Hypotheses testing problems for inhomogeneous Poisson processes

Lin Yang

Thèse de Doctorat

Lin YANG

Mémoire présenté en vue de l'obtention du
grade de Docteur de l'Université du Maine
sous le label de L'Université Nantes Angers Le Mans

École doctorale : *STIM*

Discipline : *26*
Spécialité : *mathématique*
Unité de recherche : *LMM*

Soutenue le *22 janvier 2014*
Thèse N° : *(10)*

Tests d'Hypothèses pour les Processus de Poisson dans les Cas non Réguliers

JURY

Rapporteurs : **Alexander NOVIKOV**, Professeur, University of Technology, Sydney, Australie
Mikhail NIKULIN, Professeur Emérite, Université Victor Segalen, Bordeaux

Examinateurs : **Dominique DEHAY**, Professeur, Université Haute Bretagne, Rennes
Marina KLEPTSYNA, Professeur, Université du Maine, Le Mans
Saïd HAMADENE, Professeur, Université du Maine, Le Mans

Directeur de Thèse : **Yury A. KUTOYANTS**, Professeur, Université du Maine, Le Mans

Co-directeur de Thèse : **Sergueï DACHIAN**, MCF-HDR, Université Blaise Pascal, Clermont-Ferrand

Remerciements

Je tiens à adresser en premier lieu mes plus chaleureux remerciements à mes directeurs de thèse M. Serguei Dachian et M. Yury A. Kutoyants, qui m'ont donnés les conseils et le courage. Ils m'ont transmis non seulement la passion, mais aussi l'esprit et la patience de la recherche mathématique. En particulier je voulais remercier M. Kutoyants, qui m'a ouvert la porte de la recherche. J'en profite pour leurs exprimer ici ma plus profonde gratitude.

Je remercie très sincèrement M. Alexander Novikov et M. Mikhail Nikulin d'avoir accepté de rapporter ma thèse.

Je suis très honorée que M. Dominique Dehay, M. Saïd Hamadène et Mme Marina Kleptsyna aient accepté de faire partie de mon jury.

Je tiens également à remercier Maroua, Li Zhou, Rui Mu et tous les collègues qui m'ont accompagnés au cours de trois années de thèse. Leurs conseils et aides m'ont donnés les bonnes conditions du travail et aussi la plus meilleure vie en France.

Enfin, je souhaite remercier ma famille pour leur soutien et support constant. Il m'est impossible de trouver des mots pour dire à quel point je suis fiere d'eux, et à quel point je les aime.

Table des matières

Chapitre 1

Introduction

Le processus de Poisson est l'un des processus aléatoires les plus importants. Il est souvent utilisé pour modéliser des points aléatoires dans le temps et dans l'espace, tels que les temps d'émissions radioactives, les temps d'arrivée des clients à un centre de service, les positions de fissures dans une pièce de matériau, *etc.* Plusieurs lois de probabilité importantes proviennent naturellement du processus de Poisson : la loi de Poisson, la loi exponentielle, la loi Gamma, *etc.* Le processus possède une belle structure mathématique et sert de base de construction de nombreux autres processus aléatoires plus complexes.

Un processus $X = (X_t,\ t \geq 0)$ est dit un processus de Poisson non homogène de fonction d'intensité $\lambda(t)$, $t \geq 0$, si $X_0 = 0$, les accroissements de X sur des intervalles disjoints sont indépendants et suivent la loi de Poisson :

$$\mathbf{P}\left\{X_t - X_s = k\right\} = \frac{\left(\int_s^t \lambda(v)\,\mathrm{d}v\right)^k}{k!} \exp\left\{-\int_s^t \lambda(v)\,\mathrm{d}v\right\}.$$

Tous les problèmes statistiques considérés dans ce travail concernent le cas où ma fonction d'intensité dépend d'un certain paramètre unidimensionnel, c'est-à-dire $\lambda(t) = \lambda(\vartheta, t)$, $\vartheta \in \Theta$. La théorie d'estimation paramétrique pour le processus de Poisson non homogène aussi que les propriétés asymptotiques des estimateurs dans certains cas sont décrites dans [18] et [21].

La théorie des tests d'hypothèses est une branche bien développée des statistiques mathématiques [24]. L'approche asymptotique permet de trouver des solutions satisfaisantes dans beaucoup de situations différentes. Les problèmes les plus simples, comme celui de test de deux hypothèses simples, ont des solutions bien connues. Rappelons que si nous fixons dans ce problème le risque de première espèce et cherchons le test qui maximise la puissance, nous obtenons immédiatement (d'après le lemme de Neyman-Pearson) le test le plus puissant basé sur la statistique du rapport de vraisemblance.

Le cas d'une hypothèse alternative composée considéré dans ce travail est plus difficile à traiter. Le sujet principal de ce travail est l'étude du comportement de différents tests dans des cas des modèles statistiques singuliers.

L'évolution de la singularité de la fonction d'intensité est comme suit : régulière (l'information de Fisher finie), continue mais non différentiable (singularité de type "cusp"), discontinue (singularité de type "saut") et discontinue (plus précisément, on considère ici un modèle de type "change-point" ou de rupture) avec un saut de taille variable. Dans tous les cas on décrit analytiquement les tests (dans le cas d'un saut de taille variable on présente également les propriétés asymptotiques des estimateurs). En particulier, on décrit les statistiques de tests, le choix des seuils et le comportement des fonctions de puissance sous les alternatives locales. Le problème initial est toujours le test d'une hypothèse simple $\vartheta = \vartheta_1$ contre une alternative unilatérale $\vartheta > \vartheta_1$. Notons que la notion des *alternatives locales* est différente selon le cas de régularité/singularité. Dans le cas régulier, les alternatives locales peuvent être introduites comme $\vartheta = \vartheta_1 + \frac{u}{\sqrt{n}}$, $u > 0$. Dans le cas d'une singularité de type "cusp" d'ordre κ, nous introduisons les alternatives locales comme $\vartheta = \vartheta_1 + \frac{u}{n^{1/(2\kappa+1)}}$, $u > 0$. Dans le cas d'une discontinuité on pose plutôt $\vartheta = \vartheta_1 + \frac{u}{n}$, $u > 0$. Et dans le cas d'un saut de taille variable $r_n \to r$, on pose $\vartheta = \vartheta_1 + \frac{u}{n}$, $u > 0$ ou $\vartheta = \vartheta_1 + \frac{u}{nr_n^2}$, $u > 0$ selon que r soit différent ou pas de zéro (dans ce dernier cas on suppose également que $nr_n^2 \to +\infty$).

Dans tous ces problèmes, le plus intéressant pour nous est de comparer les fonctions de puissance des différents tests. Dans les situations singulières, cette comparaison est faite avec l'aide de simulations numériques. Notons aussi que les résultats principaux concernent les rapports de vraisemblance limites dans des situations non régulières, et que les mêmes rapports de vraisemblance limites apparaissent dans beaucoup d'autres modèles, tels que les observations i.i.d., les séries temporelles, les processus de diffusion, *etc.* (voir, par exemple, [17], [4]). Par conséquent, les résultats présentés ici sont de nature beaucoup plus universelle et sont valables pour tout autre modèle (non nécessairement Poissonnien) où les mêmes rapports de vraisemblance limites apparaissent.

On suppose dans la suite que l'on observe n processus de Poisson non homogènes indépendants de fonction d'intensité $\lambda(t) = \lambda(\vartheta, t)$, $\vartheta \in \Theta$, sur l'intervalle $[0, \tau]$. On note les observations $X^n = (X_1, \ldots, X_n)$, où $X_j = \{X_j(t), 0 \le t \le \tau\}$. L'hypothèse nulle et l'hypothèse alternative sont toujours les suivantes :

$$\begin{aligned} \mathscr{H}_1 \quad &: \quad \vartheta = \vartheta_1, \\ \mathscr{H}_2 \quad &: \quad \vartheta > \vartheta_1. \end{aligned}$$

1.1 Tests d'hypothèses pour les processus de Poisson. Cas régulier

Dans Chapitre 2, on considère le cas régulier lorsque la fonction d'intensité $\lambda(\vartheta, t)$ est continûment différentiable par rapport au paramètre ϑ et l'information de Fisher $\mathrm{I}(\vartheta_1) = \int_0^\tau \frac{\dot\lambda(\vartheta_1,t)^2}{\lambda(\vartheta_1,t)}\,\mathrm{d}t$, où $\dot\lambda(\vartheta, t)$ est la dérivée de $\lambda(\vartheta, t)$ par rapport à ϑ, est finie.

Rappelons que dans ce cas, la famille des mesures induites par les processus de Poisson non homogènes correspondants $\{\mathbf{P}_{\vartheta}^{(n)}, \vartheta \in \Theta\}$ est localement asymptotiquement normale (LAN) au point ϑ_1. Cela nous permet d'introduire les alternatives locales $\mathscr{H}_2 : u > 0$ en posant $\vartheta = \vartheta_1 + \frac{u}{\sqrt{n\mathrm{I}(\vartheta_1)}}$ et d'introduire le test de fonction de score (SFT) définie comme suit :

$$\hat{\psi}_n\left(X^n\right) = \mathbb{1}_{\{\Delta_n(\vartheta_1) > z_\varepsilon\}}$$

où

$$\Delta_n\left(\vartheta_1\right) = \frac{1}{\sqrt{n\mathrm{I}\left(\vartheta_1\right)}} \sum_{j=1}^{n} \int_0^\tau \frac{\dot{\lambda}\left(\vartheta_1, t\right)}{\lambda\left(\vartheta_1, t\right)} \left[\mathrm{d}X_j(\mathrm{d}t) - \lambda\left(\vartheta_1, t\right) \mathrm{d}t\right],$$

et $\varepsilon \in (0,1)$ est le seuil de signification (ou le niveau) du test, et z_ε est le quantile d'ordre $1-\varepsilon$ de la loi normale centrée réduite.

Rappelons également la vraisemblance donnée par :

$$L\big(\vartheta, X^n\big) = \exp\left\{\sum_{j=1}^{n} \int_0^\tau \ln \lambda(\vartheta, t)\, \mathrm{d}X_j\left(t\right) - n \int_0^\tau \left[\lambda\left(\vartheta, t\right) - 1\right] \mathrm{d}t\right\}$$

et le rapport de vraisemblance donné par :

$$\begin{aligned} L\left(\vartheta, \vartheta_1, X^n\right) &= \frac{L\big(\vartheta, X^n\big)}{L\big(\vartheta_1, X^n\big)} \\ &= \exp\left\{\sum_{j=1}^{n} \int_0^\tau \ln \frac{\lambda\left(\vartheta, t\right)}{\lambda\left(\vartheta_1, t\right)}\, \mathrm{d}X_j\left(t\right) - n \int_0^\tau \left[\lambda\left(\vartheta, t\right) - \lambda\left(\vartheta_1, t\right)\right] \mathrm{d}t\right\}, \end{aligned}$$

à partir duquel on construit quatre autre tests : le test du rapport de vraisemblance (GLRT), le test de Wald(WT) et les tests bayésiens (BT1) et (BT2).

Le test du rapport de vraisemblance (GLRT) est défini par

$$\hat{\psi}_n\left(X^n\right) = \mathbb{1}_{\{Q(X^n) > h_\varepsilon\}}, \qquad h_\varepsilon = \exp\{z_\varepsilon^2/2\}$$

avec

$$Q\left(X^n\right) = \sup_{\vartheta > \vartheta_1} L\left(\vartheta, \vartheta_1, X^n\right) = L\left(\hat{\vartheta}_n, \vartheta_1, X^n\right),$$

où l'estimateur du maximum de vraisemblance (EMV) $\hat{\vartheta}_n$ est défini par :

$$L\left(\hat{\vartheta}_n, \vartheta_1, X^n\right) = \sup_{\vartheta \geq \vartheta_1} L\left(\vartheta, \vartheta_1, X^n\right).$$

Le test de Wald est basé sur $\hat{\vartheta}_n$ et est défini par

$$\psi_n^o\left(X^n\right) = \mathbb{1}_{\left\{\varphi_n^{-1}\left(\hat{\vartheta}_n - \vartheta_1\right) > z_\varepsilon\right\}}, \qquad \varphi_n = \frac{1}{\sqrt{n\mathrm{I}\left(\vartheta_1\right)}}.$$

Le test BT1 est basé sur l'estimateur bayésien (EB). On suppose que le paramètre inconnu ϑ est une variable aléatoire avec la densité *a priori* $p(\theta)$, $\vartheta_1 \leq \theta < b$. Ici $p(\cdot)$ est une fonction continue connue satisfaisant $p(\vartheta_1) > 0$. Le test BT1 est défini par

$$\tilde{\psi}_n(X^n) = \mathbb{1}_{\left\{\varphi_n^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > g_\varepsilon\right\}}$$

où $\tilde{\vartheta}_n$ est l'EB pour la fonction de perte quadratique, c'est-à-dire

$$\tilde{\vartheta}_n = \int_{\vartheta_1}^{b} \theta\, p(\theta|X^n)\,\mathrm{d}\theta = \frac{\int_{\vartheta_1}^{b} \theta\, p(\theta)\, L(\theta, X^n)\,\mathrm{d}\theta}{\int_{\vartheta_1}^{b} p(\theta)\, L(\theta, X^n)\,\mathrm{d}\theta}$$

et g_ε est la solution de l'équation

$$\mathbf{P}\left\{\frac{f(\zeta)}{F(\zeta)} + \zeta > g_\varepsilon\right\} = \varepsilon$$

où ζ est une variable aléatoire de la loi normale centrale reduite et $f(\cdot)$, $F(\cdot)$ sont respectivement la densité de probabilité et la fonction de répartition de la loi normale standard.

Pour introduire le test BT2, on note $\beta\left(\bar{\psi}_n, \theta\right) = \mathbf{E}_\theta\, \bar{\psi}_n(X^n)$ la fonction de puissance d'un test $\bar{\psi}_n$ et $\alpha\left(\bar{\psi}_n, \theta\right) = 1 - \beta\left(\bar{\psi}_n, \theta\right)$ son risque de deuxième espèce. Le test BT2 est défini à la manière de l'approche bayésienne, qui consiste à définir ce test comme le test qui minimise l'erreur moyenne

$$\alpha\left(\bar{\psi}_n\right) = \int_{\vartheta_1}^{b} \alpha\left(\bar{\psi}_n, \theta\right)\, p(\theta)\,\mathrm{d}\theta.$$

La puissance moyenne $\beta\left(\bar{\psi}_n\right) = 1 - \alpha\left(\bar{\psi}_n\right) = \int_{\vartheta_1}^{b} \mathbf{E}_\theta^{(n)} \bar{\psi}_n\; p(\theta)\;\mathrm{d}\theta = \tilde{\mathbf{E}}^{(n)}\, \bar{\psi}_n(X^n)$ s'avère être la même que si nous avions deux hypothèses simples : sous l'hypothèse $\mathscr{H}_1$ on observe un processus de Poisson de la fonction d'intensité $\lambda(\vartheta_1, \cdot)$, et sous l'alternative $\mathscr{H}_2$ le processus ponctuel observé est d'intensité aléatoire : c'est un mélange (selon la densité $p(\theta)$) des processus de Poisson de fonctions d'intensité $\lambda\,(\theta, \cdot)$, $\vartheta_1 \leq \theta < b$. La mesure correspondante est notée $\tilde{\mathbf{P}}^{(n)}$. Le test le plus puissant est

$$\tilde{\psi}_n = \mathbb{1}_{\left\{\tilde{L}(X^n) > c_\varepsilon\right\}}, \qquad \mathbf{E}_{\vartheta_1} \tilde{\psi}_n(X^n) = \varepsilon,$$

ou $\tilde{L}(X^n)$ est le rapport de vraisemblance donné par

$$\tilde{L}(X^n) = \frac{\mathrm{d}\tilde{\mathbf{P}}^{(n)}}{\mathrm{d}\mathbf{P}_{\vartheta_1}^{(n)}}(X^n) = \int_{\theta_1}^{b} \frac{\mathrm{d}\mathbf{P}_\theta^{(n)}}{\mathrm{d}\mathbf{P}_{\vartheta_1}^{(n)}}(X^n)\; p(\theta)\;\mathrm{d}\theta.$$

On note

$$Z_n(u) = L\left(\vartheta_1 + \frac{u}{\sqrt{n \mathrm{I}(\vartheta_1)}}, \vartheta_1, X^n\right),$$

le rapport de vraisemblance normalisé. À l'aide de la convergence faible du processus stochastique $Z_n(\cdot)$, on montre que tout les tests introduits ci-dessus sont dans la classe

$$\mathcal{K}_\varepsilon = \{\psi_n(X^n) : \mathbf{E}_{\vartheta_1}\psi_n(X^n) \to \varepsilon\}$$

de tests de niveau ε. On introduit également la fonction de puissance d'un test $\psi_n(X^n)$ par

$$\beta_n(u) = \mathbf{E}_{\vartheta_1+\frac{u}{\sqrt{n\mathrm{I}(\vartheta_1)}}}\psi_n(X^n).$$

On montre également que tous nos tests sont localement asymptotiquement uniformément les plus puissant (LAUMP) dans la classe $\mathcal{K}_\varepsilon$. On rappelle qu'un test $\psi_n^\star \in \mathcal{K}_\varepsilon$ et LAUMP dans $\mathcal{K}_\varepsilon$ si pour tout autre test $\bar{\psi}_n(X^n) \in \mathcal{K}_\varepsilon$ et tout $K > 0$ on a

$$\varliminf_{n\to\infty} \inf_{0<u\leq K} \left[\beta_n(\psi_n^\star, u) - \beta_n(\bar{\psi}_n, u)\right] \geq 0.$$

En fait, dans le cas régulier, les limites des fonctions de puissance de nos cinq tests sont égales à $\mathbf{P}(\zeta > z_\varepsilon - u)$, avec $\zeta \sim \mathcal{N}(0,1)$, qui est aussi égale à la limite de la puissance du test de Neyman-Pearson pour une alternative locale fixe (cette limite est dite enveloppe des fonctions de puissance).

De plus, le test SFT est efficace à l'ordre deux, c'est-à-dire que la différence entre la fonction de puissance de $\hat{\psi}_n(X^n)$ et l'enveloppe des fonctions de puissance est inférieure à $\int_0^\tau \frac{\dot{\lambda}(\vartheta_1,t)^3}{\mathrm{I}(\vartheta_1)^{3/2}}\,\mathrm{d}t / \left(\sqrt{n}\lambda(\vartheta_1,t)^2\right)$, et cette propriété est uniforme par rapport à $u \in (0, K]$ pour tout $K > 0$ (voir [13] pour plus de détails).

Malheureusement, nous n'avons pas d'expressions explicites pour les autres tests. Pour cette raison, dans ce chapitre, on donne également un exemple numérique pour comparer l'efficacité de SFT, WT, GLRT et BT1.

Dans l'exemple numérique, nous prenons $\lambda(\vartheta, t) = 3\cos^2(\vartheta t) + 1$, $t \in [0, 3)$ (modulation de fréquence). Lorsque n est suffisamment grand, toutes les fonctions de puissance "collent" à l'enveloppe des fonctions de puissance. Lorsque n est petit, on voit que les niveaux des tests (les valeurs des fonctions de puissance en $u = 0$) de GLRT, de WT et de BT1 sont supérieurs à $\varepsilon = 0{,}05$, et que le niveau de SFT est bien proche de 0,05. Mais quand u augmente, le SFT perd un peu de "puissance" et arrive à 1 un peu moins rapidement que les autres tests. On voit aussi que WT est plus sensible que GLRT, même s'ils utilisent respectivement le point ϑ où $L(\vartheta, \vartheta_1, X^n)$ est maximal et la valeur de ce maximum. Le test BT1 est le plus sensible parmi nos tests dans le cas régulier, mais on verra qu'il est plus utile dans les cas non régulier que l'on va décrire par la suite.

1.2 Tests d'hypothèses pour les processus de Poisson. Cas singuliers

Dans Chapitre 3, nous étudions le comportement asymptotique du GLRT, du WT, du BT1 et du BT2 dans deux situations non régulières (non lisses). Plus précisément, on étudie les tests lorsque la fonction d'intensité a une singularité de type "cusp" ou de type "saut". Dans les deux cas, l'information de Fisher est infinie. Notre but est de décrire le choix des seuils et d'étudier les fonctions de puissance lorsque $n \to \infty$. Une différence importante entre les cas non réguliers et le cas régulier est que dû à l'absence d'un critère d'optimalité, le choix d'un test qui est le "meilleur" asymptotiquement est toujours une question ouverte. Ici on utilisera la même terminologie et les mêmes notations que précédemment, comme l'enveloppe des fonctions de puissance, la classe $\mathcal{K}_\varepsilon$, la fonction de puissance $\beta_n(u)$, *etc.*

On étudie d'abord le cas d'une singularité de type "cusp", c'est-à-dire lorsque la fonction d'intensité est continue, mais pas différentiable partout, de la forme

$$\lambda\left(\vartheta,t\right)=a\left|t-\vartheta\right|^{\kappa}+h\left(t\right),\qquad 0\leq t\leq\tau,\qquad \vartheta\in\Theta=\left(\alpha,\beta\right),$$

où $\kappa\in(0,1/2)$, $\alpha>0$, $\beta<\tau$ et $h\left(\cdot\right)$ est une fonction connue, positive et bornée.

En notant

$$\Gamma_{\vartheta_1}^2=\frac{2a^2B\left(\kappa+1,\kappa+1\right)}{h\left(\vartheta_1\right)}\left[\frac{1}{\cos\left(\pi\kappa\right)}-1\right],$$

les alternatives locales sont obtenues par la re-paramétrisation suivante : $\vartheta=\vartheta_1+u\varphi_n$, $u>0$, avec $\varphi_n=\left(\Gamma_{\vartheta_1}^{1/H}n^{\frac{1}{2\kappa+1}}\right)^{-1}$. Vu que la limite (dans le sens de la convergence faible des processus stochastiques) du rapport de vraisemblance normalisé sous $\mathscr{H}_1$ (lorsque $\vartheta=\vartheta_1$) est donné par

$$Z_n\left(u\right)=L\big(\vartheta_1+u\varphi_n,\vartheta_1,X^n\big)\Longrightarrow Z\left(u\right)=\exp\left\{W^H\left(u\right)-\frac{\left|u\right|^{2H}}{2}\right\},$$

où $W^H\left(\cdot\right)$ est un mouvement brownien fractionnaire (fBm) de paramètre de Hurst $H=\kappa+\frac{1}{2}$, on construit les GLRT, WT and BT1 de la manière suivante.

Le GLRT est construit comme suit :

$$\hat{\psi}_n\left(X^n\right)=\mathbb{1}_{\{Q(X^n)>h_\varepsilon\}},$$

où

$$Q\left(X^n\right)=\sup_{\vartheta>\vartheta_1}L\left(\vartheta,\vartheta_1,X^n\right)=L\left(\hat{\vartheta}_n,\vartheta_1,X^n\right)$$

et le seuil $h_\varepsilon=h_\varepsilon\left(H\right)$ est la solution de l'équation

$$\mathbf{P}\left\{\sup_{v>0}\left[W^H\left(v\right)-\frac{v^{2H}}{2}\right]>\ln h_\varepsilon\right\}=\varepsilon.$$

Le WT est de la forme

$$\psi_n^o\left(X^n\right)=\mathbb{1}_{\left\{\varphi_n^{-1}\left(\hat{\vartheta}_n-\vartheta_1\right)>m_\varepsilon\right\}},$$

où m_ε est défini par l'équation

$$\mathbf{P}\left\{\hat{u} > m_\varepsilon\right\} = \varepsilon, \qquad \hat{u} = \arg\sup_{v>0}\left[W^H\left(v\right) - \frac{v^{2H}}{2}\right].$$

En supposant que le paramètre ϑ est une variable aléatoire d'une densité de probabilité *a priori* donnée $p\left(\theta\right)$, $\vartheta_1 \leq \theta < b$, le BT1 est basé sur l'EB $\tilde{\vartheta}_n$:

$$\tilde{\psi}_n\left(X^n\right) = \mathbb{1}_{\left\{\varphi_n^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\}}, \qquad \tilde{\vartheta}_n = \frac{\int_{\vartheta_1}^{b}\theta\, p(\theta) L\left(\theta, X^n\right)\, \mathrm{d}\theta}{\int_{\vartheta_1}^{b} p(\theta) L\left(\theta, X^n\right)\, \mathrm{d}\theta},$$

où k_ε est défini par

$$\mathbf{P}\left\{\tilde{u} > k_\varepsilon\right\} = \varepsilon, \qquad \tilde{u} = \frac{\int_0^\infty v\, Z\left(v\right) \mathrm{d}v}{\int_0^\infty Z\left(v\right) \mathrm{d}v}.$$

Notons que les seuils de ces trois tests dépendent du paramètre H et peuvent être obtenus par des simulations numériques. Les fonctions de puissance peuvent également être obtenus par des simulations numériques en notant que sous $\vartheta = \vartheta_1 + u\varphi_n$, on a

$$Z_n(v) \Longrightarrow Z_u\left(v\right) = \exp\left\{W^H\left(v\right) - \frac{\left|u - v\right|^{2H}}{2} + \frac{\left|u\right|^{2H}}{2}\right\}.$$

Par exemple, pour le GLRT, la fonction de puissance $\beta_n(u) = \mathbf{E}_{\vartheta_1 + u\varphi_n}\hat{\psi}_n\left(X^n\right)$ tend vers $\beta(u) = \mathbf{P}\left\{\sup\limits_{v>0} Z_u\left(v\right) > h_\varepsilon\right\}$.

Dans la section numérique, on présente les réalisations de $Z_n(\cdot)$ pour des différentes valeurs de κ. On voit que κ influence la “régularité” de $Z_n(\cdot)$. Pour simplifier le modèle, on choisit le cas lorsque l'intensité $\lambda\left(\vartheta, t\right) = 2 - \left|t - \vartheta\right|^\kappa$, dans lequel nous pouvons voir que $Z_n(\cdot)$ atteint la valeur maximale sur une des occurrences d'un des processus de Poissons. Dans le cas lorsque a est positive, $Z_n(\cdot)$ atteint la valeur maximale entre deux occurrences, ce qui est plus compliqué à calculer.

De la même manière que dans le cas régulier, quand le niveau est petit (égal à 0,05) le WT est plus sensible que le GLRT, c'est-à-dire, les fonctions de puissance du WT et du BT1, lorsque n et u sont petits, varient plus que celle du GLRT. Quand n augmente, les fonctions puissances et leur limites deviennent de plus en plus proche. Les fonctions de puissance du WT et du BT1 s'approche de leur limites plus rapidement que celle du GLRT.

On compare aussi les limites des fonctions de puissance des différents tests. On voit que dans ce cas, la limite de la fonction de puissance du GLRT est plus proche de l'enveloppe des fonctions de puissance que les autres. La fonction de puissance du WT est plus petite que celles du BT1 lorsque ε est petit et devient plus grande (presque la même que celle du GLRT) lorsque ε deviens assez grand. On remarque également que la fonction de puissance du BT1 est celle qui arrive toujours le plus rapidement à 1 (elle est la première à quitter la “zone dangereuse”).

On étudie ensuite le cas d'une singularité de type “saut” (discontinuité), où la fonction d'intensité $\lambda(\vartheta, t) = \lambda(t - \vartheta)$ possède un (unique) saut unique de taille finie sur sa période $(0, \tau)$ et $\vartheta \in (\alpha, \beta) \subset (0, \tau)$. On note λ_- et λ_+ les limites respectives à gauche et à droite de la fonction d'intensité au point de discontinuité, et on pose $\rho = \lambda_-/\lambda_+ \neq 1$. Dans ce cas, l'alternative locale est obtenue par la re-paramétrisation $\vartheta = \vartheta_1 + u\varphi_n$, $u > 0$, avec $\varphi_n = \frac{1}{n\lambda_+}$ et, sous l'hypothèse nulle, la limite du rapport de vraisemblance normalisé est

$$Z_*(v) = \exp\{\ln \rho\, x_*(v) - (\rho - 1)\, v\}, \qquad v \geq 0,$$

où $x_*(\cdot)$ est un processus de Poisson d'intensité 1.

On construit les tests GLRT, WT, BT1 de la même manière que dans le cas d'un “cusp”.

Le GLRT est construit comme suit :

$$\hat{\psi}_n(X^n) = 1\!\!1_{\{Q(X^n) > h_\varepsilon\}},$$

où

$$Q(X^n) = \sup_{\vartheta > \vartheta_1} \frac{L(\vartheta, X^n)}{L(\vartheta_1, X^n)} = \max\left[\frac{L\left(\hat{\vartheta}_n+, X^n\right)}{L(\vartheta_1, X^n)}, \frac{L\left(\hat{\vartheta}_n-, X^n\right)}{L(\vartheta_1, X^n)}\right],$$

et le seuil $h_\varepsilon = h_\varepsilon(\rho)$ est la solution de l'équation

$$\mathbf{P}\left\{\sup_{v>0} Z_*(v) > h_\varepsilon\right\} = \varepsilon.$$

Le WT est de la forme

$$\psi_n^o(X^n) = 1\!\!1_{\left\{\varphi_n^{-1}\left(\hat{\vartheta}_n - \vartheta_1\right) > m_\varepsilon\right\}},$$

où m_ε est défini par l'équation

$$\mathbf{P}\{\hat{u} > m_\varepsilon\} = \varepsilon, \qquad \hat{u} = \arg\sup_{v>0} Z_*(v).$$

En supposant que le paramètre ϑ est une variable aléatoire d'une densité de probabilité *a priori* donnée $p(\theta)$, $\vartheta_1 \leq \theta < b$, le BT1 est basé sur l'EB $\tilde{\vartheta}_n$:

$$\tilde{\psi}_n(X^n) = 1\!\!1_{\left\{\varphi_n^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\}}, \qquad \tilde{\vartheta}_n = \frac{\int_{\vartheta_1}^b \theta\, p(\theta) L(\theta, X^n)\, \mathrm{d}\theta}{\int_{\vartheta_1}^b p(\theta) L(\theta, X^n)\, \mathrm{d}\theta}$$

où k_ε est défini par

$$\mathbf{P}_{\vartheta_1}\{\tilde{u} > k_\varepsilon\} = \varepsilon, \qquad \tilde{u} = \frac{\int_0^\infty v Z_*(v)\, \mathrm{d}v}{\int_0^\infty Z_*(v)\, \mathrm{d}v}.$$

Pour obtenir les puissance, on note que sous l'alternative, la limite du rapport de vraisemblance normalisé est donnée par la même formule que sous l'hypothèse nulle, mais avec $x_*(\cdot)$ qui est maintenant un processus de Poisson de fonction d'intensité

$$\mu(u,v) = \rho \mathbb{1}_{\{v<u\}} + \mathbb{1}_{\{v\geq u\}}, \qquad v \geq 0.$$

Comme les seuils et les puissances dépendent de ρ, on peut les obtenir par des simulations numériques. Notons que dans le cas $\rho > 1$, les solutions explicites pour les seuils du GLRT et du WT sont obtenus dans [32] et [31]. Dans l'exemple numérique nous prenons $\rho = 3$ et $\lambda_+ = 1$.

Lorsque n est petit, tous les tests perdent un peu de "puissance" lorsque u augmente. Cet effet disparaît lorsque n est grand. La fonction de puissance du BT1 s'approche de sa limite plus rapidement que les autres. Comme avant, le WT et le BT1 sont plus sensibles que le GLRT.

On se concentre également sur la comparaison des limites des fonctions de puissance avec l'enveloppe des fonctions de puissance. On constate que pour certaines valeurs de u, la dérivée à gauche de l'enveloppe des fonctions de puissance n'est pas égale à sa dérivée à droite, ce qui fait apparaître des points non lisses dans son graphe. On voit que dans cet exemple, aucun des tests que l'on a construit n'est asymptotiquement optimal. Lorsque le niveau est petit (égale à 0,05 dans cet exemple), la limite de la fonction de puissance du WT est la plus petite. La limite de la fonction de puissance du GLRT est le plus élevée et la plus proche de l'enveloppe des fonctions de puissance jusqu'à $u \approx 7$, au delà duquel la limite de la fonction de puissance du BT1 est légèrement plus élevée. Lorsque $\varepsilon = 0{,}4$, les limites des fonctions de puissance du WT et de GLRT sont plus proches l'une de l'autre, et celle du BT1 devient la plus petite au début et la plus élevée quand $u > 3$. Comme dans la cas $\varepsilon = 0{,}05$, la limite de la fonction de puissance du BT1 est celle qui arrive le plus rapidement à 1.

1.3 Modèle de rupture avec un saut de taille variable. Estimation de paramètres et tests d'hypothèses

Dans Chapitre 4, on décrit un modèle de type "change-point" ou de rupture avec un saut de taille variable et on considère les problèmes d'estimation des paramètres et de tests d'hypothèses correspondants.

Rappelons que dans les modèles de rupture pour les processus de Poisson avec un saut de taille fixe (singularité de type "saut"), le rapport de vraisemblance limite est un log-processus de Poisson. Notons également, que dans les modèles de rupture pour les processus de diffusion, et en particulier pour le modèle de signal dans un bruit blanc gaussien, le rapport de vraisemblance limite est un log-processus de Wiener (voir [17],[20]). Il est intéressant d'étudier les relations entre les différents rapports de vraisemblance limites. Cette étude a été commencée dans les travaux [5] et [11].

Ce travail s'inscrit dans le cadre de cette étude, car dans le cas des processus de Poisson avec un saut de taille variable on obtient deux rapports de vraisemblance limites différents en fonction de la façon dont la taille du saut varie.

On considère deux cas. Dans le premier cas la taille de sauts converge vers une limite non nulle, et dans le second cas vers zéro. Les rapports de vraisemblance limites dans ces deux cas sont très différents. Dans le premier cas, comme dans le cas d'un saut de taille fixe, le rapport de vraisemblance normalisé converge vers un log-processus de Poisson. Dans le deuxième cas, le rapport de vraisemblance normalisé converge vers un log-processus de Wiener, c'est-à-dire que les problèmes statistiques d'estimation de paramètres et de tests d'hypothèses sont asymptotiquement équivalentes aux problèmes bien connus d'estimation et de tests pour un signal dans un bruit blanc gaussien. En plus de la convergence des rapports de vraisemblance normalisés, on montre la convergence des moments des estimateurs. Cette dernière convergence nous permet d'approcher les erreurs moyennes quadratiques limites de l'estimateur du maximum de vraisemblance et des estimateurs bayésien dans le cas des observations poissonniennes par les erreurs moyennes quadratiques limites bien connues de ces estimateurs calculées pour un signal dans un bruit blanc gaussien. Les résultats théoriques obtenus sont illustrés par des simulations numériques.

Les propriétés asymptotiques des estimateurs et des tests sont obtenus à l'aide de la méthode d'Ibragimov-Khasminskii basée sur l'étude du processus de rapport de vraisemblance normalisé. Dans tous les problèmes, on vérifie la convergence faible du processus de rapport de vraisemblance normalisé vers un processus de rapport de vraisemblance limite dans un espace métrique adapté. En particulier, on vérifie la convergence des distributions fini-dimensionnelles et la tension de la famille de mesures correspondante dans l'espace de Skorohod $\mathscr{D}_0(\mathbb{R})$.

On considère le modèle suivant de type "change-point" ou de rupture avec un saut de taille variable $r_n \to r$. On suppose que la fonction d'intensité est continue à l'exception du point de rupture $\vartheta \in \Theta = (\alpha, \beta) \subset (0, \tau)$, où elle "se décale" de r_n, c'est-à-dire la fonction d'intensité est de la forme $\lambda_\vartheta^{(n)}(t) = \psi_n(t) + r_n \mathbb{1}_{\{t>\vartheta\}}$ où la fonction ψ_n est continue et uniformément convergeante sur $[0, \tau]$.

La vraisemblance est donné par :

$$L_n\big(\vartheta, X^{(n)}\big) = \exp\left\{\sum_{j=1}^{n} \int_{[0,\tau]} \ln \lambda_\vartheta^{(n)}(t)\ X_j^{(n)}(\mathrm{d}t) - n\int_0^\tau \big[\lambda_\vartheta^{(n)}(t) - 1\big]\,\mathrm{d}t\right\}.$$

L'estimateur du maximum de vraisemblance (EMV) $\widehat{\vartheta}_n$ est introduit par l'équation

$$\max\left\{L_n\big(\,\widehat{\vartheta}_n+, X^{(n)}\big), L_n\big(\,\widehat{\vartheta}_n-, X^{(n)}\big)\right\} = \sup_{\vartheta\in\Theta} L_n\big(\vartheta, X^{(n)}\big).$$

L'estimateur bayésien (EB) $\widetilde{\vartheta}_n$ pour la densité *a priori* $p(\theta)$ sur $\alpha < \theta < \beta$ peut être écrit comme suit :

$$\widetilde{\vartheta}_n = \frac{\int_\alpha^\beta \theta\, p(\theta) L_n\big(\theta, X^{(n)}\big)\,\mathrm{d}\theta}{\int_\alpha^\beta p(\theta) L_n\big(\theta, X^{(n)}\big)\,\mathrm{d}\theta}\,.$$

Premièrement, on considère le cas où la taille du saut converge vers une limite non nulle : $r_n \to r \neq 0$. Dans ce cas, on trouve que le rapport de vraisemblance limite est le même que dans le cas d'une singularité de type "saut". Plus précisément, en notant $\varphi_n = \frac{1}{n}$, on obtient que la limite du rapport de vraisemblance normalisé $Z_{n,\vartheta}(u) = \frac{L_n\left(\vartheta+u\varphi_n, X^{(n)}\right)}{L_n\left(\vartheta, X^{(n)}\right)}$ est de la forme

$$Z_\vartheta(u) = \begin{cases} \exp\Big\{\ln \frac{\psi(\vartheta)}{\psi(\vartheta)+r} X^+(u) + ru\Big\}, & \text{if } u \geq 0, \\ \exp\Big\{\ln \frac{\psi(\vartheta)+r}{\psi(\vartheta)} X^-\big((-u)-\big) + ru\Big\}, & \text{if } u < 0, \end{cases}$$

où X^+ et X^- sont des processus de Poisson indépendants sur $\mathbb{R}_+$ d'intensités respectives $\psi(\vartheta) + r$ et $\psi(\vartheta)$ (ψ est la limite de ψ_n).

On démontre que le processus $Z_{n,\vartheta}$ converge faiblement dans l'espace $\mathscr{D}_0(\mathbb{R})$ vers le processus Z_ϑ et que cette convergence est uniforme par rapport à $\vartheta \in \mathbb{K}$ pour tout compact $\mathbb{K} \subset \Theta$. Les propriétés asymptotiques des estimateurs en découlent directement.

Dans le problème de tests d'hypothèses on considère le cas $r < 0$. Le cas $r > 0$ peut être traité d'une manière similaire. En effectuant le changement de variable $\vartheta = \vartheta_1 + u\varphi_n^*$ avec $\varphi_n^* = \frac{1}{n|r|}$, on construit l'alternative locale $\mathscr{H}_2 \ : \ u > 0$.

Sous l'hypothèse, le rapport de vraisemblance limite est

$$Z_\rho^*(v) = \exp\left\{\rho Y^+(v) - v\right\}, \, v \geq 0,$$

où $\rho = \left|\ln \frac{\psi(\vartheta_1)}{\psi(\vartheta_1)+r}\right|$ et Y^+ est un processus de Poisson d'intensité $(e^\rho - 1)^{-1}$.

Sous l'alternative, le rapport de vraisemblance limite est donné par la même formule, mais avec Y^* (au lieu de Y^+) qui est un processus de Poisson de fonction d'intensité

$$\lambda_\rho^*(v) = (1 - e^{-\rho})^{-1} \mathbb{1}_{\{v<u\}} + (e^\rho - 1)^{-1} \mathbb{1}_{\{v \geq u\}}.$$

Le GLRT est construit comme suit :

$$\hat{\psi}_n\left(X^n\right) = \mathbb{1}_{\{Q(X^n) > h_\varepsilon\}},$$

où

$$Q\left(X^{(n)}\right) = \sup_{\vartheta > \vartheta_1} \frac{L_n\left(\vartheta, X^{(n)}\right)}{L_n\left(\vartheta_1, X^{(n)}\right)} = \max\left[\frac{L_n\left(\hat{\vartheta}_n+, X^{(n)}\right)}{L_n\left(\vartheta_1, X^{(n)}\right)}, \frac{L_n\left(\hat{\vartheta}_n-, X^{(n)}\right)}{L_n\left(\vartheta_1, X^{(n)}\right)}\right]$$

et le seuil $h_\varepsilon = h_\varepsilon\left(\rho\right)$ est la solution de l'équation

$$\mathbf{P}\left\{\sup_{v>0} Z_\rho^*(v) > h_\varepsilon\right\} = \varepsilon.$$

Le WT est de la forme

$$\psi_n^o\left(X^n\right) = \mathbb{1}_{\left\{\varphi_n^{-1}\left(\hat{\vartheta}_n-\vartheta_1\right)>m_\varepsilon\right\}},$$

où m_ε est défini par l'équation

$$\mathbf{P}\left\{\hat{u} > m_\varepsilon\right\} = \varepsilon, \qquad \hat{u} = \arg\sup_{v>0} Z_\rho^*(v).$$

En supposant que le paramètre ϑ est une variable aléatoire d'une densité de probabilité *a priori* donnée $p\left(\theta\right)$, $\vartheta_1 \leq \theta < b$, le BT1 est basé sur l'EB $\tilde{\vartheta}_n$:

$$\tilde{\psi}_n\left(X^{(n)}\right) = \mathbb{1}_{\left\{(\varphi_n^*)^{-1}\left(\tilde{\vartheta}_n-\vartheta_1\right)>k_\varepsilon\right\}}, \qquad \tilde{\vartheta}_n = \frac{\int_{\vartheta_1}^b \theta\, p(\theta) L_n\left(\theta, X^{(n)}\right)\,\mathrm{d}\theta}{\int_{\vartheta_1}^b p(\theta) L_n\left(\theta, X^{(n)}\right)\,\mathrm{d}\theta}$$

où k_ε est défini par

$$\mathbf{P}_{\vartheta_1}\left\{\tilde{u} > k_\varepsilon\right\} = \varepsilon, \qquad \tilde{u} = \frac{\int_0^\infty v Z_\rho^*\left(v\right)\mathrm{d}v}{\int_0^\infty Z_\rho^*\left(v\right)\mathrm{d}v}.$$

On décrit également les fonctions de puissance des trois tests et leur limites à l'aide des simulations numériques de la même maniére que dans le cas d'une singularité de type “saut”.

Deuxièmement, on considère le cas où la taille du saut converge vers zéro : $r_n \to 0$. Il s'avère que dans ce cas, r_n est l'un des facteurs de la vitesse de convergence. De plus, le rapport de vraisemblance limite est similaire à celui du cas d'une singularité de type “cusp”, mais avec un mouvement brownien bilatéral $W\left(\cdot\right)$ à la place du mouvement brownien fractionnaire $W^H\left(\cdot\right)$ (autrement dit, le rapport de vraisemblance limite est le même, mais avec $H = 1/2$). Plus précisément, en notant $\varphi_n = \frac{1}{n\, r_n^2}$, on obtient que la limite du rapport de vraisemblance normalisé $Z_{n,\vartheta}(u)$ est de la forme

$$Z_\vartheta(u) = Z^*\big(u/\psi(\vartheta)\big) \stackrel{d}{=} \exp\left\{\psi^{-1/2}(\vartheta)\, W(u) - \frac{|u|}{2\psi(\vartheta)}\right\},$$

où

$$Z^*(u) = \exp\left\{W(u) - \frac{|u|}{2}\right\}$$

et $W(u)$ est un mouvement brownien bilatéral.

On démontre que le processus $Z_{n,\vartheta}$ converge faiblement dans l'espace $\mathscr{D}_0(\mathbb{R})$ vers le processus Z_ϑ et que cette convergence est uniforme par rapport à $\vartheta \in \mathbb{K}$ pour tout compact $\mathbb{K} \subset \Theta$. Les propriétés asymptotiques des estimateurs en découlent directement. Notons que, bien que les réalisation du processus limite Z_ϑ appartiennent à l'espace $\mathscr{C}_0(\mathbb{R}) \subset \mathscr{D}_0(\mathbb{R})$, les trajectoires de $Z_{n,\vartheta}$ ne le font pas nécessairement, et il s'agit donc bien de la convergence faible dans l'espace $\mathscr{D}_0(\mathbb{R})$.

Dans le problème de tests d'hypothèses, le changement de variable $\vartheta = \vartheta_1 + u\varphi_n^*$ avec $\varphi_n^* = \frac{\psi(\vartheta_1)}{n\, r_n^2}$ nous permet de construire l'alternative locale $\mathscr{H}_2 \ : \ u > 0$.

Les limites du rapport de vraisemblance normalisé sous l'hypothèse nulle et sous l'alternative sont respectivement données par

$$Z^*(v) = \exp\left\{W(v) - \frac{|v|}{2}\right\}$$

et

$$Z_u^*(v) = \exp\left\{W(v) - \frac{|v-u|}{2} + \frac{|u|}{2}\right\}.$$

Le GLRT est construit comme suit :

$$\hat{\psi}_n\left(X^n\right) = \mathbb{1}_{\{Q(X^n) > h_\varepsilon\}} + q_\varepsilon \mathbb{1}_{\{Q(X^n) = h_\varepsilon\}},$$

où

$$Q\left(X^{(n)}\right) = \sup_{\vartheta > \vartheta_1} \frac{L_n\left(\vartheta, X^{(n)}\right)}{L_n\left(\vartheta_1, X^{(n)}\right)} = \frac{L_n\left(\hat{\vartheta}_n, X^{(n)}\right)}{L_n\left(\vartheta_1, X^{(n)}\right)}$$

$q_\varepsilon = 0$ et le seuil r_ε est la solution de l'équation

$$\mathbf{P}\left\{\sup_{v>0}\left[W\left(v\right) - \frac{v}{2}\right] > \ln h_\varepsilon\right\} = \varepsilon.$$

La variable aléatoire $\sup_{v>0}\left[W\left(v\right) - \frac{v}{2}\right]$ suit une loi exponentielle de paramètre 1, donc le seuil r_ε est calculable explicitement.

Le WT est de la forme

$$\psi_n^o\left(X^n\right) = \mathbb{1}_{\left\{\varphi_n^{-1}\left(\hat{\vartheta}_n - \vartheta_1\right) > m_\varepsilon\right\}}.$$

m_ε est défini par l'équation

$$\mathbf{P}\left\{\hat{u} > m_\varepsilon\right\} = \varepsilon$$

avec $\hat{u} = \arg\sup_{v>0}\left[W\left(v\right) - \frac{v}{2}\right]$ de la densité

$$f(t) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-t/8\right\} - \frac{1}{2}\Phi\left(-\sqrt{t}/2\right).$$

En supposant que le paramètre ϑ est une variable aléatoire d'une densité de probabilité *a priori* donnée $p\left(\theta\right)$, $\vartheta_1 \leq \theta < b$, le BT1 est basé sur l'EB $\tilde{\vartheta}_n$:

$$\tilde{\psi}_n\left(X^{(n)}\right) = \mathbb{1}_{\left\{(\varphi_n^*)^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\}}, \qquad \tilde{\vartheta}_n = \frac{\int_{\vartheta_1}^b \theta\, p(\theta) L_n\left(\theta, X^{(n)}\right)\, \mathrm{d}\theta}{\int_{\vartheta_1}^b p(\theta) L_n\left(\theta, X^{(n)}\right)\, \mathrm{d}\theta}$$

où k_ε est défini par

$$\mathbf{P}_{\vartheta_1}\left\{\tilde{u} > k_\varepsilon\right\} = \varepsilon, \qquad \tilde{u} = \frac{\int_0^\infty v Z\left(v\right) \mathrm{d}v}{\int_0^\infty Z\left(v\right) \mathrm{d}v}.$$

On décrit également les fonctions de puissance des trois tests et leur limites à l'aide des simulations numériques de la même maniére que dans le cas d'une singularité de type "cusp", mais avec $H = 1/2$.

Pour conclure, notons que les problèmes de tests d'hypothèses pour les processus de Poisson non homogènes ont été étudiés par de nombreux auteurs, comme Brown [3], Liese [25], Léger and Wolfson [23], Fazli and Kutoyants [13], Dachian and Kutoyants [8] et les références qui y figurent.

En particulier, le problème de tests d'hypothèses simples est étudié par Fazli et Kutoyants dans [13], où le test le plus puissant est construit pour le cas où les mesures correspondantes ne sont pas nécessairement équivalentes. Le choix du seuil et les propriétés du test le plus puissant sont décrits à l'aide du principe de grandes déviations.

Chapter 2

On hypotheses testing for Poisson processes. Regular case

We consider the problem of hypothesis testing in the situation when the first hypothesis is simple and the second one is local composite. We describe the choice of thresholds and the power functions of Score-function test, General Likelihood Ratio test, Bayesian tests and Wald's test in the situation when the intensity function of inhomogeneous Poisson process is smooth with respect to the parameter. It is shown that all tests are asymptotically uniformly most powerful. The results of numerical simulations are presented.

2.1 Introduction

The hypotheses testing theory is well developed branch of the mathematical statistics [24]. The asymptotic approach allows to find satisfactory solutions in many different statements. The simplest problems like the testing of two simple hypotheses have well known solution. Remind that if we fix the first type error and seek the test which maximizes the power, then we obtain immediately (by Neyman-Pearson Lemma) the most powerful test based on the likelihood ratio statistic. The case of composite alternative is more difficult to treat and here the asymptotic solution is available in the regular case. It is possible, using, for example, the score-function test (SFT) to construct the asymptotically (locally) most powerful test. Moreover, the General Likelihood Ratio Test (GLRT) and the Wald's test (WT), based on the maximum likelihood estimator, are asymptotically most powerful in the same sense. In the non regular cases the situation became much more complicate. First of all there are different non regular (singular) situations and in all these situations the choice of the asymptotically the best test is always an open question.

This work is an attempt to study all these situations on the model of inhomogeneous Poisson processes of intensity function $\lambda(t), 0 \leq t \leq \tau$. This model is sufficiently simple to allow us to realize first the well known tests in the regular case and to verify that for this model too the construction of the asymptotically most powerful tests (SFT, GLRT, WT) is possible. In the next chapter we study the behavior of these tests in the case of singular statistical models. The "evolution of singularity" of the intensity function is the following: regular (finite Fisher information, this paper), continuous but not differentiable (cusp-type singularity), discontinuous intensity function [10]. In all three cases we describe analytically the tests. This means that we describe the test statistics, the choice of thresholds and the form of the power functions for local alternatives. Note that the notion of *local alternative* is different following the type of regularity-singularity. In the regular case and the simple hypothesis $\vartheta = \vartheta_1$ against one-sided alternative $\vartheta > \vartheta_1$, the local alternative can be $\vartheta = \vartheta_1 + \frac{u}{\sqrt{n}}$, $u > 0$. In the cusp-type singularity, the local alternative is $\vartheta = \vartheta_1 + \frac{u}{n^{\frac{1}{2\kappa+1}}}$, $u > 0$ and in the case of discontinuous intensity function we put $\vartheta = \vartheta_1 + \frac{u}{n}$, $u > 0$. In all these problems the most interesting for us question is the comparison of the power functions of different tests. In singular situations these comparison is done with the help of numerical simulations. Note that the main results concern the limit likelihood ratios in non-regular situations and the same limits have likelihood ratios in the many other models of observations (i.i.d., time series, diffusion processes etc.) see, e.g., [17], [7]. Therefore the presented here results are of more universal nature and are valid for any other model (non Poissonian) with the same limit likelihood ratios.

We recall that $X = (X_t, t \geq 0)$ is an inhomogeneous Poisson process with intensity function $\lambda(t)$, if $X_0 = 0$, the increments of X on disjoint intervals are independent and distributed according to the Poisson law

$$\mathbf{P}\{X_t - X_s = k\} = \frac{\left(\int_s^t \lambda(v)\,\mathrm{d}v\right)^k}{k!} \exp\left\{-\int_s^t \lambda(v)\,\mathrm{d}v\right\}.$$

All statistical problems considered in this work concerned the intensity functions depending on some one-dimensional parameter, i.e., $\lambda(t) = \lambda(\vartheta, t)$ and the basic hypothesis is always the same : $\vartheta = \vartheta_1$ and the alternative $\vartheta > \vartheta_1$. The diversity of the statements corresponds to the different types of regularity of the function $\lambda(\vartheta, t)$.

The hypotheses testing problems for inhomogeneous Poisson processes were studied by many authors, see, for example, Brown [3], Liese [25], Léger and Wolfson [23], Fazli and Kutoyants [13], Dachian and Kutoyants [8] and the references therein.

At particularly, the problem of two simple hypotheses testing was studied by Fazli and Kutoyants [13], where the construction of the most powerful test is done in the situation when the corresponding measures are not necessary equivalent. The choice of the threshold and the behavior of the power of the most powerful test are described with the help of large deviation principle.

2.1.1 Preliminaries

For simplicity of exposition we consider the model of n independent observations of inhomogeneous Poisson processes $X^n = (X_1, \ldots, X_n)$, where $X_j = \{X_j(t), 0 \leq t \leq \tau\}$.

We have

$$\mathbf{E}_\vartheta X_j(t) = \Lambda(\vartheta, t) = \int_0^t \lambda(\vartheta, s)\ \mathrm{d}s.$$

Here ϑ is one-dimensional parameter and $\mathbf{E}_\vartheta$ is the mathematical expectation, when the true value is ϑ. Note that this model is equivalent the observations of inhomogeneous Poisson process $X^T = [X_t, 0 \leq t \leq T]$ with periodic intensity $\lambda(\vartheta, t + j\tau) = \lambda(\vartheta, t)$, $j = 1, 2, \ldots, n$ and $T = n\tau$ (the period τ is supposed to be known). Indeed, if we put $X_j(s) = X_{s+\tau(j-1)} - X_{\tau(j-1)}, s \in [0, \tau], j = 1, \ldots, n$, then the observation of one trajectory X^T is equivalent to n independent observations $X_1, \ldots, X_n$.

Therefore, we suppose that we observe n copies of inhomogeneous Poisson process $X^n = (X_1, \ldots, X_n)$ with the intensity function $\lambda(\vartheta, t), 0 \leq t \leq \tau$. The intensity function is supposed in our work separated from zero on $[0, \tau]$, the measures corresponding to Poisson processes with the different values of ϑ are equivalent. The likelihood function is defined by the equality (see the details, for example, in [21])

$$L(\vartheta, X^n) = \exp\left\{\sum_{j=1}^n \int_0^\tau \ln \lambda(\vartheta, t)\, \mathrm{d}X_j(t) - n\int_0^\tau [\lambda(\vartheta, t) - 1]\, \mathrm{d}t\right\}.$$

Let us consider the problem of the testing of two simple hypotheses

$$\begin{aligned} \mathscr{H}_1 &\quad : \quad \vartheta = \vartheta_1, \\ \mathscr{H}_2 &\quad : \quad \vartheta = \vartheta_2 \qquad (\vartheta_2 > \vartheta_1). \end{aligned}$$

We define a test $\hat{\psi}_n$, which is the probability to accept the hypothesis $\mathscr{H}_2$ and its power function $\beta\left(\hat{\psi}_n, \vartheta_2\right) = \mathbf{E}_{\vartheta_2}\hat{\psi}_n(X^n)$.

Definition 2.1. *We call a test $\hat{\psi}_n$ the most powerful in the class*

$$\mathcal{K}_\varepsilon^o = \left\{\bar{\psi}_n : \mathbf{E}_{\vartheta_1}\hat{\psi}_n(X^n) = \varepsilon\right\}$$

if for any other test $\bar{\psi}_n \in \mathcal{K}_\varepsilon^o$, the relation

$$\beta\left(\hat{\psi}_n, \vartheta_2\right) - \beta\left(\bar{\psi}_n, \vartheta_2\right) \geq 0$$

holds.

Let us denote the likelihood ratio in this problem as

$$L(\vartheta_2, \vartheta_1, X^n) = L(\vartheta_2, X^n) / L(\vartheta_1, X^n),$$

then by Neyman-Pearson lemma, the test

$$\hat{\psi}_n(X^n) = \begin{cases} 1, & \text{if} \quad L(\vartheta_2, \vartheta_1, X^n) > b_\varepsilon, \\ q_\varepsilon, & \text{if} \quad L(\vartheta_2, \vartheta_1, X^n) = b_\varepsilon, \\ 0, & \text{if} \quad L(\vartheta_2, \vartheta_1, X^n) < b_\varepsilon. \end{cases}$$

is the most powerful in the class $\mathcal{K}_\varepsilon^o$. The constants b_ε and q_ε are solutions of the equation

$$\mathbf{P}_{\vartheta_1}\left(L\left(\vartheta_2,\vartheta_1,X^n\right)>b_\varepsilon\right)+q_\varepsilon\mathbf{P}_{\vartheta_1}\left(L\left(\vartheta_2,\vartheta_1,X^n\right)=b_\varepsilon\right)=\varepsilon.$$

This equation is equivalent to the condition $\hat{\psi}_n\left(X^n\right)\in\mathcal{K}_\varepsilon^o$.

Suppose now that we observe n independent inhomogeneous Poisson processes X^n with intensity function $\lambda\left(\vartheta,t\right)$ and we have to test the following two hypotheses

$$\begin{aligned}\mathscr{H}_1 &\quad : \quad \vartheta=\vartheta_1,\\ \mathscr{H}_2 &\quad : \quad \vartheta>\vartheta_1.\end{aligned}$$

In this case the alternative is composite and the power of tests are the functions of ϑ, i.e., for any tests $\bar{\psi}_n\left(X^n\right)$ the power $\beta\left(\bar{\psi}_n,\vartheta\right),\vartheta>\vartheta_1$.

Definition 2.2. *We call a test $\hat{\psi}_n$ uniformly most powerful (UMP) in the class $\mathcal{K}_\varepsilon^o$ if for any other test $\bar{\psi}_n\in\mathcal{K}_\varepsilon^o$ the relation*

$$\inf_{\vartheta>\vartheta_1}\left[\beta\left(\hat{\psi}_n\left(X^n\right),\vartheta\right)-\beta\left(\bar{\psi}_n\left(X^n\right),\vartheta\right)\right]\geq 0$$

holds.

The likelihood ratio function can be written as

$$\begin{aligned}L\left(\vartheta,\vartheta_1,X^n\right)&=L\left(\vartheta,X^n\right)/L\left(\vartheta_1,X^n\right)\\ &=\exp\left\{\sum_{j=1}^n\int_0^\tau\ln\frac{\lambda\left(\vartheta,t\right)}{\lambda\left(\vartheta_1,t\right)}\mathrm{d}X_j\left(t\right)-n\int_0^\tau\left[\lambda\left(\vartheta,t\right)-\lambda\left(\vartheta_1,t\right)\right]\mathrm{d}t\right\}.\end{aligned}$$

Theorem 2.1. *If the function $L(\vartheta,\vartheta_1,X^n)$ admits a representation*

$$L\left(\vartheta,\vartheta_1,X^n\right)=\Psi\left(\vartheta,\vartheta_1,S\left(X^n\right)\right)\tag{2.1}$$

where $\Psi\left(\vartheta,\vartheta_1,S\right)$ is a monotone, say, increasing function of the statistics $S\left(X^n\right)$ for all $\vartheta>\vartheta_1$, then the test

$$\hat{\psi}_n(X^n)=\begin{cases}1, & \text{if}\quad S\left(X^n\right)>c_\varepsilon,\\ q_\varepsilon, & \text{if}\quad S\left(X^n\right)=c_\varepsilon,\\ 0, & \text{if}\quad S\left(X^n\right)<c_\varepsilon.\end{cases}$$

with the constants $c_\varepsilon,q_\varepsilon$ - solution of the equation

$$\mathbf{P}_{\vartheta_1}\left(S\left(X^n\right)>c_\varepsilon\right)+q_\varepsilon\mathbf{P}_{\vartheta_1}\left(S\left(X^n\right)=c_\varepsilon\right)=\varepsilon.\tag{2.2}$$

is uniformly most powerful in the class $\mathcal{K}_\varepsilon^o$.

For any fixed alternative the most powerful test has the form $L(\vartheta, \vartheta_1, X^n) > b_\varepsilon$ but this relation is equivalent to $S(X^n) > c_\varepsilon$, where the constant ε is obtained from the condition $\hat{\psi}_n \in \mathcal{K}_\varepsilon^o$. Note that solution of the equation (2.2) does not depend on the alternative ϑ. Hence the obtained test is the most powerful for any alternative. This proves that the test $\hat{\psi}_n$ is UMP.

Example 2.1. *Suppose that the intensity function is*

$$\lambda(\vartheta, t) = \exp\{\vartheta h(t)\}, \quad 0 \le t \le \tau,$$

where $h(t) > 0$. *Then the likelihood ratio*

$$\ln L(\vartheta, \vartheta_1, X^n) = (\vartheta - \vartheta_1) \sum_{j=1}^{n} \int_0^\tau h(t)\, \mathrm{d}X_j(t) - n \int \left[e^{\vartheta h(t)} - e^{\vartheta_1 h(t)}\right] \mathrm{d}t$$

for $\vartheta > \vartheta_1$ *is a monotone increasing function of the statistic*

$$S(X^n) = \sum_{j=1}^{n} \int_0^\tau h(t)\, \mathrm{d}X_j(t)$$

and the test $\hat{\psi}_n(X^n)$ *given above is UMP.*

If the likelihood ratio is not a monotone function of some sufficient statistics, then the construction of the UMP test is a complicate problem. Some general results can be obtained if we consider an asymptotic approach $n \to \infty$. Below we consider the same one-sided alternatives but in the asymptotic statement.

Denote by $\mathcal{K}_\varepsilon$ the class of test functions $\bar{\psi}_n$ of asymptotic size ε

$$\mathcal{K}_\varepsilon = \left\{\bar{\psi}_n \quad : \quad \lim_{n\to\infty} \mathbf{E}_{\vartheta_1} \bar{\psi}_n(X^n) = \varepsilon\right\}.$$

Our goal is to construct tests which belong to this class and have some proprieties of asymptotic optimality. The comparison of tests can be done by their power functions. Unfortunately for any reasonable test for any fixed alternative under natural conditions the power function tends to 1 and the comparison of the limits is **useless**. To avoid this difficulty it is interesting to consider *close* or *contiguous* alternatives. Let us put $\vartheta = \vartheta_1 + \varphi_n u$, where $\varphi_n = \varphi_n(\vartheta_1) > 0$ and $\varphi_n \to 0$. Then the initial problem of hypotheses testing can be rewritten as

$$\begin{aligned} \mathscr{H}_1 \quad &: \quad u = 0, \\ \mathscr{H}_2 \quad &: \quad u > 0. \end{aligned}$$

The considered tests are usually of the form

$$\bar{\psi}_n = \mathbb{1}_{\{Y_n(X^n) > c_\varepsilon\}} + q_\varepsilon \mathbb{1}_{\{Y_n(X^n) = c_\varepsilon\}},$$

where the constant c_ε is defined with the help of the limit random variable Y (suppose that $Y_n \Longrightarrow Y$ under hypothesis) by the following relation

$$\mathbf{E}_{\vartheta_1}\bar{\psi}_n = \mathbf{P}_{\vartheta_1}\left\{Y_n\left(X^n\right) > c_\varepsilon\right\} + q_\varepsilon \mathbf{P}_{\vartheta_1}\left\{Y_n\left(X^n\right) = c_\varepsilon\right\} \longrightarrow \mathbf{P}_{\vartheta_1}\left\{Y > c_\varepsilon\right\} = \varepsilon$$

if the limit random variable Y is continuous and by

$$\mathbf{P}_{\vartheta_1}\left\{Y > c_\varepsilon\right\} + q_\varepsilon \mathbf{P}_{\vartheta_1}\left\{Y = c_\varepsilon\right\} = \varepsilon$$

if Y has distribution function with jumps.

Hence, $\bar{\psi}_n \in \mathcal{K}_\varepsilon$.

The corresponding power function is denoted as

$$\beta_n\left(\bar{\psi}_n, u\right) = \mathbf{E}_{\vartheta_1+\varphi_n u}\,\bar{\psi}_n, \qquad u > 0.$$

We introduce the asymptotic optimality of tests with the help of the following definition.

Definition 2.3. *We call a test $\psi_n^\star\left(X^n\right) \in \mathcal{K}_\varepsilon$ locally asymptotically uniformly most powerful (LAUMP) in the class $\mathcal{K}_\varepsilon$ if its power function $\beta_n\left(\psi_n^\star, u\right)$ satisfies the relation: for any other test $\bar{\psi}_n\left(X^n\right) \in \mathcal{K}_\varepsilon$ and any $K > 0$ we have*

$$\varliminf_{n\to\infty} \inf_{0<u\leq K}\left[\beta_n\left(\psi_n^\star, u\right) - \beta_n\left(\bar{\psi}_n, u\right)\right] \geq 0. \tag{2.3}$$

Below we show that in the regular case many tests can be LAUMP. In the singular situations the definition of the reasonable optimality of tests is an open question and we turn to the methods of numerical simulations.

2.1.2 Regular case

Let us denote $\dot{\lambda}(\vartheta, t)$ the derivative of $\lambda(\vartheta, t)$ w.r.t. ϑ. We assume that the following *Regularity conditions* are satisfied.

Smoothness. *The intensity function $\lambda\left(\vartheta, t\right), 0 \leq t \leq \tau$ of the observed Poisson process X^n is two times continuously differentiable w.r.t. ϑ, separated from zero uniformly on $\vartheta \geq \vartheta_1$ and the Fisher information*

$$\mathrm{I}\left(\vartheta_1\right) = \int_0^\tau \frac{\dot{\lambda}\left(\vartheta_1, t\right)^2}{\lambda\left(\vartheta_1, t\right)}\mathrm{d}t > 0.$$

Distinguishability. *For any $\nu > 0$*

$$\inf_{\vartheta-\vartheta_1>\nu}\left\|\sqrt{\lambda\left(\vartheta,\cdot\right)} - \sqrt{\lambda\left(\vartheta_1,\cdot\right)}\right\|_{\mathbb{L}^2([0,\tau))} > 0.$$

In this case the natural normalization function is $\varphi_n = n^{-1/2}$ and the change of variables is $\vartheta = \vartheta_1 + \frac{u}{\sqrt{n}}$. The key propriety of the statistical problems in regular case is the *local asymptotic normality* (LAN) of the family of measures of corresponding inhomogeneous Poisson processes at the point ϑ_1.

This means that the normalized likelihood ratio

$$\tilde{Z}_n(u) = L\left(\vartheta_1 + \frac{u}{\sqrt{n}}, \vartheta_1, X^n\right)$$

admits the representation

$$\tilde{Z}_n(u) = \exp\left\{u\widetilde{\Delta}_n(\vartheta_1, X^n) - \frac{u^2}{2}\,\mathrm{I}(\vartheta_1) + r_n\right\},$$

where by the central limit theorem, we have

$$\widetilde{\Delta}_n(\vartheta_1, X^n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\int_0^\tau \frac{\dot{\lambda}(\vartheta_1,t)}{\lambda(\vartheta_1,t)}\left[\mathrm{d}X_j(t) - \lambda(\vartheta_1,t)\,\mathrm{d}t\right] \Longrightarrow \widetilde{\Delta}$$

with $\widetilde{\Delta} \sim \mathcal{N}(0, \mathrm{I}(\vartheta_1))$ and

$$r_n = r_n(\vartheta_1, u, X^n) \xrightarrow{p} 0.$$

Moreover, the convergence is uniform on $0 \le u < K$ for any $K > 0$.

Let us show how this representation was obtained. Denoting $\lambda_0 = \lambda(\vartheta_1, t)$ and $\lambda_u = \lambda\left(\vartheta_1 + \frac{u}{\sqrt{n}}, t\right)$, with the help of Taylor series expansion we can write

$$\begin{aligned}
\ln Z_n(u) &= \sum_{j=1}^{n}\int_0^\tau \ln\frac{\lambda_u}{\lambda_0}\left[\mathrm{d}X_j(t) - \lambda_0\mathrm{d}t\right] - n\int_0^\tau\left[\lambda_u - \lambda_0 - \lambda_0\ln\frac{\lambda_u}{\lambda_0}\right]\mathrm{d}t \\
&= \frac{u}{\sqrt{n}}\sum_{j=1}^{n}\int_0^\tau \frac{\dot{\lambda}_0}{\lambda_0}\mathrm{d}\pi_j(t) - \frac{u^2}{2}\int_0^\tau \frac{\dot{\lambda}_0^2}{\lambda_0}\mathrm{d}t + r_n \\
&= u\widetilde{\Delta}_n(\vartheta_1, X^n) - \frac{u^2}{2}\mathrm{I}(\vartheta_1) + r_n \Longrightarrow \widetilde{\Delta} - \frac{u^2}{2}\mathrm{I}(\vartheta_1)
\end{aligned}$$

where π_j, $j = 1, \ldots, n$ is the centred Poisson process.

Here and in the sequel we choose the reparametrization which leads to *universal* in some sense limits. For example, in regular case we can put

$$\varphi_n = \frac{1}{\sqrt{n\mathrm{I}(\vartheta_1)}}.$$

With such change of variables the

$$\Delta_n(\vartheta_1, X^n) = \frac{1}{\sqrt{\mathrm{I}(\vartheta_1)}}\widetilde{\Delta}_n \Longrightarrow \Delta \sim \mathcal{N}(0,1).$$

and also

$$Z_n(u) = L(\vartheta_1 + u\varphi_n, \vartheta_1, X^n) = \exp\left\{u\Delta_n(\vartheta_1, X^n) - \frac{u^2}{2} + r_n\right\}.$$

The LAN families have many remarkable properties, which we will use in the sequel.

Let us remind here one general result which is valid for the wider class of distributions. We suppose only that the normalized likelihood ratio converges to some limit in distribution. Such situations we have in all our regular and singular problems. This property allows us to calculate the distribution under local alternative if we know the distribution under the $\mathscr{H}_1$ hypothesis. Moreover, it gives an efficient algorithm for the calculation of the power functions during the numerical simulations. Let us denote $Z(u)$ the limit process of $Z_n(u)$.

Lemma 2.1. (Le Cam's Third Lemma) *Suppose that the random vector* $(Z_n(u), Y_n)$ *converges in distribution under measure* $\mathbf{P}_{\vartheta_1}^{(n)}$*:*

$$(Z_n(u), Y_n) \Longrightarrow (Z(u), Y).$$

Then for any bounded continuous function $g(\cdot)$

$$\mathbf{E}_{\vartheta_1+\varphi_n u}\left[g(Y_n)\right] \longrightarrow \mathbf{E}\left[Z(u)\, g(Y)\right].$$

For the proof see [22].

In the regular case the limit of $Z_n(u)$ is the random function

$$Z(u) = \exp\left\{u\,\Delta - \frac{u^2}{2}\right\}, \qquad u \geq 0.$$

This is a Radon-Nikodim density of the gaussian family $\mathcal{N}(u,1)$. We have (for any fixed $u > 0$) the convergence

$$Z_n(u) \Longrightarrow Z(u).$$

According to this lemma we can write for characteristic function of $\Delta_n = \Delta_n(\vartheta_1, X^n)$ the following relations

$$\mathbf{E}_{\vartheta_1+\varphi_n u} e^{i\mu\Delta_n} \to \mathbf{E} Z(u)\, e^{i\mu u\Delta} = e^{-\frac{u^2}{2}} \mathbf{E} e^{u\Delta + i\mu\Delta} = e^{i\mu u - \frac{\mu^2}{2}} = \mathbf{E} e^{i\mu(u+\zeta)}$$

which yields, under alternative, the distribution of $\Delta_n(\vartheta_1, X^n) \Rightarrow \mathcal{N}(u,1)$.

2.2 Weak convergence

All tests which we study are functionals of the normalized likelihood-ratio $Z_n(\cdot)$. For each test we have to evaluate two quantities. The first one is the threshold which provides asymptotically the guaranteed type one error and the second is the power function, which has to be calculated under alternative. Our study is based on the weak convergence of the likelihood ratio $Z_n(\cdot)$ under the hypothesis $\mathscr{H}_1$ (to calculate the threshold) and under alternative $\mathscr{H}_2$(to calculate the limit power function). Note that the test statistics of all tests are continuous functionals of $Z_n(\cdot)$, that is why we verify the weak convergence of $Z_n(\cdot)$ under hypothesis and under alternative and these allow us to obtain the limit distributions of the statistics of the proposed tests.

Let $\mathbb{C}_0$ be the space of functions $f(\cdot)$ continuous on $[a,+\infty)$ such that $\lim\limits_{u\to+\infty} f(u) = 0$. Here a is some constant. Introduce the Introduce the metric of $\mathbb{C}_0$ by the formula

$$\rho(f,g) = \sup_{u\geq a} |f(u) - g(u)|$$

and the function

$$W_h(f) = \sup_{u\geq a} \sup_{|u-u'|\leq h} |f(u) - f(u')| + \sup_{u>1/h} |f(u)|.$$

Let the trajectories of the process $Y_n = \{Y_n(u), u \in [a,+\infty)\}$ and $Y = \{Y(u), u \in [a,+\infty)\}$ belong to the space $\mathbb{C}_0$ with probability one and denote as $\mu_\vartheta^{(n)}$ and μ_ϑ the distributions of these processes on the measurable space $(\mathbb{C}_0, \mathbb{B})$. Here $\mathbb{C}_0$ is complete separable space with metric $\rho(\cdot,\cdot)$ and $\mathbb{B}$ is the σ-algebra of Borel subsets of $\mathbb{C}_0$. Let $\mathbb{K}$ be an arbitrary compact set in $\Theta = [\vartheta_1, b)$.

Theorem 2.2. (Yu. V. Prohorov). *Let the finite dimensional distributions of the process Y_n converge to the finite dimensional distributions of the process Y uniformly in $\vartheta \in \mathbb{K}$ as $n \to +\infty$ and the following conditions hold:*

$$\lim_{N\to+\infty} \sup_{\vartheta\in\Theta} \mu_\vartheta^{(n)} \{|Y_n(a)| > N\} = 0$$

and

$$\lim_{h\to 0} \lim_{n\to+\infty} \sup_{\vartheta\in\Theta} \mu_\vartheta^{(n)}\{W_h(Y_n) > \delta\} = 0$$

for every $\delta > 0$. Then $\mu_\vartheta^{(n)}$ converges weakly to μ_ϑ uniformly in ϑ as $n \to +\infty$.

Theorem 2.3. *Let the finite dimensional distributions of the process Y_n converge to the finite dimensional distributions of the process Y uniformly in $\vartheta \in \mathbb{K}$ as $n \to +\infty$ and the following conditions hold:*

$$\lim_{N\to+\infty} \sup_{\vartheta\in\mathbb{K}} \mu_\vartheta^{(n)} \{|Y_n(a)| > N\} = 0$$

and

$$\lim_{h\to 0} \lim_{n\to+\infty} \sup_{\vartheta\in\mathbb{K}} \mu_\vartheta^{(n)}\{W_h(Y_n) > \delta\} = 0$$

for every $\delta > 0$. Then $\mu_\vartheta^{(n)}$ converges weakly to μ_ϑ uniformly in $\vartheta \in \mathbb{K}$ as $n \to +\infty$.

The observed inhomogeneous Poisson processes X^n has the distribution $\mathbf{P}_\vartheta^{(n)}$ induced on the measurable space of its realizations. The measures of the family $\left\{\mathbf{P}_\vartheta^{(n)},\ \vartheta \geq \vartheta_1\right\}$ are equivalent, we have the following theorem (detail see [17, Theorem 1.10.1]).

Theorem 2.4. *Let the following conditions be satisfied.*

1. *The family of measures* $\left\{\mathbf{P}_\vartheta^{(n)},\ \vartheta \in \Theta\right\}$ *is locally asymptotically normal uniformly for* $\vartheta \in \mathbb{K}$ *as* $n \to +\infty$.
2. *The inequality*
$$\sup_{\vartheta\in\mathbb{K}} \mathbf{E}_\vartheta^{(n)} \left|Z_n^{\frac{1}{2}}(u_2) - Z_n^{\frac{1}{2}}(u_1)\right|^2 \leq C\left|u_2 - u_1\right|^2$$
holds for every $u_1, u_2 \in \mathbb{U}_n^+ = [0, \varphi_n^{-1}(b - \vartheta_1))$ *and some constant* $C > 0$.
3. *There exists* $d > 0$, $C > 0$ *and* $\gamma > 0$ *such that one can find some finite constant* $n_0 > 0$ *such that, for all* $n \geq n_0$,
$$\sup_{\vartheta\in\mathbb{K}} \mathbf{P}_\vartheta^{(n)} \left\{Z_n(u) > e^{-d|u|^\gamma}\right\} \leq Ce^{-d|u|^\gamma}.$$

Then the weak convergence of Z_n *to* Z *holds as* $n \to +\infty$ *uniformly for* $\vartheta \in \mathbb{K}$.

So to prove the weak convergence of Z_n, we just need to verify these three conditions under alternatives.

Lemma 2.2. *Let the Regularity conditions be fulfilled. Then the family of measures* $\left\{\mathbf{P}_\vartheta^{(n)},\ \vartheta \in \Theta\right\}$ *is locally asymptotically normal.*

Proof. We have

$$\begin{aligned}
\ln Z_n(v) &= \varphi_n \sum_{j=1}^{n} \int_0^\tau \ln \frac{\lambda\left(\vartheta_1 + v\varphi_n, t\right)}{\lambda\left(\vartheta_1, t\right)} \,\mathrm{d}X_j(t) \\
&\qquad - n\int_0^\tau \left(\lambda\left(\vartheta_1 + v\varphi_n, t\right) - \lambda\left(\vartheta_1, t\right)\right) \mathrm{d}t \\
&= \varphi_n \sum_{j=1}^{n} \int_0^\tau \ln \frac{\lambda\left(\vartheta_1 + v\varphi_n, t\right)}{\lambda\left(\vartheta_1, t\right)} \left[\mathrm{d}X_j(t) - \lambda\left(\vartheta_1 + u\varphi_n, t\right)\,\mathrm{d}t\right] \\
&\quad - n\int_0^\tau \Big(\frac{\lambda\left(\vartheta_1 + v\varphi_n, t\right)}{\lambda\left(\vartheta_1, t\right)} - 1 \\
&\qquad - \frac{\lambda\left(\vartheta_1 + u\varphi_n, t\right)}{\lambda\left(\vartheta_1, t\right)} \ln \frac{\ln\lambda\left(\vartheta_1 + v\varphi_n, t\right)}{\lambda\left(\vartheta_1, t\right)}\Big)\lambda\left(\vartheta_1, t\right)\,\mathrm{d}t.
\end{aligned}$$

For the first term, by Taylor series expansion, we obtain

$$\lambda\left(\vartheta_1+v\varphi_n,t\right)=\lambda\left(\vartheta_1,t\right)+v\varphi_n\dot{\lambda}\left(\vartheta_1,t\right)+o(\varphi_n).$$

This allows us to use the expansion $\ln x=x-1+o(x-1)$ with $x=\frac{\lambda(\vartheta_1+v\varphi_n,t)}{\lambda(\vartheta_1,t)}$ and hence to obtain the following convergence

$$\begin{aligned}&\sum_{j=1}^{n}\int_0^{\tau}\ln\frac{\lambda\left(\vartheta_1+v\varphi_n,t\right)}{\lambda\left(\vartheta_1,t\right)}\left[\mathrm{d}X_j(t)-\lambda\left(\vartheta_1+u\varphi_n,t\right)\,\mathrm{d}t\right]\\&=v\varphi_n\sum_{j=1}^{n}\int_0^{\tau}\frac{\dot{\lambda}\left(\vartheta_1,t\right)}{\lambda\left(\vartheta_1,t\right)}\left[\mathrm{d}X_j(t)-\lambda\left(\vartheta_1+u\varphi_n,t\right)\,\mathrm{d}t\right]+o(1)\Longrightarrow v\zeta,\ \zeta\sim\mathcal{N}\left(0,1\right).\end{aligned}$$

Similarly, using the Taylor series expansion, we get

$$\lambda\left(\vartheta_1+u\varphi_n,t\right)=\lambda\left(\vartheta_1,t\right)+u\varphi_n\dot{\lambda}\left(\vartheta_1,t\right)+o(\varphi_n).$$

and (to simplify the notations, we put $\lambda_u=\lambda\left(\vartheta_1+u\varphi_n,t\right)$, $\lambda_v=\lambda\left(\vartheta_1+v\varphi_n,t\right)$, $\lambda_1=\lambda\left(\vartheta_1,t\right)$ and $\dot{\lambda}_1=\dot{\lambda}\left(\vartheta_1,t\right)$)

$$\begin{aligned}&-n\int_0^{\tau}\Big(\frac{\lambda_v}{\lambda_1}-1-\frac{\lambda_u}{\lambda_1}\ln\frac{\lambda_v}{\lambda_1}\Big)\lambda_1\,\mathrm{d}t\\&=n\int_0^{\tau}\left(\left(u\varphi_n\dot{\lambda}_1/\lambda_1+1\right)\ln\frac{\lambda_v}{\lambda_1}-\left(\lambda_v/\lambda_1-1\right)\right)\lambda_1\,\mathrm{d}t+o(1)\\&=n\int_0^{\tau}\left(\ln\frac{\lambda_v}{\lambda_1}-\left(\lambda_v/\lambda_1-1\right)\right)\lambda_1\,\mathrm{d}t+n\int_0^{\tau}u\varphi_n\frac{\dot{\lambda}_1}{\lambda_1}\ln\frac{\lambda_v}{\lambda_1}\lambda_1\,\mathrm{d}t+o(1)\\&=-\frac{n}{2}\int_0^{\tau}\lambda_1\left(\frac{\lambda_v}{\lambda_1}-1\right)^2\mathrm{d}t+nu\varphi_n\int_0^{\tau}\dot{\lambda}_1\left(\frac{\lambda_v}{\lambda_1}-1\right)\mathrm{d}t+o(1)\\&=-\frac{u^2}{2}n\varphi_n^2\int_0^{\tau}\lambda_1\frac{\dot{\lambda}_1^2}{\lambda_1^2}\,\mathrm{d}t+uvn\varphi_n^2\int_0^{\tau}\frac{\dot{\lambda}_1^2}{\lambda_1}\,\mathrm{d}t+o(1)\\&\longrightarrow-\frac{u^2}{2}+uv\end{aligned}$$

where we used the same expansions of λ_v and $\ln x$ as mentioned above. Finally, the lemma is proved with the limit

$$Z(u,v)=\exp\left\{\zeta-\frac{u^2-2uv}{2}\right\},\ \zeta\sim\mathcal{N}\left(0,1\right).$$

□

Let us write the random function Z_n under alternative $\vartheta=\vartheta_1+u\varphi_n$ as follows:

$$Z_n\left(v\right)=L\left(\vartheta_1+v\varphi_n,\vartheta_1,X^n\right)=L\left(\vartheta_1+u\varphi_n,\vartheta_1,X^n\right)\,L\left(\vartheta_1+v\varphi_n,\vartheta_1+u\varphi_n,X^n\right).$$

As the first term

$$L\left(\vartheta_1+u\varphi_n,\vartheta_1,X^n\right)\Longrightarrow Z(u)=\exp\left\{u\Delta+\frac{u^2}{2}\right\},$$

we only need to check the rest conditions of the Theorem for the term

$$\overline{Z}_n(v) = L\left(\vartheta_1 + v\varphi_n, \vartheta_1 + u\varphi_n, X^n\right).$$

Lemma 2.3. *Let the Regularity conditions be fulfilled. Then there exists a constant* $C > 0$*,*

$$\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\left|\overline{Z}_n^{1/2}(v_1) - \overline{Z}_n^{1/2}(v_2)\right|^2 \le C\left|v_1 - v_2\right|^2$$

for all $v_1, v_2 \in \mathbb{U}_n^+$ *and sufficiently large values of* n*.*

Proof. According to [19, Lemma 1.1.5], we have, for $v_1 > v_2 > 0$ (the other cases can be treated in the similar way), using the Taylor series expansion,

$$\begin{aligned}
&\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\left|\overline{Z}_n^{1/2}(v_1) - \overline{Z}_n^{1/2}(v_2)\right|^2 \\
&\le n\int_0^\tau \Big(\frac{\lambda^{1/2}(\vartheta_1 + v_1\varphi_n, t)}{\lambda^{1/2}(\vartheta_1 + u\varphi_n, t)} - \frac{\lambda^{1/2}(\vartheta_1 + v_2\varphi_n, t)}{\lambda^{1/2}(\vartheta_1 + u\varphi_n, t)}\Big)^2 \lambda(\vartheta_1 + u\varphi_n, t)\,\mathrm{d}t \\
&= n\int_0^\tau \big(\lambda^{1/2}(\vartheta_1 + v_1\varphi_n, t) - \lambda^{1/2}(\vartheta_1 + v_2\varphi_n, t)\big)^2\,\mathrm{d}t \\
&\le \frac{n}{4}\varphi_n^2\left(v_2 - v_1\right)^2 \int_0^\tau \frac{\dot\lambda\left(\widetilde\vartheta_v, t\right)^2}{\lambda\left(\vartheta_1, t\right)}\,\mathrm{d}t \\
&\le \frac{n}{2}\varphi_n^2\left(v_2 - v_1\right)^2 \int_0^\tau \frac{\dot\lambda\left(\vartheta_1, t\right)^2}{\lambda\left(\vartheta_1, t\right)}\,\mathrm{d}t \le C\left(v_2 - v_1\right)^2
\end{aligned}$$

and the lemma is proved. □

Lemma 2.4. *Let the Regularity conditions be fulfilled. Then there exist constants* $C, d > 0$*, such that*

$$\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\left\{\overline{Z}_n(v) > e^{-d|v-u|^2}\right\} \le e^{-d|v-u|^2} \tag{2.4}$$

for all $v \in \mathbb{U}_n^+$ *and sufficiently large value of* n*.*

Proof. Using the Markov inequality, we get

$$\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\left\{\overline{Z}_n(v) > e^{-d|v-u|^2}\right\} \le e^{\frac{1}{2}d|v-u|^2}\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\overline{Z}_n^{1/2}(v).$$

Consider the case when $v > u$ (the case when $v < u$ can be treated in the similar way). according to [19, Lemma 1.1.5], we have,

$$\begin{aligned}
&\mathbf{E}_{\vartheta_1+u\varphi_n}\overline{Z}_n^{1/2}(v) \\
&= \exp\Big\{-\frac{1}{2}\int_0^{n\tau}\Big(\frac{\lambda^{1/2}(\vartheta_1 + v\varphi_n, t)}{\lambda^{1/2}(\vartheta_1 + u\varphi_n, t)} - 1\Big)^2 \lambda(\vartheta_1 + u\varphi_n, t)\,\mathrm{d}t\Big\} \\
&= \exp\left\{-\frac{1}{2}n\int_0^\tau \left(\lambda^{1/2}(\vartheta_1 + v\varphi_n, t) - \lambda^{1/2}(\vartheta_1 + u\varphi_n, t)\right)^2\mathrm{d}t\right\},
\end{aligned}$$

Using the Taylor's expansion we get

$$\lambda^{1/2}(\vartheta_1 + v\varphi_n, t) = \lambda^{1/2}(\vartheta_1 + u\varphi_n, t) + \frac{\varphi_n(v-u)}{2} \frac{\dot\lambda\left(\tilde\vartheta, t\right)}{\lambda^{1/2}\left(\tilde\vartheta, t\right)}.$$

Hence, for sufficiently large n providing $|v-u|\,\varphi_n \leq \gamma$ we have
$\mathrm{I}\left(\tilde\vartheta\right) \geq \frac{1}{2}\mathrm{I}\left(\vartheta_1\right)$, and we obtain

$$\mathbf{E}_{\vartheta_1+u\varphi_n} \overline{Z}_n^{1/2}(v) \leq \exp\left\{-\frac{1}{8\mathrm{I}\left(\vartheta_1\right)} \left|v-u\right|^2 \mathrm{I}\left(\tilde\vartheta\right)\right\} \leq \exp\left\{-\frac{|v-u|^2}{16}\right\}. \tag{2.5}$$

By Distinguishability condition, we have

$$g(\gamma) = \inf_{\varphi_n|v-u|>\gamma} \int_0^\tau \left(\lambda^{1/2}(\vartheta_1 + v\varphi_n, t) - \lambda^{1/2}(\vartheta_1 + u\varphi_n, t)\right)^2 \mathrm{d}t > 0$$

and hence, we can write

$$\int_0^\tau \left(\lambda^{1/2}(\vartheta_1 + v\varphi_n, t) - \lambda^{1/2}(\vartheta_1 + u\varphi_n, t)\right)^2 \mathrm{d}t \geq g(\gamma) \geq g(\gamma)\frac{\varphi_n^2(u-v)^2}{\left(b-\vartheta_1\right)^2}$$

and

$$\mathbf{E}_{\vartheta_1+u\varphi_n} \overline{Z}_n^{1/2}(v) \leq \exp\left\{-\frac{|v-u|^2}{2\mathrm{I}\left(\vartheta_1\right)\left(b-\vartheta_1\right)^2}\right\}. \tag{2.6}$$

Now (2.4) follows from (2.5) and (2.6). □

The weak convergence now follows from the Theorem 2.2.

2.3 Hypothesis testing

2.3.1 Score function test

Let us introduce *score function test* (SFT)

$$\hat\psi_n\left(X^n\right) = \mathbb{1}_{\{\Delta_n(\vartheta_1, X^n) > z_\varepsilon\}}$$

where z_ε is the $(1-\varepsilon)$-quantile of the standard normal distribution $\mathcal{N}\left(0,1\right)$ and the statistic $\Delta_n\left(\vartheta_1, X^n\right)$ is

$$\Delta_n\left(\vartheta_1, X^n\right) = \frac{1}{\sqrt{n\mathrm{I}\left(\vartheta_1\right)}} \sum_{j=1}^n \int_0^\tau \frac{\dot\lambda\left(\vartheta_1, t\right)}{\lambda\left(\vartheta_1, t\right)} \left[\mathrm{d}X_j\left(t\right) - \lambda\left(\vartheta_1, t\right)\mathrm{d}t\right].$$

The SFT has the following properties.

Proposition 2.1. *The test $\hat\psi_n\left(X^n\right)\in\mathcal{K}_\varepsilon$ and is LAUMP. Its power function*

$$\beta_n\left(\hat\psi_n,u\right)\longrightarrow\beta^\star\left(u\right)=\mathbf{P}\left(\zeta>z_\varepsilon-u\right),\quad \zeta\sim\mathcal{N}\left(0,1\right). \tag{2.7}$$

Proof. The property $\hat\psi_n\left(X^n\right)\in\mathcal{K}_\varepsilon$ follows immediately from the asymptotic normality

$$\Delta_n\left(\vartheta_1,X^n\right)\Longrightarrow\Delta.$$

Further, we have (under alternative $\vartheta_u=\vartheta_1+u\varphi_n$) the convergence

$$\beta_n\left(\hat\psi_n,u\right)\longrightarrow\mathbf{P}\left(\Delta+u>z_\varepsilon\right)=\beta^\star\left(u\right).$$

This follows from the Third Le Cam's Lemma and can be shown directly as follows. Suppose the the intensity of the observed Poisson process is $\lambda\left(\vartheta_1+u\varphi_n,t\right)$, then we can write

$$\begin{aligned}\Delta_n\left(\vartheta_1,X^n\right)&=\frac{1}{\sqrt{n\mathrm{I}\left(\vartheta_1\right)}}\sum_{j=1}^n\int_0^\tau\frac{\dot\lambda\left(\vartheta_1,t\right)}{\lambda\left(\vartheta_1,t\right)}\left[\mathrm{d}X_j\left(t\right)-\lambda\left(\vartheta_1+u\varphi_n,t\right)\mathrm{d}t\right]\\&\quad+\frac{1}{\sqrt{n\mathrm{I}\left(\vartheta_1\right)}}\sum_{j=1}^n\int_0^\tau\frac{\dot\lambda\left(\vartheta_1,t\right)}{\lambda\left(\vartheta_1,t\right)}\left[\lambda\left(\vartheta_1+u\varphi_n,t\right)-\lambda\left(\vartheta_1,t\right)\right]\mathrm{d}t\\&=\Delta_n^*\left(\vartheta_1,X^n\right)+\frac{u}{n\mathrm{I}\left(\vartheta_1\right)}\sum_{j=1}^n\int_0^\tau\frac{\dot\lambda\left(\vartheta_1,t\right)^2}{\lambda\left(\vartheta_1,t\right)}\mathrm{d}t+o\left(1\right)\\&=\Delta_n^*\left(\vartheta_1,X^n\right)+u+o\left(1\right)\Longrightarrow\Delta+u.\end{aligned}$$

The LRT is defined by

$$\psi_n^\star\left(X^n\right)=\mathbb{1}_{\left\{Z_n(u)>d_\varepsilon\right\}},$$

where the threshold d_ε is chosen from the condition $\psi_n^\star\left(X^n\right)\in\mathcal{K}_\varepsilon$, i.e.,

$$\mathbf{P}_{\vartheta_1}\left\{Z\left(u\right)>d_\varepsilon\right\}=\varepsilon.$$

The threshold d_ε can be found as follows. The LAN of the family of measures at the point ϑ_1 allows us to write

$$\begin{aligned}\mathbf{P}_{\vartheta_1}\left(Z_n\left(u\right)>d_\varepsilon\right)&=\mathbf{P}_{\vartheta_1}\left(u\Delta_n\left(\vartheta_1,X^n\right)-\frac{u^2}{2}+r_n>\ln d_\varepsilon\right)\\&\longrightarrow\mathbf{P}\left(u\Delta-\frac{u^2}{2}>\ln d_\varepsilon\right)=\mathbf{P}\left(\Delta>\frac{\ln d_\varepsilon}{u}+\frac{u}{2}\right)=\varepsilon.\end{aligned}$$

Hence we have

$$\frac{\ln d_\varepsilon}{u}+\frac{u}{2}=z_\varepsilon\qquad\text{and}\qquad d_\varepsilon=\exp\left\{uz_\varepsilon-\frac{u^2}{2}\right\}.$$

Therefore the test

$$\psi_n^\star\left(X^n\right)=\mathbb{1}_{\left\{Z_n(u)>\exp\left\{uz_\varepsilon-\frac{u^2}{2}\right\}\right\}}$$

belongs to $\mathcal{K}_\varepsilon$.

For the power function of this test we have (below $\vartheta_u = \vartheta_1 + u\varphi_n$)

$$\begin{aligned}\beta_n\left(\psi_n^\star, u\right) &= \mathbf{P}_{\vartheta_u}\left(Z_n\left(u\right) > d_\varepsilon\right) = \mathbf{P}_{\vartheta_u}\left(u\Delta_n\left(\vartheta_1, X^n\right) + r_n > uz_\varepsilon\right) \\ &= \mathbf{P}_{\vartheta_u}\left(\Delta_n\left(\vartheta_1, X^n\right) + \frac{r_n}{u} > z_\varepsilon\right) \\ &\longrightarrow \mathbf{P}\left(\Delta + u > z_\varepsilon\right) = \beta^\star\left(u\right).\end{aligned}$$

Therefore the limits of these two tests coincide and the score-function test is asymptotically as good as the Neyman-Pearson optimal one. Note that the limits are valid for any sequence of $0 \leq u \leq K$ and for any $K > 0$ and we can choose a sequence $\hat{u}_n \in [0, K]$ such that

$$\sup_{0\leq u\leq K}\left|\beta_n\left(\psi_n^\star, u\right) - \beta_n\left(\hat{\psi}_n, u\right)\right| = \left|\beta_n\left(\psi_n^\star, \hat{u}_n\right) - \beta_n\left(\hat{\psi}_n, \hat{u}_n\right)\right| \to 0$$

in obvious notations, which represents the asymptotic coincidence of two tests.

Nevertheless we need to verify (2.3). When n is sufficiently large, we can find out a constant $\bar{u}_n \in (0, K]$, where k is any positive constant, such that it is the solution of the equation

$$\varliminf_{n\to\infty}\inf_{0\leq u\leq K}\left[\beta_n\left(\psi_n^\star, u\right) - \beta_n\left(\bar{\psi}_n, u\right)\right] = \varliminf_{n\to\infty}\left[\beta_n\left(\psi_n^\star, \bar{u}_n\right) - \beta_n\left(\bar{\psi}_n, \bar{u}_n\right)\right]$$

where $\bar{\psi}_n$ is any other test in the class $\mathcal{K}_\varepsilon$. As the Neyman-Pearson test is the most powerful in the class $\mathcal{K}_\varepsilon$, we obtain that the last inferior limit is greater than zero and the theorem is proved. □

2.3.2 GLRT and Wald's test

Let us remind the definition of the MLE $\hat{\vartheta}_n$:

$$L\left(\hat{\vartheta}_n, \vartheta_1, X^n\right) = \sup_{\vartheta\geq\vartheta_1} L\left(\vartheta, \vartheta_1, X^n\right).$$

where the likelihood-ratio function is

$$\begin{aligned}L\left(\vartheta, \vartheta_1, X^n\right) = \exp\Bigg\{&\sum_{j=1}^n \int_0^\tau \ln\frac{\lambda\left(\vartheta, t\right)}{\lambda\left(\vartheta_1, t\right)}\,\mathrm{d}X_j\left(t\right) \\ &-n\int_0^\tau \left[\lambda\left(\vartheta, t\right) - \lambda\left(\vartheta_1, t\right)\right]\mathrm{d}t\Bigg\}, \qquad \vartheta \geq \vartheta_1.\end{aligned}$$

The GLRT is

$$\hat{\psi}_n\left(X^n\right) = \mathbb{1}_{\{Q(X^n)>h_\varepsilon\}}, \qquad h_\varepsilon = \exp\{z_\varepsilon^2/2\},$$

where

$$Q\left(X^n\right)=\sup_{\vartheta\geq\vartheta_1}L\left(\vartheta,\vartheta_1,X^n\right)=L\left(\hat\vartheta_n,\vartheta_1,X^n\right).$$

The Wald's test is based on the maximum likelihood estimator $\hat\vartheta_n$ and is defined as follows

$$\psi_n^o\left(X^n\right)=\mathbb{1}_{\left\{\varphi_n^{-1}\left(\hat\vartheta_n-\vartheta_1\right)>z_\varepsilon\right\}}.$$

The properties of these tests are given in the following Proposition.

Proposition 2.2. *The tests $\hat\psi_n\left(X^n\right),\ \psi_n^o\left(X^n\right)\in\mathcal{K}_\varepsilon$, their power functions $\beta(\hat\psi_n,u)$ and $\beta\left(\psi_n^o,u\right)$ converge to $\beta^\star\left(u\right)$ and therefore are LAUMP.*

Proof. Let us put $\vartheta=\vartheta_1+\frac{v}{\sqrt{n\mathrm{I}(\vartheta_1)}}$. We denote, correspondingly, $\hat\vartheta_n=\vartheta_1+\hat v_n\varphi_n$. We have

$$\begin{aligned}\mathbf{P}_{\vartheta_1}\left\{\sup_{\vartheta>\vartheta_1}L\left(\vartheta,\vartheta_1,X^n\right)>h_\varepsilon\right\}&=\mathbf{P}_{\vartheta_1}\left\{\sup_{v>0}L\left(\vartheta_1+u\varphi_n,\vartheta_1,X^n\right)>h_\varepsilon\right\}\\&=\mathbf{P}_{\vartheta_1}\left\{\sup_{v>0}Z_n\left(v\right)>h_\varepsilon\right\}.\end{aligned}$$

As the family of measures is LAN we have the convergence of finite-dimensional distributions of $\left\{Z_n^{\frac12}\left(v\right),v\geq0\right\}$ to the distributions of the limit process $\left\{Z^{\frac12}\left(v\right),v\geq0\right\}$.

Further, we can write

$$\begin{aligned}&\mathbf{E}_{\vartheta_1}\left|Z_n^{1/2}\left(v_2\right)-Z_n^{1/2}\left(v_1\right)\right|^2\\&\quad=2-2\mathbf{E}_{\vartheta_1}Z_n^{1/2}\left(v_2\right)Z_n^{1/2}\left(v_1\right)=2-2\mathbf{E}_{\vartheta_1+v_1\varphi_n}\left(\frac{Z_n\left(v_2\right)}{Z_n\left(v_1\right)}\right)^{1/2}\\&\quad=2-2\exp\left\{-\frac{n}{2}\int_0^\tau\left[\sqrt{\lambda\left(\vartheta_1+v_2\varphi_n,t\right)}-\sqrt{\lambda\left(\vartheta_1+v_1\varphi_n,t\right)}\right]^2\mathrm{d}t\right\}\\&\quad\leq\frac{n}{2}\int_0^\tau\left[\sqrt{\lambda\left(\vartheta_1+v_2\varphi_n,t\right)}-\sqrt{\lambda\left(\vartheta_1+v_1\varphi_n,t\right)}\right]^2\mathrm{d}t\leq C\left|v_2-v_1\right|^2.\end{aligned}$$

Similarly we obtain the estimate

$$\mathbf{E}_{\vartheta_1}Z_n^{1/2}\left(v\right)=\exp\left\{-\frac{n}{2}\int_0^\tau\left[\sqrt{\lambda\left(\vartheta_1+v\varphi_n,t\right)}-\sqrt{\lambda\left(\vartheta_1,t\right)}\right]^2\mathrm{d}t\right\}\leq e^{-\kappa u^2}$$

where $\kappa>0$ is some constant.

Therefore, we have the weak convergence of the measures of the random processes $\left\{Z_n^{1/2}\left(v\right),v\geq0\right\}$ to the measure of the process $\left\{Z^{1/2}\left(v\right),v\geq0\right\}$ at the point ϑ_1. This provides the convergence of the distributions of all continuous in uniform metric functionals. Hence

$$Q\left(X^n\right)=\sup_{v>0}Z_n\left(v\right)\Rightarrow\sup_{v>0}Z\left(v\right)$$

$$= \sup_{v>0} \exp\left\{v\Delta - \frac{v^2}{2}\right\} = \exp\left\{\frac{\Delta^2}{2}\mathbb{1}_{\{\Delta\geq 0\}}\right\}.$$

This provides the convergence

$$\mathbf{E}_{\vartheta_1}\hat\psi_n\left(X^n\right) \longrightarrow \mathbf{P}\left\{\Delta > z_\varepsilon\right\} = \varepsilon.$$

Remind that for $\varepsilon < \frac{1}{2}$

$$\mathbf{P}\left\{\Delta\mathbb{1}_{\{\Delta\geq 0\}} > z_\varepsilon\right\} = \mathbf{P}\left\{\Delta > z_\varepsilon\right\} = \varepsilon.$$

Using the same weak convergence we obtain the asymptotic normality of the MLE

$$\hat v_n = \frac{\hat\vartheta_n - \vartheta_1}{\varphi_n} \Longrightarrow \hat v = \Delta\mathbb{1}_{\{\Delta\geq 0\}}.$$

The limit behavior of the power functions we study under alternative $\vartheta_u = \vartheta_1 + u\varphi_n$. Let us fix $u > 0$.

We can write

$$\begin{aligned}\sup_{v>0} Z_n\left(v\right) &= \sup_{v>0}\frac{L\left(\vartheta_1+v\varphi_n,X^n\right)}{L\left(\vartheta_1,X^n\right)} = \frac{L\left(\vartheta_u,X^n\right)}{L\left(\vartheta_1,X^n\right)}\sup_{v>0}\frac{L\left(\vartheta_1+v\varphi_n,X^n\right)}{L\left(\vartheta_u,X^n\right)}\\ &= \frac{L\left(\vartheta_u,X^n\right)}{L\left(\vartheta_1,X^n\right)}\sup_{v>0}\frac{L\left(\vartheta_u+(v-u)\varphi_n,X^n\right)}{L\left(\vartheta_u,X^n\right)}.\end{aligned}$$

Note that as

$$\left(\frac{L\left(\vartheta_u,X^n\right)}{L\left(\vartheta_1,X^n\right)}\right)^{-1} = \frac{L\left(\vartheta_u-u\varphi_n,X^n\right)}{L\left(\vartheta_u,X^n\right)} \Rightarrow Z(-u) = \exp\left\{-u\Delta - \frac{u^2}{2}\right\}$$

and

$$\frac{L\left(\vartheta_u+(v-u)\varphi_n,X^n\right)}{L\left(\vartheta_u,X^n\right)} \Rightarrow \exp\left\{\left(v-u\right)\Delta - \frac{\left(v-u\right)^2}{2}\right\},$$

we obtain

$$\sup_{v>0} Z\left(v\right) \to \sup_{v>0} Z\left(v,u\right) = \sup_{v>0}\exp\left\{v\Delta - \frac{\left(v^2-2vu\right)}{2}\right\}.$$

Therefore,

$$\begin{aligned}\beta\left(\psi_n^o,u\right) &\to \mathbf{P}_u\left\{\left(\Delta+u\right)\mathbb{1}_{\{\Delta+u\geq 0\}} > z_\varepsilon\right\}\\ &= \mathbf{P}\left\{\max\left[\Delta+u,\,0\right] > z_\varepsilon\right\}\\ &= \mathbf{P}\left\{\max\left[\zeta,\,-u\right] > z_\varepsilon - u\right\}\mathbb{1}_{\{z_\varepsilon\geq u\}}\\ &\qquad + \mathbb{1}_{\{z_\varepsilon<u\}}\Big[\mathbf{P}\left\{\zeta < -u\right\}\mathbb{1}_{\{z_\varepsilon-u<-u\}} + \mathbf{P}\left\{\zeta > z_\varepsilon-u,\,\zeta > -u\right\}\Big]\\ &= \mathbf{P}\left\{\zeta > z_\varepsilon - u\right\} = \beta^\star\left(u\right)\end{aligned}$$

and similarly

$$\begin{aligned}\mathbf{P}_{\vartheta_u}^{(n)}\left\{Q\left(X^n\right) > h_\varepsilon\right\} &\longrightarrow \mathbf{P}_{\vartheta_1}\left\{\left(\Delta+u\right)^2\mathbb{1}_{\{\Delta+u\geq 0\}} > z_\varepsilon^2\right\}\\ &= \mathbf{P}\left\{\zeta > z_\varepsilon - u\right\} = \beta^\star\left(u\right).\end{aligned}$$

Therefore the tests are LAUMP. □

This asymptotic equivalence and optimality of these tests is a well known property of the tests in regular statistical experiences (see, e.g. [21]). We remind this here to show the difference between regular and non regular situations below. At particularly, we will see that the asymptotic properties of these tests in non regular situations will be quite different.

2.3.3 Bayesian test

Suppose now that the unknown parameter ϑ is a random variable with an *a priori* density $p(\theta)$, $\theta \in [\vartheta_1, b]$. Here $p(\cdot)$ is a known continuous function satisfying condition $p(\vartheta_1) > 0$.

There are at least two possibilities here. One is to use the Bayes estimator (BE) in the construction of the test :

$$\tilde{\psi}_n(X^n) = \mathbb{1}_{\left\{\varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_1) > g_\varepsilon\right\}}.$$

Remind that the BE for quadratic loss function is

$$\tilde{\vartheta}_n = \int_{\vartheta_1}^{b} \theta\, p(\theta|X^n)\,\mathrm{d}\theta = \frac{\int_{\vartheta_1}^{b} \theta\, p(\theta)\, L(\theta, \vartheta_1, X^n)\,\mathrm{d}\theta}{\int_{\vartheta_1}^{b} p(\theta)\, L(\theta, \vartheta_1, X^n)\,\mathrm{d}\theta}.$$

The threshold g_ε is obtained with the help of the following convergence:

$$\begin{aligned}
\frac{\tilde{\vartheta}_n - \vartheta_1}{\varphi_n} \Longrightarrow \tilde{u} &= \frac{\int_0^\infty u Z(u)\,\mathrm{d}u}{\int_0^\infty Z(u)\,\mathrm{d}u} \\
&= \frac{e^{\Delta^2/2}\int_0^\infty (u-\Delta)\exp\left\{-\frac{(v-\Delta)^2}{2}\right\}\mathrm{d}u}{e^{\Delta^2/2}\int_0^\infty \exp\left\{-\frac{(v-\Delta)^2}{2}\right\}\mathrm{d}u} + \Delta \\
&= \frac{-\exp\left\{-\frac{(v-\Delta)^2}{2}\right\}\Big|_{v=0}^{+\infty}}{\sqrt{2\pi}\frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left\{-\frac{(v-\Delta)^2}{2}\right\}\mathrm{d}u} + \Delta \\
&= \frac{\exp\left\{-\frac{\Delta^2}{2}\right\}}{\sqrt{2\pi}\left(1 - F(-\Delta)\right)} + \Delta = \frac{f(\Delta)}{F(\Delta)} + \Delta
\end{aligned}$$

where $f(\cdot)$, $F(\cdot)$ are the density and distribution function of the standard normal distribution.

The similar calculation under alternatives allows us to write the limit power function of $\tilde{\psi}_n$ as follows.

$$\beta\left(\tilde{\psi}_n, u\right) = \mathbf{P}_{\vartheta_1 + u\varphi_n}\left\{\varphi_n^{-1}\left(\tilde{\vartheta}_n - \vartheta_u\right) + u > g_\varepsilon\right\}$$

$$\longrightarrow \mathbf{P}_u \left\{ \frac{\int_{-u}^{\infty} vZ(v)\,\mathrm{d}v}{\int_{-u}^{\infty} Z(v)\,\mathrm{d}v} + u > g_\varepsilon \right\}$$

$$= \mathbf{P}_u \left\{ \frac{\exp\left\{-\frac{(\Delta+u)^2}{2}\right\}}{\sqrt{2\pi}F(\Delta+u)} + \Delta + u > g_\varepsilon \right\}$$

$$= \mathbf{P}_u \left\{ \frac{f(\Delta+u)}{F(\Delta+u)} + \Delta + u > g_\varepsilon \right\}.$$

Another possibility in bayesian approach is to define the test with the minimal mean error. Denote $\alpha\left(\bar\psi_n,\theta\right) = 1-\beta\left(\bar\psi_n,\theta\right)$ the type two error under alternative and introduce the mean error

$$\alpha\left(\bar\psi_n\right) = \int_{\vartheta_1}^{b} \alpha\left(\bar\psi_n,\theta\right)\, p(\theta)\,\mathrm{d}\theta.$$

The bayesian test $\tilde\psi_n(X^n)$ is defined as the test which minimizes the mean error

$$\alpha\left(\tilde\psi_n\right) = \inf_{\bar\psi_n\in\mathcal{K}_\varepsilon} \alpha\left(\bar\psi_n\right).$$

The integral we can write as follows

$$\int_{\vartheta_1}^{b} \mathbf{E}_\theta \bar\psi_n(X^n)\, p(\theta)\,\mathrm{d}\theta = \int_{\vartheta_1}^{b}\int \bar\psi_n(x^n)\,\mathrm{d}\mathbf{P}_\theta^{(n)}\, p(\theta)\,\mathrm{d}\theta$$
$$= \int \bar\psi_n(x^n)\,\mathrm{d}\tilde{\mathbf{P}}^{(n)} = \tilde{\mathbf{E}}^{(n)}\,\bar\psi_n(X^n),$$

where we denoted

$$\tilde{\mathbf{P}}^{(n)}(A) = \int_{\vartheta_1}^{b} \mathbf{P}_\theta^{(n)}(A)\, p(\theta)\,\mathrm{d}\theta.$$

The power $\beta\left(\tilde\psi_n\right) = \tilde{\mathbf{E}}^{(n)}\,\bar\psi_n(X^n)$ is the same as if we have the following two simple hypotheses. Under $\mathscr{H}_1$ we observe a Poisson process of intensity function $\lambda(\vartheta_1,\cdot)$, and under alternative $\mathscr{H}_2$ the observed point process has random intensity and its measure is $\tilde{\mathbf{P}}^{(n)}$. This process is a mixture (according to the density $p(\theta)$) of inhomogeneous Poisson processes with intensities $\lambda(\theta,\cdot)$, $\theta\in[\vartheta_1,b]$. This means that we have two simple hypotheses and the most powerful test by Neyman-Pearson lemma is of the form

$$\tilde\psi_n = \mathbb{1}_{\left\{\tilde L(X^n)>k_\varepsilon\right\}}, \qquad \mathbf{E}_{\vartheta_1}\tilde\psi_n(X^n) = \varepsilon,$$

where the likelihood ratio ratio

$$\tilde L(X^n) = \frac{\mathrm{d}\tilde{\mathbf{P}}^{(n)}}{\mathrm{d}\mathbf{P}_{\vartheta_1}^{(n)}}(X^n) = \int_{\vartheta_1}^{b} \frac{\mathrm{d}\mathbf{P}_\theta^{(n)}}{\mathrm{d}\mathbf{P}_{\vartheta_1}^{(n)}}(X^n)\, p(\theta)\,\mathrm{d}\theta.$$

To study this test under hypothesis we change the variables

$$\tilde L(X^n) = \int_{\vartheta_1}^{b} L(\theta,\vartheta_1,X^n)\, p(\theta)\,\mathrm{d}\theta = \varphi_n \int_0^{\varphi_n^{-1}(b-\vartheta_1)} Z_n(v)\, p(\vartheta_1+v\varphi_n)\,\mathrm{d}v.$$

The last integral was already studied when the properties of the BE were described. At particularly we have the following limit

$$\begin{aligned}\widetilde{R}_n &= \frac{1}{p(\vartheta_1)}\int_0^{\varphi_n^{-1}(b-\vartheta_1)} e^{v\Delta_n-\frac{v^2}{2}+r_n}\, p(\vartheta_1+v\varphi_n)\,\mathrm{d}v\\ &\Longrightarrow \int_0^\infty e^{v\Delta-\frac{v^2}{2}}\,\mathrm{d}v = e^{\frac{\Delta^2}{2}}\int_{-\Delta}^\infty e^{\frac{y^2}{2}}\,\mathrm{d}y\\ &= e^{\frac{\Delta^2}{2}}\left(1-F(-\Delta)\right) = \frac{F(\Delta)}{f(\Delta)},\end{aligned}$$

where $F(\cdot)$ is the distribution function of the standard normal distribution. Hence k_ε is solution of the equation

$$\mathbf{P}\left\{\frac{F(\Delta)}{f(\Delta)} > k_\varepsilon\right\} = \varepsilon. \tag{2.8}$$

Therefore we slightly modify the test and put

$$\tilde{\psi}_n(X^n) = \mathbb{1}_{\{R_n>k_\varepsilon\}}, \qquad R_n = \frac{\varphi_n^{-1}\tilde{L}_n(X^n)}{p(\vartheta_1)}.$$

The similar calculation yields the power function

$$\mathbf{P}_{\vartheta_u}^{(n)}\{R_n > k_\varepsilon\} \longrightarrow \mathbf{P}\left\{\frac{F(\Delta+u)}{f(\Delta+u)} > k_\varepsilon\right\}.$$

Remark 2.1. To simplify the calculation of the threshold k_ε we can, for example, modify the test as follows. Let us put

$$\tilde{R}_n = \frac{\varphi_n^{-1}\tilde{L}_n(X^n)}{p(\vartheta_1)\sqrt{2\pi}}\, e^{-\frac{\varphi_n^{-2}(\hat{\vartheta}_n-\vartheta_1)^2}{2}}.$$

Then under hypothesis we have the limit

$$\tilde{R}_n \Longrightarrow F(\Delta) = \eta, \quad \eta \sim \mathcal{U}[0,1]$$

and the threshold $k_\varepsilon = \varepsilon$.

2.4 Numerical simulation

Below we present the results of numerical simulations of the realizations of the likelihood ratio and show the approximation of the power functions of the tests. We observe n independent realizations $X_j = \{X_j(t), t \in [0,3]\}$; $j = 1, ..., n$ of inhomogeneous Poisson process of intensity function

$$\lambda(\vartheta, t) = 3\cos^2(\vartheta t) + 1,\ 0 \le t \le 3, \qquad \vartheta \in [3, 7).$$

where $\vartheta_1 = 3$. The Fisher information at the point ϑ_1 is $\mathrm{I}(\vartheta_1) \approx 19.8244$. As all tests in regular case are LAUMP and have the same limit power function, we just comparer the power functions of different tests for finite n to see the convergence to the limit curve.

The realizations of the log likelihood $\ln L(\vartheta, X^n), \vartheta \in (1, 7)$ are shown on the Fig. 1 for $n = 100$ and $n = 1000$. It is clearly seen that the max of these curves is near the true value $\vartheta = 3$. For the normalized likelihood ratio $Z_n(u)$ we have the expression :

$$Z_n(u) = \exp\left\{\varphi_n \sum_{j=1}^{n} \int_0^3 \ln \frac{3\cos^2((3 + u\varphi_n)\, t) + 1}{3\cos^2(3t) + 1}\, \mathrm{d}X_j(t) \right.$$
$$\left. - \frac{3n}{4(3 + u\varphi_n)} \sin(6(3 + u\varphi_n)) + \frac{n}{4}\sin(18)\right\}.$$

where $\varphi_n = (19.82\, n)^{-1/2}$.

The realizations of the random functions $Z_n(u)$ are given on the Fig. 2.2.

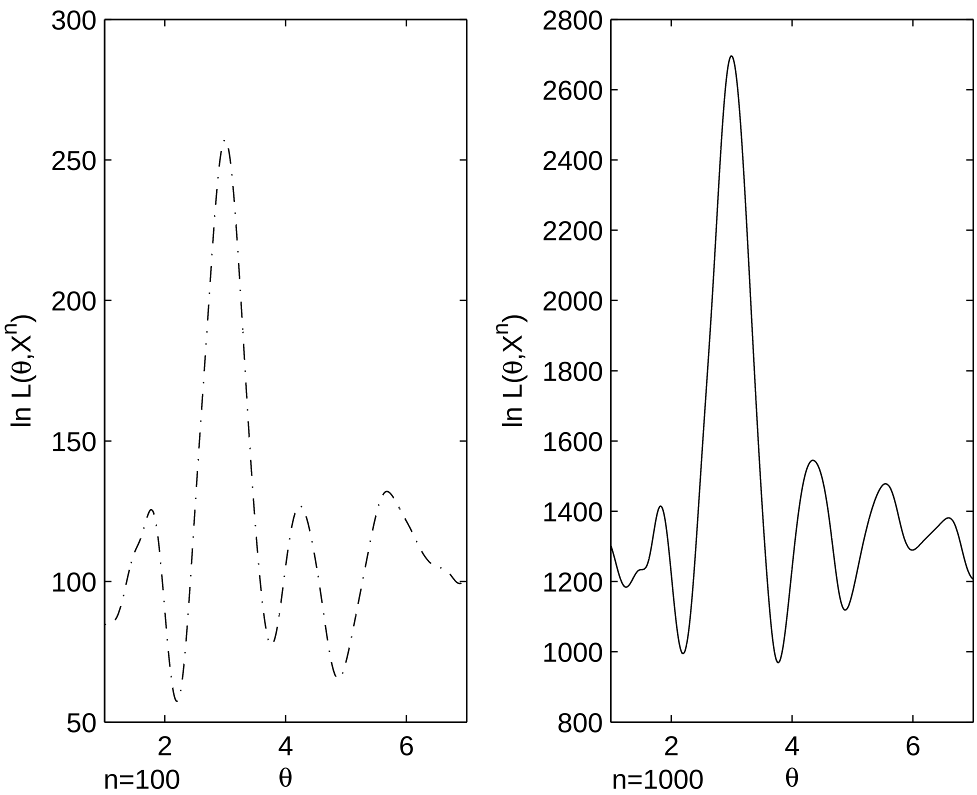

Figure 2.1: Some realizations of $\ln L(\vartheta, X^n)$ with $n = 100$ and $n = 1000$

We obtained analytically the thresholds of the studied tests by the central limit theorem. The convergence of the power functions shows how good is the approximation in the case of finite n.

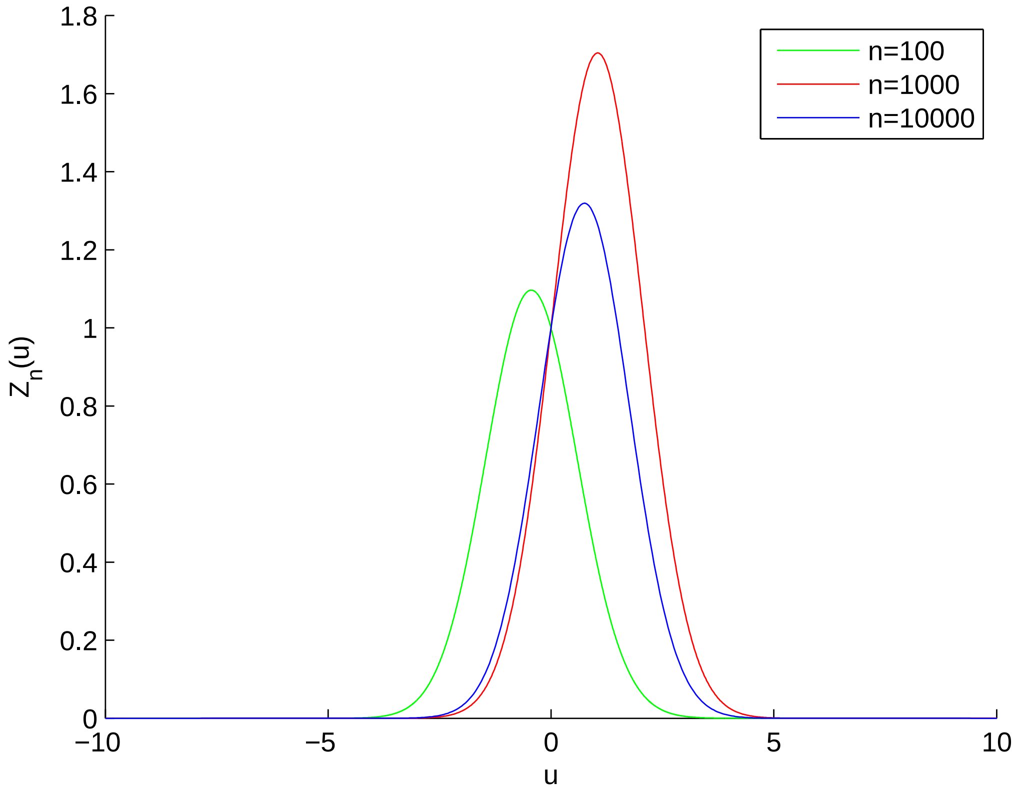

Figure 2.2: Some realizations of $Z_n(u) = L(\vartheta, \vartheta_1, X^n)$ with different n

We calculate the power function of the score function test as follows. We define a increasing sequence u beginning at 0. For every u, we simulate N observations i.i.d of $Y_i = X_i^n$, $i = 1, ..., N$ of the intensity function $\lambda\,(3 + u\varphi_n, t)$ and correspondingly $\Delta_{n,i}(3, Y_i)$, $i = 1, ..., N$, calculate the empirical frequency of acceptance of the alternative hypothesis

$$\beta_n(u) \approx \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{\Delta_{n,i}(3,Y_i)>z_\varepsilon\}}$$

and repeat it with the step $\Delta u = 0.1$ until $\beta^*(u)$ takes the value closed to 1.

In the calculation of the power function of the Bayesian estimator test(BT1), we defined the density *a priori* of the continuous uniform distribution $p\,(\vartheta) \sim \mathcal{U}([3, 7])$. The thresholds of the BT1 are obtained by simulating $M = 10^5$ r.v.s of $\zeta_i \sim \mathcal{N}\,(0, 1)$, $i = 1, \ldots, M$, calculating for each of them the quantity $\frac{f(\zeta_i)}{F(\zeta_i)} + \zeta_i$ and taking $(1-\varepsilon)$M-th greatest between them.

We note that when n is small, under alternative, we discover that the power function of SFT starts to decrease. This interesting fact can be explain by the strongly non linear dependence of the likelihood ratio of the parameter. The test statistics

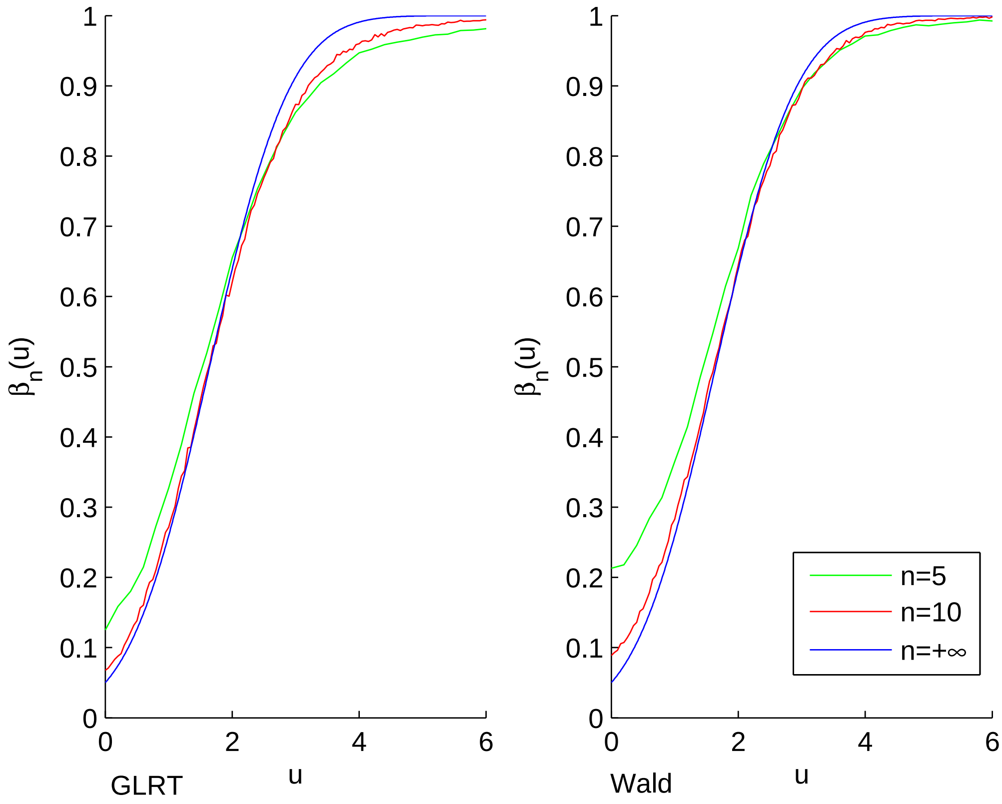

Figure 2.3: Power functions of GLRT and Wald's test for the regular case

ε	0.01	0.05	0.10	0.2	0.4	0.5
g_ε	2.325	1.751	1.478	1.193	0.895	0.794

Table 2.1: The thresholds of the BT1.

$\Delta_n = \Delta_n(3, X^n)$ under alternative can be written as follows

$$\begin{aligned}\Delta_n &= \varphi_n \sum_{j=1}^{n} \int_0^T \frac{\dot\lambda(\vartheta_1, t)}{\lambda(\vartheta_1, t)} \left[\mathrm{d}X_j(t) - \lambda(\vartheta_1 + u\varphi_n, t)\,\mathrm{d}t\right] \\ &\quad + \sqrt{\frac{n}{\mathrm{I}(\vartheta_1)}} \int_0^T \frac{\dot\lambda(\vartheta_1, t)}{\lambda(\vartheta_1, t)} \left[\lambda(\vartheta_1 + u\varphi_n, t) - \lambda(\vartheta_1, t)\right] \mathrm{d}t \\ &= -3\varphi_n \sum_{j=1}^{n} \int_0^3 \frac{t\sin(6t)}{3\cos^2(3\,t) + 1} \left[\mathrm{d}X_t - \left(3\cos^2((3 + u\varphi_n)\,t) + 1\right) \mathrm{d}t\right] \\ &\quad + 9\sqrt{\frac{n}{\mathrm{I}(\vartheta_1)}} \int_0^3 \frac{t\sin(6t)}{3\cos^2(3\,t) + 1} \times \left[\cos^2(3\,t) - \cos^2((3 + u\varphi_n)\,t)\right] \mathrm{d}t\end{aligned} \tag{2.9}$$

and so the value of the second integral in the r.h.s. of the equation (2.9) becomes

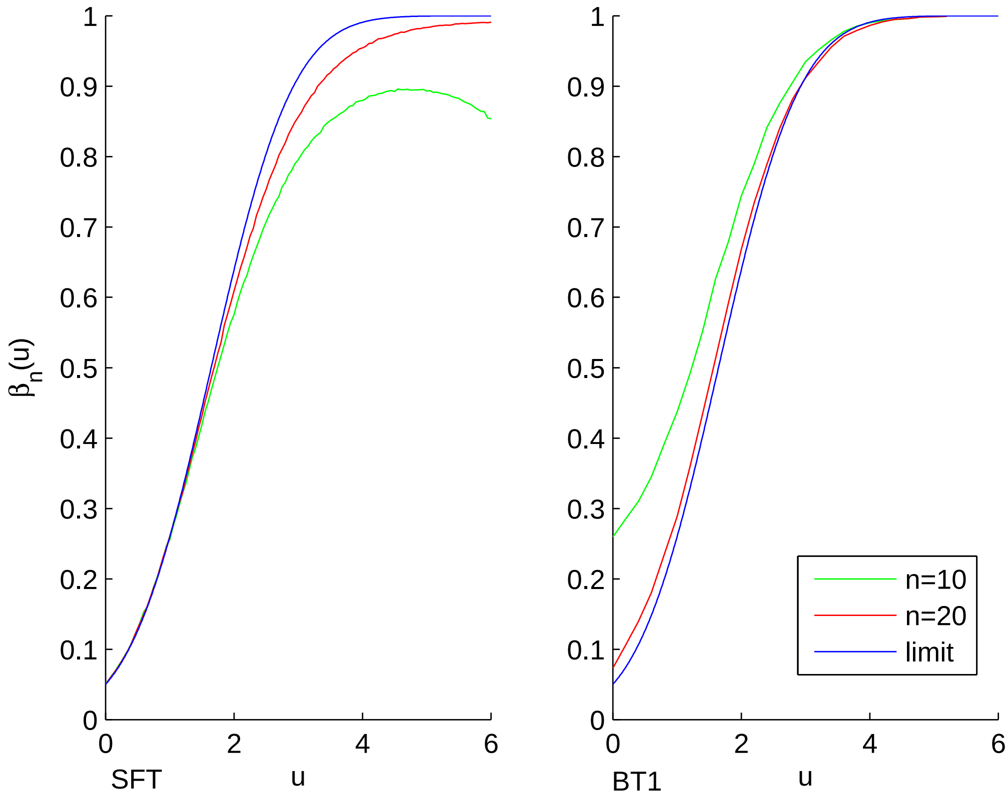

Figure 2.4: Power functions of SFT and BT1 in regular case

negative, and this leads to decreasing of the power function of SFT for the value $n = 10$, in Fig. 2.4.

With the help of 4 graphics, we see that, the power function of WT is more sensible than those of GLRT when u is small. And except BT1, the power functions began to be stable after $n = 20$ and will be coincident with the limit after $n = 60$. In regular case, the limit power function of BT1 is asymptotically equivalent to the NP-T. But its power functions are more sensible than others in the simulations of the thresholds. In singular case it can be much better.

Chapter 3

On hypotheses testing for Poisson processes. Singular cases

We consider the problem of hypothesis testing in the situation when the first hypothesis is simple and the second one is local composite. We describe the choice of thresholds and the power functions of different tests when the intensity has two different types of singularity: cusp and discontinuity. The asymptotic results are illustrated by the numerical simulations.

3.1 Introduction

This is the second part of the study devoted to hypotheses testing problems in the situations when the basic hypothesis is simple ($\vartheta = \vartheta_1$) and the alternative is local one-sided ($\vartheta > \vartheta_1$). As the model of observations we choose the inhomogeneous Poisson process. The first part [9] treat this problem in the *smooth case*, when the intensity function is continuously differentiable with respect to parameter and the Fisher information is finite. In the first part it was shown that the score-function test (SCF), general likelihood ratio test (GLRT), Wald's test (WT) and Bayesian test (BT1) are locally asymptotically uniformly most powerful (LAUMP). The second part of this study is the present work. We investigate the asymptotic behavior of the GLRT, WT, BT1 and BT2 in two non regular (non smooth) situation. At particularly, we study the tests when the intensity functions has cusp-type singularity and jumps-type singularity. In both cases the Fisher information is infinite. The local alternative is obtained by the following re-parametrization $\vartheta = \vartheta_1 + u\varphi_n, u > 0$. The rate of convergence $\varphi_n \to 0$ depends on the order of singularity. In the cusp case $\varphi_n \sim n^{-\frac{1}{2\kappa+1}}$ and in the discontinuous case $\varphi_n \sim n^{-1}$. Our goal is to describe the choice of the thresholds and the behavior of the power functions as $n \to \infty$. The important difference between smooth and non-smooth cases is due to the absence of the criteria of optimality. This leads to the situation when the comparison of the power functions can be done only numerically. That is why the main contribution of this work are the results of numerical simulations of the limit power functions and the comparison them with the power functions with finite small and large values of n.

We recall that $X = (X_t, t \geq 0)$ is an inhomogeneous Poisson process with intensity function $\lambda(t)$, if $X_0 = 0$, the increments of X on disjoint intervals are independent and distributed according to the Poisson law

$$\mathbf{P}\left\{X_t - X_s = k\right\} = \frac{\left(\int_s^t \lambda(v)\,\mathrm{d}v\right)^k}{k!} \exp\left\{-\int_s^t \lambda(v)\,\mathrm{d}v\right\}.$$

We suppose that the intensity function depends on some one-dimensional parameter, i.e., $\lambda(t) = \lambda(\vartheta, t)$ and the basic hypothesis is simple : $\vartheta = \vartheta_1$. The alternative is one-sided composite $\vartheta > \vartheta_1$.

The hypotheses testing problems for inhomogeneous Poisson processes were studied by many authors, see, for example, [9] and the references therein.

3.1.1 Preliminaries

We consider the model of n independent observations of inhomogeneous Poisson processes $X^n = (X_1, \ldots, X_n)$, where $X_j = \{X_j(t), 0 \leq t \leq \tau\}$. We have

$$\mathbf{E}_\vartheta X_j(t) = \Lambda(\vartheta, t) = \int_0^t \lambda(\vartheta, s)\ \mathrm{d}s.$$

We recall the notations introduced in [9]. Here ϑ is one-dimensional parameter and $\mathbf{E}_\vartheta$ is the mathematical expectation, when the true value is ϑ.

Therefore, we suppose that we observe n copies of inhomogeneous Poisson process $X^n = (X_1, \ldots, X_n)$ with the intensity function $\lambda(\vartheta, t), 0 \leq t \leq \tau$. The intensity function is supposed in our work separated from zero on $[0, \tau]$, the measures corresponding to Poisson processes with the different values of ϑ are equivalent and the likelihood function is defined by the equality

$$L(\vartheta, X^n) = \exp\left\{\sum_{j=1}^n \int_0^\tau \ln\lambda(\vartheta, t)\,\mathrm{d}X_j(t) - n\int_0^\tau \left[\lambda(\vartheta, t) - 1\right]\mathrm{d}t\right\}.$$

In non-regular situation we have no UMP test and it is interesting to compare the power functions of the studied tests with the power function of the Neyman-Pearson test (N-PT) for each fixed local alternative. Of course all power functions have to be below the power of N-PT. To remind its definition, let us denote the likelihood ratio for fixed alternative $\vartheta = \vartheta_2 > \vartheta_1$ as $L(\vartheta_2, \vartheta_1, X^n) = L(\vartheta_2, X^n)/L(\vartheta_1, X^n)$. Then by Neyman-Pearson Lemma the test is

$$\hat{\psi}_n(X^n) = \begin{cases} 1, & \text{if} \quad L(\vartheta_2, \vartheta_1, X^n) > b_\varepsilon, \\ q_\varepsilon, & \text{if} \quad L(\vartheta_2, \vartheta_1, X^n) = b_\varepsilon, \\ 0, & \text{if} \quad L(\vartheta_2, \vartheta_1, X^n) < b_\varepsilon. \end{cases}$$

The constants b_ε and q_ε are solutions of the equation

$$\mathbf{P}_{\vartheta_1}\left(L\left(\vartheta_2,\vartheta_1,X^n\right)>b_\varepsilon\right)+q_\varepsilon\mathbf{P}_{\vartheta_1}\left(L\left(\vartheta_2,\vartheta_1,X^n\right)=b_\varepsilon\right)=\varepsilon.$$

This equation is equivalent to the condition $\hat{\psi}_n\left(X^n\right)\in\mathcal{K}_\varepsilon^o$.

Suppose now that we observe n independent inhomogeneous Poisson processes with intensity function $\lambda\left(\vartheta,t\right)$ and we have to test the following two hypotheses

$$\begin{aligned}\mathscr{H}_1 \quad &: \quad \vartheta=\vartheta_1,\\ \mathscr{H}_2 \quad &: \quad \vartheta>\vartheta_1.\end{aligned}$$

In this case the alternative is composite and the power of tests are the functions of ϑ, i.e., for any test $\bar{\psi}_n\left(X^n\right)$ the power is $\beta\left(\bar{\psi}_n,\vartheta\right),\vartheta>\vartheta_1$.

The log likelihood ratio function can be written as

$$\ln L(\vartheta,\vartheta_1,X^n)=\sum_{j=1}^n\int_0^\tau\ln\frac{\lambda\left(\vartheta,t\right)}{\lambda\left(\vartheta_1,t\right)}\mathrm{d}X_j\left(t\right)-n\int_0^\tau\left[\lambda\left(\vartheta,t\right)-\lambda\left(\vartheta_1,t\right)\right]\mathrm{d}t.$$

Denote by $\mathcal{K}_\varepsilon$ the class of test functions $\bar{\psi}_n$ of asymptotic size ε

$$\mathcal{K}_\varepsilon=\left\{\bar{\psi}_n \quad : \quad \lim_{n\to\infty}\mathbf{E}_{\vartheta_1}\bar{\psi}_n\left(X^n\right)=\varepsilon\right\}.$$

We consider *close* or *contiguous* alternatives. Let us put $\vartheta=\vartheta_1+\varphi_n u$, where $\varphi_n=\varphi_n\left(\vartheta_1\right)>0$ and $\varphi_n\to 0$. Then the initial problem of hypotheses testing can be rewritten as

$$\begin{aligned}\mathscr{H}_1 \quad &: \quad u=0,\\ \mathscr{H}_2 \quad &: \quad u>0.\end{aligned}$$

The considered tests are usually of the form

$$\bar{\psi}_n=\mathbb{1}_{\{Y_n(X^n)>c_\varepsilon\}}+q_\varepsilon\mathbb{1}_{\{Y_n(X^n)=c_\varepsilon\}},$$

where the constant c_ε is defined with the help of the limit random variable Y (suppose that $Y_n\Longrightarrow Y$ under hypothesis) by the following relation

$$\mathbf{E}_{\vartheta_1}\bar{\psi}_n=\mathbf{P}_{\vartheta_1}\left\{Y_n\left(X^n\right)>c_\varepsilon\right\}+q_\varepsilon\mathbf{P}_{\vartheta_1}\left\{Y_n\left(X^n\right)=c_\varepsilon\right\}\longrightarrow\mathbf{P}_{\vartheta_1}\left\{Y>c_\varepsilon\right\}=\varepsilon$$

if the limit random variable Y is continuous and by

$$\mathbf{P}_{\vartheta_1}\left\{Y>c_\varepsilon\right\}+q_\varepsilon\mathbf{P}_{\vartheta_1}\left\{Y=c_\varepsilon\right\}=\varepsilon$$

if Y has distribution function with jumps.

Hence, $\bar{\psi}_n\in\mathcal{K}_\varepsilon$.

The corresponding power function is denoted as

$$\beta_n\left(\bar{\psi}_n,u\right)=\mathbf{E}_{\vartheta_1+\varphi_n u}\,\bar{\psi}_n,\qquad u>0.$$

3.1.2 Non regular cases

We consider two different models of close alternatives in non smooth case. The observed Poisson process $X^n = (X_1, \ldots, X_n)$, where the observation is defined by $X_j = \{X_j(t), 0 \leq t \leq \tau\}$ has intensity function $\lambda(\vartheta, t), 0 \leq t \leq \tau$ and we consider the same (simple against composite) hypotheses testing problem

$$\begin{aligned} \mathscr{H}_1 &: \quad \vartheta = \vartheta_1, \\ \mathscr{H}_2 &: \quad \vartheta > \vartheta_1, \end{aligned}$$

but now the function $\lambda(\vartheta, t)$ is not differentiable everywhere and Fisher information is infinite. At particularly, we study the behavior of the tests in two situations. The first one is *cusp* case when the intensity function is continuous but not differentiable everywhere and the second is discontinuous intensity case. In both cases it corresponds to the location parameter and the intensity function $\lambda(\vartheta, t)$ has no derivative at the point $t = \vartheta$. Both situations were already discussed in parameter estimation problems (see [19] and [4]) and here we will show the properties of the tests. The main tool, of course, is the limit behavior of the normalized likelihood ratio function.

In non regular cases presented below there is no LAUMP tests for the limit model and by this reason why the special attention is paid to the numerical simulations.

3.2 Cusp

The intensity function is supposed to be of the form

$$\lambda(\vartheta, t) = a\,|t - \vartheta|^{\kappa} + h(t), \qquad 0 \leq t \leq \tau, \qquad \vartheta \in \Theta = [\vartheta_1, b),$$

where $\kappa \in (0, 1/2)$, $0 < \vartheta_1 < b < \tau$ and $h(\cdot)$ is a known positive bounded function. Remind that if we put $\vartheta = \vartheta_1 + \frac{u}{n^{\frac{1}{2\kappa+1}}}$, then the normalized likelihood ratio

$$\tilde{Z}_n(u) = L\left(\vartheta_1 + \frac{u}{n^{\frac{1}{2\kappa+1}}}, \vartheta_1, X^n\right) = \frac{L\left(\vartheta_1 + \frac{u}{n^{\frac{1}{2\kappa+1}}}, X^n\right)}{L(\vartheta_1, X^n)}$$

has a non degenerate limit (detail see [4])

$$\tilde{Z}_n(u) \Longrightarrow \tilde{Z}(u) = \exp\left\{\Gamma_{\vartheta_1} W^H(u) - \frac{|u|^{2H}}{2}\Gamma_{\vartheta_1}^2\right\}, \qquad u \in \mathbb{R}_+,$$

where $W^H(\cdot)$ is a fractional Brownian motion (fBm), $H = \kappa + \frac{1}{2}$ is the *Hurst parameter* and the constant

$$\Gamma_{\vartheta_1}^2 = \frac{2a^2 B(\kappa + 1, \kappa + 1)}{h(\vartheta_1)}\left[\frac{1}{\cos(\pi\kappa)} - 1\right].$$

To study the limit in more *universal* form we change the parameter as follows

$$\vartheta = \vartheta_1 + u\varphi_n, \qquad \varphi_n = \left(\Gamma_{\vartheta_1}^{1/H} n^{\frac{1}{2\kappa+1}}\right)^{-1}.$$

The corresponding normalized likelihood ratio converges to the limit which does not depend on the constant Γ:

$$Z_n(u) = \frac{L(\vartheta_1 + u\varphi_n, X^n)}{L(\vartheta_1, X^n)} \Longrightarrow Z(u) = \exp\left\{W^H(u) - \frac{|u|^{2H}}{2}\right\}.$$

As before for the comparison of powers of different tests we replace the initial hypotheses testing problem by the following one

$$\begin{aligned} \mathscr{H}_1 &: \quad u = 0, \\ \mathscr{H}_2 &: \quad u > 0. \end{aligned}$$

The score-function test does not exist and we study the others. The limit normalized likelihood ratio $Z(u)$ is the same as the likelihood ratio of the similar hypothesis problem in the case of observations $(Y(v), v \geq 0)$ of the following type

$$Y(v) = u\, \mathbb{1}_{\{v<u\}} + W^H(v), \qquad v \geq 0.$$

The UMP test in this problem does not exists and we have no asymptotically UMP tests.

3.2.1 GLRT

Note that the construction of the tests is almost the same as in regular case and the main difference is in the properties of these tests. For example, the GLRT is defined by the same relations

$$\hat\psi_n(X^n) = \mathbb{1}_{\{Q(X^n)>h_\varepsilon\}},$$

where

$$Q(X^n) = \sup_{\vartheta>\vartheta_1} L(\vartheta, \vartheta_1, X^n) = L\left(\hat\vartheta_n, \vartheta_1, X^n\right).$$

To choose the threshold h_ε we need the solution of the following equation (under hypothesis $\mathscr{H}_1$)

$$\begin{aligned} \mathbf{P}_{\vartheta_1}^{(n)}\left\{\sup_{\vartheta>\vartheta_1} L(\vartheta, \vartheta_1, X^n) > h_\varepsilon\right\} &= \mathbf{P}_{\vartheta_1}^{(n)}\left\{\sup_{v>0} Z_n(v) > h_\varepsilon\right\} \\ \longrightarrow \mathbf{P}_{\vartheta_1}\left\{\sup_{v>0} Z(v) > h_\varepsilon\right\} &= \mathbf{P}\left\{\sup_{v>0}\left[W^H(v) - \frac{v^{2H}}{2}\right] > \ln h_\varepsilon\right\} = \varepsilon. \end{aligned}$$

As we know there is no analytical solution of this equation that is why we turn to the simulation method. Note that $h_\varepsilon = h_\varepsilon(H)$ and does not depend on Γ_{ϑ_1}.

The power function has the following limit

$$\begin{aligned}
&\mathbf{P}^{(n)}_{\vartheta_u}\left\{\sup_{v>0} Z_n\left(v\right)>h_\varepsilon\right\}=\mathbf{P}^{(n)}_{\vartheta_u}\left\{Z_n\left(u\right)\sup_{v>0}\frac{L\left(\vartheta_u+(v-u)\varphi_n,X^n\right)}{L\left(\vartheta_u,X^n\right)}>h_\varepsilon\right\}\\
&\longrightarrow \mathbf{P}_{\vartheta_1}\left\{\left(Z\left(-u\right)\right)^{-1}\sup_{v>0}\exp\left\{W^H\left(v-u\right)-\frac{\left|v-u\right|^{2H}}{2}\right\}>h_\varepsilon\right\}\\
&=\mathbf{P}_u\left\{\sup_{s>0}\left[-W^H\left(-u\right)+W^H\left(s-u\right)-\frac{\left|s-u\right|^{2H}}{2}+\frac{\left|u\right|^{2H}}{2}\right]>\ln h_\varepsilon\right\}\\
&=\mathbf{P}_u\left\{\sup_{s>0}\left[W^H\left(s\right)-\frac{\left|s-u\right|^{2H}}{2}\right]>\ln h_\varepsilon-\frac{\left|u\right|^{2H}}{2}\right\}\equiv\beta\left(u\right).
\end{aligned}$$

This power function is obtained below with the help of numerical simulations.

3.2.2 Wald's test

We already know that the MLE converges in distribution

$$\varphi_n^{-1}\left(\hat\vartheta_n-\vartheta_1\right)\Longrightarrow\hat u$$

where the random variable $\hat u$ is solution of the equation

$$Z\left(\hat u\right)=\sup_{v>0}Z\left(v\right),\qquad Z\left(v\right)=e^{W^H(v)-\frac{v^{2H}}{2}}.$$

Therefore if we put the test

$$\psi_n^o\left(X^n\right)=1\!\!1_{\left\{\varphi_n^{-1}\left(\hat\vartheta_n-\vartheta_1\right)>m_\varepsilon\right\}}+q_\varepsilon 1\!\!1_{\left\{\varphi_n^{-1}\left(\hat\vartheta_n-\vartheta_1\right)=m_\varepsilon\right\}},$$

where q_ε is supposed to be zero and m_ε is defined by the equation

$$\mathbf{P}\left\{\hat u>m_\varepsilon\right\}=\varepsilon,$$

then $\psi_n^o\in\mathcal{K}_\varepsilon$.

The limit of the power function of the test for the local alternative $\vartheta_u=\vartheta_1+u\varphi_n$ is the following

$$\begin{aligned}
\beta\left(\psi_n^o,u\right)=\mathbf{E}_{\vartheta_u}\psi_n\left(X^n\right)=\mathbf{P}_{\vartheta_u}&\left\{\varphi_n^{-1}\left(\hat\vartheta_n-\vartheta_u\right)+u>m_\varepsilon\right\}\\
&\longrightarrow\mathbf{P}_u\left\{\hat u_*>m_\varepsilon-u\right\}=\beta^o\left(u\right),
\end{aligned}$$

where the random variable $\hat u_*$ is solution of the equation

$$Z\left(\hat u_*\right)=\sup_{v>-u}Z\left(v\right)$$

and $W^H(v)$, $v>-u$ is the fBm. The threshold m_ε and the power function $\beta^o\left(u\right)$ are obtained by the numerical simulations.

3.2.3 Bayesian approach

We suppose that the parameter ϑ is a random variable with the density *a priori* $p(\theta), \vartheta_1 \leq \theta < b$. This function is supposed to be continuous and positive. We consider two tests.

The first one is based on the BE

$$\tilde{\psi}_n(X^n) = \mathbb{1}_{\left\{\varphi_n^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\}},$$

and as before we have the limit

$$\mathbf{E}_{\vartheta_1}\mathbb{1}_{\left\{\varphi_n^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\}} \longrightarrow \mathbf{P}_{\vartheta_1}\left\{\tilde{u} > k_\varepsilon\right\}, \qquad \tilde{u} = \frac{\int_0^\infty vZ(v)\,\mathrm{d}v}{\int_0^\infty Z(v)\,\mathrm{d}v}.$$

For the power function the limit is

$$\beta\left(\tilde{\psi}_n, u\right) = \mathbf{P}_{\vartheta_u}\left\{\varphi_n^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\} = \mathbf{P}_{\vartheta_u}\left\{\frac{\int_0^\infty vZ_u(v)\,\mathrm{d}v}{\int_0^\infty Z_u(v)\,\mathrm{d}v} > k_\varepsilon\right\},$$

where

$$Z_u(v) = \exp\left\{W^H(v) - \frac{|u-v|^{2H}}{2} + \frac{|u|^{2H}}{2}\right\}.$$

The threshold and power function are obtained by the numerical simulations.

The second test is to minimize the mean error, with the same construction as in regular case (see [9]). The likelihood ratio in the bayesian is

$$\tilde{L}(X^n) = \int_{\vartheta_1}^{b} L(\theta, \vartheta_1, X^n)\,p(\theta)\,\mathrm{d}\theta = \varphi_n \int_0^{\varphi_n^{-1}(\beta - \vartheta_1)} Z_n(v)\,p(\vartheta_1 + u\varphi_n)\,\mathrm{d}v$$

Hence we have the following limit

$$\begin{aligned}\varphi_n^{-1}\tilde{L}(X^n) \Longrightarrow & p(\vartheta_1)\int_0^\infty \exp\left\{W^H(v) - \frac{v^{2H}}{2}\right\}\mathrm{d}v\\ &= p(\vartheta_1)\int_0^\infty \exp\left\{W^H(v) - \frac{v^{2H}}{2}\right\}\mathrm{d}v.\end{aligned}$$

Therefore if we take b_ε as solution of the equation

$$\mathbf{P}\left\{\int_0^\infty \exp\left\{W^H(v) - \frac{v^{2H}}{2}\right\}\mathrm{d}v > b_\varepsilon\right\} = \varepsilon$$

then the test

$$\tilde{\psi}_n(X^n) = \mathbb{1}_{\{R_n > b_\varepsilon\}}, \qquad R_n = \frac{\varphi_n^{-1}\tilde{L}(X^n)}{p(\vartheta_1)}$$

belongs to the class $\mathcal{K}_\varepsilon$.

3.2.4 Simulations

Let us consider the following example. We observe n independent realizations $X^n = (X_1, \ldots, X_n)$ of inhomogeneous Poisson process

$$X_j = \{X_j(t), t \in [0,2]\}\,; \qquad j = 1, ..., n.$$

The intensity function of this processes is

$$\lambda(\vartheta, t) = 2 - |t - \vartheta|^{2/5}\,; \qquad 0 \leq t \leq 2,$$

where the parameter $\vartheta \in \left[\frac{1}{2}, 2\right)$. We take $\vartheta_1 = 1.5$ as the value of the basic hypothesis. The Hurst parameter is $H = 0.9$ and the constant

$$\Gamma_{\vartheta_1}^2 = B(1.4, 1.4)\left[\frac{1}{\cos(0.4\pi)} - 1\right] \approx 1.027.$$

The realizations of the normalized likelihood ratio $Z_n(u)$ under hypothesis are given on the Fig. 1 for the values $n = 100, n = 1000$ and $n = 10000$. It is easy to

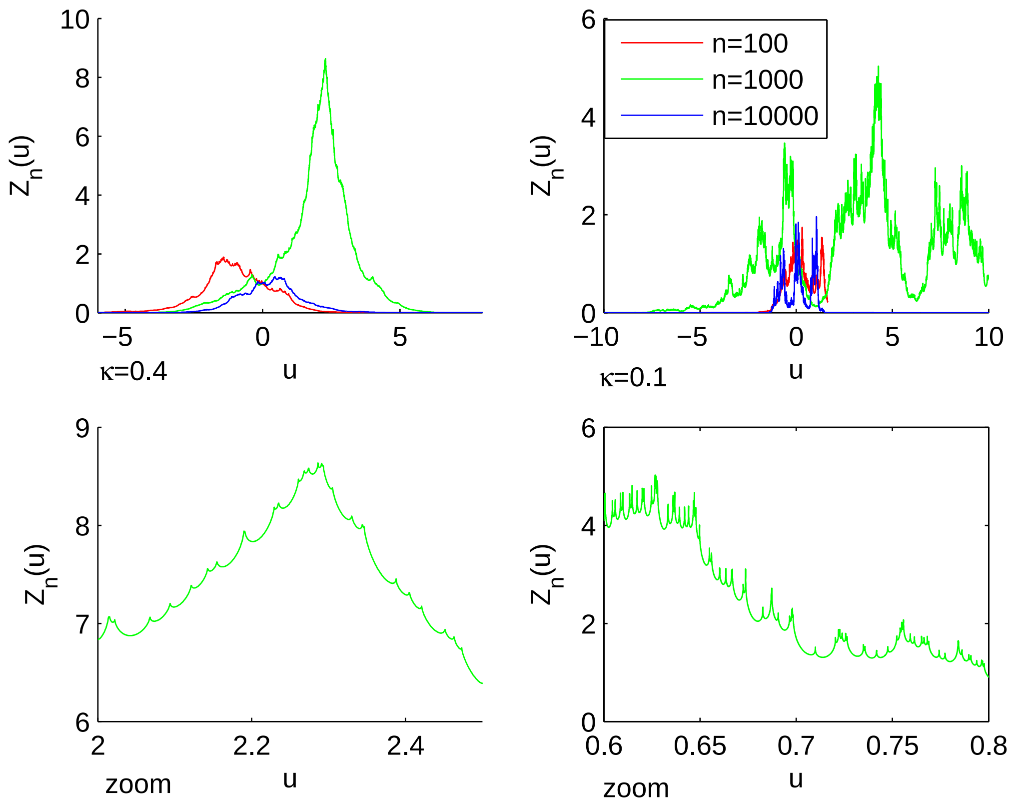

Figure 3.1: Realizations of $Z_n(u)$ with $\lambda(\vartheta, t) = 2 - |t - \vartheta|^{\kappa}$.

see the difference of the trajectories for the two values of κ. As the case $\kappa = 0.4$

corresponds $H = 0.9$ which is close to $H = 1$ we see that the trajectories are "almost smooth".

To find the thresholds of the GLRT and WT we need to calculate the maximal point and the maximal value of this functions. In the case of the chosen intensity function the maximum is attained at one of the cusp points of the observations,shown in the Fig. 3.1.

It is interesting to note that if the intensity function is

$$\lambda(\vartheta, t) = 0.5 + |t - \vartheta|^{\kappa}$$

then to find the maximum is much more difficult, because the function is not differentiable at the cusp-points and the maximum is always between such points. The realizations are given on the Fig. 3.2. The threshold of the GLRT are obtained by

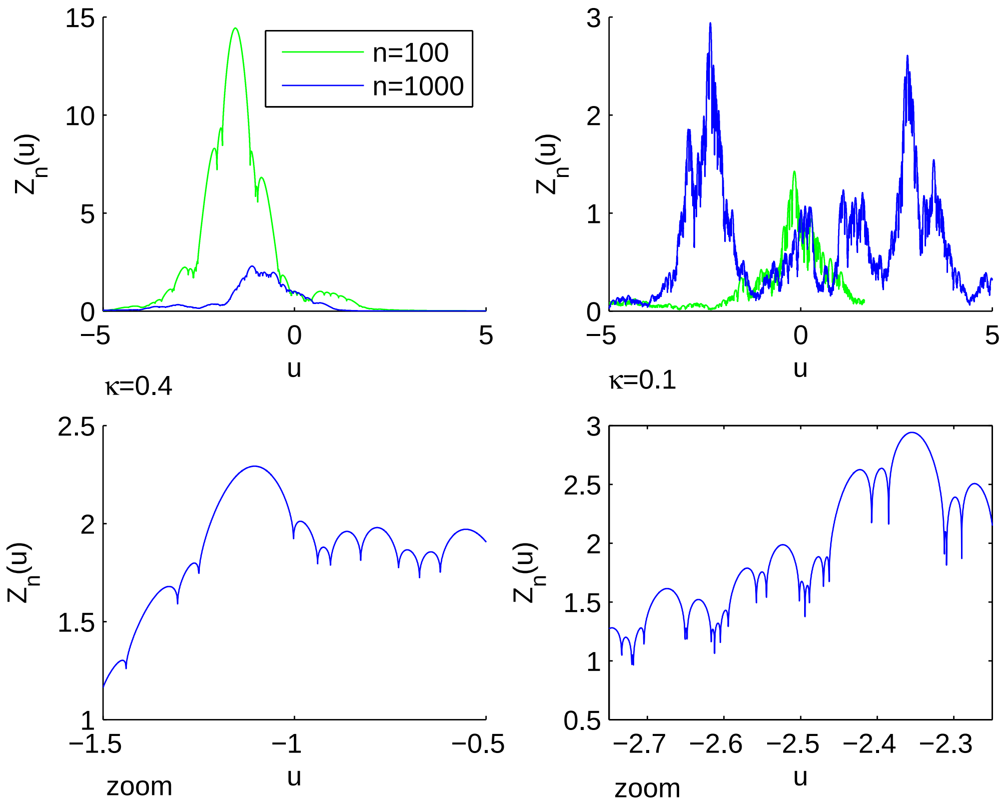

Figure 3.2: Some realizations of $Z_n(u)$. $\lambda(\vartheta, t) = 0.5 + |t - \vartheta|^{\kappa}$.

simulating $M = 10^5$ r.v.s of $Z^i(v)$, $v \in [0, 20]$, $i = 1, \ldots, M$ (when $v > 20$ the value of $Z(v)$ is negligible) and calculating for each of them the quantity $\sup_u Z^i(v)$ and taking $(1 - \varepsilon)$M-th greatest between them.

We simulate the power functions by choosing different n and see the tendency of the convergence. For example, the power function of GLRT can be simulated

ε	0.01	0.05	0.10	0.2	0.4	0.5
$\ln h_\varepsilon$	2.959	1.641	1.081	0.559	0.159	0.068
m_ε	3.041	1.996	1.521	0.950	0.333	0.166
k_ε	2.864	2.0776	1.720	1.365	1.005	0.885

Table 3.1: Thresholds of GLRT, WT and BT1 in cusp case.

as follows. We define a increasing sequence u beginning at 0 such that for every u we simulate N observations i.i.d of X_i^n, $i = 1, ..., N$ of the intensity function $\lambda(\vartheta, t)$ and correspondingly $\sup_{v>0} Z_{n,i}(v)$, $i = 1, ..., N$, calculate the empirical frequency of acceptance of the alternative hypothesis

$$\beta_n(u) \approx \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\left\{\sup\limits_{v>0} Z_{n,i}(v) > h_\varepsilon\right\}}$$

and repeat it with the step $\Delta u = 0.1$ until $\beta_n(u)$ takes the value closed to 1.

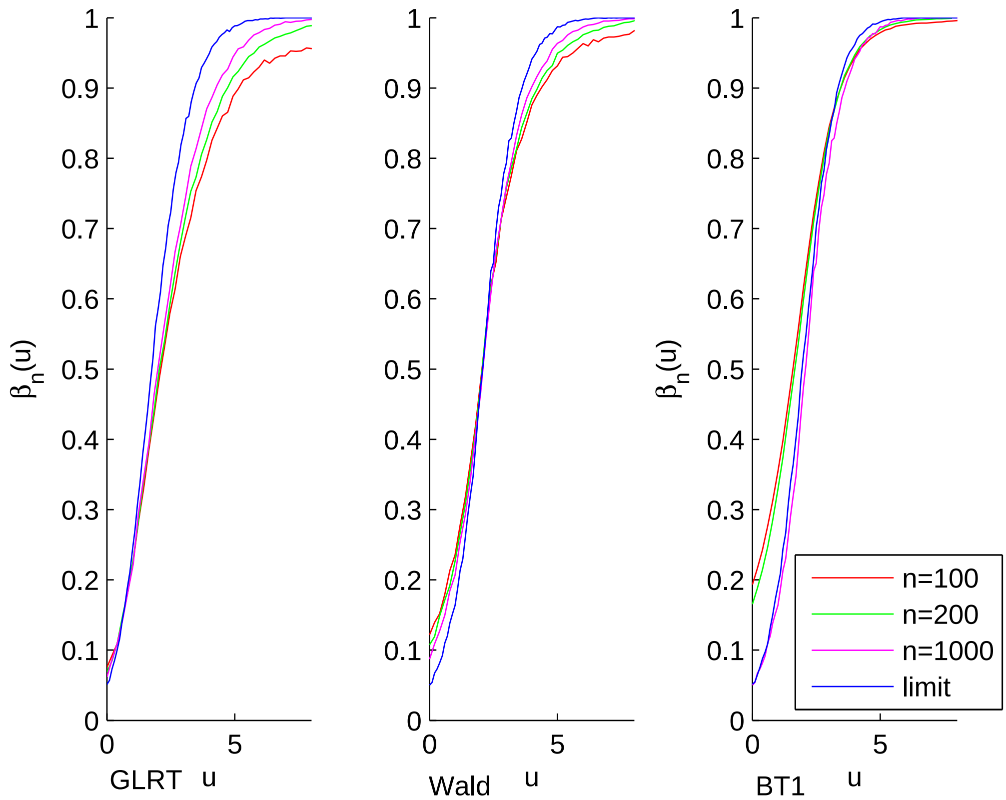

Figure 3.3: Power functions of GLRT, WT and BT1 in cusp case. $\lambda(\vartheta, t) = 2 - |t - \vartheta|^{0.4}$.

In this case, we can see that, like in the regular case, in the beginning when u is small, the power function of WT converge more slowly than that of GLRT, but still quicker than that of BT1. When u is large, the power function of BT1 converge more quickly than WT, and the power function of GLRT converge the most slowly.

3.2.5 Comparison of the limit power functions

Our goal is to compare the limit power functions of all studied tests with the help of numerical simulations because the analytic expressions for these functions are not yet available. It will be interesting to see as well the limit power function of the Neyman-Pearson Test (N-PT) constructed in the problem of testing two simple hypotheses as follows. Let us fix an alternative $\vartheta_2 > \vartheta_1$ and consider the hypotheses testing problem

$$\begin{aligned} \mathscr{H}_1 \quad &: \quad \vartheta = \vartheta_1, \\ \mathscr{H}_2 \quad &: \quad \vartheta = \vartheta_2. \end{aligned}$$

As usual we need the local alternatives to have non degenerate limit of the power functions therefore we introduce the change of variables $\vartheta_2 = \vartheta_1 + u_1\varphi_n$. This leads us to the problem

$$\begin{aligned} \mathscr{H}_1 \quad &: \quad u = 0, \\ \mathscr{H}_2 \quad &: \quad u = u_1 \qquad (u_1 > 0). \end{aligned}$$

The Neyman-Pearson test is

$$\psi_n^* (X^n) = \mathbb{1}_{\{Z_n(u_1) > d_\varepsilon\}},$$

where the threshold d_ε is la solution of the equation

$$\mathbf{P}_{\vartheta_1} (Z(u_1) > d_\varepsilon) = \varepsilon.$$

Remind that

$$Z(u_1) = \exp\left\{ W^H(u_1) - \frac{u_1^{2H}}{2} \right\}.$$

Hence

$$\mathbf{P}_{\vartheta_1} (Z(u_1) > d_\varepsilon) = \mathbf{P}\left\{ W^H(u_1) - \frac{u_1^{2H}}{2} > \ln d_\varepsilon \right\} = \mathbf{P}\left(\zeta > \frac{\ln d_\varepsilon + \frac{u_1^{2H}}{2}}{u_1^H} \right),$$

where

$$d_\varepsilon = e^{z_\varepsilon u_1^H - \frac{u_1^{2H}}{2}}, \qquad \mathbf{P}(\zeta > z_\varepsilon) = \varepsilon;\ \zeta \sim \mathcal{N}(0,1).$$

Of course, it is impossible indeed to have N-PT because the value of parameter ϑ or u_1 under alternative is unknown, but as this test is the most powerful in the class

$\mathcal{K}_\varepsilon$ its power function shows an upper bound for powers of all tests. The distance between it and the power functions of studied tests provides useful information.

To study the likelihood ratio function under alternative we write

$$Z_n(u_1) = \frac{L(\vartheta_1 + u_1\varphi_n, X^n)}{L(\vartheta_1, X^n)} = \left(\frac{L(\vartheta_1 + u_1\varphi_n - u_1\varphi_n, X^n)}{L(\vartheta_1 + u_1\varphi_n, X^n)}\right)^{-1}.$$

For the power function of N-PT we obtain the following convergence

$$\beta_n(u_1) = \beta(\psi_n^*(X^n), u_1) = \mathbf{P}_{\vartheta_1 + u_1\varphi_n}(Z_n(u_1) > d_\varepsilon)$$
$$\longrightarrow \beta(u_1) = \mathbf{P}_{\vartheta_1}\left((Z(-u_1))^{-1} > d_\varepsilon\right) = \mathbf{P}_{\vartheta_1}\left(\exp\left\{-W^H(-u_1) + \frac{u_1^{2H}}{2}\right\} > d_\varepsilon\right).$$

and hence

$$\beta(u_1) = \mathbf{P}\left(\zeta > \frac{\ln d_\varepsilon - \frac{u_1^{2H}}{2}}{u_1^H}\right) = \mathbf{P}\left(\zeta > z_\varepsilon - u_1^H\right).$$

This allows us to calculate the power function with the help of the distribution function of the standard normal law.

Note that the Third Le Cam's Lemma allows to calculate this power function according to the following relations:

$$\beta_n(u) = \beta(\psi_n^*(X^n), u) = \mathbf{E}_{\vartheta_1 + u_1\varphi_n}\psi_n^*(X^n)$$
$$= \mathbf{E}_{\vartheta_1} Z_n(u_1)\,\psi_n^*(X^n) \longrightarrow \mathbf{E}_{\vartheta_1} Z(u_1)\,\mathbb{1}_{\{Z(u_1) > d_\varepsilon\}}.$$

Of course, we need not use it here, but in the case of other tests this relation simplifies significantly the calculations.

The limit power functions are calculated by Monte-Carlo method by the same way as it was explained as above.

To compare the limit power functions, we calculate two distances between the limit power function and envelope power function of NP-T: the Minkowski distance of order 2 (2-norm distance) which describes the sum of the distance of each point between two lines,and the Chebyshev distance, which describes the maximal distance between two lines.

In this example, we calculate, the difference of the points u for each limit power function $\beta(u)$ and the envelope power function $\beta^*(u)$; that is,

$$D_{Minkowski} = \sqrt{\sum_i \left(\beta(u^i) - \beta^*(u^i)\right)^2}$$

and

$$D_{Chebyshev} = \sup_i \left|\beta(u^i) - \beta^*(u^i)\right|.$$

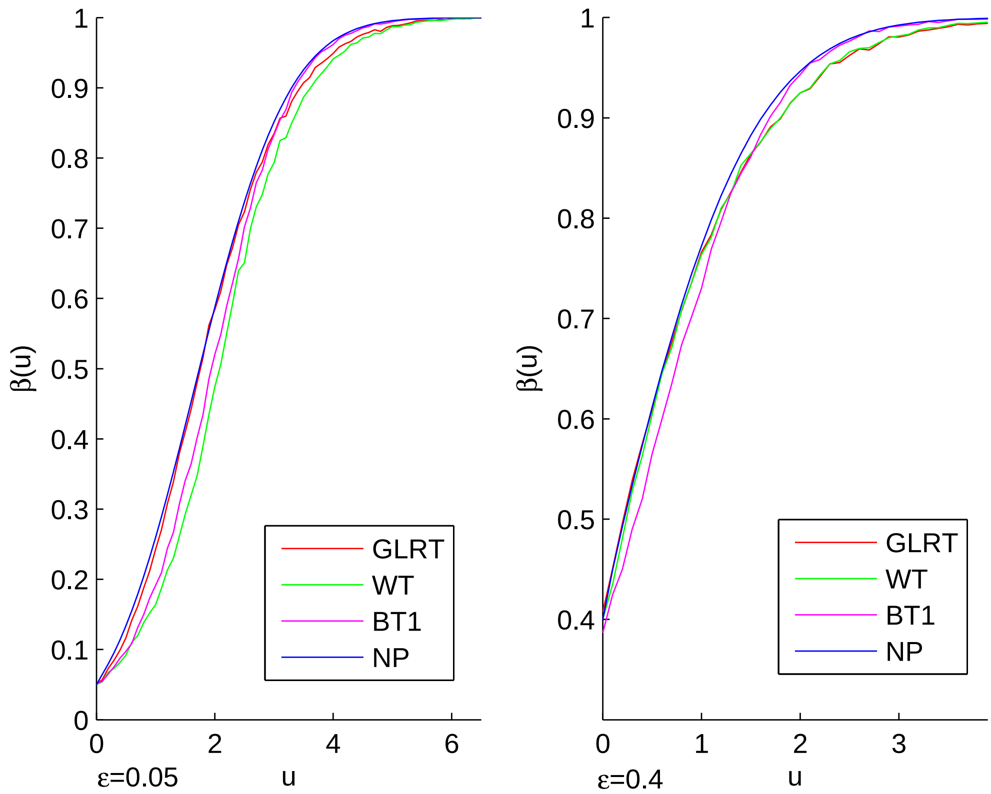

Figure 3.4: Comparison of limit power functions in cusp case with $\lambda(\vartheta, t) = 2 - |t - \vartheta|^{0.4}$ and $\varepsilon = 0.05$.

	$\varepsilon = 0.05$			$\varepsilon = 0.4$		
	GLRT	WT	BT1	GLRT	WT	BT1
$D_{Minkowski}$	0.0990	0.5109	0.3205	0.0886	0.0882	0.1501
$D_{Chebyshev}$	0.0259	0.1399	0.0888	0.0265	0.0256	0.0535

Table 3.2: Distances between each limit power function and envelop power function with $\varepsilon = 0.05$ and $\varepsilon = 0.4$.

We can see that, the power function of GLRT is the closest one to the power function of NP-T. When ε is small, the power function of WT is lower than BT1. It becomes coincident with that of GLRT when ε increases. At the same time, the power function of BT1 will become the lowest one. We also mention that for the power function of BT1, it arrives more quickly to 1 than the others, which we can see in Fig. 3.4.

3.3 Discontinuous intensity

We consider the model of inhomogeneous Poisson process with discontinuous intensity function. The Fisher information is equal infinity and we have singular statistical problem. The properties of estimators for such models are well known in [19].

Condition ***L***. *Suppose that the intensity $\lambda(\vartheta,t)=\lambda(t-\vartheta)$, where the function $\lambda(t)$ is continuously differentiable everywhere except at the point t_*. The parameter $\vartheta\in\Theta\subset(0,\tau)$. At the point t_* this function has a jump $r=\lambda(t_*+)-\lambda(t_*-)=\lambda_+-\lambda_-\neq 0$.*

The intensity function $\lambda(\vartheta,t)$ has a jump at the instant $t=t_*+\vartheta$ and the parameter $\vartheta_1\in(-t_*,\tau-t_*)$. We have to test the hypotheses

$$\mathscr{H}_1 \quad : \quad \vartheta=\vartheta_1,$$
$$\mathscr{H}_2 \quad : \quad \vartheta>\vartheta_1.$$

Of course, under alternative $\vartheta\in(\vartheta_1,\tau-t_*)$. We are interested by the close alternatives and in this case the change of variables $\vartheta=\vartheta_1+u\varphi_n$; $\varphi_n=\frac{1}{n\lambda_+}$ reduces the problem to the following one

$$\mathscr{H}_1 \quad : \quad u=0,$$
$$\mathscr{H}_2 \quad : \quad u>0.$$

Recall that the normalized likelihood ratio

$$L\left(\vartheta_1+\frac{v}{n},\vartheta_1,X^n\right)=\frac{L\left(\vartheta_1+\frac{v}{n},X^n\right)}{L\left(\vartheta_1,X^n\right)},\qquad v\in\left(0,n\left(\tau-t_*-\vartheta_1\right)\right)$$

converges to the process

$$Z\left(v\right)=\exp\left\{\ln\frac{\lambda_-}{\lambda_+}x\left(v\right)-\left(\lambda_--\lambda_+\right)v\right\}\qquad v\geq 0,$$

where $x\left(v\right)$ is Poisson process with constant intensity λ_+ (the details see [19]).

And hence, denoting $\rho=\frac{\lambda_-}{\lambda_+}$, we have

$$Z_n(v)=\frac{L\left(\vartheta_1+v\varphi_n,X^n\right)}{L\left(\vartheta_1,X^n\right)},\qquad v\in\mathbb{U}_n^+=\left(0,\varphi_n^{-1}\left(\tau-t_*-\vartheta_1\right)\right)$$

converges to the process

$$Z_*(v) = \exp\left\{\ln\rho\, x_*(v) - (\rho-1)\, v\right\}, \qquad v \geq 0,$$

where $x_*(v), v \geq 0$ is the Poisson process of unit intensity.

The limit likelihood ratio $Z_*(v)$ under alternative $\mathscr{H}_1$ is the same as in the problem of hypotheses testing by observations of Poisson process $x_*(v), v \geq 0$ with the switching intensity function

$$\mu(u,v) = \rho \mathbb{1}_{\{v<u\}} + \mathbb{1}_{\{v\geq u\}}, \qquad v \geq 0. \tag{3.1}$$

To compare the power functions of different tests, we consider this likelihood ratio under (close) alternative $u > 0$. Then the process $x_*(v), v \geq 0$ has intensity function ρ for $v < u$ and 1 for $v \geq u$.

3.3.1 Weak convergence

The GLRT, WT, BT are some functionals of the likelihood function $L(\vartheta, X^n)$. It was shown above all these tests can be written as functionals of the normalized likelihood ratio $Z_n(v), v \geq 0$. Therefore as in regular case we have to prove the weak convergence of the measures induced by the random functions $Z_n(\cdot)$.

Let $\mathbb{D}$ be the space of functions $f(\cdot)$ on $\mathbb{R}_a = [a, +\infty)$ which do not have discontinuities of the second kind such that $\lim\limits_{u\to\infty} f(u) = 0$. We suppose that the functions $f(\cdot)$ are cadlag; that is, the left limit $f(t-) = \lim\limits_{s\nearrow t} f(s)$ exists and the right limit $f(t-) = \lim\limits_{s\swarrow t} f(s)$ exists and equals to $f(t)$. Introduce the metric

$$d(f,g) = \inf_{\lambda}\Big[\sup_{u\in\mathbb{R}_a}\left|f(u) - g\big(\lambda(u)\big)\right| + \sup_{u\in\mathbb{R}_a}\left|u - \lambda(u)\right|\Big]$$

in the space $\mathbb{D}$ where inf is taken over all monotone, continuous, one-to-one mappings $\lambda(\cdot) : \mathbb{R}_a \to \mathbb{R}_a$. Suppose

$$\Delta_h(f) = \sup_{u\in\mathbb{R}_a}\sup_{u\in\delta}\left\{\min\Big[\left|f(u') - f(u)\right|, \left|f(u'') - f(u)\right|\Big]\right\} + \sup_{|u|>1/h}\left|f(u)\right|$$

here the interval $\delta = \left[u', u''\right) \subseteq [u-h, u+h)$.

Let the trajectories of the process $Y_n = \{Y_n(u), u \in [a, +\infty)\}$ and the process $Y = \{Y(u), u \in [a, +\infty)\}$ belong to the space $\mathbb{D}$ with probability one with distributions $\mu_\vartheta^{(n)}$ and μ_ϑ on the measurable space $(\mathbb{D}, \mathbb{B})$ depending on the parameter $\vartheta \in \Theta$. Here $\mathbb{D}$ is complete separable space with metric $d(\cdot,\cdot)$ and $\mathbb{B}$ is the σ-algebra of Borel subsets of $\mathbb{D}$.

Theorem 3.1. (A.V.Skorohod) *Let the finite dimensional distributions of the process* Y_n *converge to the finite dimensional distributions of the process* Y *as* $n \to +\infty$ *uniformly for* $\vartheta \in \mathbb{K}$*, where* $\mathbb{K}$ *is arbitrary compact in* Θ*, and for* $\delta > 0$

$$\lim_{h\to 0} \lim_{n\to+\infty} \sup_{\vartheta\in\mathbb{K}} \mu_\vartheta^{(n)}\{\Delta_h(Y_n) > \delta\} = 0. \tag{3.2}$$

Then $\mu_\vartheta^{(n)}$ *converges weakly to* μ_ϑ *uniformly in* $\vartheta \in \mathbb{K}$ *as* $n \to +\infty$*.*

Lemma 3.1. *Let condition* **L** *be fulfilled. Then the finite-dimensional distributions of the process* Z_n *converge to those of the process* Z *under alternative.*

Proof. The characteristic function of $\ln Z_n$ can be written as follows (see [18]):

$$\begin{aligned}
&\mathbf{E}_{\vartheta_1+u\varphi_n} \exp\{i\mu \ln Z_n(v)\}\\
&= \exp\Bigg[n\int_0^\tau \Big(\exp\Big(i\mu\ln\frac{\lambda(t-\vartheta_1-v\varphi_n)}{\lambda(t-\vartheta_1)}\Big)-1\Big)\lambda(t-\vartheta_1-u\varphi_n)\,\mathrm{d}t-\\
&\qquad - ni\mu\int_0^\tau \big(\lambda(t-\vartheta_1-v\varphi_n)-\lambda(t-\vartheta_1)\big)\,\mathrm{d}t\Bigg]\\
&= \exp\Big(n\int_0^\tau A_n(v,t)\,\mathrm{d}t\Big)
\end{aligned}$$

where we denoted

$$A_n(v,t) = \Big(\exp\big\{i\mu\ln(g(v,t)+1)\big\}-1\Big)\lambda(t-\vartheta_1-u\varphi_n) - i\mu h(v,t)$$

with

$$g(v,t) = \frac{\lambda(t-\vartheta_1-v\varphi_n)}{\lambda(t-\vartheta_1)} - 1$$

and

$$h(v,t) = \lambda(t-\vartheta_1-v\varphi_n) - \lambda(t-\vartheta_1).$$

To prove it, we consider two case, $v \le u$ and $v > u$.

When $v \le u < (\tau - t_* - \vartheta_1)\varphi_n^{-1}$, the functions $\lambda(t-\vartheta_1)$ and $\lambda(t-\vartheta_1-v\varphi_n)$ are continuous on the intervals $(0, t_*+\vartheta_1)$ and $(t_*+\vartheta_1+u\varphi_n, \tau)$. Using the inequality $e^{i\mu x} \le 1 + i\mu x$ and $|\ln x| \le |x-1| + o(|x-1|)$, which are valid for the sufficiently large n and applying the Taylor series expansion, we obtain

$$\begin{aligned}
n\big|A_n(v,t)\big| &\le n\Big|i\mu\ln\frac{\lambda(t-\vartheta_1-v\varphi_n)}{\lambda(t-\vartheta_1)}\lambda(t-\vartheta_1-u\varphi_n) - i\mu h(v,t)\Big|\\
&\le n\Big|i\mu\Big(\frac{\lambda(t-\vartheta_1-v\varphi_n)}{\lambda(t-\vartheta_1)}-1\Big)\lambda(t-\vartheta_1-u\varphi_n) - i\mu h(v,t)\Big| + o(1)\\
&= n\Big|i\mu h(v,t)\Big(\frac{\lambda(t-\vartheta_1-u\varphi_n)}{\lambda(t-\vartheta_1)}-1\Big)\Big| + o(1)\\
&\le n\Big|v\varphi_n\dot\lambda(t-\vartheta_1)u\varphi_n\frac{\dot\lambda(t-\vartheta_1)}{\lambda(t-\vartheta_1)}\Big| + o(1) \longrightarrow 0.
\end{aligned}$$

Here $\dot{\lambda}(\vartheta, t)$ is denoted by the derivative of the absolute continuous component of the function $\lambda(\vartheta, t)$.

For sufficiently large n, we obtain that, in the sub-intervals $(t_* + \vartheta_1, t_* + \vartheta_1 + v\varphi_n)$ and $(t_* + \vartheta_1 + v\varphi_n, t_* + \vartheta_1 + u\varphi_n)$,

$$\begin{aligned} n \int_{t_*+\vartheta_1}^{t_*+\vartheta_1+v\varphi_n} & A_n(v, t) \,\mathrm{d}t \\ &= \frac{v}{\lambda_+}\Big(\exp\Big(i\mu \ln \frac{\lambda(t_* - v\varphi_n)}{\lambda(t_*)}\Big) - 1\Big)\lambda(t_* - u\varphi_n) \\ &\qquad - i\mu\big(\lambda(t_* - v\varphi_n) - \lambda(t_*)\big)\Big) + o(1) \\ &\longrightarrow \frac{v}{\lambda_+}\Big(\exp\Big(i\mu \ln \frac{\lambda_-}{\lambda_+}\Big) - 1\Big)\lambda_- - i\mu\big(\lambda_- - \lambda_+\big)\Big) \end{aligned}$$

and

$$\begin{aligned} n \int_{t_*+\vartheta_1+v\varphi_n}^{t_*+\vartheta_1+u\varphi_n} & A_n(v, t) \,\mathrm{d}t \\ &= \frac{u - v}{\lambda_+}\Big(\exp\Big(i\mu \ln \frac{\lambda(t_*)}{\lambda(t_* + v\varphi_n)}\Big) - 1\Big)\lambda(t_* + (v - u)\varphi_n) - \\ &\qquad - i\mu\big(\lambda(t_*) - \lambda(t_* + (v - u)\varphi_n)\big)\Big) + o(1) \\ &\longrightarrow \frac{u - v}{\lambda_+}\Big(\exp\Big(i\mu \ln \frac{\lambda_+}{\lambda_+}\Big) - 1\Big)\lambda_- - i\mu\big(\lambda_+ - \lambda_+\big)\Big) = 0. \end{aligned}$$

So we get, for $v \leq u$,

$$\begin{aligned} &\mathbf{E}_{\vartheta_1+u\varphi_n} \exp\left\{i\mu \ln Z_n(v)\right\} \\ &\longrightarrow \exp\Big\{v\Big(\exp\Big(i\mu \ln \rho\Big) - 1\Big)\rho - i\mu\big(\rho - 1\big)\Big)\Big\} \\ &= \mathbf{E}_u \exp\left\{i\mu \ln Z(v)\right\}. \end{aligned}$$

Now we consider the case when $(\tau - t_* - \vartheta_1)\varphi_n^{-1} > v > u > 0$. Similarly as before, we obtain on the continuous sub-intervals $(0, t_* + \vartheta_1 + u\varphi_n)$ and $(t_* + \vartheta_1 + v\varphi_n, \tau)$,

$$n\Big(\int_0^{t_*+\vartheta_1} + \int_{t_*+\vartheta_1+v\varphi_n}^{\tau}\Big) \left|A_n(v, t)\right| dt \longrightarrow 0.$$

For the sub-intervals $(t_* + \vartheta_1, t_* + \vartheta_1 + u\varphi_n)$ and $(t_* + \vartheta_1 + u\varphi_n, t_* + \vartheta_1 + v\varphi_n)$,

$$\begin{aligned} n \int_{t_*+\vartheta_1}^{t_*+\vartheta_1+u\varphi_n} & A_n(v, t) \mathrm{d}t \\ &= \frac{u}{\lambda_+}\Big(\exp\Big(i\mu \ln \frac{\lambda(t_* - v\varphi_n)}{\lambda(t_*)}\Big) - 1\Big)\lambda(t_* - u\varphi_n) - \end{aligned}$$

$$
\begin{aligned}
&\qquad - i\mu\big(\lambda(t_* - v\varphi_n) - \lambda(t_*)\big)\Big) + o(1)\\
&\longrightarrow \frac{u}{\lambda_+}\Big(\exp\Big(i\mu\ln\frac{\lambda_-}{\lambda_+}\Big) - 1\Big)\lambda_- - i\mu\big(\lambda_- - \lambda_+\big)\Big)
\end{aligned}
$$

and also

$$
\begin{aligned}
&n\int_{t_*+\vartheta_1+u\varphi_n}^{t_*+\vartheta_1+v\varphi_n} A_n(v,t)\mathrm{d}t\\
&= \frac{v-u}{\lambda_+}\Big(\exp\Big(i\mu\ln\frac{\lambda(t_*+(u-v)\varphi_n)}{\lambda(t_*+u\varphi_n)}\Big) - 1\Big)\lambda(t_*) -\\
&\qquad - i\mu\big(\lambda(t_*+(u-v)\varphi_n) - \lambda(t_*+u\varphi_n)\big)\Big) + o(1)\\
&\longrightarrow \frac{v-u}{\lambda_+}\Big(\Big(\exp\Big(i\mu\ln\frac{\lambda_-}{\lambda_+}\Big) - 1\Big)\lambda_+ - i\mu\big(\lambda_- - \lambda_+\big)\Big).
\end{aligned}
$$

So we get for $v > u$,

$$
\begin{aligned}
&\mathbf{E}_{\vartheta_1+u\varphi_n}\exp\left\{i\mu\ln Z_n(v)\right\}\\
&\longrightarrow \exp\Big(u\left(\exp\left\{i\mu\ln\rho\right\} - 1\right)\rho - i\mu\big(\rho-1\big)\Big) +\\
&\qquad + (v-u)\Big(\left(\exp\left\{i\mu\ln\rho\right\} - 1\right) - i\mu\left(\rho-1\right)\Big)\Big)\\
&= \mathbf{E}_u\exp\left\{i\mu\ln Z(v)\right\}.
\end{aligned}
$$

So the one-dimensional distributions of the random processes Z_n converge to those of Z. Similarly we obtain the convergence of the multi-dimensional distribution. For example, when $v_1 < v_2 < u$,

$$
\begin{aligned}
&\mathbf{E}_{\vartheta_1+u\varphi_n}\exp\big\{it_1\ln Z_n(v_1) + it_2\ln Z_n(v_2)\big\}\\
&\quad\to \exp\Big\{(v_2-v_1)\Big[\rho\big(\exp\left\{it_2\ln\rho\right\} - 1\big) - it_2(\rho-1)\Big]\\
&\qquad + v_1\Big[\rho\Big(\exp\big\{i\left(t_1+t_2\right)\ln\rho\big\} - 1\Big) - i\left(t_1+t_2\right)\left(\rho-1\right)\Big]\Big\}\\
&\quad= \mathbf{E}_u\exp\big\{it_1\ln Z(v_1) + it_2\ln Z(v_2)\big\}.
\end{aligned}
$$

□

To check the week convergence of the random field Z_n to the random field Z in the Skorohod space $\mathbb{D}$, we also need to prove the following lemmas.

We can write (under the alternative),

$$
Z_n(v) = Z_n(u)\,\widetilde{Z}_n(v),
$$

where

$$\widetilde{Z}_n(v) = \frac{\mathrm{d}\mathbf{P}_{\vartheta_1+v\varphi_n}}{\mathrm{d}\mathbf{P}_{\vartheta_1+u\varphi_n}}, \qquad Z_n(u) = \frac{\mathrm{d}\mathbf{P}_{\vartheta_1+u\varphi_n}}{\mathrm{d}\mathbf{P}_{\vartheta_1}}.$$

Note that $Z_n(u)$ does not depend of v. As we proved in lemma 3.1, $Z_n(u)$ converge to $Z(u)$, we only study the weak convergence of $\widetilde{Z}_n(v)$ in the following lemmas.

Lemma 3.2. *Let condition* **L** *be fulfilled. Then there exists a constant $C > 0$, the inequality*

$$\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\left|\widetilde{Z}_n^{1/2}(v_1) - \widetilde{Z}_n^{1/2}(v_2)\right|^2 \leq C\left|v_1 - v_2\right|$$

holds for all $v_1, v_2 \in \mathbb{U}_n^+$ and sufficiently large values of n.

Proof. According to [19, Lemma 1.1.5], we have, for $v_1 > v_2 > 0$,

$$\begin{aligned}
&\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\left|\widetilde{Z}_n^{1/2}(v_1) - \widetilde{Z}_n^{1/2}(v_2)\right|^2 \\
&\leq \int_0^{n\tau}\Big(\frac{\lambda^{1/2}(t-\vartheta_1-v_1\varphi_n)}{\lambda^{1/2}(t-\vartheta_1-u\varphi_n)} - \frac{\lambda^{1/2}(t-\vartheta_1-v_2\varphi_n)}{\lambda^{1/2}(t-\vartheta_1-u\varphi_n)}\Big)^2\lambda(t-\vartheta_1-u\varphi_n)\,\mathrm{d}t \\
&= n\int_0^{\tau}\big(\lambda^{1/2}(x-v_1\varphi_n) - \lambda^{1/2}(x-v_2\varphi_n)\big)^2\,\mathrm{d}x \\
&= n\Big(\int_0^{t_*+v_2\varphi_n} + \int_{t_*+v_2\varphi_n}^{t_*+v_1\varphi_n} + \int_{t_*+v_1\varphi_n}^{\tau}\Big)\big(\lambda^{1/2}(x-v_1\varphi_n) - \lambda^{1/2}(x-v_2\varphi_n)\big)^2\,\mathrm{d}x \\
&= n(I_1 + I_2 + I_3).
\end{aligned}$$

As $\lambda(t)$ is continuously differentiable on $(0, t_* + v_2\varphi_n)$ and $(t_* + v_1\varphi_n, \tau)$, we apply the Taylor series expansion and obtain

$$\lambda^{\frac{1}{2}}(\vartheta_1 + v_1\varphi_n, t) = \lambda^{\frac{1}{2}}(\vartheta_1 + v_2\varphi_n, t) + \frac{(v_1 - v_2)\,\varphi_n}{2}\frac{\dot{\lambda}(\vartheta_v, t)}{\lambda^{\frac{1}{2}}(\vartheta_v, t)}$$

where $\dot{\lambda}(\vartheta_v, t)$ is the derivative of the absolute continuous component of λ w.r.t ϑ_v; $\vartheta_v \in (\vartheta_1 + v_2\varphi_n, \vartheta_1 + v_1\varphi_n)$. We get that for sufficiently large n,

$$\begin{aligned}
I_1 + I_3 &\leq n\varphi_n^2\frac{(v_1 - v_2)^2}{4}\Big(\int_0^{t_*+v_2\varphi_n} + \int_{t_*+v_1\varphi_n}^{\tau}\Big)\frac{\dot{\lambda}^2(\vartheta_v, t)}{\lambda(\vartheta_v, t)}\,\mathrm{d}t \\
&\leq \frac{4C_1}{n\lambda_+^2}\left|v_1 - v_2\right|^2 \leq \frac{4C_1}{\lambda_+^2}\left|v_1 - v_2\right|
\end{aligned}$$

with

$$C_1 = \int_0^{\tau}\frac{\dot{\lambda}^2(\vartheta_1,\, t)}{\lambda(\vartheta_1,\, t)}\,\mathrm{d}t.$$

For the integral I_2, as λ is a bounded function, we obtain

$$I_2 \leq n\frac{|v_1 - v_2|}{n\lambda_+}C_2 = \frac{C_2}{\lambda_+}\left|v_1 - v_2\right|$$

where the inequality holds with some constant $C = 4C_1/\lambda_+^2 + C_2/\lambda_+$.

□

Lemma 3.3. *Let condition $\boldsymbol{L}$ be fulfilled. Then there exists a constant $k^*>0$ such that*

$$\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\widetilde{Z}_n^{1/2}(v)\leq \exp\{-k^*\left|v-u\right|\}$$

for all $v\in\mathbb{U}_n^+$ and sufficiently large values of n.

Proof. According to [19, Lemma 1.1.5], we have

$$\begin{aligned}
&\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\widetilde{Z}_n^{1/2}(v)\\
&=\exp\Big\{-\frac{1}{2}\int_0^{n\tau}\big(\frac{\lambda^{1/2}(t-\vartheta_1-v\varphi_n)}{\lambda^{1/2}(t-\vartheta_1-u\varphi_n)}-1\big)^2\lambda(t-\vartheta_1-u\varphi_n)\,\mathrm{d}t\Big\}\\
&=\exp\left\{-\frac{1}{2}n\int_0^{\tau}\Big(\lambda^{1/2}\big(x-(v-u)\varphi_n\big)-\lambda^{1/2}(x)\Big)^2\mathrm{d}x\right\},
\end{aligned}$$

where we denoted

$$F(u,v)=\int_0^{\tau}\big(\lambda^{1/2}\big(x-(v-u)\varphi_n\big)-\lambda^{1/2}(x)\big)^2\,\mathrm{d}x.$$

Here we suppose that $0<\left|v-u\right|\varphi_n<\tau$, while the other cases can be treated in a similar way. To prove it, we consider two cases $\left|v-u\right|<\delta$ and $\left|v-u\right|\geq\delta$, where δ is some positive constant in vicinity of zero.

For the case $\left|v-u\right|<\delta$, in the vicinity of $v=u$, using the mean value theorem, we can find out some constant $\widetilde{x}\in(t_*,t_*+\left|v-u\right|\varphi_n)\to t_*$ such that for sufficiently large n

$$\begin{aligned}
F(u,v)&\geq\int_{t_*}^{t_*+|v-u|\varphi_n}\big(\lambda^{1/2}\big(x-(v-u)\varphi_n\big)-\lambda^{1/2}(x)\big)^2\,\mathrm{d}x\\
&=\left|v-u\right|\varphi_n\big(\lambda^{1/2}\big(\widetilde{x}-(v-u)\varphi_n\big)-\lambda^{1/2}(\widetilde{x})\big)^2\\
&\geq\frac{1}{2}\left|v-u\right|\varphi_n\left(\sqrt{\lambda_-}-\sqrt{\lambda_+}\right)^2
\end{aligned}$$

which imply that the inequality holds for $k_1=\frac{1}{4}(\sqrt{\rho}-1)^2$.

For the case $\left|v-u\right|\geq\delta$, we have $\inf\limits_{|u-v|\geq\delta}F(u,v)>0$, because if it is not so, then there exists the point $v^*\neq u$ such that $F(u,v^*)=0$, that is, the intensity has jumps in two different instants on the interval $[0,\tau)$, which contradicts the condition $\boldsymbol{L}$.

Let us denote $k_2=\inf\limits_{|u-v|\geq\delta}F(u,v)>0$, then we obtain

$$F(u,v)\geq k_2\geq k_2\frac{\left|u-v\right|\varphi_n}{b-\vartheta_1}$$

and finally the inequality holds for $k^*=\min\{k_1/(b-\vartheta_1),k_2/\lambda_+\}$ □

In order to check (3.2) in the Theorem 3.1, we introduce some more notations. For $p=1,2$, we denote $A_p = A_p(v, v+h)$ the event that $\widetilde{Z}_n$ has at least p jumps on the interval $(v, v+h)$. $\widetilde{Z}_{n,a}$ is denoted by the absolute continuous component of the function $\widetilde{Z}_n$.

Lemma 3.4. *Let condition* **L** *be fulfilled. Then the inequalities*

$$\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\left|\widetilde{Z}^{1/2}_{n,a}(v+h)-\widetilde{Z}^{1/2}_{n,a}(v)\right|^2 \leq Ch^2, \tag{3.3}$$

$$\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}(A_1) \leq D_1 h \tag{3.4}$$

and

$$\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}(A_2) \leq D_2 h^2 \tag{3.5}$$

hold with certain constants $D_1, D_2 > 0$ (independent of ϑ, u and h).

Proof. To show the inequality (3.3), we follow the proof of [18, Lemma 4.4.3].We have

$$\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\left|\widetilde{Z}^{1/2}_{n,a}(v+h)-\widetilde{Z}^{1/2}_{n,a}(v)\right|^2 = \mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\left|\int_v^{v+h}\frac{\partial}{\partial s}\widetilde{Z}^{1/2}_{n,a}(s)\,\mathrm{d}s\right|^2.$$

Denote $\dot\lambda(t-\vartheta_1-v\varphi_n)$ the derivative of the absolute continuous component of the function $\lambda(t-\vartheta_1-v\varphi_n)$ w.r.t $v\varphi_n$, simple calculation shows that

$$\begin{aligned}
&\frac{\partial}{\partial s}\widetilde{Z}^{1/2}_{n,a}(s) = \frac{\partial}{\partial s}\widetilde{Z}^{1/2}_{n}(s)\\
&= -\frac{\widetilde{Z}^{1/2}_{n}(s)}{2}\left[\varphi_n\sum_{j=0}^{n}\int_0^\tau\frac{\dot\lambda(t-\vartheta_1-s\varphi_n)}{\lambda(t-\vartheta_1-s\varphi_n)}\,\mathrm{d}X_j(\mathrm{d}t) - n\varphi_n\int_0^\tau\dot\lambda(t-\vartheta_1-s\varphi_n)\,\mathrm{d}t\right]\\
&= -\varphi_n\frac{\widetilde{Z}^{1/2}_{n}(s)}{2}\sum_{j=0}^{n}\int_0^\tau\frac{\dot\lambda(t-\vartheta_1-s\varphi_n)}{\lambda(t-\vartheta_1-s\varphi_n)}\left[\mathrm{d}X_j(\mathrm{d}t)-\lambda(t-\vartheta_1-s\varphi_n)\,\mathrm{d}t\right].
\end{aligned}$$

Denoting the last sum of the integrals by $I(s)$, we obtain

$$\begin{aligned}
\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\left|\widetilde{Z}^{1/2}_{n,a}(v+h)-\widetilde{Z}^{1/2}_{n,a}(v)\right|^2 &= \frac{1}{4n^2\lambda_+^2}\,\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\left|\int_v^{v+h}\widetilde{Z}^{1/2}_{n}(s)\,I(s)\,\mathrm{d}s\right|^2\\
&\leq \frac{1}{4n^2\lambda_+^2}\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\left[\int_v^{v+h}\widetilde{Z}_n(s)I^2(s)\mathrm{d}s\int_v^{v+h}\mathrm{d}s\right]\\
&= \frac{h}{4n^2\lambda_+^2}\int_v^{v+h}\mathbf{E}^{(n)}_{\vartheta+s\varphi_n}I^2(s)\,\mathrm{d}s\\
&= \frac{nh}{4n^2\lambda_+^2}\int_v^{v+h}\left[\int_0^\tau\frac{\dot\lambda(t-\vartheta_1-s\varphi_n)^2}{\lambda(t-\vartheta_1-s\varphi_n)}\mathrm{d}t\right]\mathrm{d}s\\
&\leq \frac{h}{4n\lambda_+^2}\,hC_1,
\end{aligned}$$

where
$$C_1 = \sup_{s\in[v,v+h)} \int_0^{\tau} \frac{\dot\lambda(t-\vartheta_1-s\varphi_n)^2}{\lambda(t-\vartheta_1-s\varphi_n)}\mathrm{d}t$$
and hence the inequality (3.3) holds with $C = \frac{C_1}{4\lambda_+^2}$.

In order to establish the inequalities (3.4) and (3.5), we follow the proof of [18, Lemma 4.4.4]. The pure jump component of the function $\ln \widetilde{Z}_n(\cdot)$ is given by
$$\sum_{i=1}^{n} \sum_{0<t_{j,i}<\tau} \ln \lambda(t_{j,i}-\vartheta_1-\cdot\varphi_n),$$
where $t_{j,i}$ are the jump times of the process X_i, $i=1,\dots,n$. So, the process $\widetilde{Z}_n$ has its jumps in the points $v_{j,i}=(t_{j,i}-t_*-\vartheta_1)\varphi_n^{-1}$, where $t_{j,i}-t_*\in(\vartheta_1,b)$.

The event A_1 is equivalent to the event $v_{j,i}\in(v,v+h)$ for (at least) some j and some i. The event $v_{j,i}\in(v,v+h)$ is, in turn, equivalent to the inequality
$$t_*+\vartheta_1+v\varphi_n < t_{j,i} < t_*+\vartheta_1+(v+h)\varphi_n.$$

We denote $a_k = t_*+\vartheta_1+v\varphi_n$ and let $B_p^{(k)}$, $p=1,2$, be the event that the process X_1 has at least p jumps on the interval $(a_k, a_k+h\varphi_n)$. We have
$$\begin{aligned}
\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\big(B_1^{(k)}\big) &= 1-\exp\Big\{-\int_{a_k}^{a_k+h\varphi_n} \lambda\big(t-\vartheta_1-u\varphi_n\big)\,\mathrm{d}t\Big\}\\
&= 1-\exp\Big\{-\int_{t_*+v\varphi_n}^{t_*+(v+h)\varphi_n} \lambda(x)\,\mathrm{d}x\Big\}\\
&\le \int_{t_*+v\varphi_n}^{t_*+(v+h)\varphi_n} \lambda_n(x)\,\mathrm{d}x \le h\varphi_n\, L
\end{aligned}$$
where $\lambda_n(x)$ is bounded function such that $\lambda_n(x)\le L$. Note that in the first equality we assumed that $(a_k,a_k+h\varphi_n)\subset[0,\tau]$. However, if this is not the case, the value of $\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\big(B_1^{(k)}\big)$ will be even smaller.

As $A_1\subset \bigcup_{k=1}^{n} B_1^{(k)}$, we obtain
$$\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}(A_1) \le \sum_{k=1}^{n} \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\big(B_1^{(k)}\big) \le nh\varphi_n\, L \le D_1 h$$
with $D_1 = L/\lambda_+$.

Further, as $A_2 \subset \Big(\bigcup_{j=1}^{n-1}\bigcup_{k=j+1}^{n} B_1^{(j)}\cap B_1^{(k)}\Big) \cup \Big(\bigcup_{k=1}^{n} B_2^{(k)}\Big)$, and since the numbers of jumps of a Poisson process on disjoint intervals are independent, we get
$$\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}(A_2) \le \sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\big(B_1^{(j)}\big)\,\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\big(B_1^{(k)}\big) + \sum_{k=1}^{n} \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\big(B_2^{(k)}\big)$$

$$\leq \frac{n(n+1)}{2}\Big(h\varphi_n\, L\Big)^2 + (n+1)\Big(h\varphi_n\, L\Big)^2$$
$$= \frac{n+1}{2n}\,\frac{n+2}{n}\,\frac{L^2}{\lambda_+^2}h^2 \leq D_2 h^2.$$

Here we denoted $D_2 = 3L^2/\lambda_+^2$ and used the inequality

$$\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\big(B_2^{(k)}\big) = 1 - \exp\Big\{-\int_{t_*+v\varphi_n}^{t_*+(v+h)\varphi_n}\lambda(x)\,\mathrm{d}x\Big\} -$$
$$-\Big(\int_{t_*+v\varphi_n}^{t_*+(v+h)\varphi_n}\lambda(x)\,\mathrm{d}x\Big)\ \exp\Big\{-\int_{t_*+v\varphi_n}^{t_*+(v+h)\varphi_n}\lambda(x)\,\mathrm{d}x\Big\}$$
$$\leq \Big(\int_{t_*+v\varphi_n}^{t_*+(v+h)\varphi_n}\lambda(x)\,\mathrm{d}x\Big)^2 \leq \Big(h\varphi_n\, L\Big)^2.$$

So, the lemma is proved. □

Let us denote the function

$$\Delta_h^l(\widetilde Z_n^{1/2}(v)) = \sup_v \sup_{v',v''\in\delta_l}\Big\{\min\Big[\big|\widetilde Z_n^{1/2}(v') - \widetilde Z_n^{1/2}(v)\big|, \big|\widetilde Z_n^{1/2}(v'') - \widetilde Z_n^{1/2}(v)\big|\Big]\Big\}$$

where $\delta_l = \big(v', v''\big] \subseteq [v-h, v+h) \subseteq [l, l+1]$, l arbitrary integer.

We have the following lemma.

Lemma 3.5. *Let condition* **L** *be fulfilled. Then the inequality*

$$P^{(n)}_{\vartheta_1+u\varphi_n}\{\Delta_h^l(\widetilde Z_n^{1/2}(v)) > h^{1/4})\} \leq Ch^{1/2}$$

holds with any l.

Proof. Define the event $\mathcal{G}$ such that whenever $\mathcal{G}^c$ occurs, each interval $(v-h, v+h)$ contains not more than one discontinuity point of the function $\widetilde Z_n(\cdot)$ such that it is contained either in $(v-h, v)$ or in $(v, v+h)$. And we assume that the discontinuity be on the interval $(v-h, v)$. The probability of the discontinuity in the interval $(v-h, v)$ is the same as in $(v, v+h)$. Then for $(v, v+h)$, the function $\widetilde Z_n(\cdot)$ is continuous,

$$\sup_{v<v''<v+h}\big|\widetilde Z_n^{1/2}(v) - \widetilde Z_n^{1/2}(v'')\big| \leq \sup_{|v_2-v_1|\leq h}\big|\widetilde Z_n^{1/2}(v_2) - \widetilde Z_n^{1/2}(v_1)\big|$$

where the latter supper is taken over $v_1, v_2 \in (v, v+h)$. For the continuous part of $\widetilde Z_n(v)$, that is, using [18, Lemma 5.3.1] and the inequality (3.3), the inequality

$$\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{\sup_{|v_2-v_1|\leq h}|\widetilde Z_n^{1/2}(v_2) - \widetilde Z_n^{1/2}(v_1)| > h^{1/4}\}$$
$$= \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{\sup_{|v_2-v_1|\leq h}|\widetilde Z_{n,a}^{1/2}(v_2) - \widetilde Z_{n,a}^{1/2}(v_1)| > h^{1/4}\} \leq Ch^{1-1/4} \leq Ch^{1/2}$$

is valid for $h < 1$. Thus,

$$\sup_{u\in\mathbb{U}_n} \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\left\{\sup_{v\in\delta_l}\min\Big[\big|\widetilde Z_n^{1/2}(v^{'})-\widetilde Z_n^{1/2}(v)\big|,\big|\widetilde Z_n^{1/2}(v^{''})-\widetilde Z_n^{1/2}(v)\big|\Big]>h^{\frac14},\mathcal{G}^c\right\}$$
$$\leq 2\sup_{u\in\mathbb{U}_n} \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\Big\{\sup_{|v_2-v_1|\leq h}\big|\widetilde Z_{n,a}^{1/2}(v_2)-\widetilde Z_{n,a}^{1/2}(v_1)\big|>h^{1/4}\Big\}\leq Ch^{1/2}.$$

Hence,

$$\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}(\Delta_h^l(\widetilde Z_n^{1/2}(\cdot))>h^{1/4})$$
$$\leq \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\left\{\sup_{v\in\delta_l}\min\Big[\big|\widetilde Z_n^{1/2}(v^{'})-\widetilde Z_n^{1/2}(v)\big|,\big|\widetilde Z_n^{1/2}(v^{''})-\widetilde Z_n^{1/2}(v)\big|\Big]>h^{1/4},\mathcal{G}^c\right\}$$
$$+\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{\mathcal{G}\}$$
$$\leq Ch^{1/2}+D_1h\leq Ch^{1/2}$$

with h sufficiently small such that $h < h^{1/2}$. □

Lemma 3.6. *Let the condition* **L** *holds. Then*

$$\sup_{u\in\mathbb{U}_n^+} \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\Big\{\sup_{l\leq|v|\leq l+1}\widetilde Z_n(v)>\exp\big[-\frac{c}{4}\,|v-u|\,\big]\Big\}\leq c\exp\big[-\frac{c}{4}\,|v-u|\,\big]$$

with some constant $c > 0$.

Proof. For definitely, we suppose here $v > u$ (The case when $v < u$ can be treated similarly). We divide $[l, l+1]$ into $\gamma = \Big[\exp\{\frac{c}{4}\,|l-u|\,\}\Big]+1$ of length $h=\gamma^{-1}$, here $[a]$ denote the inter part of a. Then the inequality can be written as following

$$\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\Big\{\sup_{l\leq|v|\leq l+1}\widetilde Z_n(v)>\exp\big[-\frac{c}{4}\,|v-u|\,\big]\Big\}$$
$$=\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\Big\{\sup_{0\leq k\leq\gamma}\widetilde Z_n^{1/2}\big(l+\frac{k}{\gamma}\big)$$
$$+\sup_{l+\frac{k}{\gamma}\leq v\leq l+\frac{k+1}{\gamma}}\Big[\widetilde Z_n^{1/2}(v)-\widetilde Z_n^{1/2}\big(l+\frac{k}{\gamma}\big)\Big]>\exp\big[-\frac{c}{8}\,|v-u|\,\big]\Big\}$$
$$\leq \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\Big\{\sup_{0\leq k\leq\gamma}\widetilde Z_n\big(l+\frac{k}{\gamma}\big)>\exp\big[-c\,|l-u|\,\big]\Big\}+$$
$$+\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\Big\{\sup_{|v_1-v_2|<\gamma^{-1}}\Big|\widetilde Z_n^{1/2}(v_2)-\widetilde Z_n^{1/2}(v_1)\Big|>\exp\big[-\frac{c}{4}\,|l-u|\,\big]\Big\}$$
$$=P_1+P_2.$$

Here the inequality holds for $l \geq l_0$ such that

$$l_0=\inf_{l\geq 0}\left\{\exp\left\{-\frac{c}{8}\,|l-u|\right\}-\exp\left\{-\frac{c}{2}\,|l-u|\right\}>\exp\left\{-\frac{c}{4}\,|l-u|\right\}\right\}.$$

For the first probability P_1, using the result of the [9, Lemma 4], we obtain that

$$
\begin{aligned}
P_1 &\le \sum_{k=0}^{\gamma} \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\Big\{\widetilde{Z}_n\big(l+\frac{k}{\gamma}\big) > \exp\big[-c\,|l-u|\,\big]\Big\}\\
&\le (\gamma+1)\exp\big[\frac{c}{2}\,|l-u|\,\big]\mathbf{E}^{(n)}_{\vartheta_1+u\varphi_n}\widetilde{Z}_n^{1/2}\big(l+\frac{k}{\gamma}\big)\\
&\le (\gamma+1)\exp\big[\frac{c}{2}\,|l-u|\,\big]\exp\Big[-k_*\big|l+\frac{k}{\gamma}-u\big|\,\Big]\\
&\le (\gamma+1)\exp\big[-(k^*-\frac{c}{2})\,|l-u|\,\big] < c\exp\big\{-c\,|l-u|\big\}
\end{aligned}
$$

holds with some constant $c>0$.

For the second Probability P_2, using the same way as in [18, Lemma 5.3.3] we have

$$
P_2 = \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{\sup_{|v_2-v_1|\le h} |\widetilde{Z}_n^{1/2}(v_2)-\widetilde{Z}_n^{1/2}(v_1)| > h^{1/4}\} \le Ch^{1/2} < c\exp\big\{-\frac{c}{8}\,|v-u|\,\big\}.
$$

So the inequality holds for $l\ge 0$. The case when $l<0$ can be treated by changing the valued of c. □

Lemma 3.7. *If the condition* **L** *holds, then*

$$
\sup_{u\in\mathbb{U}_n^+}\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\Big\{\sup_{|v|\ge L}\widetilde{Z}_n(v) > \exp\big[-\frac{c}{4}\,|L-u|\,\big]\Big\} \le d\exp\big[-\frac{c}{4}\,|L-u|\,\big]
$$

with some constant $c,d>0$.

Proof. We have

$$
\begin{aligned}
&\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\Big\{\sup_{|v|\ge L}\widetilde{Z}_n(v) > \exp\big[-\frac{c}{4}\,|L-u|\,\big]\Big\}\\
&\qquad = \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\Big\{\max_{l\ge L}\sup_{l\le|v|\le l+1}\widetilde{Z}_n(v) > \exp\big[-\frac{c}{4}\,|L-u|\,\big]\Big\}\\
&\qquad \le \sum_{l=L}^{+\infty}\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\Big\{\sup_{l\le|v|\le l+1}\widetilde{Z}_n(v) > \exp\big[-\frac{c}{4}\,|l-u|\,\big]\Big\}\\
&\qquad \le c\sum_{l=L}^{+\infty}\exp\big[-\frac{c}{4}\,|l-u|\,\big]\\
&\qquad = c\sum_{l=0}^{+\infty}\exp\big[-\frac{c}{4}\,|L+l-u|\,\big]\\
&\qquad \le c\int_0^{+\infty}\exp\big[-\frac{c}{4}\,|L+x-u|\,\big]\,\mathrm{d}x\\
&\qquad \le c\exp\big[-\frac{c}{4}\,|L-u|\,\big].
\end{aligned}
$$

□

Finally to proof the Theorem 3.1, we need to prove that, for any $\varepsilon > 0$

$$\lim_{h\to 0}\lim_{n\to+\infty}\sup_{u\in\mathbb{U}_n^+}\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\left\{\Delta_h\big(\widetilde{Z}_n(\cdot)\big) > \varepsilon\right\} = 0.$$

Let $M_n = \sup_{|v|\leq L}\widetilde{Z}_n^{1/2}(v)$, then $\Delta_h^l(\widetilde{Z}_n) \leq M_n\Delta_h^l(\widetilde{Z}_n^{1/2})$ and

$$\begin{aligned}
&\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\left\{\Delta_h\big(\widetilde{Z}_n(\cdot)\big) > \varepsilon\right\}\\
&\quad\leq \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{\sum_{l=-L}^{L}\Delta_h^l(\widetilde{Z}_n(\cdot)) > \varepsilon/2\} + \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{\sup_{|v|>L}\widetilde{Z}_n(v) > \varepsilon/2\}\\
&\quad\leq \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{M_n\sum_{l=-L}^{L}\Delta_h^l(\widetilde{Z}_n^{1/2}(\cdot)) > \varepsilon/2\} + \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{\sup_{|v|>L}\widetilde{Z}_n(v) > \varepsilon/2\}
\end{aligned}$$

The first term tends to zero as $n \to +\infty$, because, for sufficiently small h,

$$\begin{aligned}
&\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{M_n\sum_{l=-L}^{L}\Delta_h^l(\widetilde{Z}_n^{1/2}(\cdot)) > \varepsilon/2\}\\
&\quad\leq \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{\sum_{l=-L}^{L}\Delta_h^l(\widetilde{Z}_n^{1/2}(\cdot)) > \varepsilon/2\} + \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{M_n > \varepsilon/2\}
\end{aligned}$$

where

$$\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{\sum_{l=-L}^{L}\Delta_h^l(\widetilde{Z}_n^{1/2}(\cdot)) > \frac{\varepsilon}{2}\} \leq \sum_{l=-L}^{L}\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{\Delta_h^l(\widetilde{Z}_n^{1/2}(\cdot)) > \frac{h\varepsilon}{2}\} \leq \frac{C(L+1)}{4}\varepsilon^2h^2$$

and similarly as above

$$\begin{aligned}
\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{M_n > \varepsilon/2\} &= \mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{\max_{0\leq l\leq L}\sup_{l\leq|v|\leq l+1}\widetilde{Z}_n^{1/2}(v) > \frac{h\varepsilon}{2}\}\\
&\leq \sum_{l=0}^{L}\mathbf{P}^{(n)}_{\vartheta_1+u\varphi_n}\{\sup_{l\leq|v|\leq l+1}\widetilde{Z}_n^{1/2}(v) > \frac{h\varepsilon}{2}\}\\
&< (L+1)\frac{C}{4}\varepsilon^2h^2.
\end{aligned}$$

Hence the limit holds and the weak convergence is proved.

3.3.2 GLRT

The GLRT is based on the statistic

$$Q_n\left(X^n\right) = \sup_{\vartheta>\vartheta_1} L\left(\vartheta,\vartheta_1,X^n\right) = \max\left[L\left(\hat{\vartheta}_n+,\vartheta_1,X^n\right),\, L\left(\hat{\vartheta}_n-,\vartheta_1,X^n\right)\right]$$

and is of the form

$$\psi_n\left(X^n\right) = \mathbb{1}_{\left\{Q_n\left(X^n\right) > C_\varepsilon\right\}}.$$

We define the threshold C_ε with the help of the convergence (under $\mathscr{H}_1$)

$$Q_n\left(X^n\right) = \sup_{v \in \mathbb{U}_n^+} Z_n\left(v\right) \Longrightarrow \sup_{v>0} Z_*\left(v\right) = \hat{Z}_*.$$

Hence $C_\varepsilon = C_\varepsilon\left(\rho\right)$ is solution of the equation

$$\mathbf{P}\left\{\hat{Z}_* > C_\varepsilon\right\} = \varepsilon$$

which depends on ρ.

Let us fix an alternative $u > 0$, then for the power function we have

$$\begin{aligned}\beta\left(\psi_n, u\right) = \mathbf{E}_{\vartheta_1 + u\varphi_n}\psi_n\left(X^n\right) = \mathbf{P}_{\vartheta_1 + u\varphi_n}&\left\{\sup_{v>0} Z_n\left(v\right) > C_\varepsilon\right\} \\ &\rightarrow \mathbf{P}_u\left\{\sup_{v>0} Z_*\left(v\right) > C_\varepsilon\right\},\end{aligned}$$

where the Poisson process $x_*\left(v\right), v \geq 0$ in $Z_*\left(v\right)$ according to (3.1) has the intensity function

$$\mu\left(u, v\right) = \rho\,\mathbb{1}_{\left\{v<u\right\}} + \mathbb{1}_{\left\{v \geq u\right\}}, \qquad v \geq 0.$$

Let us put $Y\left(v\right) = \ln\rho\, x_*\left(v\right) - \left(\rho - 1\right)v$, then we can write

$$\begin{aligned}&\sup_{v>0}\left[\ln\rho\; x_*\left(v\right) - \left(\rho-1\right)v\right] \\ &\qquad = \max\left(\sup_{0<v<u} Y\left(v\right), Y\left(u\right) + \sup_{v \geq u}\left[Y\left(v\right) - Y\left(u\right)\right]\right).\end{aligned}$$

Note that the Poisson process $\tilde{x}\left(v-u\right) = x_*\left(v\right) - x_*\left(u\right), v \geq u$ is independent of $x_*\left(u\right)$ and $x_*\left(v\right), 0 \leq v \leq u$. Hence the presentation of the limit power

$$\beta\left(\hat{\psi}, u\right) = \mathbf{P}_u\left\{\max\left(\sup_{0<v<u} Z_*\left(v\right), Z_*\left(u\right) + \tilde{Z}_*\right) > C_\varepsilon\right\} \tag{3.6}$$

the random variable

$$\tilde{Z}_* = \sup_{v \geq 0}\exp\left\{\ln\rho\,\tilde{x}_*\left(v\right) - \left(\rho - 1\right)v\right\}$$

is independent of $Z_*\left(v\right)$, $0 \leq v \leq u$. (This expression for the power can be used for numerical simulation similarly as in the continuous case.)

3.3.3 Wald's test

The Wald's test is based on the MLE $\hat{\vartheta}_n$. We already know that

$$\varphi_n^{-1}\left(\hat{\vartheta}_n-\vartheta_1\right)\Rightarrow\hat{v},$$

where $\hat{v}$ is solution of the equation

$$\max\left[Z_*\left(\hat{v}+\right),Z_*\left(\hat{v}-\right)\right]=\sup_{v>0}Z_*\left(v\right).$$

(see [19]). The Wald's test is

$$\psi_n\left(X^n\right)=\mathbb{1}_{\left\{\varphi_n^{-1}\left(\hat{\vartheta}_n-\vartheta_1\right)>c_\varepsilon\right\}}.$$

The threshold $c_\varepsilon=c_\varepsilon\left(\rho\right)$ is solution of the equation

$$\mathbf{P}\left\{\hat{v}>c_\varepsilon\right\}=\varepsilon$$

and depend on ρ too.

For the power function we have (below $\vartheta_u=\vartheta_1+u\varphi_n$)

$$\begin{aligned}\beta\left(\psi_n,u\right)=\mathbf{E}_{\vartheta_u}\psi_n\left(X^n\right)=\mathbf{P}_{\vartheta_u}&\left\{\varphi_n^{-1}\left(\hat{\vartheta}_n-\vartheta_u\right)+u>c_\varepsilon\right\}\\&\longrightarrow\mathbf{P}\left\{\hat{v}_*>c_\varepsilon-u\right\}\end{aligned}$$

where $\hat{v}$ is the solution of the equation

$$\max\left[Z_*\left(\hat{v}_*+\right),Z_*\left(\hat{v}_*-\right)\right]=\sup_{v>-u}Z_*\left(v\right).$$

and where the Poisson process $x_*\left(v\right),v\geq0$ in $Z_*\left(v\right)$ has the intensity $\rho\,\mathbb{1}_{\{v<0\}}+\mathbb{1}_{\{v\geq0\}}$.

And specially we remark that under alternatives, we can also use the analytical formula in the problem of GLRT and obtain the limit of the power function

$$\mathbf{P}\left\{\arg\sup_{v>0}Z_*(v)>c_\varepsilon\right\}.$$

and the Poisson process $x_*\left(v\right),v\geq0$ with the switching intensity function (3.1).

3.3.4 Bayesian approach

Suppose that the parameter ϑ is a random variable with known probability density $p\left(\theta\right)$, $\vartheta_1\leq\theta<b$. This function is supposed to be continuous and positive. We consider two tests.

The first one is based on the BE

$$\tilde\psi_n\left(X^n\right)=\mathbb{1}_{\left\{\varphi_n^{-1}\left(\tilde\vartheta_n-\vartheta_1\right)>k_\varepsilon\right\}},\qquad \tilde\vartheta_n=\frac{\int_{\vartheta_1}^b\theta p\left(\theta\right)L\left(\theta,\vartheta_1,X^n\right)\mathrm{d}\theta}{\int_{\vartheta_1}^b\theta L\left(\theta,\vartheta_1,X^n\right)\mathrm{d}\theta}$$

and as before we have the limit

$$\mathbf{E}_{\vartheta_1}\mathbb{1}_{\left\{\varphi_n^{-1}\left(\tilde\vartheta_n-\vartheta_1\right)>k_\varepsilon\right\}}\longrightarrow\mathbf{P}_{\vartheta_1}\left\{\tilde u>k_\varepsilon\right\},\qquad \tilde u=\frac{\int_0^\infty vZ_*\left(v\right)\mathrm{d}v}{\int_0^\infty Z_*\left(v\right)\mathrm{d}v}$$

where the Poisson process $x_*\left(v\right),v\geq 0$ in $Z_*\left(v\right)$ has the unit intensity.

For the power function the limit is obtained by the following convergence (denoting $\vartheta_u=\vartheta_1+u\varphi_n$):

$$\varphi_n^{-1}\left(\tilde\vartheta_n-\vartheta_1\right)=\varphi_n^{-1}\left(\tilde\vartheta_n-\vartheta_u\right)+u\Longrightarrow\frac{\int_{-u}^\infty vZ_*\left(v\right)\mathrm{d}v}{\int_{-u}^\infty Z_*\left(v\right)\mathrm{d}v}+u$$

where the Poisson process $x_*\left(v\right),v\geq 0$ in $Z_*\left(v\right)$ has the intensity $\rho\,\mathbb{1}_{\{v<0\}}+\mathbb{1}_{\{v\geq 0\}}$.

Or with the help of the convergence of Z_n to Z_* under alternatives, we also get that

$$\beta\left(\tilde\psi_n,u\right)=\mathbf{P}_{\vartheta_u}\left\{\varphi_n^{-1}\left(\tilde\vartheta_n-\vartheta_1\right)>k_\varepsilon\right\}=\mathbf{P}_{\vartheta_u}\left\{\frac{\int_0^\infty vZ_*\left(v\right)\mathrm{d}v}{\int_0^\infty Z_*\left(v\right)\mathrm{d}v}>k_\varepsilon\right\},$$

where the Poisson process $x_*\left(v\right),v\geq 0$ in $Z_*\left(v\right)$ has the intensity (3.1).

The thresholds and power function are obtained by the numerical simulations.

he second test minimizes the mean error. For the normalized likelihood ratio we have

$$\varphi_n^{-1}\tilde L\left(X^n\right)\Longrightarrow p\left(\vartheta_1\right)\int_0^\infty\exp\left\{\ln\rho\,x_*\left(v\right)-\left(\rho-1\right)v\right\}\mathrm{d}v.$$

Hence the test

$$\tilde\psi_n\left(X^n\right)=\mathbb{1}_{\{R_n>h_\varepsilon\}},\qquad R_n=\frac{\varphi_n^{-1}\tilde L\left(X^n\right)}{p\left(\vartheta_1\right)}$$

with the threshold h_ε satisfying equation

$$\mathbf{P}\left\{\int_0^\infty\exp\left\{\ln\rho\,x_*\left(v\right)-\left(\rho-1\right)v\right\}\mathrm{d}v>h_\varepsilon\right\}=\varepsilon$$

is bayesian and belongs to the class $\mathcal{K}_\varepsilon$.

Note that in all cases the limit random variables

$$Z=\int_0^\infty Z(v)\,\mathrm{d}v$$

This test has *heavy tails* because

$$\mathbf{E}\int_0^\infty Z\left(v\right)\,\mathrm{d}v=\infty.$$

3.3.5 Simulations

We consider n independent observations $X_j = \{X_j(t), t \in [0,4]\}$; $j = 1, ..., n$ of a Poisson process of intensity function

$$\lambda(t,\vartheta) = \lambda(t-\vartheta) = 3 - 2\cos^2(t-\vartheta)\mathbb{1}_{\{t>\vartheta\}}, \qquad 0 \le t \le 4.$$

As in this example the instant of jump $t = \vartheta$, we have the interval of ϑ is $\vartheta \in (3,4)$ by supposing $\vartheta_1 = 3$. We get that $\lambda_+ = 1$, $\rho = \frac{\lambda_-}{\lambda_+} = 3$ and

$$\begin{aligned}\ln Z_n(v) = &\sum_{j=1}^{n}\int_{\vartheta_1}^{\vartheta_1+v/n} \ln \frac{3}{3-2\cos^2(t-\vartheta_1)}\,\mathrm{d}X_j(t)\\ &+\sum_{j=1}^{n}\int_{\vartheta_1+v/n}^{4} \ln \frac{3-2\cos^2(t-\vartheta_1-v/n)}{3-2\cos^2(t-\vartheta_1)}\,\mathrm{d}X_j(t)\\ &- v - \frac{n}{2}\sin\bigl(2(4-\vartheta_1)\bigr) + \frac{n}{2}\sin\bigl(2(4-\vartheta_1-v/n)\bigr).\end{aligned}$$

We recall that in this case the limit likelihood ratio is

$$Z_*(v) = \exp\{\ln 3\, x_*(v) - 2v\}$$

where $x_*(v), v \ge 0$ is the Poisson process of unit intensity.

Using the limit expression Z_*, we obtain the thresholds c_ε of the GLRT as solution of the equation

$$\mathbf{P}\left\{\hat{Z}_* > c_\varepsilon\right\} = \varepsilon.$$

In the numerical section, we also obtain the expression equal in distribution to Z_* by following way

$$\exp\left[\ln 3\left(x_*(v) - \frac{2}{\ln 3}v\right)\right] = \exp\left\{\ln 3\left[x_{**}\left(\frac{2}{\ln 3}v\right) - \frac{2}{\ln 3}v\right]\right\}$$

where x_{**} is the Poisson process of constant intensity $\frac{\ln 3}{2} < 1$.

And hence we can choose the thresholds of GLRT by following formula

$$\begin{aligned}&\mathbf{P}\left\{\sup_{v>0} Z_*(v) > C_\varepsilon\right\}\\ &= \mathbf{P}\left\{\sup_{v>0}\exp\left\{\ln 3 \sup_{v>0}\left[x_{**}\left(\frac{2}{\ln 3}v\right) - \frac{2}{\ln 3}v\right]\right\} > \ln C_\varepsilon\right\}\\ &= \mathbf{P}\left\{\sup_{t>0}\left[x_{**}(t) - t\right] > \frac{\ln C_\varepsilon}{\ln 3}\right\}.\end{aligned}$$

The distribution of $\max_{t>0}[\Pi(t) - t]$, where Π is the Poisson process proceeding the intensity γ smaller than 1, is

$$\mathbf{P}\left\{\sup_{t>0}[\Pi(t) - t] \ge x\right\} = \sum_{m>x}\frac{(m-x)^m}{m!}(\gamma e^{-\gamma})^m e^{\gamma x}(1-\gamma), \qquad (3.7)$$

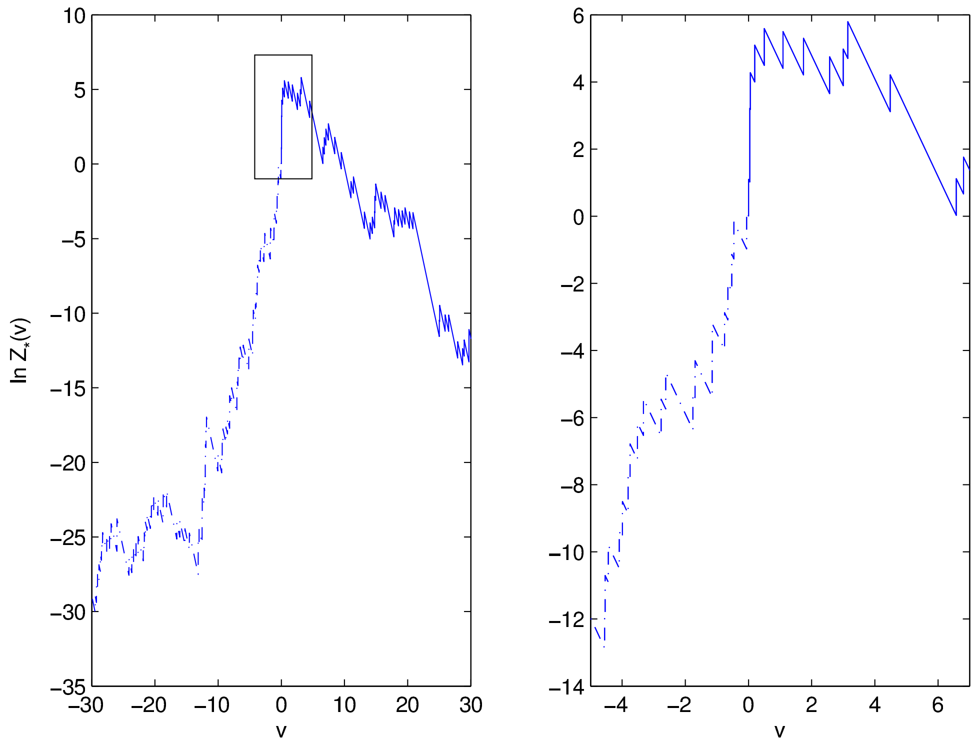

Figure 3.5: Realization of $\ln Z_*$

which was found by Pyke [32]. So using the equation (3.7), posing $\gamma = \frac{1}{2}\ln 3$ and $x = \frac{\ln C_\varepsilon}{\ln 3}$ we obtain the thresholds.

Denoting that using the same formula as in GLRT, we have (the detail see [31], by Pflug)

$$\mathbf{P}\left\{\hat{v}_* > c_\varepsilon\right\} = \mathbf{P}\left\{\arg\sup_{t\geq 0}\left[\Pi\left(t\right) - t\right] > \frac{\ln 3}{2}c_\varepsilon\right\} = \varepsilon.$$

Here the statistic $\arg\sup_{t\geq 0}\left[\Pi\left(t\right) - t\right]$ coincide with the random sum $\sum_{k=1}^{\nu} Q_k$, where ν is a geometric random variable, independent of Q_k:

$$\mathbf{P}\left\{\nu = i\right\} = (1-\gamma)\gamma^i;\ i = 0, 1...$$

($\sum_{k=1}^{0} Q_k$ is set to zero) and $\{Q_k\}; k = 1, 2...$ is an i.i.d sequence with common distribution

$$F(x) = \mathbf{P}\left\{Q_k \leq x\right\} = \frac{1}{\gamma}\left[1 - (1-\gamma)e^{-\gamma x}\sum_{j=0}^{[x]-1}\frac{(\gamma x)^j}{j\,!} - e^{-\gamma x}\frac{(\gamma x)^{[x]}}{[x]\,!}\right]. \tag{3.8}$$

Since the threshold of the Wald's test is difficult to calculate, we will simulate them by the Monte Carlo method, similarly as in the problem of the Bayesian test (BT1) in Regular case (detail see [9]).

We also calculate the quantile of BT1 by the following way. We divide the interval $[0, +\infty)$ to the sub-intervals $[v_i, v_{i+1})$ such that the Poisson process x_* does not have any even occurring on $[v_i, v_{i+1})$, $i = 0, 1, \ldots$; that is, $x_*(v_{i+1}-) - x_*(v_i) = 0$. Supposing $x_*(v_i) = N_i$, $i = 0, 1, \ldots$, we have

$$\begin{aligned}
&\mathbf{P}_{\vartheta_1}\left(\frac{\int_0^{+\infty} vZ_*(v)\,\mathrm{d}v}{\int_0^{+\infty} Z_*(v)\,\mathrm{d}v} > k_\varepsilon\right)\\
&= \mathbf{P}_{\vartheta_1}\left(\frac{\int_0^{+\infty} v\exp\left\{\ln\rho x_*(v) - (\rho-1)v\right\}\,\mathrm{d}v}{\int_0^{+\infty} \exp\left\{\ln\rho x_*(v) - (\rho-1)v\right\}\,\mathrm{d}v} > k_\varepsilon\right)\\
&= \mathbf{P}_{\vartheta_1}\left(\frac{\sum_{i=0}^{+\infty}\rho N_i\int_{v_i}^{v_{i+1}} v\exp\left\{-(\rho-1)v\right\}\,\mathrm{d}v}{\sum_{i=0}^{+\infty}\rho N_i\int_{v_i}^{v_{i+1}} \exp\left\{-(\rho-1)v\right\}\,\mathrm{d}v} > k_\varepsilon\right)\\
&= \mathbf{P}_{\vartheta_1}\left(\frac{\sum_{i=0}^{+\infty} N_i\left[v_i\exp\left\{-(\rho-1)v_i\right\} - v_{i+1}\exp\left\{-(\rho-1)v_{i+1}\right\}\right]}{\sum_{i=0}^{+\infty} N_i\left[\exp\left\{-(\rho-1)v_i\right\} - \exp\left\{-(\rho-1)v_{i+1}\right\}\right]} + \frac{1}{\rho-1} > k_\varepsilon\right)\\
&= \varepsilon,
\end{aligned} \tag{3.9}$$

which allow us to calculate the threshold of BT1 by numeric simulation.

ε	0.01	0.05	0.10	0.20	0.40	0.50
$\ln C_\varepsilon$	4.242	2.607	1.922	1.120	0.573	0.191
c_ε	5.990	3.556	2.078	1.045	0.329	0.099
k_ε	6.669	3.937	2.983	2.132	1.402	1.196

Table 3.3: Thresholds of GLRT, WT and BT1 in discontinuous case.

We also calculate the power function using the similar way as in regular case. We remark that when $n = 10$, $u > 10$ is exceed the interval of $\vartheta = \vartheta_1 + u\varphi_n \in (3, 4)$.

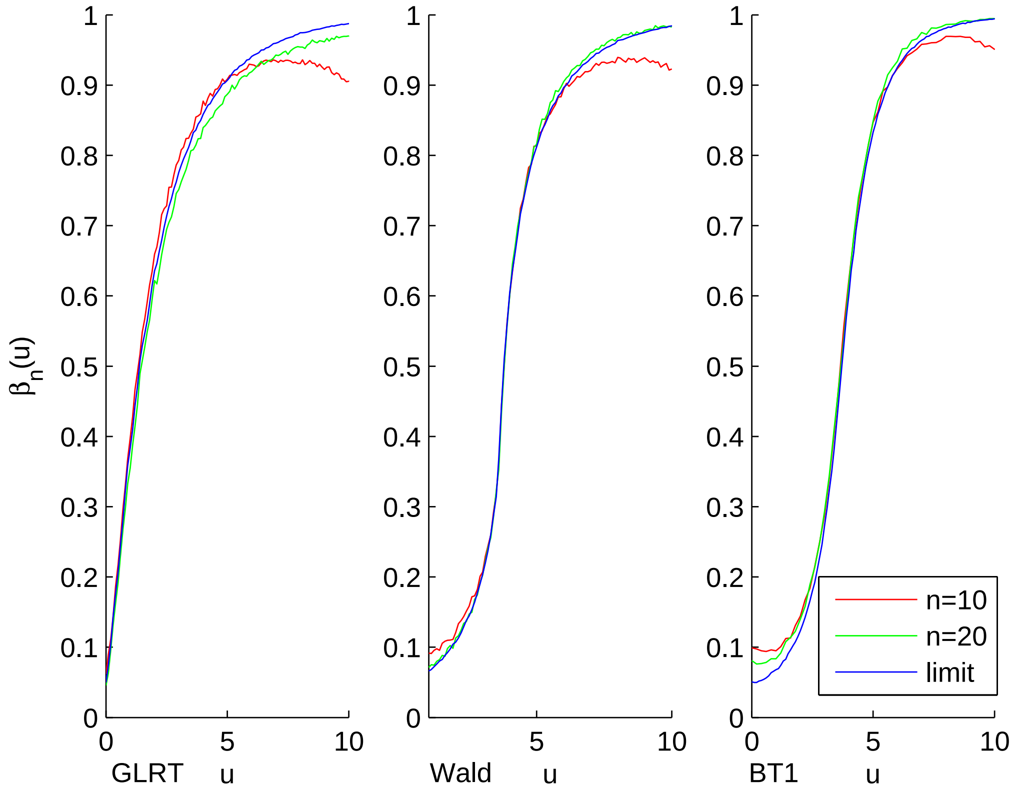

Figure 3.6: Power functions of GLRT, Wald's and BT1 in discontinuous density case

3.3.6 Comparison of the limit power functions

Let us fix an alternative $\vartheta_2 > \vartheta_1$. The hypotheses are defined by

$$\begin{aligned} \mathscr{H}_1 &: \quad \vartheta = \vartheta_1, \\ \mathscr{H}_2 &: \quad \vartheta = \vartheta_2. \end{aligned}$$

Using the notation $\vartheta_2 = \vartheta_1 + u_1\varphi_n$, we construct the correspondent close alternative and the problem became

$$\begin{aligned} \mathscr{H}_1 &: \quad u = 0, \\ \mathscr{H}_2 &: \quad u = u_1 \qquad (u_1 > 0). \end{aligned}$$

It is interesting to compare the studied tests with the Neyman-Pearson test. Of course, it is impossible to apply N-PT because the value under alternative is unknown, but its power function shows an upper bound and the distance between it and the power functions of studied tests provides an important information.

$$\psi_n^*\left(X^n\right) = \mathbb{1}_{\{Z_n(u_1) > d_\varepsilon\}} + q_\varepsilon \mathbb{1}_{\{Z_n(u_1) = d_\varepsilon\}},$$

where d_ε, q_ε is la solution of the equation

$$\mathbf{P}_{\vartheta_1}\left(Z_*\left(u_1\right) > d_\varepsilon\right) + q_\varepsilon \mathbf{P}_{\vartheta_1}\left(Z_*\left(u_1\right) = d_\varepsilon\right) = \varepsilon,$$

which can be calculated as follows, $\left(\text{denoting } M_\varepsilon = \frac{\ln d_\varepsilon + (\rho-1)u_1}{\ln \rho}\right)$

$$\mathbf{P}_{\vartheta_1}\left(x_*(u_1) > M_\varepsilon\right) + q_\varepsilon \mathbf{P}_{\vartheta_1}\left(x_*(u_1) = M_\varepsilon\right) = \varepsilon.$$

We have

$$\mathbf{P}_{\vartheta_1}\left(x_*(u_1) = M_\varepsilon\right) = \mathbf{P}_{\vartheta_1}\left(x_*(u_1) > M_\varepsilon-\right) - \mathbf{P}_{\vartheta_1}\left(x_*(u_1) > M_\varepsilon\right),$$

where x_* has the unit intensity.

Similar calculate yields the power function

$$\beta\left(\psi_n^*, u_1\right) \to \mathbf{P}_{\vartheta_1}\left(x_*(u_1) > M_\varepsilon\right) + q_\varepsilon \mathbf{P}_{\vartheta_1}\left(x_*(u_1) = M_\varepsilon\right).$$

and the Poisson process $x_*\left(v\right), v \geq 0$ has the switching intensity function (3.1).

Here we discuss the derivative of $\beta_n = \beta\left(\psi_n^*, u_1\right)$ w.r.t u_1. For each point u_j^*, $j = 1, \ldots$ such that $M_\varepsilon = [M_\varepsilon]$, we have the left derivative of the power function defined as follows

$$\begin{aligned} \frac{\partial \beta_n}{\partial u_1}\Big|_{u_1 = u_j^*-} &= \lim_{h\to 0} \frac{\beta\left(\psi_n^*, u_j^* - h\right) - \beta\left(\psi_n^*, u_j^*\right)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left\{\sum_{k > u_j^* - h} \frac{\left(\rho\left(u_j^* - h\right)\right)^k}{k!} e^{-\rho\left(u_j^* - h\right)} - \sum_{k > u_j^*} \frac{\left(\rho u_j^*\right)^k}{k!} e^{-\rho u_j^*}\right\} \end{aligned}$$

$$= \lim_{h\to 0} \frac{1}{h} \left\{ \frac{\left(\rho\left(u_j^* - h\right)\right)^{u_j^*}}{u_j^*!} e^{-\rho\left(u_j^*-h\right)} - \sum_{k>u_j^*} \left[\frac{\left(\rho u_j^*\right)^k}{k!} e^{-\rho u_j^*} \right.\right.$$

$$\left.\left. - \frac{\left(\rho\left(u_j^* - h\right)\right)^k}{k!} e^{-\rho\left(u_j^*-h\right)} \right] \right\} \longrightarrow +\infty$$

and the right derivative

$$\frac{\partial \beta_n}{\partial u_1}\Big|_{u_1=u_j^*+} = \lim_{h\to 0} \frac{\beta\left(\psi_n^*, u_j^* + h\right) - \beta\left(\psi_n^*, u_j^*\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \sum_{k>u_j^*} \left[\frac{\left(\rho\left(u_j^* + h\right)\right)^k}{k!} e^{-\rho\left(u_j^*+h\right)} - \frac{\left(\rho u_j^*\right)^k}{k!} e^{-\rho u_j^*} \right]$$

$$= \lim_{h\to 0} \sum_{k>u_j^*} \left((k-1)\left(\rho\left(u_j^* + h\right)\right)^{k-1} - \rho\left(\rho\left(u_j^* + h\right)\right)^k \right) \frac{e^{-\rho\left(u_j^*+h\right)}}{k!}$$

$$= \sum_{k>u_j^*} \left((k-1)\left(\rho u_j^*\right)^{k-1} - \rho\left(\rho u_j^*\right)^k \right) \frac{e^{-\rho u_j^*}}{k!}$$

which yields certain cusps in the envelope power function. The points u_1 such that $M_\varepsilon \neq [M_\varepsilon]$, we have

$$\frac{\partial \beta\left(\psi_n^*, u_1\right)}{\partial u_1} = \sum_{k>[M_\varepsilon]} \left((k-1)\left(\rho u_j^*\right)^{k-1} - \rho\left(\rho u_j^*\right)^k \right) \frac{e^{-\rho u_j^*}}{k!}$$

where $[A]$ present the integer part of A.

In Fig 3.7, the curves of BT1 tends to 1 the most quickly. The limit power function of GLRT is the closest one to the envelope power function. WT has the lowest limit power function when $\varepsilon = 0.05$ and the curve becomes better than that of BT1 when $\varepsilon = 0.4$ and u is small.

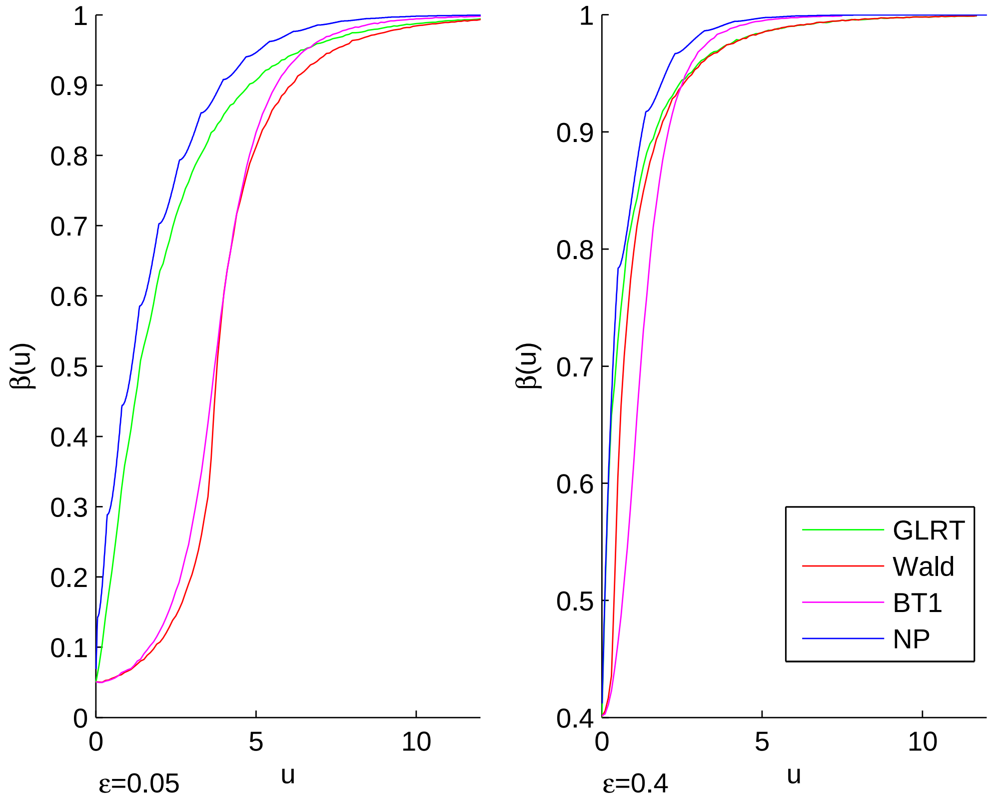

Figure 3.7: Comparison of different power functions with $\rho = 3$.

Chapter 4

Change-point model with variable jump size. Parameter estimation and hypotheses testing

4.1 Introduction

In regular statistical experiments, the limit of the normalized likelihood ratio is always the same, because the families are LAN (see, [17]). In the case of non regular statistical models for Poisson processes, there exists a large diversity of limiting likelihood ratio processes (change-point type models lead to a log Poisson process, "cusp" type singularities provide a log fBm process, in the models with discontinuous intensity function for Poisson field observations model the limits are more sophisticated, which can be found in [19], [4], [6]). Note that in the models of change-point type for diffusion processes, and particularly in the model of signal in white Gaussian noise (WGN), the limiting likelihood ratio is a log Wiener process (see also in [17], [20]). It is interesting to investigate the relations between the different limit processes. This study was initiated in the works [5] and [11]. The present work is a part of this investigation since in the case of Poisson processes with variable jump size we obtain two different limits depending on the way the jump size is varying.

We consider two cases. The first one corresponds to the situation when the jump size converges to a non-zero limit, while in the second one the limit is zero. The limiting likelihood ratios in these two cases are quite different. In the first case, like in the case of a fixed jump size, the normalized likelihood ratio converges to a log Poisson process. In the second case, the normalized likelihood ratio converges to a log Wiener process, i.e., the statistical problems of parameter estimation and hypotheses testing are asymptotically equivalent to the well known problems of change-point estimation and testing for signal in WGN. We show not only the convergence of normalized likelihood ratios, but also the convergence of the moments of the estimators. This last convergence allows us to approximate the limiting mean square errors of the maximum likelihood and Bayeasian estimators in the case of Poisson observations by the well known limiting mean square errors of these estimators calculated for signal in WGN model. The obtained theoretical results are illustrated by numerical simulations.

The properties of the estimators and tests are obtained with the help of the Ibragimov-Khasminskii method based on the study of the normalized likelihood ratio process. In all problems we verify the weak convergence of the normalized likelihood ratio process to the limiting likelihood ratio process in a suitable metric space. In particular, we check the convergence of finite-dimensional distributions and the tightness of the corresponding family of measures in the Skorohod space $\mathscr{D}_0(\mathbb{R})$.

4.2 Change-point model with variable jump size converging to a non-zero limit

Suppose we observe n independent realizations $X_j^{(n)} = \{X_j^{(n)}(t),\ t \in [0,\tau]\}$, $j = 1,\dots,n$, of an inhomogeneous Poisson process on the interval $[0,\tau]$ (the constant $\tau > 0$ is supposed to be known) of intensity measure

$$\Lambda_\vartheta^{(n)}(A) = \int_A \lambda_\vartheta^{(n)}(t)\,\mathrm{d}t, \qquad A \in \mathscr{B}\big([0,\tau]\big),$$

with intensity function $\lambda_\vartheta^{(n)}$, where $\vartheta \in \Theta = (\alpha,\beta)$, $0 < \alpha < \beta < \tau$, is some unknown parameter. The observation will be denoted $X^{(n)} = \{X_1^{(n)},\dots,X_n^{(n)}\}$ and the corresponding probability distribution will be denoted $\mathbf{P}_\vartheta^{(n)}$.

Let us note that this model of observation is equivalent to observing a single realization on the interval $[0,n\tau]$ of an inhomogeneous Poisson process with the τ-periodic intensity function coinciding with $\lambda_\vartheta^{(n)}$ on $[0,\tau]$.

The parameter ϑ corresponds to the location of a jump in the (elsewhere continuous) intensity function $\lambda_\vartheta^{(n)}$. The size of the jump (depending on n) will be denoted r_n and will be supposed converging to some $r \neq 0$.

More precisely, we assume that the following conditions are satisfied.

(C1) The intensity function $\lambda_\vartheta^{(n)}(t)$ can be written as $\lambda_\vartheta^{(n)}(t) = \psi_n(t) + r_n \mathbb{1}_{\{t>\vartheta\}}$, where the function ψ_n is continuous on $[0,\tau]$.

(C2) For all $t \in [0,\tau]$, there exist the $\lim\limits_{n\to+\infty} \psi_n(t) = \psi(t) > 0$ and, moreover, this convergence is uniform with respect to t.

(C3) The jump size r_n is such that $r_n \to r \neq 0$ as $n \to +\infty$.

(C4) The family of functions $\{\lambda_\vartheta^{(n)}\}_{n\in\mathbb{N},\vartheta\in\Theta}$ is uniformly strictly positive and uniformly bounded, that is, there exist some constants $\ell, L > 0$ such that

$$\ell \leq \lambda_\vartheta^{(n)}(t) \leq L$$

for all $n \in \mathbb{N}$, $\vartheta \in \Theta$ and $t \in [0,\tau]$.

Note that the conditions **C1** – **C3**, together with the natural condition

$$r > -\min_{t\in[0,\tau]} \psi(t), \tag{4.1}$$

easily imply that the condition **C4** holds for the family $\{\lambda_\vartheta^{(n)}\}_{n\geq n_0,\vartheta\in\Theta}$ with some $n_0 \in \mathbb{N}$. So, in the asymptotic setting ($n \to +\infty$), the condition **C4** can be replaced by (4.1), and we assume **C4** instead of the latter only for convenience (as well as in order for our model to be well defined for all $n \in \mathbb{N}$).

An important particular case of this model is when only the jump size (and not the regular part of $\lambda_\vartheta^{(n)}$) depend on n. More precisely, the conditions **C1** – **C2** will be clearly met if we assume that the following condition is satisfied.

(J) The intensity function $\lambda_\vartheta^{(n)}(t)$ can be written as $\lambda_\vartheta^{(n)}(t) = \psi(t) + r_n \mathbb{1}_{\{t>\vartheta\}}$, where the function ψ is strictly positive and continuous on $[0,\tau]$.

4.2.1 Asymptotic behavior of the likelihood ratio

The likelihood of our model is given by (see, for example, [18])

$$\begin{aligned} L_n\big(\vartheta, X^{(n)}\big) &= \exp\Bigg\{\sum_{j=1}^n \int_{[0,\tau]} \ln \lambda_\vartheta^{(n)}(t)\ X_j^{(n)}(\mathrm{d}t) - n\int_0^\tau \big[\lambda_\vartheta^{(n)}(t) - 1\big]\,\mathrm{d}t\Bigg\} \\ &= \exp\Bigg\{\sum_{j=1}^n \sum_{i\in I_j^{(n)}} \ln \lambda_\vartheta^{(n)}(t_{j,i}) - n\int_0^\tau \big[\lambda_\vartheta^{(n)}(t) - 1\big]\,\mathrm{d}t\Bigg\}, \end{aligned} \tag{4.2}$$

where $t_{j,i}$, $i \in I_j^{(n)}$, are the jump times of the process $X_j^{(n)}$. Note that as function of ϑ, each $\lambda_\vartheta^{(n)}(t_{j,i})$ is discontinuous (has a jump and is right continuous) at $\vartheta = t_{j,i}$. So, $L_n\big(\,\cdot\,, X^{(n)}\big)$ is a random process with càdlàg (continuous from the right and having finite limits from the left) trajectories.

We introduce the normalized likelihood ratio

$$\begin{aligned} Z_{n,\vartheta}(u) &= \frac{L_n\big(\vartheta + u/n, X^{(n)}\big)}{L_n\big(\vartheta, X^{(n)}\big)} \\ &= \exp\Bigg\{\sum_{j=1}^n \int_{[0,\tau]} \ln \frac{\lambda_{\vartheta+u/n}^{(n)}(t)}{\lambda_\vartheta^{(n)}(t)}\ X_j^{(n)}(\mathrm{d}t) - n\int_0^\tau \big(\lambda_{\vartheta+u/n}^{(n)}(t) - \lambda_\vartheta^{(n)}(t)\big)\,\mathrm{d}t\Bigg\} \end{aligned}$$

$$= \exp\Bigg\{ \sum_{j=1}^{n} \sum_{i \in I_j^{(n)}} \ln \frac{\lambda_{\vartheta+u/n}^{(n)}(t_{j,i})}{\lambda_{\vartheta}^{(n)}(t_{j,i})} - n \int_0^{\tau} \big(\lambda_{\vartheta+u/n}^{(n)}(t) - \lambda_{\vartheta}^{(n)}(t)\big)\, \mathrm{d}t \Bigg\},$$

where $u \in U_n = \big(n(\alpha - \vartheta), n(\beta - \vartheta)\big)$.

Note that if $u > 0$, we can rewrite $Z_{n,\vartheta}(u)$ as

$$\begin{aligned} Z_{n,\vartheta}(u) &= \exp\Bigg\{ \sum_{j=1}^{n} \int_{(\vartheta\,,\,\vartheta+u/n]} \ln \frac{\psi_n(t)}{\psi_n(t) + r_n}\, X_j^{(n)}(\mathrm{d}t) + n \int_{\vartheta}^{\vartheta+u/n} r_n \,\mathrm{d}t \Bigg\} \\ &= \exp\Bigg\{ \sum_{j=1}^{n} \sum_{i} \ln \frac{\psi_n(t_{j,i})}{\psi_n(t_{j,i}) + r_n} + u r_n \Bigg\}, \end{aligned} \tag{4.3}$$

where the last sum is taken over the set $\big\{ i \in I_j^{(n)} \,:\, \vartheta < t_{j,i} \le \vartheta + u/n \big\}$.

Similarly, if $u < 0$, we have

$$\begin{aligned} Z_{n,\vartheta}(u) &= \exp\Bigg\{ \sum_{j=1}^{n} \int_{(\vartheta+u/n\,,\,\vartheta]} \ln \frac{\psi_n(t) + r_n}{\psi_n(t)}\, X_j^{(n)}(\mathrm{d}t) - n \int_{\vartheta+u/n}^{\vartheta} r_n \,\mathrm{d}t \Bigg\} \\ &= \exp\Bigg\{ \sum_{j=1}^{n} \sum_{i} \ln \frac{\psi_n(t_{j,i}) + r_n}{\psi_n(t_{j,i})} + u r_n \Bigg\}, \end{aligned} \tag{4.4}$$

where the last sum is taken over the set $\big\{ i \in I_j^{(n)} \,:\, \vartheta + u/n < t_{j,i} \le \vartheta \big\}$.

Note also, that the process $\ln Z_{n,\vartheta}$ has independent increments. Indeed, its increments on disjoint intervals involve stochastic integrals on disjoint intervals, and hence are independent.

Note finally, that the trajectories of the process $Z_{n,\vartheta}$ are càdlàg functions. Moreover, correctly extending these trajectories to the whole real line, one can consider that they belong to the Skorohod space $\mathscr{D}_0(\mathbb{R})$. This space is defined as the space of functions f on $\mathbb{R}$ which do not have discontinuities of the second kind and which are vanishing at infinity, that is, such that $\lim_{u \to \pm\infty} f(u) = 0$. We assume that all the functions $f \in \mathscr{D}_0(\mathbb{R})$ are continuous from the right (are càdlàg).

Let us recall that the Skorohod metric on the space $\mathscr{D}_0(\mathbb{R})$ is introduced by

$$d(f,g) = \inf_{\lambda} \Big[\sup_{u \in \mathbb{R}} \big| f(u) - g\big(\lambda(u)\big) \big| + \sup_{u \in \mathbb{R}} |u - \lambda(u)| \Big],$$

where the inf is taken over all strictly increasing continuous one-to-one mappings $\lambda : \mathbb{R} \to \mathbb{R}$.

Let us also recall a criterion of weak convergence in $\mathscr{D}_0(\mathbb{R})$. We put

$$\Delta_h(f) = \sup_{u \in \mathbb{R}} \sup_{u', u''} \Big[\min\big\{ |f(u') - f(u)|\,, |f(u'') - f(u)| \big\} \Big] + \sup_{|u| > 1/h} |f(u)|,$$

where the inner sup is over all u', u'' such that $u - h \le u' < u \le u'' < u + h$. A criterion of weak convergence in $\mathscr{D}_0(\mathbb{R})$ is given in the following lemma (see [14] for more details).

Lemma 4.1. *Let $z_{n,\vartheta}$, $n \in \mathbb{N}$, and z_ϑ be random processes with realizations belonging to $\mathscr{D}_0(\mathbb{R})$ with probability 1. If, as $n \to +\infty$, the finite dimensional distributions of $z_{n,\vartheta}$ converge uniformly in $\vartheta \in \mathbb{K}$ to the finite dimensional distributions of z_ϑ, and if for any $\delta > 0$*

$$\lim_{h \to 0} \sup_{n \in \mathbb{N}, \vartheta \in \mathbb{K}} \mathbf{P}\big\{\Delta_h(z_{n,\vartheta}) > \delta\big\} = 0, \tag{4.5}$$

then, uniformly in $\vartheta \in \mathbb{K}$, the process $z_{n,\vartheta}$ converges weakly in the space $\mathscr{D}_0(\mathbb{R})$ to the process z_ϑ.

Note that here and in the sequel $\mathbb{K}$ denotes an arbitrary compact in Θ.

The main objective of this section is the study of the asymptotic behavior (in the sense of the weak convergence in the space $\mathscr{D}_0(\mathbb{R})$ as $n \to \infty$) of the above introduced normalized likelihood ratio $Z_{n,\vartheta}$.

For this, we introduce the random process

$$Z_\vartheta(u) = \begin{cases} \exp\Big\{\ln \frac{\psi(\vartheta)}{\psi(\vartheta)+r} X^+(u) + ru\Big\}, & \text{if } u \geq 0, \\ \exp\Big\{\ln \frac{\psi(\vartheta)+r}{\psi(\vartheta)} X^-\big((-u)-\big) + ru\Big\}, & \text{if } u < 0, \end{cases} \tag{4.6}$$

where X^+ and X^- are independent Poisson processes on $\mathbb{R}_+$ of constant intensities $\psi(\vartheta) + r$ and $\psi(\vartheta)$ respectively.

Let us note that $Z_\vartheta(u) \overset{d}{=} Z^*_\rho(-ru)$ with the constant $\rho = \big|\ln \frac{\psi(\vartheta)}{\psi(\vartheta)+r}\big|$ and the process Z^*_ρ defined by

$$Z^*_\rho(x) = \begin{cases} \exp\big\{\rho Y^+(x) - x\big\}, & \text{if } x \geq 0, \\ \exp\big\{-\rho Y^-\big((-x)-\big) - x\big\}, & \text{if } x < 0, \end{cases} \tag{4.7}$$

where Y^+ and Y^- are independent Poisson processes on $\mathbb{R}_+$ of constant intensities $\frac{1}{e^\rho - 1}$ and $\frac{1}{1-e^{-\rho}}$ respectively.

Note also that the process Z^*_ρ was recently studied in [5] and that its trajectories (as well as those of the process Z_ϑ) almost surely belong to the space $\mathscr{D}_0(\mathbb{R})$.

Now we can state the following theorem about the asymptotic behavior of the normalized likelihood ratio.

Theorem 4.1. *Let the conditions* **C1** – **C4** *be fulfilled. Then, uniformly in $\vartheta \in \mathbb{K}$, the process $Z_{n,\vartheta}$ converges weakly in the space $\mathscr{D}_0(\mathbb{R})$ to the process Z_ϑ.*

Let us also remark, that sometimes it may be more convenient to use the rate $-\frac{1}{rn}$ when $r < 0$ (rather than $\frac{1}{n}$) for introducing the normalized likelihood ratio. That is, one can put $\varphi_n = -\frac{1}{rn}$ and introduce the normalized likelihood ratio $Z^*_{n,\vartheta}$ (instead of $Z_{n,\vartheta}$) as

$$Z^*_{n,\vartheta}(v) = \frac{L_n\big(\vartheta + v\,\varphi_n, X^{(n)}\big)}{L_n\big(\vartheta, X^{(n)}\big)} = Z_{n,\vartheta}(-v/r).$$

Then, Theorem 4.1 will be clearly transformed to the following (equivalent) statement.

Theorem 4.2. *Let the conditions* **C1** – **C4** *be fulfilled. Then, uniformly in* $\vartheta \in \mathbb{K}$, *the process* $Z^*_{n,\vartheta}$ *converges weakly in the space* $\mathscr{D}_0(\mathbb{R})$ *to the process* Z^*_ρ.

When $r > 0$ we can take the rate $\varphi_n = \frac{1}{rn}$ such that the normalized likelihood ratio $Z^*_{n,\vartheta}$ as

$$Z^*_{n,\vartheta}(v) = \frac{L_n\big(\vartheta + v\,\varphi_n, X^{(n)}\big)}{L_n\big(\vartheta, X^{(n)}\big)} = Z_{n,\vartheta}(v/r)$$

which has the limit normalized likelihood ratio $Z^*_\rho(-v)$ and obtain the similar result.

The proof of Theorem 4.1 consist in checking the criterion of week convergence given in Lemma 4.1. For this, we follow the methods and ideas used in [17, Chapters 5.3 and 5.4] and establish several lemmas.

Lemma 4.2. *Let the conditions* **C1** – **C4** *be fulfilled. Then the finite-dimensional distributions of the process* $Z_{n,\vartheta}$ *converge to those of the process* Z_ϑ, *and this convergence is uniform with respect to* $\vartheta \in \mathbb{K}$.

Proof. We follow the proof of [18, Theorem 4.4.4]. First we study the convergence of 2-dimensional distributions. For this, consider the distribution of the vector $\big(Z_n(u_1), Z_n(u_2)\big)$ with some fixed $u_1, u_2 \in \mathbb{R}$. The characteristic function of the natural logarithm of this vector can be written as follows $\big($see, for example, [18]$\big)$:

$$\begin{aligned}
&\mathbf{E}^{(n)}_\vartheta \exp\big(it_1 \ln Z_{n,\vartheta}(u_1) + it_2 \ln Z_{n,\vartheta}(u_2)\big)\\
&\quad= \exp\Bigg\{n\int_0^\tau \Bigg(\exp\Bigg\{it_1 \ln \frac{\lambda^{(n)}_{\vartheta+u_1/n}(t)}{\lambda^{(n)}_\vartheta(t)} + it_2 \ln \frac{\lambda^{(n)}_{\vartheta+u_2/n}(t)}{\lambda^{(n)}_\vartheta(t)}\Bigg\} - 1\\
&\qquad\qquad - it_1\Bigg(\frac{\lambda^{(n)}_{\vartheta+u_1/n}(t)}{\lambda^{(n)}_\vartheta(t)} - 1\Bigg) - it_2\Bigg(\frac{\lambda^{(n)}_{\vartheta+u_2/n}(t)}{\lambda^{(n)}_\vartheta(t)} - 1\Bigg)\Bigg)\lambda^{(n)}_\vartheta(t)\,\mathrm{d}t\Bigg\}\\
&\quad= \exp\Big\{A_{n,\vartheta}(u_1,u_2,t)\Big\}
\end{aligned}$$

with an evident notation.

We will consider the case $u_2 > u_1 \geq 0$ only (the other cases can be treated in a similar way). In this case, we have

$$\begin{aligned}
A_{n,\vartheta}(u_1,u_2,t) &= n\int_\vartheta^{\vartheta+u_1/n}\Bigg(\exp\Bigg\{(it_1+it_2)\ln\frac{\psi_n(t)}{\psi_n(t)+r_n}\Bigg\} - 1\\
&\qquad - (it_1+it_2)\Bigg(\frac{\psi_n(t)}{\psi_n(t)+r_n} - 1\Bigg)\Bigg)\Big(\psi_n(t)+r_n\Big)\,\mathrm{d}t\\
&\quad + n\int_{\vartheta+u_1/n}^{\vartheta+u_2/n}\Bigg(\exp\Bigg\{it_2\ln\frac{\psi_n(t)}{\psi_n(t)+r_n}\Bigg\} - 1\\
&\qquad - it_2\Bigg(\frac{\psi_n(t)}{\psi_n(t)+r_n} - 1\Bigg)\Bigg)\Big(\psi_n(t)+r_n\Big)\,\mathrm{d}t\\
&= nI_1 + nI_2
\end{aligned}$$

with evident notations.

Using the mean value theorem for the integrals I_1 and I_2, it is possible to find some $s_n \in (\vartheta, \vartheta + u_1/n)$ and $v_n \in (\vartheta + u_1/n, \vartheta + u_2/n)$ such that

$$nI_1 = u_1\Big(\exp\{i(t_1+t_2)\ln g_n(s_n)\} - 1 - i(t_1+t_2)\big(g_n(s_n)-1\big)\Big)\Big(\psi_n(s_n)+r_n\Big)$$

and

$$nI_2 = (u_2-u_1)\Big(\exp\{it_2\ln g_n(v_n)\} - 1 - it_2\big(g_n(v_n)-1\big)\Big)\Big(\psi_n(v_n)+r_n\Big),$$

where we have denoted $g_n(t) = \frac{\psi_n(t)}{\psi_n(t)+r_n}$.

As $s_n \to \vartheta$, according to the condition **C2**, we clearly have $\psi_n(s_n) \to \psi(\vartheta)$. Taking into account the condition **C4** $\big($or (4.1)$\big)$, we have $\psi(\vartheta) + r > 0$, and hence $g_n(s_n) \to \frac{\psi(\vartheta)}{\psi(\vartheta)+r}$. So,

$$nI_1 \to u_1\big(\psi(\vartheta)+r\big)\left(\exp\Big\{i(t_1+t_2)\ln\frac{\psi(\vartheta)}{\psi(\vartheta)+r}\Big\} - 1 - i(t_1+t_2)\Big(\frac{\psi(\vartheta)}{\psi(\vartheta)+r}-1\Big)\right).$$

Similarly, we can show that

$$nI_2 \to (u_2-u_1)\big(\psi(\vartheta)+r\big)\left(\exp\Big\{it_2\ln\frac{\psi(\vartheta)}{\psi(\vartheta)+r}\Big\} - 1 - it_2\Big(\frac{\psi(\vartheta)}{\psi(\vartheta)+r}-1\Big)\right).$$

Hence,

$$\begin{aligned}
&\mathbf{E}_\vartheta^{(n)} \exp\big(it_1\ln Z_{n,\vartheta}(u_1) + it_2\ln Z_{n,\vartheta}(u_2)\big)\\
&\qquad \to \exp\Big\{u_1\big(\psi(\vartheta)+r\big)\Big[\exp\Big\{i(t_1+t_2)\ln\frac{\psi(\vartheta)}{\psi(\vartheta)+r}\Big\}-1\Big] + i(t_1+t_2)u_1 r\\
&\qquad\qquad + (u_2-u_1)\big(\psi(\vartheta)+r\big)\Big[\exp\Big\{it_2\ln\frac{\psi(\vartheta)}{\psi(\vartheta)+r}\Big\}-1\Big] + it_2(u_2-u_1)r\Big\}\\
&\qquad = \mathbf{E}\exp\big(it_1\ln Z_\vartheta(u_1) + it_2\ln Z_\vartheta(u_2)\big),
\end{aligned}$$

where the last equality is due to [18, Lemma 4.4.1].

So, the convergence of 2-dimensional distributions is proved. The convergence of three and more dimensional distributions can be carried out in a similar way, and the uniformity with respect to ϑ is obvious. □

Lemma 4.3. *Let the conditions* **C1** – **C4** *be fulfilled. Then there exists a constant* $C > 0$ *such that*

$$\mathbf{E}_\vartheta^{(n)}\big|Z_{n,\vartheta}^{1/2}(u_1) - Z_{n,\vartheta}^{1/2}(u_2)\big|^2 \le C\,|u_1-u_2|$$

for all $n \in \mathbb{N}$, $u_1, u_2 \in U_n$ *and* $\vartheta \in \mathbb{K}$.

Proof. We will consider the case $u_2 \geq u_1 \geq 0$ only (the other cases can be treated in a similar way). According to [19, Lemma 1.1.5], we have

$$\begin{aligned}\mathbf{E}_\vartheta^{(n)} \left|Z_{n,\vartheta}^{1/2}(u_1) - Z_{n,\vartheta}^{1/2}(u_2)\right|^2 &\leq n \int_0^\tau \left(\sqrt{\lambda_{\vartheta+u_1/n}^{(n)}(t)} - \sqrt{\lambda_{\vartheta+u_2/n}^{(n)}(t)}\right)^2 \mathrm{d}t \\ &= n \int_{\vartheta+u_1/n}^{\vartheta+u_2/n} \left(\sqrt{\psi_n(t) + r_n} - \sqrt{\psi_n(t)}\right)^2 \mathrm{d}t.\end{aligned}$$

Using the elementary inequality $\left|\sqrt{a} - \sqrt{b}\right| \leq \sqrt{|a-b|}$ and the fact that the sequence $\{r_n\}_{n\in\mathbb{N}}$ is convergent, and hence bounded, we get

$$\mathbf{E}_\vartheta^{(n)} \left|Z_{n,\vartheta}^{1/2}(u_1) - Z_{n,\vartheta}^{1/2}(u_2)\right|^2 \leq n \int_{\vartheta+u_1/n}^{\vartheta+u_2/n} |r_n| \,\mathrm{d}t = |r_n|\,(u_2 - u_1) \leq C(u_2 - u_1),$$

which is the required inequality. □

Lemma 4.4. *Let the conditions* **C1** – **C4** *be fulfilled. Then there exists a constant* $k_* > 0$ *such that*

$$\mathbf{E}_\vartheta^{(n)} Z_{n,\vartheta}^{1/2}(u) \leq \exp\bigl\{-k_*\,|u|\bigr\}$$

for all $u_1, u_2 \in U_n$, $\vartheta \in \mathbb{K}$ *and sufficiently large values of* n.

Proof. We will consider the case $u \geq 0$ only (the other case can be treated in a similar way). According to [19, Lemma 1.1.5], we have

$$\begin{aligned}\mathbf{E}_\vartheta^{(n)} Z_{n,\vartheta}^{1/2}(u) &= \exp\left\{-\frac{n}{2}\int_0^\tau \left(\sqrt{\lambda_{\vartheta+u/n}^{(n)}(t)} - \sqrt{\lambda_\vartheta^{(n)}(t)}\right)^2 \mathrm{d}t\right\} \\ &= \exp\left\{-\frac{n}{2}\int_\vartheta^{\vartheta+u/n} \left(\sqrt{\psi_n(t)} - \sqrt{\psi_n(t)+r_n}\right)^2 \mathrm{d}t\right\} \\ &= \exp\left\{-\frac{n}{2}\int_\vartheta^{\vartheta+u/n} \frac{r_n^2}{\left(\sqrt{\psi_n(t)} + \sqrt{\psi_n(t)+r_n}\right)^2} \,\mathrm{d}t\right\}.\end{aligned}$$

As $\lambda_\vartheta^{(n)}$ is uniformly bounded,

$$\left(\sqrt{\psi_n(t)+r_n} + \sqrt{\psi_n(t)}\right)^2 \leq \left(\sqrt{L} + \sqrt{L}\right)^2 = 4L,$$

and hence, for sufficiently large values of n we get

$$\mathbf{E}_\vartheta^{(n)} Z_{n,\vartheta}^{1/2}(u) \leq \exp\left\{-\frac{n}{2}\int_\vartheta^{\vartheta+u/n} \frac{r_n^2}{4L}\,\mathrm{d}t\right\} = \exp\left\{-\frac{r_n^2}{8L}\,u\right\} \leq \exp\left\{-\frac{r^2}{16L}\,|u|\right\},$$

which is the required inequality with $k_* = \frac{r^2}{16L}$. □

In order to check the condition (4.5) (for $z_{n,\vartheta} = Z_{n,\vartheta}$), we need to introduce some more notations. We denote $Z^{1/2}_{n,\vartheta;\text{a.c.}}$ the absolutely continuous component of the function $Z^{1/2}_{n,\vartheta}$ and, for $p = 1, 2$, we denote $A_p = A_p(u, u+h)$ the event that $Z_{n,\vartheta}$ has at least p jumps on the interval $(u, u+h)$.

Lemma 4.5. *Let the conditions* **C1** – **C4** *be fulfilled. Then the inequalities*

$$\mathbf{E}^{(n)}_{\vartheta}\big|Z^{1/2}_{n,\vartheta;\text{a.c.}}(u+h) - Z^{1/2}_{n,\vartheta;\text{a.c.}}(u)\big|^2 \leq Ch^2, \tag{4.8}$$

$$\mathbf{P}^{(n)}_{\vartheta}(A_1) \leq D_1 h \tag{4.9}$$

and

$$\mathbf{P}^{(n)}_{\vartheta}(A_2) \leq D_2 h^2 \tag{4.10}$$

hold with certain constants $C, D_1, D_2 > 0$ (independent of n, ϑ, u and h).

Proof. To show the inequality (4.8), we follow the proof of [18, Lemma 4.4.3]. Denoting $\dot Z^{1/2}_{n,\vartheta}$ the derivative of $Z^{1/2}_{n,\vartheta;\text{a.c.}}$ (or, equivalently, of $Z^{1/2}_{n,\vartheta}$), we get

$$\mathbf{E}^{(n)}_{\vartheta}\big|Z^{1/2}_{n,\vartheta;\text{a.c.}}(u+h) - Z^{1/2}_{n,\vartheta;\text{a.c.}}(u)\big|^2 = \mathbf{E}^{(n)}_{\vartheta}\bigg|\int_u^{u+h} \dot Z^{1/2}_{n,\vartheta}(v)\,\mathrm{d}v\bigg|^2.$$

Further, we have

$$Z^{1/2}_{n,\vartheta}(v) = \exp\bigg\{\frac{1}{2}\sum_{j=1}^{n}\int_{[0,\tau]} \ln\frac{\lambda^{(n)}_{\vartheta+v/n}(t)}{\lambda^{(n)}_{\vartheta}(t)}\, X^{(n)}_j(\mathrm{d}t) - \frac{n}{2}\int_0^{\tau}\big(\lambda^{(n)}_{\vartheta+u/n}(t) - \lambda^{(n)}_{\vartheta}(t)\big)\,\mathrm{d}t\bigg\}.$$

Looking at the expressions (4.3) and (4.4) it becomes clear that the first term in the above exponential is a piecewise constant function with respect to v, and the second one equals $vr_n/2$. So,

$$\dot Z^{1/2}_{n,\vartheta}(v) = \frac{r_n}{2}\, Z^{1/2}_{n,\vartheta}(v),$$

and hence

$$\begin{aligned}\mathbf{E}^{(n)}_{\vartheta}\big|Z^{1/2}_{n,\vartheta;\text{a.c.}}(u+h) - Z^{1/2}_{n,\vartheta;\text{a.c.}}(u)\big|^2 &= \mathbf{E}^{(n)}_{\vartheta}\bigg|\frac{r_n}{2}\int_u^{u+h} Z^{1/2}_{n,\vartheta}(v)\,\mathrm{d}v\bigg|^2 \\ &\leq \frac{r_n^2}{4}\,\mathbf{E}^{(n)}_{\vartheta}\bigg[\int_u^{u+h} Z_{n,\vartheta}(v)\,\mathrm{d}v \int_u^{u+h}\mathrm{d}v\bigg] \\ &= \frac{r_n^2}{4}\,h^2 \leq Ch^2\end{aligned}$$

since the sequence $\{r_n\}_{n\in\mathbb{N}}$ is convergent, and hence bounded.

In order to establish the remaining inequalities (4.9) and (4.10), we follow the proof of [18, Lemma 4.4.4]. The pure jump component of the function $\ln Z_{n,\vartheta}(\cdot)$ is given by

$$\sum_{j=1}^{n}\sum_{i\in I_j^{(n)}} \ln \lambda^{(n)}_{\vartheta+\cdot/n}(t_{j,i}),$$

where $t_{j,i}$, $i \in I_j^{(n)}$, are (as before) the jump times of the process $X_j^{(n)}$. So, the process $Z_{n,\vartheta}$ has its jumps in the points $u_{j,i} = (t_{j,i} - \vartheta)n$ (where j and i are such that $u_{j,i} \in U_n$).

The event A_1 is equivalent to the event $u_{j,i} \in (u, u+h)$ for (at least) some j and some i. The event $u_{j,i} \in (u, u+h)$ is, in turn, equivalent to the inequality

$$\vartheta + u/n < t_{j,i} < \vartheta + (u+h)/n.$$

We denote $a_n = \vartheta + u/n$ and let $B_p^{(j)}$, $p = 1, 2$, be the event that the process $X_j^{(n)}$ has at least p jumps on the interval $(a_n, a_n + h/n)$. We have

$$\begin{aligned} \mathbf{P}_\vartheta^{(n)}\bigl(B_1^{(j)}\bigr) &= 1 - \exp\Bigl\{-\int_{a_n}^{a_n+h/n} \lambda_\vartheta^{(n)}(t)\,\mathrm{d}t\Bigr\} \\ &\leq \int_{a_n}^{a_n+h/n} \lambda_\vartheta^{(n)}(t)\,\mathrm{d}t \leq \frac{h}{n}\,L. \end{aligned}$$

Note that in the first equality we assumed that $(a_n, a_n + h/n) \subset [0, \tau]$. However, if this is not the case, the value of $\mathbf{P}_\vartheta^{(n)}\bigl(B_1^{(j)}\bigr)$ will be even smaller.

As $A_1 \subset \bigcup_{j=1}^{n} B_1^{(j)}$, we obtain

$$\mathbf{P}_\vartheta^{(n)}(A_1) \leq \sum_{j=1}^{n} \mathbf{P}_\vartheta^{(n)}\bigl(B_1^{(j)}\bigr) \leq n\,\frac{h}{n}\,L = D_1 h$$

with $D_1 = L$.

Further, as $A_2 \subset \Bigl(\bigcup_{j=1}^{n-1} \bigcup_{k=j+1}^{n} B_1^{(j)} \cap B_1^{(k)}\Bigr) \cup \Bigl(\bigcup_{j=1}^{n} B_2^{(j)}\Bigr)$, we get

$$\begin{aligned} \mathbf{P}_\vartheta^{(n)}(A_2) &\leq \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \mathbf{P}_\vartheta^{(n)}\bigl(B_1^{(j)}\bigr)\,\mathbf{P}_\vartheta^{(n)}\bigl(B_1^{(k)}\bigr) + \sum_{j=1}^{n} \mathbf{P}_\vartheta^{(n)}\bigl(B_2^{(j)}\bigr) \\ &\leq \frac{n(n-1)}{2}\Bigl(\frac{h}{n}\,L\Bigr)^2 + n\Bigl(\frac{h}{n}\,L\Bigr)^2 \\ &= \frac{n+1}{2n}\,L^2 h^2 \leq D_2 h^2. \end{aligned}$$

Here we denoted $D_2 = L^2$ and used the inequality

$$\begin{aligned} \mathbf{P}_\vartheta^{(n)}\bigl(B_2^{(j)}\bigr) &= 1 - \exp\Bigl\{-\int_{a_n}^{a_n+h/n} \lambda_\vartheta^{(n)}(t)\,\mathrm{d}t\Bigr\} - \\ &\qquad - \int_{a_n}^{a_n+h/n} \lambda_\vartheta^{(n)}(t)\,\mathrm{d}t\,\exp\Bigl\{-\int_{a_n}^{a_n+h/n} \lambda_\vartheta^{(n)}(t)\,\mathrm{d}t\Bigr\} \\ &\leq \int_{a_n}^{a_n+h/n} \lambda_\vartheta^{(n)}(t)\,\mathrm{d}t\,\Biggl(1 - \exp\Bigl\{-\int_{a_n}^{a_n+h/n} \lambda_\vartheta^{(n)}(t)\,\mathrm{d}t\Bigr\}\Biggr) \end{aligned}$$

$$\leq \Big(\int_{a_n}^{a_n+h/n} \lambda_\vartheta^{(n)}(t)\,\mathrm{d}t\Big)^2 \leq \Big(\frac{h}{n}\,L\Big)^2.$$

So, the lemma is proved. □

Now, with the help of the above lemmas, we can finish the proof of Theorem 4.1 following the standard argument of [17, Chapters 5.3 and 5.4]. More precisely, the weak convergence in $\mathscr{D}_0(\mathbb{R})$ of the processes $Z_{n,\vartheta}$ to the process Z_ϑ follows from Theorem 5.4.2 of [18], which is, in fact, contained in [17] (without being formulated there). Note, that the conditions of this theorem are nothing but Lemmas 4.2, 4.4 and 4.5, and that its proof consist in verifying the condition (4.5). Let us mention, that one of the ingredients obtained during this verifying are the inequalities

$$\mathbf{P}_\vartheta^{(n)}\bigg(\sup_{D\leq|u|\leq D+1} Z_{n,\vartheta}(u) > e^{-bD}\bigg) \leq Ce^{-bD}$$

and

$$\mathbf{P}_\vartheta^{(n)}\bigg(\sup_{|u|>D} Z_{n,\vartheta}(u) > e^{-bD}\bigg) \leq Ce^{-bD}. \tag{4.11}$$

These inequalities (particularly the last one) are needed to control the second term of the modulus of continuity $\Delta_h(Z_{n,\vartheta})$, but will also be useful for the study of the maximum likelihood estimator in the next section.

4.2.2 Parameter estimation

In this section we are interested in the estimation of the unknown parameter ϑ in our model of observations.

Recall that as function of ϑ, the likelihood of our model given by (4.2) is discontinuous (has jumps). So, the maximum likelihood estimator is introduced through the equation

$$\max\Big\{L_n\big(\widehat{\vartheta}_n+, X^{(n)}\big), L_n\big(\widehat{\vartheta}_n-, X^{(n)}\big)\Big\} = \sup_{\vartheta\in\Theta} L_n\big(\vartheta, X^{(n)}\big).$$

The Bayesian estimator for a given prior density p and for square loss is defined by

$$\widetilde{\vartheta}_n = \frac{\int_\alpha^\beta \vartheta\, p(\vartheta) L_n\big(\vartheta, X^{(n)}\big)\,\mathrm{d}\vartheta}{\int_\alpha^\beta p(\vartheta) L_n\big(\vartheta, X^{(n)}\big)\,\mathrm{d}\vartheta}.$$

We are interested in the asymptotic properties of the maximum likelihood and Bayesian estimators of ϑ as $n \to +\infty$. To describe the properties of the estimators we need some additional notations.

We introduce the random variables ξ_ϑ and ζ_ϑ by the equations

$$\max\bigl\{Z_\vartheta(\xi_\vartheta+), Z_\vartheta(\xi_\vartheta-)\bigr\} = \sup_{u\in\mathbb{R}} Z_\vartheta(u)$$

and

$$\zeta_\vartheta = \frac{\int_{-\infty}^{+\infty} u\, Z_\vartheta(u)\,\mathrm{d}u}{\int_{-\infty}^{+\infty} Z_\vartheta(u)\,\mathrm{d}u}\,,$$

where Z_ϑ is the process introduced in (4.6).

Note that introducing the random variables ξ_ρ^* and ζ_ρ^* by the equations

$$\max\bigl\{Z_\rho^*(\xi_\rho^*+), Z_\rho^*(\xi_\rho^*-)\bigr\} = \sup_{u\in\mathbb{R}} Z_\rho^*(u)$$

and

$$\zeta_\rho^* = \frac{\int_{-\infty}^{+\infty} u\, Z_\rho^*(u)\,\mathrm{d}u}{\int_{-\infty}^{+\infty} Z_\rho^*(u)\,\mathrm{d}u}\,,$$

where Z_ρ^* is the process introduced in (4.7), we equally have $\xi_\vartheta \overset{d}{=} -\xi_\rho^*/r$ and $\zeta_\vartheta \overset{d}{=} -\zeta_\rho^*/r$.

Now we can state the following theorem giving an asymptotic lower bound on the risk of all the estimators of ϑ.

Theorem 4.3. *Let the conditions* **C1** – **C4** *be fulfilled. Then, for any $\vartheta_0 \in \Theta$, we have*

$$\lim_{\delta\to 0}\ \varliminf_{n\to+\infty}\ \inf_{\overline{\vartheta}_n}\ \sup_{|\vartheta-\vartheta_0|<\delta} n^2\,\mathbf{E}_\vartheta^{(n)}(\overline{\vartheta}_n-\vartheta)^2 \geq \mathbf{E}\zeta_{\vartheta_0}^2 = \mathbf{E}(\zeta_{\rho_0}^*)^2/r^2,$$

where $\rho_0 = \left|\ln \frac{\psi(\vartheta_0)}{\psi(\vartheta_0)+r}\right|$ and the inf *is taken over all possible estimators $\overline{\vartheta}_n$ of the parameter ϑ.*

This theorem allows us to introduce the following definition.

Definition 4.1. *Let the conditions* **C1** – **C4** *be fulfilled. We say that an estimator ϑ_n^* is asymptotically efficient if*

$$\lim_{\delta\to 0}\ \lim_{n\to+\infty}\ \sup_{|\vartheta-\vartheta_0|<\delta} n^2\,\mathbf{E}_\vartheta^{(n)}(\vartheta_n^*-\vartheta)^2 = \mathbf{E}\zeta_{\vartheta_0}^2 = \mathbf{E}(\zeta_{\rho_0}^*)^2/r^2$$

for all $\vartheta_0 \in \Theta$.

Now, we can state the two following theorems giving the asymptotic properties of the maximum likelihood and Bayesian estimators.

Theorem 4.4. *Let the conditions* **C1** – **C4** *be fulfilled. Then the maximum likelihood estimator $\widehat{\vartheta}_n$ satisfies uniformly on $\vartheta \in \mathbb{K}$ the relations*

$$\mathbf{P}_\vartheta^{(n)} - \lim_{n\to+\infty} \widehat{\vartheta}_n = \vartheta,$$

$$\mathcal{L}_{\vartheta}^{(n)}\left\{n(\widehat{\vartheta}_n-\vartheta)\right\} \Rightarrow \mathcal{L}(\xi_\vartheta)=\mathcal{L}(-\xi_\rho^*/r)$$

and

$$\lim_{n\to+\infty} \mathbf{E}_{\vartheta}^{(n)} n^p \left|\widehat{\vartheta}_n-\vartheta\right|^p = \mathbf{E}\left|\xi_\vartheta\right|^p = \mathbf{E}\left|\xi_\rho^*\right|^p / \left|r\right|^p \quad \textit{for any} \quad p>0.$$

In particular, the relative asymptotic efficiency of $\widehat{\vartheta}_n$ *is* $\mathbf{E}(\zeta_\rho^*)^2/\mathbf{E}(\xi_\rho^*)^2$.

Theorem 4.5. *Let the conditions* **C1** – **C4** *be fulfilled. Then, for any continuous strictly positive density, the Bayesian estimator* $\widetilde{\vartheta}_n$ *satisfies uniformly on* $\vartheta \in \mathbb{K}$ *the relations*

$$\mathbf{P}_{\vartheta}^{(n)} - \lim_{n\to+\infty} \widetilde{\vartheta}_n = \vartheta,$$
$$\mathcal{L}_{\vartheta}^{(n)}\left\{n(\widetilde{\vartheta}_n-\vartheta)\right\} \Rightarrow \mathcal{L}(\zeta_\vartheta)=\mathcal{L}(-\zeta_\rho^*/r)$$

and

$$\lim_{n\to+\infty} \mathbf{E}_{\vartheta}^{(n)} n^p \left|\widetilde{\vartheta}_n-\vartheta\right|^p = \mathbf{E}\left|\zeta_\vartheta\right|^p = \mathbf{E}\left|\zeta_\rho^*\right|^p / \left|r\right|^p \quad \textit{for any} \quad p>0.$$

In particular, $\widetilde{\vartheta}_n$ *is asymptotically efficient.*

Theorems 4.3–4.5 follow from the properties of the normalized likelihood ratio established in the previous section. More precisely, Theorem 4.5 follows from Lemmas 4.2–4.4 and [17, Theorem 1.10.2]. Having the properties of the Bayesian estimators given in Theorem 4.5, we can cite [17, Theorem 1.9.1] to provide the proof of Theorem 4.3. Finally, the proof of Theorem 4.4 can be carried out following the standard argument of [17, Chapters 5.3 and 5.4] which is based on the weak convergence established in Theorem 4.1 together with the inequality (4.11).

4.2.3 Hypothesis testing

Suppose that we observe n independent inhomogeneous Poisson processes with intensity function $\lambda_{\vartheta}^{(n)}(t)=\psi_n(t)+r_n\mathbb{1}_{\{t>\vartheta\}}$, where the function $\psi_n(\cdot)$ is continuous on the interval $[0,\tau]$, the parameter $\vartheta\in[\vartheta_1,b)\subseteq[0,\tau)$ and the conditions **(C2)**–**(C4)** are fulfilled.

We have to test the following two hypotheses

$$\mathscr{H}_1 \quad : \quad \vartheta=\vartheta_1,$$
$$\mathscr{H}_2 \quad : \quad \vartheta>\vartheta_1.$$

As usual, we consider *close* alternatives. We put $\vartheta=\vartheta_1+\varphi_n u$, where $\varphi_n=\frac{1}{n}$. Then the initial problem of hypotheses testing can be rewritten as

$$\mathscr{H}_1 \quad : \quad u=0,$$

$$\mathscr{H}_2 \quad : \qquad u > 0.$$

In particular, when $r < 0$, if we change φ_n by $\varphi_n^* = \frac{1}{n|r|} = -\frac{1}{nr}$, using Theorem 4.2 we obtain that, under hypothesis $\mathscr{H}_1$,

$$Z^*_{n,\vartheta_1}(v) = \frac{L_n\left(\vartheta_1 + v\varphi_n^*, X^{(n)}\right)}{L_n\left(\vartheta_1, X^{(n)}\right)} \Longrightarrow Z^*_\rho(v) = \exp\left\{\rho Y^+(v) - v\right\}, \quad v \geq 0,$$

where $\rho = \left|\ln \frac{\psi(\vartheta_1)}{\psi(\vartheta_1)+r}\right|$ and Y^+ is the Poisson process of constant intensity $(e^\rho - 1)^{-1}$.

For simplicity of exposition we suppose that $r < 0$. The case $r > 0$ can be treated in a similar way with $\varphi_n^* = \frac{1}{nr}$. All results can be reformulated with the help of the normalized likelihood ratio $Z^*_{n,\vartheta_1}(v)$ and its limit $Z^*_\rho(-v)$.

As $\frac{1}{1-e^{-\rho}} = \frac{\psi(\vartheta_1)}{|r|}$ and $\frac{1}{e^\rho - 1} = \frac{\psi(\vartheta_1)+r}{|r|}$, the limit under alternatives is obtained as follows (we put $\vartheta_u = \vartheta_1 + u\varphi_n^*$, $\vartheta_v = \vartheta_1 + v\varphi_n^*$ and suppose that $v \geq u$):

$$\begin{aligned}
&\mathbf{E}^{(n)}_{\vartheta_u} \exp\left\{i\nu \ln Z^*_{n,\vartheta_1}(v)\right\} \\
&\quad = \exp\Bigg\{n\int_0^\tau \left(\exp\left\{i\nu\ln\frac{\lambda^{(n)}_{\vartheta_v}(t)}{\lambda^{(n)}_{\vartheta_1}(t)}\right\} - 1\right)\lambda^{(n)}_{\vartheta_u}(t)\,\mathrm{d}t \\
&\qquad\qquad - n\int_0^\tau \left(\lambda^{(n)}_{\vartheta_v}(t) - \lambda^{(n)}_{\vartheta_1}(t)\right)\,\mathrm{d}t\Bigg\} \\
&\quad = \exp\Bigg\{n\int_{\vartheta_1}^{\vartheta_u} \left(\exp\left\{i\nu\ln\frac{\psi_n(t)}{\psi_n(t)+r_n}\right\} - 1\right)\psi_n(t)\,\mathrm{d}t \\
&\qquad + n\int_{\vartheta_1}^{\vartheta_u} \left(\exp\left\{i\nu\ln\frac{\psi_n(t)}{\psi_n(t)+r_n}\right\} - 1\right)\left(\psi_n(t)+r_n\right)\,\mathrm{d}t + nvr_n\varphi_n^*\Bigg\} \\
&\quad \to \exp\Bigg\{\frac{u\psi\left(\vartheta_1\right)}{|r|}\left(\exp\left\{i\nu\ln\frac{\psi(\vartheta_1)}{\psi(\vartheta_1)+r}\right\} - 1\right) \\
&\qquad + \frac{(v-u)\left(\psi\left(\vartheta_1\right)+r\right)}{|r|}\left(\exp\left\{i\nu\ln\frac{\psi(\vartheta_1)}{\psi(\vartheta_1)+r}\right\} - 1\right)\,\mathrm{d}t + v\frac{r}{|r|}\Bigg\} \\
&\quad = \mathbf{E}\exp\left\{i\nu\big(\rho Y_*(v) - v\big)\right\},
\end{aligned}$$

where Y_* is the Poisson process with the intensity function

$$\lambda^*_\rho(v) = \left(1 - e^{-\rho}\right)^{-1}\mathbb{1}_{\{v<u\}} + \left(e^\rho - 1\right)^{-1}\mathbb{1}_{\{v\geq u\}}. \tag{4.12}$$

Hence, under alternative,

$$Z^*_{n,\vartheta_1}(v) \longrightarrow Z^*_\rho(v) = \exp\left\{\rho Y_*(v) - v\right\}$$

where Y_* is the Poisson process of switching intensity function (4.12).

Now we can construct the tests GLRT, WT, BT1 and BT2.

The GLRT test is based on the statistic

$$Q_n\left(X^{(n)}\right)=\sup_{\vartheta>\vartheta_1}\frac{L_n\left(\vartheta,X^{(n)}\right)}{L_n\left(\vartheta_1,X^{(n)}\right)}=\max\left[\frac{L_n\left(\hat\vartheta_n+,X^{(n)}\right)}{L_n\left(\vartheta_1,X^{(n)}\right)},\frac{L_n\left(\hat\vartheta_n-,X^{(n)}\right)}{L_n\left(\vartheta_1,X^{(n)}\right)}\right].$$

and is of the form

$$\hat\phi_n\left(X^{(n)}\right)=\mathbb{1}_{\left\{Q_n\left(X^{(n)}\right)>h_\varepsilon\right\}}.$$

The threshold h_ε is defined with the help of the convergence (under $\mathscr{H}_1$)

$$Q_n\left(X^{(n)}\right)=\sup_{v\in\mathbb{U}_n^+}Z_{n,\vartheta_1}^*\left(v\right)\Longrightarrow\sup_{v>0}Z_\rho^*\left(v\right)=\hat Z_\rho^*.$$

Hence $h_\varepsilon=h_\varepsilon\left(\rho\right)$ is solution of the equation

$$\mathbf{P}\left\{\hat Z_\rho^*>h_\varepsilon\right\}=\varepsilon$$

and depend on ρ. Note that as it was proved by Pyke [32] that the random variable $\hat Z_\rho^*$ has continuous distribution function (3.7), therefore this equation has solution for any ε.

Let us fix an alternative $u>0$. Then, for the power function we have

$$\beta\left(\hat\phi_n,u\right)=\mathbf{E}_{\vartheta_1+u\varphi_n^*}^{(n)}\hat\phi_n\left(X^{(n)}\right)=\mathbf{P}_{\vartheta_1+u\varphi_n^*}^{(n)}\left\{\sup_{v>0}Z_{n,\vartheta_1}^*\left(v\right)>h_\varepsilon\right\}$$
$$\rightarrow\mathbf{P}_u\left\{\sup_{v>0}Z_\rho^*\left(v\right)>h_\varepsilon\right\},$$

where the Poisson process $Y_*(v)$, $v\geq 0$ in $Z_\rho^*\left(v\right)$ has the intensity function (4.12).

The Wald's test is based on the MLE $\hat\vartheta_n$. We already know that

$$\left(\varphi_n^*\right)^{-1}\left(\hat\vartheta_n-\vartheta_1\right)\Longrightarrow\xi_{\rho,+}^*,$$

where $\xi_{\rho,+}^*$ is solution of the equation

$$\max\left[Z_\rho^*\left(\xi_{\rho,+}^*+\right),Z_\rho^*\left(\xi_{\rho,+}^*-\right)\right]=\sup_{v>0}Z_\rho^*\left(v\right)$$

(see [19]). The Wald's Test is

$$\phi_n^o\left(X^{(n)}\right)=\mathbb{1}_{\left\{\left(\varphi_n^*\right)^{-1}\left(\hat\vartheta_n-\vartheta_1\right)>m_\varepsilon\right\}}.$$

The threshold $m_\varepsilon=m_\varepsilon\left(\rho\right)$ is solution of the equation

$$\mathbf{P}\left\{\xi_{\rho,+}^*>m_\varepsilon\right\}=\varepsilon$$

and depend also on ρ. Note as well, that the random variable $\xi_{\rho,+}^*$ has continuous distribution function (3.8) (see Pflug [31]).

For the power function we have (below $\vartheta_u = \vartheta_1 + u\varphi_n^*$)

$$\beta\left(\phi_n^o, u\right) = \mathbf{E}_{\vartheta_u}^{(n)} \phi_n^o\left(X^{(n)}\right) = \mathbf{P}_{\vartheta_u}^{(n)}\left\{(\varphi_n^*)^{-1}\left(\hat{\vartheta}_n - \vartheta_u\right) + u > c_\varepsilon\right\} \\ \longrightarrow \mathbf{P}\left\{\xi_{\rho,u}^* > c_\varepsilon - u\right\}$$

where $\xi_{\rho,u}^*$ is the solution of the equation

$$\max\left[Z_\rho^*\left(\xi_{\rho,u}^*+\right), Z_\rho^*\left(\xi_{\rho,u}^*-\right)\right] = \sup_{v>-u} Z_\rho^*(v)$$

and Y_* is the Poisson process with the intensity function

$$\lambda_\rho^*(v) = (1-e^{-\rho})^{-1}\mathbb{1}_{\{v<0\}} + (e^\rho - 1)^{-1}\mathbb{1}_{\{v\geq 0\}}.$$

To construct bayesian tests we suppose that the parameter ϑ is a random variable with the *a priori* density $p(\theta)$, $\vartheta_1 \leq \theta < b$. This function is supposed to be continuous and positive. We consider two tests.

The first one is based on the BE :

$$\tilde{\phi}_n\left(X^{(n)}\right) = \mathbb{1}_{\left\{(\varphi_n^*)^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\}}, \qquad \tilde{\vartheta}_n = \frac{\int_{\vartheta_1}^b \theta\, p(\theta) L_n(\theta, X^{(n)})\, \mathrm{d}\theta}{\int_{\vartheta_1}^b p(\theta) L_n(\theta, X^{(n)})\, \mathrm{d}\theta}.$$

As we know the asymptotic behavior of this estimator (Theorem 4.5) we can write the following limit (under hypothesis) :

$$\mathbf{E}_{\vartheta_1}^{(n)} \mathbb{1}_{\left\{(\varphi_n^*)^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\}} \longrightarrow \mathbf{P}_{\vartheta_1}\left\{\tilde{u} > k_\varepsilon\right\}, \qquad \tilde{u} = \frac{\int_0^\infty v Z_\rho^*(v)\, \mathrm{d}v}{\int_0^\infty Z_\rho^*(v)\, \mathrm{d}v},$$

where the Poisson process Y^+ in the definition of $Z_\rho^*(\cdot)$ has the constant intensity $(e^\rho - 1)^{-1}$.

For the power function the limit is

$$\beta\left(\tilde{\phi}_n, u\right) = \mathbf{P}_{\vartheta_u}^{(n)}\left\{(\varphi_n^*)^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\} = \mathbf{P}_{\vartheta_u}^{(n)}\left\{(\varphi_n^*)^{-1}\left(\tilde{\vartheta}_n - \vartheta_u\right) > k_\varepsilon - u\right\} \\ = \mathbf{P}\left\{\frac{\int_{-u}^\infty v Z_\rho^*(v)\, \mathrm{d}v}{\int_{-u}^\infty Z_\rho^*(v)\, \mathrm{d}v} > k_\varepsilon - u\right\} = \mathbf{P}\left\{\tilde{v} > k_\varepsilon - u\right\}$$

where the Poisson process Y_* in the definition of $Z_\rho^*(\cdot)$ has the switching intensity function

$$\lambda_\rho^*(v) = (1-e^{-\rho})^{-1}\mathbb{1}_{\{v<0\}} + (e^\rho - 1)^{-1}\mathbb{1}_{\{v\geq 0\}}.$$

Here, the calculation of $(\varphi_n^*)^{-1}\left(\tilde{\vartheta}_n - \vartheta_u\right)$ is carried in the following way. Changing the variable $\vartheta = \vartheta_u + v\varphi_n^*$ where $\vartheta_u = \vartheta_1 + u\varphi_n^*$, the Bayesian estimator can be written as

$$\begin{aligned}\tilde{\vartheta}_n &= \frac{\int_{-u}^{b} \left(\vartheta_u + v\varphi_n^*\right) p\left(\vartheta_u + v\varphi_n^*\right) Z_{n,\vartheta_u}^*(v)\, \mathrm{d}v}{\int_{-u}^{b} p\left(\vartheta_u + v\varphi_n^*\right) Z_{n,\vartheta_u}^*(v)\, \mathrm{d}v} \\ &= \vartheta_u + \varphi_n^* \frac{\int_{-u}^{b} vp\left(\vartheta_u\right) Z_{n,\vartheta_u}^*(v)\, \mathrm{d}v}{\int_{-u}^{b} p\left(\vartheta_u\right) Z_{n,\vartheta_u}^*(v)\, \mathrm{d}v} + o(\varphi_n^*),\end{aligned} \tag{4.13}$$

which yields

$$(\varphi_n^*)^{-1}\left(\tilde{\vartheta}_n - \vartheta_u\right) \Rightarrow \frac{\int_{-u}^{\infty} vZ_\rho^*(v)\, \mathrm{d}v}{\int_{-u}^{\infty} Z_\rho^*(v)\, \mathrm{d}v} = \tilde{v}.$$

The analytical solutions of the equations for m_ε and k_ε are not available and therefore these thresholds and the corresponding power functions are obtained by the numerical simulations (see below).

The second Bayesian approach is based on replacing the composite alternative by a simple one as follows. Let us introduce the measure

$$\tilde{\mathbf{P}}_2^{(n)}(A) = \int_{\vartheta_1}^{b} \mathbf{P}_\theta^{(n)}(A)\, p(\theta)\; \mathrm{d}\theta.$$

Then the likelihood ratio is

$$\tilde{L}\left(X^{(n)}\right) = \frac{\mathrm{d}\tilde{\mathbf{P}}_2^{(n)}}{\mathrm{d}\tilde{\mathbf{P}}_{\vartheta_1}^{(n)}}\left(X^{(n)}\right) = \int_{\vartheta_1}^{b} \frac{L_n\left(\theta, X^{(n)}\right)}{L_n\left(\vartheta_1, X^{(n)}\right)} p(\theta)\; \mathrm{d}\theta$$

Note that, as it follows from the proof given above for the normalized likelihood ratio, we have the following limit

$$(\varphi_n^*)^{-1}\tilde{L}\left(X^{(n)}\right) \Longrightarrow p(\vartheta_1)\int_0^\infty \exp\left\{\rho Y^+(v) - v\right\} \mathrm{d}v.$$

Hence the test

$$\tilde{\phi}_n\left(X^{(n)}\right) = \mathbb{1}_{\{R_n > g_\varepsilon\}}, \qquad R_n = \frac{(\varphi_n^*)^{-1}\tilde{L}\left(X^{(n)}\right)}{p(\vartheta_1)}$$

with the threshold g_ε satisfying equation

$$\mathbf{P}\left\{\int_0^\infty \exp\left\{\rho Y^+(v) - v\right\} \mathrm{d}v > g_\varepsilon\right\} = \varepsilon$$

is bayesian and belongs to the class $\mathcal{K}_\varepsilon$.

Note that in all cases the limit random variables

$$Z = \int_0^\infty Z_\rho^*(v)\; \mathrm{d}v$$

have *heavy tails* because

$$\mathbf{E}\, Z = \int_0^\infty \mathbf{E}\, Z_\rho^*(v)\; \mathrm{d}v = \int_0^\infty 1\, \mathrm{d}v = \infty.$$

4.2.4 Simulations

We consider n independent observations $X^{(n)} = \left\{X_j^{(n)}(t), t \in [0,4]\right\}; \, j = 1, ..., n$ of a Poisson process of intensity function

$$\lambda_\vartheta^{(n)}(t) = 3\cos^2(t) + 2 - \exp\left\{-\frac{1}{n}\right\} \mathbb{1}_{\{t>\vartheta\}}, \qquad 0 \le t \le 4$$

with $\vartheta \in [2,4)$ and $r = -1$. Let us take $\vartheta_1 = 2$. We have

$$\ln Z_n(v) = \sum_{j=1}^{n} \int_{\vartheta_1}^{\vartheta_1 + v/n} \ln \frac{3\cos^2(t) + 2}{3\cos^2(t) + 2 - \exp\left\{-\frac{1}{n}\right\}} \mathrm{d}X_j(t) - ve^{-\frac{1}{n}}.$$

Here we present the results of simulations for the choice of the thresholds and the power functions. In this case $\rho = 0.5057$.

ε	0.01	0.05	0.10	0.20	0.40	0.50
$\ln C_\varepsilon$	4.253	2.631	1.943162	1.229	1.120	0.237
c_ε	14.886	7.282	4.531	2.236	0.685	0.248
k_ε	24.906	15.805	12.104	8.690	5.704	4.793

Table 4.1: Thresholds of GLRT, WT and BT1.

Firstly, we calculate the thresholds of GLRT. Using the distribution function (3.7), we obtain the quantile of GLRT by the following equation

$$\begin{aligned}
\mathbf{P}_{\vartheta_1}\left(\sup_{v>0}\left[\rho Y^+(v) - v\right] > \ln C_\varepsilon\right) &= \mathbf{P}_{\vartheta_1}\left(\sup_{v>0}\left[Y^+(v) - v/\rho\right] > \frac{1}{\rho}\ln C_\varepsilon\right)\\
&= \mathbf{P}_{\vartheta_1}\left(\sup_{v>0}\left[Y_*^+(v/\rho) - v/\rho\right] > \frac{1}{\rho}\ln C_\varepsilon\right)\\
&= \mathbf{P}_{\vartheta_1}\left(\sup_{s>0}\left[Y_*^+(s) - s\right] > \frac{1}{\rho}\ln C_\varepsilon\right) = \varepsilon,
\end{aligned}$$

where Y_*^+ is the Poisson process of the constant intensity $\gamma = \frac{\rho}{e^\rho - 1} = 0.7684$.

The similar calculation as (3.9) yields the quantile of BT1 calculated by the following equation

$$\mathbf{P}_{\vartheta_1}\left(\frac{\sum_{i=0}^{+\infty}\exp\{\rho N_i\}\left[v_i\exp\{-v_i\} - v_{i+1}\exp\{-v_{i+1}\}\right]}{\sum_{i=0}^{+\infty}\exp\{\rho N_i\}\left[\exp\{-v_i\} - \exp\{-v_{i+1}\}\right]} + 1 > k_\varepsilon\right) = \varepsilon.$$

As when n is sufficiently large, the value of r_n change so little, we just discuss the case when $n = 10,\, 50,\, 100$ ($r_n = 09048,\, 0.9802,\, 0.9900$). When $n > 100$ (for example, $r_{200} = 0.995$), $r_n \approx 1$ and the tendency of the power functions are similar as in the jump-type singularity.

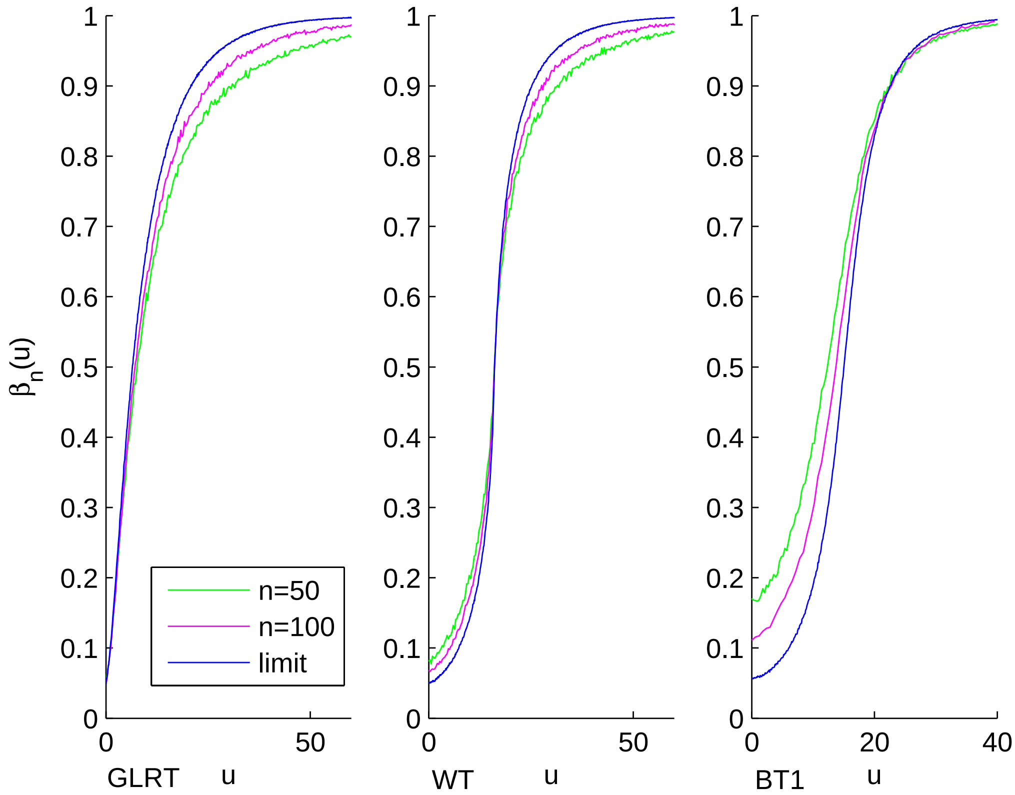

Figure 4.1: Power functions of GLRT, WT and BT1 in change-point model with $r_n \rightarrow r$

In this model, we have two factors of varying the tendency of the convergence: n and r_n. Comparing the curves in Fig. 3.6 in the discontinuous intensity case (in [10]), we can see that the curves in Fig. 4.1 when $n = 10$ is much lower than those in Fig. 3.6, which present the influence of the factor r_n. We note also that when $n = 10$, $\vartheta = \vartheta_1 + u\varphi_n^*$ exceed the interval $\vartheta \in [2, 4)$ when $u > 20$.

4.2.5 Comparison of the limit power functions

Fixing an alternative $\vartheta_2 > \vartheta_1$, the hypotheses are defined by

$$\begin{aligned} \mathscr{H}_1 \quad &: \quad \vartheta = \vartheta_1, \\ \mathscr{H}_2 \quad &: \quad \vartheta = \vartheta_2. \end{aligned}$$

Using the notation $\vartheta_2 = \vartheta_1 + u_1\varphi_n$, we construct the correspondent close alternative and the problem became

$$\mathscr{H}_1 \quad : \quad u = 0,$$

$$\mathscr{H}_2 \quad : \qquad u = u_1 \qquad (u_1 > 0).$$

It is interesting to compare the studied tests with the Neyman-Pearson test. Of course, it is impossible to apply N-PT because the value under alternative is unknown, but its power function shows an upper bound and the distance between it and the power functions of studied tests provides an important information.

$$\psi_n^*(X^n) = \mathbb{1}_{\left\{Z^*_{n,\vartheta_1}(u_1) > d_\varepsilon\right\}} + q_\varepsilon \mathbb{1}_{\left\{Z^*_{n,\vartheta_1}(u_1) = d_\varepsilon\right\}},$$

where d_ε, q_ε is la solution of the equation

$$\mathbf{P}_{\vartheta_1}\left(Z^*_\rho(u_1) > d_\varepsilon\right) + q_\varepsilon \mathbf{P}_{\vartheta_1}\left(Z^*_\rho(u_1) = d_\varepsilon\right) = \varepsilon,$$

which can be calculated as follows, $\left(\text{which we denote } M_\varepsilon = \frac{\ln d_\varepsilon + u_1}{\rho}\right)$

$$\mathbf{P}_{\vartheta_1}\left(Y^+(u_1) > M_\varepsilon\right) + q_\varepsilon \mathbf{P}_{\vartheta_1}\left(Y^+(u_1) = M_\varepsilon\right) = \varepsilon.$$

We have

$$\mathbf{P}_{\vartheta_1}\left(Y^+(u_1) = M_\varepsilon\right) = \mathbf{P}_{\vartheta_1}\left(Y^+(u_1) > M_\varepsilon -\right) - \mathbf{P}_{\vartheta_1}\left(Y^+(u_1) > M_\varepsilon\right),$$

where Y^+ is the Poisson process of the constant intensity $\frac{1}{e^\rho - 1}$.

Under alternatives, we have

$$\beta(\psi_n^*, u_1) = \mathbf{P}^{(n)}_{\vartheta_1 + u_1\varphi_n}\left(Z^*_{n,\vartheta_1}(u_1) > d_\varepsilon\right) + q_\varepsilon \mathbf{P}^{(n)}_{\vartheta_1 + u_1\varphi_n}\left(Z^*_{n,\vartheta_1}(u_1) = d_\varepsilon\right)$$

And similar calculate yields the power function

$$\beta(\psi_n^*, u_1) \longrightarrow \mathbf{P}_{u_1}\left(Y^+(u_1) > M_\varepsilon\right) + q_\varepsilon \mathbf{P}_{u_1}\left(Y^+(u_1) = M_\varepsilon\right)$$

where Y^+ is the Poisson process of the constant intensity $\frac{1}{1 - e^{-\rho}}$. In the Fig. 4.2, the "cusps" in the limit powers of N-PT follows the reason that the right derivative is not equal to the left's at these points, which was described exactly in [9]

The curves of BT1 tends to 1 the most quickly. The limit power function of GLRT is the closest one to the envelope power function. WT has the lowest limit power function when $\varepsilon = 0.05$ and the curve becomes better than that of BT1 when $\varepsilon = 0.4$ and u is small.

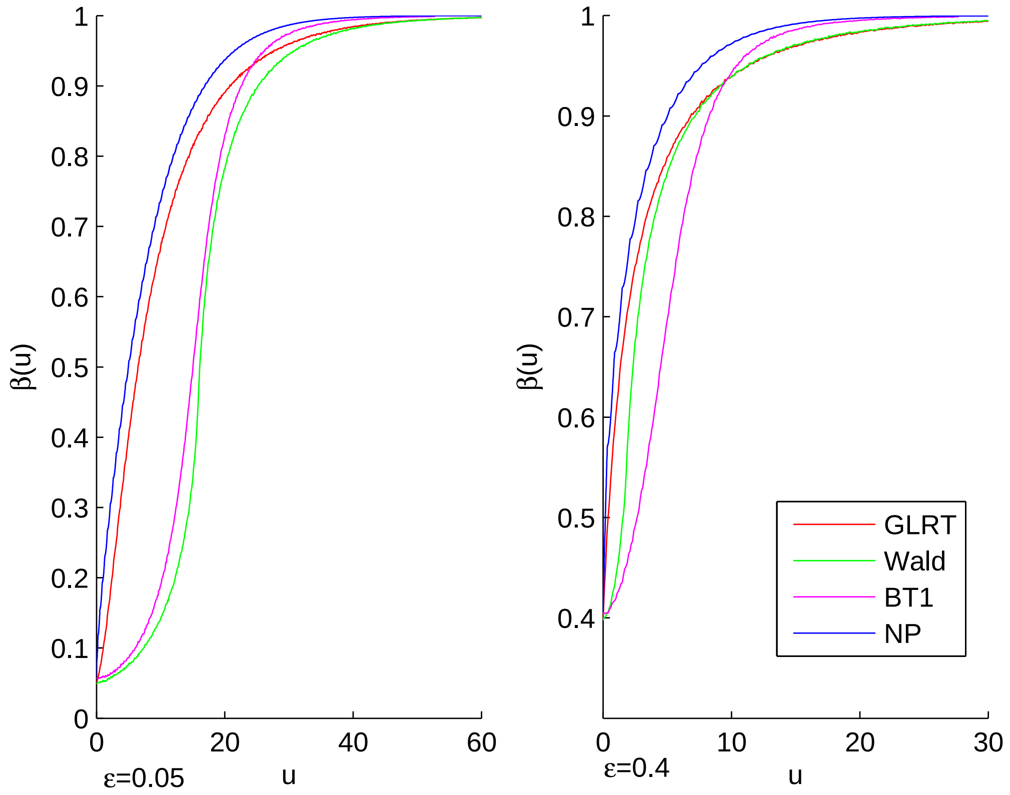

Figure 4.2: Comparison of limit power functions for change-point type model with $r = -1$ and $\varepsilon = 0.05$ and $\varepsilon = 0.4$.

4.3 Change-point model with variable jump size converging to zero

Suppose we observe n independent realizations $X_j^{(n)} = \{X_j^{(n)}(t),\ t \in [0,\tau]\}$, $j = 1,\ldots,n$, of an inhomogeneous Poisson process on the interval $[0,\tau]$ (the constant $\tau > 0$ is supposed to be known) of intensity measure

$$\Lambda_\vartheta^{(n)}(A) = \int_A \lambda_\vartheta^{(n)}(t)\,\mathrm{d}t, \qquad A \in \mathscr{B}([0,\tau]),$$

with intensity function $\lambda_\vartheta^{(n)}$, where $\vartheta \in \Theta = (\alpha,\beta)$, $0 < \alpha < \beta < \tau$, is some unknown parameter. The observation will be denoted $X^{(n)} = \{X_1^{(n)},\ldots,X_n^{(n)}\}$ and the corresponding probability distribution will be denoted $\mathbf{P}_\vartheta^{(n)}$.

Let us note that this model of observation is equivalent to observing a single realization on the interval $[0,n\tau]$ of an inhomogeneous Poisson process with the τ-periodic intensity function coinciding with $\lambda_\vartheta^{(n)}$ on $[0,\tau]$.

The parameter ϑ corresponds to the location of a jump in the (elsewhere continuous) intensity function $\lambda_\vartheta^{(n)}$. The size of the jump (depending on n) will be denoted r_n and will be supposed converging to 0.

More precisely, we assume that the following conditions are satisfied.

(I1) The intensity function $\lambda_\vartheta^{(n)}(t)$ can be written as $\lambda_\vartheta^{(n)}(t) = \psi_n(t) + r_n \mathbb{1}_{\{t>\vartheta\}}$, where the function ψ_n is continuous on $[0,\tau]$.

(I2) For all $t \in [0,\tau]$, there exist the $\lim\limits_{n\to+\infty} \psi_n(t) = \psi(t) > 0$ and, moreover, this convergence is uniform with respect to t.

(I3) As $n \to +\infty$, the jump size r_n converges to zero ($r_n \to 0$), but this convergence is slower than $n^{-1/2}$, that is, we have $n\, r_n^2 \to +\infty$.

(I4) The family of functions $\{\lambda_\vartheta^{(n)}\}_{n\in\mathbb{N},\vartheta\in\Theta}$ is uniformly strictly positive and uniformly bounded, that is, there exist some constants $\ell, L > 0$ such that

$$\ell \le \lambda_\vartheta^{(n)}(t) \le L$$

for all $n \in \mathbb{N}$, $\vartheta \in \Theta$ and $t \in [0,\tau]$.

Note that the conditions **I1** – **I3** easily imply that the condition **I4** holds for the family $\{\lambda_\vartheta^{(n)}\}_{n\ge n_0,\vartheta\in\Theta}$ with some $n_0 \in \mathbb{N}$. So, in the asymptotic setting ($n \to +\infty$), the condition **I4** can be omitted, and we assume it only for convenience (as well as in order for our model to be well defined for all $n \in \mathbb{N}$).

An important particular case of this model is when only the jump size (and not the regular part of $\lambda_\vartheta^{(n)}$) depend on n. More precisely, the conditions **I1** – **I2** will be clearly met if we assume that the following condition is satisfied.

(J) The intensity function $\lambda_\vartheta^{(n)}(t)$ can be written as $\lambda_\vartheta^{(n)}(t) = \psi(t) + r_n \mathbb{1}_{\{t>\vartheta\}}$, where the function ψ is strictly positive and continuous on $[0,\tau]$.

4.3.1 Asymptotic behavior of the likelihood ratio

The likelihood of our model is given by (see, for example, [18])

$$\begin{aligned} L_n\big(\vartheta, X^{(n)}\big) &= \exp\biggl\{\sum_{j=1}^{n} \int_{[0,\tau]} \ln \lambda_\vartheta^{(n)}(t)\; X_j^{(n)}(\mathrm{d}t) - n \int_0^\tau \big[\lambda_\vartheta^{(n)}(t) - 1\big]\,\mathrm{d}t\biggr\} \\ &= \exp\biggl\{\sum_{j=1}^{n} \sum_{i\in I_j^{(n)}} \ln \lambda_\vartheta^{(n)}(t_{j,i}) - n \int_0^\tau \big[\lambda_\vartheta^{(n)}(t) - 1\big]\,\mathrm{d}t\biggr\}, \end{aligned} \tag{4.14}$$

where $t_{j,i}$, $i \in I_j^{(n)}$, are the jump times of the process $X_j^{(n)}$. Note that as function of ϑ, each $\lambda_\vartheta^{(n)}(t_{j,i})$ is discontinuous (has a jump and is right continuous) at $\vartheta = t_{j,i}$. So, $L_n\bigl(\,\cdot\,, X^{(n)}\bigr)$ is a random process with càdlàg (continuous from the right and having finite limits from the left) trajectories.

We denote $\varphi_n = \frac{1}{n\,r_n^2}$ and introduce the normalized likelihood ratio

$$\begin{aligned} Z_{n,\vartheta}(u) &= \frac{L_n\bigl(\vartheta + u\varphi_n, X^{(n)}\bigr)}{L_n\bigl(\vartheta, X^{(n)}\bigr)} \\ &= \exp\biggl\{\sum_{j=1}^{n} \int_{[0,\tau]} \ln \frac{\lambda_{\vartheta+u\varphi_n}^{(n)}(t)}{\lambda_\vartheta^{(n)}(t)}\, X_j^{(n)}(\mathrm{d}t) - n\int_0^\tau \bigl(\lambda_{\vartheta+u\varphi_n}^{(n)}(t) - \lambda_\vartheta^{(n)}(t)\bigr)\,\mathrm{d}t\biggr\} \\ &= \exp\biggl\{\sum_{j=1}^{n} \sum_{i\in I_j^{(n)}} \ln \frac{\lambda_{\vartheta+u\varphi_n}^{(n)}(t_{j,i})}{\lambda_\vartheta^{(n)}(t_{j,i})} - n\int_0^\tau \bigl(\lambda_{\vartheta+u\varphi_n}^{(n)}(t) - \lambda_\vartheta^{(n)}(t)\bigr)\,\mathrm{d}t\biggr\}, \end{aligned}$$

where $u \in U_n = \bigl(\varphi_n^{-1}(\alpha - \vartheta), \varphi_n^{-1}(\beta - \vartheta)\bigr)$.

Note that by the condition **I3**, we have $\varphi_n \to 0$.

Note also that if $u > 0$, we can rewrite $Z_{n,\vartheta}(u)$ as

$$\begin{aligned} Z_{n,\vartheta}(u) &= \exp\biggl\{\sum_{j=1}^{n} \int_{(\vartheta,\vartheta+u\varphi_n]} \ln \frac{\psi_n(t)}{\psi_n(t) + r_n}\, X_j^{(n)}(\mathrm{d}t) + n\int_\vartheta^{\vartheta+u\varphi_n} r_n\,\mathrm{d}t\biggr\} \\ &= \exp\biggl\{\sum_{j=1}^{n} \sum_{i} \ln \frac{\psi_n(t_{j,i})}{\psi_n(t_{j,i}) + r_n} + \frac{u}{r_n}\biggr\}, \end{aligned} \tag{4.15}$$

where the last sum is taken over the set $\bigl\{i \in I_j^{(n)} \,:\, \vartheta < t_{j,i} \le \vartheta + u\varphi_n\bigr\}$.

Similarly, if $u < 0$, we have

$$\begin{aligned} Z_{n,\vartheta}(u) &= \exp\biggl\{\sum_{j=1}^{n} \int_{(\vartheta+u\varphi_n,\vartheta]} \ln \frac{\psi_n(t) + r_n}{\psi_n(t)}\, X_j^{(n)}(\mathrm{d}t) - n\int_{\vartheta+u\varphi_n}^{\vartheta} r_n\,\mathrm{d}t\biggr\} \\ &= \exp\biggl\{\sum_{j=1}^{n} \sum_{i} \ln \frac{\psi_n(t_{j,i}) + r_n}{\psi_n(t_{j,i})} + \frac{u}{r_n}\biggr\}, \end{aligned}$$

where the last sum is taken over the set $\bigl\{i \in I_j^{(n)} \,:\, \vartheta + u\varphi_n < t_{j,i} \le \vartheta\bigr\}$.

Note also, that the process $\ln Z_{n,\vartheta}$ has independent increments. Indeed, its increments on disjoint intervals involve stochastic integrals on disjoint intervals, and hence are independent.

Note finally, that the trajectories of the process $Z_{n,\vartheta}$ are càdlàg functions. Moreover, correctly extending these trajectories to the whole real line, one can consider that they belong to the Skorohod space $\mathscr{D}_0(\mathbb{R})$.

We note that a criterion of weak convergence in $\mathscr{D}_0(\mathbb{R})$ is given in the lemma 4.1 (see [14] for more details) where $\mathbb{K}$ denotes an arbitrary compact in Θ.

The main objective of this section is the study of the asymptotic behavior $\big($in the sense of the weak convergence in the space $\mathscr{D}_0(\mathbb{R})$ as $n \to \infty\big)$ of the above introduced normalized likelihood ratio $Z_{n,\vartheta}$. For this, we introduce the random processes

$$Z^*(u) = \exp\left\{W(u) - \frac{|u|}{2}\right\}, \quad u \in \mathbb{R},$$

where $W(u)$, $u \in \mathbb{R}$, is a double-sided Brownian motion, and

$$Z_\vartheta(u) = Z^*\big(u/\psi(\vartheta)\big) \stackrel{d}{=} \exp\left\{\psi^{-1/2}(\vartheta)\, W(u) - \frac{|u|}{2\psi(\vartheta)}\right\}, \quad u \in \mathbb{R}.$$

Note that the trajectories of the processes Z^* and Z_ϑ almost surely belong to the space $\mathscr{C}_0(\mathbb{R})$ of continuous functions on $\mathbb{R}$ vanishing at infinity, and that $\mathscr{C}_0(\mathbb{R}) \subset \mathscr{D}_0(\mathbb{R})$.

Now we can state the following theorem about the asymptotic behavior of the renormalized likelihood ratio.

Theorem 4.6. *Let the conditions* **I1** – **I4** *be fulfilled. Then, uniformly in* $\vartheta \in \mathbb{K}$, *the process* $Z_{n,\vartheta}$ *converges weakly in the space* $\mathscr{D}_0(\mathbb{R})$ *to the process* Z_ϑ.

Let us also remark, that sometimes it may be more convenient to use the rate $\frac{\psi(\vartheta)}{n\,r_n^2}$ (rather than $\frac{1}{n\,r_n^2}$) for introducing the normalized likelihood ratio. That is, one can put $\varphi_n^* = \frac{\psi(\vartheta)}{n\,r_n^2}$ and introduce the normalized likelihood ratio $Z_{n,\vartheta}^*$ (instead of $Z_{n,\vartheta}$) as

$$Z_{n,\vartheta}^*(v) = \frac{L_n\big(\vartheta + v\,\varphi_n^*, X^{(n)}\big)}{L_n\big(\vartheta, X^{(n)}\big)} = Z_{n,\vartheta}\big(v\,\psi(\vartheta)\big).$$

Then, Theorem 4.6 will be clearly transformed to the following (equivalent) statement.

Theorem 4.7. *Let the conditions* **I1** – **I4** *be fulfilled. Then, uniformly in* $\vartheta \in \mathbb{K}$, *the process* $Z_{n,\vartheta}^*$ *converges weakly in the space* $\mathscr{D}_0(\mathbb{R})$ *to the process* Z^*.

To prove Theorem 4.6, we need to establish several lemmas.

Lemma 4.6. *Let the conditions* **I1** – **I4** *be fulfilled. Then the finite-dimensional distributions of the process* $Z_{n,\vartheta}$ *converge to those of the process* Z_ϑ, *and this convergence is uniform with respect to* $\vartheta \in \mathbb{K}$.

Proof. First we study the convergence of 2-dimensional distributions. For this, consider the distribution of the vector $\big(Z_{n,\vartheta}(u_1), Z_{n,\vartheta}(u_2)\big)$ with some fixed $u_1, u_2 \in \mathbb{R}$. The characteristic function of the natural logarithm of this vector can be written as follows $\big($see, for example, [18]$\big)$:

$$\mathbf{E}_\vartheta^{(n)} \exp\big(it_1 \ln Z_{n,\vartheta}(u_1) + it_2 \ln Z_{n,\vartheta}(u_2)\big)$$

$$
\begin{aligned}
&= \exp\Big\{n \int_0^\tau \Big(\exp\Big\{it_1 \ln \frac{\lambda^{(n)}_{\vartheta+u_1\varphi_n}(t)}{\lambda^{(n)}_{\vartheta}(t)} + it_2 \ln \frac{\lambda^{(n)}_{\vartheta+u_2\varphi_n}(t)}{\lambda^{(n)}_{\vartheta}(t)}\Big\} - 1 \\
&\qquad\qquad - it_1\Big(\frac{\lambda^{(n)}_{\vartheta+u_1\varphi_n}(t)}{\lambda^{(n)}_{\vartheta}(t)} - 1\Big) - it_2\Big(\frac{\lambda^{(n)}_{\vartheta+u_2\varphi_n}(t)}{\lambda^{(n)}_{\vartheta}(t)} - 1\Big)\Big)\lambda^{(n)}_{\vartheta}(t)\,\mathrm{d}t\Big\} \\
&= \exp\Big\{A_{n,\vartheta}(u_1,u_2,t)\Big\}
\end{aligned}
$$

with an evident notation.

We will consider the case $u_2 > u_1 \geq 0$ only (the other cases can be treated in a similar way). In this case, we have

$$
\begin{aligned}
A_{n,\vartheta}(u_1,u_2,t) &= n\int_{\vartheta}^{\vartheta+u_1\varphi_n}\Big(\exp\Big\{(it_1+it_2)\ln\frac{\psi_n(t)}{\psi_n(t)+r_n}\Big\} - 1 \\
&\qquad\qquad - (it_1+it_2)\Big(\frac{\psi_n(t)}{\psi_n(t)+r_n} - 1\Big)\Big)\Big(\psi_n(t)+r_n\Big)\,\mathrm{d}t \\
&\quad + n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n}\Big(\exp\Big\{it_2\ln\frac{\psi_n(t)}{\psi_n(t)+r_n}\Big\} - 1 \\
&\qquad\qquad - it_2\Big(\frac{\psi_n(t)}{\psi_n(t)+r_n} - 1\Big)\Big)\Big(\psi_n(t)+r_n\Big)\,\mathrm{d}t \\
&= nI_1 + nI_2
\end{aligned}
$$

with evident notations.

Using the mean value theorem for the integrals I_1 and I_2, it is possible to find some $s_n \in (\vartheta, \vartheta+u_1\varphi_n)$ and $v_n \in (\vartheta+u_1\varphi_n, \vartheta+u_2\varphi_n)$ such that

$$
nI_1 = \frac{u_1}{r_n^2}\Big(\exp\{i(t_1+t_2)\ln g_n(s_n)\} - 1 - i(t_1+t_2)\big(g_n(s_n)-1\big)\Big)\Big(\psi_n(s_n)+r_n\Big)
$$

and

$$
nI_2 = \frac{u_2-u_1}{r_n^2}\Big(\exp\{it_2\ln g_n(v_n)\} - 1 - it_2\big(g_n(v_n)-1\big)\Big)\Big(\psi_n(v_n)+r_n\Big),
$$

where we have denoted $g_n(t) = \frac{\psi_n(t)}{\psi_n(t)+r_n} = 1 - \frac{r_n}{\psi_n(t)+r_n}$.

As $s_n \to \vartheta$, using the condition **I3** we obtain $\lim\limits_{n\to+\infty}\psi_n(s_n) = \psi(\vartheta)$. So,

$$
nI_1 \sim \frac{u_1\psi(\vartheta)}{r_n^2}\Big(\exp\{i(t_1+t_2)\ln g_n(s_n)\} - 1 - i(t_1+t_2)\big(g_n(s_n)-1\big)\Big).
$$

As $r_n \to 0$ and $\ell \leq \psi_n(t)+r_n \leq L$, we have $g_n(s_n) - 1 = O(r_n) \to 0$. So, using Taylor expansion we get

$$
\ln g_n(s_n) = \ln\Big(1 + \big(g_n(s_n)-1\big)\Big)
$$

$$
\begin{aligned}
&= g_n(s_n) - 1 - \frac{1}{2}\big(g_n(s_n)-1\big)^2 + o\bigg(\frac{r_n^2}{(\psi_n(s_n)+r_n)^2}\bigg)\\
&= g_n(s_n) - 1 - \frac{1}{2}\big(g_n(s_n)-1\big)^2 + o(r_n^2).
\end{aligned}
$$

In particular, $\ln g_n(s_n) = O(r_n)$ and $\big(\ln g_n(s_n)\big)^2 = \big(g_n(s_n)-1\big)^2 + o(r_n^2)$.

Using Taylor expansion once more, we obtain

$$
\exp\big(it\ln g_n(s_n)\big) = 1 + it\ln g_n(s_n) - \frac{t^2}{2}\big(\ln g_n(s_n)\big)^2 + o(r_n^2).
$$

So,

$$
\begin{aligned}
nI_1 &\sim \frac{u_1\psi(\vartheta)}{r_n^2}\bigg(-i(t_1+t_2)\frac{\big(g_n(s_n)-1\big)^2}{2} - \frac{(t_1+t_2)^2}{2}\big(g_n(s_n)-1\big)^2 + o(r_n^2)\bigg)\\
&= \frac{u_1\psi(\vartheta)}{r_n^2}\bigg(-\frac{i(t_1+t_2)r_n^2}{2(\psi(s_n)+r_n)^2} - \frac{(t_1+t_2)^2\,r_n^2}{2(\psi(s_n)+r_n)^2} + o(r_n^2)\bigg)\\
&\to \frac{u_1}{\psi(\vartheta)}\bigg(-\frac{i(t_1+t_2)}{2} - \frac{(t_1+t_2)^2}{2}\bigg).
\end{aligned}
$$

Similarly, we can show that

$$
nI_2 \to \frac{u_2-u_1}{\psi(\vartheta)}\Big(-\frac{it_2}{2} - \frac{t_2^2}{2}\Big).
$$

Hence,

$$
\begin{aligned}
&\mathbf{E}_\vartheta^{(n)} \exp\big(it_1\ln Z_{n,\vartheta}(u_1) + it_2\ln Z_{n,\vartheta}(u_2)\big)\\
&\qquad\to \exp\Big\{-\frac{u_2-u_1}{2\psi(\vartheta)}(it_2+t_2^2) - \frac{u_1}{2\psi(\vartheta)}\big(i(t_1+t_2)+(t_1+t_2)^2\big)\Big\}.
\end{aligned}
\tag{4.16}
$$

For all $u>0$, we introduce the σ-algebra $\mathbb{F}_u = \sigma\big\{W(v),\ 0\le v\le u\big\}$ and write

$$
\begin{aligned}
&\mathbf{E}\exp\big(it_1\ln Z_\vartheta(u_1) + it_2\ln Z_\vartheta(u_2)\big)\\
&\qquad= \mathbf{E}\bigg(\exp\big\{i(t_1+t_2)\ln Z_\vartheta(u_1)\big\}\mathbf{E}\Big(\exp\big\{it_2\big(\ln Z_\vartheta(u_2) - \ln Z_\vartheta(u_1)\big)\big\}\ \Big|\ \mathbb{F}_{u_1}\Big)\bigg)\\
&\qquad= \exp\Big\{-\frac{(t_1+t_2)^2}{2\psi(\vartheta)}u_1 - \frac{i(t_1+t_2)}{2\psi(\vartheta)}u_1 - \frac{t_2^2}{2\psi(\vartheta)}(u_2-u_1) - \frac{it_2}{2\psi(\vartheta)}(u_2-u_1)\Big\}.
\end{aligned}
$$

Combining this with (4.16), we obtain the convergence of 2-dimensional distributions. The convergence of three and more dimensional distributions can be carried out in a similar way, and the uniformity with respect to ϑ is obvious. □

Lemma 4.7. *Let the conditions* **I1** – **I4** *be fulfilled. Then there exists a constant* $C>0$ *such that*

$$
\mathbf{E}_\vartheta^{(n)}\big|Z_{n,\vartheta}^{1/2}(u_1) - Z_{n,\vartheta}^{1/2}(u_2)\big|^2 \le C\,|u_1-u_2|
$$

for all $n\in\mathbb{N}$, $u_1,u_2\in U_n$ *and* $\vartheta\in\mathbb{K}$.

Proof. We will consider the case $u_2 \geq u_1 \geq 0$ only (the other cases can be treated in a similar way). According to [19, Lemma 1.1.5], we have

$$\begin{aligned}\mathbf{E}_\vartheta^{(n)}\left|Z_{n,\vartheta}^{1/2}(u_1) - Z_{n,\vartheta}^{1/2}(u_2)\right|^2 &\leq n\int_0^\tau \left(\sqrt{\lambda_{\vartheta+u_1\varphi_n}^{(n)}(t)} - \sqrt{\lambda_{\vartheta+u_2\varphi_n}^{(n)}(t)}\right)^2 \mathrm{d}t \\ &= n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \left(\sqrt{\psi_n(t)+r_n} - \sqrt{\psi_n(t)}\right)^2 \mathrm{d}t \\ &= n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \frac{r_n^2}{\left(\sqrt{\psi_n(t)+r_n} + \sqrt{\psi_n(t)}\right)^2}\,\mathrm{d}t.\end{aligned}$$

As $\lambda_\vartheta^{(n)}$ is uniformly separated from zero, we have

$$\left(\sqrt{\psi_n(t)+r_n} + \sqrt{\psi_n(t)}\right)^2 \geq \left(\sqrt{\ell} + \sqrt{\ell}\right)^2 = 4\ell,$$

and hence

$$\mathbf{E}_\vartheta^{(n)}\left|Z_{n,\vartheta}^{1/2}(u_1) - Z_{n,\vartheta}^{1/2}(u_2)\right|^2 \leq n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \frac{r_n^2}{4\ell}\,\mathrm{d}t = \frac{1}{4\ell}\left|u_1 - u_2\right|.$$

So, the required inequality holds with $C = \frac{1}{4\ell}$. □

Lemma 4.8. *Let the conditions* **I1** – **I4** *be fulfilled. Then there exists a constant* $k_* > 0$ *such that*

$$\mathbf{E}_\vartheta^{(n)} Z_{n,\vartheta}^{1/2}(u) \leq \exp\left\{-k_*\left|u\right|\right\}$$

for all $n \in \mathbb{N}$, $u \in U_n$ *and* $\vartheta \in \mathbb{K}$.

Proof. We will consider the case $u \geq 0$ only (the other case can be treated in a similar way). According to [19, Lemma 1.1.5], we have

$$\begin{aligned}\mathbf{E}_\vartheta^{(n)} Z_{n,\vartheta}^{1/2}(u) &= \exp\left\{-\frac{n}{2}\int_0^\tau \left(\sqrt{\lambda_{\vartheta+u\varphi_n}^{(n)}(t)} - \sqrt{\lambda_\vartheta^{(n)}(t)}\right)^2 \mathrm{d}t\right\} \\ &= \exp\left\{-\frac{n}{2}\int_\vartheta^{\vartheta+u\varphi_n} \left(\sqrt{\psi_n(t)} - \sqrt{\psi_n(t)+r_n}\right)^2 \mathrm{d}t\right\} \\ &= \exp\left\{-\frac{n}{2}\int_\vartheta^{\vartheta+u\varphi_n} \frac{r_n^2}{\left(\sqrt{\psi_n(t)} + \sqrt{\psi_n(t)+r_n}\right)^2}\,\mathrm{d}t\right\}.\end{aligned}$$

As $\lambda_\vartheta^{(n)}$ is uniformly bounded, we have

$$\left(\sqrt{\psi_n(t)+r_n} + \sqrt{\psi_n(t)}\right)^2 \leq \left(\sqrt{L} + \sqrt{L}\right)^2 = 4L,$$

and hence

$$\mathbf{E}_\vartheta^{(n)} Z_{n,\vartheta}^{1/2}(u) \leq \exp\left\{-\frac{n}{2}\int_\vartheta^{\vartheta+\varphi_n u} \frac{r_n^2}{4L}\,\mathrm{d}t\right\} = \exp\left\{-\frac{1}{8L}\left|u\right|\right\}.$$

So, the required inequality holds with $k_* = \frac{1}{8L}$. □

Since the increments of the process $\ln Z_{n,\vartheta}$ are independent, the convergence of its restrictions (and hence of those of $Z_{n,\vartheta}$) on finite intervals $[A,B]\subset\mathbb{R}$ $\big($that is, convergence in the Skorohod space $\mathscr{D}\big([A,B]\big)$ of functions on $[A,B]$ without discontinuities of the second kind$\big)$ follows from Gihman and Skorohod [15, Theorem 6.5.5], Lemma 4.6 and the following lemma.

Lemma 4.9. *Let the conditions* **I1** – **I4** *be fulfilled. Then for any* $\varepsilon>0$ *we have*

$$\lim_{h\to 0}\ \lim_{n\to+\infty}\ \sup_{|u_1-u_2|<h}\ \mathbf{P}_\vartheta^{(n)}\big(|\ln Z_{n,\vartheta}(u_1)-\ln Z_{n,\vartheta}(u_2)|>\varepsilon\big)=0.$$

for all $u_1,u_2\in U_n$ *and* $\vartheta\in\mathbb{K}$.

Proof. Using Markov inequality, we get

$$\mathbf{P}_\vartheta^{(n)}\big(|\ln Z_{n,\vartheta}(u_1)-\ln Z_{n,\vartheta}(u_2)|>\varepsilon\big)\leq\frac{1}{\varepsilon^2}\,\mathbf{E}_\vartheta^{(n)}\big(\ln Z_{n,\vartheta}(u_1)-\ln Z_{n,\vartheta}(u_2)\big)^2.$$

First we consider the case $u_1,u_2\geq 0$ (and say $u_2\geq u_1$). In this case, we have

$$\begin{aligned}
&\ln Z_{n,\vartheta}(u_2)-\ln Z_{n,\vartheta}(u_1)\\
&\qquad=\sum_{j=1}^n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n}\ln\frac{\psi_n(t)}{\psi_n(t)+r_n}\,\mathrm{d}X_j^{(n)}(t)+n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n}r_n\,\mathrm{d}t\\
&\qquad=\sum_{j=1}^n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n}\ln\frac{\psi_n(t)}{\psi_n(t)+r_n}\,\mathrm{d}Y_j^{(n)}(t)\\
&\qquad\quad+n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n}\Big(\big(\psi_n(t)+r_n\big)\ln\frac{\psi_n(t)}{\psi_n(t)+r_n}+r_n\Big)\,\mathrm{d}t,
\end{aligned}$$

where $Y_j^{(n)}$ is the centered version of the process $X_j^{(n)}$.

Since the stochastic integrals with respect to $Y_j^{(n)}$, $j=1,\ldots,n$, are independent and has mean zero, we obtain

$$\begin{aligned}
&\mathbf{E}_\vartheta^{(n)}\big(\ln Z_{n,\vartheta}(u_1)-\ln Z_{n,\vartheta}(u_2)\big)^2\\
&\qquad=n\,\mathbf{E}_\vartheta^{(n)}\bigg(\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n}\ln\frac{\psi_n(t)}{\psi_n(t)+r_n}\,\mathrm{d}Y_j^{(n)}(t)\bigg)^2\\
&\qquad\quad+n^2\bigg(\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n}\Big(\big(\psi_n(t)+r_n\big)\ln\frac{\psi_n(t)}{\psi_n(t)+r_n}+r_n\Big)\,\mathrm{d}t\bigg)^2\\
&\qquad=E_1+E_2
\end{aligned}$$

with obvious notations.

Using elementary inequalities $\ln(1+x)\leq x$ and $\ln(1+x)\geq x-x^2/2$ for $|x|<1/2$, for sufficiently large values of n (such that $\frac{r_n}{\psi_n(t)+r_n}<\frac{r_n}{\ell}<1/2$) we obtain

$$-\frac{r_n}{\psi_n(t)+r_n}-\frac{r_n^2}{2\big(\psi_n(t)+r_n\big)^2}\leq\ln\frac{\psi_n(t)}{\psi_n(t)+r_n}\leq-\frac{r_n}{\psi_n(t)+r_n}\,. \tag{4.17}$$

For E_1, if $r_n \le 0$, we obtain

$$E_1 = n \int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \left(\ln \frac{\psi_n(t)}{\psi_n(t)+r_n}\right)^2 (\psi_n(t)+r_n)\,\mathrm{d}t$$
$$\le n \int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \frac{r_n^2}{\psi_n(t)+r_n}\,\mathrm{d}t \le n\,\frac{(u_2-u_1)r_n^2\varphi_n}{\ell} = \frac{|u_1-u_2|}{\ell}.$$

As to the case $r_n \ge 0$, as $\frac{r_n}{\psi_n(t)+r_n} < 1/2$, we have

$$E_1 = n \int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \left(\ln \frac{\psi_n(t)}{\psi_n(t)+r_n}\right)^2 (\psi_n(t)+r_n)\,\mathrm{d}t$$
$$\le n \int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \left[\frac{r_n^2}{\psi_n(t)+r_n} + \frac{r_n^3}{(\psi_n(t)+r_n)^2} + \frac{r_n^4}{4(\psi_n(t)+r_n)^3}\right]\,\mathrm{d}t$$
$$\le n \int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \frac{r_n^2}{\psi_n(t)+r_n}\left[1+\frac{1}{2}+\frac{1}{16}\right]\,\mathrm{d}t \le \frac{25\,|u_1-u_2|}{16\ell}.$$

For E_2, we have

$$-\frac{r_n^2}{2\ell} \le -\frac{r_n^2}{2(\psi_n(t)+r_n)} \le \big(\psi_n(t)+r_n\big)\ln\frac{\psi_n(t)}{\psi_n(t)+r_n} + r_n \le 0,$$

and hence

$$E_2 = n^2\left(\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \left(\big(\psi_n(t)+r_n\big)\ln\frac{\psi_n(t)}{\psi_n(t)+r_n} + r_n\right)\mathrm{d}t\right)^2$$
$$\le n^2\left(\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \frac{r_n^2}{2\ell}\,\mathrm{d}t\right)^2 = \frac{(u_2-u_1)^2}{4\ell^2}.$$

Thus, for sufficiently large values of n, we have

$$\mathbf{E}_\vartheta^{(n)}\big(\ln Z_{n,\vartheta}(u_1) - \ln Z_{n,\vartheta}(u_2)\big)^2 \le \frac{25\,|u_1-u_2|}{16\ell} + \frac{(u_2-u_1)^2}{4\ell^2}.$$

In the case $u_1, u_2 \le 0$, proceeding similarly, we obtain the same inequality.

Finally, in the case $u_1u_2 < 0$ (say $u_1 < 0$ and $u_2 > 0$), we obtain

$$\begin{aligned}
\mathbf{E}_\vartheta^{(n)}\big(\ln Z_{n,\vartheta}(u_1) - \ln Z_{n,\vartheta}(u_2)\big)^2 &\le 2\,\mathbf{E}_\vartheta^{(n)}\big(\ln Z_{n,\vartheta}(u_1)\big)^2 + 2\,\mathbf{E}_\vartheta^{(n)}\big(\ln Z_{n,\vartheta}(u_2)\big)^2\\
&\le \frac{25\,|u_1|}{8\ell} + \frac{u_1^2}{2\ell^2} + \frac{25\,|u_2|}{8\ell} + \frac{u_2^2}{2\ell^2}\\
&= \frac{25}{8\ell}\big(|u_1|+|u_2|\big) + \frac{1}{2\ell^2}\big(u_1^2+u_2^2\big)\\
&\le \frac{25\,|u_1-u_2|}{8\ell} + \frac{(u_2-u_1)^2}{\ell^2}.
\end{aligned}$$

Note that this final inequality holds for all the three cases, and so

$$\mathbf{P}_{\vartheta}^{(n)}\big(\left|\ln Z_{n,\vartheta}(u_1)-\ln Z_{n,\vartheta}(u_2)\right|>\varepsilon\big)\leq\frac{25\left|u_1-u_2\right|}{8\varepsilon^2\ell}+\frac{(u_2-u_1)^2}{\varepsilon^2\ell^2} \tag{4.18}$$

for all $u_1, u_2 \in U_n$ and sufficiently large values of n. Hence,

$$\lim_{n\to+\infty}\sup_{|u_1-u_2|<h}\mathbf{P}_{\vartheta}^{(n)}\big(\left|\ln Z_{n,\vartheta}(u_1)-\ln Z_{n,\vartheta}(u_2)\right|>\varepsilon\big)\leq\frac{25h}{8\varepsilon^2\ell}+\frac{h^2}{\varepsilon^2\ell^2}\to 0$$

as $h \to 0$.

So, the lemma is proved. □

Let us note that we have proved even a stronger result: for any $\varepsilon > 0$ we have

$$\lim_{h\to 0}\lim_{n\to+\infty}\sup_{\vartheta\in\mathbb{K}}\sup_{|u_1-u_2|<h}\mathbf{P}_{\vartheta}^{(n)}\big(\left|\ln Z_{n,\vartheta}(u_1)-\ln Z_{n,\vartheta}(u_2)\right|>\varepsilon\big)=0$$

for all $u_1, u_2 \in U_n$ and $\vartheta \in \mathbb{K}$, which allow us to conclude that the convergence of the restriction of $Z_{n,\vartheta}$ on finite intervals $[A, B] \subset \mathbb{R}$ in the Skorohod space $\mathcal{D}\big([A, B]\big)$ to those of the process Z_ϑ is uniform with respect to $\vartheta \in \mathbb{K}$.

In order to conclude the proof of Theorem 4.6 applying the criterion of week convergence in $\mathscr{D}_0(\mathbb{R})$ given in Lemma 4.1, we need to check the condition (4.5). Since we have already established the convergence of the restrictions on finite intervals $[A, B] \subset \mathbb{R}$, it remains to control the second term of the modulus of continuity $\Delta_h(Z_{n,\vartheta})$ (see, for example, [17, Chapters 5.3 and 5.4]). So, the final ingredient of the proof of Theorem 4.1 is the following estimate on the tails of the process $Z_{n,\vartheta}$.

Lemma 4.10. *Let the conditions* **I1** – **I4** *be fulfilled. Then there exist some constants* $b, C > 0$ *such that*

$$\mathbf{P}_{\vartheta}^{(n)}\left(\sup_{|u|>D} Z_{n,\vartheta}(u) > e^{-bD}\right) \leq Ce^{-bD} \tag{4.19}$$

for all $D \geq 0$, $\vartheta \in \mathbb{K}$ *and sufficiently large values of* n.

To prove this lemma, first we establish the following one.

Lemma 4.11. *Let the conditions* **I1** – **I4** *be fulfilled. Then there exist some constants* $b, C > 0$ *such that*

$$\mathbf{P}_{\vartheta}^{(n)}\left(\sup_{D\leq|u|\leq D+1} Z_{n,\vartheta}(u) > e^{-bD}\right) \leq Ce^{-bD} \tag{4.20}$$

for all $D \geq 0$, $\vartheta \in \mathbb{K}$ *and sufficiently large values of* n.

Proof. It is sufficient to establish the inequality (4.20) with the sup taken over $u > 0$ only. Indeed, the case $u < 0$ can be treated similarly, and then the lemma will hold (with two times grater C) since

$$\begin{aligned}
&\mathbf{P}_\vartheta^{(n)}\Big(\sup_{D\le|u|\le D+1} Z_{n,\vartheta}(u) > e^{-bD}\Big)\\
&\qquad\le \mathbf{P}_\vartheta^{(n)}\Big(\sup_{D\le u\le D+1} Z_{n,\vartheta}(u) > e^{-bD}\Big) + \mathbf{P}_\vartheta^{(n)}\Big(\sup_{-D-1\le u\le -D} Z_{n,\vartheta}(u) > e^{-bD}\Big)\\
&\qquad\le 2Ce^{-bD}.
\end{aligned}$$

Note also, that it is sufficient to prove the lemma for all $D \ge D_0$ only, where $D_0 > 0$ is some fixed constant, the choice of which will be specified later. Indeed, for the case $0 \le D \le D_0$, we can write

$$\mathbf{P}_\vartheta^{(n)}\Big(\sup_{|u|>D} Z_{n,\vartheta}(u) > e^{-bD}\Big) \le 1 \le e^{bD_0}e^{-bD},$$

and so the lemma will hold for all $D \ge 0$ by adapting, if necessary, the constant C so that $C \ge e^{bD_0}$.

We fix equally some constant $b > 0$. The choice of this constant will also be specified later. Denoting $\lfloor A\rfloor$ the integer part of A, we split the interval $[D, D+1]$ into $\gamma = \lfloor e^{bD}\rfloor + 1$ parts with the length of each part equal to $h = \gamma^{-1} \le e^{-bD}$. We have the inequality

$$\begin{aligned}
&\mathbf{P}_\vartheta^{(n)}\Big(\sup_{D\le u\le D+1} Z_{n,\vartheta}(u) > e^{-bD}\Big)\\
&\qquad= \mathbf{P}_\vartheta^{(n)}\bigg(\max_{0\le k\le\gamma-1}\Big[\ln Z_{n,\vartheta}(D+kh)\\
&\qquad\qquad+ \sup_{u\in[D+kh\,,\,D+(k+1)h]}\big(\ln Z_{n,\vartheta}(u) - \ln Z_{n,\vartheta}(D+kh)\big)\Big] > -bD\bigg)\\
&\qquad\le \mathbf{P}_\vartheta^{(n)}\Big(\max_{0\le k\le\gamma-1} \ln Z_{n,\vartheta}(D+kh) > -2bD\Big)\\
&\qquad\qquad+ \mathbf{P}_\vartheta^{(n)}\Big(\max_{0\le k\le\gamma-1}\ \sup_{u\in[D+kh\,,\,D+(k+1)h]}\big(\ln Z_{n,\vartheta}(u) - \ln Z_{n,\vartheta}(D+kh)\big) > bD\Big)\\
&\qquad= P_1 + P_2
\end{aligned}$$

with obvious notations.

For the term P_1, using Lemma 4.8, we have

$$\begin{aligned}
P_1 &\le \sum_{k=0}^{\gamma-1} \mathbf{P}_\vartheta^{(n)}\Big(Z_{n,\vartheta}^{\frac12}(D+kh) > e^{-bD}\Big)\\
&\le \sum_{k=0}^{\gamma-1} e^{bD}\mathbf{E}_\vartheta^{(n)} Z_{n,\vartheta}^{\frac12}(D+kh)
\end{aligned}$$

$$\leq \sum_{k=0}^{\gamma-1} e^{bD} e^{-k_*(D+kh)}$$
$$\leq (e^{bD}+1)\, e^{bD} e^{-k_* D} \leq 2e^{2bD} e^{-k_* D} \leq 2e^{-bD}$$

if we choose $b \leq k_*/3$.

For the term P_2, we have

$$\begin{aligned} P_2 &\leq \sum_{k=0}^{\gamma-1} \mathbf{P}_\vartheta^{(n)}\biggl(\sup_{u\in[D+kh\,,\,D+(k+1)h]} \bigl(\ln Z_{n,\vartheta}(u) - \ln Z_{n,\vartheta}(D+kh)\bigr) > bD\biggr) \\ &\leq \sum_{k=0}^{\gamma-1} \mathbf{P}_\vartheta^{(n)}\biggl(\sup_{u\in[D+kh\,,\,D+(k+1)h]} \bigl|\ln Z_{n,\vartheta}(u) - \ln Z_{n,\vartheta}(D+kh)\bigr| > bD\biggr) \\ &= \sum_{k=0}^{\gamma-1} P_{2,k} \end{aligned}$$

with obvious notations.

Looking at the representation (4.15), it becomes clear that the process $\ln Z_{n,\vartheta}$ has its jumps at the points $u_{j,i} = \varphi_n^{-1}(t_{j,i} - \vartheta)$ and the size of each jump is given by $\bigl|\ln \frac{\psi_n(t_{j,i})}{\psi_n(t_{j,i})+r_n}\bigr|$. Hence, using the inequalities (4.17), we see that for sufficiently large values of n (both in the cases $r_n > 0$ and $r_n < 0$) the size of the jumps of the process $\ln Z_{n,\vartheta}$ is bounded by

$$\sup_{t\in[0,\tau]} \left|\ln \frac{\psi_n(t)}{\psi_n(t)+r_n}\right| \leq \frac{|r_n|}{\ell} + \frac{r_n^2}{2\ell^2} \longrightarrow 0.$$

So, we can find $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the jumps of the process $\ln Z_{n,\vartheta}$ are all smaller than $bD_0/3$ (and, in particular, smaller than $bD/3$ for all $D > D_0$). Hence,

$$\begin{aligned} P_{2,k} &= \mathbf{P}_\vartheta^{(n)}\biggl(\sup_{u\in[D+kh,D+(k+1)h]} \bigl|\ln Z_{n,\vartheta}(u) - \ln Z_{n,\vartheta}(D+kh)\bigr| > bD\biggr) \\ &\leq \mathbf{P}_\vartheta^{(n)}\biggl(\sup_{u,v,w} \min\Bigl\{\bigl|\ln Z_{n,\vartheta}(v) - \ln Z_{n,\vartheta}(w)\bigr|, \bigl|\ln Z_{n,\vartheta}(u) - \ln Z_{n,\vartheta}(v)\bigr|\Bigr\} > \frac{bD}{3}\biggr) \end{aligned}$$

where the supremum is taken over the set

$$\bigl\{u,v,w \;:\; D+kh \leq w < v < u \leq D+(k+1)h\bigr\}.$$

The last probability will be estimated with the help of the corollary of [15, Lemma 6.5.3] (page 432). For this, we introduce

$$\alpha_n(h,\varepsilon) = \sup \mathbf{P}_\vartheta^{(n)}\Bigl(\bigl|\ln Z_{n,\vartheta}(u) - x\bigr| > \varepsilon \;\Big|\; \ln Z_{n,\vartheta}(v) = x\Bigr)$$

where the supremum is taken over $u, v \in U_n$ such that $v < u \leq v + h$ and $x \in \mathbb{R}$.

As $\ln Z_{n,\vartheta}(u)$ has independent increments, and using the inequality (4.18), we have

$$\begin{aligned}\alpha_n(h,\varepsilon) &= \sup_{u,v\,:\,|u-v|\leq h} \mathbf{P}_\vartheta^{(n)}\Big(\big|\ln Z_{n,\vartheta}(u) - \ln Z_{n,\vartheta}(v)\big| > \varepsilon\Big)\\ &\leq \sup_{u,v\,:\,|u-v|\leq h}\left(\frac{25\,|u-v|}{8\varepsilon^2\ell} + \frac{(u-v)^2}{\varepsilon^2\ell^2}\right) \leq \frac{25h}{8\varepsilon^2\ell} + \frac{h^2}{\varepsilon^2\ell^2}.\end{aligned}$$

If we suppose $D \geq D_1 = 12/b$, we have $bD/12 \geq 1$ and, noting that $h \leq 1$, we obtain

$$\alpha_n^* = \alpha_n\Big(h, \frac{bD}{12}\Big) \leq \frac{25h}{8\ell} + \frac{h^2}{\ell^2} \leq \left(\frac{25}{8\ell} + \frac{1}{\ell^2}\right)h \leq \left(\frac{25}{8\ell} + \frac{1}{\ell^2}\right)e^{-bD}.$$

If we suppose, moreover, $D \geq D_2 = \frac{1}{b}\ln\left(\frac{25}{4\ell} + \frac{2}{\ell^2}\right)$, we will also have $\alpha_n^* \leq 1/2 < 1$.

Now, using the above mentioned corollary, we have

$$\begin{aligned}P_{2,k} &\leq \frac{\alpha_n^*}{(1-\alpha_n^*)^2}\mathbf{P}_\vartheta^{(n)}\left(\Big|\ln Z_{n,\vartheta}\big(D+(k+1)h\big) - \ln Z_{n,\vartheta}(D+kh)\Big| > \frac{bD}{12}\right)\\ &\leq \frac{\alpha_n^{*\,2}}{(1-\alpha_n^*)^2} \leq 4\alpha_n^{*\,2}.\end{aligned}$$

Returning to the term P_2, we obtain

$$P_2 \leq \sum_{k=0}^{\gamma-1} P_{2,k} \leq 4\gamma\alpha_n^{*\,2} \leq 4\gamma\left(\frac{25}{8\ell} + \frac{1}{\ell^2}\right)^2 h^2 = \left(\frac{25}{4\ell} + \frac{2}{\ell^2}\right)^2 h \leq C_1 e^{-bD}$$

where we denoted $C_1 = \left(\frac{25}{4\ell} + \frac{2}{\ell^2}\right)^2$.

So, fixing an arbitrary $b \in (0, k_*/3]$ and putting $D_0 = \max\{D_1, D_2\}$, we have

$$\mathbf{P}_\vartheta^{(n)}\left(\sup_{D\leq u\leq D+1} Z_{n,\vartheta}(u) > e^{-bD}\right) \leq (2 + C_1)e^{-bD}$$

for all $D \geq D_0$, $\vartheta \in \mathbb{K}$ and sufficiently large values of n, and Lemma 4.11 is henceforth proved. □

Proof of Lemma 4.10. We have

$$\begin{aligned}\mathbf{P}_\vartheta^{(n)}\left(\sup_{|u|>D} Z_{n,\vartheta}(u) > e^{-bD}\right) &= \mathbf{P}_\vartheta^{(n)}\left(\max_{k\in\mathbb{N}}\ \sup_{D+k\leq|u|\leq D+k+1} Z_{n,\vartheta}(u) > e^{-bD}\right)\\ &\leq \sum_{k=0}^{+\infty}\mathbf{P}_\vartheta^{(n)}\left(\sup_{D+k\leq|u|\leq D+k+1} Z_{n,\vartheta}(u) > e^{-bD}\right)\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{+\infty} \mathbf{P}_\vartheta^{(n)}\Bigg(\sup_{D+k\leq|u|\leq(D+k)+1} Z_{n,\vartheta}(u) > e^{-b(D+k)}\Bigg) \\
&\leq \sum_{k=0}^{+\infty} C\, e^{-b(D+k)} \leq C\, e^{-bD} \sum_{k=0}^{+\infty} e^{-bk} \\
&\leq C\, e^{-bD} \int_{-1}^{+\infty} e^{-bx}\, \mathrm{d}x = \frac{e^b}{b}\, C\, e^{-bD} = C' e^{-bD}
\end{aligned}$$

where we denoted $C' = e^b C/b$, and so, Lemma 4.10 holds with $C = C'$. □

4.3.2 Parameter estimation

In this section we are interested in the estimation of the unknown parameter ϑ in our model of observations.

Recall that as function of ϑ, the likelihood of our model given by (4.14) is discontinuous (has jumps). So, the maximum likelihood estimator is introduced through the equation

$$\max\Big\{L_n\big(\,\widehat{\vartheta}_n+, X^{(n)}\big), L_n\big(\,\widehat{\vartheta}_n-, X^{(n)}\big)\Big\} = \sup_{\vartheta\in\Theta} L_n\big(\vartheta, X^{(n)}\big).$$

The Bayesian estimator for a given prior density p and for square loss is defined by

$$\widetilde{\vartheta}_n = \frac{\int_\alpha^\beta \vartheta\, p(\vartheta) L_n\big(\vartheta, X^{(n)}\big)\, \mathrm{d}\vartheta}{\int_\alpha^\beta p(\vartheta) L_n\big(\vartheta, X^{(n)}\big)\, \mathrm{d}\vartheta}\,.$$

We are interested in the asymptotic properties of the maximum likelihood and Bayesian estimators of ϑ as $n \to +\infty$. To describe the properties of the estimators we need some additional notations.

We introduce the random variables ξ^*, ξ_ϑ, ζ^* and ζ_ϑ by the equations

$$\begin{aligned}
Z^*(\xi^*) &= \sup_{u\in\mathbb{R}} Z^*(u), \\
Z_\vartheta(\xi_\vartheta) &= \sup_{u\in\mathbb{R}} Z_\vartheta(u),
\end{aligned}$$

$$\zeta^* = \frac{\int_{-\infty}^{+\infty} u\, Z^*(u)\, \mathrm{d}u}{\int_{-\infty}^{+\infty} Z^*(u)\, \mathrm{d}u}$$

and

$$\zeta_\vartheta = \frac{\int_{-\infty}^{+\infty} u\, Z_\vartheta(u)\, \mathrm{d}u}{\int_{-\infty}^{+\infty} Z_\vartheta(u)\, \mathrm{d}u}\,.$$

Let us note that $\xi_\vartheta \overset{d}{=} \psi(\vartheta)\,\xi^*$ and $\zeta_\vartheta \overset{d}{=} \psi(\vartheta)\,\zeta^*$.

Now we can state the following theorem giving an asymptotic lower bound on the risk of all the estimators of ϑ.

Theorem 4.8. *Let the conditions* **I1** – **I4** *be fulfilled. Then, for any* $\vartheta_0 \in \Theta$, *we have*

$$\lim_{\delta\to 0}\ \varliminf_{n\to+\infty}\ \inf_{\overline{\vartheta}_n}\ \sup_{|\vartheta-\vartheta_0|<\delta} \varphi_n^{-2}\,\mathbf{E}_\vartheta^{(n)}(\overline{\vartheta}_n-\vartheta)^2 \geq \mathbf{E}\zeta_{\vartheta_0}^2 = \psi^2(\vartheta_0)\,\mathbf{E}(\zeta^*)^2,$$

where the inf *is taken over all possible estimators* $\overline{\vartheta}_n$ *of the parameter* ϑ.

This theorem allows us to introduce the following definition.

Definition 4.2. *Let the conditions* **I1** – **I4** *be fulfilled. We say that an estimator* ϑ_n^* *is asymptotically efficient if*

$$\lim_{\delta\to 0}\ \lim_{n\to+\infty}\ \sup_{|\vartheta-\vartheta_0|<\delta} \varphi_n^{-2}\,\mathbf{E}_\vartheta^{(n)}(\vartheta_n^*-\vartheta)^2 = \mathbf{E}\zeta_{\vartheta_0}^2 = \psi^2(\vartheta_0)\,\mathbf{E}(\zeta^*)^2$$

for all $\vartheta_0 \in \Theta$.

Now, we can state two following theorems giving the asymptotic properties of the maximum likelihood and Bayesian estimators.

Theorem 4.9. *Let the conditions* **I1** – **I4** *be fulfilled. Then the maximum likelihood estimator* $\widehat{\vartheta}_n$ *satisfies uniformly on* $\vartheta \in \mathbb{K}$ *the relations*

$$\mathbf{P}_\vartheta^{(n)} - \lim_{n\to+\infty} \widehat{\vartheta}_n = \vartheta,$$

$$\mathcal{L}_\vartheta^{(n)}\big\{\varphi_n^{-1}(\widehat{\vartheta}_n-\vartheta)\big\} \Rightarrow \mathcal{L}(\xi_\vartheta) = \mathcal{L}\big(\psi(\vartheta)\xi^*\big)$$

and

$$\lim_{n\to+\infty} \mathbf{E}_\vartheta^{(n)}\varphi_n^{-p}\big|\widehat{\vartheta}_n-\vartheta\big|^p = \mathbf{E}\,|\xi_\vartheta|^p = \psi^p(\vartheta)\,\mathbf{E}\,|\xi^*|^p \quad \textit{for any} \quad p>0.$$

In particular, the relative asymptotic efficiency of $\widehat{\vartheta}_n$ *is* $\mathbf{E}(\zeta^*)^2/\mathbf{E}(\xi^*)^2$.

Theorem 4.10. *Let the conditions* **I1** – **I4** *be fulfilled. Then, for any continuous strictly positive density, the Bayesian estimator* $\widetilde{\vartheta}_n$ *satisfies uniformly on* $\vartheta \in \mathbb{K}$ *the relations*

$$\mathbf{P}_\vartheta^{(n)} - \lim_{n\to+\infty} \widetilde{\vartheta}_n = \vartheta,$$

$$\mathcal{L}_\vartheta^{(n)}\big\{\varphi_n^{-1}(\widetilde{\vartheta}_n-\vartheta)\big\} \Rightarrow \mathcal{L}(\zeta_\vartheta) = \mathcal{L}\big(\psi(\vartheta)\zeta^*\big)$$

and

$$\lim_{n\to+\infty} \mathbf{E}_\vartheta^{(n)}\varphi_n^{-p}\big|\widetilde{\vartheta}_n-\vartheta\big|^p = \mathbf{E}\,|\zeta_\vartheta|^p = \psi^p(\vartheta)\,\mathbf{E}\,|\zeta^*|^p \quad \textit{for any} \quad p>0.$$

In particular, $\widetilde{\vartheta}_n$ *is asymptotically efficient.*

Theorems 4.8–4.10 follow from the properties of the normalized likelihood ratio established in the previous section. More precisely, Theorem 4.10 follows from Lemmas 4.6–4.8 and [17, Theorem 1.10.2]. Having the properties of the Bayesian estimators given in Theorem 4.10, we can cite [17, Theorem 1.9.1] to provide the proof of Theorem 4.8. Finally, the proof of Theorem 4.9 can be carried out following the standard argument of [17, Chapters 5.3 and 5.4] which is based on the weak convergence established in Theorem 4.6 together with the inequality (4.19).

4.3.3 Hypotheses testing

Suppose that we observe n independent inhomogeneous Poisson processes with intensity function $\lambda_\vartheta^{(n)}(t) = \psi_n(t) + r_n \mathbb{1}_{\{t>\vartheta\}}$, where the function $\psi_n(\cdot)$ is continuous on the interval $[0,\tau]$, the parameter $\vartheta \in [\vartheta_1, b) \subseteq [0,\tau)$ and the conditions **(I2)**–**(I4)** are fulfilled.

As before, for the comparison of powers of different tests, we denote $\vartheta = \vartheta_1 + u\varphi_n^*$, where $\varphi_n^* = \frac{\psi(\vartheta_1)}{nr_n^2}$, and we replace the initial hypotheses testing problem by the following one

$$\begin{aligned} \mathscr{H}_1 &: \quad u = 0, \\ \mathscr{H}_2 &: \quad u > 0. \end{aligned}$$

The score-function test does not exist and we study the GLRT, Wald's test and bayesian tests. This study is essentially based on the properties of the normalized likelihood ratio established above. Note that the limit of the normalized likelihood ratio at the point $\vartheta = \vartheta_1$ (under hypothesis $\mathscr{H}_1$) is the following:

$$Z_{n,\vartheta_1}^*(v) = \frac{L_n\left(\vartheta_1 + v\varphi_n^*, X^{(n)}\right)}{L_n\left(\vartheta_1, X^{(n)}\right)} \Longrightarrow Z^*(v) = \exp\left\{W(v) - \frac{|v|}{2}\right\},$$

where $W(v)$, $v \geq 0$, is a Brownian motion.

Under alternatives, similar calculation yields (below we put $\vartheta_u = \vartheta_1 + u\varphi_n^*$):

$$\begin{aligned} Z_{n,\vartheta_1}^*(v) &= \frac{L_n\left(\vartheta_1 + v\varphi_n^*, X^{(n)}\right)}{L_n\left(\vartheta_1, X^{(n)}\right)} \\ &= \left(\frac{L_n\left(\vartheta_1, X^{(n)}\right)}{L_n\left(\vartheta_u, X^{(n)}\right)}\right)^{-1} \frac{L_n\left(\vartheta_1 + v\varphi_n^*, X^{(n)}\right)}{L_n\left(\vartheta_u, X^{(n)}\right)} \\ &= \left(\frac{L_n\left(\vartheta_u - u\varphi_n^*, X^{(n)}\right)}{L_n\left(\vartheta_u, X^{(n)}\right)}\right)^{-1} \frac{L_n\left(\vartheta_u + (v-u)\varphi_n^*, X^{(n)}\right)}{L_n\left(\vartheta_u, X^{(n)}\right)} \\ &\Longrightarrow \left(Z^*(-u)\right)^{-1} Z^*(v-u) \overset{d}{=} Z_u^*(v) = \exp\left\{W(v) - \frac{|v-u|}{2} + \frac{|u|}{2}\right\}. \end{aligned}$$

Note that the construction of the tests is almost the same as before and that the main difference is in the properties of these tests.

The GLRT is defined by the relations

$$\hat{\phi}_n\left(X^{(n)}\right)=\mathbb{1}_{\left\{Q\left(X^{(n)}\right)>h_\varepsilon\right\}},$$

where

$$Q\left(X^{(n)}\right)=\sup_{\vartheta>\vartheta_1}\frac{L_n\left(\vartheta,X^{(n)}\right)}{L_n\left(\vartheta_1,X^{(n)}\right)}=\max\left[\frac{L_n\left(\hat{\vartheta}_n+,X^{(n)}\right)}{L_n\left(\vartheta_1,X^{(n)}\right)},\frac{L_n\left(\hat{\vartheta}_n-,X^{(n)}\right)}{L_n\left(\vartheta_1,X^{(n)}\right)}\right].$$

To choose the threshold h_ε such that $\hat{\phi}_n\big(X^{(n)}\big)\in\mathcal{K}_\varepsilon$ we need the solution of the following equation (under hypothesis $\mathscr{H}_1$)

$$\mathbf{P}_{\vartheta_1}^{(n)}\left\{Q\left(X^{(n)}\right)>h_\varepsilon\right\}=\mathbf{P}_{\vartheta_1}^{(n)}\left\{\sup_{v>0}Z_{n,\vartheta_1}^*\left(v\right)>h_\varepsilon\right\}\longrightarrow\mathbf{P}\left\{\sup_{v>0}Z^*\left(v\right)>h_\varepsilon\right\}=\varepsilon.$$

Note that $\sup_{v>0}\ln Z^*\left(v\right)$ follows the exponential distribution with mean 1 $\big($see [1]$\big)$.

The power function has the following limit

$$\mathbf{P}_{\vartheta_u}^{(n)}\left\{\sup_{v>0}Z_{n,\vartheta_1}\left(v\right)>h_\varepsilon\right\}\longrightarrow\mathbf{P}\left\{\sup_{v>0}Z_u^*\left(v\right)>h_\varepsilon\right\}.$$

This power function is obtained below with the help of numerical simulations.

To define Wald's test, let us note that we already know that the MLE converges in distribution

$$\left(\varphi_n^*\right)^{-1}\left(\hat{\vartheta}_n-\vartheta_1\right)\Longrightarrow\xi_+^*$$

where the random variable ξ_+^* is solution of the equation

$$Z^*(\xi_+^*)=\sup_{v>0}Z^*(v).$$

Therefore, if we put

$$\phi_n^o\left(X^{(n)}\right)=\mathbb{1}_{\left\{\left(\varphi_n^*\right)^{-1}\left(\hat{\vartheta}_n-\vartheta_1\right)>m_\varepsilon\right\}},$$

where m_ε is defined by the equation

$$\mathbf{P}\left\{\xi_+^*>m_\varepsilon\right\}=\varepsilon,$$

then $\phi_n^o\in\mathcal{K}_\varepsilon$.

We recall the result of [34], that the joint distribution of $\left(\ln Z^*\left(\xi_+^*\right),\xi_+^*\right)$ has the density

$$f(y,t)=\frac{y}{\sqrt{2\pi t^3}}\exp\left\{-\frac{\left(y+\frac{t}{2}\right)^2}{2t}\right\},$$

which allows us to calculate the marginal density of ξ_+^* as follows:

$$
\begin{aligned}
f(t) &= \int_0^{+\infty} f(y,t)\,\mathrm{d}y = \int_0^{+\infty} \frac{\frac{y}{\sqrt{t}}}{\sqrt{2\pi t}} \exp\left\{-\frac{1}{2}\left(\frac{y}{\sqrt{t}}+\frac{\sqrt{t}}{2}\right)^2\right\} \mathrm{d}\left(\frac{y}{\sqrt{t}}\right) \\
&= \int_0^{+\infty} \frac{z}{\sqrt{2\pi t}} \exp\left\{-\frac{1}{2}\left(z+\frac{\sqrt{t}}{2}\right)^2\right\} \mathrm{d}z \\
&= \int_{\frac{\sqrt{t}}{2}}^{+\infty} \frac{x-\frac{\sqrt{t}}{2}}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2}\right\} \mathrm{d}x \\
&= -\int_{\frac{\sqrt{t}}{2}}^{+\infty} \frac{1}{\sqrt{2\pi t}} \mathrm{d}\exp\left\{-\frac{x^2}{2}\right\} - \int_{\frac{\sqrt{t}}{2}}^{+\infty} \frac{\frac{\sqrt{t}}{2}}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2}\right\} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{t}{8}\right\} - \frac{1}{2}\Phi\left(-\frac{\sqrt{t}}{2}\right),
\end{aligned}
$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution.

The limit of the power function of Wald's test for the local alternatives is the following

$$
\begin{aligned}
\beta\left(\phi_n^o, u\right) = \mathbf{E}_{\vartheta_u}^{(n)} \phi_n^o\left(X^{(n)}\right) &= \mathbf{P}_{\vartheta_u}^{(n)}\left\{\left(\varphi_n^*\right)^{-1}\left(\hat{\vartheta}_n - \vartheta_u\right) + u > m_\varepsilon\right\} \\
&\longrightarrow \mathbf{P}\left\{\xi_{+,u}^* > m_\varepsilon - u\right\},
\end{aligned}
$$

where the random variable $\xi_{+,u}^*$ is solution of the equation

$$
Z\left(\xi_{+,u}^*\right) = \sup_{v>-u} Z^*(v).
$$

Suppose now that the parameter ϑ is a random variable with the *a priori* density $p(\theta)$, $\vartheta_1 \le \theta < b$. This function is supposed to be continuous and positive. We consider two tests.

The first one is based on the BE

$$
\tilde{\phi}_n\left(X^{(n)}\right) = \mathbb{1}_{\left\{\left(\varphi_n^*\right)^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\}}.
$$

As above we have the limit

$$
\mathbf{E}_{\vartheta_1}^{(n)} \mathbb{1}_{\left\{\left(\varphi_n^*\right)^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\}} \longrightarrow \mathbf{P}\left\{\zeta_+^* > k_\varepsilon\right\}, \qquad \zeta_+^* = \frac{\int_0^\infty v Z^*(v)\,\mathrm{d}v}{\int_0^\infty Z^*(v)\,\mathrm{d}v}
$$

which help us to calculate the threshold such that $\tilde{\phi}_n \in \mathcal{K}_\varepsilon$.

$$
\beta\left(\tilde{\phi}_n, u\right) = \mathbf{P}_{\vartheta_u}^{(n)}\left\{\left(\varphi_n^*\right)^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\} = \mathbf{P}_{\vartheta_u}^{(n)}\left\{\left(\varphi_n^*\right)^{-1}\left(\tilde{\vartheta}_n - \vartheta_u\right) + u > k_\varepsilon\right\}
$$

$$\longrightarrow \mathbf{P}\left\{\zeta^*_{+,u} > k_\varepsilon - u\right\}$$

where $\zeta^*_{+,u}$ is obtained by the similar calculation as (4.13) and of the form

$$\zeta^*_{+,u} = \frac{\int_{-u}^{\infty} v Z^*(v)\,\mathrm{d}v}{\int_{-u}^{\infty} Z^*(v)\,\mathrm{d}v}.$$

We also denote one other expression of the limit power function of BT1 as follows

$$\beta\left(\tilde{\phi}_n, u\right) = \mathbf{P}^{(n)}_{\vartheta_u}\left\{(\varphi^*_n)^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\} \longrightarrow \mathbf{P}\left\{\frac{\int_0^{\infty} v Z^*_u(v)\,\mathrm{d}v}{\int_0^{\infty} Z^*_u(v)\,\mathrm{d}v} > k_\varepsilon\right\}.$$

The threshold and power function are obtained by the numerical simulations.

The second test minimizes the mean error. The likelihood ratio is

$$\tilde{L}\left(X^{(n)}\right) = \int_{\vartheta_1}^{b} \frac{L_n\left(\theta, X^{(n)}\right)}{L_n\left(\vartheta_1, X^{(n)}\right)} p(\theta)\,\mathrm{d}\theta = \varphi^*_n \int_0^{(\varphi^*_n)^{-1}(b-\vartheta_1)} Z^*_{n,\vartheta_1}(v)\, p\left(\vartheta_1 + u\varphi^*_n\right)\mathrm{d}v.$$

Hence, we have the following limit:

$$(\varphi^*_n)^{-1}\tilde{L}\left(X^{(n)}\right) \Longrightarrow p(\vartheta_1)\int_0^{\infty} \exp\left\{W(v) - \frac{v}{2}\right\}\mathrm{d}v.$$

Therefore, if we take g_ε as solution of the equation

$$\mathbf{P}\left\{\int_0^{\infty} \exp\left\{W(v) - \frac{v}{2}\right\}\mathrm{d}v > g_\varepsilon\right\} = \varepsilon,$$

then the test

$$\tilde{\phi}_n\left(X^{(n)}\right) = \mathbb{1}_{\{R_n > g_\varepsilon\}}, \qquad R_n = \frac{(\varphi^*_n)^{-1}\tilde{L}\left(X^{(n)}\right)}{p(\vartheta_1)}$$

belongs to the class $\mathcal{K}_\varepsilon$.

4.3.4 Simulations

We consider n independent observations $X^{(n)} = \left\{X_j^{(n)}(t), t \in [0,4]\right\}; j = 1, ..., n$ of the Poisson process of intensity function

$$\lambda^{(n)}_\vartheta(t) = 3\cos^2(t) + 1 + \frac{1}{n^{1/4}}\mathbb{1}_{\{t>\vartheta\}}, \qquad 0 \le t \le 4$$

with $\vartheta \in [2,4)$. Let us take $\vartheta_1 = 2$ and put

$$\varphi^*_n = \frac{\psi(\vartheta_1)}{n r_n^2} = \frac{3\cos^2(2) + 1}{\sqrt{n}}.$$

Then we have

$$\ln Z^*_{n,\vartheta_1}(v) = \sum_{j=1}^{n} \int_{\vartheta_1}^{\vartheta_1+v\varphi_n^*} \ln \frac{3\cos^2(t)+1}{3\cos^2(t)+1+\frac{1}{n^{1/4}}} \,\mathrm{d}X_j(t) + v\,\left(3\cos^2(2)+1\right) n^{1/4}.$$

We note that, under $\mathscr{H}_1$, the random variable $\sup_{v>0} \ln Z^*(v)$ has the exponential distribution withe parameter 1. This allows us to calculate the threshold of GLRT as solution h_ε of the equation $1-e^{-\ln h_\varepsilon} = 1-\varepsilon$. Hence $h_\varepsilon = 1/\varepsilon$.

The threshold m_ε of WT can be obtained by the numerical solution of the equation

$$\int_{c_\varepsilon}^{+\infty} \left(\frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{t}{8}\right\} - \frac{1}{2}\Phi\left(-\frac{\sqrt{t}}{2}\right)\right) \mathrm{d}t = \varepsilon.$$

Here $\Phi(\cdot)$ is the distribution function of the standard normal law.

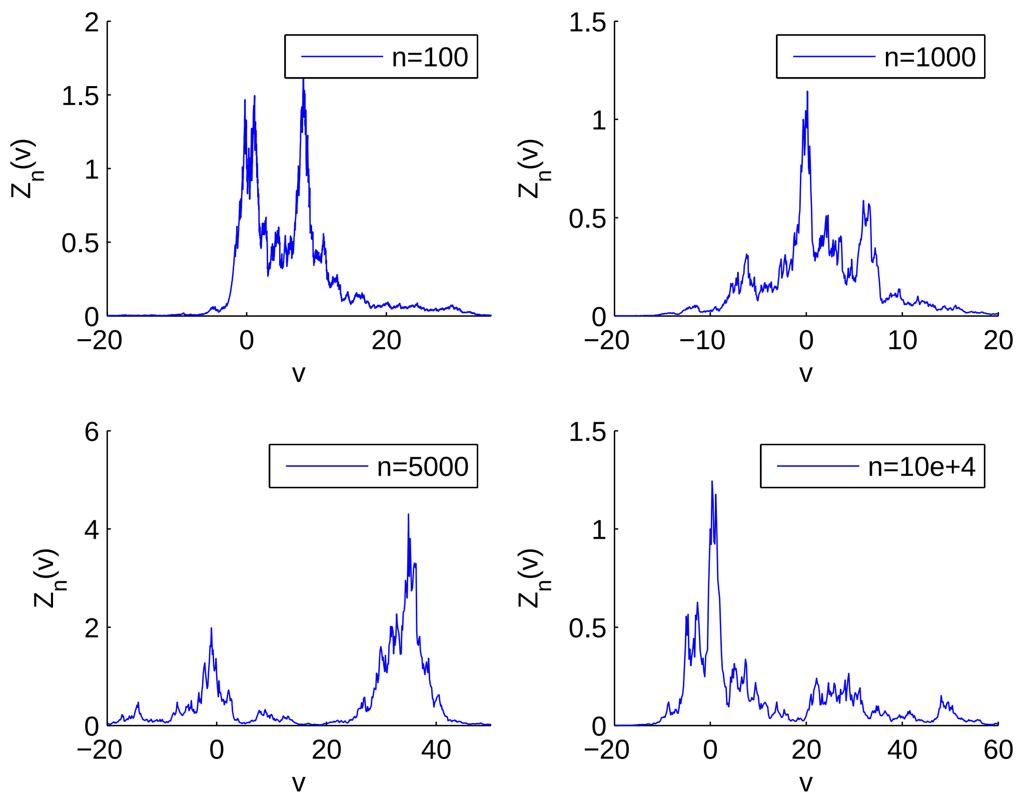

Figure 4.3: Some realization of $Z^*_{n,\vartheta_1}(v)$ for change-point model with $r_n = r^{-0.25}$.

We will also discuss for $n = 300, 1000$ (respectively $r_n = 0.2403, 0.1778$), the tendency when r_n is not sufficiently small. Some realizations of $Z^*_{n,\vartheta_1}(v)$ are shown in the Fig. 4.3. We mention that when $n = 10$ ($r_n = 0.5623$), as the factor r_n still large, the power functions of WT and BT1 are almost equal to zero. Even when $n = 50$, the power function of BT1 is almost equal to zero which apply that the BT1 is the most sensible than the others. When $r_n \approx 0$, the tendency of the power functions in this case will be similar as in the cusp-type singularity but specially when $\kappa = 0.5$.

ε	0.01	0.05	0.10	0.20	0.40	0.50
m_ε	14.886	7.282	4.531	2.236	0.685	0.248
k_ε	16.782	8.582	5.573	3.024	1.102	0.657

Table 4.2: Thresholds of GLRT, WT and BT1.

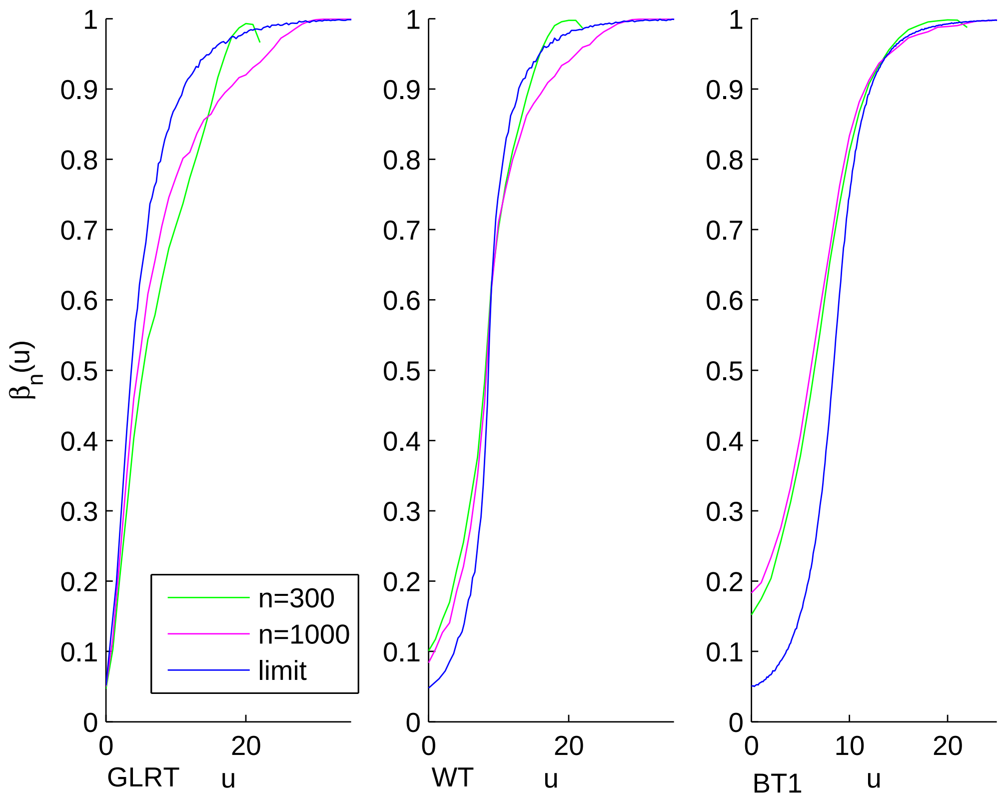

Figure 4.4: Power functions of GLRT and WT for change-point model with $r_n = r^{-0.25}$.

4.3.5 Comparison of the limit power functions

Le us fix an alternative $\vartheta_2 > \vartheta_1$ and consider the testing problem with two simple hypotheses

$$\begin{aligned} \mathscr{H}_1 &: \quad \vartheta = \vartheta_1, \\ \mathscr{H}_2 &: \quad \vartheta = \vartheta_2. \end{aligned}$$

Using the notation $\vartheta_2 = \vartheta_1 + u_1\varphi_n$, we construct the correspondent close alternative and the problem became

$$\begin{aligned} \mathscr{H}_1 &: \quad u = 0, \\ \mathscr{H}_2 &: \quad u = u_1 \qquad (u_1 > 0). \end{aligned}$$

Remind that in this situation we have the most powerful test, called Neyman-Pearson test (N-PT) given by the relation

$$\phi_n^*\left(X^n\right) = \mathbb{1}_{\left\{Z_{n,\vartheta_1}^*(u_1) > d_\varepsilon\right\}},$$

where d_ε and q_ε are defined as follows. The likelihood ratio $Z_{n,\vartheta_1}^*(u_1)$ under hypothesis $\mathscr{H}_1$ converges to following limit (see above)

$$Z_{n,\vartheta_1}^*(u_1) \Longrightarrow Z^*(u_1) = \exp\left\{W\left(u_1\right) - \frac{u_1}{2}\right\}.$$

Hence $q_\varepsilon \to 0$ and we can put $q_\varepsilon = 0$. The threshold d_ε is the solution of the equation

$$\mathbf{P}_{\vartheta_1}\left(Z^*\left(u_1\right) > d_\varepsilon\right) = \varepsilon.$$

Note that

$$\mathbf{P}_{\vartheta_1}\left(Z^*\left(u_1\right) > d_\varepsilon\right) = \mathbf{P}_{\vartheta_1}\left(W(u_1) > \ln d_\varepsilon + \frac{u_1}{2}\right) = \mathbf{P}_{\vartheta_1}\left(\zeta > \frac{\ln d_\varepsilon + \frac{u_1}{2}}{\sqrt{u_1}}\right) = \varepsilon$$

where $\zeta \sim \mathcal{N}\left(0,1\right)$. Therefore, if we denote z_ε the $1-\varepsilon$ quantile of the standard Gaussian law ($\mathbf{P}\left(\zeta > z_\varepsilon\right) = \varepsilon$) then the threshold d_ε is

$$d_\varepsilon = e^{z_\varepsilon\sqrt{u_1} - \frac{u_1}{2}}.$$

Of course, it is impossible to construct N-PT because the value u_1 under alternative is unknown, but its power function gives an upper bound for the power functions of all other tests. Moreover the distance between it and the power functions of studied tests provides an important information. Therefore it is interesting to compare the powers of the studied tests with the power of the N-PT.

The normalized likelihood ratio under alternative $\mathscr{H}_2$ we write as

$$Z_{n,\vartheta_1}^*(u_1) = \frac{L_n\left(\vartheta_1 + u_1\varphi_n, X^n\right)}{L_n\left(\vartheta_1, X^n\right)} = \left(\frac{L_n\left(\vartheta_1 + u_1\varphi_n - u_1\varphi_n, X^n\right)}{L_n\left(\vartheta_1 + u_1\varphi_n, X^n\right)}\right)^{-1},$$

and for the power function of the N-PT we obtain

$$\beta_n(u_1) = \beta(\phi_n^*(X^n), u_1) = \mathbf{P}_{\vartheta_1+u_1\varphi_n}\left(Z_{n,\vartheta_1}^*(u_1) > d_\varepsilon\right)$$
$$\longrightarrow \mathbf{P}_{\vartheta_1}\left(\left(Z^*(-u_1)\right)^{-1} > d_\varepsilon\right) = \mathbf{P}_{\vartheta_1}\left(\exp\left\{-W(-u_1) + \frac{u_1}{2}\right\} > d_\varepsilon\right)$$
$$= \mathbf{P}_{\vartheta_1}\left(W(u_1) > \ln d_\varepsilon - \frac{u_1}{2}\right) = \mathbf{P}_{\vartheta_1}\left(\zeta > \frac{\ln d_\varepsilon - \frac{u_1}{2}}{\sqrt{u_1}}\right) = \mathbf{P}\left(\zeta > z_\varepsilon - \sqrt{u_1}\right).$$

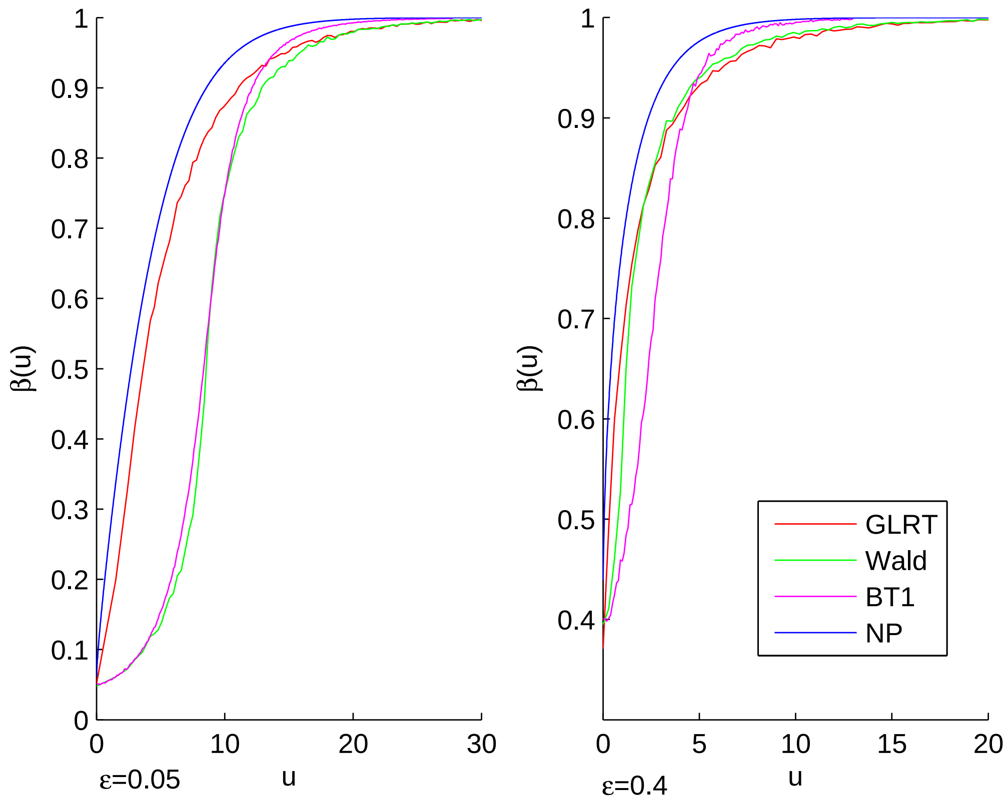

Figure 4.5: Comparison of limit power functions for change-point type model with $r_n = n^{-0.25}$ and $\varepsilon = 0.05$ and $\varepsilon = 0.4$.

The curves of BT1 are always the most quickly tends to 1. And we see that the curves of WT is close to BT1 when u is small and $\varepsilon = 0.04$. When $\varepsilon = 0.4$, the curve of GLRT and WT are coincident. Finally we need to say that as all these limit power functions are not close to the power of NP-T, the choice of the asymptotic optimal test is until an open question.

Bibliographie

[1] Baskakova, P. E., Borodin, A. N., Tables of distribution of functionals of Brownian motion, *J. Math. Sci.*, vol. 75, no. 5, pp. 1873–1883, 1995.

[2] Bagdonavičius, V., Nikulin, M., Goodness-of-fit test for homogeneous Markov processes, *C. R., Math., Acad. Sci. Paris*, vol. 351, no. 3-4, pp. 149-154, New York, 2013.

[3] Brown, M., Discrimination of Poisson processes, *Ann. Math. Statist.*, vol. 42, pp. 773–776, 1971.

[4] Dachian, S., Estimation of cusp location by Poisson observation, *Stat. Inference Stoch. Process.*, vol. 6, no. 1, pp. 1–14, 2003.

[5] Dachian, S., On Limiting Likelihood Ratio Processes of Some Change-Point Type Statistical Models, *J. Statist. Plann. Inference*, vol. 140, no. 9, pp. 2682–2692, 2010.

[6] Dachian, S., Estimation of the location of a 0-type or ∞-type singularity by Poisson observations, *Statistics*, vol. 45, no. 5, pp. 509–523, 2011.

[7] Dachian, S., Kutoyants, Yu. A., On cusp estimation of ergodic diffusion process, *J. Statist. Plann. Inference*, vol. 117, pp. 153–199, 2003.

[8] Dachian, S., Kutoyants Yu. A., Hypotheses testing : Poisson versus stress-release, *J. Statist. Plan. Inference*, vol. 139, pp. 1668–1684, 2009.

[9] Dachian, S., Kutoyants, Yu. A., Yang, L., On hypotheses testing for Poisson processes. Regular case, submitted, 2013.

[10] Dachian, S., Kutoyants, Yu. A., Yang, L., On hypotheses testing for Poisson processes. Singular cases, submitted, 2013.

[11] Dachian, S., Negri, I., On compound Poisson processes arising in change-point type statistical models as limiting likelihood ratios, *Stat. Inference Stoch. Process.*, vol. 14, no. 3, pp. 255–271, 2011.

[12] Dachian, S., Yang, L., Change-point model with variable jump size. Parameter estimation and hypothesis testing, submitted, 2013.

[13] Fazli, K., Kutoyants, Yu. A., Two simple hypotheses testing for Poisson process, *Far East J. of Theoretical Statistics*, vol. 15, no. 2, pp. 251–290, 2005.

[14] Gikhman, I. I., Skorohod, A. V., "*Introduction to the Theory of Random Processes*", Saunders, Philadelphia, 1969.

[15] Gihman, I. I., Skorohod, A. V., "*The theory of stochastic processes I.*", Springer-Verlag, New York, 1974.

[16] Greenwood, P., Nikulin, M., "*A Guide to Chi-Squared Testing*", John Wiley and Sons, New York, 1996.

[17] Ibragimov, I. A., Hasminskii, R. Z., "*Statistical Estimation. Asymptotic Theory*", Springer, New York, 1981.

[18] Kutoyants, Yu. A., "*Parameter Estimation for Stochastic Processes*", Armenian Academy of Sciences, Yerevan, 1980 (in Russian), translation of revised version, Heldermann-Verlag, Berlin, 1984.

[19] Kutoyants, Yu. A., "*Statistical Inference for Spatial Poisson Processes*", Lect. Notes Statist. **134**, Springer-Verlag, New York, 1998.

[20] Kutoyants, Yu. A., "*Statistical Inference for Ergodic Diffusion Processes*", Springer-Verlag, New York, 2004.

[21] Kutoyants, Yu. A., "*Introduction to Statistics of Poisson Processes*", to appear, 2013.

[22] Le Cam, L., Yang, G. "*Asymptotic Methods in Statistical Decision Theory*", Springer, New York, 1986.

[23] Léger, C., Wolfson, D. B., Hypothesis testing for a non-homogeneous Poisson process, *Communications in Statistics. Stochastic Models*, vol. 3, no. 3, pp. 439–455, 1987.

[24] Lehmann, E., Romano, J., "*Testing Statistical Hypotheses*", Springer, Heidelberg, 2005.

[25] Liese, F., Eine informationstheoretische Bedingung fű̃r die Ã̂quivalenz unbegrenzt teilbarer Punktprozesse, "*Math. Nachr.*", vol. 70, pp. 183–196, 1975.

[26] Liese, F., Lorz, U., Contiguity and LAN-property of sequences of Poisson processes, preprint 97/6, Universität Rostock, Rostocker Mathematiches Kolloquium, Heft 51, 1997.

[27] Liptser, R.S., Shiryayev, A.N., "*Statistics of Random Processes. II. Applications*", second ed. Springer, New York, 2001.

[28] Novikov, A., Kordzahiya, N., Pitman estimators : an asymptotic variance revisited, *Theory Probab. Appl.*, vol. 57, no. 3, pp. 521–529, 2013.

[29] Novikov, A., Kordzakhia, N., Ling, T., On moments of the limit distribution of Pitman estimators : the case of fractional Brownian motion, submitted, 2013.

[30] Pflug, G. C. The limiting log-likelihood process for discontinuous density families, *Z. Wahrsch. verw. Geb.*, vol. 64, pp. 15–35, 1983.

[31] Pflug, G. C., On an argmax-distribution connected to the Poisson process, Proceedings of the *Fifth Prague Conference on Asymtotic Statistics*, P. Mandl and M. Huskova eds, pp. 123–130, 1993.

[32] Pyke, R., The supremum and infimum of the Poisson process, *Ann. Math. Statist*, vol. 30, pp. 568–576, 1959.

[33] Roussas, G. G., "*Contiguity of Probability Measures : Some Applications in Statistics*", Cambridge Univ. Press, Cambridge, U.K., 1972.

[34] Shepp, L. A., The joint density of the maximum and its location for a Wiener process with drift, *J. Appl. Probab.*, vol. 16, pp. 423–427, 1979.

[35] Yang, L., Hypotheses testing for Poisson processes in singular cases (joint work with Dachian, S. and Kutoyants, Yu. A.), *Conference : Ph.D. Students Day*, poster, Nantes, 2012.

[36] Yang, L., Hypotheses testing for Poisson processes in singular cases (joint work with Dachian, S. and Kutoyants, Yu. A.), *Conference : Statistical Methods for Dynamical Stochastic Models*, poster, Paris, 2012.

[37] Yang, L., Some examples of the hypothesis testing for Poisson processes (joint work with Dachian, S. and Kutoyants, Yu. A.), *Conference : Statistique Asymptotique des Processus Stochastiques IX*, poster, Le Mans, 2013.

Thèse de Doctorat

Lin YANG

Tests d'Hypothèses pour les Processus de Poisson dans les Cas non Réguliers

Résumé

Ce travail est consacré aux problèmes de test d'hypothèses pour les processus de Poisson non homogènes.
L'objectif principal de ce travail est l'étude de comportement des différents tests dans le cas des modèles statistiques singuliers. L'évolution de la singularité de la fonction d'intensité est comme suit : régulière (l'information de Fisher finie), continue mais non différentiable (singularité de type "cusp"), discontinue (singularité de type saut) et discontinue avec un saut de taille variable. Dans tous les cas on décrit analytiquement les tests. Dans le cas d'un saut de taille variable, on présente également les propriétés asymptotiques des estimateurs.
En particulier, on décrit les statistiques de tests, le choix des seuils et le comportement des fonctions de puissance sous les alternatives locales. Le problème initial est toujours le test d'une hypothèse simple contre une alternative unilatérale. La méthode principale est la théorie de la convergence faible dans l'espace des fonctions discontinues. Cette théorie est appliquée à l'étude des processus de rapport de vraisemblance normalisé dans les modèles singuliers considérés. La convergence faible du rapport de vraisemblance sous l'hypothèse et sous les alternatives vers les processus limites correspondants nous permet de résoudre les problèmes mentionnés précédemment.
Les résultats asymptotiques sont illustrés par des simulations numériques contenant la construction des tests, le choix des seuils et les fonctions de puissances sous les alternatives locales.

Mots clés

Tests d'hypothèses, processus de Poisson non homogène, théorie asymptotique, alternatives composées, modèles statistiques singuliers.

Abstract

This work is devoted to the hypotheses testing problems for inhomogeneous Poisson processes.
The main object of the work is the study of the behaviour of different tests in the case of singular statistical models. The "evolution of singularity" of the intensity function is the following: regular (finite Fisher information), continuous but not differentiable ("cusp" type singularity), discontinuous (jump type singularity) and discontinuous with variable jump size. In all the cases we describe analytically the tests. In the case of variable jump size we present as well the asymptotic properties of the estimators.
In particular we describe the test statistics, the choice of thresholds and the form of the power functions for the local alternatives. The initial problem is always the test of a simple hypothesis against a one-sided alternative. The main tool is the weak convergence theory in the space of discontinuous functions. This theory is applied to the study of the normalized likelihood ratio processes in the considered singular models. The weak convergence of the likelihood ratio processes under hypothesis and under alternatives to the corresponding limit processes allows us to solve the mentioned above problems.
The asymptotic results are illustrated by numerical simulations which contain the construction of the tests, the choice of the thresholds, and the power functions for local alternatives.

Keywords

Hypotheses testing, inhomogeneous Poisson processes, asymptotic theory, composite alternatives, singular statistical models.