic subgroups of the additive group in [Cas72], Lemma 11. We define $\mathcal{M}_D[y_1 \dots, y_\nu]_{\Delta_t}$, as the ring of linear homogeneous differential polynomials. There exists $P_1, \dots P_m \in \mathcal{M}_D[y]_{\Delta_t}$ such that

$$A = \{a(t) \in \mathcal{M}_D | P_1(a(t)) = \dots = P_m(a(t)) = 0\}.$$

We recall that $g(z,t)\int_{u=0}^{z} g(u,t)^{-2}du$ is another solution. We can choose $\beta(t) \in \mathcal{M}_D$ such that in the basis formed by the solutions $g(z,t)$ and $g(z,t)\int_{u=0}^{z} g(u,t)^{-2}du + \beta(t)g(z,t)$, G is equal to $\left\{ \begin{pmatrix} m(t) & a(t) \\ 0 & m(t)^{-1} \end{pmatrix}, \text{ where } m(t) \in M, a(t) \in A \right\}$. Let $G^g \subset G$ be the subfield of elements that fix $g(z,t)$ and let $\sigma \in G^g$. Let $a(t) \in A$ be such that

$$\sigma\left(g(z,t)\int_{u=0}^{z} g(u,t)^{-2}du + \beta(t)g(z,t)\right) = \left(g(z,t)\int_{u=0}^{z} g(u,t)^{-2}du + \beta(t)g(z,t)\right) + a(t)g(z,t).$$

Since

$$\sigma\left(g(z,t)\int_{u=0}^{z} g(u,t)^{-2}du + \beta(t)g(z,t)\right) = g(z,t)\left(\sigma\left(\int_{u=0}^{z} g(u,t)^{-2}du\right) + \beta(t)\right),$$

we deduce that

$$\sigma\left(\int_{u=0}^{z} g(u,t)^{-2}du\right) - \int_{u=0}^{z} g(u,t)^{-2}du = a(t) \in A.$$

Therefore, the differential polynomials P_i satisfy $\forall\sigma \in G^g$:

$$\begin{aligned}\sigma\Big(P_i\left(\textstyle\int_{u=0}^{z} g(u,t)^{-2}du\right)\Big) &= P_i\Big(\sigma\left(\textstyle\int_{u=0}^{z} g(u,t)^{-2}du\right)\Big)\\ &= P_i\Big(\textstyle\int_{u=0}^{z} g(u,t)^{-2}du + a(t)\Big)\\ &= P_i\Big(\textstyle\int_{u=0}^{z} g(u,t)^{-2}du\Big).\end{aligned}$$

Since $P_i\Big(\int_{u=0}^{z} g(u,t)^{-2}du\Big)$ is fixed by the elements of G^g, we deduce by the Galois correspondence in the parameterized differential Galois theory (see [CS07], Theorem 9.5) that

$$P_i(a(t)) = 0 \Longleftrightarrow P_i\left(\int_{u=0}^{z} g(u,t)^{-2}du\right) \in \mathcal{M}_D(z)\langle g(z,t)\rangle_{\partial_z,\Delta_t},$$

where $\mathcal{M}_D(z)\langle g(z,t)\rangle_{\partial_z,\Delta_t}$ denotes the $(\partial_z,\Delta_t)-$differential field generated by $\mathcal{M}_D(z)$ and $g(z,t)$.

2.2.2 Case number 2

Let us consider the case number 2. There exists a Liouvillian solution of the form $e^{\int_0^z f(u,t)du}$, such that $f(z,t)$ satisfies $f(z,t)^2 + a(z,t)f(z,t) + b(z,t) = 0$, where $a(z,t), b(z,t) \in \mathcal{M}_D(z)$. There exists $\varepsilon \in \{\pm 1\}$ such that

$$f(z,t) = \frac{-a(z,t) + \varepsilon\sqrt{a(z,t)^2 - 4b(z,t)}}{2}.$$

By computing the action of G on $e^{\int_0^z \frac{-a(u,t)+\varepsilon\sqrt{a(u,t)^2-4b(u,t)}}{2}du}$, we find that $e^{\int_0^z \frac{-a(u,t)-\varepsilon\sqrt{a(u,t)^2-4b(u,t)}}{2}du}$ is another Liouvillian solution which is linearly independent of the first one. By computing the action of G on the second Liouvillian solution we find the existence of M, a linear differential algebraic subgroup of the multiplicative group G_m such that, in the basis formed by the two Liouvillian solutions

$$G \simeq \left\{\begin{pmatrix} a(t) & 0\\ 0 & a^{-1}(t)\end{pmatrix} \bigcup \begin{pmatrix} 0 & b^{-1}(t)\\ -b(t) & 0\end{pmatrix}, \text{ where } a(t), b(t) \in M\right\}.$$

We are now interested in the computation of M. A direct computation shows that if there exists $\sigma \in G$ such that $\sigma\left(e^{\int_0^z f(u,t)du}\right) = \alpha(t)e^{\int_0^z f(u,t)du}$, then for all $i \leq n$, $\alpha(t)$ satisfies the parameterized differential equation

$$\partial_{t_i}\alpha(t) + \alpha(t)\left(\partial_{t_i}\int_0^z f(u,t)du\right) = \alpha(t)\sigma\left(\partial_{t_i}\int_0^z f(u,t)du\right).$$

Let $\widetilde{\partial_{t_i}}\alpha(t) = \frac{\partial_{t_i}\alpha(t)}{\alpha(t)}$ be the logarithm derivation. In [Cas72], Chapter 4, we see that there exist $P_1, \dots, P_k \in \mathcal{M}_D[y_1 \dots, y_n]_{\Delta_t}$ such that

$$M \simeq \left\{\alpha(t) \middle| P_1\left(\widetilde{\partial_{t_i}}\alpha(t)\right) = \dots = P_k\left(\widetilde{\partial_{t_i}}\alpha(t)\right) = 0\right\}.$$

The polynomial P_j satisfies, for all $\sigma \in G$,

$$P_j\left(\partial_{t_i}\int_0^z f(u,t)du\right) = \sigma\left(P_j\left(\partial_{t_i}\int_0^z f(u,t)du\right)\right)$$

and then

$$P_j\left(\widetilde{\partial_{t_i}}\alpha(t)\right) = 0 \Longleftrightarrow P_j\left(\partial_{t_i}\int_0^z f(u,t)du\right) \in \mathcal{M}_D(z).$$

2.2.3 Case number 3

In the third case, G is finite, because whose Zariski closure is finite. Since all finite linear differential algebraic subgroups of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ are finite linear algebraic subgroups of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$, G is equal to the unparameterized differential Galois group. This is the same problem as in the unparameterized case. See [vHW05] for the computation of G.

2.2.4 Case number 4

We now consider the case where no Liouvillian solutions are found. We have seen in the introduction that in this case, the unparameterized differential Galois group is $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$. Therefore, G is Zariski dense in $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$.

The classification of the Zariski dense subgroup of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ has been made in [Cas72], Proposition 42. Let $\mathbf{D}$ be the $\mathcal{M}_D$-vectorial space of derivations of the form

$$\left\{\sum_{i=0}^{n} a_i(t)\partial_{t_i}, \text{ where } a_i(t) \in \mathcal{M}_D\right\},$$

and $\mathbb{D}$ a vectorial subspace of $\mathbf{D}$. Let $\mathcal{M}_D^{\mathbb{D}}$ be the elements of $\mathcal{M}_D$ that are constant for the derivations in $\mathbb{D}$. Remark that if $\mathbb{D} = \{0\}$, then $\mathcal{M}_D^{\mathbb{D}} = \mathcal{M}_D$. The linear differential algebraic subgroups of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ that are Zariski dense in $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ are conjugated over $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ to the groups of the form $\mathrm{SL}_2\left(\mathcal{M}_\mathrm{D}^{\mathbb{D}}\right)$, with $\mathbb{D}$ a vectorial subspace of $\mathbf{D}$.

Let $\mathbb{D} \subset \mathbf{D}$ be such that the group G is conjugated over $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ to $\mathrm{SL}_2\left(\mathcal{M}_\mathrm{D}^{\mathbb{D}}\right)$. We want to compute explicitly $\mathbb{D}$. This leads us to the notion of integrable systems. Let $A_0(z,t), \dots, A_k(z,t)$, $m \times m$ matrices with entries in $\mathcal{M}_D(z)$ and $\partial'_{t_1}, \dots, \partial'_{t_k} \in \mathbf{D}$. The following system

$$[S] : \begin{cases} \partial_z Y(z,t) &= A_0(z,t)Y(z,t) \\ \partial'_{t_1} Y(z,t) &= A_1(z,t)Y(z,t) \\ &\vdots \\ \partial'_{t_k} Y(z,t) &= A_k(z,t)Y(z,t) \end{cases}$$

is integrable if and only if, for all $0 \le i,j \le k$,

$$\partial'_{t_j}A_i(z,t) - \partial'_{t_i}A_j(z,t) = A_j(z,t)A_i(z,t) - A_i(z,t)A_j(z,t),$$

where $\partial'_{t_0} = \partial_z$. We recall here [CS07], Proposition 6.3, which relates the integrable system and the parameterized differential Galois group in the case where the field of the ∂_z-constants is differentially closed.

Proposition 2.2.7. *Let $\{\partial'_{t_1}, \ldots, \partial'_{t_k}\}$ be a commuting basis of $\mathbb{D}$, a vectorial subspace of $\boldsymbol{D}$. G is conjugated to $\mathrm{SL}_2\left(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}}\right)$ over $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$ if and only if there exist $A_1(z,t), \ldots, A_k(z,t)$, $m \times m$ matrices with entries in $\mathcal{M}_D(z)$ †, such that the following system is integrable:*

$$[S] : \begin{cases} \partial_z Y(z,t) &= A(z,t)Y(z,t) \\ \partial'_{t_1} Y(z,t) &= A_1(z,t)Y(z,t) \\ &\vdots \\ \partial'_{t_k} Y(z,t) &= A_k(z,t)Y(z,t). \end{cases}$$

We want to give simpler necessary and sufficient conditions for the integrability of the system in Proposition 2.2.7. First, we will write a necessary and sufficient condition for the integrability of

$$[S'] : \begin{cases} \partial_z Y(z,t) &= A(z,t)Y(z,t) \\ \\ \partial' Y(z,t) &= A'(z,t)Y(z,t), \end{cases}$$

where $A'(z,t) = \begin{pmatrix} a(z,t) & b(z,t) \\ c(z,t) & d(z,t) \end{pmatrix}$ is an $m \times m$ matrix with entries in $\mathcal{M}_D(z)$ and $\partial' \in \mathbf{D}$. The fact that $[S']$ is integrable is equivalent to the solution in $(\mathcal{M}_D(z))^4$ of the parameterized differential system:

$$\begin{cases} \partial_z a(z,t) &= c(z,t) - b(z,t)r(z,t) \\ \partial_z b(z,t) &= d(z,t) - a(z,t) \\ \partial_z c(z,t) &= (a(z,t) - d(z,t))r(z,t) + \partial' r(z,t) \\ \partial_z d(z,t) &= b(z,t)r(z,t) - c(z,t) \end{cases}$$

$$\iff \begin{cases} \partial_z a(z,t) &= -\partial_z d(z,t) \\ \partial_z^2 b(z,t) &= 2\partial_z d(z,t) \\ \partial_z c(z,t) &= -\partial_z b(z,t)r(z,t) + \partial' r(z,t) \\ \frac{\partial_z^2 b(z,t)}{2} &= b(z,t)r(z,t) - c(z,t) \end{cases}$$

$$\iff \begin{cases} \partial_z a(z,t) &= -\partial_z d(z,t) \\ \partial_z^2 b(z,t) &= 2\partial_z d(z,t) \\ \partial_z c(z,t) &= -\partial_z b(z,t)r(z,t) + \partial' r(z,t) \\ \frac{\partial_z^3 b(z,t)}{2} &= 2\partial_z b(z,t)r(z,t) + b(z,t)\partial_z r(z,t) - \partial' r(z,t). \end{cases}$$

We can easily see that the existence of $b(z,t) \in \mathcal{M}_D(z)$ as a solution of

$$\frac{\partial_z^3 b(z,t)}{2} = 2\partial_z b(z,t)r(z,t) + b(z,t)\partial_z r(z,t) - \partial' r(z,t)$$

is equivalent to the fact that the system $[S']$ is integrable. There exists an algorithm to determine if such a system has a solution (see [vdPS03], p. 100). We obtain a necessary

†. Using the result of Seidenberg, we can identify the matrices as elements of $\mathrm{GL}_2(\mathcal{M}_{\mathrm{D}}(\mathrm{z}))$ because their entries involve a finite number of elements of the fields of the ∂_z-constants.

and sufficient condition on ∂' for the integrability condition of the system $[S']$. Let $\mathbb{D}$ be the maximal vectorial subspace of $\mathbf{D}$ such that for all derivations ∂' in $\mathbb{D}$, there exists $A'(z,t)$, $m \times m$ matrix with entries in $\mathcal{M}_D(z)$ such that the following system is integrable:

$$[S'] : \begin{cases} \partial_z Y(z,t) &= A(z,t)Y(z,t) \\ \partial' Y(z,t) &= A'(z,t)Y(z,t). \end{cases}$$

We want to prove that the parameterized differential Galois group of the equation $\partial_z Y(z,t) = A(z,t)Y(z,t)$ is conjugated to $\mathrm{SL}_2\left(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}}\right)$ over $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$. Assume that this is not the case. Then by Proposition 2.2.7, there exists $\mathbb{D}_1, \mathbb{D}_2 \subsetneq \mathbb{D}$, having at least dimension 1, with $\mathbb{D}_1 \neq \mathbb{D}_2$ such that G is conjugated to $\mathrm{SL}_2\left(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}_1}\right)$ and $\mathrm{SL}_2\left(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}_2}\right)$. In this case, $\mathrm{SL}_2\left(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}_1}\right)$ is conjugated to $\mathrm{SL}_2\left(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}_2}\right)$ over $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$. The fact that $\mathbb{D}_1 = \mathbb{D}_2$ is proved in [Sit75], Theorem 1.2, Chapter 2, but we will recall the proof here. Let $\alpha \in \mathcal{M}_D^{\mathbb{D}_1}$ and consider the diagonal matrix $M = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in \mathrm{SL}_2\left(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}_1}\right)$. Since similar matrices have the same set of eigenvalues and $\mathcal{M}_D^{\mathbb{D}_2}$ is algebraically closed, we obtain that $\alpha(t) \in \mathcal{M}_D^{\mathbb{D}_2}$. Therefore $\mathcal{M}_D^{\mathbb{D}_1} \subset \mathcal{M}_D^{\mathbb{D}_2}$ and, by symmetry, $\mathcal{M}_D^{\mathbb{D}_1} = \mathcal{M}_D^{\mathbb{D}_2}$. We then deduce $\mathbb{D}_1 = \mathbb{D}_2 = \mathbb{D}$. We have proved:

Proposition 2.2.8. *We have the following equivalences:*

(1) G is conjugated to $\mathrm{SL}_2\left(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}}\right)$ over $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$.

(2) For all ∂' that belongs in a commuting basis of $\mathbb{D}$, the following parameterized differential equation has a solution in $\mathcal{M}_D(z)$:

$$\frac{\partial_z^3 b(z,t)}{2} = 2\partial_z b(z,t) r(z,t) + b(z,t)\partial_z r(z,t) - \partial' r(z,t).$$

(3) For all $\partial' \in \mathbb{D}$, the following parameterized differential equation has a solution in $\mathcal{M}_D(z)$:

$$\frac{\partial_z^3 b(z,t)}{2} = 2\partial_z b(z,t) r(z,t) + b(z,t)\partial_z r(z,t) - \partial' r(z,t).$$

Remark 2.2.9. In the case where $n = 1$, i.e, there is only one parameter, the Zariski dense subgroups of $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$ are (up to conjugation over $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$) $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$ and $\mathrm{SL}_2(\mathbb{C})$. Then we only have to check whether

$$\frac{\partial_z^3 b(z,t)}{2} = 2\partial_z b(z,t) r(z,t) + b(z,t)\partial_z r(z,t) - \partial_t r(z,t),$$

has a solution in $\mathcal{M}_D(z)$.

2.2.5 Summary

We summarize in the next theorem the results of this section.

Theorem 2.2.10. *Let us consider $\partial_z^2 Y(z,t) = r(z,t)Y(z,t)$ with $r(z,t) \in \mathcal{M}_D(z)$ and let G be the parameterized differential Galois group, seen as a linear differential algebraic subgroup of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$. There are four possibilities:*

1. *There exists a Liouvillian solution of the form $g(z,t) = e^{\int_0^z f(u,t)du}$, with $f(z,t) \in \mathcal{M}_D(z)$. There are two possibilities:*
   - *(a) If $g(z,t) \in \mathcal{M}_D$, then we can compute explicitly another solution $g(z,t)\int_{u=0}^{z} g(u,t)^{-2}du$ which is linearly independent with $g(z,t)$. In this basis of solutions we can compute explicitly G.*
   - *(b) In the other case, G is conjugated to:*
$$\left\{\begin{pmatrix} m(t) & a(t) \\ 0 & m(t)^{-1}\end{pmatrix}, \text{ where } m(t) \in M, a(t) \in A\right\},$$
     where:
$$M = \left\{g(z,t)^{-1}\sigma(g(z,t)), \sigma \in G\right\},$$
$$A = \left\{ a(t) \in \mathcal{M}_D \;\middle|\; \begin{array}{l} \forall P \in \mathcal{M}_D[y]_{\Delta_t}, \\ P\left(\int_{u=0}^{z} g(u,t)^{-2}du\right) \in \mathcal{M}_D(z)\langle g(z,t)\rangle_{\partial_z,\Delta_t} \Longleftrightarrow P(a(t)) = 0 \end{array}\right\}.$$
2. *There exists a Liouvillian solution of the form $g(z,t) = e^{\int_0^z f(u,t)du}$, where $f(z,t)$ is algebraic over $\mathcal{M}_D(z)$ of degree two and $f(z,t) \notin \mathcal{M}_D(z)$. In this case, G is conjugated to*
$$\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \bigcup \begin{pmatrix} 0 & b^{-1} \\ -b & 0\end{pmatrix}, \text{ where } a,b \in M\right\}, \text{ where}$$
$$M = \left\{ f(t) \in \mathcal{M}_D \;\middle|\; \begin{array}{l} \forall P \in \mathcal{M}_D[y_1 \dots, y_n]_{\Delta_t}, \\ P\left(\partial_{t_i}\int_{u=0}^{z} f(u,t)du\right) \in \mathcal{M}_D(z) \Leftrightarrow P\left(\widetilde{\partial}_{t_i} f(t)\right) = 0 \end{array}\right\}.$$
3. *G is finite. In this case, G is equal to the unparameterized differential Galois group.*
4. *There are no Liouvillian solutions. In this case, there exists $\mathbb{D}$, a $\mathcal{M}_D$-vectorial space of derivations spanned by Δ_t, such that G is conjugated to $\mathrm{SL}_2\left(\mathcal{M}_\mathrm{D}^{\mathbb{D}}\right)$. Moreover, $\partial_t' \in \mathbb{D}$ if and only if the following parameterized differential equation has a solution in $\mathcal{M}_D(z)$:*
$$\frac{\partial_z^3 b(z,t)}{2} = 2\partial_z b(z,t)r(z,t) + b(z,t)\partial_z r(z,t) - \partial_t' r(z).$$

Notice that the computation of the Liouvillian solutions and the unparameterized differential Galois group are already known. Our results compute the parameterized differential Galois group in the cases 1,2 and 4. The classification of the Zariski dense linear differential algebraic subgroup of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ and the link with integrable systems were already known (see [Cas72, CS07]), but we give here an effective way to compute the Galois group in the case number 4 and we decrease the number of integrability conditions.

2.3 Examples

In the following examples, we will consider equations having coefficients in $\mathcal{M}_D(z)$ and we will compute G as a linear differential algebraic subgroup of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$. In the three first examples, we are in the case where no Liouvillian solutions are found. In the fourth example, we are in the case number 1 and in the last example, we are in the case number 2.

Example 2.3.1 (Schrodinger equation with rational potential of odd degree). Let us consider $r(z,t) = z^{2n+1} + \sum_{i=0}^{2n} t_i z^i$. There are no Liouvillian solutions. The parameterized linear differential equation

$$\frac{\partial_z^3 b(z,t)}{2} = 2\partial_z b(z,t) r(z,t) + b(z,t)\partial_z r(z,t) - \sum_{i=0}^{2n} a_i(t) z^i$$

has a rational solution if and only if there exists $c(t) \in \mathcal{M}_D$ such that

$$\left\{ \begin{array}{rrcl} & a_{2n}(t) & = & c(t)(2n+1) \\ i < 2n: & a_i(t) & = & c(t)(i+1)t_{i+1}. \end{array} \right.$$

Then

$$G \simeq \mathrm{SL}_2\left(\mathcal{M}_\mathrm{D}^{\partial_{t'}}\right), \text{ where } \partial_{\mathrm{t}'} = (2\mathrm{n}+1)\partial_{\mathrm{t}_{2\mathrm{n}}} + \sum_{\mathrm{i}=0}^{2\mathrm{n}-1} (\mathrm{i}+1)\mathrm{t}_{\mathrm{i}+1}\partial_{\mathrm{t}_\mathrm{i}}.$$

Example 2.3.2 (Bessel equation). Let $r(z,t) = \frac{4t^2-1}{4z^2} - 1$. In [Kov86], §4.2, Example 2, we see that if $t \notin \frac{1}{2} + \mathbb{Z}$, this parameterized linear differential equation has no Liouvillian solution. We can choose D such that $\{D \cap (\frac{1}{2} + \mathbb{Z})\} = \varnothing$. We obtain that G is Zariski dense in $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$. With Remark 2.2.9, we have to see whether the parameterized linear differential equation

$$\frac{\partial_z^3 b(z,t)}{2} = 2\partial_z b(z,t)\left(\frac{4t^2-1}{4z^2} - 1\right) + b(z,t)\frac{1-4t^2}{2z^3} - \frac{2t}{z} \tag{2.3.1}$$

has a solution in $\mathcal{M}_D(z)$. Suppose that there exists $b(z,t) \in \mathcal{M}_D(z)$ satisfying such an equation. We can see directly that if $b(z,t)$ has a pole, then it is $z=0$. Assume that $b(z,t)$ has a pole of order ν at $z=0$ and let $0 \neq f(t) \in \mathcal{M}_D$ equal the value at $(0,t)$ of $z^\nu b(z,t)$. Since $b(z,t)$ satisfies the equation (2.3.1), we find for all $t \in D$:

$$\frac{-f(t)\nu(\nu-1)(\nu-2)}{2} = -f(t)\nu\frac{4t^2-1}{2} + f(t)\frac{1-4t^2}{2}.$$

For all ν, there is no $0 \neq f(t)$ satisfying this equality and we find that $b(z,t) \in \mathcal{M}_D[z]$. Let ν be its degree and $f(t)$ its leading term. The equation (2.3.1) has no constant solution, and we can assume $\nu > 1$. We find that for all $t \in D$,

$$0 = -2\nu f(t),$$

which implies that the equation (2.3.1) has no solutions in $\mathcal{M}_D(z)$ and then

$$G \simeq \mathrm{SL}_2(\mathcal{M}_\mathrm{D}).$$

Example 2.3.3 (Harmonic oscillator). Let $r(z,t)=\frac{z^2}{4}+t$. There are no Liouvillian solutions. With Remark 2.2.9, we have to check whether the parameterized linear differential equation

$$\frac{\partial_z^3 b(z,t)}{2}=2\partial_z b(z,t)\left(\frac{z^2}{4}+t\right)+b(z,t)\frac{z}{2}-1$$

has a solution in $\mathcal{M}_D(z)$. We can see directly that if $b(z,t)\in\mathcal{M}_D(z)$ is a solution, then it has no poles, which means that $b(z,t)\in\mathcal{M}_D[z]$. Let ν be its degree and $0\neq f(t)$ be its leading term. We find that $\frac{(\nu+1)f(t)}{2}=0$, which admits no solution different from 0. Then

$$G\simeq \mathrm{SL}_2(\mathcal{M}_\mathrm{D}).$$

Example 2.3.4. If $r(z,t)=\frac{t}{z^2}$, then we have two Liouvillian solutions

$$f_1(z,t)=\sqrt{z}z^{\frac{\sqrt{1+4t}}{2}} \text{ and } f_2(z,t)=\sqrt{z}z^{-\frac{\sqrt{1+4t}}{2}}.$$

We can compute the parameterized differential Galois group for the fundamental solution

$$\begin{pmatrix} f_1(z,t) & f_2(z,t)\\ \partial_z f_1(z,t) & \partial_z f_2(z,t)\end{pmatrix}:$$

$$G\simeq\left\{\begin{pmatrix}\alpha e^{a(\sqrt{1+4t})} & 0\\ 0 & \alpha^{-1}e^{-a(\sqrt{1+4t})}\end{pmatrix}, \text{ where } a\in\mathbb{C}\ ,\alpha\in\mathbb{C}^*\right\}.$$

Viewed as a linear differential algebraic subgroup $\mathrm{GL}_2(\mathcal{M}_\mathrm{D})$,

$$G\simeq\left\{\begin{pmatrix}\alpha(t) & 0\\ 0 & \alpha^{-1}(t)\end{pmatrix}, \text{ where } \partial_t\left(\frac{\sqrt{1+4t}\partial_t\alpha(t)}{\alpha(t)}\right)=0\right\}.$$

Example 2.3.5. If $r(z,t)=\frac{t}{z}-\frac{3}{16z^2}$, then we have two Liouvillian solutions

$$f_1(z,t)=(z)^{1/4}e^{2(tz)^{1/2}} \text{ and } f_2(z,t)=(z)^{1/4}e^{-2(tz)^{1/2}}.$$

We can compute the parameterized differential Galois group for the fundamental solution

$$\begin{pmatrix} f_1(z,t) & f_2(z,t)\\ \partial_z f_1(z,t) & \partial_z f_2(z,t)\end{pmatrix}:$$

$$G\simeq\left\{\begin{pmatrix}a(t) & 0\\ 0 & a^{-1}(t)\end{pmatrix}\bigcup\begin{pmatrix}0 & b^{-1}(t)\\ -b(t) & 0\end{pmatrix}, \text{ where } a(t),b(t)\in\mathbb{C}^*\right\}.$$

We can remark that we have an integrable system

$$\begin{cases}\partial_z Y(z,t) &= A(z,t)Y(z,t)\\ \partial_t Y(z,t) &= B(z,t)Y(z,t)\end{cases}$$

with

$$A(z,t)=\begin{pmatrix}0 & 1\\ \frac{t}{z}-\frac{3}{16z^2} & 0\end{pmatrix} \text{ and } B(z,t)=\begin{pmatrix}-\frac{1}{4t} & \frac{z}{t}\\ 1-\frac{3}{16tz} & \frac{3}{4t}\end{pmatrix}.$$

Chapitre 3

A density theorem in parameterized differential Galois theory.

Sommaire

Résumé: We study parameterized linear differential equations with coefficients depending meromorphically upon the parameters. As a main result, analogously to the unparameterized density theorem of Ramis, we show that the parameterized monodromy, the parameterized exponential torus and the parameterized Stokes operators are topological generators in Kolchin topology, for the parameterized differential Galois group introduced by Cassidy and Singer. We prove an analogous result for the global parameterized differential Galois group, which generalizes a result by Mitschi and Singer. These authors give also a necessary condition on a group for being a global parameterized differential Galois group: as a corollary of the density theorem, we prove that their condition is also sufficient. As an application, we give a characterization of completely integrable equations, and we give a partial answer to a question of Sibuya about the transcendence properties of a given Stokes matrix. Moreover, using a parameterized Hukuhara-Turrittin theorem, we show that the Galois group descends to a smaller field, whose field of constants is not differentially closed.

Introduction

Let us consider a linear differential system of the form

$$\partial_z Y(z) = A(z)Y(z),$$

where $\partial_z = \frac{d}{dz}$, and $A(z)$ is an $m \times m$ matrix whose entries are germs of meromorphic functions in a neighborhood of a point, say 0 to fix ideas. The differential Galois group, which measures the algebraic dependencies among the solutions, can be viewed as an algebraic subgroup of $\mathrm{GL}_m(\mathbb{C})$ via the injective group morphism

$$\begin{array}{rccc} \rho_U : & Gal & \longrightarrow & \mathrm{GL}_m(\mathbb{C}) \\ & \sigma & \mapsto & U(z)^{-1}\sigma(U(z)), \end{array}$$

where $U(z)$ is some arbitrary fundamental solution, i.e., an invertible solution matrix.

Let $U(z)$ be a fundamental solution contained in a Picard-Vessiot extension of $\partial_z Y(z) = A(z)Y(z)$. The linear differential equation is said to be regular singular at 0 if there exists an invertible matrix $P(z)$ whose entries are germs of meromorphic functions such that $W(z) = P(z)U(z)$ satisfies

$$\partial_z W(z) = \frac{A_0}{z} W(z),$$

where A_0 is a matrix with constant complex entries. In this case, $W(z)$ usually involves multivalued functions. Analytic continuation of $W(z)$ along any simple loop γ around 0 yields another fundamental solution $W(z)M_\gamma$, and the matrix M_γ with complex entries, which is a monodromy matrix, does not depend on the choice of the homotopy class of γ. The Schlesinger theorem says that the Zariski closure of the group generated by the monodromy matrix is the Galois group. In the general case, i.e., in presence of irregular singularity, the monodromy is no longer sufficient to provide a complete collection of topological generators. Ramis has shown that the group generated by the monodromy, the exponential torus and the Stokes operators, which is defined in a transcendental way as a subgroup of the differential Galois group, is dense in the latter in the Zariski topology.

More recently, a Galois theory for parameterized linear differential equations of the form

$$\partial_z Y(z,t) = A(z,t)Y(z,t),$$

where $t = (t_1, \dots, t_n)$ are parameters, has been developed in [CS07]. See also [HS08, Lan08, Rob59, Ume96b]. Namely, the Galois group, which measures the $(\partial_{t_1}, \dots, \partial_{t_n})$-differential and algebraic dependencies among the solutions, can be seen as a differential group in the sense of Kolchin, that is a group of matrices whose entries lie in a differential field and satisfy a set of polynomial differential equations in the variables $t_1, \dots, t_n$. See [Cas72, Cas89, Kol73, Kol85, MO11]. To be applied, the theory from [CS07] requires the field of constants with respect to ∂_z to be of characteristic 0 and differentially closed (see §3.2.1). The drawback of this latter assumption is that a differentially closed field is a very big field, and cannot be interpreted as a field of functions.

There is a link between the parameterized differential Galois theory and isomonodromy for equations with only regular singular poles (see [CS07, MS12, MS13]). Let $\mathcal{D}(t_0, r) = \left\{(z_1, \dots, z_n) \in \mathbb{C}^n \middle| \forall i \leq n, |z_i - t_{0,i}| < r\right\}$ be an open polydisc in $\mathbb{C}^n$, let $\mathcal{D}$ be an open subset of $\mathbb{C}$, and let $A(z,t)$ be a matrix whose entries are analytic on $\mathcal{D} \times \mathcal{D}(t_0, r)$. We consider open disks D_j that cover $\mathcal{D}$, and a solution $U_j(z,t)$

of the differential equation $\partial_z Y(z,t) = A(z,t)Y(z,t)$ analytic on $D_j \times \mathcal{D}(t_0,r)$. If $D_i \cap D_j \neq \varnothing$, we define $C_{i,j}(t) = U_i(z,t)^{-1}U_j(z,t)$, the connection matrices. Following Definition 5.2 in [CS07] (see also [Bol97, Mal83]), the parameterized linear differential equation $\partial_z Y(z,t) = A(z,t)Y(z,t)$ is said to be isomonodromic if, there is a choice of (D_i) covering $\mathcal{D}$, and of the solutions $U_i(z,t)$ of $\partial_z Y(z,t) = A(z,t)Y(z,t)$ analytic on $D_i \times \mathcal{D}(t_0,r)$ such that the connection matrices are independent of t. In this case, the matrix of the monodromy is constant on the polydisc $\mathcal{D}(t_0,r)$. When $A(z,t)$ is of the form $\sum_{i=1}^{s} \frac{A_i(t)}{z-u_i}$, such that all the $A_i(t)$ have analytic entries on U and $u_i \in \mathcal{D}$, the following statements are equivalent (see [CS07], Propositions 5.3 and 5.4).

- The Galois group is conjugate, over a differentially closed field (see Definition 3.2.2), to a group of constant matrices.
- The parameterized linear differential equation is isomonodromic in the above sense.
- The parameterized linear differential equation is completely integrable (see Definition 3.3.1).

We are interested in the case where the parameterized linear differential equation may have irregular singularities, in a sense we are going to explain. The main result of this chapter is a parameterized analogue of the density theorem of Ramis: we give topological generators for the Galois group in the Kolchin topology (in which closed sets are zero sets of differential algebraic polynomials). As an application of our main result, we improve Proposition 3.9 in [CS07] (see Remark 3.3.4): a parameterized linear differential equation is completely integrable if and only if the topological generators for the Galois group just mentioned are conjugate to constant matrices over a field of meromorphic functions. Notice that the latter is not differentially closed.

* * *

The chapter is organized as follows. In the first section we study parameterized linear differential systems from an analytic point of view. The parameters will vary in U, a non-empty polydisc of $\mathbb{C}^n$. Let $t = (t_1, \dots, t_n) \in U$ denote the multiparameter. Let $\mathcal{M}_U$ be the field of meromorphic functions on U and let $\hat{K}_U = \mathcal{M}_U[[z]][z^{-1}]$. The Hukuhara-Turrittin theorem in this case gives the following result (see Remark 3.1.6 for a discussion of a similar result present in [Sch01]):

Proposition (see Proposition 3.1.3 below).
Let $\partial_z Y(z,t) = A(z,t)Y(z,t)$, with $A(z,t) \in \mathrm{M}_m\left(\hat{K}_U\right)$ (that is a $m \times m$ matrix with entries in $\hat{K}_U$). Then, there exists a non empty polydisc $U' \subset U$, $\nu \in \mathbb{N}^*$, such that we have a fundamental solution $F(z,t)$ of the form:

$$F(z,t) = \hat{H}(z,t)z^{L(t)}e^{Q(z,t)},$$

where:

- $\hat{H}(z,t) \in \mathrm{GL}_m\left(\hat{K}_{U'}[z^{1/\nu}]\right)$.
- $L(t) \in \mathrm{M}_m(\mathcal{M}_{U'})$.
- $e^{Q(z,t)} = \mathrm{Diag}(e^{q_i(z,t)})$, *with* $q_i(z,t) \in z^{-1/\nu}\mathcal{M}_{U'}[z^{-1/\nu}]$.
- *Moreover, we have* $z^{L(t)}e^{Q(z,t)} = e^{Q(z,t)}z^{L(t)}$.

See Remark 3.1.4 for a discussion about the uniqueness of a fundamental solution of $\partial_z Y(z,t) = A(z,t)Y(z,t)$ written in the above way.

In §3.1.3, we briefly review the Stokes phenomenon in the unparameterized case. We have solutions, which are analytic in some sector and Gevrey asymptotic to the formal part of the solution in the Hukuhara-Turrittin canonical form. The fact that various asymptotic solutions do not glue to a single solution on the Riemann surface of the logarithm is called the Stokes phenomenon.

Let U be a non empty polydisc of $\mathbb{C}^n$ and let $f(z,t) = \sum f_i(t)z^i \in \hat{K}_U$. We say that $f(z,t)$ belongs to $\mathcal{O}_U(\{z\})$ if for all $t \in U$, $z \mapsto \sum f_i(t)z^i$ is a germ of meromorphic function at 0. Remark that if $f(z,t) \in \mathcal{O}_U(\{z\}) \subset \mathcal{M}_U[[z]][z^{-1}] = \hat{K}_U$, then the z-coefficients $f_i(t)$ of $f(z,t)$ are analytic on U.

In §3.1.4, we study the Stokes phenomenon of equations of the form $\partial_z Y(z,t) = A(z,t)Y(z,t)$, with $A(z,t) \in \mathrm{M}_m(\mathcal{O}_U(\{z\}))$. In particular, we prove that the asymptotic solutions depend analytically (under mild conditions) upon the parameters.

In the second section, we use the parameterized Hukuhara-Turrittin theorem to deduce some Galois theoretic properties of parameterized linear differential equations in coefficients in $\mathcal{O}_U(\{z\})$. We first recall some facts from [CS07] about parameterized differential Galois theory. The problem is that the theory in [CS07] cannot be applied here, since $\mathcal{M}_U$, our field of constants with respect to ∂_z, is a field of functions that are meromorphic in $t_1, \dots, t_n$, and this field is not differentially closed (see §3.2.1). In the papers [GGO13, Wib12], the authors prove the existence of parameterized Picard-Vessiot extensions under weaker assumptions than in [CS07]. See also [CHS08, PN11]. We do not use these latter results because we need a parameterized Hukuhara-Turrittin theorem, which proves directly that a parameterized Picard-Vessiot extension exists, not necessarily unique, in order to study the parameterized Stokes phenomenon. This allow us to define a group, we will call by abuse of language, see Remark 3.2.8, the parameterized differential Galois group. In §3.2.4, we consider the local case $\partial_z Y(z,t) = A(z,t)Y(z,t)$, with $A(z,t) \in \mathrm{M}_m(\mathcal{O}_U(\{z\}))$. We state and show the main result:

Parameterized analogue of the density theorem of Ramis (Theorem 3.2.20). *The group generated by the parameterized monodromy, the parameterized exponential torus and the parameterized Stokes operators is dense in the parameterized differential Galois group for the Kolchin topology.*

Then, we turn to the global case. We consider equations with coefficients in $\mathcal{M}_U(z)$ and study their global Galois group. We prove a density theorem in this global setting, see Theorem 3.2.24. The proof in the unparameterized case can be found in [Mit96]. In §3.2.6, we give various examples of calculations.

In the third section, we give three applications. First, we prove a criterion for the integrability of differential systems (see Definition 3.3.1):

Proposition (see Proposition 3.3.2 below). *Let $A(z,t) \in \mathrm{M}_m(\mathcal{M}_U(z))$. The linear differential equation $\partial_z Y(z,t) = A(z,t)Y(z,t)$ is completely integrable if and only if there exists a fundamental solution such that the matrices of the parameterized monodromy, the parameterized exponential torus and the parameterized Stokes operators for all the singularities are constant, i.e., do not depend on z.*

As a second application, we give a partial answer to a question of Sibuya (see [Sib75]), regarding the differential transcendence properties of a Stokes matrix of the parameterized linear differential equation:

$$\begin{pmatrix} \partial_z Y(z,t) \\ \partial_z^2 Y(z,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ z^3+t & 0 \end{pmatrix} \begin{pmatrix} Y(z,t) \\ \partial_z Y(z,t) \end{pmatrix}.$$

Sibuya was asking whether an entry of a given Stokes matrix at infinity is ∂_t-differentially transcendental, i.e., satisfies no differential polynomial equation. We prove that it is at least not ∂_t-finite, i.e., that it satisfies no linear differential equations.

As a last application, we deal with the inverse problem. We prove that if G is the global parameterized differential Galois group of some equation having coefficients in $k(z)$ (see §3.3.3), then G contains a finitely generated Kolchin dense subgroup. The converse of this latter assertion has been proved in Corollary 5.2 in [MS12], and we obtain a result on the inverse problem:

Theorem (see Theorem 3.3.11 below). *The group G is the global parameterized differential Galois group of some equation having coefficients in $k(z)$ if and only if G contains a finitely generated Kolchin-dense subgroup.*

In the appendix, we prove the following result.

Theorem (see Theorem A.1 below). *Let us consider $\partial_z Y(z,t) = A(z,t)Y(z,t)$, with $A(z,t) \in \mathrm{M}_m\left(\hat{K}_U\right)$. Then, there exists a non empty polydisc $U' \subset U$, such that we have a fundamental solution $F(z,t)$ of the form:*

$$F(z,t) = \hat{P}(z,t) z^{C(t)} e^{Q(z,t)},$$

where:

- $\hat{P}(z,t) \in \mathrm{GL}_m\left(\hat{K}_{U'}\right)$.
- $C(t) \in \mathrm{M}_m(\mathcal{M}_{U'})$.
- $e^{Q(z,t)} = \mathrm{Diag}(e^{q_i(z,t)})$*, with* $q_i(z,t) \in z^{-1/\nu}\mathcal{M}_{U'}[z^{-1/\nu}]$*, for some* $\nu \in \mathbb{N}^*$.

Remark that contrary to Proposition 3.1.3, the entries of the formal part are not ramified. On the other hand, $z^{C(t)}$ and $e^{Q(z,t)}$ do not commute anymore. This theorem is not necessary for the proof of the main result of the chapter, this is the reason why we give the proof in the appendix. However, this result is important since it permits to determine the equivalence classes (see [vdPS03], Page 7) of parameterized linear differential systems in coefficients in $\hat{K}_U$.

3.1 Local analytic linear differential systems depending upon parameters.

In §3.1.1, we define the field to which the entries of the fundamental solution, in the Hukuhara-Turrittin canonical form, will belong. In §3.1.2, we prove a parameterized version of the Hukuhara-Turrittin theorem. In §3.1.3, we briefly review the Stokes phenomenon in the unparameterized case. In §3.1.4, we study the Stokes phenomenon in the parameterized case.

3.1.1 Definition of the fields.

Let us consider a linear differential system of the form $\partial_z Y(z) = A(z)Y(z)$, where $A(z)$ is a $m \times m$ matrix whose entries belongs to $\mathbb{C}[[z]][z^{-1}]$. We know we can find a formal fundamental solution in the Hukuhara-Turrittin canonical form $\hat{H}(z)z^L e^{Q(z)}$, where:

- $\hat{H}(z)$ is a matrix of formal power series in $z^{1/\nu}$ for some $\nu \in \mathbb{N}^*$.
- $L \in \mathrm{M}_m(\mathbb{C})$.
- $Q(z) = \mathrm{Diag}(q_i(z))$, with $q_i(z) \in z^{-1/\nu}\mathbb{C}\left[z^{-1/\nu}\right]$.
- Moreover, we have $z^L e^{Q(z)} = e^{Q(z)} z^L$.

Notice that this formulation is trivially equivalent to Theorem 3.1 in [vdPS03].

Let U be a non empty polydisc of $\mathbb{C}^n$, let $\hat{K}_U$ and $\mathcal{M}_U$ defined in page 68. We want to construct a field containing a fundamental set of solutions of

$$\partial_z Y(z,t) = A(z,t)Y(z,t),$$

where $A(z,t) \in \mathrm{M}_m\left(\hat{K}_U\right)$. Let $\Delta_t = \{\partial_{t_1}, \dots, \partial_{t_n}\}$ and let

$$\mathbf{E}_U = \bigcup_{\nu \in \mathbb{N}^*} z^{\frac{-1}{\nu}} \mathcal{M}_U\left[z^{\frac{-1}{\nu}}\right].$$

We define formally the (∂_z, Δ_t)-ring, i.e., a ring equipped with $n+1$ derivations $\partial_z, \partial_{t_1}, \dots, \partial_{t_n}$ a priori not required to commute with each other,

$$R_U := \hat{K}_U\left[\log, \left(z^{a(t)}\right)_{a(t)\in\mathcal{M}_U}, \left(e(q(z,t))\right)_{q(z,t)\in\mathbf{E}_U}\right],$$

with the following rules:

1. The symbols log, $\left(z^{a(t)}\right)_{a(t)\in\mathcal{M}_U}$ and $\left(e(q(z,t))\right)_{q(z,t)\in\mathbf{E}_U}$ only satisfy the following relations:

$$\begin{aligned} z^{a(t)+b(t)} &= z^{a(t)} z^{b(t)}, & e(q_1(z,t)+q_2(z,t)) &= e(q_1(z,t))e(q_2(z,t)), \\ z^a &= z^a \in \hat{K}_U \text{ for } a \in \mathbb{Z}, & e(0) &= 1. \end{aligned}$$

2. The following rules of differentiation

$$\begin{array}{llllll} \partial_z \log & = & z^{-1}, & \partial_{t_i} \log & = & 0, \\ \partial_z z^{a(t)} & = & \frac{a(t)}{z} z^{a(t)}, & \partial_{t_i} z^{a(t)} & = & \partial_{t_i}(a(t)) \log z^{a(t)}, \\ \partial_z e(q(z,t)) & = & \partial_z(q(z,t)) e(q(z,t)), & \partial_{t_i} e(q(z,t)) & = & \partial_{t_i}(q(z,t)) e(q(z,t)), \end{array}$$

equip the ring with a (∂_z, Δ_t)-differential structure, since these rules go to the quotient as can be readily checked.

The intuitive interpretation of these symbols are $\log = \log(z)$, $z^{a(t)} = e^{a(t)\log(z)}$ and $e(q(z,t)) = e^{q(z,t)}$. Let $f(z,t)$ be one these latter functions. Then $f(z,t)$ has a natural interpretation as an analytic function on $\widetilde{\mathbb{C}} \times U'$, where $\widetilde{\mathbb{C}}$ is the Riemann surface of the logarithm and U' is some non empty polydisc contained in U. We will use the analytic function instead of the symbol when we will consider asymptotic solutions (see §3.1.3 and §3.1.4). For the time being, however, we see them only as symbols.

Let $\overline{\mathcal{M}_U}$ be the algebraic closure of $\mathcal{M}_U$. In the same way than for R_U, we construct the (∂_z, Δ_t)-ring

$$\overline{R}_U := \overline{\mathcal{M}_U}[[z]][z^{-1}] \left[\log, \left(z^{a(t)}\right)_{a(t) \in \overline{\mathcal{M}_U}}, \left(e(q(z,t))\right)_{q(z,t) \in \bigcup_{\nu \in \mathbb{N}^*} z^{\frac{-1}{\nu}} \overline{\mathcal{M}_U}\left[z^{\frac{-1}{\nu}}\right]} \right].$$

We can see (Proposition 3.22 in [vdPS03]) that this latter is an integral domain and its field of fractions has field of constants with respect to ∂_z equal to $\overline{\mathcal{M}_U}$. Since $R_U \subset \overline{R}_U$, R_U is also an integral domain. Therefore, we may consider the (∂_z, Δ_t)-fields:

$$K_{F,U} = \mathcal{M}_U \left(\log, \left(z^{a(t)}\right)_{a(t) \in \mathcal{M}_U} \right),$$

$$\hat{K}_{F,U} = \hat{K}_U \left(\log, \left(z^{a(t)}\right)_{a(t) \in \mathcal{M}_U} \right),$$

and

$$\widehat{\mathbf{K}_U} = \hat{K}_U \left(\log, \left(z^{a(t)}\right)_{a(t) \in \mathcal{M}_U} \left(e(q(z,t))\right)_{q(z,t) \in \mathbf{E}_U} \right).$$

In the definition of the fields $K_{F,U}$ and $\hat{K}_{F,U}$, the subscript F stands for Fuchsian. Since $\widehat{\mathbf{K}_U}$ is contained in the field of fractions of $\overline{R}_U$, it has field of constants with respect to ∂_z equal to $\overline{\mathcal{M}_U} \cap \widehat{\mathbf{K}_U} = \mathcal{M}_U$.

We have defined (∂_z, Δ_t)-fields where all the derivations commute with each other. We have the following inclusions of (∂_z, Δ_t)-fields:

$$\begin{array}{ccccccc} & & & K_{F,U} & & & \\ & & \nearrow & & \searrow & & \\ \mathcal{M}_U & \to & \hat{K}_U & & \to & \hat{K}_{F,U} & \to & \widehat{\mathbf{K}_U}. \end{array}$$

Remark 3.1.1. Any algebraic function over $\mathcal{M}_U$ can be seen as an element of $\mathcal{M}_{U'}$, for some non-empty $U' \subset U$. Therefore, a finite extension of $\mathcal{M}_U$ can be embedded in $\mathcal{M}_{U'}$ for a convenient choice of $U' \subset U$. We will use this fact in the rest of the chapter.

Lemma 3.1.2. *Let U be a non empty polydisc of $\mathbb{C}^n$ and let $L(t) \in \mathrm{M}_m\Big(\overline{\mathcal{M}_U}\Big)$, where $\overline{\mathcal{M}_U}$ is the algebraic closure of $\mathcal{M}_U$. There exists a non empty polydisc $U' \subset U$, and $z^{L(t)} \in \mathrm{GL}_m(K_{F,U'})$ satisfying*

$$\partial_z z^{L(t)} = \frac{L(t)}{z} z^{L(t)} = z^{L(t)} \frac{L(t)}{z}.$$

Proof. Let us write $L(t) = P(t)(D(t) + N(t))P^{-1}(t)$, with

- $D(t) = \mathrm{Diag}(d_i(t))$, $d_i(t) \in \overline{\mathcal{M}_U}$,
- $N(t)$ nilpotent,
- $D(t)N(t) = N(t)D(t)$,
- $P(t) \in \mathrm{GL}_m\Big(\overline{\mathcal{M}_U}\Big)$,

be the Jordan decomposition of $L(t)$.

Due to Remark 3.1.1, there exists a non empty polydisc $U' \subset U$, such that $d_i(t) \in \mathcal{M}_{U'}$ and $P(t) \in \mathrm{GL}_m(\mathcal{M}_{U'})$. We may restrict U' and assume that $N(t)$ does not depend upon t in U'. Let us write $N := N(t)$. Then, the matrix $z^{L(t)} = P(t)\mathrm{Diag}(z^{d_i(t)})e^{N\log}P^{-1}(t)$ belongs to $\mathrm{GL}_m(K_{F,U'})$ and $z^{L(t)}$ satisfies

$$\partial_z z^{L(t)} = \frac{L(t)}{z} z^{L(t)} = z^{L(t)} \frac{L(t)}{z}.$$

□

Let $a(t) \in \mathcal{M}_U$ and let $(a(t)) \in \mathrm{M}_1(\mathcal{M}_U)$ be the corresponding matrix. Then, we have $z^{a(t)} = z^{(a(t))}$.

3.1.2 The Hukuhara-Turrittin theorem in the parameterized case.

The goal of this subsection is to give the parameterized version of the Hukuhara-Turrittin theorem. In the appendix, we prove a slightly different result, which is not needed in the chapter. See Theorem A.1.

Proposition 3.1.3. *Let U be a non empty polydisc of $\mathbb{C}^n$ and consider*

$$\partial_z Y(z,t) = A(z,t)Y(z,t),$$

with $A(z,t) \in \mathrm{M}_m\Big(\hat{K}_U\Big)$. There exists a non empty polydisc $U' \subset U$ such that we have a fundamental solution $F(z,t) \in \mathrm{GL}_m\Big(\widehat{\boldsymbol{K}_{U'}}\Big)$ of the form

$$F(z,t) = \hat{H}(z,t)z^{L(t)}e\Big(Q(z,t)\Big),$$

where:

- $\hat{H}(z,t) \in \mathrm{GL}_m\Big(\hat{K}_{U'}\left[z^{1/\nu}\right]\Big)$, *for some $\nu \in \mathbb{N}^*$.*
- $L(t) \in \mathrm{M}_m(\mathcal{M}_{U'})$.
- $e\Big(Q(z,t)\Big) = \mathrm{Diag}\Big(e(q_i(z,t))\Big)$, *with $q_i(z,t) \in \boldsymbol{E}_{U'}$.*

– *Moreover, we have* $e\Big(Q(z,t)\Big)z^{L(t)} = z^{L(t)}e\Big(Q(z,t)\Big)$.

Furthermore, if $A(z,t) \in \mathrm{M}_m\Big(\mathcal{O}_U(\{z\})\Big)$, *there exists a non empty polydisc* $U'' \subset U'$ *such that we may assume that the* z*-coefficients of* $\hat{H}(z,t)$ *are all analytic on* U''.

Remark 3.1.4. Remark that we have no uniqueness of the fundamental solution written in the same way as above, since for all $\kappa \in \mathbb{Z}$, $z^{\kappa}\hat{H}(z,t)z^{L(t)-\kappa}e^{Q(z,t)}$ is also a fundamental solution. However, because of the construction of $\widehat{\mathbf{K}_{U'}}$, we obtain that if for $i \in \{1,2\}$, $\hat{H}_i(z,t)z^{L_i(t)}e\Big(Q_i(z,t)\Big)$ is a fundamental solution of $\partial_z Y(z,t) = A(z,t)Y(z,t)$ written in the same way as above, then, up to a permutation, Q_1 and Q_2 have the same entries.

Example 3.1.5 ([Sch01], Introduction). If we consider

$$z^2\partial_z Y(z,t) = \begin{pmatrix} t & 1 \\ z & 0 \end{pmatrix} Y(z,t),$$

we get the solution

$$\left(\begin{pmatrix} 1 & 1 \\ 0 & -t \end{pmatrix} + O(z)\right)\begin{pmatrix} z^{\frac{1}{t}}e^{\frac{-t}{z}} & 0 \\ 0 & z^{\frac{-1}{t}} \end{pmatrix}, \qquad (3.1.1)$$

for $t \neq 0$ and the solution

$$\begin{pmatrix} 1 & 1 \\ z^{1/2} & -z^{1/2} \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(z^{1/2})\right)\begin{pmatrix} z^{\frac{1}{4}}e^{-z^{-1/2}} & 0 \\ 0 & z^{\frac{1}{4}}e^{z^{-1/2}} \end{pmatrix},$$

for $t = 0$. The latter is not the specialization of (3.1.1) at $t = 0$. The problem is that the level of the unparameterized system (see §3.1.3 for the definition) at $t = 0$ is 1 and the level of the unparameterized system for $t \neq 0$ is $\frac{1}{2}$. This example shows that we cannot get a solution in the parameterized Hukuhara-Turrittin form, that remains valid for all values of the parameter t. This is the reason why we have to restrict the subset of the parameter-space.

Remark 3.1.6. Similar results to Proposition 3.1.3 have been proved in Theorem 4.2 of [Sch01]. We now explain the result of Schäfke. Let U be an open connected subset of $\mathbb{C}^n$ that contains 0 and let $A(z,t) = \sum_{l=s}^{\infty} A_l(t)$, with $s \in \mathbb{Z}$, and $A_l(t)$ analytic in U. In particular, $A(z,t) \in \mathrm{M}_m\Big(\hat{K}_U\Big)$. Let us consider $\partial_z Y(z,t) = A(z,t)Y(z,t)$ and assume that for all $t \in U$, there exists a solution $\hat{H}_t(z)z^{L_t}e\Big(Q(z,t)\Big)$, given in the classical Hukuhara-Turrittin canonical form such that:

- The z-coefficients of the $q_i(z,t)$ are analytic functions in $t \in U$.
- The degree in z^{-1} of $q_i(z,t) - q_j(z,t)$ is independent of t in U.
- If $q_i(z,t) \not\equiv q_j(z,t)$, then $q_i(z,0) \neq q_0(z,0)$.

Under these assumptions, Schäfke concludes that, there exists an open neighborhood $U' \subset U$ of 0 in the t-plane such that there exists a fundamental solution $\hat{H}(z,t)z^{L(t)}e(Q(z,t)) \in \mathrm{GL_m}\Big(\widehat{\mathbf{K}_{U'}}\Big)$ with $\hat{H}(z,t) = \sum_{l=0}^{\infty}\hat{H}_l(t)$ and $t \mapsto \hat{H}_l(t), L(t)$ are analytic. Notice that Schäfke gives a necessary and sufficient condition, that can be algorithmically checked, for well behaved exponential part. See [Sch01], Theorem 5.2.

Using Schäfke's theorem, we can deduce Proposition 3.1.3 only in the particular case where $A(z,t)$ has entries with z-coefficients analytic in U. Note that [Sch01] does not allow us to deduce the general case. See also [BV85], § 10, Theorem 1, for another result of this nature.

Proof of Proposition 3.1.3. Let $K = C\left[[z]\right]\left[z^{-1}\right]$, where C is an algebraically closed fields of characteristic 0 equipped with a derivation ∂_z that acts trivially on C and with $\partial_z(z) = 1$. The Hukuhara-Turrittin theorem (see Theorem 3.1 in [vdPS03]) is valid for linear differential system with entries in K. We apply it with $C = \overline{\mathcal{M}_U}$, the algebraic closure of $\mathcal{M}_U$.

Let us consider the matrix $L(t) \in \mathrm{M}_m\Big(\overline{\mathcal{M}_U}\Big)$ and $Q(z,t) = \mathrm{Diag}\Big(q_i(z,t)\Big)$, with $q_i(z,t) \in z^{-1/\nu}\overline{\mathcal{M}_U}\left[z^{-1/\nu}\right]$ for some $\nu \in \mathbb{N}$. Because of Remark 3.1.1 and Lemma 3.1.2, there exists a non empty polydisc $U' \subset U$, such that we may define $z^{L(t)} \in \mathrm{GL}_m(K_{F,U'})$ satisfying $\partial_z z^{L(t)} = \frac{L(t)}{z} z^{L(t)} = z^{L(t)}\frac{L(t)}{z}$, $L(t) \in \mathrm{M}_m(\mathcal{M}_{U'})$ and $q_i(z,t) \in \mathbf{E}_{U'}$. Hence, there exists a non empty polydisc $U' \subset U$ such that the Hukuhara-Turrittin theorem gives a fundamental solution
$$F'(z,t) = \hat{H}'(z,t) z^{L(t)} e\Big(Q(z,t)\Big),$$
where:

- $\hat{H}'(z,t) \in \mathrm{GL}_m\Big(\overline{\mathcal{M}_{U'}}\left[\left[z^{1/\nu}\right]\right]\left[z^{-1/\nu}\right]\Big)$, for some $\nu \in \mathbb{N}$.
- $L(t) \in \mathrm{M}_m(\mathcal{M}_{U'})$.
- $e\Big(Q(z,t)\Big) = \mathrm{Diag}\Big(e(q_i(z,t))\Big)$, with $q_i(z,t) \in \mathbf{E}_{U'}$.
- Moreover, we have $e\Big(Q(z,t)\Big) z^{L(t)} = z^{L(t)} e\Big(Q(z,t)\Big)$.

Let us prove now that we may find a matrix $\hat{H}(z,t) \in \mathrm{GL}_m\Big(\hat{K}_{U'}\left[z^{1/\nu}\right]\Big)$, such that $F(z,t) = \hat{H}(z,t) z^{L(t)} e\Big(Q(z,t)\Big)$ is a fundamental solution. The matrix

$$F'(z,t) = \hat{H}'(z,t) z^{L(t)} e\Big(Q(z,t)\Big)$$

satisfies the parameterized linear differential equation

$$\partial_z F'(z,t) = A(z,t) F'(z,t),$$

and the matrix $z^{L(t)} e\Big(Q(z,t)\Big)$ satisfies parameterized linear differential equation:

$$\begin{aligned} \partial_z z^{L(t)} e\Big(Q(z,t)\Big) &= \left(z^{-1}L(t) + \partial_z Q(z,t)\right) z^{L(t)} e\Big(Q(z,t)\Big) \\ &= z^{L(t)} e\Big(Q(z,t)\Big) \left(z^{-1}L(t) + \partial_z Q(z,t)\right). \end{aligned}$$

Hence,
$$\partial_z \hat{H}'(z,t) = A(z,t)\hat{H}'(z,t) - \hat{H}'(z,t)\left(z^{-1}L(t) + \partial_z Q(z,t)\right).$$

We write $\hat{H}'(z,t)$ as a column vector $\widetilde{H}'(z,t)$ of size m^2. Let $C(z,t) \in \mathrm{M}_{m^2}\Big(\hat{K}_{U'}\left[z^{1/\nu}\right]\Big)$, with $\nu \in \mathbb{N}^*$ such that $\widetilde{H}'(z,t)$ satisfies the parameterized linear differential system:

$$\partial_z \widetilde{H}'(z,t) = C(z,t)\widetilde{H}'(z,t).$$

Let us write $\widetilde{H}'(z,t) = \sum_{i\geq N} \widetilde{H}'_i(t) z^{i/\nu}$ and $C(z,t) = \sum_{i\geq M} C_i(t) z^{i/\nu}$, where $M, N \in \mathbb{Z}$. Then, by identifying the coefficients of the $z^{i/\nu}$-terms of the power series in the equation $\partial_z \widetilde{H}'(z,t) = C(z,t)\widetilde{H}'(z,t)$, we find that:

$$\left(\frac{i}{\nu}+1\right)\widetilde{H}'_{i+\nu}(t) = \sum_{l=N}^{i-M} C_{i-l}(t)\widetilde{H}'_l(t).$$

We recall that because of the definition of $\hat{K}_{U'}\left[z^{1/\nu}\right]$, every $C_i(t)$ belongs to $\mathrm{M}_m\left(\mathcal{M}_{U'}\right)$. The fact that there exists a fundamental solution $\hat{H}(z,t)z^{L(t)}e\Big(Q(z,t)\Big)$, with $\hat{H}(z,t) \in \mathrm{GL}_m\Big(\hat{K}_{U'}\left[z^{1/\nu}\right]\Big)$ is now clear.

Assume now that $A(z,t) \in \mathrm{M}_m\Big(\mathcal{O}_U(\{z\})\Big)$. Let U'' be a non empty polydisc with $U'' \subset U'$ such that for $z \neq 0$ fixed, the entries of the z-coefficients of $z^{-1}L(t) + \partial_z Q(z,t)$ are analytic on U''. Then, the entries of the z-coefficients of $C(z,t)$ are all analytic on U''. Hence, we may assume that the entries of the z-coefficients of $\hat{H}(z,t)$ are all analytic on U''. □

Remark 3.1.7. If we take a smaller non empty polydisc U, we may assume that if we consider $\partial_z Y(z,t) = A(z,t)Y(z,t)$, with $A(z,t) \in \mathrm{M}_m\Big(\mathcal{O}_U(\{z\})\Big)$, then the fundamental solution of Proposition 3.1.3 belongs to $\mathrm{GL}_m\Big(\widehat{\mathbf{K}_U}\Big)$, and the entries of the z-coefficients of $\hat{H}(z,t)$ are all analytic on U.

3.1.3 Review of the Stokes phenomenon in the unparameterized case.

In this subsection we will briefly review the Stokes phenomenon in the unparameterized case. See [CR, Eca, Éca81, LR90, LR94, LR95, LRR11, Mal91, Mal95, MR92, Ram80, Ram85, Ras10, Rem12, RS89, Sin09, Was87] and in particular Chapter 8 of [vdPS03] for more details. We will generalize some results concerning the summation of divergent series in the parameterized case in §3.1.4. First we treat the example of the Euler equation:

$$z^2\partial_z Y(z) + Y(z) = z,$$

which admits as a solution the formal series: $\hat{f}(z) = \sum_{n=0}^{\infty}(-1)^n n! z^{n+1}$. Classical methods of differential equations give another solution:

$$f(z) = \int_0^z e^{1/z}e^{-1/t}\frac{dt}{t} = \int_0^\infty \frac{1}{1+u}e^{-u/z}du,$$

where $1/t - 1/z = u/z$. The solution $\hat{f}(z)$ is divergent and the solution $f(z)$ can be extended to an analytic function on the sector:

$$V = \left\{ z \in \widetilde{\mathbb{C}} \,\middle|\, \arg(z) \in \left]\frac{-3\pi}{2}, \frac{+3\pi}{2}\right[\right\}.$$

On this sector, $f(z)$ is 1-Gevrey asymptotic to $\hat{f}(z)$: for every closed subsector W of V, there exists $A_W \in \mathbb{R}$, $\varepsilon > 0$ such that for all N and all $z \in W$ with $|z| < \varepsilon$,

$$\left| f(z) - \sum_{n=0}^{N-1}(-1)^n n! z^{n+1} \right| \leq (A_W)^{N+1}(N+1)!|z|^{N+1}.$$

We can also consider $f(e^{2i\pi}z)$, which is an asymptotic solution on the sector:

$$V' = \left\{ z \in \widetilde{\mathbb{C}} \middle| \arg(z) \in \left] \frac{\pi}{2}, \frac{7\pi}{2} \right[\right\}.$$

The two asymptotic solutions do not glue to a single asymptotic solution on $V \cup V'$. In fact, the residue theorem implies that the difference in $V \cap V'$ of the two asymptotic solutions is:

$$2i\pi e^{1/z}.$$

The fact that various asymptotic solutions do not glue to a single analytic solution is called Stokes phenomenon.

More generally, let us consider a linear differential equation $\partial_z Y(z) = A(z)Y(z)$ such that the entries of $A(z)$ are germs of meromorphic functions in a neighborhood of 0. Let $\hat{H}(z)z^L e\Big(Q(z)\Big)$, with $Q(z) = \text{Diag}\Big(q_i(z)\Big)$, be a fundamental solution in the Hukuhara-Turrittin canonical form. Since for all $k \in \mathbb{N}$,

$$\hat{H}(z)z^L e\Big(Q(z)\Big) = \hat{H}(z)\text{Diag}(z^k)z^{L-k\text{Id}}e\Big(Q(z)\Big),$$

we may assume that $\hat{H}(z)$ has no pole at $z = 0$.

The levels of $\partial_z Y(z) = A(z)Y(z)$ are the degrees in z^{-1} of the $q_i(z) - q_j(z)$ (the levels are positive rational numbers and are well defined because of Remark 3.1.4). Consider $q(z) = q_k z^{-k/\nu} + \cdots + q_1 z^{-1/\nu} \in z^{-1/\nu}\mathbb{C}\left[z^{-1/\nu}\right]$ with $\nu \in \mathbb{N}$. The real number d is called singular for $q(z)$ if $q_k e^{-idk/\nu}$ is a positive real number. This correspond to the arguments d such that $r \mapsto e^{q\left(re^{id}\right)}$ increases fastest as r tends to 0^+. The singular directions of $\partial_z Y(z) = A(z)Y(z)$ (we will write singular directions when no confusion is likely to arise) are the real number that are singular for one of the $q_i(z) - q_j(z)$, with $i \neq j$. Notice that the set of singular directions is finite modulo $2\pi\nu$ for some $\nu \in \mathbb{N}$. Let $k_1 < \cdots < k_r$ be the levels of the linear differential equation. There exists a decomposition $\hat{H}(z) = \hat{H}_{k_1}(z) + \cdots + \hat{H}_{k_r}(z)$, such that for d not a singular direction, there exists an unique r-tuple of matrices $\Big(H^d_{k_1}(z), \dots, H^d_{k_r}(z)\Big)$, such that $H^d_{k_i}(z)$ is analytic on the sector

$$V_d = \left\{ z \in \widetilde{\mathbb{C}} \middle| \arg(z) \in \left] d - \frac{\pi}{2k_i}, d + \frac{\pi}{2k_i} \right[\right\},$$

and is k_i-Gevrey asymptotic to $\hat{H}_{k_i}(z) = \sum_{n \in \mathbb{N}} \hat{H}_{n,k_i} z^n$ on V_d: for every closed subsector W of V_d, there exists $A_W \in \mathbb{R}$, $\varepsilon > 0$ such that for all N and all $z \in W$ with $|z| < \varepsilon$,

$$\left| H^d_{k_i}(z) - \sum_{n=0}^{N-1} \hat{H}_{n,k_i} z^n \right| \leq (A_W)^N \Gamma\left(1 + \frac{N}{k_i}\right) |z|^N,$$

where Γ denotes the Gamma function. Until the end of the chapter, we will denote a fixed determination of the complex logarithm by $\log(z)$. Furthermore, the matrix

$$\Big(H^d_{k_1}(z) + \cdots + H^d_{k_r}(z)\Big)e^{L\log(z)}e^{Q(z)} = H^d(z)e^{L\log(z)}e^{Q(z)}, \tag{3.1.2}$$

which is analytic on the sector $\left\{ z \in \widetilde{\mathbb{C}} \middle| \arg(z) \in \left] d - \frac{\pi}{2k_r}, d + \frac{\pi}{2k_r} \right[\right\}$, is a solution of $\partial_z Y(z) = A(z)Y(z)$. As a matter of fact, $H^d_{k_i}(z)$ is k_i-Gevrey asymptotic to $\hat{H}_{k_i}(z)$

on the larger sector:

$$\left\{ z \in \widetilde{\mathbb{C}} \middle| \arg(z) \in \left] d_l - \frac{\pi}{2k_i}, d_{l+1} + \frac{\pi}{2k_i} \right[\right\},$$

where d_l, d_{l+1} are two singular directions and such that $]d_l, d_{l+1}[$ contains no singular directions. Therefore, we can construct an analytic solution on the sector $\left\{ z \in \widetilde{\mathbb{C}} \middle| \arg(z) \in \left] d_l - \frac{\pi}{2k_r}, d_{l+1} + \frac{\pi}{2k_r} \right[\right\}$. Let $d \in \mathbb{R}$, and let:

$$d - \frac{\pi}{2k_r} < d^- < d < d^+ < d + \frac{\pi}{2k_r},$$

such that there are no singular directions in $[d^-, d[\bigcup]d, d^+]$. We get two matrices $H^{d^+}(z)e^{L\log(z)}e^{Q(z)}$ and $H^{d^-}(z)e^{L\log(z)}e^{Q(z)}$ which are germs of analytic solutions on the sectors

$$\left\{ z \in \widetilde{\mathbb{C}} \middle| \arg(z) \in \left] d^- - \frac{\pi}{2k_r}, d + \frac{\pi}{2k_r} \right[\right\} \text{ and } \left\{ z \in \widetilde{\mathbb{C}} \middle| \arg(z) \in \left] d - \frac{\pi}{2k_r}, d^+ + \frac{\pi}{2k_r} \right[\right\}.$$

The two matrices are in particular germs of solutions of $\partial_z Y(z) = A(z)Y(z)$ on the sector

$$\left\{ z \in \widetilde{\mathbb{C}} \middle| \arg(z) \in \left] d - \frac{\pi}{2k_r}, d + \frac{\pi}{2k_r} \right[\right\}.$$

A computation shows that there exists a matrix $St_d \in \mathrm{GL}_m(\mathbb{C})$, which we call the Stokes matrix in the direction d, such that:

$$H^{d^+}(z)e^{L\log(z)}e^{Q(z)} = H^{d^-}(z)e^{L\log(z)}e^{Q(z)}St_d.$$

Proposition 3.1.8. *The following statements are equivalent.*

1. *The entries of $\hat{H}(z)$ converges.*
2. $St_d = \mathrm{Id}$ *for all* $d \in \mathbb{R}$.
3. $St_d = \mathrm{Id}$ *for all singular directions.*

Proof. From what is preceding, we deduce that if d is not a singular direction, then $St_d = \mathrm{Id}$. Therefore, the statements 2 and 3 are equivalents. If the entries of $\hat{H}(z)$ converges, then, since $\hat{H}(z)$ is Gevrey asymptotic to itself on every sector of $\widetilde{\mathbb{C}}$, for all $d \in \mathbb{R}$, $H^d(z) = \hat{H}(z)$ and (2) holds. Assume now that $St_d = \mathrm{Id}$ for all singular directions. From the proof of [vdPS03], Theorem 8.10, we obtain that the entries of $\hat{H}(z)$ converge. □

We can compute the asymptotic solutions using the Laplace and the Borel transformations. See Chapters 2 and 3 of [Bal94] for more details.

Definition 3.1.9. (1) Let $k \in \mathbb{Q}$. The formal Borel transform $\hat{\mathcal{B}}_k$ is the map that transforms the formal power series $\sum a_n z^n$ into the formal power series:

$$\hat{\mathcal{B}}_k \left(\sum a_n z^n \right) = \sum \frac{a_n}{\Gamma(1 + \frac{n}{k})} z^n.$$

(2) Let $d\in\mathbb{R}$, $k\in\mathbb{Q}$, $\varepsilon>0$ and let f analytic on the sector $\Big\{z\in\widetilde{\mathbb{C}}\Big|\arg(z)\in]d-\varepsilon,d+\varepsilon[\Big\}$. We assume that there exists $A,B>0$ such that for $\arg(z)=d$,

$$|f(z)|\leq Ae^{B|z|^k}.$$

Then, the following integral is the germ of an analytic function on $\Big\{z\in\widetilde{\mathbb{C}}\Big|\arg(z)\in \,]d-\frac{\pi}{2k},d+\frac{\pi}{2k}[\Big\}$ (see [Bal94], page 13 for a proof), and is called the Laplace transform of order k in the direction d of f:

$$\mathcal{L}_{k,d}(f)(z)=\int_0^{\infty e^{id}} f(u)e^{-\left(\frac{u}{z}\right)^k}d\left(\left(\frac{u}{z}\right)^k\right).$$

For a proof of the following proposition, see Section 7.2 of [Bal94].

Proposition 3.1.10. *Let $k_1<\cdots<k_r$ be the levels of $\partial_z Y(z)=A(z)Y(z)$ and set $k_{r+1}=+\infty$. Suppose that $d\in\mathbb{R}$ is not a singular direction, and let $\hat{h}(z)$ be an entry of $\hat{H}(z)$. Let $(\kappa_1,\dots,\kappa_r)$ defined as:*

$$\kappa_i^{-1}=k_i^{-1}-k_{i+1}^{-1}.$$

The series $\hat{\mathcal{B}}_{\kappa_r}\circ\cdots\circ\hat{\mathcal{B}}_{\kappa_1}(\hat{h})$ converges and there exist $\varepsilon_1,A_1,B_1>0$ such that it has an analytic continuation h_1 on the sector $\Big\{z\in\widetilde{\mathbb{C}}\Big|\arg(z)\in]d-\varepsilon_1,d+\varepsilon_1[\Big\}$, and in this sector,

$$|h_1(z)|\leq A_1e^{B_1|z|^{\kappa_1}}.$$

Moreover, for $j=2$ (resp. $j=3,\dots,j=r$), there exist $\varepsilon_j,A_j,B_j>0$ such that the function $h_{j+1}=\mathcal{L}_{\kappa_j,d}(h_j)$ is analytic on the sector $\Big\{z\in\widetilde{\mathbb{C}}\Big|\arg(z)\in]d-\varepsilon_j,d+\varepsilon_j[\Big\}$ and on this sector

$$|h_j(z)|\leq A_je^{B_j|z|^{\kappa_j}}.$$

Therefore, we may apply $\mathcal{L}_{\kappa_r,d}\circ\cdots\circ\mathcal{L}_{\kappa_1,d}\circ\hat{\mathcal{B}}_{\kappa_r}\circ\cdots\circ\hat{\mathcal{B}}_{\kappa_1}$ to every entries of $\hat{H}(z)$. We have the following equality:

$$H^d(z)=\mathcal{L}_{\kappa_r,d}\circ\cdots\circ\mathcal{L}_{\kappa_1,d}\circ\hat{\mathcal{B}}_{\kappa_r}\circ\cdots\circ\hat{\mathcal{B}}_{\kappa_1}\Big(\hat{H}\Big).$$

3.1.4 Stokes phenomenon in the parameterized case.

Let us consider $\partial_z Y(z,t)=A(z,t)Y(z,t)$, with $A(z,t)\in \mathrm{M}_m(\mathcal{O}_U\Big(\{z\}\Big))$ (see page 69), where U is a non empty polydisc of $\mathbb{C}^n$, and consider $F(z,t)=\hat{H}(z,t)z^{L(t)}e\Big(Q(z,t)\Big)$, with $Q(z,t)=\mathrm{Diag}\Big(q_i(z,t)\Big)$, the fundamental solution of Proposition 3.1.3. Since for all $k\in\mathbb{N}$, $F(z,t)$ is equal to $\hat{H}(z,t)\mathrm{Diag}(z^k)z^{L(t)-k\mathrm{Id}}e\Big(Q(z,t)\Big)$, we may assume that $\hat{H}(z,t)$ has no pole at $z=0$.

We define the levels of the system $\partial_z Y(z,t)=A(z,t)Y(z,t)$ as the levels of the specialized system. The levels may depend upon t, but they are invariant on the complementary of a closed set with empty interior. We want to extend the definition of the singular directions to the parameterized case. Consider $q(z,t)=q_k(t)z^{-k/\nu}+\cdots+q_1(t)z^{-1/\nu}\in\mathbf{E}_U$. A continuous function $d:U\rightarrow\mathbb{R}$ is called singular for $q(z,t)$ if

$$\forall t\in U,\quad q_k(t)e^{-id(t)k/\nu}\in\mathbb{R}^{\geq 0}.$$

In general, if $d(t)$ is a singular direction for $q(z,t)$, the positive number $q_k(t)e^{-id(t)k/\nu}$ depends on t. The singular directions of $\partial_z Y(z,t) = A(z,t)Y(z,t)$ (we will write singular directions when no confusion is likely to arise) are the directions that are singular for one of the $q_i(z,t) - q_j(z,t)$, with $i \neq j$.

Remark 3.1.11. (1) It may happen that for some $t_0 \in U$, the singular directions of $\partial_z Y(z,t) = A(z,t)Y(z,t)$ evaluated at t_0 are not equal to the singular directions of the specialized system $\partial_z Y(z,t_0) = A(z,t_0)Y(z,t_0)$. Let us take for example $n = 1$, $U = \mathbb{C}$, $t_0 = 0$ and $A(z,t) = \mathrm{Diag}\Big(-2tz^{-3} - z^{-2}, 2tz^{-3} + z^{-2}\Big)$. The two exponentials are $e(q_1(z,t)) = e(tz^{-2} + z^{-1})$ and $e(q_2(z,t)) = e(-tz^{-2} - z^{-1})$. However, there exists $V \subset U$, a closed set with empty interior, such that for all t_0 in $U \setminus V$, the singular directions of $\partial_z Y(z,t) = A(z,t)Y(z,t)$ evaluated at t_0 are equal to the singular directions of the specialized system $\partial_z Y(z,t_0) = A(z,t_0)Y(z,t_0)$.

(2) Unfortunately, two different singular directions may be equal on a subset of U. For example, for $n = 1$, $U = \mathbb{C}^*$, and $A(z,t) = \mathrm{Diag}\Big(z^{-2}, tz^{-2}, -tz^{-2}\Big)$ we find three exponentials: $e^{-1/z}, e^{t/z}$ and $e^{-t/z}$. For $t \in \mathbb{R}^{>0}$, the singular directions of $(2t)z^{-1}$ are the same as singular directions of $(t+1)z^{-1}$.

Let $(d_i(t))_{i\in\mathbb{N}}$ be the singular directions, and

$$\mathcal{D} = \Big\{ t \in U \Big| \exists j, j' \in \mathbb{N}, \text{ such that } d_j \not\equiv d_{j'} \text{ and } d_j(t) = d_{j'}(t) \Big\}.$$

Lemma 3.1.12. *$\mathcal{D}$ is a closed subset of U with empty interior.*

Proof. Assume that there exist a non empty polydisc $D \subset \mathcal{D}$, and two singular directions $d_j(t), d_{j'}(t)$ such that $d_j(t) = d_{j'}(t)$ on D. Then, there exist a non empty polydisc $D' \subset D$, $q(t), q'(t) \in \mathcal{M}_{D'}$ that do not vanish on D' such that $q(t)/q'(t)$ has constant argument on D'. An analytic function with constant argument on a polydisc is constant. Hence, we deduce that $d_j(t) = d_{j'}(t)$ on a polydisc, which implies that $d_j(t) = d_{j'}(t)$ on U. Since the set of singular directions is finite modulo $2\pi\nu$ with $\nu \in \mathbb{N}^*$, $\mathcal{D}$ has empty interior. □

Thus, if we take a smaller non empty polydisc U, we may assume the following:

- $\mathcal{D} = \varnothing$.
- The levels of $\partial_z Y(z,t) = A(z,t)Y(z,t)$ are independent of t.
- For all $t_0 \in U$, the singular directions of $\partial_z Y(z,t) = A(z,t)Y(z,t)$ evaluated at t_0 are equal to the singular directions of the specialized system $\partial_z Y(z,t_0) = A(z,t_0)Y(z,t_0)$.

We still consider $\partial_z Y(z,t) = A(z,t)Y(z,t)$ a parameterized linear differential system with $A(z,t) \in \mathrm{M}_m\Big(\mathcal{O}_U(\{z\})\Big)$ and $\hat{H}(z,t)z^{L(t)}e\Big(Q(z,t)\Big) \in \mathrm{GL}_m\left(\widehat{\mathbf{K}_U}\right)$ the fundamental solution in the same form as in Proposition 3.1.3. Let $d(t)$ be a singular direction, and let $k_1 < \cdots < k_r$ be the levels of $\partial_z Y(z,t) = A(z,t)Y(z,t)$. For t belonging to U, we define the parameterized Stokes matrix $St_{d(t)}$ (we will just call it Stokes matrix when no confusion is likely to arise) as $t \mapsto St_{d(t)}$, where $St_{d(t)}$ is the Stokes matrix of the specialized system defined just before Proposition 3.1.8.

Proposition 3.1.13. *Let $d(t)$ continuous in t such that for all t_0 in U, $d(t_0)$ is not a singular direction of the unparameterized linear differential equation $\partial_z Y(z,t_0)=A(z,t_0)Y(z,t_0)$. We define $t\mapsto H^{d(t)}(z,t)e^{L(t)\log(z)}e^{Q(z,t)}$, as the solution (3.1.2), of the specialized system. Let $d_1(t),d_2(t)$ be two singular directions such that for all $t\in U$, $d_1(t)<d(t)<d_2(t)$ and $]d_1(t),d_2(t)[$ contains no singular directions. Then, there exists a map $U\to\mathbb{R}^{>0}$, $t\mapsto\varepsilon(t)$, which is not necessary continuous, such that $H^{d(t)}(z,t)e^{L(t)\log(z)}e^{Q(z,t)}$ is meromorphic in (z,t) for*

$$(z,t)\in\left\{z\in\widetilde{\mathbb{C}}\Big|\arg(z)\in\left]d_1(t)-\frac{\pi}{2k_r},d_2(t)+\frac{\pi}{2k_r}\right[,\ \text{and } 0<|z|<\varepsilon(t)\right\}\times U.$$

Notice that the existence of $d(t)$ continuous in t such that for all t_0 in U, $d(t_0)$ is not a singular direction of the unparameterized linear differential equation $\partial_z Y(z,t_0)=A(z,t_0)Y(z,t_0)$ is a direct consequence of the fact that $\mathcal{D}=\varnothing$, and the fact that the singular directions are continuous in t.

Proof. We recall that we have assumed that for all $t_0\in U$, the singular directions of $\partial_z Y(z,t)=A(z,t)Y(z,t)$ evaluated at t_0 are equal to the singular directions of the specialized system $\partial_z Y(z,t_0)=A(z,t_0)Y(z,t_0)$. We have seen in §3.1.3, that for t fixed, the asymptotic solution is a germ of meromorphic function on the sector

$$\left\{z\in\widetilde{\mathbb{C}}\Big|\arg(z)\in\left]d_1(t)-\frac{\pi}{2k_r},d_2(t)+\frac{\pi}{2k_r}\right[\right\}.$$

We may replace $d(t)$ by any function, possibly non continuous, such that for all $t\in U$, $d_1(t)<d(t)<d_2(t)$. Since the singular directions are continuous in t, we may assume that $d(t)$ is locally constant. Since for $z\neq 0$, $t\mapsto e^{L(t)\log(z)}e^{Q(z,t)}\in\mathcal{M}_U$, this is now a consequence of Proposition 3.1.10 and Lemma 3.1.14 below. □

Lemma 3.1.14. *We keep the same notations as in Definition 3.1.9 and Proposition 3.1.10. Let $\hat{h}(z,t)$ be one of the entries of $\hat{H}(z,t)$. Let $V\subset U$ be a non empty polydisc, and let $d\in\mathbb{R}$ such that for all $t\in V$, d is not an unparameterized singular direction of $\partial_z Y(z,t)=A(z,t)Y(z,t)$. Then, there exists a map $U\to\mathbb{R}^{>0}$, $t\mapsto\varepsilon(t)$, which is not necessary continuous such that*

$$\mathcal{L}_{\kappa_r,d}\circ\cdots\circ\mathcal{L}_{\kappa_1,d}\circ\hat{\mathcal{B}}_{\kappa_r}\circ\cdots\circ\hat{\mathcal{B}}_{\kappa_1}\left(\hat{h}\right)$$

is meromorphic in (z,t) on

$$(z,t)\in\left\{z\in\widetilde{\mathbb{C}}\Big|\arg(z)\in\left]d-\frac{\pi}{2k_r},d+\frac{\pi}{2k_r}\right[,\ \text{and } 0<|z|<\varepsilon(t)\right\}\times V.$$

Moreover, for all $j\leq n$:

$$\mathcal{L}_{\kappa_r,d}\circ\cdots\circ\mathcal{L}_{\kappa_1,d}\circ\hat{\mathcal{B}}_{\kappa_r}\circ\cdots\circ\hat{\mathcal{B}}_{\kappa_1}\left(\partial_{t_j}\hat{h}\right)=\partial_{t_j}\left(\mathcal{L}_{\kappa_r,d}\circ\cdots\circ\mathcal{L}_{\kappa_1,d}\circ\hat{\mathcal{B}}_{\kappa_r}\circ\cdots\circ\hat{\mathcal{B}}_{\kappa_1}\left(\hat{h}\right)\right).$$

Proof. We will proceed in two steps.

(1) We recall that $\hat{h}(z,t)\in\hat{K}_U\left[z^{1/\nu}\right]$ ($\nu\in\mathbb{N}^*$ has been defined in Proposition 3.1.3) and (see Remark 3.1.7) all the z-coefficients are analytic on U. Because of Proposition 3.1.10, for t fixed, $\hat{\mathcal{B}}_{\kappa_r}\circ\cdots\circ\hat{\mathcal{B}}_{\kappa_1}\left(\hat{h}\right)$, is a germ of meromorphic function. Therefore, it belongs to $\mathcal{O}_U(\{z\})\left[z^{1/\nu}\right]$. Let h_1 be the analytic continuation defined in Proposition 3.1.10. In

particular, for all $z \in \widetilde{\mathbb{C}}$ with $\arg(z) = d$, $t \mapsto h_1(z,t) \in \mathcal{M}_V$. The fact that we have a meromorphic function allows us to differentiate termwise and for all $j \leq n$, $\partial_{t_j} h_1$ is equal to the analytic continuation of:

$$\hat{\mathcal{B}}_{\kappa_r} \circ \cdots \circ \hat{\mathcal{B}}_{\kappa_1}\Big(\partial_{t_j} \hat{h}\Big).$$

(2) Let $h_2, \dots, h_r$ be the successive Laplace transforms defined in Proposition 3.1.10. Let $t_0 \in V$, let W_{t_0} be a compact neighborhood of t_0 in V, let $i \leq r$, and assume that for $z \in \widetilde{\mathbb{C}}$ with $\arg(z) = d$, $t \mapsto h_i(z,t)$ is meromorphic on W_{t_0}. It is sufficient to prove that for all $z \in \widetilde{\mathbb{C}}$ with $\arg(z) \in \Big]d - \frac{\pi}{2\kappa_i}, d + \frac{\pi}{2\kappa_i}\Big[$ and $|z|$ sufficiently small, $t \mapsto h_{i+1}(z,t)$ is meromorphic on W_{t_0} and for all $j \leq n$:

$$\mathcal{L}_{\kappa_i,d}\Big(\partial_{t_j} h_i\Big) = \partial_{t_j}\Big(\mathcal{L}_{\kappa_i,d}(h_i)\Big) = \partial_{t_j} h_{i+1}.$$

The function $\mathcal{L}_{\kappa_i,d}(h_i)$ is an integral of a meromorphic function depending analytically upon parameters, and we just have to prove that it is possible to find a function f such that, for all $t \in W_{t_0}$, $|h_i(u,t)| < |f(u)|$ and for $\arg(z) \in \Big]d - \frac{\pi}{2\kappa_i}, d + \frac{\pi}{2\kappa_i}\Big[$, $|z|$ sufficiently small, $\mathcal{L}_{\kappa_i,d}(|f|)(z) < \infty$. From Proposition 3.1.10, we obtain the existence of $A(t), B(t) > 0$ such that for $\arg(u) = d$,

$$|h_i(u,t)| \leq A(t) e^{B(t)|u|^{\kappa_i}}.$$

Since $h_i(u,t)$ is meromorphic, we may assume that $A(t)$ and $B(t)$ are continuous on W_{t_0}. The functions $A(t)$ and $B(t)$ admit a maximum A and B on the compact set W_{t_0}. Finally for $\arg(z) \in \Big]d - \frac{\pi}{2\kappa_i}, d + \frac{\pi}{2\kappa_i}\Big[$ and $|z|$ sufficiently small,

$$\begin{aligned} |\mathcal{L}_{\kappa_i,d} h_i| &= \left| \int_0^{\infty e^{id}} h_i(u,t) e^{-\left(\frac{u}{z}\right)^{\kappa_i}} d\left(\left(\frac{u}{z}\right)^{\kappa_i}\right) \right| \\ &\leq \int_0^{\infty} A e^{B|u|^{\kappa_i}} \left| e^{-\left(\frac{u}{z}\right)^{\kappa_i}} \right| d\left(\left(\frac{u}{z}\right)^{\kappa_i}\right) \\ &< \infty. \end{aligned}$$

□

3.2 Parameterized differential Galois theory

In this section we are interested in the parameterized differential Galois theory: this is a generalization of the differential Galois theory for parameterized linear differential equations. In §3.2.1, we review the parameterized differential Galois theory developed in [CS07]. In §3.2.2, we prove that some of the results of §3.2.1 stay valid without the assumption that the field of constants is differentially closed. This will help us in § 3.2.3 to prove that the local analytic parameterized differential Galois group descends to a smaller field, whose field of constants is not differentially closed. In §3.2.4, we explain the main result of the chapter: we show an analogue of the density theorem of Ramis in the parameterized case. In §3.2.5, we give a similar result for the global parameterized differential Galois group. We end by giving various examples of computation of parameterized differential Galois groups using the parameterized density theorem.

3.2.1 Basic facts.

We recall some facts from [CS07] about Galois theory of parameterized linear differential equations. Classical Galois theory of unparameterized linear differential equation is presented in some books such as [vdPS03] and [Mag94].

Let K be a differential field of characteristic 0 with $n+1$ commuting derivations: $\partial_0, \dots, \partial_n$. We want to study differential equations of the form $\partial_0 Y = AY$, with $A \in \mathrm{M}_m(K)$. Let C_K be the field of constants with respect to ∂_0. Since all the derivations commute with ∂_0, $(C_K, \partial_1, \dots, \partial_n)$ is a differential field. By abuse, we will sometimes start from a $(\partial_1, \dots, \partial_n)$-differential field C_K and build a $(\partial_0, \dots, \partial_n)$-differential field extension K of C_K, such that C_K is the field of constants with respect to ∂_0.

Example 3.2.1. If $K = \hat{K}_U$, then $\partial_0 = \partial_z$, $\{\partial_1, \dots, \partial_n\} = \Delta_t$, and $C_K = \mathcal{M}_U$.

A parameterized Picard-Vessiot extension for the parameterized linear differential equation $\partial_0 Y = AY$ on K is a $(\partial_0, \dots, \partial_n)$-differential field extension $\widetilde{K}\big|K$ with the following properties:

- There exists a fundamental solution for $\partial_0 Y = AY$ in $\widetilde{K}$, i.e., an invertible matrix $U = (u_{i,j})$, with entries in $\widetilde{K}$, such that $\partial_0 U = AU$.
- $\widetilde{K} = K\langle u_{i,j}\rangle_{\partial_0,\dots,\partial_n}$, i.e., $\widetilde{K}$ is the $(\partial_0, \dots, \partial_n)$-differential field generated by K and the $u_{i,j}$.
- The field of constants of $\widetilde{K}$ with respect to ∂_0 is C_K.

Let L be a $(\partial_1, \dots, \partial_n)$-field of characteristic 0 with commuting derivations. The $(\partial_1, \dots, \partial_n)$-differential ring $L\{y_1, \dots, y_k\}_{\partial_1,\dots,\partial_n}$ of differential polynomials in k indeterminates over L is the usual polynomial ring in the infinite set of variables

$$\{\partial_1^{\nu_1} \dots \partial_n^{\nu_n} y_j\}_{j \leq k}^{\nu_i \in \mathbb{N}},$$

and with derivations extending those in $\{\partial_1, \dots, \partial_n\}$ on L, defined by:

$$\partial_i \left(\partial_1^{\nu_1} \dots \partial_n^{\nu_n} y_j\right) = \partial_1^{\nu_1} \dots \partial_i^{\nu_i+1} \dots \partial_n^{\nu_n} y_j.$$

Definition 3.2.2 ([CS07], Definition 3.2)**.** We say that $(C_K, \partial_1, \dots, \partial_n)$ is differentially closed if it has the following property: For any $k, l \in \mathbb{N}$ and for all $P_1, \dots, P_k \in C_K\{y_1, \dots, y_l\}_{\partial_1, \dots, \partial_n}$, the system

$$\begin{cases} P_1(\alpha_1, \dots, \alpha_l) & = & 0 \\ & \vdots & \\ P_{k-1}(\alpha_1, \dots, \alpha_l) & = & 0 \\ P_k(\alpha_1, \dots, \alpha_l) & \neq & 0, \end{cases}$$

has a solution in C_K as soon as it has a solution in a $(\partial_1, \dots, \partial_n)$-differential field containing C_K.

For the simplicity of the notations, we will say that C_K differentially closed rather than $(C_K, \partial_1, \dots, \partial_n)$ is differentially closed. Note that there exists a differentially closed extension of C_K, see [CS07], Section 9.1. By definition, a differentially closed field is algebraically closed.

Proposition 3.2.3 ([CS07], Theorem 9.5)**.** *Assume that C_K is differentially closed. Then, we have existence of the parameterized Picard-Vessiot extension for $\partial_0 Y = AY$. We have also the uniqueness of the parameterized Picard-Vessiot extension for $\partial_0 Y = AY$, up to $(\partial_0, \dots, \partial_n)$-differential isomorphism.*

Until the end of the subsection 3.2.1, we assume that C_K is differentially closed.

Let us consider $\partial_0 Y = AY$, with $A \in \mathrm{M}_m(K)$ and let $\widetilde{K}\big|K$ be a parameterized Picard-Vessiot extension. The parameterized differential Galois group $Gal_{\partial_0}^{\partial_1, \dots, \partial_n}\left(\widetilde{K}\big|K\right)$ is the group of field automorphisms of $\widetilde{K}$ which induce the identity on K and commute with all the derivations. This latter is independent of the choice of the parameterized Picard-Vessiot extension, since all the parameterized Picard-Vessiot extensions are $(\partial_0, \dots, \partial_n)$-differentially isomorphic. In the unparameterized case, the differential Galois group is an algebraic subgroup of $\mathrm{GL}_m(C_K)$. In the parameterized case, we find a linear differential algebraic subgroup:

Definition 3.2.4. Let us consider m^2 indeterminates $(X_{i,j})_{i,j \leq m}$. We say that a subgroup G of $\mathrm{GL}_m(C_K)$ is a linear differential algebraic group if there exist $P_1, \dots, P_k \in C_K\{X_{i,j}\}_{\partial_1, \dots, \partial_n}$ such that for $A = (a_{i,j}) \in \mathrm{GL}_m(C_K)$,

$$A \in G \Longleftrightarrow P_1(a_{i,j}) = \dots = P_k(a_{i,j}) = 0.$$

Let U be a fundamental solution of $\partial_0 Y = AY$. One proves directly that the map:

$$\begin{array}{rccc} \rho_U : & Gal_{\partial_0}^{\partial_1, \dots, \partial_n}\left(\widetilde{K}\big|K\right) & \longrightarrow & \mathrm{GL}_m(C_K) \\ & \varphi & \longmapsto & U^{-1}\varphi(U), \end{array}$$

is an injective group morphism. A fundamental fact is that

$$\mathrm{Im}\ \rho_U = \left\{U^{-1}\varphi(U), \varphi \in Gal_{\partial_0}^{\partial_1, \dots, \partial_n}\left(\widetilde{K}\big|K\right)\right\}$$

is a linear differential algebraic subgroup of $\mathrm{GL}_m(C_K)$ (see Theorem 9.5 in [CS07]). If we take a different fundamental solution in $\widetilde{K}$, we obtain a conjugate linear differential algebraic subgroup of $\mathrm{GL}_m(C_K)$. We will identify $Gal_{\partial_0}^{\partial_1, \dots, \partial_n}\left(\widetilde{K}\big|K\right)$ with a linear differential algebraic subgroup of $\mathrm{GL}_m(C_K)$ for a chosen fundamental solution. We put a topology on $\mathrm{GL}_m(C_K)$, called Kolchin topology, for which the closed sets are defined as the zero loci of finite sets of differential polynomials with coefficients in C_K.

Example 3.2.5. (Example 3.1 in [CS07]) Let $n = 1$, let (C_K, ∂_t) be a differentially closed ∂_t-field that contains $(\mathbb{C}(t), \partial_t)$, and let us consider $K = C_K(z)$, the (∂_z, ∂_t)-differential field of rational functions in the indeterminate z, with coefficients in C_K, where z is a ∂_t-constant with $\partial_z z = 1$, C_K is the field of constants with respect to ∂_z, and ∂_z commutes with ∂_t. Let us consider the parameterized differential equation

$$\partial_z Y(z,t) = \frac{t}{z} Y(z,t).$$

The fundamental solution is (z^t) and $K(z^t, \log)$ is a Parameterized Picard-Vessiot extension (see §3.1.1 for the notations). Here, we have added log because we want the extension to be closed under the derivations ∂_z and ∂_t. Using the fact the Galois group commutes with ∂_z and ∂_t, we find that the Galois group is given by:

$$\left\{ f \in C_K \middle| f \neq 0 \text{ and } f\partial_t^2 f - (\partial_t f)^2 = 0 \right\}.$$

We can see that if we take $C_K = \mathbb{C}(t)$ or $C_K = \mathcal{M}_{\mathbb{C}}$ (see page 68), which are not differentially closed, then we can find two different groups of differential automorphisms:

$$\left\{ f \in \mathbb{C}(t) \middle| f \neq 0 \text{ and } f\partial_t^2 f - (\partial_t f)^2 = 0 \right\} = \mathbb{C}^*$$

and

$$\left\{ f \in \mathcal{M}_{\mathbb{C}} \middle| f \neq 0 \text{ and } f\partial_t^2 f - (\partial_t f)^2 = 0 \right\} = \left\{ ce^{bt}, b \in \mathbb{C}, c \in \mathbb{C}^* \right\},$$

which shows the importance of considering a Galois group defined over a differentially closed field. See Example 3.2.26 for the resolution of this ambiguity using the parameterized density theorem.

There is a Galois correspondence theorem for parameterized differential Galois theory, see Theorem 9.5 in [CS07]. For G subgroup of $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$, let:

$$\widetilde{K}^G = \left\{ a \in \widetilde{K} \middle| \sigma(a) = a,\ \forall \sigma \in G \right\}.$$

Then, the theorem says that the Kolchin closed subgroups of $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$ are in bijection with the $(\partial_0, \dots, \partial_n)$-differential subfields of $\widetilde{K}$ containing K, via the map:

$$G \mapsto \widetilde{K}^G.$$

The inverse map is given by:

$$M \mapsto Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|M\right),$$

where $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|M\right)$ denotes the set of elements of $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$ inducing identity on M. In particular, we have the following corollary:

Corollary 3.2.6. *Let G be an arbitrary subgroup of $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$. Then, $\widetilde{K}^G = K$ if and only if G is dense for Kolchin topology in $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$.*

Let $L|M|K$ be $(\partial_1,\dots,\partial_n)$-differential field extensions. Notice that we do not exclude $L = M = K$. All the definitions we are going to give before the next Proposition come from [HS08], § 6.2.3.

Given $a_1,\dots,a_k \in L$ and $P \in M\{X_1,\dots,X_k\}_{\partial_1,\dots,\partial_n}$, we remark that $P(a_1,\dots,a_n)$ is well defined. Then, we may define the $(\partial_1,\dots,\partial_n)$-differential transcendence degree of L over M as the maximum number of elements $a_1,\dots,a_k$ of L such that:

$$P(a_1,\dots,a_k) \neq 0,$$

for all non-zero $(\partial_1,\dots,\partial_n)$-differential polynomials P with coefficients in M. The $(\partial_1,\dots,\partial_n)$-differential transcendence degree of an integral domain over another integral domain is defined to be the $(\partial_1,\dots,\partial_n)$-differential transcendence degree of the fraction field of the first one over the fraction field of the second one.

Let us consider m^2 indeterminates $(X_{i,j})_{i,j\leq m}$. Let (p) be a prime $(\partial_1,\dots,\partial_n)$-differential ideal of $C_K\{X_{i,j}\}_{\partial_1,\dots,\partial_n}$, i.e., a prime ideal stable under the derivations $\partial_1,\dots,\partial_n$. The $(\partial_1,\dots,\partial_n)$-dimension of (p) over C_K is defined to be the $(\partial_1,\dots,\partial_n)$-differential transcendence degree of the quotient ring $C_K\{X_{i,j}\}_{\partial_1,\dots,\partial_n}/(p)$ over C_K.

Let (r) be a radical $(\partial_1,\dots,\partial_n)$-differential ideal of $C_K\{X_{i,j}\}_{\partial_1,\dots,\partial_n}$, i.e., a radical ideal stable under the derivations $\partial_1,\dots,\partial_n$. Let $(p_1),\dots,(p_\nu)$ with $\nu \in \mathbb{N}^*$ be the prime $(\partial_1,\dots,\partial_n)$-differential ideals such that $(r) = \bigcap_{k\leq\nu}(p_k)$. The $(\partial_1,\dots,\partial_n)$-dimension of (r) over C_K is defined to be the maximum in k of the $(\partial_1,\dots,\partial_n)$-dimension of (p_k) over C_K.

Assume that $M \subset \widetilde{K}$. Let (q) be the radical $(\partial_1,\dots,\partial_n)$-differential ideal of $C_K\{X_{i,j}\}_{\partial_1,\dots,\partial_n}$ that defines $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|M\right)$ (see the proof of Proposition 9.10 in [CS07]). We define the $(\partial_1,\dots,\partial_n)$-differential dimension of $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|M\right)$ over C_K as the $(\partial_1,\dots,\partial_n)$-dimension of (q) over C_K.

Proposition 3.2.7 ([HS08], Proposition 6.26)**.** *The $(\partial_1,\dots,\partial_n)$-differential transcendence degree of $\widetilde{K}$ over M is equal to the $(\partial_1,\dots,\partial_n)$-differential dimension of $Gal_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|M\right)$ over C_K.*

Example. 3.2.5 (bis). Let us keep the same notations as in Example 3.2.5. The parameterized Picard-Vessiot extension is $K(z^t,\log)$ and the Galois group is: $\left\{f \in C_K \middle| f \neq 0 \text{ and } f\partial_t^2 f - (\partial_t f)^2 = 0\right\}$. We may directly check that the ∂_t-differential dimension of the Galois group is 0 and therefore, z^t satisfies a ∂_t-differential polynomial equation in coefficients in C_K.

3.2.2 Parameterized differential Galois theory for a non-differentially closed field of constants.

Let K be a differential field of characteristic 0 with $n+1$ commuting derivations: $\partial_0,\dots,\partial_n$. Let C_K be the field of constants with respect to ∂_0. Note that we do not assume C_K to be differentially closed. Consider $\partial_0 Y = AY$, with $A \in \mathrm{M}_m(K)$, and assume the existence of $\widetilde{K}\middle|K$, a parameterized Picard-Vessiot extension for $\partial_0 Y = AY$

(see § 3.2.1). This means in particular that the field of constants of $\widetilde{K}$ with respect to ∂_0 is C_K. Let $F = (F_{i,j}) \in \mathrm{GL}_m\left(\widetilde{K}\right)$ be a fundamental solution such that $\widetilde{K} = K\langle F_{i,j}\rangle_{\partial_0,\ldots,\partial_n}$ (see §3.2.1 for the notation). Let $Aut_{\partial_0}^{\partial_1,\ldots,\partial_n}\left(\widetilde{K}\middle|K\right)$ be the group of $(\partial_0,\ldots,\partial_n)$-differential field automorphisms of $\widetilde{K}$ letting K invariant

Remark 3.2.8. We avoid here the notation $Gal_{\partial_0}^{\partial_1,\ldots,\partial_n}\left(\widetilde{K}\middle|K\right)$, because we have no theorem that guarantees the uniqueness of the parameterized Picard-Vessiot extension $\widetilde{K}\middle|K$, since C_K is not differentially closed. However we will call it the parameterized differential Galois group, or Galois group, if no confusion is likely to arise.

We extend Definition 3.2.4 for the field C_K. Let us consider m^2 indeterminates $(X_{i,j})_{i,j\leq m}$. We say that a subgroup G of $\mathrm{GL}_m(C_K)$ is a linear differential algebraic group if there exist $P_1,\ldots,P_k \in C_K\{X_{i,j}\}_{\partial_1,\ldots,\partial_n}$ such that for $A=(a_{i,j}) \in \mathrm{GL}_m(C_K)$,

$$A \in G \Longleftrightarrow P_1(a_{i,j}) = \cdots = P_k(a_{i,j}) = 0.$$

The goal of the subsection is to prove:

Proposition 3.2.9. *(1) Let us consider the injective group morphism:*

$$\begin{array}{rccc} \rho_F: & Aut_{\partial_0}^{\partial_1,\ldots,\partial_n}\left(\widetilde{K}\middle|K\right) & \longrightarrow & \mathrm{GL}_m(C_K) \\ & \varphi & \longmapsto & F^{-1}\varphi(F). \end{array}$$

Then,

$$Im\ \rho_F = \left\{F^{-1}\varphi(F), \varphi \in Aut_{\partial_0}^{\partial_1,\ldots,\partial_n}\left(\widetilde{K}\middle|K\right)\right\}$$

is a linear differential algebraic subgroup of $\mathrm{GL}_m(C_K)$*. We will identify* $Aut_{\partial_0}^{\partial_1,\ldots,\partial_n}\left(\widetilde{K}\middle|K\right)$ *with a linear differential algebraic subgroup of* $\mathrm{GL}_m(C_K)$ *for a chosen fundamental solution. The image is independent of this choice, up to conjugacy by an element of* $\mathrm{GL}_m(C_K)$*.*

(2) Let G be a subgroup of $Aut_{\partial_0}^{\partial_1,\ldots,\partial_n}\left(\widetilde{K}\middle|K\right)$*. If* $\widetilde{K}^G = K$*, then G is dense for Kolchin topology in* $Aut_{\partial_0}^{\partial_1,\ldots,\partial_n}\left(\widetilde{K}\middle|K\right)$*.*

Remark that, contrary to Corollary 3.2.6, the converse of (2) is false when C_K is not differentially closed. See [CS07], Example 3.1. Before showing the proposition, we point out two facts we will use in the proof. Let $L|K$ be a $(\partial_0,\ldots,\partial_n)$-differential field extension and $a_1,\ldots,a_k \in L$.

- As in the case where C_K is differentially closed (see §3.2.1), if $P \in K\{X_1,\ldots,X_k\}_{\partial_1,\ldots,\partial_n}$, then $P(a_1,\ldots,a_k)$ is well defined.
- The set $\{P(a_1,\ldots,a_k) | P \in K\{X_1,\ldots,X_k\}_{\partial_1,\ldots,\partial_n}\}$ is a $(\partial_0,\ldots,\partial_n)$-differential field extension we will denote by $L\{a_1,\ldots,a_k\}_{\partial_1,\ldots,\partial_n}|L$.

Proof of Proposition 3.2.9. (1) We follow here the proof of Proposition 9.10 in [CS07]. We consider the differential polynomial ring:

$$R = K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\ldots,\partial_n},$$

and endow it with the ∂_0-differential structure defined by $\partial_0(X_{i,j}) = A(X_{i,j})$. Let us consider:

$$S = K\{F_{i,j}, 1/\det(F_{i,j})\}_{\partial_0,\ldots,\partial_n},$$

the $(\partial_0, \dots, \partial_n)$-differential subring of $\widetilde{K}$ generated over K by the $F_{i,j}$ and $1/\det(F_{i,j})$. It is an integral domain. Let q be the obvious prime $(\partial_0, \dots, \partial_n)$-differential ideal such that $R/q \simeq S$. Let $Z_{i,j}$ be the image of $X_{i,j}$ in $S \subset \widetilde{K}$, so that $(Z_{i,j})$ is a fundamental solution for $\partial_0 Y = AY$ in S. Consider the following rings:

$$\begin{array}{ccc} \widetilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\dots,\partial_n} & = & \widetilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\dots,\partial_n} \\ \cup & & \cup \\ K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\dots,\partial_n} & & C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\dots,\partial_n}, \end{array}$$

where the indeterminates $Y_{i,j}$ are defined by $(X_{i,j}) = (Z_{i,j})(Y_{i,j})$. We remark that $\partial_0(Y_{i,j}) = 0$. Since we consider fields that are of characteristic 0, the differential ideal:

$$q\widetilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\dots,\partial_n} \subset \widetilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\dots,\partial_n} = \widetilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\dots,\partial_n},$$

is a radical $(\partial_0, \dots, \partial_n)$-differential ideal (see Corollary A.17 in [vdPS03]). The next lemma is an adaptation of Lemma 9.8 in [CS07] without the assumption that the field of constants is differentially closed.

Lemma 3.2.10. *The $(\partial_0, \dots, \partial_n)$-ideal $q\widetilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\dots,\partial_n}$ is generated by:*

$$I = q\widetilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\dots,\partial_n} \cap C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\dots,\partial_n}.$$

Proof. Let $(e_i)_{i\in B}$ be a basis of $C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\dots,\partial_n}$ over C_K. Let:

$$f = \sum_{i=1}^{n} m_i e_i \in q\widetilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\dots,\partial_n},$$

with $m_i \in \widetilde{K}$. By induction on n we will show that $f \in I$. If $n = 0$ or 1 there is nothing to prove. We assume that $n > 1$. We can suppose that $m_1 = 1$ and $m_2 \notin C_K$. Then, because of the fact that the field of constants of $\widetilde{K}$ with respect to ∂_z is C_K:

$$\partial_0(f) = \sum_{i=2}^{n} \partial_0(m_i) e_i \neq 0 \text{ and } f \in q\widetilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\dots,\partial_n}.$$

Then, by induction, $\partial_0(f) \in I$. With the same argument:

$$\partial_0(m_2^{-1} f) \in I.$$

Then, $\partial_0(m_2^{-1})f = \partial_0(m_2^{-1}f) - m_2^{-1}\partial_0 f \in I$. Since $\partial_0(m_2^{-1}) \neq 0$, we obtain that $f \in I$. □

By Lemma 3.2.10, $q\widetilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\dots,\partial_n}$ is generated by:

$$I = q\widetilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\dots,\partial_n} \cap C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\dots,\partial_n}.$$

Clearly I is a $(\partial_1, \dots, \partial_n)$-radical ideal of $C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\dots,\partial_n}$. Let $C = (C_{i,j}) \in \mathrm{GL}_m(C_K)$. The following statements are equivalent:

1. $(C_{i,j}) \in Aut_{\partial_0}^{\partial_1,\dots,\partial_n}\left(\widetilde{K}\middle|K\right)$.
2. The map $K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\dots,\partial_n} \to K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\dots,\partial_n}$ defined by $(X_{i,j}) \mapsto (X_{i,j})(C_{i,j}) := (\sum_{k=1}^{m} X_{i,k}C_{k,j})$ leaves q invariant.

3. The map $K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\ldots,\partial_n} \to \widetilde{K}$ defined by $(X_{i,j}) \mapsto (Z_{i,j})(C_{i,j})$ sends q to 0.
4. The map $\widetilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\ldots,\partial_n} \to \widetilde{K}$ defined by $(X_{i,j}) \mapsto (Z_{i,j})(C_{i,j})$ sends

$$q\widetilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\ldots,\partial_n} = q\widetilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\ldots,\partial_n} \text{ to } 0.$$

5. The map $\widetilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\ldots,\partial_n} \to \widetilde{K}$ defined by $(Y_{i,j}) \mapsto (C_{i,j})$ sends

$$q\widetilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\ldots,\partial_n} \text{ to } 0.$$

The theorem is now a consequence of the fact that $q\widetilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\ldots,\partial_n}$ is generated by I, a $(\partial_1,\ldots,\partial_n)$-radical ideal of $C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\ldots,\partial_n}$.

(2) We follow the proof of Proposition 9.10 in [CS07], and use the same notations as before. By construction, the ideal I of Lemma 3.2.10 above is the differential ideal that defines the Galois group. Assume that the Kolchin closure of G is not the whole Galois group. Then, there exists $P \in C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\ldots,\partial_n}$ such that $P \notin I$ and $P(g) = 0$ for all $g \in G$. Lemma 3.2.10 implies that

$$P \notin J = q\widetilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1,\ldots,\partial_n}.$$

Let

$$T = \Big\{Q \in \widetilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\ldots,\partial_n} \Big| Q \notin J \text{ and } Q\Big((Z_{i,j})(g_{i,j})\Big) = 0, \forall g = (g_{i,j}) \in G\Big\}.$$

Since $P \in T$, $T \neq \{0\}$. An element $Q \in T$ can be written as:

$$Q = f_1 Q_1 + \cdots + f_\nu Q_\nu,$$

where $f_i \in \widetilde{K}$ and $Q_i \in K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\ldots,\partial_n}$. Let $Q = f_1 Q_1 + \cdots + f_\nu Q_\nu \in T$ such that:

- $f_1 = 1$.
- All the f_i are non-zero.
- ν is minimal.

For all $g \in G$, let $Q^g = f_1^g Q_1 + \cdots + f_\nu^g Q_\nu \in T$. Let $g \in G$. Since $Q - Q^g$ is shorter than Q, and satisfies $(Q - Q^g)((Z_{i,j})(g_{i,j})) = 0$, we have $Q - Q^g \in J$. If $Q - Q^g \neq 0$, there exists $l \in \widetilde{K}$ such that $Q - l(Q - Q^g)$ is shorter than Q. Since $Q - l(Q - Q^g) \in T$, this is not possible unless $Q - Q^g = 0$. Therefore, $Q = Q^g$, for all $g \in G$, and so $Q \in K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1,\ldots,\partial_n}$. Since $Q(Z_{i,j}) = 0$, we have $Q \in J$. This yields the result.

□

3.2.3 A result of descent for the local analytic parameterized differential Galois group.

We keep the notations of Section 3.1. Let us consider $\partial_z Y(z,t) = A(z,t)Y(z,t)$, with $A(z,t) \in \mathrm{M}_m\Big(\mathcal{O}_U(\{z\})\Big)$, where U is a non empty polydisc of $\mathbb{C}^n$ and $\mathcal{O}_U(\{z\})$ has been defined in Page 69.

Remark 3.2.11. Note that $\mathcal{O}_U(\{z\})$ is a ring but not a field in general. For example, if $n=1$, $(z-t)^{-1}\notin\mathcal{O}_U(\{z\})$. However, we have $(z-t)^{-1}\in\mathcal{O}_{\mathbb{C}^*}(\{z\})$. More generally let $\alpha(z,t)\in\mathcal{O}_U(\{z\})$. For $t\in U$, let $R(t)$ minimal such that $|\alpha(z,t)|\neq 0$ for $0<|z|<R(t)$. There exists a non empty polydisc U' such that there exists $\varepsilon>0$ with $R(t)>\varepsilon$ on U'. In particular, we have $\alpha(z,t)^{-1}\in\mathcal{O}_{U'}(\{z\})$.

Since $\mathcal{O}_U(\{z\})\subset\hat{K}_U$, which is a field, $\mathcal{O}_U(\{z\})$ is an integral domain, and we can define K_U as the fraction field of $\mathcal{O}_U(\{z\})$. We have

$$\{a\in K_U|\partial_z a=0\}=\{a\in\hat{K}_U|\partial_z a=0\}=\mathcal{M}_U.$$

Let:

$$F(z,t)=(F_{i,j})=\hat{H}(z,t)z^{L(t)}e\Big(Q(z,t)\Big)\in\mathrm{GL}_m\Big(\widehat{\mathbf{K}_U}\Big), \text{(see §3.1.1)}$$

be the fundamental solution given in Proposition 3.1.3. Let us note $K_U\langle F_{i,j}\rangle_{\partial_z,\Delta_t}=\widetilde{K_U}$, which is a (∂_z,Δ_t)-differential subfield of $\widehat{\mathbf{K}_U}$. We have seen in § 3.1.1, that $\widehat{\mathbf{K}_U}$ has field of constants with respect to ∂_z equal to $\mathcal{M}_U$. Then, we deduce that $\widetilde{K_U}\Big|K_U$ is a parameterized Picard-Vessiot extension. Therefore, the results of §3.2.2 may be applied here; and we can define a parameterized differential Galois group $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\Big|K_U\right)$, which will be identified with a linear differential algebraic subgroup of $\mathrm{GL}_m(\mathcal{M}_U)$. We want to prove now that it is the "same" as the one of §3.2.1.

Let C be a (Δ_t)-differentially closed field that contains $\mathcal{M}_U$. Let us define $C[[z]][z^{-1}]$, the (∂_z,Δ_t)-differential field, where z is a (Δ_t)-constant with $\partial_z z=1$, C is the field of constants with respect to ∂_z, and ∂_z commutes with all the derivations. We define the ring $K_U\otimes_{\mathcal{M}_U}C$ with the differential structure given by:

$$\forall a\in K_U,\forall c\in C,\forall\partial\in\{\partial_z,\Delta_t\},\quad \partial(a\otimes_{\mathcal{M}_U}c)=\partial a\otimes_{\mathcal{M}_U}c+a\otimes_{\mathcal{M}_U}\partial c.$$

This (∂_z,Δ_t)-differential ring can be naturally embedded into $C[[z]][z^{-1}]$, which implies that it is an integral domain. Therefore, we may define $\mathcal{K}_{C,U}$, the field of fractions of $K_U\otimes_{\mathcal{M}_U}C$. We see now $\mathcal{K}_{C,U}$ (resp. $K_U\otimes_{\mathcal{M}_U}C$) as a subfield (resp. subring) of $C[[z]][z^{-1}]$.

Proposition 3.2.12. *Let us keep the same notations. Let $\partial_z Y(z,t)=A(z,t)Y(z,t)$, with $A(z,t)\in\mathrm{M}_m\Big(\mathcal{O}_U(\{z\})\Big)$. The extension field $\mathcal{K}_{C,U}\langle F_{i,j}\rangle_{\partial_z,\Delta_t}\Big|\mathcal{K}_{C,U}=\widetilde{\mathcal{K}_{C,U}}\Big|\mathcal{K}_{C,U}$ is a parameterized Picard-Vessiot extension for $\partial_z Y(z,t)=A(z,t)Y(z,t)$. Moreover, there exist $P_1,\dots,P_k\in\mathcal{M}_U\{X_{i,j}\}_{\Delta_t}$ such that the image of the representation of $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{K}_{C,U}}\Big|\mathcal{K}_{C,U}\right)$ (resp. $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\Big|K_U\right)$) associated to $F(z,t)$ is the set of C-rational points (resp. $\mathcal{M}_U$-rational points) of the linear differential algebraic subgroup of $\mathrm{GL}_m(C)$ (resp. $\mathrm{GL}_m(\mathcal{M}_U)$) defined by $P_1,\dots,P_k$. More explicitly:*

$$\begin{array}{rl} & \left\{F^{-1}\varphi(F),\varphi\in Gal_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{K}_{C,U}}\Big|\mathcal{K}_{C,U}\right)\right\}\\ = & \left\{A=(a_{i,j})\in\mathrm{GL}_m(C)\Big|P_1(a_{i,j})=\dots=P_k(a_{i,j})=0\right\}\end{array}$$

$$\begin{array}{rl} & \left\{F^{-1}\varphi(F),\varphi\in Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\Big|K_U\right)\right\}\\ = & \left\{A=(a_{i,j})\in\mathrm{GL}_m(\mathcal{M}_U)\Big|P_1(a_{i,j})=\dots=P_k(a_{i,j})=0\right\}.\end{array}$$

Proof. We follow the proof of [MS12], Proposition 3.3. Let (d_k) be a $\mathcal{M}_U$-basis of C. Let us prove that the d_k are linearly independent over $\widetilde{K_U}$. Write $\sum_{k\leq\kappa} d_k P_k = 0$ with $0 \neq P_k \in \widetilde{K_U}$, $\kappa \geq 2$ minimal and $P_\kappa = 1$. We have $\sum_{k\leq\kappa-1} d_k \partial_z P_k = 0$. If $\kappa = 2$, $\partial_z P_1 = 0$. If $\kappa > 2$, we have that for all k, $\partial_z P_k = 0$, because of the minimality of κ. Since $\widetilde{K_U}\Big|K_U$ is a parameterized Picard-Vessiot extension, for all k, $P_k \in \mathcal{M}_U$, and the d_k are linearly independent over $\widetilde{K_U}$.

Now, we prove that $\mathcal{K}_{C,U}\langle F_{i,j}\rangle_{\partial_z,\Delta_t}\Big|\mathcal{K}_{C,U}$ is a parameterized Picard-Vessiot extension for $\partial_z Y(z,t) = A(z,t)Y(z,t)$. Let $\alpha \in \mathcal{K}_{C,U}\langle F_{i,j}\rangle_{\partial_z,\Delta_t}$ with $\partial_z\alpha = 0$. We may assume that $\alpha = \sum d_k P_k$, where $P_k \in \widetilde{K_U}$. We have $\partial_z\alpha = \sum d_k \partial_z P_k = 0$. Since the d_k are linearly independent over $\widetilde{K_U}$, we find $\partial_z P_k = 0$. Hence, $P_k \in \mathcal{M}_U$, because $\widetilde{K_U}\Big|K_U$ is a parameterized Picard-Vessiot extension. Therefore, $\alpha \in C$ and $\mathcal{K}_{C,U}\langle F_{i,j}\rangle_{\partial_z,\Delta_t}\Big|\mathcal{K}_{C,U}$ is a parameterized Picard-Vessiot extension for $\partial_z Y(z,t) = A(z,t)Y(z,t)$.

Let $Y_{i,j}$ be a set of m^2 indeterminates and let I_0, I_1, (∂_z,Δ_t)-differential ideals such that:

$$\begin{array}{rcccl} R_0 & = & K_U\{F_{i,j}\}_{\partial_z,\Delta_t} & = & K_U\{Y_{i,j}\}_{\partial_z,\Delta_t}/I_0 \\ R_1 & = & \mathcal{K}_{C,U}\{F_{i,j}\}_{\partial_z,\Delta_t} & = & \mathcal{K}_{C,U}\{Y_{i,j}\}_{\partial_z,\Delta_t}/I_1. \end{array}$$

The group $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\Big|K_U\right)$ (resp. $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{K}_{C,U}}\Big|\mathcal{K}_{C,U}\right)$) is the set of $B \in \mathrm{GL}_m(\mathcal{M}_U)$ (resp. $B \in \mathrm{GL}_m(C)$) such that $(F_{i,j})B$ is again a zero of I_0 (resp. I_1). We just have to prove that $I_1 = CI_0$. The inclusion $CI_0 \subset I_1$ is clear. Let us prove the other inclusion. Let $P \in I_1$. Without loss of generality, we may assume that $P \in (K_U \otimes_{\mathcal{M}_U} C)[Y_{i,j}]$. Let us write $P = \sum d_k P_k$, where $P_k \in K_U[Y_{i,j}]$. One finds:

$$P(F_{i,j}) = \sum d_k P_k(F_{i,j}) = 0.$$

Since the d_k are linearly independent over $\widetilde{K_U}$, one finds, $P_k(F_{i,j}) = 0$, and therefore $I_1 = CI_0$.

□

3.2.4 An analogue of the density theorem in the parameterized case.

Let us consider $\partial_z Y(z,t) = A(z,t)Y(z,t)$, with $A(z,t) \in \mathrm{M}_m\Big(\mathcal{O}_U(\{z\})\Big)$, where U is a non empty polydisc of $\mathbb{C}^n$. We want to find topological generators for $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\Big|K_U\right)$ for Kolchin topology.

We define now the parameterized monodromy. The notion of monodromy in the unparameterized case is well explained in [vdPS03]. For more details about parameterized monodromy, see [CS07, MS12, MS13, Sib90].

Definition 3.2.13. The notations are introduced in § 3.1.1. We define $\hat{m}$, the formal parameterized monodromy, as follows:

- $\forall \hat{H}(z,t) \in \hat{K}_U, \hat{m}\Big(\hat{H}(z,t)\Big) = \hat{H}(z,t)$.
- $\forall a(t) \in \mathcal{M}_U, \hat{m}(z^{a(t)}) = e^{2i\pi a(t)} z^{a(t)}$.
- $\hat{m}(\log) = 2i\pi + \log$.
- For all $q(z,t) = \sum a_n z^{-n} \in \mathbf{E}_U = \bigcup\limits_{\nu\in\mathbb{Q}^{>0}} z^{\frac{-1}{\nu}}\mathcal{M}_U\left[z^{\frac{-1}{\nu}}\right]$, we define

$$\hat{m}\Big(e(q(z,t))\Big) = e\Big(\sum a_n e^{-2i\pi n} z^{-n}\Big).$$

From the construction of $\hat{K}_U\left[\log, \left(z^{a(t)}\right)_{a(t)\in\mathcal{M}_U}\left(e(q(z,t))\right)_{q(z,t)\in\mathbf{E}_U}\right]$, it is easy to check that $\hat{m}$ induces a well defined (∂_z,Δ_t)-differential ring automorphism of $\hat{K}_U\left[\log, \left(z^{a(t)}\right)_{a(t)\in\mathcal{M}_U}\left(e(q(z,t))\right)_{q(z,t)\in\mathbf{E}_U}\right]$, and then it can be extended as a (∂_z,Δ_t)-differential field automorphism of $\widehat{\mathbf{K}_U}$ letting K_U invariant. Since $\widetilde{K_U}\subset\widehat{\mathbf{K}_U}$, and since $\widetilde{K_U}$ is stable by $\hat{m}$, $\hat{m}$ induces an element of $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\middle|K_U\right)$.

Remark 3.2.14. In the regular singular case with one singularity at 0, the definition of formal parameterized monodromy restricts to the definition given in [MS12].

We now introduce the parameterized exponential torus, which is a subgroup of $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\middle|K_U\right)$ consisting of elements that act on the $e\Big(q(z,t)\Big)$, with $q(z,t)\in\mathbf{E}_U$.

Definition 3.2.15. Let α be a character of $\mathbf{E}_U$. We define τ_α as the map

– τ_α is the identity on $\hat{K}_{F,U}$.
– $\forall q(z,t)\in\mathbf{E}_U, \tau_\alpha(e(q(z,t)))=\alpha(q(z,t))e(q(z,t))$.

From the construction of $\hat{K}_U\left[\log, \left(z^{a(t)}\right)_{a(t)\in\mathcal{M}_U}\left(e(q(z,t))\right)_{q(z,t)\in\mathbf{E}_U}\right]$, it is easy to check that τ_α induces a well defined (∂_z,Δ_t)-differential ring automorphism of $\hat{K}_U\left[\log, \left(z^{a(t)}\right)_{a(t)\in\mathcal{M}_U}\left(e(q(z,t))\right)_{q(z,t)\in\mathbf{E}_U}\right]$, and then it can be extended to a (∂_z,Δ_t)-differential field automorphism of $\widehat{\mathbf{K}_U}$ letting K_U invariant. Since $\widetilde{K_U}\subset\widehat{\mathbf{K}_U}$, and since $\widetilde{K_U}$ is stable by τ_α, τ_α induces an element of $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\middle|K_U\right)$.

The parameterized exponential torus (or simply, the exponential torus) is the subgroup of $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\middle|K_U\right)$ consisting of the τ_α, where α is a character of $\mathbf{E}_U$. Notice that the matrices of the exponential torus belongs to $\mathrm{GL}_m(\mathbb{C})$, while the coefficients of the matrix of $\hat{m}$ depend upon t.

Example 3.2.16. Let $t=(t_1,t_2)$ and let us consider

$$\partial_z\begin{pmatrix}Y_1(z,t)\\Y_2(z,t)\end{pmatrix}=\begin{pmatrix}-t_1z^{-2}&0\\0&-t_2z^{-2}\end{pmatrix}\begin{pmatrix}Y_1(z,t)\\Y_2(z,t)\end{pmatrix},$$

which admits $\begin{pmatrix}e^{t_1/z}&0\\0&e^{t_2/z}\end{pmatrix}$ as fundamental solution. The parameterized exponential torus and the parameterized differential Galois group are both equal to

$$\left\{\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}, \text{ where } \alpha,\beta\in\mathbb{C}^*\right\}.$$

Remark that the unparameterized exponential torus (see p.80 of [vdPS03]) and the unparameterized differential Galois group are isomorphic to $(\mathbb{C}^*)^2$ if and only if t_1 and t_2 are linearly independant over $\mathbb{Q}$. In particular, the matrices of the parameterized exponential torus evaluated at a specialized value (u,v) of the parameter are not always equal to the matrices of the unparameterized exponential torus of the system

$$\partial_z\begin{pmatrix}Y_1(z,u,v)\\Y_2(z,u,v)\end{pmatrix}=\begin{pmatrix}-uz^{-2}&0\\0&-vz^{-2}\end{pmatrix}\begin{pmatrix}Y_1(z,u,v)\\Y_2(z,u,v)\end{pmatrix}.$$

This is a difference between the exponential torus and the two others generators of the parameterized differential Galois group: the monodromy and the Stokes operators (see Definition 3.2.18 below).

Lemma 3.2.17. *Let $d(t)$ be a singular direction of $\partial_z Y(z,t) = A(z,t)Y(z,t)$ (see* §3.1.4*). The Stokes matrix $St_{d(t)}$ induces an element of $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\middle|K_U\right)$.*

Proof. Let us recall the construction of the Stokes matrices. Let $d(t)$ be a singular direction and let k_r be the biggest level of $\partial_z Y(z,t) = A(z,t)Y(z,t)$. The assumption we have made on $\mathcal{D}$ (see §3.1.4) tells us that there exists $t \mapsto d^{\pm}(t)$ continuous in t such that

$$d(t) - \frac{\pi}{2k_r} < d^-(t) < d(t) < d^+(t) < d(t) + \frac{\pi}{2k_r},$$

with no singular directions in $[d^-(t), d(t)[\bigcup]d(t), d^+(t)]$. From the construction of $St_{d(t)}$, and §3.1.3, we know that

$$H^{d^+(t)}(z,t)e^{L(t)\log(z)}e^{Q(z,t)} = H^{d^-(t)}(z)e^{L(t)\log(z)}e^{Q(z,t)}St_{d(t)}.$$

By construction, the Stokes matrix induces identity on K_U. To prove that the Stokes matrices are elements of $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\middle|K_U\right)$, we have to prove that the maps $i^{\pm}$, that send $\hat{H}(z,t)z^{L(t)}e\Big(Q(z,t)\Big)$ to $H^{d^{\pm}(t)}(z,t)e^{L(t)\log(z)}e^{Q(z,t)}$, induce (∂_z, Δ_t)-field isomorphisms. From the unparameterized case (see Theorem 2, Section 6.4 of [Bal94]), and the relations satisfied by the symbols log, $\left(z^{a(t)}\right)_{a(t)\in\mathcal{M}_U}$ and $\Big(e(q(z,t))\Big)_{q(z,t)\in\mathbf{E}_U}$ (see §3.1.1), $i^{\pm}$ induce ∂_z-field isomorphisms.

We want now to prove that if $\hat{H}(z,t)$ admits, $H^{d^{\pm}(t)}(z,t)$ as asymptotic sum in the direction $d^{\pm}(t)$, then $\partial_{t_i}\hat{H}(z,t)$ admits $\partial_{t_i}H^{d^{\pm}(t)}(z,t)$ as asymptotic sum in the direction $d^{\pm}(t)$, for all $i \leq n$. This is a consequence of Lemma 3.1.14 and the fact that we may assume that the $d^{\pm}(t)$ are locally constant. Hence $i^{\pm}$ commute with ∂_{t_i} and $i^{\pm}$ induce (∂_z, Δ_t)-field isomorphisms. □

Definition 3.2.18. Let $d(t)$ be a singular direction of $\partial_z Y(z,t) = A(z,t)Y(z,t)$. The element of $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\middle|K_U\right)$ induced by the Stokes matrix in the direction $d(t)$ is the Stokes operator in the direction $d(t)$. For simplicity of notation, we write $St_{d(t)}$ for both Stokes operator and the Stokes matrix in the direction $d(t)$.

Proposition 3.2.19. *If $g(z,t) \in \widetilde{K_U}$ is fixed by all the Stokes operators $St_{d(t)}$, the monodromy and the exponential torus, then $g(z,t) \in K_U$.*

Proof. Let $\overline{\mathcal{M}_U}$ be the algebraic closure of $\mathcal{M}_U$. Proposition 3.25 of [vdPS03] implies that if $g(z,t) \in \widehat{\mathbf{K}_U}$ is fixed by the monodromy and the exponential torus, then $g(z,t) \in \widehat{\mathbf{K}_U} \cap \overline{\mathcal{M}_U}[[z]][z^{-1}] = \hat{K}_U$. Since $\widetilde{K_U} \subset \widehat{\mathbf{K}_U}$, we have to prove that if $g(z,t) \in \widetilde{K_U} \cap \hat{K}_U$ is fixed by all the Stokes operators, then $g(z,t) \in K_U$. Let $g(z,t) \in \widetilde{K_U} \cap \hat{K}_U$ fixed by all the Stokes operators. Let $F(z,t) = \hat{H}(z,t)z^{L(t)}e\Big(Q(z,t)\Big)$ be the fundamental solution defined in Proposition 3.1.3 and let $\left(\hat{H}_{i,j}\right)$ be the entries of the matrix $\hat{H}(z,t)$. There exists $P \in K_U\langle X_{i,j}\rangle_{\partial_z,\Delta_t}$ such that $P(\hat{H}_{i,j}) = g(z,t)$. Let $d(t)$ that satisfies the same properties as in Proposition 3.1.13. Because of Proposition 3.1.13, there exists a map $U \to \mathbb{R}^{>0}$, $t \mapsto \varepsilon(t)$, which is not necessarily continuous such that $P\Big(H_{i,j}^{d(t)}\Big)$ is meromorphic in (z,t) for

$$(z,t) \in \left\{z \in \widetilde{\mathbb{C}}\middle| \arg(z) \in \left]d_1(t) - \frac{\pi}{2k_r}, d_2(t) + \frac{\pi}{2k_r}\right[\text{ and } 0 < |z| < \varepsilon(t)\right\} \times U,$$

where $d_1(t), d_2(t)$ are two singular directions. Since the matrix $P\Big(\hat{H}_{i,j}\Big)$ is fixed by all the Stokes operators, $P\Big(H_{i,j}^{d(t)}\Big)$ is meromorphic in (z,t) for $0<|z|<\varepsilon(t)$ and $(z,t)\in\widetilde{\mathbb{C}}\times U$. Moreover, $P\Big(H_{i,j}^{d(t)}\Big)(z,t)=P\Big(H_{i,j}^{d(t)}\Big)(e^{2i\pi}z,t)$ on his domain of definition, which means that $P\Big(H_{i,j}^{d(t)}\Big)$ is meromorphic in (z,t) for $0<|z|<\varepsilon(t)$ and $(z,t)\in\mathbb{C}\times U$. We recall that K_U consists of elements $f(z,t)\in\hat{K}_U$ such that for $0<|z|<\varepsilon(t)$, $t\mapsto f(z,t)\in\mathcal{M}_U$. We have, $P\left(H_{i,j}^{d(t)}\right)\in K_U$. We have seen in Lemma 3.2.17 that the map that sends $\hat{H}(z,t)z^{L(t)}e\Big(Q(z,t)\Big)$ to $H^{d(t)}(z,t)e^{L(t)\log(z)}e^{Q(z,t)}$ induces a (∂_z,Δ_t)-field isomorphism. Since the above map leaves K_U invariant, this implies that $P\Big(\hat{H}_{i,j}\Big)=g(z,t)\in K_U$. □

We can now prove the main theorem of this chapter. We recall some notations. Let $\partial_z Y(z,t)=A(z,t)Y(z,t)$, with $A(z,t)\in \mathrm{M}_m\Big(\mathcal{O}_U(\{z\})\Big)$ (see page 69), let K_U be the fraction field of $\mathcal{O}_U(\{z\})$, and let $\widetilde{K_U}\Big|K_U$ be the parameterized Picard-Vessiot extension defined in the beginning of §3.2.4. Let $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\Big|K_U\right)$ be the field automorphisms of $\widetilde{K_U}$ which commute with all the derivations and leave K_U invariant.

Theorem 3.2.20 (Parameterized analogue of the density theorem of Ramis)**.** *The group generated by the monodromy, the exponential torus and the Stokes operators is dense for the Kolchin topology in* $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\Big|K_U\right)$.

Proof. First of all, we have already pointed out that the monodromy, the exponential torus and the Stokes operators are elements of $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\Big|K_U\right)$. Using Proposition 3.2.9, we just have to prove that if $\alpha(z,t)\in\widetilde{K_U}$ is fixed by the monodromy, the exponential torus and the Stokes operators, then it belongs to K_U. This is exactly Proposition 3.2.19. □

Remark 3.2.21. (1) Let $\mathbb{C}(t)\{z\}$ be the subset of $\mathcal{O}_U(\{z\})$ consisting of elements of the form $\sum_{i>N}a_i(t)z^i$, with $a_i(t)\in\mathbb{C}(t)$ and $N\in\mathbb{Z}$. Let us consider $\partial_z Y(z,t)=A(z,t)Y(z,t)$, with $A(z,t)\in\mathrm{M}_m\Big(\mathbb{C}(t)\{z\}\Big)$. Even if we would be able to define a parameterized Picard-Vessiot extension over $\mathbb{C}(t)\{z\}$, we would not have a parameterized analogue of the density theorem of Ramis, because the monodromy is not defined in this case. In general, we have

$$\hat{m}(z^{\alpha(t)})=e^{2i\pi\alpha(t)}z^{\alpha(t)}\notin\mathbb{C}(t)\{z\}(z^{\alpha(t)}).$$

This is why we take a larger field of constants with respect to ∂_z.
(2) Similarly, we can prove that the group generated by the monodromy and the exponential torus is dense for Kolchin topology in $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\Big|\hat{K}_U\cap\widetilde{K_U}\right)$.

Corollary 3.2.22. $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}\Big|K_U\right)$ *contains a finitely generated Kolchin-dense subgroup.*

Proof. Let $q_1(z,t),\dots,q_\beta(z,t)\in\mathbf{E}_U$, $\mathbb{Q}$-linearly independent such that

$$\widetilde{K_U}\subset\hat{K}_{F,U}\Big(e(q_1(z,t)),\dots,e\left(q_\beta(z,t)\right)\Big).$$

Let τ_i be an element of the exponential torus that fixes the $e(q_j(z,t))$ for $j\neq i$, and that sends $e(q_i(z,t))$ to $ae(q_i(z,t))$, with a not a root of unity.

By the definition of the singular directions (see §3.1.4), there exists $\nu \in \mathbb{N}^*$ such that the singular directions modulo $2\nu\pi$ are in finite number. Let $d_1(t),\dots,d_k(t)$ be continuous singular direction such that, if $d(t)$ is a singular direction, then $d(t)$ is equal to one of the $d_i(t)$ modulo $2\nu\pi$. Let $g(z,t) \in \widetilde{K_U}$ be fixed by the monodromy, $\tau_1,\dots,\tau_\beta$, and $St_{d_1(t)},\dots,St_{d_k(t)}$. Using (2) of Proposition 3.2.9, it is sufficient to prove that $g(z,t) \in K_U$.

We can write $g(z,t)$ as an element of:

$$\hat{K}_{F,U}\Big(e(q_1(z,t)),\dots,e(q_{\beta-1}(z,t)))(e(q_\beta(z,t))\Big).$$

Since the elements $q_i(z,t) \in \mathbf{E}_U$ are $\mathbb{Q}$-linearly independent, we know by construction that the $e(Nq_\beta(z,t))$, with $N \in \mathbb{Z}$, are $\mathbb{C}$-linearly independent over $\hat{K}_{F,U}\Big(e(q_1(z,t)),\dots,e(q_{\beta-1}(z,t))\Big)$. If we add the fact that $g(z,t)$ is fixed by τ_β, we obtain:

$$g(z,t) \in \hat{K}_{F,U}\Big(e(q_1(z,t)),\dots,e(q_{\beta-1}(z,t))\Big).$$

We apply the same argument β times to conclude that $g(z,t) \in \hat{K}_{F,U} \cap \widetilde{K_U}$. By the construction of the Stokes operators, we have that $St_{d(t)} = \mathrm{Id}$ if and only if $St_{2\nu\pi+d(t)} = \mathrm{Id}$, where $\nu \in \mathbb{N}^*$ has been defined in the proof. Proposition 3.2.19 allows us to conclude that $g(z,t) \in K_U$. $\square$

3.2.5 The density theorem for the global parameterized differential Galois group.

In this subsection, we consider parameterized linear differential equation of the form:

$$\partial_z Y(z,t) = A(z,t)Y(z,t),$$

with $A(z,t) \in \mathrm{M}_m\Big(\mathcal{M}_U(z)\Big)$. We want to prove a density theorem for the global parameterized differential Galois group. The result in the unparameterized case is due to Ramis and a proof can be found for instance in [Mit96], Proposition 1.3. The parameterized singularities of $\partial_z Y(z,t) = A(z,t)Y(z,t)$ (that is the poles, including maybe ∞, of $A(z,t)$, as a rational function in z) belong to the algebraic closure of $\mathcal{M}_U$. Because of Remark 3.1.1, after taking a smaller non empty polydisc U, we may assume that the set of parameterized singularities belongs to $\mathcal{M}_U$. We will write singularity instead of parameterized singularity when no confusion is likely to arise. Let $S = \{\alpha_1(t),\dots,\alpha_k(t)\} \subset \mathbb{P}_1(\mathcal{M}_U)$ be the set of the singularities of $\partial_z Y(z,t) = A(z,t)Y(z,t)$. For any singularity $\alpha(t)$ of $\partial_z Y(z,t) = A(z,t)Y(z,t)$, we may define the levels and the set of singular directions of $\alpha(t)$ by considering

$$\partial_z Y(z-\alpha(t),t) = A(z-\alpha(t),t)Y(z-\alpha(t),t) \text{ if } \infty \not\equiv \alpha(t) \in S$$

and

$$\partial_z Y(z^{-1},t) = A(z^{-1},t)Y(z^{-1},t) \text{ if } \infty \equiv \alpha(t) \in S.$$

Let $(d_{i,j}(t))$ be the singular directions $\alpha_i(t)$. As in §3.1.4, we define:

$$\mathcal{D}_{\alpha_i(t)} = \Big\{t \in U \Big| \exists j,j' \in \mathbb{N}, \text{ such that } d_{i,j} \not\equiv d_{i,j'} \text{ and } d_{i,j}(t) = d_{i,j'}(t)\Big\}.$$

From Lemma 3.1.12, all the $\mathcal{D}_{\alpha_i(t)}$ are closed set with empty interior. After taking a smaller non empty polydisc U, we may assume that:

- There exists $\varepsilon > 0$ such that for all $t \in U$ and for all $i \neq j$:

$$|\alpha_i(t) - \alpha_j(t)| > \varepsilon.$$

- $\mathcal{D}_{\alpha_i(t)} = \varnothing$ for all $i \leq k$.
- For all the singularities of $\partial_z Y(z,t) = A(z,t)Y(z,t)$, the levels are independent of t.
- For all $t_0 \in U$, for all the singularities $\infty \not\equiv \alpha(t) \in S$ (resp. for the singularity ∞), the singular directions of

$$\partial_z Y(z-\alpha(t),t) = A(z-\alpha(t),t)Y(z-\alpha(t),t)$$
$$\text{resp. } \partial_z Y(z^{-1},t_0) = A(z^{-1},t_0)Y(z^{-1},t_0)$$

evaluated at t_0 are equal to the singular directions of the specialized system

$$\partial_z Y(z-\alpha(t),t_0) = A(z-\alpha(t),t_0)Y(z-\alpha(t),t_0)$$
$$\text{resp. } \partial_z Y(z^{-1},t_0) = A(z^{-1},t_0)Y(z^{-1},t_0).$$

- Every entry of every z-coefficients of $A(z,t)$ is analytic on U.

Let $x_0(t) \in \mathcal{M}_U$ and let $\varepsilon > 0$ such that:

$$\forall t \in U, \forall i < j \leq k, \quad |x_0(t) - \alpha_j(t)| > \varepsilon \text{ and } |\alpha_i(t) - \alpha_j(t)| > \varepsilon.$$

For all $i \leq k$ and all $t \in U$, we define $U_{\alpha_i(t)}$, the polydisc in the z-plane with center $\alpha_i(t)$ and with radius ε. Let $d_{\alpha_i}(t)$ be a continuous ray from $\alpha_i(t)$ in $U_{\alpha_i(t)}$, $b_{\alpha_i}(t)$ be the continuous point of $d_{\alpha_i}(t)$ with $|b_{\alpha_i}(t) - \alpha_i(t)| = \varepsilon$ and $\gamma_{\alpha_i}(t)$ be a continuous path in $\mathbb{P}_1(\mathcal{M}_U)$ from $x_0(t)$ to $b_{\alpha_i}(t)$ such that for all $t \in U$ and all $j \leq k$, $|\gamma_{\alpha_i}(t) - \alpha_j(t)| > \varepsilon/2$. Analytic continuation of $F(z,t) = (F_{i,j})$, a germ of solution at $x_0(t)$ with the path $\gamma_{\alpha_i}(t)$ and $d_{\alpha_i}(t)$ provides a fundamental solution $F^{d_{\alpha_i}(t)}(z,t)$ on a germ of open sector with vertex $\alpha_i(t)$ bisected by $d_{\alpha_i}(t)$.

Let $\widetilde{\mathcal{M}_U(z)} = \mathcal{M}_U(X)\langle F_{i,j}\rangle_{\partial_z,\Delta_t}$. From the assumptions we have made on $x_0(t)$, we deduce that this field has a field of constants with respect to ∂_z equal to $\mathcal{M}_U$. Therefore, we deduce that $\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)$ is a parameterized Picard-Vessiot extension. The results of §3.2.2 may be applied here and we can define a parameterized differential Galois group $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)\right)$, which will be identified with a linear differential algebraic subgroup of $\mathrm{GL}_m(\mathcal{M}_U)$. We will make the same abuse of language as in the local case (see Remark 3.2.8) and call it the parameterized linear differential Galois group, or Galois group, if no confusion is likely to arise. As in Proposition 3.2.12, we want to prove now that it is the "same" as the one of §3.2.1.

Let C be a (Δ_t)-differentially closed field that contains $\mathcal{M}_U$ and let $C(z)$ denotes the (∂_z, Δ_t)-differential field of rational functions in the indeterminate z, with coefficients in C, where z is a (Δ_t)-constant with $\partial_z z = 1$, C is the field of constants with respect to ∂_z, and ∂_z commutes with all the derivations. The next proposition is the analogue in the global case of Proposition 3.2.12.

Proposition 3.2.23. *Let us keep the same notations. Let $\partial_z Y(z,t) = A(z,t)Y(z,t)$, with $A(z,t) \in \mathrm{M}_m(\mathcal{M}_U(z))$. The extension field $C(z)\langle F_{i,j}\rangle_{\partial_z,\Delta_t}\Big|C(z) := \widetilde{C(z)}\Big|C(z)$ is a parameterized Picard-Vessiot extension for $\partial_z Y(z,t) = A(z,t)Y(z,t)$. Moreover, there exist $P_1,\dots,P_k \in \mathcal{M}_U\{X_{i,j}\}_{\Delta_t}$ such that the image of the representation of $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{C(z)}\Big|C(z)\right)$ (resp. $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)\right)$) associated to $F(z,t)$ is the set* *of C-rational points (resp. $\mathcal{M}_U$-rational points) of the linear differential algebraic subgroup of* $\mathrm{GL}_m(C)$ *(resp.* $\mathrm{GL}_m(\mathcal{M}_U)$*) defined by $P_1, \dots, P_k$. More explicitly:*

$$\begin{aligned} & \left\{F^{-1}\varphi(F), \varphi \in Gal_{\partial_z}^{\Delta_t}\left(\widetilde{C(z)}\middle|C(z)\right)\right\} \\ = & \left\{A = (a_{i,j}) \in \mathrm{GL}_m(C)\middle|P_1(a_{i,j}) = \dots = P_k(a_{i,j}) = 0\right\} \end{aligned}$$

$$\begin{aligned} & \left\{F^{-1}\varphi(F), \varphi \in Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right)\right\} \\ = & \left\{A = (a_{i,j}) \in \mathrm{GL}_m(\mathcal{M}_U)\middle|P_1(a_{i,j}) = \dots = P_k(a_{i,j}) = 0\right\}. \end{aligned}$$

Proof. This is exactly the same reasoning as in Proposition 3.2.12. □

We want to find topological generators for $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right)$ for the Kolchin topology.

For $\alpha(t) \in \mathcal{M}_U$, let

$$K_{U,\alpha(t)} = \{f(z-\alpha(t), t) \mid f(z,t) \in K_U\},$$

and let

$$K_{U,\infty} = \{f(z^{-1}, t) \mid f(z,t) \in K_U\}.$$

Let $\alpha(t) \in S$ and let $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|K_{U,\alpha(t)} \cap \widetilde{\mathcal{M}_U(z)}\right)$ be the local Galois group for the fundamental solution $F^{d_{\alpha}(t)}(z,t)$ described above. If we conjugate $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|K_{U,\alpha(t)} \cap \widetilde{\mathcal{M}_U(z)}\right)$ by the differential isomorphism defined by analytic continuation of $F(z,t)$ described above, we get an injective morphism of linear differential algebraic groups:

$$Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|K_{U,\alpha(t)} \cap \widetilde{\mathcal{M}_U(z)}\right) \hookrightarrow Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right).$$

Using the maps $i^{\pm}$ defined in the proof of Lemma 3.2.17 and the injection above, we can define the monodromy, the exponential torus, and the Stokes operators for any singularities in S, as elements of

$$Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right).$$

Theorem 3.2.24 (Global parameterized analogue of the density theorem of Ramis)**.** *Let $\partial_z Y(z,t) = A(z,t)Y(z,t)$, where $A(z,t) \in \mathrm{M}_m(\mathcal{M}_U(z))$. For $\alpha(t) \in S$, let $G_{\alpha(t)}$ be the subgroup of:*

$$Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|K_{U,\alpha(t)} \cap \widetilde{\mathcal{M}_U(z)}\right),$$

generated by the monodromy, the exponential torus and the Stokes operators. Let G be the subgroup of $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right)$ generated by the $G_{\alpha(t)}$, with $\alpha(t) \in S$. Then G in dense for Kolchin topology in

$$Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right).$$

Proof. We use (2) of Proposition 3.2.9. We have to prove that the subfield of $\widetilde{\mathcal{M}_U(z)}$ fixed by G is $\mathcal{M}_U(z)$. Let $f(z,t) \in \widetilde{\mathcal{M}_U(z)}$ be fixed by G. Then, by the same reasoning as in Proposition 3.2.19, it follows that $f(z,t)$ belongs to $K_{U,\alpha(t)}$, for $\alpha(t) \in S$. Therefore, we deduce that $f(z,t)$ is meromorphic in (z,t) on $\mathbb{P}_1(\mathbb{C}) \times U$, and has a finite number of poles in the z-plane for t fixed. Hence, $f(z,t) \in \mathcal{M}_U(z)$. □

In particular, this generalizes Theorem 4.2 in [MS12] which says that, if the equation has only regular singular poles, then the group generated by the monodromy at each pole is dense for Kolchin topology in $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right)$.

Corollary 3.2.25. *$Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right)$ contains a finitely generated Kolchin-dense subgroup.*

Proof. In the proof of Theorem 3.2.24, we see that the global parameterized differential Galois group is generated by all local parameterized differential Galois groups. Since there is a finite number of singularities, this is a consequence of Corollary 3.2.22. □

3.2.6 Examples.

In all the examples, we will compute the global parameterized differential Galois group. This means that the base field is $\mathcal{M}_U(z)$.

Example 3.2.26. Let us consider $\partial_z Y(z,t) = \frac{t}{z} Y(z,t)$. This example was considered by direct computations in Example 3.2.5 but we will compute here $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right)$ using the parameterized density theorem. The fundamental solution is (z^t) and the parameterized Picard-Vessiot extension over $\mathcal{M}_U(z)$ is $\mathcal{M}_U(z, z^t, \log)$ (we want the extension to be closed under the derivations ∂_z and ∂_t). The exponential torus and the Stokes matrices are trivial. The monodromy sends z^t to $e^{2i\pi t}z^t$. The element $e^{2i\pi t}$ satisfies the differential equation

$$\partial_t \left(\frac{\partial_t e^{2i\pi t}}{e^{2i\pi t}} \right) = 0.$$

Therefore, the Kolchin closure of the monodromy is contained in:

$$\left\{ a \in \mathcal{M}_U \middle| \partial_t \left(\frac{\partial_t a}{a} \right) \right\} = \{ce^{bt}, b \in \mathbb{C}, c \in \mathbb{C}^*\}.$$

Conversely, the map that sends z^t to $ce^{bt}z^t$ is an element of $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right)$. Finally, viewed as a linear differential algebraic subgroup of $\mathrm{GL}_1(\mathcal{M}_U)$,

$$\begin{aligned} Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right) &\simeq \left\{ a \in \mathcal{M}_U \middle| \partial_t \left(\tfrac{\partial_t a}{a}\right) = 0 \right\} \\ &= \{a \in \mathcal{M}_U | a \neq 0 \text{ and } a\partial_t^2 a - (\partial_t a)^2 = 0\} \\ &\subseteq \mathrm{GL}_1(\mathcal{M}_U). \end{aligned}$$

Example 3.2.27 (Parameterized Euler equation). Let $f(t)$ be an analytic function different from 0, and let us consider:

$$\partial_z^2 Y(z,t) + \left(\frac{1}{z} - \frac{1}{f(t)z^2}\right)\partial_z Y(z,t) + \frac{1}{f(t)z^3} Y(z,t) = 0,$$

which can be seen as a system:

$$\partial_z \begin{pmatrix} Y(z,t) \\ \partial_z Y(z,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-1}{f(t)z^3} & \frac{1}{f(t)z^2} - \frac{1}{z} \end{pmatrix} \begin{pmatrix} Y(z,t) \\ \partial_z Y(z,t) \end{pmatrix}.$$

If $f \equiv 1$, we recognize the Euler equation. A fundamental solution is:

$$\begin{pmatrix} 1 & \hat{F}(z,t) \\ \frac{1}{f(t)z^2} & \partial_z \hat{F}(z,t) \end{pmatrix} \begin{pmatrix} e\left(\frac{-1}{f(t)z}\right) & 0 \\ 0 & 1 \end{pmatrix},$$

where $\hat{F}(z,t) = -\sum_{n\geq 0} n!(f(t)z)^{n+1}$. The only singularity is 0. The monodromy is trivial. Let τ be an element of the exponential torus. Then, the image of the fundamental solution under τ is:

$$\begin{pmatrix} 1 & \hat{F}(z,t) \\ \frac{1}{f(t)z^2} & \partial_z \hat{F}(z,t) \end{pmatrix} \begin{pmatrix} \alpha e\left(\frac{-1}{f(t)z}\right) & 0 \\ 0 & 1 \end{pmatrix},$$

with $\alpha \in \mathbb{C}^*$. Therefore, the matrices of the elements of the exponential torus are of the form $\mathrm{Diag}(\alpha, 1)$, with $\alpha \in \mathbb{C}^*$. The only level of the system is 1 and the singular directions are the $\arg\left(f(t)^{-1}\right) + 2k\pi$, with $k \in \mathbb{Z}$. As we have seen in Proposition 3.1.10, we can compute the Stokes matrix with the Laplace and the Borel transforms. It follows from the definition of the formal Borel transform that

$$\hat{\mathcal{B}}_1\left(\hat{F}(z,t)\right) \equiv \log(1 - f(t)z).$$

Let $0 < \varepsilon < \frac{\pi}{2}$, such that there are no singular directions in:

$$\left[\arg\left(f(t)^{-1}\right) - \varepsilon, \arg\left(f(t)^{-1}\right)\right[\ \bigcup \ \left]\arg\left(f(t)^{-1}\right), \arg\left(f(t)^{-1}\right) + \varepsilon\right].$$

Then, the following matrices are fundamental solution:

$$\begin{pmatrix} 1 & \mathcal{L}_{1,\arg(f(t)^{-1})+\varepsilon}(\log(1-f(t)z)) \\ \frac{1}{f(t)z^2} & \partial_z \mathcal{L}_{1,\arg(f(t)^{-1})+\varepsilon}(\log(1-f(t)z)) \end{pmatrix} \begin{pmatrix} e^{\frac{-1}{f(t)z}} & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \mathcal{L}_{1,\arg(f(t)^{-1})-\varepsilon}(\log(1-f(t)z)) \\ \frac{1}{f(t)z^2} & \partial_z \mathcal{L}_{1,\arg(f(t)^{-1})-\varepsilon}(\log(1-f(t)z)) \end{pmatrix} \begin{pmatrix} e^{\frac{-1}{f(t)z}} & 0 \\ 0 & 1 \end{pmatrix}.$$

To compute the Stokes matrix in the direction $\arg\left(f(t)^{-1}\right)$, we have to compute:

$$\mathcal{L}_{1,\arg(f(t)^{-1})+\varepsilon}(\log(1-f(t)z)) - \mathcal{L}_{1,\arg(f(t)^{-1})-\varepsilon}(\log(1-f(t)z)).$$

We have

$$\begin{aligned} & \mathcal{L}_{1,\arg(f(t)^{-1})+\varepsilon}(\log(1-f(t)z)) - \mathcal{L}_{1,\arg(f(t)^{-1})-\varepsilon}(\log(1-f(t)z)) \\ = \quad & z^{-1}\int_0^{\infty i(\arg\left(f(t)^{-1}\right)+\varepsilon)} \log(1-f(t)u)e^{-\left(\frac{u}{z}\right)}d(u) \\ - \quad & z^{-1}\int_0^{\infty i(\arg\left(f(t)^{-1}\right)-\varepsilon)} \log(1-f(t)u)e^{-\left(\frac{u}{z}\right)}d(u). \end{aligned}$$

Integration by parts and the residue theorem imply that:

$$\mathcal{L}_{1,\arg(f(t)^{-1})+\varepsilon}(\log(1-f(t)z)) - \mathcal{L}_{1,\arg(f(t)^{-1})-\varepsilon}(\log(1-f(t)z)) = 2i\pi f(t)e^{-\left(\frac{1}{f(t)z}\right)}.$$

Therefore, the Stokes matrix in this direction is $\begin{pmatrix} 1 & 2i\pi f(t) \\ 0 & 1 \end{pmatrix}$. Finally we obtain:

$$\begin{aligned} Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right) &\simeq \left\{\begin{pmatrix} \alpha & bf \\ 0 & 1 \end{pmatrix}, \text{ where } \alpha \in \mathbb{C}^* \text{ and } b \in \mathbb{C}\right\} \\ &\simeq \left\{\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}, \text{ where } \partial_t \alpha = 0, \alpha \neq 0 \text{ and } \partial_t\left(\frac{\beta}{f}\right) = 0\right\}. \end{aligned}$$

Example 3.2.28 (Bessel equation). We are interested in the parameterized linear differential equation:

$$\partial_z \begin{pmatrix} Y(z,t) \\ \partial_z Y(z,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{(t^2-z^2)}{z^2} & \frac{-1}{z} \end{pmatrix} \begin{pmatrix} Y(z,t) \\ \partial_z Y(z,t) \end{pmatrix}.$$

This equation has two singularities: 0 and ∞. Let U be a non empty disc such that $U \cap (1/2 + \mathbb{Z}) = \varnothing$. First, we will compute the local group at 0:

$$Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|K_{U,0} \cap \widetilde{\mathcal{M}_U(z)}\right).$$

If $t + 1/2 \notin \mathbb{Z}$, the two solutions:

$$J_t(z) = \left(\frac{z}{2}\right)^t \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k!\Gamma(t+k+1)2^k}$$

$$J_{-t}(z) = \left(\frac{z}{2}\right)^{-t} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k!\Gamma(-t+k+1)2^k},$$

are linearly independent (see [Wat95] Page 43) and we have a fundamental solution of the specialized system. The equation is regular singular at $z = 0$, therefore, the group generated by the monodromy $\hat{m}$ is dense for Kolchin topology in the parameterized differential Galois group $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|K_{U,0} \cap \widetilde{\mathcal{M}_U(z)}\right)$. By the same reasoning as in Example 3.2.26:

$$Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|K_{U,0} \cap \widetilde{\mathcal{M}_U(z)}\right) \simeq \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \text{ where } \alpha \neq 0 \text{ and } \alpha\partial_t^2\alpha - (\partial_t\alpha)^2 = 0\right\}.$$

We now turn to the singularity at infinity. We have:

$$\partial_z \begin{pmatrix} Y(z^{-1},t) \\ \partial_z Y(z^{-1},t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{t^2}{z^2} - \frac{1}{z^4} & \frac{-1}{z} \end{pmatrix} \begin{pmatrix} Y(z^{-1},t) \\ \partial_z Y(z^{-1},t) \end{pmatrix}.$$

In order to compute the matrices of the monodromy, the elements of the exponential torus, and the Stokes operators, we make use of another basis of solutions:

$$H_t^{(1)}(z^{-1}) = \frac{J_{-t}(z^{-1}) - e^{-it\pi}J_t(z^{-1})}{i\sin(t\pi)}$$

$$H_t^{(2)}(z^{-1}) = \frac{J_{-t}(z^{-1}) - e^{it\pi} J_t(z^{-1})}{-i\sin(t\pi)}.$$

In [Wat95] page 198, we find that on the sector $]-\pi, 2\pi[$, $H_t^{(1)}(z^{-1})$ is asymptotic to:

$$\widetilde{H}_t^{(1)}(z^{-1}) = \left(\frac{2z}{\pi}\right)^{1/2} e^{i(z^{-1}-t\pi/2-\pi/4)} \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma(t+k+1/2)z^k}{(2i)^k k!\Gamma(t-k+1/2)}.$$

The same holds for $H_t^{(2)}(z^{-1})$ on the sector $]-2\pi, \pi[$:

$$\widetilde{H}_t^{(2)}(z^{-1}) = \left(\frac{2z}{\pi}\right)^{1/2} e^{-i(z^{-1}-t\pi/2-\pi/4)} \sum_{k=0}^{\infty} \frac{\Gamma(t+k+1/2)z^k}{(2i)^k k!\Gamma(t-k+1/2)}.$$

It follows that in the basis $\left(H_t^{(1)}(z^{-1}), H_t^{(2)}(z^{-1})\right)$, the matrix of the monodromy is:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the matrices of the elements of the exponential torus are of the form:

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \text{ where } \alpha \in \mathbb{C}^* \right\}.$$

The only level is 1 and due to the expression of $\widetilde{H}_t^1(z^{-1})$ and $\widetilde{H}_t^2(z^{-1})$, the singular directions are the directions $\frac{\pi}{2} + k\pi$, with $k \in \mathbb{Z}$. By definition, the Stokes matrix in the direction $\frac{\pi}{2} + k\pi$ is the matrix that sends the asymptotic representation defined on the sector $](k-1)\pi, (k+1)\pi[$ to the asymptotic representation defined on the sector $]k\pi, (k+2)\pi[$. In [RM90], 3.4.12 (see also [Ber92]), we find that in the basis $(H_t^1(z^{-1}), H_t^2(z^{-1}))$ the Stokes matrix in the direction $\frac{\pi}{2} + 2k\pi$ is

$$\begin{pmatrix} 1 & 0 \\ 2e^{2i\pi t}\cos(\pi t) & 1 \end{pmatrix},$$

and the Stokes matrix in the direction $-\frac{\pi}{2} + 2k\pi$ is

$$\begin{pmatrix} 1 & -2e^{-2i\pi t}\cos(\pi t) \\ 0 & 1 \end{pmatrix}.$$

An application of the local and global density theorems (Theorems 3.2.24 and 3.2.20) gives that

$$Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\Big| K_{U,\infty} \cap \widetilde{\mathcal{M}_U(z)}\right) \text{ and } Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\Big| \mathcal{M}_U(z)\right)$$

are linear differential algebraic subgroups of $\mathrm{SL}_2(\mathcal{M}_U)$, because all the matrices we have computed are in $\mathrm{SL}_2(\mathcal{M}_U)$, which is closed in the Kolchin topology.

Let C be a differentially closed field that contains $\mathcal{M}_U$ and consider $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{C(z)}\Big|C(z)\right)$, the parameterized differential Galois group defined in Proposition 3.2.23. We are going first to compute the Zariski closure G of $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{C(z)}\Big|C(z)\right)$. Let $C^* = C \setminus \{0\}$. From the classification of linear algebraic subgroup of $\mathrm{SL}_2(C)$ (see [vdPS03], Theorem 4.29), there are four possibilities:

1. G is conjugate to a subgroup of

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \text{ where } a \in C^*, b \in \mathbb{C} \right\}.$$

2. G is conjugate to a subgroup of

$$D_\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \bigcup \begin{pmatrix} 0 & b^{-1} \\ -b & 0 \end{pmatrix}, \text{ where } a, b \in C^* \right\}.$$

3. G is finite.

4. $G = \mathrm{SL}_2(\mathrm{C})$.

From Proposition 3.2.23, every matrix that belongs to $Aut^{\Delta_t}_{\partial_z}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right)$ belongs also to $Gal^{\Delta_t}_{\partial_z}\left(\widetilde{C(z)}\middle|C(z)\right)$. Since G must contain

$$\begin{pmatrix} 1 & 0 \\ 2e^{2i\pi t}\cos(\pi t) & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -2e^{-2i\pi t}\cos(\pi t) \\ 0 & 1 \end{pmatrix},$$

we find that the only possibility is that $Gal^{\Delta_t}_{\partial_z}\left(\widetilde{C(z)}\middle|C(z)\right)$ is Zariski dense in $SL_2(C)$. In [Cas72], Proposition 42, Cassidy classifies the Zariski-dense differential algebraic subgroups of $\mathrm{SL}_2(C)$. Finally, we have two possibilities:

- The group $Gal^{\Delta_t}_{\partial_z}\left(\widetilde{C(z)}\middle|C(z)\right)$ is conjugate to $\mathrm{SL}_2(C_0)$ over $\mathrm{SL}_2(C)$, where

$$C_0 = \{a \in C(z) | \partial_z a = \partial_t a = 0\}.$$

- The group $Gal^{\Delta_t}_{\partial_z}\left(\widetilde{C(z)}\middle|C(z)\right) = \mathrm{SL}_2(C)$.

If $Gal^{\Delta_t}_{\partial_z}\left(\widetilde{C(z)}\middle|C(z)\right)$ is conjugate to $\mathrm{SL}_2(C_0)$ over $\mathrm{SL}_2(C)$, the matrix of the monodromy of the singularity 0 is conjugate to a matrix $M \in \mathrm{SL}_2(C_0)$ over $\mathrm{SL}_2(C)$. Similar matrices have the same eigenvalues, then the eigenvalues of M are $e^{2i\pi t}$ and $e^{-2i\pi t}$, which is not possible if M belongs to $\mathrm{SL}_2(C_0)$. Because of Proposition 3.2.23, we find that

$$Aut^{\Delta_t}_{\partial_z}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right) = \mathrm{SL}_2(\mathcal{M}_U).$$

3.3 Applications.

We now give three applications of the parameterized differential Galois theory. In §3.3.1, we deal with linear differential equations that are completely integrable (see Definition 3.3.1). It has been proved in [CS07] that an equation is completely integrable if and only if its parameterized differential Galois group is conjugate over a differentially closed field to a group of constant matrices. We use the global density theorem (Theorem 3.2.24) to prove that the equation is completely integrable if and only if there exists a fundamental solution such that the matrices of the topological generators for the Galois group appearing in the global density theorem (Theorem 3.2.24) are constant matrices. As a corollary, we deduce that the equation is completely integrable if and only if the matrices of the topological generators for the Galois group given in the parameterized density theorem are conjugate over $\mathrm{GL}_m(\mathcal{M}_U)$ to constant matrices. In §3.3.2, we study an entry of a Stokes operator at the singularity at infinity of the equation:

$$\partial_z^2 Y(z,t) = (z^3+t)Y(z,t).$$

In particular, we prove that it is not ∂_t-finite: it satisfies no parameterized linear differential equation. This partially answers a question by Sibuya. In §3.3.3, we deal with the inverse problem in the parameterized differential Galois theory. Let k be a so-called universal (Δ_t)-field (see §3.3.2). We give a necessary condition on a linear differential algebraic subgroup of $\mathrm{GL}_m(k)$ for being the global parameterized differential Galois group for some equation having coefficients in $k(z)$. The corresponding sufficient condition has been proved in [MS12], Corollary 5.2.

3.3.1 Completely integrable equations.

In this subsection, we study completely integrable equations. See also [GO12] for an approach from the point of view of differential Tannakian categories.

Definition 3.3.1. Let $A_0 \in \mathrm{M}_m\Big(\mathcal{M}_U(z)\Big)$. We say that the linear differential equation $\partial_0 Y = A_0 Y$ is completely integrable if there exist $A_1,\dots,A_n \in \mathrm{M}_m\Big(\mathcal{M}_U(z)\Big)$ such that, for all $0 \leq i,j \leq n$,

$$\partial_{t_i} A_j - \partial_{t_j} A_i = A_i A_j - A_j A_i,$$

with $\partial_{t_0} = \partial_z$.

Sibuya shows in [Sib90], Theorem A.5.2.3, that if the parameterized linear differential equation $\partial_z Y(z,t) = A(z,t)Y(z,t)$ is regular singular, then it is isomonodromic (see page 68 for the definition) if and only if it is completely integrable. This result is not true in the irregular case. The main reason is the fact that there are more topological generators in the parameterized differential Galois group.

Proposition 3.3.2. *Let $A_0(z,t) \in \mathrm{M}_m\Big(\mathcal{M}_U(z)\Big)$ and let $\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)$ be the parameterized Picard-Vessiot extension for $\partial_z Y(z,t) = A_0(z,t)Y(z,t)$ defined in §3.2.5. The linear differential equation $\partial_z Y(z,t) = A_0(z,t)Y(z,t)$ is completely integrable if and only if there exists a fundamental solution $F(z,t)$ in $\widetilde{\mathcal{M}_U(z)}$ such that the images of the topological generators of $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)\right)$ (see Theorem 3.2.24), with respect to the representation associated to $F(z,t)$, belongs to $\mathrm{GL}_m(\mathbb{C})$.*

Proof. Let C be a differentially closed field that contains $\mathcal{M}_U$ and let us consider $C(z)$ as in §3.2.5. Let $\widetilde{C(z)}\Big|C(z)$ be the parameterized Picard-Vessiot extension for $\partial_z Y(z,t) = A_0(z,t)Y(z,t)$, and let $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{C(z)}\Big|C(z)\right)$ be the parameterized differential Galois group defined in §3.2.1. We recall that if we take a different fundamental solution in $\widetilde{\mathcal{M}_U(z)}$ to compute the Galois group, we obtain a conjugate linear differential algebraic subgroup of $\mathrm{GL}_m(\mathcal{M}_U)$.

Using the global density theorem (Theorem 3.2.24), we find that there exists a fundamental solution such that the matrices of the topological generators for the Galois group appearing in Theorem 3.2.24 are constant if and only if $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)\right)$ is conjugate over $\mathrm{GL}_m(\mathcal{M}_U)$ to a subgroup of $\mathrm{GL}_m(\mathbb{C})$. Using Proposition 3.2.23, we find that $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)\right)$ is conjugate over $\mathrm{GL}_m(\mathcal{M}_U)$ to a subgroup of $\mathrm{GL}_m(\mathbb{C})$ if and only if $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{C(z)}\Big|C(z)\right)$ is conjugate over $\mathrm{GL}_m(C)$ to a subgroup of $\mathrm{GL}_m(C_0)$, where

$$C_0 = \{a \in C(z) | \partial_z a = \partial_{t_1} a = \dots = \partial_{t_n} a = 0\}.$$

Proposition 3.9, [CS07] says that this occurs if and only if there exist $A_1, \dots, A_n \in \mathrm{M}_m(C(z))$ such that, for all $0 \leq i,j \leq n$,

$$\partial_{t_i} A_j - \partial_{t_j} A_i = A_i A_j - A_j A_i,$$

with $\partial_{t_0} = \partial_z$. To finish, we follow the proof of Proposition 1.24 in [DVH12]. Let $0 < i \leq n$ and let us consider

$$\partial_z A_i - \partial_{t_i} A_0 = A_0 A_i - A_i A_0.$$

By clearing the denominators, we obtain that every entry of every z-coefficient of A_i satisfies a finite set of polynomial equations with coefficients in $\mathcal{M}_U$. Since the polynomial equations have a solution in C, they must have a solution in the algebraic closure of $\mathcal{M}_U$. Using Remark 3.1.1, we find the existence of a non empty polydisc $U' \subset U$ such that all the A_i belongs to $\mathrm{M}_m\Big(\mathcal{M}_{U'}(z)\Big)$. This concludes the proof. □

In the proof of Proposition 3.3.2, we have proved:

Corollary 3.3.3. *Let $A(z,t) \in \mathrm{M}_m\Big(\mathcal{M}_U(z)\Big)$. The parameterized linear differential equation $\partial_z Y(z,t) = A(z,t)Y(z,t)$, is completely integrable if and only if the matrices of the topological generators for the Galois group appearing in Theorem 3.2.24 are conjugate over $\mathrm{GL}_m(\mathcal{M}_U)$ to constant matrices.*

Remark 3.3.4. This corollary improves Proposition 3.9 in [CS07]. The conjugation occurs in a field that is not differentially closed. Furthermore, we do not need for the entire parameterized differential Galois group to be conjugate to a group of constant matrices in order to deduce that the equation $\partial_z Y(z,t) = A(z,t)Y(z,t)$ is completely integrable.

In [GO12], the authors study completely integrable parameterized linear differential equations using differential Tannakian categories. In particular, they prove that the notion of integrability with respect to all the parameters is equivalent to the notion of integrability with respect to each parameter separately, which generalizes [Dre12], Proposition 9. Furthermore, they improve Proposition 3.9 in [CS07] by avoiding the assumption that the field of constants is differentially closed.

3.3.2 On the hyper transcendence of a Stokes matrix.

In this subsection, we will study the parameterized linear differential equation:

$$\partial_z^2 Y(z,t) = (z^3+t)Y(z,t). \tag{3.3.1}$$

Sibuya, in Chapter 2 of [Sib75], shows that there exists a formal solution $y_0(z,t)$ which admits an asymptotic representation $\widetilde{y}_0(z,t)$ on the sector (see Theorem 6.1 in [Sib75]):

$$\left\{z \in \widetilde{\mathbb{C}} \middle| \arg(z) \in \left]\frac{-3\pi}{5}, \frac{3\pi}{5}\right[\right\}.$$

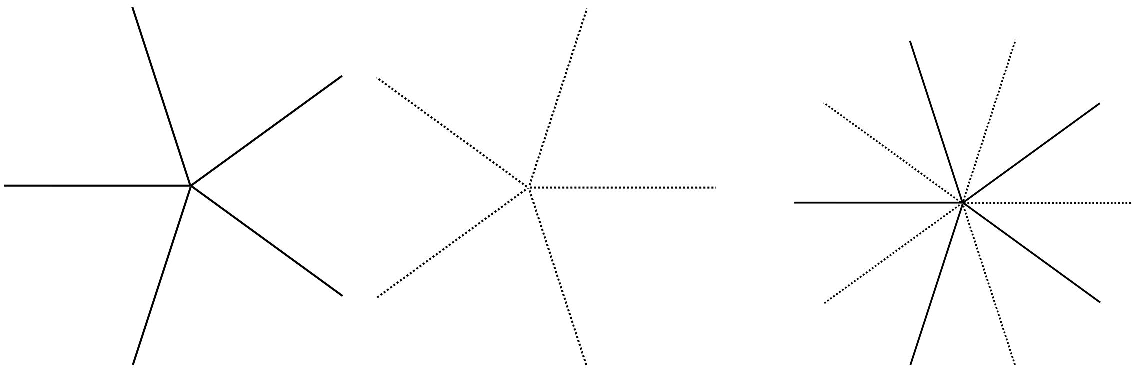
We easily check that for $k \in \mathbb{Z}$,

$$y_k(z,t) = y_0\left(e^{\frac{-2ki\pi}{5}}z, e^{\frac{-6ki\pi}{5}}t\right)$$

is a solution of (3.3.1) which has the asymptotic representation $\widetilde{y}_k(z,t) = \widetilde{y}_0\left(e^{\frac{-2ki\pi}{5}}z, e^{\frac{-6ki\pi}{5}}t\right)$ on the sector $S_{k-1} \cup \bar{S}_k \cup S_{k+1}$, where

$$S_k = \left\{z \in \widetilde{\mathbb{C}} \middle| \arg(z) \in \left]\frac{(2k-1)\pi}{5}, \frac{(2k+1)\pi}{5}\right[\right\},$$

and $\bar{S}_k$ is its closure.



The sectors S_k The singular directions Sectors S_k and singular directions

The asymptotic representation $\widetilde{y}_k(z,t)$ is bounded uniformly on each compact set in the t-plane as $|z|$ tends to infinity on the sector S_k, and tends to infinity uniformly on each compact set in the t-plane as $|z|$ tends to infinity on the sectors S_{k-1} and S_{k+1}. As we see in [Sib75], page 83, $y_{k+1}(z,t)$ and $y_{k+2}(z,t)$ are linearly independent and we can write $y_k(z,t)$ as a $\mathcal{M}_{\mathbb{C}}$-linear combination of $y_{k+1}(z,t)$ and $y_{k+2}(z,t)$:

$$\forall k \in \mathbb{N}, \forall z,t \in \mathbb{C},\ y_k(z,t) = C_k(t)y_{k+1}(z,t) + \widetilde{C}_k(t)y_{k+2}(z,t), \tag{3.3.2}$$

where $\widetilde{C}_k(t), C_k(t) \in \mathcal{M}_{\mathbb{C}}$. By Theorem 21.1 in [Sib75], we obtain that

$$\widetilde{C}_k(t) = -e^{\frac{2i\pi}{5}} \text{ and } C_k(t) = C_0\left(e^{\frac{-6ki\pi}{5}}t\right).$$

In [Sib75], the author asks if $C_0(t)$ is differentially transcendental, i.e., satisfies no differential polynomial equations. We will use Galois theory to prove that for every non empty polydisc U, $C_0(t)$ is not ∂_t-finite over $\mathcal{M}_U$, i.e., satisfies no linear differential equations in

coefficients in $\mathcal{M}_U$.
The singularity of the system is at infinity. Let $W(z,t) = zY(z^{-1},t)$. We obtain the parameterized linear differential equation

$$z^7\partial_z^2 W(z,t) = (1+tz^3)W(z,t), \tag{3.3.3}$$

which can be written in the form

$$\partial_z \begin{pmatrix} W(z,t) \\ \partial_z W(z,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1+tz^3}{z^7} & 0 \end{pmatrix} \begin{pmatrix} W(z,t) \\ \partial_z W(z,t) \end{pmatrix}.$$

Let k be a so-called universal (Δ_t)-field of characteristic 0: for any (Δ_t)-field $k_0 \subset k$, (Δ_t)-finitely generated over $\mathbb{Q}$, and any (Δ_t)-finitely generated extension k_1 of k_0, there is a (Δ_t)-differential k_0-isomorphism of k_1 into k. See Chapter 3, Section 7 of [Kol73] for more details. In particular, k is (Δ_t)-differentially closed. Let $k(z)$ denotes the (∂_z, Δ_t)-differential field of rational functions in the indeterminate z, with coefficients in k, where z is a (Δ_t)-constant with $\partial_z z = 1$, k is the field of constants with respect to ∂_z, and ∂_z commutes with all the derivations.

Let $A(z,t) = \begin{pmatrix} 0 & 1 \\ \frac{1+tz^3}{z^7} & 0 \end{pmatrix}$. The two solutions $zy_1(z^{-1},t)$, $zy_2(z^{-1},t)$ admit asymptotic representation and the only singularity is 0. Therefore,

$$\mathcal{M}_U(z)\Big\langle y_1(z^{-1},t), y_2(z^{-1},t)\Big\rangle_{\partial_z,\partial_t}\Big|\mathcal{M}_U(z) = \widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)$$

is a parameterized Picard-Vessiot extension for $\partial_z W(z,t) = A(z,t)W(z,t)$. Because of Proposition 3.2.23, $\widetilde{k(z)}\Big|k(z) = k(z)\Big\langle y_1(z^{-1},t), y_2(z^{-1},t)\Big\rangle_{\partial_z,\partial_t}\Big|k(z)$ is a parameterized Picard-Vessiot extension.

Lemma 3.3.5. *$Gal_{\partial_z}^{\Delta_t}\left(\widetilde{k(z)}\Big|k(z)\right) = \mathrm{SL}_2(k)$.*

Notice that the differential equation is of the form $\partial_z^2 W(z,t) = r(z,t)W(z,t)$, where $r(z,t) \in k(z)$. In this case, we can compute the Galois group using a parameterized version of Kovacic's algorithm, see [Arr12, Dre12]. In order to have a self contained proof, we will perform the calculations explicitly.

Proof. If we apply Kovacic's algorithm (see [Kov86]), we find that the unparameterized differential Galois group $Gal_{\partial_z}\left(\widetilde{k(z)}\Big|k(z)\right)$ is equal to $\mathrm{SL}_2(k)$. We apply Proposition 6.26 in [HS08], to deduce that $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{k(z)}\Big|k(z)\right)$ is Zariski-dense in $\mathrm{SL}_2(k)$. By Proposition 42 in [Cas72], we deduce that there are two possibilities:

- $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{k(z)}\Big|k(z)\right) = \mathrm{SL}_2(k)$
- $Gal_{\partial_z}^{\Delta_t}\left(\widetilde{k(z)}\Big|k(z)\right)$ is conjugate to $\mathrm{SL}_2(k_0)$ over $\mathrm{SL}_2(k)$, where

$$k_0 = \{a \in k(z) | \partial_z a = \partial_t a = 0\}.$$

We see in [Dre12], Remark 4.4, that the last case occurs if and only if the following parameterized differential equation has a solution in $\mathcal{M}_U(z)$, for some non empty polydisc U of $\mathbb{C}^n$:

$$\partial_z^3 y(z,t) = \partial_z y(z,t)\frac{4+4tz^3}{z^7} + y(z,t)\partial_z\frac{4+4tz^3}{z^7} - \partial_t\frac{4+4tz^3}{z^7}.$$

With the algorithm presented in [vdPS03] p.100, we find that this does not happen and then:
$$Gal_{\partial_z}^{\Delta_t}\left(\widetilde{k(z)}\middle|k(z)\right)=\mathrm{SL}_2(k).$$

□

Lemma 3.3.6. *The singular directions of the equation (3.3.3) are:*

$$\frac{2k\pi}{5}, \textit{with } k\in\mathbb{Z}.$$

Proof. Let $k\in\mathbb{Z}$. The matrix

$$\begin{pmatrix} zy_k(z^{-1},t) & zy_{k+1}(z^{-1},t)\\ \partial_z zy_k(z^{-1},t) & \partial_z zy_{k+1}(z^{-1},t)\end{pmatrix},$$

is a fundamental solution for the equation

$$\partial_z\begin{pmatrix} W(z,t)\\ \partial_z W(z,t)\end{pmatrix}=\begin{pmatrix} 0 & 1\\ \frac{1+tz^3}{z^7} & 0\end{pmatrix}\begin{pmatrix} W(z,t)\\ \partial_z W(z,t)\end{pmatrix}.$$

The fundamental solution admits an asymptotic representation on the sectors:

$$\left\{z\in\widetilde{\mathbb{C}}\middle|\arg(z)\in\left]\frac{(2k-1)\pi}{5},\frac{(2k+3)\pi}{5}\right[\right\}.$$

The only level is $\frac{5}{2}$. From Proposition 3.1.13 and the construction of the singular directions, we find that the singular directions are

$$\frac{2k\pi}{5}, \text{ with } k\in\mathbb{Z}.$$

□

Example 3.3.7. We want to compute the Stokes matrix in the direction $\frac{8\pi}{5}$ for the fundamental solution:
$$\begin{pmatrix} zy_1(z^{-1},t) & zy_2(z^{-1},t)\\ \partial_z zy_1(z^{-1},t) & \partial_z zy_2(z^{-1},t)\end{pmatrix}.$$

We recall the construction of the Stokes matrices. See §3.1.3 for the notations. Let $\hat{H}(z,t)z^{L(t)}e\Big(Q(z,t)\Big)$ be a fundamental solution in the parameterized Hukuhara-Turrittin canonical form. Let $H^-(z,t)$ (resp. $H^+(z,t)$) be the matrix such that

$$H^-(z,t)e^{L(t)\log(z)}e^{Q(z,t)}\quad\Big(\text{resp. } H^+(z,t)e^{L(t)\log(z)}e^{Q(z,t)}\Big)$$

is the germ of an asymptotic solution on the sector

$$\left\{z\in\widetilde{\mathbb{C}}\middle|\arg(z)\in\left]\pi,\tfrac{9\pi}{5}\right[\right\}\quad\left(\text{resp. }\left\{z\in\widetilde{\mathbb{C}}\middle|\arg(z)\in\left]\tfrac{7\pi}{5},\tfrac{11\pi}{5}\right[\right\}\right)$$

The Stokes matrix in the direction $\frac{8\pi}{5}$ is the matrix that sends

$$H^-(z,t)e^{L(t)\log(z)}e^{Q(z,t)} \text{ to } H^+(z,t)e^{L(t)\log(z)}e^{Q(z,t)}.$$

With the domain of definition of the asymptotic representation of $z\widetilde{y}_1(z^{-1},t)$, we deduce from the definition of the Stokes operators that:

$$St_{\frac{8\pi}{5}}\Big(zy_1(z^{-1},t)\Big) = zy_1(z^{-1},t). \tag{3.3.4}$$

We first write $St_{\frac{8\pi}{5}}\Big(zy_2(z^{-1},t)\Big)$ in the basis

$$\Big(zy_0(z^{-1},t), zy_1(z^{-1},t)\Big).$$

There exist $a(t)$ and $b(t) \in \mathcal{M}_U$ such that:

$$St_{\frac{8\pi}{5}}\Big(zy_2(z^{-1},t)\Big) = a(t)zy_0(z^{-1},t) + b(t)zy_1(z^{-1},t).$$

By the construction of the asymptotic solutions with Laplace and Borel transforms (see Proposition 3.1.10), the asymptotic representation of $St_{\frac{8\pi}{5}}\Big(zy_2(z^{-1},t)\Big)$ has to be bounded in some sector of $\Big]\frac{7\pi}{5}, \frac{11\pi}{5}\Big[$, which means that there exist $\frac{7\pi}{5} < \alpha < \beta < \frac{11\pi}{5}$ and $\varepsilon > 0$ such that $St_{\frac{8\pi}{5}}\Big(zy_2(z^{-1},t)\Big)$ is uniformly bounded for $\arg(z) \in]\alpha,\beta[$ and $z < |\varepsilon|$. Therefore, $a(t) = 0$ or $b(t) = 0$. Since the Stokes operators are automorphisms, we get $b(t) = 0$. Lemma 3.3.5 says that the parameterized differential Galois group is $\mathrm{SL}_2(k)$. Therefore, because of Proposition 3.2.23 and Lemma 3.2.17, the determinant of the matrix has to be 1. Thus by (3.3.2), we get that the Stokes matrix in direction $\frac{8\pi}{5}$ is:

$$St_{\frac{8\pi}{5}} = \begin{pmatrix} 1 & -C_0(t)e^{\frac{3i\pi}{5}} \\ 0 & 1 \end{pmatrix}.$$

Lemma 3.3.8. *Let $C_0(t)$ be defined as above. Assume that $C_0(t)$ is ∂_t-finite over k. Then, the ∂_t-differential transcendence degree (see* §3.2.1 *for definition) of $\widetilde{k(z)}$ over $k(z)$ is at most* 2.

Proof. The extension field $\widetilde{k(z)}$ is generated over $k(z)$ by $y_1(z^{-1},t)$ and $y_2(z^{-1},t)$. By the parameterized differential Galois correspondence (see Theorem 9.5 in [CS07]), the Kolchin closure of the group generated by $St_{\frac{8\pi}{5}}$ is equal to

$$Gal_{\partial_z}^{\Delta_t}\left(\widetilde{k(z)}\Big|F\right),$$

where F is the subfield of $\widetilde{k(z)}$ fixed by $St_{\frac{8\pi}{5}}$. Using (3.3.4), we deduce that F contains

$$k(z)\Big\langle y_1(z^{-1},t)\Big\rangle_{\partial_z,\partial_t}.$$

Because $C_0(t)$ satisfies a linear differential equation with coefficients in k, there exists P, a linear differential polynomial such that this group is of the form

$$\left\{\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \text{ with } P(\alpha) = 0 = P(C_0(t))\right\},$$

and has ∂_t-differential dimension over k equals to 0. Therefore by Proposition 3.2.7, the ∂_t-differential transcendence degree of $\widetilde{k(z)}$ over F is equal to 0. Because of the fact that F

contains $k(z)\Big\langle y_1(z^{-1},t)\Big\rangle_{\partial_z,\partial_t}$, there exists a differential polynomial Q with coefficients in $k(z)$ such that:

$$Q\Big(y_1(z^{-1},t), y_2(z^{-1},t)\Big) = 0 = Q\Big(\partial_z(y_1(z^{-1},t)), \partial_z(y_2(z^{-1},t))\Big).$$

Therefore, the ∂_t-differential transcendence degree of $\widetilde{k(z)}$ over $k(z)$ is at most 2, because $\widetilde{k(z)}$ is generated as a ∂_t-differential field over $k(z)$ by

$$\Big\{y_1(z^{-1},t), y_2(z^{-1},t), \partial_z(y_1(z^{-1},t)), \partial_z(y_2(z^{-1},t))\Big\}.$$

□

Theorem 3.3.9. *The function $C_0(t)$ is not ∂_t-finite over k.*

Proof. As we see from Lemma 3.3.5,

$$Gal_{\partial_z}^{\Delta_t}\left(\widetilde{k(z)}\middle|k(z)\right) = \mathrm{SL}_2(k).$$

Therefore, by Proposition 3.2.7, the ∂_t-differential transcendence degree of $\widetilde{k(z)}$ over $k(z)$ is 3. If $C_0(t)$ was ∂_t-finite over k, because of Lemma 3.3.8, the ∂_t-differential transcendence degree of $\widetilde{k(z)}$ over $k(z)$ would be smaller than 3. Therefore, $C_0(t)$ is not ∂_t-finite over k. □

3.3.3 Which linear differential algebraic groups are parameterized differential Galois groups?

As in §3.3.2, let k be a so-called universal (Δ_t)-field of characteristic 0. Let us consider an equation $\partial_z Y(z,t) = A(z,t)Y(z,t)$, with $A(z,t) \in \mathrm{M}_m(k(z))$, let $\widetilde{k(z)}\Big|k(z)$ be the parameterized Picard-Vessiot extension, and let $G = Gal_{\partial_z}^{\Delta_t}\left(\widetilde{k(z)}\middle|k(z)\right) \subset \mathrm{GL}_m(k)$ be the parameterized differential Galois group defined in §3.2.1. The following theorem of Seidenberg, applied with $K_0 = \mathbb{Q}$ and K_1, the (Δ_t)-field generated by $\mathbb{Q}$ and the z-coefficients of $A(z,t)$, tells us that there exists a non empty polydisc U such that $A(z,t)$ may be seen as an element of $\mathrm{M}_m(\mathcal{M}_U(z))$.

Theorem 3.3.10 (Seidenberg,[Sei58, Sei69])**.** *Let $\mathbb{Q} \subset K_0 \subset K_1$ be finitely generated (Δ_t)-differential extensions of $\mathbb{Q}$, and assume that K_0 consists of meromorphic functions on some domain U of $\mathbb{C}^n$. Then, K_1 is isomorphic to the field K_1^* of meromorphic functions on a non empty polydisc $U' \subset U$ such that $K_0|_{U'} \subset K_1^*$, and the derivations in Δ_t can be identified with the derivations with respect to the coordinates on U'.*

Let $\widetilde{\mathcal{M}_U(z)}\Big|\mathcal{M}_U(z)$ be the parameterized Picard-Vessiot extension defined in §3.2.5 and let $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right)$ be the parameterized differential Galois group. Using Corollary 3.2.25, we find that $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right)$ contains a finitely generated subgroup that is Kolchin-dense in $Aut_{\partial_z}^{\Delta_t}\left(\widetilde{\mathcal{M}_U(z)}\middle|\mathcal{M}_U(z)\right)$. With Proposition 3.2.23, we find that G contains a finitely generated subgroup that is Kolchin-dense in G. Combined with Corollary 5.2 in [MS12], which gives the sufficiency of the condition, this yields the following result:

Theorem 3.3.11 (Inverse problem)**.** *Let G be a linear differential algebraic subgroup of* $\mathrm{GL}_m(k)$*. Then, G is the global parameterized differential Galois group of some equation having coefficients in $k(z)$ if and only if G contains a finitely generated subgroup that is Kolchin-dense in G.*

In the unparameterized case, any linear algebraic group defined over $\mathbb{C}$ is a Galois group of a Picard-Vessiot extension (see [TT79]). In fact, every linear algebraic group defined over $\mathbb{C}$ contains a finitely generated subgroup that is Zariski-dense, which means that Theorem 3.3.11 is a generalization of the result in [TT79].

The situation is more complicated in the parameterized case. For example, the additive group:

$$\left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \text{ with } \alpha \in k \right\},$$

is not the global parameterized differential Galois group of any equation having coefficients in $k(z)$ (see Section 7 of [CS07]). In the parameterized case with only regular singular poles, the problem has been solved in [MS12], Corollary 5.2: they obtain the same necessary and sufficient condition on the group than Theorem 3.3.11. In [Sin13], the author characterizes the linear algebraic subgroups of $\mathrm{GL}_m(k)$ that appear as the global parameterized differential Galois groups of some equation having coefficients in $k(z)$: they are the groups such that the identity component has no quotient isomorphic to the additive group $(k,+)$ or multiplicative group $(k^*,\times)$ of k.

Annexe A

Appendice du Chapitre 3.

Let us keep the same notations as in §3.1.1 and §3.1.2. The goal of the appendix is to prove the following theorem. Notice that our proof follows closely the unparameterized case, see [BJL80, LR01]. See Remark 3.1.6 for a discussion of another similar result.

Theorem A.1. *Let $\partial_z Y(z,t) = A(z,t)Y(z,t)$, with $A(z,t) \in \mathrm{M}_m\Big(\hat{K}_U\Big)$. There exists a non empty polydisc $U' \subset U$ such that we have a fundamental solution of the form*

$$\hat{P}(z,t)z^{C(t)}e\Big(Q(z,t)\Big) \in \mathrm{GL}_m\left(\widehat{\boldsymbol{K}_{U'}}\right),$$

with:

- $\hat{P}(z,t) \in \mathrm{GL}_m\Big(\hat{K}_{U'}\Big)$.
- $C(t) \in \mathrm{M}_m(\mathcal{M}_{U'})$.
- $e\Big(Q(z,t)\Big) = \mathrm{Diag}\Big(e(q_i(z,t))\Big)$, *with* $q_i(z,t) \in \boldsymbol{E}_{U'}$.

Moreover, we may choose the same non empty polydisc U' as in Proposition 3.1.3. Combined with Remark 3.1.6, if $A(z,t) \in \mathrm{M}_m\Big(\mathcal{O}_U(\{z\})\Big)$, this gives a sufficient condition on $t_0 \in U$, to have a fundamental solution $\hat{P}(z,t)z^{C(t)}e\Big(Q(z,t)\Big) \in \mathrm{GL}_m\Big(\widehat{\boldsymbol{K}_{U'}}\Big)$ in the same form as above with $t_0 \in U'$.

Remark that contrary to Proposition 3.1.3, $\hat{H}(z,t) \in \mathrm{GL}_m\Big(\hat{K}_{U'}\Big)$. On the other hand, we loose the commutation between $z^{C(t)}$ and $e\Big(Q(z,t)\Big)$. Before giving the proof of the theorem, we state and prove two lemmas.

Lemma A.2. *Let $U' \subset U$ be a non empty polydisc. Let $a(t) \in \mathcal{M}_{U'}$ and $\alpha(z,t) \in \hat{K}_{F,U'}$ such that $\hat{m}(\alpha(z,t)) = a(t)\alpha(z,t)$. Then there exists $\hat{h}(z,t) \in \hat{K}_{U'}$ and $b(t) \in \mathcal{M}_{U'}$ such that $\alpha(z,t) = \hat{h}(z,t)z^{b(t)}$.*

Proof. Let $\alpha(z,t) \in \hat{K}_{F,U'}$ such that $\hat{m}(\alpha(z,t)) = a(t)\alpha(z,t)$. The element $\alpha(z,t)$ belongs to the fraction field of a free polynomial ring:

$$P = \hat{K}_{U'}\left[\log, z^{b_1(t)}, \dots, z^{b_k(t)}\right].$$

Write $\alpha(z,t) = \alpha_1(z,t)/\alpha_2(z,t)$ with gcd in P equals to 1. Using the relations in $\hat{K}_{F,U'}$, and applying $\hat{m}$ to $\alpha_1(z,t)/\alpha_2(z,t)$, we find that $\alpha(z,t)$ contains no terms in log. One can normalize $\alpha_2(z,t)$ such that it contains a term of the form $z^{n_1 b_1(t)+\cdots+n_k b_k(t)}$ with coefficient 1 and $n_i \in \mathbb{Z}$. Using $\hat{m}(\alpha_1(z,t)/\alpha_2(z,t)) = a(t)\alpha_1(z,t)/\alpha_2(z,t)$, we find that

$$\hat{m}(\alpha_2(z,t)) = e^{2i\pi(n_1 b_1(t)+\cdots+n_k b_k(t))}\alpha_2(z,t)$$

and

$$\hat{m}(\alpha_1(z,t)) = a(t)e^{2i\pi(n_1 b_1(t)+\cdots+n_1 b_1(t))}\alpha_1(z,t),$$

which is impossible, unless

$$e^{2i\pi(n_1 b_1(t)+\cdots+n_k b_k(t))} = 1.$$

This means that $\alpha_2(z,t) \in \hat{K}_{U'}$ and we may assume $\alpha_2(z,t) = 1$. Applying $\hat{m}$ to $\alpha_1(z,t)$, one finds that $\alpha_1(z,t)$ contains at most one term, that is $\alpha(z,t) = \hat{h}(z,t)z^{b(t)}$, with $\hat{h}(z,t) \in \hat{K}_{U'}$ and $b(t) \in \mathcal{M}_{U'}$ that satisfies $e^{2i\pi b(t)} = a(t)$. □

Lemma A.3. *Let $U' \subset U$ be a non empty polydisc. Let us consider $A(z,t) \in \mathrm{M}_m\Big(\hat{K}_{U'}\Big)$. Let $F_1(z,t)e\Big(Q_1(z,t)\Big)$ and $F_2(z,t)e\Big(Q_2(z,t)\Big)$ be two fundamental solutions of:*

$$\partial_z Y(z,t) = A(z,t)Y(z,t),$$

satisfying, for $i \in \{1;2\}$, $F_i(z,t) \in \mathrm{GL}_m\Big(\hat{K}_{F,U'}\Big)$ and $Q_i(z,t) = \mathrm{Diag}[q_{i,j}(z,t)]$ such that $q_{i,j}(z,t)$ belongs to $\boldsymbol{E}_{U'}$. Then, $F_1(z,t)^{-1}F_2(z,t) \in \mathrm{GL}_m(\mathcal{M}_{U'})$.

Proof. A straightforward computation shows that:

$$\partial_z\left(\Big(F_1(z,t)e(Q_1(z,t))\Big)^{-1}F_2(z,t)e(Q_2(z,t))\right) = 0.$$

By Proposition 3.2.19,

$$\Big(F_1(z,t)e(Q_1(z,t))\Big)^{-1}F_2(z,t)e(Q_2(z,t)) = C(t) \in \mathrm{GL}_m(\mathcal{M}_{U'}).$$

Hence, we have the equality:

$$e(Q_1(z,t))C(t)e(-Q_2(z,t)) = F_1(z,t)^{-1}F_2(z,t).$$

The entries of $e(Q_1(z,t))C(t)e(-Q_2(z,t))$ are of the form $C_{i,j}(t)e(q_{1,j}(z,t) - q_{2,j}(z,t))$, with $C_{i,j}(t)$ that belongs to $\mathcal{M}_{U'}$, and the matrix $F_1(z,t)^{-1}F_2(z,t)$ belongs to $\mathrm{GL}_m\Big(\hat{K}_{F,U'}\Big)$. By construction,

$$\hat{K}_{F,U'} \cap \mathcal{M}_{U'}\Big(\left(e(q(z,t))\right)_{q(z,t)\in\mathbf{E}_{U'}}\Big) = \mathcal{M}_{U'},$$

and we obtain that:

$$F_1(z,t)^{-1}F_2(z,t) \in \mathrm{GL}_m(\mathcal{M}_{U'}).$$

□

Proof of Theorem A.1. By Proposition 3.1.3, we know that we have a fundamental solution of the parameterized linear differential equation $\partial_z Y(z,t) = A(z,t)Y(z,t)$ of the form:

$$\hat{H}(z,t)z^{L(t)}e\Big(Q(z,t)\Big),$$

with $\hat{H}(z,t) \in \mathrm{GL}_m\Big(\hat{K}_{U'}[z^{1/\nu}]\Big)$ and $\nu \in \mathbb{N}^*$. From Definition 3.2.13, $\hat{m}$ commutes with the derivation ∂_z, and therefore $\hat{m}\left(\hat{H}(z,t)z^{L(t)}e(Q(z,t))\right)$ is another fundamental solution. From the construction of $\hat{m}$, we deduce that $\hat{m}\left(\hat{H}(z,t)z^{L(t)}\right) \in \mathrm{GL}_m\Big(\hat{K}_{F,U'}\Big)$, and we can apply Lemma A.3 to deduce the existence of $\hat{M}(t) \in \mathrm{GL}_m(\mathcal{M}_{U'})$ such that:

$$\hat{m}\left(\hat{H}(z,t)z^{L(t)}\right) = \hat{H}(z,t)z^{L(t)}\hat{M}(t). \tag{A.0.1}$$

Let us write $\hat{M}(t) = D(t)U(t)$, with $D(t)$ diagonalizable and $U(t)$ unipotent such that $D(t)U(t) = U(t)D(t)$, the multiplicative analogue of the Jordan decomposition of $\hat{M}(t)$. If $a(t)$ is an eigenvalue of $D(t)$ (and therefore an eigenvalue of $\hat{M}(t)$), then there exists $0 \neq \alpha(z,t) \in \hat{K}_{F,U'}$ such that $\hat{m}(\alpha(z,t)) = a(t)\alpha(z,t)$, because of the relation (A.0.1). By Lemma A.2, $\alpha(z,t)$ is equal to $\hat{h}(z,t)z^{b(t)}$, with $b(t) \in \mathcal{M}_{U'}$ satisfying $e^{2i\pi b(t)} = a(t)$ and $\hat{h}(z,t) \in \hat{K}_{U'}$. This implies that $a(t)$ and all the eigenvalues of $D(t)$ are of the form $e^{\beta(t)}$, with $\beta(t) \in \mathcal{M}_{U'}$. So we have proved the existence of $C(t) \in \mathrm{M}_m(\mathcal{M}_{U'})$ such that $e^{2i\pi C(t)} = \hat{M}(t)$. Let:

$$\hat{P}(z,t) = \hat{H}(z,t)z^{L(t)}z^{-C(t)}.$$

A computation shows that the monodromy of $z^{C(t)}$ is:

$$\hat{m}\left(z^{C(t)}\right) = e^{2i\pi C(t)}z^{C(t)} = z^{C(t)}e^{2i\pi C(t)}.$$

The matrix $\hat{P}(z,t)$ is fixed by the monodromy and therefore belongs to $\mathrm{GL}_m\Big(\hat{K}_{U'}\Big)$, because of Proposition 3.2.19. Finally,

$$\hat{P}(z,t)z^{C(t)}e\Big(Q(z,t)\Big)$$

is a fundamental solution of the parameterized linear differential equation $\partial_z Y(z,t) = A(z,t)Y(z,t)$ that has the required property. □

Deuxième partie

Équations aux q-différences

Chapitre 4

Confluence of meromorphic solutions of q-difference systems.

Sommaire

Résumé: In this chapter, we consider a q-analogue of the Borel-Laplace summation where $q > 1$ is a real parameter. In particular, we show that the Borel-Laplace summation of a divergent power series solution of a linear differential equation can be uniformly approximated on a convenient sector, by a meromorphic solution of a corresponding family of linear q-difference equations. We perform the computations for the basic hypergeometric series. Following Sauloy, we prove how a basis of solutions of a linear differential equation can be uniformly approximated on a convenient domain by a basis of solutions of a corresponding family of linear q-difference equations. This leads us to the approximations of Stokes matrices and monodromy matrices of the linear differential equation by matrices with entries that are invariants by the multiplication by q.

Introduction

When q tends to 1, the q-difference operator $d_q := f \mapsto \frac{f(qz)-f(z)}{(q-1)z}$ "tends" to the usual derivation. Hence every differential equation may be discretized by a q-difference equation. Given a linear differential equation $\widetilde{\Delta}$ and a family of linear q-difference equations Δ_q that discretize $\widetilde{\Delta}$, we wonder if there exists a basis of solutions of Δ_q, that converges as q goes to 1 to a given basis of solutions of $\widetilde{\Delta}$. This question has been studied in the Fuchsian case (see [Sau00]) and the main goal of this chapter is to consider the general situation. The problem is that for non-Fuchsian linear differential equations, the fundamental solution, i.e., the invertible solution matrix, given by the Hukuhara-Turrittin theorem involves divergent formal power series. However, we may apply to them a Borel-Laplace summation process in order to obtain a fundamental solution that is analytic on a convenient sector. To extend the work of Sauloy to the non-Fuchsian case, we have to approximate the Borel-Laplace summation of a given formal power series solution of a linear differential equation, by a q-analogue of the Borel-Laplace summation applied to a formal power series solution of a corresponding family of linear q-difference equations. Our main result, Theorem 4.4.5, gives a confluence* result of this nature. Then, we use our main result to prove that under convenient assumptions, a basis of meromorphic solutions of a linear differential equation, not necessarily Fuchsian, can be uniformly approximated on a convenient domain by a basis of solutions of a corresponding family of linear q-difference equations. This leads us to the approximations of Stokes matrices and monodromy matrices of the linear differential equation by matrices with entries that are invariants by the multiplication by q. We also perform the computations for the basic hypergeometric series.

* * *

Let $q > 1$ be a real parameter, and let us define the dilatation operator σ_q

$$\sigma_q\big(f(z)\big) := f(qz).$$

See Remark 4.4.6 for the reason why we consider q real, and not q complex number such that $|q| > 1$, like others papers present in the literature. We define $\delta_q := \frac{\sigma_q - \mathrm{Id}}{q-1}$, which converges formally to $\delta := z\frac{d}{dz}$ when $q \to 1$. Let us consider

$$\begin{cases} \delta_q Y(z,q) &=& B(z)Y(z,q) \\ \\ \delta \widetilde{Y}(z) &=& B(z)\widetilde{Y}(z), \end{cases}$$

where $B(z) \in \mathrm{M}_m\Big(\mathbb{C}(z)\Big)$, that is a m by m square matrix with coefficients in $\mathbb{C}(z)$. We are going to recall the main result of [Sau00] in the particular case where the above matrix $B(z)$ does not depend upon q and $q > 1$ is real. Notice that a part of what follows now is purely local at $z = 0$, which means that we could consider systems that have coefficients in the field of germs of meromorphic functions in the neighborhood of $z = 0$, but for the simplicity of exposition, we have assumed that the coefficients are rational. In [Sau00], Sauloy assumes that the systems are Fuchsian at 0 and the linear differential system has exponents at 0 which are non resonant (see [Sau00], §1, for a precise definition). The Frobenius algorithm provides a local fundamental solution at $z = 0$, $\widetilde{\Phi}_0(z)$, of the linear differential system $\delta\widetilde{Y}(z) = B(z)\widetilde{Y}(z)$. This solution can be analytically continued

*. Throughout the chapter, we will use the word "confluence" to describe the q-degeneracy when $q \to 1$.

into an analytic solution on $\mathbb{C}^*$, minus a finite number of lines and half lines of the form $\mathbb{R}_{>0}\alpha := \left\{x\alpha \middle| x \in]0,\infty[\right\}$ and $\mathbb{R}_{\geq 1}\beta := \left\{x\beta \middle| x \in [1,\infty[\right\}$, with $\alpha, \beta \in \mathbb{C}^*$. Notice that in Sauloy's paper, the lines and half lines are in fact respectively q-spirals and q-half-spirals since the author considers the case where q is a complex number such that $|q| > 1$.

In [Sau00], §1, the author uses a q-analogue of the Frobenius algorithm to construct a local fundamental matrix solution at $z = 0$, $\Phi_0(z,q)$, of the family of linear q-difference systems $\delta_q Y(z,q) = B(z)Y(z,q)$, which is for a fixed q, meromorphic on $\mathbb{C}^*$ and has its poles contained in a finite number of q-spirals of the form $q^{\mathbb{Z}}\alpha := \{q^n\alpha, n \in \mathbb{Z}\}$ and $q^{\mathbb{N}^*}\beta := \{q^n\beta, n \in \mathbb{N}^*\}$, with $\alpha, \beta \in \mathbb{C}^*$. Sauloy proves that $\Phi_0(z,q)$ converges uniformly to $\widetilde{\Phi}_0(z)$ when $q \to 1$, in every compact subset of its domain of definition.

Let us assume that the systems are Fuchsian at ∞ and the linear differential system has exponents at ∞ which are non resonant. Let us consider $\Phi_\infty(z,q)$ and $\widetilde{\Phi}_\infty(z)$, the corresponding fundamental solutions at infinity of the linear δ and δ_q-systems. Sauloy shows that the Birkhoff connection matrix $P(z,q) := \Big(\Phi_\infty(z,q)\Big)^{-1}\Phi_0(z,q)$, which is invariant under the action of σ_q, converges to $\widetilde{P}(z) := \Big(\widetilde{\Phi}_\infty(z)\Big)^{-1}\widetilde{\Phi}_0(z)$ when $q \to 1$. The matrix $\widetilde{P}(z)$ is locally constant and the monodromy matrices at the intermediates singularities (those different from 0 and ∞) of the linear differential system can be expressed with the values of $\widetilde{P}(z)$.

The goal of this chapter is to prove similar results in the non-Fuchsian case. The question implies difficulties of very different nature than in the Fuchsian case, since divergent formal power series may appear as solutions. The prototypical example is the Euler equation and one possible q-deformation:

$$\left\{\begin{array}{lllll} z\delta_q y(z,q) & + & y(z,q) & = & z \\ \\ z\delta\tilde{y}(z) & + & \tilde{y}(z) & = & z, \end{array}\right.$$

which admits respectively the formal divergent solutions:

$$\sum_{n=0}^{\infty}(-1)^n[n]_q^! z^{n+1}, \text{ and } \sum_{n=0}^{\infty}(-1)^n n! z^{n+1},$$

where $[n]_q^! := \prod_{l=0}^{n}[l]_q$, $[l]_q := \Big(1 + \ldots + q^{l-1}\Big)$ if $l \in \mathbb{N}^*$, and $[0]_q := 1$. In this example, the first formal power series converges coefficientwise to the second when $q \to 1$. However, there exist also analytic solutions of the linear differential equation. For example, if $d \not\equiv \pi[2\pi]$ the following functions are solutions:

$$\int_0^{\infty e^{id}} \frac{e^{-\zeta/z}}{1+\zeta}d\zeta.$$

More generally, given a formal power series solution of a linear differential equation in coefficients that are germs of meromorphic functions, it is well known (see §4.1) that we may apply to it several Borel and Laplace transformations to obtain a germ of analytic solution on a sector of the form

$$\overline{S}(a,b) := \left\{z \in \widetilde{\mathbb{C}} \middle| \arg(z) \in]a,b[\right\},$$

where $\widetilde{\mathbb{C}}$ denotes the Riemann surface of the logarithm.

The situation is similar in the q-difference case. Consider a linear q-difference system with coefficients that are germs of meromorphic functions, and assume that the slopes belongs to $\mathbb{Z}$ (see [RSZ13] for the definition). Like in the differential case, formal power series appear as solutions. The authors of [RSZ13] show how to transform a formal fundamental solution into fundamental solutions which entries are meromorphic on a punctured neighborhood of 0 in $\mathbb{C}^*$. Then, it is shown how the meromorphic fundamental solutions are linked with the local meromorphic classification of q-difference equations. It is natural to study the behavior, as q goes to 1 of their meromorphic fundamental solutions. Unfortunately, there are two difficulties for this approach:

- In [RSZ13] it is used the Birkhoff-Guenter normal form which has no known analogous in the differential case. Study the behavior of the normal form as q goes to 1 seems to be very complicated.
- Although there are several q-analogues of the Borel and Laplace transformations, see [DVZ09, MZ00, Ram92, RZ02, Zha99, Zha00, Zha01, Zha02, Zha03], we do not know how to express the meromorphic fundamental solutions using a q-analogue of the Borel-Laplace summation.

* * *

Let us state now our main result, Theorem 4.4.5, in a particular case. Let $z \mapsto \hat{h}(z,q), \widetilde{h}$ be formal power series solutions of

$$\begin{cases} b_m(z)\delta_q^m \hat{h}(z,q) & + \cdots + b_0(z)\hat{h}(z,q) & = & 0 \\ b_m(z)\delta^m \widetilde{h}(z) & + \cdots + b_0(z)\widetilde{h}(z) & = & 0, \end{cases}$$

where $b_0, \dots, b_m \in \mathbb{C}[z]$. We assume that $\hat{h}$ converges coefficientwise to $\widetilde{h}$ when $q \to 1$. We prove that for $q > 1$ sufficiently close to 1, we may apply to $\hat{h}$ several q-analogues of the Borel and Laplace transformation and obtain $S_q\left(\hat{h}\right)$, solution of the family of linear q-difference equations that is for q fixed meromorphic on $\mathbb{C}^*$. Moreover, $S_q\left(\hat{h}\right)$ converges uniformly on a convenient domain to the Borel-Laplace summation of $\widetilde{h}$ when $q \to 1$. Notice that although this theorem deal with a problem which is purely local at $z = 0$, we have assumed that the equations have coefficients in $\mathbb{C}[z]$, instead of the ring of germs of analytic functions, since we need this assumption to prove the theorem. Another result of same nature can be found in [DVZ09], Theorem 2.6. See Remark 4.4.7 for the comparison of the setting of this result and our theorem.

In the appendix, we introduce another q-Laplace transformation and prove an analogous result for the associated q-Borel-Laplace summation. See Theorem B.4.

In §4.7, we consider the basic hypergeometric series ${}_r\varphi_s$. Let us choose $r, s \in \mathbb{N}$ with $r > s+1$, $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s \in \mathbb{C} \setminus (-\mathbb{N})$ with different images in $\mathbb{C}/\mathbb{Z}$, let $\underline{p} := q^{-1/(r-s-1)}$, and consider, see [GR04],

$$\begin{aligned} &{}_r\varphi_s\left(\begin{matrix} \underline{p}^{\alpha_1}, \dots, \underline{p}^{\alpha_r} \\ \\ \underline{p}^{\beta_1}, \dots, \underline{p}^{\beta_s} \end{matrix} ; \underline{p}, \left(1-\underline{p}\right)^{1+s-r} z\right) \\ &:= \sum_{n=0}^{\infty} \frac{(\underline{p}^{\alpha_1};\underline{p})_n \dots (\underline{p}^{\alpha_r};\underline{p})_n \left(1-\underline{p}\right)^{(1+s-r)n}}{(\underline{p};\underline{p})_n(\underline{p}^{\beta_1};\underline{p})_n \dots (\underline{p}^{\beta_s};\underline{p})_n} \underline{p}^{-n(n-1)/2}(-1)^{n(1+s-r)} z^n, \end{aligned}$$

where $(a;\underline{p})_{n+1} := (1-a\underline{p}^n)(a;\underline{p})_n$ and $(a;\underline{p})_0 := 1$, for $a \in \mathbb{C}$. The above series converge coefficientwise when $q \to 1$ to

$$ {}_rF_s \left(\begin{matrix} \alpha_1, \dots, \alpha_r \\ \\ \beta_1, \dots, \beta_s \end{matrix} ; (-1)^{1+s-r} z \right) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_r)_n}{n!(\beta_1)_n \dots (\beta_s)_n} (-1)^{n(1+s-r)} z^n $$

where, $(\alpha)_{n+1} := (\alpha+n)(\alpha)_n$ and $(\alpha)_0 := 1$ for $\alpha \in \mathbb{C}^*$. We prove that the series ${}_r\varphi_s$ and ${}_rF_s$ do not satisfy the assumptions of our main result, Theorem 4.4.5. However, we perform explicitly the computation of a q-Borel-Laplace summation of ${}_r\varphi_s$, using others q-analogues of the Borel and Laplace transformations, and prove also the convergence when $q \to 1$ to the classical Borel-Laplace summation of ${}_rF_s$. See Theorem 4.7.4. See also [Zha02], §2, for the case $r=2, s=0$.

In §4.8, we apply our main result to prove that we can uniformly approximate on a convenient domain a basis of solutions of a linear differential equation by a basis of solutions of a corresponding family of linear q-difference equations. Our theorem holds in the non-Fuchsian case but does not recover Sauloy's result in the Fuchsian case. In other words, the two results are complementary.

In §4.8.2, we are interested in the case where the linear δ_q and δ-equations have formal coefficients and we want to prove the convergence, in a sense we specify later, of a basis of formal solutions of a family of linear δ_q-equations, to the Hukuhara-Turrittin solution of a linear δ-equation. A problem is the size of the field of constants. A fundamental solution of a linear differential system is defined modulo an invertible matrix with complex entries, while a fundamental solution of a linear q-difference system is defined modulo a matrix with entries in $\mathcal{M}_{\mathbb{E}}$, the field of functions invariant under the action of σ_q, i.e., the field of meromorphic functions over the torus $\mathbb{C}^* \setminus q^{\mathbb{Z}}$. This field can be identified with the field of elliptic functions. The consequence of this is that we have to choose very carefully our basis of solutions of the family of linear δ_q-equations in order to have the convergence. For example, if we consider

$$ \begin{cases} \delta_q y(z,q) &= (z^{-1}+1) y(z,q) \\ \\ \delta \widetilde{y}(z) &= (z^{-1}+1)\widetilde{y}(z), \end{cases} $$

the solutions of the linear δ-equation are of the form $\widetilde{y}(z) = a\left(e^{-z^{-1}}+z\right)$ with $a \in \mathbb{C}$. Let us introduce the Jacobi theta function

$$ \Theta_q(z) := \sum_{n\in\mathbb{Z}} q^{\frac{-n(n-1)}{2}} z^n = \prod_{n=0}^{\infty} \left(1-q^{-n-1}\right)\left(1+q^{-n-1}z\right)\left(1+q^{-n}z^{-1}\right), $$

which is analytic on $\mathbb{C}^*$, vanishes on the discrete q-spiral $-q^{\mathbb{Z}}$, with simple zeros, and satisfies:

$$ \sigma_q \Theta_q(z) = z\Theta_q(z); \qquad \Theta_q(z) = \Theta_q\left(q^{-1}z^{-1}\right). $$

The following function is solution of the δ_q-equation $y(z,q) = \dfrac{1}{\Theta_q(z)} \displaystyle\sum_{n=0}^{\infty} \frac{q^n z^n}{\prod_{k=0}^{n}(q^k - q + 1)}$, but the behavior as q goes to 1 is unclear. If we want to construct a solution of the family of linear δ_q-equations that converges to a solution of the linear δ-equation, we need to introduce the q-exponential:

$$ e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_q^!} = \prod_{n=0}^{\infty} (1+(q-1)q^{-n-1}z). $$

It is analytic on $\mathbb{C}$, with simple zeros on the discrete q-spiral $\frac{q^{\mathbb{N}^*}}{1-q}$ and satisfies $\delta_q e_q(z) = z e_q(z)$. The function, $e_q\left(qz^{-1}\right)^{-1} + z$ is solution of the family of linear δ_q-equations and converges uniformly on the compacts of $\mathbb{C}^*$ to $e^{-z^{-1}} + z$ when $q \to 1$. More generally, we will multiply a fundamental solution of the family of linear δ_q-equations by a convenient matrix with entries in $\mathcal{M}_{\mathbb{E}}$, in order to have a confluence result. See Theorem 4.8.4 for a precise statement.

In §4.8.3, we are interested in the case where the linear δ_q and δ-equations have coefficients in $\mathbb{C}(z)$. We combine our main result, Theorem 4.4.5, and what we have just mentioned above, to prove that under reasonable assumptions, we have the uniform convergence on a convenient domain of a basis of solutions of a family of linear δ_q-equations to a basis of solutions of the corresponding linear δ-equation when $q \to 1$. This leads us to the convergence of the q-Stokes matrices, that do not correspond to the q-Stokes matrices present in [RSZ13], to the Stokes matrices. See Theorem 4.8.10.

In §4.8.4, following [Sau00], we construct a locally constant matrix, and his values allow us to obtain the monodromy matrices at the intermediate singularities of the linear differential system. This result is an analogue of [Sau00], §4, in the irregular singular case. See Theorem 4.8.11. The results of §4.8.3 and §4.8.4 could be the first step to a numerical algorithm of approximation of the Stokes and monodromy matrices. See [FRJT09, FRRJT10, vdH07, LRR11, Rem12] for results of numerical approximation of the Stokes matrices and [MS10, Mez11] for results of numerical approximation of the monodromy matrices.

* * *

The chapter is organized as follows. In §4.1, we make a short overview of the Stokes phenomenon of the linear differential equations. In particular, we recall the definition of the Stokes matrices. In §4.2, we recall some results that can be found in [RSZ13] on the local formal study of linear q-difference equations. In §4.3, we introduce the q-Borel and the q-Laplace transformations.

The §4.4, is devoted to the statement of our main result, Theorem 4.4.5, while §4.5 and §4.6 are devoted to the proof of Theorem 4.4.5. In §4.5, we prove a proposition that deals with the confluence of meromorphic solutions. In §4.6.1, we study the confluence of the q-Laplace transformation. In §4.6.2, we show Theorem 4.4.5 in a particular case, and in §4.6.3, we prove Theorem 4.4.5 in the general case.

As told above, in §4.7, we study basic hypergeometric series, and in §4.8, we apply our main result to obtain the uniform convergence on a convenient domain of a basis of solutions of a family of linear δ_q-equations to a basis of solutions of the corresponding linear δ-equation when $q \to 1$.

4.1 Local analytic study of linear differential equations

In this section, we make a short overview of the Stokes phenomenon of linear differential equations. See [Bal94, vdPS03] for more details. See also [Ber92, LR90, LR95, Mal95, MR92, Ram93, RM90, Sin09].

Let $\mathbb{C}[[z]]$ be the ring of formal power series and $\mathbb{C}((z)) := \mathbb{C}[[z]][z^{-1}]$ be its fraction field. Let K be an intermediate differential field extension: $\mathbb{C}(z) \subset K \subset \bigcup_{\nu\in\mathbb{N}^*} \mathbb{C}\left(\left(z^{1/\nu}\right)\right)$. We recall that $\delta = z\frac{d}{dz}$. Let us consider the linear differential operator with coefficients in K

$$\widetilde{P} = \widetilde{b}_m\delta^m + \widetilde{b}_{m-1}\delta^{m-1} + \cdots + \widetilde{b}_0.$$

The Newton polygon of $\widetilde{P}$ is the convex hull of

$$\bigcup_{k=0}^{m} \left\{(i,j) \in \mathbb{N}^* \times \mathbb{Q} \middle| i \leq k, j \geq v_0\left(\widetilde{b}_k\right)\right\},$$

where v_0 denotes the z-adic valuation of K. Let $\{(d_1,n_1),\dots,(d_r,n_r)\}$ be a minimal subset of $\mathbb{Z}^2$ for the inclusion, with $d_1 < \cdots < d_r$, such that the Newton polygon is the convex hull of

$$\bigcup_{k=0}^{r} \left\{(i,j) \in \mathbb{N}^* \times \mathbb{Q} \middle| i \leq d_k, j \geq n_k\right\}.$$

We call slopes of the linear δ-equation the positive rational numbers $\frac{n_{i+1}-n_i}{d_{i+1}-d_i}$, and multiplicity of the slope $\frac{n_{i+1}-n_i}{d_{i+1}-d_i}$, the integer $d_{i+1}-d_i$.

Let $\widetilde{b}_0,\dots,\widetilde{b}_{m-1} \in K$ and $\widetilde{B} := \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ -\widetilde{b}_0 & \dots & \dots & -\widetilde{b}_{m-1} \end{pmatrix} \in \mathrm{M}_m(K)$ be a companion matrix. The linear differential system $\delta\widetilde{Y} = \widetilde{B}\widetilde{Y}$ is equivalent to the linear differential equation $\delta^m\widetilde{y} + \widetilde{b}_{m-1}\delta^{m-1}\widetilde{y} + \cdots + \widetilde{b}_0\widetilde{y} = 0$. Let $\widetilde{P} := \delta^m + \widetilde{b}_{m-1}\delta^{m-1} + \cdots + \widetilde{b}_0$. We define the Newton polygon of $\delta\widetilde{Y} = \widetilde{B}\widetilde{Y}$, as the Newton polygon of $\widetilde{P}$. We also define the slopes and the multiplicities of the slopes of $\delta\widetilde{Y} = \widetilde{B}\widetilde{Y}$ as the slopes and the multiplicities of the slopes of $\widetilde{P}$. Notice that if $\widetilde{B} \in \mathrm{M}_m\Big(\mathbb{C}((z))\Big)$ is not a companion matrix, we can still define the Newton polygon of $\delta\widetilde{Y} = \widetilde{B}\widetilde{Y}$, but we will not need this in this chapter.

The linear differential equations $\delta\widetilde{Y} = \widetilde{A}\widetilde{Y}$ and $\delta\widetilde{Y} = \widetilde{B}\widetilde{Y}$, with $\widetilde{A},\widetilde{B} \in \mathrm{M}_m(K)$ are said to be equivalent over K if there exists $\widetilde{H} \in \mathrm{GL}_m(K)$, that is an invertible matrix with coefficients in K, such that

$$\widetilde{A} = \widetilde{H}\left[\widetilde{B}\right]_\delta := \widetilde{H}\widetilde{B}\widetilde{H}^{-1} + \delta\widetilde{H}\widetilde{H}^{-1}.$$

Notice that in this case:

$$\delta\widetilde{Y} = \widetilde{B}\widetilde{Y} \Longleftrightarrow \delta\left(\widetilde{H}\widetilde{Y}\right) = \widetilde{A}\widetilde{H}\widetilde{Y}.$$

Conversely, if there exist $\widetilde{A},\widetilde{B} \in \mathrm{M}_m(K)$ and $\widetilde{H} \in \mathrm{GL}_m(K)$, such that $\delta\widetilde{Y} = \widetilde{B}\widetilde{Y}$, $\delta\widetilde{Z} = \widetilde{A}\widetilde{Z}$ and $\widetilde{Z} = \widetilde{H}\widetilde{Y}$, then

$$\widetilde{A} = \widetilde{H}\left[\widetilde{B}\right]_\delta.$$

One can prove that if the above matrices $\widetilde{A},\widetilde{B}\in \mathrm{M}_m(K)$ are companion matrices, then they have the same Newton polygon.

Let us consider $\delta\widetilde{Y}=\widetilde{B}\widetilde{Y}$, where $\widetilde{B}\in \mathrm{M}_m\Big(\mathbb{C}((z))\Big)$ is a companion matrix, having slopes $k_1<\cdots<k_{r-1}$ with multiplicity $m_1,\dots,m_{r-1}$, and let $\nu\in\mathbb{N}^*$ be minimal such that all the νk_i belongs to $\mathbb{N}$. The Hukuhara-Turrittin theorem (see Theorem 3.1 in [vdPS03] for a statement that is trivially equivalent to the following) says that there exist

- $\widetilde{H}\in \mathrm{GL}_m\Big(\mathbb{C}\left(\left(z^{1/\nu}\right)\right)\Big)$,
- $\widetilde{L}_i\in \mathrm{M}_{m_i}(\mathbb{C})$,
- $\widetilde{\lambda}_i\in z^{-1/\nu}\mathbb{C}\left[z^{-1/\nu}\right]$,

such that $\widetilde{B}=\widetilde{H}\left[\mathrm{Diag}_i\left(\widetilde{L}_i+\delta\widetilde{\lambda}_i\times \mathrm{Id}_{m_i}\right)\right]_{\delta}$, where

$$\mathrm{Diag}_i\left(\widetilde{L}_i+\delta\widetilde{\lambda}_i\times \mathrm{Id}_{m_i}\right):=\begin{pmatrix}\widetilde{L}_1+\delta\widetilde{\lambda}_1\times \mathrm{Id}_{m_1} & & \\ & \ddots & \\ & & \widetilde{L}_k+\delta\widetilde{\lambda}_k\times \mathrm{Id}_{m_k}\end{pmatrix}\dagger.$$

Roughly speaking, this means that if $\widetilde{B}\in \mathrm{M}_m\Big(\mathbb{C}((z))\Big)$ is a companion matrix, there exists a formal fundamental solution of $\delta\widetilde{Y}=\widetilde{B}\widetilde{Y}$, of the form

$$\widetilde{H}(z)\mathrm{Diag}\left(z^{\widetilde{L}_i}e^{\widetilde{\lambda}_i(z)\times \mathrm{Id}_{m_i}}\right).$$

Of course, written like this, this statement is not rigorous, since matrices $\widetilde{H}(z)$ and $\mathrm{Diag}\left(z^{\widetilde{L}_i}e^{\widetilde{\lambda}_i(z)\times \mathrm{Id}_{m_i}}\right)$ can not be multiplied.

Remark that for all $n\in\mathbb{Z}$, we have also

$$\widetilde{B}=\left(z^n\widetilde{H}\right)\left[\mathrm{Diag}\left(\widetilde{L}_i-n\times \mathrm{Id}+\delta\widetilde{\lambda}_i\times \mathrm{Id}_{m_i}\right)\right]_{\delta},$$

which allows us to reduce to the case where the entries of $\widetilde{H}$ belongs to $\mathbb{C}\left[\left[z^{1/\nu}\right]\right]$.

We recall that $\widetilde{\mathbb{C}}$ is the Riemann surface of the logarithm. If $a,b\in\mathbb{R}$ with $a<b$, we define $\mathcal{A}(a,b)$ as the ring of functions that are analytic in some punctured neighborhood of 0 in

$$\overline{S}(a,b):=\left\{z\in\widetilde{\mathbb{C}}\Big|\arg(z)\in]a,b[\right\}.$$

Let $\mathbb{C}\{z\}$ be the ring of germs of analytic functions in the neighborhood of $z=0$, and $\mathbb{C}(\{z\})$ be its fraction field, that is the field of germs of meromorphic functions in the neighborhood of $z=0$. Let $\widetilde{B}\in \mathrm{M}_m\Big(\mathbb{C}(\{z\})\Big)$ be a companion matrix. We are now interested in the existence of a fundamental solution of the system $\delta\widetilde{Y}=\widetilde{B}\widetilde{Y}$, we will see as an equation, that has coefficients in $\mathcal{A}(a,b)$, for some $a<b$.

Once for all, we fix a determination of the complex logarithm over $\widetilde{\mathbb{C}}$ we call log. We define the family of continuous map $(\rho_a)_{a\in\mathbb{C}}$, from the Riemann surface of the logarithm to itself, that sends z to $e^{a\log(z)}$. One has $\rho_b\circ\rho_c=\rho_{bc}$ for any $b,c\in\mathbb{C}$. For

†. If no confusions is likely to arise we will write $\mathrm{Diag}\left(\widetilde{L}_i+\delta\widetilde{\lambda}_i\times \mathrm{Id}_{m_i}\right)$ instead of $\mathrm{Diag}_i\left(\widetilde{L}_i+\delta\widetilde{\lambda}_i\times \mathrm{Id}_{m_i}\right)$. Notice that altough the index i seems here to be useless, he will be later helpfull when we will consider diagonal bloc matrices with diagonal bloc having several indexes.

$\widetilde{f} := \sum f_n z^n \in \bigcup_{\nu\in\mathbb{N}^*} \mathbb{C}\left(\left(z^{1/\nu}\right)\right)$ and $c \in \mathbb{Q}_{>0}$, we set $\rho_c\left(\widetilde{f}\right) := \sum f_n z^{nc} \in \bigcup_{\nu\in\mathbb{N}^*} \mathbb{C}\left(\left(z^{1/\nu}\right)\right)$. For $f \in \mathcal{A}(a,b)$ and $c \in \mathbb{Q}_{>0}$, we define $\rho_c(f) := f(z^c)$. Of course, the definitions of ρ_c coincide on $\mathbb{C}(\{z\})$.

Definition 4.1.1. (1) Let $k \in \mathbb{Q}_{>0}$. We define the formal Borel transform of order k, $\hat{\mathcal{B}}_k$ as follows:

$$\begin{array}{rccc} \hat{\mathcal{B}}_k : & \mathbb{C}[[z]] & \longrightarrow & \mathbb{C}[[\zeta]] \\ & \sum_{n\in\mathbb{N}} a_n z^n & \longmapsto & \sum_{n\in\mathbb{N}} \frac{a_n}{\Gamma\left(1+\frac{n}{k}\right)}\zeta^n, \end{array}$$

where Γ is the Gamma function. We remark that we have for all $k \in \mathbb{Q}_{>0}$:

$$\hat{\mathcal{B}}_k = \rho_k \circ \hat{\mathcal{B}}_1 \circ \rho_{1/k}.$$

(2) Let $d \in \mathbb{R}$ and $k \in \mathbb{Q}_{>0}$. Let f be a function such that there exists $\varepsilon > 0$, such that $f \in \mathcal{A}(d-\varepsilon, d+\varepsilon)$. We say that f belongs to $\widetilde{\mathbb{H}}_k^d$, if f admits an analytic continuation defined on $\overline{S}(d-\varepsilon, d+\varepsilon)$ that we will still call f, with exponential growth of order k at infinity. This means that there exist constants $J, L > 0$, such that for $\zeta \in \overline{S}(d-\varepsilon, d+\varepsilon)$:

$$|f(\zeta)| < J\exp\left(L|\zeta|^k\right).$$

(3) Let $d \in \mathbb{R}$ and $k \in \mathbb{Q}_{>0}$. We define the Laplace transformations of order 1 and k in the direction d as follow (see [Bal94], Page 13 for a justification that the maps are defined)

$$\begin{array}{rccc} \mathcal{L}_1^d : & \widetilde{\mathbb{H}}_1^d & \longrightarrow & \mathcal{A}\left(d-\frac{\pi}{2}, d+\frac{\pi}{2}\right) \\ & f & \longmapsto & \int_0^{\infty e^{id}} z^{-1} f(\zeta) e^{-\left(\frac{\zeta}{z}\right)} d\zeta, \end{array}$$

$$\begin{array}{rccc} \mathcal{L}_k^d : & \widetilde{\mathbb{H}}_k^d & \longrightarrow & \mathcal{A}\left(d-\frac{\pi}{2k}, d+\frac{\pi}{2k}\right) \\ & g & \longmapsto & \rho_k \circ \mathcal{L}_1^d \circ \rho_{1/k}(g). \end{array}$$

The following proposition will be needed for the proof of our main result, Theorem 4.4.5.

Proposition 4.1.2. *Let $\widetilde{f} \in \mathbb{C}[[z]]$, let $d \in \mathbb{R}$ and let $\widetilde{g} \in \widetilde{\mathbb{H}}_1^d$. Then:*

- $\hat{\mathcal{B}}_1\left(\delta\widetilde{f}\right) = \delta\hat{\mathcal{B}}_1\left(\widetilde{f}\right)$.
- $\delta\hat{\mathcal{B}}_1\left(z\widetilde{f}\right) = \zeta\hat{\mathcal{B}}_1\left(\widetilde{f}\right)$, *where* $\delta := \zeta\frac{d}{d\zeta}$.
- $\mathcal{L}_1^d\left(\delta\widetilde{g}\right) = \delta\mathcal{L}_1^d\left(\widetilde{g}\right)$.
- $z\mathcal{L}_1^d\left(\delta\widetilde{g}\right) = \mathcal{L}_1^d\left(\zeta\widetilde{g}\right) - z\mathcal{L}_1^d\left(\widetilde{g}\right)$.

Proof. The two first points are straightforward computations. Let us prove the third point. Making the variable change $\zeta \mapsto q\zeta$ in the integral, we find that for all $q > 1$, $\mathcal{L}_1^d$ commutes with σ_q. Then, for all $q > 1$, we find

$$\mathcal{L}_1^d\left(\delta_q\widetilde{g}\right) = \delta_q\mathcal{L}_1^d\left(\widetilde{g}\right).$$

Since $\widetilde{g} \in \widetilde{\mathbb{H}}_1^d$, the dominated convergence theorem allow us to take the limit as q goes to 1

$$\mathcal{L}_1^d\left(\delta\widetilde{g}\right) = \lim_{q\to 1}\mathcal{L}_1^d\left(\delta_q\widetilde{g}\right) = \lim_{q\to 1}\delta_q\mathcal{L}_1^d\left(\widetilde{g}\right) = \delta\mathcal{L}_1^d\left(\widetilde{g}\right).$$

Let us prove the last equality. Since $\widetilde{g} \in \widetilde{\mathbb{H}}_1^d$, we may perform an integration by part (let $\widetilde{g}'$ be the derivation of $\widetilde{g}$), and we obtain:

$$\begin{aligned} z\mathcal{L}_1^d\Big(\delta\widetilde{g}\Big) &= \int_0^{\infty e^{id}} \zeta\widetilde{g}'(\zeta)e^{-\left(\frac{\zeta}{z}\right)}d\zeta \\ &= \int_0^{\infty e^{id}} \widetilde{g}(\zeta)e^{-\left(\frac{\zeta}{z}\right)}\left(-1+\frac{\zeta}{z}\right)d\zeta \\ &= \mathcal{L}_1^d\Big(\zeta\widetilde{g}\Big) - z\mathcal{L}_1^d\Big(\widetilde{g}\Big). \end{aligned}$$

□

Remark 4.1.3. Let $k \in \mathbb{Q}_{>0}$, let $\widetilde{d}_0, \dots, \widetilde{d}_r \in \mathbb{C}\left[z^k\right]$ and let us consider $\widetilde{f} \in \mathbb{C}\left[\left[z^k\right]\right]$, that satisfies

$$\sum_{i=0}^{r} \widetilde{d}_i(z)\delta^i\widetilde{f} = 0. \tag{4.1.1}$$

From Proposition 4.1.2, there exist $\widetilde{c}_0, \dots, \widetilde{c}_s \in \mathbb{C}\left[z^k\right]$ with degree less or equal that the maximum of the degrees of the $\widetilde{d}_i$, such that

$$\sum_{i=0}^{s} \widetilde{c}_i(z)\delta^i\hat{\mathcal{B}}_k\left(\widetilde{f}\right) = 0.$$

Furthermore, if there exists $d \in \mathbb{R}$ such that $\hat{\mathcal{B}}_k\left(\widetilde{f}\right) \in \widetilde{\mathbb{H}}_k^d$, then we have:

$$\delta\mathcal{L}_k^d \circ \hat{\mathcal{B}}_k\left(\widetilde{f}\right) = \mathcal{L}_k^d \circ \hat{\mathcal{B}}_k\left(\delta\widetilde{f}\right) \text{ and } \delta\left(z^k\mathcal{L}_k^d \circ \hat{\mathcal{B}}_k\left(\widetilde{f}\right)\right) = \mathcal{L}_k^d \circ \hat{\mathcal{B}}_k\left(\delta\left(z^k\widetilde{f}\right)\right).$$

Hence, $\mathcal{L}_k^d \circ \hat{\mathcal{B}}_k\left(\widetilde{f}\right)$ is solution of (4.1.1). But in general, if $\widetilde{f} \in \mathbb{C}[[z]]$ is solution of a linear δ-equation with coefficients in $\mathbb{C}[z]$, then, for all $(d,k) \in \mathbb{R} \times \mathbb{Q}_{>0}$, we have $\hat{\mathcal{B}}_k\left(\widetilde{f}\right) \notin \widetilde{\mathbb{H}}_k^d$, and we must apply successively several Borel and Laplace transformations to compute an analytic solution of the same equation. See Proposition 4.1.5.

Let us consider $\delta\widetilde{Y} = \widetilde{B}\widetilde{Y}$, where $\widetilde{B} \in \mathrm{M}_m\Big(\mathbb{C}(\{z\})\Big)$ is a companion matrix and let $\widetilde{H}$ be a formal matrix obtained with the Hukuhara-Turrittin theorem. We have seen that we may assume that $\widetilde{H}$ has no poles at 0. Let $\widetilde{h} \in \mathbb{C}\left[\left[z^{1/\nu}\right]\right]$ be an entry of $\widetilde{H}$ and let us consider a linear δ-equation satisfied by $\widetilde{h}$:

$$\widetilde{b}_m\delta^m\widetilde{h} + \widetilde{b}_{m-1}\delta^{m-1}\widetilde{h} + \dots + \widetilde{b}_0\widetilde{h} = 0, \tag{4.1.2}$$

with $\widetilde{b}_m \neq 0$ and $\widetilde{b}_i \in \mathbb{C}\left(\left\{z^{1/\nu}\right\}\right)$. Assume that (4.1.2) has at least one slope different from 0. Let $d_0 := \max\Big(2, \deg\left(\widetilde{b}_0\right), \dots, \deg\left(\widetilde{b}_m\right)\Big)$, where deg denotes the degree. Let $k_1 < \dots < k_{r-1}$ be the slopes of (4.1.2) different from 0, let k_r be an integer strictly bigger than k_{r-1} and d_0, and set $k_{r+1} := +\infty$. Let $(\kappa_1, \dots, \kappa_r)$ be defined by:

$$\kappa_i^{-1} := k_i^{-1} - k_{i+1}^{-1}.$$

We define the rational numbers $(\widetilde{\kappa_1}, \dots, \widetilde{\kappa_s})$ as follows: We take $(\kappa_1, \dots, \kappa_r)$ and for $i = 1, \dots, i = r$, replace successively κ_i by α_i terms $\alpha_i\kappa_i$, where α_i is the smallest integer such that $\alpha_i\kappa_i$ is greater or equal than d_0. Therefore, by construction, all the $\widetilde{\kappa}_i$ are greater than $d_0 \geq 2$, $\widetilde{\kappa_s}$ belongs to $\mathbb{N}$, and $\widetilde{\kappa_s} = \kappa_r = k_r > k_{r-1}$.

Example 4.1.4. Assume that $\widetilde{h} \in \mathbb{C}[[z]]$ is solution of

$$\left(z^4 + z^3\right)\delta^3\widetilde{h} + z\delta^2\widetilde{h} + \delta\widetilde{h} - \widetilde{h} = 0.$$

We have $d_0 = 4$, $r = 3$ and $(k_1, k_2, k_3, k_4) = (1, 2, 5, \infty)$. Then, we find that $(\kappa_1, \kappa_2, \kappa_3) = (2, 10/3, 5)$, $s = 5$, and we obtain $(\widetilde{\kappa_1}, \dots, \widetilde{\kappa_5}) = (4, 4, 20/3, 20/3, 5)$.

We recall that $\widetilde{h} \in \mathbb{C}\left[\left[z^{1/\nu}\right]\right]$. Let us write $\widetilde{h} =: \sum\limits_{\substack{n=0\\ n\in\mathbb{N}/\nu}}^{\infty} \widetilde{h}_n z^n$. Let $\beta \in \mathbb{N}^*$ be minimal such that $\beta/\widetilde{\kappa_1}, \dots, \beta/\widetilde{\kappa_s}$ belong to $\mathbb{N}^*$ and for $l \in \{0, \dots, \beta\nu - 1\}$, let $\widetilde{h}^{(l)} := \sum\limits_{n=0}^{\infty} \widetilde{h}_{l/\nu + n\beta} z^{n\beta}$.

Proposition 4.1.5. *Let us keep the same notations as above. There exists* $\widetilde{\Sigma}_{\widetilde{h}} \subset \mathbb{R}$, *finite modulo* $2\pi\mathbb{Z}$, *such that for all* $l \in \{0, \dots, \beta\nu - 1\}$, *if* $d \in \mathbb{R} \setminus \widetilde{\Sigma}_{\widetilde{h}}$, *the series* $\widetilde{f}_{1,l} := \hat{\mathcal{B}}_{\widetilde{\kappa_1}} \circ \dots \circ \hat{\mathcal{B}}_{\widetilde{\kappa_s}}\left(\widetilde{h}^{(l)}\right)$ *converges and belongs to* $\widetilde{\mathbb{H}}^d_{\widetilde{\kappa_1}}$.

Moreover, for $j = 2$ *(resp.* $j = 3$, ..., *resp.* $j = s$*),* $\widetilde{f}_{j,l} := \mathcal{L}^d_{\widetilde{\kappa_{j-1}}}\left(\widetilde{f}_{j-1,l}\right)$ *belongs to* $\widetilde{\mathbb{H}}^d_{\widetilde{\kappa_j}}$. *Let* $\widetilde{S}^d\left(\hat{h}^{(l)}\right) := \mathcal{L}^d_{\widetilde{\kappa_s}}\left(\widetilde{f}_{s,l}\right)$. *The function*

$$\widetilde{S}^d\left(\widetilde{h}\right) := \sum_{l=0}^{\beta\nu-1} z^{l/\nu}\widetilde{S}^d\left(\widetilde{h}^{(l)}\right) \in \mathcal{A}\left(d - \frac{\pi}{2\widetilde{\kappa_s}}, d + \frac{\pi}{2\widetilde{\kappa_s}}\right) = \mathcal{A}\left(d - \frac{\pi}{2k_r}, d + \frac{\pi}{2k_r}\right),$$

is solution of the same linear δ*-equation than* $\widetilde{h}$.

Remark 4.1.6. We make a priori an abuse of notations, since $\widetilde{S}^d\left(\widetilde{h}\right)$ may depend on the choice of the linear differential equation satisfied by $\widetilde{h}$. However, we can prove that $\widetilde{S}^d\left(\widetilde{h}\right)$ is independent upon the choice of the linear differential equation satisfied by $\widetilde{h}$. Notice that we will not use this fact.

Remark 4.1.7. As we can see in Theorem 7.51 in [vdPS03], the function $\widetilde{S}^d\left(\widetilde{h}\right)$ is $\widetilde{\kappa_s}$-Gevrey asymptotic to $\widetilde{h}$ on $\overline{S}\left(d - \frac{\pi}{2\widetilde{\kappa_s}}, d + \frac{\pi}{2\widetilde{\kappa_s}}\right)$: for every closed subsector W of $\overline{S}\left(d - \frac{\pi}{2\widetilde{\kappa_s}}, d + \frac{\pi}{2\widetilde{\kappa_s}}\right)$, there exist $A_W \in \mathbb{R}$, $\varepsilon > 0$ such that for all $N \in \mathbb{N}^*$ and all $z \in W$ with $|z| < \varepsilon$,

$$\left|\widetilde{S}^d\left(\widetilde{h}\right)(z) - \sum_{n=0}^{N-1} \widetilde{h}_n z^n\right| \leq (A_W)^N \Gamma\left(1 + \frac{N}{\widetilde{\kappa_s}}\right)|z|^N.$$

Proof of Proposition 4.1.5. Let $\widetilde{g} := \rho_\nu \widetilde{h} \in \mathbb{C}[[z]]$. For all $l \in \{0, \dots, \beta\nu - 1\}$, we have

$$z^{l/\nu}\widetilde{h}^{(l)}(z, q) = \rho_{1/\nu} \sum_{j=0}^{\beta\nu-1} \frac{\widetilde{g}\left(e^{2i\pi lj/\beta\nu} z\right)}{e^{2i\pi lj/\beta\nu}\beta\nu}.$$

It follows that there exists $\widetilde{\Sigma}_{\widetilde{h}} \subset \mathbb{R}$, finite modulo $2\pi\mathbb{Z}$, such that for all $l \in \{0, \dots, \beta\nu - 1\}$, if $d \in \mathbb{R} \setminus \widetilde{\Sigma}_{\widetilde{h}}$, then

- $\widetilde{f}_1 := \hat{\mathcal{B}}_{\widetilde{\kappa_1}} \circ \dots \circ \hat{\mathcal{B}}_{\widetilde{\kappa_s}}\left(\widetilde{h}\right) \in \widetilde{\mathbb{H}}^d_{\widetilde{\kappa_1}}$ if and only if for all integers $l \in \{0, \dots, \beta\nu - 1\}$, we have $\widetilde{f}_{1,l} := \hat{\mathcal{B}}_{\widetilde{\kappa_1}} \circ \dots \circ \hat{\mathcal{B}}_{\widetilde{\kappa_s}}\left(\widetilde{h}^{(l)}\right) \in \widetilde{\mathbb{H}}^d_{\widetilde{\kappa_1}}$.

– For $j=2$ (resp. $j=3,\ldots$, resp. $j=s$), $\widetilde{f}_j := \mathcal{L}^d_{\widetilde{\kappa_{j-1}}}\left(\widetilde{f}_{j-1}\right) \in \widetilde{\mathbb{H}}^d_{\widetilde{\kappa_j}}$ if and only if for all $l \in \{0,\ldots,\beta\nu-1\}$, $\widetilde{f}_{j,l} := \mathcal{L}^d_{\widetilde{\kappa_{j-1}}}\left(\widetilde{f}_{j-1,l}\right) \in \widetilde{\mathbb{H}}^d_{\widetilde{\kappa_j}}$.

Let $d \in \mathbb{R} \setminus \widetilde{\Sigma}_{\widetilde{h}}$ and let $(\kappa'_1,\ldots,\kappa'_{r-1})$ defined as:

$$\kappa'_{r-1} := k_{r-1} \text{ and for } i<r-1, \frac{1}{\kappa'_i} := \frac{1}{k_i} - \frac{1}{k_{i+1}}.$$

Due to Theorem 7.51 in [vdPS03] and [Bal94], §7.2, $\widetilde{f}'_1 := \hat{\mathcal{B}}_{\kappa'_1} \circ \cdots \circ \hat{\mathcal{B}}_{\kappa'_r}\left(\widetilde{h}\right) \in \widetilde{\mathbb{H}}^d_{\kappa'_1}$, and for $j=2$ (resp. $j=3,\ldots$, resp. $j=r-1$), $\widetilde{f}'_j := \mathcal{L}^d_{\kappa'_{j-1}}\left(\widetilde{f}'_{j-1}\right) \in \widetilde{\mathbb{H}}^d_{\kappa'_j}$. With Lemma 2 in [Bal94], §6.2, this implies that $\widetilde{f}_1 := \hat{\mathcal{B}}_{\widetilde{\kappa_1}} \circ \cdots \circ \hat{\mathcal{B}}_{\widetilde{\kappa_s}}\left(\widetilde{h}^{(l)}\right) \in \widetilde{\mathbb{H}}^d_{\widetilde{\kappa_1}}$ and for $j=2$ (resp. $j=3,\ldots$, resp. $j=s$), $\widetilde{f}_j := \mathcal{L}^d_{\widetilde{\kappa_{j-1}}}\left(\widetilde{f}_{j-1,l}\right) \in \widetilde{\mathbb{H}}^d_{\widetilde{\kappa_j}}$. With the equivalence we have written in the beginning of the proof, we may apply successively the Borel and Laplace transformations of the required order to each series $\hat{h}^{(l)}$.

To finish, we have to prove that $\widetilde{S}^d\left(\widetilde{h}\right)$ is solution of the same linear δ-equation than $\widetilde{h}$. This is a direct consequence of Theorem 2 in [Bal94], §6.4. □

As a matter of fact, as we can see in Page 239 of [vdPS03], $\widetilde{S}^d\left(\widetilde{h}\right)$ belongs to $\mathcal{A}\left(d_l - \frac{\pi}{2k_r}, d_{l+1} + \frac{\pi}{2k_r}\right)$, where $d_l, d_{l+1} \in \widetilde{\Sigma}_{\widetilde{h}}$ are chosen such that $]d_l, d_{l+1}[\bigcap \widetilde{\Sigma}_{\widetilde{h}} = \varnothing$.

If (4.1.2) has only slope 0, then $\widetilde{h} \in \mathbb{C}\left\{z^{1/\nu}\right\}$. In this case we set $\widetilde{\Sigma}_{\widetilde{h}} := \varnothing$, and for all $d \in \mathbb{R}$ we set

$$\widetilde{S}^d\left(\widetilde{h}\right) := \widetilde{h}.$$

We recall that we consider the equation $\delta\widetilde{Y} = \widetilde{B}\widetilde{Y}$, where $\widetilde{B} \in \mathrm{M}_m\Big(\mathbb{C}(\{z\})\Big)$ is a companion matrix and $\widetilde{H} := \left(\widetilde{h}_{i,j}\right) \in \mathrm{M}_m\left(\mathbb{C}\left[\left[z^{1/\nu}\right]\right]\right)$ is a formal matrix given by the Hukuhara-Turrittin theorem. The entries of $\widetilde{H}$ satisfy linear δ-equations with coefficients in $\mathbb{C}\left[z^{1/\nu}\right]$ for some ν. We may assume that for a given entry, the coefficients of the δ-equation are relatively prime. Let d_0 be the maximum among 2 and the degrees of the coefficients of the equations. Let $\widetilde{\Sigma}_{\widetilde{H}}$ be the union of the $\widetilde{\Sigma}_{\widetilde{h}_{i,j}}$, where $\widetilde{\Sigma}_{\widetilde{h}_{i,j}}$ has been defined in Proposition 4.1.5; $k_{i,j} \in \mathbb{Q}$ be the biggest slope of the equation satisfied by $\widetilde{h}_{i,j}$; k' be the maximum of the $k_{i,j}$; and k be an integer strictly bigger than k' and d_0. Let $d, d^{\pm} \in \mathbb{R}\setminus\widetilde{\Sigma}_{\widetilde{H}}$, with

$$d - \frac{\pi}{2k} < d^- < d < d^+ < d + \frac{\pi}{2k},$$

and such that $\left([d^-, d[\bigcup]d, d^+]\right)\bigcap \widetilde{\Sigma}_{\widetilde{H}} = \varnothing$. Let $\widetilde{S}^{d^{\pm}}\left(\widetilde{H}\right) := \widetilde{S}^{d^{\pm}}\left(\widetilde{h}_{i,j}\right)$. We get two analytic solutions,

$$\widetilde{S}^{d^-}\left(\widetilde{H}\right)\mathrm{Diag}\left(e^{\widetilde{L}_i \log(z)} e^{\widetilde{\lambda}_i \times \mathrm{Id}_{m_i}}\right) \in \mathrm{GL}_m\left(\mathcal{A}\left(d^- - \frac{\pi}{2k}, d + \frac{\pi}{2k}\right)\right),$$

and

$$\widetilde{S}^{d^+}\left(\widetilde{H}\right)\mathrm{Diag}\left(e^{\widetilde{L}_i \log(z)} e^{\widetilde{\lambda}_i \times \mathrm{Id}_{m_i}}\right) \in \mathrm{GL}_m\left(\mathcal{A}\left(d - \frac{\pi}{2k}, d^+ + \frac{\pi}{2k}\right)\right).$$

Note that by definition, the analyticity holds on a subset of $\widetilde{\mathbb{C}}$. A computation shows that there exists a matrix $\widetilde{ST}_d \in \mathrm{GL}_m(\mathbb{C})$, we call the Stokes matrix in the direction d, such that:

$$\widetilde{S}^{d^+}\left(\widetilde{H}\right)\mathrm{Diag}\left(e^{\widetilde{L}_i \log(z)} e^{\widetilde{\lambda}_i \times \mathrm{Id}_{m_i}}\right) = \widetilde{S}^{d^-}\left(\widetilde{H}\right)\mathrm{Diag}\left(e^{\widetilde{L}_i \log(z)} e^{\widetilde{\lambda}_i \times \mathrm{Id}_{m_i}}\right)\widetilde{ST}_d.$$

4.2 Local formal study of q-difference equations

In this section, we summarize results about formal classification of linear q-difference equations. See in particular [RSZ13] for more details. Let $q>1$ be fixed. We extend the action of σ_q to $\bigcup_{\nu\in\mathbb{N}^*}\mathbb{C}\left(\left(z^{1/\nu}\right)\right)$ by $\sigma_q z^{1/\nu}=e^{\log(q)/\nu}z^{1/\nu}$, for $\nu\in\mathbb{N}^*$. Let K be an intermediate field extension: $\mathbb{C}(z)\subset K\subset \bigcup_{\nu\in\mathbb{N}^*}\mathbb{C}\left(\left(z^{1/\nu}\right)\right)$, stable by σ_q.

Let us consider the q-difference operator:

$$P=\sum_{i=l}^{m} b_i\sigma_q^i,$$

where $b_i\in K$, $l,m\in\mathbb{Z}$ and $l<m$. The Newton polygon of P is the convex hull of

$$\bigcup_{k=l}^{m}\left\{(i,j)\in\mathbb{Z}\times\mathbb{Q}\Big|j\geq v_0\left(b_k\right)\right\},$$

where v_0 denotes the z-adic valuation of K. Let $\{(d_1,n_1),\dots,(d_r,n_r)\}$ be a minimal subset of $\mathbb{Z}^2$ for the inclusion, with $d_1<\dots<d_r$, such that the Newton polygon is the convex hull of

$$\bigcup_{k=1}^{r}\left\{(d_k,j)\in\mathbb{Z}\times\mathbb{Q}\Big|j\geq n_k\right\}.$$

We call slopes of the linear δ-equation the rational numbers $\frac{n_{i+1}-n_i}{d_{i+1}-d_i}$, and multiplicity of the slope $\frac{n_{i+1}-n_i}{d_{i+1}-d_i}$, the integer $d_{i+1}-d_i$.

Like in §4.1, let $B\in\mathrm{GL}_m(K)$ be a companion matrix. As in the differential case, we can naturally associate to the linear σ_q-equation $\sigma_q Y=BY$ a unitary q-difference operator $P=\sigma_q^m+b_{m-1}\sigma_q^{m-1}+\dots+b_0$ with coefficients in K. We define the Newton polygon of $\sigma_q Y=BY$, as the Newton polygon of P. We also define the slopes and the multiplicities of the slopes of $\sigma_q Y=BY$ as the slopes and the multiplicities of the slopes of P. Notice that if $B\in\mathrm{GL}_m\Big(\mathbb{C}((z))\Big)$ is not a companion matrix, we can still define the Newton polygon of $\sigma_q Y=BY$, but we will not need this in this chapter.

Let $A,B\in\mathrm{GL}_m(K)$. The two q-difference systems, $\sigma_q Y=AY$ and $\sigma_q Y=BY$ are equivalent over K, if there exists $P\in\mathrm{GL}_m(K)$, called gauge transformation, such that

$$A=P[B]_{\sigma_q}:=(\sigma_q P)BP^{-1}.$$

In particular,

$$\sigma_q Y=BY\Longleftrightarrow \sigma_q\left(PY\right)=APY.$$

Conversely, if there exist $A,B,P\in\mathrm{GL}_m(K)$ such that $\delta Y=BY$, $\delta Z=AZ$ and $Z=PY$, then

$$A=P\left[B\right]_{\sigma_q}.$$

If the above matrices $A,B\in\mathrm{GL}_m(K)$ are companion matrices, then, see [RSZ13], Theorem 2.2.1, they have the same Newton polygon.

Theorem 4.2.1 ([RSZ13], §2.2). *Let $B \in \mathrm{GL}_m\Big(\mathbb{C}((z))\Big)$ be a companion matrix and let us consider $\sigma_q Y = BY$. Let $\mu_1, \dots, \mu_k$ be the slopes of the q-difference equation, let $m_1 \dots, m_k$ be their multiplicities and assume that the μ_i belong to $\mathbb{Z}$. Then, we have existence of $B_i \in \mathrm{GL}_{m_i}(\mathbb{C})$, $\hat{H} \in \mathrm{GL}_m\Big(\mathbb{C}((z))\Big)$, such that:*

$$B = \hat{H}\Big[\mathrm{Diag}\left(z^{-\mu_i} B_i\right)\Big]_{\sigma_q}.$$

See [vdPR07] for a more general result that works for q-difference equation with arbitrary slopes. Notice that for all $n \in \mathbb{Z}$, we have also

$$B = \left(z^n \hat{H}\right)\Big[\mathrm{Diag}\left(B_i \times q^{-n} z^{-\mu_i}\right)\Big]_{\sigma_q},$$

which allow us to reduce to the case where $\hat{H}$ has entries in $\mathbb{C}[[z]]$.

We want to determine the eigenvalues of the B_i and the z-valuation of the entries of $\hat{H}$. Let $b_0, \dots, b_{m-1} \in \mathbb{C}((z))$, and let us consider the q-difference equation:

$$\sigma_q^m y + b_{m-1}\sigma_q^{m-1} y + \dots + b_0 y = 0. \tag{4.2.1}$$

Let $\Big\{(d_1, n_1), \dots, (d_r, n_r)\Big\}$ be a minimal subset of $\mathbb{Z}^2$ for the inclusion, with $d_1 < \dots < d_r$, such that the Newton polygon is the convex hull of $\displaystyle\bigcup_{k=1}^{r} \Big\{(d_k, j) \in \mathbb{Z} \times \mathbb{Z} \Big| j \geq n_k\Big\}$. Let $\mu_1, \dots, \mu_k$ be the slopes of the q-difference equation, $m_1 \dots, m_k$ be their multiplicities and assume that the slopes $\mu_i = \frac{n_{i+1} - n_i}{d_{i+1} - d_i}$ belongs to $\mathbb{Z}$.

For $d_i \leq j \leq d_{i+1}$, let a_j be the value at $z = 0$ of $b_j(z) z^{-n_i - \mu_i (j - d_i)}$. We define the characteristic polynomial associated to the slope μ_i as follows:

$$\begin{array}{lcll} P^{(\mu_i)}(X) & := & \left(a_{d_{i+1}} q^{d_{i+1}(d_{i+1}-1)/2\mu_i} X^{d_{i+1} - d_i} + \dots + a_{d_i} q^{d_i(d_i - 1)/2\mu_i}\right) & \text{if} \quad \mu_i \neq 0. \\ P^{(\mu_i)}(X) & := & \left(a_{d_{i+1}} X^{d_{i+1} - d_i} + \dots + a_{d_i}\right) & \text{if} \quad \mu_i = 0. \end{array}$$

From [MZ00], Theorem 3.2.3, we deduce directly the following:

Theorem 4.2.2. *Let $B \in \mathrm{GL}_m\Big(\mathbb{C}((z))\Big)$ be a companion matrix, such that $\sigma_q Y = BY$ is the linear σ_q-system equivalent to (4.2.1). There exist*

- *$B_i \in \mathrm{GL}_{m_i}(\mathbb{C})$, which are of the form $\mathrm{Diag}_l\left(T_{i,l}\right)$, where $T_{i,l}$ are upper triangular matrices with diagonal terms that are equal to the roots of the characteristic polynomial associated to the slope μ_i,*
- *$\hat{H} \in \mathrm{GL}_m\Big(\mathbb{C}((z))\Big)$, whose entries of the first row of $\hat{H}$ have z-valuation equal to 0,*

such that

$$B = \hat{H}\Big[\mathrm{Diag}\left(z^{-\mu_i} B_i\right)\Big]_{\sigma_q}.$$

4.3 Definition of q-Borel and q-Laplace transformations.

The goal of this section is to define q-analogues of the Borel and Laplace transformations. We will study their behavior as q goes to 1 in §4.4.2. Remark that there are several possible definitions of q-analogues of Borel and Laplace transformations. See [DVZ09, MZ00, Ram92, RZ02, Zha99, Zha00, Zha01, Zha02, Zha03] for example. Following [DVZ09], we begin by defining a q-Borel transformation we are going to study. In this section, $q>1$ is fixed. Let us recall that for all $n\in\mathbb{N}$, $[n]_q^! = \displaystyle\prod_{l=1}^{n}\frac{q^l-1}{q-1}$.

Definition 4.3.1. Let $k\in\mathbb{Q}_{>0}$ and let $\nu\in\mathbb{N}^*$ minimal such that $\nu k\in\mathbb{N}^*$. We define $\hat{\mathcal{B}}_{q,k}$ as follows:

$$\begin{array}{rccc}\hat{\mathcal{B}}_{q,k}: & \mathbb{C}\left[\left[z^{\nu k}\right]\right] & \longrightarrow & \mathbb{C}\left[\left[\zeta^{\nu k}\right]\right]\\ & \displaystyle\sum_{l\in\mathbb{N}} a_l z^l & \longmapsto & \displaystyle\sum_{l\in\mathbb{N}}\frac{a_l}{[l/k]_q^!}\zeta^l,\end{array}$$

Let $k\in\mathbb{Q}_{>0}$, let $\nu\in\mathbb{N}^*$ minimal such that $\nu k\in\mathbb{N}^*$ and let $\rho_k,\rho_{1/k}$ be the maps defined in §4.1. We remark that we have:

$$\hat{\mathcal{B}}_{q,k}=\rho_k\circ\hat{\mathcal{B}}_{q,1}\circ\rho_{1/k}.$$

Definition 4.3.2. Let $d\in\mathbb{R}$ and let $k\in\mathbb{Q}_{>0}$. Let f be a function such that there exists $\varepsilon>0$, such that $f\in\mathcal{A}(d-\varepsilon,d+\varepsilon)$. We say that f belongs to $\mathbb{H}^d_{q,k}$, if f admits an analytic continuation defined on $\overline{S}(d-\varepsilon,d+\varepsilon)$, that we will still call f, such that there exist constants $J,L>0$, such that for $\zeta\in\overline{S}(d-\varepsilon,d+\varepsilon)$ (see the introduction for the definition of e_q):

$$|f(\zeta)|<Je_q\left(L|\zeta|^k\right).$$

For all $d\in\mathbb{R}$, we write $[d]:=q^{\mathbb{Z}}e^{id}$ the discrete logarithmic q-spiral through the point $e^{id}\in\mathbb{C}^*$. For $d\in\mathbb{R}$ we set the Jackson integral:

$$\int_{[d]} f(\zeta)d_q\zeta := (q-1)\sum_{l\in\mathbb{Z}} f\left(q^l e^{id}\right)q^l e^{id},$$

whenever the right hand side converges. Roughly speaking, Jackson integral degenerates into classical integral when q goes to 1, which means that for a convenient choice of function f, we have on a convenient domain

$$\int_{[d]} f(\zeta)d_q\zeta \underset{q\to 1}{\longrightarrow}\int_0^{\infty e^{id}} f(\zeta)d\zeta.$$

From now, let $p:=1/q\in]0,1[$. Let $\mathcal{M}(\mathbb{C}^*,0)$ be the field of functions that are meromorphic on some punctured neighborhood of 0 in $\mathbb{C}^*$. We define now the q-Laplace transformation.

Definition 4.3.3. Let $k\in\mathbb{Q}_{>0}$ and let $\rho_k\left(\mathcal{M}(\mathbb{C}^*,0)\right):=\{\rho_k(f)|f\in\mathcal{M}(\mathbb{C}^*,0)\}$. Let $d\in\mathbb{R}$. As we can see in [DVZ09], §4.2, the following maps are well defined and we call them the q-Laplace transformation of order 1 and k respectively:

$$\begin{array}{rccc}\mathcal{L}^{[d]}_{q,1}: & \mathbb{H}^d_{q,1} & \longrightarrow & \mathcal{M}(\mathbb{C}^*,0)\\ & f & \longmapsto & \displaystyle\int_{[d]}\frac{f(\zeta)}{ze_q\left(\frac{q\zeta}{z}\right)}d_q\zeta,\end{array}$$

$$\begin{array}{rccc}\mathcal{L}^{[d]}_{q,k}: & \mathbb{H}^d_{q,k} & \longrightarrow & \rho_k\left(\mathcal{M}(\mathbb{C}^*,0)\right)\\ & g & \longmapsto & \rho_k\circ\mathcal{L}^{[d]}_{q,1}\circ\rho_{1/k}(g).\end{array}$$

For $|z|$ small, the function $\mathcal{L}_{q,1}^{[d]}(f)(z)$ has poles of order at most 1 that are contained on the q-spiral $(q-1)[d+\pi] := q^{\mathbb{Z}}(1-q)e^{id}$. The following proposition is the q-analogue of Proposition 4.1.2.

Proposition 4.3.4. *Let $\hat{f} \in \mathbb{C}[[z]]$, let $d \in \mathbb{R}$, and let $g \in \mathbb{H}_{q,1}^{d}$. Then*

- $\hat{\mathcal{B}}_{q,1}\left(\delta_q \hat{f}\right) = \delta_q \hat{\mathcal{B}}_{q,1}\left(\hat{f}\right)$.
- $\delta_q \hat{\mathcal{B}}_{q,1}\left(z\hat{f}\right) = \zeta \hat{\mathcal{B}}_{q,1}\left(\hat{f}\right)$.
- $\mathcal{L}_{q,1}^{[d]}\left(\delta_q g\right) = \delta_q \mathcal{L}_{q,1}^{[d]}\left(g\right)$.
- $z\mathcal{L}_{q,1}^{[d]}\left(\delta_q g\right) = p\mathcal{L}_{q,1}^{[d]}\left(\zeta g\right) - pz\mathcal{L}_{q,1}^{[d]}\left(g\right)$.

Proof. The three first points are straightforward computations. Let us prove the last equality. Let $z \in \mathbb{C}^*$. It is a well known fact and easy to verify that $\sigma_q\Big(e_q(z)e_p(-z)\Big) = e_q(z)e_p(-z)$. Since $e_q(z)e_p(-z)$ is a formal power series with constant term equals to 1, $e_q(z)e_p(-z) = 1$. We have the equalities:

$$\begin{array}{rcl}
z\mathcal{L}_{q,1}^{[d]}\left(\delta_q g\right) & = & (q-1)e^{id} \displaystyle\sum_{l\in\mathbb{Z}} \frac{\delta_q g\left(q^l e^{id}\right)}{e_q\left(\frac{q^{l+1}e^{id}}{z}\right)} q^l \\
& = & (q-1)e^{id} \displaystyle\sum_{l\in\mathbb{Z}} \delta_q g\left(q^l e^{id}\right) e_p\left(\frac{-q^{l+1}e^{id}}{z}\right) q^l \\
& = & e^{id} \displaystyle\sum_{l\in\mathbb{Z}} g\left(q^{l+1}e^{id}\right) e_p\left(\frac{-q^{l+1}e^{id}}{z}\right) - g\left(q^l e^{id}\right) e_p\left(\frac{-q^{l+1}e^{id}}{z}\right) q^l \\
& = & e^{id} \displaystyle\sum_{l\in\mathbb{Z}} g\left(q^l e^{id}\right) \left(e_p\left(\frac{-q^{l+1}e^{id}}{qz}\right) p - e_p\left(\frac{-q^{l+1}e^{id}}{z}\right)\right) q^l \\
& = & (p-1)e^{id} \displaystyle\sum_{l\in\mathbb{Z}} g\left(q^l e^{id}\right) e_p\left(\frac{-q^{l+1}e^{id}}{z}\right) \left(\frac{-q^l e^{id}}{z} + 1\right) q^l \\
& = & p(q-1)e^{id} \displaystyle\sum_{l\in\mathbb{Z}} \frac{g\left(q^l e^{id}\right)}{e_q\left(\frac{q^{l+1}e^{id}}{z}\right)} \left(\frac{q^l e^{id}}{z} - 1\right) q^l \\
& = & p\mathcal{L}_{q,1}^{[d]}\left(\zeta g\big(\zeta\big)\right) \; - \; pz\mathcal{L}_{q,1}^{[d]}\Big(g(\zeta)\Big).
\end{array}$$

□

Remark 4.3.5. Let $k \in \mathbb{N}^*$ and let $d \in \mathbb{R}$. If we consider $\hat{f} \in \mathbb{C}\left[\left[z^k\right]\right]$, solution of a linear δ_q-equation with coefficients in $\mathbb{C}\left[z^k\right]$ with $\hat{\mathcal{B}}_{q,k}\left(\hat{f}\right) \in \mathbb{H}_{q,k}^{d}$, then we have:

$$\delta_q\left(\mathcal{L}_{q,k}^{[d]} \circ \hat{\mathcal{B}}_{q,k}\left(\hat{f}\right)\right) = \mathcal{L}_{q,k}^{[d]} \circ \hat{\mathcal{B}}_{q,k}\left(\delta_q \hat{f}\right) \text{ and } \delta_q\left(z^k \mathcal{L}_{q,k}^{[d]} \circ \hat{\mathcal{B}}_{q,k}\left(\hat{f}\right)\right) = \mathcal{L}_{q,k}^{[d]} \circ \hat{\mathcal{B}}_{q,k}\left(\delta_q\left(z^k \hat{f}\right)\right).$$

Hence, $\mathcal{L}_{q,k}^{[d]} \circ \hat{\mathcal{B}}_{q,k}\left(\hat{f}\right)$ is solution of the same linear δ_q-equation than $\hat{f}$. But in general, if $\hat{f} \in \mathbb{C}[[z]]$ is solution of a linear δ_q-equation with coefficients in $\mathbb{C}[z]$, we will have to apply successively several q-Borel and q-Laplace transformations in order to compute an analytic solution of the same equation than $\hat{f}$. See Theorem 4.4.5.

In §4.7, we will use other q-analogue of the Borel (resp. Laplace) transformation that has been originally introduced by Ramis (resp. Zhang). See [Zha02], §1 for the justification of the convergence of the q-Laplace transformation.

Definition 4.3.6. (1) We define $\hat{B}_q$ as follows:

$$\begin{array}{rccc} \hat{B}_q : & \mathbb{C}[[z]] & \longrightarrow & \mathbb{C}[[\zeta]] \\ & \displaystyle\sum_{l\in\mathbb{N}} a_l z^l & \longmapsto & \displaystyle\sum_{l\in\mathbb{N}} \frac{a_l}{q^{l(l-1)/2}}\zeta^l. \end{array}$$

(2) Let $d\in\mathbb{R}$. We define the map $L_q^{[d]}$ as follows:

$$\begin{array}{rccc} L_q^{[d]} : & \mathbb{H}_{q,1}^d & \longrightarrow & \mathcal{M}(\mathbb{C}^*,0) \\ & f & \longmapsto & \displaystyle\sum_{n\in\mathbb{Z}} \frac{f\left(q^n(q-1)e^{id}\right)}{\Theta_q\left(\frac{q^{n+1}(q-1)e^{id}}{z}\right)}. \end{array}$$

For $|z|$ small, the function $L_q^{[d]}(f)(z)$ admits a spiral of poles of order at most 1 that are contained in the q-spiral $(q-1)[d+\pi]$.

Remark 4.3.7. Let $d\in\mathbb{R}$. The maps $\hat{\mathcal{B}}_{q,1}$, $\mathcal{L}_{q,1}^{[d]}$, $\hat{B}_q$ and $L_q^{[d]}$ are very similar to the q-Borel and the "discrete" q-Laplace transformations introduced in [DVZ09], §4.2. Let $\hat{f}\in\mathbb{C}[[z]]$ such that there exists $d\in\mathbb{R}$ with $g:=\hat{\mathcal{B}}_{q,1}\left(\hat{f}\right)\in\mathbb{H}_{q,1}^d$ (resp. $h:=\hat{B}_q\left(\hat{f}\right)\in\mathbb{H}_{q,1}^d$). By a straightforward computation, we find that $\mathcal{L}_{q,1}^{[d]}(g)$ and $L_q^{[d]}(h)$ are respectively equal to the two "discrete" q-Borel-Laplace summation defined in [DVZ09], Definition 4.12, (1).

We can compare the two q-Borel-Laplace summation processes for formal power series solutions of a linear σ_q-equation with coefficients in $\mathbb{C}(\{z\})$ with only slope 1. From [DVZ09], Theorem 4.14, and Remark 4.3.7, we deduce directly the following:

Theorem 4.3.8. *Let $\hat{h}(z)\in\mathbb{C}[[z]]$ be a formal power series solution of a linear σ_q-equation with coefficients in $\mathbb{C}(\{z\})$ with only slope 1 and let $d\in\mathbb{R}$. Then, the series $\hat{B}_q\left(\hat{h}\right)$ converges and admits an analytic continuation $f\in\mathbb{H}_{q,1}^d$ if and only if $\hat{\mathcal{B}}_{q,1}\left(\hat{h}\right)$ converges and admits an analytic continuation $g\in\mathbb{H}_{q,1}^d$. Moreover for such a $d\in\mathbb{R}$, $L_q^{[d]}(f)=\mathcal{L}_{q,1}^{[d]}(g)$ on a convenient domain.*

4.4 Statement of the main result.

From now, we see q as a parameter in $]1,\infty[$. We recall that when we say that q is close to 1, we mean that q will be in the neighborhood of 1 in $]1,\infty[$. In §4.4.1, we prove two preliminaries lemmas that deal with the confluence of formal solutions of family of linear σ_q-equations. In §4.4.2, we state our main result. We consider $\left(\hat{h}(z,q)\right)_{q>1}$ (resp. $\widetilde{h}(z)$), formal power series solutions of a family of linear δ_q-equations (resp. δ-equation) with coefficients in $\mathbb{C}[z]$. We assume that $\hat{h}(z,q)$ converges coefficientwise to $\widetilde{h}(z)$ when $q\to 1$. We state that under reasonable assumptions, for q close to 1, we may apply several q-Borel and q-Laplace transformations to $\hat{h}(z,q)$, and obtain a solution of the family of linear δ_q-equations, that is for q fixed, meromorphic on some punctured neighborhood of 0 in $\mathbb{C}^*$. Moreover, the latter converges as q goes to 1, to the solution of the linear δ-equation, computed with the classical Borel and Laplace transformations.

4.4.1 Preliminaries on confluence of formal solutions.

Lemma 4.4.1. *Let us consider*

$$\begin{cases} \Delta_q & := & b_m(z,q)\delta_q^m & + & b_{m-1}(z,q)\delta_q^{m-1} & + & \dots & + & b_0(z,q) \\ \widetilde{\Delta} & := & \widetilde{b}_m(z)\delta^m & + & \widetilde{b}_{m-1}(z)\delta^{m-1} & + & \dots & + & \widetilde{b}_0(z), \end{cases}$$

with $z\mapsto b_i(z,q),\widetilde{b}_i(z)\in\mathbb{C}[[z]]$, and the b_i converge coefficientwise to the $\widetilde{b}_i$ when $q\to 1$. We assume that the $\mathbb{C}$-vector subspace $\widetilde{F}\subset\mathbb{C}((z))$, of solutions of $\widetilde{\Delta}\left(\widetilde{F}\right)=0$ has dimension 1. Let $\kappa\in\mathbb{Z}$ be the z-valuation of the elements of $\widetilde{F}\setminus\{0\}$. Let $\hat{h}(z,q):=\sum_{n=\kappa}^{\infty}\hat{h}_n(q)z^n$ be a solution of $\Delta_q\left(\hat{h}\right)=0$, such that $\lim_{q\to 1}\hat{h}_\kappa(q)=\widetilde{h}_\kappa\neq 0$. Let $\widetilde{h}(z):=\sum_{n=\kappa}^{\infty}\widetilde{h}_n z^n\in\widetilde{F}\setminus\{0\}$, which is uniquely determined by assumption. Then, for all $n\geq\kappa$,

$$\lim_{q\to 1}\hat{h}_n(q)=\widetilde{h}_n.$$

Proof. We will prove by an induction on n that for all $n\geq\kappa$, $\hat{h}_n(q)$ converges as q goes to 1 to $\widetilde{h}_n$. By assumption, $\hat{h}_\kappa(q)$ converges to $\widetilde{h}_\kappa$.

Let $n\geq\kappa$. Induction hypothesis: assume that for all $k\in\{\kappa,\dots,n-1\}$, $\lim_{q\to 1}\hat{h}_k(q)=\widetilde{h}_k$. Let us prove that $\hat{h}_n(q)$ converges to $\widetilde{h}_n$. Looking at the linear σ_q-equation (resp. the linear δ-equation) satisfied by $\hat{h}(z,q)$ (resp. $\widetilde{h}(z)$), we find a relation of the form:

$$\begin{array}{lllllll} c_n(q)\hat{h}_n(q) & = & c_{n-1}(q)\hat{h}_{n-1}(q) & + & \dots & + & c_\kappa(q)\hat{h}_\kappa(q), \\ \widetilde{c}_n\widetilde{h}_n & = & \widetilde{c}_{n-1}\widetilde{h}_{n-1} & + & \dots & + & \widetilde{c}_\kappa\widetilde{h}_\kappa, \end{array}$$

where $c_i(q),\widetilde{c}\in\mathbb{C}$. Since the b_i converge coefficientwise to the $\widetilde{b}_i$ when $q\to 1$, we find that for all $k\in\{\kappa,\dots,n\}$, $\lim_{q\to 1}c_k(q)=\widetilde{c}_k$.

If $\widetilde{c}_n=0$, then we obtain a formal solution of the same linear δ-equation than $\widetilde{h}$ with z-valuation equal to n. This is in contradiction with the assumptions of the lemma. Therefore, $\widetilde{c}_n\neq 0$. Using the convergence of $c_n(q)$ to $\widetilde{c}_n$, $c_n(q)$ is not vanishing in the neighborhood of 1. Because of the induction hypothesis and the convergence of the $c_i(q)$, we obtain

$$\lim_{q\to 1}\hat{h}_n(q)=\widetilde{h}_n.$$

By induction, we have proved that for all $n \geq \kappa$, $\hat{h}_n(q)$ converges as q goes to 1 to $\widetilde{h}_n$. □

If A and B are matrices with coefficients in $\mathbb{C}$ and $R \in \mathbb{R}_{>0}$, we say that $|A| < |B|$ (resp. $|A| < R$) if every entry of A has modulus bounded by the modulus of the corresponding entry of B (resp. by R).

Following §3.3.1 of [Sau00], we prove:

Lemma 4.4.2. *Let us consider $z \mapsto \hat{h}(z,q), \widetilde{h}(z) \in \mathbb{C}\{z\}$, solution of*

$$\begin{cases} b_m(z,q)\delta_q^m \hat{h}(z,q) + b_{m-1}(z,q)\delta_q^{m-1}\hat{h}(z,q) + \dots + b_0(z,q)\hat{h}(z,q) = 0 \\ \widetilde{b}_m(z)\delta^m \widetilde{h}(z) + \widetilde{b}_{m-1}(z)\delta^{m-1}\widetilde{h}(z) + \dots + \widetilde{b}_0(z)\widetilde{h}(z) = 0, \end{cases}$$

with $z \mapsto b_i(z,q), \widetilde{b}_i(z) \in \mathbb{C}[z]$ and assume that

- *The b_i converge coefficientwise to the $\widetilde{b}_i$ when $q \to 1$.*
- *The series $\hat{h}$ converges coefficientwise to $\widetilde{h}$ when $q \to 1$.*

Then, we have

$$\lim_{q \to 1} \hat{h}(z,q) = \widetilde{h}(z),$$

uniformly on a closed disk centered at 0.

Proof. Let us consider the equations as systems:

$$\delta_q Y(z,q) = B(z,q)Y(z,q) \text{ and } \delta \widetilde{Y}(z) = \widetilde{B}(z)\widetilde{Y}(z).$$

Let $\kappa \in \mathbb{Z}$ and let us write the vector solutions $Y(z,q) =: \sum_{k=\kappa}^{\infty} Y_k(q)z^k$, $\widetilde{Y}(z) =: \sum_{k=\kappa}^{\infty} \widetilde{Y}_k z^k$ and the matrices $B(z,q) =: \sum_{k=\kappa}^{\infty} B_k(q)z^k$, $\widetilde{B}(z) =: \sum_{k=\kappa}^{\infty} \widetilde{B}_k z^k$. For all $k \geq \kappa$, we have the relation:

$$\Big([k]_q \times \mathrm{Id} - B_0(q)\Big)Y_k(q) = \sum_{i \neq k} B_i(q)Y_{k-i}(q) \text{ and } \Big(k \times \mathrm{Id} - \widetilde{B}_0\Big)\widetilde{Y}_k = \sum_{i \neq k} \widetilde{B}_i \widetilde{Y}_{k-i}. \quad (4.4.1)$$

There exist $k_0 \geq \kappa$, $C \in \mathbb{R}_{>0}$, such that for all $k \geq k_0$, for all q close to 1, for all $Y \in \mathbb{C}^m$,

$$\Big([k]_q \times \mathrm{Id} - B_0(q)\Big) \in \mathrm{GL}_m(\mathbb{C}) \text{ and } \left|\Big([k]_q \times \mathrm{Id} - B_0(q)\Big)^{-1} Y\right| = \left|\sum_{l=0}^{\infty} ([k]_q)^{-1} \left(\frac{B_0(q)}{[k]_q}\right)^l Y\right| < C|Y|$$

resp.

$$\Big(k \times \mathrm{Id} - \widetilde{B}_0\Big) \in \mathrm{GL}_m(\mathbb{C}) \text{ and } \left|\Big(k \times \mathrm{Id} - \widetilde{B}_0\Big)^{-1} Y\right| = \left|\sum_{l=0}^{\infty} k^{-1} \left(\frac{\widetilde{B}_0}{k}\right)^l Y\right| < C|Y|.$$

Since the equations have coefficients in $\mathbb{C}[z]$, the first assumption implies the existence of $C_0 > 0$ such that for all $k \geq \kappa$, for all q close to 1, $|B_k(q)| < C_0^k$ and $\left|\widetilde{B}_k(q)\right| < C_0^k$. Using additionally (4.4.1), we can prove by an induction that there exists $C_1 > 0$, such that for all $k \geq \kappa$, for all q close to 1, we have:

$$|Y_k(q)| = \left|\Big([k]_q \times \mathrm{Id} - B_0(q)\Big)^{-1} \sum_{i \neq k} B_i(q)Y_{k-i}(q)\right| < C_1^k$$

and

$$\left|\widetilde{Y}_k\right| = \left|\left(k \times \mathrm{Id} - \widetilde{B}_0\right)^{-1} \sum_{i\neq k} \widetilde{B}_i \widetilde{Y}_{k-i}\right| < C_1^k.$$

Using the dominated convergence theorem, and the second assumption of the lemma, we obtain the result. □

4.4.2 Confluence of a "discrete" q-Borel-Laplace summation.

The goal of the subsection is to state our main result, Theorem 4.4.5. See §4.5, §4.6 for the proof. We begin with a definition.

Definition 4.4.3. Let $d \in \mathbb{R}$ and let $k \in \mathbb{Q}_{>0}$. Let f be a function such that there exists $\varepsilon > 0$, such that for q close to 1, $z \mapsto f(z,q) \in \mathcal{A}(d-\varepsilon, d+\varepsilon)$. We say that f belongs to $\overline{\mathbb{H}}_k^d$, if for q close to 1, $z \mapsto f(z,q)$ admits an analytic continuation defined on $\overline{S}(d-\varepsilon, d+\varepsilon)$, that we will still call f, such that there exist constants $J, L > 0$, that do not depend upon q, such that for all $z \in \mathbb{R}_{>0}$:

$$\left|f\left(e^{id}z, q\right)\right| < J e_q\left(Lz^k\right).$$

Let us consider $z \mapsto \hat{h}(z,q) \in \mathbb{C}[[z]]$, that converges coefficientwise to $\widetilde{h}(z) \in \mathbb{C}[[z]]$ when $q \to 1$. We make the following assumptions:

(A1) There exist

$$z \mapsto b_0(z,q), \dots, b_m(z,q) \in \mathbb{C}[z],$$

with z-coefficients that converge as q goes to 1, such that for all q close to 1, $\hat{h}(z,q)$ is solution of:

$$b_m(z,q)\delta_q^m(y(z,q)) + \cdots + b_0(z,q)y(z,q) = 0. \tag{4.4.2}$$

Let $\widetilde{b}_0(z), \dots, \widetilde{b}_m(z) \in \mathbb{C}[z]$, be the limit as q tends to 1 of the $b_0(z,q), \dots, b_m(z,q)$.

(A2) For q close to 1, the slopes of (4.4.2) are independent of q, and the set of slopes of (4.4.2) that are positive coincides with the set of slopes of

$$\widetilde{b}_m(z)\delta^m\left(\widetilde{y}(z)\right) + \cdots + \widetilde{b}_0(z)\widetilde{y}(z) = 0. \tag{4.4.3}$$

Notice that the series $\widetilde{h}(z)$ is solution of (4.4.3).

(A3) There exists $c_1 > 0$, such that for all $i \leq m$ and q close to 1:

$$\left|b_i(z,q) - \widetilde{b}_i(z)\right| < (q-1)c_1\left(\left|\widetilde{b}_i(z)\right| + 1\right).$$

Remark 4.4.4. (1) Conversely, given equations like (4.4.2) and (4.4.3) that satisfies the assumptions **(A2)** and **(A3)**, we would like to know if there exists $z \mapsto \hat{h}(z,q) \in \mathbb{C}[[z]]$, solution of (4.4.2), which converges coefficientwise to $\widetilde{h}(z) \in \mathbb{C}[[z]]$, solution of (4.4.3). The answer is in general no, but Lemma 4.4.1 gives a sufficient condition.

(2) If for q close to 1, the only slope of (4.4.3) is 0 then, $z \mapsto \hat{h}(z,q), \widetilde{h}(z) \in \mathbb{C}\{z\}$ and we set for all $d \in \mathbb{R}$, $S_q^{[d]}\left(\hat{h}\right) := \hat{h}$. Remember that we have set in §4.1, $\widetilde{S}^d\left(\widetilde{h}\right) := \widetilde{h}$. In this particular case, applying Lemma 4.4.2, we obtain that

$$\lim_{q\to 1} S_q^{[d]}\left(\hat{h}\right) = \widetilde{S}^d\left(\widetilde{h}\right),$$

uniformly on a closed disk centered at 0.

From now, we are going to assume that (4.4.3) has at least one slope strictly bigger than 0. Let $d_0 := \max\left(2, \deg \widetilde{b}_0, \dots, \deg \widetilde{b}_m\right)$. Let $k_1 < \dots < k_{r-1}$ be the slopes of (4.4.3) different from 0, let k_r be an integer strictly bigger than k_{r-1} and d_0, and set $k_{r+1} := +\infty$. Let $(\kappa_1, \dots, \kappa_r)$ defined as:

$$\kappa_i^{-1} := k_i^{-1} - k_{i+1}^{-1}.$$

As in Proposition 4.1.5, we define the $(\widetilde{\kappa_1}, \dots, \widetilde{\kappa_s})$ as follows: we take $(\kappa_1, \dots, \kappa_r)$ and for $i = 1, \dots, i = r$, replace successively κ_i by α_i terms $\alpha_i \kappa_i$, where α_i is the smallest integer such that $\alpha_i \kappa_i$ is greater or equal than d_0. See Example 4.1.4. Therefore, by construction, each of the $\widetilde{\kappa_i}$ is a rational number greater than d_0 and $\widetilde{\kappa_s} \in \mathbb{N}^*$.

Let $\beta \in \mathbb{N}^*$ be minimal, such that for all $i \in \{1, \dots, s\}$, $\beta / \widetilde{\kappa_i} \in \mathbb{N}^*$. Let us write $\hat{h}(z,q) =: \sum_{n=0}^{\infty} \hat{h}_n(q) z^n$ and, for $l \in \{0, \dots, \beta - 1\}$, let $\hat{h}^{(l)}(z,q) := \sum_{n=0}^{\infty} \hat{h}_{l+n\beta}(q) z^{n\beta}$.

The main result of the chapter is the following. See §4.1, §4.3 for the notations, and §4.5, §4.6 for the proof. See also Theorem B.4 in the appendix for a similar result with a "continuous" q-Laplace transformation. We recall that the series $\hat{h}, \widetilde{h}$ satisfies the assumptions **(A1)** to **(A3)**.

Theorem 4.4.5. *There exists $\Sigma_{\widetilde{h}} \subset \mathbb{R}$ finite modulo $2\pi\mathbb{Z}$, that contains the set of singular directions defined in Proposition 4.1.5, such that if $d \in \mathbb{R} \setminus \Sigma_{\widetilde{h}}$ and $l \in \{0, \dots, \beta - 1\}$, then the series $g_{1,l} := \hat{\mathcal{B}}_{q,\widetilde{\kappa_1}} \circ \dots \circ \hat{\mathcal{B}}_{q,\widetilde{\kappa_s}} \left(\hat{h}^{(l)}\right)$ converges and belongs to $\overline{\mathbb{H}}^d_{\widetilde{\kappa_1}}$.*

Moreover, for $j = 2$ (resp. $j = 3, \dots,$ resp. $j = s$), $g_{j,l} := \mathcal{L}^{[d]}_{q,\widetilde{\kappa_{j-1}}}(g_{j-1,l})$ belongs to $\overline{\mathbb{H}}^d_{\widetilde{\kappa_j}}$. Let $S_q^{[d]}\left(\hat{h}^{(l)}\right) := \mathcal{L}^{[d]}_{q,\widetilde{\kappa_s}}(g_{r,l})$. The function

$$S_q^{[d]}\left(\hat{h}\right) := \sum_{l=0}^{\beta-1} z^l S_q^{[d]}\left(\hat{h}^{(l)}\right) \in \mathcal{A}\left(d - \frac{\pi}{k_r}, d + \frac{\pi}{k_r}\right)$$

is solution of (4.4.2). Furthermore, we have

$$\lim_{q \to 1} S_q^{[d]}\left(\hat{h}\right) = \widetilde{S}^d\left(\widetilde{h}\right),$$

uniformly on the compacts of $\overline{S}\left(d - \frac{\pi}{2k_r}, d + \frac{\pi}{2k_r}\right) \setminus \bigcup \mathbb{R}_{\geq 1}\alpha_i$, where α_i are the roots of $\widetilde{b}_m \in \mathbb{C}[z]$, and $\widetilde{S}^d\left(\widetilde{h}\right)$ is the asymptotic solution of (4.4.3) that has been defined in Proposition 4.1.5.

Remark 4.4.6. After some arrangements, we could probably state and show a similar result for q not real. As [Sau00], we should make q goes to 1 following a q-spiral of the form $\left\{q_0^{\lambda}, \lambda \in \mathbb{R}_{>0}\right\}$, for some $q_0 \in \mathbb{C}$ fixed with modulus strictly bigger than 1. The problem here, is that we would obtain at the limit, a solution of the differential equation that is not classic, since at the limit, we would obtain integrals of the form $\int_{q_0^{\mathbb{R}} e^{id}} z^{-k} f(\zeta) e^{-\left(\frac{\zeta}{z}\right)^k} d\zeta^k$, instead of Laplace transformations. In order to interpret the limit as the classical Borel-Laplace summation, we have to consider q real.

Remark 4.4.7. A confluence result of this nature can also be found in [DVZ09], Theorem 2.6. We are going now to state [DVZ09], Corollary 2.9, which is the particular case where the coefficients of the family of linear q-difference equations do not depend upon q. Let $p = 1/q$ and let $\delta_p := \frac{\sigma_q^{-1} - \mathrm{Id}}{p-1}$, which converges formally to δ when $p \to 1$.

Let $z \mapsto \hat{h}(z,q) \in \mathbb{C}\{z\}$ that converges coefficientwise to $\widetilde{h}(z) \in \mathbb{C}[[z]]$ when $p \to 1$. Assume the existence of $b_0, \dots, b_m \in \mathbb{C}[z]$, such that for all p close to 1, we have

$$\begin{cases} b_m(z)\delta_p^m \hat{h}(z,q) & + \quad \dots + b_0(z)\hat{h}(z,q) & = & 0 \\ & & & \\ b_m(z)\delta^m \widetilde{h}(z) & + \quad \dots + b_0(z)\widetilde{h}(z) & = & 0. \end{cases}$$

Moreover, assume that the series $\hat{\mathcal{B}}_1\left(\widetilde{h}\right)$ belongs to $\mathbb{C}\{z\}$ and is solution of a linear differential equation which is Fuchsian at 0 and infinity and has non resonant exponents at ∞.

Let $\widetilde{\Sigma}_{\widetilde{h}} \subset \mathbb{R}$ be the set of singular directions that has been defined in Proposition 4.1.5. The authors of [DVZ09] conclude that for all $d \notin \widetilde{\Sigma}_{\widetilde{h}}$, the series $\hat{\mathcal{B}}_1\left(\widetilde{h}\right)$ belongs to $\widetilde{\mathbb{H}}_1^d$, and

$$\lim_{p\to 1} \hat{h}(z,q) = \widetilde{S}^d\left(\widetilde{h}\right)(z),$$

uniformly on the compacts of $\overline{S}\left(d-\frac{\pi}{2}, d+\frac{\pi}{2}\right)$, where $\widetilde{S}^d\left(\widetilde{h}\right)$ is the asymptotic solution of the linear differential equation that has been defined in Proposition 4.1.5. Notice that Theorem 4.4.5 and this theorem have not the same setting, since we consider δ_q-equations and not δ_p-equations. In particular, in our case $z \mapsto \hat{h}(z,q)$ might be divergent and we have to replace $\hat{h}$ by $S_q^{[d]}\left(\hat{h}\right)$ in order to have the convergence.

4.5 Lemmas on meromorphic solutions.

The goal of this section is to prove lemmas on meromorphic solutions that will be used in the proof of Theorem 4.4.5 in §4.6. See §4.4 for the notations. If $D(z) \in \mathrm{GL}_m\Big(\mathbb{C}(z)\Big)$, we define $\mathbf{S}^q(D(z))$ as the union of the $q^{\mathbb{N}^*}x_i$, where the x_i are the poles of $D(z)$ or $D(z)^{-1}$.

Lemma 4.5.1. *Let $a<b$. Let us consider $\sigma_q M(z) = D(z)M(z)$ with $D(z) \in \mathrm{GL}_m\Big(\mathbb{C}(z)\Big)$ and $M(z)$ is s solution in $\Big(\mathcal{A}(a,b)\Big)^m$. Then, the entries of $M(z)$ are meromorphic on $\overline{S}(a,b)$, with poles contained in $\mathbf{S}^q(D(z))$.*

Proof. Let $z \in \mathbb{C}^* \setminus \mathbf{S}^q(D(z))$. We use the fact that $M(qz) = D(z)M(z)$ to deduce that if the entries of M are analytic on a domain U, then there are analytic on the domain $qU := \{qz, z \in U\}$. We use the existence of $\varepsilon > 0$, such that the entries of $M(z)$ are analytic for all $|z| < \varepsilon$ and $z \in \overline{S}(a,b)$, to obtain that the entries of $M(z)$ are meromorphic on $\overline{S}(a,b)$, with poles contained in $\mathbf{S}^q(D(z))$. □

If $\widetilde{D}(z) \in \mathrm{M}_m\Big(\mathbb{C}(z)\Big)$, we define $\mathbf{S}^1\left(\widetilde{D}(z)\right)$ as the union of the $\mathbb{R}_{\geq 1}x_i$, where the x_i are the poles of $\widetilde{D}(z)$. We define also $\mathbb{R}_{>0}[z]$ as the set of polynomials with coefficients that are strictly positive real numbers. We recall that if A and B are matrices with coefficients in $\mathbb{C}$ and $R \in \mathbb{R}_{>0}$, we say that $|A| < |B|$ (resp. $|A| < R$) if every entry of A has modulus bounded by the modulus of the corresponding entry of B (resp. by R).

Proposition 4.5.2. *Let $a<b$, $z \mapsto \mathrm{Id} + (q-1)D(z,q) \in \mathrm{GL}_m\Big(\mathbb{C}(z)\Big), \widetilde{D}(z) \in \mathrm{M}_m\Big(\mathbb{C}(z)\Big)$ and let C be a convex set with non empty interior contained in $\overline{S}(a,b) \setminus \mathbf{S}^1\left(\widetilde{D}(z)\right)$ such that 0 does not belong to its closure. Let us consider $z \mapsto M(z,q), \widetilde{M}(z)$, $1 \times m$ matrices with entries continuous on C and analytic in the interior of C, solutions of*

$$\begin{cases} \delta_q M(z,q) &= D(z,q)M(z,q) \\ \delta \widetilde{M}(z) &= \widetilde{D}(z)\widetilde{M}(z). \end{cases}$$

We assume that:

(i) There exists $c_1 > 0$, such that for all q close to 1 and for all $z \in C$,

$$\left|D(z,q) - \widetilde{D}(z)\right| < (q-1)c_1\left(\left|\widetilde{D}(z)\right| + |\mathbf{1}_m|\right),$$

where $\mathbf{1}_m$ denotes the square matrix of size m with 1 entries everywhere. Notice that this condition implies that for q close to 1, the entries of $D(z,q)$ have no poles in C.

(ii) There exists $w_0 \in C$, such that for all q close to 1, $M(w_0,q) = \widetilde{M}(w_0)$. Moreover, we have $\lim\limits_{q \to 1} M(w,q) = \widetilde{M}(w)$ uniformly on a compact K contained in C.

(iii) There exists $R \in \mathbb{C}[z]$, such that for all $z \in C$, $\left|\widetilde{M}(z)\right| < |R(z)|$.

Let κ be the maximum of the degrees of the numerators and the denominators of the entries of $\widetilde{D}(z)$, written as the quotient of two coprime polynomials. Let $S \in \mathbb{R}_{>0}[z]$ be a polynomial of degree κ, such that for all $z \in C$, $S(|z|) > \left|\widetilde{D}(z)\right| + |\mathbf{1}_m|$. Under those assumptions, there exist

– *$\delta(q) > 0$ that converges to 1 as $q \to 1$,*

– $\varepsilon(q) > 0$ *that converges to* 0 *as* $q \to 1$,

– $S_0 \in \mathbb{R}_{>0}[z]$ *which has degree* κ *and satisfies* $S_0(|z|) > S(|z|)$ *for all* $z \in C$,

such that for all $z \in C \cap \mathbb{R}_{\geq 1}K := \Big\{xw \in C \Big| x \in [1, \infty[, w \in K\Big\}$, *we have*

$$\Big|M(z,q) - \widetilde{M}(z)\Big| < (q-1)\delta(q)e_{q^\kappa}\Big(S_0\left(|z|\right)\Big) + \varepsilon(q)\left|R(z)\right|.$$

In particular,

$$\lim_{q\to 1} M\left(z,q\right) = \widetilde{M}\left(z\right),$$

uniformly on the compacts of $C \cap \mathbb{R}_{\geq 1}K$.

Remark 4.5.3. The polynomial S_0 does not depend upon w and q.

Before proving the proposition, we need to prove a technical lemma.

Lemma 4.5.4. *Let* $z \mapsto \mathrm{Id} + (q-1)\,D(z,q) \in \mathrm{GL}_m\Big(\mathbb{C}(z)\Big), \widetilde{D}(z) \in \mathrm{M}_m\Big(\mathbb{C}(z)\Big)$ *that satisfies assumption (i) of Proposition 4.5.2. Let* C *be the corresponding convex set and let* K *be the corresponding compact set defined in Proposition 4.5.2. Let* $\Big(z \mapsto M_w\left(z,q\right)\Big)_{w\in K}, \Big(\widetilde{M}_w(z)\Big)_{w\in K}$, *be a family of* $1 \times m$ *matrices with entries continuous on* C *and analytic in the interior of* C, *solutions of*

$$\begin{cases} \delta_q M_w(z,q) &= D(z,q)M_w(z,q) \\ \delta \widetilde{M}_w(z) &= \widetilde{D}(z)\widetilde{M}_w(z). \end{cases}$$

We assume that the matrices $\Big(M_w(z,q)\Big)_{w\in K}, \Big(\widetilde{M}_w(z)\Big)_{w\in K}$ *satisfy:*

(a) For all q *close to* 1, *for all* $w \in K$, $M_w\left(w,q\right) = \widetilde{M}_w\left(w\right)$.

(b) There exists $R \in \mathbb{C}[z]$, *such that for all* $z \in C$, *for all* $w \in K$:

$$\Big|\widetilde{M}_w\left(z\right)\Big| < \left|R\left(z\right)\right|.$$

Under those assumptions, there exists a polynomial S_0 *that satisfies the same properties than the one in Proposition 4.5.2, such that for all* $w \in K$, *for all* q *close to* 1, *for all* $N \in \mathbb{N}$ *with* $q^N w \in C$:

$$\left|\frac{M_w\left(q^N w, q\right) - \widetilde{M}_w\left(q^N w\right)}{q-1}\right| < e_{q^\kappa}\Big(S_0\left(\left|q^N w\right|\right)\Big). \tag{4.5.1}$$

Proof of Lemma 4.5.4. For the reader's convenience, we will decompose the proof in four steps.

Step 1: Find another expression of $\frac{\widetilde{M}_w(q^n w) - M_w(q^n w, q)}{q-1}$.

Let f be a function continuous on C, that is analytic in the interior of C, and let $z_0, z_1 \in C$. The generalized mean value theorem (see §1.4 of [KM97]) says that there exists $c \in C$ that belongs to the convex hull of

$$\Big\{f'(z_0 + x(z_1 - z_0))\Big| x \in [0,1]\Big\},$$

such that:

$$\frac{f(z_1)-f(z_0)}{z_1-z_0}=c.$$

For all $q>1, w\in K, n\in\mathbb{N}$ with $q^n w\in C$, let us define the $A_{w,q,n-1}$ as the convex hull of

$$\left\{\frac{\widetilde{D}\left(q^{n-1}wx\right)\widetilde{M}\left(q^{n-1}wx\right)}{q^{n-1}wx}\middle| x\in[1,q]\right\}.$$

Because of the generalized mean value theorem, for all $n\in\mathbb{N}$, for all $q>1$, for all $w\in K$, with $q^n w\in C$, there exists $\widetilde{D}_{w,q,n-1}$ that belongs to $A_{w,q,n-1}$, such that:

$$\frac{\widetilde{M}_w\left(q^n w\right)-\widetilde{M}_w\left(q^{n-1}w\right)}{q^{n-1}w(q-1)}=\widetilde{D}_{w,q,n-1}.$$

The linear δ_q-equation satisfied by $M_w(z,q)$ gives that for all $n\in\mathbb{N}$, for all $q>1$, for all $w\in K$, with $q^n w\in C$:

$$\frac{M_w\left(q^n w,q\right)-M_w\left(q^{n-1}w,q\right)}{q-1}=D\left(q^{n-1}w,q\right)M_w\left(q^{n-1}w,q\right).$$

In particular, we have

$$\begin{aligned}\frac{\widetilde{M}_w(q^n w)-M_w(q^n w,q)}{q-1} &= \frac{\widetilde{M}_w\left(q^{n-1}w\right)-M_w\left(q^{n-1}w,q\right)}{q-1}\\ &+ q^{n-1}w\widetilde{D}_{w,q,n-1}-D\left(q^{n-1}w,q\right)M_w\left(q^{n-1}w,q\right).\end{aligned}\tag{4.5.2}$$

Step 2: Bound the expression of $\frac{\widetilde{M}_w(q^n w)-M_w(q^n w,q)}{q-1}$.

Let $q_0>1$ sufficiently close to 1. Let us prove the existence of $R_1,R_2\in\mathbb{C}[z]$, such that for all $n\in\mathbb{N}$, $q\in]1,q_0[$, $w\in K$, with $q^n w\in C$,

$$\begin{aligned}&\left|\frac{\widetilde{M}_w(q^n w)-M_w(q^n w,q)}{q-1}\right|\\ \leq\ &\left|\frac{\widetilde{M}_w\left(q^{n-1}w\right)-M_w\left(q^{n-1}w,q\right)}{q-1}\right| + (q-1)\Big(\left|R_1\left(q^n w\right)\right|+\left|R_2\left(q^n w\right)\right|\Big)\\ +\ &\left|\frac{\widetilde{M}_w\left(q^{n-1}w\right)-M_w\left(q^{n-1}w,q\right)}{q-1}\right| \times (q-1)(1+(q-1)c_1)mS\left(\left|q^n w\right|\right),\end{aligned}\tag{4.5.3}$$

where $S,c_1>0$ are given by Proposition 4.5.2. Using the triangular inequality and (4.5.2), it is sufficient to prove the existence of $R_1,R_2\in\mathbb{C}[z]$, such that we have for all $n\in\mathbb{N}$, $q\in]1,q_0[$, $w\in K$, with $q^n w\in C$,

$$\begin{aligned}&\left|q^{n-1}w\widetilde{D}_{w,q,n-1}-D\left(q^{n-1}w,q\right)M_w\left(q^{n-1}w,q\right)\right|\\ &\leq (q-1)\Big(\left|R_1\left(q^n w\right)\right|+\left|R_2\left(q^n w\right)\right|\Big)\\ &+\left|\frac{\widetilde{M}_w\left(q^{n-1}w\right)-M_w\left(q^{n-1}w,q\right)}{q-1}\right|\times(q-1)(1+(q-1)c_1)mS\left(\left|q^n w\right|\right).\end{aligned}\tag{4.5.4}$$

We have for all $n \in \mathbb{N}$, $q \in]1, q_0[$, $w \in K$, with $q^n w \in C$,

$$\begin{aligned} & \left| q^{n-1} w \widetilde{D}_{w,q,n-1} - D\left(q^{n-1}w, q\right) M_w\left(q^{n-1}w, q\right) \right| \\ \leq & \left| q^{n-1} w \widetilde{D}_{w,q,n-1} - \widetilde{D}\left(q^{n-1}w\right) \widetilde{M}_w\left(q^{n-1}w\right) \right| \\ + & \left| \widetilde{D}\left(q^{n-1}w\right) \widetilde{M}_w\left(q^{n-1}w\right) - D\left(q^{n-1}w, q\right) \widetilde{M}_w\left(q^{n-1}w\right) \right| \\ + & \left| D\left(q^{n-1}w, q\right) \widetilde{M}_w\left(q^{n-1}w\right) - D\left(q^{n-1}w, q\right) M_w\left(q^{n-1}w, q\right) \right| . \end{aligned}$$

Let

$$\begin{aligned} \tau_1 &:= \left| q^{n-1} w \widetilde{D}_{w,q,n-1} - \widetilde{D}\left(q^{n-1}w\right) \widetilde{M}_w\left(q^{n-1}w\right) \right|, \\ \tau_2 &:= \left| \widetilde{D}\left(q^{n-1}w\right) \widetilde{M}_w\left(q^{n-1}w\right) - D\left(q^{n-1}w, q\right) \widetilde{M}_w\left(q^{n-1}w\right) \right|, \\ \tau_3 &:= \left| D\left(q^{n-1}w, q\right) \widetilde{M}_w\left(q^{n-1}w\right) - D\left(q^{n-1}w, q\right) M_w\left(q^{n-1}w, q\right) \right|. \end{aligned}$$

Let us bound τ_1. The entries of $q^{n-1} w \widetilde{D}_{w,q,n-1}$ and $\widetilde{D}\left(q^{n-1}w\right) \widetilde{M}_w\left(q^{n-1}w\right)$ belong to the convex hull of $\left\{ \frac{\widetilde{D}(q^{n-1}wx)\widetilde{M}(q^{n-1}wx)}{x} \middle| x \in [1, q] \right\}$. The entries of the elements of this set of matrices are bounded by a polynomial, because of the assumption (b) and the fact that the entries of $\widetilde{D}$ are bounded by polynomials. This provides $R_1 \in \mathbb{C}[z]$, such that for all $q \in]1, q_0[$, for all $n \in \mathbb{N}$, for all $w \in K$, with $q^n w \in C$:

$$\tau_1 = \left| q^{n-1} w \widetilde{D}_{w,q,n-1} - \widetilde{D}\left(q^{n-1}w\right) \widetilde{M}_w\left(q^{n-1}w\right) \right| < (q-1) \left| R_1\left(q^n w\right) \right|.$$

Let us bound τ_2. Due to the assumptions (i) and (b), there exists $R_2 \in \mathbb{C}[z]$ such that for all $q \in]1, q_0[$, for all $n \in \mathbb{N}$, for all $w \in K$, with $q^n w \in C$:

$$\tau_2 = \left| \left(\widetilde{D}\left(q^n w\right) - D\left(q^n w, q\right) \right) \widetilde{M}_w\left(q^n w\right) \right| < (q-1) \left| R_2\left(q^n w\right) \right|.$$

Let us bound the quantity τ_3. By assumption (i) and the fact that for all $z \in \mathbb{C}$, $\left|\widetilde{D}(z)\right| + |\mathbf{1}_m| < S(|z|)$, we obtain that for all $q \in]1, q_0[$, $n \in \mathbb{N}$, $w \in K$, with $q^{n-1} w \in C$:

$$\begin{aligned} \tau_3 &= \left| D\left(q^{n-1}w, q\right) \left(\widetilde{M}_w\left(q^{n-1}w\right) - M_w\left(q^{n-1}w, q\right) \right) \right| \\ &\leq \left(\left|\widetilde{D}\left(q^{n-1}w\right)\right| + (q-1)c_1 \left(\left|\widetilde{D}\left(q^{n-1}w\right)\right| + |\mathbf{1}_m| \right) \right) \left| \widetilde{M}_w\left(q^{n-1}w\right) - M_w\left(q^{n-1}w, q\right) \right| \\ &\leq (1 + (q-1)c_1) m S\left(\left|q^{n-1}w\right|\right) \left| \widetilde{M}_w\left(q^{n-1}w\right) - M_w\left(q^{n-1}w, q\right) \right|. \end{aligned}$$

Since the polynomial S has real positive coefficients, $S\left(\left|q^{n-1}w\right|\right) \leq S\left(\left|q^n w\right|\right)$. In particular, for all $q \in]1, q_0[$, $n \in \mathbb{N}$, $w \in K$ with $q^{n-1} w \in C$:

$$\tau_3 \leq (q-1)(1 + (q-1)c_1) m S\left(\left|q^n w\right|\right) \left| \frac{\widetilde{M}_w\left(q^{n-1}w\right) - M_w\left(q^{n-1}w, q\right)}{q-1} \right|.$$

This concludes the proof of (4.5.4) and yields (4.5.3), because of the triangular inequality.

Step 3: Construction of S_0.

We recall that $\kappa\in\mathbb{N}$ is the degree of S. Before constructing S_0, we are going to prove that for all $b>0$ sufficiently big, for all $z\in C\cap\mathbb{R}_{\geq 1}K$ and for all q close to 1

$$\begin{aligned}&e_{q^{\kappa}}\Big(b\left|z\right|^{\kappa}\Big)+(q-1)\Big(\left|R_1(qz)\right|+\left|R_2(qz)\right|\Big)\\&+(q-1)(1+(q-1)c_1)mS(q|z|)e_{q^{\kappa}}\Big(b\left|z\right|^{\kappa}\Big)\quad\leq e_{q^{\kappa}}\Big(b\left|qz\right|^{\kappa}\Big).\end{aligned}\tag{4.5.5}$$

Using the q-difference equation satisfied by the q-exponential, we find that this inequality is equivalent to:

$$\begin{aligned}1+(q-1)\frac{|R_1(qz)|+|R_2(qz)|}{e_{q^{\kappa}}\Big(b|z|^{\kappa}\Big)}+(q-1)(1+(q-1)c_1)mS(q|z|)\quad&\leq\quad\frac{e_{q^{\kappa}}\Big(b|qz|^{\kappa}\Big)}{e_{q^{\kappa}}\Big(b|z|^{\kappa}\Big)}\\&=\quad 1+(q^{\kappa}-1)b\left|z\right|^{\kappa}.\end{aligned}$$

This inequality is equivalent to the following:

$$\frac{|R_1(qz)|+|R_2(qz)|}{e_{q^{\kappa}}\Big(b\left|z\right|^{\kappa}\Big)}+(1+(q-1)c_1)mS(q|z|)\leq b[\kappa]_q\left|z\right|^{\kappa}.$$

Since R_1,R_2 are polynomials, for all $b>0$ sufficiently big, for all $q\in]1,q_0[$ and for all $z\in C\cap\mathbb{R}_{\geq 1}K$, this latter inequality is true. This proves (4.5.5).

We recall that by assumption, 0 does not belong to the closure of C. Using (4.5.5), we obtain the existence of a polynomial $S_0\in\mathbb{R}_{>0}[z]$ of degree κ, such that for all $z\in C$, $S_0(|z|)>S(|z|)$, and such that for all $z\in C\cap\mathbb{R}_{\geq 1}K$, for all q close to 1

$$\begin{aligned}&e_{q^{\kappa}}\Big(S_0(|z|)\Big)+(q-1)\Big(\left|R_1(qz)\right|+\left|R_2(qz)\right|\Big)+\\&(q-1)(1+(q-1)c_1)mS(q|z|)e_{q^{\kappa}}\Big(S_0(|z|)\Big)\leq e_{q^{\kappa}}\Big(S_0(q|z|)\Big).\end{aligned}\tag{4.5.6}$$

Step 4 : Conclusion.

We are going now to prove (4.5.1) with the polynomial S_0 we have defined in Step 3. We will proceed by an induction on n. The step $n=0$ is true because of the assumption (a).

Induction hypothesis: let us fix $n\in\mathbb{N}$, and assume that if $q\in]1,q_0[$, $w\in K$, with $q^{n+1}w\in C$,

$$\left|\frac{\widetilde{M}_w\left(q^n w\right)-M_w\left(q^n w,q\right)}{q-1}\right|<e_{q^{\kappa}}\Big(S_0\left(\left|q^n w\right|\right)\Big).$$

From (4.5.3), we obtain that

$$\begin{aligned}\left|\frac{\widetilde{M}_w\left(q^{n+1}w\right)-M_w\left(q^{n+1}w,q\right)}{q-1}\right|\quad&\leq\quad e_{q^{\kappa}}\Big(S_0\left(\left|q^n w\right|\right)\Big)+(q-1)\Big(\left|R_1\left(q^{n+1}w\right)\right|+\left|R_2\left(q^{n+1}w\right)\right|\Big)\\&+\quad e_{q^{\kappa}}\Big(S_0\left(\left|q^n w\right|\right)\Big)\times(q-1)(1+(q-1)c_1)mS\left(\left|q^{n+1}w\right|\right).\end{aligned}$$

Using additionally (4.5.6), we find that

$$\left|\frac{\widetilde{M}_w\left(q^{n+1}w\right)-M_w\left(q^{n+1}w,q\right)}{q-1}\right|<e_{q^\kappa}\Big(S_0\left(\left|q^{n+1}w\right|\right)\Big).$$

This concludes the proof of (4.5.1). □

Proof of Proposition 4.5.2. Let K be the compact considered in hypothesis (ii), with $w_0\in K\subset C$, so that we have

$$\lim_{q\to 1}\Big(M(w,q)\Big)=\Big(\widetilde{M}(w)\Big),$$

uniformly on K. Let $N(w,q)$ be the matrix, such that $N(w,q)$ has entries that are equal to the entrywise division of $\widetilde{M}(w)$ by $M(w,q)$. Due to the uniform convergence on K (assumption (ii)), the entries of $N(w,q)$ converge uniformly on K to 1, as q goes to 1. We are going to apply Lemma 4.5.4, with

$$\Big(M_w(z,q)\Big)_{w\in K}:=\Big(M(z,q)\times_h N(w,q)\Big)_{w\in K},\text{ and }\Big(\widetilde{M}_w(z)\Big)_{w\in K}:=\Big(\widetilde{M}(z)\Big)_{w\in K},\quad (4.5.7)$$

where $\times_h$ denotes the Hadamard product, that is $(a_i)\times_h(b_i):=(a_ib_i)$. If $a,b,c\in\mathbb{C}$, we have:

$$|a-b|<|c|^{-1}|a\times c-b|+\left|c^{-1}-1\right|\times|b|.$$

We are going to apply this inequality entrywise, to the entries of $M\left(q^nw,q\right)$, $\widetilde{M}\left(q^nw\right)$ and $N(w,q)$. Since the entries of $N(w,q)$ tend to 1, we find that there exists $\delta(q)>0$, (resp. $\varepsilon(q)>0$) that converges to 1 (resp. converges to 0) as q goes to 1, such that for all $w\in K$ and $n\in\mathbb{N}$, with $q^nw\in C$:

$$\left|M\left(q^nw,q\right)-\widetilde{M}\left(q^nw\right)\right|<\delta(q)\left|M\left(q^nw,q\right)\times_t N(w,q)-\widetilde{M}\left(q^nw\right)\right|+\varepsilon(q)\left|\widetilde{M}(q^nw)\right|.$$

Using the assumption (iii), there exists $R\in\mathbb{C}[z]$, such that for all $z\in C\cap\mathbb{R}_{\geq 1}K$, $\left|\widetilde{M}(z)\right|<|R(z)|$. Lemma 4.5.4 applied to (4.5.7), gives the existence of a polynomial S_0, that does not depend upon w, such that for all q close to 1, for all $w\in K$, for all $n\in\mathbb{N}$, with $q^nw\in C$, we obtain:

$$\left|M\left(q^nw,q\right)-\widetilde{M}\left(q^nw\right)\right|<(q-1)\delta(q)e_{q^\kappa}\Big(S_0\left(\left|q^nw\right|\right)\Big)+\varepsilon(q)\left|R\left(q^nw\right)\right|.$$

In other words, for q close to 1 and for all $z\in C\cap\mathbb{R}_{\geq 1}K$, we have

$$\left|M(z,q)-\widetilde{M}(z)\right|<(q-1)\delta(q)e_{q^\kappa}\Big(S_0\left(|z|\right)\Big)+\varepsilon(q)\left|R(z)\right|.$$

The uniform convergence follows immediately. □

4.6 Proof of Theorem 4.4.5.

The goal of this section is to prove Theorem 4.4.5. In §4.6.1, we treat the confluence of the "discrete" q-Laplace transformation. In §4.6.2 we prove Theorem 4.4.5 in a particular case. In §4.6.3, we prove Theorem 4.4.5 in the general case.

4.6.1 Confluence of the "discrete" q-Laplace transformation.

Lemma 4.6.1. *Let $a \in \mathbb{C}$ and $k \in \mathbb{Q}$. Then, for any $q > 1$ and $z \in \mathbb{C}^*$, the following inequality is true $\left|e_q\left(az^k\right)\right| \leq \exp\left|az^k\right|$. Moreover, we have*

$$\lim_{q\to 1} e_q\left(az^k\right) = \exp\left(az^k\right),$$

uniformly on the compacts of $\mathbb{C}^*$.

Proof. The coefficients of the series of function defining $e_q\left(az^k\right)$ depend upon the parameter q. By construction, we have for all $n \in \mathbb{N}$ and all $q > 1$, $n \leq [n]_q$, and therefore $n! \leq [n]_q^!$. Then, for all $q > 1$ and $z \in \mathbb{C}^*$, we have the following inequalities:

$$\left|e_q\left(az^k\right)\right| \leq \sum_{n=0}^{\infty}\left|\frac{a^n z^{nk}}{[n]_q^!}\right| \leq \sum_{n=0}^{\infty}\left|\frac{a^n z^{nk}}{n!}\right| = \exp\left|az^k\right|.$$

The convergence is then a direct consequence of the dominated convergence theorem, since the series defining $e_q\left(az^k\right)$ is termwise dominated by the series defining $\exp\left|az^k\right|$. □

Let $d \in \mathbb{R}$, let $k \in \mathbb{Q}_{>0}$ and let f be a function that belongs to $\overline{\mathbb{H}}_k^d$, see Definition 4.4.3, $g := \rho_{1/k}(f)$, $\widetilde{f} \in \widetilde{\mathbb{H}}_k^d$, see Definition 4.1.1, and $\widetilde{g} := \rho_{1/k}\left(\widetilde{f}\right)$. For the reader's convenience, we recall the expressions of the Laplace transformations of order 1 and k that come from Definitions 4.1.1 and 4.3.3:

$$\begin{aligned}
&\mathcal{L}_{q,1}^{[d]}(g)(z,q) = (1-q)e^{id}\sum_{l\in\mathbb{Z}}\frac{q^l g\left(q^l e^{id},q\right)}{z e_q\left(\frac{q^{l+1}e^{id}}{z}\right)}, && \mathcal{L}_1^d\left(\widetilde{g}\right)(z) = \int_0^{\infty e^{id}}\frac{\widetilde{g}(\zeta)}{z\exp\left(\frac{\zeta}{z}\right)}d\zeta,\\
&\mathcal{L}_{q,k}^{[d]}(f) = \rho_k \circ \mathcal{L}_{q,1}^{[d]} \circ \rho_{1/k}(f), && \mathcal{L}_k^d\left(\widetilde{f}\right) = \rho_k \circ \mathcal{L}_1^d \circ \rho_{1/k}\left(\widetilde{f}\right).
\end{aligned}$$

Since $f \in \overline{\mathbb{H}}_k^d$, there exist $\varepsilon > 0$, constants $J, L > 0$, such that for all q close to 1, $\zeta \mapsto f(\zeta,q)$ is analytic on $\overline{S}(d-\varepsilon, d+\varepsilon)$, and for all $\zeta \in \mathbb{R}_{>0}$:

$$\left|f\left(e^{id}\zeta,q\right)\right| < Je_q\left(L\zeta^k\right). \tag{4.6.1}$$

Lemma 4.6.2. *In the notation introduced above, let us assume that we have $\lim\limits_{q\to 1} f := \widetilde{f}$, uniformly on the compacts of $\overline{S}(d-\varepsilon, d+\varepsilon)$. Then, we have*

$$\lim_{q\to 1}\mathcal{L}_{q,k}^{[d]}\left(f\right)(z,q) = \mathcal{L}_k^d\left(\widetilde{f}\right)(z),$$

uniformly on the compacts of $\left\{z \in \overline{S}\left(d - \frac{\pi}{2k\pi}, d + \frac{\pi}{2k\pi}\right)\Big||z| < 1/L\right\}$.

Proof. The expressions of the Laplace transformations of order k allow us to reduce to the case $k=1$. The variable change $\zeta \mapsto \zeta e^{-id}$ allows us to reduce to the case $d=0$. Let us fix a an arbitrary compact subset K of $\left\{z \in \overline{S}\left(-\frac{\pi}{2\pi},+\frac{\pi}{2\pi}\right) \middle| |z|<1/L\right\}$, and let us prove the uniform convergence on K.

The q-Laplace transformation can be seen as a Riemann sum with associated partition $\left(q^l\right)_{l\in\mathbb{Z}}$. Moreover, on every compact of $]0,\infty[$, the mesh of the partition tends to 0 as q goes to 1. Using the dominated convergence theorem, it is sufficient to prove the existence of $(h_l) \in (\mathbb{R}_{>0})^{\mathbb{Z}}$ that satisfies $\sum_{l\in\mathbb{Z}} h_l < \infty$, such that for all q close to 1, $l \in \mathbb{Z}$ and $z \in K$,

$$\left|\frac{(q-1)q^l f\left(q^l,q\right)}{z e_q\left(q^{l+1}/z\right)}\right| < h_l.$$

By definition of the q-Laplace transformation and (4.6.1), we have for all $z \in K$,

$$\left|\mathcal{L}_{q,k}^{[d]}(f)(z,q)\right| \leq (q-1)\sum_{l\in\mathbb{Z}}\left|\frac{q^l J}{z}\frac{e_q\left(Lq^l\right)}{e_q\left(q^{l+1}/z\right)}\right|.$$

For all $l \in \mathbb{Z}$, $z \in K$, $q>1$, we have:

$$\left|\frac{q^{l+1}J}{z}\frac{e_q\left(Lq^{l+1}\right)}{e_q\left(q^{l+2}/z\right)}\right| = \left|q\frac{1+(q-1)Lq^l}{1+(q-1)q^{l+1}/z}\right|\left|\frac{q^l J}{z}\frac{e_q\left(Lq^l\right)}{e_q\left(q^{l+1}/z\right)}\right|. \tag{4.6.2}$$

Let $R \in \mathbb{R}_{>0}$, $M_1<1$, $q_0>1$, such that for all $x \geq R$, for all $z \in K$, and for all $q \in]1,q_0[$,

$$\left|q\frac{1+(q-1)Lx}{1+(q-1)qx/z}\right| < M_1. \tag{4.6.3}$$

Let $q \mapsto l_0(q) \in \mathbb{Z}$ be the smallest integer that satisfies

$$q^{l_0(q)} \geq R.$$

We will break the series into two parts, and start by treating the convergence of $(q-1)\sum_{l=l_0(q)}^{\infty}\frac{q^l}{z}\frac{f\left(q^l,q\right)}{e_q\left(q^{l+1}/z\right)}$ to $\int_R^{\infty} z^{-1}\widetilde{f}(\zeta)e^{-\frac{\zeta}{z}}d\zeta$. Because of (4.6.2) and (4.6.3), for all $q \in]1,q_0[$, $l \geq l_0(q)$ and $z \in K$, we have

$$\left|\frac{q^{l+1}J}{z}\frac{e_q\left(Lq^{l+1}\right)}{e_q\left(q^{l+2}/z\right)}\right| < M_1\left|\frac{q^l J}{z}\frac{e_q\left(Lq^l\right)}{e_q\left(q^{l+1}/z\right)}\right|.$$

By iteration, we find that

$$(q-1)\sum_{l=l_0(q)}^{\infty}\left|\frac{q^l J}{z}\frac{e_q\left(Lq^l\right)}{e_q\left(q^{l+1}/z\right)}\right| \leq (q-1)\sum_{l=0}^{\infty}\left|\frac{q^{l_0(q)}J}{z}\frac{e_q\left(Lq^{l_0(q)}\right)}{e_q\left(q^{l_0(q)+1}/z\right)}\right|\left(M_1\right)^l$$

Using Lemma 4.6.1, we obtain that $|e_q(z)|$ can be bounded for $(z,q) \in K_0 \times]1,q_0[$, where K_0 is an arbitrary compact of $\mathbb{C}$. Moreover, the fact that $e_q(z)$ vanishes only on $\frac{q^{\mathbb{N}^*}}{1-q}$, implies that $\frac{1}{|e_q(z)|}$ can also be bounded for $(z,q)\in K_1\times]1,q_0[$, where K_1 is an arbitrary compact of $\mathbb{C}\setminus\mathbb{R}_{<0}$. In particular, we find that for all $R_0\in\mathbb{R}_{>0}$

$$\sup_{\substack{x\in[1,R_0],\\ q\in]1,q_0[,\\ z\in K}}\left|\frac{e_q(Lx)}{e_q(qx/z)}\right|<\infty. \tag{4.6.4}$$

Then, we obtain that for all $q\in]1,q_0[$ and for all $l\geq l_0(q)$,

$$\begin{aligned}(q-1)\sum_{l=0}^{\infty}\left|\frac{q^{l_0(q)}J}{z}\frac{e_q\left(Lq^{l_0(q)}\right)}{e_q\left(q^{l_0(q)+1}/z\right)}\right|\left(M_1\right)^l &\leq \sup_{\substack{x\in[1,q_0],\\ q\in]1,q_0[,\\ z\in K}}\left|(q-1)\frac{RxJ}{z}\frac{e_q(LRx)}{e_q(qRx/z)}\right|\sum_{l=0}^{\infty}\left(M_1\right)^l\\ &= \frac{M_2}{1-M_1},\end{aligned}$$

where $M_2:=\sup_{\substack{x\in[1,q_0],\\ q\in]1,q_0[,\\ z\in K}}\left|(q-1)\frac{RxJ}{z}\frac{e_q(LRx)}{e_q(qRx/z)}\right|$ is a real positive constant. Hence, we have

$$(q-1)\sum_{l=l_0(q)}^{\infty}\left|\frac{q^lf\left(q^l,q\right)}{ze_q\left(q^{l+1}/z\right)}\right|\leq\frac{M_2}{1-M_1}<\infty,$$

and the dominated convergence theorem gives

$$\lim_{q\to1}(q-1)\sum_{l=l_0(q)}^{\infty}\frac{q^l}{z}\frac{f\left(q^l,q\right)}{e_q\left(q^{l+1}/z\right)}=\int_R^{\infty}z^{-1}\widetilde{f}(\zeta)e^{-\frac{\zeta}{z}}d\zeta, \tag{4.6.5}$$

uniformly on K.

Let us now treat $(q-1)\sum_{l=-\infty}^{l_0(q)-1}\frac{q^l}{z}\frac{f\left(q^l,q\right)}{e_q\left(q^{l+1}/z\right)}$. Because of (4.6.4), we may define

$$M_3:=\sup_{\substack{x\in[0,R],\\ q\in]1,q_0[,\\ z\in K}}\left|\frac{J}{z}\frac{e_q(Lx)}{e_q(qx/z)}\right|<\infty.$$

Therefore, for all q close to 1 and for all $z\in K$, we have

$$\left|(q-1)\sum_{l=-\infty}^{l_0(q)-1}\frac{q^l}{z}\frac{f\left(q^l,q\right)}{e_q\left(q^{l+1}/z\right)}\right|\leq(q-1)\sum_{l=-\infty}^{l_0(q)-1}q^lM_3\leq\frac{(q-1)RM_3}{1-1/q}\leq qRM_3.$$

Consequently, due to the dominated convergence theorem, we have

$$\lim_{q\to1}(q-1)\sum_{l=-\infty}^{l_0(q)-1}\frac{q^l}{z}\frac{f\left(q^l,q\right)}{e_q\left(q^{l+1}/z\right)}=\int_0^{R}z^{-1}\widetilde{f}(\zeta)e^{-\frac{\zeta}{z}}d\zeta,$$

uniformly on K. This limit combined with (4.6.5) yields the result. □

4.6.2 Proof of Theorem 4.4.5 in a particular case.

In this subsection, we are going to prove Theorem 4.4.5 in a particular case. Let us start by recalling some notations. See §4.1 to §4.4 for rest of the notations. We consider (4.4.3), that admits $\widetilde{h} \in \mathbb{C}[[z]]$ as solution and $\widetilde{b}_0, \dots, \widetilde{b}_m$ as coefficients. In other words, we have

$$\widetilde{b}_m(z)\delta^m\left(\widetilde{h}(z)\right) + \dots + \widetilde{b}_0(z)\widetilde{h}(z) = 0.$$

Let $d_0 := \max\left(2, \deg\left(\widetilde{b}_0\right), \dots, \deg\left(\widetilde{b}_m\right)\right)$. Let $k_1 < \dots < k_{r-1}$ be the slopes of (4.4.3) different from 0, let k_r be an integer strictly bigger than k_{r-1} and d_0, and set $k_{r+1} := +\infty$. Let $(\kappa_1, \dots, \kappa_r)$ defined as:

$$\kappa_i^{-1} := k_i^{-1} - k_{i+1}^{-1}.$$

We define the $(\widetilde{\kappa_1}, \dots, \widetilde{\kappa_s})$ as follows: We take $(\kappa_1, \dots, \kappa_r)$ and for $i = 1, ..., r$, replace successively κ_i by α_i terms $\alpha_i\kappa_i$, where α_i is the smallest integer such that $\alpha_i\kappa_i$ is greater or equal than $d_0 \geq 2$. See Example 4.1.4. Let $\beta \in \mathbb{N}^*$ be minimal, such that for all $i \in \{1, \dots, s\}$, $\beta/\widetilde{\kappa_i} \in \mathbb{N}^*$.

In this subsection 4.6.2, we are going to assume that $z \mapsto \hat{h}(z,q), \widetilde{h}(z) \in \mathbb{C}\left[\left[z^\beta\right]\right]$. Ramification in §4.6.3 will allow us to reduce to the case tackled in the present subsection. Note that in this case, we have $\hat{h} = \hat{h}^{(0)}$. For the reader's convenience, we will decompose the proof of Theorem 4.4.5 in four steps.

Step 1: Construction of $\Sigma_{\widetilde{h}}$.

Let us consider a general formal power series $\hat{f} \in \mathbb{C}\left[\left[z^\beta\right]\right]$ (resp. $\widetilde{f} \in \mathbb{C}\left[\left[z^\beta\right]\right]$) that satisfy a linear δ_q-equation (resp. δ-equation) of order m_0 with coefficients in $\mathbb{C}\left[z^\beta\right]$. Then, for all $i \in \{1, \dots, s\}$, $\rho_{1/\widetilde{\kappa_i}}\left(\hat{f}\right)$ (resp. $\rho_{1/\widetilde{\kappa_i}}\left(\widetilde{f}\right)$) satisfies a linear δ_q-equation (resp. δ-equation) with coefficients in $\mathbb{C}[z]$. Therefore, Propositions 4.1.2 and 4.3.4, combined with the definition of the Borel transformations (see Definitions 4.1.1 and 4.3.1) imply that for all $i \in \{1, \dots, s\}$, $\hat{\mathcal{B}}_{q,\widetilde{\kappa_i}}\left(\hat{f}\right)$ (resp. $\hat{\mathcal{B}}_{\widetilde{\kappa_i}}\left(\widetilde{f}\right)$) satisfies a linear δ_q-equation (resp. δ-equation) of order independent of q (resp. of the same order than the δ_q-equation satisfied by $\hat{\mathcal{B}}_{q,\widetilde{\kappa_i}}\left(\hat{f}\right)$) with coefficients in $\mathbb{C}\left[z^\beta\right]$.

In particular, for $j \in \{1, \dots, s\}$, $\hat{\mathcal{B}}_{\widetilde{\kappa_j}} \circ \dots \circ \hat{\mathcal{B}}_{\widetilde{\kappa_s}}\left(\widetilde{h}\right)$ satisfies a linear δ-equation that we will see as a system. We define $\Sigma_{\widetilde{h}}$ as the union of $\widetilde{\Sigma}_{\widetilde{h}}$, the set of its singular direction that has been defined in Proposition 4.1.5, and the argument of the poles of the differential system satisfied by the successive Borel transformations. The set $\Sigma_{\widetilde{h}} \subset \mathbb{R}$ is finite modulo $2\pi\mathbb{Z}$.

Step 2: Local convergence of the q-Borel transformations.

From what is preceding, $\hat{\mathcal{B}}_{q,\widetilde{\kappa_1}} \circ \dots \circ \hat{\mathcal{B}}_{q,\widetilde{\kappa_s}}\left(\hat{h}\right)$ (resp. $\hat{\mathcal{B}}_{\widetilde{\kappa_1}} \circ \dots \circ \hat{\mathcal{B}}_{\widetilde{\kappa_s}}\left(\widetilde{h}\right)$) satisfies a linear δ_q-equation of order $m_1 \in \mathbb{N}$, that we will see as a system $\delta_q Y(\zeta,q) = E(\zeta,q)Y(\zeta,q)$ with $\zeta \mapsto \mathrm{Id} + (q-1)E(\zeta,q) \in \mathrm{GL}_{m_1}\left(\mathbb{C}\left(\zeta^\beta\right)\right)$ (resp. a linear δ-equation of order m_1 we will see as a system $\delta\widetilde{Y}(\zeta) = \widetilde{E}(\zeta)\widetilde{Y}(\zeta)$ with $\widetilde{E}(\zeta) \in \mathrm{M}_{m_1}\left(\mathbb{C}\left(\zeta^\beta\right)\right)$).

Because of Proposition 4.1.5, $\hat{\mathcal{B}}_{\widetilde{\kappa_1}} \circ \dots \circ \hat{\mathcal{B}}_{\widetilde{\kappa_s}}\left(\widetilde{h}\right)$ is convergent. Let us prove that $\zeta \mapsto \hat{\mathcal{B}}_{q,\widetilde{\kappa_1}} \circ \dots \circ \hat{\mathcal{B}}_{q,\widetilde{\kappa_s}}\left(\hat{h}\right) \in \mathbb{C}\left\{\zeta^\beta\right\}$. Due to **(A2)**, the slopes of the σ_q-equation satisfied by $\hat{h}$ are independent of q, and the smallest positive slope is k_1. As we can see in [Ram92], Theorem 4.8, (see also [Béz92]), there exist $C_1(q), C_2(q) > 0$, such that for all $l \in \mathbb{N}$, for all $q > 1$

$$\left|\hat{h}_l(q)\right| < C_1(q)C_2(q)^l\left([l]_q^!\right)^{1/k_1},$$

where $\hat{h}(z,q) = \sum \hat{h}_l(q)z^l$. By construction of the $\widetilde{\kappa}_i$, we have $\sum_{i=1}^{s} \widetilde{\kappa}_i^{-1} = \sum_{i=1}^{r} \kappa_i^{-1} = k_1^{-1}$. Since for all $l, k \in \mathbb{N}^*$, and for all $q > 1$, $\left([kl]_q^!\right)^{1/k} \leq [l]_q^!$, we find that for all $l \in \mathbb{N}$, for all $q > 1$,

$$\left|\hat{h}_l(q)\right| < C_1(q)C_2(q)^l \prod_{i=1}^{s} [l/\widetilde{\kappa}_i]_q^!.$$

Hence, we obtain that $\zeta \mapsto \hat{\mathcal{B}}_{q,\widetilde{\kappa}_1} \circ \cdots \circ \hat{\mathcal{B}}_{q,\widetilde{\kappa}_s}\left(\hat{h}\right) \in \mathbb{C}\left\{\zeta^{\beta}\right\}$. Applying Lemma 4.4.2, we find

$$\lim_{q\to 1} \hat{\mathcal{B}}_{q,\widetilde{\kappa}_1} \circ \cdots \circ \hat{\mathcal{B}}_{q,\widetilde{\kappa}_s}\left(\hat{h}\right) = \hat{\mathcal{B}}_{\widetilde{\kappa}_1} \circ \cdots \circ \hat{\mathcal{B}}_{\widetilde{\kappa}_s}\left(\widetilde{h}\right), \tag{4.6.6}$$

uniformly on a closed disk centered at 0.

Step 3: Local convergence of the q-Borel-Laplace summation.

Let $d \in \mathbb{R} \setminus \Sigma_{\widetilde{h}}$. The variable change $\zeta \mapsto \zeta e^{-id}$ allows us to reduce to the case $d = 0$. By construction of $\Sigma_{\widetilde{h}}$, $\widetilde{E}(\zeta)$ has no poles for $\zeta \in \overline{S}(-\varepsilon, +\varepsilon)$. Because of the assumption **(A3)**, Propositions 4.3.4 and 4.1.2, we deduce that $E(\zeta, q)$ has no poles for $\zeta \in \overline{S}(-\varepsilon, \varepsilon)$ and for q close to 1. Because of Lemma 4.5.1, the series $z \mapsto \hat{\mathcal{B}}_{q,\widetilde{\kappa}_1} \circ \cdots \circ \hat{\mathcal{B}}_{q,\widetilde{\kappa}_s}\left(\hat{h}\right)(z,q)$ admits, for q close to 1, an analytic continuation $f_1(\zeta, q)$ defined on $\overline{S}(-\varepsilon, \varepsilon)$. We want now to prove that $f_1(\zeta, q)$ converges to $\widetilde{f}_1(\zeta)$ on a convenient domain.

Due to Proposition 4.1.5, there exists $B_1 > 0$ such that the functions $\dfrac{\widetilde{f}_1(\zeta)}{\exp\left(B_1\zeta^{\widetilde{\kappa}_1}\right)}, \dots, \dfrac{\delta^{m_1-1}\widetilde{f}_1(\zeta)}{\exp\left(B_1\zeta^{\widetilde{\kappa}_1}\right)}$ tend to 0 as $\zeta \in \overline{S}(-\varepsilon, \varepsilon)$ tends to infinity. Using

$$\delta_q\left(e_{q^{\widetilde{\kappa}_1}}\left(B_1\zeta^{\widetilde{\kappa}_1}\right)^{-1}\right) = \frac{-\left[\widetilde{\kappa}_1\right]_q B_1\zeta^{\widetilde{\kappa}_1}}{1+(q-1)\left[\widetilde{\kappa}_1\right]_q B_1\zeta^{\widetilde{\kappa}_1}} e_q\left(B_1\zeta^{\widetilde{\kappa}_1}\right)^{-1},$$

we obtain that $\dfrac{f_1(\zeta,q)}{e_{q^{\widetilde{\kappa}_1}}\left(B_1\zeta^{\widetilde{\kappa}_1}\right)}$ (resp. $\widetilde{f}_1(\zeta)\exp\left(-B_1\zeta^{\widetilde{\kappa}_1}\right)$) satisfies a linear δ_q-equation of order m_1+1 (resp. a linear δ-equation of order m_1+1) with coefficients in $\mathbb{C}(z)$. Because of (4.6.6), there exists $\zeta_0 > 0$, such that $f_1(\zeta_0, q)$ converges to $\widetilde{f}_1(\zeta_0)$ as q goes to 1. Let

$$Y(\zeta,q) := \begin{pmatrix} \dfrac{f_1(\zeta,q)}{e_{q^{\widetilde{\kappa}_1}}\left(B_1\zeta^{\widetilde{\kappa}_1}\right)}F_0(q) \\ \vdots \\ \dfrac{\delta_q^{m_1}f_1(\zeta,q)}{e_{q^{\widetilde{\kappa}_1}}\left(B_1\zeta^{\widetilde{\kappa}_1}\right)}F_{m_1}(q) \end{pmatrix}, \quad \widetilde{Y}_{B_1}(\zeta) := \begin{pmatrix} \dfrac{\widetilde{f}_1(\zeta)}{\exp\left(B_1\zeta^{\widetilde{\kappa}_1}\right)} \\ \vdots \\ \dfrac{\delta^{m_1}\widetilde{f}_1(\zeta)}{\exp\left(B_1\zeta^{\widetilde{\kappa}_1}\right)} \end{pmatrix},$$

where the $F_i(q) \in \mathbb{C}$ are defined by:

$$\left.\frac{\delta_q^{i}f_1(\zeta,q)}{e_{q^{\widetilde{\kappa}_1}}\left(B_1\zeta^{\widetilde{\kappa}_1}\right)}F_i(q)\right|_{\zeta=\zeta_0} = \left.\frac{\delta^{i}\widetilde{f}_1(\zeta)}{\exp\left(B_1\zeta^{\widetilde{\kappa}_1}\right)}\right|_{\zeta=\zeta_0}. \tag{4.6.7}$$

From what is preceding, there exist $\zeta \mapsto \mathrm{Id} + (q-1)D(\zeta,q) \in \mathrm{GL}_{m_1+1}\Big(\mathbb{C}(\zeta)\Big)$ and $\widetilde{D}(\zeta) \in \mathrm{M}_{m_1+1}\Big(\mathbb{C}(\zeta)\Big)$, such that

$$\begin{cases} \delta_q Y(\zeta,q) &= D(\zeta,q)Y(\zeta,q) \\ \delta \widetilde{Y}(\zeta) &= \widetilde{D}(\zeta)\widetilde{Y}(\zeta). \end{cases}$$

Lemma 4.6.3. *Let us consider C, a convex subset of $\overline{S}(-\varepsilon,\varepsilon)$, that contains $\Big\{\zeta \in \overline{S}(-\varepsilon_1,\varepsilon_1)\Big||\zeta| > \zeta_0/2\Big\}$, for some $\varepsilon_1 \in]0,\varepsilon[$, such that 0 does not belong to its closure. Then, the systems*

$$\begin{cases} \delta_q Y(\zeta,q) &= D(\zeta,q)Y(\zeta,q) \\ \delta \widetilde{Y}(\zeta) &= \widetilde{D}(\zeta)\widetilde{Y}(\zeta), \end{cases}$$

satisfy the assumptions of Proposition 4.5.2, with $K := \Big\{\zeta \in \widetilde{\mathbb{C}}\Big||\zeta - \zeta_0| \leq \varepsilon_0\Big\}$ and $\varepsilon_0 > 0$ is a real positive constant sufficiently small.

Proof of the lemma. We are going to check separately the three assumptions of Proposition 4.5.2.

(i) Because of the assumption **(A3)**, Propositions 4.3.4 and 4.1.2, we obtain the existence of $c_1 > 0$, such that for all $\zeta \in C$,

$$\Big|E(\zeta,q) - \widetilde{E}(\zeta)\Big| < (q-1)c_1\left(\Big|\widetilde{E}(\zeta)\Big| + \mathbf{1}_{m_1}\right).$$

With the q-difference equation satisfied by $e_{\widetilde{q^{\kappa_1}}}\left(B_1\zeta^{\widetilde{\kappa_1}}\right)$, this implies that we have the existence of $c_2 > 0$, such that for q close to 1, for $\zeta \in C$,

$$\Big|D(\zeta,q) - \widetilde{D}(\zeta)\Big| < (q-1)c_2\left(\Big|\widetilde{D}(\zeta)\Big| + \mathbf{1}_{m_1+1}\right).$$

(ii) Let $i \in \{0,\dots,m_1\}$. Due to (4.6.6) and Lemma 4.6.1, $F_i(q)$ converges to 1 as q goes to 1. Then, we have for all $i \in \{0,\dots,m_1\}$

$$\lim_{q\to 1} \frac{\delta_q^i f_1(\zeta,q)}{e_{\widetilde{q^{\kappa_1}}}\left(B_1\zeta^{\widetilde{\kappa_1}}\right)} F_i(q) = \frac{\delta^i \widetilde{f}_1(\zeta)}{\exp\left(B_1\zeta^{\widetilde{\kappa_1}}\right)},$$

uniformly on a compact set with non empty interior containing ζ_0. Let us choose $\varepsilon_0 > 0$ small enough, such that we have the uniform convergence on $K := \Big\{\zeta \in \widetilde{\mathbb{C}}\Big||\zeta - \zeta_0| \leq \varepsilon_0\Big\}$ and such that K is included in C. Because of (4.6.7), $\dfrac{\delta_q^i f_1(\zeta,q)}{e_{\widetilde{q^{\kappa_1}}}\left(B_1\zeta^{\widetilde{\kappa_1}}\right)} F_i(q)$ and $\dfrac{\delta^i \widetilde{f}_1(\zeta)}{\exp\left(B_1\zeta^{\widetilde{\kappa_1}}\right)}$ are equal at ζ_0.

(iii) From the choice of B_1, we have the existence of $R \in \mathbb{C}[\zeta]$, such that for $\zeta \in C$, for all $i \in \{0,\dots,m_1-1\}$:

$$\left|\delta^i\left(\widetilde{f}_1(\zeta)\right)\exp\left(-B_1\zeta^{\widetilde{\kappa_1}}\right)\right| < |R(\zeta)|.$$

□

We need now the following elementary lemma.

Lemma 4.6.4. *For all $z\in\mathbb{C}$, for all $q>1$, we have $e_{q^2}(|z|)^2\leq e_q(|(1+q)z|)$.*

Proof of the lemma. Let us remark that the two functions are equal at $z=0$. The lemma is now a direct consequence of the q-difference equation

$$\sigma_q^2\left(\frac{e_{q^2}(|z|)^2}{e_q(|(1+q)z|)}\right)=\frac{1+2(q^2-1)|z|+(q^2-1)^2|z|^2}{1+(1+q)(q^2-1)|z|+(q^2-1)^2q|z|^2}\frac{e_{q^2}(|z|)^2}{e_q(|(1+q)z|)},$$

since $\frac{1+2(q^2-1)|z|+(q^2-1)^2|z|^2}{1+(1+q)(q^2-1)|z|+(q^2-1)^2q|z|^2}\leq 1$. □

We finish now the proof of Theorem 4.4.5, in the particular case $z\mapsto \hat{h}(z,q),\widetilde{h}(z)\in\mathbb{C}\left[\left[z^{\beta}\right]\right]$. Let us define $\widetilde{d}$ (resp. $\widetilde{e}$) as the maximum of the degrees of the numerators and the denominators of the entries of $\widetilde{D}(\zeta)$ (resp. $\widetilde{E}(\zeta)$), written as the quotient of two coprime polynomials. Using the differential equation satisfied by $\exp\left(-B_1\zeta^{\widetilde{\kappa_1}}\right)$, we find that $\widetilde{d}\leq\max(\widetilde{e},\widetilde{\kappa_1})$. Because of Remark 4.1.3, and the definition of d_0 and $\widetilde{\kappa_1}$ (see the beginning of the subsection), $\widetilde{e}\leq d_0\leq\widetilde{\kappa_1}$. Hence $\widetilde{d}\leq\widetilde{\kappa_1}$. Proposition 4.5.2 applied to the systems

$$\begin{cases}\delta_q Y(\zeta,q) &= D(\zeta,q)Y(\zeta,q)\\ \delta\widetilde{Y}(\zeta) &= \widetilde{D}(\zeta)\widetilde{Y}(\zeta),\end{cases}$$

implies that there exist $R,S_0\in\mathbb{C}[z]$, $\delta(q),\varepsilon(q)$ that converge respectively to 1 and 0 as $q\to 1$, such that

$$\left|\frac{f_1(\zeta,q)F_0(q)}{e_{q^{\widetilde{\kappa_1}}}\left(B_1\zeta^{\widetilde{\kappa_1}}\right)}-\frac{\widetilde{f}_1(\zeta)}{\exp\left(B_1\zeta^{\widetilde{\kappa_1}}\right)}\right|<(q-1)\delta(q)e_{q^{\widetilde{d}}}\Big(S_0(|\zeta|)\Big)+\varepsilon(q)\left|R(\zeta)\right|.$$

There exists a polynomial S_1 with degree $\widetilde{\kappa_1}$, such that for $|\zeta|$ sufficiently big and for all q close to 1,

$$\left|e_{q^{\widetilde{\kappa_1}}}\left(B_1\zeta^{\widetilde{\kappa_1}}\right)e_{q^{\widetilde{d}}}\Big(S_0(|\zeta|)\Big)\right|\leq e_{q^{\widetilde{\kappa_1}}}\Big(|S_1(\zeta)|\Big)^2.$$

By construction, $\widetilde{\kappa_1}\geq 2$, (see the beginning of the subsection). Using Lemma 4.6.4, we obtain that for $|\zeta|$ sufficiently big,

$$e_{q^{\widetilde{\kappa_1}}}\Big(|S_1(\zeta)|\Big)^2\leq e_{q^2}\Big(|S_1(\zeta)|\Big)^2\leq e_q\Big((1+q)\left|S_1(\zeta)\right|\Big).$$

Since $F_0(q)$ converges to 1 and the fact that $\widetilde{f}_1(\zeta)\exp\left(-B_1\zeta^{\widetilde{\kappa_1}}\right)$ is bounded by a polynomial, the triangular inequality yields

$$f_1\in\overline{\mathbb{H}}^0_{\widetilde{\kappa_1}}.$$

Moreover, due to Proposition 4.5.2, we have $\lim\limits_{q\to 1}f_1=\widetilde{f}_1$, uniformly on the compacts of $C\cap\mathbb{R}_{\geq 1}K:=\left\{xw\in C\middle|x\in[1,\infty[,w\in K\right\}$. Hence, we find that there exists $\varepsilon_2\in]0,\varepsilon_1[$, such that $\lim\limits_{q\to 1}f_1=\widetilde{f}_1$, uniformly on the compacts of $\overline{S}(-\varepsilon_2,\varepsilon_2)$. We may now apply Lemma 4.6.2 to obtain the existence of $L_0>0$, such that we have

$$\lim_{q\to 1}\mathcal{L}^{[0]}_{q,\widetilde{\kappa_1}}\Big(f_1\Big)(\zeta,q)=\mathcal{L}^0_{\widetilde{\kappa_1}}\left(\widetilde{f}_1\right)(\zeta),$$

uniformly on the compacts of $\left\{\zeta \in \overline{S}\left(-\frac{\pi}{2\widetilde{\kappa_1}},+\frac{\pi}{2\widetilde{\kappa_1}}\right)\Big||\zeta|<L_0\right\}$.

If $s>1$, we apply for $j=2$ (resp. $j=3,\ldots$, resp. $j=s$) the same reasoning with the analytic continuation of

$$f_j(\zeta,q)e_{q^{\widetilde{\kappa_j}}}\left(B_j\zeta^{\widetilde{\kappa_j}}\right)^{-1} := \mathcal{L}^{[0]}_{q,\widetilde{\kappa_{j-1}}}\left(f_{j-1}\right)e_{q^{\widetilde{\kappa_j}}}\left(B_j\zeta^{\widetilde{\kappa_j}}\right)^{-1}$$

and

$$\widetilde{f}_j(\zeta)\exp\left(-B_j\zeta^{\widetilde{\kappa_j}}\right) := \mathcal{L}^{0}_{\widetilde{\kappa_{j-1}}}\left(\widetilde{f}_{j-1}\right)\exp\left(-B_j\zeta^{\widetilde{\kappa_j}}\right),$$

where $B_j>0$ are chosen sufficiently large. We again use Propositions 4.1.2 and 4.3.4 to prove that they satisfy linear δ_q and δ-equations with coefficients in $\mathbb{C}(\zeta)$, which are the same as the linear δ_q and δ-equations satisfied by $\hat{\mathcal{B}}_{q,\widetilde{\kappa_j}}\circ\cdots\circ\hat{\mathcal{B}}_{q,\widetilde{\kappa_s}}\left(\hat{h}\right)e_{q^{\widetilde{\kappa_j}}}\left(B_j\zeta^{\widetilde{\kappa_j}}\right)^{-1}$ and $\hat{\mathcal{B}}_{\widetilde{\kappa_j}}\circ\cdots\circ\hat{\mathcal{B}}_{\widetilde{\kappa_s}}\left(\widetilde{h}\right)\exp\left(-B_j\zeta^{\widetilde{\kappa_j}}\right)$.

We have proved the existence of $L_1>0$, such that we have

$$\lim_{q\to1} S_q^{[0]}\left(\hat{h}\right)=\widetilde{S}^0\left(\widetilde{h}\right),$$

uniformly on the compacts of $\left\{z\in\overline{S}\left(-\frac{\pi}{2\kappa_r},+\frac{\pi}{2\kappa_r}\right)\Big||z|<L_1\right\}$.

Step 4: Global convergence of the q-Borel-Laplace summation.

To finish the proof in the particular case $z\mapsto\hat{h}(z,q),\widetilde{h}(z)\in\mathbb{C}\left[\left[z^{\beta}\right]\right]$, we have to prove that

$$\lim_{q\to1} S_q^{[0]}\left(\hat{h}\right)=\widetilde{S}^0\left(\widetilde{h}\right),$$

uniformly on the compacts of $\overline{S}\left(-\frac{\pi}{2k_r},+\frac{\pi}{2k_r}\right)\setminus\bigcup\mathbb{R}_{\geq1}\alpha_i$, where α_i are the roots of $\widetilde{b}_m\in\mathbb{C}[z]$. Let K_0 be an arbitrary compact of $\overline{S}\left(-\frac{\pi}{2k_r},+\frac{\pi}{2k_r}\right)\setminus\bigcup\mathbb{R}_{\geq1}\alpha_i$, and let us prove the uniform convergence on K_0. Without loss of generality, we may assume that K_0 is convex and has non empty intersection with the open disk of radius L_1 (we recall that L_1 was defined in the end of Step 3) centered at 0.

From Remark 4.3.5 (resp. Proposition 4.1.5), we deduce that $S_q^{[0]}\left(\hat{h}\right)$ (resp. $\widetilde{S}^0\left(\widetilde{h}\right)$) is solution of the same linear δ_q-equation than $\hat{h}$ (resp. the same linear δ-equation than $\widetilde{h}$).

Let $|z_0|<L_1$ with $z_0\in K_0$. We are going to use Proposition 4.5.2 with $C=K_0$ and with the systems

$$\begin{cases}\delta_q Y(z,q) &= F(z,q)Y(z,q)\\[2ex] \delta\widetilde{Y}(z) &= \widetilde{F}(z)\widetilde{Y}(z),\end{cases}$$

where

$$Y(\zeta,q):=\left(\delta_q^i S_q^{[0]}\left(\hat{h}\right)G_i(q)\right)_{i\in\{0,\ldots,m-1\}},\quad \widetilde{Y}(\zeta):=\left(\delta^i\widetilde{S}^0\left(\widetilde{h}\right)\right)_{i\in\{0,\ldots,m-1\}},$$

$$z\mapsto \mathrm{Id}+(q-1)F(z,q)\in\mathrm{GL}_m\Big(\mathbb{C}(z)\Big),\widetilde{F}(z)\in\mathrm{M}_m\Big(\mathbb{C}(z)\Big),$$

and $G_i(q)\in\mathbb{C}$ are defined such that:

$$\delta_q^i S_q^{[0]}\left(\hat{h}\right)G_i(q)\Big|_{z=z_0}=\delta^i\widetilde{S}^0\left(\widetilde{h}\right)\Big|_{z=z_0}.$$

The assumption (i) of Proposition 4.5.2 is satisfied because of the assumption **(A3)**, and the two others are trivially satisfied, since K_0 is bounded.

This yields $\lim\limits_{q\to 1} S_q^{[0]}\left(\hat{h}\right)=\widetilde{S}^0\left(\widetilde{h}\right)$ uniformly on K_0, and completes the proof in the particular case $z\mapsto \hat{h}(z,q),\widetilde{h}(z)\in\mathbb{C}\left[\left[z^{\beta}\right]\right]$.

4.6.3 Proof of Theorem 4.4.5 in the general case.

In this subsection, we are going to prove Theorem 4.4.5 in the general case. See §4.1 to §4.4 for the notations. We recall that for all $l\in\{0,\dots,\beta-1\}$, we define $\hat{h}^{(l)}\in\mathbb{C}[[z^{\beta}]]$, so that $\hat{h}=\sum\limits_{l=0}^{\beta-1} z^l\hat{h}^{(l)}$. Let us set $\Sigma_{\widetilde{h}}:=\bigcup\limits_{l=0}^{\beta-1}\Sigma_{\widetilde{h}^{(l)}}$ (see Step 1 in §4.6.2). Let $d\in\mathbb{R}\setminus\Sigma_{\widetilde{h}}$. After considering $z\mapsto ze^{-id}$, we may assume that $d=0$.

Looking at the term with z-degree congruent to j modulo β , for $j=0,\dots,j=\beta-1$, we find that the equation satisfied by $\hat{h}$ is equivalent to the following family of δ_q-linear equations:

$$\left\{\begin{array}{ccl} 0 & = & \sum\limits_{k,l} d_{0,k,l}(z,q)\delta_q^k\hat{h}^{(l)}(z,q)\\ \vdots & & \\ 0 & = & \sum\limits_{k,l} d_{\beta-1,k,l}(z,q)\delta_q^k\hat{h}^{(l)}(z,q),\end{array}\right.$$

where $z\mapsto d_{j,k,l}\in\mathbb{C}\left[z^{\beta}\right]$. Let $l\in\{0,\dots,\beta-1\}$. Following the equalities

$$z^l\hat{h}^{(l)}(z,q)=\sum_{j=0}^{\beta-1}\frac{\hat{h}\left(e^{2i\pi lj/\beta}z,q\right)}{e^{2i\pi lj/\beta}\beta},\quad z^l\widetilde{h}^{(l)}(z)=\sum_{j=0}^{\beta-1}\frac{\widetilde{h}\left(e^{2i\pi lj/\beta}z\right)}{e^{2i\pi lj/\beta}\beta},$$

we obtain that for all $l\in\{0,\dots,\beta-1\}$, $\hat{h}^{(l)}(z,q)$ (resp. $\widetilde{h}^{(l)}$) satisfies a linear q-difference (resp. differential) equation with coefficients in $\mathbb{C}[z^{\beta}]$. Moreover, for all $l\in\{0,\dots,\beta-1\}$, $\hat{h}^{(l)}$, converges coefficientwise to $\widetilde{h}^{(l)}$ and the equations they satisfy have coefficients that check the assumptions **(A2)** and **(A3)**.

Because of the fact that $0\in\mathbb{R}\setminus\Sigma_{\widetilde{h}}$, Proposition 4.1.5 implies that for all $l\in\{0,\dots,\beta-1\}$, there exists $\widetilde{S}^0\left(\widetilde{h}^{(l)}\right)$, asymptotic solution of the same linear δ-equation than $\widetilde{h}^{(l)}$. These latter can be computed with Laplace and Borel transformations.

Using the case $z\mapsto\hat{h}(z,q),\widetilde{h}(z)\in\mathbb{C}\left[\left[z^{\beta}\right]\right]$ (see §4.6.2), we can compute for q close to 1, and $l\in\{0,\dots,\beta-1\}$, $z\mapsto S_q^{[0]}\left(\hat{h}^{(l)}\right)\in\mathcal{M}(\mathbb{C}^*,0)$, solution of the same family of linear δ_q-equations than $\hat{h}^{(l)}$. Because of Remark 4.3.5, we find:

$$\left\{\begin{array}{ccl} 0 & = & \sum\limits_{k,l} d_{0,k,l}(z,q)\delta_q^k S_q^{[0]}\left(\hat{h}^{(l)}\right)\\ \vdots & & \\ 0 & = & \sum\limits_{k,l} d_{\beta-1,k,l}(z,q)\delta_q^k S_q^{[0]}\left(\hat{h}^{(l)}\right).\end{array}\right.$$

Hence, we obtain that for q close to 1, $S_q^{[0]}\left(\hat{h}\right):=\sum\limits_{l=0}^{\beta-1} z^l S_q^{[0]}\left(\hat{h}^{(l)}\right)$ satisfies the same linear δ_q-equation than $\hat{h}$. We apply now the theorem in the case $z\mapsto\hat{h}(z,q),\widetilde{h}(z)\in\mathbb{C}\left[\left[z^{\beta}\right]\right]$

previously treated, to prove the existence of $L_2 > 0$, such that we have

$$\lim_{q\to 1} S_q^{[0]}\left(\hat{h}\right) = \widetilde{S}^0\left(\widetilde{h}\right) := \sum_{l=0}^{\beta-1} z^l \widetilde{S}^0\left(\widetilde{h}^{(l)}\right),$$

uniformly on the compacts of $\left\{z \in \overline{S}\left(-\frac{\pi}{2\kappa_r}, +\frac{\pi}{2\kappa_r}\right) \middle| |z| < L_2\right\}$. To conclude, we have to prove

$$\lim_{q\to 1} S_q^{[0]}\left(\hat{h}\right) = \widetilde{S}^0\left(\widetilde{h}\right),$$

uniformly on the compacts of $\overline{S}\left(-\frac{\pi}{2k_r}, +\frac{\pi}{2k_r}\right) \setminus \bigcup \mathbb{R}_{\geq 1}\alpha_i$, where α_i are the roots of $\widetilde{b}_m \in \mathbb{C}[z]$. This is the same reasoning than for the particular case $z \mapsto \hat{h}(z,q), \widetilde{h}(z) \in \mathbb{C}\left[\left[z^{\beta}\right]\right]$ (see Step 4 in §4.6.2). This completes the proof of our main result, Theorem 4.4.5.

4.7 Basic hypergeometric series.

We refer the reader to [GR04] for more details about basic hypergeometric series. We recall that $p = 1/q$. In this section, we will say that two functions are equal if their analytic continuations coincide. Let $r, s \in \mathbb{N}$, let $a_1, \dots, a_r, b_1, \dots, b_s \in \mathbb{C}\backslash q^{\mathbb{N}}$, with different images in $\mathbb{C}^*/q^{\mathbb{Z}}$, and let us consider the formal power series

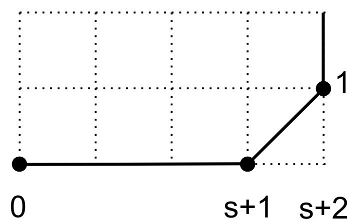
$$ {}_r\varphi_s\left(\begin{matrix} a_1,\dots,a_r \\ \\ b_1,\dots,b_s\end{matrix} ;p,z\right) := \sum_{n=0}^{\infty} \frac{(a_1;p)_n \dots (a_r;p)_n}{(p;p)_n(b_1;p)_n \dots (b_s;p)_n}\left((-1)^n p^{n(n-1)/2}\right)^{1+s-r} z^n, $$

where $(a;p)_{n+1} := (1-ap^n)(a;p)_n$ and $(a;p)_0 := 1$, for $a \in \mathbb{C}$. Assume now that $r > s+1$ and $\prod_{i=1}^{r} a_i \neq 0$. In this case, the formal power series is divergent. Let us put $\underline{p} := q^{-1/(r-s-1)}$ and $\underline{q} := q^{1/(r-s-1)}$.

Lemma 4.7.1. *The series* ${}_r\varphi_s\left(\begin{matrix} a_1,\dots,a_r \\ \\ b_1,\dots,b_s\end{matrix} ;\underline{p},z\right)$ *satisfies the linear* $\sigma_{\underline{q}}$*-equation*

$$ \left((\sigma_{\underline{q}}-1)\prod_{i=1}^{s}(\sigma_{\underline{q}}-b_i\underline{q}) + z(-1)^{s-r}\underline{q}^{1+s}\sigma_{\underline{q}}^{2+s-r}\prod_{i=1}^{r}(\sigma_{\underline{q}}-a_i)\right)\left({}_r\varphi_s\left(\begin{matrix} a_1,\dots,a_r \\ \\ b_1,\dots,b_s\end{matrix} ;\underline{p},z\right)\right) = 0, $$

which admits 0 *and* 1 *as non negative slopes.*



Proof. Let us define $(u_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that ${}_r\varphi_s\left(\begin{matrix} a_1,\dots,a_r \\ \\ b_1,\dots,b_s\end{matrix} ;\underline{p},z\right) = \sum_{n\in\mathbb{N}} u_n z^n$. Let us fix $n \in \mathbb{N}$. We find that

$$ u_{n+1}(1-\underline{p}^{n+1})\prod_{i=1}^{s}(1-b_i\underline{p}^n) = u_n(-1)^{1+s-r}\underline{q}^n\prod_{i=1}^{r}(1-a_i\underline{p}^n) $$

and then

$$ u_{n+1}(\underline{q}^{n+1}-1)\prod_{i=1}^{s}(\underline{q}^{n+1}-b_i\underline{q}) = u_n(-1)^{1+s-r}\underline{q}^{1+s}\underline{q}^{n(2+s-r)}\prod_{i=1}^{r}(\underline{q}^n-a_i). $$

Multiplying the two sides of the equality by z^{n+1}, we obtain the result. □

Let $\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_s \in \mathbb{C}\setminus(-\mathbb{N})$ with different images in $\mathbb{C}/\mathbb{Z}$. If we put $x := z\left(1-\underline{p}\right)^{1+s-r}$, we have the following convergence coefficientwise when $\underline{p}$ goes to 1:

$$ {}_r\varphi_s\left(\begin{matrix}\underline{p}^{\alpha_1},\dots,\underline{p}^{\alpha_r}\\ \\ \underline{p}^{\beta_1},\dots,\underline{p}^{\beta_s}\end{matrix};\underline{p},x\right)\to {}_rF_s\left(\begin{matrix}\alpha_1,\dots,\alpha_r\\ \\ \beta_1,\dots,\beta_s\end{matrix};(-1)^{1+s-r}z\right), $$

where

$$ {}_rF_s\left(\begin{matrix}\alpha_1,\dots,\alpha_r\\ \\ \beta_1,\dots,\beta_s\end{matrix};z\right):=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\dots(\alpha_r)_n}{n!(\beta_1)_n\dots(\beta_s)_n}z^n, $$

and $(\alpha)_0:=1;(\alpha)_{n+1}:=(\alpha+n)(\alpha)_n$ for $\alpha\in\mathbb{C}$. Applying Lemma 4.7.1, we obtain that the first series satisfies $\Delta_{\underline{q}}\left({}_r\varphi_s\left(\begin{matrix}\underline{p}^{\alpha_1},\dots,\underline{p}^{\alpha_r}\\ \\ \underline{p}^{\beta_1},\dots,\underline{p}^{\beta_s}\end{matrix};\underline{p},x\right)\right)=0$ where

$$ \Delta_{\underline{q}}:=\delta_{\underline{q}}\prod_{i=1}^{s}\left(\delta_{\underline{q}}+\frac{1-\underline{p}^{\beta_i-1}}{1-\underline{q}}\right)+z(-1)^{s-r}\underline{q}^{2+2s-r}\sigma_{\underline{q}}^{2+s-r}\prod_{i=1}^{r}\left(\delta_{\underline{q}}+\frac{1-\underline{p}^{\alpha_i}}{1-\underline{q}}\right). $$

Using the same reasoning, one can prove that the second series satisfies $\widetilde{\Delta}\left({}_rF_s\left(\begin{matrix}\alpha_1,\dots,\alpha_r\\ \\ \beta_1,\dots,\beta_s\end{matrix};(-1)^{1+s-r}z\right)\right)=0$, where

$$ \widetilde{\Delta}:=\delta\prod_{i=1}^{s}(\delta+\beta_i-1)+z(-1)^{s-r}\prod_{i=1}^{r}(\delta+\alpha_i). $$

The above series do not satisfy the assumptions of Theorem 4.4.5, since the slopes of $\widetilde{\Delta}$, do not correspond to the slopes of $\Delta_{\underline{q}}$ that are positive.

The goal of this section is to show that if $d\not\equiv(r-s-1)\pi[2\pi]$, we may apply successively $\hat{B}_{\underline{q}}$ and $L_{\underline{q}}^{[d]}$, see Definition 4.3.6, to ${}_r\varphi_s\left(\begin{matrix}\underline{p}^{\alpha_1},\dots,\underline{p}^{\alpha_r}\\ \\ \underline{p}^{\beta_1},\dots,\underline{p}^{\beta_s}\end{matrix};\underline{p},x\right)$, and prove, by making explicitly the computations, that we obtain a function that converges, as $\underline{q}$ goes to 1, to $\widetilde{S}^d\left({}_rF_s\left(\begin{matrix}\alpha_1,\dots,\alpha_r\\ \\ \beta_1,\dots,\beta_s\end{matrix};(-1)^{1+s-r}z\right)\right)$. The case $r=2$ and $s=0$ has been treated in [Zha02], §2.

First, we are going to consider ${}_r\varphi_s\left(\begin{matrix}a_1,\dots,a_r\\ \\ b_1,\dots,b_s\end{matrix};\underline{p},z\right)$, which satisfies, see Lemma 4.7.1, a linear $\sigma_{\underline{q}}$-equation with non negative slopes 0 and 1. As we can see in [Zha02], §1, if $d\not\equiv(r-s-1)\pi[2\pi]$, we can compute a solution of the same linear $\sigma_{\underline{q}}$-equation than ${}_r\varphi_s\left(\begin{matrix}a_1,\dots,a_r\\ \\ b_1,\dots,b_s\end{matrix};\underline{p},z\right)$ applying successively to it $\hat{B}_{\underline{q}}$ and $L_{\underline{q}}^{[d]}$. Applying

$\hat{B}_{\underline{q}}$ to ${}_r\varphi_s\left(\begin{matrix} a_1,\dots,a_r \\ \\ b_1,\dots,b_s\end{matrix};\underline{p},z\right)$, we obtain for all $d\not\equiv (r-s-1)\pi[2\pi]$

$$h(\zeta):={}_r\varphi_{r-1}\left(\begin{matrix} a_1,\dots,a_r \\ \\ b_1,\dots,b_s,0,\dots,0\end{matrix};\underline{p},(-1)^{1+s-r}\zeta\right)\in\mathbb{H}^d_{\underline{q},1}.$$

For $a,a_1,\dots,a_k\in\mathbb{C}$, let $(a;\underline{p})_\infty:=\prod_{n=0}^{\infty}(1-a\underline{p}^n)$, and $(a_1,\dots,a_k;\underline{p})_\infty:=\prod_{i=1}^{k}(a_i;\underline{p})_\infty$. For all $j\in\{1,\dots,k\}$ and $a_1,\dots,a_k\in\mathbb{C}$, let $(a_1,\dots,\widehat{a_j},\dots,a_k)$ be equals to the finite sequence $(a_1,\dots,a_k)$ after the withdrawn of the element a_j. As we can see in Page 121 of [GR04], the convergent series ${}_r\varphi_{r-1}$ may be expressed with connection formula at infinity:

$$\begin{aligned}{}_r\varphi_{r-1}\left(\begin{matrix} a_1,\dots,a_r \\ \\ b_1,\dots,b_{r-1}\end{matrix};\underline{p},z\right)&=\sum_{j=1}^{r}\frac{(a_1,\dots,\widehat{a_j},\dots,a_r,b_1/a_j,\dots,b_{r-1}/a_j,a_jz,\underline{p}/a_jz;\underline{p})_\infty}{(b_1,\dots,b_{r-1},a_1/a_j,\dots,\widehat{a_j/a_j},\dots,a_r/a_j,z,\underline{p}/z;\underline{p})_\infty}\\ &\times{}_r\varphi_{r-1}\left(\begin{matrix} a_j,a_j\underline{p}/b_1,\dots,a_j\underline{p}/b_{r-1} \\ \\ a_j\underline{p}/a_1,\dots,\widehat{a_j\underline{p}/a_j},\dots,a_j\underline{p}/a_r\end{matrix};\underline{p},\frac{\underline{p}\prod_{i=1}^{r-1}b_i}{z\prod_{i=1}^{r}a_i}\right).\end{aligned}$$

Making $b_{s+1},\dots,b_{r-1}$ goes to 0, we find:

$$\begin{aligned}h(\zeta)&=\sum_{j=1}^{r}\frac{(a_1,\dots,\widehat{a_j},\dots,a_r,b_1/a_j,\dots,b_s/a_j;\underline{p})_\infty\Theta_{\underline{q}}\Big((-1)^{s-r}a_j\zeta\Big)}{(b_1,\dots,b_s,a_1/a_j,\dots,\widehat{a_j/a_j},\dots,a_r/a_j;\underline{p})_\infty\Theta_{\underline{q}}\Big((-1)^{s-r}\zeta\Big)}\\ &\times{}_{s+1}\varphi_{r-1}\left(\begin{matrix} a_j,a_j\underline{p}/b_1,\dots,a_j\underline{p}/b_s \\ \\ a_j\underline{p}/a_1,\dots,\widehat{a_j\underline{p}/a_j},\dots,a_j\underline{p}/a_r\end{matrix};\underline{p},\frac{(-1)^{1+s-r}\underline{p}a_j^{r-s-1}\prod_{i=1}^{s}b_i}{\zeta\prod_{i=1}^{r}a_i}\right).\end{aligned}$$

The next lemma gives the expression of the $\underline{q}$-Laplace transformation of the first term of the sum of h. The expression of the $\underline{q}$-Laplace transformation of h will follows directly.

Lemma 4.7.2. *Let $d\not\equiv(r-s-1)\pi[2\pi]$, $\lambda:=(\underline{q}-1)e^{id}$ and $\alpha:=\dfrac{(-1)^{1+s-r}\underline{p}a_1^{r-s-2}b_1\dots b_s}{a_2\dots a_r}$. Then,*

$$L^{[d]}_{\underline{q}}\left(\frac{\Theta_{\underline{q}}\Big((-1)^{s-r}a_1\zeta\Big)}{\Theta_{\underline{q}}\Big((-1)^{s-r}\zeta\Big)}{}_{s+1}\varphi_{r-1}\left(\begin{matrix} a_1,a_1\underline{p}/b_1,\dots,a_1\underline{p}/b_s \\ \\ a_1\underline{p}/a_2,\dots,a_1\underline{p}/a_r\end{matrix};\underline{p},\tfrac{\alpha}{\zeta}\right)\right)$$

is equal to

$$\frac{\Theta_{\underline{q}}\Big((-1)^{s-r}a_1\lambda\Big)}{\Theta_{\underline{q}}\Big((-1)^{s-r}\lambda\Big)}\frac{\Theta_{\underline{q}}\Big(a_1z/\lambda\Big)}{\Theta_{\underline{q}}\Big(z/\lambda\Big)}{}_{s+2}\varphi_{r-1}\left(\begin{matrix} a_1,a_1\underline{p}/b_1,\dots,a_1\underline{p}/b_s,0 \\ \\ a_1\underline{p}/a_2,\dots,a_1\underline{p}/a_r\end{matrix};\underline{p},-\tfrac{\alpha}{a_1\underline{p}z}\right).$$

Proof. Using the expression of $\Theta_{\underline{q}}$, we find that for all $k \in \mathbb{Z}$,

$$\Theta_{\underline{q}}(\underline{q}^k z) = \underline{q}^{k(k-1)/2} z^k \Theta_{\underline{q}}(z).$$

Let us write

$$\begin{aligned} f(\zeta) &:= {}_{s+1}\varphi_{r-1}\left(\begin{matrix} a_1, a_1\underline{p}/b_1, \dots, a_1\underline{p}/b_s \\ \\ a_1\underline{p}/a_2, \dots, a_1\underline{p}/a_r \end{matrix} ; \underline{p}, \frac{\alpha}{\zeta}\right) \\ &=: \sum_{l=0}^{\infty} f_l \zeta^{-l} \end{aligned}$$

and

$$\begin{aligned} g(z) &:= {}_{s+2}\varphi_{r-1}\left(\begin{matrix} a_1, a_1\underline{p}/b_1, \dots, a_1\underline{p}/b_s, 0 \\ \\ a_1\underline{p}/a_2, \dots, a_1\underline{p}/a_r \end{matrix} ; \underline{p}, -\frac{\alpha}{a_1\underline{p}z}\right) \\ &=: \sum_{l=0}^{\infty} g_l z^{-l}. \end{aligned}$$

Then,

$$\begin{aligned} & L_{\underline{q}}^{[d]}\left(\frac{\Theta_{\underline{q}}\Big((-1)^{s-r}a_1\zeta\Big)}{\Theta_{\underline{q}}\Big((-1)^{s-r}\zeta\Big)} {}_{s+1}\varphi_{r-1}\left(\begin{matrix} a_1, a_1\underline{p}/b_1, \dots, a_1\underline{p}/b_s \\ \\ a_1\underline{p}/a_2, \dots, a_1\underline{p}/a_r \end{matrix} ; \underline{p}, \frac{\alpha}{\zeta}\right)\right) \\ = & \frac{\Theta_{\underline{q}}\Big((-1)^{s-r}a_1\lambda\Big)}{\Theta_{\underline{q}}\Big((-1)^{s-r}\lambda\Big)} \frac{1}{\Theta_{\underline{q}}\Big(\lambda\underline{q}/z\Big)} \sum_{n\in\mathbb{Z}} \left(\frac{a_1 z}{\lambda}\right)^n \underline{q}^{-n(n-1)/2} f(\underline{q}^n\lambda) \\ = & \frac{\Theta_{\underline{q}}\Big((-1)^{s-r}a_1\lambda\Big)}{\Theta_{\underline{q}}\Big((-1)^{s-r}\lambda\Big)} \frac{1}{\Theta_{\underline{q}}\Big(\lambda\underline{q}/z\Big)} \sum_{n\in\mathbb{Z}} \sum_{l=0}^{\infty} \left(\frac{a_1 z}{\lambda}\right)^n \underline{q}^{-n(n-1)/2} f_l \underline{q}^{-ln}\lambda^{-l}. \end{aligned}$$

We apply now Fubini's Theorem to conclude that

$$\begin{aligned}
&\frac{\Theta_{\underline{q}}\Big((-1)^{s-r}a_1\lambda\Big)}{\Theta_{\underline{q}}\Big((-1)^{s-r}\lambda\Big)}\frac{1}{\Theta_{\underline{q}}\Big(\lambda\underline{q}/z\Big)} \sum_{n\in\mathbb{Z}}\sum_{l=0}^{\infty}\left(\frac{a_1 z}{\lambda}\right)^n \underline{q}^{-n(n-1)/2} f_l \underline{q}^{-ln}\lambda^{-l}\\
=\ &\frac{\Theta_{\underline{q}}\Big((-1)^{s-r}a_1\lambda\Big)}{\Theta_{\underline{q}}\Big((-1)^{s-r}\lambda\Big)}\frac{1}{\Theta_{\underline{q}}\Big(\lambda\underline{q}/z\Big)} \sum_{l=0}^{\infty}\sum_{n\in\mathbb{Z}}\left(\frac{a_1 z}{\lambda}\right)^n \underline{q}^{-n(n-1)/2} f_l \underline{q}^{-ln}\lambda^{-l}\\
=\ &\frac{\Theta_{\underline{q}}\Big((-1)^{s-r}a_1\lambda\Big)}{\Theta_{\underline{q}}\Big((-1)^{s-r}\lambda\Big)}\frac{1}{\Theta_{\underline{q}}\Big(\lambda\underline{q}/z\Big)} \sum_{l=0}^{\infty}\Theta_{\underline{q}}\left(\frac{a_1 z\underline{q}^{-l}}{\lambda}\right) f_l\lambda^{-l}\\
=\ &\frac{\Theta_{\underline{q}}\Big((-1)^{s-r}a_1\lambda\Big)}{\Theta_{\underline{q}}\Big((-1)^{s-r}\lambda\Big)}\frac{\Theta_{\underline{q}}\Big(a_1 z/\lambda\Big)}{\Theta_{\underline{q}}\Big(\lambda\underline{q}/z\Big)} \sum_{l=0}^{\infty} f_l \underline{p}^{-l(l-1)/2} a_1^{-l}\underline{q}^{l} z^{-l}\\
=\ &\frac{\Theta_{\underline{q}}\Big((-1)^{s-r}a_1\lambda\Big)}{\Theta_{\underline{q}}\Big((-1)^{s-r}\lambda\Big)}\frac{\Theta_{\underline{q}}\Big(a_1 z/\lambda\Big)}{\Theta_{\underline{q}}\Big(z/\lambda\Big)} \sum_{l=0}^{\infty} g_l z^{-l}.
\end{aligned}$$

□

We have proved:

Theorem 4.7.3. *Let $d \not\equiv (r-s-1)\pi[2\pi]$ and let $\mathbb{S}_{\underline{q}}^{[d]}\left({}_r\varphi_s\right)$ be the function obtained applying successively $\hat{B}_{\underline{q}}$ and $L_{\underline{q}}^{[d]}$ to ${}_r\varphi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s\end{matrix};\underline{p},z\right)$. Then*

$$\mathbb{S}_{\underline{q}}^{[d]}\left({}_r\varphi_s\right)=\sum_{j=1}^{r}\frac{(a_1,\dots,\widehat{a_j},\dots,a_r,b_1/a_j,\dots,b_s/a_j;\underline{p})_\infty\Theta_{\underline{q}}\Big((-1)^{s-r}a_j(1-\underline{q})e^{id}\Big)\Theta_{\underline{q}}\Big(\frac{a_j z}{(1-\underline{q})e^{id}}\Big)}{(b_1,\dots,b_s,a_1/a_j,\dots,\widehat{a_j/a_j},\dots,a_r/a_j;\underline{p})_\infty\Theta_{\underline{q}}\Big((-1)^{s-r}(1-\underline{q})e^{id}\Big)\Theta_{\underline{q}}\Big(\frac{z}{(1-\underline{q})e^{id}}\Big)}$$

$$\times{}_{s+2}\varphi_{r-1}\left(\begin{matrix} a_j,a_j\underline{p}/b_1,\dots,a_j\underline{p}/b_s,0 \\ a_j\underline{p}/a_1,\dots,\widehat{a_j\underline{p}/a_j},\dots,a_1\underline{p}/a_r\end{matrix};\underline{p},\frac{(-1)^{s-r}a_1^{r-s-2}\prod_{i=1}^{s}b_i}{z\prod_{i=1}^{r}a_i}\right).$$

Let $\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_s\in\mathbb{C}\setminus -\mathbb{N}$ with different images in $\mathbb{C}/\mathbb{Z}$. We replace now a_i by $\underline{p}^{\alpha_i}$, b_i by $\underline{p}^{\beta_i}$, z by $x=z\left(1-\underline{p}\right)^{1+s-r}$ and consider the limit as $\underline{p}$ goes to 1. It is clear that for all $j\in\{1,\dots,r\}$, we have the uniform convergence on the compacts of $\mathbb{C}^*$

$$\lim_{\underline{p}\to 1} {}_{s+2}\varphi_{r-1}\left(\begin{matrix}\underline{p}^{\alpha_j}, \underline{p}^{\alpha_j-\beta_1+1}, \dots, \underline{p}^{\alpha_j-\beta_s+1}, 0 \\ \\ \underline{p}^{\alpha_j-\alpha_1+1}, \dots, \widehat{\underline{p}^{\alpha_j-\alpha_j+1}}, \dots, \underline{p}^{\alpha_j-\alpha_r+1}\end{matrix} ; \underline{p}, \frac{(-1)^{s-r}\underline{p}^{\alpha_j(r-s-2)+\beta_1+\dots+\beta_s}}{x\underline{p}^{\alpha_1+\dots+\alpha_r}}\right)$$

$$= {}_{s+1}F_{r-1}\left(\begin{matrix}\alpha_j, \alpha_j-\beta_1+1, \dots, \alpha_j-\beta_s+1 \\ \\ \alpha_j-\alpha_1+1, \dots, \alpha_j \widehat{-\alpha_j}+1, \dots, \alpha_j-\alpha_r+1\end{matrix} ; \tfrac{(-1)^{s-r}}{z}\right).$$

As we can see in [Zha02], §2.3,

– For $\gamma\in\mathbb{C}$,

$$\lim_{\underline{p}\to 1}\frac{(\underline{p}^{\gamma}, \underline{p})_{\infty}(1-\underline{p})^{\gamma-1}}{(\underline{p}, \underline{p})_{\infty}} = \Gamma(\gamma)^{-1}.$$

– We have

$$\lim_{\underline{p}\to 1}\frac{\Theta_{\underline{q}}(\underline{p}^{\gamma}u)}{\Theta_{\underline{q}}(u)} = u^{-\gamma},$$

uniformly on the compacts of $\{z\in\mathbb{C}^* | \arg(-z)\neq\pi\}$.

We have proved:

Theorem 4.7.4. *Let $d\not\equiv(r-s-1)\pi[2\pi]$. Then,*

$$\lim_{\underline{p}\to 1}\mathbb{S}_{\underline{q}}^{[d]}\left({}_{r}\varphi_{s}\left(\begin{matrix}\underline{p}^{\alpha_1}, \dots, \underline{p}^{\alpha_r} \\ \\ \underline{p}^{\beta_1}, \dots, \underline{p}^{\beta_s}\end{matrix} ; \underline{p}, x\right)\right) = \sum_{j=1}^{r}\frac{\displaystyle\prod_{i=1}^{s}\Gamma(\beta_i)\prod_{\substack{i=1\\ i\neq j}}^{r}\Gamma(\alpha_i-\alpha_j)\Big((-1)^{s-r}z\Big)^{-\alpha_j}}{\displaystyle\prod_{\substack{i=1\\ i\neq j}}^{r}\Gamma(\alpha_i)\prod_{i=1}^{s}\Gamma(\beta_i-\alpha_j)}$$

$$\times {}_{s+1}F_{r-1}\left(\begin{matrix}\alpha_j, \alpha_j-\beta_1+1, \dots, \alpha_j-\beta_s+1 \\ \\ \alpha_j-\alpha_1+1, \dots, \alpha_j \widehat{-\alpha_j}+1, \dots, \alpha_j-\alpha_r+1\end{matrix} ; \tfrac{(-1)^{s-r}}{z}\right),$$

uniformly on the compacts of $\{z\in\mathbb{C}^* | \arg(-z)\neq d\}$.

Remark 4.7.5. The right hand side of the limit equals to the function $\widetilde{S}^{d}\left({}_{r}F_{s}\left(\begin{matrix}\alpha_1, \dots, \alpha_r \\ \\ \beta_1, \dots, \beta_s\end{matrix} ; (-1)^{1+s-r}z\right)\right)$.

4.8 Application: Confluence of a basis of meromorphic solutions

We study a family of linear δ_q-equations that discretize a linear δ-equation, and the behavior of the solutions as q goes to 1. After introducing some notations in §4.8.1, we prove in §4.8.2, that a basis of local formal solutions of the family of linear δ_q-equations converges to the Hukuhara-Turrittin solution of the differential equation in a sense that we are going to explain. We apply this and our main result, Theorem 4.4.5, to prove the convergence of the q-Stokes matrices to the Stokes matrices of the linear differential equation in §4.8.3. In §4.8.4, we show how to find the monodromy matrices of the differential equation, as limit of q-solutions when q tends to 1. When q is real, this extends the results in §4 of [Sau00] in the irregular singular case ‡.

4.8.1 Notations

Some of the notations below were already introduced before, see the introduction, but we recall them for the reader's convenience. For $a \in \mathbb{C}^*$ and $n \in \mathbb{N}^*$, let us consider $\Theta_q(z) = \displaystyle\sum_{n\in\mathbb{Z}} q^{\frac{-n(n-1)}{2}} z^n$, $l_q(z) := \dfrac{\delta\left(\Theta_q(z)\right)}{\Theta_q(z)}$, $\Lambda_{q,a}(z) := \frac{\Theta_q(z)}{\Theta_q(z/a)}$, $e_{q^n}\left(az^n\right)$ and $e_{q^n}\left(az^{-n}\right)$. They satisfy the q-difference equations:

- $\sigma_q\Theta_q(z) = z\Theta_q(z)$.
- $\sigma_q l_q = l_q + 1$.
- $\sigma_q\Lambda_{q,a}(z) = a\Lambda_{q,a}(z)$.
- $\delta_q e_{q^n}\left(az^n\right) = a[n]_q z^n e_{q^n}\left(az^n\right)$.
- $\delta_q e_{q^n}\left(az^{-n}\right) = \dfrac{-a[n]_q q^{-n} z^{-n}}{1+(q-1)a[n]_q q^{-n}z^{-n}} e_{q^n}\left(az^{-n}\right)$.

Let A be an invertible matrix with complex coefficients and consider now the decomposition in Jordan normal form $A = P(DN)P^{-1}$, where $D := \mathrm{Diag}(d_i)$ is diagonal, N is a nilpotent upper triangular matrix with $DN = ND$, and P is an invertible matrix with complex coefficients. Following [Sau00], we construct the matrix:

$$\Lambda_{q,A} := P\left(\mathrm{Diag}\left(\Lambda_{q,d_i}\right) e^{\log(N) l_q}\right) P^{-1} \in \mathrm{GL}_m\Big(\mathbb{C}\left(l_q, \left(\Lambda_{q,a}\right)_{a\in\mathbb{C}^*}\right)\Big)$$

that satisfies:

$$\sigma_q\Lambda_{q,A} = A\Lambda_{q,A} = \Lambda_{q,A}A.$$

Let $a \in \mathbb{C}^*$ and consider the corresponding matrix $(a) \in \mathrm{GL}_1(\mathbb{C})$. By construction, we have $\Lambda_{q,a} = \Lambda_{q,(a)}$.

We now introduce the q-exponential of matrices. For $A \in \mathrm{M}_m\Big(\mathbb{C}(z)\Big)$, we define:

$$e_q(A) := \sum_{n\in\mathbb{N}} \frac{A^n}{[n]_q^!} \in \mathrm{GL}_m\Big(\mathcal{M}(\mathbb{C}^*, 0)\Big).$$

‡. Notice that the results of this section do not allow us to recover Sauloy's Theorem, but are to be considered as an analogous result in a different situation.

4.8.2 Confluence of a basis of local formal solutions

Formally, we have the convergence $\lim\limits_{q\to 1}\delta_q=\delta$. We want to prove the formal convergence of a basis of solutions of a family of linear δ_q-equations to the Hukuhara-Turrittin solution of the corresponding linear δ-equation. First, we will consider the family of equations,

$$\begin{cases} \Delta_q & := & b_m(z,q)\delta_q^m & + & b_{m-1}(z,q)\delta_q^{m-1} & + & \dots & + & b_0(z,q) \\ \widetilde{\Delta} & := & \widetilde{b}_m(z)\delta^m & + & \widetilde{b}_{m-1}(z)\delta^{m-1} & + & \dots & + & \widetilde{b}_0(z), \end{cases}$$

that satisfies the following assumptions:

(H1) For all i, for all q close to 1, $z\mapsto b_i(z,q),\widetilde{b}_i(z)\in\mathbb{C}[[z]]$.

(H2) For all i, $b_i(z,q)$ converges coefficientwise to $\widetilde{b}_i(z)$ when $q\to 1$.

(H3) Viewed as a linear σ_q-equation, Δ_q has slopes that belongs to $\mathbb{Z}$. For q close to 1, the Newton polygon of Δ_q is independent of q.

(H4) The slopes of $\widetilde{\Delta}$ belongs to $\mathbb{N}$.

We consider now the associated systems

$$\begin{cases} \delta_q Y(z,q) & = & B(z,q)Y(z,q) \\ \delta\widetilde{Y}(z) & = & \widetilde{B}(z)\widetilde{Y}(z), \end{cases} \tag{4.8.1}$$

with $z\mapsto \mathrm{Id}+(q-1)B(z,q)\in\mathrm{GL}_m\Big(\mathbb{C}((z))\Big)$, $\widetilde{B}(z)\in\mathrm{M}_m\Big(\mathbb{C}((z))\Big)$. From Theorem 4.2.2 and the Hukuhara-Turrittin theorem (see §4.1), we have the existence of

- $z\mapsto \hat{H}(z,q),\widetilde{H}(z)\in\mathrm{GL}_m\Big(\mathbb{C}((z))\Big)$, such that the entries of the first row of $\hat{H}(z,q)$ have z-valuation equal to 0,
- $\mu_i\in\mathbb{Z}$, and matrices $B_i(q)\in\mathrm{GL}_{m_i}(\mathbb{C})$, which are of the form $\mathrm{Diag}_l\Big(T_{i,l}(q)\Big)$ where $T_{i,l}(q)$ are upper triangular matrices with diagonal terms equal to the roots of the characteristic polynomial associated to the slope μ_i,
- $\widetilde{\lambda}_i(z)\in z^{-1}\mathbb{C}[z^{-1}]$, $\widetilde{L}_i\in\mathrm{M}_{m_i'}(\mathbb{C})$,

such that

$$\begin{cases} \hat{H}(z,q)\left[\mathrm{Diag}\Big(B_i(q)z^{-\mu_i}\Big)\right]_{\sigma_q}=\mathrm{Id}+(q-1)B(z,q) \\ \widetilde{H}(z)\left[\mathrm{Diag}\left(\widetilde{L}_i+\delta\widetilde{\lambda}_i(z)\times\mathrm{Id}_{m_i}\right)\right]_{\delta}=\widetilde{B}(z). \end{cases} \tag{4.8.2}$$

We make two more assumptions:

(H5) For q close to 1, $\mathrm{Diag}\Big(B_i(q)z^{-\mu_i}\Big)$ commutes with $\mathrm{Diag}\left(\widetilde{L}_i+\delta\widetilde{\lambda}_i(z)\times\mathrm{Id}_{m_i}\right)$.

(H6) If $\widetilde{H}'(z)$ is any formal matrix solution of the differential system of (4.8.2), then the entries of the first row of $\widetilde{H}'(z)$ have necessarily z-valuation equal to 0. Moreover, we assume that the term of lower degree of each entry of the first row of $\hat{H}(z,q)$ converges as q goes to 1, to the term of lower degree of the corresponding entry of $\widetilde{H}(z)$.

Remark 4.8.1. (1) Assumptions **(H1)** to **(H4)** are satisfied if the $b_i(z,q) \in \mathbb{C}[[z]]$ are independent of q and if the slopes of Δ_q, viewed as a linear σ_q-equation, belong to $\mathbb{Z}$.

(2) As we can see in [RSZ13], Theorem 2.2.1, up to a ramification, we may always reduce to the case where the slopes of Δ_q, viewed as a linear σ_q-equation, belong to $\mathbb{Z}$. Up to a ramification, we may also reduce to the case where **(H4)** is satisfied.

(3) Assumption **(H5)** is satisfied if and only if, for all q close to 1, $\mathrm{Diag}\Big(B_i(q)\Big)$ commutes with $\mathrm{Diag}\left(\widetilde{L}_i\right)$. If Assumption **(H5)** is satisfied, then the the blocks of $\mathrm{Diag}\Big(B_i(q)z^{-\mu_i}\Big)$ and $\mathrm{Diag}\left(\widetilde{L}_i + \delta\widetilde{\lambda}_i(z) \times \mathrm{Id}_{m_i}\right)$ have the same size.

(4) If, for q close to 1, the $B_i(q)$ and $\widetilde{L}_i$ are diagonal, we may perform shearing transformations on the differential system (resp. a diagonal gauge transformation that depends only upon q on the q-difference system), in order to change the entries $\widetilde{l}_{i,j}$ of $\widetilde{L}_i$ by $\widetilde{l}_{i,j} + k_{i,j}$ where $k_{i,j} \in \mathbb{Z}$ (resp. multiply to the right $\hat{H}(z,q)$ by a diagonal complex matrix), and to reduce to the case where **(H6)** is satisfied. Notice that in this case, **(H5)** was already satisfied because of the point (3) of the remark. The $B_i(q)$ and $\widetilde{L}_i$ are diagonal if for q close to 1, the multiplicities of the slopes of Δ_q, viewed as a linear σ_q-equation, (resp. the multiplicities of the slopes of $\widetilde{\Delta}$) are all equal to 1. A weaker condition for $B_i(q)$ and $\widetilde{L}_i$ being diagonal is to assume that Δ_q, viewed as a linear σ_q-equation, and $\widetilde{\Delta}$ have exponents at 0 which are not resonant.

(5) If Assumption **(H6)** is satisfied, then the $\mathbb{C}$-vectorial subspace of $\mathrm{M}_m\Big(\mathbb{C}((z))\Big)$ of solutions of the differential system of (4.8.2) has dimension 1. Remark that the converse in not true.

(6) The slopes of $\widetilde{\Delta}$ and Δ_q, viewed as a linear σ_q-equation, may be different. We will make assumptions on the slopes in §4.8.3 and §4.8.4.

Definition 4.8.2. We say that the $m \times m$ invertible square matrix $F(z,q)$ belongs to $\mathcal{O}_m^*$, if for q close to 1, the entries of $z \mapsto F(z,q)$ are meromorphic on $\mathbb{C}^*$, and $F(z,q)$ satisfies

- We have the uniform convergence $\lim\limits_{q\to 1}\Big(\delta_q F(z,q)\Big)F(z,q)^{-1} = 0$, on the compacts of $\mathbb{C}^*$.
- We have the uniform convergence $\lim\limits_{q\to 1} F(z,q) = \mathrm{Id}$, on the compacts of $\mathbb{C}^*$.

Remark 4.8.3. Roughly speaking, the matrices $\hat{H}(z,q)\mathrm{Diag}\Big(\Lambda_{q,B_i(q)}\Theta_q(z)^{-\mu_i}\Big)$ and $\widetilde{H}(z)\mathrm{Diag}\left(e^{\log(z)\widetilde{L}_i}e^{\widetilde{\lambda}_i(z)\times \mathrm{Id}_{m_i}}\right)$ are fundamental solutions of the systems (4.8.1). Let us write $\widetilde{\lambda}_i(z) := \displaystyle\sum_{j=1}^{k_i} \widetilde{\lambda}_{i,j} z^{-j}$ with $k_i \in \mathbb{N}$. The next theorem says that there exists a fundamental solution of $\delta_q Y(z,q) = B(z,q)Y(z,q)$ of the form:

$$\hat{H}(z,q)F_1(z,q)F_2(z,q)\mathrm{Diag}_i\left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i}\prod_{j=1}^{k_i} e_{q^j}\left(\widetilde{\lambda}_{i,j}z^{-j}\times \mathrm{Id}_{m_i}\right)\right),$$

such that:

- $z \mapsto F_1(z,q) \in \mathrm{GL}_m\Big(\mathbb{C}\{z\}\Big)$ and the matrix $\hat{H}(z,q)F_1(z,q) \in \mathrm{GL}_m\Big(\mathbb{C}((z))\Big)$ converge entrywise and coefficientwise to $\widetilde{H}(z)$ when $q \to 1$.
- The matrix $F_2(z,q)$ belongs to $\mathcal{O}_m^*$ and therefore, for $z \in \mathbb{C}^*$, $\lim\limits_{q\to 1} F_2(z,q) = \mathrm{Id}$.

– Because of what is written in Page 1048 of [Sau00] and Lemma 4.6.1, for all $z \in \mathbb{C}^* \setminus \mathbb{R}_{<0}$, we have the convergence

$$\lim_{q\to 1} \mathrm{Diag}_i \left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i} \prod_{j=1}^{k_i} e_{q^j}\left(\widetilde{\lambda}_{i,j} z^{-j} \times \mathrm{Id}_{m_i}\right)\right) = \mathrm{Diag}\left(e^{\log(z)\widetilde{L}_i} e^{\widetilde{\lambda}_i(z)\times \mathrm{Id}_{m_i}}\right).$$

In other words, the above fundamental solution of $\delta_q Y(z,q) = B(z,q)Y(z,q)$ formally converges to the fundamental solution $\widetilde{H}(z)\mathrm{Diag}\left(e^{\log(z)\widetilde{L}_i} e^{\widetilde{\lambda}_i(z)\times \mathrm{Id}_{m_i}}\right)$ of $\delta\widetilde{Y}(z) = \widetilde{B}(z)\widetilde{Y}(z)$ given by the Hukuhara-Turrittin Theorem. Of course, written like this, this statement is not rigorous since the matrices can not be multiplied among them, see §4.1.

Theorem 4.8.4. *Let us consider the systems (4.8.1) that satisfies the assumptions **(H1)** to **(H6)**. Let $B_i(q)$, $\widetilde{L}_i$ and $\widetilde{\lambda}_i(z) = \sum_{j=1}^{k_i} \widetilde{\lambda}_{i,j} z^{-j}$ that come from (4.8.2).*

(1) There exist $z \mapsto F_1(z,q) \in \mathrm{GL}_m\left(\mathbb{C}\{z\}\right)$, $F_2(z,q) \in \mathcal{O}_m^*$, $z \mapsto N(z,q) \in \mathrm{M}_m\left(\mathbb{C}(z)\right)$ such that

$$F_1(z,q)\Big[\mathrm{Id} + (q-1)N(z,q)\Big]_{\sigma_q} = \mathrm{Diag}\Big(B_i(q)z^{-\mu_i}\Big),$$

where $N(z,q)$ satisfies:

$$\delta_q \left(F_2(z,q)\mathrm{Diag}_i \left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i} \prod_{j=1}^{k_i} e_{q^j}\left(\widetilde{\lambda}_{i,j} z^{-j} \times \mathrm{Id}_{m_i}\right)\right)\right) =$$

$$N(z,q)F_2(z,q)\mathrm{Diag}_i \left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i} \prod_{j=1}^{k_i} e_{q^j}\left(\widetilde{\lambda}_{i,j} z^{-j} \times \mathrm{Id}_{m_i}\right)\right).$$

(2) The matrix $\hat{H}(z,q)F_1(z,q)$ converges entrywise to $\widetilde{H}(z)$ when $q \to 1$. Moreover, there exists $N \in \mathbb{N}$, such that for q close to 1, $z \mapsto z^N \hat{H}(z,q)F_1(z,q)$ belongs to $\mathrm{M}_m\Big(\mathbb{C}[[z]]\Big)$.

Notice that the point (2) implies in particular that $z^N \widetilde{H}(z) \in \mathrm{M}_m\Big(\mathbb{C}[[z]]\Big)$. Before proving the theorem, we state and prove a lemma:

Lemma 4.8.5. *Let us consider an invertible complex matrix that depends upon q, $A(q)$, and assume the existence of $k \in \mathbb{N}^*$, such that we have the simple convergence $\lim_{q\to 1} A(q)^{-1}(q-1)^k = 0 \in \mathrm{M}_m(\mathbb{C})$. Let $n \in \mathbb{Z}$. There exist*

– $z \mapsto E_1(z,q) \in \mathrm{GL}_m\Big(\mathbb{C}\{z\}\Big)$
– $F_2(z,q) \in \mathcal{O}_m^*$

such that

$$\sigma_q\Big(E_1(z,q)F_2(z,q)\Big) = z^n A(q)E_1(z,q)F_2(z,q) = E_1(z,q)F_2(z,q)A(q)z^n.$$

Example 4.8.6. Let us solve $\sigma_q Y(z,q) = \frac{z}{(q-1)^2} Y(z,q)$ with solution in the same form as in the lemma. The trick of the proof of the lemma is the following identity that is valid for all $z \in \mathbb{C}^*$:

$$\frac{z}{(q-1)^2} = \frac{1 + \frac{z}{(q-1)^2}}{1 + \frac{(q-1)^2}{z}}.$$

We may take $E_1(z,q) := e_q\left(\frac{z}{(q-1)^3}\right)$ that satisfies $\sigma_q\left(\frac{z}{(q-1)^3}\right) = \left(1 + \frac{z}{(q-1)^2}\right) e_q\left(\frac{z}{(q-1)^3}\right)$ and $F_2(z,q) := e_q\left(\frac{q(q-1)}{z}\right)$ that satisfies $\sigma_q e_q\left(\frac{q(q-1)}{z}\right) = \frac{1}{1+\frac{(q-1)^2}{z}} e_q\left(\frac{q(q-1)}{z}\right)$.

Proof of Lemma 4.8.5. For, $l, d \in \mathbb{N}^*$ with $l \geq 2$, let us define the function $f_{l,d} := e_{q^d}\left(\frac{z^d}{(q-1)^{l+1}[d]_q}\right) e_{q^d}\left(\frac{q^d(q-1)^{l-1}}{[d]_q z^d}\right)$, that satisfies:

$$\sigma_q f_{l,d} = \frac{z^d}{(q-1)^l} f_{l,d} = f_{l,d} \frac{z^d}{(q-1)^l},$$

with $z \mapsto e_{q^d}\left(\frac{z^d}{(q-1)^{l+1}[d]_q}\right) \in \mathbb{C}\{z\}$ and $e_{q^d}\left(\frac{q^d(q-1)^{l-1}}{[d]_q z^d}\right) \in \mathcal{O}_1^*$. Let us also consider $z \mapsto e_q\left(\frac{zA(q)}{(q-1)^{k+2}}\right) \in \mathrm{GL}_m\Big(\mathbb{C}\{z\}\Big)$ and $e_q\left(\frac{q(q-1)^k A^{-1}(q)}{z}\right) \in \mathrm{GL}_m\Big(\mathbb{C}\{z^{-1}\}\Big)$. We can prove that they satisfy

$$\sigma_q e_q\left(\frac{zA(q)}{(q-1)^{k+2}}\right) = e_q\left(\frac{zA(q)}{(q-1)^{k+2}}\right)\left(\mathrm{Id} + \frac{zA(q)}{(q-1)^{k+1}}\right) = \left(\mathrm{Id} + \frac{zA(q)}{(q-1)^{k+1}}\right) e_q\left(\frac{zA(q)}{(q-1)^{k+2}}\right)$$

and

$$\begin{aligned}\sigma_q e_q\left(\frac{q(q-1)^k A^{-1}(q)}{z}\right) &= \left(\mathrm{Id} + \frac{(q-1)^{k+1}A^{-1}(q)}{z}\right)^{-1} e_q\left(\frac{q(q-1)^k A^{-1}(q)}{z}\right)\\ &= e_q\left(\frac{q(q-1)^k A^{-1}(q)}{z}\right)\left(\mathrm{Id} + \frac{(q-1)^{k+1}A^{-1}(q)}{z}\right)^{-1}.\end{aligned}$$

Hence, $e_q\left(\frac{q(q-1)^k A^{-1}(q)}{z}\right) \in \mathcal{O}_m^*$ and we have:

$$\begin{aligned}&\sigma_q\left(e_q\left(\frac{zA(q)}{(q-1)^{k+2}}\right) e_q\left(\frac{q(q-1)^k A^{-1}(q)}{z}\right)\right)\\ =\ & \frac{zA(q)}{(q-1)^{k+1}} e_q\left(\frac{zA(q)}{(q-1)^{k+2}}\right) e_q\left(\frac{q(q-1)^k A^{-1}(q)}{z}\right)\\ =\ & e_q\left(\frac{zA(q)}{(q-1)^{k+2}}\right) e_q\left(\frac{q(q-1)^k A^{-1}(q)}{z}\right) \frac{zA(q)}{(q-1)^{k+1}}.\end{aligned}$$

Let us choose $d_1, d_2, l_1, l_2 \in \mathbb{N}^*$ with $l_1, l_2 \geq 2$, such that $d_1 - d_2 + 1 = n$ and $l_1 + (k+1) = l_2$. Then,

$$f_{l_1,d_1}\left(f_{l_2,d_2}\right)^{-1} e_q\left(\frac{zA(q)}{(q-1)^{k+2}}\right) e_q\left(\frac{q(q-1)^k A^{-1}(q)}{z}\right),$$

is solution of $\sigma_q Y(z,q) = z^n A(q) Y(z,q) = Y(z,q) A(q) z^n$ and admits a decomposition that has the required property. □

Proof of Theorem 4.8.4. (1) Let us define

$$W_1(z,q) := \mathrm{Diag}_i\left(\prod_{j=1}^{k_i} e_{q^j}\left(\frac{q^j z^j}{\widetilde{\lambda}_{i,j}(q-1)^2[j]_q^2}\times \mathrm{Id}_{m_i}\right)\right)$$

and

$$W_2(z,q) := \mathrm{Diag}_i\left(\prod_{j=1}^{k_i} e_{q^j}\left(\widetilde{\lambda}_{i,j} z^{-j}\times \mathrm{Id}_{m_i}\right)\right),$$

which satisfy

$$\sigma_q\Big(W_1(z,q)W_2(z,q)\Big) = \mathrm{Diag}_i\left(\prod_{j=1}^{k_i}\frac{q^j z^j}{(q-1)[j]_q\widetilde{\lambda}_{i,j}}\times \mathrm{Id}_{m_i}\right)W_1(z,q)W_2(z,q).$$

Because of (4.8.2), $\mathrm{Diag}_i\left(\prod_{j=1}^{k_i}\frac{q^j z^j}{(q-1)[j]_q\widetilde{\lambda}_{i,j}}\times \mathrm{Id}_{m_i}\right)$ commutes with $\mathrm{Diag}\left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i}\right)$ and we obtain that:

$$\sigma_q\left(\mathrm{Diag}\left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i}\right)W_1(z,q)W_2(z,q)\right) =$$

$$\mathrm{Diag}_i\left(\left(\mathrm{Id}+(q-1)\widetilde{L}_i\right)\prod_{j=1}^{k_i}\frac{q^j z^j}{(q-1)[j]_q\widetilde{\lambda}_{i,j}}\right)\mathrm{Diag}\left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i}\right)W_1(z,q)W_2(z,q).$$

Let

$$\begin{aligned} C(z,q) &:= \mathrm{Diag}\Big(B_i(q)z^{-\mu_i}\Big)\mathrm{Diag}_i\left(\left(\left(\mathrm{Id}+(q-1)\widetilde{L}_i\right)\prod_{j=1}^{k_i}\frac{q^j z^j}{(q-1)[j]_q\widetilde{\lambda}_{i,j}}\right)^{-1}\right)\\ &=: \mathrm{Diag}\Big(C_i(q)z^{n_i}\Big). \end{aligned}$$

If we are able to construct $z\mapsto E_1(z,q)\in \mathrm{GL}_m\Big(\mathbb{C}\{z\}\Big)$ and $F_2(z,q)\in\mathcal{O}_m^*$, that commute with $\mathrm{Diag}\Big(B_i(q)z^{-\mu_i}\Big)$ and are solution of

$$\sigma_q\Big(E_1(z,q)F_2(z,q)\Big) = C(z,q)E_1(z,q)F_2(z,q) = E_1(z,q)F_2(z,q)C(z,q),$$

then the following matrix would be a fundamental solution of the linear σ_q-equation $\sigma_q Y(z,q) = \mathrm{Diag}\Big(B_i(q)z^{-\mu_i}\Big)Y(z,q)$:

$$E_1(z,q)F_2(z,q)\mathrm{Diag}\left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i}\right)W_1(z,q)W_2(z,q). \tag{4.8.3}$$

Let us construct the matrices E_1 and F_2 using Lemma 4.8.5 applied on each block $C_i(q)$. Let us check that the matrices $q\mapsto C_i(q)$ satisfies the assumptions of Lemma 4.8.5. Since the matrices $\left(\left(\mathrm{Id}+(q-1)\widetilde{L}_i\right)\prod_{j=1}^{k_i}\frac{q^j z^j}{(q-1)[j]_q\widetilde{\lambda}_{i,j}}\right)^{-1}$ satisfy the assumptions of Lemma 4.8.5, it is sufficient to prove that the matrices $B_i(q)$ satisfy the assumptions of

Lemma 4.8.5. Using Theorem 4.2.2, the $B_i(q)$ are of the form $\mathrm{Diag}_l\Big(T_{i,l}(q)\Big)$ where $T_{i,l}(q)$ are upper triangular matrices with diagonal terms equal to the roots of the characteristic polynomial associated to the slope μ_i. We recall that the linear δ_q-equation is

$$\Delta_q := b_m(z,q)(\delta_q)^m + b_{m-1}(z,q)(\delta_q)^{m-1} + \dots + b_0(z,q),$$

where the b_i converge coefficientwise when $q \to 1$. Since for all $n \in \mathbb{N}$,

$$\delta_q^n = (q-1)^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sigma_q^k,$$

a straightforward computation shows that each root of the characteristic polynomial associated to a slope different from zero (resp. to the slope zero) is of the form $\alpha(q)(q-1)$ (resp. $\alpha(q)$), where $\alpha(q)$ converges to a non zero complex number. Therefore, each diagonal term of a $B_i(q)$ is of the form $\alpha(q)(q-1)$ or $\alpha(q)$, where $\alpha(q)$ converges to a non zero complex number. We recall, see (4.8.2), that the matrix $\hat{H}(z,q)$ satisfies

$$\sigma_q\left(\hat{H}(z,q)\right)\mathrm{Diag}\Big(B_i(q)z^{-\mu_i}\Big) = \Big(\mathrm{Id} + (q-1)B(z,q)\Big)\hat{H}(z,q).$$

Using the convergence of the constant terms of the entries in the first row of $\hat{H}(z,q)$, see the assumption **(H6)**, and the behavior of the diagonal terms of the $B_i(q)$, we find that each non diagonal term of a triangular matrix $B_i(q)$ is of the form $\alpha(q)(q-1)$ or $\alpha(q)$, where $\alpha(q)$ converges to a non zero complex number. Hence, for all i, $B_i(q)^{-1}(q-1)^2$ simply converges to 0 as q goes to 1.

Applying Lemma 4.8.5, there exist $z \mapsto E_1(z,q) \in \mathrm{GL}_m\Big(\mathbb{C}\{z\}\Big)$ and $F_2(z,q) \in \mathcal{O}_m^*$ that satisfy

$$\sigma_q\Big(E_1(z,q)F_2(z,q)\Big) = \mathrm{Diag}\Big(C_i(q)z^{n_i}\Big)E_1(z,q)F_2(z,q) = E_1(q)F_2(z,q)\mathrm{Diag}\Big(C_i(q)z^{n_i}\Big).$$

Because of **(H5)** and the construction of $E_1(z,q)$ and $F_2(z,q)$ (see the proof of Lemma 4.8.5), we obtain that they commute with $\mathrm{Diag}\Big(B_i(q)z^{-\mu_i}\Big)$.

We have proved that the matrix (4.8.3) is a fundamental solution of the system

$$\sigma_q Y(z,q) = \mathrm{Diag}\Big(B_i(q)z^{-\mu_i}\Big)Y(z,q).$$

We have the following relation:

$$\sigma_q\left(\mathrm{Diag}\left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i}\right)W_2(z,q)\right) =$$

$$\mathrm{Diag}_i\left(\left(\mathrm{Id}+(q-1)\widetilde{L}_i\right)\prod_{j=1}^{k_i}\left(1+\frac{q^j z^j}{(q-1)[j]_q\widetilde{\lambda}_{i,j}}\right)\right)\mathrm{Diag}\left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i}\right)W_2(z,q).$$

Using **(H5)** and the construction of $F_2(z,q)$, we find that $F_2(z,q)$ commutes with $\mathrm{Diag}_i\left(\left(\mathrm{Id}+(q-1)\widetilde{L}_i\right)\prod_{j=1}^{k_i}\left(1+\frac{q^j z^j}{(q-1)[j]_q\widetilde{\lambda}_{i,j}}\right)\right)$. From the construction of $F_2(z,q)$, we find also that $\sigma_q\Big(F_2(z,q)\Big)F_2(z,q)^{-1} \in \mathrm{GL}_m\Big(\mathbb{C}(z)\Big)$. Let

$$U(z,q) := F_2(z,q)\mathrm{Diag}\left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i}W_2(z,q)\right).$$

From what is preceding, we obtain the existence of $z \mapsto N(z,q) \in \mathrm{M}_m\Big(\mathbb{C}(z)\Big)$, such that $\delta_q U(z,q) = N(z,q)U(z,q)$.

Because of (4.8.2), $W_1(z,q)$ commutes with $\mathrm{Diag}\left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i}\right)$. Because of **(H5)**, and the construction of $F_2(z,q)$, $W_1(z,q)$ commutes also with $F_2(z,q)$. Let $F_1(z,q) := E_1(z,q)W_1(z,q)$. Then, by construction,

$$F_1(z)\Big[\mathrm{Id} + (q-1)N(z,q)\Big]_{\sigma_q} = \mathrm{Diag}\Big(B_i(q)z^{-\mu_i}\Big),$$

and the matrices $N(z,q)$, $F_1(z,q)$ and $F_2(z,q)$ have entries in the good fields.

(2) We recall that the matrix $U(z,q)$ satisfies the linear δ_q-equation:

$$\delta_q U(z,q) = N(z,q)U(z,q).$$

Let $\widetilde{N}(z) := \mathrm{Diag}\left(\widetilde{L}_i + \delta\widetilde{\lambda}_i(z) \times \mathrm{Id}_{m_i}\right)$ which satisfies

$$\delta\left(\mathrm{Diag}\left(e^{\log(z)\widetilde{L}_i}e^{\widetilde{\lambda}_i(z)\times \mathrm{Id}_{m_i}}\right)\right) = \widetilde{N}(z)\mathrm{Diag}\left(e^{\log(z)\widetilde{L}_i}e^{\widetilde{\lambda}_i(z)\times \mathrm{Id}_{m_i}}\right).$$

From what is preceding, we deduce the following relations:

$$\begin{array}{rcl} \delta\widetilde{H}(z) &=& \widetilde{B}(z)\widetilde{H}(z) - \widetilde{H}(z)\widetilde{N}(z) \\ \sigma_q\left(\hat{H}(z,q)F_1(z,q)\right)\left(\mathrm{Id} + (q-1)N(z,q)\right) &=& \left(\mathrm{Id} + (q-1)B(z,q)\right)\left(\hat{H}(z,q)F_1(z,q)\right). \end{array}$$

This implies that

$$\delta_q\left(\hat{H}(z,q)F_1(z,q)\right) = B(z,q)\hat{H}(z,q)F_1(z,q) - \sigma_q\left(\hat{H}(z,q)F_1(z,q)\right)N(z,q),$$

and finally

$$\delta_q\left(\hat{H}(z,q)F_1(z,q)\right)\left(\mathrm{Id} + (q-1)N(z,q)\right) = \widetilde{B}(z)\hat{H}(z,q)F_1(z,q) - \hat{H}(z,q)F_1(z,q)N(z,q). \tag{4.8.4}$$

We are going now to prove that the entries that belong to the first row of $\hat{H}(z,q)F_1(z,q)$ converge coefficientwise to the corresponding entries of $\widetilde{H}(z)$ when $q \to 1$.

Let $\hat{h}(z,q) := \sum_{n=0}^{\infty} \hat{h}_n(q)z^n$ be an entry of the first row of $\hat{H}(z,q)F_1(z,q)$ and let $\widetilde{h}(z) := \sum_{n=0}^{\infty} \widetilde{h}_n z^n$ be the corresponding entry of $\widetilde{H}(z)$. We want to use Lemma 4.4.1 to prove that for all $n \in \mathbb{N}$, $\hat{h}_n(q)$ converges as q goes to 1 to $\widetilde{h}_n$. We are going to prove now that the assumptions of Lemma 4.4.1 are satisfied.

- The matrices $B(z,q)$ and $N(z,q)$ converge entrywise and coefficientwise to $\widetilde{B}(z)$ and $\widetilde{N}(z)$ when $q \to 1$. Therefore, using additionally (4.8.4), we find that there exists a δ_q-equation with coefficient in $\mathbb{C}[[z]]$ that is satisfied by $\hat{h}(z,q)$, with z-coefficients that converge to the z-coefficients of a δ-equation with coefficient in $\mathbb{C}[[z]]$, that is satisfied by $\widetilde{h}(z)$.
- As we can see in Remark 4.8.1 (5), the vector space of Lemma 4.4.1 has dimension 1.

- By construction, $F_1(z,q)$ is of the form $\mathrm{Id}+zG_1(z,q)$, where $z\mapsto G_1(z,q)\in \mathrm{M}_m\Big(\mathbb{C}\{z\}\Big)$. Hence for q close to 1, the entries of the first row of $\hat{H}(z,q)F_1(z,q)$ have z-valuation equal to the entries of the first row of $\hat{H}(z,q)$, which are 0 (see the paragraph just below **(H4)**). Due to **(H6)**, the entries of the first row of $\widetilde{H}(z)$ have z-valuation equal to 0.
- Let us prove the convergence of $\hat{h}_0(q)$ to $\widetilde{h}_0$. Since $F_1(z,q)$ is of the form $\mathrm{Id}+zG_1(z,q)$, it is sufficient to prove that the constant term of the entries of the first row of $\hat{H}(z,q)$ converges to the constant term of the corresponding entry of $\widetilde{H}(z)$. This is guaranteed by **(H6)**.

We can apply Lemma 4.4.1, which gives that the first row of $\hat{H}(z,q)F_1(z,q)$ converges entrywise and coefficientwise to the first row of $\widetilde{H}(z)$ when $q\to 1$.

Let us prove now the convergence of the other rows. Let $\hat{h}(z,q)$ be an entry of $\hat{H}(z,q)F_1(z,q)$ and let $\widetilde{h}(z)$ be the corresponding entry of $\widetilde{H}(z)$. Let $\hat{h}_1(z,q),\dots,\hat{h}_m(z,q)$ be the entries of the first row of $\hat{H}(z,q)F_1(z,q)$ and let $\widetilde{h}_1(z),\dots,\widetilde{h}_m(z)$ be the corresponding entries of $\widetilde{H}(z)$. From (4.8.4), we find that there exist $r\in\mathbb{N}$, $z\mapsto \Big(d_{i,j}(z,q)\Big)_{i\leq r,j\leq m}, \Big(\widetilde{d}_{i,j}\Big)_{i\leq r,j\leq m}\in\mathbb{C}((z))$, such that:

$$\left\{\begin{array}{lcl}\sum_{i,j} d_{i,j}(z,q)\delta_q^i\Big(\hat{h}_j(z,q)\Big) & = & \hat{h}(z,q)\\ \\ \sum_{i,j}\widetilde{d}_{i,j}(z)\delta^i\Big(\widetilde{h}_j(z)\Big) & = & \widetilde{h}(z),\end{array}\right. \tag{4.8.5}$$

and such that for all i,j, $d_{i,j}(z,q)$ converges entrywise to $\widetilde{d}_{i,j}(z)$ when $q\to 1$. The entrywise convergence of $\hat{h}(z,q)$ to $\widetilde{h}(z,q)$ when $q\to 1$ follows immediately from the case of the first row.

Using (4.8.5) and the fact that for all q close to 1, the z-valuation of the entry of the first row of $\hat{H}(z,q)F_1(z,q)$ are 0, we obtain the existence of $N'\in\mathbb{N}$, such that for all q close to 1, $z\mapsto z^{N'}\hat{h}(z,q)\in\mathbb{C}[[z]]$. We apply the same reasoning on the other entries of $\hat{H}(z,q)F_1(z,q)$ to conclude the existence of $N\in\mathbb{N}$, such that for q close to 1, $z\mapsto z^N\hat{H}(z,q)F_1(z,q)\in\mathrm{M}_m\Big(\mathbb{C}[[z]]\Big)$.

□

4.8.3 Confluence of the Stokes matrices

In this subsection, we combine Theorems 4.4.5 and 4.8.4, to prove the convergence of a basis of meromorphic solutions of a family of linear δ_q-equations to a basis of meromorphic solutions of the corresponding linear δ-equation. We consider the family of equations

$$\left\{\begin{array}{lcccccccc}\Delta_q & := & b_m(z,q)\delta_q^m & + & b_{m-1}(z,q)\delta_q^{m-1} & + & \dots & + & b_0(z,q)\\ \\ \widetilde{\Delta} & := & \widetilde{b}_m(z)\delta^m & + & \widetilde{b}_{m-1}(z)\delta^{m-1} & + & \dots & + & \widetilde{b}_0(z),\end{array}\right.$$

and assume that they satisfy the assumptions **(H2)** to **(H6)** of §4.8.2 and the two following assumptions:

(H1') For all $i\leq m$, $z\mapsto b_i(z,q),\widetilde{b}_i(z)\in\mathbb{C}[z]$.

(H7) Every entry $\hat{h}$ of the matrix $z^N\hat{H}(z,q)F_1(z,q)$ given by Theorem 4.8.4 (resp. every entry $\widetilde{h}$ of the matrix $z^N\widetilde{H}(z)$), satisfies a family of δ_q-equations (resp. δ-equation) that verifies the assumptions **(A2)** and **(A3)** detailed §4.4.2.

As in §4.8.2, we consider the associated systems:

$$\begin{cases} \delta_q Y(z,q) &= B(z,q)Y(z,q) \\ \\ \delta\widetilde{Y}(z) &= \widetilde{B}(z)\widetilde{Y}(z). \end{cases}$$

The next lemma gives a sufficient condition for the assumption **(H7)** to be satisfied. See Remark 4.8.1 for the discussion about the cases where the other assumptions are satisfied.

Lemma 4.8.7. *If the $b_i(z,q)$ are independent of q, and if **(H1')**, **(H2)** to **(H6)** hold, then **(H7)** is satisfied.*

Proof. The matrix $z^N\widetilde{H}(z)$ satisfies the equation

$$\delta\left(z^N\widetilde{H}(z)\right) = \widetilde{B}(z)z^N\widetilde{H}(z) - z^N\widetilde{H}(z)\left(\widetilde{N}(z) - N\times \mathrm{Id}\right),$$

where $\widetilde{N}(z) = \mathrm{Diag}\left(\widetilde{L}_i + \delta\widetilde{\lambda}_i(z)\times \mathrm{Id}_{m_i}\right)$ has entries in $\mathbb{C}[z^{-1}]$. From (4.8.4), we obtain

$$\begin{aligned} &\delta_q\left(z^N\hat{H}(z,q)F_1(z,q)\right)\left(\mathrm{Id}+(q-1)N(z,q)\right) \\ = \;& q^N\widetilde{B}(z)z^N\hat{H}(z,q)F_1(z,q) - z^N\hat{H}(z,q)F_1(z,q)\Big(q^N N(z,q) - [N]_q\times \mathrm{Id}\Big), \end{aligned} \tag{4.8.6}$$

where $N(z,q)$ converges to $\widetilde{N}(z)$. Let $\hat{h}(z,q)$ be an entry of $z^N\hat{H}(z,q)F_1(z,q)$ and let $\widetilde{h}(z)$ be the corresponding entry of $z^N\widetilde{H}(z)$. Using (4.8.6), we obtain the existence of $r\in\mathbb{N}^*$, $z\mapsto d_1(z,q),\dots,d_r(z,q),\widetilde{d}_1,\dots,\widetilde{d}_r\in\mathbb{C}[z]$, $c>0$, such that for all $i\leq r$, for all $q>1$ sufficiently close to 1, $\left|d_i(z,q)-\widetilde{d}_i(z)\right| < (q-1)c\left(\left|\widetilde{d}_i(z)\right|+1\right)$, and such that

$$\begin{cases} \sum_{i\leq r} d_i(z,q)\delta_q^i\Big(\hat{h}(z,q)\Big) &= 0 \\ \\ \sum_{i\leq r}\widetilde{d}_i(z)\delta^i\left(\widetilde{h}(z)\right) &= 0. \end{cases}$$

In particular, $\hat{h}$ satisfies the assumptions **(A1)** and **(A3)**, with formal limit the formal power series $\widetilde{h}(z)$.

Moreover, the z-valuations of the $b_i(z,q)$ are independent of q and are equal to the z-valuations of the $\widetilde{b}_i(z)$. Therefore, the z-valuations of the $d_i(z,q)$ are independent of q and are equal to the z-valuations of the $\widetilde{d}_i(z)$. Since the slopes of the equation depend only on the z-valuation, we obtain that $\hat{h}$ satisfies the assumption **(A2)**, with formal limit the formal power series $\widetilde{h}(z)$. □

We recall that if $\widetilde{D}(z)\in \mathrm{M}_m\Big(\mathbb{C}(z)\Big)$, we define $\mathbf{S}^1\left(\widetilde{D}(z)\right)$ as the union of the $\mathbb{R}_{\geq 1}x_i$, where x_i are the poles of $\widetilde{D}(z)$. Let $\Sigma_{\widetilde{H}}$ be the union of the $\Sigma_{\widetilde{h}_{i,j}}$, that have been defined in §4.6.2, Step 1, where $\widetilde{h}_{i,j}$ are the entries of $\widetilde{H}$. Due to **(H7)**, we may apply Theorem 4.4.5 to the divergent entries of $z^N\hat{H}(z,q)F_1(z,q)$ and $z^N\widetilde{H}(z)$. Using additionally

Remark 4.4.4, (2), and the reasoning in §4.6.2, Step 4, we may prove a similar result for the convergent entries, and we find the existence of $k \in \mathbb{N}^*$, such that for all $d \in \mathbb{R} \setminus \Sigma_{\widetilde{H}}$,

$$\lim_{q \to 1} \mathcal{S}_q^{[d]} \left(z^N \hat{H} F_1 \right) = \widetilde{\mathcal{S}}^d \left(z^N \widetilde{H} \right),$$

uniformly on the compacts of $\overline{S} \left(d - \frac{\pi}{2k}, d + \frac{\pi}{2k} \right) \setminus \mathbf{S}^1 \left(\widetilde{B}(z) \right)$. From Theorem 4.4.5 and Theorem 4.8.4, there exists $F_2(z,q) \in \mathcal{O}_m^*$, such that

$$\Phi_0^{[d]}(z,q) := z^{-N} \mathcal{S}_q^{[d]} \left(z^N \hat{H} F_1 \right) F_2(z,q) \mathrm{Diag}_i \left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i} \prod_{j=1}^{k_i} e_{q^j} \left(\widetilde{\lambda}_{i,j} z^{-j} \times \mathrm{Id}_{m_i} \right) \right)$$

$$\in \mathrm{GL}_m \Big(\mathcal{M}(\mathbb{C}^*, 0) \Big),$$

is a fundamental solution of $\delta_q Y(z,q) = B(z,q) Y(z,q)$. From §4.1, we recall that

$$\widetilde{\Phi}_0^d(z) := z^{-N} \widetilde{\mathcal{S}}^d \left(z^N \widetilde{H}(z) \right) \mathrm{Diag} \left(e^{\widetilde{L}_i \log(z) + \widetilde{\lambda}_i(z) \times \mathrm{Id}_{m_i}} \right) \in \mathcal{A} \left(d - \frac{\pi}{2k}, d + \frac{\pi}{2k} \right)$$

is a fundamental solution of $\delta \widetilde{Y}(z) = \widetilde{B}(z) \widetilde{Y}(z)$.

Lemma 4.8.8. *We have*

$$\lim_{q \to 1} \Phi_0^{[d]}(z,q) = \widetilde{\Phi}_0^d(z),$$

uniformly on the compacts of $\overline{S} \left(d - \frac{\pi}{2k}, d + \frac{\pi}{2k} \right) \setminus \left(\mathbf{S}^1 \left(\widetilde{B}(z) \right) \bigcup \{ \mathbb{R}_{<0} \} \right)$.

Proof. Due to the preceding discussion and the definition of $\mathcal{O}_m^*$, we only have to prove the convergence

$$\lim_{q \to 1} \mathrm{Diag}_i \left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i} \prod_{j=1}^{k_i} e_{q^j} \left(\widetilde{\lambda}_{i,j} z^{-j} \times \mathrm{Id}_{m_i} \right) \right) = \mathrm{Diag} \left(e^{\widetilde{L}_i \log(z) + \widetilde{\lambda}_i(z) \times \mathrm{Id}_{m_i}} \right).$$

The fact that

$$\lim_{q \to 1} \mathrm{Diag} \left(\Lambda_{q,\mathrm{Id}+(q-1)\widetilde{L}_i} \right) = \mathrm{Diag} \left(e^{\widetilde{L}_i \log(z)} \right),$$

uniformly on the compacts of a convenient domain has been proved in a more generalize case in Page 1048 of [Sau00]. See Lemma 4.6.1, for the convergence of the q-exponential part. □

Let $d^- < d^+$ with $d^\pm \in \mathbb{R} \setminus \Sigma_{\widetilde{H}}$, so that we can define $\Phi_0^{[d^\pm]}(z,q)$. We define the q-Stokes matrix $ST^{[d^-],[d^+]}(z,q) \in \mathrm{GL}_m \Big(\mathcal{M}_{\mathbb{E}} \Big)$ (we recall that $\mathcal{M}_{\mathbb{E}}$ is the field of functions invariant under the action of σ_q, see the introduction) as follows:

$$\Phi_0^{[d^+]}(z,q) = \Phi_0^{[d^-]}(z,q) ST^{[d^-],[d^+]}(z,q).$$

Let $d - \frac{\pi}{2k} < d^- < d < d^+ < d + \frac{\pi}{2k}$ such that

$$\Big([d^-, d[\bigcup]d, d^+] \Big) \bigcap \Sigma_{\widetilde{H}} = \varnothing.$$

Let us recall that by construction, $\Sigma_{\widetilde{H}}$ contains $\widetilde{\Sigma}_{\widetilde{H}}$, the set of singular directions that has been defined in Proposition 4.1.5. Therefore, following §4.1, we may define the Stokes matrix in the direction d, $\widetilde{ST}^d \in \mathrm{GL}_m(\mathbb{C})$, as follows:

$$\widetilde{\Phi_0}^{d^+}(z) = \widetilde{\Phi_0}^{d^-}(z) \widetilde{ST}^d.$$

Remark 4.8.9. If d is not a singular direction (see Proposition 4.1.5), then although $\widetilde{ST}^d = \text{Id}$, the entries of $\widetilde{H}(z)$ might be divergent. In fact, see [vdPS03] Page 247, the entries of $\widetilde{H}(z)$ are convergent if and only if $\widetilde{ST}^d = \text{Id}$ for all $d \in \mathbb{R}$. On the other hand, the principle of analytic continuation implies that if $ST^{[d^-],[d^+]}(z,q_0) = \text{Id}$ for some $d^- < d^+$ and for some $q_0 > 1$, then $z \mapsto z^N \hat{H}(z,q_0)F_1(z,q_0) \in \text{M}_m\Big(\mathbb{C}\{z\}\Big)$.

Using Lemma 4.8.8, we prove:

Theorem 4.8.10. *Let $d - \frac{\pi}{2k} < d^- < d < d^+ < d + \frac{\pi}{2k}$ such that*

$$\left(\Big[d^-, d\Big[\bigcup\Big]d, d^+\Big]\right)\bigcap \Sigma_{\widetilde{H}} = \varnothing.$$

Then, for q close to 1, we can define $ST^{[d^-],[d^+]}(z,q)$ and we have

$$\lim_{q\to 1} ST^{[d^-],[d^+]}(z,q) = \widetilde{ST}^d,$$

uniformly on the compacts of $\overline{S}\left(d - \frac{\pi}{2k}, d + \frac{\pi}{2k}\right) \setminus \left(\mathbf{S}^1\left(\widetilde{B}(z)\right)\bigcup\{\mathbb{R}_{<0}\}\right)$.

4.8.4 Confluence to the monodromy

In this subsection, we show how a basis of meromorphic solutions of a family of linear δ_q-equations at 0 and at ∞ can help us to find the monodromy matrices of the corresponding differential equation. We consider the family of equations

$$\begin{cases} \Delta_q := b_m(z,q)\delta_q^m + b_{m-1}(z,q)\delta_q^{m-1} + \ldots + b_0(z,q) \\ \widetilde{\Delta} := \widetilde{b}_m(z)\delta^m + \widetilde{b}_{m-1}(z)\delta^{m-1} + \ldots + \widetilde{b}_0(z), \end{cases}$$

that satisfies the assumptions **(H1')**,**(H2)** to **(H7)** of §4.8.2, §4.8.3 and the following assumptions:

(H8) The zeros of $\widetilde{b}_m(z)$ have different arguments and there is no zero which has an argument equal to π.

(H9) The assumptions **(H1')**,**(H2)** to **(H8)** are satisfied with the linear δ_q and δ-equation at infinity, obtained by considering $z \mapsto z^{-1}$.

As in §4.8.2, §4.8.3, we consider the associated systems:

$$\begin{cases} \delta_q Y(z,q) = B(z,q)Y(z,q) \\ \delta\widetilde{Y}(z) = \widetilde{B}(z)\widetilde{Y}(z). \end{cases}$$

Let $d \in \mathbb{R} \setminus \Sigma_{\widetilde{H}}$. Due to Lemma 4.8.8, there exists $k \in \mathbb{N}^*$ such that for we have

$$\lim_{q\to 1} \Phi_0^{[d]}(z,q) = \widetilde{\Phi}_0^d(z),$$

uniformly on the compacts of

$$\widetilde{\Omega}_0 := \overline{S}\left(d - \frac{\pi}{2k}, d + \frac{\pi}{2k}\right) \setminus \left(\mathbf{S}^1\left(\widetilde{B(z)}\right)\bigcup \mathbb{R}_{<0}\right).$$

We are now interested in the domain of definition of the fundamental solution $\Phi_0^{[d]}(z,q)$ for q close to 1 fixed. We recall that if $D(z) \in \mathrm{GL}_m\Big(\mathbb{C}(z)\Big)$, we define $\mathbf{S}^q(D(z))$ as the union of the $q^{\mathbb{N}^*}x_i$, where x_i is a pole of $D(z)$ or $D^{-1}(z)$. Following Page 1035 in [Sau00], we obtain that $\Lambda_{q,\mathrm{Id}+(q-1)\mathrm{Diag}\left(\widetilde{L}_i\right)}$ has poles contained in a finite number of q-discrete spiral of the form $q^{\mathbb{Z}}\beta_i(q)$, that converge to the spiral $\mathbb{R}_{<0}$ as q tends to 1. By construction, for q fixed, the domain of definition of the matrices $\mathcal{S}_q^{[d]}\left(z^N\hat{H}F_1\right)$, $F_2(z,q)$ and $\mathrm{Diag}_i\left(\displaystyle\prod_{j=1}^{k_i} e_{q^j}\left(\widetilde{\lambda}_{i,j}z^{-j}\times \mathrm{Id}_{m_i}\right)\right)$, intersected with $\overline{S}\left(d-\frac{\pi}{2k},d+\frac{\pi}{2k}\right)$ is $\overline{S}\left(d-\frac{\pi}{2k},d+\frac{\pi}{2k}\right)\setminus\left(\mathbf{S}^q\left(\mathrm{Id}+(q-1)\widetilde{B}(z)\right)\right)$. Notice that, $\mathbf{S}^q\left(\mathrm{Id}+(q-1)\widetilde{B}(z)\right)$ tends to $\mathbf{S}^1\left(\widetilde{B}(z)\right)$ as q goes to 1. We have proved that for q fixed close to 1, the domain of definition of $\Phi_0^{[d]}(z,q)$ intersected with $\overline{S}\left(d-\frac{\pi}{2k},d+\frac{\pi}{2k}\right)$ is:

$$\overline{S}\left(d-\frac{\pi}{2k},d+\frac{\pi}{2k}\right)\setminus\left(\mathbf{S}^q\left(\mathrm{Id}+(q-1)\widetilde{B}(z)\right)\bigcup q^{\mathbb{Z}}\beta_i(q)\right).$$

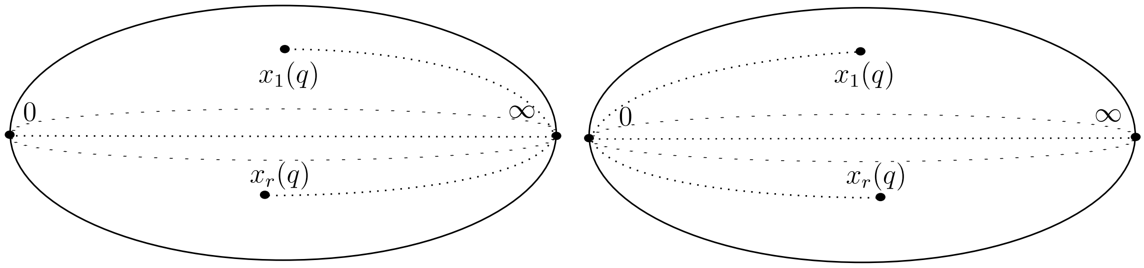


Figure 4.1: Intersection of $\overline{S}\left(d-\frac{\pi}{2k},d+\frac{\pi}{2k}\right)$ and the domain of definition of $\Phi_0^{[d]}(z,q)$ (left) and $\Phi_\infty^{[d]}(z,q)$ (right).

We consider now the singularity at ∞ putting $z\mapsto z^{-1}$. After taking a larger set finite modulo $2\pi\mathbb{Z}$, $\Sigma_{\widetilde{H}}\subset\mathbb{R}$, we may assume that for all $d\notin\Sigma_{\widetilde{h}}$, we can also compute a fundamental solution at infinity $\Phi_\infty^{[d]}(z,q)$ in the same way than $\Phi_0^{[d]}(z,q)$. Let $p=q^{-1}$. Similarly to $\widetilde{\Omega}_0$, let us define $\widetilde{\Omega}_\infty$, such that

$$\lim_{q\to1}\Phi_\infty^{[d]}(z,q)=\widetilde{\Phi}_\infty^d(z),$$

uniformly on the compacts of $\widetilde{\Omega}_\infty$, where $\widetilde{\Phi}_\infty^d(z)$ is the fundamental solution of the linear δ-system at infinity computed with Borel and Laplace transformations. More precisely, there exists $k'\in\mathbb{N}^*$, such that $\widetilde{\Omega}_\infty:=\overline{S}\left(d-\frac{\pi}{2k'},d+\frac{\pi}{2k'}\right)\setminus\left\{\mathbb{R}_{<0},t\widetilde{x}_1,\dots,t\widetilde{x}_r\Big|t\in]0,1]\right\}$, where the $\widetilde{x}_i$ satisfies $\widetilde{\Omega}_0=\overline{S}\left(d-\frac{\pi}{2k},d+\frac{\pi}{2k}\right)\setminus\{\mathbb{R}_{<0},\mathbb{R}_{\geq1}\widetilde{x}_1,\dots,\mathbb{R}_{\geq1}\widetilde{x}_r\}$. If we restrict the domain of convergence, we may assume that $k=k'$.

The Birkhoff matrix $\left(\Phi_\infty^{[d]}(z,q)\right)^{-1}\Phi_0^{[d]}(z,q)$ is invariant under the action of σ_q and tends to

$$\lim_{q\to1}\left(\Phi_\infty^{[d]}(z,q)\right)^{-1}\Phi_0^{[d]}(z,q)=\left(\widetilde{\Phi}_\infty^d(z)\right)^{-1}\widetilde{\Phi}_0^d(z)=:\widetilde{P}^d,$$

uniformly on the compacts of $\widetilde{\Omega}_\infty\cap\widetilde{\Omega}_0$.

Since $\left(\Phi_\infty^{[d]}(z,q)\right)^{-1}\Phi_0^{[d]}(z,q)$ is invariant under the action of σ_q, we obtain that $\widetilde{P}^d$ is locally constant.

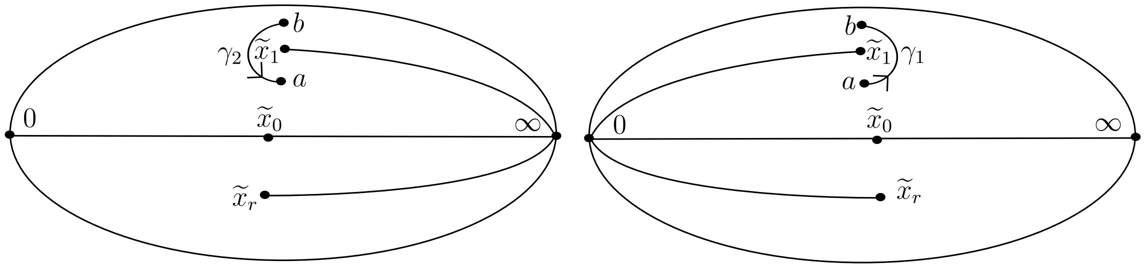


Figure 4.2: Domain of definition of $\widetilde{\Phi}_0^d(z)$ (left) and $\widetilde{\Phi}_\infty^d(z)$ (right).

Let $\widetilde{x}_0 = -1$. We order the $\widetilde{x}_i$ by increasing arguments in $]d - \frac{\pi}{2k}, d + \frac{\pi}{2k}[$. The connected component of the domain of definition of $\widetilde{P}^d$ are the $\widetilde{U}_j$, where

$$\widetilde{U}_j := \overline{S}\left(d - \frac{\pi}{2k}, d + \frac{\pi}{2k}\right) \bigcap \overline{S}\Big(\arg(\widetilde{x}_j), \arg(\widetilde{x}_{j+1})\Big).$$

Let $\widetilde{P}_j^d \in \mathrm{GL}_m(\mathbb{C})$ be the value of $\widetilde{P}^d$ in $\widetilde{U}_j$. Let us chose $\widetilde{x}_j$ such

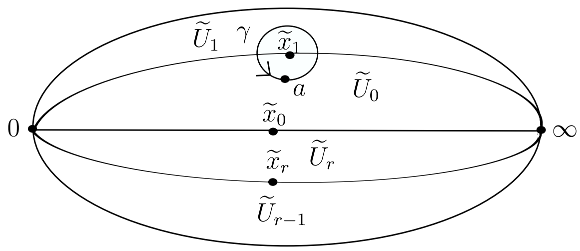


Figure 4.3: Domain of definition of $\widetilde{P}^d$.

that $\widetilde{x}_j \in \overline{S}\left(d - \frac{\pi}{2k}, d + \frac{\pi}{2k}\right)$. Let us consider a little positive path γ around $\widetilde{x}_j$ starting from $a \in \widetilde{U}_{j-1}$. We may choose γ such that we can decompose γ into γ_1 and γ_2 such that γ_1 comes from a to $b \in \widetilde{U}_j$ in $\widetilde{\Omega}_\infty$ and γ_2 comes from b to a in $\widetilde{\Omega}_0$. The analytic continuation along γ_1 transforms $\widetilde{\Phi}_0^d(z)$ into $\widetilde{\Phi}_\infty^d(z)\widetilde{P}_{j-1}^d$, and the analytic continuation along γ_2 transforms $\widetilde{\Phi}_\infty^d(z)$ into $\widetilde{\Phi}_0^d(z)\left(\widetilde{P}_j^d\right)^{-1}$. We have proved the following theorem, which extends when q is real, the theorem of the §4 in [Sau00] in the non-Fuchsian case:

Theorem 4.8.11. *The monodromy matrix of the δ-equation $\delta\widetilde{Y}(z) = \widetilde{B}(z)\widetilde{Y}(z)$ in the basis $\widetilde{\Phi}_0^d(z)$ around the singularity $\widetilde{x}_j$ is $\left(\widetilde{P}_j^d\right)^{-1}\widetilde{P}_{j-1}^d$.*

Annexe B

Appendice du Chapitre 4 : Confluence of a "continuous" q-Borel-Laplace summation.

The goal of this appendix is to prove the equivalent of Theorem 4.4.5 for a "continuous" q-Borel-Laplace summation. We introduce now the "continuous" q-Laplace transformation. See §4.3 for the notations.

Definition B.1. Let $k \in \mathbb{Q}_{>0}$ and let $d \in \mathbb{R}$. As we can see in [DVZ09], §4.2, the following maps are defined and we call them the "continuous" q-Laplace transformation of order 1 and k:

$$\begin{array}{rccl}\mathcal{L}_{q,1}^{d}: & \mathbb{H}_{q,1}^{d} & \longrightarrow & \mathcal{A}(d-\pi, d+\pi) \\ & f & \longmapsto & \frac{q-1}{\log(q)} \displaystyle\int_{0}^{\infty e^{id}} \frac{f(\zeta)}{z e_q\left(\frac{q\zeta}{z}\right)} d\zeta,\end{array}$$

$$\begin{array}{rccl}\mathcal{L}_{q,k}^{d}: & \mathbb{H}_{q,k}^{d} & \longrightarrow & \displaystyle\bigcup_{\nu=0}^{k-1} \mathcal{A}\left(\frac{2\pi\nu(d-\pi)}{k}, \frac{2\pi(\nu+1)(d-\pi)}{k}\right) \\ & g & \longmapsto & \rho_k \circ \mathcal{L}_{q,1}^{[d]} \circ \rho_{1/k}(g).\end{array}$$

Remark B.2. We say that the q-Laplace transformation is "continuous" because it is defined with a "continuous" integral, in opposition to the q-Laplace transformation of §4.3, which involves a "discrete" Jackson integral. Notice that the term "continuous" q-Borel-Laplace summation is an abuse of language since the q-Borel transformation we use in this summation process is the same as in the "discrete" q-Borel-Laplace summation.

Theorem 4.14 of [DVZ09] compares the "discrete" and the "continuous" q-Borel-Laplace summation for the case of formal power series solutions of a linear σ_q-equation with coefficients in $\mathbb{C}(\{z\})$ with only slope 1. The next proposition is the analogue of Proposition 4.3.4 of the present chapter.

Proposition B.3. *Let* $g \in \mathbb{H}_{q,1}^{d}$. *Then*

– $\mathcal{L}_{q,1}^{d}\left(\delta_q g\right) = \delta_q \mathcal{L}_{q,1}^{d}\left(g\right)$.

– $z\mathcal{L}^d_{q,1}\left(\delta_q g\right) = p\mathcal{L}^d_{q,1}(\zeta g) - pz\mathcal{L}^d_{q,1}(g)$.

Proof. To prove the first equality, it is sufficient to prove that the "continuous" q-Laplace transformation commutes with σ_q. To do this, we just have to perform the variable change $\zeta \mapsto q\zeta$ in the integral.

Let us prove the last equality. We recall that $\sigma_q\left(e_q\left(\frac{q\zeta}{z}\right)\right) = \frac{e_q\left(\frac{q\zeta}{z}\right)}{1+(q-1)\zeta/z}$. Let $p = 1/q$. Then,

$$\begin{aligned} z\mathcal{L}^d_{q,1}(\delta_q g) &= z\int_0^{\infty e^{id}} \frac{g(\zeta)}{ze_q\left(\frac{q\zeta}{z}\right)} \frac{p-1+\frac{(q-1)\zeta}{qz}}{q-1} d\zeta \\ &= \int_0^{\infty e^{id}} \frac{g(\zeta)}{e_q\left(\frac{q\zeta}{z}\right)}(-p+p\zeta/z) \\ &= p\mathcal{L}^d_{q,1}(\zeta g) - pz\mathcal{L}^d_{q,1}(g). \end{aligned}$$

□

Let $k \in \mathbb{N}^*$. If we consider $\hat{f} \in \mathbb{C}\left[\left[z^k\right]\right]$, solution of a linear δ_q-equation with coefficients in $\mathbb{C}\left[z^k\right]$, with $\hat{\mathcal{B}}_{q,k}\left(\hat{f}\right) \in \mathbb{H}^d_{q,k}$, then we have:

$$\delta_q\left(\mathcal{L}^d_{q,k}\circ\hat{\mathcal{B}}_{q,k}\left(\hat{f}\right)\right) = \mathcal{L}^d_{q,k}\circ\hat{\mathcal{B}}_{q,k}\left(\delta_q\hat{f}\right) \text{ and } \delta_q\left(z^k\mathcal{L}^d_{q,k}\circ\hat{\mathcal{B}}_{q,k}\left(\hat{f}\right)\right) = \mathcal{L}^d_{q,k}\circ\hat{\mathcal{B}}_{q,k}\left(\delta_q\left(z^k\hat{f}\right)\right).$$

Hence, $\mathcal{L}^d_{q,k}\circ\hat{\mathcal{B}}_{q,k}\left(\hat{f}\right)$ is solution of the same linear δ_q-equation than $\hat{f}$. But in general, if $\hat{f} \in \mathbb{C}[[z]]$ is solution of a linear δ_q-equation with coefficients in $\mathbb{C}[z]$, we will have to apply successively several q-Borel and "continuous" q-Laplace transformations in order to compute an analytic solution of the same equation than $\hat{f}$. See Theorem B.4.

As in §4.4.2, let $z \mapsto \hat{h}(z,q) \in \mathbb{C}[[z]]$ that converges coefficientwise to $\widetilde{h}(z) \in \mathbb{C}[[z]]$ when $q \to 1$. We make the following assumptions:

– There exists
$$z \mapsto b_0(z,q),\dots,b_m(z,q) \in \mathbb{C}[z],$$
with z-coefficients that converge as q goes to 1, such that for all q close to 1, $\hat{h}(z,q)$ is solution of:
$$b_m(z,q)\delta_q^m\left(\hat{h}(z,q)\right) + \dots + b_0(z,q)\hat{h}(z,q) = 0.$$
Let $\widetilde{b}_0(z),\dots,\widetilde{b}_m(z) \in \mathbb{C}[z]$ be the limit as q tends to 1 of the $b_0(z,q),\dots,b_m(z,q)$. Notice that the series $\widetilde{h}(z)$ is solution of:
$$\widetilde{b}_m(z)\delta^m\left(\widetilde{h}(z)\right) + \dots + \widetilde{b}_0(z)\widetilde{h}(z) = 0.$$

– For q close to 1, the slopes of the linear q-difference equation satisfied by $\hat{h}$ are independent of q, and the set of slopes of the latter that are positive coincides with the set of slopes of the linear differential equation satisfied by $\widetilde{h}$.
– There exists $c_1 > 0$, such that for all $i \leq m$ and q close to 1:
$$\left|b_i(z,q) - \widetilde{b}_i(z)\right| < (q-1)c_1\left(\left|\widetilde{b}_i(z)\right| + 1\right).$$

– The differential equation has at least one slope strictly bigger than 0.

Let $d_0 := \max\left(2, \deg\left(\widetilde{b}_0\right), \dots, \deg\left(\widetilde{b}_m\right)\right)$. Let $k_1 < \dots < k_{r-1}$ be the slopes of (4.4.3) different from 0, let k_r be an integer strictly bigger than k_{r-1} and d_0, and set $k_{r+1} := +\infty$. Let $(\kappa_1, \dots, \kappa_r)$ defined as:

$$\kappa_i^{-1} := k_i^{-1} - k_{i+1}^{-1}.$$

As in Proposition 4.1.5, we define the $(\widetilde{\kappa_1}, \dots, \widetilde{\kappa_s})$ as follows: we take $(\kappa_1, \dots, \kappa_r)$ and for $i = 1, ..., i = r$, we replace successively κ_i by α_i terms $\alpha_i \kappa_i$, where α_i is the smallest integer such that $\alpha_i \kappa_i$ is greater or equal than d_0. See Example 4.1.4. Therefore, by construction, each of the $\widetilde{\kappa_i}$ are rational number greater than d_0. Let $\beta \in \mathbb{N}^*$ be minimal, such that for all $i \in \{1, \dots, s\}$, $\beta/\widetilde{\kappa_i} \in \mathbb{N}^*$. Let us write $\hat{h}(z,q) =: \sum_{n=0}^{\infty} \hat{h}_n(q) z^n$ and for $l \in \{0, \dots, \beta-1\}$, let $\hat{h}^{(l)}(z,q) := \sum_{n=0}^{\infty} \hat{h}_{l+n\beta}(q) z^{n\beta}$.

Theorem B.4. *There exists $\Sigma_{\widetilde{h}} \subset \mathbb{R}$ finite modulo $2\pi\mathbb{Z}$, such that if $d \in \mathbb{R} \setminus \Sigma_{\widetilde{h}}$ then for all $l \in \{0, \dots, \beta-1\}$, the series $g_{1,l} := \hat{\mathcal{B}}_{q,\widetilde{\kappa_1}} \circ \dots \circ \hat{\mathcal{B}}_{q,\widetilde{\kappa_s}}\left(\hat{h}^{(l)}\right)$ converges and belongs to $\overline{\mathbb{H}}^d_{\widetilde{\kappa_1}}$ (see Definition 4.4.3).*

Moreover, for $j = 2$ (resp. $j = 3$, …, resp. $j = r$), $g_{j,l} := \mathcal{L}^d_{q,\widetilde{\kappa_{j-1}}}(g_{j-1,l})$ belongs to $\overline{\mathbb{H}}^d_{\widetilde{\kappa_j}}$. Let $S^d_q\left(\hat{h}^{(l)}\right) := \mathcal{L}^d_{q,\widetilde{\kappa_s}}(g_{r,l})$. The function

$$S^d_q\left(\hat{h}\right) := \sum_{l=0}^{\beta-1} z^l S^d_q\left(\hat{h}^{(l)}\right) \in \mathcal{A}\left(d - \frac{\pi}{k_r}, d + \frac{\pi}{k_r}\right),$$

is solution of (4.4.2). Furthermore, we have

$$\lim_{q \to 1} S^d_q\left(\hat{h}\right) = \widetilde{S}^d\left(\widetilde{h}\right),$$

uniformly on the compacts of $\overline{S}\left(d - \frac{\pi}{2k_r}, d + \frac{\pi}{2k_r}\right) \setminus \bigcup \mathbb{R}_{\geq 1}\alpha_i$, where α_i are the roots of $\widetilde{b}_m \in \mathbb{C}[z]$ and $\widetilde{S}^d\left(\widetilde{h}\right)$ is the asymptotic solution of the same linear δ-equation than $\widetilde{h}$ that has been defined in Proposition 4.1.5.

The proof of this theorem is basically the same as the proof of Theorem 4.4.5. The only difference is that we can not use Lemma 4.6.2, so we state and prove a similar result for the "continuous" summation.

Let $d \in \mathbb{R}$, let $k \in \mathbb{Q}_{>0}$ and let f be a function that belongs to $\overline{\mathbb{H}}^d_k$. By definition (see Definition 4.4.3), there exist $\varepsilon > 0$, constants $J, L > 0$, such that for all q close to 1, $\zeta \mapsto f(\zeta, q)$ is analytic on $\overline{S}(d-\varepsilon, d+\varepsilon)$, and for all $\zeta \in \mathbb{R}_{>0}$:

$$\left|f(e^{id}\zeta, q)\right| < J e_q\left(L\zeta^k\right).$$

Lemma B.5. *In the notations introduced above, let us assume that $\lim_{q \to 1} f := \widetilde{f} \in \widetilde{\mathbb{H}}^d_k$ uniformly on the compacts of $\overline{S}(d-\varepsilon, d+\varepsilon)$. Then, we have*

$$\lim_{q \to 1} \mathcal{L}^d_{q,k}\left(f\right)(z) = \mathcal{L}^d_k\left(\widetilde{f}\right)(z),$$

uniformly on the compacts of $\left\{z \in \overline{S}\left(d - \frac{\pi}{2k\pi}, d + \frac{\pi}{2k\pi}\right) \middle| |z| < 1/L\right\}$.

Proof. For the same reasons as in the proof of Lemma 4.6.2, we may assume that $d=0$ and $k=1$.

Let us fix a compact K of $\left\{z\in \overline{S}\left(-\frac{\pi}{2\pi},+\frac{\pi}{2\pi}\right)\Big| |z|<1/L\right\}$. Using the dominated convergence theorem, it is sufficient to prove the existence of a positive integrable function h, such that for all q close to 1, $\zeta\in\mathbb{R}_{>0}$ and $z\in K$, $\left|\frac{f(\zeta,q)}{ze_q\left(\frac{q\zeta}{z}\right)}\right|<h(\zeta)$.

Let $J>0$ be the constant that comes from Definition 4.4.3 and let $z\in K$. By definition of the "continuous" q-Laplace transformation

$$\left|\mathcal{L}^d_{q,k}(f)(z)\right|\leq \int_0^{\infty}\left|\frac{J}{z}\frac{e_q(L\zeta)}{e_q(q\zeta/z)}\right|d\zeta.$$

Let us fix $q_0>1$. Let $S\in\mathbb{R}$, such that for all $z\in K$, $q>1$ and $\zeta>S$, $\zeta\mapsto\left|\frac{J}{z}\frac{e_q(L\zeta)}{e_q(q\zeta/z)}\right|$ is decreasing. The convergence $\lim\limits_{q\to 1}\int_0^S \frac{f(\zeta,q)}{ze_q\left(\frac{q\zeta}{z}\right)}d\zeta=\int_0^S\frac{f(\zeta)}{z\exp\left(\frac{q\zeta}{z}\right)}d\zeta$ is clear. Moreover, we have for all $q\in]1,q_0[$ and $z\in K$:

$$\int_S^{\infty}\left|\frac{J}{z}\frac{e_q(L\zeta)}{e_q(q\zeta/z)}\right|d\zeta\leq (q-1)\sum_{l=0}^{\infty}\left|\frac{q^lSJ}{z}\frac{e_q\left(Lq^lS\right)}{e_q\left(q^{l+1}S/z\right)}\right|.$$

We have seen in the proof of Lemma 4.6.2, than we can bound this latter quantity uniformly in q and $z\in K$. This yields the result. □

Bibliographie

[AAR99] George E. Andrews, Richard Askey et Ranjan Roy : *Special functions*, volume 71 de *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[AB94] D. V. Anosov et A. A. Bolibruch : *The Riemann-Hilbert problem*. Aspects of Mathematics, E22. Friedr. Vieweg & Sohn, Braunschweig, 1994.

[Ada31] C. R. Adams : Linear q-difference equations. *Bull. Amer. Math. Soc.*, 37(6):361–400, 1931.

[Ada29] C. R. Adams : On the linear ordinary q-difference equation. *Ann. of Math. (2)*, 30(1-4):195–205, 1928/29.

[AH09] Primitivo B Acosta-Humanez : *Galoisian Approach to Supersymmetric Quantum Mechanics, Phd Dissertation*. Thèse de doctorat, Universitat Politecnica de Catalunya, 2009.

[AHMRW11] Primitivo B. Acosta-Humánez, Juan J. Morales-Ruiz et Jacques-Arthur Weil : Galoisian approach to integrability of Schrödinger equation. *Rep. Math. Phys.*, 67(3):305–374, 2011.

[AMBSW11] Ainhoa Aparicio-Monforte, Moulay Barkatou, Sergi Simon et Jacques-Arthur Weil : Formal first integrals along solutions of differential systems I. *In ISSAC 2011—Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation*, pages 19–26. ACM, New York, 2011.

[AMCW13] Ainhoa Aparicio-Monforte, Elie Compoint et Jacques-Arthur Weil : A characterization of reduced forms of linear differential systems. *J. Pure Appl. Algebra*, 217(8):1504–1516, 2013.

[And00] Yves André : Séries Gevrey de type arithmétique. II. Transcendance sans transcendance. *Ann. of Math. (2)*, 151(2):741–756, 2000.

[And01] Yves André : Différentielles non commutatives et théorie de Galois différentielle ou aux différences. *Ann. Sci. École Norm. Sup. (4)*, 34(5):685–739, 2001.

[Arr12] Carlos E. Arreche : Computing the differential galois group of a one-parameter family of second order linear differential equations. *Preprint available on arkiv*, 2012.

[Bal94] Werner Balser : *From divergent power series to analytic functions*, volume 1582 de *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994. Theory and application of multisummable power series.

[Bal00] Werner Balser : *Formal power series and linear systems of meromorphic ordinary differential equations*. Universitext. Springer-Verlag, New York, 2000.

[Ban80] Steven B. Bank : Some results on hypertranscendental meromorphic functions. *Monatsh. Math.*, 90(4):267–289, 1980.

[Bar08] E. W. Barnes : A New Development of the Theory of the Hypergeometric Functions. *Proc. London Math. Soc.*, S2-6(1):141, 1908.

[BD94] A. Barkatou et A. Duval : Sur les séries formelles solutions d'équations aux différences polynomiales. *Ann. Inst. Fourier (Grenoble)*, 44(2):495–524, 1994.

[BD97] Moulay A. Barkatou et Anne Duval : Sur la somme de certaines séries de factorielles. *Ann. Fac. Sci. Toulouse Math. (6)*, 6(1):7–58, 1997.

[Bea93] Arnaud Beauville : Monodromie des systèmes différentiels linéaires à pôles simples sur la sphère de Riemann (d'après A. Bolibruch). *Astérisque*, (216):Exp. No. 765, 4, 103–119, 1993. Séminaire Bourbaki, Vol. 1992/93.

[Ber92] Daniel Bertrand : Groupes algébriques et équations différentielles linéaires. *Astérisque*, (206):Exp. No. 750, 4, 183–204, 1992. Séminaire Bourbaki, Vol. 1991/92.

[Béz92] Jean-Paul Bézivin : Sur les équations fonctionnelles aux q-différences. *Aequationes Math.*, 43(2-3):159–176, 1992.

[Bir30] George D. Birkhoff : Formal theory of irregular linear difference equations. *Acta Math.*, 54(1):205–246, 1930.

[BJL80] W. Balser, W. B. Jurkat et D. A. Lutz : A general theory of invariants for meromorphic differential equations. III. Applications. *Houston J. Math.*, 6(2):149–189, 1980.

[Blu77] Lenore Blum : Differentially closed fields : a model-theoretic tour. *In Contributions to algebra (collection of papers dedicated to Ellis Kolchin)*, pages 37–61. Academic Press, New York, 1977.

[BMM06] A. A. Bolibruch, S. Malek et C. Mitschi : On the generalized Riemann-Hilbert problem with irregular singularities. *Expo. Math.*, 24(3):235–272, 2006.

[Bol97] A. A. Bolibruch : On isomonodromic deformations of Fuchsian systems. *J. Dynam. Control Systems*, 3(4):589–604, 1997.

[Bug11] Virginie Bugeaud : Classification analytique et théorie de Galois locales des modules aux q-différences à pentes non entières. *C. R. Math. Acad. Sci. Paris*, 349(19-20):1037–1039, 2011.

[Bug12] Virginie Bugeaud : *Groupe de Galois local des équations aux q-différences irrégulières*. Thèse de doctorat, Institut de Mathématiques de Toulouse, 2012.

[Bui97] Alexandru Buium : Differential algebraic geometry and Diophantine geometry : an overview. *In Arithmetic geometry (Cortona, 1994)*, Sympos. Math., XXXVII, pages 87–98. Cambridge Univ. Press, Cambridge, 1997.

[BV85] Donald G. Babbitt et V. S. Varadarajan : Deformations of nilpotent matrices over rings and reduction of analytic families of meromorphic differential equations. *Mem. Amer. Math. Soc.*, 55(325):iv+147, 1985.

[Car12] R. D. Carmichael : The General Theory of Linear q-Difference Equations. *Amer. J. Math.*, 34(2):147–168, 1912.

[Cas72] P. J. Cassidy : Differential algebraic groups. *Amer. J. Math.*, 94:891–954, 1972.

[Cas89] Phyllis Joan Cassidy : The classification of the semisimple differential algebraic groups and the linear semisimple differential algebraic Lie algebras. *J. Algebra*, 121(1):169–238, 1989.

[Cas07] Guy Casale : The Galois groupoid of Picard-Painlevé VI equation. *In Algebraic, analytic and geometric aspects of complex differential equations and their deformations. Painlevé hierarchies*, RIMS Kôkyûroku Bessatsu, B2, pages 15–20. Res. Inst. Math. Sci. (RIMS), Kyoto, 2007.

[Cas09] Guy Casale : Une preuve galoisienne de l'irréductibilité au sens de Nishioka-Umemura de la première équation de Painlevé. *Astérisque*, (323): 83–100, 2009.

[CHS08] Zoé Chatzidakis, Charlotte Hardouin et Michael F. Singer : On the definitions of difference Galois groups. *In Model theory with applications to algebra and analysis. Vol. 1*, volume 349 de *London Math. Soc. Lecture Note Ser.*, pages 73–109. Cambridge Univ. Press, Cambridge, 2008.

[Chy98] F Chyzak : *Fonctions holonomes en calcul formel, Thèse de doctorat,*. Thèse de doctorat, Ecole polytechnique, 1998.

[CL09] Serge Cantat et Frank Loray : Dynamics on character varieties and Malgrange irreducibility of Painlevé VI equation. *Ann. Inst. Fourier (Grenoble)*, 59(7):2927–2978, 2009.

[Com13] Thierry Combot : Integrability conditions at order 2 for homogeneous potentials of degree -1. *Nonlinearity*, 26(1):95–120, 2013.

[CR] J Cano et Jean-Pierre. Ramis : Théorie de galois différentielle, multisommabilité et phénomène de stokes. Preprint.

[CS99] Elie Compoint et Michael F. Singer : Computing Galois groups of completely reducible differential equations. *J. Symbolic Comput.*, 28(4-5):473–494, 1999. Differential algebra and differential equations.

[CS07] Phyllis J. Cassidy et Michael F. Singer : Galois theory of parameterized differential equations and linear differential algebraic groups. *In Differential equations and quantum groups*, volume 9 de *IRMA Lect. Math. Theor. Phys.*, pages 113–155. Eur. Math. Soc., Zürich, 2007.

[Del70] Pierre Deligne : *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin, 1970.

[Del90] P. Deligne : Catégories tannakiennes. *In The Grothendieck Festschrift, Vol. II*, volume 87 de *Progr. Math.*, pages 111–195. Birkhäuser Boston, Boston, MA, 1990.

[DLR92] Anne Duval et Michèle Loday-Richaud : Kovačič's algorithm and its application to some families of special functions. *Appl. Algebra Engrg. Comm. Comput.*, 3(3):211–246, 1992.

[DMR07] Pierre Deligne, Bernard Malgrange et Jean-Pierre Ramis : *Singularités irrégulières*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 5. Société Mathématique de France, Paris, 2007. Correspondance et documents. [Correspondence and documents].

[DR08] Anne Duval et Julien Roques : Familles fuchsiennes d'équations aux (q-)différences et confluence. *Bull. Soc. Math. France*, 136(1):67–96, 2008.

[Dre12] T. Dreyfus : Computing the galois group of some parameterized linear differential equation of order two. *To appear in Proceedings of the American Mathematical Society*, 2012.

[Dre13] T. Dreyfus : A density theorem for parameterized differential galois theory. *To appear in Pacific Journal of Mathematics*, 2013.

[Dre14] T. Dreyfus : Confluence of meromorphic solutions of q-difference equations. *Preprint available on arkiv*, 2014.

[DSK05] Alberto De Sole et Victor G. Kac : On integral representations of q-gamma and q-beta functions. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 16(1):11–29, 2005.

[DV02] Lucia Di Vizio : Arithmetic theory of q-difference equations : the q-analogue of Grothendieck-Katz's conjecture on p-curvatures. *Invent. Math.*, 150(3): 517–578, 2002.

[DV09] Lucia Di Vizio : Local analytic classification of q-difference equations with $|q| = 1$. *J. Noncommut. Geom.*, 3(1):125–149, 2009.

[DVH12] Lucia Di Vizio et Charlotte Hardouin : Descent for differential Galois theory of difference equations : confluence and q-dependence. *Pacific J. Math.*, 256(1):79–104, 2012.

[DVRSZ03] L. Di Vizio, J.-P. Ramis, J. Sauloy et C. Zhang : Équations aux q-différences. *Gaz. Math.*, (96):20–49, 2003.

[DVZ09] Lucia Di Vizio et Changgui Zhang : On q-summation and confluence. *Ann. Inst. Fourier (Grenoble)*, 59(1):347–392, 2009.

[Eca] J. Ecalle. : Résurgence et accélération. Cours.

[Éca81] Jean Écalle : *Les fonctions résurgentes. Tome I*, volume 5 de *Publications Mathématiques d'Orsay 81 [Mathematical Publications of Orsay 81]*. Université de Paris-Sud Département de Mathématique, Orsay, 1981. Les algèbres de fonctions résurgentes. [The algebras of resurgent functions], With an English foreword.

[FRJT09] F. Fauvet, F. Richard-Jung et J. Thomann : Automatic computation of Stokes matrices. *Numer. Algorithms*, 50(2):179–213, 2009.

[FRRJT10] F. Fauvet, J.-P. Ramis, F. Richard-Jung et J. Thomann : Stokes phenomenon for the prolate spheroidal wave equation. *Appl. Numer. Math.*, 60(12):1309–1319, 2010.

[GGO13] Henri Gillet, Sergey Gorchinskiy et Alexey Ovchinnikov : Parameterized Picard–Vessiot extensions and Atiyah extensions. *Adv. Math.*, 238:322–411, 2013.

[GO12] Sergey Gorchinskiy et Alexey Ovchinnikov : Isomonodromic differential equations and differential tannakian categories. *To appear in Journal de Mathématiques Pures et Appliquées*, 2012.

[GR04] George Gasper et Mizan Rahman : *Basic hypergeometric series*, volume 96 de *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second édition, 2004. With a foreword by Richard Askey.

[Gra11] Anne Granier : A Galois D-groupoid for q-difference equations. *Ann. Inst. Fourier (Grenoble)*, 61(4):1493–1516 (2012), 2011.

[Har10] Charlotte Hardouin : Iterative q-difference Galois theory. *J. Reine Angew. Math.*, 644:101–144, 2010.

[Hes01] Sabrina A. Hessinger : Computing the Galois group of a linear differential equation of order four. *Appl. Algebra Engrg. Comm. Comput.*, 11(6):489–536, 2001.

[Heu09] Viktoria Heu : Stability of rank 2 vector bundles along isomonodromic deformations. *Math. Ann.*, 344(2):463–490, 2009.

[Hru02] Ehud Hrushovski : Computing the Galois group of a linear differential equation. *In Differential Galois theory (Będlewo, 2001)*, volume 58 de *Banach Center Publ.*, pages 97–138. Polish Acad. Sci., Warsaw, 2002.

[HS08] Charlotte Hardouin et Michael F. Singer : Differential Galois theory of linear difference equations. *Math. Ann.*, 342(2):333–377, 2008.

[HvdP95] Peter A. Hendriks et Marius van der Put : Galois action on solutions of a differential equation. *J. Symbolic Comput.*, 19(6):559–576, 1995.

[KM97] Andreas Kriegl et Peter W. Michor : *The convenient setting of global analysis*, volume 53 de *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[Kol73] E. R. Kolchin : *Differential algebra and algebraic groups*. Academic Press, New York, 1973. Pure and Applied Mathematics, Vol. 54.

[Kol74] E. R. Kolchin : Constrained extensions of differential fields. *Advances in Math.*, 12:141–170, 1974.

[Kol85] E. R. Kolchin : *Differential algebraic groups*, volume 114 de *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1985.

[Kov86] Jerald J. Kovacic : An algorithm for solving second order linear homogeneous differential equations. *J. Symbolic Comput.*, 2(1):3–43, 1986.

[Lan08] Peter Landesman : Generalized differential Galois theory. *Trans. Amer. Math. Soc.*, 360(8):4441–4495, 2008.

[Lor02] Frank Loray : Towards the Galois groupoid of nonlinear O.D.E. *In Differential equations and the Stokes phenomenon*, pages 203–275. World Sci. Publ., River Edge, NJ, 2002.

[Lor06] Frank Loray : A preparation theorem for codimension-one foliations. *Ann. of Math. (2)*, 163(2):709–722, 2006.

[LR90] Michèle Loday-Richaud : Introduction à la multisommabilité. *Gaz. Math.*, (44):41–63, 1990.

[LR94] Michèle Loday-Richaud : Stokes phenomenon, multisummability and differential Galois groups. *Ann. Inst. Fourier (Grenoble)*, 44(3):849–906, 1994.

[LR95] Michèle Loday-Richaud : Solutions formelles des systèmes différentiels linéaires méromorphes et sommation. *Exposition. Math.*, 13(2-3):116–162, 1995.

[LR01] Michèle Loday-Richaud : Rank reduction, normal forms and Stokes matrices. *Expo. Math.*, 19(3):229–250, 2001.

[LRR11] Michèle Loday-Richaud et Pascal Remy : Resurgence, Stokes phenomenon and alien derivatives for level-one linear differential systems. *J. Differential Equations*, 250(3):1591–1630, 2011.

[Mag94] Andy R. Magid : *Lectures on differential Galois theory*, volume 7 de *University Lecture Series*. American Mathematical Society, Providence, RI, 1994.

[Mal83] B. Malgrange : Sur les déformations isomonodromiques. II. Singularités irrégulières. *In Mathematics and physics (Paris, 1979/1982)*, volume 37 de *Progr. Math.*, pages 427–438. Birkhäuser Boston, Boston, MA, 1983.

[Mal91] B. Malgrange : *Équations différentielles à coefficients polynomiaux*, volume 96 de *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1991.

[Mal95] Bernard Malgrange : Sommation des séries divergentes. *Exposition. Math.*, 13(2-3):163–222, 1995.

[Mal02] B. Malgrange : On nonlinear differential Galois theory. *Chinese Ann. Math. Ser. B*, 23(2):219–226, 2002. Dedicated to the memory of Jacques-Louis Lions.

[Mal12] Stéphane Malek : On the parametric Stokes phenomenon for solutions of singularly perturbed linear partial differential equations. *Abstr. Appl. Anal.*, pages Art. ID 930385, 86, 2012.

[Mar00] David Marker : Model theory of differential fields. *In Model theory, algebra, and geometry*, volume 39 de *Math. Sci. Res. Inst. Publ.*, pages 53–63. Cambridge Univ. Press, Cambridge, 2000.

[McG00] Tracey McGrail : The model theory of differential fields with finitely many commuting derivations. *J. Symbolic Logic*, 65(2):885–913, 2000.

[Mez11] Marc Mezzarobba : *Autour de l'évaluation numérique des fonctions D-finies. Thèse de doctorat.* Thèse de doctorat, Ecole polytechnique, 2011.

[Mit96] Claude Mitschi : Differential Galois groups of confluent generalized hypergeometric equations : an approach using Stokes multipliers. *Pacific J. Math.*, 176(2):365–405, 1996.

[MO11] Andrey Minchenko et Alexey Ovchinnikov : Zariski closures of reductive linear differential algebraic groups. *Adv. Math.*, 227(3):1195–1224, 2011.

[Mor13] Takeshi Morita : A connection formula of the q-confluent hypergeometric function. Preprint, 2013.

[MR91] Jean Martinet et Jean-Pierre Ramis : Elementary acceleration and multisummability. I. *Ann. Inst. H. Poincaré Phys. Théor.*, 54(4):331–401, 1991.

[MR92] B. Malgrange et J.-P. Ramis : Fonctions multisommables. *Ann. Inst. Fourier (Grenoble)*, 42(1-2):353–368, 1992.

[MR99] Juan J. Morales Ruiz : *Differential Galois theory and non-integrability of Hamiltonian systems*, volume 179 de *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[MRR01a] Juan J. Morales-Ruiz et Jean Pierre Ramis : Galoisian obstructions to integrability of Hamiltonian systems. I, II. *Methods Appl. Anal.*, 8(1):33–95, 97–111, 2001.

[MRR01b] Juan J. Morales-Ruiz et Jean Pierre Ramis : A note on the non-integrability of some Hamiltonian systems with a homogeneous potential. *Methods Appl. Anal.*, 8(1):113–120, 2001.

[MRR10] Juan J. Morales-Ruiz et Jean-Pierre Ramis : Integrability of dynamical systems through differential Galois theory : a practical guide. *In Differential algebra, complex analysis and orthogonal polynomials*, volume 509 de *Contemp. Math.*, pages 143–220. Amer. Math. Soc., Providence, RI, 2010.

[MRRS07] Juan J. Morales-Ruiz, Jean-Pierre Ramis et Carles Simo : Integrability of Hamiltonian systems and differential Galois groups of higher variational equations. *Ann. Sci. École Norm. Sup. (4)*, 40(6):845–884, 2007.

[MS96a] C. Mitschi et M. F. Singer : On Ramis's solution of the local inverse problem of differential Galois theory. *J. Pure Appl. Algebra*, 110(2):185–194, 1996.

[MS96b] Claude Mitschi et Michael F. Singer : The inverse problem in differential Galois theory. *In The Stokes phenomenon and Hilbert's 16th problem (Groningen, 1995)*, pages 185–196. World Sci. Publ., River Edge, NJ, 1996.

[MS10] Marc Mezzarobba et Bruno Salvy : Effective bounds for P-recursive sequences. *J. Symbolic Comput.*, 45(10):1075–1096, 2010.

[MS12] Claude Mitschi et Michael F. Singer : Monodromy groups of parameterized linear differential equations with regular singularities. *Bull. Lond. Math. Soc.*, 44(5):913–930, 2012.

[MS13] Claude Mitschi et Michael F. Singer : Projective isomonodromy and Galois groups. *Proc. Amer. Math. Soc.*, 141(2):605–617, 2013.

[MZ00] F. Marotte et C. Zhang : Multisommabilité des séries entières solutions formelles d'une équation aux q-différences linéaire analytique. *Ann. Inst. Fourier (Grenoble)*, 50(6):1859–1890 (2001), 2000.

[Ngu11] Pierre Nguyen : Hypertranscedance de fonctions de Mahler du premier ordre. *C. R. Math. Acad. Sci. Paris*, 349(17-18):943–946, 2011.

[Pil97] Anand Pillay : Differential Galois theory. II. *Ann. Pure Appl. Logic*, 88(2-3):181–191, 1997. Joint AILA-KGS Model Theory Meeting (Florence, 1995).

[Pil98] Anand Pillay : Differential Galois theory. I. *Illinois J. Math.*, 42(4):678–699, 1998.

[PN11] Ana Peón Nieto : On $\sigma\delta$-Picard-Vessiot extensions. *Comm. Algebra*, 39(4):1242–1249, 2011.

[Ram80] J.-P. Ramis : Les séries k-sommables et leurs applications. *In Complex analysis, microlocal calculus and relativistic quantum theory (Proc. Internat. Colloq., Centre Phys., Les Houches, 1979)*, volume 126 de *Lecture Notes in Phys.*, pages 178–199. Springer, Berlin, 1980.

[Ram85] Jean-Pierre Ramis : Phénomène de Stokes et filtration Gevrey sur le groupe de Picard-Vessiot. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(5):165–167, 1985.

[Ram92] Jean-Pierre Ramis : About the growth of entire functions solutions of linear algebraic q-difference equations. *Ann. Fac. Sci. Toulouse Math. (6)*, 1(1):53–94, 1992.

[Ram93] Jean-Pierre Ramis : Séries divergentes et théories asymptotiques. *Bull. Soc. Math. France*, 121(Panoramas et Syntheses, suppl.):74, 1993.

[Ras10] Jean-Marc Rasoamanana : Résurgence-sommabilité de séries formelles ramifiées dépendant d'un paramètre et solutions d'équations différentielles linéaires. *Ann. Fac. Sci. Toulouse Math. (6)*, 19(2):303–343, 2010.

[Rem12] Pascal Remy : Matrices de Stokes-Ramis et constantes de connexion pour les systèmes différentiels linéaires de niveau unique. *Ann. Fac. Sci. Toulouse Math. (6)*, 21(1):93–150, 2012.

[RM90] J.-P. Ramis et J. Martinet : Théorie de Galois différentielle et resommation. *In Computer algebra and differential equations*, Comput. Math. Appl., pages 117–214. Academic Press, London, 1990.

[Rob59] Abraham Robinson : On the concept of a differentially closed field. *Bull. Res. Council Israel Sect. F*, 8F:113–128 (1959), 1959.

[Roq06] Julien Roques : Classification rationnelle et confluence des systèmes aux différences singuliers réguliers. *Ann. Inst. Fourier (Grenoble)*, 56(6):1663–1699, 2006.

[Roq08] Julien Roques : Galois groups of the basic hypergeometric equations. *Pacific J. Math.*, 235(2):303–322, 2008.

[Roq11] Julien Roques : Generalized basic hypergeometric equations. *Invent. Math.*, 184(3):499–528, 2011.

[RS89] J.-P. Ramis et Y. Sibuya : Hukuhara domains and fundamental existence and uniqueness theorems for asymptotic solutions of Gevrey type. *Asymptotic Anal.*, 2(1):39–94, 1989.

[RS07] J.-P. Ramis et J. Sauloy : The q-analogue of the wild fundamental group. I. *In Algebraic, analytic and geometric aspects of complex differential equations and their deformations. Painlevé hierarchies*, RIMS Kôkyûroku Bessatsu, B2, pages 167–193. Res. Inst. Math. Sci. (RIMS), Kyoto, 2007.

[RS09] Jean-Pierre Ramis et Jacques Sauloy : The q-analogue of the wild fundamental group. II. *Astérisque*, (323):301–324, 2009.

[RSZ13] J.-P. Ramis, J. Sauloy et C. Zhang : Local analytic classification of q-difference equations. *Astérisque*, (355), 2013.

[RZ02] Jean-Pierre Ramis et Changgui Zhang : Développement asymptotique q-Gevrey et fonction thêta de Jacobi. *C. R. Math. Acad. Sci. Paris*, 335(11): 899–902, 2002.

[Sab93] Claude Sabbah : Introduction to algebraic theory of linear systems of differential equations. *In Éléments de la théorie des systèmes différentiels. D-modules cohérents et holonomes (Nice, 1990)*, volume 45 de *Travaux en Cours*, pages 1–80. Hermann, Paris, 1993.

[Sau00] Jacques Sauloy : Systèmes aux q-différences singuliers réguliers : classification, matrice de connexion et monodromie. *Ann. Inst. Fourier (Grenoble)*, 50(4):1021–1071, 2000.

[Sau04] Jacques Sauloy : La filtration canonique par les pentes d'un module aux q-différences et le gradué associé. *Ann. Inst. Fourier (Grenoble)*, 54(1):181–210, 2004.

[Sch01] Reinhard Schäfke : Formal fundamental solutions of irregular singular differential equations depending upon parameters. *J. Dynam. Control Systems*, 7(4):501–533, 2001.

[Sei58] A. Seidenberg : Abstract differential algebra and the analytic case. *Proc. Amer. Math. Soc.*, 9:159–164, 1958.

[Sei69] A. Seidenberg : Abstract differential algebra and the analytic case. II. *Proc. Amer. Math. Soc.*, 23:689–691, 1969.

[Sib75] Yasutaka Sibuya : *Global theory of a second order linear ordinary differential equation with a polynomial coefficient.* North-Holland Publishing Co., Amsterdam, 1975. North-Holland Mathematics Studies, Vol. 18.

[Sib90] Yasutaka Sibuya : *Linear differential equations in the complex domain : problems of analytic continuation*, volume 82 de *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 1990. Translated from the Japanese by the author.

[Sin09] Michael F. Singer : Introduction to the Galois theory of linear differential equations. *In Algebraic theory of differential equations*, volume 357 de *London Math. Soc. Lecture Note Ser.*, pages 1–82. Cambridge Univ. Press, Cambridge, 2009.

[Sin13] Michael F. Singer : Linear algebraic groups as parameterized Picard-Vessiot Galois groups. *J. Algebra*, 373:153–161, 2013.

[Sit75] William Yu Sit : Differential algebraic subgroups of SL(2) and strong normality in simple extensions. *Amer. J. Math.*, 97(3):627–698, 1975.

[SU93a] Michael F. Singer et Felix Ulmer : Galois groups of second and third order linear differential equations. *J. Symbolic Comput.*, 16(1):9–36, 1993.

[SU93b] Michael F. Singer et Felix Ulmer : Liouvillian and algebraic solutions of second and third order linear differential equations. *J. Symbolic Comput.*, 16(1):37–73, 1993.

[Trj33] W. J. Trjitzinsky : Analytic theory of linear q-difference equations. *Acta Math.*, 61(1):1–38, 1933.

[TT79] Carol Tretkoff et Marvin Tretkoff : Solution of the inverse problem of differential Galois theory in the classical case. *Amer. J. Math.*, 101(6):1327–1332, 1979.

[Ulm94] Felix Ulmer : Irreducible linear differential equations of prime order. *J. Symbolic Comput.*, 18(4):385–401, 1994.

[Ume96a] Hiroshi Umemura : Differential Galois theory of infinite dimension. *Nagoya Math. J.*, 144:59–135, 1996.

[Ume96b] Hiroshi Umemura : Galois theory of algebraic and differential equations. *Nagoya Math. J.*, 144:1–58, 1996.

[Ume96c] Hiroshi Umemura : Galois theory of algebraic and differential equations. *Nagoya Math. J.*, 144:1–58, 1996.

[UW96] Felix Ulmer et Jacques-Arthur Weil : Note on Kovacic's algorithm. *J. Symbolic Comput.*, 22(2):179–200, 1996.

[vdH07] Joris van der Hoeven : Around the numeric-symbolic computation of differential Galois groups. *J. Symbolic Comput.*, 42(1-2):236–264, 2007.

[vdP99] Marius van der Put : Symbolic analysis of differential equations. *In Some tapas of computer algebra*, volume 4 de *Algorithms Comput. Math.*, pages 208–236. Springer, Berlin, 1999.

[vdPR07] Marius van der Put et Marc Reversat : Galois theory of q-difference equations. *Ann. Fac. Sci. Toulouse Math. (6)*, 16(3):665–718, 2007.

[vdPS03] Marius van der Put et Michael F. Singer : *Galois theory of linear differential equations*, volume 328 de *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2003.

[vHW97] Mark van Hoeij et Jacques-Arthur Weil : An algorithm for computing invariants of differential Galois groups. *J. Pure Appl. Algebra*, 117/118:353–379, 1997. Algorithms for algebra (Eindhoven, 1996).

[vHW05] M. van Hoeij et J.-A. Weil : Solving second order linear differential equations with Klein's theorem. *In ISSAC'05*, pages 340–347. ACM, New York, 2005.

[Was87] Wolfgang Wasow : *Asymptotic expansions for ordinary differential equations.* Dover Publications Inc., New York, 1987. Reprint of the 1976 edition.

[Wat95] G. N. Watson : *A treatise on the theory of Bessel functions.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995. Reprint of the second (1944) edition.

[Wib12] Michael Wibmer : Existence of ∂-parameterized Picard-Vessiot extensions over fields with algebraically closed constants. *J. Algebra*, 361:163–171, 2012.

[WW96] E. T. Whittaker et G. N. Watson : *A course of modern analysis.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1996. An introduction to the general theory of infinite processes and of analytic functions ; with an account of the principal transcendental functions, Reprint of the fourth (1927) edition.

[Zha95] Alexey Zharkov : Coefficient fields of solutions in Kovacic's algorithm. *J. Symbolic Comput.*, 19(5):403–408, 1995.

[Zha99] Changgui Zhang : Développements asymptotiques q-Gevrey et séries Gq-sommables. *Ann. Inst. Fourier (Grenoble)*, 49(1):vi–vii, x, 227–261, 1999.

[Zha00] Changgui Zhang : Transformations de q-Borel-Laplace au moyen de la fonction thêta de Jacobi. *C. R. Acad. Sci. Paris Sér. I Math.*, 331(1):31–34, 2000.

[Zha01] Changgui Zhang : Sur la fonction q-gamma de Jackson. *Aequationes Math.*, 62(1-2):60–78, 2001.

[Zha02] Changgui Zhang : Une sommation discrète pour des équations aux q-différences linéaires et à coefficients analytiques : théorie générale et exemples. *In Differential equations and the Stokes phenomenon*, pages 309–329. World Sci. Publ., River Edge, NJ, 2002.

[Zha03] Changgui Zhang : Sur les fonctions q-Bessel de Jackson. *J. Approx. Theory*, 122(2):208–223, 2003.

Index des notations