Quelques résultats mathématiques sur les gaz à faible nombre de Mach

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THÈSE

pour obtenir le grade de

Docteur de l'Université Paris-Est

Spécialité : **Mathématiques**

préparée au laboratoire **LAMA : Laboratoire d'Analyse et de Mathématiques Appliquées**

dans le cadre de l'École Doctorale **MSTIC : Mathématiques et Sciences et Technologies de l'Information et de la Communication**

présentée et soutenue publiquement
par

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le 24 Avril 2013

Titre:
Quelques résultats mathématiques sur les gaz à faible nombre de Mach

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Etudier abondamment, enquêter précisément,
méditer attentivement, distinguer clairement,
et exécuter sérieusement.

– Confucius

Résumé

Cette thèse est consacrée à l'étude de la dynamique des gaz à faible nombre de Mach. Le modèle étudié provient des équations de Navier-Stokes complètes lorsque le nombre de Mach tend vers zéro. On cherche à montrer que le problème de Cauchy correspondant est bien posé. Les cas visqueux et non visqueux sont tous deux considérés. Les coefficients physiques peuvent dépendre de la densité (ou de la température) inconnue. En particulier, nous prenons en compte les effets de conductivité thermique et on autorise de grandes variations d'entropie. Rappelons qu'en absence de diffusion thermique, la limite à faible nombre de Mach implique la condition d'incompressibilité. Dans le cadre étudié ici, en introduisant un nouveau champ de vitesses à divergence nulle, le système devient un couplage non linéaire entre une équation quasi-parabolique pour la densité et un système de type Navier-Stokes (ou Euler) pour la vitesse et la pression.

Pour le cas avec viscosité, on établit le résultat classique, à savoir qu'il existe une solution forte existant localement (resp. globalement) en temps pour des données initiales grandes (resp. petites). On considère ici le problème de Cauchy avec données initiales dans des espaces de Besov *critiques*.

Lorsque les coefficients physiques du système vérifient une relation spéciale, le système se simplifie considérablement, et on peut alors établir qu'il existent des solutions faibles globales en temps à énergie finie. Par un argument d'unicité fort-faible, on en déduit que en dimension deux, les solutions fortes à énergie finie existent pour tous les temps positifs.

Pour le cas sans viscosité, on montre d'abord le caractère bien posé dans des espaces de Besov limites, qui s'injectent dans l'espace des fonctions lipschitziennes. Des critères de prolongement et des estimations du temps de vie sont établis. Si l'on suppose la donnée initiale à énergie finie dans l'espace de Besov limite à exposant de Lebesgue infinie, on a également un résultat d'existence locale. En dimension deux, le temps de vie tend vers l'infini quand la densité tend vers une constante positive.

Des estimations de produits et de commutateurs, ainsi que des estimations a priori pour les équations paraboliques et pour le système de Stokes (ou d'Euler) à coefficients variables, se trouvent dans l'annexe. Ces estimations reposent sur la théorie de Littlewood-Paley et le calcul paradifférentiel.

Mot-clefs

Nombre de Mach, système de Navier-Stokes, système d'Euler, équations paraboliques, caractère bien posé, critère de prolongement, temps de vie, solutions fortes, solutions faibles, espaces de Besov, théorie de Littlewood-Paley, calcul paradifférentiel.

Abstract

This thesis is devoted to the study of the dynamics of the gases with small Mach number. The model comes from the complete Navier-Stokes equations when the Mach number goes to zero, and we aim at showing that it is well-posed. The viscous and inviscid cases are both considered. The physical coefficients may depend on the unknown density (or on the unknown temperature). In particular, we consider the effects of the thermal conductivity and hence large variations of entropy are allowed. Recall that if there is no thermal diffusion, then the low Mach number limit just implies the incompressibility condition. In the framework considered here, by introducing a new solenoidal velocity field, the system becomes a nonlinear coupling between a quasi-parabolic equation for the density and an evolutionary Stokes (or Euler) system for the velocity and the pressure.

For the case with viscosity, we establish classical results, namely the strong solutions exist locally (resp. globally) in time for big (resp. small) initial data. We consider the Cauchy problem in the critical Besov spaces with the lowest regularity. Under a special relationship between the physical coefficients, the system recasts in a simpler form and one may prove that there exist weak solutions with finite energy. In dimension two, by a weak-strong uniqueness argument, this implies that strong solutions with finite energy exist for all positive times.

In the inviscid case, we first prove the well-posedness result in endpoint Besov spaces, which can be embedded into the set of Lipschitzian functions. Continuation criterions and estimates for the lifespan are both established. If we suppose the initial data to be in the borderline Besov spaces with infinite Lebesgue exponent and to be of finite energy, we also have a local existence result. In dimension two, the lifespan goes to infinity when the density tends to a positive constant.

Estimates for products and commutators, together with a priori estimates for the parabolic equations and for the Stokes (or Euler) system with variable coefficients, are postponed in the appendix. These estimates are based on the Littlewood-Paley theory and the paradifferential calculus.

Keywords

Mach number, Navier-Stokes system, Euler system, parabolic equations, wellposedness, continuation criterion, lifespan, strong solutions, weak solutions, Besov spaces, Littlewood-Paley theory, paradifferential calculus.

Remerciements

Je souhaite exprimer mon amitié à tous ceux qui ont rendu ce travail possible. Je voudrais en premier lieu remercier **Raphaël Danchin** pour avoir dirigé ma thèse et m'avoir soumis une série de problèmes très intéressants qui m'ont amenée à apprendre de nombreuses techniques mathématiques fort utiles. Il m'a témoignée sa confiance et m'a soutenue tout au long de cette thèse. Ses remarques, toujours simples et claires, m'ont beaucoup aidée à trouver une solution aux questions posées.

Je suis très reconnaissante à **Thierry Goudon** et **Frédéric Rousset**, qui ont bien voulu rédiger un rapport sur ma thèse. Je remercie également les autre membres du jury qui ont accepté avec gentillesse de juger ce travail : **Thomas Alazard**, **Didier Bresch** et **Ping Zhang**.

Durant toute ma thèse, j'ai bénéficié de conditions de travail très avantageuses au **Laboratoire d'Analyse et de Mathématiques Appliquées**. Je voudrais à cette occasion remercier tous les membres du Laboratoire et en particulier **Frédéric Charve**, **Anais Delgado**, **Yuxin Ge** et **Lingmin Liao**. J'aimerais aussi exprimer ma gratitude au **China Scholarship Council** pour son soutien financier pendant ces années passées en France.

Je voudrais remercier du fond du coeur mes camarades thésards de l'UPEC et d'ailleurs, qui m'ont accompagnée pendant ces années : **Antoine**, **Cecilia**, **David**, **Eduardo**, **Houda**, **Jean-Maxime**, **Johann**, **Laurent**, **Omar**, **Rémy**, **Salwa**, **Zeina**, ... Un remerciement particulier s'adresse à mon collaborateur **Francesco Fanelli**, pour sa gentillesse, les discussions partagées et son soutien et encouragements constants. Les thésards chinois en France m'ont également beaucoup aidée, aussi bien en mathématiques que pour la vie quotidienne : **Deng Wen**, **Huang Guan**, **Huang Xiangdi**, **Jiang Kai**, **Lu Yong**, **Tang Shuangshuang**, **Wei Qiaoling**, **Zhang Peng**, **Zhu Lu**, ...

Enfin, je tiens à exprimer ma profonde affection à mon petit ami **Zhao Lei**, qui m'a toujours soutenue.

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Introduction

Nous nous proposons d'étudier les équations de la dynamique des gaz à faible nombre de Mach. Ce système vient des équations de Navier-Stokes complètes lorsque le nombre de Mach tend vers 0, mais peut aussi décrire l'écoulement d'un mélange, par exemple, la propagation d'un polluant. On cherche à montrer que le problème de Cauchy associé est bien-posé, comme première étape de justification de cette asymptotique. Les cas visqueux et non visqueux sont tous deux considérés. En particulier, nous considérons des effets de conductivité thermique actifs et donc on autorise de grandes variations d'entropie. S'il n'y a pas de diffusion thermique, la limite à faible nombre de Mach implique juste la condition d'incompressibilité.

Cette introduction comporte quatre parties. Les grandes lignes de la présentation de cette thèse se trouveront dans la première section, incluant les résultats classiques, le modèle étudié, les références et le but de la thèse. Après, on détaillera la formulation du système étudié dans la section §0.2. Dans la section §0.3, on présentera la théorie classique d'analyse de Fourier, ainsi que les notations utilisées dans cette thèse. Enfin on énoncera les principaux résultats et donnera quelques explications sur leurs démonstrations, dans la section §0.4.

0.1 Présentation

0.1.1 Quelques résultats classiques

Les équations de Navier-Stokes

L'étude des solutions des équations de Navier-Stokes dépend fortement du choix des espaces fonctionnels.

Le premier résultat d'existence de *solutions faibles* des équations de Navier-Stokes *incompressibles* remonte au travail de J. Leray [77] en 1934. Environ 60 ans plus tard, P.-L. Lions [82] a étendu l'analyse aux fluides *compressibles* avec pression ne dépendant que de la densité. Au tout début de ce siècle, E. Feireisl *et al.* [48, 51] ont affaibli la restriction sur le coefficient adiabatique de sorte que le cas du gaz monoatomique a été traité. Une technique similaire lui permet d'envisager aussi le cas polytropique avec diffusion de chaleur dans [49]. Le cas de données initiales discontinues a été traité en détail dans les travaux de D. Hoff [59, 60]. En 2007, D. Bresch et B. Desjardins [16] ont également examiné le système de Navier-Stokes *polytropique*, lorsque les deux coefficients de viscosité vérifient une relation spéciale. Faisons remarquer ici que, dans tous les cas (compressible ou incompressible), même si des solutions faibles existent pour tous les temps positifs, on ne sait pas démontrer l'unicité de celles-ci en grande dimension.

Une approche différente consiste à supposer les données initiales lisses et petites. Alors le problème de Cauchy du système de Navier-Stokes a des *solutions fortes* qui sont uniques. Par example, en 1964, H. Fujita et T. Kato [53] sont arrivés à l'existence globale en temps et l'unicité de solutions des équations de Navier-Stokes incompressibles dans des espaces de Sobolev dits critiques avec des données initiales petites. Voir aussi [20, 73, 91] pour l'étude dans des espaces fonctionnels plus faibles. Pour des données initiales régulières et assez proches de l'équilibre, A. Matsumura et T. Nishida [87] ont généralisé ces résultats au cas de fluides compressibles, visqueux et conducteurs de la chaleur. Au début de ce siècle, R. Danchin [27, 28] a résolu le problème dans les espaces fonctionnels dits *critiques* qui demandent moins de régularité.

En revanche, pour des *grandes* données initiales, nous ne savons pas obtenir des solutions fortes sur un intervalle de temps infini pour l'écoulement à trois dimensions. Les premiers travaux sur l'existence locale et l'unicité du problème de Cauchy d'un fluide *général* sont dûs à J. Nash [90] et J. Serrin [97]. Nous renvoyons aussi à [64] pour des résultats similaires. Y. Cho et H. Kim [24, 25] ont étudié le cas du vide. Mais s'il n'y a pas de conductivité thermique, on ne peut pas s'attendre à des solutions classiques existant globalement en temps avec la densité initiale à support compact, à cause des résultats dans [104] par Z. Xin. Citons aussi certains résultats en dimensions un et deux [58, 69, 71, 96]. Pour l'étude des solutions définies dans un domaine borné, voir [26, 75, 100].

Dans le cas particulier de la dimension deux, le terme non-linéaire du système de Navier-Stokes *incompressible* peut être majoré par l'énergie associée à l'équation. Donc on s'attend à l'existence et l'unicité globale en temps de solutions à énergie finie. En fait, l'article [76] par J. Leray évoque pour la première fois ce type de résultat. Dans le cas de domaines bornés, J.-L. Lions et G. Prodi [80] donnent aussi le résultat d'unicité des solutions faibles. Une autre observation en dimension deux est que l'on peut également considérer l'inégalité d'énergie concernant les dérivées des solutions. Par conséquent, grâce à un argument d'unicité fort-faible, on espère que la caractère bien-posé globalement est vérifié pour les équations de Navier-Stokes *non-homogènes.* Citons R. Danchin [33] par exemple. En introduisant un nouveau triplet inconnu pour des équations de Navier-Stokes *barotropique*, A.V. Kazhikhov et V.A. Vaĭgant [71] énoncent que, si les coefficients de viscosité sont de forme spéciale, alors il existe une solution forte globale.

Les équations d'Euler

Par contre, on n'obtient que l'existence locale en temps de solutions des équations d'Euler en dimension trois. La résolution locale remonte aux travaux de L. Lichtenstein dans [79] en 1925, en adoptant le point du vue lagrangien. Les grandes lignes de la présentation des équations d'Euler ont été exposées dans [4, 43]. Sur l'espace $\mathbb{R}^n$ tout entier, par des techniques d'équations aux dérivées partielles non linéaires hyperboliques, T. Kato [67] en 1972, T. Kato et G. Ponce [68] en 1988 ont montré l'unicité et l'existence locale en temps des solutions dans des espaces de Sobolev. J.-Y. Chemin [22] a étendu l'étude aux espaces de Hölder. Citons aussi la généralisation dans l'espace de Besov dans le livre [5] et le critère d'explosion de Beale-Kato-Majda dans [8].

En dimension deux, nous renvoyons le lecteur aux travaux [76] de J. Leray et [103] de W. Wolibner en 1933 pour l'existence globale en temps de solutions régulières. Parce que le tourbillon *scalaire* en dimension deux satisfait simplement une équation de transport sans terme de force, sous la seule hypothèse du tourbillon initial borné, V. Yudovich [66] et J.-Y. Chemin [21] ont démontré l'existence globale dans un domaine borné et dans l'espace $\mathbb{R}^2$ entier respectivement. Le cas particulier où le tourbillon initial est la mesure de surface d'une courbe régulière, dit le problème des nappes de tourbillon, a aussi attiré beaucoup d'attention. Voir par exemple [38, 46, 86].

0.1.2 Limite à faible nombre de Mach

Formellement, s'il n'y a pas de grande variation de température (ou de densité), la condition d'incompressibilité apparaît et la pression tend vers une constante lorsque le nombre de Mach tend vers zero. Ce phénomène est justifié mathématiquement dans les articles [42, 62, 63, 72, 74, 89, 92, 101] pour le modèle sans viscosité, et [10, 30, 31, 40, 41, 56, 61, 83, 95] pour le système de Navier-Stokes.

Au contraire, si la diffusion thermique entre en jeu, on trouve une relation non linéaire entre $\operatorname{div} u$ et la température (voir $(0.1)_3$ ci-dessous). Nous renvoyons le lecteur aux travaux concernant ce comportement asymptotique *formel* : P.-L. Lions [81], A. Majda [84], R.Kh. Zeytounian [106], etc. Récemment, T. Alazard [2] a justifié cette asymptotique à faible nombre de Mach pour de grandes variations de température, dans des espaces de Sobolev régulières.

D'autres asymptotiques qui ne seront pas étudiées ici présentent un grand intérêt : on peut par exemple simultanément considérer le petit nombre de Mach, le grand nombre de Reynolds et le grand nombre de Péclet (i.e. la limite non visqueuse et incompressible). Dans le cadre des solutions variationnelles avec énergie finie, le modèle limite s'avère être l'équation d'Euler incompressible couplée avec une équation de transport sur la température. Voir le livre [50] de E. Feireisl et A. Novotný.

Pour d'autres systèmes limites, on peut aussi citer les références [6, 7, 88, 105].

0.1.3 But de la thèse

Dans cette thèse, on cherche à étendre les résultats classiques ci-dessus d'existence et d'unicité de solutions des équations de Navier-Stokes ou d'Euler au système de la dynamique des gaz avec nombre de Mach nul, loin du vide, qui inclut la diffusion thermique. Comme nos résultats seront principalement basés sur l'analyse de Fourier, nous nous concentrons sur des solutions définies sur l'espace $\mathbb{R}^d$, $d \geq 2$ tout entier.

Le modèle limite

Dans le cas des gaz parfaits, le modèle étudié dans cette thèse s'écrit :

$$\begin{cases} \partial_t\rho + \operatorname{div}_x(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) - \operatorname{div}_x\sigma + \nabla_x\Pi = 0, \\ \operatorname{div}_x v - \operatorname{div}_x(\alpha k \nabla\vartheta) = 0. \end{cases} \tag{0.1}$$

Les inconnues sont la densité de masse ρ, le champ de vitesse v et la pression Π. La variable de temps t est positive et la variable d'espace x appartient à $\mathbb{R}^d$ avec $d \geq 2$. On désigne par σ le tenseur des contraintes visqueuses, donné par $\sigma = 2\mu Sv + \nu \operatorname{div} v \operatorname{Id}$, où Id est la matrice identité d'ordre d, μ et ν sont les coefficients de viscosité, et $Sv := \frac{1}{2}(\nabla v + Dv)$ [1] est le tenseur des déformations. Pour un fluide sans viscosité, σ est nul partout. La fonction positive k représente le coefficient de conductivité thermique. Parce que pour des gaz parfaits, la pression $P_0 = R\rho\vartheta$ (R étant la constante des gaz parfaits) est une constante positive lorsque le nombre de Mach s'annule, la température ϑ peut être vue comme une fonction régulière de ρ. Par conséquent, on a les relations :

$$\mu = \mu(\rho),\ \nu = \nu(\rho),\ k = k(\rho) \ \text{ et } \ \rho\nabla\vartheta \equiv -\vartheta\nabla\rho.$$

La notation α désigne une constante physique $R/(C_P P_0)$, avec C_P étant la capacité thermique à pression constante.

Les deux premières équations du système (0.1) représentent les lois de conservation de masse et de la quantité de mouvement respectivement. La troisième équation vient de l'équation de conservation de l'énergie quand le nombre de Mach tend vers 0.

1. Dans toute cette thèse, on convient que pour un quelconque vecteur $v = (v^1, \cdots, v^d)$, $(Dv)_{ij} := \partial_j v^i$ et $(\nabla v)_{ij} := \partial_i v^j$. On note aussi toujours $Sv := \frac{1}{2}(\nabla v + Dv)$ et $Av := \frac{1}{2}(\nabla v - Dv)$.

S'il n'y a pas de diffusion thermique, c'est-à-dire $k \equiv 0$, le système (0.1) est juste celui des fluides inhomogènes et incompressibles. Dans ce cadre, nous renvoyons au livre [3] pour l'étude des solutions dans un domaine borné. Pour le cas de l'espace entier, citons les articles [1, 23, 33, 39, 65] en ce qui concerne les solutions fortes et [81, 98] les solutions faibles.

Vue l'équation $(0.1)_3$, il est naturel d'introduire le champ de vitesses *modifié* suivant :

$$u = v - \alpha k \nabla \vartheta = v + \kappa \nabla \ln \rho, \quad \text{avec} \quad \kappa = \kappa(\rho) = \alpha k \vartheta,$$

tel que $\operatorname{div} u = 0$. Après introduction de ce nouveau champ de vitesses à divergence nulle, le système limite se réduit au couplage non linéaire entre l'équation *parabolique* vérifiée par la densité et une équation de Stokes à densité variable :

$$\left\{\begin{array}{rcl} \partial_t \rho + \operatorname{div}(\rho v) = \partial_t \rho + \operatorname{div}(\rho u) - \operatorname{div}(\kappa \nabla \rho) & = & 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho v \otimes u) - \operatorname{div}(\mu \nabla u) + \nabla \Pi' & = & h(\rho, u), \\ \operatorname{div} u & = & 0. \end{array}\right. \tag{0.2}$$

La quantité Π' est « la nouvelle pression » inconnue. Le terme source h est compliqué mais *quadratique* :

$$h = -\operatorname{div}(\beta \kappa \nabla \ln \rho \otimes \nabla \rho) - (\beta + \mu'(\rho)) \nabla u \cdot \nabla \rho, \quad \text{avec} \quad \beta = \beta(\rho) := \kappa(\rho) - 2\mu'(\rho).$$

Ce système modifié ressemble beaucoup aux équations pour les fluides incompressibles et non-homogènes (c.-à-d. système (0.1) avec k nul), sauf que l'équation pour la densité et le terme source diffèrent.

Remarquons que si $\beta \equiv 0$, à savoir que la relation suivante entre les coefficients de viscosité μ et de conductivité k vérifie :

$$\kappa(\rho) - 2\mu'(\rho) = 0, \quad \text{c.-à-d.} \quad k(\vartheta) + 2C_P\, \vartheta\, \mu'(\vartheta) = 0,$$

alors le terme source h s'écrit $-\nabla u \cdot \nabla \mu$, qui est équivalent à $-\operatorname{div}(\mu D u)$ car $\operatorname{div} u = 0$. Ceci nous permet de récrire le système (0.2) sous une forme très simple :

$$\left\{\begin{array}{rcl} \partial_t \rho + \operatorname{div}(\rho v) = \partial_t \rho + \operatorname{div}(\rho u) - \operatorname{div}(\kappa \nabla \rho) & = & 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho v \otimes u) - \operatorname{div}(2\mu A u) + \nabla \pi & = & 0, \\ \operatorname{div} u & = & 0, \end{array}\right. \tag{0.3}$$

avec $Au := \frac{1}{2}(\nabla u - Du)$ et $\pi = \Pi'$.

Dans le cas sans viscosité, c'est-à-dire $\mu \equiv \nu \equiv 0$, le système limite (0.2) s'apparente aux équations d'Euler à densité variable (*i.e.* le système (0.1) avec σ, k nuls)

$$\left\{\begin{array}{rcl} \partial_t \rho + \operatorname{div}(\rho v) = \partial_t \rho + \operatorname{div}(\rho u) - \operatorname{div}(\kappa \nabla \rho) & = & 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho v \otimes u) + \nabla \Pi' & = & h(\rho, u), \\ \operatorname{div} u & = & 0. \end{array}\right. \tag{0.4}$$

Notons que système (0.1) est une généralisation du modèle de Kazhikhov-Smagulov, qui décrit l'écoulement d'un mélange. En fait, dans ce cas-là, les deux lois de conservation $(0.1)_1$ et $(0.1)_2$ sont satisfaites, mais la divergence de la vitesse v est liée aux dérivées de la densité ρ par la loi de Fick :

$$\operatorname{div}_x v + \operatorname{div}_x(\kappa_0 \nabla_x \ln \rho) = 0, \tag{0.5}$$

où κ_0 désigne le coefficient de diffusion qui est une constante positive. Cette loi est introduite dans [70] et décrit des flux diffusifs d'un fluide dans l'autre. Donc, en particulier, si on prend le coefficient thermique $k(\vartheta)$ égal à $\kappa_0(\alpha\vartheta)^{-1}$, alors l'équation (0.5) devient l'équation $(0.1)_3$, à cause de l'identité $\rho \nabla \vartheta = -\vartheta \nabla \rho$. Autrement dit, le modèle pour des mélanges avec le coefficient diffusif κ_0 coïncide avec celui des gaz parfaits à faible nombre de Mach et coefficient thermique k, à condition que $\alpha k \vartheta = \kappa_0$.

Quelques résultats pour le système limite

Il y a eu quelques travaux montrant le caractère bien posé du système (0.1). Dans le travail pionnier de A.V. Kazhikhov et Sh. Smagulov [70], il est considéré un modèle plus simple que système (0.1), qui décrit un mélange dans un domaine régulier avec diffusion entre les deux composants. Plus précisément, le coefficient de diffusion κ_0 dans la loi de Fick (0.5) est petit par rapport au coefficient de viscosité constant μ. De plus, le terme d'ordre κ_0^2 dans le terme source h du système (0.2) est négligé. Finalement, pour une inhomogénéité initiale $\rho_0 - 1$ dans $L^\infty \cap H^1$ et une vitesse initiale u_0 dans L^2, on obtient une solution faible globale en temps. Si de plus, u_0 appartient à H^1, cette solution globale est unique en dimension deux. Sous une hypothèse de petitesse similaire sur κ_0, P. Secchi [94] a obtenu une solution classique et globale en temps pour le système (0.1) en dimension deux, ainsi qu'une description du comportement asymptotique quand κ_0 tend vers zero.

Dans le cadre des solutions régulières définies dans un domaine, H. Beirão da Veiga *et al.* ont établit l'existence locale d'une solution unique dans le cas sans viscosité [11] (i.e. le système (0.4)) et avec viscosité [9] (i.e. le système (0.2)) respectivement. Peu après, P. Secchi [93] a étendu l'analyse dans l'espace tout entier. Citons également ici que pour un modèle à faible nombre de Mach proposé dans [84], qui prend en compte une équation de réaction-diffusion pour les différentes espèces en plus du système (0.1), P. Embid [44, 45] a montré l'existence locale de solutions régulières.

Dans le cadre des solutions faibles, à la fin de son livre [82], P.-L. Lions a indiqué qu'en dimension deux, une *petite* perturbation régulière initiale engendre une solution faible et globale pour le système (0.1), si le coefficient de conductivité thermique k est une constante positive. Pour le système limite spécial (0.3), M. Sy *et al.* [17, 99] ont prouvé l'existence globale dans un domaine régulier. Récemment, X. Cai, L. Liao et Y. Sun [18] ont démontré que cette solution est unique en dimension deux, si la condition initiale est celle de [70].

Enfin, citons également [19, 54, 55] pour l'étude numérique.

But de la thèse

Dans cette thèse, on va résoudre le problème de Cauchy associé au système (0.1) dans des espaces de Besov critiques. Notre approche reposera sur la technique de décomposition de Littlewood-Paley pour obtenir des estimations optimales pour les équations linéarisées et les termes non-linéaires.

En s'inspirant de l'analyse du système de Navier-Stokes incompressible à densité variable (i.e. le système (0.1) avec $k \equiv 0$) par R. Danchin dans [32], on résoudra le système (0.1) avec viscosité. Une difficulté supplémentaire est que dans notre cas $k = k(\rho) > 0$, on n'a pas la condition d'incompressibilité. De plus, les coefficients physiques dépendent de la densité inconnue et ceci rend les estimations plus délicates. Motivé par l'article de H. Abidi et M. Paicu [1], on peut également considérer les deux variables (la densité et la vitesse) dans des espaces de Besov avec des exposants de Lebesgue différents.

Par ailleurs, pour le système limite spécial (0.3), on obtiendra l'existence globale de solutions faibles similaires à celles de P.-L. Lions [81] pour le système de Navier-Stokes incompressible à densité variable. De plus, en s'inspirant de l'article de R. Danchin [33], grâce à un argument d'uncité fort-faible, on prouvera l'unicité de cette solution faible dans l'espace de Besov critique en dimension deux.

On étudiera également le système limite sans viscosité (0.4), en suivant l'approche de [35] par R. Danchin. Le critère de prolongement et l'estimation du temps de vie seront tout deux considérés. Comme dans l'article [36], le cas limite à exposant de Lebesgue infini sera également traité. De plus, en dimension deux, on montrera que le temps de vie tend vers l'infini lorsque la densité tend vers une constante positive.

0.2 Dérivation des équations

Dans cette section, on dérive *formellement* les équations de la dynamique des gaz à faible nombre de Mach avec viscosité (*i.e.* les systèmes (0.2) et (0.3)) dans le premier paragraphe et sans viscosité (*i.e.* le système (0.4)) dans le deuxième. Un exemple de modèle de mélange sera présenté dans la sous-section §0.2.3.

0.2.1 Le cas visqueux

Le système de Navier-Stokes complet

$$\left\{\begin{array}{rcl} \partial_t\rho + \mathrm{div}\,(\rho v) & = & 0, \\ \partial_t(\rho v) + \mathrm{div}\,(\rho v\otimes v) - \mathrm{div}\,\sigma + \nabla p & = & 0, \\ \partial_t(\rho e) + \mathrm{div}\,(\rho v e) - \mathrm{div}\,(k\nabla\vartheta) + p\,\mathrm{div}\,v & = & \sigma\cdot Dv, \end{array}\right. \tag{0.6}$$

régit l'évolution libre d'un fluide compressible, visqueux et conducteur de chaleur.

Ici, $\rho = \rho(t,x)$ désigne la densité de masse, $v = v(t,x)$, le champ de vitesses et $e = e(t,x)$, l'énergie interne par unité de masse. La variable de temps t appartient à $\mathbb{R}^+$ ou $[0,T]$ et la variable d'espace x appartient à $\mathbb{R}^d$ avec $d \geq 2$. Les fonctions scalaires $p = p(t,x)$ et $\vartheta = \vartheta(t,x)$ représentent la pression et la température respectivement. Le tenseur des contraintes visqueuses σ s'écrit

$$\sigma = 2\mu Sv + \nu\,\mathrm{div}\,v\,\mathrm{Id}\,. \tag{0.7}$$

Le coefficient de conductivité k et les coefficients de Lamé (ou viscosité) μ et ν dépendent de ρ et de ϑ régulièrement, et vérifient

$$k > 0, \quad \mu > 0, \quad \text{et} \quad \nu + 2\mu > 0. \tag{0.8}$$

Deux *équations d'état* supplémentaires mettant en jeu les états thermodynamiques p, ρ, e et ϑ doivent être ajoutées au système (0.6) (voir (0.10), (0.23) ci-dessous).

Dans cette thèse, on va étudier le système limite du système complet (0.6) quand le nombre de Mach tend vers zero. Du point de vue heuristique, cela revient à négliger la compression due aux variations de la pression, une hypothèse commune pour décrire des fluides fortement subsoniques. En suivant l'introduction du livre [81] de P.-L. Lions (voir aussi les livres [85, 106]), nous expliquons comment le système limite est dérivé *formellement.*

Définissons d'abord le nombre de Mach (sans dimension) ε comme le rapport entre la vitesse du flot et la vitesse du son dans le fluide. Supposons que (ρ, v, e) soit une certaine solution du système (0.6) avec $p = p(\rho, e)$, $\vartheta = \vartheta(\rho, e)$, dépendant du petit paramètre ε. Alors le triplet rééchelonné

$$\Big(\rho_\varepsilon(t,x) = \rho\Big(\frac{t}{\varepsilon}, x\Big), \quad v_\varepsilon(t,x) = \frac{1}{\varepsilon}v\Big(\frac{t}{\varepsilon}, x\Big), \quad e_\varepsilon(t,x) = e\Big(\frac{t}{\varepsilon}, x\Big)\Big)$$

vérifie

$$\left\{\begin{array}{rcl} \partial_t\rho_\varepsilon + \mathrm{div}\,(\rho_\varepsilon v_\varepsilon) & = & 0, \\ \partial_t(\rho_\varepsilon v_\varepsilon) + \mathrm{div}\,(\rho_\varepsilon v_\varepsilon\otimes v_\varepsilon) - \mathrm{div}\,\sigma_\varepsilon + \frac{\nabla p_\varepsilon}{\varepsilon^2} & = & 0, \\ \partial_t(\rho_\varepsilon e_\varepsilon) + \mathrm{div}\,(\rho_\varepsilon e_\varepsilon v_\varepsilon) - \mathrm{div}\,(k_\varepsilon\nabla\vartheta_\varepsilon) + p_\varepsilon \mathrm{div}\,v_\varepsilon & = & \varepsilon^2\sigma_\varepsilon\cdot Dv_\varepsilon, \end{array}\right. \tag{0.9}$$

avec $p_\varepsilon = p(\rho_\varepsilon, e_\varepsilon)$ et $\vartheta_\varepsilon = \vartheta(\rho_\varepsilon, e_\varepsilon)$. Les coefficients physiques redimensionnés s'écrivent

$$\mu_\varepsilon = \frac{1}{\varepsilon}\mu(\rho_\varepsilon, \vartheta_\varepsilon), \quad \nu_\varepsilon = \frac{1}{\varepsilon}\nu(\rho_\varepsilon, \vartheta_\varepsilon), \quad k_\varepsilon = \frac{1}{\varepsilon}k(\rho_\varepsilon, \vartheta_\varepsilon).$$

Et la notation σ_ε représente

$$\sigma_\varepsilon = 2\mu_\varepsilon S v_\varepsilon + \nu_\varepsilon \mathrm{div}\, v_\varepsilon \mathrm{Id}\,.$$

Lorsque le nombre de Mach tend vers 0, l'équation $(0.9)_2$ implique que

$$p_\varepsilon = P(t) + \Pi(t,x)\varepsilon^2 + o(\varepsilon^2).$$

Supposons que div v_ε, $\nabla\vartheta_\varepsilon$, ∇e_ε tendent vers zéro et ρ_ε tende vers une constante positive à l'infini, alors $\partial_t\rho_\varepsilon$, $\partial_t e_\varepsilon$ s'annulent à l'infini, d'après le système (0.9) ci-dessus. On va faire partout cette hypothèse dans cette thèse, en un sens à préciser. Ici, on arrive à $P'(t) = 0$ et donc $P(t)$ est indépendant de t. Cette pression constante positive sera désormais désignée par P_0.

Système limite (0.1) pour les gaz parfaits

On a déjà obtenu les deux équations limites $(0.1)_1 - (0.1)_2$ pour les inconnues $(\rho, v, \nabla\Pi)$. On se concentre maintenant sur la dérivation de l'équation limite $(0.1)_3$ de l'équation $(0.9)_3$.

Considérons le cas de gaz *parfaits*, à savoir que l'on suppose les relations d'état suivantes :

$$p = R\rho\vartheta, \quad e = C_V\vartheta, \tag{0.10}$$

où les deux constantes positives R, C_V désignent la constante des gaz parfaits et la capacité thermique à volume constant, respectivement.

Notons $\gamma = 1 + R/C_V$ le coefficient adiabatique et donc

$$\rho e = \frac{1}{\gamma - 1} p.$$

Remarquons que $p_\varepsilon \to P_0$ quand $\varepsilon \to 0$. Donc, en notant $C_P = \gamma C_V = \gamma R/(\gamma-1)$ la capacité thermique à pression constante, l'équation $(0.9)_3$ à faible nombre de Mach s'écrit $(0.1)_3$ avec la constante physique α définie par

$$\alpha = \frac{\gamma - 1}{\gamma P_0} = \frac{R}{C_P P_0}. \tag{0.11}$$

Donc, le système (0.1) est vérifié pour les gaz parfaits à nombre de Mach s'annulant.

Reformulation des systèmes (0.2) et (0.3)

On cherche à récrire le système limite (0.1), afin de transformer l'équation $(0.1)_3$ en une condition d'incompressibilité. Pour cela, il est naturel d'introduire la nouvelle vitesse à divergence nulle suivante :

$$u = v - \alpha k \nabla\vartheta = v + \kappa\nabla \ln\rho, \quad \text{avec} \quad \kappa(\rho) = \alpha k \vartheta. \tag{0.12}$$

On va alors chercher à établir les équations vérifiées par les inconnues ρ, u, du système (0.1). En fait, nous démontrerons que le système (0.2) (ou le système (0.3)) est vérifié.

Premièrement, l'équation $(0.1)_1$ donne une équation de transport-diffusion sur la densité ρ, à savoir équation $(0.2)_1$, car

$$\mathrm{div}\,(\rho v) = \mathrm{div}\,(\rho u) + \mathrm{div}\,(\alpha k\rho\nabla\vartheta) = \mathrm{div}\,(\rho u) - \mathrm{div}\,(\kappa\nabla\rho).$$

Ensuite, on établit l'équation $(0.2)_2$ et pour ce faire, on écrit les termes avec des dérivées d'ordre élevé comme somme d'un gradient et d'une autre quantité avec des dérivées d'ordre plus bas, c'est-à-dire comme somme d'une « pression » et d'un « terme source ».

Observant que $\rho v = \rho u - \kappa\nabla\rho$ et que

$$\partial_t(\kappa\nabla\rho) = \nabla(\kappa\partial_t\rho) \quad \text{et} \quad \mathrm{div}\,(\nu\,\mathrm{div}\,v\,\mathrm{Id}\,) = \nabla(\nu\,\mathrm{div}\,v),$$

on récrit $(0.1)_2$ comme

$$\partial_t(\rho u) + \mathrm{div}\,(\rho v \otimes u) - \Big(\mathrm{div}\,(v \otimes \kappa\nabla\rho) + \mathrm{div}\,(2\mu S v)\Big) + \nabla Q_1 = 0, \tag{0.13}$$

avec $Q_1 = \Pi - \kappa\partial_t\rho - \nu\mathrm{div}\,v$.

Remarquons l'identité suivante :

$$Sv = Au + Dv.$$

Cela implique que

$$\mathrm{div}\,(v \otimes \kappa\nabla\rho) + \mathrm{div}\,(2\mu S v) = \mathrm{div}\,\big(\beta v \otimes \nabla\rho\big) + \mathrm{div}\,(2\mu A u) + \nabla\mathrm{div}\,(2\mu\, v),$$

avec la fonction scalaire $\beta(\rho) = \kappa - 2\mu'(\rho)$.

- Si $\beta \equiv 0$, c'est-à-dire si la relation suivante est vérifiée :

  $$\kappa(\rho) - 2\mu'(\rho) \equiv 0, \tag{0.14}$$

  ou encore,

  $$k(\vartheta) + 2C_P\vartheta\mu'(\vartheta) \equiv 0, \tag{0.15}$$

  alors (0.13) donne l'équation $(0.3)_2$ avec une nouvelle pression

  $$\pi = Q_1 - \mathrm{div}\,(2\mu\, v) = \Pi - \kappa\partial_t\rho - \nu\mathrm{div}\,v - \mathrm{div}\,(2\mu\, v).$$

- Ensuite, en utilisant le fait que $\mathrm{div}\,u = 0$, on arrive à (désignant par B une primitive de β)

  $$\begin{aligned} \mathrm{div}\,(\beta v \otimes \nabla\rho) &= \mathrm{div}\,(\beta u \otimes \nabla\rho) - \mathrm{div}\,(\beta\kappa\nabla\ln\rho \otimes \nabla\rho), \\ &= -\beta\nabla u \cdot \nabla\rho + \nabla\mathrm{div}\,(B(\rho)u) - \mathrm{div}\,(\beta\kappa\nabla\ln\rho \otimes \nabla\rho). \end{aligned}$$

  De plus, on peut aussi récrire $\mathrm{div}\,(2\mu A u)$ comme

  $$\mathrm{div}\,(2\mu A u) = \mathrm{div}\,(\mu\nabla u) - \mu'(\rho)\nabla u \cdot \nabla\rho.$$

  Donc, en vertu de (0.13), l'équation $(0.2)_2$ sort immédiatement, avec une nouvelle pression Π' et un terme source h donnés par

  $$\Pi' = \pi - \mathrm{div}\,(B(\rho)u), \quad h = -\mathrm{div}\,(\beta\kappa\nabla\ln\rho \otimes \nabla\rho) - (\beta + \mu'(\rho))\nabla u \cdot \nabla\rho.$$

Donc, on a démontré que le système (0.1) est équivalent au système (0.2), ou également au système (0.3) sous la condition (0.14).

En fait, il est aisé de vérifier que si le triplet $(\rho, u, \nabla\pi)$ vérifie (0.3) au sens des distributions avec de plus (pour trois constantes positives $\underline{\rho}, \overline{\rho}, c_0$) :

$$\begin{aligned} &\rho(t,x) \in [\underline{\rho}, \overline{\rho}],\ \forall t \in [0,+\infty),\ x \in \mathbb{R}^d, \\ &\rho - c_0 \in L^\infty\Big([0,+\infty); L^2(\mathbb{R}^d)\Big) \cap L^2_{\mathrm{loc}}\Big([0,+\infty); H^1(\mathbb{R}^d)\Big), \\ &\partial_t\rho \in L^2_{\mathrm{loc}}\Big([0,+\infty); H^{-1}(\mathbb{R}^d)\Big), \\ &u,\ v := u - \kappa\nabla\ln\rho \in L^2_{\mathrm{loc}}\Big([0,+\infty); (L^2(\mathbb{R}^d))^d\Big), \end{aligned} \tag{0.16}$$

alors le calcul du système (0.1) à partir du système (0.3) sous la condition (0.14) a un sens. Mais la justification de la continuité à l'instant initial dans le cadre des solutions faibles est délicate.

Si on résout le système (0.2) dans le cadre des solutions fortes, e.g. dans l'espace de Besov critique (voir Chapitre §1), alors l'équivalence entre les deux systèmes (0.1) et (0.2) s'obtient facilement, après vérification minutieuse.

Formulation du système par rapport à la température

Parce que le fait que le nombre de Mach s'annule implique que la pression p est une constante positive P_0, les états ρ, e et les coefficients physiques μ, ν, k peuvent être vus comme des fonctions de la température ϑ seulement. Plus précisément, quand on considère des solutions régulières du système (0.1) pour des gaz parfaits, grâce à

$$v = u + \alpha k \nabla \vartheta \quad \text{et} \quad \alpha C_P \rho \vartheta = 1,$$

on peut récrire $(0.1)_1$ comme une équation sur la température ϑ :

$$\partial_t \vartheta + u \cdot \nabla \vartheta - \mathrm{div}\,(\kappa \nabla \vartheta) = h_1(\vartheta),$$

avec le terme source h_1 défini par

$$h_1(\vartheta) = -2\alpha k |\nabla \vartheta|^2. \tag{0.17}$$

On peut aussi récrire la loi de conservation $(0.1)_2$ ainsi :

$$\partial_t u + u \cdot \nabla u - \mathrm{div}\,(\eta \nabla u) + \vartheta \nabla Q = h_2(\vartheta, u),$$

avec un nouveau coefficient de viscosité

$$\eta(\vartheta) = \alpha \mu \vartheta C_P, \tag{0.18}$$

et une nouvelle pression Q. Le nouveau terme source h_2 est donné par

$$h_2(\vartheta, u) = A_1 |\nabla \vartheta|^2 \nabla \vartheta + A_2 \Delta \vartheta \nabla \vartheta + A_3 \nabla \vartheta \cdot \nabla^2 \vartheta + A_4 \nabla u \cdot \nabla \vartheta + A_5 Du \cdot \nabla \vartheta, \tag{0.19}$$

avec

$$\begin{gathered} A_1 = -\rho^{-1}\Big(\alpha k \rho(\alpha k + 2\mu' \rho^{-1})\Big)'(\vartheta), \quad A_2 = A_3 = -\alpha k(\alpha k + 2\mu'\rho^{-1})(\vartheta), \\ A_4 = (\alpha k + \mu' \rho^{-1})(\vartheta), \quad A_5 = -\alpha k - \mu(\partial \rho^{-1}/\partial \vartheta) = -\alpha k - \alpha C_P \mu. \end{gathered} \tag{0.20}$$

En effet, grâce à l'équation $(0.1)_1$ sur la densité, on écrit le terme de convection comme

$$\partial_t(\rho v) + \mathrm{div}\,(\rho v \otimes v) = \rho \partial_t u + \rho v \cdot \nabla u + \nabla(\kappa \partial_t \rho) + \mathrm{div}\,(\rho v \otimes \alpha k \nabla \vartheta),$$

où $\mathrm{div}\,(\rho v \otimes \alpha k \nabla \vartheta)$ s'écrit pour une fonction Q_2 dépendant de u et ϑ :

$$\mathrm{div}\,(u \otimes \alpha k \rho \nabla \vartheta) + \mathrm{div}\,(\alpha^2 k^2 \rho \nabla \vartheta \otimes \nabla \vartheta) = \nabla Q_2 - \nabla u \cdot \alpha k \rho \nabla \vartheta + \mathrm{div}\,(\alpha^2 k^2 \rho \nabla \vartheta \otimes \nabla \vartheta).$$

De plus, pour une certaine fonction Q_3 dépendant de ϑ, on récrit le terme de diffusion comme

$$\begin{aligned} -\mathrm{div}\,(\mu S v) &= -\mathrm{div}\,(\mu \nabla u) - \mathrm{div}\,(\mu D u) - 2\mathrm{div}\,(\mu \nabla(\alpha k \nabla \vartheta)) \\ &= -\mathrm{div}\,(\mu \nabla u) - \nabla u \cdot \nabla \mu - \nabla Q_3 + 2\mathrm{div}\,(\alpha k \nabla \vartheta \otimes \nabla \mu). \end{aligned}$$

Donc, l'équation $(0.1)_2$ devient

$$\begin{aligned} &\rho(\partial_t u + u \cdot \nabla u) - \mathrm{div}\,(\mu \nabla u) + \nabla\Big(\Pi + \kappa \partial_t \rho - \nu \mathrm{div}\, v + Q_2 - Q_3\Big) \\ &\quad + \rho\Big(Du \cdot \alpha k \nabla \vartheta - \nabla u \cdot \alpha k \nabla \vartheta - \rho^{-1} \nabla u \cdot \nabla \mu\Big) + \mathrm{div}\,\Big((\alpha^2 k^2 \rho + 2\alpha k \mu'(\vartheta)) \nabla \vartheta \otimes \nabla \vartheta\Big) = 0. \end{aligned} \tag{0.21}$$

On introduit une nouvelle pression $Q = \Pi + \kappa \partial_t \rho - \nu \mathrm{div}\, v + Q_2 - Q_3$ et multiplie l'équation ci-dessus par $\rho^{-1} = \alpha C_P \vartheta$. Finalement on obtient le système suivant pour les inconnues $(\vartheta, u, \nabla Q)$:

$$\left\{ \begin{array}{ccc} \partial_t \vartheta + u \cdot \nabla \vartheta - \mathrm{div}\,(\kappa \nabla \vartheta) & = & h_1(\vartheta), \\ \partial_t u + u \cdot \nabla u - \mathrm{div}\,(\eta \nabla u) + \vartheta \nabla Q & = & h_2(\vartheta, u), \\ \mathrm{div}\, u & = & 0. \end{array} \right. \tag{0.22}$$

La résolution de ce dernier système fera l'objet du premier chapitre de cette thèse.

Le système limite (0.33) **pour les gaz généraux**

Pour des *gaz généraux*, on suppose les relations d'état suivantes à la place de (0.10) :

$$\rho = \rho(p, \vartheta), \quad e = e(p, \vartheta), \tag{0.23}$$

avec ρ et e deux fonctions scalaires régulières.

D'après la loi de Gibbs, il existe une variable s s'appelant l'entropie, qui vérifie

$$ds = \frac{1}{\vartheta}\Big(de \,+\, p\, d\Big(\frac{1}{\rho}\Big)\Big). \tag{0.24}$$

La relation de compatibilité $\frac{\partial}{\partial p}(\frac{\partial s}{\partial \vartheta}) = \frac{\partial}{\partial \vartheta}(\frac{\partial s}{\partial p})$ implique immédiatement :

$$p\frac{\partial \rho}{\partial p} + \vartheta\frac{\partial \rho}{\partial \vartheta} \;=\; \rho^2\frac{\partial e}{\partial p}. \tag{0.25}$$

De plus, on suppose les inégalités strictes

$$\frac{\partial \rho}{\partial p} > 0, \quad \frac{\partial \rho}{\partial \vartheta} < 0 \;\text{ et }\; \frac{\partial e}{\partial \vartheta}\frac{\partial \rho}{\partial p} > \frac{\partial e}{\partial p}\frac{\partial \rho}{\partial \vartheta}, \tag{0.26}$$

afin que les coefficients de compressibilité isotherme K_T, d'expansion thermique K_P et de capacité C_V suivants :

$$K_T := \frac{1}{\rho}\frac{\partial \rho}{\partial p}, \quad K_P := -\frac{1}{\rho}\frac{\partial \rho}{\partial \vartheta}, \quad C_V := \vartheta\frac{(\partial s/\partial \vartheta)(\partial \rho/\partial p) \,-\, (\partial s/\partial p)(\partial s/\partial \vartheta)}{\partial \rho/\partial p},$$

soient positifs (voir [2] pour plus de détails). Clairement, l'équation d'état (0.10) pour les gaz parfaits satisfait les deux conditions (0.25) et (0.26).

Dans la suite, on va chercher à dériver le système limite sur les inconnues $(\vartheta, u, \nabla Q)$ pour les gaz généraux, d'après le système rééchelonné (0.9) et les équations d'état (0.23)-(0.25)-(0.26). Pour simplifier, on va utiliser les mêmes notations que pour les gaz parfaits (e.g. α, η, h_1, h_2, Q) pour désigner des quantités correspondantes pour les gaz généraux. On vérifie aisément que pour les équations d'état (0.10), les valeurs de ces quantités définies ci-dessous correspondent bien à celles des gaz parfaits.

Comme la pression p devient constante lorsque le nombre de Mach s'annule, les états ρ, e et les coefficients physiques μ, ν, k sont fonctions de la température ϑ seulement. L'équation limite de $(0.9)_1$ peut donc s'écrire comme une équation dont l'inconnue est ϑ :

$$\partial_t \vartheta + v\cdot\nabla\vartheta \,+\, \rho\Big(\frac{\partial \rho}{\partial \vartheta}\Big)^{-1}\mathrm{div}\, v = 0. \tag{0.27}$$

Et en vertu de l'équation ci-dessus, l'équation limite de $(0.9)_3$ se réduit à

$$\mathrm{div}\, v - \alpha(\vartheta)\mathrm{div}\,(k\nabla\vartheta) = 0, \tag{0.28}$$

avec le coefficient α dépendant de ϑ tel que :

$$\alpha^{-1}(\vartheta) = P_0 - \rho^2\frac{\partial e}{\partial \vartheta}\Big(\frac{\partial \rho}{\partial \vartheta}\Big)^{-1} \quad \text{avec} \quad P_0 \quad \text{constante positive.} \tag{0.29}$$

En raison des deux conditions (0.25) et (0.26), on a $\alpha^{-1}(\partial\rho/\partial p) > 0$ et donc $\alpha > 0$.

Comme dans le cas des gaz parfaits, en vertu de l'équation (0.28), on introduit un nouveau champ de vitesses

$$u = v - \alpha(\vartheta)k(\vartheta)\nabla\vartheta,$$

tel que

$$\operatorname{div} u = -\alpha'(\vartheta)k(\vartheta)|\nabla\vartheta|^2. \tag{0.30}$$

Alors l'équation limite (0.27) sur la température ϑ se récrit

$$\partial_t\vartheta + u\cdot\nabla\vartheta - \operatorname{div}(\kappa\nabla\vartheta) = h_1(\vartheta),$$

avec

$$\kappa(\vartheta) = -\rho(\vartheta)\Big(\frac{\partial\rho}{\partial\vartheta}\Big)^{-1}\alpha(\vartheta)k(\vartheta) > 0, \quad h_1(\vartheta) = -\alpha k|\nabla\vartheta|^2 + k\nabla\Big(\alpha\rho\Big(\frac{\partial\rho}{\partial\vartheta}\Big)^{-1}\Big)\cdot\nabla\vartheta. \tag{0.31}$$

En suivant les calculs qui donnent l'équation limite (0.21), l'équation limite $(0.1)_2$ devient

$$\rho(\partial_t u + u\cdot\nabla u) - \operatorname{div}(\mu\nabla u) + \nabla\Big(\Pi + \kappa\partial_t\rho - \nu\operatorname{div} v + Q_2 - Q_3\Big) + (\operatorname{div} u)(\alpha k\rho\nabla\vartheta + \nabla\mu)$$
$$+ \rho\Big(Du\cdot\alpha k\nabla\vartheta - \nabla u\cdot\alpha k\nabla\vartheta - \rho^{-1}\nabla u\cdot\nabla\mu\Big) + \operatorname{div}\Big((\alpha^2k^2\rho + 2\alpha k\mu'(\vartheta))\nabla\vartheta\otimes\nabla\vartheta\Big) = 0.$$

En fait, parce que $\operatorname{div} u \neq 0$, on récrit le terme de convection $\operatorname{div}(u\otimes\alpha k\rho\nabla\vartheta)$ comme

$$\nabla Q_2 + (\operatorname{div} u)(\alpha k\rho\nabla\vartheta) - \nabla u\cdot\alpha k\rho\nabla\vartheta,$$

et le terme de diffusion $-\operatorname{div}(\mu Du)$ comme

$$-Q_3 + (\operatorname{div} u)\nabla\mu - \nabla u\cdot\nabla\mu.$$

Par conséquent, l'équation (0.21) donne $(0.1)_2$ avec le terme supplémentaire $(\operatorname{div} u)(\alpha k\rho\nabla\vartheta + \nabla\mu)$ dans le membre de gauche. Finalement, en vertu de (0.30), l'équation limite $(0.2)_2$ sur la vitesse u se récrit

$$\partial_t u + u\cdot\nabla u - \operatorname{div}(\eta\nabla u) + \rho^{-1}\nabla Q = h_2,$$

avec $\eta = \mu\rho^{-1}$. Le terme source h_2 est également défini par (0.19), avec

$$A_1 = -\rho^{-1}\Big(\alpha k\rho(\alpha k + 2\mu'\rho^{-1})\Big)'(\vartheta) + \alpha' k(\alpha k + \mu'\rho^{-1})(\vartheta) \tag{0.32}$$

et les mêmes A_i, $i = 2, \cdots, 5$ que dans (0.20).

Finalement, le système limite sur les inconnues $(\vartheta, u, \nabla Q)$ pour un gaz général s'écrit donc

$$\begin{cases} \partial_t\vartheta + u\cdot\nabla\vartheta - \operatorname{div}(\kappa\nabla\vartheta) & = & h_1(\vartheta), \\ \partial_t u + u\cdot\nabla u - \operatorname{div}(\eta\nabla u) + \vartheta\nabla Q & = & h_2(\vartheta, u), \\ \operatorname{div} u & = & -\alpha' k|\nabla\vartheta|^2. \end{cases} \tag{0.33}$$

Ce système sera étudié dans le premier chapitre.

0.2.2 Le cas non visqueux

L'évolution libre d'un fluide compressible, conducteur de chaleur mais non visqueux obéit aux équations suivantes :

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) & = & 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v\otimes v) + \nabla p & = & 0, \\ \partial_t(\rho e) + \operatorname{div}(\rho v e) - \operatorname{div}(k\nabla\vartheta) + p\operatorname{div} v & = & 0. \end{cases} \tag{0.34}$$

Il s'agit donc du système (0.6) avec tenseur de viscosité σ nul. Les inconnues ρ, u, e, p, ϑ et les variables t, x sont identiques à celles du cas visqueux.

On adopte également les équations d'état (0.10) des gaz parfaits reliant les quatre variables thermodynamiques ρ, e, p, ϑ. Le coefficient thermique k est une fonction $\mathcal{C}^\infty$ de ρ et ϑ.

Formellement, en considérant le système redimensionné (0.9) avec $\sigma_\varepsilon \equiv 0$, on obtient les équations à faible nombre de Mach sans viscosité suivantes :

$$\left\{\begin{array}{rcl} \partial_t\rho + \mathrm{div}\,(\rho v) & = & 0, \\ \partial_t(\rho v) + \mathrm{div}\,(\rho v \otimes v) + \nabla\Pi & = & 0, \\ \mathrm{div}\, v - \mathrm{div}\,(\alpha k \nabla\vartheta) & = & 0, \end{array}\right. \tag{0.35}$$

où $\Pi = \Pi(t, x)$ est une fonction inconnue, α est une constante physique définie par (0.11) et $\vartheta = (P_0/R)\rho^{-1}$.

Comme dans le dernier paragraphe (§0.2.1), on peut introduire la nouvelle vitesse $u := v - \alpha k \nabla\vartheta$ donnée par (0.12) à divergence nulle. Pour cette vitesse modifiée, l'équation sur la densité est

$$\partial_t\rho + u \cdot \nabla\rho - \mathrm{div}\,(\kappa\nabla\rho) = 0 \quad \text{avec} \quad \kappa(\rho) := \alpha k \vartheta.$$

Pour simplifier, on définit les primitives $a(\rho)$ et $b(\rho)$ de $\kappa(\rho)$ et $-\kappa(\rho)\rho^{-1}$, telles que $a(1) = b(1) = 0$. Il est aisé de vérifier que

$$\nabla a \;=\; \kappa\nabla\rho \;=\; -\rho\nabla b \;\text{ et }\; u = v - \nabla b = v + \rho^{-1}\nabla a. \tag{0.36}$$

De plus, l'équation $(0.35)_2$ s'écrit

$$\partial_t(\rho u) - \partial_t\nabla a + \mathrm{div}\,(\rho v \otimes u) - \mathrm{div}\,(v \otimes \nabla a) + \nabla\Pi = 0.$$

Grâce à l'équation $(0.35)_1$, l'équation ci-dessus devient

$$\rho(\partial_t u + v \cdot \nabla u) + \nabla q = \mathrm{div}\,(v \otimes \nabla a),$$

où $q = \Pi - \partial_t a$.

Par conséquent, le système (0.35) est équivalent au système suivant sur les inconnues ρ, u, q :

$$\left\{\begin{array}{c} \partial_t\rho + u \cdot \nabla\rho - \mathrm{div}\,(\kappa\nabla\rho) = 0, \\ \partial_t u + (u + \nabla b) \cdot \nabla u + \lambda\nabla q = h_3(\rho, u), \\ \mathrm{div}\, u = 0, \end{array}\right. \tag{0.37}$$

avec le coefficient variable $\lambda = \lambda(\rho) = \rho^{-1}$ et le terme source

$$h_3 \;=\; \lambda \mathrm{div}\,(v \otimes \nabla a) \;=\; -u \cdot \nabla^2 b - (u \cdot \nabla\lambda)\nabla a - (\nabla b \cdot \nabla\lambda)\nabla a - \mathrm{div}\,(\nabla b \otimes \nabla b)\,. \tag{0.38}$$

Nous étudierons ce système au chapitre §3.

Il est aisé d'écrire le système limite sans viscosité pour les gaz généraux comme le modèle (0.33) ci-dessus, avec coefficient de viscosité $\mu \equiv 0$. Les détails sont laissés au lecteur.

0.2.3 Un modèle de mélange à deux composants

Dans ce paragraphe, d'après [70], on dérive le système (0.1) pour un mélange de deux composants incompressibles. On suppose que le mélange se compose d'une phase diluée interagissant avec une phase dense, et l'on va adopter une description moyenne de l'écoulement. Ces deux fluides sont caractérisés par leurs densités constantes de référence : on note $\overline{\rho}_f$ la densité de la phase dense et $\overline{\rho}_d$ la densité de la phase diluée respectivement. Les deux viscosités cinématiques constantes correspondantes sont $\overline{\mu}_f$ et $\overline{\mu}_d$. Deux champs de vecteurs $v_f(t, x)$ et $v_d(t, x)$ désignent les champs de vitesses correspondants.

On définit $\phi(t,x) \in [0,1]$ comme la fraction volumique de la phase diluée :

$$\phi(t,x) = \lim_{r\to 0} \frac{\text{le volume occupé par la phase diluée dans la boule } B(x,r) \text{ au moment } t}{\text{le volume de la boule } B(x,r)}.$$

Supposons que les deux phases sont incompressibles et que les densités associées restent constantes. La densité du mélange est alors définie par

$$\rho(t,x) = (1-\phi(t,x))\,\overline{\rho}_f \,+ \phi(t,x)\,\overline{\rho}_d.$$

Notons $\rho_f(t,x) = \overline{\rho}_f(1-\phi)$ et $\rho_d(t,x) = \overline{\rho}_d\phi$, alors les lois de conservation de masse pour les deux phases s'écrivent

$$\partial_t\rho_f + \mathrm{div}\,_x(\rho_f v_f) = 0, \quad \partial_t\rho_d + \mathrm{div}\,_x(\rho_d v_d) = 0.$$

Donc la loi de conservation de masse du mélange s'écrit

$$\partial_t\rho + \mathrm{div}\,_x(\rho v) = 0, \quad \text{avec} \quad (\rho v)(t,x) := \big(\rho_f v_f + \rho_d v_d\big)(t,x). \tag{0.39}$$

De même, la viscosité dynamique μ s'écrit

$$\mu = (1-\phi(t,x))\overline{\rho}_f\overline{\mu}_f \,+\, \phi(t,x)\overline{\rho}_d\overline{\mu}_d \equiv \frac{\overline{\rho}_f\overline{\rho}_d}{\overline{\rho}_d - \overline{\rho}_f}(\overline{\mu}_f - \overline{\mu}_d) \,+\, \Big(\frac{\overline{\rho}_d\overline{\mu}_d - \overline{\rho}_f\overline{\mu}_f}{\overline{\rho}_d - \overline{\rho}_f}\Big)\,\rho.$$

Alors on écrit la loi de conservation de la quantité du mouvement ρv :

$$\partial_t(\rho v) + \mathrm{div}\,(\rho v \otimes v) - \mathrm{div}\,(2\mu S v) + \nabla P = 0, \tag{0.40}$$

avec une pression $P(t,x)$ inconnue.

Notons que même si les deux composants sont incompressibles, la vitesse à masse moyenne v n'est pas nécessairement à divergence nulle. Définissons la vitesse à volume moyen

$$u(t,x) := (1-\phi(t,x))v_f(t,x) + \phi(t,x)v_d(t,x).$$

Alors, le champ de vitesse u est solénoïdal et l'on a

$$0 \equiv \partial_t\Big((1-\phi)+\phi\Big) = \partial_t\Big(\frac{\rho_f}{\overline{\rho}_f} + \frac{\rho_d}{\overline{\rho}_d}\Big) = -\mathrm{div}\,\Big(\frac{\rho_f v_f}{\overline{\rho}_f} + \frac{\rho_d v_d}{\overline{\rho}_d}\Big) \equiv -\mathrm{div}\,u.$$

Selon Kazhikhov-Smagulov [70], u et v sont reliés par

$$v = u - \kappa_0\nabla\ln\rho \quad \text{avec} \quad \kappa_0 > 0.$$

Cette loi de Fick décrit les flux diffusifs d'un fluide dans l'autre.

Donc finalement on arrive au système (0.1) avec $\alpha k\nabla\vartheta = \kappa_0\nabla\ln\rho$.

0.3 Le cadre fonctionnel

Dans cette section on va définir les espaces de Besov et présenter les techniques utilisées dans la thèse. Les estimations de produit, de commutateur et de composition seront essentielles pour l'obtention d'estimations a priori des solutions des équations linéarisées. Sauf mention contraire, les résultats de cette section sont classiques et on peut trouver leurs démonstrations dans le livre [5]. Dans l'appendice on va démontrer des résultats plus généraux.

Avant tout, on fixe quelques notations utilisées dans cette thèse :

(i) C désigne une constante inoffensive. Nous allons utiliser alternativement la notation $A \lesssim B$ au lieu de $A \leq CB$ de temps en temps. De plus, $A \approx B$ signifie que $A \lesssim B$ et $B \lesssim A$.

(ii) Soit X, Y deux espaces de Banach, la notation $X \hookrightarrow Y$ signifie que l'espace X est continûment inclus dans l'espace Y tandis que $X \hookrightarrow\hookrightarrow Y$ veut dire que l'inclusion de X dans Y est compacte.

(iii) Soit X un espace de Banach, $T > 0$ et $p \in [1, +\infty]$, $L^p_T(X)$ désigne l'ensemble des fonctions f mesurables au sens de Lebesgue, de $[0, T)$ dans X telles que $t \mapsto \|f(t)\|_X$ appartient à $L^p([0, T))$. Si $T = +\infty$, alors l'espace est désigné simplement par $L^p(X)$. Finalement, si I est un intervalle de $\mathbb{R}$ alors la notation $\mathcal{C}(I; X)$ décrit l'ensemble des fonctions continues de I à X. De même, $\mathcal{C}_w(I; X)$ décrit l'ensemble des fonctions continues pour la topologie faible de X.

(iv) On utilisera la même notation X pour désigner les champs de vecteurs avec des composantes dans X.

(v) Les indices p_1, p_2, p, r_1, r_2, r prennent des valeurs dans l'intervalle $[1, +\infty]$. Et l'indice p' représente l'exposant conjugué de p tel que $\frac{1}{p} + \frac{1}{p'} = 1$.

(vi) Soit p un réel et f une fonction de L^p, on note $\|D^k f\|_{L^p} = \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^p}$.

(vii) Les fonctions sont définies sur l'espace entier $\mathbb{R}^d$, $d \geq 2$ étant la dimension.

(viii) La densité ρ et la température ϑ sont toujours positives (par l'hypothèse initiale et la nature de l'équation parabolique). Sauf indication contraire, on suppose que ces deux quantités tendent vers 1 à l'infini et on note « la densité modifiée » par $\varrho = \rho - 1$ et « la température modifiée » par $\theta = \vartheta - 1$.

(ix) Sauf indication contraire, pour les inconnues f, les fonctions de la forme f_0 désignent toujours les données initiales. Et pour les coefficients g, les nombres réels positifs de la forme $\bar{g}$ désignent la valeur de g en 1, à savoir $g(1)$; et les fonctions scalaires de la forme g^δ, sont juste $g(\rho^\delta)$.

0.3.1 Espaces de Besov

Tout d'abord, on rappelle brièvement la définition de la décomposition de Littlewood-Paley : une partition dyadique de l'unité par rapport à la variable de Fourier, désignée par ξ dans toute cette thèse. Plus précisément, soit $\chi(\xi)$ une fonction régulière, radiale, décroissante, à support compact dans la boule $B(0, \frac{4}{3})$ et valant 1 près de la boule $B(0, 1)$, définissons

$$\varphi(\xi) = \chi(\xi/2) - \chi(\xi), \quad \varphi_j(\xi) = \varphi(2^{-j}\xi), \forall j \in \mathbb{Z}.$$

Alors on a

$$\chi(\xi) + \sum_{j \geq 0} \varphi_j(\xi) \equiv 1 \text{ pour tout } \xi \in \mathbb{R}^d, \text{ et } \sum_{j \in \mathbb{Z}} \varphi_j(\xi) \equiv 1 \text{ pour } \xi \in \mathbb{R}^d \backslash \{0\}.$$

Posons $h_j = \mathcal{F}^{-1}\varphi_j$, et $\check{h} = \mathcal{F}^{-1}\chi$. Les *blocs dyadiques inhomogènes* $(\Delta_j)_{j \in \mathbb{Z}}$ sont définis par

$$\begin{aligned}
&\Delta_j u = 0 \quad \text{si} \quad j \leq -2,\\
&\Delta_{-1} u = \chi(D)u = \int_{\mathbb{R}^d} \check{h}(y) u(x-y)\, dy,\\
&\Delta_j u = \varphi_j(D)u = \int_{\mathbb{R}^d} h_j(y) u(x-y)\, dy \quad \text{si} \quad j \geq 0,
\end{aligned}$$

et on introduit la troncature à basse fréquence :

$$S_j u = \sum_{k \le j-1} \Delta_k u.$$

Notons que $S_j u = \chi(2^{-j}D)u$ si $j \geq 0$. Il est important de souligner que la décomposition dyadique ci-dessus vérifie

$$\Delta_k \Delta_j u \equiv 0 \quad \text{si} \quad |k-j| \geq 2 \quad \text{et} \quad \Delta_k(S_{j-1}u\Delta_j u) \equiv 0 \quad \text{si} \quad |k-j| \geq 5.$$

De plus, pour toute distribution tempérée u, on peut écrire

$$u = \sum_{j \in \mathbb{Z}} \Delta_j u.$$

L'intérêt de décomposer l'espace de Fourier en couronnes de taille 2^j réside dans le comportement vis-à-vis de la dérivation des fonctions dont la transformée de Fourier est à support compact. Plus précisément, on a les inégalités de Bernstein suivantes (voir e.g. [5], Chap. 2),

$$\begin{aligned}
&\|\Delta_j u\|_{L^{p_1}(\mathbb{R}^d)} \lesssim 2^{d(\frac{1}{p_2}-\frac{1}{p_1})}\|\Delta_j u\|_{L^{p_2}(\mathbb{R}^d)} \quad \text{si } p_1 \geq p_2,\\
&\|D^k(\Delta_j u)\|_{L^p(\mathbb{R}^d)} \lesssim 2^{kj}\|\Delta_j u\|_{L^p(\mathbb{R}^d)}, \quad \forall j \geq -1,\\
&\|D^k(\Delta_j u)\|_{L^p(\mathbb{R}^d)} \approx 2^{kj}\|\Delta_j u\|_{L^p(\mathbb{R}^d)}, \quad \forall j \geq 0.
\end{aligned}$$

De plus, on a l'inégalité de Bernstein suivante, concernant à la fois le bloc dyadique lui-même et sa dérivée (voir Appendice B dans [35]) :

$$\int |\Delta_j u|^{p-2}|D(\Delta_j u)|^2 \gtrsim 2^{2j}\int |\Delta_j u|^p\,dx \quad \text{pour} \;\; j \geq 0, \quad p \in (1,\infty). \tag{0.41}$$

Rappelons aussi l'action du semi-groupe de l'équation de la chaleur :

$$\|e^{t\Delta}\Delta_j u\|_{L^p} \lesssim e^{-Ct2^{2j}}\|\Delta_j u\|_{L^p} \quad \text{pour} \;\; j \geq 0, \quad p \in [1,+\infty]. \tag{0.42}$$

Nous pouvons maintenant définir l'espace de Besov $B^s_{p,r}$ comme suit :

Definition 0.1. *Soit $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$, et $u \in \mathcal{S}'(\mathbb{R}^d)$, on pose*

$$\|u\|_{B^s_{p,r}} = \left(\sum_{j \geq -1} 2^{jsr}\|\Delta_j u\|^r_{L^p}\right)^{1/r} \quad \text{si} \;\; r < \infty, \quad \text{et} \quad \|u\|_{B^s_{p,\infty}} := \sup_{j \geq -1}\left\{2^{js}\|\Delta_j u\|_{L^p}\right\}.$$

On définit alors

$$B^s_{p,r} = B^s_{p,r}(\mathbb{R}^d) := \left\{u \in \mathcal{S}'(\mathbb{R}^d), \|u\|_{B^s_{p,r}} < \infty\right\}.$$

Dans la suite, on utilisera souvent les propriétés classiques suivantes des espaces de Besov.

Proposition 0.1. *Les propriétés suivantes sont vérifiées :*

(i) Action des dérivées : $\|\nabla u\|_{B^{s-1}_{p,r}} \lesssim \|u\|_{B^s_{p,r}}$.

(ii) Inclusion : $B^{s_1}_{p_1,r_1} \hookrightarrow B^{s_2}_{p_2,r_2}$ *si* $s_1 - \frac{d}{p_1} > s_2 - \frac{d}{p_2}$ *et* $p_1 \leq p_2$, *ou si* $s_1 = s_2$, $p_1 = p_2$ *et* $r_1 \leq r_2$. *En particulier, l'espace de Sobolev* H^s *s'écrit* $B^s_{2,2}$ *et donc* $H^{s+1} \hookrightarrow B^s_{2,1}$, $B^s_{2,1} \hookrightarrow H^s$. *De plus,* $B^{\frac{d}{p}}_{p,1}(\mathbb{R}^d) \hookrightarrow L^\infty$, $\forall p \in [1,\infty]$.

(iii) Interpolation réelle : $(B^{s_1}_{p,r_1}, B^{s_2}_{p,r_2})_{\theta,r'} = B^{(1-\theta)s_1+\theta s_2}_{p,r'}$.

Comme les espaces invariants d'échelle doivent être homogènes, on introduit aussi les espaces de Besov homogènes. Pour cela, on définit les *blocs dyadiques homogènes* $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ par

$$\dot{\Delta}_j u = \int_{\mathbb{R}^d} h_j(y)u(x-y)dy, \quad \forall j \in \mathbb{Z},$$

et l'opérateur de troncature homogène à basse fréquence est

$$\dot{S}_j := \chi(2^{-j}D), \quad \forall j \in \mathbb{Z}. \tag{0.43}$$

Nous définissons alors les semi-normes homogènes :

$$\|u\|_{\dot{B}^s_{p,r}} = \left\|(2^{js}\|\dot{\Delta}_j u\|_{L^p})_{j\in\mathbb{Z}}\right\|_{\ell^r}.$$

Remarquons que pour $u \in \mathcal{S}'(\mathbb{R}^d)$, l'égalité

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u$$

est vraie seulement modulo les polynômes. Afin d'obtenir de « vrais » espaces de distributions, on définit, d'après [5], les espaces de Besov homogènes de la façon suivante :

Definition 0.2. *L'espace de Besov homogène $\dot{B}^s_{p,r}$ est l'ensemble des distributions tempérées u telles que*

$$\|u\|_{\dot{B}^s_{p,r}} < \infty \quad et \quad \lim_{j\to-\infty} \|\dot{S}_j u\|_{L^\infty} = 0.$$

La définition précédente implique que $\dot{B}^s_{p,r}(\mathbb{R}^d)$ est un espace de Banach à condition que

$$s < d/p \quad \text{ou} \quad s \leq d/p \ \text{ si } \ r = 1. \tag{0.44}$$

Les propriétés de la Proposition 0.1 restent vraies pour les espaces de Besov homogènes. De plus, si $u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u$ et $\|u\|_{\dot{B}^s_{p,r}}$ est fini pour un triplet (s,p,r) satisfaisant (0.44), alors u appartient à $\dot{B}^s_{p,r}(\mathbb{R}^d)$, grâce aux inégalités de Bernstein mentionnées ci-dessus. Ce fait servira plusieurs fois.

0.3.2 Estimations classiques

Lorsqu'on estime des produits dans l'espace de Besov, il est souvent pratique d'utiliser le calcul paradifférentiel, un outil qui a été introduit par J.-M. Bony dans [15]. Rappelons que le paraproduit de v par u est défini par

$$T_u v = \sum_j S_{j-1} u \, \Delta_j v,$$

et on appelle reste du produit uv, noté $R(u,v)$, l'opérateur bilinéaire symétrique suivant :

$$R(u,v) = \sum_j \Delta_j u \, \widetilde{\Delta_j} v \quad \text{avec} \quad \widetilde{\Delta_j} v = (\Delta_{j-1} + \Delta_j + \Delta_{j+1})v.$$

Il est immédiat, par définition des opérateurs de paraproduit et de reste, que l'on a la décomposition de Bony suivante du produit uv :

$$uv = T_u v + R(u,v) + T_v u := T'_u v + T_v u. \tag{0.45}$$

On a l'estimation classique suivante dans les espaces de Besov pour les opérateurs de paraproduit et de reste :

Proposition 0.2. *Soit $s, s_1, s_2 \in \mathbb{R}$, $1 \leq r, r_1, r_2, p \leq \infty$ tels que $\frac{1}{r} \leq \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}$,*
– *pour l'opérateur de paraproduit, on a les deux estimations suivantes :*

$$\|T_u v\|_{B^{s_1+s_2-\frac{d}{p}}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p,r_1}} \|v\|_{B^{s_2}_{p,r_2}} \quad si \quad s_1 < \tfrac{d}{p},$$
$$\|T_u v\|_{B^s_{p,r}} \lesssim \|u\|_{L^\infty} \|v\|_{B^s_{p,r}};$$

– *pour l'opérateur de reste, si $s_1 + s_2 + d\min\{0, 1 - \frac{2}{p}\} > 0$, alors on a*

$$\|R(u,v)\|_{B^{s_1+s_2-\frac{d}{p}}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p,r_1}} \|v\|_{B^{s_2}_{p,r_2}}.$$

Par conséquent, on en déduites estimations suivantes pour le produit :

Proposition 0.3. *Soit $s, s_1, s_2 \in \mathbb{R}$, $1 \leq r, r_1, r_2, p \leq \infty$ tels que $\frac{1}{r} \leq \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}$,*
– *si $s_1, s_2 < \frac{d}{p}$, $s_1 + s_2 + d\min\{0, 1 - \frac{2}{p}\} > 0$, alors on a*

$$\|uv\|_{B^{s_1+s_2-\frac{d}{p}}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p,r_1}} \|v\|_{B^{s_2}_{p,r_2}};$$

– *si $s > 0$, alors on a*

$$\|uv\|_{B^s_{p,r}} \lesssim \|u\|_{L^\infty} \|v\|_{B^s_{p,r}} + \|u\|_{B^s_{p,r}} \|v\|_{L^\infty};$$

– *si $-\min\{\frac{d}{p}, \frac{d}{p'}\} < s \leq \frac{d}{p}$ et $r = 1$ quand $s = \frac{d}{p}$, alors on a*

$$\|uv\|_{B^s_{p,r}} \lesssim \|u\|_{B^s_{p,r}} \|v\|_{B^{\frac{d}{p}}_{p,1}}.$$

Lorsqu'on résout des équations d'évolution, il est naturel d'utiliser des espaces de type $L^\rho_T(X) = L^\rho(0,T;X)$ avec X représentant un espace de Banach. Dans notre cas, X sera un espace de Besov. Donc on doit localiser les équations à l'aide du découpage de Littlewood-Paley. Cette technique nous donne l'estimation de chaque bloc dyadique dans l'espace de Lebesgue *avant* l'intégration en temps. Ceci motive la définition suivante :

Definition 0.3. *Soit $s \in \mathbb{R}$, $(\rho, p, r) \in [1, +\infty]^3$ et $T \in [0, +\infty]$, on définit*

$$\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} = \left\| \left(2^{js} \|\Delta_j u(t)\|_{L^\rho_T(L^p(\mathbb{R}^d))} \right)_{j \geq -1} \right\|_{\ell^r}.$$

On note aussi $\widetilde{\mathcal{C}}_T(B^s_{p,r}) = \widetilde{L}^\infty_T(B^s_{p,r}) \cap \mathcal{C}([0,T]; B^s_{p,r})$.

Remark 0.1. *(i) Grâce à l'inégalité de Minkowski, on a*

$$\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \leq \|u\|_{L^\rho_T(B^s_{p,r})} \quad si \ \rho \leq r,$$
$$\|u\|_{L^\rho_T(B^s_{p,r})} \leq \|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \quad si \ \rho \geq r,$$

et donc en particulier $\|u\|_{\widetilde{L}^1_T(B^s_{p,1})} = \|u\|_{L^1_T(B^s_{p,1})}$ est vraie.

(ii) Les propositions 0.1 et 0.3 se généralisent à ces espaces fonctionnels. Par exemple, on a pour $\theta \in [0,1]$, $\frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2}$, et $s = \theta s_1 + (1-\theta)s_2$, l'inégalité d'interpolation suivante :

$$\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \leq \|u\|^\theta_{\widetilde{L}^{\rho_1}_T(B^{s_1}_{p,r})} \|u\|^{1-\theta}_{\widetilde{L}^{\rho_2}_T(B^{s_2}_{p,r})}.$$

(iii) Dans les estimations de produit, les exposants de Lebesgue par rapport à la variable de temps se comportent conformément à l'inégalité de Hölder. Par exemple, on a :

$$\|uv\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \lesssim \|u\|_{L^{\rho_1}_T(L^\infty)}\|v\|_{\widetilde{L}^{\rho_4}_T(B^s_{p,r})} + \|u\|_{\widetilde{L}^{\rho_2}_T(B^s_{p,r})}\|v\|_{L^{\rho_3}_T(L^\infty)}, \tag{0.46}$$

quand $s > 0$, $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_4} = \frac{1}{\rho_2} + \frac{1}{\rho_3}$.

S'il y a un terme de transport $\varphi \cdot \nabla\psi$ dans l'équation d'évolution sur ψ, alors appliquer l'opérateur de localisation Δ_j donne le terme de commutateur

$$[\varphi, \Delta_j] \cdot \nabla\psi := \varphi \cdot \nabla\Delta_j\psi - \Delta_j(\varphi \cdot \nabla\psi),$$

qui vérifie l'estimation suivante :

Proposition 0.4. *Soit le triplet* (s, p, r) *satisfaisant*

$$s > -d\min\{\frac{1}{p}, \frac{1}{p'}\} \quad \text{et} \;\; (p,r) \in [1,+\infty]^2\,, \;\; \text{avec} \;\; r = 1 \quad \text{si} \;\; s = 1 + \frac{d}{p}\,.$$

Alors on a (pour une certaine constante C *ne dépendant que de* d, s, p, r*)*

$$\int_0^t \left\|2^{js}\,\|[\varphi,\Delta_j]\nabla\psi\|_{L^p}\right\|_{\ell^r} d\tau \;\leq\; C\int_0^t \Phi^s(\tau)\|\nabla\psi\|_{B^{s-1}_{p,r}}\,d\tau, \tag{0.47}$$

où $\{\Phi^s\}_s$ *est défini par*

$$\Phi^s(t) = \begin{cases} \|\nabla\varphi(t)\|_{B^{\frac{d}{p}}_{p,1}} & \text{si } s \in (-d\min\{\frac{1}{p},\frac{1}{p'}\}, 1+\frac{d}{p})\,, \\ \|\nabla\varphi(t)\|_{B^{s-1}_{p,r}} & \text{sinon.} \end{cases}$$

De plus, si $s > 0$ *on a également*

$$\int_0^t \left\|2^{js}\,\|[\varphi,\Delta_j]\nabla\psi\|_{L^p}\right\|_{\ell^r} d\tau \;\leq\; C\int_0^t \Big(\|\nabla\varphi\|_{L^\infty}\;\|\psi\|_{B^s_{p,r}} \;+\; \|\nabla\varphi\|_{B^{s-1}_{p,r}}\;\|\nabla\psi\|_{L^\infty}\Big)\,d\tau. \tag{0.48}$$

On a aussi le résultat suivant pour la composition par une fonction régulière :

Proposition 0.5. *Soit* $(\rho, p, r) \in [1,+\infty]^3$ *et* $s > 0$*. Soit* f *une fonction régulière sur* $\mathbb{R}$*.*

– *Si* $f(0) = 0$ *alors pour tout* $u \in B^s_{p,r} \cap L^\infty$ *on a*

$$\|f \circ u\|_{B^s_{p,r}} \leq C(f', \|u\|_{L^\infty})\|u\|_{B^s_{p,r}}. \tag{0.49}$$

– *Si* $f'(0) = 0$ *alors pour tout couple* (u, v) *dans* $B^s_{p,r} \cap L^\infty$*, on a*

$$\|f \circ u - f \circ v\|_{B^s_{p,r}} \leq C(f'', \|u\|_{L^\infty\cap B^s_{p,r}}, \|v\|_{L^\infty\cap B^s_{p,r}})\|u - v\|_{B^s_{p,r}}. \tag{0.50}$$

Soit I *un intervalle ouvert de* $\mathbb{R}$ *et* $F : I \to \mathbb{R}$*, une fonction régulière. Alors pour tout sous-ensemble compact* $J \subset I$*, et pour toute fonction* u *valant dans* J*, on a*

$$\|\nabla(F(u))\|_{\widetilde{L}^\rho_T(B^{s-1}_{p,r})} \leq C\|\nabla u\|_{\widetilde{L}^\rho_T(B^{s-1}_{p,r})}.$$

0.3.3 Estimations a priori pour des équations d'évolution

Rappelons d'abord quelques estimations a priori dans les espaces de Besov pour l'équation de transport :

$$(T) \qquad \begin{cases} \partial_t a + v \cdot \nabla a = f, \\ a_{|t=0} = a_0. \end{cases}$$

En utilisant la proposition 0.4, on montre aisément que

Proposition 0.6. *Soit* $(p,r) \in [1,+\infty]^2$ *et*

$$s > -d \min\{\frac{1}{p}, \frac{1}{p'}\} \quad (s > -1 - d \min\{\frac{1}{p}, \frac{1}{p'}\} \quad \text{si} \ \operatorname{div} v = 0\,), \tag{0.51}$$

avec une inégalité stricte si $r < +\infty$. *Soit* $a_0 \in B^s_{p,r}$, $f \in L^1([0,T]; B^s_{p,r})$. *Soit* v *un champ de vecteurs dépendant de temps dans* $L^q([0,T]; B^{-M}_{\infty,\infty})$ *avec* $q > 1$ *et* $M > 0$ *tel que*

$$\begin{aligned} \nabla v &\in L^1([0,T]; B^{d/p}_{p,\infty} \cap L^\infty) \quad \text{si} \quad s < 1 + d/p, \\ \nabla v &\in L^1([0,T]; B^{s-1}_{p,r}) \qquad\quad \text{si} \quad s > 1 + d/p, \quad \text{ou} \quad s = 1 + d/p \quad \text{et} \quad r = 1. \end{aligned}$$

L'équation (T) *a une unique solution* a *dans*

- *l'espace* $\mathcal{C}([0,T]; B^s_{p,r})$ *si* $r < \infty$,
- *l'espace* $\Big(\bigcap_{s'<s} \mathcal{C}([0,T]; B^{s'}_{p,\infty})\Big) \bigcap \mathcal{C}_w([0,T]; B^s_{p,\infty})$ *si* $r = \infty$.

De plus, pour tout $t \in [0,T]$, *on a*

$$e^{-CV(t)} \|a(t)\|_{B^s_{p,r}} \le \|a_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(t')} \|f(t')\|_{B^s_{p,r}}\, dt' \tag{0.52}$$

avec $V(0) = 0$ *et*

$$V'(t) := \begin{cases} \|\nabla v(t)\|_{B^{d/p}_{p,\infty} \cap L^\infty} \quad si \ \ s < 1 + d/p, \\ \|\nabla v(t)\|_{B^{s-1}_{p,r}} \quad si \quad s > 1 + d/p, \quad ou \quad s = 1 + d/p \quad \text{et} \quad r = 1, \end{cases} \tag{0.53}$$

et par ailleurs, si l'égalité est vérifiée dans (0.51) *et* $r = \infty$, *alors* $V'(t) = \|\nabla v\|_{B^{d/p}_{p,1}}$.
Si $a = v$ *alors, pour tout* $s > 0$ *(*$s > -1$ *si* $\operatorname{div} v = 0$*), l'estimation* (0.52) *est vérifiée avec* $V'(t) := \|\nabla v(t)\|_{L^\infty}$.

Pour le système linéaire parabolique avec un terme de transport qui s'écrit

$$\begin{cases} \partial_t \theta \ + \ v \cdot \nabla \theta - \bar{\kappa} \Delta \theta = f, \\ \theta_{|t=0} = \theta_0, \end{cases} \tag{0.54}$$

et le système linéarisé de Navier-Stokes

$$\begin{cases} \partial_t u + v \cdot \nabla u - \bar{\eta} \Delta u + \nabla Q = h, \\ \operatorname{div} u = 0, \\ u_{|t=0} = u_0, \end{cases} \tag{0.55}$$

le lecteur est renvoyé à [34] pour la démonstration des deux résultats suivants :

Proposition 0.7. *Soit* $1 \le p \le p_1 \le \infty$ *et* $1 \le r \le \infty$. *Soit* $s \in \mathbb{R}$ *tel que*

$$\begin{cases} s < 1 + \dfrac{d}{p_1}, \quad \text{ou} \quad s \le 1 + \dfrac{d}{p_1} \quad si \;\; r = 1, \\ s > -d \min\Big\{\dfrac{1}{p_1}, \dfrac{1}{p'}\Big\}, \quad \text{ou} \quad s > -1 - d\min\Big\{\dfrac{1}{p_1}, \dfrac{1}{p'}\Big\} \quad \text{si} \;\; \operatorname{div} v = 0. \end{cases}$$

Il existe une constante C *dépendant seulement de* d, r, s *et* $s - 1 - \frac{d}{p_1}$ *telle que pour toutes les solutions régulières* θ *de* (0.54) *avec* $\bar\kappa \ge 0$, *et* $q \in [1, \infty]$, *on ait l'estimation a priori suivante :*

$$\bar\kappa^{\frac{1}{q}} \|\theta\|_{\widetilde L^q_T(\dot B^{s+\frac{2}{q}}_{p,r})} \le e^{C\dot V_{p_1}(T)} \Big(\|\theta_0\|_{\dot B^s_{p,r}} + \|f\|_{\widetilde L^1_T(\dot B^s_{p,r})} \Big)$$

avec
$$\begin{cases} \dot V_{p_1}(T) = \displaystyle\int_0^T \|\nabla v(t)\|_{\dot B^{\frac{d}{p_1}}_{p_1,\infty} \cap L^\infty} \, dt \quad si \quad s < \frac{d}{p_1} + 1, \\ \dot V_{p_1}(T) = \displaystyle\int_0^T \|\nabla v(t)\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}} \, dt \quad si \quad s = \frac{d}{p_1} + 1. \end{cases}$$

Proposition 0.8. *Soit* p, p_1, r, s *et* $\dot V_{p_1}$ *comme dans la proposition 0.7. Il existe une constante* C *dépendant seulement de* d, r, s *et* $s - 1 - \frac{d}{p_1}$ *telle que pour toutes les solutions régulières* $(u, \nabla Q)$ *de* (0.55) *avec* $\bar\eta \ge 0$, *et* $q \in [1, \infty]$, *on ait l'estimation a priori suivante :*

$$\bar\eta^{\frac{1}{q}} \|u\|_{\widetilde L^q_T(\dot B^{s+\frac{2}{q}}_{p,r})} \le e^{C\dot V_{p_1}(T)} \Big(\|u_0\|_{\dot B^s_{p,r}} + \|\mathcal{P}h\|_{\widetilde L^1_T(\dot B^s_{p,r})} \Big),$$
$$\|\nabla Q - \mathcal{Q}h\|_{\widetilde L^1_T(\dot B^s_{p,r})} \le C\Big(e^{C\dot V_{p_1}(T)} - 1\Big)\Big(\|u_0\|_{\dot B^s_{p,r}} + \|\mathcal{P}h\|_{\widetilde L^1_T(\dot B^s_{p,r})} \Big).$$

Ci-dessus, $\mathcal{P}$ et $\mathcal{Q}$ représentent les projecteurs orthogonaux sur les champs de vecteurs à divergence nulle et les champs de vecteurs potentiels.

Finalement, nous allons souvent considérer l'équation elliptique suivante (pour la pression Π) :

$$-\operatorname{div}(\lambda \nabla \Pi) = \operatorname{div} L \quad \text{sur} \;\; \mathbb{R}^d, \tag{0.56}$$

où $\lambda = \lambda(x)$ est une fonction régulière et bornée, vérifiant $\lambda_* := \inf_{x \in \mathbb{R}^d} \lambda(x) > 0$. On va utiliser beaucoup l'estimation d'énergie suivante (voir e.g. [35] pour la preuve) :

Proposition 0.9. *Pour tous les champs de vecteurs* L *à coefficients dans* L^2, *il existe une distribution temperée* Π, *unique à constante près, vérifiant l'équation* (0.56). *De plus, on a*

$$\lambda_* \|\nabla \Pi\|_{L^2} \le \|L\|_{L^2}.$$

0.4 Résultats et démonstrations

Dans cette section, on présentera les résultats principaux obtenus dans cette thèse et les méthodes utilisées. Dans la première sous-section (resp. dans la troisième sous-section), on parlera du système limite avec viscosité (resp. du système sans viscosité) qui a été traité dans mon travail en collaboration avec R. Danchin [37] (resp. avec F. Fanelli [47]). Les résolutions correspondantes en détail se trouveront dans le premier chapitre et dans le troisième chapitre de la thèse respectivement. Le deuxième chapitre §2 correspond à l'article [78] et on énoncera ses résultats pour le système limite spécial (0.3) dans la deuxième sous-section §0.4.2.

0.4.1 Etude du système limite avec viscosité

On a déjà vu que le nouveau système (0.22) avec les inconnues $(\vartheta, u, \nabla Q)$ incluant la condition d'incompressibilité était équivalent au système original (0.1) avec les inconnues $(\rho, v, \nabla\Pi)$, au moins pour des solutions régulières. On va donc d'abord résoudre le problème de Cauchy associé à (0.22) dans un espace fonctionnel *optimal*. La résolution en détail se trouvera dans le premier chapitre §1.

Motivé par les travaux sur les équations de Navier-Stokes incompressibles ou compressibles à densité variable (voir cf. [27, 28, 29, 32]), on va utiliser des arguments d'échelle afin de déterminer le cadre fonctionnel optimal. On note que si $(\vartheta, u, \nabla Q)$ est une solution de (0.22), alors il en va de même pour

$$(\vartheta(\ell^2 t, \ell x), \ell u(\ell^2 t, \ell x), \ell^3 \nabla Q(\ell^2 t, \ell x)) \quad \text{pour tout} \quad \ell > 0. \tag{0.57}$$

Donc, l'espace critique pour les données initiales (ϑ_0, u_0) doit être à norme invariante par la transformation

$$(\vartheta_0, u_0)(x) \to (\vartheta_0(\ell x), \ell u_0(\ell x)) \quad \text{pour tout} \quad \ell > 0. \tag{0.58}$$

Remarquons que l'espace fonctionnel suivant [2] :

$$\Big(L^\infty(\mathbb{R}^+; \dot{B}^{d/p_1}_{p_1,r_1}(\mathbb{R}^d)) \cap L^1(\mathbb{R}^+; \dot{B}^{d/p_1+2}_{p_1,r_1}(\mathbb{R}^d))\Big)$$
$$\times \Big(L^\infty(\mathbb{R}^+; \dot{B}^{d/p_2-1}_{p_2,r_2}(\mathbb{R}^d)) \cap L^1(\mathbb{R}^+; \dot{B}^{d/p_2+1}_{p_2,r_2}(\mathbb{R}^d))\Big)^d \times \Big(L^1(\mathbb{R}^+; \dot{B}^{d/p_3-1}_{p_3,r_3}(\mathbb{R}^d))\Big)^d$$

vérifie la condition d'échelle (0.57) pour tout $1 \le p_1, p_2, p_3, r_1, r_2, r_3 \le \infty$.

Néanmoins, comme on a besoin de coefficients (κ, η) bornés et, d'autre part, l'inclusion $\dot{B}^{d/p}_{p,r} \hookrightarrow L^\infty$ est vérifiée si et seulement si $r = 1$, on va alors fixer $r_1 = 1$. De plus, à plusieurs reprises, $\nabla u \in L^1([0,T]; L^\infty)$ sera nécessaire. Cela nous conduit à choisir $r_2 = 1$ et $r_3 = 1$ en conséquence. Donc finalement, on cherche à résoudre le système (0.22) dans l'espace $\dot{E}^{p_1,p_2}_T(\mathbb{R}^d)$ suivant :

$$\Big(\widetilde{\mathcal{C}}_T(\dot{B}^{d/p_1}_{p_1,1}) \cap L^1_T(\dot{B}^{d/p_1+2}_{p_1,1})\Big) \times \Big(\widetilde{\mathcal{C}}_T(\dot{B}^{d/p_2-1}_{p_2,1}) \cap L^1_T(\dot{B}^{d/p_2+1}_{p_2,1})\Big)^d \times \Big(L^1_T(\dot{B}^{d/p_2-1}_{p_2,1})\Big)^d, \tag{0.59}$$

où $\widetilde{\mathcal{C}}_T(\dot{B}^s_{p,1})$ est un (grand) sous-espace de $\mathcal{C}([0,T]; \dot{B}^s_{p,1})$ (voir Définition 0.3).

Le cas presque homogène

On commence par le cas plus simple où la température initiale ϑ_0 est proche d'une constante (disons 1 pour simplifier la présentation). Posons $\theta = \vartheta - 1$, alors le système (0.22) se réécrit

$$\left\{\begin{array}{lcl} \partial_t\theta + u\cdot\nabla\theta - \bar\kappa\Delta\theta &=& r_1(\theta), \\ \partial_t u + u\cdot\nabla u - \bar\eta\Delta u + \nabla Q &=& r_2(\theta, u, \nabla Q), \\ \operatorname{div} u &=& 0, \end{array}\right. \tag{0.60}$$

où $\bar\kappa = \kappa(1)$, $\bar\eta = \eta(1)$ et

$$r_1(\theta) = \operatorname{div}\big((\kappa(1+\theta) - \bar\kappa)\nabla\theta\big) + h_1(1+\theta),$$
$$r_2(\theta, u, \nabla Q) = \operatorname{div}\big((\eta(1+\theta) - \bar\eta)\nabla u\big) - \theta\nabla Q + h_2(1+\theta, u).$$

Notons que le système linéarisé de (0.60) est juste l'équation parabolique (0.54) à coefficients constants sur la « température » θ, et le système de Stokes (0.55) sur la « vitesse »

2. Nous renvoyons le lecteur à la Définition 0.2 pour la définition de l'espace de Besov homogène.

u et la « pression » Q. Donc, les deux propositions relatives s'appliquent et nous donnent des estimations a priori. Une autre observation importante pour résoudre (0.60) est que les deux « termes sources » r_1 et r_2 sont au moins quadratiques. Dans l'espace fonctionnel $\dot{E}_T^{p_1,p_2}$, on s'attend à ce qu'ils soient négligeables si des données initiales sont petites. Par conséquent, l'estimation a priori linéaire et l'estimation de produit (voir l'annexe §A.1 pour l'estimation dans des espaces de Besov plus généraux) suffisent pour borner la solution pour tous les temps positifs à condition que les données initiales soient petites. Cela nous permet de démontrer l'existence globale en temps.

Par ailleurs, un argument classique similaire à celui qui s'applique dans le cas de la densité constante nous permet de traiter le cas u_0 grand. On va présenter ce résultat précisément dans le théorème 1.1 et le prouver dans la section §1.2.

Le cas véritablement non homogène

Si l'on considère le cas véritablement non homogène, c'est-à-dire que l'on suppose juste que la température initiale est positive et tend vers 1 à l'infini (donc elle varie arbitrairement), alors les estimations a priori données par la proposition 0.7 et la proposition 0.8 ne suffisent pas pour borner la solution de (0.22), même en temps petit. En fait, les « termes sources » $\nabla u \cdot \nabla\theta$, $\theta\nabla Q$, etc. peuvent être du même ordre que ceux du membre de gauche et donc ne peuvent pas être absorbés.

Donc, dans le cas véritablement non homogène, il faut étudier les équations linéarisées avec coefficients *variables* dans leurs termes à ordre principal. Dans la section §1.3.1, on obtiendra des nouvelles estimations a priori appropriées, dont les preuves se trouvent dans l'annexe §B.1 et l'annexe §B.4. Remarquons que la démonstration repose sur des estimations délicates de produits et commutateurs (voir l'annexe §A) sous une condition « adéquate » entre les exposants de Lebesgue p_1, p_2.

La pression Q, quant à elle, vérifie une équation elliptique sous forme divergence :

$$\operatorname{div}(\vartheta\nabla Q) = \operatorname{div} L_1 \quad \text{avec} \quad L_1 := -u\cdot\nabla u + Du\cdot\nabla\eta + h_2. \tag{0.61}$$

Dans le cadre d'énergie (disons l'espace de Sobolev H^s ou l'espace de Besov $B^{d/p_2}_{p_2,1}$ avec $p_2 = 2$), la démonstration est classique. Mais si $p_2 \neq 2$, on a besoin d'informations sur les basses fréquences. L'estimation a priori $\|\nabla Q\|_{L^2} \lesssim \|L_1\|_{L^2}$ nous aide à les contrôler dans le cas $p_2 \geq 2$, grâce à l'inclusion de la proposition 0.1. Pour majorer $\|L_1\|_{L^2}$, on peut simplement appliquer l'inégalité de Hölder $\|fg\|_{L^2} \leq \|f\|_{L^4}\|g\|_{L^4}$ aux termes de L_1. Ceci impose aux exposants de Lebesgue de prendre leur valeur dans l'intervalle $[2,4]$ et donc donne une restriction supplémentaire sur p_1 et p_2.

Pour conclure, on cherche à résoudre le système limite dans l'espace fonctionnel *non homogène* $E_T^{p_1,p_2}(\mathbb{R}^d)$ suivant (avec quelques restrictions sur les exposants de Lebesgue p_1 et p_2) :

$$\Big(\widetilde{\mathcal{C}}_T(B^{d/p_1}_{p_1,1}) \cap L^1_T(B^{d/p_1+2}_{p_1,1})\Big) \times \Big(\widetilde{\mathcal{C}}_T(B^{d/p_2-1}_{p_2,1}) \cap L^1_T(B^{d/p_2+1}_{p_2,1})\Big)^d \times \Big(L^1_T(B^{d/p_2-1}_{p_2,1} \cap L^2)\Big)^d, \tag{0.62}$$

qui est critique par rapport à la régularité mais impose des conditions de décroissance plus fortes à l'infini.

Afin de traiter les grandes données, on va décomposer la solution en deux parties : une grande partie "libre" (θ_L, u_L) correspondant à la donné initiale (θ_0, u_0), autrement dit, la solution de

$$\begin{cases} \partial_t\theta_L - \bar\kappa\Delta\theta_L &= 0, \\ \partial_t u_L - \bar\eta\Delta u_L &= 0, \\ (\theta_L, u_L)|_{t=0} &= (\theta_0, u_0), \end{cases} \tag{0.63}$$

avec $\bar{\kappa} = \kappa(1)$ et $\bar{\eta} = \eta(1)$; et un terme de reste $(\bar{\theta}, \bar{u}, \nabla Q)$. On applique l'estimation a priori classique sur la partie libre (θ_L, u_L) et la nouvelle estimation a priori sur la petite partie $(\bar{\theta}, \bar{u}, \nabla Q)$, ce qui donne une estimation uniforme sur un petit intervalle de temps. L'existence et l'unicité suivent par la même technique. Ce résultat se trouvera dans le théorème 1.2 et on le prouvera en détail dans la section §1.3.

Remarquons que dans le cadre des solutions fortes, le système (0.1) et le système (0.22) sont équivalents et donc $(\rho, v) := (P_0/(R\vartheta), u + \alpha k \nabla\vartheta)$ satifait le système (0.1) avec une pression $\nabla\Pi \in L^1_T(B^{d/p_2-1}_{p_2,1})$ si $p_1 \geq p_2$.

À la fin du premier chapitre, on considèrera également le système (0.33) pour les gaz généraux. L'analyse est similaire et on esquissera la preuve.

0.4.2 Un cas spécial

Le deuxième chapitre §2 est consacré à l'étude du système limite (0.1) sous la condition algébrique (0.15), à savoir le système (0.3).

Grâce à sa forme spéciale, on obtient formellement les identités d'énergie suivantes, en prenant le produit scalaire $L^2(\mathbb{R}^d)$ entre les équations et les inconnues $\varrho = \rho - 1$ (voir la notation (viii)) et u, respectivement,

$$\int_{\mathbb{R}^d} |\varrho(t)|^2 + 2\int_0^t \int_{\mathbb{R}^d} \kappa |\nabla\varrho|^2 = \int_{\mathbb{R}^d} |\varrho_0|^2, \quad \forall\, t > 0\,, \tag{0.64}$$

et

$$\int_{\mathbb{R}^d} \rho(t)|u(t)|^2 + 4\int_0^t \int_{\mathbb{R}^d} \mu |Au|^2 = \int_{\mathbb{R}^d} \rho_0 |u_0|^2, \quad \forall\, t > 0\,. \tag{0.65}$$

Une autre observation est que si la densité initiale ρ_0 prend ses valeurs entre deux constantes positives ρ_* et ρ^*, alors l'application du principe de maximum à l'équation parabolique sur la densité implique que la solution ρ reste dans l'intervalle $[\rho_*, \rho^*]$. Donc il existe deux constantes positives κ_* et μ_* telles que (remarquons $\operatorname{div} u = 0$)

$$\int_0^t \int_{\mathbb{R}^d} \kappa(\rho)|\nabla\varrho|^2 \geq \kappa_* \|\nabla\varrho\|^2_{L^2_t(L^2(\mathbb{R}^d))}, \int_0^t \int_{\mathbb{R}^d} \mu |Au|^2 \geq \mu_* \|Au\|^2_{L^2_t(L^2)} \equiv \mu_* \|\nabla u\|^2_{L^2_t(L^2(\mathbb{R}^d))}.$$

Par conséquent, les quantités $\|(\varrho, u)\|_{L^\infty_t(L^2)}$ et $\|(\nabla\varrho, \nabla u)\|_{L^2_t(L^2)}$ sont bornées par les données initiales (i.e. par les quantités $\|(\varrho_0, u_0)\|_{L^2}$, ρ_*, ρ_*) pour tous les temps positifs.

En général, (0.65) est vérifié sous forme d'inégalité, ce qui assure toutefois un effet régularisant sur les deux inconnues, suffisant pour obtenir des solutions faibles de type Leray via des méthodes de compacité. En effet, en s'inspirant de l'étude sur les équations de Navier-Stokes à densité variable dans le livre [81], on régularise avec soin d'abord les coefficients κ, μ et aussi les données initiales, afin de trouver une suite de solutions approchées satisfaisant les inégalités d'énergie, en utilisant le théorème du point fixe de Schauder sur un opérateur compact (voir la proposition 2.2). Enfin, la méthode de compacité classique marche bien et on montrera qu'il existe une solution faible à énergie finie globalement en temps dans la section §2.2.1.

Il est aisé de contrôler (en vertu de la condition (0.16)) que ρ et $v := u - \kappa\nabla \ln\rho \in L^2([0, +\infty); (L^2(\mathbb{R}^d))^d)$ vérifient le système (0.1) au sens des distributions. Mais en raison d'un manque de régularité sur la densité initiale ρ_0, on ne sait pas établir la continuité de v à l'instant initial, parce qu'on ne peut même pas définir la quantité $\kappa(\rho_0)\nabla\ln\rho_0$.

Dans la section §2.2.2 on démontrera que : supposons plutôt $\varrho_0 := \rho_0 - 1 \in H^1(\mathbb{R}^d)$, c.-à-d. la vitesse originale v_0 appartient à $(L^2(\mathbb{R}^d))^d$ aussi, alors $(\rho - 1, v) \in H^1(\mathbb{R}^d) \times (L^2(\mathbb{R}^d))^d$ pour tous les temps positifs si $d = 2$ ou 3. En fait, pour simplifier, supposons que $\kappa \equiv 1$, alors en prenant le produit scalaire $L^2(\mathbb{R}^d)$ entre l'équation (0.3) et $\Delta\rho$ donne l'inégalité d'énergie sur $\nabla\rho$ si l'on peut majorer le produit scalaire L^2 entre le terme de transport et $\Delta\rho$: $|\int_{\mathbb{R}^d} \operatorname{div}(\rho u)\Delta\rho|$. Parce que a priori on a (en remarquant $\operatorname{div} u = 0$)

$$\left|\int_{\mathbb{R}^d} \operatorname{div}(\rho u)\Delta\rho\right| = \left|\int_{\mathbb{R}^d} \nabla\rho\cdot\nabla u\cdot\nabla\rho\right| \leq \|\nabla u\|_{L^2(\mathbb{R}^d)}\|\nabla\rho\|^2_{L^4(\mathbb{R}^d)},$$

et en particulier en dimension deux et trois les deux inégalités suivantes sont vérifiées :

$$\|\nabla\rho\|^2_{L^4(\mathbb{R}^2)} \leq C\|\Delta\rho\|_{L^2(\mathbb{R}^2)}\|\nabla\rho\|_{L^2(\mathbb{R}^2)}, \quad \|\nabla\rho\|^2_{L^4(\mathbb{R}^3)} \leq C\|\Delta\rho\|_{L^2(\mathbb{R}^3)}\|\rho\|_{L^\infty(\mathbb{R}^3)}, \qquad (0.66)$$

le terme de transport $\operatorname{div}(\rho u)$ peut être majoré et donc $\|\nabla\rho\|_{L^\infty_t(L^2)}$ et $\|\Delta\rho\|_{L^2_t(L^2)}$ restent bornés pour tous les temps positifs. Pour le cas général où κ dépend de ρ, on considère la fonction scalaire $a = a(\rho)$ avec $\nabla a = \kappa\nabla\rho$ et $a(1) = 0$ (voir (0.36)) à la place de ρ lui-même. Pour conclure, la solution faible obtenue (ρ, v) satisfait le système (0.1) avec valeur initiale $(\varrho_0, v_0) \in H^1 \times (L^2)^d$ en dimension deux et trois.

Dans le cas particulier de la dimension deux, on espère construire une seule solution forte (ϱ, u) dans l'espace de Besov critique $\mathcal{C}(B^1_{2,1}(\mathbb{R}^2)) \times \mathcal{C}(B^0_{2,1}(\mathbb{R}^2))$, globale en temps. Pour cela, il faut démontrer que la solution forte et les solutions faibles coïncident et donc la solution existe globalement et uniquement. Plus précisément, d'abord, on a une solution forte maximale (ρ_1, u_1) sur un intervalle de temps $[0, T^*)$, $T^* \leq +\infty$. Remarquons que l'espace des solutions fortes $E^{2,2}_{T^*}(\mathbb{R}^2)$ implique qu'il existe un temps positif $T_0 < T^*$ tel que $(\varrho_1, u_1)|_{T_0} \in B^3_{2,1}(\mathbb{R}^2) \times B^2_{2,1}(\mathbb{R}^2)$ et, il existe des solutions faibles et globales (ρ_2, u_2) à donnée initiale en temps T_0. Par conséquent, la pseudo loi de conservation des normes de (ϱ_2, u_2) dans l'espace de Sobolev $H^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ assure que, les solutions faibles (ϱ_2, u_2) sont toutes finies dans l'espace de Sobolev précédent sur l'intervalle $[T_0, +\infty)$, et donc finies dans l'espace de Besov critique, par inclusion. Pour conclure, les solutions faibles mais régulières prolongent la solution forte pour tous les temps positifs.

0.4.3 Etude du système limite sans viscosité

Dans le troisième chapitre, nous nous concentrons sur le système limite sans viscosité (0.35), ou encore le système (0.37).

Grossièrement, le système (0.37) se compose de deux types d'équations : une équation parabolique sur la densité ρ (la même que dans le système avec viscosité (0.2)) et un système d'Euler à coefficients variables pour la vitesse modifiée u et la pression q.

Contrairement à l'équation avec viscosité $(0.22)_2$, l'équation $(0.37)_2$ ne régularise pas la vitesse u au cours du temps. Donc afin de conserver la régularité initiale, on suppose la vitesse initiale u_0 dans un espace de Besov $B^s_{p,r}$ qui s'injecte dans la classe des fonctions globalement lipschitziennes. C'est-à-dire que le triplet $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ doit satisfaire

$$s > 1 + \frac{d}{p}, \qquad \text{ou} \qquad s = 1 + \frac{d}{p}, \ r = 1. \qquad (0.67)$$

Afin de propager la régularité initiale, il faut en plus que le terme source $(h_3 - \lambda\nabla q)$ soit dans $L^1([0, T]; B^s_{p,r})$. En vertu de la définition de h_3 (0.38), il faut donc au moins

$$\nabla^2\rho \in L^1([0, T]; B^s_{p,r}), \quad \nabla\rho \in L^\infty([0, T]; L^\infty), \quad \text{et} \quad \nabla q \in L^1([0, T]; B^s_{p,r}).$$

Comme ρ vérifie une équation parabolique à coefficients variables, on s'attend, au moins sur un intervalle de temps petit, à maintenir la régularité initiale et à un gain de deux dérivées en espace lors de l'intégration en temps, comme dans la proposition 0.7 de la section §0.4.1. Alors il suffit que l'on considère la densité initiale $\varrho_0 = \rho_0 - 1$ dans le même espace fonctionnel $B^s_{p,r}$ que u_0.

Le cas $p \in [2,4]$

Pour traiter la pression ∇q satisfaisant l'équation suivante (similaire à (0.61))

$$\operatorname{div}(\lambda \nabla q) = \operatorname{div} L_2 \quad \text{avec} \quad L_2 = -(u + \nabla b) \cdot \nabla u + h_3, \tag{0.68}$$

on applique l'estimation $\|\nabla q\|_{L^2} \leq \|L_2\|_{L^2}$ pour estimer les basses fréquences quand $p \geq 2$. De plus, l'application de Hölder $\|fg\|_{L^2} \leq \|f\|_{L^4} \|g\|_{L^4}$ à L_2 demande $p \leq 4$.

L'analyse de l'équation d'Euler linéarisée dans l'espace généralisé (voir l'annexe §B.3) nous permet d'obtenir une unique solution $(\varrho, u, \nabla q)$ (voir le théorème 3.1) dans l'espace fonctionnel $E^s_{p,r}(T)$ suivant :

$$\left(\widetilde{\mathcal{C}}_T(B^s_{p,r}) \cap \widetilde{L}^1_T(B^{s+2}_{p,r})\right) \times \left(\widetilde{\mathcal{C}}_T(B^s_{p,r})\right)^d \times \left(\widetilde{L}^1_T(B^s_{p,r}) \cap L^1_T(L^2)\right)^d. \tag{0.69}$$

Lorsque l'on cherche à établir un critère de prolongement de type de Beale-Kato-Majda pour le système (0.37), la structure des non linéarités h_3 et $\nabla b \cdot \nabla u$ entre en jeu. En fait, par exemple, en vertu de l'espace (0.69), la norme $\widetilde{L}^1_T(B^s_{p,r})$ de $u \cdot \nabla^2 b$, $\Delta b \nabla b$ et $\nabla b \cdot \nabla u$ nécessite que $\|(u, \nabla b)\|_{L^\infty_T(L^\infty)}$ et $\|(\Delta b, \nabla u)\|_{L^2_T(L^\infty)}$ soient finis, car on ne sait pas que ∇b appartient à l'espace $\widetilde{L}^2_T(B^s_{p,r})$. Finalement, par des estimations de commutateur et de produit plus fines, c'est-à-dire que l'on considère la norme de L^∞ à la place de $B^{s-1}_{p,r}$, on arrive à un critère de prolongement qui est présenté dans le théorème 3.2.

Si l'on prend la norme de $B^{s-1}_{p,r}$ à la place de L^∞ et L^4, on obtient une borne inférieure pour le temps de vie en terme des données initiales, dans le théorème 3.3. De plus, observons une relation entre la taille des données initiales et le temps de vie : soit $(\rho^\varepsilon, u^\varepsilon, \nabla q^\varepsilon)$ une solution du système (0.37) avec la donnée initiale $(\rho^\varepsilon_0, u^\varepsilon_0)(x)$ sur un intervalle de temps $[0, T_\varepsilon[$, alors le triplet $(\rho, u, \nabla q)$ est également une solution, sur un intervalle de temps $[0, T[$ tel que $T \geq T_\varepsilon\, \varepsilon^{-2}$. La relation entre les deux solutions est la suivante :

$$\left(\rho^\varepsilon,\, u^\varepsilon, \nabla q^\varepsilon\right)(t,x) := \left(\rho, \varepsilon^{-1} u, \varepsilon^{-3} \nabla q\right)(\varepsilon^{-2} t\,, \varepsilon^{-1}\, x), \quad (\rho^\varepsilon_0, u^\varepsilon_0)(x) := (\rho_0, \varepsilon^{-1} u_0)(\varepsilon^{-1} x).$$

Par conséquent, fixons $\varepsilon^2 = \|u_0\|_{B^s_{p,r}}$ et supposons que la densité initiale ϱ_0 soit d'ordre ε, alors le temps de vie est d'ordre ε^{-2}.

Le cas à énergie finie

Notons que la restriction $p \in [2,4]$ assure que h_3 et $(u + \nabla b) \cdot \nabla u$ sont dans L^2, ce qui permet de résoudre l'équation de la pression (0.68). Donc pour éliminer cette restriction, on demande à la donnée initiale (ϱ_0, u_0) d'être d'énergie finie et on espère obtenir $\nabla q \in L^1_T(L^p)$ pour tout $p \in]1, \infty]$. En fait, formellement, on a l'identité d'énergie (0.64) pour la densité et l'égalité d'énergie suivante pour u (similaire à (0.65)) :

$$\left(\int_{\mathbb{R}^d} \rho |u|^2\right)\Big|_0^t = \int_0^t \int_{\mathbb{R}^d} u \cdot \operatorname{div}(v \otimes \nabla a) \equiv \int_0^t \int_{\mathbb{R}^d} (u + \nabla b) \cdot \nabla^2 a \cdot u + (u \cdot \nabla a)\, \Delta b. \tag{0.70}$$

Donc

$$\|u(t)\|^2_{L^2(\mathbb{R}^d)} \leq C e^{C\Theta(t)} \left(\|u_0\|^2_{L^2} + \|\rho_0 - 1\|^2_{L^2}\right),$$

avec

$$\Theta(t) = \int_0^t \left(\|\nabla \rho\|^2_{L^\infty} + \|\nabla \rho\|^4_{L^\infty} + \|\nabla^2 \rho\|_{L^\infty} + \|\nabla^2 \rho\|^2_{L^\infty}\right) d\tau\,.$$

Ceci donne $\nabla q \in L^1_T(L^2)$ immédiatement, à condition que $\varrho \in L^\infty_T(B^s_{p,r}) \cap \widetilde{L}^1_T(B^{s+2}_{p,r})$ et $\nabla u \in L^\infty_T(L^\infty)$. Donc si $p \geq 2$, alors en raison de l'injection, on a déjà obtenu l'information de basses fréquences sur la pression. Sinon, on applique l'inégalité $\|fg\|_{L^p} \leq \|f\|_{L^2} \|g\|_{L^{p*}}$

avec $\frac{1}{p^*} + \frac{1}{2} = \frac{1}{p}$, afin d'obtenir $\nabla q \in L^1_T(L^p)$. Plus précisément, en multipliant l'équation $(0.4)_2$ par ρ et en appliquant l'opérateur div , on arrive à l'équation de Poisson suivante (remarquant $\operatorname{div} \partial_t u \equiv 0$) :

$$\Delta q = \operatorname{div} L_3, \quad \text{avec} \quad L_3 = -(\rho - 1)\partial_t u - (\rho u - \nabla a) \cdot \nabla u + \rho h_3.$$

Parce que $B^{s-1}_{p,r}$ s'injecte dans L^{p^*} et de plus, $\partial_t u \in L^1_T(L^2)$ qui résulte de l'équation $(0.4)_2$, on a L_3 et donc ∇q appartenant à $L^1_T(L^p)$. L'étude des hautes fréquences est plus classique : on applique la décomposition de Littlewood-Paley et après l'estimation de commutateur.

Dans l'annexe §B.2, on établit une nouvelle estimation pour l'équation parabolique à coefficients variables dans les espaces de Besov avec exposant de Lebesgue infini. Ceci nous permet de traiter le cas limite $s = 1$, $p = \infty$, $r = 1$. Lorsqu'on considère le nouveau système vérifié par la différence $(\delta\rho, \delta u, \nabla\delta q)$ de deux solutions $(\rho^i, u^i, \nabla q^i)$, $i = 1, 2$ de (0.37), il y a une perte de régularité, à cause du caractère hyperbolique de l'équation sur u. Ceci nous conduit à démontrer la convergence de la suite de solutions approchées (et l'unicité) dans un espace moins régulier. Donc l'espace candidat naturel est $B^0_{\infty,1}$ au lieu de $B^s_{p,r}$. Ceci requiert $\nabla\delta q \in L^1_T(B^0_{\infty,1})$ et en conséquence, ceci demande $\operatorname{div}(u^1 \cdot \nabla\delta u) \in L^1_T(B^{-1}_{\infty,1})$, en raison de l'équation (0.68). Mais l'estimation

$$\|\operatorname{div}(u^1 \cdot \nabla\delta u)\|_{B^{d/p-1}_{p,1}} \leq C\|u^1\|_{B^{d/p+1}_{p,1}}\|\nabla\delta u\|_{B^{d/p-1}_{p,1}}$$

est vérifiée seulement si $p < \infty$. Donc, on est conduit à prouver que la suite de solutions approchées est une suite de Cauchy dans l'espace $\mathcal{C}_T(L^2)$. Comme le terme source de l'équation de $\delta\rho$ appartient à $L^2_T(L^2)$ au lieu de $L^1_T(L^2)$, il permet d'augmenter la régularité de $\delta\rho$ tel que $\delta\rho \in L^\infty_T(H^1) \cap L^2_T(H^2)$. De là, le terme source de l'équation de δu est dans $L^1_T(L^2)$ et donc l'estimation d'énergie de δu sort. Voir la section §3.4.2 pour l'analyse en détail et le théorème 3.6 pour le résultat.

Signalons enfin que si la densité est proche de 1 alors on peut écrire

$$\Delta q = \operatorname{div} L_4 \quad \text{avec} \quad L_4 = L_2 + \frac{\varrho}{\rho}\nabla q,$$

où $(\varrho/\rho)\nabla q$ est vue comme une perturbation. La théorie de Caldéron-Zygmund nous permet de traiter alors tout $p \in]1, \infty[$.

Minoration du temps de vie en dimension deux

En dimension deux, il est naturel de supposer que le tourbillon $\omega := \partial_{x_1} u^2 - \partial_{x_2} u^1 \equiv \partial_{x_1} v^2 - \partial_{x_2} v^1$ est dans l'espace de Besov limite $B^0_{\infty,1}$. Cette information permet de majorer les hautes fréquences de la vitesse u. D'un autre côté, les basses fréquences de u sont contrôlées par l'inégalité d'énergie (0.70) en considérant la densité $\varrho := \rho - 1$ comme des *petits* coefficients. Donc, il reste à majorer le tourbillon $\|\omega\|_{B^0_{\infty,1}}$ par la vitesse $\|u\|_{B^1_{\infty,1}}$ *linéairement* sur un intervalle de temps (qui peut dépendre de la taille de la densité), afin de borner u sur cet intervalle. Et ceci implique que le temps de vie tend vers l'infini à condition que la densité initiale ϱ_0 soit petite.

Plus précisément, en vertu de l'équation (0.35), ω vérifie l'équation suivante :

$$\partial_t\omega + v \cdot \nabla\omega + \omega\,\Delta b + (\partial_1\lambda\,\partial_2\Pi - \partial_2\lambda\,\partial_1\Pi) = 0 \quad \text{avec} \quad b = b(\rho), \quad \lambda = \lambda(\rho).$$

Conformément à la proposition 3.2 (qui est une version précisée de la proposition 0.6 où la croissance exponentielle est remplacée par une croissance linéaire) pour l'estimation a priori de l'équation de transport, $\|\omega\|_{B^0_{\infty,1}}$ est majorée par $\|\nabla v\|_{L^\infty}$ et $\|\operatorname{div} v\|_{C^\beta} \equiv \|\Delta b\|_{C^\beta}$:

$$\Big(\|\omega_0\|_{B^0_{\infty,1}} + \|\omega\Delta b + (\partial_1\lambda\,\partial_2\Pi - \partial_2\lambda\,\partial_1\Pi)\|_{B^0_{\infty,1}}\Big)\Big(1 + \int_0^t (\|\nabla v\|_{L^\infty} + \|\Delta b\|_{C^\beta})\Big).$$

Donc, l'estimation de produit et l'estimation de la pression ci-dessus permet de majorer ω par $C(\rho)\|u\|_{B^1_{\infty,1}}$, avec le coefficient $C(\rho)$ dépendant de ρ. Par conséquent, u reste bornée sur un intervalle de temps qui dépend de la densité et de plus, le temps de vie tend vers l'infini si la densité ρ tend vers une constante positive.

Chapitre 1

Low-Mach number limit system

This chapter is devoted to the well-posedness issue for the low-Mach number limit system (0.1) (or more generally, the system (0.33)) obtained from the full compressible Navier-Stokes system, in the whole space $\mathbb{R}^d$ with $d \geq 2$. The physical coefficients (μ, ν, k) in (0.1) are assumed to be regular functions of the temperature ϑ, and we consider only the viscous and heat-conducting case, namely Condition (0.8) is verified. In the case where the initial temperature (or density) is close to a positive constant, we establish the local existence and uniqueness of a solution in critical homogeneous Besov spaces of type $\dot{B}^s_{p,1}$. If, in addition, the initial velocity is small then we show that the solution exists for all positive time. In the fully nonhomogeneous case, we establish the local well-posedness in nonhomogeneous Besov spaces $B^s_{p,1}$ (still with critical regularity) for arbitrarily large data with positive initial temperature. As in the recent work by Abidi-Paicu [1] concerning the density dependent incompressible Navier-Stokes equations, the Lebesgue exponents of the Besov spaces for the temperature and the (modified) velocity, need not be the same. This enables us to consider initial data in Besov spaces with a *negative* index of regularity. At the end of this chapter, we will sketch the proof of the well-posedness results for the low-Mach number limit system (0.33) in the case of general gases.

Except for Section §1.4, the content of this chapter has been published in [37].

1.1 Main results

As showed in Section §0.2.1, by introducing a new divergence-free velocity vector $u := v - \alpha k \nabla \vartheta$, the low Mach number limit system (0.1) for perfect gases can be rewritten into System (0.22), concerning the new unknowns $(\vartheta, u, \nabla Q)$:

$$\left\{ \begin{array}{rcl} \partial_t \vartheta + u \cdot \nabla \vartheta - \mathrm{div}\,(\kappa \nabla \vartheta) &=& h_1(\vartheta), \\ \partial_t u + u \cdot \nabla u - \mathrm{div}\,(\eta \nabla u) + \vartheta \nabla Q &=& h_2(\vartheta, u), \\ \mathrm{div}\, u &=& 0. \end{array} \right. \tag{1.1}$$

Here κ, h_1, η, h_2 are defined by (0.12), (0.17), (0.18) and (0.19) respectively :

$$\kappa(\vartheta) = \alpha k \vartheta, \quad h_1(\vartheta) = -2\alpha k |\nabla \vartheta|^2, \quad \eta(\vartheta) = \alpha \mu \vartheta C_P,$$

and

$$h_2(\vartheta, u) = A_1 |\nabla \vartheta|^2 \nabla \vartheta + A_2 \Delta \vartheta \nabla \vartheta + A_3 \nabla \vartheta \cdot \nabla^2 \vartheta + A_4 \nabla u \cdot \nabla \vartheta + A_5 Du \cdot \nabla \vartheta,$$

with $A_i, i = 1, \cdots, 5$ are all regular functions of ϑ, given by (0.20). We will see that our solutions are regular enough to return from the above system (1.1) to the original system (0.1).

Recall that in the slightly nonhomogeneous case (that is under a smallness condition on the "temperature" $\theta := \vartheta - 1$), the optimal functional space $\dot{E}_T^{p_1,p_2}(\mathbb{R}^d)$ is defined by (0.59) in introduction part

$$\Big(\widetilde{\mathcal{C}}_T(\dot{B}^{d/p_1}_{p_1,1}) \cap L^1_T(\dot{B}^{d/p_1+2}_{p_1,1})\Big) \times \Big(\widetilde{\mathcal{C}}_T(\dot{B}^{d/p_2-1}_{p_2,1}) \cap L^1_T(\dot{B}^{d/p_2+1}_{p_2,1})\Big)^d \times \Big(L^1_T(\dot{B}^{d/p_2-1}_{p_2,1})\Big)^d.$$

In what follows, we shall denote

$$\begin{aligned}
\|\theta\|_{\dot{X}^{p_1}(T)} &= \|\theta\|_{\widetilde{L}^\infty_T(\dot{B}^{d/p_1}_{p_1,1})} + \|\theta\|_{L^1_T(\dot{B}^{d/p_1+2}_{p_1,1})},\\
\|u\|_{\dot{Y}^{p_2}(T)} &= \|u\|_{\widetilde{L}^\infty_T(\dot{B}^{d/p_2-1}_{p_2,1})} + \|u\|_{L^1_T(\dot{B}^{d/p_2+1}_{p_2,1})},\\
\|\nabla Q\|_{\dot{Z}^{p_2}(T)} &= \|\nabla Q\|_{L^1_T(\dot{B}^{d/p_2-1}_{p_2,1})},
\end{aligned}$$

and we will drop T in $\dot{X}^{p_1}(T)$, $\dot{Y}^{p_2}(T)$, $\dot{Z}^{p_3}(T)$ if $T = +\infty$.

In such case, for convenience, we can rewrite System (1.1) into System (0.60), *i.e.*

$$\left\{\begin{array}{rcl}
\partial_t\theta + u\cdot\nabla\theta - \bar{\kappa}\Delta\theta &=& r_1(\theta),\\
\partial_t u + u\cdot\nabla u - \bar{\eta}\Delta u + \nabla Q &=& r_2(\theta, u, \nabla Q),\\
\operatorname{div} u &=& 0,
\end{array}\right. \tag{1.2}$$

where $\bar{\kappa} = \kappa(1)$, $\bar{\eta} = \eta(1)$ and

$$\begin{aligned}
r_1(\theta) &= \operatorname{div}\big((\kappa(1+\theta) - \bar{\kappa})\nabla\theta\big) + h_1(1+\theta),\\
r_2(\theta, u, \nabla Q) &= \operatorname{div}\big((\eta(1+\theta) - \bar{\eta})\nabla u\big) - \theta\nabla Q + h_2(1+\theta, u).
\end{aligned}$$

Let us state the main well-posedness result for the slightly nonhomogeneous case :

Theorem 1.1. *Let* $\theta_0 \in \dot{B}^{d/p_1}_{p_1,1}$ *and* $u_0 \in \dot{B}^{d/p_2-1}_{p_2,1}$ *with* $\operatorname{div} u_0 = 0$. *Assume that*

$$\begin{gathered}
1 \le p_1 < 2d, \quad 1 \le p_2 < \infty, \quad p_1 \le 2p_2,\\
\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{d}, \quad \frac{1}{p_2} + \frac{1}{d} \ge \frac{1}{p_1} \quad \text{and} \quad \frac{1}{p_1} + \frac{1}{d} \ge \frac{1}{p_2}\cdot
\end{gathered} \tag{1.3}$$

There exist two constants τ and K depending only on the coefficients of System (1.2) *and on d, p_1, p_2, and satisfying the following properties :*

– *If*
$$\|\theta_0\|_{\dot{B}^{d/p_1}_{p_1,1}} \le \tau, \tag{1.4}$$
then there exists $T \in (0, +\infty]$ such that System (1.2) *has a unique solution* $(\theta, u, \nabla Q)$ *in $\dot{E}_T^{p_1,p_2}$, which satisfies*
$$\|\theta\|_{\dot{X}^{p_1}(T)} \le K\|\theta_0\|_{\dot{B}^{d/p_1}_{p_1,1}}$$
and
$$\|u\|_{\dot{Y}^{p_2}(T)} + \|\nabla Q\|_{\dot{Z}^{p_2}(T)} \le K\big(\|\theta_0\|_{\dot{B}^{d/p_1}_{p_1,1}} + \|u_0\|_{\dot{B}^{d/p_2-1}_{p_2,1}}\big).$$

– *If*
$$\|\theta_0\|_{\dot{B}^{d/p_1}_{p_1,1}} + \|u_0\|_{\dot{B}^{d/p_2-1}_{p_2,1}} \le \tau, \tag{1.5}$$
then $T = +\infty$ and the unique global solution satisfies
$$\|\theta\|_{\dot{X}^{p_1}} + \|u\|_{\dot{Y}^{p_2}} + \|\nabla Q\|_{\dot{Z}^{p_2}} \le K\big(\|\theta_0\|_{\dot{B}^{d/p_1}_{p_1,1}} + \|u_0\|_{\dot{B}^{d/p_2-1}_{p_2,1}}\big). \tag{1.6}$$

In addition, the flow map $(\theta_0, u_0) \mapsto (\theta, u, \nabla Q)$ is Lipschitz continuous from $\dot{B}^{d/p_1}_{p_1,1} \times \dot{B}^{d/p_2-1}_{p_2,1}$ to $\dot{E}_T^{p_1,p_2}$.

Remark 1.1. A similar statement for the nonhomogeneous incompressible Navier-Stokes equations has been obtained by Abidi-Paicu in [1]. There, the conditions over p_1 and p_2 (which stem from the structure of the nonlinearities) are not exactly the same as ours for there is no gain of regularity over the density and the right-hand side of the momentum equation in (0.60) is simpler.

Let us stress that in the above statement, the homogeneous Besov spaces for the velocity are almost the same as for the standard incompressible Navier-Stokes equation (except that in this latter case, one may take *any* space $\dot{B}^{d/p_2-1}_{p_2,r}$ with $1 \le p_2 < \infty$ and $1 \le r \le \infty$). In particular, here one may take p_2 as large as we want hence the regularity exponent $d/p_2 - 1$ may be negative and our result ensures that suitably oscillating *large* velocities give rise to a global solution.

The important observation for solving (1.2) is that all the "source terms" (that is the terms on the right-hand side) are at least quadratic. In a suitable functional framework – $\dot{E}^{p_1,p_2}_T$ given by our scaling considerations, we thus expect them to be negligible if the initial data are small. Hence, appropriate a priori estimates for the linearized system pertaining to (1.2) and suitable product estimates suffice to control the solution for all time if the data are small. This will enable us to prove the global existence. In addition, a classical argument borrowed from the one that is used in the constant density case will enable us to consider large u_0.

Let us now turn to the fully nonhomogeneous case. Then, in order to ensure the ellipticity of the second order operators in the left-hand side of (1.1), we have to assume that ϑ_0 is bounded by below by some positive constant. Proving a priori estimates for the heat or Stokes equations with variable time-dependent rough coefficients will be the key to our local existence statement. Bounding the gradient of the pressure (namely ∇Q) is the main difficulty. To achieve it, we will have to consider the elliptic equation

$$\operatorname{div}(\vartheta \nabla Q) = \operatorname{div} L \quad \text{with} \quad L := -u\cdot\nabla u + Du\cdot\nabla\eta + h_2. \tag{1.7}$$

In the energy framework (that is in Sobolev spaces H^s or in Besov spaces $B^{d/p_2}_{p_2,1}$ with $p_2 = 2$), this is quite standard. At the same time, if $p_2 \neq 2$, estimating ∇Q in $B^{d/p_2-1}_{p_2,1}$ requires some low order information in L^{p_2} over ∇Q. Thanks to suitable functional embedding, we shall see that if $p_2 \ge 2$ then it suffices to bound ∇Q in L^2, an information which readily stems from the standard L^2 elliptic theory. As a consequence, we will have to restrict our attention to a functional framework which ensures that L belongs to L^2. This will induce us to make further assumptions on p_1 and p_2 (compared to (1.3) in Theorem 1.1) so as to ensure in particular that ∇Q is in L^2. Consequently, the homogeneous critical framework is no longer appropriate since some additional control will be required over the low frequencies of the solution.

More precisely, we shall prove the existence of a solution in the nonhomogeneous space $E^{p_1,p_2}_T(\mathbb{R}^d)$ which is defined by (0.62) in introduction part. And hence by convenience, in what follows, we denote

$$\begin{aligned}
\|\theta\|_{X^{p_1}(T)} &= \|\theta\|_{\widetilde{L}^\infty_T(B^{d/p_1}_{p_1,1})} + \|\theta\|_{L^1_T(B^{d/p_1+2}_{p_1,1})},\\
\|u\|_{Y^{p_2}(T)} &= \|u\|_{\widetilde{L}^\infty_T(B^{d/p_2-1}_{p_2,1})} + \|u\|_{L^1_T(B^{d/p_2+1}_{p_2,1})},\\
\|\nabla Q\|_{Z^{p_2}(T)} &= \|\nabla Q\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}.
\end{aligned}$$

Let us state our main local-in-time existence result for the fully nonhomogeneous case.

Theorem 1.2. *Let* (p_1, p_2) *satisfy*

$$1 < p_1 \leq 4, \quad 2 \leq p_2 \leq 4, \quad \frac{1}{p_2} + \frac{1}{d} > \frac{1}{p_1} \tag{1.8}$$

with in addition

$$p_1 < 4 \ \text{ if } \ d = 2, \quad \text{and} \quad \frac{1}{p_1} + \frac{1}{d} > \frac{1}{p_2} \ \text{ if } \ d \geq 3.$$

For any initial temperature $\vartheta_0 = 1 + \theta_0$ *and velocity field* u_0 *which satisfy*

$$0 < m \leq \vartheta_0, \quad \operatorname{div} u_0 = 0 \quad \text{and} \quad \|\theta_0\|_{B^{d/p_1}_{p_1,1}} + \|u_0\|_{B^{d/p_2-1}_{p_2,1}} \leq M, \tag{1.9}$$

for some positive constants m, M, *there exists a positive time* T *depending only on* m, M, p_1, p_2, d, *and on the parameters of the system such that* (1.1) *has a unique solution* $(\vartheta, u, \nabla Q)$ *with* $(\theta, u, \nabla Q)$ *in* $E_T^{p_1,p_2}$. *Furthermore, for some constant* $C = C(d, p_1, p_2)$, *we have*

$$m \leq \vartheta \quad \text{and} \quad \|\theta\|_{X^{p_1}(T)} + \|u\|_{Y^{p_2}(T)} + \|\nabla Q\|_{Z^{p_2}(T)} \leq CM, \tag{1.10}$$

and the flow map $(\theta_0, u_0) \mapsto (\theta, u, \nabla Q)$ *is Lipschitz continuous.*

Remark 1.2. *The above theorems 1.1, 1.2 and the transformation* (0.12) $u = v - \alpha k \nabla \vartheta$ *ensure that the original system* (0.1) *is well-posed. More precisely, in the case* $1 \leq p_1 = p_2 < 2d$ *for the initial data* (ϑ_0, v_0) *satisfying the third equation, and* $(\vartheta_0 - 1, v_0)$ *in* $\dot{B}^{d/p_1}_{p_1,1} \times \dot{B}^{d/p_1-1}_{p_1,1}$ *with* (1.4), *we get a local solution* $(\vartheta, v, \nabla \Pi)$ *of* (0.1) *such that* $(\theta, u, \nabla Q) \in \dot{E}_T^{p_1,p_1}$ *and the solution is global if* (1.5) *holds. Under the same regularity assumptions with in addition* $2 \leq p_1 \leq 4$ *then if* ϑ_0 *is just bounded from below, we get a local solution in* $E_T^{p_1,p_1}$. *In the case* $p_1 \neq p_2$, *a similar result holds true. It is more complicated to state, though.*

Let us end this section with a few comments and a short list of open questions that we plan to address in the future.

- To simplify the presentation, we restricted to the *free* evolution of a solution to (1.1). As in e.g. [29, 32], our methods enable us to treat the case where the fluid is subject to some external body force.
- We expect similar results for reacting flows as in [44] (as it only introduces coupling with parabolic equations involving reactants, the scaling of which is the same as that of ϑ). We here restricted our analysis to gases for simplicity only.
- Granted with the above results, it is natural to study the asymptotics ε going to 0 of the rescaled system (0.9) in the above functional framework. This would extend some of the results of Alazard in [2] to the case of rough data.

The rest of this chapter unfolds as follows. In Section 1.2, we focus on the proof of our first well-posedness result (pertaining to the case where the initial temperature is close to a constant) whereas our second well-posedness result is proved in Section 1.3. In the last section §1.4, we will consider the general gases case briefly.

1.2 Slightly nonhomogeneous case

This section is devoted to the well-posedness issue for System (1.2) in the slightly nonhomogeneous case, i.e. Theorem 1.1. The proof strongly relies on a priori estimates for the linearized equations about zero which are already given by Proposition 0.7 and Proposition 0.8 in the introduction part. Besides, the estimates for the products given by Proposition A.2 and the estimates for the actions given by Proposition 0.5 also play an important role on controlling the "source" terms. In fact, we will thoroughly use the following inequalities, which come from the usage of Bony's decomposition and estimates in Proposition A.1 :

– For any $p\in[1,\infty)$, the homogeneous Besov space $\dot B^{\frac dp}_{p,1}$ is an algebra. Hence

$$\|uv\|_{\dot B^{\frac dp}_{p,1}} \lesssim \|u\|_{\dot B^{\frac dp}_{p,1}}\|v\|_{\dot B^{\frac dp}_{p,1}}, \tag{1.11}$$

and moreover for any $f\in\mathcal C^1:\mathbb R^d\to\mathbb R$ with 0 as a fixed point, we have

$$\|f(u)\|_{\dot B^{\frac dp}_{p,1}} \le C(\|u\|_{L^\infty})\|u\|_{\dot B^{\frac dp}_{p,1}}. \tag{1.12}$$

– If $\frac1{p_1}\le\frac1{p_2}+\frac1d$, then

$$\|uv\|_{\dot B^{\frac d{p_1}}_{p_1,1}} \lesssim \|u\|_{\dot B^{\frac d{p_2}}_{p_2,1}}\|v\|_{\dot B^{\frac d{p_1}}_{p_1,1}} + \|u\|_{\dot B^{\frac d{p_2}+1}_{p_2,1}}\|v\|_{\dot B^{\frac d{p_1}-1}_{p_1,1}}. \tag{1.13}$$

– If $p_1<2d$, $p_1\le 2p_2$, $\frac1{p_2}\le\frac1{p_1}+\frac1d$, then

$$\|uv\|_{\dot B^{\frac d{p_2}-1}_{p_2,1}} \lesssim \|u\|_{\dot B^{\frac d{p_1}-1}_{p_1,1}}\|v\|_{\dot B^{\frac d{p_1}}_{p_1,1}} + \|u\|_{\dot B^{\frac d{p_1}}_{p_1,1}}\|v\|_{\dot B^{\frac d{p_1}-1}_{p_1,1}}. \tag{1.14}$$

In fact, in the case $p_1\le p_2$, then if $p_1<2d$, we have

$$\|uv\|_{\dot B^{\frac d{p_2}-1}_{p_2,1}} \lesssim \|uv\|_{\dot B^{\frac d{p_1}-1}_{p_1,1}} \lesssim \|u\|_{\dot B^{\frac d{p_1}-1}_{p_1,1}}\|v\|_{\dot B^{\frac d{p_1}}_{p_1,1}},$$

In the case $p_1>p_2$, then noticing that

$$\|T_uv\|_{\dot B^{\frac d{p_2}-1}_{p_2,1}} \lesssim \|u\|_{\dot B^{\frac d{p_3}-1}_{p_3,1}}\|v\|_{\dot B^{\frac d{p_1}}_{p_1,1}}, \quad \text{if}\quad \frac1{p_3}=\frac1{p_2}-\frac1{p_1}\le\min\{\frac1{p_1},\frac1d\},$$

Estimate (1.14) also holds.

– If $\frac1{p_1}+\frac1{p_2}>\frac1d$, $\frac1{p_2}\le\frac1{p_1}+\frac1d$, then

$$\|uv\|_{\dot B^{\frac d{p_2}-1}_{p_2,1}} \lesssim \|u\|_{\dot B^{\frac d{p_1}}_{p_1,1}}\|v\|_{\dot B^{\frac d{p_2}-1}_{p_2,1}}. \tag{1.15}$$

For proving existence, we will follow a standard procedure, first we construct a sequence of approximate solutions, second, we prove uniform bounds for them, and finally we show the convergence to some solution of the system. In the case of *large* initial velocity, we will have to split the constructed velocity into the free solution of the Stokes system with initial data u_0, and the discrepancy to this free velocity. Stability estimates and uniqueness will be obtained afterward by the same argument as the convergence of the sequence.

Step 1. Approximate solutions

Solving System (1.2) will be based on an iterative scheme : first we set the solution to the following linear system [1] :

$$\begin{cases} \partial_t\theta^{n+1}+u^n\cdot\nabla\theta^{n+1}-\bar\kappa\Delta\theta^{n+1} &= r_1^n,\\ \partial_tu^{n+1}+u^n\cdot\nabla u^{n+1}-\bar\eta\Delta u^{n+1}+\nabla Q^{n+1} &= r_2^n,\\ \operatorname{div}u^{n+1} &= 0,\\ (\theta^{n+1},u^{n+1})_{|t=0} &= (\dot S_{n+1}\theta_0,\dot S_{n+1}u_0),\end{cases} \tag{1.16}$$

with $\dot S_{n+1}:=\chi(2^{-(n+1)}D)$ defined in (0.43) and, denoting $\vartheta^n:=1+\theta^n$,

$r_1^n:=r_1^n(\theta^n)=\operatorname{div}((\kappa(\vartheta^n)-\bar\kappa)\nabla\theta^n)-\kappa'(\vartheta^n)|\nabla\theta^n|^2$,

$$r_2^n:=r_2^n(\theta^n,u^n,\nabla Q^n)=\operatorname{div}((\eta(\vartheta^n)-\bar\eta)\nabla u^n)-\theta^n\nabla Q^n+A_1^n|\nabla\theta^n|^2\nabla\theta^n \\ +A_2^n\Delta\theta^n\nabla\theta^n+A_3^n\nabla^2\theta^n\cdot\nabla\theta^n+A_4^n\nabla u^n\cdot\nabla\theta^n+A_5^nDu^n\cdot\nabla\theta^n.$$

Above, it is understood that $A_i^n:=A_i(\vartheta^n)$ with A_i defined by (0.20).

1. Note that the existence of solution for this system may be deduced from the case with no convection. Indeed, considering the convection terms as source terms, it is not difficult to construct an iterative scheme the convergence of which is based on the estimates given by Proposition 0.7 and Proposition 0.8.

Step 2. Uniform bounds

In order to bound $(\theta^{n+1}, u^{n+1}, \nabla Q^{n+1})$, one may take advantage of Proposition 0.7 with $s = d/p_1$ and Lebesgue exponents (p_1, p_2) (here comes the assumption that $1/p_1 \leq 1/d + 1/p_2$), and of Proposition 0.8 with $s = d/p_2 - 1$ and exponents (p_2, p_2). Concerning θ^{n+1}, if $p_2 \leq p_1$ then we use the embedding $\dot{B}^{\frac{d}{p_2}+1}_{p_2,1} \hookrightarrow \dot{B}^{\frac{d}{p_1}+1}_{p_1,1}$. We eventually get

$$\|\theta^{n+1}\|_{\dot{X}^{p_1}(t)} \lesssim e^{\|u^n\|_{L^1_t(\dot{B}^{d/p_2+1}_{p_2,1})}} \Big(\|\dot{S}_{n+1}\theta_0\|_{\dot{B}^{d/p_1}_{p_1,1}} + \|r_1^n\|_{L^1_t(\dot{B}^{d/p_1}_{p_1,1})}\Big), \tag{1.17}$$

$$\|u^{n+1}\|_{\dot{Y}^{p_2}(t)} + \|\nabla Q^{n+1}\|_{\dot{Z}^{p_2}(t)} \lesssim e^{\|u^n\|_{L^1_t(\dot{B}^{d/p_2+1}_{p_2,1})}} \Big(\|\dot{S}_{n+1}u_0\|_{\dot{B}^{d/p_2-1}_{p_2,1}} + \|r_2^n\|_{L^1_t(\dot{B}^{d/p_2-1}_{p_2,1})}\Big). \tag{1.18}$$

Let us now bound r_1^n and r_2^n. Using (1.11) and (1.12), we easily get

$$\|r_1^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} \lesssim \left(1 + \|\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}}\right)\left(\|\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} \|\theta^n\|_{\dot{B}^{\frac{d}{p_1}+2}_{p_1,1}} + \|\nabla\theta^n\|^2_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}}\right). \tag{1.19}$$

As regards r_2^n, it is mostly a matter of bounding the following terms in $L^1_t(\dot{B}^{\frac{d}{p_2}-1}_{p_2,1})$ (noticing $\operatorname{div} u = 0$) :

$$\nabla^2\theta^n \cdot \nabla\theta^n, \quad |\nabla\theta^n|^2\nabla\theta^n, \quad \operatorname{div}(\theta^n\nabla u^n), \quad \nabla\theta^n \otimes \nabla u^n \ \text{ and } \ \theta^n\nabla Q^n.$$

Indeed, on any interval $[0, T]$, taking the ϑ^n dependency of the coefficients into account will only multiply the estimates by some continuous function of $\|\theta^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p_1}}_{p_1,1})}$, by (1.12) and (1.15). In what follows, this function will be denoted by C_{θ^n}.

Now, by (1.14), we have

$$\|\nabla^2\theta^n \cdot \nabla\theta^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} \lesssim \|\nabla^2\theta^n\|_{\dot{B}^{\frac{d}{p_1}-1}_{p_1,1}}\|\nabla\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} + \|\nabla^2\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}}\|\nabla\theta^n\|_{\dot{B}^{\frac{d}{p_1}-1}_{p_1,1}}, \tag{1.20}$$

and

$$\||\nabla\theta^n|^2\nabla\theta^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} \lesssim \|\nabla\theta^n\|_{\dot{B}^{\frac{d}{p_1}-1}_{p_1,1}}\|\nabla\theta^n\|^2_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}}. \tag{1.21}$$

We also easily get

$$\|\operatorname{div}(\theta^n\nabla u^n)\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} \lesssim \|\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}}\|\nabla u^n\|_{\dot{B}^{\frac{d}{p_2}}_{p_2,1}} + \|\theta^n\|_{\dot{B}^{\frac{d}{p_1}+1}_{p_1,1}}\|\nabla u^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}}. \tag{1.22}$$

Finally, according to Inequality (1.15),

$$\begin{aligned}\|\nabla u^n \otimes \nabla\theta^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} &\lesssim \|\nabla\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}}\|\nabla u^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}},\\ \|\theta^n\nabla Q^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} &\lesssim \|\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}}\|\nabla Q^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}}.\end{aligned}$$

So, plugging all the above inequalities in (1.17), (1.18) finally implies that

$$\|\theta^{n+1}\|_{\dot{X}^{p_1}(t)} \lesssim e^{C\|u^n\|_{L^1_t(\dot{B}^{\frac{d}{p_2}+1}_{p_2,1})}}\Big(\|\theta_0\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} + C_{\theta^n}\|\theta^n\|^2_{\dot{X}^{p_1}(t)}\Big), \tag{1.23}$$

$$\begin{aligned}\|u^{n+1}\|_{\dot{Y}^{p_2}(t)} + \|\nabla Q^{n+1}\|_{\dot{Z}^{p_2}(t)} \lesssim e^{C\|u^n\|_{L^1_t(\dot{B}^{\frac{d}{p_2}+1}_{p_2,1})}}\Big(\|u_0\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}}\\ + C_{\theta^n}\|\theta^n\|_{\dot{X}^{p_1}(t)}\big(\|\theta^n\|_{\dot{X}^{p_1}(t)} + \|u^n\|_{\dot{Y}^{p_2}(t)} + \|\nabla Q^n\|_{\dot{Z}^{p_2}(t)}\big)\Big).\end{aligned} \tag{1.24}$$

Note that the right-hand sides involves only initial data and at least quadratic combinations of the norms of $(\theta^n, u^n, \nabla Q^n)$. From a standard induction argument, it is thus easy to find some small constant τ such that if

$$\|\theta_0\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} + \|u_0\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} \leq \tau \tag{1.25}$$

then, for all $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$, we have for some $K > 0$ depending only on the parameters of the system and on d, p_1, p_2,

$$\|\theta^n\|_{\dot{X}^{p_1}(t)} + \|u^n\|_{\dot{Y}^{p_2}(t)} + \|\nabla Q^n\|_{\dot{Z}^{p_2}(t)} \leq K\big(\|\theta_0\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} + \|u_0\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}}\big). \tag{1.26}$$

This completes the proof of uniform estimates in the case where both θ_0 and u_0 are small.

Let us now concentrate on the case where only θ_0 is small. Assuming that T has been chosen so that

$$\exp\bigg(C\int_0^T \|u^n\|_{\dot{B}^{\frac{d}{p_2}+1}_{p_2,1}}\,dt\bigg) \leq 2, \tag{1.27}$$

and that θ_0 is small enough, Inequality (1.23) still implies that

$$\|\theta^{n+1}\|_{\dot{X}^{p_1}(T)} \leq K\|\theta_0\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} \tag{1.28}$$

if (1.4) is satisfied and if θ^n also satisfies (1.28).

However, if u_0 is large then Inequality (1.24) is not enough to bound u^{n+1}. Therefore we introduce the "free" solution u_L to the heat equation

$$\left\{\begin{array}{rcl} \partial_t u_L - \bar{\eta}\Delta u_L & = & 0, \\ u_L|_{t=0} & = & u_0, \end{array}\right. \tag{1.29}$$

and define $u_L^n := \dot{S}_n u_L$. Of course that $\operatorname{div} u_0 \equiv 0$ implies that $\operatorname{div} u_L \equiv 1$. Now, $\bar{u}^{n+1} := u^{n+1} - u_L^{n+1}$ satisfies

$$\left\{\begin{array}{rcl} \partial_t \bar{u}^{n+1} + u^n \cdot \nabla \bar{u}^{n+1} - \bar{\eta}\Delta\bar{u}^{n+1} + \nabla Q^{n+1} & = & \bar{r}_2^n, \\ \operatorname{div} \bar{u}^{n+1} & = & 0, \\ \bar{u}^{n+1}|_{t=0} & = & 0, \end{array}\right.$$

with $\bar{r}_2^n = r_2^n - u^n \cdot \nabla u_L^{n+1}$.

Note that $u^n \cdot \nabla u_L^{n+1} = \operatorname{div}(u^n \otimes u_L^{n+1})$. Hence, by (1.11) we have

$$\|\bar{r}_2^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} \lesssim \|r_2^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} + \|u^n\|_{\dot{B}^{\frac{d}{p_2}}_{p_2,1}} \|u_L^{n+1}\|_{\dot{B}^{\frac{d}{p_2}}_{p_2,1}}. \tag{1.30}$$

Hence, bounding r_2^n as above but splitting u^n into $\bar{u}^n + u_L^n$ when dealing with the terms $\nabla u^n \cdot \nabla\theta^n$ or $Du^n \cdot \nabla\theta^n$, we get under hypothesis (1.27), for all $t \in [0, T]$,

$$\begin{aligned} \|\bar{u}^{n+1}\|_{\dot{Y}^{p_2}(t)} + \|\nabla Q^{n+1}\|_{\dot{Z}^{p_2}(t)} \leq C\Big(&\|\theta^n\|_{\dot{X}^{p_1}(t)}\big(\|\theta^n\|_{\dot{X}^{p_1}(t)} + \|\bar{u}^n\|_{\dot{Y}^{p_2}(t)} + \|\nabla Q^n\|_{\dot{Z}^{p_2}(t)}\big) \\ &+\|u_L\|_{L^2_t(\dot{B}^{\frac{d}{p_2}}_{p_2,1})}\big(\|u^n\|_{L^2_t(\dot{B}^{\frac{d}{p_2}}_{p_2,1})} + \|\nabla\theta^n\|_{L^2_t(\dot{B}^{\frac{d}{p_1}}_{p_1,1})}\big) + \|\theta^n\|_{L^\infty_t(\dot{B}^{\frac{d}{p_1}}_{p_1,1})}\|\nabla u_L\|_{L^1_t(\dot{B}^{\frac{d}{p_2}}_{p_2,1})}\Big). \end{aligned}$$

Therefore, if we assume in addition that T has been chosen so that

$$\|u_L\|_{L^2_T(\dot{B}^{\frac{d}{p_2}}_{p_2,1}) \cap L^1_T(\dot{B}^{\frac{d}{p_2}+1}_{p_2,1})} \leq \tau \tag{1.31}$$

and if

$$\|\bar{u}^n\|_{\dot{Y}^{p_2}(T)} + \|\nabla Q^n\|_{\dot{Z}^{p_2}(T)} \le \tau \tag{1.32}$$

then we have also (taking τ smaller if needed) (1.27) and

$$\|\bar{u}^{n+1}\|_{\dot{Y}^{p_2}(T)} + \|\nabla Q^{n+1}\|_{\dot{Z}^{p_2}(T)} \le \tau.$$

Now, an elementary induction argument enables us to conclude that both (1.27) and (1.32) are satisfied (for all $n \in \mathbb{N}$) if T has been chosen so that (1.31) holds.

Step 3. Convergence of the scheme

Let us just treat the case where only local existence is expected (that is u_0 may be large). We fix some time T such that (1.31) is fulfilled. Let $(\delta\theta^{n+1}, \delta u^{n+1}, \delta Q^{n+1}) := (\theta^{n+1} - \theta^n, u^{n+1} - u^n, Q^{n+1} - Q^n)$. We have

$$\begin{cases} \partial_t \delta\theta^{n+1} + u^n \cdot \nabla \delta\theta^{n+1} - \bar{\kappa}\Delta\delta\theta^{n+1} & = & -\delta u^n \cdot \nabla\theta^n + r_1^n - r_1^{n-1}, \\ \partial_t \delta u^{n+1} + u^n \cdot \nabla\delta u^{n+1} - \bar{\eta}\Delta\delta u^{n+1} + \nabla\delta Q^{n+1} & = & -\delta u^n \cdot \nabla u^n + r_2^n - r_2^{n-1}, \\ \operatorname{div} \delta u^{n+1} & = & 0, \\ (\delta\theta^{n+1}, \delta u^{n+1})|_{t=0} & = & (\dot{\Delta}_n\theta_0, \dot{\Delta}_n u_0). \end{cases}$$

By arguing exactly as in the proof of the stability estimates below, it is not difficult to establish that if τ has been chosen small enough in (1.31) then for all $n \ge 1$,

$$\begin{aligned} \|\delta\theta^{n+1}\|_{\dot{X}^{p_1}(T)} + \|\delta u^{n+1}\|_{\dot{Y}^{p_2}(T)} + \|\nabla\delta Q^{n+1}\|_{\dot{Z}^{p_2}(T)} \le C\big(2^{n\frac{d}{p_1}}\|\dot{\Delta}_n\theta_0\|_{L^{p_1}} + 2^{n(\frac{d}{p_2}-1)}\|\dot{\Delta}_n u_0\|_{L^{p_2}}\big) \\ + \frac{1}{2}\big(\|\delta\theta^n\|_{\dot{X}^{p_1}(T)} + \|\delta u^n\|_{\dot{Y}^{p_2}(T)} + \|\nabla\delta Q^n\|_{\dot{Z}^{p_2}(T)}\big). \end{aligned}$$

Hence $(\theta^n, u^n, \nabla Q^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\dot{E}_T^{p_1,p_2}$. The limit $(\theta, u, \nabla Q)$ belongs to $\dot{E}_T^{p_1,p_2}$ and obviously satisfies System (1.2).

Step 4. Uniqueness and stability estimates

Let us consider two solutions $(\theta^1, u^1, \nabla Q^1)$ and $(\theta^2, u^2, \nabla Q^2)$ of System (1.2), in the space $\dot{E}_T^{p_1,p_2}$ with (p_1, p_2) satisfying (1.3). The difference $(\delta\theta, \delta u, \delta Q) := (\theta^2 - \theta^1, u^2 - u^1, Q^2 - Q^1)$ between these two solutions satisfies

$$\begin{cases} \partial_t\delta\theta + u^1 \cdot \nabla\delta\theta - \bar{\kappa}\Delta\delta\theta = -\delta u \cdot \nabla\theta^2 + r_1(\theta^2) - r_1(\theta^1), \\ \partial_t \delta u + u^1 \cdot \nabla\delta u - \bar{\eta}\Delta\delta u + \nabla\delta Q = -\delta u \cdot \nabla u^2 + r_2(\theta^2, u^2, \nabla Q^2) - r_2(\theta^1, u^1, \nabla Q^1), \\ \operatorname{div} \delta u = 0. \end{cases}$$

Therefore, according to Propositions 0.7 and 0.8, we have for all $t \in [0, T]$,

$$\|\delta\theta\|_{\dot{X}^{p_1}(t)} \lesssim e^{\|u^1\|_{L_t^1(\dot{B}_{p_2,1}^{\frac{d}{p_2}+1})}} \Big(\|\delta\theta_0\|_{\dot{B}_{p_1,1}^{\frac{d}{p_1}}} + \|\delta u \cdot \nabla\theta^2\|_{L_t^1(\dot{B}_{p_1,1}^{\frac{d}{p_1}})} + \|r_1(\theta^2) - r_1(\theta^1)\|_{L_t^1(\dot{B}_{p_1,1}^{\frac{d}{p_1}})}\Big),$$

$$\begin{aligned} \|\delta u\|_{\dot{Y}^{p_2}(t)} + \|\nabla\delta Q\|_{\dot{Z}^{p_2}(t)} \lesssim e^{\|u^1\|_{L_t^1(\dot{B}_{p_2,1}^{\frac{d}{p_2}+1})}} \Big(\|\delta u_0\|_{\dot{B}_{p_1,1}^{\frac{d}{p_2}-1}} + \|\delta u \cdot \nabla u^2\|_{L_t^1(\dot{B}_{p_2,1}^{\frac{d}{p_2}-1})} \\ + \|r_2(\theta^2, u^2, \nabla Q^2) - r_2(\theta^1, u^1, \nabla Q^1)\|_{L_t^1(\dot{B}_{p_2,1}^{\frac{d}{p_2}-1})}\Big). \end{aligned}$$

The nonlinear terms in the right-hand side may be handled exactly as in the proof of the uniform estimates (as the norms which are involved are the same, there are no further conditions on p_1 and p_2). For instance, by (1.13), we have for $\frac{1}{p_1} \le \frac{1}{p_2} + \frac{1}{d}$,

$$\|\delta u \cdot \nabla\theta^2\|_{L^1_t(\dot B^{\frac{d}{p_1}}_{p_1,1})} \lesssim \|\delta u\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})}\|\nabla\theta^2\|_{L^2_t(\dot B^{\frac{d}{p_1}}_{p_1,1})} + \|\delta u\|_{L^1_t(\dot B^{\frac{d}{p_2}+1}_{p_2,1})}\|\nabla\theta^2\|_{L^\infty_t(\dot B^{\frac{d}{p_1}-1}_{p_1,1})},$$

and, because

$$r_1(\theta^2) - r_1(\theta^1) = \operatorname{div}\Big(\big(\kappa(\vartheta^2)-\kappa(\vartheta^1)\big)\nabla\theta^2 + \big(\kappa(\vartheta^1)-\bar\kappa\big)\nabla\delta\theta\Big)$$
$$-\big(\kappa'(\vartheta^2)-\kappa'(\vartheta^1)\big)|\nabla\theta^2|^2 - \kappa'(\vartheta^1)\big(\nabla(\theta^1+\theta^2)\cdot\nabla\delta\theta\big),$$

we have, according to (1.11) and (1.12),

$$\|r_1(\theta^2)-r_1(\theta^1)\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}} \lesssim C_{\theta^1,\theta^2}\Big(\big(\|\nabla\theta^1\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}} + \|\nabla\theta^2\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\big)\|\nabla\delta\theta\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}$$
$$+\big(\|\nabla\theta^2\|^2_{\dot B^{\frac{d}{p_1}}_{p_1,1}} + \|\nabla^2\theta^2\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\big)\|\delta\theta\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}} + \|\theta^1\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\|\Delta\delta\theta\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\Big).$$

We may proceed similarly in order to bound the right-hand side of the inequality for δu. We eventually get for all $t \in [0,T]$,

$$\|\delta\theta\|_{\dot X^{p_1}(t)} \le C_{\theta^1,\theta^2,u^1}\Big(\|\delta\theta_0\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}} + \|\theta^2\|_{\dot X^{p_1}(t)}\|\delta u\|_{\dot Y^{p_2}(t)} + (\|\theta^1\|_{\dot X^{p_1}(t)} + \|\theta^2\|_{\dot X^{p_1}(t)})\|\delta\theta\|_{\dot X^{p_1}(t)}\Big),$$

$$\|\delta u\|_{\dot Y^{p_2}(t)} + \|\nabla\delta Q\|_{\dot Z^{p_2}(t)} \le C_{\theta^1,\theta^2,u^1}\Big(\|\delta u_0\|_{\dot B^{\frac{d}{p_2}-1}_{p_2,1}} + \|u^2\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})}(\|\delta\theta\|_{\dot X^{p_1}(t)} + \|\delta u\|_{\dot Y^{p_2}(t)})$$
$$+\big(\|\nabla u^2\|_{L^1_t(\dot B^{\frac{d}{p_2}}_{p_2,1})} + \|\nabla Q^2\|_{L^1_t(\dot B^{\frac{d}{p_2}-1}_{p_2,1})}\big)\|\delta\theta\|_{\dot X^{p_1}(t)}$$
$$+\big(\|\theta^1\|_{\dot X^{p_1}(t)} + \|\theta^2\|_{\dot X^{p_1}(t)}\big)\big(\|\delta\theta\|_{\dot X^{p_1}(t)} + \|\delta u\|_{\dot Y^{p_2}(t)} + \|\nabla\delta Q\|_{\dot Z^{p_2}(t)}\big)\Big).$$

In the case where $(\theta^1, u^1, \nabla Q^1)$ and $(\theta^2, u^2, \nabla Q^2)$ are small enough on $[0,T]$, all the terms involving $(\delta\theta, \delta u)$ in the right-hand side may be absorbed by the left-hand side. This yields stability estimates on the whole interval $[0,T]$, and implies uniqueness.

The case where the velocity is large requires more care for it is not clear that the terms corresponding to $\operatorname{div}(\delta\theta\cdot\nabla u^2)$, $\nabla u^2\cdot\nabla\delta\theta$, $\delta u\cdot\nabla u^2$ and $\delta\theta\nabla Q^2$ are small compared to $\|\delta\theta\|_{\dot X^{p_1}(t)} + \|\delta u\|_{\dot Y^{p_2}(t)} + \|\nabla\delta Q\|_{\dot Z^{p_2}(t)}$. However, we notice that they may be bounded in $L^1_t(\dot B^{\frac{d}{p_2}-1}_{p_2,1})$ by

$$(\|\nabla\delta\theta\|_{L^2_t(\dot B^{\frac{d}{p_1}}_{p_1,1})} + \|\delta u\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})})\|u^2\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})} + \|\delta\theta\|_{L^\infty_t(\dot B^{\frac{d}{p_1}}_{p_1,1})}\Big(\|\nabla u^2\|_{L^1_t(\dot B^{\frac{d}{p_2}}_{p_2,1})} + \|\nabla Q^2\|_{L^1_t(\dot B^{\frac{d}{p_2}-1}_{p_2,1})}\Big).$$

Obviously the terms corresponding to u^2 and ∇Q^2 go to zero when t tends to 0. If both solutions coincide initially, this implies uniqueness on a small enough time interval. Then uniqueness on the whole interval $[0,T]$ follows from standard continuity arguments.

For proving stability estimates, one may further decompose u^1 and u^2 into

$$u^1 = \bar u^1 + u_L \quad\text{and}\quad u^2 = \bar u^2 + u_L,$$

where u_L stands for the free solution to the Stokes system that has been defined in (1.29). We can thus write

$$\|u^2\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})\cap L^1_t(\dot B^{\frac{d}{p_2}+1}_{p_2,1})} \le \|u_L\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})\cap L^1_t(\dot B^{\frac{d}{p_2}+1}_{p_2,1})} + \|\bar u^2\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})\cap L^1_t(\dot B^{\frac{d}{p_2}+1}_{p_2,1})}.$$

If T has been chosen so that (1.31) holds true and if $(\theta^2, u^2, \nabla Q^2)$ is the solution that has been constructed above then we conclude that the above terms may be bounded by τ. So they may be absorbed by the left-hand side, and it is thus possible to get the continuity of the flow map on $[0, T]$ for T satisfying (1.31). The details are omitted.

1.3 Fully nonhomogeneous case

In this section we establish local well-posedness results in the fully nonhomogeneous case (i.e. Theorem 1.2) : we just assume that the initial temperature is positive and tends to some positive constant at infinity (we take 1 for notational simplicity). In this framework, the a priori estimates for the linear equations considered in Section §1.2, i.e. Proposition 0.7 and Proposition 0.8, are not sufficient to bound the solutions to (1.1) even at small time. The reason why is that some quadratic terms such as $\nabla u \cdot \nabla\theta$ or $\theta\nabla Q$ may be of the same order as the terms of the left-hand side hence cannot be absorbed any longer. In the fully nonhomogeneous case, the appropriate linear equations that have to be considered have *variable coefficients* in their main order terms.

In the first part of this section, we present new a priori estimates for these linear equations in the solution space $E_T^{p_1,p_2}$. The proofs can be found in Appendix §B.1 and §B.4 and as we believe this type of estimates to be of interest in other contexts, we provide there the statements for a wider range of Lebesgue and regularity exponents than the solution space $E_T^{p_1,p_2}$ here. The second part of this section is devoted to the proof of Theorem 1.2. We remark here that the nonhomogeneous version of the product estimates such as (1.11)-(1.15) also holds true. Hence we will also use them to bound the nonlinearities.

1.3.1 Linearized equations

In order to bound the temperature, we shall establish a priori estimates in nonhomogeneous Besov norms for the solutions to

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta - \operatorname{div}(\kappa\nabla\theta) &= f, \\ \theta|_{t=0} &= \theta_0. \end{cases} \tag{1.33}$$

Here the conductivity function κ is strictly positive such that

$$\kappa_* := \min_{(t,x)\in[0,T]\times\mathbb{R}^d} \kappa(t,x) > 0. \tag{1.34}$$

Besides, the scalar function $\kappa - 1$, the divergence free vector-field u, the initial data θ_0 and the source term f are smooth enough and decay at infinity.

We will have the following estimate for θ in the Banach space $X^{p_1}(T)$, the proof (and also the generalized version) of which can be found in Appendix §B.1 :

Proposition 1.1. *Let θ satisfy (1.33) on $[0,T]\times\mathbb{R}^d$. Let $(p_1,p_2)\in(1,\infty)^2$ fulfill*

$$\frac{1}{p_1} \le \frac{1}{p_2} + \frac{1}{d}\cdot \tag{1.35}$$

Then there exists a constant C_1 depending only on d, p_1, p_2, κ_* such that θ satisfies for all $t\in[0,T]$:

$$\begin{aligned} \|\theta\|_{X^{p_1}(t)} \le \exp\left\{C_1\left(\|\nabla u\|_{L^1_t(B^{d/p_2}_{p_2,1})} + \|\nabla\kappa\|^2_{L^2_t(B^{d/p_1}_{p_1,1})}\right)\right\} \\ \times C_1\left(\|\theta_0\|_{B^{d/p_1}_{p_1,1}} + \|\Delta_{-1}\theta\|_{L^1_t(L^{p_1})} + \|f\|_{L^1_t(B^{d/p_1}_{p_1,1})}\right). \end{aligned} \tag{1.36}$$

In the fully nonhomogeneous case, the appropriate linearized momentum equation turns out to be

$$\begin{cases} \partial_t u + v\cdot\nabla u - \mathrm{div}\,(\eta\nabla u) + \lambda\nabla Q &= h, \\ \mathrm{div}\,u &= 0, \\ u|_{t=0} &= u_0, \end{cases} \tag{1.37}$$

where v is a given divergence free vector-field, and (η, λ) are given smooth positive functions. Suppose that v, $\eta-1$, $\lambda-1$, the initial data u_0 and the source term h are smooth and decay at infinity. We have the following a priori estimate for this linear system (see Appendix §B.4 for the proof) :

Proposition 1.2. *Let $T>0$ and $(u,\nabla Q)$ be a solution to (1.37) on $[0,T]\times\mathbb{R}^d$. Let $p_1\in[1,\infty)$ and $p_2\in[2,4]$ satisfy*

$$p_2 \le \frac{2p_1}{p_1-2} \ \text{ if } \ p_1>2, \quad \frac{1}{p_2}\le\frac{1}{p_1}+\frac{1}{d}, \quad \text{and} \quad (p_1,p_2)\neq(4,4) \ \text{ if } \ d=2. \tag{1.38}$$

Assume that there exist some constants $0<m<M$, $c_\lambda(d,p_1,p_2)$ small enough, and $N\in\mathbb{N}$ such that

$$\begin{aligned} &\min(\eta,\lambda)\ge m, \quad \|\lambda\|_{L^\infty([0,T]\times\mathbb{R}^d)} + \|\nabla\lambda\|_{L^\infty_T(B^{d/p_1-1}_{p_1,1})} \le M, \\ &\inf_{x\in\mathbb{R}^d, t\in[0,T]} S_N\lambda(t,x)\ge m/2, \quad \|\lambda - S_N\lambda\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\le c_\lambda m. \end{aligned} \tag{1.39}$$

Then there exists a constant C_2 depending on d,p_1,p_2,m,M,N such that the following a priori estimates hold :

$$\begin{aligned} &\|u\|_{Y^{p_2}(t)} + \|\nabla Q\|_{Z^{p_2}(t)} \\ &\quad \le e^{C_2\left(\|\nabla v\|_{L^1_t(B^{d/p_2}_{p_2,1})} + \|\nabla v\|^{4/3}_{L^{4/3}_t(B^{d/p_2-1/2}_{p_2,1})} + \|\nabla\eta\|^2_{L^2_t(B^{d/p_1}_{p_1,1})} + \|\nabla\eta\|^{2/(1-\delta)}_{L^{2/(1-\delta)}_t(B^{d/p_1-\delta}_{p_1,1})}\right)} \\ &\quad \times C_2\Big(\|u_0\|_{B^{d/p_2}_{p_2,1}} + \|\Delta_{-1}u\|_{L^1_t(L^{p_2})} + \|h\|_{L^1_t(L^2\cap B^{d/p_2-1}_{p_2,1})}\Big), \end{aligned} \tag{1.40}$$

with $\delta=\min(1/2, d/p_1)$.

Remark 1.3. *As pointed in Remarks B.2 and B.8, the quantities $C_1\|\Delta_{-1}\theta\|_{L^1_t(L^{p_1})}$ and $C_2\|\Delta_{-1}u\|_{L^1_t(L^{p_2})}$ in the a priori estimates (1.36) and (1.40) respectively can be absorbed if the time t is small. Hence in the following proof of the local existence result we will always assume the existing time sufficiently small such that this statement holds.*

Remark 1.4. *Compared to the statement of Proposition 0.8 in the case $s=d/p_2-1$ one has to assume in addition that $p_2\le 4$ and also that $p_2\le 2p_1/(p_1-2)$ (if $p_1\ge 2$). This is due to the fact that bounding ∇Q, through the elliptic equation (1.7), requires a L^2 information over the right-hand side. The naive idea is just that, according to Hölder's inequality, L^4 bounds over ∇v, u, ∇u and $\nabla\eta$ provides this L^2 bound. This is the key to go beyond the energy framework for (1.37). At the same time, we do not know how to treat the case $p_2>4$.*

1.3.2 Proof of Theorem 1.2

We follow the same procedure as in the proof of Theorem 1.1 : first we construct a sequence of approximate solutions, then we prove uniform bounds for this sequence and finally, we show the convergence to some solution of (1.1). It is easy to check that the inequalities such as (1.11)-(1.15) all hold true in nonhomogeneous case. And moreover we can generalize (1.13) into

$$\|uv\|_{B^{d/p_1}_{p_1,1}} \lesssim \|u\|_{B^{d/p_2}_{p_2,1}}\|v\|_{B^{d/p_1}_{p_1,1}} + \|u\|_{B^{d/p_2+\varepsilon}_{p_2,1}}\|v\|_{B^{d/p_1-\varepsilon}_{p_1,1}}, \quad \text{if} \quad \varepsilon\in[0,1],\ \frac{1}{p_1}\leq\frac{\varepsilon}{d}+\frac{1}{p_2}. \tag{1.41}$$

We also use frequently the following :

$$\|uv\|_{L^2} \lesssim \|u\|_{B^{d/p_1-1/2}_{p_1,1}}\|v\|_{B^{d/p_2-1/2}_{p_2,1}}, \quad \text{if} \quad p_i\leq 4,\ i=1,2. \tag{1.42}$$

And without loss of generality, we can suppose the functions $\kappa=\kappa(\vartheta), \eta=\eta(\vartheta)$ to have the same bounds as ϑ. That is, if $0<m\leq\vartheta$, $\|\vartheta\|_{L^\infty}+\|\nabla\vartheta\|_{B^{d/p_1}_{p_1,1}}\leq M$, then so do κ,η.

Compared to the almost homogeneous case, the main difference is that our estimates rely mostly on Propositions 1.1 and 1.2. Furthermore, in order to handle large data, we will have to introduce the “free solution” (θ_L,u_L) corresponding to data (θ_0,u_0), namely the solution to (0.63) :

$$\begin{cases} \partial_t\theta_L-\bar\kappa\Delta\theta_L &= 0,\\ \partial_t u_L-\bar\eta\Delta u_L &= 0,\\ (\theta_L,u_L)|_{t=0} &= (\theta_0,u_0), \end{cases} \tag{1.43}$$

with $\bar\kappa=\kappa(1)$ et $\bar\eta=\eta(1)$.

Step 1. Construction of a sequence of approximate solutions

As θ_0 is in $B^{d/p_1}_{p_1,1}$ and u_0, in $B^{d/p_2-1}_{p_2,1}$, the above System (1.43) has a unique global solution (θ_L,u_L) with

$$\theta_L\in\widetilde{\mathcal C}_T(B^{d/p_1}_{p_1,1})\cap L^1_T(B^{d/p_1+2}_{p_1,1}) \quad\text{and}\quad u_L\in\widetilde{\mathcal C}_T(B^{d/p_2-1}_{p_2,1})\cap L^1_T(B^{d/p_2+1}_{p_2,1}) \quad\text{for all}\quad T>0,$$

and we have (if T is small enough and with C depending only on d,p_1,p_2)

$$\begin{aligned} \|\theta_L\|_{\widetilde L^\infty_T(B^{d/p_1}_{p_1,1})}+\bar\kappa\|\theta_L\|_{L^1_T(B^{d/p_1+2}_{p_1,1})} &\leq C\|\theta_0\|_{B^{d/p_1}_{p_1,1}},\\ \|u_L\|_{\widetilde L^\infty_T(B^{d/p_2-1}_{p_2,1})}+\bar\eta\|u_L\|_{L^1_T(B^{d/p_2+1}_{p_2,1})} &\leq C\|u_0\|_{B^{d/p_2-1}_{p_2,1}}. \end{aligned} \tag{1.44}$$

Note also that the divergence free property for the initial velocity is conserved during the evolution. Another important feature is that, owing to $\theta_L=e^{\bar\kappa t\Delta}\theta_0\in\widetilde L^\infty_T(B^{d/p_1}_{p_1,1})$, we have from Inequality (0.42) that, for any $T>0$,

$$\lim_{N\to+\infty}\|\theta_L-S_N\theta_L\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}=0. \tag{1.45}$$

Let us fix some small enough positive time T. Given (1.45), we see that for any positive constant c, there exists some positive integer N_0 so that

$$\|\theta_L-S_{N_0}\theta_L\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\leq cm. \tag{1.46}$$

In addition, if the data satisfy (1.9) then one may assume that we have (changing N_0 and C if need be)

$$\frac{m}{2}\leq S_n\vartheta_0\leq CM, \quad \|S_n\theta_0\|_{B^{d/p_1}_{p_1,1}}+\|S_nu_0\|_{B^{d/p_2-1}_{p_2,1}}\leq CM \quad\text{for all}\quad n\geq N_0. \tag{1.47}$$

In order to define our approximate solutions, we use the following iterative scheme : first we set $(\theta^0,u^0,\nabla Q^0)=(S_{N_0}\theta_0,S_{N_0}u_0,0)$ (this is obviously a smooth stationary function with decay at infinity) then, assuming that the approximate solution $(\theta^n, u^n, \nabla Q^n)$ has been constructed over $\mathbb{R}^+ \times \mathbb{R}^d$, we set $\vartheta^n = 1 + \theta^n$ and define $(\theta^{n+1}, u^{n+1}, \nabla Q^{n+1})$ to be the unique solution of the system

$$\left\{\begin{array}{ccc} \partial_t \theta^{n+1} + u^n \cdot \nabla \theta^{n+1} - \mathrm{div}\,(\kappa^n \nabla \theta^{n+1}) & = & h_1^n, \\ \partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} - \mathrm{div}\,(\eta^n \nabla u^{n+1}) + \vartheta^n \nabla Q^{n+1} & = & h_2^n, \\ \mathrm{div}\, u^{n+1} & = & 0, \\ (\theta^{n+1}, u^{n+1})|_{t=0} & = & (S_{N_0+n+1}\theta_0, S_{N_0+n+1}u_0), \end{array}\right. \tag{1.48}$$

where

$$\kappa^n = \kappa(\vartheta^n), \quad \eta^n = \eta(\vartheta^n), \quad h_1^n = h_1(1+\theta^n, u^n) \;\text{ and }\; h_2^n = h_2(1+\theta^n, u^n).$$

Note that the existence and uniqueness of a global smooth solution for the above system is ensured by the standard theory of parabolic equations (concerning θ^{n+1}) and by (a slight modification of) Theorem 2.10 in [2] (concerning u^{n+1}) whenever (θ^n, u^n) is suitably smooth and the coefficients κ^n, η^n are bounded by above and by below. In fact, given (1.47), the maximum principle ensures that

$$m/2 \leq \vartheta^n \leq CM. \tag{1.49}$$

Hence κ^n and η^n are bounded by above and from below independently of n.

Next, we notice that if we set

$$(\theta_L^n, u_L^n) = (S_{N_0+n}\theta_L, S_{N_0+n}u_L)$$

then the equation for $(\bar\theta^{n+1}, \bar u^{n+1}, \nabla Q^{n+1}) = (\theta^{n+1} - \theta_L^{n+1}, u^{n+1} - u_L^{n+1}, \nabla Q^{n+1})$ reads

$$\left\{\begin{array}{ccc} \partial_t \bar\theta^{n+1} + u^n \cdot \nabla \bar\theta^{n+1} - \mathrm{div}\,(\kappa^n \nabla \bar\theta^{n+1}) & = & H_1^n, \\ \partial_t \bar u^{n+1} + u^n \cdot \nabla \bar u^{n+1} - \mathrm{div}\,(\eta^n \nabla \bar u^{n+1}) + (1+\theta_L^n) \nabla Q^{n+1} & = & H_2^n, \\ \mathrm{div}\, \bar u^{n+1} & = & 0, \\ (\bar\theta^{n+1}, \bar u^{n+1})|_{t=0} & = & (0,0), \end{array}\right.$$

where

$$H_1^n = -u^n \cdot \nabla \theta_L^{n+1} + \mathrm{div}\,((\kappa^n - \bar\kappa)\nabla \theta_L^{n+1}) + h_1^n,$$
$$H_2^n = -u^n \cdot \nabla u_L^{n+1} + \mathrm{div}\,((\eta^n - \bar\eta)\nabla u_L^{n+1}) - \bar\theta^n \nabla Q^{n+1} + h_2^n.$$

Let us point out that, given (1.46), we have

$$\|\theta_L^n - S_{N_0}\theta_L^n\|_{L_T^\infty(B^{d/p_1}_{p_1,1})} \leq cm \quad \text{for all} \;\; n \in \mathbb{N}. \tag{1.50}$$

Step 2. Uniform bounds

Bounding $(\bar\theta^{n+1}, \bar u^{n+1}, \nabla Q^{n+1})$ in terms of the free solution (θ_L, u_L) and of $(\bar\theta^n, \bar u^n, \nabla Q^n)$ relies on Propositions 1.1 and 1.2 with $s = d/p_1$ and $s = d/p_2 - 1$, respectively, and $N = N_0$ (here we have to take c small enough in (1.50)). Using the fact that, as pointed out by Remark 1.3, taking T smaller if needed allows to discard $C_1\|\Delta_{-1}\bar\theta^{n+1}\|_{L_t^1(L^{p_1})}$ and $C_2\|\Delta_{-1}\bar u^{n+1}\|_{L_t^1(L^{p_2})}$ in the estimates, we get, under the condition (1.8),

$$\|\bar\theta^{n+1}\|_{X^{p_1}(T)} \leq C \exp\left\{ C_1 \left(\|\nabla u^n\|_{L_T^1(B^{d/p_2}_{p_2,1})} + \|\nabla \kappa^n\|^2_{L_T^2(B^{d/p_1}_{p_1,1})} \right) \right\} \|H_1^n\|_{L_T^1(B^{d/p_1}_{p_1,1})},$$

$$\|\bar u^{n+1}\|_{Y^{p_2}(T)} + \|\nabla Q^{n+1}\|_{Z^{p_2}(T)} \leq C \|H_2^n\|_{L_T^1(B^{d/p_2-1}_{p_2,1} \cap L^2)}$$
$$\times \exp\left\{ C_2 \left(\|\nabla u^n\|_{L_T^1(B^{d/p_2}_{p_2,1})} + \|\nabla u^n\|^{4/3}_{L_T^{4/3}(B^{d/p_2-1/2}_{p_2,1})} \right.\right.$$
$$\left.\left. + \|\nabla \eta^n\|^2_{L_T^2(B^{d/p_1}_{p_1,1})} + \|\nabla \eta^n\|^4_{L_T^4(B^{d/p_1-1/2}_{p_1,1})} \right) \right\}.$$

From Proposition 0.5 and elementary interpolation inequalities, we gather that all the terms in the exponential may be bounded by $\|\theta^n\|_{X^{p_1}(T)} + \|u^n\|_{Y^{p_2}(T)}$ to some power. Therefore, if we assume that

$$\|\theta^n\|_{X^{p_1}(T)} + \|u^n\|_{Y^{p_2}(T)} \leq 2CM \tag{1.51}$$

then we have [2]

$$\|\bar{\theta}^{n+1}\|_{X^{p_1}(T)} \leq C_M\Big(\|h_1^n\|_{L^1_T(B^{d/p_1}_{p_1,1})} + \|u^n\cdot\nabla\theta_L^{n+1}\|_{L^1_T(B^{d/p_1}_{p_1,1})} + \|(\kappa^n-\bar{\kappa})\nabla\theta_L^{n+1}\|_{L^1_T(B^{d/p_1+1}_{p_1,1})}\Big), \tag{1.52}$$

$$\begin{aligned}\|\bar{u}^{n+1}\|_{Y^{p_2}(T)} + \|\nabla Q^{n+1}\|_{Z^{p_2}(T)} \leq C_M\Big(&\|h_2^n\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)} + \|u^n\cdot\nabla u_L^{n+1}\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}\\ &+\|\mathrm{div}\,((\eta^n-\bar{\eta})\nabla u_L^{n+1})\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)} + \|\bar{\theta}^n\nabla Q^{n+1}\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}\Big).\end{aligned} \tag{1.53}$$

So bounding the right-hand sides of (1.52) and of (1.53) is our next task. Given (1.51), we easily get

$$\begin{aligned}\|h_1^n\|_{L^1_T(B^{d/p_1}_{p_1,1})} &\leq C_M\|\nabla\theta^n\|^2_{L^2_T(B^{d/p_1}_{p_1,1})},\\ \|(\kappa^n-\bar{\kappa})\nabla\theta_L^{n+1}\|_{L^1_T(B^{d/p_1+1}_{p_1,1})} &\leq C_M\big(\|\theta^n\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\|\nabla\theta_L\|_{L^1_T(B^{d/p_1+1}_{p_1,1})}\\ &\qquad+\|\nabla\theta_L\|_{L^2_T(B^{d/p_1}_{p_1,1})}\|\theta^n\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\big).\end{aligned}$$

By view of Estimate (1.41), it is not difficult to get that

$$\begin{aligned}\|u^n\cdot\nabla\theta_L^{n+1}\|_{L^1_T(B^{d/p_1}_{p_1,1})} \lesssim\, &\|u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})}\|\nabla\theta_L\|_{L^2_T(B^{d/p_1}_{p_1,1})}\\ &+\|u^n\|_{L^{2/(2-\varepsilon)}_T(B^{d/p_2+1-\varepsilon}_{p_2,1})}\|\nabla\theta_L\|_{L^{2/\varepsilon}_T(B^{d/p_1-1+\varepsilon}_{p_1,1})}\end{aligned}$$

whenever $\varepsilon\in[0,1]$ and $d/p_1\leq 1-\varepsilon+d/p_2$.

Next, computations similar to those that enable us to bound h_2^n in the homogeneous framework lead to

$$\begin{aligned}\|h_2^n\|_{L^1_T(B^{d/p_2-1}_{p_2,1})} \leq C_M\Big(&\|\nabla\theta^n\|^2_{L^2_T(B^{d/p_1}_{p_1,1})} + \|\nabla^2\theta^n\|_{L^2_T(B^{d/p_1-1}_{p_1,1})}\|\nabla\theta^n\|_{L^2_T(B^{d/p_1}_{p_1,1})}\\ &+\|\nabla^2\theta^n\|_{L^{2/(2-\varepsilon)}_T(B^{d/p_1-\varepsilon}_{p_1,1})}\|\nabla\theta^n\|_{L^{2/\varepsilon}_T(B^{d/p_1-1+\varepsilon}_{p_1,1})} + \|\nabla\theta^n\|_{L^2_T(B^{d/p_1}_{p_1,1})}\|\nabla u^n\|_{L^2_T(B^{d/p_2-1}_{p_2,1})}\Big),\end{aligned}$$

provided

$$p_1<2d,\quad p_1\leq 2p_2,\quad \frac{1}{p_1}+\frac{1}{p_2}>\frac{1}{d}\ \text{ and }\ \frac{1}{p_2}\leq\frac{1}{p_1}+\frac{1}{d}-\frac{\varepsilon}{d}\ \text{ for some }\ \varepsilon\in[0,1],$$

and for $p_1\leq 4$,

$$\begin{aligned}\|h_2^n\|_{L^1_T(L^2)} \leq C_M\Big(&\|\nabla\theta^n\|_{L^2_T(L^\infty)}\|\nabla\theta^n\|^2_{L^4_T(B^{d/p_1-1/2}_{p_1,1})}\\ &+\|\nabla^2\theta^n\|_{L^{4/3}_T(B^{d/p_1-1/2}_{p_1,1})}\|\nabla\theta^n\|_{L^4_T(B^{d/p_1-1/2}_{p_1,1})} + \|\nabla\theta^n\|_{L^4_T(B^{d/p_1-1/2}_{p_1,1})}\|\nabla u^n\|_{L^{4/3}_T(B^{d/p_2-1/2}_{p_2,1})}\Big).\end{aligned}$$

2. In all that follows, we denote by C_M a suitable increasing function of M. To simplify the notation, we omit the dependency with respect to d, N, p_1, p_2, etc.

Under Condition (1.8), we have furthermore

$$\|u^n\cdot\nabla u_L^{n+1}\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}\lesssim\|u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})}\|u_L\|_{L^2_T(B^{d/p_2}_{p_2,1})}+\|u^n\|_{L^4_T(B^{d/p_2-1/2}_{p_2,1})}\|\nabla u_L\|_{L^{4/3}_T(B^{d/p_2-1/2}_{p_2,1})}$$

$$\begin{aligned}\|\mathrm{div}\,\big((\eta^n-\bar\eta)\nabla u_L^{n+1}\big)\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}&\lesssim\|\nabla\theta^n\|_{L^2_T(B^{d/p_1}_{p_1,1})}\|\nabla u_L\|_{L^2_T(B^{d/p_2-1}_{p_2,1})}\\&+\|\theta^n\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\|u_L\|_{L^1_T(B^{d/p_2+1}_{p_2,1})}+\|\nabla\theta^n\|_{L^4_T(B^{d/p_1-1/2}_{p_1,1})}\|\nabla u_L\|_{L^{4/3}_T(B^{d/p_2-1/2}_{p_2,1})},\end{aligned}$$

$$\|\bar\theta^n\nabla Q^{n+1}\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}\lesssim\|\bar\theta^n\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\|\nabla Q^{n+1}\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}.$$

Let us fix some small positive constant τ_M that we shall specify later on and let us assume that T has been chosen so that

$$\|\theta_L\|_{L^{2/\varepsilon}_T(B^{d/p_1+\varepsilon}_{p_1,1})\cap L^4_T(B^{d/p_1+1/2}_{p_1,1})\cap L^1_T(B^{d/p_1+2}_{p_1,1})}+\|u_L\|_{L^4_T(B^{d/p_2-1/2}_{p_2,1})\cap L^1_T(B^{d/p_2+1}_{p_2,1})}\leq\tau_M.\quad(1.54)$$

Note that, in order that the above condition is satisfied for some positive T even if θ_0 is large, we have to rule out the case $\varepsilon=0$. This accounts for the strict inequality in the conditions

$$d/p_1<1+d/p_2\quad\text{and}\quad d/p_2<1+d/p_1,$$

that we did not have in the statement of Theorem 1.2.

Now, plugging all the above estimates in (1.52) and (1.53) yields (up to a harmless change of C_M)

$$\begin{aligned}&\|\bar\theta^{n+1}\|_{X^{p_1}(T)}+\|\bar u^{n+1}\|_{Y^{p_2}(T)}+\|\nabla Q^{n+1}\|_{Z^{p_2}(T)}\\&\quad\leq C_M\Big(\|\bar\theta^n\|_{X^{p_1}(T)}(\|\bar\theta^n\|_{X^{p_1}(T)}+\|\bar u^n\|_{Y^{p_2}(T)}+\|\nabla Q^{n+1}\|_{Z^{p_2}(T)})\\&\qquad+\tau_M\big(\|\bar\theta^n\|_{X^{p_1}(T)}+\|\bar u^n\|_{Y^{p_2}(T)}\big)+\tau_M^2+\tau_M\|\theta_L\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\Big).\end{aligned}$$

Using (1.44) so as to bound the last term, we see that if we assume that

$$\|\bar\theta^n\|_{X^{p_1}(T)}+\|\bar u^n\|_{Y^{p_2}(T)}+\|\nabla Q^n\|_{Z^{p_2}(T)}\leq K\tau_M\quad(1.55)$$

for some $K=K(M)$ that we shall choose below, and take τ_M so that

$$C_M\tau_MK\leq1/2$$

then

$$\|\bar\theta^{n+1}\|_{X^{p_1}(T)}+\|\bar u^{n+1}\|_{Y^{p_2}(T)}+\|\nabla Q^{n+1}\|_{Z^{p_2}(T)}\leq2C_M\tau_M\big(M+(K^2+K+1)\tau_M\big).$$

Hence $(\bar\theta^{n+1},\bar u^{n+1},\nabla\bar Q^{n+1})$ satisfies (1.55) too if we take $K=2C_M(1+M)$ and assume that τ_M also satisfies

$$(1+K+K^2)\tau_M\leq1.$$

This completes the proof of a priori estimates on any interval $[0,T]$ such that (1.54) is fulfilled.

Step 3. Convergence

The equation for $(\delta\theta^{n+1},\delta u^{n+1},\nabla\delta Q^{n+1})=(\theta^{n+1}-\theta^n,u^{n+1}-u^n,\nabla Q^{n+1}-\nabla Q^n)$ reads

$$\left\{\begin{array}{ccc}\partial_t\delta\theta^{n+1}+u^n\cdot\nabla\delta\theta^{n+1}-\mathrm{div}\,(\kappa^n\nabla\delta\theta^{n+1})&=&I^n,\\\partial_t\delta u^{n+1}+u^n\cdot\nabla\delta u^{n+1}-\mathrm{div}\,(\eta^n\nabla\delta u^{n+1})+(1+\theta_L^n)\nabla\delta Q^{n+1}&=&J^n,\\\mathrm{div}\,\delta u^{n+1}&=&0,\\(\delta\theta^{n+1},\delta u^{n+1})|_{t=0}&=&(\Delta_{N_0+n}\theta_0,\Delta_{N_0+n}u_0),\end{array}\right.$$

where

$$I^n = -\,\delta u^n\cdot\nabla\theta^n + \mathrm{div}\,(\delta\kappa^n\nabla\theta^n) + h_1^n - h_1^{n-1},$$
$$J^n = -\,\delta u^n\cdot\nabla u^n + \mathrm{div}\,(\delta\eta^n\nabla u^n) - \delta\theta^n\nabla Q^n - \bar\theta^n\nabla\delta Q^{n+1} + h_2^n - h_2^{n-1}.$$

Let $b_n = 2^{(N_0+n)d/p_1}\|\Delta_{N_0+n}\theta_0\|_{L^{p_1}}$ and $d_n = 2^{(N_0+n)(d/p_2-1)}\|\Delta_{N_0+n}u_0\|_{L^{p_2}}$. Since $\theta_0\in B^{d/p_1}_{p_1,1}$ and $u_0\in B^{d/p_2-1}_{p_2,1}$, we have $(b_n)\in\ell^1$ and $(d_n)\in\ell^1$.

To simplify the presentation, we assume that $p_1\geq p_2$. Then, applying Propositions 1.1 and 1.2 and using the bounds of the previous step, we get (bearing in mind that if τ_M is sufficiently small in (1.55) then one may absorb $\bar\theta^n\nabla\delta Q^{n+1}$) :

$$\begin{aligned}\|\delta\theta^{n+1}\|_{X^{p_1}(T)} \leq Cb_n + C_M\Big(&\big(\|\delta u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})} + \|\delta\theta^n\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\big)\|\theta^n\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\\ &+\|\delta\theta^n\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\big(\|\theta^n\|_{L^1_T(B^{d/p_1+2}_{p_1,1})} + \|\theta^n\|^2_{L^2_T(B^{d/p_1+1}_{p_1,1})}\big)\\ &+\|\delta\theta^n\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\|\theta^{n-1}\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\Big),\end{aligned}$$

$$\begin{aligned}&\|\delta u^{n+1}\|_{Y^{p_2}(T)} + \|\nabla\delta Q^{n+1}\|_{Z^{p_2}(T)} \leq Cd_n\\ &+C_M\Big(\|\delta\theta^n\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\big(\|(\theta^{n-1},\theta^n)\|^2_{L^2_T(B^{d/p_1+1}_{p_1,1})} + \|\theta^{n-1}\|_{L^1_T(B^{d/p_1+2}_{p_1,1})} + \|\nabla Q^n\|_{Z^{p_2}(T)}\big)\\ &+\|\delta\theta^n\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\big(\|(\theta^n,\theta^{n-1})\|_{L^2_T(B^{d/p_1+1}_{p_1,1})} + \|(\theta^n,\theta^{n-1})\|^2_{L^4_T(B^{d/p_1+1/2}_{p_1,1})} + \|u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})}\big)\\ &+\|\delta\theta^n\|_{L^4_T(B^{d/p_1+1/2}_{p_1,1})}\big(\|u^n\|_{L^{4/3}_T(B^{d/p_2+1/2}_{p_2,1})} + \|\theta^{n-1}\|_{L^{4/3}_T(B^{d/p_1+3/2}_{p_1,1})}\big)\\ &+\|\delta\theta^n\|_{L^{4/3}_T(B^{d/p_1+3/2}_{p_1,1})}\|\theta^n\|_{L^4_T(B^{d/p_1+1/2}_{p_1,1})} + \|\delta\theta^n\|_{L^{2/(2-\varepsilon)}_T(B^{d/p_1+2-\varepsilon}_{p_1,1})}\|\theta^n\|_{L^{2/\varepsilon}_T(B^{d/p_1+\varepsilon}_{p_1,1})}\\ &+\|\delta u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})}\big(\|u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})} + \|\theta^{n-1}\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\big)\\ &+\|\delta u^n\|_{L^4_T(B^{d/p_2-1/2}_{p_2,1})}\|u^n\|_{L^{4/3}_T(B^{d/p_2+1/2}_{p_2,1})} + \|\delta u^n\|_{L^{4/3}_T(B^{d/p_2+1/2}_{p_2,1})}\|\theta^{n-1}\|_{L^4_T(B^{d/p_1+1/2}_{p_1,1})}\Big)\end{aligned}$$

where $C = C(d,p_1,p_2)$.

Let us emphasize that, according to (1.54) and (1.55), the previous inequalities imply that, up to a change of C_M, we have

$$B^{n+1}(T)\leq C_M\tau_M B^n(T) + C(b_n+d_n).$$

with $B^n(T) = \|\delta\theta^n\|_{X^{p_1}(T)} + \|\delta u^n\|_{Y^{p_2}(T)} + \|\nabla\delta Q^n\|_{Z^{p_2}(T)}$.

Therefore, taking τ_M small enough, we end up with

$$B^{n+1}(T)\leq \frac{1}{2}B^n(T) + C(b_n+d_n).$$

As (b_n) and (d_n) are in ℓ^1, one may thus conclude that $\sum(B^n(T))<\infty$, which is to say $(\theta^n,u^n,\nabla Q^n)_{n\in\mathbb{N}}$ is a Cauchy sequence and converges to a solution $(\theta,u,\nabla Q)$ of the system (1.1) in the space E^{p_1,p_2}_T which also satisfies the estimates (1.10) (that $\vartheta\geq m$ is a consequence of the maximum principle for the parabolic equation satisfied by ϑ).

Step 4. Stability estimates and uniqueness

To prove the stability, i.e. the continuity of the flow map, and the uniqueness, we consider two solutions $(\theta^1,u^1,\nabla Q^1)$ and $(\theta^2,u^2,\nabla Q^2)$ of System (1.1) in E^{p_1,p_2}_T with initial data

(θ_0^1, u_0^1) and (θ_0^2, u_0^2), respectively. Let $\vartheta^1 = 1+\theta^1$ and $\vartheta^2 = 1+\theta^2$. We assume in addition that

$$\vartheta^1, \vartheta^2 \geq m$$

and we fix some large enough integer N_1 so that

$$m/2 \leq 1 + S_{N_1}\theta^1 \quad \text{and} \quad \|\theta^1 - S_{N_1}\theta^1\|_{L_T^\infty(B^{d/p_1}_{p_1,1})} \leq cm$$

with c given by Condition (1.39). Finally, we denote by M a common bound for the two solutions in $E_T^{p_1,p_2}$.

The proof goes from arguments similar to those of the previous step : we notice that the difference of the two solutions $(\delta\theta, \delta u, \nabla\delta Q) := (\theta^1-\theta^2, u^1-u^2, \nabla Q^1 - \nabla Q^2)$ satisfies

$$\begin{cases} \partial_t\delta\theta + u^1\cdot\nabla\delta\theta - \operatorname{div}(\kappa(\theta^1)\nabla\delta\theta) &= I, \\ \partial_t\delta u + u^1\cdot\nabla\delta u - \operatorname{div}(\eta(\theta^1)\nabla\delta u) + \vartheta^1\nabla\delta Q &= J, \\ \operatorname{div}\delta u &= 0, \\ (\delta\theta, \delta u)|_{t=0} &= (\delta\theta_0, \delta u_0), \end{cases}$$

where

$$\begin{aligned} I &= -\,\delta u\cdot\nabla\theta^2 + \nabla\cdot(\delta\kappa\nabla\theta^2) + h_1(1+\theta^1) - h_1(1+\theta^2), \\ J &= -\,\delta u\cdot\nabla u^2 + \nabla\cdot(\delta\eta\nabla u^2) - \delta\theta\nabla Q^2 + h_2(1+\theta^1, u^1) - h_2(1+\theta^2, u^2). \end{aligned}$$

Let $B(t) = \|\delta\theta\|_{X^{p_1}(t)} + \|\delta u\|_{Y^{p_2}(t)} + \|\nabla\delta Q\|_{Z^{p_2}(t)}$. Then arguing exactly as in the previous step, we get for small enough t,

$$\begin{aligned} B(t) \leq C_{M,N_1}\Big(&\|\delta\theta_0\|_{B^{d/p_1}_{p_1,1}} + \|\delta u_0\|_{B^{d/p_2-1}_{p_2,1}} + \big(\|(\theta^1,\theta^2)\|_{L^2_t(B^{d/p_1+1}_{p_1,1})\cap L^4_t(B^{d/p_1+1/2}_{p_1,1})} \\ &+\|\theta^1\|_{L^{4/3}_t(B^{d/p_1+3/2}_{p_1,1})\cap L^4_t(B^{d/p_1+1/2}_{p_1,1})} + \|\theta^2\|_{L^1_t(B^{d/p_1+2}_{p_1,1})\cap L^{2/\varepsilon}_t(B^{d/p_1+\varepsilon}_{p_1,1})} \\ &+\|u^2\|_{L^2_t(B^{d/p_2}_{p_2,1})\cap L^{4/3}_t(B^{d/p_2+1/2}_{p_2,1})} + \|\nabla Q^2\|_{Z^{p_2}(t)}\big)B(t)\Big). \end{aligned}$$

If the initial data coincide then we have $B(0) = 0$. Given that the factors of $B(t)$ in the right-hand side go to 0 when t goes to 0, we thus get $B \equiv 0$ on a small enough time interval. Then, from standard continuation arguments, we conclude to uniqueness on the whole time interval $[0,T]$.

In the more general case where the initial data do not coincide, then one may split both solutions into

$$\theta^i = \theta^i_L + \bar\theta^i \quad \text{and} \quad u^i = u^i_L + \bar u^i,$$

where (θ^i_L, u^i_L) stands for the free solution of (1.43) pertaining to data (θ^i_0, u^i_0).

If $\|\delta\theta_0\|_{B^{d/p_1}_{p_1,1}} + \|\delta u_0\|_{B^{d/p_2-1}_{p_2,1}} \leq \delta$, then we have

$$\|\theta^2_L - \theta^1_L\|_{X^{p_1}(T)} + \|u^2_L - u^1_L\|_{Y^{p_2}(T)} \leq C\delta.$$

By arguing as in step two, one may also prove that if T is so small as to satisfy (1.54) for (say) (θ^1_L, u^1_L) and $\tau_M = \delta$ (with δ small enough) then $(\bar\theta^1, \bar u^1, \nabla Q^1)$ satisfies (1.55). Therefore, from the above inequality, one may conclude that $B(T) \leq 2C_{M,N_1}\delta$. This completes the proof of the continuity of the flow map.

1.4 General gases

At the end of this chapter, let us consider the low Mach number limit system of the general gases (0.33) :

$$\begin{cases} \partial_t\vartheta + u\cdot\nabla\vartheta - \operatorname{div}(\kappa\nabla\vartheta) &= h_1(\vartheta), \\ \partial_t u + u\cdot\nabla u - \operatorname{div}(\eta\nabla u) + \vartheta\nabla Q &= h_2(\vartheta, u), \\ \operatorname{div} u &= -\alpha' k|\nabla\vartheta|^2, \end{cases} \tag{1.56}$$

where the coefficients $\kappa(\vartheta), h_1(\vartheta), \eta(\vartheta), h_2(\vartheta, u), \alpha(\vartheta), k(\vartheta)$ are defined in Section §0.2.

The main difference of the above system from System (1.1) is that the divergence of the velocity $\operatorname{div} u$ depends strongly on the unknown temperature ϑ. Therefore, in the sequel, it is convenient to consider the following system instead of (1.56) :

$$\begin{cases} \partial_t\vartheta + u\cdot\nabla\vartheta - \operatorname{div}(\kappa\nabla\vartheta) = h_1(\vartheta), \\ \operatorname{div}\big(\vartheta\nabla Q) = \operatorname{div}\big(h_2 - u\cdot\nabla u + Du\cdot\nabla\eta - \eta\nabla(\alpha' k|\nabla\vartheta|^2)\big) + \partial_t(\alpha' k|\nabla\vartheta|^2), \\ \partial_t u + u\cdot\nabla u - \operatorname{div}(\eta\nabla u) = h_2 - \vartheta\nabla Q. \end{cases} \tag{1.57}$$

In fact, on one side, applying the operator div to Equation $(1.56)_2$ (noticing Equation $(1.56)_3$) gives the equation $(1.57)_2$ for ∇Q. On the other side, applying div to $(1.57)_3$ (noticing Equation $(1.57)_2$) issues

$$\partial_t\big(\operatorname{div} u + \alpha' k|\nabla\vartheta|^2\big) - \operatorname{div}\big(\eta\nabla\big(\operatorname{div} u + \alpha' k|\nabla\vartheta|^2\big)\big) = 0.$$

If one supposes the initial data (ϑ_0, u_0) to satisfy

$$\operatorname{div} u_0 + \big(\alpha' k\big)(\vartheta_0)|\nabla\vartheta_0|^2 = 0, \tag{1.58}$$

then according to Proposition 1.1, Equation $(1.56)_3$ follows immediately provided that the solutions are regular enough and decay at infinity sufficiently.

Slightly nonhomogeneous case

Just as in the ideal gas case, we begin with the small initial data case and rewrite the system for the unknowns $(\theta, u, \nabla Q)$ with $\theta = \vartheta - 1$ as follows :

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta - \bar\kappa\Delta\theta &= r_1(\theta), \\ \partial_t u + u\cdot\nabla u - \bar\eta\Delta u + \nabla Q &= r_2(\theta, u, \nabla Q), \\ \operatorname{div} u &= -\alpha' k|\nabla\theta|^2, \\ (\theta, u)|_{t=0} &= (\theta_0, u_0), \end{cases} \tag{1.59}$$

with $\bar\kappa$, $\bar\eta$, r_1 and r_2 defined by (1.2). In the following, we will sketch the proof of the following well-posedness result for the above system (1.59) :

Theorem 1.3. *Let (p_1, p_2) satisfy $p_1 < d$, $p_2 < 2d$, in addition to Condition (1.3). Let the initial data $\theta_0 \in \dot B^{d/p_1}_{p_1,1}$ and $u_0 \in \dot B^{d/p_2-1}_{p_1,1}$ satisfy (1.58).*
Then if the initial condition (1.4) is satisfied, then there exists a unique solution $(\theta, u, \nabla Q) \in \dot E^{p_1,p_2}_T$, $T \in (0, +\infty]$ to System (1.59) ; if Condition (1.5) is satisfied additionally, then $T = +\infty$. Furthermore, the flow map $(\theta_0, u_0) \mapsto (\theta, u, \nabla Q)$ is Lipschitz continuous.

The key to the proof is the a priori estimates for the following linearized system of (1.59) :

$$\begin{cases} \partial_t\theta + v\cdot\nabla\theta - \bar\kappa\Delta\theta &= r_1, \\ \partial_t u + v\cdot\nabla u - \bar\eta\Delta u + \nabla Q &= r_2, \\ \operatorname{div} u &= -\alpha' k|\nabla\theta|^2, \\ (\theta, u)|_{t=0} &= (\theta_0, u_0). \end{cases} \tag{1.60}$$

In the above, v, r_1, r_2, θ_0 and u_0 are known smooth functions which decay rapidly at infinity ; $\bar\kappa$ and $\bar\eta$ are two positive constants ; α and k depend regularly on the unknown θ.

Firstly, it is easy to apply Proposition 0.7 to get the a priori estimate for θ :

$$\|\theta\|_{\dot X^{p_1}(t)} \lesssim e^{C\|v\|_{L^1_t(\dot B^{d/p_2+1}_{p_2,1})}} \Big(\|\theta_0\|_{\dot B^{d/p_1}_{p_1,1}} + \|r_1\|_{L^1_t(\dot B^{d/p_1}_{p_1,1})}\Big). \tag{1.61}$$

A direct computation implies that $\|\partial_t\theta\|_{L^1_t(\dot B^{d/p_1}_{p_1,1})}$ is bounded by (if $\frac{1}{p_1} \le \frac{1}{p_2} + \frac{1}{d}$)

$$\Big(1 + \|v\|_{L^2_t(\dot B^{d/p_2}_{p_2,1})}\Big) e^{C\|v\|_{L^1_t(\dot B^{d/p_2+1}_{p_2,1})}} \Big(\|\theta_0\|_{\dot B^{d/p_1}_{p_1,1}} + \|r_1\|_{L^1_t(\dot B^{d/p_1}_{p_1,1})}\Big). \tag{1.62}$$

However, since we don't have the divergence-free condition for the velocity vector u, we have to get a new a priori estimate as in Proposition 0.8 for u and ∇Q. Notice that the system (1.60) is equivalent to the following Cauchy problem (similar to System (1.57))

$$\begin{cases} \partial_t\theta + v\cdot\nabla\theta - \bar\kappa\Delta\theta = r_1, \\ \Delta Q = \mathrm{div}\,(r_2 - v\cdot\nabla u) + \Big(\partial_t(\alpha' k|\nabla\theta|^2) - \bar\eta\Delta(\alpha' k|\nabla\theta|^2)\Big), \\ \partial_t u - \bar\eta\Delta u = r_2 - \nabla Q - v\cdot\nabla u, \\ (\theta, u)|_{t=0} = (\theta_0, u_0). \end{cases} \tag{1.63}$$

Let us focus on ∇Q for a while. Except for the estimates (1.11), (1.12), (1.13), (1.14) and (1.15), we use the following product estimate

$$\|fg\|_{\dot B^{\frac{d}{p_2}-2}_{p_2,1}} \lesssim \|f\|_{\dot B^{\frac{d}{p_1}-1}_{p_1,1}} \|g\|_{\dot B^{\frac{d}{p_1}-1}_{p_1,1}}, \text{ if } p_1 < d,\, p_1 \le 2p_2,\, \frac{1}{p_2} \le \frac{1}{p_1} + \frac{1}{d}. \tag{1.64}$$

Finally, $\|\partial_t(\alpha' k|\nabla\theta|^2) - \bar\eta\Delta(\alpha' k|\nabla\theta|^2)\|_{L^1_t(\dot B^{d/p_2-2}_{p_2,1})}$ can be controlled by Quantity (1.62), multiplied by a coefficient C_θ which may depend on $\|\theta\|_{L^\infty_t(\dot B^{d/p_1}_{p_1,1})}$. Besides, we have

$$\|\mathrm{div}\,(r_2 - v\cdot\nabla u)\|_{L^1_t(\dot B^{d/p_2-2}_{p_2,1})} \lesssim \|r_2\|_{L^1_t(\dot B^{d/p_2-1}_{p_2,1})} + \int_0^t \|u\|_{\dot B^{d/p_2}_{p_2,1}} \|v\|_{\dot B^{d/p_2}_{p_2,1}}, \quad \text{if} \quad p_2 < 2d.$$

Until now, we have got the estimate for $\|\Delta Q\|_{L^1_t(\dot B^{d/p_2-2}_{p_2,1})}$, and hence the following holds for ∇Q :

$$\begin{aligned} \|\nabla Q\|_{L^1_t(\dot B^{d/p_2-1}_{p_2,1})} &\le C\Big(\|r_2\|_{L^1_t(\dot B^{d/p_2-1}_{p_2,1})} + \int_0^t \|u\|_{\dot B^{d/p_2}_{p_2,1}} \|v\|_{\dot B^{d/p_2}_{p_2,1}}\Big) \\ &+ C_\theta(\|\theta\|_{L^\infty_t(\dot B^{d/p_1}_{p_1,1})}) \left(e^{C\|v\|_{L^1_t(\dot B^{d/p_2+1}_{p_2,1})}} \Big(1 + \|v\|_{L^2_t(\dot B^{d/p_2}_{p_2,1})}\Big) \Big(\|\theta_0\|_{\dot B^{d/p_1}_{p_1,1}} + \|r_1\|_{L^1_t(\dot B^{d/p_1}_{p_1,1})}\Big)\right). \end{aligned} \tag{1.65}$$

Now applying Proposition 0.7 to the Cauchy problem $(1.63)_3 - (1.63)_4$ yields the following estimate for u by use of the interpolation inequality $\|u\|_{\dot B^{d/p_2}_{p_2,1}} \lesssim \|u\|^{1/2}_{\dot B^{d/p_2-1}_{p_2,1}} \|u\|^{1/2}_{\dot B^{d/p_2+1}_{p_2,1}}$ and Gronwall's inequality :

$$\begin{aligned} \|u\|_{\dot Y^{p_2}(t)} &\le C_\theta(\|\theta\|_{L^\infty_t(\dot B^{d/p_1}_{p_1,1})}) \exp\Big\{C(\|v\|^2_{L^2_t(\dot B^{d/p_2}_{p_2,1})} + \|v\|_{L^1_t(\dot B^{d/p_2+1}_{p_2,1})})\Big\} \\ &\times \left(\|u_0\|_{\dot B^{d/p_2-1}_{p_2,1}} + \|r_2\|_{L^1_t(\dot B^{d/p_2-1}_{p_2,1})} + \|\theta_0\|_{\dot B^{d/p_1}_{p_1,1}} + \|r_1\|_{L^1_t(\dot B^{d/p_1}_{p_1,1})}\right). \end{aligned} \tag{1.66}$$

With the a priori estimates (1.61), (1.65) and (1.66) in hand, we can follow the standard stategy as in Section §1.2 to get the well-posedness result for the slightly nonhomogeneous system (1.59). For example, we construct a sequence of approximated solutions $(\theta^{n+1}, u^{n+1}, \nabla Q^{n+1})$ which satisfies Equation (1.63) (instead of System (1.16)) with the initial data $(\dot{S}_{n+1}\theta_0, \dot{S}_{n+1}u_0)$ and the known functions v, r_1, r_2 taking value at $(\theta^n, u^n, \nabla Q^n)$. Making use of the above a priori estimets and almost the same estimates as in Section §1.2 for the nonlinearities, the uniform estimates for the solution sequence hold. Then, denoting the primitive of the scalar function $(\alpha' k)(\vartheta)$ by $\zeta(\vartheta)$, one arrives at the following system for the sequence of the differences of the approximated solutions $(\delta\theta^{n+1}, \delta u^{n+1}, \delta Q^{n+1}) := (\theta^{n+1} - \theta^n, u^{n+1} - u^n, Q^{n+1} - Q^n)$

$$\begin{cases} \partial_t \delta\theta^{n+1} + u^n \cdot \nabla \delta\theta^{n+1} - \bar{\kappa}\Delta\delta\theta^{n+1} = -\delta u^n \cdot \nabla\theta^n + r_1^n - r_1^{n-1}, \\ \Delta\delta Q^{n+1} = \mathrm{div}\left(r_2^n - r_2^{n-1} - \delta u^n \cdot \nabla u^n - u^n \cdot \nabla \delta u^{n+1}\right) \\ \qquad +(\partial_t - \bar{\eta}\Delta)\left(\nabla\zeta^{n+1} \cdot \nabla\delta\theta^{n+1} + \nabla\delta\zeta^{n+1} \cdot \nabla\theta^n\right), \\ (\partial_t - \bar{\eta}\Delta)\delta u^{n+1} = r_2^n - r_2^{n-1} - \nabla\delta Q^{n+1} - \delta u^n \cdot \nabla u^n - u^n \cdot \nabla\delta u^{n+1}, \\ (\delta\theta^{n+1}, \delta u^{n+1})|_{t=0} = (\dot{\Delta}_n\theta_0, \dot{\Delta}_n u_0). \end{cases}$$

The above a priori estimates still work for these differences. And one has constructed a Cauchy sequence in $\dot{E}_T^{p_1,p_2}$ whose limit solves Equation (1.59). Details are omitted.

Fully nonhomogeneous case

Let us move to the well-posedness result in the fully nonhomogeneous case :

Theorem 1.4. *Let (p_1, p_2) satisfies*

$$d \geq 2, \quad \frac{1}{2} < \frac{1}{p_1} < \frac{1}{p_2} + \frac{1}{d}, \quad p_2 \in [2,4] \quad (\textit{with} \quad p_2 < 4 \quad \textit{if} \quad d = 2). \tag{1.67}$$

Let $\vartheta_0 \geq m > 0$ and $u_0 \in B_{p_1,1}^{d/p_2-1}$ satisfy $\theta_0 = \vartheta_0 - 1 \in B_{p_1,1}^{d/p_1}$ and (1.58). Then System (1.56) has a unique solution $(\vartheta, u, \nabla Q)$ with $(\theta, u, \nabla Q) \in E_T^{p_1,p_2}$, $T \in (0, +\infty]$. Furthermore, the flow map $(\theta_0, u_0) \mapsto (\theta, u, \nabla Q)$ is Lipschitz continuous.

As in the slightly nonhomogeneous case, we mainly strive for getting a priori estimates for the linearized equations of System (1.56) :

$$\begin{cases} \partial_t\theta + v \cdot \nabla\theta - \mathrm{div}\,(\kappa\nabla\theta) = h_1, \\ \mathrm{div}\left(\vartheta\nabla Q\right) = \mathrm{div}\left(h_2 - v \cdot \nabla u + Du \cdot \nabla\eta - \eta\nabla(\alpha' k|\nabla\theta|^2)\right) + \partial_t(\alpha' k|\nabla\theta|^2), \\ \partial_t u + v \cdot \nabla u - \mathrm{div}\,(\eta\nabla u) = h_2 - \vartheta\nabla Q, \\ (\theta, u)|_{t=0} = (\theta_0, u_0). \end{cases} \tag{1.68}$$

In the above, v, h_1, h_2, θ_0 and u_0 are given smooth functions which decay rapidly at infinity ; $\kappa, \eta, \vartheta \in [m, M]$ are positive smooth functions with $\kappa - 1, \eta - 1, \vartheta - 1 \in \mathcal{C}_0^\infty(\mathbb{R}^d)$; α, k are positive scalar functions depending on θ ; there exist a small enough constant c_m depending only on d, p_1, p_2 (see Proposition B.8), an integer $N > 0$ and a positive time T such that

$$\inf_{x\in\mathbb{R}^d,\, t\in[0,T]} S_N\vartheta(t,x) \geq m/2 > 0, \quad \|\vartheta - S_N\vartheta\|_{L_T^\infty(B_{p_1,1}^{d/p_1})} \leq c_m. \tag{1.69}$$

Firstly, Proposition 1.1 still works for θ :

$$\begin{aligned} \|\theta\|_{X^{p_1}(t)} \leq \exp&\left\{C_1\left(\|\nabla v\|_{L_t^1(B_{p_2,1}^{d/p_2})} + \|\nabla\kappa\|^2_{L_t^2(B_{p_1,1}^{d/p_1})}\right)\right\} \\ &\times C_1\left(\|\theta_0\|_{B_{p_1,1}^{d/p_1}} + \|\Delta_{-1}\theta\|_{L_t^1(L^{p_1})} + \|h_1\|_{L_t^1(B_{p_1,1}^{d/p_1})}\right). \end{aligned} \tag{1.70}$$

Consequently, by use of the product estimates, the following holds for $\partial_t\theta$:

$$\begin{aligned}\|\partial_t\theta\|_{L^1_t(B^{\frac{d}{p_1}}_{p_1,1})\cap L^2_t(B^{\frac{d}{p_1}-1}_{p_1,1})} \leq \exp\left\{C\left(\|\nabla v\|_{L^1_t(B^{\frac{d}{p_2}}_{p_2,1})}+\|\nabla\kappa\|^2_{L^2_t(B^{\frac{d}{p_1}}_{p_1,1})}\right)\right\}\\ \times C\Big(1+\|v\|_{L^2_t(B^{\frac{d}{p_2}}_{p_2,1})\cap L^\infty_t(B^{\frac{d}{p_2}-1}_{p_2,1})}+\|\nabla\kappa\|_{L^\infty_t(B^{\frac{d}{p_1}-1}_{p_1,1})}\Big)\\ \times\left(\|\theta_0\|_{B^{\frac{d}{p_1}}_{p_1,1}}+\|\Delta_{-1}\theta\|_{L^1_t(L^{p_1})}+\|h_1\|_{L^1_t(B^{\frac{d}{p_1}}_{p_1,1})\cap L^2_t(B^{\frac{d}{p_1}-1}_{p_1,1})}\right).\end{aligned} \tag{1.71}$$

It rests to show a similar estimate as in Proposition 1.2 for u and ∇Q. We will follow the proof of Proposition B.8 in the appendix. Firstly, in order to apply Proposition 0.9 to Equation $(1.68)_2$, let us consider the following Poisson's equation

$$\Delta Q_\theta = L_\theta := \partial_t(\alpha' k|\nabla\theta|^2) \equiv \partial_t\theta\big(\nabla(\alpha'k)\cdot\nabla\theta\big)\ +\ 2\partial_t\nabla\theta\cdot(\alpha'k\nabla\theta).$$

Following the arguments in estimating the pressure term in Chemin's book [22], one defines the following four quantites by use of the paraproduct operator T and the remainder operator R (see (0.45) for the definition) :

$$\begin{aligned}\nabla Q^1_\theta &:= \nabla\Delta^{-1}L^T_\theta,\\ \nabla Q^2_\theta &:= \nabla\Delta^{-1}(\mathrm{Id}\,-\Delta_{-1})L^R_\theta,\\ \nabla Q^3_\theta &= (\chi T_d)*\nabla\Delta_{-1}L^R_\theta,\\ \nabla Q^4_\theta &= \nabla((1-\chi)T_d)*\Delta_{-1}L^R_\theta.\end{aligned}$$

In the above, χ is the smooth positive function defined in Section §0.3.1, T_d is the fundamental solution of the Laplace operator Δ in the space $\mathbb{R}^d$, L^T_θ denotes the paraproduct parts of L_θ :

$$T_{(\partial_t\theta)}\big(\nabla(\alpha'k)\cdot\nabla\theta\big)+T_{(\nabla(\alpha'k)\cdot\nabla\theta)}(\partial_t\theta)+T_{(2\partial_t\nabla\theta)}(\alpha'k\nabla\theta)+T_{(\alpha'k\nabla\theta)}(2\partial_t\nabla\theta),$$

and L^R_θ represents the remainder parts of L_θ :

$$R(\partial_t\theta,\nabla(\alpha'k)\cdot\nabla\theta)\ +\ R(2\partial_t\nabla\theta,\alpha'k\nabla\theta).$$

Since $T_{(\partial_t\theta)}(\nabla(\alpha'k)\cdot\nabla\theta)$ is spectrally supported away from the origin, one deduces from Proposition A.1 that if $d\geq 2$, $p_1\leq 4$ and $\frac{1}{p_1}\geq\frac{1}{2}-\frac{1}{d}$, then

$$\begin{aligned}\|\nabla\Delta^{-1}T_{(\partial_t\theta)}\big(\nabla(\alpha'k)\cdot\nabla\theta\big)\|_{L^2} &\lesssim \|T_{(\partial_t\theta)}\big(\nabla(\alpha'k)\cdot\nabla\theta\big)\|_{B^{-1}_{2,2}}\\ &\lesssim \|T_{(\partial_t\theta)}\big(\nabla(\alpha'k)\cdot\nabla\theta\big)\|_{B^{d/2-2}_{2,2}} \lesssim \|\partial_t\theta\|_{B^{d/p_1-1}_{p_1,1}}\|\nabla(\alpha'k)\cdot\nabla\theta\|_{B^{d/p_1-1}_{p_1,1}}.\end{aligned}$$

Similarly, noticing that all the members of ∇Q^1_θ and ∇Q^2_θ are spectrally supported away from the origin, according to Proposition A.1, one obtains the following estimate (i.e. Estimate (1.64) with $p_2=2$) if $d\geq 2$, $p_1<d$, $p_1\leq 4$, $\frac{1}{p_1}\geq\frac{1}{2}-\frac{1}{d}$:

$$\|(\nabla Q^1_\theta,\nabla Q^2_\theta)\|_{L^2}\lesssim\|\partial_t\theta\|_{B^{\frac{d}{p_1}-1}_{p_1,1}}\|\nabla(\alpha'k)\cdot\nabla\theta\|_{B^{\frac{d}{p_1}-1}_{p_1,1}}+\|\partial_t\nabla\theta\|_{B^{\frac{d}{p_1}-1}_{p_1,1}}\|\alpha'k\nabla\theta\|_{B^{\frac{d}{p_1}-1}_{p_1,1}}. \tag{1.72}$$

It is easy to see that χT_d is integrable. Thus

$$\|\nabla Q^3_\theta\|_{L^2}\lesssim\|\nabla\Delta_{-1}L^R_\theta\|_{L^2}\lesssim\|\Delta_{-1}L^R_\theta\|_{B^{-1}_{2,2}}.$$

Hence (1.72) holds for ∇Q^3_θ too. It rests to study ∇Q^4_θ :

- firstly, it is easy to check that $\nabla((1-\chi)T_d) \in L^a$, $\forall a > \frac{d}{d-1}$;
- then, $\|\nabla Q_\theta^4\|_{L^2}$ is controlled by $\|\Delta_{-1} L_\theta^R\|_{L^p}$, $\forall \frac{1}{p} > \frac{1}{2} + \frac{1}{d}$;
- finally, if $p_1 < d$, $\frac{2}{p_1} \geq \frac{1}{p}$ and $\frac{1}{p_1} \geq \frac{1}{p} - \frac{1}{d}$, i.e. if $p_1 < 2 \leq d$, then $\|\Delta_{-1} L_\theta^R\|_{L^p}$ can be bounded by the right-hand side of (1.72), up to a constant.
- Hence, (1.72) holds for $\|\nabla Q_\theta^4\|_{L^2}$.

To conclude, (1.72) holds for ∇Q_θ, provided $p_1 < 2 \leq d$.

Now, applying Proposition 0.9 to Equation $(1.68)_2$ issues that

$$\|\nabla Q\|_{L^2} \lesssim \|h_2 - v\cdot\nabla u + Du\cdot\nabla\eta - \eta\nabla(\alpha' k|\nabla\theta|^2)\|_{L^2} + \|\nabla Q_\theta\|_{L^2}.$$

The same argument as to get Estimate (1.72) implies the following estimates if $p_1 < 2$, $p_2 \leq 4$, $p_2 < 2d$, $\frac{1}{p_2} \geq \frac{1}{2} - \frac{1}{d} + \frac{\delta}{d}$, $\forall \delta \geq 0$:

$$\begin{aligned}
&\|v\cdot\nabla u\|_{L^2} \lesssim \|v\cdot\nabla u\|_{B^{d/2-1}_{2,2}} \lesssim \|v\|_{B^{d/p_2-1+\delta}_{p_2,1}} \|\nabla u\|_{B^{d/p_2-\delta}_{p_2,1}} + \|v\|_{B^{d/p_2}_{p_2,1}} \|\nabla u\|_{B^{d/p_2-1}_{p_2,1}},\\
&\|Du\cdot\nabla\eta\|_{L^2} \lesssim \|\nabla u\|_{B^{d/p_2-1}_{p_2,1}} \|\nabla\eta\|_{B^{d/p_1}_{p_1,1}} + \|\nabla u\|_{B^{d/p_2-\delta}_{p_2,1}} \|\nabla\eta\|_{B^{d/p_1-1+\delta}_{p_1,1}},\\
&\|\nabla(\alpha' k|\nabla\theta|^2)\|_{L^2} \lesssim \|\nabla(\alpha' k|\nabla\theta|^2)\|_{B^{d/p_1-1}_{p_1,1}} \leq C_\theta(\|\theta\|_{B^{d/p_1}_{p_1,1}}) \|\nabla\theta\|^2_{B^{d/p_1}_{p_1,1}}.
\end{aligned}$$

Therefore, we arrive at the energy estimate for ∇Q :

$$\begin{aligned}
\|\nabla Q\|_{L^1_t(L^2)} &\leq \|h_2\|_{L^1_t(L^2)} + C_\theta(d, p_1, M, \|\theta\|_{L^\infty_t(B^{d/p_1}_{p_1,1})})\\
&\quad\times\Big(\|\nabla\theta\|^2_{L^2_t(B^{\frac{d}{p_1}}_{p_1,1})} + \|\nabla\theta\|_{L^2_t(B^{\frac{d}{p_1}}_{p_1,1})} \|\partial_t\theta\|_{L^2_t(B^{\frac{d}{p_1}-1}_{p_1,1})} + \|\partial_t\theta\|_{L^1_t(B^{\frac{d}{p_1}}_{p_1,1})}\Big)\\
&+ C_\theta \int_0^t \big(\|v\|_{B^{d/p_2-1+\delta}_{p_2,1}} + \|\nabla\eta\|_{B^{d/p_1-1+\delta}_{p_1,1}}\big)\|\nabla u\|_{B^{d/p_2-\delta}_{p_2,1}} + \big(\|v\|_{B^{d/p_2}_{p_2,1}} + \|\nabla\eta\|_{B^{d/p_1}_{p_1,1}}\big)\|\nabla u\|_{B^{d/p_2-1}_{p_2,1}}.
\end{aligned} \tag{1.73}$$

Now we can proceed exactly as in the proof of Proposition B.8. Thanks to Condition (1.69), one deduces from Equation $(1.68)_2$ that if $p_2 \geq 2$ then

$$\|\nabla Q\|_{B^{\frac{d}{p_2}-1}_{p_2,1}} \leq C_\vartheta\Big(\|h_2 - v\cdot\nabla u + Du\cdot\nabla\eta - \eta\nabla(\alpha' k|\nabla\theta|^2)\|_{B^{\frac{d}{p_2}-1}_{p_2,1}} + \|L_\theta\|_{B^{\frac{d}{p_2}-2}_{p_2,1}} + \|\nabla Q\|_{L^2}\Big),$$

for some constant C_ϑ depending on $N, d, p_1, p_2, m, \|\nabla\vartheta\|_{B^{d/p_1-1}_{p_1,1}}$. Noticing $p_1 < 2 \leq p_2$, one applies the product estimates (e.g. (1.15), (1.64)) to the right-hand side. It entails that (1.73) also holds for $\|\nabla Q\|_{L^1_t(B^{d/p_2-1}_{p_2,1})}$, with C_θ depending additionally on N, p_2, m, $\|(\nabla\vartheta, \nabla\eta)\|_{L^\infty_t(B^{d/p_1-1}_{p_1,1})}$ and $\|h_2\|_{L^1_t(L^2)}$ replaced by $C_\theta\|h_2\|_{L^1_t(L^2\cap B^{d/p_2-1}_{p_2,1})}$.

Therefore, applying Proposition 1.1, interpolation inequalities and Gronwall's inequality to the equation for u $(1.68)_3$, one arrives at the a priori estimate for u and ∇Q (with $\delta \in (0, d(\frac{1}{p_2} - \frac{1}{2} + \frac{1}{d})])$:

$$\begin{aligned}
&\|u\|_{Y^{p_2}(t)} + \|\nabla Q\|_{Z^{p_2}(t)}\\
&\leq \exp\Big\{C_\theta\Big(\|v\|^2_{L^2_t(B^{d/p_2}_{p_2,1})} + \|v\|^{2/\delta}_{L^2_t(B^{d/p_2-1+\delta}_{p_2,1})} + \|\nabla\eta\|^2_{L^2_t(B^{d/p_1}_{p_1,1})} + \|\nabla\eta\|^{2/\delta}_{L^2_t(B^{d/p_1-1+\delta}_{p_1,1})}\Big)\Big\}\\
&\quad\times\Big(\|u_0\|_{B^{d/p_2}_{p_2,1}} + C_\theta\Big(\|\Delta_{-1}u\|_{L^1_t(L^{p_2})} + \|h_2\|_{L^1_t(L^2\cap B^{d/p_2-1}_{p_2,1})}\\
&\qquad + \|\nabla\theta\|^2_{L^2_t(B^{\frac{d}{p_1}}_{p_1,1})} + \|\nabla\theta\|_{L^2_t(B^{\frac{d}{p_1}}_{p_1,1})} \|\partial_t\theta\|_{L^2_t(B^{\frac{d}{p_1}-1}_{p_1,1})} + \|\partial_t\theta\|_{L^1_t(B^{\frac{d}{p_1}}_{p_1,1})}\Big)\Big).
\end{aligned} \tag{1.74}$$

Since we have obtained the a priori estimates (1.70), (1.71) and (1.74), the same strategy as in Section §1.3.2 works well, with the approximated solutions satisfying Equation (1.68) (instead of (1.48)). Theorem 1.4 thus follows. Details are left to the reader.

Chapitre 2

Global existence results for a special case

2.1 Introduction

This chapter is to study global-in-time existence of weak solutions to zero Mach number system (0.1) :

$$\begin{cases} \partial_t\rho + \mathrm{div}_x(\rho v) & = & 0, \\ \partial_t(\rho v) + \mathrm{div}_x(\rho v\otimes v) - \mathrm{div}_x(\mu(\nabla v + Dv) + \nu \mathrm{div}\, v\mathrm{Id}\,) + \nabla_x\Pi & = & 0, \\ \mathrm{div}_x(v+\kappa\nabla \ln\rho) & = & 0, \end{cases} \tag{2.1}$$

under the following relationship between the viscosity coefficient μ and the diffusion coefficient κ

$$\kappa(\rho) - 2\mu'(\rho) = 0. \tag{2.2}$$

Roughly speaking, this relation implies that the source term in the equation for the newly introduced divergence-free velocity vector field

$$u := v + \kappa\nabla \ln\rho \tag{2.3}$$

vanishes. In fact, according to the calculation in Subsection §0.2.1 in the introduction, one arrives at the following system (i.e. System (0.3)) concerning the unknowns $(\rho, u, \nabla\pi)$:

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho v) = \partial_t\rho + \mathrm{div}\,(\rho u) - \mathrm{div}\,(\kappa\nabla\rho) & = & 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho v\otimes u) - \mathrm{div}\,(2\mu A u) + \nabla\pi & = & 0, \\ \mathrm{div}\, u & = & 0. \end{cases} \tag{2.4}$$

Let's first analyze System (2.4) *formally.* If we assume the initial density $\rho_0(x)$ to be bounded from below and above by two positive constants ρ_* and ρ^* respectively, then the maximum principle for the parabolic equation $(2.4)_1$ ensures the density to satisfy the following uniform bound :

$$\rho(t,x) \in [\rho_*, \rho^*], \quad \forall\, t\geq 0,\, x\in\mathbb{R}^d. \tag{2.5}$$

Furthermore, if ρ is close to a constant, say "1", at infinity, then we expect the solution (ρ, u) to satisfy the following two energy equalities which come from taking the $L^2(\mathbb{R}^d)$-inner product between $\rho - 1$ (resp. u) and $(2.4)_1$ (resp. $(2.4)_2$) :

$$\int_{\mathbb{R}^d} |\rho(t)-1|^2 + 2\int_0^t\int_{\mathbb{R}^d} \kappa|\nabla\rho|^2 = \int_{\mathbb{R}^d} |\rho_0 - 1|^2, \quad \forall\, t>0\,, \tag{2.6}$$

and

$$\int_{\mathbb{R}^d} \rho(t)|u(t)|^2 + 4\int_0^t \int_{\mathbb{R}^d} \mu|Au|^2 = \int_{\mathbb{R}^d} \rho_0|u_0|^2, \quad \forall\, t > 0\,. \tag{2.7}$$

Therefore, we complement System (2.4) with initial data (ρ_0, u_0) verifying

$$\rho - 1|_{t=0} = \rho_0 - 1 \in L^2(\mathbb{R}^d), \quad u|_{t=0} = u_0 \in (L^2(\mathbb{R}^d))^d, \quad 0 < \rho_* \leq \rho_0 \leq \rho^*, \quad \operatorname{div} u_0 = 0, \tag{2.8}$$

hoping that the obtained solution (ρ, u) satisfies (2.6) and (2.7), at least in inequality form.

Moreover, we assume that under the bounds (2.5) imposed on the density, there exist positive constants $\kappa_*, \kappa^*, \mu_*, \mu^*$ depending only on ρ_*, ρ^* such that the physical coefficients κ and μ also have positive lower and upper bounds :

$$0 < \kappa_* \leq \kappa(\rho) \leq \kappa^*, \quad 0 < \mu_* \leq \mu(\rho) \leq \mu^*, \quad \forall\, t \geq 0,\, x \in \mathbb{R}^d. \tag{2.9}$$

In this chapter, we shall establish the following global existence result for System (2.4) :

Theorem 2.1. *There exists a global-in-time weak solution (ρ, u) to Cauchy problem* (2.4)-(2.8) *in the following sense :*

- $\rho - 1 \in \mathcal{C}([0, +\infty); L^p(\mathbb{R}^d)) \cap L^2_{\text{loc}}([0, +\infty); H^1(\mathbb{R}^d))$, $\forall\, p \in [2, \infty)$.
  ρ satisfies $(2.4)_1$ *in* $L^2_{\text{loc}}([0, +\infty); H^{-1}(\mathbb{R}^d))$ *and* $\rho(0) = \rho_0$.
  The uniform bound (2.5) *and* Energy Equality (2.6) *both hold.*
- *For any $t > 0$ and any test function $\phi(t, x) \in (\mathcal{C}^\infty([0, +\infty) \times \mathbb{R}^d))^d$ with compact support such that* $\operatorname{div} \phi = 0$*, the following holds :*

$$\int_{\mathbb{R}^d} \rho(t)u(t) \cdot \phi - \int_{\mathbb{R}^d} \rho_0 u_0 \cdot \phi|_{t=0}$$
$$- \int_0^t \int_{\mathbb{R}^d} [\rho u \cdot \partial_t \phi + (\rho u - \kappa\nabla\rho) \cdot \nabla\phi \cdot u - 2\mu Au : A\phi] = 0. \tag{2.10}$$

- $u \in \mathcal{C}([0, \infty); (L^2_w(\mathbb{R}^d))^d) \cap L^2_{\text{loc}}([0, +\infty); (H^1(\mathbb{R}^d))^d)$, $\lim_{t\to 0_+} u(t) = u_0$ *in* $L^2(\mathbb{R}^d)$ *and* $\operatorname{div} u = 0$ *in* $L^2_w(\mathbb{R}^+ \times \mathbb{R}^d)$ *with* L^2_w *denoting the Lebesgue space* L^2 *endowed with weak topology.*
  There exists a positive constant C depending only on ρ_*, ρ^* such that u verifies the energy inequality

$$\int_{\mathbb{R}^d} |u(t)|^2 + \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 \leq C \int_{\mathbb{R}^d} |u_0|^2, \quad \forall t > 0. \tag{2.11}$$

In the above theorem, it is easy to check (according to Condition (0.16)) that ρ and $v := u - \kappa\nabla\ln\rho \in L^2([0, +\infty); (L^2(\mathbb{R}^d))^d)$ satisfy System (2.1)-(2.2) in distribution sense. But owing to a lack of high regularity assumption on the initial density ρ_0, we don't know the continuity of v at the initial instant, since we can't even define the quantity $\kappa(\rho_0)\nabla\ln\rho_0$.

Instead, if we assume $\rho_0 - 1 \in H^1(\mathbb{R}^d)$ in addition to the initial condition (2.8), that is, the initial original velocity field v_0 belongs to $(L^2(\mathbb{R}^d))^d$ too, then we expect that there exists a global weak solution $(\rho - 1, v) \in H^1(\mathbb{R}^d) \times (L^2(\mathbb{R}^d))^d$ to System (2.1)-(2.2), if $d = 2$ or 3. In fact, if $\kappa \equiv 1$, then taking $L^2(\mathbb{R}^d)$-inner product between Equation $(2.4)_1$ and $\Delta\rho$ yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} + \|\Delta\rho\|^2_{L^2(\mathbb{R}^d)} \leq \left|\int_{\mathbb{R}^d} \operatorname{div}(\rho u)\Delta\rho\right|. \tag{2.12}$$

Since a priori we have the estimate (noticing $\operatorname{div} u = 0$)

$$\left|\int_{\mathbb{R}^d} \operatorname{div}(\rho u)\Delta\rho\right| = \left|\int_{\mathbb{R}^d} \nabla\rho \cdot \nabla u \cdot \nabla\rho\right| \leq \|\nabla u\|_{L^2(\mathbb{R}^d)}\|\nabla\rho\|^2_{L^4(\mathbb{R}^d)}, \tag{2.13}$$

and the following two interpolation inequalities :

$$\|\nabla\rho\|^2_{L^4(\mathbb{R}^2)} \leq C\|\Delta\rho\|_{L^2(\mathbb{R}^2)}\|\nabla\rho\|_{L^2(\mathbb{R}^2)}, \quad \|\nabla\rho\|^2_{L^4(\mathbb{R}^3)} \leq C\|\Delta\rho\|_{L^2(\mathbb{R}^3)}\|\rho\|_{L^\infty(\mathbb{R}^3)}, \quad (2.14)$$

we have from Young's Inequality and Estimate (2.11) the following two energy estimates :

$$\begin{aligned}
&\|\nabla\rho\|_{L^\infty_t(L^2(\mathbb{R}^2))} + \|\Delta\rho\|_{L^2_t(L^2(\mathbb{R}^2))} \leq C\|\nabla\rho_0\|_{L^2}\, e^{C\int_0^t \|\nabla u\|^2_{L^2}} \leq C\|\nabla\rho_0\|_{L^2(\mathbb{R}^2)}\, e^{C\|u_0\|^2_{L^2(\mathbb{R}^2)}}, \\
&\|\nabla\rho\|_{L^\infty_t(L^2(\mathbb{R}^3))} + \|\Delta\rho\|_{L^2_t(L^2(\mathbb{R}^3))} \leq C(\|\nabla\rho_0\|_{L^2} + \|\nabla u\|_{L^2_t(L^2)}) \leq C\|(\nabla\rho_0, u_0)\|_{L^2(\mathbb{R}^3)},
\end{aligned} \quad (2.15)$$

for some constant C depending only on ρ_*, ρ^*. In the general case where κ depends on ρ, we consider, instead, the equation for the scalar function $a = a(\rho)$ with $\nabla a = \kappa\nabla\rho$ and $a(1) = 0$ (see (0.36) for the definition). See Section §2.2.2 in the following for more details.

But it is not clear that we can still have $v \in L^\infty_t((L^2(\mathbb{R}^d))^d)$, in dimension $d \geq 4$, due to a lack of an interpolation inequality like (2.14) which can be used to control the convection term $u \cdot \nabla\rho$ in the equation of the density. Anyway, we have

Theorem 2.2. *Let $d = 2, 3$ and Relation (2.2) hold. For any initial data (ρ_0, v_0) such that*

$$\rho_0 - 1 \in H^1(\mathbb{R}^d), \quad v_0 \in (L^2(\mathbb{R}^d))^d, \quad 0 < \rho_* \leq \rho_0 \leq \rho^*, \quad \mathrm{div}\,(v_0 + \kappa(\rho_0)\nabla\ln\rho_0) = 0, \quad (2.16)$$

there exists a global-in-time weak solution (ρ, v) to Cauchy problem (2.1)-(2.16) in the sense given in Theorem 2.1, except with Equality (2.10) replaced by

$$\int_{\mathbb{R}^d} \rho(t)v(t)\cdot\phi - \int_{\mathbb{R}^d} \rho_0 v_0 \cdot \phi|_{t=0} - \int_0^t \int_{\mathbb{R}^d} [\rho v \cdot \partial_t\phi + \rho v \cdot \nabla\phi \cdot v - \sigma : D\phi] = 0. \quad (2.17)$$

Furthermore, the following properties hold :

- *In dimension two, the density ρ satisfies Equation $(2.1)_1$ in $L^2([0, +\infty); L^2)$ and the "velocity" $u = v - \kappa\nabla\ln\rho$ verifies Energy Equality (2.7).*
- *The solution (ρ, v) satisfies the following continuity property :*

$$\begin{aligned}
&(\rho - 1, v) \in \mathcal{C}\Big([0, +\infty); H^1(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2\Big), \ \text{if } d = 2, \quad (2.18)\\
&\rho - 1 \in \mathcal{C}\Big([0, +\infty); H^s(\mathbb{R}^3)\Big), \ \forall\, s < 1, \ \text{and } v \in \mathcal{C}\Big([0, +\infty); (L^2_w(\mathbb{R}^3))^3\Big), \ \text{if } d = 3.
\end{aligned}$$

- *The following energy estimate holds true :*

$$\|(\rho-1, v)\|_{L^\infty_t([0,\infty); H^1\times L^2)} + \|(\nabla\rho, \nabla v)\|_{L^2_t([0,\infty); H^1\times L^2)} \leq C(\rho_*, \rho^*, \|(\rho_0 - 1, v_0)\|_{H^1\times L^2}). \quad (2.19)$$

Next we want to show that the global solutions constructed above are actually unique for some reasonably smooth initial data when $d = 2$. We notice that even if $\rho - 1 \in L^\infty(H^1)$, it is difficult to show the uniqueness. In fact, if we consider the system for the difference of any two solutions, then the nonlinear terms in System (2.1) ask the $L^\infty(L^\infty)$-norm control on the difference of two densities. It is unknown a priori because the difference does not (at least not obviously) satisfy any parabolic equation and another unlucky thing is that we can't embed $H^1(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$. Therefore, we have to resort to Besov functional space $B^1_{2,1}(\mathbb{R}^2)$ which can be embedded both in H^1 and in L^∞ in dimension 2 and furthermore, we have already the following existence result, as a special case of Theorem 1.2 :

Theorem 2.3. *In dimension 2, for any initial density ρ_0 and velocity field u_0 which satisfy*

$$0 < \rho_* \leq \rho_0 \leq \rho^*, \quad \mathrm{div}\, u_0 = 0 \quad \textit{and} \quad \|\rho_0 - 1\|_{B^1_{2,1}} + \|u_0\|_{B^0_{2,1}} \leq M, \quad (2.20)$$

for some positive constants ρ_*, ρ^*, M, there exists a positive time T_c depending only on ρ_*, ρ^*, M such that System (2.4) *has a unique solution $(\rho, u, \nabla\pi)$ with $\rho \in [\rho_*, \rho^*]$ and $(\rho - 1, u, \nabla\pi) \in E_{T_c}$, where the solution space $E_T := E_T^{2,2}(\mathbb{R}^2)$ is the following critical nonhomogeneous Besov spaces in dimension* 2 *:*

$$\Big(\mathcal{C}([0,T]; B^1_{2,1}) \cap L^1([0,T]; B^3_{2,1})\Big) \times \Big(\mathcal{C}([0,T]; B^0_{2,1}) \cap L^1([0,T]; B^2_{2,1})\Big)^2 \times \Big(L^1([0,T]; B^0_{2,1})\Big)^2. \tag{2.21}$$

Hence, System (2.4)-(2.20) admits a unique local strong solution $(\rho, u, \nabla\pi)$ on its lifespan $[0, T^*)$, $T^* > T_c$, with $(\rho - 1, u, \nabla\pi) \in E_t$ for any $t < T^*$. Moreover, there exists a positive time $T_0 < T^*$ such that $(\rho - 1, u)|_{t=T_0} \in B^3_{2,1} \times (B^2_{2,1})^2 \subset H^2 \times (H^1)^2$. Just as in [33], we therefore consider an extra pseudo-conservation law concerning $L^\infty(H^2) \times (L^\infty(H^1))^2$-norm of the weak solutions which evolve from the initial moment T_0. By virtue of the embedding $H^2 \times (H^1)^2 \subset B^1_{2,1} \times (B^0_{2,1})^2$, this law ensures that these global weak solutions also belong to the above solution space E_t, for all $t \geq T_0$ (see Lemma 2.4 for more details). To conclude, thanks to the uniqueness result on the time interval $[T_0, T^*)$, the Cauchy problem (2.4)-(2.20) has a unique global strong solution $(\rho, u, \nabla\pi)$ with $(\rho - 1, u, \nabla\pi)$ in E_T for all $T \in [0, +\infty)$.

The following estimates in Besov spaces hold (see Proposition 0.3 and Proposition 0.5) :

$$\|f(\rho) - f(1)\|_{B^s_{2,1}} \leq C(\rho_*, \rho^*)\|\rho - 1\|_{B^s_{2,1}}, \ \forall f \in \mathcal{C}^1, \ s > 0, \tag{2.22}$$

$$\|gh\|_{B^i_{2,1}} \leq C\|g\|_{B^1_{2,1}}\|h\|_{B^i_{2,1}}, \ \text{if } i = 0, \ 1. \tag{2.23}$$

Hence, by use of Equation $(2.4)_1$, it is easy to find that $\partial_t\rho \in L^1([0, +\infty); B^1_{2,1}(\mathbb{R}^2))$ and furthermore, according to the calculation from System (2.1) to System (2.4) in Section §0.2.1, we have

Theorem 2.4. *Let $d = 2$ and Relation* (2.2) *hold. For any initial data (ρ_0, v_0) satisfying*

$$0 < \rho_* \leq \rho_0 \leq \rho^*, \quad \operatorname{div}(v_0 + \kappa_0 \nabla \ln \rho_0) = 0, \quad \textit{and} \quad \rho_0 - 1 \in B^1_{2,1}(\mathbb{R}^2), \quad v_0 \in (B^0_{2,1}(\mathbb{R}^2))^2, \tag{2.24}$$

System (2.1) *has a unique global strong solution $(\rho, v, \nabla\Pi)$ with $(\rho - 1, v, \nabla\Pi) \in E_T$, $\forall T \in [0, +\infty)$.*

Remark 2.1. *Theorem 2.4 implies that in dimension two, for any initial datum $\rho_0 \geq \rho_* > 0$ such that $\rho_0 - 1 \in B^1_{2,1}$, there exists a unique global-in-time solution to the quasilinear heat equation $\partial_t\rho - \operatorname{div}(\kappa(\rho)\nabla\rho) = 0$ in functional space $\mathcal{C}_b([0, +\infty); B^1_{2,1}) \cap L^1([0, +\infty); B^3_{2,1})$.*

We will use the following a priori estimate for parabolic systems with elliptic operator of divergence form in Besov spaces, in dimension 2 (see Proposition 1.1 or Proposition B.1) :

Proposition 2.1. *Let $s \in (-1, 1]$. Let $a(t, x) \in \mathcal{S}([0, T] \times \mathbb{R}^2)$ satisfy*

$$\begin{cases} \partial_t a - \operatorname{div}(\kappa \nabla a) &= f, \\ a|_{t=0} &= a_0, \end{cases} \tag{2.25}$$

where $\kappa = \kappa(t, x) \geq \kappa_* > 0$ is known. Then there exists a constant C depending on κ_*, s such that for any $t \in [0, T]$, we have the following a priori estimate :

$$\begin{aligned} &\|a\|_{L^\infty_t(B^s_{2,1}(\mathbb{R}^2)) \cap L^1_t(B^{s+2}_{2,1}(\mathbb{R}^2))} \\ &\leq \Big(\|a_0\|_{B^s_{2,1}(\mathbb{R}^2)} + C\|\Delta_{-1}a\|_{L^1_t(L^2(\mathbb{R}^2))} + \|f\|_{L^1_t(B^s_{2,1}(\mathbb{R}^2))}\Big) \times \exp\Big\{C\|\nabla\kappa\|^2_{L^2_t(B^1_{2,1}(\mathbb{R}^2))}\Big\}. \end{aligned} \tag{2.26}$$

Notations : Let us fix some notations which will be used throughout in the present chapter :

- We always take $\varrho = \rho - 1$ in any environment, that is, $\varrho_0 = \rho_0 - 1$, $\varrho^\varepsilon = \rho^\varepsilon - 1$, etc.
- All the physical coefficients will always be viewed as the functions of the density ρ for simplicity. For example, $\kappa' \triangleq \frac{d\kappa}{d\rho}$, $\kappa^\varepsilon \triangleq \kappa(\rho^\varepsilon)$, etc.
- Functions of the form $\langle f \rangle_\varepsilon$ will always be viewed as the regularized functions of f, in the sense specified in Section §2.2.1 (see (2.27)).
- We write $u_n \to u$ in some Banach space X to represent the strong convergence of the sequence $\{u_n\}_n$ to u in space X such that $\|u_n - u\|_X \to 0$, while $u_n \rightharpoonup u$ and $u_n \overset{*}{\rightharpoonup} u$ in X mean that $\{u_n\}_n$ converges to u in the associated weak and weak-$*$ topology of space X respectively.
- $L^2_w(X)$ denotes the Lebesgue space $L^2(X)$ endowed with weak topology.

The rest of this chapter unfolds as follows. The next section is devoted to proving Theorem 2.1 and Theorem 2.2 whereas the proof of Theorem 2.4 is left in the third section.

2.2 Global existence of weak solutions

In this section we will prove the global-in-time existence of weak solutions, i.e. Theorem 2.1 and Theorem 2.2. The first paragraph is devoted to the case when the initial density ρ_0 satisfies $\rho_0 - 1 \in L^2(\mathbb{R}^d)$ and the second, is to the case where $\rho_0 - 1 \in H^1(\mathbb{R}^2)$ or $H^1(\mathbb{R}^3)$.

2.2.1 The case with the density of lower regularity

In this subsection we will prove Theorem 2.1 in two steps. The first step is to solve the regularized system of the Cauchy problem (2.4)-(2.8) while the second, is devoted to show that the convergent limit of the obtained regular solution sequence is indeed a weak solution of this Cauchy problem.

Regularized system

In this paragraph we will consider the regularized system of (2.4)-(2.8). More precisely, let us fix a nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that

$$\text{Supp}\ \varphi \in B(0,1), \quad 0 \leq \varphi \leq 1, \quad \int_{\mathbb{R}^d} \varphi = 1,$$

and consequently define a sequence of functions $\{\varphi_\varepsilon\}_\varepsilon$ such that $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(\varepsilon^{-1}x)$, $\forall x \in \mathbb{R}^d$. Given any $f \in \mathcal{D}'(\mathbb{R}^d)$, we set the regularized functions $\{\langle f \rangle_\varepsilon\}_\varepsilon$ as

$$\langle f \rangle_\varepsilon \triangleq \varphi_\varepsilon * f. \tag{2.27}$$

Now we regularize the Cauchy problem as following [1]

$$\begin{cases} \partial_t \rho + \text{div}\,(\rho\,\langle u \rangle_\varepsilon) - \text{div}\,(\langle \kappa \rangle_\varepsilon \nabla \rho) & = & 0, \\ \partial_t(\rho u) + \text{div}\,\Big((\rho \langle u \rangle_\varepsilon - \langle \kappa \rangle_\varepsilon \nabla \rho) \otimes u\Big) - \text{div}\,(2\mu A u) + \nabla \pi & = & 0, \\ \text{div}\, u & = & 0, \\ \rho|_{t=0} & = & \langle \rho_0 \rangle_\varepsilon, \\ u|_{t=0} & = & \langle u_0 \rangle_\varepsilon. \end{cases} \tag{2.28}$$

1. We point out that we don't have to regularize the coefficient μ since the density ρ as a solution of the regularized equation $(2.28)_1$ is already smooth, whereas we remain the regularized form of the coefficient κ to keep uniform with $(2.28)_1$. This implies the uniform energy bound for u.

It is easy to see that if initial data (ρ_0, u_0) satisfies (2.8), then we have the following properties (keep in mind that $\varrho_0 = \rho_0 - 1$) :

$$\langle \varrho_0 \rangle_\varepsilon, \langle u_0 \rangle_\varepsilon \in H^\infty, \quad \langle \varrho_0 \rangle_\varepsilon \to \varrho_0 \text{ in } L^2, \quad \langle u_0 \rangle_\varepsilon \to u_0 \text{ in } L^2,$$
$$0 < \rho_* \le \langle \rho_0 \rangle_\varepsilon \le \rho^*, \quad \operatorname{div} \langle u_0 \rangle_\varepsilon = 0.$$

In the following, for any ε fixed, we will apply Schauder fixed point theorem to show the existence of a solution to System (2.28). More precisely, for some positive time $T \in (0, \infty)$ fixed, let $E_{R_0,T}$ be the convex subset in the Banach space $\{(\rho, u) \,|\, \rho \in \mathcal{C}([0,T] \times \mathbb{R}^d), (\rho - 1, u) \in \mathcal{C}([0,T]; H^1)\}$, defined by

$$\begin{aligned} E_{R_0,T} := \Big\{ &(\rho, u) \in \mathcal{C}([0,T] \times \mathbb{R}^d) \times \mathcal{C}([0,T]; H^1) \,|\, \rho - 1 \in \mathcal{C}([0,T]; H^1), \\ &0 < \rho_* \le \rho \le \rho^*, \quad \operatorname{div} u = 0 \text{ a.e. on } [0,T] \times \mathbb{R}^d, \\ &\|(\rho - 1, u)\|_{L^\infty(0,T;L^2)}, \, \|(\nabla \rho, \nabla u)\|_{L^2(0,T;L^2)} \le R_0 \Big\}, \end{aligned} \tag{2.29}$$

with R_0 depending only on the initial data, to be determined later. Then, we can define a map F from $E_{R_0,T}$ to itself : given any element $(\widetilde{\rho}, \widetilde{u}) \in E_{R_0,T}$, F maps it to the unique smooth solution $(\rho, u) \in E_{R_0,T}$ of the following system (with $\widetilde{\kappa} = \kappa(\widetilde{\rho})$)

$$\left\{ \begin{array}{rcl} \partial_t \rho + \operatorname{div} (\rho \, \langle \widetilde{u} \rangle_\varepsilon) - \operatorname{div} (\langle \widetilde{\kappa} \rangle_\varepsilon \nabla \rho) & = & 0, \\ \partial_t (\rho u) + \operatorname{div} \Big((\rho \langle \widetilde{u} \rangle_\varepsilon - \langle \widetilde{\kappa} \rangle_\varepsilon \nabla \rho) \otimes u \Big) - \operatorname{div} (2\mu(\rho) A u) + \nabla \pi & = & 0, \\ \operatorname{div} u & = & 0, \\ \rho|_{t=0} & = & \langle \rho_0 \rangle_\varepsilon, \\ u|_{t=0} & = & \langle u_0 \rangle_\varepsilon. \end{array} \right. \tag{2.30}$$

We will furthermore show that this mapping is continuous and its image $F(E_{R_0,T})$ is contained in some compact subset of $E_{R_0,T}$. Hence F has a fixed point which is the solution to System (2.28). At last, the uniqueness of the solution will be proven.

Now we state the well-posedness result for System (2.28) :

Proposition 2.2. *For any positive time $T \in (0, +\infty)$, there exists a unique smooth solution $(\rho, u) \in E_{R_0,T}$ to System* (2.28)*, such that $(\rho - 1, u) \in \mathcal{C}([0,T]; H^\infty)$.*

Proof. Throughout the proof, we will use frequently the notation C_ε to denote the constants which may depend on $\varepsilon, T, \rho_*, \rho^*$ and R_0. Let $m \in \mathbb{N}$, then the notation $C_\varepsilon(m)$ denotes those constants C_ε depending additionally on m.

We consider first the following linear equation for $\varrho := \rho - 1$

$$\left\{ \begin{array}{rcl} \partial_t \varrho + \langle \widetilde{u} \rangle_\varepsilon \cdot \nabla \varrho - \operatorname{div} (\langle \widetilde{\kappa} \rangle_\varepsilon \nabla \varrho) & = & 0, \\ \varrho|_{t=0} & = & \langle \varrho_0 \rangle_\varepsilon. \end{array} \right. \tag{2.31}$$

To solve it, we will use Friedrich's method. For any $n \in \mathbb{N}$, we define the space L^2_n to be the closed set of L^2 functions with Fourier transform supported in the ball of center 0 and radius n and the associated orthogonal projector P_n is defined by $\widehat{P_n f}(\xi) = 1_{|\xi| \le n} \widehat{f}(\xi)$, then we immediately get a unique solution $\varrho_n \in \mathcal{C}([0,T]; L^2_n) \cap \mathcal{C}^1((0,T); L^2_n)$ to the following system :

$$\left\{ \begin{array}{rcl} \partial_t \varrho_n + P_n (\langle \widetilde{u} \rangle_\varepsilon \cdot \nabla \varrho_n) - P_n \operatorname{div} (\langle \widetilde{\kappa} \rangle_\varepsilon \nabla \varrho_n) & = & 0, \\ \varrho_n|_{t=0} & = & P_n \langle \varrho_0 \rangle_\varepsilon. \end{array} \right. \tag{2.32}$$

In fact, it is easy to see that the above equation is a linear ordinary differential equation on L^2_n.

Now taking the $L^2(\mathbb{R}^d)$ inner product between (2.32) and ϱ_n and integrating by parts give the following a priori estimate (noticing that $\mathrm{div}\,\langle\widetilde{u}\rangle_\varepsilon = 0$, $P_n\varrho_n = \varrho_n$ and $\langle P_n f, g\rangle_{L^2} = \langle f, P_n g\rangle_{L^2}$) :

$$\frac{1}{2}\frac{d}{dt}\|\varrho_n\|_{L^2}^2 + \int_{\mathbb{R}^d}\langle\widetilde{\kappa}\rangle_\varepsilon|\nabla\varrho_n|^2 = 0,$$

that is,

$$\|\varrho_n\|_{L^\infty_T(L^2)}^2 + C\|\nabla\varrho_n\|_{L^2_T(L^2)}^2 \le \|\varrho_n|_{t=0}\|_{L^2}^2 \le \|\varrho_0\|_{L^2}^2. \tag{2.33}$$

Similarly, we can multiply (2.32) by $\Delta\varrho_n$, $\Delta^2\varrho_n$, $\cdots$ and integrate on the whole space $\mathbb{R}^d$, to get

$$\|\varrho_n\|_{L^\infty_T(H^m)}^2 + \|\varrho_n\|_{L^2_T(H^{m+1})}^2 \le C_\varepsilon(m), \quad \forall m \in \mathbb{N}. \tag{2.34}$$

Hence, the fact that ϱ_n solves (2.32) implies that

$$\|\partial_t\varrho_n\|_{L^\infty_T(H^m)} \le \|\langle\widetilde{u}\rangle_\varepsilon\cdot\nabla\varrho_n\|_{L^\infty_T(H^m)} + \|\langle\widetilde{\kappa}\rangle_\varepsilon\nabla\varrho_n\|_{L^\infty_T(H^{m+1})} \le C_\varepsilon(m), \quad \forall m\in\mathbb{N}. \tag{2.35}$$

Thus by Inequality (2.34), Inequality (2.35) and Arzelà-Ascoli Theorem, there exists a unique $\varrho \in \mathcal{C}([0,T]; H^\infty)$ such that for any fixed $j \in \mathbb{N}$, one has a convergent subsequence $\{\varrho_{n_j}\} \subset \cdots \subset \{\varrho_{n_1}\} \subset \{\varrho_n\}$ with

$$\varrho_{n_j} \to \varrho \text{ in } L^\infty_T(H^j_{\mathrm{loc}}). \tag{2.36}$$

Now we rewrite (2.32) as

$$\partial_t\varrho_n + \langle\widetilde{u}\rangle_\varepsilon\cdot\nabla\varrho_n - \mathrm{div}\,(\langle\widetilde{\kappa}\rangle_\varepsilon\nabla\varrho_n) = (\mathrm{Id}\,-P_n)\mathrm{div}\,(\langle\widetilde{u}\rangle_\varepsilon\varrho_n - \langle\widetilde{\kappa}\rangle_\varepsilon\nabla\varrho_n). \tag{2.37}$$

Since $\forall s\in\mathbb{R}$, we have

$$\|(\mathrm{Id}\,-P_n)f\|_{H^s} \le \frac{1}{n}\|f\|_{H^{s+1}}, \tag{2.38}$$

thus let n_j go to ∞, then the above uniform bounds and the convergence result (2.36) imply that the limit $\varrho\in\mathcal{C}([0,T];H^\infty)$ really solves (2.31) and satisfies Estimates (2.33), (2.34) and (2.35). Moreover, by maximum principle, we have

$$0 < \rho_* \le \varrho + 1 \le \rho^*. \tag{2.39}$$

On the other hand, the linearity of the equation (2.31) and the fact that the solution depends continuously on the initial data (see (2.33)) yield the uniqueness of the solution.

Now we move to solve the following Cauchy problem in $\mathcal{C}([0,T];H^\infty)$ with $\varrho\in\mathcal{C}([0,T];H^\infty)$ satisfying (2.31) given above (which amounts to solving $(2.30)_2 - (2.30)_3 - (2.30)_5$) :

$$\begin{cases} \rho\partial_t u + (\rho\langle\widetilde{u}\rangle_\varepsilon - \langle\widetilde{\kappa}\rangle_\varepsilon\nabla\rho)\cdot\nabla u - \mathrm{div}\,(2\mu Au) + \nabla\pi &= 0, \\ \mathrm{div}\,u &= 0, \\ u|_{t=0} &= \langle u_0\rangle_\varepsilon. \end{cases} \tag{2.40}$$

We will proceed exactly as above. Firstly, we look for $u_n \in \mathcal{C}([0,T];L^2_n)\cap\mathcal{C}^1((0,T);L^2_n)$ satisfying

$$\begin{cases} \partial_t u_n + P_n(\mathcal{L}u_n) &= 0, \\ u_n|_{t=0} &= P_n\langle u_0\rangle_\varepsilon, \end{cases} \tag{2.41}$$

where the linear operator $\mathcal{L}$ is defined by [2]

$$\mathcal{L}u_n = \Big(\langle\widetilde{u}\rangle_\varepsilon - \langle\widetilde{\kappa}\rangle_\varepsilon\rho^{-1}\nabla\rho\Big)\cdot\nabla u_n + \rho^{-1}\nabla\mu\cdot Du_n - \rho^{-1}\mathrm{div}\,(\mu\nabla u_n) + \rho^{-1}\nabla\pi_n, \tag{2.42}$$

2. We multiply $(2.40)_1$ by ρ^{-1} and rewrite the quantity $-\rho^{-1}\mathrm{div}\,(2\mu Au)$ into a summation of one 2-order term and one 1-order term by use of $\mathrm{div}\,u = 0$.

with $\nabla\pi_n$ satisfying

$$\mathrm{div}\,(\rho^{-1}\nabla\pi_n) = -\mathrm{div}\,\Big((\langle\widetilde{u}\rangle_\varepsilon - \rho^{-1}\langle\widetilde{\kappa}\rangle_\varepsilon\nabla\rho)\cdot\nabla u_n + 2\mu\nabla\rho^{-1}\cdot Au_n\Big). \tag{2.43}$$

We point out here that the following equality holds true :

$$\mathrm{div}\,(\rho^{-1}\mathrm{div}\,(2\mu Au)) \equiv -\mathrm{div}\,(2\mu\nabla\rho^{-1}\cdot Au). \tag{2.44}$$

It is easy to see that the unique map from u_n to $\nabla\pi_n$ defined by the equation (2.43) is continuous such that

$$\|\nabla\pi_n\|_{L^2} \le C_\varepsilon\|\nabla u_n\|_{L^2} \le C_\varepsilon(n)\|u_n\|_{L^2_n}. \tag{2.45}$$

Therefore the linear map $u_n \mapsto P_n(\mathcal{L}u_n)$ is continuous on L^2_n which ensures a unique solution $u_n \in \mathcal{C}([0,T];L^2_n)\cap\mathcal{C}^1((0,T);L^2_n)$ of System (2.41).

Now we are at the point to get the uniform estimates for u_n to show the convergence. Taking the L^2 inner product between (2.41) and u_n implies that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|u_n|^2 + \int_{\mathbb{R}^d}\mu\rho^{-1}|\nabla u_n|^2 \le C_\varepsilon\|\nabla u_n\|_{L^2}\|u_n\|_{L^2}.$$

Thus by Hölder's inequality the following uniform estimate for u_n holds :

$$\|u_n\|_{L^\infty_T(L^2)} + \|\nabla u_n\|_{L^2_T(L^2)} \le C_\varepsilon\|u_0\|_{L^2}. \tag{2.46}$$

Since by induction we have from (2.43) that

$$\|\nabla\pi_n\|_{H^m} \le C_\varepsilon(m)\|\nabla u_n\|_{H^m},$$

we can multiply (2.41) by $\Delta^m u_n$, with $m\in\mathbb{N}$ to derive

$$\|u_n\|_{L^\infty_T(H^m)} + \|\partial_t u_n\|_{L^\infty_T(H^m)} + \|\nabla\pi_n\|_{L^\infty_T(H^m)} \le C_\varepsilon(m), \quad \forall m\in\mathbb{N}. \tag{2.47}$$

Hence there exist $u\in\mathcal{C}([0,T];H^\infty)$ and $\nabla\pi\in L^\infty((0,T];H^\infty)$ which is given by (2.43) with u_n replaced by u, such that $u(0)=\langle u_0\rangle_\varepsilon$ and for any fixed $j\in\mathbb{N}$, there exists a subsequence $\{(u_{n_j},\nabla\pi_{n_j})\}$ verifying

$$u_{n_j}\to u \text{ in } L^\infty_T(H^j_{\mathrm{loc}}), \quad \nabla\pi_{n_j}\rightharpoonup\nabla\pi \text{ in } L^\infty_T(H^j_{\mathrm{loc}}).$$

Moreover, (2.38) entails

$$\partial_t u + \mathcal{L}u = 0. \tag{2.48}$$

Applying the divergence operator div to it yields [3]

$$\partial_t(\mathrm{div}\,u) - \mathrm{div}\,\Big(\mu\rho^{-1}\nabla\mathrm{div}\,u\Big) = 0. \tag{2.49}$$

Thus the parabolic equation (2.49) ensures that $\mathrm{div}\,u=0$. Therefore u truly solves (2.40). Furthermore, we take the inner product between (2.40) and u, issuing the Energy Equality (2.7) by use of Equation (2.31) for ρ, which together with the identity $\|Au\|_{L^2}=\|\nabla u\|_{L^2}$ (by $\mathrm{div}\,u=0$) entails

$$\|u\|_{L^\infty([0,t];L^2)} + \|\nabla u\|_{L^2((0,t];L^2)} \le C(\rho_*,\rho^*)\|u_0\|_{L^2}, \quad \forall t>0. \tag{2.50}$$

Finally, the uniqueness of the solution follows similarly as in the density case.

3. By Definition (2.42) of the operator $\mathcal{L}$, Equation (2.43) of π and Equality (2.44), we have

$$\begin{aligned}\mathrm{div}\,\mathcal{L}u &= \mathrm{div}\,(\rho^{-1}\nabla\mu\cdot Du - \rho^{-1}\mathrm{div}\,(\mu\nabla u) + \rho^{-1}\mathrm{div}\,(2\mu Au))\\ &= \mathrm{div}\,(\rho^{-1}\nabla\mu\cdot Du - \rho^{-1}\mathrm{div}\,(\mu Du)) = -\mathrm{div}\,(\mu\rho^{-1}\nabla\mathrm{div}\,u).\end{aligned}$$

Noticing (2.33) and (2.50), we just have to choose R_0 depending on $\|\varrho_0\|_{L^2}$, $\|u_0\|_{L^2}$, ρ_*, ρ^* such that the operator $F : (\widetilde{\rho}, \widetilde{u}) \mapsto (\rho, u)$ maps from $E_{R_0,T}$ to $E_{R_0,T}$. Furthermore, the boundedness (2.34), (2.35) and (2.47) ensures that

$$F : E_{R_0,T} \mapsto E_{R_0,T} \cap \Big\{(\rho, u) \,|\, \|(\varrho, \partial_t\rho, u, \partial_t u)\|_{L^\infty_T(H^m)} \le C_\varepsilon(m),\, \forall m \in \mathbb{N}\Big\},$$

which implies that the image of F (i.e. $F(E_{R_0,T})$) is contained in a compact subset of $E_{R_0,T}$.

In order to show the continuity of the mapping $F : (\widetilde{\rho}, \widetilde{u}) \to (\rho, u)$, let the sequence $\{(\widetilde{\rho}_k, \widetilde{u}_k)\}$ converge to $(\widetilde{\rho}, \widetilde{u})$ in $E_{R_0,T}$ and we will show that the corresponding smooth solution sequence $\{(\rho_k, u_k)\}$ converges to (ρ, u). Indeed, one can argue just as in the proof of the existence result. More precisely, the solution difference $(\bar{\rho}_k, \bar{u}_k, \nabla\bar{\pi}_k) := (\rho - \rho_k, u - u_k, \nabla\pi - \nabla\pi_k)$ solves the following system :

$$\begin{cases} \partial_t\bar{\rho}_k + \langle\widetilde{u}\rangle_\varepsilon \cdot \nabla\bar{\rho}_k - \operatorname{div}(\langle\widetilde{\kappa}\rangle_\varepsilon\nabla\bar{\rho}_k) = J_k^1 := -\langle\widetilde{u} - \widetilde{u}_k\rangle_\varepsilon \cdot \nabla\rho_k + \operatorname{div}\Big(\langle\widetilde{\kappa} - \widetilde{\kappa}_k\rangle_\varepsilon\nabla\rho_k\Big), \\ \rho\partial_t\bar{u}_k + (\rho\langle\widetilde{u}\rangle_\varepsilon - \langle\widetilde{\kappa}\rangle_\varepsilon\nabla\rho) \cdot \nabla\bar{u}_k - \operatorname{div}(2\mu A\bar{u}_k) + \nabla\bar{\pi}_k = J_k^2 := -\bar{\rho}_k\partial_t u_k \\ \quad -\Big(\rho\langle\widetilde{u} - \widetilde{u}_k\rangle_\varepsilon + \bar{\rho}_k\langle\widetilde{u}_k\rangle_\varepsilon - \langle\widetilde{\kappa}\rangle_\varepsilon\nabla\bar{\rho}_k - \langle\widetilde{\kappa} - \widetilde{\kappa}_k\rangle_\varepsilon\nabla\rho_k\Big) \cdot \nabla u_k + \operatorname{div}\Big(2(\mu - \mu_k)Au_k\Big), \\ \operatorname{div}\bar{u}_k = 0, \\ (\bar{\rho}_k, \bar{u}_k)|_{t=0} = (0, 0). \end{cases} \tag{2.51}$$

It is easy to see that for any fixed $i \in \mathbb{N}$, $\|\nabla\rho_k\|_{H^i}$ is uniformly bounded and hence $\|J_k^1\|_{H^{i-1}} \to 0$ as $k \to \infty$. Taking $L^2(\mathbb{R}^d)$-inner product between $(2.51)_1$ and $\bar{\rho}_k, \Delta\bar{\rho}_k, \Delta^2\bar{\rho}_k, \cdots$ yields that for any $j \in \mathbb{N}$, $\|\bar{\rho}_k\|_{L^\infty_T(H^j)} \to 0$, and hence $\|J_k^2\|_{H^j} \to 0$. Then, obviously the pressure difference $\bar{\pi}_k$ satisfies Equation (2.43), with u_n replaced by $\bar{u}_k$ and an additional term $\operatorname{div}(\rho^{-1}J_k^2)$ on the right hand side. Hence by iteration argument, one has for any $m \in \mathbb{N}$,

$$\|\nabla\bar{\pi}_k\|_{H^m} \le C_\varepsilon(m)(\|\nabla\bar{u}_k\|_{H^m} + \|J_k^2\|_{H^m}).$$

By the induction argument as before, one arrives at $\|\bar{u}_k\|_{L^\infty_T(H^m)} \to 0$. This completes the continuity proof.

Therefore, Schauder fixed point theorem applies and there exists a fixed point (ρ, u) of the map F in $E_{R_0,T}$ which is also a smooth solution to System (2.28). In order to show the uniqueness, let (ρ_1, u_1, π_1) and $(\rho_2, u_2, \nabla\pi_2)$ be two smooth solutions to System (2.28). Then the difference $(\delta\rho, \delta u) := (\rho_1 - \rho_2, u_1 - u_2) \in \mathcal{C}([0, T]; H^\infty)$ satisfies the following system (with $\nabla\delta\pi := \nabla\pi_1 - \nabla\pi_2$)

$$\begin{cases} \partial_t\delta\rho + \langle u_1\rangle_\varepsilon \cdot \nabla\delta\rho - \operatorname{div}(\langle\kappa_1\rangle_\varepsilon\nabla\delta\rho) = -\langle\delta u\rangle_\varepsilon \cdot \nabla\rho_2 + \nabla\rho_2 \cdot \nabla\langle\delta\kappa\rangle_\varepsilon + \Delta\rho_2\langle\delta\kappa\rangle_\varepsilon, \\ \rho_1\partial_t\delta u + (\rho_1\langle u_1\rangle_\varepsilon - \langle\kappa_1\rangle_\varepsilon\nabla\rho_1) \cdot \nabla\delta u - \operatorname{div}(2\mu_1 A\delta u) + \nabla\delta\pi = -\delta\rho\partial_t u_2 \\ \quad -\Big(\rho_1\langle\delta u\rangle_\varepsilon + \delta\rho\langle u_2\rangle_\varepsilon - \langle\kappa_1\rangle_\varepsilon\nabla\delta\rho - \langle\delta\kappa\rangle_\varepsilon\nabla\rho_2\Big) \cdot \nabla u_2 + \operatorname{div}\Big(2\delta\mu Au_2\Big), \\ \operatorname{div}\delta u = 0, \\ (\delta\rho, \delta u)|_{t=0} = (0, 0). \end{cases} \tag{2.52}$$

Taking L^2-inner product between $(2.52)_1$ (resp. $(2.52)_2$) and $\delta\rho$ (resp. δu) entails the following estimates (with $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^d)}$ and some constant C depending on the two solutions)

$$\frac{d}{dt}\|\delta\rho\|^2 + \|\nabla\delta\rho\|^2 \le C(\|\delta u\| + \|\nabla\delta\rho\|)\|\delta\rho\|,$$

$$\frac{d}{dt}\|\delta u\|^2 + \|\nabla\delta u\|^2 \le C(\|\delta u\| + \|\delta\rho\| + \|\nabla\delta u\| + \|\nabla\delta\rho\|)\|\delta u\|.$$

Hence, summing up the above two estimates and performing Young's Inequality and then Gronwall's Inequality yields

$$\|\delta\rho\|_{L^\infty_T(L^2)} = \|\delta u\|_{L^\infty_T(L^2)} = \|\nabla\delta\rho\|_{L^2_T(L^2)} = \|\nabla\delta u\|_{L^2_T(L^2)} \equiv 0.$$

An induction argument ensures that $(\delta\rho, \delta u) \equiv 0$ in $\mathcal{C}([0,T]; H^\infty)$. This ends the proof of Proposition 2.2. □

Remark 2.2. *Since in Banach space $E_{R_0,T}$, the bound R_0 is independent of the time T, Proposition 2.2 actually permits that for any ε, there exists a unique globally-in-time existing smooth solution $(\varrho^\varepsilon, u^\varepsilon) \in \mathcal{C}([0,+\infty); H^\infty)$ to System (2.28) such that $\forall\varepsilon > 0$, $\rho^\varepsilon \in [\rho_*, \rho^*]$, and the following energy inequalities hold true :*

$$\|\varrho^\varepsilon\|_{L^\infty(L^2)} + \|\nabla\rho^\varepsilon\|_{L^2(L^2)} \le C\|\varrho_0\|_{L^2(\mathbb{R}^d)}, \; \|u^\varepsilon\|_{L^\infty(L^2)} + \|\nabla u^\varepsilon\|_{L^2(L^2)} \le C\|u_0\|_{L^2(\mathbb{R}^d)}, \quad (2.53)$$

with C being a constant depending only on ρ_*, ρ^*.

Convergence to a weak solution

We now are at the point to show that the solution sequence given above converges to a weak solution to Cauchy problem (2.4)-(2.8). The smoothing effect on both variables is useful to use the compactness methods and the strategy is hence quite standard. So let's just sketch the proof.

By Remark 2.2, we may assume that there exist subsequences $\{\rho^{\varepsilon_n}\}_n$ and $\{u^{\varepsilon_n}\}_n$ of the solution sequences $\{\rho^\varepsilon\}_\varepsilon$ and $\{u^\varepsilon\}_\varepsilon$ respectively such that

$$\varrho^{\varepsilon_n} \overset{*}{\rightharpoonup} \varrho \text{ in } L^\infty(L^2 \cap L^\infty), \quad u^{\varepsilon_n} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(L^2), \quad \nabla\varrho^{\varepsilon_n} \rightharpoonup \nabla\varrho, \quad \nabla u^{\varepsilon_n} \rightharpoonup \nabla u \text{ in } L^2(L^2),$$

with the limit (ρ, u) verifying Estimate (2.53) too.

According to Equation $(2.28)_1$ for the density, $\{\partial_t\varrho^{\varepsilon_n}\}$ is uniformly bounded in $L^2_{\text{loc}}(H^{-1})$. By the classical compactness results, e.g. Aubin-Lions lemma (see also Corollary 4 in Simon [?]), one deduces easily that $\varrho^{\varepsilon_n} \to \varrho$ in $\mathcal{C}([0,T]; H^{-1} \cap L^2_w)$. Thus up to an extraction of subsequence, we may assume $\varrho^{\varepsilon_n} \to \varrho$ in $L^2_T(H^s)$, $s < 1$ and a.e. on $[0,\infty) \times \mathbb{R}^d$.

Since $\{\|\varrho^{\varepsilon_n}\|_{L^2_T(H^1)}\}_n$ is uniformly bounded, we have for all $n \in \mathbb{N}$, $\|\langle\varrho^{\varepsilon_n}\rangle_\eta - \varrho^{\varepsilon_n}\|_{L^2_T(L^2)} \to 0$ when $\eta \to 0$, and hence in particular $\|\langle\varrho^{\varepsilon_n}\rangle_{\varepsilon_n} - \varrho^{\varepsilon_n}\|_{L^2_T(L^2)} \to 0$ whenever $\varepsilon_n \to 0$. Furthermore, we have $\|\langle\varrho^{\varepsilon_n}\rangle_{\varepsilon_n} - \varrho^{\varepsilon_n}\|_{L^2_T(H^s)} \to 0$, if $0 < s < 1$. Similarly, we can assume $\|\langle u^{\varepsilon_n}\rangle_{\varepsilon_n} - u^{\varepsilon_n}\|_{L^2_T(H^s)} \to 0$ and $\|\langle\kappa^{\varepsilon_n}\rangle_{\varepsilon_n} - \kappa^{\varepsilon_n}\|_{L^2_T(H^s)} \to 0$ by considering $\kappa(1+\varrho^{\varepsilon_n}) - \kappa(1)$ instead of κ^{ε_n}. Therefore up to an extraction, we may assume (noticing $\kappa^{\varepsilon_n} \to \kappa$ a.e.)

$$\langle\varrho^{\varepsilon_n}\rangle_{\varepsilon_n} \to \varrho \text{ in } L^2_T(H^s), \quad \langle u^{\varepsilon_n}\rangle_{\varepsilon_n} \rightharpoonup u \text{ in } L^2_T(H^s) \quad \text{for } s<1 \quad \text{and} \quad \langle\kappa^{\varepsilon_n}\rangle_{\varepsilon_n} \to \kappa \text{ a.e.}$$

Thanks to the uniform bound on $\nabla\rho$ and by use of the regularized system $(2.28)_1 - (2.28)_4$, we can easily show that ρ solves Equation $(2.4)_1$ in distribution sense. In fact, we will use the following convergence result

$$f_n g_n \to (\text{resp.} \rightharpoonup)\, fg \quad \text{in} \quad L^2, \quad \text{if} \quad f_n \to (\text{resp.} \rightharpoonup) f \quad \text{in} \quad L^2 \quad \text{and} \quad g_n \to g \text{ a.e.}$$

For example, one has $\langle\kappa^{\varepsilon_n}\rangle_{\varepsilon_n}\nabla\rho^{\varepsilon_n} \rightharpoonup \kappa\nabla\rho$ in $L^2(L^2(\mathbb{R}^d))$.

The above bound (2.53) furthermore ensures that Equation $(2.4)_1$ holds in $L^2_{\text{loc}}(H^{-1})$. Thus, we can test it by the solution $\varrho \in L^2_{\text{loc}}(H^1)$ itself such that Energy Equality (2.6) hold for ϱ^ε and ϱ both (notice that $\operatorname{div} u = 0$ in $L^2_w([0,\infty) \times \mathbb{R}^d)$), i.e. for all $t \in [0,\infty)$,

$$\frac{1}{2}\|\varrho^\varepsilon(t)\|^2_{L^2} + \|\langle\kappa^\varepsilon\rangle^{\frac{1}{2}}_\varepsilon\nabla\varrho^\varepsilon\|^2_{L^2_t(L^2)} = \frac{1}{2}\|\langle\varrho_0\rangle_\varepsilon\|^2_{L^2}, \quad \frac{1}{2}\|\varrho(t)\|^2_{L^2} + \|\kappa^{\frac{1}{2}}\nabla\varrho\|^2_{L^2_t(L^2)} = \frac{1}{2}\|\varrho_0\|^2_{L^2}. \quad (2.54)$$

Now we consider the quantity

$$\frac{1}{2}\|\varrho^{\varepsilon_n}(t) - \varrho(t)\|^2_{L^2(\mathbb{R}^d)} + \|\langle\kappa^{\varepsilon_n}\rangle^{\frac{1}{2}}_{\varepsilon_n}\nabla\varrho^{\varepsilon_n} - \kappa^{\frac{1}{2}}\nabla\varrho\|^2_{L^2_t(L^2)},$$

which by Energy Equality (2.54), is equal to

$$\frac{1}{2}\|\langle\varrho_0\rangle_{\varepsilon_n}\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|\varrho_0\|^2_{L^2(\mathbb{R}^d)} - \Big\langle \varrho^{\varepsilon_n}(t), \varrho(t)\Big\rangle_{L^2(\mathbb{R}^d)} - 2\Big\langle \langle\kappa^{\varepsilon_n}\rangle^{\frac{1}{2}}_{\varepsilon_n}\nabla\varrho^{\varepsilon_n}, \kappa^{\frac{1}{2}}\nabla\varrho\Big\rangle_{L^2_t(L^2)}.$$

Since we have also $\langle\kappa^{\varepsilon_n}\rangle^{\frac{1}{2}}_{\varepsilon_n}\nabla\varrho^{\varepsilon_n} \rightharpoonup \kappa^{\frac{1}{2}}\nabla\varrho$ in $L^2(L^2)$, the above quantity converges to

$$\|\varrho_0\|^2_{L^2} - \|\varrho(t)\|^2_{L^2} - 2\|\kappa^{\frac{1}{2}}\nabla\varrho\|^2_{L^2_t(L^2)} = 0.$$

This implies that

$$\varrho^{\varepsilon_n} \to \varrho \text{ in } L^\infty(L^2) \quad \text{and} \quad \langle\kappa^{\varepsilon_n}\rangle^{\frac{1}{2}}_{\varepsilon_n}\nabla\varrho^{\varepsilon_n} \to \kappa^{\frac{1}{2}}\nabla\varrho \text{ in } L^2(L^2).$$

Thus, by the boundedness of $\|\varrho^{\varepsilon_n}\|_{L^\infty([0,\infty)\times\mathbb{R}^d)}$, we have $\varrho^{\varepsilon_n} \to \varrho$ in $L^\infty(L^p)$, $\forall p \in [2,\infty)$. Therefore, $\varrho|_{t=0} = \varrho_0$ in L^p for all $p \in [2,\infty)$.

The following statement concerning the velocity u follows exactly Proof of Theorem 2.4 in the P.-L. Lions's book [81]. Let us recall it briefly for the reader's convenience. Let us first observe that the Sobolev embedding ensures that $\{u^\varepsilon\}_\varepsilon$ is bounded in $L^\infty_T(L^2) \cap L^2_T(L^{\frac{2d}{d-2}})$ (or $L^2_T(L^p)$ with $p \in [2,\infty)$ if $d = 2$) for any positive finite time T and hence

$$\{\langle u^\varepsilon\rangle_\varepsilon \otimes u^\varepsilon\} \text{ is bounded in } L^\infty_T(L^1) \cap L^2_T(L^{\frac{d}{d-1}}) \,(\text{or } L^2_T(L^p) \text{ with } p \in [1,2) \text{ if } d = 2),$$
$$\text{and } \{\nabla\rho^\varepsilon \otimes u^\varepsilon\} \text{ is bounded in } L^2_T(L^1) \cap L^1_T(L^{\frac{d}{d-1}}) \,(\text{or } L^1_T(L^p) \text{ with } p \in [1,2) \text{ if } d = 2).$$

Therefore, in view of the equation $(2.28)_2$ for u^ε, there exist constants $p \in (2,\infty)$, $m > 1$ [4] and M depending on T, R_0 such that for all divergence-free function $\phi \in L^p_T(H^m)$ we have

$$\left|\Big\langle \partial_t(\rho^\varepsilon u^\varepsilon), \phi\Big\rangle_{\mathcal{D}',\mathcal{D}}\right| = \left|\Big\langle -\Big(\rho^\varepsilon\langle u^\varepsilon\rangle_\varepsilon - \langle\kappa^\varepsilon\rangle_\varepsilon\nabla\rho^\varepsilon\Big)\otimes u^\varepsilon + 2\mu^\varepsilon A u^\varepsilon, \nabla\phi\Big\rangle_{\mathcal{D}',\mathcal{D}}\right| \le M\|\phi\|_{L^p_T(H^m)}. \tag{2.55}$$

Let us notice that the Leray projector $\mathcal{P} := \mathrm{Id} + \nabla(-\Delta)^{-1}\mathrm{div}$ is bounded on each Sobolev space H^s. Hence from (2.55), we actually have $\partial_t(\mathcal{P}(\rho^\varepsilon u^\varepsilon))$ is bounded in $L^{p'}_T(H^{-m})$. Since $\rho^{\varepsilon_n}u^{\varepsilon_n}$ converges weakly to ρu, the boundedness of $\{\rho^\varepsilon u^\varepsilon\}$ in $L^\infty_T(L^2)$ implies the existence of a convergent subsequence (still denoted by $\rho^{\varepsilon_n}u^{\varepsilon_n}$) such that $\mathcal{P}(\rho^{\varepsilon_n}u^{\varepsilon_n}) \to \mathcal{P}(\rho u)$ in $\mathcal{C}([0,T]; L^2_w)$. Hence, we have for any $t > 0$,

$$\int_0^t\int_{\mathbb{R}^d}\rho^{\varepsilon_n}|u^{\varepsilon_n}|^2 = \int_0^t \langle\mathcal{P}(\rho^{\varepsilon_n}u^{\varepsilon_n}), u^{\varepsilon_n}\rangle \to \int_0^t\langle\mathcal{P}(\rho u), u\rangle = \int_0^t\int_{\mathbb{R}^d}\rho|u|^2.$$

Thus $\rho^{\varepsilon_n}u^{\varepsilon_n} \to \rho u$ and $u^{\varepsilon_n} \to u$ in $L^2_{\mathrm{loc}}(L^2)$. It is easy to find that

$$\rho^{\varepsilon_n}\langle u^{\varepsilon_n}\rangle_{\varepsilon_n}\otimes u^{\varepsilon_n} \to \rho u\otimes u \text{ in } L^2_{\mathrm{loc}}(L^1), \quad \mu^{\varepsilon_n}Au^{\varepsilon_n} \rightharpoonup \mu A u \text{ in } L^2_{\mathrm{loc}}(L^2),$$

and $\langle\kappa^{\varepsilon_n}\rangle_{\varepsilon_n}\nabla\rho^{\varepsilon_n}\otimes u^{\varepsilon_n} \to \kappa\nabla\rho\otimes u$ in $L^1_{\mathrm{loc}}(L^1)$. Thus observing Equation $(2.28)_2$, there exists some distribution of gradient form $\nabla\pi$ such that Equation $(2.4)_2$ holds for the above limit u at least in distribution sense and hence (2.10) holds. At last, since we restrict ourselves in the case away from vacuum and we assume initially $\mathrm{div}\, u_0 = 0$, one applies Theorem 2.2 in [81] to conclude that $u \in \mathcal{C}([0,\infty); L^2_w)$ and $u(0) = u_0$ in $L^2(\mathbb{R}^d)$.

This completes the proof of Theorem 2.1.

4. We notice that $H^{m_1} \hookrightarrow W^{m_2,q}$ if $m_1 - \frac{d}{2} \ge m_2 - \frac{d}{q}$, $q \ge 2$.

2.2.2 The case with the density of higher regularity

In this subsection we will tackle the case with smoother density. It is easy to see that proving Theorem 2.2 amounts to proving the following :
Let $d = 2, 3$. For any initial data (ρ_0, u_0) such that

$$0 < \rho_* \leq \rho_0 \leq \rho^*, \quad \rho_0 - 1 \in H^1(\mathbb{R}^d), \quad u_0 \in L^2(\mathbb{R}^d), \quad \mathrm{div}\, u_0 = 0, \tag{2.56}$$

System (2.4) has a weak solution (ρ, u) satisfying (2.18) and (2.19).

Thus in the first paragraph of this subsection, by establishing a new a priori estimate in smoother functional space in dimension 2, we deduce that if (2.56) holds for the initial data, then the weak solutions (ρ, u) got in the last subsection satisfy (2.18) and (2.19). However in dimension 3, since $u \notin L^4(L^4)$, the L^2 inner product between the convection term $\mathrm{div}\,(\rho u)$ and $\Delta\rho$ doesn't make sense a priori. Therefore, we will reprove the existence of weak solutions by regularizing the system in two levels which, ensures us to get the uniform estimate (2.15) when the transport velocity is still regularized. This will be done in the second paragraph.

2D case

We will establish two lemmas (Lemma 2.1 and Lemma 2.2), in order to show that the global weak solutions $(\varrho, u, \nabla\pi) = (\rho - 1, u, \nabla\pi)$ given by Theorem 2.1 but with smoother initial density $\varrho_0 \in H^1(\mathbb{R}^2)$, are bounded only by initial data in the Banach space $X_2(T) \times X_1(T) \times X_{-1}(T)$ for all $T \in [0, \infty]$ in dimension 2 with

$$\begin{aligned}
&X_2(T) := \mathcal{C}([0,T]; H^1(\mathbb{R}^2)) \cap L^2_T(\dot{H}^1 \cap \dot{H}^2(\mathbb{R}^2)),\\
&X_1(T) := \mathcal{C}([0,T]; L^2(\mathbb{R}^2)) \cap L^2_T(\dot{H}^1(\mathbb{R}^2)),\\
&X_{-1}(T) := (X_1(T))' : \text{ the dual of the Banach space } X_1(T).
\end{aligned}$$

In fact, the density ϱ always belongs to the following convex subspace of $X_2(T)$:

$$X_2^*(T) := X_2(T) \cap \{\varrho \in L^\infty([0,T] \times \mathbb{R}^d) \,|\, 0 < \rho_* \leq \varrho(t,x) + 1 \leq \rho^*,\ t \in [0,T],\ x \in \mathbb{R}^d \text{ a.e.}\}.$$

However, in order to prove uniqueness and stability, it is not enough to just consider the solutions in $X_2(T) \times X_1(T) \times X_{-1}(T)$, since $H^1(\mathbb{R}^2)$ can not be embedded into $L^\infty(\mathbb{R}^2)$, which is needed in estimating the nonlinear terms. Therefore we will consider the initial data in the critical Besov spaces, in order to get a unique global strong solution. This will be done in next section.

In the following, we will use frequently (explicitly or implicitly) the following Gagliardo-Nirenberg inequality in dimension 2 :

$$\|f\|_{L^4(\mathbb{R}^2)} \lesssim \|f\|^{1/2}_{L^2(\mathbb{R}^2)} \|\nabla f\|^{1/2}_{L^2(\mathbb{R}^2)}. \tag{2.57}$$

We notice that by (2.57), one has

$$\|h\|_{L^4_T(L^4)} \lesssim \|h\|_{X_1(T)} \text{ and } \|\varrho\|_{L^4_T(L^4)}, \|\nabla\varrho\|_{L^4_T(L^4)} \lesssim \|\varrho\|_{X_2(T)}.$$

Furthermore, if $\varrho \in X_2^*(T)$, then the mapping $h \mapsto f(\rho)h$ is an isomorphism on $X_1(T)$ (and hence on $X_{-1}(T)$) for any diffeomorphism f from $[\rho_*, \rho^*]$ to $\mathbb{R}$: in fact, we have

$$\begin{aligned}
\|f(\rho)h\|_{X_1(T)} &\leq C(f, \|\rho\|_{L^\infty_T(L^\infty)})(\|h\|_{X_1(T)} + \|\nabla\rho\|_{L^4_T(L^4)}\|h\|_{L^4_T(L^4)})\\
&\leq C(f, \|\varrho\|_{X_2(T)}, \rho_*, \rho^*)\|h\|_{X_1(T)}.
\end{aligned}$$

We now prove first that $\nabla\pi \in X_{-1}(T)$, provided that $\varrho \in X_2^*(T), u \in X_1(T)$. Indeed, we just have to show that the convergent regular sequence $\nabla\pi^{\varepsilon_n}$ which are solutions of the following equation (see Equation (2.43))

$$\operatorname{div}(\rho^{-1}\nabla\pi) = -\operatorname{div}\left((\langle u\rangle_\varepsilon - \rho^{-1}\langle\kappa\rangle_\varepsilon\nabla\rho)\cdot\nabla u + 2\mu\nabla\rho^{-1}\cdot Au\right), \tag{2.58}$$

are uniformly bounded in $X_{-1}(T)$. We introduce the following lemma :

Lemma 2.1. *Suppose that the smooth triplet $(\varrho, u, \nabla\pi)$ satisfies Equation (2.58) such that $\varrho \in X_2^*(T)$, $u \in X_1(T)$, then there exists a constant D_1 depending only on $\rho_*, \rho^*, \|\varrho\|_{X_2(T)}$, $\|u\|_{X_1(T)}$ such that*

$$\|\nabla\pi\|_{X_{-1}(T)} \le D_1. \tag{2.59}$$

Proof. The proof is very similar to the proof of Lemma 2.1 in [81]. Given any $h \in X_1(T)$, we first claim that we have the decomposition $h = h_1 + h_2$ such that

$$\mathcal{R}\wedge(\rho h_1) = 0, \quad \operatorname{div} h_2 = 0, \quad \|h_i\|_{X_1(T)} \le C(\rho_*, \rho^*, \|\varrho\|_{X_2(T)})\|h\|_{X_1(T)}, \quad i = 1, 2, \tag{2.60}$$

where $\mathcal{R}_i = (-\Delta)^{-1/2}\frac{\partial}{\partial x_i}$, $i = 1, 2$ denotes the usual Riesz transform and $f\wedge g := f_1g_2 - f_2g_1$.

In fact, we decompose the function $\rho h \in X_1(T)$ into $\widetilde{h}_1 + \widetilde{h}_2$ with

$$\widetilde{h}_1 = -\mathcal{R}(\mathcal{R}\cdot(\rho h)), \quad \widetilde{h}_2 = (\mathrm{Id} + \mathcal{R}(\mathcal{R}\cdot))(\rho h), \quad \mathcal{R}\wedge\widetilde{h}_1 = 0, \quad \mathcal{R}\cdot\widetilde{h}_2 = 0,$$

such that

$$\|\widetilde{h}_i\|_{X_1(T)} \le C\|\rho h\|_{X_1(T)} \le C(\rho_*, \rho^*, \|\varrho\|_{X_2(T)})\|h\|_{X_1(T)},\ i = 1, 2.$$

To prove (2.60) then amounts to searching for a unique function $\mathcal{R}v \in X_1(T)$ such that

$$h_1 = \rho^{-1}\widetilde{h}_1 - \rho^{-1}\mathcal{R}v, \quad h_2 = \rho^{-1}\widetilde{h}_2 + \rho^{-1}\mathcal{R}v,$$

with

$$\operatorname{div}(\rho^{-1}\nabla V + \rho^{-1}\widetilde{h}_2) = 0, \quad V = (-\Delta)^{-1/2}v. \tag{2.61}$$

According to Proposition 0.9, Equation (2.61) admits a unique solution $\nabla V = \mathcal{R}v$ such that

$$\|\mathcal{R}v\|_{L^\infty_T(L^2)} \le C(\rho_*, \rho^*)\|\widetilde{h}_2\|_{L^\infty_T(L^2)} \le C(\rho_*, \rho^*)\|\rho h\|_{L^\infty_T(L^2)} \le C(\rho_*, \rho^*)\|h\|_{L^\infty_T(L^2)}.$$

Now we take the derivative ∇ to (2.61) to arrive at

$$\operatorname{div}(\rho^{-1}\nabla^2 V + \nabla V\otimes\nabla\rho^{-1} + (\nabla(\rho^{-1}\widetilde{h}_2))^T) = 0,$$

which similarly gives

$$\|\nabla\mathcal{R}v\|_{L^2_T(L^2)} \le C(\rho_*, \rho^*)\Big(\|\nabla V\|_{L^4_T(L^4)}\|\nabla\rho^{-1}\|_{L^4_T(L^4)} + \|\rho^{-1}\widetilde{h}_2\|_{X_1(T)}\Big).$$

Thus by (2.57) and Young's inequality, (2.60) follows.

By the decomposition (2.60), we have for any $h \in X_1(T)$,

$$\langle\nabla\pi, h\rangle_{X_{-1}(T), X_1(T)} = \langle\nabla\pi, h_1\rangle = \langle\rho^{-1}\nabla\pi, \rho h_1\rangle.$$

Therefore Equation (2.58) for $\nabla\pi$ yields

$$\begin{aligned}|\langle\nabla\pi, h\rangle| &= \Big|\Big\langle(\langle u\rangle_\varepsilon - \rho^{-1}\langle\kappa\rangle_\varepsilon\nabla\rho)\cdot\nabla u + 2\mu\nabla\rho^{-1}\cdot Au, \rho h_1\Big\rangle\Big| \\ &\le C(\|u\|_{L^4_T(L^4)} + \|\nabla\rho\|_{L^4_T(L^4)})\|\nabla u\|_{L^2_T(L^2)}\|\rho h_1\|_{L^4_T(L^4)} \\ &\le C(\rho_*, \rho^*, \|\varrho\|_{X_2(T)}, \|u\|_{X_1(T)})\|h_1\|_{X_1(T)},\end{aligned}$$

which together with (2.60) gives the lemma. □

Remark 2.3. *We point out here that,* $L^2_T(H^{-1}(\mathbb{R}^2)) \subset X_{-1}(T)$ *by definition and* $\Delta u \in L^2_T(H^{-1}(\mathbb{R}^2))$ *since* $\nabla u \in L^2_T(L^2(\mathbb{R}^2))$*. But since the "low" regularity* $\varrho \in X_2(T)$ *can not permit that* $h \mapsto \rho h$ *is an isomorphism on* $L^2_T(H^1(\mathbb{R}^2))$ *due to a lack of control on* $\|\nabla\rho \otimes h\|_{L^2_T(L^2)}$*, it is not clear that* $\|\nabla\pi\|_{L^2_T(H^{-1}(\mathbb{R}^2))}$ *is bounded in Lemma 2.1. Even if* $\rho \equiv 1$*, that is in the classical incompressible Navier-Stokes equation case, it is only known that the pressure term* $\nabla\pi$ *can be bounded in* $L^2_T(\mathcal{M}(\mathbb{R}^2))$ *with* $\mathcal{M}(\mathbb{R}^2)$ *the measure space, denoting the dual space of* $\mathcal{C}_0(\mathbb{R}^2)$*.*

It is easy to check that the following lemma concerning the unknowns $\rho, u, \nabla\pi$ immediately yields Theorem 2.2 when $d = 2$. Details are omitted here.

Lemma 2.2. *Let* $d = 2$*. For any initial data* (ρ_0, u_0) *satisfying (2.56), the global weak solution* $(\varrho, u, \nabla\pi)$ *given by Theorem 2.1 satisfies*

$$\|\varrho\|_{X_2(T)} + \|u\|_{X_1(T)} + \|\nabla\pi\|_{X_{-1}(T)} + \|\partial_t\rho\|_{L^2_T(L^2)} + \|\partial_t u\|_{X_{-1}(T)} \le D_2, \tag{2.62}$$

with D_2 *depending only on* $\rho_*, \rho^*, \|\varrho_0\|_{H^1(\mathbb{R}^2)}, \|u_0\|_{L^2(\mathbb{R}^2)}$*. Moreover, Energy Equality (2.7) holds and* $\varrho \in \mathcal{C}([0,\infty); H^1(\mathbb{R}^2))$*,* $u \in \mathcal{C}([0,\infty); L^2(\mathbb{R}^2))$*.*

Proof. By Theorem 2.1, Equation $(2.4)_1$ and Lemma 2.1, to prove (2.62) it rests to prove $\varrho \in X_2(T)$ and $\partial_t u \in X_{-1}(T)$. In fact, since if $u, \nabla\rho \in X_1(T)$, then $\rho u, \kappa\nabla\rho \in X_1(T)$ and hence

$$\|\partial_t\rho\|_{L^2_T(L^2(\mathbb{R}^2))} = \| - \operatorname{div}(\rho u - \kappa\nabla\rho)\|_{L^2_T(L^2(\mathbb{R}^2))} \le C(\rho_*, \rho^*, \|\varrho\|_{X_2(T)}, \|u\|_{X_1(T)}). \tag{2.63}$$

Assume a priori that $\nabla\varrho \in X_1(T)$. As in (0.36) in the introduction part, set a to be an antiderivative of κ such that $a(1) = 0$. Since $\nabla a = \kappa\nabla\rho \in X_1(T)$, then $a \in L^\infty_T(H^1 \cap L^\infty)$ and $\nabla a \in L^2_T(H^1)$. Multiplying $(2.4)_1$ by $\kappa = \kappa(\rho)$ yields

$$\partial_t a + u \cdot \nabla a - \kappa\Delta a = 0 \text{ in } L^2_T(L^2). \tag{2.64}$$

Taking the $L^2(\mathbb{R}^2)$ inner product with Δa issues

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla a|^2 - \int_{\mathbb{R}^2} u \cdot \nabla a \Delta a + \int_{\mathbb{R}^2} \kappa|\Delta a|^2 = 0. \tag{2.65}$$

For any $\varepsilon > 0$, we have from Inequality (2.57) and Young's Inequality

$$\Big|\int_{\mathbb{R}^2} u \cdot \nabla a \Delta a\Big| \le \|u\|_{L^4}\|\nabla a\|_{L^4}\|\Delta a\|_{L^2} \le \varepsilon\|\Delta a\|^2_{L^2} + C_\varepsilon\|u\|^4_{L^4}\|\nabla a\|^2_{L^2},$$

for some constant C_ε. Let us choose sufficiently small ε, then we have shown (by (2.11))

$$\sup_{0\le t\le T}\|\nabla a\|^2_{L^2} + \int_0^T \|\Delta a\|^2_{L^2} \le e^{C\int_0^T \|u\|^4_{L^4}} C\|\nabla\varrho_0\|^2_{L^2} \le e^{C\|u_0\|^4_{L^2}} C\|\nabla\varrho_0\|^2_{L^2}, \quad \forall T \ge 0. \tag{2.66}$$

It is also easy to see from (2.65) that $\nabla a \in \mathcal{C}([0,\infty); L^2)$. Thus $\varrho \in \mathcal{C}([0,\infty); H^1)$.

Now since $\Delta a = \kappa\Delta\varrho + \nabla\kappa \cdot \nabla\varrho$, we have

$$\|\Delta\varrho\|_{L^2} \le C(\|\Delta a\|_{L^2} + \|\nabla a\|^2_{L^4}) \le C\|\Delta a\|_{L^2}(1 + \|\nabla a\|_{L^2}),$$

which already gives the estimate for ϱ :

$$\sup_{0\le t\le T}\|\nabla\varrho(t)\|^2_{L^2} + \int_0^T \|\nabla^2\varrho\|^2_{L^2} \le e^{C\|u_0\|^4_{L^2}} C\|\nabla\varrho_0\|^2_{L^2}(1 + \|\nabla\varrho_0\|^2_{L^2}), \forall T \ge 0. \tag{2.67}$$

Now we turn to the equation for u. It is easy to find that

$$\partial_t u = -\rho^{-1}\Big(\partial_t\rho\, u + \mathrm{div}\,((\rho u - \kappa\nabla\rho)\otimes u) - \mathrm{div}\,(2\mu Au) + \nabla\pi\Big).$$

Thus for any $h \in X_1(T)$, we have by (2.63)

$$\begin{aligned}|\langle\partial_t u, h\rangle_{X_{-1}(T),X_1(T)}| &\le \|\partial_t\rho\|_{L^2_T(L^2)}\|u\|_{L^4_T(L^4)}\|\rho^{-1}h\|_{L^4_T(L^4)} + |\langle\nabla\pi, \rho^{-1}h\rangle| \\ &\quad + \|(\rho u - \kappa\nabla\rho)\otimes u - 2\mu Au\|_{L^2_T(L^2)}\|\nabla(\rho^{-1}h)\|_{L^2_T(L^2)} \\ &\le C(\rho_*, \rho^*, \|\varrho\|_{X_2(T)}, \|u\|_{X_1(T)}, \|\nabla\pi\|_{X_{-1}(T)})\|h\|_{X_1(T)}.\end{aligned}$$

Hence (2.62) follows and $(2.4)_2$ holds in $X_{-1}(T)$.

In order to show the Energy Equality (2.7), we take the $\langle\cdot,\cdot\rangle_{X_{-1}(T),X_1(T)}$ inner product between Equation $(2.4)_2$ and u to arrive at (notice $(2.4)_1$ holding in $L^2_T(L^2)$)

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\rho|u|^2 + 2\int_{\mathbb{R}^2}\mu|Au|^2 = 0,$$

which gives (2.7) immediately. This implies $u \in \mathcal{C}([0,\infty); L^2)$. □

Remark 2.4. *Here we have to consider the function a instead of directly the density ρ, in order to get estimates on $\|\nabla\rho\|_{L^\infty_T(L^2)\cap L^2_T(H^1)}$. In fact, if we directly take the derivative on $(2.4)_1$, then whether the quantity $\nabla\rho\Delta\rho$ issuing from the divergence term* $\mathrm{div}\,(\kappa\nabla\rho)$ *can be killed by the "good" term $\Delta\nabla\rho$ is not clear.*

3D case

Unlike the last paragraph, we do not have Inequality (2.57) in dimension 3. Thus $u \notin L^4(L^4(\mathbb{R}^3))$ and hence the quantity $\int_{\mathbb{R}^3} u\cdot\nabla a\Delta a$ in (2.65) doesn't make sense. However, inspired by the computations before Theorem 2.2, we can show first the uniform bounds as (2.15) for the regular approximated solutions ρ^ε and then, by lower semi-continuity of the L^2-norm, it holds for the weak solution ρ.

More precisely, for any $\varepsilon > 0$ and any $\delta > 0$, we will consider the following regularized system of Cauchy problem (2.4)-(2.56), instead of System (2.28) [5] :

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho\,\langle u\rangle_\varepsilon) - \mathrm{div}\,(\langle\kappa\rangle_\delta\nabla\rho) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,\Big((\rho\langle u\rangle_\varepsilon - \langle\kappa\rangle_\delta\nabla\rho)\otimes u\Big) - \mathrm{div}\,(2\mu Au) + \nabla\pi = 0, \\ \mathrm{div}\, u = 0, \\ (\rho, u)|_{t=0} = (\langle\rho_0\rangle_\varepsilon, \langle u_0\rangle_\varepsilon). \end{cases} \tag{2.68}$$

Following the same argument as in Subsection 2.2.1, the above system has a unique solution $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ with $\rho^{\varepsilon,\delta} - 1, u^{\varepsilon,\delta} \in \mathcal{C}([0,\infty); H^\infty)$, such that Estimate (2.53) holds uniformly in ε and δ. Letting $\delta \to 0$, then there exists a global-in-time weak solution $(\rho^\varepsilon, u^\varepsilon)$ to the following system

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho\,\langle u\rangle_\varepsilon) - \mathrm{div}\,(\kappa\nabla\rho) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,\Big((\rho\langle u\rangle_\varepsilon - \kappa\nabla\rho)\otimes u\Big) - \mathrm{div}\,(2\mu Au) + \nabla\pi = 0, \\ \mathrm{div}\, u = 0, \\ (\rho, u)|_{t=0} = (\langle\rho_0\rangle_\varepsilon, \langle u_0\rangle_\varepsilon). \end{cases} \tag{2.69}$$

5. Due to Remark 2.4 and $u \notin L^4(L^4)$, we regularize the system in two levels in order to get the bound for the H^1-norm of the density ρ by considering the equation for the scalar function $a = a(\rho)$ with $\nabla a = \kappa\nabla\rho$ where the transport velocity is regularized.

Thanks to the smooth transport velocity, by a similar method as in the proof of Lemma 2.2, the equation for the density ρ^ε holds in the following sense :

$$\partial_t\rho + \langle u^\varepsilon\rangle_\varepsilon \cdot \nabla\rho - \mathrm{div}\,(\kappa\nabla\rho) = 0 \quad \text{in} \quad L^2(L^2).$$

In fact, we suppose a priori $\rho^\varepsilon - 1 \in L^\infty_T(H^1 \cap L^\infty) \cap L^2_T(H^2)$ for any $T \in (0,\infty)$. By use of the interpolation inequality in dimension 3

$$\|\nabla\rho\|_{L^4} \lesssim \|\Delta\rho\|^{1/2}_{L^2}\|\rho\|^{1/2}_{L^\infty}, \tag{2.70}$$

one has $\nabla\rho^\varepsilon \in L^4_T(L^4)$. Hence the scalar function $a^\varepsilon := a(\rho^\varepsilon) \in L^\infty_T(H^1 \cap L^\infty) \cap L^2_T(H^2)$ with the function a as defined in the proof of Lemma 2.2 satisfies Equation (2.64) with the transport velocity $\langle u^\varepsilon\rangle_\varepsilon$. Taking the L^2-inner product between it and Δa^ε, applying the following inequality (noticing $\mathrm{div}\,(\langle u^\varepsilon\rangle_\varepsilon) = 0$) :

$$\left|\int_{\mathbb{R}^3} \langle u^\varepsilon\rangle_\varepsilon \cdot \nabla a^\varepsilon \Delta a^\varepsilon\right| = \left|\int_{\mathbb{R}^3} \nabla a^\varepsilon \cdot \nabla\langle u^\varepsilon\rangle_\varepsilon \cdot \nabla a^\varepsilon\right| \le \|\nabla u^\varepsilon\|_{L^2}\|\nabla a^\varepsilon\|^2_{L^4} \le C\|\nabla u^\varepsilon\|_{L^2}\|\Delta a^\varepsilon\|_{L^2},$$

and then performing Young's Inequality and Estimate (2.53), we arrive at

$$\|\nabla a^\varepsilon\|_{L^\infty_T(L^2)} + \|\Delta a^\varepsilon\|_{L^2_T(L^2)} \le C(\rho_*,\rho^*)\Big(\|\nabla a^\varepsilon(0)\|_{L^2} + \|u_0\|_{L^2}\Big) \le C\Big(\|\nabla\rho_0\|_{L^2} + \|u_0\|_{L^2}\Big).$$

One easily finds that the above estimate also holds for ρ^ε. Therefore, taking into account also Energy Identity (2.6), we arrive at for all $T \in (0,\infty)$, $\varepsilon > 0$,

$$\|\rho^\varepsilon - 1\|_{L^\infty_T(H^1(\mathbb{R}^3))} + \|\nabla\rho^\varepsilon\|_{L^2_T(H^1(\mathbb{R}^3))} \le C\|(\rho_0 - 1, u_0)\|_{H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)}. \tag{2.71}$$

Now we let $\varepsilon \to 0$, then the same argument as in Subsection 2.2.1 ensures that there exists a global-in-time weak solution (ρ, u) to System (2.4) such that Estimate (2.19) holds and $\nabla\rho \in \mathcal{C}([0,\infty); L^2_w)$. Define $v = u - \kappa\nabla\ln\rho$, then one easily checks that Equality (2.17) holds for some divergence-free test function ϕ by virtue of (2.10). This completes the proof of Theorem 2.2.

2.3 Well-posedness in dimension two

In this section we aim at proving Theorem 2.4, by the arguments before which, we just have to prove the global-in-time existence of a unique strong solution (ρ, u) to Cauchy problem (2.4)-(2.20). Indeed, by Theorem 2.3, it rests to show a pseudo-conservation law concerning $L^\infty(H^2)\times L^\infty(H^1)$-norm of its weak solutions which, should be proved to belong to (strong) solution space E_t (see (2.21) for the definition).

The following lemma supplies the needed conservation law :

Lemma 2.3. *We assume $(\varrho, u, \nabla\pi)$ to be a weak solution to System* (2.4) *with the initial data ϱ_0, u_0 satisfying the following condition :*

$$0 < \rho_* \le \varrho_0 + 1 \le \rho^*, \quad \varrho_0 \in H^2(\mathbb{R}^2), \quad u_0 \in H^1(\mathbb{R}^2), \quad \mathrm{div}\,u_0 = 0, \tag{2.72}$$

then there exists a constant D_3 depending only on $\rho_*, \rho^*, \|\varrho_0\|_{H^2(\mathbb{R}^2)}, \|u_0\|_{H^1(\mathbb{R}^2)}$ such that the following a priori estimate holds true :

$$\sup_{0\le t\le T}(\|\varrho\|^2_{H^2} + \|u\|^2_{H^1}) + \int_0^T (\|\nabla\varrho\|^2_{H^2} + \|\nabla u\|^2_{H^1} + \|\partial_t\varrho\|^2_{H^1} + \|\partial_t u\|^2_{L^2} + \|\nabla\pi\|^2_{L^2}) \le D_3. \tag{2.73}$$

Proof. It is easy to see that (2.62) already holds by Lemma 2.2. As usual, we can assume a priori that $\varrho \in L^\infty(H^2)$, $\nabla\varrho \in L^2(H^2)$ and $u \in L^\infty(H^1)$, $\nabla u \in L^2(H^1)$. In the following we will use thoroughly the Gagliardo-Nirenberg inequality (2.57) and the following interpolation inequality

$$\|f\|_{L^\infty(\mathbb{R}^2)} \lesssim \|f\|^{1/2}_{L^2(\mathbb{R}^2)}\|\Delta f\|^{1/2}_{L^2(\mathbb{R}^2)}. \tag{2.74}$$

We first consider the equation for u. We take the L^2 inner product between Equation $(2.4)_2$ and $\partial_t u$ to arrive at

$$0 = \int_{\mathbb{R}^2} \rho|\partial_t u|^2 + \frac{d}{dt}\int_{\mathbb{R}^2} 2\mu|Au|^2 + \int_{\mathbb{R}^2} \rho u\cdot\nabla u\cdot\partial_t u - \kappa\nabla\rho\cdot\nabla u\cdot\partial_t u - 2\mu'|Au|^2\partial_t\rho. \tag{2.75}$$

On the other hand, from the equation $(2.4)_2 - (2.4)_3$ we have

$$\Delta u = \mu^{-1}\Big(\rho\partial_t u + \rho u\cdot\nabla u - \kappa\nabla\rho\cdot\nabla u - 2\mu'\nabla\rho\cdot Au + \nabla\pi\Big), \tag{2.76}$$

and the elliptic equation for π (similar to (2.43))

$$\mathrm{div}\,(\rho^{-1}\nabla\pi) = -\mathrm{div}\,\Big((u-\rho^{-1}\kappa\nabla\rho)\cdot\nabla u + 2\mu\nabla\rho^{-1}\cdot Au\Big). \tag{2.77}$$

Equation (2.77) above gives us the estimate for $\nabla\pi$:

$$\|\nabla\pi\|_{L^2} \le C(\|u\|_{L^4} + \|\nabla\rho\|_{L^4})\|\nabla u\|_{L^4}. \tag{2.78}$$

Therefore Equality (2.76) entails

$$\|\Delta u\|_{L^2} \le C\Big(\|\partial_t u\|_{L^2} + (\|u\|_{L^4} + \|\nabla\rho\|_{L^4})\|\nabla u\|_{L^4}\Big),$$

which implies, by applying (2.57) on ∇u and Young's Inequality,

$$\|\Delta u\|_{L^2} \le C\Big(\|\partial_t u\|_{L^2} + (\|u\|^2_{L^4} + \|\nabla\rho\|^2_{L^4})\|\nabla u\|_{L^2}\Big), \tag{2.79}$$

and hence

$$\|\nabla u\|^2_{L^4} \le C\Big(\|\partial_t u\|_{L^2}\|\nabla u\|_{L^2} + (\|u\|^2_{L^4} + \|\nabla\rho\|^2_{L^4})\|\nabla u\|^2_{L^2}\Big). \tag{2.80}$$

Set

$$\mathcal{I}_1 \triangleq \int_{\mathbb{R}^2} \rho u\cdot\nabla u\cdot\partial_t u - \kappa\nabla\rho\cdot\nabla u\cdot\partial_t u - 2\mu'|Au|^2\partial_t\rho,$$

then

$$\|\mathcal{I}_1\|_{L^1([0,T])} \le C\int_0^T (\|u\|_{L^4} + \|\nabla\rho\|_{L^4})\|\nabla u\|_{L^4}\|\partial_t u\|_{L^2} + \|Au\|^2_{L^4}\|\partial_t\rho\|_{L^2}.$$

Thus by Estimate (2.80) and Young's Inequality, we finally arrive at

$$\|\mathcal{I}_1\|_{L^1([0,T])} \le \varepsilon\int_0^T \|\partial_t u\|^2_{L^2} + C_\varepsilon\int_0^T (\|u\|^4_{L^4} + \|\nabla\rho\|^4_{L^4} + \|\partial_t\rho\|^2_{L^2})\|\nabla u\|^2_{L^2}.$$

Therefore by the equality $\|\nabla u\|^2_{L^2} = \int |Au|^2$ and Estimate (2.62), we can choose sufficiently small ε to deduce from (2.75) that

$$\sup_{0\le t\le T}\|\nabla u\|^2_{L^2} + \int_0^T \|\partial_t u\|^2_{L^2} \le Ce^{D_2}\|\nabla u_0\|^2_{L^2}. \tag{2.81}$$

Moreover, by estimates (2.78), (2.79), (2.80) and (2.81), we derive

$$\int_0^T \|\Delta u\|_{L^2}^2 + \|\nabla u\|_{L^4}^4 + \|\nabla \pi\|_{L^2}^2 \leq C(\|\varrho_0\|_{H^2}, \|u_0\|_{H^1}). \tag{2.82}$$

Now we turn to the density ρ. We further apply "Δ" to Equation (2.64) of a, yielding the equation for the scalar function $\mathfrak{a} = \Delta a \in L^\infty(L^2) \cap L^2(H^1)$:

$$\partial_t \mathfrak{a} + 2\nabla u : \nabla^2 a + u \cdot \nabla \mathfrak{a} + \Delta u \cdot \nabla a - \Delta(\kappa \mathfrak{a}) = 0. \tag{2.83}$$

Hence again taking the L^2 inner product between (2.83) and $\mathfrak{a}$ shows

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \mathfrak{a}^2 + \int_{\mathbb{R}^2} \kappa |\nabla \mathfrak{a}|^2 + \int_{\mathbb{R}^2} 2(\nabla u : \nabla^2 a)\,\mathfrak{a} + (\Delta u \cdot \nabla a)\mathfrak{a} + \kappa' \mathfrak{a} \nabla \rho \cdot \nabla \mathfrak{a} = 0. \tag{2.84}$$

Set

$$\mathcal{I}_2 \triangleq \int_{\mathbb{R}^2} 2(\nabla u : \nabla^2 a)\,\mathfrak{a} + (\Delta u \cdot \nabla a)\mathfrak{a} + \kappa' \mathfrak{a} \nabla \rho \cdot \nabla \mathfrak{a},$$

then noticing $\|\nabla \rho\|_{L^4} \leq C\|\nabla a\|_{L^4}$, we have

$$\|\mathcal{I}_2\|_{L^1([0,T])} \leq C \int_0^T (\|\nabla u\|_{L^2} \|\nabla^2 a\|_{L^4} + \|\Delta u\|_{L^2} \|\nabla a\|_{L^4} + \|\nabla a\|_{L^4} \|\nabla \mathfrak{a}\|_{L^2}) \|\mathfrak{a}\|_{L^4}.$$

By use of $\|\nabla^2 a\|_{L^4}, \|\mathfrak{a}\|_{L^4} \lesssim \|\mathfrak{a}\|_{L^2}^{1/2} \|\nabla \mathfrak{a}\|_{L^2}^{1/2}$, we have from above that

$$\|\mathcal{I}_2\|_{L^1([0,T])} \leq \varepsilon \int_0^T \|\nabla \mathfrak{a}\|_{L^2}^2 + C_\varepsilon \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla a\|_{L^4}^4) \|\mathfrak{a}\|_{L^2}^2 + \|\Delta u\|_{L^2}^2.$$

Therefore by (2.62), (2.66), (2.82) and Gronwall's Inequality, Inequality (2.84) gives

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\mathfrak{a}(t)\|_{L^2}^2 + \int_0^T \|\nabla \mathfrak{a}\|_{L^2}^2 &\leq C \exp\Big\{\int_0^T \|\nabla u\|_{L^2}^2 + \|\nabla a\|_{L^4}^4\Big\} \Big(\|\mathfrak{a}(0)\|_{L^2}^2 + \int_0^T \|\Delta u\|_{L^2}^2\Big) \\ &\leq C(\|\varrho_0\|_{H^2}, \|u_0\|_{H^1}), \end{aligned} \tag{2.85}$$

and hence

$$\sup_{0 \leq t \leq T} \|\nabla a(t)\|_{L^4} \leq \sup_{0 \leq t \leq T} \|\nabla a(t)\|_{L^2}^{1/2} \|\mathfrak{a}(t)\|_{L^2}^{1/2} \leq C(\|\varrho_0\|_{H^2}, \|u_0\|_{H^1}). \tag{2.86}$$

Furthermore, according to the two identities

$$\kappa \Delta \varrho = \mathfrak{a} - \nabla \kappa \cdot \nabla \varrho \text{ and } \kappa \nabla \Delta \varrho = \nabla \mathfrak{a} - \nabla \kappa \Delta \varrho - \nabla \varrho \cdot \nabla^2 \kappa - \nabla \kappa \cdot \nabla^2 \varrho,$$

we get from (2.57) and (2.74) that

$$\sup_{0 \leq t \leq T} \|\Delta \varrho(t)\|_{L^2}^2 + \int_0^T \|\nabla \Delta \varrho\|_{L^2}^2 \leq C(\|\varrho_0\|_{H^2}, \|u_0\|_{H^1}). \tag{2.87}$$

Moreover, Equation $(2.4)_1$ ensures

$$\nabla(\partial_t \rho) = -\nabla \rho \cdot Du - u \cdot \nabla^2 \rho + \nabla \mathfrak{a},$$

which by above yields

$$\int_0^T \|\nabla \partial_t \varrho\|_{L^2}^2 \leq C(\|\varrho_0\|_{H^2}, \|u_0\|_{H^1}), \tag{2.88}$$

which together with the estimates (2.62), (2.81), (2.82) and (2.87) give (2.73). □

Remark 2.5. *Compared to Remark 2.3, thanks to the high regularity of the density, we have control on* $\|\nabla\pi\|_{L^2(L^2)}$ *and moreover, we have* $\Delta\pi \in L^1(L^2)$. *Indeed, it is easy to deduce from* (2.77) *that*

$$-\rho^{-1}\Delta\pi = \nabla\rho^{-1}\cdot\nabla\pi + (\nabla u - \kappa\rho^{-1}\nabla\rho) : \nabla u - \mu\nabla\rho^{-1}\cdot\Delta u.$$

Thus by the uniform bound (2.73), *one gets* $\|\Delta\pi\|_{L^1(L^2)} \leq C(\|\varrho_0\|_{H^2(\mathbb{R}^2)}, \|u_0\|_{H^1(\mathbb{R}^2)})$.

The next lemma is devoted to a weak-strong uniqueness result under the initial condition (2.72), by use of the estimates (2.22) and (2.23).

Lemma 2.4. *Under the same hypotheses of Lemma 2.3, suppose* $(\rho, u, \nabla\pi)$ *to be a weak solution to System* (2.4), *then* $(\rho - 1, u, \nabla\pi) \in E_T$, $\forall T \in [0, +\infty)$, *and hence is unique globally in time.*

Proof. In this proof, we will always denote by C_0 some harmless constant depending only on the initial data $\rho_*, \rho^*, \|\varrho_0\|_{H^2(\mathbb{R}^2)}, \|u_0\|_{H^1(\mathbb{R}^2)}$, which may vary from time to time.

Let's first rewrite the equations for ϱ and u in System (2.4) as following :

$$\begin{cases} \partial_t\varrho - \operatorname{div}(\kappa\nabla\varrho) = -u\cdot\nabla\varrho, \\ \partial_t u - \operatorname{div}(\mu\rho^{-1}\nabla u) = -(u - (\kappa\rho^{-1} + \mu\rho^{-2})\nabla\rho)\cdot\nabla u - \rho^{-1}\nabla\mu\cdot Du - \rho^{-1}\nabla\pi. \end{cases} \tag{2.89}$$

By (2.22) and (2.23), the $L^1_T(B^1_{2,1})$ (resp. $L^1_T(B^0_{2,1})$)-norm of the right hand side of Equation $(2.89)_1$ (resp. $(2.89)_2$) can be bounded by the following two quantities respectively :

$$C\int_0^T \|u\|_{B^1_{2,1}}\|\nabla\rho\|_{B^1_{2,1}}\,dt \quad \text{and} \quad C\int_0^T \Big((\|u\|_{B^1_{2,1}} + \|\varrho\|_{B^2_{2,1}})\|\nabla u\|_{B^0_{2,1}} + \|\varrho\|_{B^1_{2,1}}\|\nabla\pi\|_{B^0_{2,1}}\Big)\,dt.$$

Since $\|\varrho\|_{B^1_{2,1}} \leq \|\varrho\|_{H^2}$ and $\|\nabla\pi\|_{B^0_{2,1}} \lesssim \|\nabla\pi\|_{L^2} + \|\Delta\pi\|_{L^2}$, we have from the uniform estimate (2.73) and the elliptic equation (2.58) for π

$$\operatorname{div}(\rho^{-1}\nabla\pi) = -\operatorname{div}((u - \rho^{-1}\kappa\nabla\rho)\cdot\nabla u + 2\mu\nabla\rho^{-1}\cdot Au),$$

that

$$\int_0^T \|\varrho\|_{B^1_{2,1}}\|\nabla\pi\|_{B^0_{2,1}}\,dt \lesssim \int_0^T C_0\Big(\|\Delta\pi\|_{L^2} + \|(u - \kappa\rho^{-1}\nabla\rho)\cdot\nabla u + 2\mu\nabla\rho^{-1}\cdot Au\|_{L^2}\Big)\,dt.$$

Because $B^0_{2,1} \hookrightarrow L^2$, the product estimate (2.23) entails

$$\int_0^T \|\varrho\|_{B^1_{2,1}}\|\nabla\pi\|_{B^0_{2,1}}\,dt \leq C_0\int_0^T \|\Delta\pi\|_{L^2} + C_0\int_0^T (\|u\|_{B^1_{2,1}} + \|\varrho\|_{B^2_{2,1}})\|\nabla u\|_{B^0_{2,1}}\,dt.$$

On the other side, it is easy to check that for any $\mathcal{C}^1(\mathbb{R},\mathbb{R})$-function f with $f(1) = 0$, one has

$$\|f(\rho)\|_{L^\infty(H^2)} + \|\nabla f(\rho)\|_{L^2(H^2)} \leq C_0.$$

Therefore, for any solution (ϱ, u) to System (2.4), Proposition 2.1 tells us that

$$\begin{aligned} &\|\varrho\|_{L^\infty_T(B^1_{2,1})\cap L^1_T(B^3_{2,1})} + \|u\|_{L^\infty_T(B^0_{2,1})\cap L^1_T(B^2_{2,1})} \\ &\leq C_0\Big(1 + \int_0^T \|(\varrho, u)\|_{L^2} + \int_0^T \|u\|_{B^1_{2,1}}\|\nabla\rho\|_{B^1_{2,1}} + (\|u\|_{B^1_{2,1}} + \|\varrho\|_{B^2_{2,1}})\|\nabla u\|_{B^0_{2,1}} + \|\Delta\pi\|_{L^2}\Big). \end{aligned}$$

As we have interpolation inequalities

$$\|u\|_{B^1_{2,1}} \lesssim \|u\|^{1/2}_{B^0_{2,1}}\|u\|^{1/2}_{B^2_{2,1}}, \quad \|\varrho\|_{B^2_{2,1}} \lesssim \|\varrho\|^{1/2}_{B^1_{2,1}}\|\varrho\|^{1/2}_{B^3_{2,1}},$$

by Young's Inequality and Gronwall's Inequality again, the last three inequalities imply

$$\|\varrho\|_{L^\infty_T(B^1_{2,1})\cap L^1_T(B^3_{2,1})} + \|u\|_{L^\infty_T(B^0_{2,1})\cap L^1_T(B^2_{2,1})} \\ \leq C_0 \exp\Big\{\int_0^T \|\nabla\rho\|^2_{B^1_{2,1}} + \|\nabla u\|^2_{B^0_{2,1}}\Big\}\Big(1 + \|(\varrho, u)\|_{L^1_T(L^2)} + \|\Delta\pi\|_{L^1_T(L^2)}\Big).$$

It is easy to find from Estimate (2.73) and Remark 2.5 that

$$\|\varrho\|_{L^\infty_T(B^1_{2,1})\cap L^1_T(B^3_{2,1})} + \|u\|_{L^\infty_T(B^0_{2,1})\cap L^1_T(B^2_{2,1})} \leq C_0\ \Big(1 + \|(\varrho, u)\|_{L^1_T(L^2)}\Big).$$

This completes the proof. □

Now with Lemma 2.4 in hand, we can easily check that there exists a global-in-time strong solution to Cauchy problem (2.4)-(2.20). Firstly, Theorem 2.3 ensures a unique strong solution $(\rho_1, u_1, \nabla\pi_1)$ with lifespan T^* for some positive time $T^* > T_c$. Hence there exists $T_0 \in (0, T^*)$ such that $\varrho_1(T_0) \in B^2_{2,1} \hookrightarrow H^2$ and $u_1(T_0) \in B^1_{2,1} \hookrightarrow H^1$. Let $(\rho_2, u_2, \nabla\pi_2)$ be a weak solution which evolves from the initial data $\rho_1(T_0), u_1(T_0)$. Thus, by Lemma 2.4, $(\rho_2, u_2, \nabla\pi_2) \in E_T$, $\forall T \geq T_0$. Hence, it coincides with $(\rho_1, u_1, \nabla\pi_1)$ on the time interval $[T_0, T^*)$ by the uniqueness result and moreover, we can choose $T^* = +\infty$. In fact, if $T^* < +\infty$, then by the global-in-time boundedness of $\|\rho_2 - 1\|_{L^\infty(B^1_{2,1})}$ and $\|u_2\|_{L^\infty(B^0_{2,1})}$ given by (2.73), the lifespan of $(\rho_1, u_1, \nabla\pi_1)$ should go beyond T^*. This is a contradiction.

Chapitre 3

Inviscid low-Mach number limit system

This chapter is devoted to the well-posedness issue for a low-Mach number limit system with heat conduction but no viscosity, that is, System (0.35) :

$$\left\{\begin{array}{ccc}\partial_t\rho+\mathrm{div}\,(\rho v) & = & 0,\\ \partial_t(\rho v)+\mathrm{div}\,(\rho v\otimes v)+\nabla\Pi & = & 0,\\ \mathrm{div}\,v-\mathrm{div}\,(\alpha k\nabla\vartheta) & = & 0,\end{array}\right. \tag{3.1}$$

or equivalently System (0.37) :

$$\left\{\begin{array}{c}\partial_t\rho+u\cdot\nabla\rho-\mathrm{div}\,(\kappa\nabla\rho)=0,\\ \partial_t u+(u+\nabla b)\cdot\nabla u+\lambda\nabla q=h(\rho,u),\\ \mathrm{div}\,u=0,\end{array}\right. \tag{3.2}$$

where the coefficients $\kappa=\kappa(\rho)=k\rho^{-1}C_P^{-1}$, $\lambda=\lambda(\rho)=\rho^{-1}$ and the source term h (corresponding to h_3 in the introduction part) is defined by the following

$$h\;=\;\lambda\mathrm{div}\,(v\otimes\nabla a)\;=\;-u\cdot\nabla^2 b-(u\cdot\nabla\lambda)\nabla a-(\nabla b\cdot\nabla\lambda)\nabla a-\mathrm{div}\,(\nabla b\otimes\nabla b)\,. \tag{3.3}$$

The relation between $u,v,a=a(\rho),b=b(\rho)$ is given by

$$u=v-\nabla b,\quad \nabla a=\kappa\nabla\rho=-\rho\nabla b, \tag{3.4}$$

while the relation between the two pressures $\nabla\Pi$ and ∇q is

$$\nabla\Pi=\nabla q+\partial_t\nabla a. \tag{3.5}$$

Notice also that Equation $(3.2)_2$ can be written as the following equation for u :

$$\partial_t(\rho u)+\mathrm{div}\,(\rho v\otimes u)+\nabla q=\mathrm{div}\,(v\otimes\nabla a). \tag{3.6}$$

Let us note that if $\kappa\equiv 0$, then $a\equiv b\equiv 0$ and hence System (3.2) becomes the so-called density-dependent Euler equations

$$\left\{\begin{array}{c}\partial_t\rho+v\cdot\nabla\rho=0,\\ \partial_t v+v\cdot\nabla v+\lambda\nabla q=0,\\ \mathrm{div}\,v=0.\end{array}\right. \tag{3.7}$$

We can see that due to the null heat conduction, the density ρ satisfies a transport equation, the “source” term h vanishes and the velocity v itself is divergence free. In System (3.2) instead, one encounters a quasilinear parabolic-type equation for ρ, the transport velocity for u is no longer solenoidal and the "source" term h is a little complicated, as it involves two derivatives of ρ. As early as in 1980, H. Beirão da Veiga and A. Valli in [12, 13, 14] have investigated the local well-posedness issue in some smooth bounded domain of the nonhomogeneous Euler system (3.7). We also cite here the book [3] as a good survey of the boundary-value problems in mechanics for nonhomogeneous fluids. By use of an energy identity, in [35] R. Danchin studied System (3.7) in the framework of nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ which can be embedded in $\mathcal{C}^{0,1}$. Recently in [36], R. Danchin and F. Fanelli has treated the end point case where the Lebesgue exponent p in the Besov space $B^s_{p,r}$ is chosen to be ∞. We notice that if morever $\rho \equiv 1$ is a constant density state, then System (3.7) reduces to the classical Euler system, which has been deeply studied.

However, to our knowledge, there are not so many theoretical works devoted to heat-conducting inviscid zero Mach number system (3.1), or equivalently System (3.2). It is interesting to view System (3.1) as the model for an inviscid fluid consisting of two components, both incompressible, with a diffusion effect obeying Fick's law. In fact, we can write Equation $(3.1)_3$ as the following Fick's law :

$$\operatorname{div}(v + \kappa \nabla \ln \rho) = 0,$$

with κ denoting the diffusion coefficient. Now ρ, u and v are considered to be the mean density, the mean-volume velocity and the mean-mass velocity of the mixture respectively. As usual, $\nabla\Pi$ denotes some unknown pressure. For more physical backgrounds of this model, see [52]. One can also see [11] for a local existence and uniqueness result of the initial boudary value problem for this model, in the framework of classical solutions.

In this chapter, we aim at showing some similar results as in [35, 36] about it. We will work in the framework of general Besov spaces $B^s_{p,r}(\mathbb{R}^d)$, which can be embedded into the class of Lipschitz functions. Firstly, we consider the case of $p \in [2, 4]$, with no further restrictions on the initial data. Then we tackle the case of any $p \in]1, \infty]$, but requiring also a finite energy assumption, or a smallness condition on the initial density. The extreme value $p = \infty$ can be treated due to a new a priori estimate for parabolic equations. A continuation criterion and a lower bound for the lifespan of the solution are proved as well.

This chapter is organized in the following way.
In the first section we will present our first main results on System (3.2) (the corresponding results for System (3.1) will be pointed out, in Remark 3.1 for example), i.e. the local in time well-posedness in Besov spaces $B^s_{p,r} \hookrightarrow \mathcal{C}^{0,1}$. Firstly, we focus on the case when $p \in [2, 4]$ (see Theorem 3.1) ; then, we consider also the general instance $p \in]1, +\infty]$, under the additional assumptions of data in L^2 (see Theorem 3.4) or when the initial density is a small perturbation of a constant state (Theorem 3.6). We will also state a continuation criterion for solutions to our system, and a lower bound for the lifespan of the solution in terms of the norms of the initial data only : this will be done in Theorems 3.2 and 3.3.

In Section §3.2 we will tackle the proof of Theorem 3.1 : it will be carried out in a standard way. First of all, we will show a priori estimates for the linearized equations. Then, we will construct inductively a sequence of smooth approximated solutions. Finally, we will show its convergence to a "real solution" to the original equations.

Section §3.3, instead, is devoted to the proof of the continuation criterion (Theorem 3.2) and of the lower bound for the lifespan of solutions to system (3.2) (Theorem 3.3).

The finite energy case is presented in Section §3.4. A new estimate for parabolic equations in Besov spaces $B^s_{p,\infty}$ will be the key to tackle the case up to this endpoint space, and so to prove Theorem 3.4.

We will sketch the proof of Theorem 3.6 when the inital density is near a constant state, in the last section §3.5.

3.1 Main results

Now let us analyze System (3.2) to introduce our main results. Just as in Chapter 2, in view of the parabolic equation $(3.2)_1$ for the density, by maximum principle, we can assume that the density ρ (if it exists on the time interval $[0, T]$) has the same positive upper and low bounds as initial density ρ_0 :

$$0 < \rho_* \leq \rho(t, x) \leq \rho^*, \quad \forall\, t \in [0, T],\, x \in \mathbb{R}^d. \tag{3.8}$$

Correspondingly, the coefficients κ and λ can always be assumed to have positive upper and lower bounds too, which ensures that the pressure q satisfies an elliptic equation. As a matter of fact, applying operator "div" to Equation $(3.2)_2$ gives the following elliptic equation in divergence form :

$$\mathrm{div}\,(\lambda \nabla q) = \mathrm{div}\,(h - v \cdot \nabla u). \tag{3.9}$$

Since by Equation $(3.2)_2$, there is no gain of regularity for the velocity u as time goes by, we suppose the initial divergence-free "velocity" field u_0 to belong to some space $B^s_{p,r}$ which can be continuously embedded in $\mathcal{C}^{0,1}$, i.e. the triplet $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ only has to satisfy the following condition :

$$s > 1 + \frac{d}{p}\,, \qquad \text{or} \qquad s = 1 + \frac{d}{p}\,,\ r = 1\,. \tag{3.10}$$

This assumption will be enough to have the velocity field u Lipschitz continuous, and so to preserve the initial regularity as a transport velocity. This at least requires the "source" term $h - \lambda \nabla q$ to belong to $L^1([0, T]; B^s_{p,r})$ which, according to definition (3.3) of h and bound (3.8) for ρ, asks at least

$$\nabla^2 \rho \in L^1([0, T]; B^s_{p,r}), \quad \nabla \rho \in L^\infty([0, T]; L^\infty), \quad \text{and} \quad \nabla q \in L^1([0, T]; B^s_{p,r}).$$

Keeping in mind that ρ satisfies the parabolic type equation $(3.2)_1$, we expect to maintain the initial regularity and to gain also two orders of regularity with respect to the space variable when integrating in time (see e.x. Proposition 0.7), at least on a small time interval. We thus have to assume the difference of the initial density and a constant, say $\rho_0 - 1$, to be in the same space $B^s_{p,r}$ as above. However in general, we only get $\nabla^2 \rho$ in time-dependent Besov space $\widetilde{L}^1_T(B^s_{p,r})$, which is a little larger than $L^1_T(B^s_{p,r})$, see Definition 0.3. Therefore in the whole context we will deal with the spaces $\widetilde{L}^\infty_T(B^s_{p,r})$ and $\widetilde{L}^1_T(B^s_{p,r})$ instead.

Let us pay attention to the estimate of the pressure q which requires restrictions on the Lebesgue exponent p, just as in the nonhomogeneous case in Chapter 1 (see (1.8)). Firstly, as $h - v \cdot \nabla u$ is already in space $\widetilde{L}^1_T([0, T]; B^s_{p,r})$, then since applying operator "Δ_j" to Equation (3.9) gives

$$\mathrm{div}\,(\lambda \Delta_j \nabla q) = -\mathrm{div}\, \Delta_j (h - v \cdot \nabla u) + \mathrm{div}\,([\lambda, \Delta_j] \nabla q),$$

it remains to controlling the commutator $\mathrm{div}\,([\lambda, \Delta_j]\nabla q)$ and hence the low frequency of ∇q, by the classical commutator estimation. Due to the fact that we have the a priori estimate

$$\|\nabla q\|_{L^p} \leq C \|h - v \cdot \nabla u\|_{L^p}$$

only when $p = 2$ (see Lemma 0.9) or very close to 2, but depending strongly on λ, for Equation (3.9) above, we have to make sure that $h - v \cdot \nabla u \in L^2$. Hence the fact that h is composed of quadratic forms entails that p has to verify

$$p \in [2, 4]. \tag{3.11}$$

Finally, we establish the following theorem, whose proof is shown in Section §3.2.

Theorem 3.1. *Let the triplet* $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ *satisfy conditions* (3.10) *and* (3.11)*. Let us take an initial density state* ρ_0 *and an initial divergence-free velocity field* u_0 *such that*

$$0 < \rho_* \leq \rho_0 \leq \rho^* , \qquad \|\rho_0 - 1\|_{B^s_{p,r}} + \|u_0\|_{B^s_{p,r}} \leq M , \tag{3.12}$$

for some positive constants ρ_**,* ρ^* *and* M*.*

Then there exist a positive time T *(depending only on* $\rho_*, \rho^*, M, d, s, p, r$*) and a unique solution* $(\rho, u, \nabla q)$ *to System* (3.2) *such that* $(\varrho, u, \nabla q) = (\rho - 1, u, \nabla q)$ *belongs to the space* $E^s_{p,r}(T)$*, defined as the set of triplet* $(\varrho, u, \nabla\pi)$ *such that*

$$\begin{cases} \varrho & \in & \widetilde{\mathcal{C}}([0,T]; B^s_{p,r}) \cap \widetilde{L}^1([0,T]; B^{s+2}_{p,r}) , \\ u & \in & \widetilde{\mathcal{C}}([0,T]; B^s_{p,r}) , \\ \nabla q & \in & \widetilde{L}^1([0,T]; B^s_{p,r}) \cap L^1([0,T]; L^2) , \end{cases} \tag{3.13}$$

with $\widetilde{\mathcal{C}}_w([0,T]; B^s_{p,r})$ *if* $r = +\infty$ *(see also Definition 0.3).*

Remark 3.1. *Let us state briefly here the corresponding wellposedness result for the original system* (3.1)*. According to the change of variables* (3.4)*, we have* $u = \mathcal{P}v$*,* $\nabla b = \mathcal{Q}v$*, where* $\mathcal{P}$ *denotes the Leray projector over divergence-free vector fields and* $\mathcal{Q} = \mathrm{Id} - \mathcal{P}$*. Assume Conditions* (3.10) *and* (3.11)*, and the initial datum* (ρ_0, v_0) *to satisfy*

$$0 < \rho_* \leq \rho_0 \leq \rho^* , \quad \nabla b(\rho_0) = \mathcal{Q}v_0, \qquad \|\rho_0 - 1\|_{B^s_{p,r}} + \|\mathcal{P}v_0\|_{B^s_{p,r}} \leq M .$$

Then, there exist a positive time T *(depending only on* $\rho_*, \rho^*, M, d, s, p, r$*) and a unique solution* $(\rho, v, \nabla q)$ *to System* (3.1) *such that*

$$\begin{cases} \rho - 1 \in \widetilde{\mathcal{C}}([0,T]; B^s_{p,r}) \cap \widetilde{L}^1([0,T]; B^{s+2}_{p,r}) , \\ v \in \widetilde{\mathcal{C}}([0,T]; B^{s-1}_{p,r}) \cap \widetilde{L}^2([0,T]; B^s_{p,r}) , \quad \mathcal{P}v \in \widetilde{\mathcal{C}}([0,T]; B^s_{p,r}) , \\ \nabla q \in \widetilde{L}^1([0,T]; B^s_{p,r}) , \end{cases}$$

with $\widetilde{\mathcal{C}}_w([0,T]; B^s_{p,r})$ *if* $r = +\infty$*.*

One notices from above that, the initial velocity v_0 *needs not to be in* $B^s_{p,r}$ *but the velocity* $v(t)$ *will be in it for almost every* $t \in [0,T]$*. That is, the initial non-Lipschitzian velocity may evolve in a unique Lipschitzian velocity. However, Theorem 3.1 doesn't say that if the initial datum* (ρ_0, v_0) *satisfies* (3.12)*, then the local existence result holds. In fact, we just can say that, if initially* $\rho_0 - 1 \in B^{s+1}_{p,r}$ *and* $v_0 \in B^s_{p,r}$*, then there exists a unique local solution to System* (3.1)*.*

Let us also point out that since $\partial_t \rho \notin L^1([0,T]; L^2)$ *in general, we do not know whether* $\nabla q \in L^1([0,T]; L^2)$*. Hence it seems not convenient to deal with System* (3.1) *directly since the low frequences of* ∇q *can not be controlled a priori.*

Next, one can furthermore get a Beale-Kato-Majda type continuation criterion (see [8] for the original version) for solutions to System (3.2), similar as in [35] or [36] for the density-dependent Euler System (3.7). As it is a coupling of a parabolic equation (for the density) and an Euler-type equation (for the velocity field and the pressure), one may expect, a priori, to impose conditions similar, or even weaker, to those found in paper [35], because the regularity of the density improves in time evolution. Actually, this is not the case : the criterion holds true under additional conditions, which are motivated by the structure of the nonlinearities in h and $v \cdot \nabla u$. In fact, for example, we consider $\widetilde{L}^1_T(B^s_{p,r})$-norm of h and $\nabla b \cdot \nabla u$:

– the term $u \cdot \nabla^2 b$ requires at least $\|\nabla^2 b\|_{L^1_T(L^\infty)}$ and $\|u\|_{L^\infty_T(L^\infty)}$;

– the term $\Delta b \nabla b$ requires $\|\nabla b\|_{L^\infty_T(L^\infty)}$ and $\|\Delta b\|_{L^2_T(L^\infty)}$;
– the transport term $\nabla b \cdot \nabla u$ needs control on $\|\nabla u\|_{L^2_T(L^\infty)}$, since we only have $\nabla b \in \widetilde{L}^2_T(B^s_{p,r})$.

Finally, we have the following statement.

Theorem 3.2. [Continuation Criterion] *Let the triplet* $(s,p,r) \in \mathbb{R} \times [1,+\infty]^2$ *satisfy conditions* (3.10) *and* (3.11). *Let* $(\rho, u, \nabla q)$ *be a solution of* (3.2) *on* $[0,T[\,\times\mathbb{R}^d$ *such that it is of the following regularity and satisfies the following conditions :*

(i) $\rho - 1 \in \widetilde{\mathcal{C}}([0,T[\,; B^s_{p,r}) \cap \widetilde{L}^1_{loc}([0,T[\,; B^{s+2}_{p,r})$ *and it satisfies*

$$\sup_{t\in[0,T[} \|\nabla\rho(t)\|_{L^\infty} + \int_0^T \left\|\nabla^2\rho(t)\right\|^2_{L^\infty} dt \; < \; +\infty\,; \tag{3.14}$$

(ii) $u \in \widetilde{\mathcal{C}}([0,T[\,; B^s_{p,r})$ *and it satisfies*

$$\sup_{t\in[0,T[} \|u(t)\|_{L^\infty} + \int_0^T \|\nabla u(t)\|^2_{L^\infty} dt \; < \; +\infty\,; \tag{3.15}$$

(iii) $\nabla q \in \widetilde{L}^1_{loc}([0,T[\,; B^s_{p,r})$ *verifies, for some* $\sigma > 0$,

$$\int_0^T \|\nabla q(t)\|_{B^{-\sigma}_{p,\infty}\cap L^\infty} dt \; < \; +\infty\,. \tag{3.16}$$

Then $(\rho, u, \nabla q)$ *could be continued beyond* T *(if* T *is finite) into a solution of* (3.2) *with the same regularity.*

The proof of Theorem 3.2 issues from the following fundamental lemma whose proof can be found in Section §3.3.1 :

Lemma 3.1. *Let* $s > 0$, $p \in (1,+\infty)$ *and* $r \in [1,+\infty]$. *Let* $(\rho, u, \nabla q)$ *be a solution of* (3.2) *over* $[0,T[\,\times\mathbb{R}^d$ *such that Hypotheses (i), (ii), (iii) in Theorem 3.2 hold true. If* T *is finite, then one gets*

$$\|\rho - 1\|_{\widetilde{L}^\infty_T(B^s_{p,r})\cap\widetilde{L}^1_T(B^{s+2}_{p,r})} + \|u\|_{\widetilde{L}^\infty_T(B^s_{p,r})} + \|\nabla q(t)\|_{\widetilde{L}^1_T(B^s_{p,r})} \; < \; +\infty\,. \tag{3.17}$$

In fact, from Theorem 3.1, one knows that once Conditions (3.10), (3.11) and (3.12) hold, the initial data will evolve into a unique solution at least on a positive time interval, which depends only on the parameters in these conditions. Therefore, once we have (3.17) and besides Conditions (3.10), (3.11) hold true, then there exists a time t_0 such that, for any $\widetilde{T} < T$, System (3.2) with initial data $\left(\rho(\widetilde{T}), u(\widetilde{T})\right)$ has a unique solution until the time $\widetilde{T} + t_0$. Thus, if we take, for instance, $\widetilde{T} = T - (t_0/2)$, then we get a solution until the time $T + (t_0/2)$, which is, by uniqueness, the continuation of $(\rho, u, \nabla q)$. Theorem 3.2 follows.

Let us analyze the scaling property of our system in order to show an explicit relationship between the size of the initial data and the lifespan. Recalling (3.3), it's easy to see that, if $(\rho, u, \nabla q)$ is a solution of (3.2) with initial data (ρ_0, u_0), then

$$\left(\rho^\varepsilon\,,\, u^\varepsilon\,,\, \nabla q^\varepsilon\right)(t\,,\, x) \;:=\; \left(\rho\,,\, \varepsilon^{-1}\, u\,,\, \varepsilon^{-3}\,\nabla q\right)(\varepsilon^{-2}\, t\,, \varepsilon^{-1}\, x) \tag{3.18}$$

is still a solution of (3.2), with initial data $(\rho^\varepsilon_0, u^\varepsilon_0)(x) := (\rho_0, \varepsilon^{-1}u_0)(\varepsilon^{-1}x)$. So, the following result immediately follows.

Proposition 3.1. *Let $(\rho^\varepsilon, u^\varepsilon, \nabla q^\varepsilon)$ be a solution of System (3.2) with initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$ on a time interval $[0, T_\varepsilon[$. Then the triplet $(\rho\,,\, u\,,\, \nabla q)$, given by (3.18), is still a solution of (3.2), defined on a time interval $[0, T[$, with*

$$T \;\geq\; T_\varepsilon\, \varepsilon^{-2}\,.$$

Remark 3.2. *In the homogeneous case, we know that if the initial velocity is of order ε then the lifespan is at least of order ε^{-1}. But here for our system (3.2) (and hence for System (3.1)), we have from the above proposition that if the initial data $\rho_0 - 1, u_0$ are of order $\varepsilon^{s-d/p}$ and $\varepsilon^{s+1-d/p}$ respectively, then the lifespan is at least of order ε^{-2}, due to the fact that*

$$\|\rho_0^\varepsilon - 1\|_{B^s_{p,r}} = \varepsilon^{d/p-s}\|\rho_0 - 1\|_{B^s_{p,r}}, \quad \|u_0^\varepsilon\|_{B^s_{p,r}} = \varepsilon^{d/p-s-1}\|u_0\|_{B^s_{p,r}}.$$

In particular if $s = 1 + d/p$, then the lifespan is of order $O(\varepsilon^{-2})$ for the initial data $\rho_0 - 1 = O(\varepsilon)$, $u_0 = O(\varepsilon^2)$.

Even in the two-dimensional case, it's hard to expect global in time well-posedness for this system : the parabolic equation $(3.2)_1$ allows to improve regularity for the density term, but such a gain is (roughly speaking) deleted by the nonlinear term in the momentum equation $(3.2)_2$. Hence, the obstacles to global existence are the same as those one has to face in considering the density-dependent Euler system (3.7), which has already been dealt in [36]. However, we manage to establish an explicit lower bound for the lifespan of the solution, depending only on the norms of the initial data, in any dimension $d \geq 2$, in Section §3.3.2 :

Theorem 3.3. *Under the hypotheses of Theorem 3.1, there exists a positive constant K_1, depending only on d, p, r, ρ_* and ρ^*, such that, the lifespan T of the solution to System (3.2) given by Theorem 3.1 is bounded from below by the quantity*

$$\frac{K_1}{\|u_0\|_{B^s_{p,r}}}\ \log\left(\frac{K_1}{1 + \left(\|\rho_0 - 1\|_{B^s_{p,r}} \,/\, \|u_0\|^{1/2}_{B^s_{p,r}}\right)^{\mathfrak{X}}}\right), \tag{3.19}$$

where $\mathfrak{X} > 2$ is a constant big enough, depending only on s, d, p.

Remark 3.3. *Thanks to Theorem 3.2, the lifespan is independent of the regularity. Hence, if we want to get the lower bound of the lifespan, we only deal with the endpoint case $s = 1 + d/4$, $p = 4$, $r = 1$, whose lifespan is the largest. Therefore, the $B^s_{p,r}$-norm in (3.19) can be replaced by $B^{1+d/4}_{4,1}$-norm.*

In the following Theorem 3.5 we will improve the previous result for 2-D fluids. Under the additional requirements of finite energy initial data (see also below) and taking $p = +\infty$, we will show that the lifespan tends to $+\infty$, i.e. the solution tends to be global in time, if ρ_0 is "close" to (say) 1.

As we have already pointed out, Hypothesis (3.11) over the index p is imposed to get $h \in L^2$, and so to solve the elliptic equation for the pressure term. In fact, we can remove it and consider any $p \in]1, +\infty]$, provided that the following energy estimates hold : this immediately gives $\nabla q \in L^1_T(L^2)$.

First of all, by taking the $L^2(\mathbb{R}^d)$-inner product between $\rho - 1$ and Equation $(3.2)_1$ and integrating in time variable, we arrive at the following energy identity (i.e. (2.6) in Chapter 2) :

$$\frac{1}{2}\int_{\mathbb{R}^d} |\rho(t) - 1|^2 \,+\, \int_0^t \int_{\mathbb{R}^d} \kappa|\nabla\rho|^2 \;=\; \frac{1}{2}\|\rho_0 - 1\|^2_{L^2}. \tag{3.20}$$

Such an energy identity requires the following equality to hold true :

$$\int_0^t \int_{\mathbb{R}^d} (u \cdot \nabla\rho)\,(\rho - 1) = 0.$$

In fact for instance, if $u \in L^\infty_T(L^\infty)$, which is always the case under our hypothesis, then for any finite t it holds, according to $\operatorname{div} u = 0$ and the previous Identity (3.20).

In the same way, formally, if we take L^2-inner product between Equation (3.6) and u, then thanks to Equation $(3.1)_1$ and integration by parts, the first two terms entail

$$\int_{\mathbb{R}^d} (\rho\partial_t u + \rho v \cdot \nabla u) \cdot u \;=\; \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} \rho|u|^2. \tag{3.21}$$

Thanks to the divergence-free condition of u, the inner product $\langle \nabla q, u\rangle_{L^2(\mathbb{R}^d)}$ vanishes. It rests to dealing with $\langle \operatorname{div}(v \otimes \nabla a), u\rangle_{L^2(\mathbb{R}^d)}$. In fact, from Relation (3.4) we have

$$\operatorname{div}(v \otimes \nabla a) = \Delta b \nabla a + u \cdot \nabla^2 a + \nabla b \cdot \nabla^2 a.$$

The boundedness (3.8) implies that (with C depending on ρ_*, ρ^*)

$$\|\Delta b\|_{L^\infty},\, \|\nabla^2 a\|_{L^\infty} \le C\left(\|\nabla\rho\|^2_{L^\infty} + \|\nabla^2\rho\|_{L^\infty}\right) \text{ and } \|\nabla b\|_{L^2},\, \|\nabla a\|_{L^2} \le C\|\nabla\rho\|_{L^2}.$$

Thus

$$\begin{aligned} |\langle \operatorname{div}(v \otimes \nabla a), u\rangle_{L^2(\mathbb{R}^d)}| &\le C(\|\nabla\rho\|^2_{L^\infty} + \|\nabla^2\rho\|_{L^\infty})(\|\nabla\rho\|_{L^2} + \|u\|_{L^2})\|u\|_{L^2} \\ &\le C\left(\Theta'(t)\|u\|^2_{L^2} + \|\nabla\rho\|^2_{L^2}\right), \end{aligned} \tag{3.22}$$

where we have defined

$$\Theta(t) = \int_0^t \left(\|\nabla\rho\|^2_{L^\infty} + \|\nabla\rho\|^4_{L^\infty} + \|\nabla^2\rho\|_{L^\infty} + \|\nabla^2\rho\|^2_{L^\infty}\right) d\tau\,.$$

Hence Identity (3.21), Hölder's Inequality, Gronwall's Inequality and Energy Equality (3.20) give

$$\|u(t)\|^2_{L^2} \le Ce^{C\Theta(t)}\left(\|u_0\|^2_{L^2} + \int_0^t \|\nabla\rho\|^2_{L^2}\right) \le Ce^{C\Theta(t)}\left(\|u_0\|^2_{L^2} + \|\rho_0 - 1\|^2_{L^2}\right). \tag{3.23}$$

On the other hand, if there exists $(\rho - 1, u)$ such that $(3.13)_1$ and $(3.13)_2$ hold for some finite time $T > 0$, with (s, p, r) verifying (3.10), then we have also (3.14) and (3.15). Thus, if we assume the initial data (ρ_0, u_0) to satisfy

$$\rho_0 - 1, u_0 \in L^2, \tag{3.24}$$

then by Energy Estimates (3.20), (3.23), we gather $u, \rho - 1 \in L^\infty_T(L^2)$ and $\nabla\rho \in L^2_T(L^2)$. Hence for any finite positive time T,

$$h - v \cdot \nabla u = \rho^{-1}\operatorname{div}(v \otimes \nabla a) - v \cdot \nabla u = \rho^{-1}\Delta b\, \nabla a + \rho^{-1} v \cdot \nabla^2 a - v \cdot \nabla u \,\in\, L^1_T(L^2),$$

which ensures $\nabla q \in L^1_T(L^2)$, according to Equation (3.9) and Lemma 0.9.

To conclude, by the arguments before Theorem 3.1 and above, and a new a priori estimate for linear parabolic equations in Besov space $B^s_{\infty,r}$ (see Proposition B.3 in Section §B.2), we have the following :

Theorem 3.4. *Let the triplet* $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ *satisfy Condition* (3.10) *and* $1 < p \leq \infty$.
Let the initial density ρ_0 *and divergence-free initial velocity* u_0 *fulfill* (3.24) *as well as* (3.12), *for some positive constants* ρ_*, ρ^* *and* M.

Then there exist a positive time $T > 0$ *and a unique solution* $(\rho, u, \nabla q)$ *to System* (3.2) *with* $(\rho - 1, u, \nabla q) \in E^s_{p,r}(T)$ *(see* (3.13) *for its definition) such that* $\rho - 1, u \in \mathcal{C}([0, T]; L^2)$, $\nabla\rho \in L^2_T(L^2)$ *also hold true.*

Remark 3.4. *Notice that we have the embedding* $B^s_{p,r}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ *if* $s > d/p - d/2$, $p \in [1, 2]$ *(see Proposition 0.1). Thus if the triplet* $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ *satisfy Condition* (3.10) *and* $1 < p \leq 2$, *then the initial data* ρ_0, u_0 *satisfying* (3.12) *verifies finite-energy condition* (3.24).

In fact, if we take $\rho \equiv 1$ constant in System (3.2), or equivalently in (3.1), we get the classical homogeneous Euler system. For this system, the global-in-time existence issue in dimension $d = 2$ has been well-known since 1933, due to the pioneering work [103] by Wolibner. For non-homogeneous perfect fluids, see system (3.7), it's still open if its solutions exist globally in time. However, in [36] it's proved that, for initial densities close to a constant state, the lifespan of the corresponding solutions tends to $+\infty$ (independently of the initial velocity field).

Actually, under the same hypothesis as in Theorem 3.4 in dimension 2, we are able to prove a similar result also for our system.

Theorem 3.5. *Let* $d = 2$. *Let us suppose the initial data* ρ_0 *and* u_0 *to be such that* $\varrho_0 := \rho_0 - 1 \in L^2 \cap B^1_{\infty,1}$, *with* $0 < \rho_* \leq \rho_0 \leq \rho^*$, *and* $u_0 \in L^2 \cap B^1_{\infty,1}$.

Then the lifespan of the solution to system (3.2), *given by Theorem 3.4, is bounded from below by the quantity*

$$\frac{K_2}{1 + \|\varrho_0\|^2_{L^2} + \|u_0\|_{L^2 \cap B^1_{\infty,1}}} \log\left(1 + \log\left(\frac{K_2}{\left(1 + \|\varrho_0\|^{\mathfrak{M}}_{B^1_{\infty,1}}\right)\|\varrho_0\|_{B^1_{\infty,1}}}\right)\right), \tag{3.25}$$

for a suitable exponent $\mathfrak{M} > 5$ *and a constant* K_2 *which depends only on* ρ_* *and* ρ^*.

In particular, the lifespan tends to $+\infty$, *i.e. the solution tends to be global, for initial densities close (in the* $B^1_{\infty,1}$ *norm) to a constant state.*

There is another way to get rid of Condition (3.11) imposed on p, that is, we assume an additional smallness condition over the initial density, which ensures that the elliptic equation for the pressure is almost a Poisson's equation, up to a perturbation term.

Theorem 3.6. *A small constant* $c > 0$ *exists such that the following statement holds true.*

Let the triplet $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ *satisfy Condition* (3.10) *and* $p \in (1, +\infty)$. *Let us take an initial density state* ρ_0 *and an initial divergence-free velocity field* u_0 *such that the bounds in* (3.12) *are true, for some positive constants* ρ_*, ρ^* *and* M. *Assume moreover* $\|\rho_0 - 1\|_{B^s_{p,r}} \leq c$.

Then there exist a positive time T *(depending on* $\rho_*, \rho^*, c, M, d, s, p, r$*) and a unique solution* $(\varrho, u, \nabla q)$ *to System* (3.2) *belonging to the space* $I^s_{p,r}(T)$, *defined as the Space* $E^s_{p,r}(T)$ *(recall* (3.13)*), but without condition* $\nabla q \in L^1([0, T]; L^2)$.

Remark 3.5. *It goes without saying that under hypotheses of Theorem 3.4 and Theorem 3.6, a continuation criterion and lifespan lower bound analogous to that of Theorem 3.2 and Theorem 3.3 respectively, can be proved. The corresponding results for the original system* (3.1) *also hold true, similar as in Remark 3.1, which are omitted here.*

3.2 Proof of the first well-posedness result

The aim of this section is to prove our first well-posedness result, i.e. Theorem 3.1, for System (3.2). The a priori estimates we establish here will be the basis of all the results of the present chapter.

The notation f_j will always denote $\Delta_j f$, unless otherwise defined. We also notice here that by embedding results stated in Proposition 0.1 and Remark 0.1, for any $\epsilon > 0$, the following chain of embeddings holds true :

$$L^1_t(B^s_{p,1}) \hookleftarrow \widetilde{L}^1_t(B^{s+\epsilon/2}_{p,1}) \hookleftarrow \widetilde{L}^1_t(B^{s+\epsilon}_{p,\infty}), \tag{3.26}$$

which will be frequently used in our computations.

3.2.1 Linearized equations

In this subsection we want to establish a priori estimates for the linearized equations associated to original System (3.2).

Let us point out here that in order to prove the uniqueness, we will consider the difference of two solutions to System (3.2). Noticing that Equation $(3.2)_2$ for the velocity u will cause one derivative loss, we therefore also have to look for a priori estimates for the unknown in $B^s_{p,r}$, under a weaker condition

$$s > \frac{d}{p}\,, \quad p \in]1,+\infty[\,, \quad r \in [1,+\infty]\,, \qquad \text{with} \qquad r = 1 \quad \text{if} \quad s = 1 + \frac{d}{p} \ \text{ or } \ \frac{d}{p}, \tag{3.27}$$

rather than (3.10).

First of all, we consider the linearized equation for the density term, which is actually the same for both ρ and ϱ. Let us suppose the right hand side of the first equation of (3.2) to be some scalar function f, a more general case which turns out to be useful in the sequel :

$$\begin{cases} \partial_t \varrho + u \cdot \nabla \varrho - \operatorname{div}\,(\kappa\,\nabla\varrho) = f\,, \\ \varrho|_{t=0} = \varrho_0. \end{cases} \tag{3.28}$$

According to Proposition B.2 in Appendix §B.1, one has the following a priori estimate (similar to Proposition 1.1 in Chapter §1) :

$$\|\varrho\|_{\widetilde{L}^\infty_t(B^s_{p,r})} + \|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})} \leq C\, e^{C\mathcal{K}_H(t)} \left(\|\varrho_0\|_{B^s_{p,r}} + \|f\|_{\widetilde{L}^1_t(B^s_{p,r})} \right), \tag{3.29}$$

where we have defined $\mathcal{K}_H(0) = 0$ and

$$\mathcal{K}'_H(t) := 1 + \|\nabla u\|_{B^{\frac{d}{p}}_{p,1} \cap B^{s-1}_{p,r}} + \|\nabla\kappa\|_{B^s_{p,r}} + \|\nabla\kappa\|^2_{L^\infty}\,. \tag{3.30}$$

Here let's just sketch the proof. If we perform the standard process : apply Δ_j to Equation (3.28), take L^2 scalar product between it and $|\varrho_j|^{p-2}\varrho_j$, use Bernstein's inequality (0.41), integrate in time variable, multiply by 2^{js} and take ℓ^r-norm, then it is easy to get for $p \in (1,\infty)$,

$$\begin{aligned} \|\varrho\|_{\widetilde{L}^\infty_t(B^s_{p,r})} + \|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})} \leq C\bigg(& \|\varrho_0\|_{B^s_{p,r}} + 2^{-(s+2)}\,\|\Delta_{-1}\varrho\|_{L^1_t(L^p)} + \|f\|_{\widetilde{L}^1_t(B^s_{p,r})} + \\ & + \left\| 2^{js} \int_0^t \left\|\mathcal{R}^1_j(\tau)\right\|_{L^p} d\tau \right\|_{\ell^r} + \left\| 2^{js} \int_0^t \left\|\mathcal{R}^2_j(\tau)\right\|_{L^p} d\tau \right\|_{\ell^r} \bigg). \end{aligned} \tag{3.31}$$

For the commutator

$$\mathcal{R}_j^1 = [u, \Delta_j] \cdot \nabla \varrho, \tag{3.32}$$

we apply Proposition 0.4 to get

$$\int_0^t \left\| 2^{js} \left\| R_j^1 \right\|_{L^p} \right\|_{\ell^r} d\tau \lesssim \int_0^t \|\nabla u\|_{B^{d/p}_{p,1} \cap B^{s-1}_{p,r}} \|\varrho\|_{B^s_{p,r}} d\tau. \tag{3.33}$$

It rests to deal with the commutator

$$\mathcal{R}_j^2 = \operatorname{div}([\kappa, \Delta_j] \nabla \varrho). \tag{3.34}$$

Since we work in the Besov space $B^s_{p,r}$ with r not necessarily being 1, we have to resort to a new commutator estimate to get the norm $\|\cdot\|_{\widetilde{L}^1_t(\cdot)}$ (instead of $\|\cdot\|_{L^1_t(\cdot)}$) of ϱ. Compared with Proposition 0.4, this is nothing but taking the interpolation before the integral with respect to the time variable. Finally, Proposition A.5 in Appendix §A.2 with $\theta = 1/2$, $\sigma = 1$ supplies us the following estimate :

$$\left\| 2^{js} \int_0^t \left\| \mathcal{R}_j^2 \right\|_{L^p} \right\|_{\ell^r} d\tau \le \frac{C}{\varepsilon} \int_0^t \Big(\|\nabla \kappa\|^2_{L^\infty} + \|\nabla \kappa\|_{B^s_{p,r}} \Big) \|\varrho\|_{B^s_{p,r}} d\tau + \varepsilon \|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}. \tag{3.35}$$

Finally, substituting (3.33), (3.35) into (3.31) gives us the a priori estimate (3.29).

The linearized equation for the velocity reads

$$\begin{cases} \partial_t u + v \cdot \nabla u + \lambda \nabla q = h, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{3.36}$$

where the initial datum u_0, the Lipschitzian transport vector field v, the coefficient λ and the source term h are all smooth and decrease rapidly at infinity.

Firstly, by classic commutator and product estimates, it is easy to find that

$$\|u(t)\|_{\widetilde{L}^\infty_t(B^s_{p,r})} \le \|u_0\|_{B^s_{p,r}} + C \int_0^t V'(\tau) \|u\|_{B^s_{p,r}} d\tau + \|h\|_{\widetilde{L}^1_t(B^s_{p,r})} + \Lambda \|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})}, \tag{3.37}$$

where $V(0) = \Lambda(0) = 0$ and

$$V'(t) = \|\nabla v\|_{B^{d/p}_{p,1} \cap B^{s-1}_{p,r}}, \text{ and } \Lambda(t) = \|\lambda\|_{L^\infty} + \|\nabla \lambda\|_{\widetilde{L}^\infty_t(B^{d/p}_{p,1} \cap B^{s-1}_{p,r})}.$$

Here we have to add the restriction

$$p \in [2, 4],$$

in order to control the pressure ∇q. In fact, we follow the "standard" process : apply div to Equation (3.36) to get elliptic equation for q :

$$\operatorname{div}(\lambda \nabla q) = \operatorname{div} h - \operatorname{div}(v \cdot \nabla u) = \operatorname{div} h - \operatorname{div}(u \cdot \nabla v - u \operatorname{div} v). \tag{3.38}$$

In the above equation, we benefits the divergence-free condition on u in putting the derivatives on the *known* Lipschitzian transport velocity v instead of the *unknown* u, such that the case $u \in B^{d/p}_{p,1}$ is included. Then, we bound the $L^2(\mathbb{R}^d)$-norm of ∇q by $\|h\|_{L^2}$ and $\|u\|_{L^4} \|\nabla v\|_{L^4}$, perform Δ_j and make use of commutator estimate, take interpolation inequality between L^2 and $B^s_{p,r}$ and Young's inequality, notice $\|u\|_{L^4} \le \|u\|_{B^s_{p,r}}$ and $\|\nabla v\|_{L^4} \le V'$, arriving at

$$\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r}) \cap L^1_t(L^2)} \le C(\Lambda) \Big(\|h\|_{\widetilde{L}^1_t(B^s_{p,r}) \cap L^1_t(L^2)} + \|\operatorname{div}(v \cdot \nabla u)\|_{L^1_t(B^{s-1}_{p,r})} + V(t) \|u\|_{\widetilde{L}^\infty_t(B^s_{p,r})} \Big).$$

Now decompose $\|\mathrm{div}\,(v\cdot\nabla u)\|_{B^{s-1}_{p,r}}$ into

$$\|T_{\partial_i v^j}\partial_j u^i + T_{\partial_j u^i}\partial_i v^j + \mathrm{div}\,\big(R(v^j,\partial_j u)\big)\|_{B^{s-1}_{p,r}}, \tag{3.39}$$

which can be controlled, according to Proposition 0.2, by $V'(t)\|\nabla u\|_{B^{s-1}_{p,r}}$.
To conclude, we sum up all the above and make use of Gronwall's inequality, obtaining

$$\|u\|_{\widetilde{L}^\infty_t(B^s_{p,r})} \leq C(\Lambda)\,e^{C(\Lambda)V(t)}\Big(\|u_0\|_{B^s_{p,r}} + \|h\|_{\widetilde{L}^1_t(B^s_{p,r})\cap L^1_t(L^2)}\Big), \tag{3.40}$$

$$\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})\cap L^1_t(L^2)} \leq C(\Lambda)\Big(\|h\|_{\widetilde{L}^1_t(B^s_{p,r})\cap L^1_t(L^2)} + V(t)\|u\|_{\widetilde{L}^\infty_t(B^s_{p,r})}\Big), \tag{3.41}$$

The detailed proof of the above estimates can be found in Appendix §B.3.

3.2.2 Proof of the existence and uniqueness

In this section we will follow the standard procedure to prove the local existence of the solution to System (3.2) : we construct a sequence of approximate solutions, we show uniform bounds and we prove the convergence to a unique solution. We note here some related key points :

- Since we consider also large initial density ρ_0, we will introduce the large linear part ρ_L of the solution ρ as the solution to the free heat equation with the same initial datum, which is explicit and has positive lower bound, such that the remainder part $\bar\rho$ is small and easy to handle.
- In the convergence part, since there are quantities like $\nabla u^n \in B^{s-1}_{p,r}$ appearing in the source term of the equation for the "difference" sequence δu^n, we first show that the built sequence converges to a solution in a space with lower regularity (i.e. in space $E^{d/p}_{p,r}(T)$, see (3.13)) and hence in the desired Besov space because of the uniform bounds for the sequence.
- For any indices s, r satisfying $s > \frac{d}{p}$ or $s \geq \frac{d}{p}$, $r = 1$, and any $\mathcal{C}^1(\mathbb{R}^+)$-function f with $f(1) = 0$, the following inequalities hold :

$$\|uv\|_{B^s_{p,r}} \leq C(d,s,p,r)\|u\|_{B^s_{p,r}}\|v\|_{B^s_{p,r}}\,, \quad \|f(\rho)\|_{B^s_{p,r}} \leq C(f',\|\rho\|_{L^\infty})\|\varrho\|_{B^s_{p,r}}, \tag{3.42}$$

  together wiht their time-dependent version

$$\|uv\|_{\widetilde{L}^\rho_t(B^s_{p,r})} \leq C(d,s,p,r)\|u\|_{\widetilde{L}^{\rho_1}_t(B^s_{p,r})}\|v\|_{\widetilde{L}^{\rho_2}(B^s_{p,r})}\,, \qquad \text{with } \frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2},$$
$$\|f(\rho)\|_{\widetilde{L}^\rho_t(B^s_{p,r})} \leq C(f',\|\rho\|_{L^\infty_t(L^\infty)})\|\varrho\|_{\widetilde{L}^\rho_t(B^s_{p,r})}.$$

In the coming proof, for simplicity we assume $\kappa(1) = \lambda(1) = 1$.

Step 1 – Construction of a sequence of approximate solutions

In this step, we take (s,p,r) such that Conditions (3.10) and (3.11) hold true. Let us define the approximate solution sequence $\{(\varrho^n, u^n, \nabla q^n)\}_{n\geq 0}$ by induction.

Without loss of generality we can assume

$$\frac{\rho_*}{2} \leq S_n\rho_0, \qquad \forall n \in \mathbb{N}\,; \tag{3.43}$$

then, first of all we set $(\varrho^0, u^0, \nabla q^0) := (S_0\varrho_0, S_0u_0, 0)$. Let us note that these functions are smooth and fast decaying at infinity.

Now, we assume by induction that the triple $(\varrho^{n-1}, u^{n-1}, \nabla q^{n-1})$ of smooth and fast decaying functions has been constructed. Let us suppose also that there exists a sufficiently small parameter τ (to be determined later), a positive time T^* (which may depend on τ) and a positive constant C_M (which may depend on M) such that

$$\frac{\rho_*}{2} \leq \rho^{n-1} := 1 + \varrho^{n-1}\,, \quad \|\varrho^{n-1}\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})} \leq C_M\,, \quad \|\varrho^{n-1}\|_{\widetilde{L}^2_{T^*}(B^{s+1}_{p,r})\cap\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} \leq \tau\,, \tag{3.44}$$

$$\|u^{n-1}\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})} \leq C_M,\ \|u^{n-1}\|_{L^2_{T^*}(B^s_{p,r})\cap L^1_{T^*}(B^s_{p,r})} \leq \tau,\ \|\nabla q^{n-1}\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})\cap L^1_{T^*}(L^2)} \leq \tau^{1/2}. \tag{3.45}$$

Let us immediately remark that the above estimates (3.44) and (3.45) obviously hold true for $(\varrho^0, u^0, \nabla q^0)$, if T^* is assumed small enough.

Now we define $(\varrho^n, u^n, \nabla q^n)$ as the unique smooth global solution of the linear system

$$\begin{cases} \partial_t \varrho^n + u^{n-1}\cdot\nabla\varrho^n - \operatorname{div}(\kappa^{n-1}\nabla\varrho^n) = 0, \\ \partial_t u^n + (u^{n-1} + \nabla b^{n-1})\cdot\nabla u^n + \lambda^{n-1}\nabla q^n = h^{n-1}, \\ \operatorname{div} u^n = 0, \\ (\varrho^n, u^n)_{|t=0} = (S_n\varrho_0, S_n u_0), \end{cases} \tag{3.46}$$

where we have set

$$b^{n-1} = b(\rho^{n-1})\,, \quad \kappa^{n-1} = \kappa(\rho^{n-1})\,, \quad \lambda^{n-1} = \lambda(\rho^{n-1})\,, \quad h^{n-1} = h(\rho^{n-1}, u^{n-1})\,.$$

We want to show that also the triplet $(\varrho^n, u^n, \nabla q^n)$ verifies (3.44) and (3.45).

First of all, keeping in mind (3.43), we apply the maximum principle to the linear parabolic equation for ϱ^n, yielding $\rho^n := 1 + \varrho^n \geq \rho_*/2$.

Now, we bring in ϱ_L as the solution of the heat equation with the initial datum $\varrho_0 \in B^s_{p,r}$:

$$\begin{cases} \partial_t\varrho_L - \Delta\varrho_L = 0 \\ (\varrho_L)_{|t=0} = \varrho_0\,. \end{cases}$$

Then, it's easy to see (e.g. applying the estimate (3.29)) that the global solution $\varrho_L = e^{t\Delta}\varrho_0$ satisfies, for any positive time $T < +\infty$ and the related constant $C_T > 0$,

$$\|\varrho_L\|_{\widetilde{L}^\infty_T(B^s_{p,r})} + \|\varrho_L\|_{\widetilde{L}^1_T(B^{s+2}_{p,r})} \leq C_T\,\|\varrho_0\|_{B^s_{p,r}}\,. \tag{3.47}$$

We claim that given τ, we can choose $T^* < +\infty$ such that one has

$$\|\varrho_L\|_{\widetilde{L}^2_{T^*}(B^{s+1}_{p,r})\cap\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} \leq \tau^2. \tag{3.48}$$

Indeed, we can write

$$\|\varrho_L\|_{\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} = \left\| \left(2^{j(s+2)} \int_0^{T^*} \|e^{t\Delta}\Delta_j\varrho_0\|_{L^p}\,dt \right)_j \right\|_{\ell^r}.$$

Let us decompose ϱ_0 into low-frequency, large part $\varrho_{0,l}$ and high-frequency, small part $\varrho_{0,h}$, such that, for some N large enough,

$$\widehat{\varrho}_{0,l} = \widehat{\varrho}_0 \text{ on } 2^N\mathcal{B}, \quad \widehat{\varrho}_{0,l} = 0 \text{ outside } 2^N\mathcal{B}, \quad \varrho_{0,l} + \varrho_{0,h} = \varrho_0, \quad \|\varrho_{0,h}\|_{B^s_{p,r}} \leq \tau^3.$$

Therefore, once one notices by (0.42) that

$$\left\| \left(\int_0^{T^*} 2^{2j} e^{-C\,t\,2^{2j}}\,dt \right)_{j\leq N} \right\|_{\ell^\infty} \leq C(1 - e^{-C2^{2N}T^*}), \quad \left\| \left(\int_0^{T^*} 2^{2j} e^{-C\,t\,2^{2j}}\,dt \right)_{j>N} \right\|_{\ell^\infty} \leq C,$$

we get

$$\begin{aligned}\|\varrho_L\|_{\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} &\leq C(1-e^{-C2^{2N}T^*}) \left\|\left(2^{js}\|\Delta_j\varrho_0\|_{L^p}\right)_{j\leq N}\right\|_{\ell^r} + C\left\|\left(2^{js}\|\Delta_j\varrho_0\|_{L^p}\right)_{j\geq N}\right\|_{\ell^r} \\ &\leq C(1-e^{-C2^{2N}T^*})\,\|\varrho_0\|_{B^s_{p,r}} + C\tau^3.\end{aligned}$$

So one can choose sufficiently small T^* such that $\|\varrho_L\|_{\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} \leq \tau^2$. The term $\|\varrho_L\|_{\widetilde{L}^2_{T^*}(B^{s+1}_{p,r})}$ can be handled in the same way or by interpolation inequality. Hence, our claim (3.48) is proved.

Now we define the sequence $\varrho^n_L = S_n\varrho_L$: it too solves the free heat equation, but with initial data $S_n\varrho_0$. Hence, it too satisfies (3.47) and (3.48), for some $T^*>0$ independent of n.

We next consider the small remainder $\bar{\varrho}^n := \varrho^n - \varrho^n_L$. We claim that it fulfills, for all $n\in\mathbb{N}$,

$$\|\bar{\varrho}^n\|_{\widetilde{L}^2_{T^*}(B^{s+1}_{p,r})} \leq \|\bar{\varrho}^n\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})} + \|\bar{\varrho}^n\|_{\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} \leq \tau^{3/2}\,. \tag{3.49}$$

In fact, $\bar{\varrho}^n = \varrho^n - \varrho^n_L$ solves

$$\begin{cases} \partial_t\bar{\varrho}^n + u^{n-1}\cdot\nabla\bar{\varrho}^n - \mathrm{div}\,(\kappa^{n-1}\nabla\bar{\varrho}^n) = -u^{n-1}\cdot\nabla\varrho^n_L + \mathrm{div}\,((\kappa^{n-1}-1)\nabla\varrho^n_L), \\ \bar{\varrho}^n|_{t=0} = 0. \end{cases} \tag{3.50}$$

So, if we define

$$K^{n-1}(t) := t + \left\|\nabla u^{n-1}\right\|_{L^1_t(B^{s-1}_{p,r})} + \left\|\nabla\kappa^{n-1}\right\|^2_{L^2_t(L^\infty)} + \left\|\nabla\kappa^{n-1}\right\|_{L^1_t(B^s_{p,r})},$$

by (3.29) we infer that

$$\|\bar{\varrho}^n\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})\cap\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} \leq C\,e^{CK^{n-1}(T^*)}\left\|-u^{n-1}\cdot\nabla\varrho^n_L + \mathrm{div}\,((\kappa^{n-1}-1)\nabla\varrho^n_L\right\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})}.$$

Now, inductive assumptions (3.44) and (3.45) tell us that $K^{n-1}(T^*) \leq C\,\tau$ if $T^*\leq\tau$ and τ is sufficiently small. One also refers to Proposition 0.3 and Proposition 0.5 to bound

$$\left\|-u^{n-1}\cdot\nabla\varrho^n_L + \mathrm{div}\,((\kappa^{n-1}-1)\nabla\varrho^n_L)\right\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})}$$

by the quantity (up to a constant factor)

$$\begin{aligned}&\|u^{n-1}\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})}\|\nabla\varrho^n_L\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})} \\ &\quad + \|\varrho^{n-1}\|_{\widetilde{L}^2_{T^*}(B^{s+1}_{p,r})}\|\nabla\varrho^n_L\|_{L^2_{T^*}(L^\infty)} + \|\varrho^{n-1}\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})}\|\nabla\varrho^n_L\|_{\widetilde{L}^1_{T^*}(B^{s+1}_{p,r})},\end{aligned}$$

and hence by $C_M\,\tau^2$. Therefore, (3.49) is proved, and hence (3.44) holds for $\varrho^n = \bar{\varrho}^n + \varrho^n_L$, for sufficiently small τ.

We now want to get (3.45). Our starting point is the a priori estimates (3.40) and (3.41). The estimates for products and action and the embedding result (3.26) give us

$$W^{n-1}(T^*) := \int_0^{T^*}\|\nabla u^{n-1} + \nabla^2 b^{n-1}\|_{B^{s-1}_{p,r}} \leq \|u^{n-1}\|_{L^1_{T^*}(B^s_{p,r})} + C\|\varrho^{n-1}\|_{L^1_{T^*}(B^{s+1}_{p,r})} \leq C\tau,$$

$$\|h^{n-1}\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})} \leq C(\|u^{n-1}\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})}+1)(\|\varrho^{n-1}\|_{\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} + \|\varrho^{n-1}\|^2_{\widetilde{L}^2_{T^*}(B^{s+1}_{p,r})}) \leq C\tau,$$

and also

$$\|h^{n-1}\|_{L^1_{T^*}(L^2)} \leq C(\|u^{n-1}\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})} + \|\varrho^{n-1}\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})})\|\varrho^{n-1}\|_{L^1_{T^*}(B^{s+1}_{p,\infty})} \leq C\tau,$$

by use of these two inequalities :

$$\|\cdot\|_{L^4} \lesssim \|\cdot\|_{B^{s-1}_{p,\infty}} \quad \text{and} \quad \|\nabla\varrho^{n-1}\|_{L^\infty} \leq \|\varrho^{n-1}\|_{B^s_{p,r}}.$$

Hence applying (3.40) and (3.41) to the system (3.46) we deduce

$$\begin{aligned}\|u^n\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})} &\leq C(\|S_n u_0\|_{B^s_{p,r}} + C\tau) \leq C_M,\\ \|\nabla q^n\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})\cap L^1_{T^*}(L^2)} &\leq C\tau + C_M C\tau.\end{aligned}$$

Hence also (3.45) holds true for small τ and T^*.

Step 2 – Convergence of the sequence

In this step we will consider the "difference" sequence

$$(\delta\varrho^n, \delta u^n, \nabla\delta q^n) := (\varrho^n - \varrho^{n-1}, u^n - u^{n-1}, \nabla q^n - \nabla q^{n-1}), \qquad \forall\, n \geq 1\,.$$

Since taking the difference of the transport term $(u+\nabla b)\cdot\nabla u$ will cause one derivative loss because of $\nabla u^n \in B^{s-1}_{p,r}$, we will consider the above difference sequence in the Banach space $E^{d/p}_{p,1}(T^*)$ (recall (3.13) for its definition).

First of all, by System (3.46), $(\delta\varrho^n, \delta u^n, \nabla\delta\pi^n)$ solves

$$\begin{cases} \partial_t\delta\varrho^n + u^{n-1}\cdot\nabla\delta\varrho^n - \mathrm{div}\,(\kappa^{n-1}\nabla\delta\varrho^n) = F^{n-1},\\ \partial_t\delta u^n + (u^{n-1}+\nabla b^{n-1})\cdot\nabla\delta u^n + \lambda^{n-1}\nabla\delta q^n = H^{n-1},\\ \mathrm{div}\,\delta u^n = 0,\\ (\delta\varrho^n, \delta u^n)|_{t=0} = (\Delta_n\varrho_0, \Delta_n u_0), \end{cases} \tag{3.51}$$

where

$$\begin{aligned} F^{n-1} &= -\delta u^{n-1}\cdot\nabla\varrho^{n-1} + \mathrm{div}\,((\kappa^{n-1}-\kappa^{n-2})\nabla\varrho^{n-1}),\\ H^{n-1} &= h^{n-1} - h^{n-2} - (\delta u^{n-1} + \nabla\delta b^{n-1})\cdot\nabla u^{n-1} - (\lambda^{n-1}-\lambda^{n-2})\nabla q^{n-1}. \end{aligned}$$

Next we apply a priori estimates (3.29), (3.40) and (3.41) with $s = d/p$, $p\in[2,4]$ and $r=1$, to $\delta\varrho^n$ and $(\delta u^n, \nabla\delta q^n)$ respectively. The use of uniform bounds (3.44) and (3.45) for the approximated solutions sequence gives us

$$\|\delta\varrho^n\|_{L^\infty_{T^*}(B^{d/p}_{p,1})\cap L^1_{T^*}(B^{d/p+2}_{p,1})} \leq C\left(\|\Delta_n\varrho_0\|_{B^{d/p}_{p,1}} + \|F^{n-1}\|_{L^1_{T^*}(B^{d/p}_{p,1})}\right), \tag{3.52}$$

and

$$\begin{aligned}\|\delta u^n\|_{\widetilde{L}^\infty_{T^*}(B^{d/p}_{p,1})} + \|\nabla\delta q^n\|_{L^1_{T^*}(B^{d/p}_{p,1}\cap L^2)} \leq C(&\|\Delta_n u_0\|_{B^{d/p}_{p,1}} + \|H^{n-1}\|_{L^1_{T^*}(B^{d/p}_{p,1}\cap L^2)}\\ &+ \|u^{n-1}+\nabla b^{n-1}\|_{L^1_{T^*}(B^{d/p}_{p,1})}\|\delta u^n\|_{\widetilde{L}^\infty_{T^*}(B^{d/p}_{p,1})}).\end{aligned} \tag{3.53}$$

Now we use the following fact, coming from Proposition 0.5 : for any smooth function $f=f(\rho)$ and a given sequence $(\rho^m)_m$, if we denote $f^m := f(\rho^m)$ and $\delta f^m := f^m - f^{m-1}$, then

$$\|\delta f^m\|_{B^s_{p,1}} \leq C(\|\varrho^m\|_{B^s_{p,1}}, \|\varrho^{m-1}\|_{B^s_{p,1}})\,\|\delta\varrho^m\|_{B^s_{p,1}}\,, \qquad \text{if} \quad s \geq \frac{d}{p}\,.$$

Therefore, one easily gets

$$\|F^{n-1}\|_{L^1_{T^*}(B^{d/p}_{p,1})} \leq C(\|\delta\varrho^{n-1}\|_{L^2_{T^*}(B^{d/p+1}_{p,1})}\|\varrho^{n-1}\|_{L^2_{T^*}(B^{d/p+1}_{p,1})} + \int_0^{T^*}\|\delta u^{n-1}\|_{B^{d/p}_{p,1}}\|\nabla\varrho^{n-1}\|_{B^{d/p}_{p,1}} + \|\delta\varrho^{n-1}\|_{B^{d/p}_{p,1}}\|\varrho^{n-1}\|_{B^{d/p+2}_{p,1}}),$$

$$\|H^{n-1}\|_{L^1_{T^*}(B^{d/p}_{p,1}\cap L^2)} \leq C(\|\delta\varrho^{n-1}\|_{L^1_{T^*}(B^{d/p+2}_{p,1})} + \|\delta\varrho^{n-1}\|_{L^2_{T^*}(B^{d/p+1}_{p,1})}(\|(\varrho^{n-1},\varrho^{n-2})\|_{L^2_{T^*}(B^{d/p+2}_{p,1})} + \|(u^{n-1},u^{n-2})\|_{L^2_{T^*}(B^{d/p+1}_{p,1})}) + \int_0^{T^*}\|\delta\varrho^{n-1}\|_{B^{d/p}_{p,1}}\|\nabla q^{n-1}\|_{B^{d/p}_{p,1}} + \|\delta u^{n-1}\|_{B^{d/p}_{p,1}}(\|\varrho^{n-1}\|_{B^{d/p+2}_{p,1}} + \|u^{n-1}\|_{B^{d/p+1}_{p,1}})).$$

Hence putting the uniform estimate (3.44) into (3.52) and making use of Gronwall's inequality entail

$$\|\delta\varrho^n\|_{L^\infty_{T^*}(B^{d/p}_{p,1})\cap L^1_t(B^{d/p+2}_{p,1})} \leq C\left(\|\Delta_n\varrho_0\|_{B^{d/p}_{p,1}} + \tau\|\delta\varrho^{n-1}\|_{L^2_{T^*}(B^{d/p+1}_{p,1})} + \int_0^{T^*}\|\delta u^{n-1}\|_{B^{d/p}_{p,1}}\|\nabla\varrho^{n-1}\|_{B^{d/p}_{p,1}}\right).$$

Apply the above estimate to the term $\|\delta\varrho^{n-1}\|_{L^1_{T^*}(B^{d/p+2}_{p,1})}$ appearing in the bound for H^{n-1}. This makes (3.53) become

$$\|\delta u^n\|_{\widetilde{L}^\infty_{T^*}(B^{d/p}_{p,1})} + \|\nabla\delta\pi^n\|_{\widetilde{L}^1_{T^*}(B^{d/p}_{p,1})} \leq C(\|(\Delta_n u_0,\Delta_{n-1}\varrho_0)\|_{B^{d/p}_{p,1}} + \tau\|(\delta\varrho^{n-1},\delta\varrho^{n-2})\|_{L^2_{T^*}(B^{d/p+1}_{p,1})} + \int_0^{T^*}\|(\delta\varrho^{n-1},\delta u^{n-1},\delta u^{n-2})\|_{B^{d/p}_{p,1}}\Big(\|\nabla q^{n-1}\|_{B^{d/p}_{p,1}} + \|(\varrho^{n-1},\varrho^{n-2})\|_{B^{d/p+2}_{p,1}} + \|u^{n-1}\|_{B^{d/p+1}_{p,1}}\Big).$$

Let us now define

$$B^n(t) := \|\delta\varrho^n\|_{L^\infty_t(B^{d/p}_{p,1})} + \|\delta\varrho^n\|_{L^1_t(B^{d/p+2}_{p,1})} + \|\delta u^n\|_{L^\infty_t(B^{d/p}_{p,1})} + \|\nabla\delta q^n\|_{L^1_t(B^{d/p}_{p,1}\cap L^2)};$$

then, from previous inequalities we gather

$$B^n(t) \leq C\,\|(\Delta_{n-1}\varrho_0,\Delta_n\varrho_0,\Delta_n u_0)\|_{B^{d/p}_{p,1}} + \tau^{\frac12}\left(B^{n-1}(t) + B^{n-2}(t)\right) + C\int_0^t\left(B^{n-1} + B^{n-2}\right)D(\sigma)\,d\sigma,$$

with $\|D(t)\|_{L^1([0,T^*])} \leq C$. Let us note that, by spectral localization, there exists a constant $C > 0$ for which, for all $n \geq 0$, we have

$$\|(\Delta_n\varrho_0,\Delta_n u_0)\|_{B^{d/p}_{p,1}} \leq C\,2^{n(d/p)}\,\|(\Delta_n\varrho_0,\Delta_n u_0)\|_{L^p}.$$

Keeping in mind this fact, we claim that the previous estimate implies $\sum_n B^n(t) < +\infty$ uniformly in $[0,T^*]$. As a matter of fact, for all $N \geq 3$ we have

$$\sum_{n=1}^N B^n(t) \leq C\,\|(\varrho_0,u_0)\|_{B^{d/p}_{p,1}} + 2\,\tau^{1/2}\sum_{n=1}^N B^n(t) + \int_0^t\sum_{n=1}^N B^n(\sigma)\,D(\sigma)\,d\sigma + \mathcal{B},$$

where we have denoted by $\mathcal{B}$ a constant which depends only on B^1 and B^2. We can suppose τ to have be chosen small enough such that, moreover, we can absorb the second term of the right hand side into the left hand side. Therefore, Gronwall's inequality entails

$$\sum_{n=1}^{N} B^n(t) \;\leq\; C_{T^*}\;\|(\varrho_0,u_0)\|_{B^{d/p}_{p,1}}\;,$$

and passing to the limit for $N\to+\infty$ we get our claim.

Hence, we gather that the sequence $(\varrho^n,u^n,\nabla q^n)$ is a Cauchy sequence in the functional space $E^{d/p}_{p,1}(T^*)$. Then, it converges to some $(\varrho,u,\nabla\pi)$, which actually belongs to the space $E^s_{p,r}(T^*)$ by Fatou property. Hence, by interpolation, we discover that convergence holds true in every intermediate space between $E^s_{p,r}(T^*)$ and $E^{d/p}_{p,1}(T^*)$, and this is enough to pass to the limit in our equations. So, $(\varrho,u,\nabla\pi)$ is actually a solution of System (3.2).

Uniqueness

Let us now prove the uniqueness part in Theorem 3.1.

We take first two solutions $(\varrho_1,u_1,\nabla q_1)$ and $(\varrho_2,u_2,\nabla q_2)$ in $E^s_{p,r}(T^*)$ with the initial data $(\varrho_{i,0},u_{i,0})$ for $i=1,2$. Then the difference $(\delta\varrho,\delta u,\nabla\delta q)=(\varrho_1-\varrho_2,u_1-u_2,\nabla q_1-\nabla q_2)$ solves

$$\begin{cases} \partial_t\delta\varrho+u_1\cdot\nabla\delta\varrho-\mathrm{div}\,(\kappa_1\nabla\delta\varrho)=-\delta u\cdot\nabla\varrho_2+\mathrm{div}\,((\kappa_1-\kappa_2)\nabla\varrho_2), \\ \partial_t\delta u+(u_1+\nabla b_1)\cdot\nabla\delta u+\lambda_1\nabla\delta q=h_1-h_2-(\delta u+\nabla\delta b)\cdot\nabla u_2-\delta\lambda\nabla q_2, \\ \mathrm{div}\,\delta u=0, \\ (\delta\varrho\,,\,\delta u)|_{t=0}=(\delta\varrho_0\,,\,\delta u_0)=(\varrho_{1,0}-\varrho_{2,0}\,,\,u_{1,0}-u_{2,0})\,, \end{cases} \tag{3.54}$$

with the notation $\kappa_i=\kappa(\varrho_i)$ and analogous for b_i, λ_i and h_i.

As $(\varrho_1,u_1,\nabla q_1),(\varrho_2,u_2,\nabla q_2)\in E^s_{p,r}(T^*)$, then for any $\varepsilon>0$ to be determined later, there exists T_ε such that

$$\|\varrho_i\|_{L^2_{T_\varepsilon}(B^{d/2+2}_{p,1})\cap L^1_{T_\varepsilon}(B^{d/p+3}_{p,1})}\,,\;\|u_i\|_{L^1_{T_\varepsilon}(B^{d/p+1}_{p,1})}\,,\;\|\nabla q_i\|_{L^1_{T_\varepsilon}(B^{d/p+1}_{p,1})}\;\leq\;\varepsilon,\qquad i=1,2.$$

Let us define (as done before)

$$B(t)\;:=\;\|\delta\varrho\|_{L^\infty_t(B^{d/p}_{p,1})}\;+\;\|\delta\varrho\|_{L^1_t(B^{d/p+2}_{p,1})}\;+\;\|\delta u\|_{L^\infty_t(B^{d/p}_{p,1})}\;+\;\|\nabla\delta q\|_{L^1_t(B^{d/p}_{p,1}\cap L^2)}\,.$$

Then, the same proof as in paragraph 3.2.2 shows that

$$B(t)\;\leq\;C\left(\|\delta\varrho_0\|_{B^{d/p}_{p,1}}+\|\delta u_0\|_{B^{d/p}_{p,1}}+\varepsilon B(t)+\int_0^t B(s)D(s)ds\right),$$

which implies, if ε is taken small enough,

$$B(t)\;\leq\;e^C\left(\|\delta\varrho_0\|_{B^{d/p}_{p,1}}\;+\;\|\delta u_0\|_{B^{d/p}_{p,1}}\right).$$

Hence, uniqueness holds for small time. Now, standard continuity arguments show uniqueness on the whole time interval $[0,T^*]$.

3.3 Continuation criterion and lifespan estimate

In this section we establish get a continuation criterion and a lower bound of the lifespan for the local-in-time solutions given by Theorem 3.1. It is only a matter of repeating a priori estimates established previously, but in a more "accurate" way (we use L^∞-norm instead of $B^{s-1}_{p,r}$-norm) for obtaining the continuation criterion, whereas in a "rough" way (we use (3.60), (3.61) below) for bounding the lifespan from below.

3.3.1 Proof of the continuation criterion

From the arguments after Theorem 3.2, it rests us to prove Lemma 3.1. As argued before Theorem 3.2, in order to ensure $u \in \widetilde{L}^\infty(B^s_{p,r})$, the source terms such as $u\cdot\nabla^2 b$, $\operatorname{div}(\nabla b \otimes \nabla b)$ require at least $u, \nabla b \in L^\infty(L^\infty)$ since $\nabla^2 b \in \widetilde{L}^1(B^s_{p,r})$ only, and also $\nabla^2 b \in L^2(L^\infty)$ since $\nabla b \in \widetilde{L}^2(B^s_{p,r})$. Similarly, the transport term $\nabla b \cdot \nabla u$ requires $\nabla u \in L^2(L^\infty)$. Therefore in the following, we do not have to strive for optimal inequalities for the commutators or products and sometimes it simplifies the argument.

Proof of Lemma 3.1. It's only a matter of estimating the commutators in a more accurate way. Roughly speaking, in the estimates we use L^∞-norm instead of Besov norm, in order to require lower regularity. Hence we will use thoroughly the product estimate in Proposition A.3 and the commutator estimate in Proposition A.5, both involving the $\widetilde{L}^1(B^s_{p,r})$-norm.

Let us consider the density term. Our starting point is (3.28), with this time $f = 0$. It is easy to see that by Proposition 0.4, we have instead of (3.33), the following :

$$\int_0^t \left\| 2^{js} \, \left\|\mathcal{R}^1_j\right\|_{L^p} \right\|_{\ell^r} \, d\tau \;\leq\; \int_0^t \Big(\|\nabla u\|_{L^\infty} \, \|\varrho\|_{B^s_{p,r}} \,+\, \|\nabla\varrho\|_{L^\infty} \, \|\nabla u\|_{B^{s-1}_{p,r}} \Big) \, d\tau \,. \tag{3.55}$$

Similarly, Proposition A.5 with $\theta = \sigma = 1/2$ issues instead of (3.35),

$$\left\| 2^{js} \int_0^t \left\|\mathcal{R}^2_j\right\|_{L^p} \right\|_{\ell^r} \, d\tau \;\leq\; \frac{C}{\varepsilon} \int_0^t \|\nabla\varrho\|^2_{L^\infty} \, \|\varrho\|_{B^s_{p,r}} \, d\tau \,+\, \varepsilon \, \|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})} \,. \tag{3.56}$$

So, if ε is small enough, $p \in (1, \infty)$, putting these inequalities in (3.31) and keeping in mind $\|\Delta_{-1}\varrho\|_{L^1_t(L^p)} \leq C \|\varrho\|_{L^1_t(B^s_{p,r})}$, we get the following estimate :

$$\begin{aligned} \|\varrho\|_{\widetilde{L}^\infty_t(B^s_{p,r}) \cap \widetilde{L}^1_t(B^{s+2}_{p,r})} \;&\leq\; C \bigg(\|\varrho_0\|_{B^s_{p,r}} \,+\, \int_0^t \|\varrho\|_{B^s_{p,r}} \, d\tau \,+\, \int_0^t \|\nabla\varrho\|_{L^\infty} \, \|\nabla u\|_{B^{s-1}_{p,r}} \, d\tau \\ &\qquad +\, \int_0^t \big(\|\nabla u\|_{L^\infty} \,+\, \|\nabla\varrho\|^2_{L^\infty} \big) \, \|\varrho\|_{B^s_{p,r}} \, d\tau \bigg) \,. \end{aligned} \tag{3.57}$$

Let us now consider velocity field and pressure term : we have

$$\|u\|_{\widetilde{L}^\infty_t(B^s_{p,r})} \;\leq\; C \left(\|u_0\|_{B^s_{p,r}} + \|\lambda \, \nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} + \|h\|_{\widetilde{L}^1_t(B^s_{p,r})} + \left\| 2^{js} \int_0^t \|\mathcal{R}_j(\tau)\|_{L^p} \, d\tau \right\|_{\ell^r} \right) , \tag{3.58}$$

where we have set, $\mathcal{R}_j \,:=\, [u + \nabla b(\rho) \,,\, \Delta_j] \cdot \nabla u$. To control the commutator term, a direct application of Proposition 0.4 yields, for $s > 0$,

$$\left\| 2^{js} \int_0^t \| \, [u, \Delta_j] \cdot \nabla u \|_{L^p} \, d\tau \right\|_{\ell^r} \;\leq\; C \int_0^t \|\nabla u\|_{L^\infty} \, \|u\|_{B^s_{p,r}} \, d\tau \,.$$

For the second part $[\nabla b(\rho) \,,\, \Delta_j] \cdot \nabla u$, as in the density case, one can resort to Proposition A.5 with $\theta = 1$, $\sigma = 1/2$ to get,

$$\begin{aligned} \left\| 2^{js} \int_0^t \| \, [\nabla b(\rho), \Delta_j] \cdot \nabla u \|_{L^p} \, d\tau \right\|_{\ell^r} \;&\leq\; C \int_0^t \left\|\nabla^2 \varrho\right\|_{L^\infty} \, \|u\|_{B^s_{p,r}} \, d\tau \,+ \\ &\qquad + C \int_0^t \|\nabla u\|^2_{L^\infty} \, \|\varrho\|_{B^s_{p,r}} \, d\tau \,+\, \varepsilon \|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})} \,. \end{aligned}$$

Let us immediately consider the pressure term :

$$\begin{aligned} \|\lambda(\rho) \, \nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} \;&\leq\; C \, \|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} \,+\, \|(\lambda(\rho) - \lambda(1)) \, \nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} \\ &\leq\; C \left(\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} \,+\, \int_0^t \|\nabla q\|_{L^\infty} \, \|\varrho\|_{B^s_{p,r}} \, d\tau \right) . \end{aligned}$$

To bound $\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})}$, we argue as follows : we take the divergence of second equation of System (3.2), we localize in frequencies by operator Δ_j and we perform a weighed summation. Hence we discover that

$$\begin{aligned}\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} &\lesssim \int_0^t \|\Delta_{-1}\nabla q\|_{L^p} + \|h\|_{\widetilde{L}^1_t(B^s_{p,r})} + \|\mathrm{div}\,((u+\nabla b)\cdot\nabla u)\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})} + \\ &\quad + \left\|2^{j(s-1)}\int_0^t \|\mathrm{div}\,([\lambda,\Delta_j]\nabla q)\|_{L^p}\,d\tau\right\|_{\ell^r}.\end{aligned}$$

Again, Inequality (0.48) in Proposition 0.4, entails the control for the commutator term :

$$\left\|2^{j(s-1)}\int_0^t \|\mathrm{div}\,([\lambda,\Delta_j]\nabla q)\|_{L^p}d\tau\right\|_{\ell^r} \lesssim \int_0^t \Big(\|\nabla\varrho\|_{L^\infty}\,\|\nabla q\|_{B^{s-1}_{p,r}} + \|\nabla\varrho\|_{B^{s-1}_{p,r}}\,\|\nabla q\|_{L^\infty}\Big)\,d\tau.$$

Accordint to (3.26), interpolation inequality helps us to control the above first term on the right hand side as follows :

$$\begin{aligned}\int_0^t \|\nabla\varrho\|_{L^\infty}\,\|\nabla q\|_{B^{s-1}_{p,r}}\,d\tau &\lesssim \|\nabla\varrho\|_{L^\infty_t(L^\infty)}\|\nabla q\|_{\widetilde{L}^1_t(B^{s-1+\epsilon}_{p,r})}, \quad \text{for some fixed} \quad \epsilon\in(0,1),\\ &\le C_\varepsilon\|\nabla\varrho\|^{(s+\sigma)/(1-\epsilon)}_{L^\infty_t(L^\infty)}\|\nabla q\|_{\widetilde{L}^1_t(B^{-\sigma}_{p,\infty})} + \varepsilon\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})}\end{aligned}$$

with any positive (large) σ. We point out here that for any $\sigma\in\mathbb{R}$, there exists C depending on σ such that the following holds :

$$\|\Delta_{-1}\nabla q\|_{L^1_t(L^p)} \le C\|\nabla q\|_{\widetilde{L}^1_t(B^{-\sigma}_{p,\infty})}.$$

Next, by the divergence-free condition over u and Proposition A.3, we infer ($\mathrm{div}\,u=0$ implies that we only need $s>0$)

$$\begin{aligned}\|\mathrm{div}\,((u+\nabla b)\cdot\nabla u)\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})} &\le \|\nabla u:\nabla u\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})} + \left\|\nabla^2 b:\nabla u\right\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})}\\ &\le C\bigg(\int_0^t \left(\|\nabla u\|_{L^\infty} + \left\|\nabla^2\varrho\right\|_{L^\infty}\right)\|u\|_{B^s_{p,r}}d\tau + \\ &\quad + \int_0^t \|\nabla u\|^2_{L^\infty}\,\|\varrho\|_{B^s_{p,r}}\,d\tau + \varepsilon\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}\bigg).\end{aligned}$$

The control of the non-linear term h is quite similar as above. We come back to Equation (3.3) and consider its pieces one by one. As usual, estimate for products in Proposition A.3 ensures that

$$\begin{aligned}\left\|u\cdot\nabla^2 b\right\|_{\widetilde{L}^1_t(B^s_{p,r})} &\lesssim \int_0^t \left\|\nabla^2\varrho\right\|_{L^\infty}\|u\|_{B^s_{p,r}}\,d\tau + \|u\|_{L^\infty_t(L^\infty)}\,\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})},\\ \|(u\cdot\nabla\lambda)\,\nabla a\|_{\widetilde{L}^1_t(B^s_{p,r})} &\lesssim \int_0^t \|u\|^2_{L^\infty}\,\|\nabla\varrho\|^2_{L^\infty}\,\|\varrho\|_{B^s_{p,r}}\,d\tau\\ &\quad + \|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})} + \int_0^t \|\nabla\varrho\|^2_{L^\infty}\,\|u\|_{B^s_{p,r}}\,d\tau,\end{aligned}$$

and

$$\|(\nabla b\cdot\nabla\lambda)\,\nabla a\|_{\widetilde{L}^1_t(B^s_{p,r})} \lesssim \int_0^t \|\nabla\varrho\|^4_{L^\infty}\,\|\varrho\|_{B^s_{p,r}}\,d\tau + \|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}.$$

Moreover, the last element of h can be treated in the following way :

$$\begin{aligned}\|\mathrm{div}\,(\nabla b\otimes\nabla b)\|_{\widetilde{L}^1_t(B^s_{p,r})} &= \left\|\Delta b\,\nabla b + \nabla b\cdot\nabla^2 b\right\|_{\widetilde{L}^1_t(B^s_{p,r})}\\ &\lesssim \int_0^t \left\|\nabla^2\varrho\right\|^2_{L^\infty}\,\|\varrho\|_{B^s_{p,r}}\,d\tau + \Big(1+\|\nabla\varrho\|_{L^\infty_t(L^\infty)}\Big)\,\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}.\end{aligned}$$

Let us collect all these informations : up to multiplication by a constant, we gather

$$\begin{aligned}\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} \;\leq\; & C(\|\nabla\varrho\|_{L^\infty_t(L^\infty)},s,\sigma)\|\nabla q\|_{L^1_t(B^{-\sigma}_{p,\infty})} \\ & + \int_0^t \left(\|\nabla u\|_{L^\infty} + \left\|\nabla^2\varrho\right\|_{L^\infty} + \|\nabla\varrho\|^2_{L^\infty} + \|\nabla\varrho\|^4_{L^\infty}\right)\|u\|_{B^s_{p,r}}d\tau \\ & + \int_0^t \left(\|\nabla u\|^2_{L^\infty} + \|u\|^2_{L^\infty}\|\nabla\varrho\|^2_{L^\infty} + \|\nabla^2\varrho\|^2_{L^\infty} + \|\nabla q\|_{L^\infty}\right)\|\varrho\|_{B^s_{p,r}}d\tau \\ & + \Big(1 + \|\nabla\varrho\|_{L^\infty_t(L^\infty)} + \|u\|_{L^\infty_t(L^\infty)}\Big)\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}\,,\end{aligned} \tag{3.59}$$

and the same control actually holds true also for $\|h\|_{\widetilde{L}^1_t(B^s_{p,r})}$.

In the end, we discover from (3.58) that $\|u\|_{\widetilde{L}^\infty_t(B^s_{p,r})}$ satisfies also Inequality (3.59), just with an additional term $\|u_0\|_{B^s_{p,r}}$ on the right-hand side. Recalling Estimate (3.57) for the density, we replace $\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}$ in Inequality (3.59) by the right-hand side of it.

Thus, we can sum up (3.57) and the (modified) estimate (3.59) for the velocity u, yielding the thesis by Gronwall's Lemma. □

3.3.2 Lower bounds for the lifespan

The aim of the present subsection is analyzing the lifespan of the solutions to system (3.2). We want to show, as carefully as possible, the dependence of the lifespan T on the initial data. This can be done repeating the a priori estimates previously established, but in a "rough" way. For example, thanks to Conditions (3.10) and (3.11), we will use extensively the following inequalities :

$$\|ab\|_{B^{s-i}_{p,r}} \lesssim \|a\|_{B^{s-i}_{p,r}}\|b\|_{B^{s-i}_{p,r}}, \quad \|a^2\|_{B^s_{p,r}} \lesssim \|a\|_{L^\infty}\|a\|_{B^s_{p,r}}, \quad \|a\|_{L^\infty}, \|\nabla a\|_{L^\infty} \lesssim \|a\|_{B^s_{p,r}}, \tag{3.60}$$

with $i = 0, 1$, and

$$\|ab\|_{L^2} \leq \|a\|_{L^4}\|b\|_{L^4} \lesssim \|a\|_{B^{s-1}_{p,r}}\|b\|_{B^{s-1}_{p,r}}. \tag{3.61}$$

In order to control the pressure term, it is useful to separate low and high frequencies. By use of Bernstein's inequalities, we get

$$\|\nabla q\|_{B^s_{p,r}} \;\leq\; \|\Delta_{-1}\nabla q\|_{B^s_{p,r}} \;+\; \|(\mathrm{Id}\,-\,\Delta_{-1})\nabla q\|_{B^s_{p,r}} \;\leq\; C\left(\|\nabla q\|_{L^2} \,+\, \|\Delta q\|_{B^{s-1}_{p,r}}\right). \tag{3.62}$$

Let us also point out that, thanks to Theorem 3.2 and embedding results, without any loss of generality from now on throughout this subsection we assume

$$s \,=\, 1 + \frac{d}{p} \qquad \text{and} \qquad p = 4, \quad r \,=\, 1.$$

In particular, for any $\sigma \in \mathbb{R}$ we have $\|\cdot\|_{\widetilde{L}^1_t(B^\sigma_{p,1})} \sim \|\cdot\|_{L^1_t(B^\sigma_{p,1})}$, and this really simplifies our computations. We also assume that C always denote some large enough constant.

For notation convenience, we define $R_0 := \|\varrho_0\|_{B^s_{p,r}}$ and $U_0 := \|u_0\|_{B^s_{p,r}}$,

$$R(t) \,=\, \|\varrho\|_{\widetilde{L}^\infty_t(B^s_{p,r})}\,, \qquad S(t) \,:=\, \|\varrho\|_{L^1_t(B^{s+2}_{p,r})} \qquad \text{and} \qquad U(t) \,:=\, \|u\|_{\widetilde{L}^\infty_t(B^s_{p,r})}\,.$$

From (3.57), we infer that

$$R(t) \,+ S(t) \,\leq\, C\left(R_0 \,+\, \int_0^t R(\tau)\big(U(\tau)+1\big)\,d\tau \,+\, \int_0^t R^3(\tau)\,d\tau\right).$$

Now, if we define

$$T_R \;:=\; \sup\left\{t\,>\,0\;\middle|\;\int_0^t R^3(\tau)\,d\tau\;\leq\;2\,R_0\right\}\,, \tag{3.63}$$

by Gronwall's Lemma, we get

$$R(t)\,+S(t)\;\leq\;C\,R_0\,\exp\left(C\left(t\,+\,\int_0^t U(\tau)\,d\tau\right)\right)\,. \tag{3.64}$$

So, from now on we work with $t\in[0,T_R]$.

Let us now focus on the velocity field and the pressure term. We immediately point out that the nonlinear term h will make density terms with critical regularity appear. Therefore, unlike what has been done in previous sections, we decided not to use systematically interpolation inequalities to isolate the term $S(t)$ and instead, we consider $S'(t)$, which controls high regularity of the density.

Our starting point is estimate (3.58) :

$$\|u\|_{\widetilde{L}^\infty_t(B^s_{p,r})}\,\leq\,C\left(\|u_0\|_{B^s_{p,r}}\,+\,\|h\,-\,\lambda\,\nabla q\|_{L^1_t(B^s_{p,r})}\,+\,\left\|2^{js}\int_0^t\|\mathcal{R}_j(\tau)\|_{L^p}\,d\tau\right\|_{\ell^r}\right)\,,$$

where, as before, $\mathcal{R}_j\;:=\;[u+\nabla b(\rho),\Delta_j]\cdot\nabla u$. Now we apply again Proposition 0.4 to control the commutator term, and we get

$$U(t)\,\leq\,C\biggl(U_0\,+\,\int_0^t\|h\|_{B^s_{p,r}}\,d\tau\,+\,\int_0^t\|\lambda\,\nabla q\|_{B^s_{p,r}}\,d\tau\,+\,\int_0^t U^2(\tau)\,d\tau\,+\,\int_0^t U(\tau)\,S'(\tau)\,d\tau\biggr). \tag{3.65}$$

Let us now establish some stationary estimates, which will be useful to complete the bound for $U(t)$. We will often use (3.60) to deal with products.

(i) Let us start with $\|h\|_{B^s_{p,r}}$. We consider its terms one by one, and finally, one infers that

$$\|h\|_{B^s_{p,r}}\;\leq\;C\left(U\,S'\,+\,U\,R\,S'\,+\,R^2\,S'\,+\,R\,S'\right).$$

(ii) Now we focus on the Besov norm of the pressure term. First of all, we have

$$\|\lambda\,\nabla q\|_{B^s_{p,r}}\;\leq\;C\,(1+R)\,\|\nabla q\|_{B^s_{p,r}}\,.$$

Now, we can handle $\|\nabla q\|_{B^s_{p,r}}$. Since $r=1$ here, we do not have to worry about the order of taking the integration with respect to time L^1_t and taking the ℓ^r norm of the sequence. Hence, we can first "forget about" the time variable and discuss $\|\nabla q\|_{B^s_{p,r}}$. According to (3.62), we first write the equation for the Laplacian of the pressure, by taking div to Equation $(3.2)_2$:

$$-\,\Delta q\;=\;\nabla(\log\lambda)\,\cdot\,\nabla q\,+\,\lambda^{-1}\,\mathrm{div}\,\left(v\cdot\nabla u\,-\,h\right)\,.$$

It immediately follows

$$\begin{aligned}\|\Delta q\|_{B^{s-1}_{p,1}}\;&\leq\;\|\nabla(\log\lambda)\,\cdot\,\nabla q\|_{B^{s-1}_{p,1}}\,+\,(\|\varrho\|_{B^{s-1}_{p,1}}+1)\left(\|\mathrm{div}\,(v\cdot\nabla u)\|_{B^{s-1}_{p,1}}+\|h\|_{B^s_{p,1}}\right)\\ &\leq\;\|\nabla\varrho\|_{B^{s-1}_{p,1}}\,\|\nabla q\|_{B^{s-1}_{p,1}}\,+\,(\|\varrho\|_{B^{s-1}_{p,1}}+1)\left(\|\nabla v:\nabla u\|_{B^{s-1}_{p,1}}+\|h\|_{B^s_{p,1}}\right)\end{aligned}$$

(where we used also that $\mathrm{div}\,u=0$). So, applying the interpolation inequality for the embeddings $B^s_{p,r}\hookrightarrow B^{s-1}_{p,1}\hookrightarrow B^{-d/2}_{p,1}$, then the inclusion $L^2\hookrightarrow B^{-d/2}_{p,1}$ and finally the Young inequality, we gather

$$\|\nabla q\|_{B^s_{p,1}}\,\leq\,C\left((1+R^\delta)\,\|\nabla q\|_{L^2}+(1+R)\|\nabla v:\nabla u\|_{B^{s-1}_{p,1}}+(1+R)\|h\|_{B^s_{p,1}}\right),$$

for some $\delta > 1$, thanks to $\operatorname{div} u \equiv 0$. Then, we have also

$$\|\lambda \nabla q\|_{B^s_{p,1}} \leq C \Big((1+R)\,(1+R^\delta)\,\|\nabla q\|_{L^2} + (1+R)^2\, U(U+S') + (1+R)^2\,\|h\|_{B^s_{p,1}} \Big) . \tag{3.66}$$

(iii) Now we focus on the L^2 norm of the pressure term. Applying Lemma 0.9 to equation (3.9) immediately gives

$$\|\nabla q\|_{L^2} \leq C \left(\|h\|_{L^2} + \|(u+\nabla b)\cdot \nabla u\|_{L^2} \right) .$$

Thanks to (3.61), we find

$$\|(u+\nabla b)\cdot\nabla u\|_{L^2} \leq C\,\|u+\nabla b\|_{B^{s-1}_{p,r}}\,\|\nabla u\|_{B^{s-1}_{p,r}} \leq C\left(U^2 + U\,R\right) .$$

Using also (3.60) and (3.61), it's easy to control the L^2 norm of h :

$$\|h\|_{L^2} \leq C\Big(US' + UR^2 + R^3 + RS'\Big).$$

Before putting all these inequalities together into (3.65), let us note the following point. Due to the fact that $\delta > 1$, for any $m \geq 0$ we have

$$R^m \left(1 + R^\delta\right) \leq C\left(1 + R^{m+\delta}\right) ,$$

and we have to deal with only a finite number of powers of R (the biggest m is actually 4, by previous stationary estimates). Hence, if we define (denoting by l some big enough exponent)

$$\mathcal{E}(t) := 1 + R_0^l \exp\!\left(C \left(t + \int_0^t U(\tau)\,d\tau \right)\right),$$

thanks to (3.64) we can bound all the terms of the form R^m and $R^m(1+R^\delta)$ which occurs in our estimate by $C\,\mathcal{E}$, with $C > 0$ suitably large.

Now, we are ready to complete the bound for $\|\lambda\,\nabla q\|_{B^s_{p,r}}$. From the previous steps, we get

$$\|h\|_{B^s_{p,r}},\, \|\lambda\,\nabla q\|_{B^s_{p,r}} \leq C\Big(\mathcal{E}\,U^2 + \mathcal{E}\,U + \mathcal{E}\,U\,S' + \mathcal{E}\,S' \Big) .$$

Now we put this last inequality in (3.65). Using again the fact that $\mathcal{E} \geq 1$, we find

$$U(t) \leq C \left(U_0 + \int_0^t \left(\mathcal{E}\,U^2 + \mathcal{E}\,U + \mathcal{E}\,U\,S' + \mathcal{E}\,S' \right) d\tau \right) \tag{3.67}$$

for all $t \in [0, T_R]$. Therefore, if we define

$$T_U := \sup \left\{ t > 0 \,\middle|\, \int_0^t \left(\mathcal{E}\,U^2 + \mathcal{E}\,U + \mathcal{E}\,U\,S' + \mathcal{E}\,S' \right) d\tau \leq 2\,U_0 \right\} , \tag{3.68}$$

then in $[0,T_R] \cap [0,T_U]$ we get $U(t) \leq C\,U_0$ and we manage to close the estimates.

So, our next goal is to prove that, if we define T as the quantity in (3.19), then $T \leq \min\{T_R, T_U\}$, i.e. both conditions in (3.63) and (3.68) are fulfilled.

Let's first tackle the case $U_0 \equiv 1$ and then we will see how to deal with the general case by use of Proposition 3.1. First of all, from (3.64), (3.67) and the definition of $\mathcal{E}$, in the interval $[0,T_R] \cap [0,T_U]$ we have (if $U_0 \equiv 1$)

$$R(t) + S(t) \leq C\,R_0\,e^{Ct} \quad \text{and} \quad \mathcal{E}(t) \leq \left(1 + R_0^l\right) e^{Ct} , \tag{3.69}$$

for some suitable positive constant C. Note that in $[0,T_R]\cap[0,T_U]$ we have

$$\int_0^t \left(\mathcal{E}\,U^2\,+\,\mathcal{E}\,U\,+\,\mathcal{E}\,U\,S'\,+\,\mathcal{E}\,S'\right)d\tau\;\leq\;2C\,(1+R_0^{l+1})\,e^{2Ct}\,.$$

Therefore, T_U, defined by (3.68), is bigger than any time t for which the quantity $2C\,(1+R_0^{l+1})\,e^{2Ct}$ is controlled by 2. Hence, for suitable values of K_1 and with $\mathfrak{X}=l+1$, $U_0\equiv 1$ in (3.19), we have that $T\leq T_U$. Now, let us consider T_R. By (3.69), it's easy to see that the left-hand side of the condition in (3.63) can be controlled in the following way :

$$\int_0^t R^3(\tau)\,d\tau\;\leq\;C\,R_0^3\int_0^t e^{3C\tau}\,d\tau\;\leq\;C\,\frac{R_0^3}{3\,C}\,e^{3Ct}\,.$$

Hence, T_R is greater than any time t for which $R_0^l\,e^{3Ct}\;\leq\;\widetilde{C}$, for some convenient constant $\widetilde{C}$. But this condition is always verified in $[0,T]$, up to change the values of K_1 in (3.19) (recall that $\mathfrak{X}>2$). Therefore, in the end we gather that $T\leq\min\{T_R,T_U\}$, and Theorem 3.3 is proved for $U_0\equiv 1$.

For arbitrary initial data ρ_0-1,u_0 in $B^{1+d/4}_{4,1}$, we have a unique local-in-time solution $(\rho,u,\nabla q)$. Let us define

$$\varepsilon^2=\|u_0\|_{B^{1+d/4}_{4,1}}\,,\qquad\Big(\rho_0^\varepsilon,u_0^\varepsilon\Big)(x):=\Big(\rho_0,\varepsilon^{-1}u_0\Big)(\varepsilon^{-1}x)\,.$$

Then $U_0^\varepsilon:=\|u_0^\varepsilon\|_{B^{1+d/4}_{4,1}}\equiv 1$ and hence the lifespan of the solution $(\rho^\varepsilon,u^\varepsilon,\nabla q^\varepsilon)$ defined by (3.18) is bigger than

$$K_1\log\Big(\frac{K_1}{1+\|\rho_0^\varepsilon-1\|^{\mathfrak{X}}_{B^{1+d/4}_{4,1}}}\Big)=K_1\log\Big(\frac{K_1}{1+\big(\varepsilon^{-1}\|\rho_0-1\|_{B^{1+d/4}_{4,1}}\big)^{\mathfrak{X}}}\Big)\,.$$

In virtue of Proposition 3.1, the lifespan of the solution $(\rho,u,\nabla q)$ is larger than

$$\frac{K_1}{\varepsilon^2}\log\Big(\frac{K_1}{1+\big(\varepsilon^{-1}\|\rho_0-1\|_{B^{1+d/4}_{4,1}}\big)^{\mathfrak{X}}}\Big)\,.$$

This completes the proof of Theorem 3.3, thanks to Remark 3.3.

3.4 Finite energy case

We want now to deal with finite energy initial data, under Condition (3.10) and $p\in(1,+\infty]$. Thanks to Proposition B.3 in the appendix, we can take the Lebesgue exponent p to be ∞ (the endpoint case) in solution space $E^s_{p,r}$. In fact, according to Remark B.6, the a priori estimate (3.29) still holds true, with $\mathcal{K}'_H(t)$ in (3.30) replaced by the following with $p=\infty$:

$$1+\|u\|^2_{L^p}+\|u\|_{B^s_{p,r}}+\|\nabla\kappa\|^2_{L^p}+\|\nabla\kappa\|_{B^s_{p,r}}.\tag{3.70}$$

And for the sake of simplicity, we from now on denote the above expression by $\mathcal{K}'(t)$.

The proof of Theorem 3.4 is just as the proof presented in Subsection §3.2.2, with some changes pertaining to the energy. We will follow the standard process in proving the local existence result also : we construct a sequence of approximate smooth solutions which have uniform bounds and then we show the convergence to a unique solution. In each step we will try to sketch the analysis first (even if a little long) and then present the proof into details.

Let us make some simplification. Sometimes a few estimates may depend on the existing time T^*, and hence a priori we suppose that $T^*\leq 1$. We also assume that all the constants appearing in the sequel, such as C,C_M,C_E, are bigger than 1. We always denote $\delta b^n=b(\rho^n)-b(\rho^{n-1})$ and $\delta a^n=a(\rho^n)-a(\rho^{n-1})$.

3.4.1 Construction of a sequence of approximate solutions

As usual, after fixing $(\varrho^0, u^0, \nabla q^0) = (\varrho_0, u_0, 0)$, we consider inductively the n-th approximate density ϱ^n to be the unique global solution of the following linear system (we do not regularize the initial data)

$$\begin{cases} \partial_t \varrho^n + u^{n-1} \cdot \nabla \varrho^n - \mathrm{div}\,(\kappa^{n-1} \nabla \varrho^n) = 0, \\ \varrho^n|_{t=0} = \varrho_0, \end{cases} \tag{3.71}$$

with $\kappa^{n-1} = \kappa(\rho^{n-1})$ and the n-th approximate velocity and pressure $(u^n, \nabla q^n)$ satisfying

$$\begin{cases} \partial_t u^n + (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1} \nabla \rho^n) \cdot \nabla u^n + \lambda^n \nabla q^n = h^{n-1}, \\ \mathrm{div}\, u^n = 0, \\ u^n|_{t=0} = u_0, \end{cases} \tag{3.72}$$

where we have set as before, $\lambda^n = \lambda(\rho^n)$ and the same h^{n-1} defined in (3.46) :

$$h^{n-1} = (\rho^{n-1})^{-1} \Big(\Delta b(\rho^{n-1}) \nabla a(\rho^{n-1}) + u^{n-1} \cdot \nabla^2 a(\rho^{n-1}) + \nabla b(\rho^{n-1}) \cdot \nabla^2 a(\rho^{n-1}) \Big). \tag{3.73}$$

We pay attention that, compared with System (3.46), the coefficients $u^{n-1} - \kappa^{n-1}(\rho^n)^{-1} \nabla \rho^n$ and λ^n of System (3.72) here are chosen to keep accordance with Equation (3.71), in order to get the energy identity for u^n. Indeed, noticing $\mathrm{div}\, u^n = 0$ and Equation (3.71) for ρ^n, we can take the $L^2(\mathbb{R}^d)$-inner product between Equation $(3.72)_1$ and $\rho^n u^n$, getting at least formally [1]

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \rho^n |u^n|^2 = \int_{\mathbb{R}^d} \rho^n h^{n-1} \cdot u^n. \tag{3.74}$$

Furthermore, the initial data for $\{\varrho^n\}$ are chosen the same. We will see later that this choice enables us to estimate the difference sequence $\{\delta \varrho^n = \varrho^n - \varrho^{n-1}\}_{n \geq 2}$ in Space $\mathcal{C}_{T^*}(H^1)$ (T^* denotes the existence time), although the initial density ϱ_0 belongs to L^2 only. This estimate makes it possible to bound the difference of the source term $h^n - h^{n-1}$ (notice it involves terms such as $\Delta \delta b^n \nabla a^n$) in Space $L^1_{T^*}(L^2)$. Therefore the convergence of the velocity sequence in Space $\mathcal{C}_{T^*}(L^2)$ follows. We will explain the convergence process in detail in Subsection §3.4.2 below.

In this paragraph, we aim at proving the existence of the solution sequence $(\rho^n, u^n, \nabla q^n)$ and uniform estimates for it. We want to show estimates (3.44) and (3.45), with a change pertaining to ∇q^n :

$$\|\nabla q^n\|_{\widetilde{L}^1_{T^*}(B^s_{p,r}) \cap L^1_{T^*}(L^2)} \leq \tau^{\frac{1}{2s_p}}, \quad \text{with} \quad s_p = \max\left\{ s\,,\, s - \frac{d}{p} + \frac{d}{2} \right\}. \tag{3.75}$$

We also prove the following inductive estimate involving energy :

$$\|\varrho^n\|_{L^\infty_{T^*}(L^2)} + \|\nabla \varrho^n\|_{L^2_{T^*}(L^2)} + \|u^n\|_{L^\infty_{T^*}(L^2)} \leq C_E E_0, \tag{3.76}$$

with $E_0 := \|\varrho_0\|_{L^2} + \|u_0\|_{L^2}$ and some constant C_E depending on d, s, ρ_*, ρ^*.

In fact, the subtlety still comes out when dealing with the pressure term ∇q^n. Getting inductive estimate (3.75) relies on the divergence-free condition of u^n, which helps us to write the equations for q^n in different forms, such as (3.77), (3.78) and (3.79) in the following. Informations on low and high frequencies issue from these equations separately. This yields estimates for ∇q itself finally, thanks to (3.62).

1. We can also follow the scheme (3.46) which, gives the following by taking L^2-inner product between $(3.46)_2$ and $\rho^{n-1} u^n$

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \rho^{n-1} |u^n|^2 + \frac{1}{2} \int_{\mathbb{R}^d} \mathrm{div}\, \Big(\rho^{n-1}(u^{n-2} - u^{n-1}) - (\kappa^{n-2} - \kappa^{n-1}) \nabla \rho^{n-1} \Big) |u^n|^2 = \int_{\mathbb{R}^d} \rho^{n-1} h^{n-1} \cdot u^n.$$

For simplicity we choose iterative linear systems (3.71) and (3.72) here.

Let us sketch the idea which supplies the control of ∇q.
In SubSection §3.2.2, where $p \in [2,4]$, according to the divergence-form elliptic equation for q^n, the quantity $\|\nabla q^n\|_{L^1_t(L^2)}$ (involving the low frequency information) can be controlled by a simple use of Hölder's Inequality $\|fg\|_{L^2} \le \|f\|_{L^4}\|g\|_{L^4}$ to the quadratic "source" terms. Now, we also apply "div" to Equation $(3.72)_1$, getting the following equation for q^n :

$$\operatorname{div}(\lambda^n \nabla q^n) = \operatorname{div}\Big(h^{n-1} - (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1}\nabla\rho^n)\cdot\nabla u^n\Big). \tag{3.77}$$

But here, thanks to Estimate (3.76), one uses $\|fg\|_{L^2} \le \|f\|_{L^2}\|g\|_{L^\infty}$ to bound $\|\nabla q^n\|_{L^1_{T^*}(L^2)}$ and hence the low frequencies, when $p \ge 2$.

If $p \in (1,2)$, instead, we multiply Equation $(3.72)_1$ by ρ^n and then apply "div" to it. Recalling that $\operatorname{div}\partial_t u^n = 0$, we get a Laplace equation for q^n :

$$\Delta q^n = -\operatorname{div}\Big((\rho^n - 1)\partial_t u^n + (\rho^n u^{n-1} - \kappa^{n-1}\nabla\rho^n)\cdot\nabla u^n\Big) + \operatorname{div}(\rho^n h^{n-1}). \tag{3.78}$$

One observes that if $p \in (1,2)$, then

$$B^{s-1-d/q}_{p,r} \hookrightarrow L^q, \quad \forall q \ge 2.$$

Thus we perform $\|fg\|_{L^p} \le \|f\|_{L^2}\|g\|_{L^{p^*}}$ with $1/p^* + 1/2 = 1/p$ to these right-hand side quadratic terms. Since ϱ^n is already in L^q for all $q \ge 2$ according to Maximum Principle and Energy Estimate, this requires $\|\partial_t u^n\|_{L^1_t(L^2)}$. Luckily it is related to $\|\nabla q^n\|_{L^1_t(L^2)}$ by Equation $(3.72)_1$.

In order to control the high frequencies, it is enough to show that Δq^n is in $\widetilde{L}^1_t(B^{s-1}_{p,r})$. Let us rewrite Equation (3.77) as

$$\Delta q^n = \nabla\log\rho^n\cdot\nabla q^n + \rho^n\operatorname{div}\Big(h^{n-1} - (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1}\nabla\rho^n)\cdot\nabla u^n\Big). \tag{3.79}$$

The quantity $\|\nabla q^n\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})}$ can be interpolated between $\widetilde{L}^1_t(B^s_{p,r})$ and $L^1_t(L^p)$ when $p \in (1,2)$ (resp. $L^1_t(L^2)$ when $p \in [2,\infty)$ since $L^2 \hookrightarrow B^{d/p-d/2}_{p,\infty}$) and the index s_p will come out from that interpolation.

As in [36], let us also notice that the limit functional space $B^{s-1}_{\infty,r}$ with $s = r = 1$ is no longer an algebra as spaces $B^{s-1}_{p,r}$ with $p < +\infty$. But we still have the following product estimates by considering paraproducts and remainder separately :

$$\|fg\|_{B^0_{\infty,1}} \le C\|f\|_{B^\epsilon_{\infty,1}}\|g\|_{B^0_{\infty,1}}, \quad \forall\epsilon > 0. \tag{3.80}$$

Thus we just have to modify the above interpolation a little to make quantity $\|\nabla q^n\|_{\widetilde{L}^1_{T^*}(B^\epsilon_{\infty,1})}$ appear. We refer also to [36] for the inequality (due to $\operatorname{div} u^n = 0$)

$$\|\operatorname{div}(u^{n-1}\cdot\nabla u^n)\|_{B^0_{\infty,1}} \le C\|u^{n-1}\|_{B^1_{\infty,1}}\|\nabla u^n\|_{B^0_{\infty,1}}. \tag{3.81}$$

Now let us realize the above analysis.
Since $(\varrho^0, u^0, \nabla q^0) = (\varrho_0, u_0, 0)$, then by choosing small T^*, inductive Estimates (3.44), (3.45), (3.75) and (3.76) all hold for $n = 0$. Next we suppose $(\varrho^{n-1}, u^{n-1}, \nabla q^{n-1})$ to belong to the functional space $\widetilde{E}^s_{p,r}$ defined by

$$\Big(\mathcal{C}(\mathbb{R}^+; B^s_{p,r}\cap L^2)\cap L^2_{\rm loc}(H^1)\cap\widetilde{L}^1_{\rm loc}(B^{s+2}_{p,r})\Big)\times\Big(\mathcal{C}(\mathbb{R}^+; B^s_{p,r}\cap L^2)\Big)^d\times\Big(\widetilde{L}^1_{\rm loc}(B^s_{p,r})\cap L^1_{\rm loc}(L^2)\Big)^d, \tag{3.82}$$

and such that the inductive assumptions hold. We just have to show that the n-th unknown $(\varrho^n, u^n, \nabla q^n)$ defined by System (3.71) and (3.72) belongs to the same space, satisfying the same conditions.

Instead of working with ϱ^n and u^n together, we first consider ϱ^n independently. By smoothing out the initial datum and coefficients, applying (3.29) (with $\mathcal{K}_H$ defined by (3.70)) and then showing the convergence, we can get the unique global solution ϱ^n of the linear system (3.71). The process is quite standard and we omit it. One observes that Estimates (3.29) imply $\varrho^n \in \widetilde{L}^\infty_t(B^s_{p,r}) \cap \widetilde{L}^1_t(B^{s+2}_{p,r})$ for any finite $t > 0$. Thus in particular $\varrho^n \in \mathcal{C}(\mathbb{R}^+; B^s_{p,r})$.

On the other hand, since $u^{n-1} \in L^\infty_{\text{loc}}(L^\infty)$ and $\operatorname{div} u^{n-1} = 0$ (noticing Equation $(3.72)_2$), energy inequality (3.20) for ϱ^n follows. Thus $\varrho^n \in \mathcal{C}(\mathbb{R}^+; L^2) \cap L^2_{\text{loc}}(H^1)$, and we necessarily have

$$\|\varrho^n\|_{L^\infty_{T^*}(L^2)} + \|\nabla \varrho^n\|_{L^2_{T^*}(L^2)} \le \frac{1}{2} C_E \|\varrho_0\|_{L^2}.$$

It is easy to see that, by Maximum Principle,

$$\rho_* \le 1 + \varrho^n(t,x) = \rho^n(t,x) \le \rho^*, \quad \forall t \in \mathbb{R}^+,\ x \in \mathbb{R}^d.$$

As in SubSection §3.2.2, we introduce ϱ_L to be the solution of the free heat equation with initial datum ϱ_0, which satisfies (3.47) and (3.48) also. Correspondingly, the remainder $\bar{\varrho}^n := \varrho^n - \varrho_L$ solves System (3.50), with ϱ_L instead of ϱ^n_L. Estimate (3.29) thus implies that for $p \in (1, \infty]$,

$$\|\bar{\varrho}^n\|_{\widetilde{L}^\infty_t(B^s_{p,r}) \cap \widetilde{L}^1_t(B^{s+2}_{p,r})} \le \left(C^{n-1}(t)\, e^{C^{n-1}(t)\mathcal{K}^{n-1}(t)} \right) \|f^n\|_{\widetilde{L}^1_t(B^s_{p,r})},$$

where

$$f^n = -u^{n-1} \cdot \nabla \bar{\varrho}^n - u^{n-1} \cdot \nabla \varrho_L + \operatorname{div}((\kappa^{n-1} - 1)\nabla \varrho_L).$$

Here $C^{n-1}(t)$ depends also on $\|\varrho^{n-1}\|_{L^\infty_t(B^s_{p,r})}$ when $p = \infty$, and by embedding result, we can take

$$\mathcal{K}^{n-1}(t) := t + \|(u^{n-1}, \nabla \kappa^{n-1})\|^2_{L^2_t(L^\infty)} + \|(u^{n-1}, \nabla \kappa^{n-1})\|_{L^1_t(B^s_{p,r})}.$$

Since inductive assumption (3.44) holds, we derive on the whole time interval $[0, T^*]$,

$$C^{n-1}(t)\, e^{C^{n-1}(t)\mathcal{K}^{n-1}(t)} \le C_{\mathcal{K}},$$

for some constant $C_{\mathcal{K}}$ depending only [2] on M. Furthermore, product estimates, interpolation inequality and Estimate (3.48) ensure that

$$\begin{aligned}
\|f^n\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})} &\le C \|u^{n-1}\|_{\widetilde{L}^2_{T^*}(B^s_{p,r})} \|\nabla \bar{\varrho}^n\|_{\widetilde{L}^2_{T^*}(B^s_{p,r})} + C C_M \tau^2 \\
&\le C_\varepsilon C_{\mathcal{K}} \|u^{n-1}\|^2_{\widetilde{L}^2_{T^*}(B^s_{p,r})} \|\bar{\varrho}^n\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})} + C_{\mathcal{K}}^{-1} \varepsilon \|\bar{\varrho}^n\|_{\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} + C C_M \tau^2.
\end{aligned}$$

Therefore the smallness statement (3.49) pertaining to $\bar{\varrho}^n$ is verified and hence inductive assumption (3.44) holds for ϱ^n.

As to solve System (3.72), a convenient way is to view it as a transport equation of the velocity u^n. For each finite time t, if there exists a constant C_t (depending on t, $\|(\varrho^k, u^{n-1}, \nabla q^{n-1})\|_{E^s_{p,r}(t)}$, $\|\varrho^k\|_{L^\infty_t(L^2) \cap L^2_t(H^1)}$, $k = n-1, n$, and $\|u^{n-1}\|_{L^\infty_t(L^2)}$) such that the $\widetilde{L}^1_t(B^s_{p,r})$-norm of the "source" term $-\lambda^n \nabla q^n + h^{n-1}$ is bounded by $C_t(1 + \mathcal{U}^n(t))$ with

$$\mathcal{U}^n(t) = \|u^n\|_{\widetilde{L}^\infty_t(B^s_{p,r})},$$

2. In fact, $C_{\mathcal{K}}$ also depends on T^*, which can be "omitted" since we have supposed a priori $T^* \le 1$.

then a standard proof gives a unique global solution $u^n \in \mathcal{C}(\mathbb{R}^+; B^s_{p,r})$ (see the proof of Theorem 3.19 in [5]). If furthermore $\nabla q^n, h^n \in L^1_{\text{loc}}(L^2)$, then $u^n \in \mathcal{C}(\mathbb{R}^+; L^2)$, according to Energy Identity (3.74). Therefore, it reduces to get a priori estimates of $\nabla q^n, h^{n-1}$ in $\widetilde{L}^1_{\text{loc}}(B^s_{p,r}) \cap L^1_{\text{loc}}(L^2)$, by use of C_t and $\mathcal{U}^n$. In fact, it will immediately follow by observing the estimates in the demonstration below of the inductive estimates for u^n and ∇q^n.

In the following, we a priori demonstrate inductive estimates for u^n and ∇q^n. The idea is that, by terms of τ and

$$\Pi^n \triangleq \|\nabla q^n\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})},$$

we are going to bound the following quantities in the following order :

$$\begin{aligned}\|u^n\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})} &\rightsquigarrow \|\nabla q^n\|_{L^1_{T^*}(L^2)} \rightsquigarrow \|\partial_t u^n\|_{L^1_{T^*}(L^2)} \\ &\rightsquigarrow \|\nabla q^n\|_{L^1_{T^*}(L^p)} \rightsquigarrow \|\nabla q^n\|_{\widetilde{L}^1_{T^*}(B^{s-1}_{p,r})} \rightsquigarrow \|\Delta q^n\|_{\widetilde{L}^1_{T^*}(B^{s-1}_{p,r})},\end{aligned}$$

which include the informations of both low frequency $\|\nabla q^n\|_{L^1_{T^*}(L^p)}$ and high frequency $\|\Delta q^n\|_{\widetilde{L}^1_{T^*}(B^{s-1}_{p,r})}$. One also notices that if $p \geq 2$, then $\|\nabla q^n\|_{L^1_{T^*}(L^2)}$ readily offers bound on $\|\Delta_{-1}\nabla q^n\|_{L^1_{T^*}(L^p)}$, as a consequence of Bernstein inequalities.

(i) First of all, according to Equation $(3.72)_1$, we have

$$\|u^n\|_{\widetilde{L}^\infty_t(B^s_{p,r})} \leq Ce^{C\mathcal{W}^{n-1}(t)}\left(\|u_0\|_{B^s_{p,r}} + \|h^{n-1} - \lambda^n \nabla q^n\|_{\widetilde{L}^1_t(B^s_{p,r})}\right),$$

where

$$\mathcal{W}^{n-1}(t) := \int_0^t \|u^{n-1} - \kappa^{n-1}(\rho^n)^{-1}\nabla\rho^n\|_{B^s_{p,r}}.$$

By virtue of inductive assumptions (3.44) for ϱ^{n-1}, ϱ^n and (3.45) for u^{n-1}, we easily derive

$$\mathcal{W}^{n-1}(T^*),\ \|h^{n-1}\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})} \leq CC_M\tau.$$

Thus (as $\|\lambda^n - 1\|_{\widetilde{L}^\infty_t(B^s_{p,r})} \leq C\,C_M$) for sufficiently small parameter τ,

$$\|u^n\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})} \leq C\Big(\|u_0\|_{B^s_{p,r}} + CC_M\tau + CC_M\|\nabla q^n\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})}\Big) \leq CC_M\Big(1 + \Pi^n\Big). \tag{3.83}$$

(ii) Secondly, inductive assumptions (3.45) and (3.76) imply

$$\|h^{n-1}\|_{L^1_{T^*}(L^2)} \leq CC_E E_0\tau.$$

Hence, according to $\|\nabla u^n\|_{L^2_{T^*}(L^\infty)} \leq (T^*)^{1/2}\|u^n\|_{L^\infty_{T^*}(B^s_{p,r})}$ and Estimate (3.83), if $(T^*)^{1/2} \leq \tau$, then Equation (3.77) entails

$$\|\nabla q^n\|_{L^1_{T^*}(L^2)} \leq CC_E E_0(\tau + \tau\|u^n\|_{L^\infty_{T^*}(B^s_{p,r})}) \leq CC_E C_M E_0\tau\Big(1 + \Pi^n\Big). \tag{3.84}$$

Correspondingly, $\|\partial_t u^n\|_{L^1_{T^*}(L^2)}$ is bounded also by above, with some change of the constant C.

(iii) Next we also want to bound $\|\nabla q^n\|_{L^1_{T^*}(L^p)}$ for $p \in (1,2)$, which controls the low frequency. It relies on Equation (3.78). Firstly, since

$$\Delta b(\rho^{n-1}),\ \nabla^2 a(\rho^{n-1}) \in \widetilde{L}^2_{T^*}(B^{s-1}_{p,r}) \hookrightarrow L^2_{T^*}(L^{p^*}), \quad \text{with} \quad p^* = \frac{2p}{2-p} \geq 2,$$

we have furthermore

$$\|h^{n-1}\|_{L^1_{T^*}(L^p)} \leq CC_E E_0\tau.$$

Similarly, we have

$$\|(\rho^n-1)\partial_t u^n\|_{L^1_{T^*}(L^p)},\ \|(\rho^n u^{n-1}-\kappa^{n-1}\nabla\rho^n)\cdot\nabla u^n\|_{L^1_{T^*}(L^p)}\leq CC_E C_M^2 E_0\tau(1+\Pi^n).$$

Hence Equation (3.78) implies, for $p\in(1,2)$,

$$\|\nabla q^n\|_{L^1_{T^*}(L^p)}\leq CC_E C_M^2 E_0\tau(1+\Pi^n)\leq C_\Pi(1+\Pi^n)\tau, \tag{3.85}$$

with notation C_Π denoting some constant depending on $s,d,p,\rho_*,\rho^*,C_E,C_M,E_0$, to be precisely determined later.

(iv) Now, one observes that Estimate (3.85) for ∇q^n implies moreover

$$\|\nabla q^n\|_{\widetilde{L}^1_{T^*}(B^{s-1}_{p,r})}\leq C\|\nabla q^n\|^{1/s}_{L^1_{T^*}(L^p)}\|\nabla q^n\|^{(s-1)/s}_{\widetilde{L}^1_{T^*}(B^s_{p,r})}\leq C_\Pi(1+\Pi^n)\tau^{1/s},\quad p\in(1,2),$$

with some change of constant C_Π. On the other hand, if $p\geq 2$, then Embedding $L^2(\mathbb{R}^d)\hookrightarrow B^{d/p-d/2}_{p,\infty}(\mathbb{R}^d)$ and (3.84) also ensure similar interpolation inequality, for any $\eta\in[0,1)$:

$$\begin{aligned}\|\nabla q^n\|_{\widetilde{L}^1_{T^*}(B^{s-1+\eta}_{p,r})}&\leq C\|\nabla q^n\|^{\frac{1-\eta}{s_p}}_{L^1_{T^*}(B^{d/p-d/2}_{p,\infty})}\|\nabla q^n\|^{\frac{s_p-1+\eta}{s_p}}_{\widetilde{L}^1_{T^*}(B^s_{p,r})}\\&\leq C\|\nabla q^n\|^{\frac{1-\eta}{s_p}}_{L^1_{T^*}(L^2)}(\Pi^n)^{\frac{s_p-1+\eta}{s_p}}\\&\leq C_\Pi(1+\Pi^n)\tau^{\frac{1-\eta}{s_p}}.\end{aligned}$$

(v) At last, Equation (3.79) and Estimates (3.80) and (3.81) ensure for some $\eta\in(0,1)$ (still with some appropriated constant C_Π)

$$\|\Delta q^n\|_{\widetilde{L}^1_{T^*}(B^{s-1}_{p,r})}\leq CC_M\|\nabla q^n\|_{\widetilde{L}^1_{T^*}(B^{s-1+\eta}_{p,r})}+CC_M\tau+CC_M\tau^2\leq C_\Pi(1+\Pi^n)\tau^{\frac{1-\eta}{s_p}},$$

which, together with (3.62), (3.84), (3.85) and the definition of Π^n, implies, for τ and T^* small enough,

$$\|\nabla q^n\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})}\leq\tau^{1/2s_p}.$$

Therefore by virtue of Estimates (3.83) and (3.84), inductive assumption (3.45) and (3.75) for u^n and ∇q^n follow respectively.

(vi) From above, Energy Identity (3.74) holds and hence we have

$$\|u^n\|_{L^\infty_{T^*}(L^2)}\leq C(\|u_0\|_{L^2}+\|h^{n-1}\|_{L^1_{T^*}(L^2)})\leq\frac{1}{2}C_E E_0.$$

Inductive assumption (3.76) is then verified.

3.4.2 Convergence Part

When we turn to establish the above sequence converging to the solution, as in SubSection §3.2.2, we introduce the difference sequence

$$(\delta\varrho^n,\ \delta u^n,\ \nabla\delta q^n)\ =\ (\varrho^n-\varrho^{n-1},\ u^n-u^{n-1},\ \nabla q^n-\nabla q^{n-1}),\quad n\geq 1.$$

When $n\geq 2$, it verifies the following system :

$$\begin{cases}\partial_t\delta\varrho^n+u^{n-1}\cdot\nabla\delta\varrho^n-\operatorname{div}(\kappa^{n-1}\nabla\delta\varrho^n)=F^{n-1},\\ \partial_t\delta u^n+(u^{n-1}-\kappa^{n-1}\nabla\log\rho^n)\cdot\nabla\delta u^n+\lambda^n\nabla\delta q^n\ =\ H_e^{n-1},\\ \operatorname{div}\delta u^n=0,\\ (\delta\varrho^n,\delta u^n)|_{t=0}=(0,0),\end{cases} \tag{3.86}$$

where

$$
\begin{aligned}
F^{n-1} &= -\delta u^{n-1}\cdot\nabla\varrho^{n-1} + \mathrm{div}\,(\delta\kappa^{n-1}\nabla\varrho^{n-1}),\\
H_e^{n-1} &= \delta h^{n-1} - (\delta u^{n-1} - \delta\kappa^{n-1}\nabla\log\rho^n \;-\; \kappa^{n-2}\nabla\delta(\log\rho)^n)\cdot\nabla u^{n-1} - \delta\lambda^n\nabla q^{n-1},
\end{aligned}
$$

with

$$
\delta\kappa^{n-1} = \kappa^{n-1}-\kappa^{n-2},\quad \delta h^{n-1} = h^{n-1}-h^{n-2},\quad \delta(\log\rho)^n = \log\rho^n-\log\rho^{n-1},\quad \delta\lambda^n = \lambda^n-\lambda^{n-1}.
$$

Firstly, since in the case $p\in(1,4]$, we have the embedding $B^s_{p,r}\hookrightarrow B^{1+d/4}_{4,1}$, thus we just have to establish that $\{(\varrho^n,u^n,\nabla q^n)_{n\geq 0}\}$ is a Cauchy sequence in the functional space $E^{d/4}_{4,1}(T^*)$ (see (3.13) for the definition). In fact, we just have to follow exactly SubSection §3.2.2, with some necessary changes in the coefficients and source terms varying from System (3.51) to System (3.86).

If p is big enough and we consider the limit case $(s,p,r)=(1,\infty,1)$, then it's no more true that $\{(\varrho^n,u^n,\nabla q^n)_{n\geq 0}\}$ is a Cauchy sequence in $E^0_{\infty,1}(T^*)$. Indeed, applying "div" to Equation $(3.86)_2$ entails

$$
\mathrm{div}\,(\lambda^n\nabla\delta q^n) = \mathrm{div}\,H_e^{n-1} \;-\; \mathrm{div}\,((u^{n-1}-\kappa^{n-1}\,\nabla\log\rho^n)\cdot\nabla\delta u^n). \tag{3.87}
$$

Since the following estimate

$$
\|\mathrm{div}\,(u^{n-1}\cdot\nabla\delta u^n)\|_{B^{d/p-1}_{p,1}} \leq \|u^{n-1}\|_{B^{d/p+1}_{p,1}}\|\nabla\delta u^n\|_{B^{d/p-1}_{p,1}}
$$

only holds for $p<\infty$. Thus we cannot have $\nabla\delta q^n\in L^1_{T^*}(B^0_{\infty,1})$ in general.

Therefore in the general case, we have to consider the difference sequence in the energy space. One wants H_e^{n-1} in $L^1_{T^*}(L^2)$. One pays attention to the following terms in H_e^{n-1} :

$$
(\rho^{n-1})^{-1}\Delta\delta b^{n-1}\nabla a^{n-1}\qquad\text{and}\qquad(\rho^{n-1})^{-1}\nabla b^{n-2}\cdot\nabla^2\delta a^{n-1}.
$$

We only have $\nabla a^{n-1},\nabla b^{n-1}$ in $L^\infty_{T^*}(L^\infty)$, and thus one requires $\Delta\delta b^{n-1},\nabla^2\delta a^{n-1}$ in $L^1_{T^*}(L^2)$ and hence $\delta\varrho^{n-1}$ in $L^1_{T^*}(H^2)$.

Firstly, it is easy to see that $F^{n-1}\in L^2_{\mathrm{loc}}(L^2)$ and hence taking L^2 inner product between Equation $(3.86)_1$ and $\delta\varrho^n$ gives controls on $\delta\varrho^n$ by $\delta\varrho^{n-1},\delta u^{n-1}$, with small coefficient if restricted on small time interval $[0,T^*]$.
Next, thanks to the null initial datum for $\delta\varrho^n$, by taking derivation of Equation $(3.86)_1$, we expect to get energy estimates for $\nabla\delta\varrho^n$. In fact, the equation for $\nabla\delta\varrho^n$, $n\geq 2$ reads

$$
\partial_t\nabla\delta\varrho^n + u^{n-1}\cdot\nabla^2\delta\varrho^n - \mathrm{div}\,(\kappa^{n-1}\nabla^2\delta\varrho^n) = -\nabla\delta\varrho^n\cdot\nabla u^{n-1} + \mathrm{div}\,(\nabla\delta\varrho^n\otimes\nabla\kappa^{n-1}) + \nabla F^{n-1}\,. \tag{3.88}
$$

The first two terms of the right-hand side are of lower order, while the third one, is in $L^2_{\mathrm{loc}}(H^{-1})$, thus taking L^2 inner product works. Therefore, $\delta\varrho^n\in L^\infty_{\mathrm{loc}}(H^1)\cap L^2_{\mathrm{loc}}(H^2)$ ensures $\delta\varrho^n\in\mathcal{C}(\mathbb{R}^+;L^2)$ and $H_e^{n-1}\in L^1_{\mathrm{loc}}(L^2)$. Thus energy inequality for δu^n also follows and its energy is bounded in terms of $\delta\varrho^{n-1},\delta\varrho^n,\delta u^{n-1}$.

Thanks to the small time T^*, we thus can demonstrate that $\{(\rho^n,u^n)\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{C}([0,T^*];L^2)$ and hence converges to some unique limit (ρ,u). Furthermore $\|\rho-\rho^n\|_{L^\infty_{T^*}(H^1)\cap L^2_{T^*}(H^2)}$ goes to 0 as n goes to ∞. Then by use of the high regularity of the solution sequence, we can show that (ρ,u) is a solution.

Now we begin to make the above analysis in detail.
Our goal is to demonstrate that $\{\rho^n-\rho_0\}_n,\{u^n\}_n$ are Cauchy sequences in $\mathcal{C}([0,T^*];L^2)$ and the limit really solves System (3.71) and System (3.72).

Since $\delta\varrho^n \in \widetilde{E}^s_{p,r}$ (see (3.82) for definition), we can take $L^2(\mathbb{R}^d)$ inner product between Equation $(3.86)_1$ and $\delta\varrho^n$, $n \geq 2$, entailing

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |\delta\varrho^n|^2 + \int_{\mathbb{R}^d} \kappa^{n-1}|\nabla\delta\varrho^n|^2 = -\int_{\mathbb{R}^d} \delta u^{n-1}\cdot\nabla\rho^{n-1}\,\delta\varrho^n - \int_{\mathbb{R}^d} \delta\kappa^{n-1}\nabla\rho^{n-1}\cdot\nabla\delta\varrho^n.$$

Thus integration in time and Young's Inequality give

$$\begin{aligned}\|\delta\varrho^n\|_{L^\infty_{T^*}(L^2)} + \|\nabla\delta\varrho^n\|_{L^2_{T^*}(L^2)} &\leq C(\|\delta u^{n-1}\|_{L^2_{T^*}(L^2)} + \|\delta\varrho^{n-1}\|_{L^2_{T^*}(L^2)})\|\nabla\varrho^{n-1}\|_{L^2_{T^*}(L^\infty)}\\ &\leq C\tau(\|\delta\varrho^{n-1}\|_{L^\infty_{T^*}(L^2)} + \|\delta u^{n-1}\|_{L^2_{T^*}(L^2)}). \qquad (3.89)\end{aligned}$$

According to the analysis above, $\delta\varrho^n \in \mathcal{C}(\mathbb{R}^+;L^2)\cap L^2_{\text{loc}}(H^2)$, $n\geq 2$, (notice that it is not clear that $\delta\varrho^1 = \rho^1 - \rho_0 \in L^2_{\text{loc}}(H^2)$), we still can take $L^2(\mathbb{R}^d)$ inner product between Equation (3.88) and $\nabla\delta\varrho^n$:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla\delta\varrho^n|^2 + \int_{\mathbb{R}^d} \kappa^{n-1}|\nabla^2\delta\varrho^n|^2\\ &\qquad = -\int \nabla\delta\varrho^n\cdot\nabla u^{n-1}\cdot\nabla\delta\varrho^n + \nabla\delta\varrho^n\cdot\nabla^2\delta\varrho^n\cdot\nabla\kappa^{n-1} + F^{n-1}\Delta\delta\varrho^n\,.\end{aligned}$$

Integrating in time also implies

$$\begin{aligned}&\|\nabla\delta\varrho^n\|_{L^\infty_{T^*}(L^2)} + \|\nabla^2\delta\varrho^n\|_{L^2_{T^*}(L^2)} \leq\\ &\quad C\Big(\|\nabla u^{n-1}\|_{L^2_{T^*}(L^\infty)}\|\nabla\delta\varrho^n\|_{L^\infty_{T^*}(L^2)} + \|\nabla\rho^{n-1}\|_{L^2_{T^*}(L^\infty)}\|\nabla\delta\varrho^n\|_{L^\infty_{T^*}(L^2)} + \|F^{n-1}\|_{L^2_{T^*}(L^2)}\Big).\end{aligned}$$

Since

$$\|F^{n-1}\|_{L^2_{T^*}(L^2)} \leq C(\tau\|\delta u^{n-1}\|_{L^2_{T^*}(L^2)} + \tau\|\delta\varrho^{n-1}\|_{L^\infty_{T^*}(L^2)} + C_M\|\nabla\delta\varrho^{n-1}\|_{L^2_{T^*}(L^2)}),$$

thus for the above small τ and T^*,

$$\|\nabla\delta\varrho^n\|_{L^\infty_{T^*}(L^2)} + \|\nabla^2\delta\varrho^n\|_{L^2_{T^*}(L^2)} \leq C\tau\|(\delta\varrho^{n-1},\delta u^{n-1})\|_{L^\infty_{T^*}(L^2)} + CC_M\|\nabla\delta\varrho^{n-1}\|_{L^2_{T^*}(L^2)}.$$

We can substitute (3.89) into the term $\|\nabla\delta\varrho^{n-1}\|_{L^2_{T^*}(L^2)}$ above, and then sum up these two inequalities, entailing

$$\|\delta\varrho^n\|_{L^\infty_{T^*}(H^1)} + \|\nabla\delta\varrho^n\|_{L^2_{T^*}(H^1)} \leq CC_M\tau\|(\delta\varrho^{n-1},\delta\varrho^{n-2},\delta u^{n-1},\delta u^{n-2})\|_{L^\infty_{T^*}(L^2)}. \qquad (3.90)$$

Now we turn to δu^n. We rewrite δh^{n-1} as

$$\begin{aligned}&\frac{1}{\rho^{n-1}}(\Delta\delta b^{n-1}\nabla a^{n-1} + \Delta b^{n-2}\nabla\delta a^{n-1} + \delta u^{n-1}\cdot\nabla^2 a^{n-1} + u^{n-2}\cdot\nabla^2\delta a^{n-1} + \nabla\delta b^{n-1}\cdot\nabla^2 a^{n-1}\\ &+\nabla b^{n-2}\cdot\nabla^2\delta a^{n-1}) + \Big((\rho^{n-1})^{-1} - (\rho^{n-2})^{-1}\Big)(\Delta b^{n-2}\cdot\nabla a^{n-2} + u^{n-2}\cdot\nabla^2 a^{n-2} + \nabla b^{n-2}\cdot\nabla^2 a^{n-2}).\end{aligned}$$

By virtue of $\|\Delta\delta b^{n-1}\|_{L^2} \leq C\|\delta\varrho^{n-1}\|_{H^2}$ and

$$\|\Delta b^{n-1}\|_{L^2_{T^*}(L^\infty)} \leq C\,\big\||\nabla\rho^{n-1}|^2\big\|_{L^2_{T^*}(L^\infty)} + C\|\Delta\rho^{n-1}\|_{L^2_{T^*}(L^\infty)} \leq CC_M\tau,$$

we have also from the above inductive estimates that

$$\|\delta h^{n-1}\|_{L^1_{T^*}(L^2)} \leq CC_M\tau(\|\delta\varrho^{n-1}\|_{L^2_{T^*}(H^2)} + \|\delta u^{n-1}\|_{L^\infty_{T^*}(L^2)}) + CC_E E_0\tau\|\delta\varrho^{n-1}\|_{L^\infty_{T^*}(L^2)}.$$

Similarly,

$$\|H_e^{n-1}\|_{L^1_{T^*}(L^2)} \leq C(C_M + C_E E_0)\tau(\|\delta\varrho^{n-1}\|_{L^2_{T^*}(H^2)} + \|\delta u^{n-1}\|_{L^\infty_{T^*}(L^2)}) + C\tau^{1/2s_p}\|\delta\varrho^n\|_{L^\infty_{T^*}(L^2)}.$$

According to Equation (3.71) and $\operatorname{div}\delta u^n = 0$, taking the $L^2(\mathbb{R}^d)$ inner product between Equation $(3.86)_2$ and $\rho^n u^n$. Then integrating on the time interval $[0,T^*]$ issues

$$\|\delta u^n\|_{L^\infty_{T^*}(L^2)} \le C\|H_e^{n-1}\|_{L^1_{T^*}(L^2)}. \tag{3.91}$$

Combining Estimate (3.90) and (3.91) entails for sufficiently small τ depending only on the space dimension d, on the indices (s,p,r) and on the constants C_M, C_E, E_0,

$$\|\delta\varrho^n\|_{L^\infty_{T^*}(H^1)\cap L^2_{T^*}(H^2)}+\|\delta u^n\|_{L^\infty_{T^*}(L^2)} \le \frac{1}{6}\|(\delta\varrho^{n-1},\delta\varrho^{n-2},\delta\varrho^{n-3},\delta u^{n-1},\delta u^{n-2},\delta u^{n-3})\|_{L^\infty_{T^*}(L^2)}.$$

Thus $\sum\|(\delta\varrho^n,\delta u^n)\|_{L^\infty_{T^*}(L^2)}$ converges. Since $\delta\varrho^n\in\mathcal{C}(\mathbb{R}^+;L^2)$, the Cauchy sequences $\{\varrho^n\}$ and $\{u^n\}$ converge to ϱ and u in $\mathcal{C}([0,T^*];L^2)$ respectively. It is also easy to see that

$$\sum_{n\ge 2}\|\delta\varrho^n\|_{L^\infty_{T^*}(H^1)\cap L^2_{T^*}(H^2)},\quad \sum_{n\ge 2}\|\delta h^n\|_{L^1_{T^*}(L^2)},\quad \sum_{n\ge 2}\|H_e^{n-1}\|_{L^1_{T^*}(L^2)} < +\infty.$$

Writing

$$\begin{aligned}\operatorname{div}\big((u^{n-1}-\kappa^{n-1}\nabla\log\rho^n)\cdot\nabla\delta u^n\big)\\ = \operatorname{div}\Big(\delta u^n\cdot\nabla(u^{n-1}-\kappa^{n-1}\nabla\log\rho^n)+\delta u^n\operatorname{div}(\kappa^{n-1}\nabla\log\rho^n)\Big),\end{aligned}$$

from Equation (3.87) we get

$$\|\nabla\delta q^n\|_{L^1_{T^*}(L^2)} \le C(\|H_e^{n-1}\|_{L^1_{T^*}(L^2)} + C_M\|\delta u^n\|_{L^\infty_{T^*}(L^2)}).$$

Thus $\sum_2^\infty\|\nabla\delta q^n\|_{L^1_{T^*}(L^2)}$ also converges and hence ∇q^n converges to the unique limit ∇q in $L^1_{T^*}(L^2)$.

Now one notices that by interpolation between L^2 and $B^s_{p,r}$, u^n converges to u in $\mathcal{C}([0,T^*];B^{1/2}_{4,1})$, for instance, and $u^{n-1}-\kappa^{n-1}(\rho^n)^{-1}\nabla\rho^n$ is at least in $L^2_{\mathrm{loc}}(B^{1/2}_{4,1})$. It entails

$$(u^{n-1}-\kappa^{n-1}(\rho^n)^{-1}\nabla\rho^n)\cdot\nabla u^n-(u-\kappa\rho\nabla\rho)\cdot\nabla u\to 0 \quad\text{in}\quad L^2_{T^*}(L^2+B^{-d/4}_{4,\infty}).$$

Thus a direct calculation ensures $(\varrho,u,\nabla q)$ solves System (3.2) and is in $E^s_{p,r}(T^*)$ by Fatou property.

3.4.3 Uniqueness part

The proof of uniqueness just follows the idea of the convergence part. More precisely, as in SubSection §3.2.2, take two solutions $(\rho_1,u_1,\nabla q_1),(\rho_2,u_2,\nabla q_2)\in E^s_{p,r}(T^*)$ of System (3.2) with the same initial data, such that $\rho_1-1,\rho_2-1,u_1,u_2\in L^\infty_{T^*}(L^2)$, $\nabla\rho_1,\nabla\rho_2\in L^2_{T^*}(L^2)$. Then the difference $(\delta\rho,\delta u,\nabla\delta q)=(\rho_1-\rho_2,u_1-u_2,\nabla q_1-\nabla q_2)$ verifies

$$\begin{cases}\partial_t\delta\varrho+u_1\cdot\nabla\delta\varrho-\operatorname{div}(\kappa_1\nabla\delta\varrho)=-\delta u\cdot\nabla\varrho_2+\operatorname{div}((\kappa_1-\kappa_2)\nabla\varrho_2),\\ \partial_t\delta u+(u_1+\nabla b_1)\cdot\nabla\delta u+\lambda_1\nabla\delta q=h_1-h_2-(\delta u+\nabla\delta b)\cdot\nabla u_2-\delta\lambda\nabla q_2,\\ \operatorname{div}\delta u=0,\\ (\delta\varrho,\delta u)|_{t=0}=(0,0),\end{cases}$$

with the notation $\kappa_i=\kappa(\varrho_i)$ and analogous for b_i, λ_i and h_i.

Similarly as Convergence Part, we can get

$$\|\delta\rho\|_{L^\infty_t(L^2)}+\|\nabla\delta\rho\|_{L^2_t(L^2)}\le C(\|\nabla\rho_2\|_{L^2_t(L^\infty)}\|\delta\rho\|_{L^\infty_t(L^2)}+\|\nabla\rho_2\|_{L^1_t(L^\infty)}\|\delta u\|_{L^\infty_t(L^2)}),$$

$$\begin{aligned}\|\nabla\delta\rho\|_{L^\infty_t(L^2)}+\|\nabla^2\delta\rho\|_{L^2_t(L^2)}\le C\Big(&(\|\nabla u_1\|_{L^1_t(L^\infty)}+\|\nabla\kappa_1\|_{L^2_t(L^\infty)})\|\nabla\delta\rho\|_{L^\infty_t(L^2)}\\ &+(\|\Delta\rho_2\|_{L^2_t(L^\infty)}+\|\nabla\rho_1\|_{L^2_t(L^\infty)}\|\nabla\rho_2\|_{L^\infty_t(L^\infty)})\|\delta\rho\|_{L^\infty_t(L^2)}\\ &+\|\nabla\rho_2\|_{L^2_t(L^\infty)}\|\delta u\|_{L^\infty_t(L^2)}+\|\nabla\rho_1\|_{L^\infty_t(L^\infty)}\|\nabla\delta\rho\|_{L^2_t(L^2)}\Big),\end{aligned}$$

and some similar estimate for $\|\delta u\|_{L_t^\infty(L^2)}$, which we omit here. Thus, on sufficiently small interval $[0,t]$, $\delta\rho \equiv \delta u \equiv 0$, the uniqueness holds. Then we recover uniqueness on the whole existence time interval $[0,T^*]$ by use of classical arguments.

3.4.4 Remark on the lifespan in dimension two

In this subsection, we want to give a better estimate for the lower bound for the lifespan of the solution in the case of dimension $d = 2$. The global-in-time existence issue in dimension $d = 2$ for the classical homogeneous Euler system (i.e. $\rho \equiv \overline{\rho}$ constant in system (3.2)) is well-known and the key to the proof is the fact that, if we define the *vorticity* of the fluid as

$$\omega := \partial_1 u^2 - \partial_2 u^1 , \tag{3.92}$$

then this quantity is conserved along the trajectories of the fluid particles, i.e. it fulfills the free transport equation

$$(V) \qquad \partial_t \omega + u \cdot \nabla \omega = 0 .$$

For non-homogeneous perfect fluids, see system (3.7), the previous relation (V) is no more true, due to a density term which comes into play combined with the pressure. Hence, it's not clear if solutions to (3.7) exist globally in time. However, in [36] it's proved that, for initial densities close to a constant state, the lifespan of the corresponding solutions tends to $+\infty$. Theorem 3.5 gives us the analogous result for our model (3.2).

The idea is to resort to the vorticity in order to control the high frequencies of the velocity field. The vorticity ω of the fluid is still defined by formula (3.92), where u solves (3.2). However, it's easy to see from relation (3.4) that actually

$$\omega \equiv \partial_1 v^2 - \partial_2 v^1 ;$$

now, from $(3.1)_2$ the equation for ω immediately follows :

$$\partial_t \omega + v \cdot \nabla \omega + \omega \, \Delta b + \nabla \lambda \wedge \nabla \Pi = 0 , \tag{3.93}$$

where we have set $\nabla\lambda \wedge \nabla\Pi = \partial_1\lambda\,\partial_2\Pi - \partial_2\lambda\,\partial_1\Pi$.

In order to bound the vorticity, it's fundamental to take advantage of a new version of refined estimates for transport equations in borderline Besov spaces of the type $B^0_{p,r}$, proved first by Vishik in [102] and then generalized by Hmidi and Keraani in [57] (see also Chapter 3 of [5]). They state that the $B^0_{\infty,1}$ norm of the vorticity grows linearly (and not exponentially, as in the general case) with respect to the Lipschitz norm of the solenoidal velocity field. Even if here we don't have the divergence-free condition for the transport velocity v, the proof in [57] still works, except that one asks for an additional regularity on $\operatorname{div} v$. More precisely, the following lemma holds.

Lemma 3.2. *Let us consider the following linear transport equation :*

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = g, \\ \omega|_{t=0} = \omega_0. \end{cases} \tag{3.94}$$

For any $\beta > 0$, there exists a constant C depending only on d, β such that the following a priori estimate holds true :

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C\Big(\|\omega_0\|_{B^0_{\infty,1}} + \|g\|_{L^1_t(B^0_{\infty,1})}\Big)\Big(1 + \mathcal{V}(t)\Big), \tag{3.95}$$

with

$$\mathcal{V}(t) := \int_0^t \|\nabla v\|_{L^\infty} + \|\operatorname{div} v\|_{B^\beta_{\infty,\infty}} \, dt'.$$

Proof. We will follow the proof of [57]. Firslty we can write the solution ω of the transport equation (3.94) as a sum : $\omega = \sum_{k\geq -1} \omega_k$, with ω_k satisfying

$$\begin{cases} \partial_t \omega_k + v\cdot\nabla\omega_k = \Delta_k g, \\ \omega_k|_{t=0} = \Delta_k\omega_0. \end{cases} \tag{3.96}$$

We obviously have from above that

$$\|\omega_k(t)\|_{L^\infty} \leq \|\Delta_k\omega_0\|_{L^\infty} + \int_0^t \|\Delta_k g\|_{L^\infty}\, dt'. \tag{3.97}$$

By Proposition 0.6, for any $\epsilon\in(0,1)$, we have the following a priori estimate in Besov space $B^{\epsilon}_{\infty,1}$:

$$\|\omega_k(t)\|_{B^{\epsilon}_{\infty,1}} \leq \Big(\|\Delta_k\omega_0\|_{B^{\epsilon}_{\infty,1}} + \|\Delta_k g\|_{L^1_t(B^{\epsilon}_{\infty,1})}\Big)\exp\Big(C\|\nabla v\|_{L^1_t(L^\infty)}\Big). \tag{3.98}$$

In order to get a priori estimate in Besov space $B^{-\epsilon}_{\infty,1}$, after applying the operator Δ_j to Equation (3.96), we write the commutator $[v,\Delta_j]\cdot\nabla\omega_k$ as follows (recalling Bony's decomposition (0.45) and denoting $\widetilde{v} := v - \Delta_{-1}v$)

$$[T_{\widetilde{v}},\Delta_j]\cdot\nabla\omega_k + T_{\Delta_j\nabla\omega_k}\widetilde{v} + R(\Delta_j\nabla\omega_k,\widetilde{v})$$
$$-\Delta_j(T_{\nabla\omega_k}\widetilde{v}) - \Delta_j\mathrm{div}\,(R(\omega_k,\widetilde{v})) + \Delta_j R(\omega_k,\mathrm{div}\,\widetilde{v}) + [\Delta_{-1}v,\Delta_j]\cdot\nabla\omega_k.$$

Then, $\forall\beta>\epsilon$, the L^∞-norm of all the above terms can be bounded by (for some nonnegative sequence $\|(c_j)\|_{\ell^1}=1$) :

$$C(d,\beta)\,2^{-j\epsilon}\,c_j\,\mathcal{V}'(t)\|\omega_k\|_{B^{-\epsilon}_{\infty,1}}.$$

Thus, we have the following a priori estimate in the space $B^{-\epsilon}_{\infty,1}$:

$$\|\omega_k(t)\|_{B^{-\epsilon}_{\infty,1}} \leq \Big(\|\Delta_k\omega_0\|_{B^{-\epsilon}_{\infty,1}} + \|\Delta_k g\|_{L^1_t(B^{-\epsilon}_{\infty,1})}\Big)\exp\Big(C\mathcal{V}(t)\Big). \tag{3.99}$$

On the other side, one has the following, for some positive integer N to be determined hereafter :

$$\|\omega\|_{B^0_{\infty,1}} \leq \sum_{j,k\geq -1}\|\Delta_j\omega_k\|_{L^\infty} = \sum_{|j-k|<N}\|\Delta_j\omega_k\|_{L^\infty} + \sum_{|j-k|\geq N}\|\Delta_j\omega_k\|_{L^\infty}.$$

Estimate (3.97) implies

$$\sum_{|j-k|<N}\|\Delta_j\omega_k\|_{L^\infty} \leq N\sum_k\Big(\|\Delta_k\omega_0\|_{L^\infty} + \|\Delta_k g\|_{L^1_t(L^\infty)}\Big) \leq N\Big(\|\omega_0\|_{B^0_{\infty,1}} + \|g\|_{L^1_t(B^0_{\infty,1})}\Big),$$

while Estimates (3.98) and (3.99) entail the following (for some nonnegative sequence $(c_j)\in\ell^1$) :

$$\|\Delta_j\omega_k\|_{L^\infty} \leq 2^{-\epsilon|k-j|}c_j\Big(\|\Delta_k\omega_0\|_{L^\infty} + \|\Delta_k g\|_{L^1_t(L^\infty)}\Big)\exp\Big(C\mathcal{V}(t)\Big),$$

which issues immediately

$$\sum_{|j-k|\geq N}\|\Delta_j\omega_k\|_{L^\infty} \leq 2^{-N\epsilon}\Big(\|\omega_0\|_{B^0_{\infty,1}} + \|g\|_{L^1_t(B^0_{\infty,1})}\Big)\exp\Big(C\mathcal{V}(t)\Big).$$

Therefore, for any $\beta>0$, we can choose $\epsilon\in(0,1)$ and $N\in\mathbb{N}$ such that $\epsilon<\beta$ and $N\epsilon\log 2\sim 1 + C\mathcal{V}(t)$. Thus the lemma follows from the above estimates. □

With the above lemma in hand, we can now begin to prove Theorem 3.5. In view of the continuation criterion, without loss of generality, we will always assume in the sequel

$$s = 1, \quad p = \infty, \quad r = 1.$$

In what follows, we will resort to the same notation in Subsection §3.3.2. That is, we set

$$R(t) = \|\varrho\|_{L_t^\infty(B^1_{\infty,1})}, \quad S(t) = \|\varrho\|_{L_t^1(B^3_{\infty,1})}, \quad U(t) = \|u\|_{L_t^\infty(B^1_{\infty,1})},$$

and the time moment T_R as defined by (3.63).

First of all, let us consider the density term. Under our hypothesis, we get the energy equality (3.20). Moreover, arguing as in the previous subsection, we recover estimate (3.64) in the time interval $[0, T_R]$.

Now, let us consider the velocity field. First of all, let us summarize the following inequalities for the nonlinear terms, which will be frequently used in the sequel :

$$\|\nabla^2 b(\rho)\|_{B^1_{\infty,1}} \lesssim \|b\|_{B^3_{\infty,1}} \lesssim \|\varrho\|_{B^3_{\infty,1}} = S'; \tag{3.100}$$

$$\|\Delta b \nabla a\|_{L^2} \lesssim \|b\|_{B^2_{\infty,1}} \|\nabla a\|_{L^2} \lesssim \|\varrho\|_{B^2_{\infty,1}} \|\nabla \rho\|_{L^2} \lesssim R^{1/2}(S')^{1/2} \|\nabla \rho\|_{L^2} \leq R\|\nabla \rho\|^2_{L^2} + S'; \tag{3.101}$$

$$\|(u + \nabla b) \cdot \nabla u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty}(\|\nabla \rho\|_{L^2} + \|u\|_{L^2}) \lesssim U(\|\nabla \rho\|_{L^2} + \|u\|_{L^2}). \tag{3.102}$$

Similarly as the above inequality (3.101), one has

$$\|\nabla b \cdot \nabla^2 a\|_{L^2} \lesssim R\|\nabla \rho\|^2_{L^2} + S', \quad \|u \cdot \nabla^2 a\|_{L^2} \lesssim R\|u\|^2_{L^2} + S'. \tag{3.103}$$

It is time to bound the velocity field u by use of the above inequalities. Firstly, by separating low and high frequencies of u and by use of Bernstein's inequalties and of a Fourier multiplier of order -1, we get

$$U(t) \,\leq\, C\left(\|u\|_{L^2} \,+\, \|\omega\|_{B^0_{\infty,1}}\right). \tag{3.104}$$

In order to handle the energy of the velocity field, one takes the L^2 scalar product between (3.6) and u and performing standard computations (recalling also (3.21) and the following arguments), getting

$$\|u(t)\|_{L^2} \,\leq\, C\left(\|u_0\|_{L^2} \,+\, \int_0^t \|\mathrm{div}\,\left(v \otimes \nabla a\right)\|_{L^2}\, d\tau\right).$$

Therefore, due to Inequalities (3.101) and (3.103), it follows that

$$\|u(t)\|_{L^2} \,\leq\, C\left(\|u_0\|_{L^2} \,+\, \int_0^t \left(R(\|\nabla \rho\|^2_{L^2} + \|u\|^2_{L^2}) \,+\, S'\right) d\tau\right). \tag{3.105}$$

Now, applying Lemma 3.2 with $\beta = 1$ to Equation (3.93), we find

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C\left(\|\omega_0\|_{B^0_{\infty,1}} \,+\, \int_0^t \left\|\nabla\lambda \wedge \nabla\Pi + \omega\,\Delta b\right\|_{B^0_{\infty,1}} d\tau\right)$$
$$\times \left(1 + \int_0^t \left(\|\nabla u\|_{L^\infty} + \|\nabla^2 b\|_{B^1_{\infty,1}}\right) d\tau\right).$$

By use of Bony's paraproduct decomposition (see also Section 4.2 of [36] for the first inequality)

$$\|\nabla\lambda \wedge \nabla\Pi\|_{B^0_{\infty,1}} \quad\lesssim\quad \|\nabla\rho\|_{B^0_{\infty,1}}\, \|\nabla\Pi\|_{B^0_{\infty,1}}$$
$$\|\omega\,\Delta b\|_{B^0_{\infty,1}} \quad\lesssim\quad \|\omega\|_{B^0_{\infty,1}}\, \|\Delta b\|_{B^1_{\infty,1}}\,.$$

Hence, by virtue of $\|\omega\|_{B^0_{\infty,1}} \lesssim U$ and Inequality (3.100), we immediately gather

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C\left(U_0 + \int_0^t \left(R\|\nabla\Pi\|_{B^0_{\infty,1}} + US'\right) d\tau\right)\left(1 + \int_0^t \|\nabla u\|_{L^\infty} d\tau + S\right). \quad (3.106)$$

It remains to deal with the pressure term. First of all, we have

$$\|\nabla\Pi\|_{B^0_{\infty,1}} \leq \|\nabla q\|_{B^0_{\infty,1}} + \|\partial_t \nabla a\|_{B^0_{\infty,1}} \lesssim \|\nabla q\|_{B^0_{\infty,1}} + \|\partial_t \varrho\|_{B^1_{\infty,1}}. \quad (3.107)$$

Thanks to equation $(3.2)_1$, we get

$$\|\partial_t \varrho\|_{B^1_{\infty,1}} \lesssim \|u \cdot \nabla\varrho\|_{B^1_{\infty,1}} + \|\kappa\nabla\varrho\|_{B^2_{\infty,1}} \lesssim U\,S' + S'.$$

Let us now focus on $\|\nabla q\|_{B^0_{\infty,1}}$. Actually, we will bound the $B^1_{\infty,1}$ norm, as it's not clear for us how to get advantage of the weaker norm in (3.106). The analysis is mostly the same performed in the previous subsection for the general case : so we cut low and high frequency, and we are reconducted to consider $\|\nabla q\|_{L^2}$ and $\|\Delta q\|_{B^0_{\infty,1}}$. Finally, as $\delta > 1$, we have

$$\|\nabla q\|_{B^1_{\infty,1}} \leq C\left(1 + R^\delta\right)\left(\|\nabla q\|_{L^2} + \|\mathrm{div}\,(v \cdot \nabla u)\|_{B^0_{\infty,1}} + \|h\|_{B^1_{\infty,1}}\right),$$

with the controls (by what we established in the case of higher dimension)

$$\|h\|_{B^1_{\infty,1}} \lesssim (1+R)\,U\,S' + (1+R)R\,S'$$

and, as $\mathrm{div}\,(u\cdot\nabla u) = \sum_{i,j} 2T_{\partial_j u^i}\partial_i u^j + \partial_j R(u^i, \partial_i u^j)$ (thanks to the divergence-free condition over u) and analogous for $\mathrm{div}\,(\nabla b \cdot \nabla u)$,

$$\|\mathrm{div}\,(v \cdot \nabla u)\|_{B^0_{\infty,1}} \lesssim U^2 + U\,S'.$$

On the other hand, observing that $\|h\|_{L^2} \lesssim \|\mathrm{div}\,(v \otimes \nabla a)\,\|_{L^2}$ and using Inequalities (3.101), (3.102) and (3.103), one has also

$$\|\nabla q\|_{L^2} \lesssim \|h\|_{L^2} + \|v \cdot \nabla u\|_{L^2} \lesssim R(\|\nabla\rho\|^2_{L^2} + \|u\|^2_{L^2}) + S' + U(\|\nabla\rho\|_{L^2} + \|u\|_{L^2}).$$

Let us define

$$X(t) := U(t) + \|u(t)\|_{L^2} = \|u(t)\|_{L^2 \cap B^1_{\infty,1}}.$$

So we get

$$\|\nabla q\|_{B^1_{\infty,1}} \leq C\left(1 + R^{\delta+2}\right)\left(\|\nabla\rho\|^2_{L^2} + S' + X^2 + XS'\right) \quad (3.108)$$

and, by (3.107), the same holds true also for $\|\nabla\Pi\|_{B^0_{\infty,1}}$.

Therefore, Estimate (3.106) for the vorticity becomes (denoting $X_0 = X(0)$)

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C\left(1 + S + \int_0^t X\,d\tau\right) \times\left(X_0 + \int_0^t (1 + R^{\delta+3})\left(R\,\|\nabla\rho\|^2_{L^2} + R\,S' + R\,X^2 + X\,S'\right)d\tau\right).$$

It's now time to insert the above estimate and (3.105) into (3.104). Keeping in mind that, in $[0, T_R]$,

$$R(t) + S(t) \leq C\,R_0 \exp\left(C\int_0^t (1 + X(\tau))d\tau\right), \quad (3.109)$$

we finally find

$$X(t) \le C\left(1+S+\int_0^t X d\tau\right)\Bigg(X_0 + \underbrace{R_0(1+R_0^{\delta+3})e^{C\int_0^t(1+X)}\int_0^t \|\nabla\rho\|_{L^2}^2}_{\Gamma_1} + \tag{3.110}$$
$$+\underbrace{R_0(1+R_0^{\delta+4})e^{C\int_0^t(1+X)}}_{\Gamma_2} + \underbrace{R_0(1+R_0^{\delta+3})e^{C\int_0^t(1+X)}\int_0^t X^2}_{\Gamma_3}$$
$$+\underbrace{(1+R_0^{\delta+3})e^{C\int_0^t(1+X)}\int_0^t XS'\,d\tau}_{\Gamma_4}\Bigg).$$

We define T_X as the following quantity (with Γ_i, $i=1,\cdots,4$ defined as above)

$$T_X := \sup\Big\{t\,|\,\Gamma_1(t)\le \|\varrho_0\|_{L^2}^2,\quad \Gamma_2(t)\le 1,\quad \Gamma_3(t),\,\Gamma_4(t)\le 1+\|\varrho_0\|_{L^2}^2+X_0\Big\}.$$

Then, noticing $S\le\Gamma_2$, one easily arrives at the following bound for $X(t)$ with t in $[0,T_R]\cap[0,T_X]$ (with some positive constant still denoted by C)

$$X(t)\ \le\ C(1+\|\varrho_0\|_{L^2}^2+X_0)\left(1+\int_0^t X(\tau)d\tau\right).$$

Hence, set $\Gamma_0 := C(1+\|\varrho_0\|_{L^2}^2+X_0)$, then by Gronwall's lemma we get

$$X(t)\ \le\ \Gamma_0\, e^{\Gamma_0 t}, \tag{3.111}$$

and the norm of the solution can be controlled by the norm of the initial data only.

Our next task, in order to complete the argument, is then to prove that T, defined by (3.25) with $\mathfrak{M}=\delta+4$ and small enough constant K_2, is smaller than both T_R and T_X. First of all, thanks to (3.111), for $t\in[0,T_X]\cap[0,T_R]$, we have

$$\int_0^t X\ \le\ e^{\Gamma_0 t},\quad \int_0^t X^2\le\frac{\Gamma_0}{2}e^{2\Gamma_0 t},\quad \int_0^t(1+X)\le 2e^{\Gamma_0 t},$$
$$e^{C\int_0^t(1+X)}\le e^{2Ce^{\Gamma_0 t}},\quad R+S\le CR_0e^{2Ce^{\Gamma_0 t}}.$$

With the above bounds in hand, one just has to show

$$\Gamma_1(T)\le\|\varrho_0\|_{L^2}^2,\quad \Gamma_2(T)\le 1,\quad \Gamma_3(T),\,\Gamma_4(T)\le 1+\|\varrho_0\|_{L^2}^2+X_0\quad\text{and}\quad \int_0^T R^3\le 2R_0.$$

We will check the above bounds one by one.
By use of the energy equality (3.20) for the density, it's then easy to see that $\Gamma_1(T)\le\|\varrho_0\|_{L^2}^2$, with T defined by (3.25). It is also easy to find that $\Gamma_2(T)\le 1$. Now noticing that ($\sigma\le e^\sigma$)

$$\Gamma_3(t)\le R_0(1+R_0^{\delta+3})e^{2Ce^{\Gamma_0 t}}\frac{\Gamma_0}{2}e^{2\Gamma_0 t}\le\frac{\Gamma_0}{2}R_0(1+R_0^{\delta+3})e^{3Ce^{2\Gamma_0 t}},$$

we hence have $\Gamma_3(T)\le 1+\|\varrho_0\|_{L^2}^2+X_0$. Similarly, since one has

$$\Gamma_4(t)\le(1+R_0^{\delta+3})e^{2Ce^{\Gamma_0 t}}\Gamma_0e^{\Gamma_0 t}\int_0^t S'\le\Gamma_0(1+R_0^{\delta+3})e^{5Ce^{\Gamma_0 t}}CR_0,$$

thus $\Gamma_4(t)\le 1+\|\varrho_0\|_{L^2}^2+X_0$. Finally, it's not hard to check that

$$\int_0^T R^3\le C^3R_0^3e^{6Ce^{\Gamma_0 T}}\le 2R_0.$$

This completes the proof of the theorem.

3.5 Slightly nonhomogeneous case

Let us give here a sketch of the proof to Theorem 3.6. In fact, we will establish only a priori estimates, the rest of the proof to existence and uniqueness being similar to that of Theorem 3.1, which is left to the reader.

Recall that $p \in (1,+\infty)$ and $\|\varrho_0\|_{B^s_{p,r}} \leq c$, where c is a small positive constant. Moreover we will use the convention that $r=1$ if $s = 1+d/p$.

We first focus on the density equation $(3.2)_1$. From Estimate (3.57), we get

$$\|\varrho\|_{\widetilde{L}^\infty_t(B^s_{p,r})} + \|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})} \leq C\,\|\varrho_0\|_{B^s_{p,r}} \exp\left(Ct + C\int_0^t \|\nabla u\|_{B^{s-1}_{p,r}} + C\|\nabla\varrho\|^2_{L^\infty}\,d\tau\right),$$

for constant C depending only on indices d, s, p, r and on ρ_*, ρ^*.

Now, if we take t small enough, such that, for instance,

$$\exp\left(Ct + C\int_0^t \|\nabla u\|_{B^{s-1}_{p,r}} + C\|\nabla\varrho\|^2_{L^\infty}\,d\tau\right) \leq \log 2\,, \tag{3.112}$$

then the previous inequality becomes

$$\|\varrho\|_{\widetilde{L}^\infty_t(B^s_{p,r})} + \|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})} \leq 2\,C\,c\,. \tag{3.113}$$

Let us now consider Equations $(3.2)_2$ and $(3.2)_3$ for the velocity field and the pressure term. Going along the lines of the proof to Estimates (3.40) and (3.41), we can see that they still hold true.

Here we handle ∇q as done in the previous section, that is, we make use of (3.62). Firstly, notice that the equation for q can be rewritten as

$$\Delta q = \operatorname{div}\left(h - (u+\nabla b)\cdot\nabla u + \frac{\varrho}{\rho}\nabla q\right), \tag{3.114}$$

which (formally) implies

$$\nabla q = \nabla(-\Delta)^{-1}\operatorname{div}\left(-h + (u+\nabla b)\cdot\nabla u - \frac{\varrho}{\rho}\,\nabla q\right).$$

Hence, by Calderon-Zygmund theory and (3.62) we infer (with $w = u+\nabla b$)

$$\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} \leq C\left(\|h\|_{\widetilde{L}^1_t(B^s_{p,r})} + \|\partial_i w^j\partial_j u^i\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})} + \|w\cdot\nabla u\|_{L^1_t(L^p)} + \left\|\frac{\varrho}{\rho}\,\nabla q\right\|_{\widetilde{L}^1_t(B^s_{p,r})}\right). \tag{3.115}$$

The term $\|\partial_i w^j\partial_j u^i\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})}$ can be controlled as done in (B.47) : recalling the divergence-free condition over u, it is easy to decompose $\|\operatorname{div}(w\cdot\nabla u)\|_{B^{s-1}_{p,r}}$ into

$$\|T_{\partial_i w^j}\partial_j u^i + T_{\partial_j u^i}\partial_i w^j + \operatorname{div}\big(R(w^j,\partial_j u)\big)\|_{B^{s-1}_{p,r}},$$

which can be controlled by $\|\nabla w\|_{B^{s-1}_{p,r}}\|\nabla u\|_{B^{s-1}_{p,r}}$.

For term $\|w\cdot\nabla u\|_{L^1_t(L^p)}$, we notice that

$$\|w\cdot\nabla u\|_{L^1_t(L^p)} \leq \int_0^t \|w\|_{L^p}\|\nabla u\|_{L^\infty} \leq \int_0^t (\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^p})\|u\|_{B^s_{p,r}}.$$

Finally, we have

$$\left\|\frac{\varrho}{\rho}\,\nabla q\right\|_{\widetilde{L}^1_t(B^s_{p,r})} \leq C\,\|\varrho\|_{\widetilde{L}^\infty_t(B^s_{p,r})}\,\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})}\,.$$

Therefore, putting all these inequalities together into (3.115) and noticing (3.113), we finally arrive at the following for sufficiently small c :

$$\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} \leq C\left(\|h\|_{\widetilde{L}^1_t(B^s_{p,r})} + \int_0^t (\|\nabla u\|_{B^{s-1}_{p,r}} + \|\nabla \rho\|_{B^s_{p,r}})\|u\|_{B^s_{p,r}} d\tau\right). \tag{3.116}$$

Now, the analysis of the nonlinear term h can be performed as before, and, up to take a smaller t than the one defined in (3.112), this allows us to close the estimates on some suitable time interval $[0, t]$, for sufficiently small c.

Annexe A

Estimates in general Besov spaces

A.1 Estimates for Products

We have the following estimates for the paraproduct and remainder operators in Besov spaces with different index.

Proposition A.1. *Let $1 \le r, r_1, r_2, p, p_1, p_2 \le \infty$ with $\frac{1}{r} \le \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}$ and $\frac{1}{p} \le \frac{1}{p_1} + \frac{1}{p_2}\cdot$*
– *If $p \le p_2$ then we have :*

$$\|T_u v\|_{B^{s_1+s_2+d(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2})}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r_1}} \|v\|_{B^{s_2}_{p_2,r_2}} \quad \textit{if} \quad s_1 < d\big(\tfrac{1}{p_1} + \tfrac{1}{p_2} - \tfrac{1}{p}\big), \tag{A.1}$$

$$\|T_u v\|_{B^{s_2}_{p,r}} \lesssim \|u\|_{L^{p_3}} \|v\|_{B^{s_2}_{p_2,r}} \quad \textit{if} \quad \tfrac{1}{p_3} = \tfrac{1}{p} - \tfrac{1}{p_2}\cdot \tag{A.2}$$

– *If $s_1 + s_2 + d\min\{0, 1 - \frac{1}{p_1} - \frac{1}{p_2}\} > 0$, then*

$$\|R(u,v)\|_{B^{s_1+s_2+d(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2})}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r_1}} \|v\|_{B^{s_2}_{p_2,r_2}}; \tag{A.3}$$

– *if $s_1 + s_2 + d\min\{0, 1 - \frac{1}{p_1} - \frac{1}{p_2}\} = 0$ and $\frac{1}{r_1} + \frac{1}{r_2} \ge 1$ then*

$$\|R(u,v)\|_{B^{s_1+s_2+d(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2})}_{p,\infty}} \lesssim \|u\|_{B^{s_1}_{p_1,r_1}} \|v\|_{B^{s_2}_{p_2,r_2}}. \tag{A.4}$$

Proof. Most of these results are classical. We just prove (A.1) and (A.2).
We write that

$$T_u v = \sum_{j \ge 1} T_j(u,v) \quad \text{with} \quad T_j(u,v) = S_{j-1} u \Delta_j v.$$

Since $\Delta_{j'}(S_{j-1}u\Delta_j v) \equiv 0$ for $|j'-j| > 4$, it suffices to show that, for some sequence $(c_j)_{j\in\mathbb{N}}$ such that $\|(c_j)\|_{\ell^r} = 1$, we have

$$\begin{aligned} \|T_j(u,v)\|_{L^p} &\lesssim c_j 2^{-js} \|u\|_{B^{s_1}_{p_1,r_1}} \|v\|_{B^{s_2}_{p_2,r_2}} \quad &\text{if} \quad s_1 < d/p_1 + d/p_2 - d/p, \\ \|T_j(u,v)\|_{L^p} &\lesssim c_j 2^{-js} \|u\|_{L^{p_3}} \|v\|_{B^{s_2}_{p_2,r_2}} \quad &\text{if} \quad s_1 = d/p_1 + d/p_2 - d/p, \end{aligned}$$

with $s = s_1 + s_2 + \frac{d}{p} - \frac{d}{p_1} - \frac{d}{p_2}$.

According to Hölder's inequality, we have

$$\|T_j(u,v)\|_{L^p} \le \|S_{j-1}u\|_{L^{p_3}} \|\Delta_j v\|_{L^{p_2}} \quad \text{with} \quad \frac{1}{p_3} = \frac{1}{p} - \frac{1}{p_2}\cdot \tag{A.5}$$

Hence, using the definition of S_{j-1} and Bernstein's inequality (here we notice that $p_1 \leq p_3$, a consequence of $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$) :

$$\|T_j(u,v)\|_{L^p} \lesssim \sum_{j' \leq j-2} 2^{j'(\frac{d}{p_1} - \frac{d}{p_3})} \|\Delta_{j'} u\|_{L^{p_1}} \|\Delta_j v\|_{L^{p_2}},$$

whence

$$2^{js}\|T_j(u,v)\|_{L^p} \leq \sum_{j' \leq j-2} 2^{(j-j')(s_1 + \frac{d}{p_3} - \frac{d}{p_1})} \big(2^{j's_1}\|\Delta_{j'} u\|_{L^{p_1}}\big)\big(2^{js_2}\|\Delta_j v\|_{L^{p_2}}\big).$$

Therefore, if $s_1 + d/p_3 - d/p_1 < 0$ then the result stems from convolution and Hölder inequalities for series. In the case where $s_1 + d/p_3 - d/p_1 = 0$, we just have to use that $\|S_{j-1}u\|_{L^{p_3}} \leq C\|u\|_{L^{p_3}}$ in (A.5).

The proof of (A.3) goes from similar arguments and is thus left to the reader. □

From the above proposition and (0.45), one may deduce a number of estimates in Besov spaces for the product of two functions. We shall use the following result :

Proposition A.2. *The following estimates hold true :*

(i) $\|uv\|_{B^s_{p,r}} \lesssim \|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|u\|_{B^s_{p,r}}\|v\|_{L^\infty}$ *if* $s > 0$.

(ii) If $s_1 < \frac{d}{p_1}$, $s_2 < d\min\{\frac{1}{p_1}, \frac{1}{p_2}\}$, $s_1 + s_2 + d\min\{0, 1 - \frac{1}{p_1} - \frac{1}{p_2}\} > 0$ *and* $\frac{1}{r} \leq \min\big\{1, \frac{1}{r_1} + \frac{1}{r_2}\big\}$ *then*

$$\|uv\|_{B^{s_1+s_2-\frac{d}{p_1}}_{p_2,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}. \tag{A.6}$$

(iii) We also have the following limit cases :

– *if* $s_1 = d/p_1$, $s_2 < \min(d/p_1, d/p_2)$ *and* $s_2 + d\min(1/p_1, 1/p_2') > 0$ *then*

$$\|uv\|_{B^{s_2}_{p_2,r}} \lesssim \|u\|_{B^{\frac{d}{p_1}}_{p_1,\infty}\cap L^\infty}\|v\|_{B^{s_2}_{p_2,r}}; \tag{A.7}$$

– *if* $s_2 = \min(d/p_1, d/p_2)$, $s_1 < d/p_1$ *and* $s_1 + s_2 + d\min\{0, 1 - \frac{1}{p_1} - \frac{1}{p_2}\} > 0$ *then*

$$\|uv\|_{B^{s_1+s_2-\frac{d}{p_1}}_{p_2,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,1}};$$

– *if* $1/r_1 + 1/r_2 \geq 1$, $s_1 < \frac{d}{p_1}$, $s_2 < d\min\{\frac{1}{p_1}, \frac{1}{p_2}\}$ *and* $s_1 + s_2 + d\min(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}) = 0$ *then*

$$\|uv\|_{B^{s_1+s_2-\frac{d}{p_1}}_{p_2,\infty}} \lesssim \|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}.$$

Sometimes we need to consider time dependent Besov spaces of type $\widetilde{L}^1_t(B^s_{p,r})$, which means taking the integral in time before summing up the L^p-norms of dyadic blocks (see Definition 0.3). The next proposition supplies an estimation in the spirit of Young's inequality.

Proposition A.3. *Let* $s > 0$ *and* $(p,r) \in [1,+\infty]^2$. *Let* $s_i \in \mathbb{R}$, $i = 1,\ldots,4$ *such that* $s_1, s_3 \leq s < s_2, s_4$. *Define* $\theta, \sigma \in]0,1]^2$ *such that*

$$s \,=\, \theta\, s_1 \,+\, (1-\theta)\, s_2 \,=\, \sigma\, s_3 \,+\, (1-\sigma)\, s_4.$$

Then for any $\varepsilon > 0$, *one has*

$$\|u\,v\|_{\widetilde{L}^1_t(B^s_{p,r})} \;\lesssim\; \frac{\theta}{\varepsilon^{(1-\theta)/\theta}}\int_0^t \|u\|^{1/\theta}_{L^\infty}\|v\|_{B^{s_1}_{p,r}}\,d\tau \,+\, \frac{\sigma}{\varepsilon^{(1-\sigma)/\sigma}}\int_0^t \|v\|^{1/\sigma}_{L^\infty}\|u\|_{B^{s_3}_{p,r}}\,d\tau \,+$$
$$+\,(1-\theta)\,\varepsilon\,\|u\|_{\widetilde{L}^1_t(B^{s_2}_{p,r})} \,+\, (1-\sigma)\,\varepsilon\,\|v\|_{\widetilde{L}^1_t(B^{s_4}_{p,r})},$$

Proof. Firstly,
$$\|\Delta_j T_u v\|_{L^p} \lesssim \|u\|_{L^\infty} \Big(\sum_{|j-\mu|\leq 1} \|\Delta_\mu v\|_{L^p} \Big) .$$

Now we integrate the above inequality on the time interval $[0,t]$, multiply it by 2^{js}, apply Young's inequality and we pass to the ℓ^r norm : using also Minkowski's inequality we gather

$$\|T_u\, v\|_{\widetilde{L}^1_t(B^s_{p,r})} \lesssim \frac{\theta}{\varepsilon^{(1-\theta)/\theta}} \int_0^t \|u\|^{1/\theta}_{L^\infty} \|v\|_{B^{s_1}_{p,r}}\, d\tau \,+\, \varepsilon\,(1-\theta)\, \|v\|_{\widetilde{L}^1_t(B^{s_2}_{p,r})} .$$

The term $T_v u$ can be treated in the same way, and this leads us to an analogous estimate, in which however the roles of u and v are reversed.

Let us now consider the remainder term. It follows that

$$\|R(u,v)\|_{\widetilde{L}^1_t(B^s_{p,r})} \lesssim \left\| \left(\sum_{j'\geq j-3} 2^{(j-j')s}\, 2^{j's} \int_0^t \|\Delta_{j'} u\|_{L^\infty} \|\Delta_{j'} v\|_{L^p}\, d\tau \right)_{j\geq -1} \right\|_{\ell^r} .$$

Now we apply Young's inequality to the product in the integral and also to the convolutions : we finally arrive at

$$\|R(u,v)\|_{\widetilde{L}^1_t(B^s_{p,r})} \lesssim \frac{\theta}{\varepsilon^{(1-\theta)/\theta}} \int_0^t \|u\|^{1/\theta}_{L^\infty} \|v\|_{B^{s_1}_{p,r}}\, d\tau \,+\, \varepsilon\,(1-\theta)\, \|v\|_{\widetilde{L}^1_t(B^{s_2}_{p,r})} ,$$

and this completes the proof of the lemma.

□

A.2 Estimates for commutators

Let us first generalize the classic estimates for the commutators of type
$$\left\| \left(2^{js} \|[\varphi, \Delta_j]\nabla\psi\|_{L^p}\right)_{j\geq -1} \right\|_{\ell^r} ,$$
presented in Proposition 0.4 in Besov spaces of low regularity. In fact, we can choose different Lebesgue exponents for the two elements φ, ψ and besides, roughly speaking, the order of gain of regularity ν for the latter element $\nabla\psi$ can be chosen to be any real number in $[0,1]$. The "commutator estimate" referring to the case $\nu = 0$ can just be viewed as a product estimate while the estimate with $\nu = 1$ recovers the classic commutator estimate. More precisely, we have the following :

Proposition A.4. *Let* $(p_1, p_2, r_1, r_2, r) \in [1,+\infty]^5$ *and* $\frac{1}{r} = \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}$. *Let* $(s,\nu) \in \mathbb{R}\times\mathbb{R}$ *satisfy*
$$-d\min\Big\{\frac{1}{p_1'}, \frac{1}{p_2}\Big\} < s < \nu + d\min\Big\{\frac{1}{p_1}, \frac{1}{p_2}\Big\} \quad \text{and} \quad -d\min\Big\{\frac{1}{p_1}, \frac{1}{p_2}\Big\} < \nu < 1. \tag{A.8}$$
For $j \geq -1$, *we have*
$$\left\| \left(2^{js} \|[\varphi, \Delta_j]\nabla\psi\|_{L^{p_1}}\right)_{j\geq -1} \right\|_{\ell^r} \lesssim \|\nabla\varphi\|_{B^{\frac{d}{p_2}+\nu-1}_{p_2,r_2}} \|\nabla\psi\|_{B^{s-\nu}_{p_1,r_1}} . \tag{A.9}$$

The following limit cases also hold true :
– *if* $s = \nu + d\min\{\frac{1}{p_1}, \frac{1}{p_2}\}$, $r_1 = 1$ *and* $r_2 = r$ *then we have*
$$\left\| \left(2^{js} \|[\varphi, \Delta_j]\nabla\psi\|_{L^{p_1}}\right)_{j\geq -1} \right\|_{\ell^r} \lesssim \|\nabla\varphi\|_{B^{\frac{d}{p_2}+\nu-1}_{p_2,r}} \|\nabla\psi\|_{B^{s-\nu}_{p_1,1}} ; \tag{A.10}$$

– *if* $\nu=1$, $r_1=r$ *and* $r_2=\infty$ *then we have*

$$\Big\|\big(2^{js}\|[\varphi,\Delta_j]\nabla\psi\|_{L^{p_1}}\big)_{j\geq-1}\Big\|_{\ell^r}\lesssim\|\nabla\varphi\|_{B^{\frac{d}{p_2}}_{p_2,\infty}\cap L^\infty}\|\nabla\psi\|_{B^{s-1}_{p_1,r}}. \tag{A.11}$$

Finally, if in addition to (A.8)*, we have* $\nu>1-d\min(1/p_1,1/p_2)$*, then*

$$\Big\|\big(2^{j(s-1)}\|\partial_k[\varphi,\Delta_j]\nabla\psi\|_{L^{p_1}}\big)_{j\geq-1}\Big\|_{\ell^r}\lesssim\|\nabla\varphi\|_{B^{\frac{d}{p_2}+\nu-1}_{p_2,r_2}}\|\nabla\psi\|_{B^{s-\nu}_{p_1,r_1}}\quad \textit{for all}\ \ k\in\{1,\cdots,d\}, \tag{A.12}$$

with the above changes in the limit cases.

Proof. Throughout, it will be understood that $\|(c_j)_{j\in\mathbb{Z}}\|_{\ell^{r_2}}=\|(d_j)_{j\in\mathbb{Z}}\|_{\ell^{r_1}}=1$.

We decompose the commutator by use of Bony's paraproduct :

$$[\varphi,\Delta_j]\nabla\psi\;=\;R^1_j(\varphi,\psi)\;+\;R^2_j(\varphi,\psi)\;+\;R^3_j(\varphi,\psi)\;+\;R^4_j(\varphi,\psi)\;+\;R^5_j(\varphi,\psi)\,, \tag{A.13}$$

where, setting $\widetilde{\varphi}=(\mathrm{Id}\,-\Delta_{-1})\varphi$, we have defined

$$\begin{aligned}
R^1_j(\varphi,\psi)\;&:=\;\big[T_{\widetilde{\varphi}},\Delta_j\big]\,\nabla\psi\\
R^2_j(\varphi,\psi)\;&:=\;T'_{\Delta_j\nabla\psi}\widetilde{\varphi}\;=\;\sum_k S_{k+2}\Delta_j\nabla\psi\,\Delta_k\widetilde{\varphi}\\
R^3_j(\varphi,\psi)\;&:=\;-\Delta_jT_{\nabla\psi}\widetilde{\varphi}\\
R^4_j(\varphi,\psi)\;&:=\;-\Delta_jR(\widetilde{\varphi},\nabla\psi)\\
R^5_j(\varphi,\psi)\;&:=\;[\Delta_{-1}\varphi,\Delta_j]\,\nabla\psi\,.
\end{aligned}$$

Let us first prove inequalities (A.9), (A.10) and (A.11). By virtue of the first-order Taylor's formula, we have

$$\begin{aligned}
R^1_j(\varphi,\psi)&=\sum_{|j-j'|\leq4}[S_{j'-1}\widetilde{\varphi},\Delta_j]\Delta_{j'}(\nabla\psi)\\
&=\sum_{|j-j'|\leq4}2^{-j'}\int_{\mathbb{R}^d}\int_0^1 h(y)y\cdot\nabla S_{j'-1}\widetilde{\varphi}(x-2^{-j}t'y)\,dt'\,\big(\Delta_{j'}\nabla\psi\big)(x-2^{-j}y)\,dy,
\end{aligned}$$

hence

$$\|R^1_j(\varphi,\psi)\|_{L^{p_1}}\lesssim2^{-j}\|\nabla S_{j-1}\widetilde{\varphi}\|_{L^\infty}\|\Delta_j\nabla\psi\|_{L^{p_1}}, \tag{A.14}$$

whence

$$\big\|(2^{js}\|R^1_j(\varphi,\psi)\|_{L^{p_1}})_{j\in\mathbb{N}}\big\|_{\ell^r}\lesssim\big\|(2^{j(\nu-1)}\|\nabla S_{j-1}\widetilde{\varphi}\|_{L^\infty}2^{j(s-\nu)}\|\Delta_j(\nabla\psi)\|_{L^{p_1}})\big\|_{\ell^r}.$$

So in the case $\nu=1$, we readily get

$$\|(2^{js}\|R^1_j(\varphi,\psi)\|_{L^{p_1}})_{j\in\mathbb{N}}\|_{\ell^r}\lesssim\|\nabla\widetilde{\varphi}\|_{L^\infty}\|\nabla\psi\|_{B^{s-1}_{p_1,r}}. \tag{A.15}$$

To handle the case $\nu<1$, we use the fact that

$$\begin{aligned}
&2^{j(\nu-1)}\|\nabla S_{j-1}\widetilde{\varphi}\|_{L^\infty}2^{j(s-\nu)}\|\Delta_j\nabla\psi\|_{L^{p_1}}\\
&\qquad\leq\sum_{j'\leq j-2}2^{(j-j')(\nu-1)}\big(2^{j'(\nu-1)}\|\nabla\Delta_{j'}\widetilde{\varphi}\|_{L^\infty}\big)\big(2^{j(s-\nu)}\|\Delta_j\nabla\psi\|_{L^{p_1}}\big).
\end{aligned}$$

Thus from Hölder and convolution inequalities for series, one may conclude that

$$\|(2^{js}\|R^1_j\|_{L^{p_1}})_{j\in\mathbb{N}}\|_{\ell^r}\lesssim\|\nabla\widetilde{\varphi}\|_{B^{\nu-1}_{\infty,r_2}}\|\nabla\psi\|_{B^{s-\nu}_{p_1,r_1}}\quad\text{if }\nu<1. \tag{A.16}$$

Concerning $R^2_j(\varphi,\psi)$, we have

$$\|R^2_j(\varphi,\psi)\|_{L^{p_1}} \le \sum_{j'\ge j-3} \|\Delta_{j'}\widetilde{\varphi}\, S_{j'+2}\Delta_j\nabla\psi\|_{L^{p_1}},$$

and we consider the following two cases :

– $p_1 \ge p_2$:

$$\begin{aligned} 2^{js}\|R^2_j(\varphi,\psi)\|_{L^{p_1}} &\le 2^{js}\sum_{j'\ge j-3}\|\Delta_{j'}\widetilde{\varphi}\|_{L^{p_1}}\|S_{j'+2}\,\Delta_j\nabla\psi\|_{L^\infty}\\ &\lesssim 2^{js}\sum_{j'\ge j-3} 2^{j'd(\frac{1}{p_2}-\frac{1}{p_1})}\|\Delta_{j'}\widetilde{\varphi}\|_{L^{p_2}}\|\Delta_j\nabla\psi\|_{L^\infty}\\ &\lesssim \sum_{j'\ge j-3} 2^{(j-j')(\nu+\frac{d}{p_1})} c_{j'}\|\widetilde{\varphi}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r_2}} d_j\|\nabla\psi\|_{B^{s-\nu-\frac{d}{p_1}}_{\infty,r_1}}, \end{aligned}$$

– $p_1 < p_2$:

$$\begin{aligned} 2^{js}\|R^2_j(\varphi,\psi)\|_{L^{p_1}} &\le 2^{js}\sum_{j'\ge j-3}\|\Delta_{j'}\widetilde{\varphi}\|_{L^{p_2}}\|S_{j'+2}\Delta_j\nabla\psi\|_{L^{\frac{p_1p_2}{p_2-p_1}}}\\ &\lesssim \sum_{j'\ge j-3} 2^{(j-j')(\nu+\frac{d}{p_2})} c_{j'}\|\widetilde{\varphi}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r_2}} d_j\|\nabla\psi\|_{B^{s-\nu-\frac{d}{p_2}}_{\frac{p_1p_2}{p_2-p_1},r_1}}. \end{aligned}$$

Hence for $\nu > -d\min\{\frac{1}{p_1},\frac{1}{p_2}\}$,

$$\|(2^{js}\|R^2_j(\varphi,\psi)\|_{L^{p_1}})_{j\in\mathbb{N}}\|_{\ell^r} \lesssim \|\widetilde{\varphi}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r_2}}\|\nabla\psi\|_{B^{s-\nu}_{p_1,r_1}}. \tag{A.17}$$

In the case $p_1 \le p_2$, bounding $R^3_j(\varphi,\psi)$ stems from (A.1), (A.2) (and an obvious embedding in the limit case). We get

$$\|(2^{js}\|R^3_j(\varphi,\psi)\|_{L^{p_1}})_{j\in\mathbb{N}}\|_{\ell^r} \lesssim \begin{cases} \|\widetilde{\varphi}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r_2}}\|\nabla\psi\|_{B^{s-\nu}_{p_1,r_1}}, & \text{if } s<\nu+d/p_2,\\ \|\widetilde{\varphi}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r}}\|\nabla\psi\|_{B^{s-\nu}_{p_1,1}}, & \text{if } s=\nu+d/p_2. \end{cases} \tag{A.18}$$

To deal with the case $p_1 > p_2$, we just have to notice that, according to (A.1), (A.2), the paraproduct operator maps $B^{s-\nu-\frac{d}{p_1}}_{\infty,r_1} \times B^{\frac{d}{p_1}+\nu}_{p_1,r_2}$ in $B^s_{p_1,r}$ provided that $s < \nu + d/p_1$ (and $L^\infty \times B^{\frac{d}{p_1}+\nu}_{p_1,r}$ in $B^s_{p_1,r}$ if $s=\nu+d/p_1$). So we still get (A.18) provided $s<\nu+d/p_1$ and $s\le\nu+d/p_1$, respectively.

As for the fourth term, it is only a matter of applying Inequality (A.3). We get

$$\|(2^{js}\|R^4_j(\varphi,\psi)\|_{L^{p_1}})_{j\in\mathbb{N}}\|_{\ell^r} \lesssim \|\widetilde{\varphi}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r_2}}\|\nabla\psi\|_{B^{s-\nu}_{p_1,r_1}},\ \text{if } s > -d\min\Big\{\frac{1}{p_1'},\frac{1}{p_2}\Big\}. \tag{A.19}$$

The term $R^5_j(\varphi, v)$ may be treated by arguing like in the proof of (A.15). One ends up with

$$\|(2^{js}\|R^5_j(\varphi,\psi)\|_{L^{p_1}})_{j\in\mathbb{N}}\|_{\ell^r} \lesssim \|\nabla\Delta_{-1}\varphi\|_{L^\infty}\|\nabla\psi\|_{B^{s-\nu}_{p_1,r}}, \qquad \text{if } \nu\le 1. \tag{A.20}$$

Given that for any (s,p,r), one has (owing to the low-frequency cut-off)

$$\|\widetilde{\varphi}\|_{B^s_{p,r}} \lesssim \|\nabla\varphi\|_{B^{s-1}_{p,r}},$$

putting together (A.15), (A.16), (A.17), (A.18), (A.19) and (A.20) completes the proof of (A.9), (A.10) and (A.11).

In order to establish (A.12), we notice that the terms $R^i_j(\varphi,\psi)$ with $i\neq 2$ are spectrally localized in balls of size 2^j. Hence Bernstein inequality together with (A.15), (A.16), (A.18), (A.19) and (A.20) ensures that they satisfy the desired inequality under Condition (A.8).

On the other hand $R^2_j(\varphi,v)$ does not have this spectral localization property. Let us just treat the case $p_1\geq p_2$ to simplify the presentation. We have

$$\|\partial_k R^2_j(\varphi,\psi)\|_{L^{p_1}}\leq \sum_{j'\geq j-3}\Big(\|\Delta_{j'}\widetilde{\varphi}\|_{L^{p_1}}\|\partial_k S_{j'+2}\Delta_j\nabla\psi\|_{L^\infty}+\|\partial_k\Delta_{j'}\widetilde{\varphi}\|_{L^{p_1}}\|S_{j'+2}\Delta_j\nabla\psi\|_{L^\infty}\Big). \tag{A.21}$$

According to Bernstein's inequality, we have

$$\sum_{j'\geq j-3}\|\Delta_{j'}\widetilde{\varphi}\|_{L^{p_1}}\|\partial_k S_{j'+2}\Delta_j\nabla\psi\|_{L^\infty}\leq C2^j\sum_{j'\geq j-3}\|\Delta_{j'}\widetilde{\varphi}\|_{L^{p_1}}\|S_{j'+2}\Delta_j\nabla\psi\|_{L^\infty}.$$

Hence this term may be bounded as desired (just follow the previous computations).

In order to handle the second term of (A.21), we write that, according to Bersntein's inequality,

$$\|\partial_k\Delta_{j'}\widetilde{\varphi}\|_{L^{p_1}}\|S_{j'+2}\Delta_j\nabla\psi\|_{L^\infty}\leq C2^{j'}\|\Delta_{j'}\widetilde{\varphi}\|_{L^{p_1}}\|S_{j'+2}\Delta_j\nabla\psi\|_{L^\infty}.$$

Hence

$$2^{j(s-1)}\sum_{j'\geq j-3}\|\partial_k\Delta_{j'}\widetilde{\varphi}\|_{L^{p_1}}\|S_{j'+2}\Delta_j\nabla\psi\|_{L^\infty}$$
$$\leq C\sum_{j'\geq j-3}2^{(j-j')(\nu+\frac{d}{p_1}-1)}c_{j'}\|\widetilde{\varphi}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r_2}}d_j\|\nabla\psi\|_{B^{s-\nu-\frac{d}{p_1}}_{\infty,r_1}},$$

which leads to the desired inequality provided that $\nu+\frac{d}{p_1}-1>0$.

□

As in Proposition A.3, we next want to study the commutator estimate in the time-involved Besov spaces of type $\widetilde{L}^1_T(B^s_{p,r})$. Finally, the following estimates hold, which also permit to choose different Lebesgue exponents in time variable for the former element φ of the commutator :

Proposition A.5. *Let the triple* $(s+1,p,r)$ *satisfy*

$$s+1>-d\min\{\frac{1}{p},\frac{1}{p'}\},\quad (p,r)\in[1,+\infty]^2,\quad \textit{and}\quad r=1\ \textit{if}\ s=\frac{d}{p}.$$

Let s_1,s_2 *be two real numbers satisfying* $s_1\leq s+1<s_2$*, and define* $\theta\in]0,1]$ *such that*

$$s+1\;=\;\theta\,s_1\;+\;(1-\theta)\,s_2.$$

Then, for any $\varepsilon>0$ *and any space derivative* ∂_k*,* $k=1,\dots,d$*, we have*

$$\left\|2^{js}\int_0^t\|\partial_k\left([\varphi,\Delta_j]\nabla\psi\right)\|_{L^p}\,d\tau\right\|_{\ell^r} \tag{A.22}$$
$$\leq\frac{C\,\theta}{\varepsilon^{(1-\theta)/\theta}}\,(\widetilde{\Phi}^{s+1}(t))^{\frac{1}{\theta}}\|\nabla\psi\|_{\widetilde{L}^\infty_t(B^{s_1-1}_{p,r})}+(1-\theta)\,\varepsilon\,\|\nabla\psi\|_{\widetilde{L}^1_t(B^{s_2-1}_{p,r})},$$

with $\widetilde{\Phi}^{s+1}$ *defined by*

$$\widetilde{\Phi}^{s+1}(t)=\begin{cases}\|\nabla\varphi(t)\|_{L^{\frac{1}{\theta}}_t(B^{\frac{d}{p}}_{p,\infty}\cap L^\infty)} & \textit{if } s\in(-1-d\min\{\frac{1}{p},\frac{1}{p'}\},\frac{d}{p}),\\ \|\nabla\varphi\|_{\widetilde{L}^{\frac{1}{\theta}}_t(B^s_{p,r})} & \textit{if } s=\frac{d}{p}\ \textit{with}\ r=1\ \textit{or}\ s>\frac{d}{p},\end{cases} \tag{A.23}$$

if in addition s_i*,* $i = 1, 2$*, satisfy*

$$s_i < 1 + \frac{d}{p} \quad \text{if} \quad s+1 < 1 + \frac{d}{p} \qquad \text{and} \qquad s_i > 1 + \frac{d}{p} \quad \text{if} \quad s+1 > 1 + \frac{d}{p}\,.$$

If moreover $s > 0$ *and, for some* s_3, s_4 *and* $\sigma \in]0,1]$ *, we have also* $s+1 \ = \ \sigma\, s_3 + (1-\sigma)\, s_4$*, then*

$$\left\| 2^{js} \int_0^t \left\|\partial_k\left([\varphi, \Delta_j]\, \nabla\psi\right)\right\|_{L^p} d\tau \right\|_{\ell^r} \leq \frac{C\,\theta}{\varepsilon^{(1-\theta)/\theta}} \int_0^t \|\nabla\varphi\|_{L^\infty}^{1/\theta} \|\psi\|_{B^{s_1}_{p,r}} d\tau \tag{A.24}$$

$$+ \frac{C\,\sigma}{\varepsilon^{(1-\sigma)/\sigma}} \int_0^t \|\nabla\psi\|_{L^\infty}^{1/\sigma} \|\nabla\varphi\|_{B^{s_3-1}_{p,r}} d\tau \, + \, (1-\theta)\,\varepsilon\, \|\psi\|_{\widetilde{L}^1_t(B^{s_2}_{p,r})} \, + \, (1-\sigma)\,\varepsilon\, \|\nabla\varphi\|_{\widetilde{L}^1_t(B^{s_4-1}_{p,r})}\,.$$

Proof. Let us first point out here that, in the following, R^i_j, $i = 1, \ldots, 5$ denote the same items as in the proof of Proposition A.4. We will derive two types of inequalities for each R^i_j, corresponding to (A.22) and (A.24) respectively.

Let us first consider the term

$$\mathcal{R}^1 \ := \ \left\| \left(2^{j\,s} \int_0^t \left\|\partial_k R^1_j\right\|_{L^p} d\tau \right)_j \right\|_{\ell^r}\,.$$

Each R^1_j is supported in a ball of radius 2^j, and hence the derivative ∂_k gives (by Bernstein's inequalities) a factor 2^j. According to (A.14), we get

$$\mathcal{R}^1 \ \lesssim \ \left\| \left(2^{js} \int_0^t \|\nabla S_{j-1}\widetilde{\varphi}\|_{L^\infty} \, \|\Delta_j\nabla\psi\|_{L^p} \, d\tau \right)_{j\geq -1} \right\|_{\ell^r}\,.$$

Since $\|\Delta_j\nabla\psi\|_{L^p} \lesssim 2^j\|\Delta_j\psi\|_{L^p}$, we can decompose the terms in the integral in the following way :

$$2^{js}\, \|\nabla S_{j-1}\widetilde{\varphi}\|_{L^\infty} \, \|\Delta_j\nabla\psi\|_{L^p} \lesssim \left(2^{js_1}\, \|\nabla S_{j-1}\widetilde{\varphi}\|_{L^\infty}^{1/\theta}\, \|\Delta_j\psi\|_{L^p}\right)^\theta \, \left(2^{js_2}\, \|\Delta_j\psi\|_{L^p}\right)^{1-\theta}.$$

Now, we apply Young's inequality to the integrant and then take ℓ^r-norm to the sequences, getting

$$\mathcal{R}^1 \ \leq \ \frac{C\,\theta}{\varepsilon^{(1-\theta)/\theta}} \int_0^t \|\nabla\varphi\|_{L^\infty}^{1/\theta} \|\psi\|_{B^{s_1}_{p,r}} d\tau \ + \ (1-\theta)\,\varepsilon\, \|\psi\|_{\widetilde{L}^1_t(B^{s_2}_{p,r})}\,. \tag{A.25}$$

Let us now handle

$$\mathcal{R}^2 \ := \ \left\| 2^{j\,s} \int_0^t \left\|\partial_k R^2_j\right\|_{L^p} d\tau \right\|_{\ell^r} \ \leq \ \left\| 2^{j\,s} \int_0^t \sum_{\mu\geq j-2} \|\partial_k S_{\mu+2}\nabla\Delta_j\psi\|_{L^\infty} \, \|\Delta_\mu\widetilde{\varphi}\|_{L^p} \, d\tau \right\|_{\ell^r}$$

$$+ \left\| 2^{j\,s} \int_0^t \sum_{\mu\geq j-2} \|S_{\mu+2}\nabla\Delta_j\psi\|_{L^\infty} \, \|\partial_k\Delta_\mu\widetilde{\varphi}\|_{L^p} \, d\tau \right\|_{\ell^r}\,.$$

One notices that, since $\mu \geq j-2$, we have

$$\|\partial_k S_{\mu+2}\nabla\Delta_j\psi\|_{L^\infty} \, \|\Delta_\mu\widetilde{\varphi}\|_{L^p} \lesssim 2^\mu \|\nabla\Delta_j\psi\|_{L^\infty} \|\Delta_\mu\widetilde{\varphi}\|_{L^p} \lesssim \|\nabla\Delta_j\psi\|_{L^\infty} \|\nabla\Delta_\mu\widetilde{\varphi}\|_{L^p}.$$

Hence

$$\begin{aligned}
\mathcal{R}^2 \ &\lesssim \ \left\| 2^{j\,s} \int_0^t \sum_{\mu\geq j-2} \|\nabla\Delta_j\psi\|_{L^\infty} \, \|\nabla\Delta_\mu\widetilde{\varphi}\|_{L^p} \, d\tau \right\|_{\ell^r} \\
&\lesssim \ \left\| \int_0^t \|\nabla\psi\|_{L^\infty} \sum_{\mu\geq j-2} 2^{(j-\mu)s} 2^{\mu s} \|\nabla\Delta_\mu\widetilde{\varphi}\|_{L^p} \, d\tau \right\|_{\ell^r}\,.
\end{aligned}$$

On one hand, we just do exactly as above (the way to obtain (A.25)) : if $s > 0$, then we have

$$\mathcal{R}^2 \le \frac{C\,\sigma}{\varepsilon^{(1-\sigma)/\sigma}} \int_0^t \|\nabla\psi\|_{L^\infty}^{1/\sigma} \|\nabla\widetilde{\varphi}\|_{B^{s_3-1}_{p,r}}\, d\tau \;+\; (1-\sigma)\,\varepsilon\,\|\nabla\widetilde{\varphi}\|_{\widetilde{L}^1_t(B^{s_4-1}_{p,r})}\,. \tag{A.26}$$

On the other hand, by the relationship $2^{-j\frac{d}{p}}\|\Delta_j f\|_{L^\infty} \lesssim \|\Delta_j f\|_{L^p}$ and Young's inequality, we further derive

$$\begin{aligned}
\mathcal{R}^2 &\lesssim \left\| \int_0^t 2^{j\,(s+1-\frac{d}{p})}\|\Delta_j\psi\|_{L^\infty} \sum_{\mu\ge j-2} 2^{(j-\mu)\frac{d}{p}} \left(2^{\mu\frac{d}{p}}\|\nabla\Delta_\mu\widetilde{\varphi}\|_{L^p}\right) d\tau \right\|_{\ell^r} \\
&\lesssim \left\| \sum_{\mu\ge j-2} 2^{(j-\mu)\frac{d}{p}} \left\|2^{j\,(s+1)}\|\Delta_j\psi\|_{L^p}\right\|_{L_t^{1/(1-\theta)}} \left\|2^{\mu\frac{d}{p}}\|\nabla\Delta_\mu\widetilde{\varphi}\|_{L^p}\right\|_{L_t^{1/\theta}} \right\|_{\ell^r} \\
&\lesssim \left\| \sum_{\mu\ge j-2} 2^{(j-\mu)\frac{d}{p}}\, c_\mu \|\nabla\varphi\|_{\widetilde{L}_t^{\frac{1}{\theta}}(B^{\frac{d}{p}}_{p,\infty})}\; 2^{j\,(\theta s_1+(1-\theta)s_2)}\|\Delta_j\psi\|^{\theta}_{L_t^\infty(L^p)}\,\|\Delta_j\psi\|^{1-\theta}_{L_t^1(L^p)} \right\|_{\ell^r}, \quad (c_j)_j\in\ell^\infty \\
&\le \frac{C\,\theta}{\varepsilon^{(1-\theta)/\theta}}\,\|\nabla\varphi\|^{\frac{1}{\theta}}_{\widetilde{L}_t^{\frac{1}{\theta}}(B^{\frac{d}{p}}_{p,\infty})}\,\|\psi\|_{\widetilde{L}^\infty_t(B^{s_1}_{p,r})} \;+\; (1-\theta)\,\varepsilon\,\|\psi\|_{\widetilde{L}^1_t(B^{s_2}_{p,r})}.
\end{aligned} \tag{A.27}$$

Moreover, since

$$\mathcal{R}^3 := \left\|2^{j\,s}\int_0^t \left\|\partial_k R^3_j\right\|_{L^p}\, d\tau\right\|_{\ell^r} \le \left\|2^{j(s+1)}\int_0^t \sum_{\mu\sim j}\|S_{\mu-1}\nabla\psi\|_{L^\infty}\,\|\Delta_\mu\widetilde{\varphi}\|_{L^p}\right\|_{\ell^r}$$

we can immediately see that (A.26) holds also for $\mathcal{R}^3$. In order to get the form like (A.27), we consider the two cases $s < \frac{d}{p}$ and $s > \frac{d}{p}$ separately. Here we notice that if $s = \frac{d}{p}$ and $r = 1$, then taking the integral in time and the ℓ^r norm can commutate and hence the classical result for commutators in Proposition 0.4 gives this lemma. Now if $s_i < 1 + \frac{d}{p}$, $i = 1, 2$, then

$$\mathcal{R}^3 \lesssim \left\|\int_0^t \left(2^{j\theta(s_1-1-\frac{d}{p})}\|S_{j-1}\nabla\psi\|^{\theta}_{L^\infty}\right)\left(2^{j(1-\theta)(s_2-1-\frac{d}{p})}\|S_{j-1}\nabla\psi\|^{1-\theta}_{L^\infty}\right) 2^{j(\frac{d}{p}+1)}\|\Delta_j\widetilde{\varphi}\|_{L^p}\right\|_{\ell^r},$$

which gives (A.27) for $\mathcal{R}^3$. On the other hand, $s_i > 1 + \frac{d}{p}$, $i = 1, 2$, then as above

$$\begin{aligned}
\mathcal{R}^3 &\lesssim \left\|\int_0^t \|\nabla\psi\|_{L^\infty}\left(2^{j(s+1)}\|\Delta_j\widetilde{\varphi}\|_{L^p}\right)\right\|_{\ell^r} \\
&\le \frac{C\,\theta}{\varepsilon^{(1-\theta)/\theta}}\,\|\widetilde{\varphi}\|^{\frac{1}{\theta}}_{\widetilde{L}_t^{\frac{1}{\theta}}(B^{s+1}_{p,r})}\,\|\nabla\psi\|_{L^\infty_t(L^\infty)} \;+\; (1-\theta)\,\varepsilon\,\|\nabla\psi\|_{L^1_t(L^\infty)} \\
&\le \frac{C\,\theta}{\varepsilon^{(1-\theta)/\theta}}\,\|\widetilde{\varphi}\|^{\frac{1}{\theta}}_{\widetilde{L}_t^{\frac{1}{\theta}}(B^{s+1}_{p,r})}\,\|\psi\|_{L^\infty_t(B^{s_1}_{p,r})} + \;(1-\theta)\,\varepsilon\|\nabla\psi\|_{\widetilde{L}^1_t(B^{s_2-1}_{p,r})}.
\end{aligned} \tag{A.28}$$

where the last inequality follows by the embeddings $L^1_t(L^\infty) \hookleftarrow \widetilde{L}^1_t(B^{d/p}_{p,1}) \hookleftarrow \widetilde{L}^1_t(B^{s_2-1}_{p,r})$.

The same holds true also for

$$\mathcal{R}^4 := \left\|2^{j\,s}\int_0^t \left\|\partial_k R^4_j\right\|_{L^p}\, d\tau\right\|_{\ell^r}.$$

Let us first sketch the case $s > \frac{d}{p}$. We can bound $\mathcal{R}^4$ in the following way :

$$\mathcal{R}^4 \lesssim \left\| \int_0^t \|\nabla\psi\|_{L^\infty} \sum_{\mu\geq j-2} 2^{(j-\mu)(s+1)} \left(2^{\mu(s+1)}\|\Delta_\mu\widetilde{\varphi}\|_{L^p}\right)\, d\tau \right\|_{\ell^r}.$$

Hence (A.26) and (A.28) follow immediately for $\mathcal{R}^4$. If $s < \frac{d}{p}$, then we consider two cases $p \leq 2$ and $p > 2$ separately, which both give (A.27). In fact, if $p \leq 2$, then $p' \geq p$ and we have for $s+1 > -\frac{d}{p'}$,

$$\begin{aligned}\mathcal{R}^4 &\lesssim \left\| \int_0^t 2^{j(s+1+\frac{d}{p'})}\|R_j^4\|_{L^1} \right\|_{\ell^r}\\ &\lesssim \left\| \int_0^t \sum_{\mu\geq j-3} 2^{(j-\mu)(s+1+\frac{d}{p'})} \left(2^{\mu(\frac{d}{p}+1)}\|\Delta_\mu\widetilde{\varphi}\|_{L^p}\right)\left(2^{\mu(s+\frac{d}{p'}-\frac{d}{p})}\|\Delta_\mu\nabla\psi\|_{L^{p'}}\right) \right\|_{\ell^r},\end{aligned}$$

which gives (A.27). Otherwise if $p > 2$, then we have for $s+1 > -\frac{d}{p}$,

$$\begin{aligned}\mathcal{R}^4 &\lesssim \left\| \int_0^t 2^{j(s+1+\frac{d}{p})}\|R_j^4\|_{L^{p/2}} \right\|_{\ell^r}\\ &\lesssim \left\| \int_0^t \sum_{\mu\geq j-3} 2^{(j-\mu)(s+1+\frac{d}{p})} \left(2^{\mu(\frac{d}{p}+1)}\|\Delta_\mu\widetilde{\varphi}\|_{L^p}\right)(2^{\mu s}\|\Delta_\mu\nabla\psi\|_{L^p}) \right\|_{\ell^r},\end{aligned}$$

which yields (A.27) also.

Finally, last term

$$\mathcal{R}^5 := \left\| 2^{j\,s}\int_0^t \left\|\partial_k R_j^5\right\|_{L^p}\, d\tau \right\|_{l^r}$$

can be handled as $\mathcal{R}^1$, leading us to the same estimate as (A.25) and so to the end of the proof. □

We also introduce the following commutator estimate, with the Besov-norm of lower regularity of the latter element $\|\nabla\psi\|_{B^{s-1}_{p,r}}$ as an integrant (in time) in Estimate (A.22). However, this requires an extra δ-order regularity on $\nabla\varphi$.

Proposition A.6. *Let*

$$q \in (1,\infty), \quad -d\min\{\frac{1}{q},\frac{1}{q'}\} < s, \quad \delta > \max\{0, -(\frac{d}{p}+1-\frac{2}{q})\}. \tag{A.29}$$

Then for any $\varepsilon \in \mathbb{R}^+$, there exists one constant C, depending on $\varepsilon, \delta, d, p, q$, such that

$$\begin{aligned}\left\| \left(2^{js}\|[\varphi,\Delta_j]\cdot\nabla\psi\|_{L^1_t(L^p)}\right)_j \right\|_{\ell^r} &\leq \varepsilon\|\nabla\psi\|_{\widetilde{L}^1_t(B^{s+1}_{p,r})}\\ &\quad + C\int_0^t \|\nabla\varphi\|^{q'}_{B^{\max\{s-1,\frac{d}{p}-\frac{2}{q}\}+\delta}_{p,\infty}}\|\nabla\psi\|_{B^{s-1}_{p,r}}\, dt',\end{aligned} \tag{A.30}$$

with the case where $s < \frac{d}{p}+1-\frac{2}{q}$ and $\delta = 0$ holding true.

Furthermore, we also have similarly

$$\begin{aligned}\left\| \left(2^{js}\|\partial_k([\varphi,\Delta_j]\cdot\nabla\psi)\|_{L^1_t(L^p)}\right)_j \right\|_{\ell^r} &\leq \varepsilon\|\nabla\psi\|_{\widetilde{L}^1_t(B^{s+1}_{p,r})}\\ &\quad + C(\varepsilon,\delta,d,p,q)\int_0^t \|\nabla^2\varphi\|^{q'}_{B^{\max\{s-1,\frac{d}{p}-\frac{2}{q}\}+\delta}_{p,\infty}}\|\nabla\psi\|_{B^{s-1}_{p,r}}\, dt',\ k=1,\cdots,d,\end{aligned} \tag{A.31}$$

for the index $(d, q, s+1, \delta)$ satisfying (A.29), with the special case where $s < \frac{d}{p} + 1 - \frac{2}{q}$ and $\delta = 0$ holding true.

Proof. Let $\mathcal{R}^i$, $i = 1, \cdots, 5$ be the same as in the proof of Proposition A.5. We treat these terms respectively as follows.

$$\begin{aligned}\mathcal{R}^1 &\lesssim \Big\| \int_0^t \Big(2^{j(s+1)/q} \|\Delta_j \nabla\psi\|_{L^p}^{1/q} \Big) \Big(2^{j(-2/q)} \|\nabla S_{j-1}\widetilde{\varphi}\|_{L^\infty} \Big) \Big(2^{j(s-1)/q'} \|\Delta_j \nabla\psi\|_{L^p}^{1/q'} \Big)\, dt' \Big\|_{\ell^r} \\ &\le \varepsilon \|\nabla\psi\|_{\widetilde{L}^1_t(B^{s+1}_{p,r})} + C(\varepsilon) \int_0^t \|\nabla\varphi\|^{q'}_{B^{-2/q}_{\infty,\infty}} \|\nabla\psi\|_{B^{s-1}_{p,r}}\, dt' ;\end{aligned}$$

$$\begin{aligned}\mathcal{R}^2 &\lesssim \Big\| \int_0^t \sum_{j' \ge j-3} \Big(2^{j(s+1-d/p)/q} \|\Delta_j \nabla\psi\|_{L^\infty}^{1/q} \Big) 2^{(j-j')(d/p+1/q'-1/q)} \\ &\quad \times \Big(2^{j'(d/p+1/q'-1/q)} \|\Delta_{j'}\widetilde{\varphi}\|_{L^p} \Big) \Big(2^{j(s-1-d/p)/q'} \|\Delta_j \nabla\psi\|_{L^\infty}^{1/q'} \Big)\, dt' \Big\|_{\ell^r} \\ &\le \varepsilon \|\nabla\psi\|_{\widetilde{L}^1_t(B^{s+1}_{p,r})} + C(\varepsilon) \int_0^t \|\nabla\varphi\|^{q'}_{B^{d/p-2/q}_{p,\infty}} \|\nabla\psi\|_{B^{s-1}_{p,r}}\, dt', \ \text{if } \frac{1}{q} < \frac{1}{2} + \frac{d}{2p},\end{aligned}$$

or

$$\begin{aligned}\mathcal{R}^2 &\lesssim \Big\| \int_0^t \sum_{j' \ge j-3} \Big(2^{j(s+1-d/p)/q} \|\Delta_j \nabla\psi\|_{L^\infty}^{1/q} \Big) 2^{(j-j')(d/p+1/q'-1/q+\delta)} \\ &\quad \times \Big(2^{j'(d/p+1/q'-1/q+\delta)} \|\Delta_{j'}\widetilde{\varphi}\|_{L^p} \Big) \Big(2^{j(s-1-d/p)/q'} \|\Delta_j \nabla\psi\|_{L^\infty}^{1/q'} \Big)\, dt' \Big\|_{\ell^r} \\ &\le \varepsilon \|\nabla\psi\|_{\widetilde{L}^1_t(B^{s+1}_{p,r})} + C(\varepsilon,\delta) \int_0^t \|\nabla\varphi\|^{q'}_{B^{d/p-2/q+\delta}_{p,\infty}} \|\nabla\psi\|_{B^{s-1}_{p,r}}\, dt', \\ &\qquad \text{if } \frac{d}{p} + 1 - \frac{2}{q} + \delta > 0, \ \text{for some } \delta \ge 0 ;\end{aligned}$$

$$\begin{aligned}\mathcal{R}^3 &\lesssim \Big\| \int_0^t \sum_{j'' \le j-3} \Big(2^{j''(s+1-d/p)/q} \|\Delta_{j''} \nabla\psi\|_{L^\infty}^{1/q} \Big) 2^{(j-j'')(s-d/p-1/q'+1/q)} \\ &\quad \times \Big(2^{j(d/p+1/q'-1/q)} \|\Delta_j\widetilde{\varphi}\|_{L^p} \Big) \Big(2^{j''(s-1-d/p)/q'} \|\Delta_{j''} \nabla\psi\|_{L^\infty}^{1/q'} \Big)\, dt' \Big\|_{\ell^r} \\ &\le \varepsilon \|\nabla\psi\|_{\widetilde{L}^1_t(B^{s+1}_{p,r})} + C(\varepsilon) \int_0^t \|\nabla\varphi\|^{q'}_{B^{d/p-2/q}_{p,\infty}} \|\nabla\psi\|_{B^{s-1}_{p,r}}\, dt', \ \text{if } s < \frac{d}{p} + 1 - \frac{2}{q},\end{aligned}$$

or

$$\begin{aligned}\mathcal{R}^3 &\lesssim \Big\| \int_0^t \sum_{j'' \le j-3} \Big(2^{j''(s+1-d/p)/q} \|\Delta_{j''} \nabla\psi\|_{L^\infty}^{1/q} \Big) 2^{(j-j'')(-\delta)} \\ &\quad \times \Big(2^{j(s+\delta)} \|\Delta_j\widetilde{\varphi}\|_{L^p} \Big) \Big(2^{j''(s-1-d/p)/q'} \|\Delta_{j''} \nabla\psi\|_{L^\infty}^{1/q'} \Big)\, dt' \Big\|_{\ell^r} \\ &\le \varepsilon \|\nabla\psi\|_{\widetilde{L}^1_t(B^{s+1}_{p,r})} + C(\varepsilon,\delta) \int_0^t \|\nabla\varphi\|^{q'}_{B^{s-1+\delta}_{p,\infty}} \|\nabla\psi\|_{B^{s-1}_{p,r}}\, dt', \ \text{if } s \ge \frac{d}{p} + 1 - \frac{2}{q} ;\end{aligned}$$

$$\begin{aligned}\mathcal{R}^4 &\lesssim \Big\| \int_0^t \sum_{j' \ge j-3} \Big(2^{j'(s+1)/q} \|\Delta_{j'} \nabla\psi\|_{L^p}^{1/q} \Big) 2^{(j-j')(s+d/p)} \\ &\quad \times \Big(2^{j'(d/p+1/q'-1/q)} \|\Delta_{j'}\widetilde{\varphi}\|_{L^p} \Big) \Big(2^{j'(s-1)/q'} \|\Delta_{j'} \nabla\psi\|_{L^p}^{1/q'} \Big)\, dt' \Big\|_{\ell^r} \\ &\le \varepsilon \|\nabla\psi\|_{\widetilde{L}^1_t(B^{s+1}_{p,r})} + C(\varepsilon) \int_0^t \|\nabla\varphi\|^{q'}_{B^{d/p-2/q}_{p,\infty}} \|\nabla\psi\|_{B^{s-1}_{p,r}}\, dt', \ \text{if } p \ge 2, \quad s > -d/p,\end{aligned}$$

or

$$\begin{aligned}\mathcal{R}^4 &\lesssim \Big\|\int_0^t \sum_{j'\geq j-3} 2^{j(s+d/p')}\|\Delta_{j'}\widetilde{\varphi}\|_{L^{p'}}\|\Delta_{j'}\nabla\psi\|_{L^p}\,dt'\Big\|_{\ell^r}\\ &\leq \varepsilon\|\nabla\psi\|_{\widetilde{L}^1_t(B^{s+1}_{p,r})}+C(\varepsilon)\int_0^t \|\nabla\varphi\|^{q'}_{B^{d/p-2/q}_{p,\infty}}\|\nabla\psi\|_{B^{s-1}_{p,r}}\,dt',\ \text{if } p\leq 2,\quad s>-d/p'\,;\end{aligned}$$

$$\begin{aligned}\mathcal{R}^5 &\lesssim \Big\|\int_0^t \|\Delta_{-1}\nabla\varphi\|_{L^\infty}2^{j(s-1)}\|\nabla\Delta_j\psi\|_{L^p}\,dt'\Big\|_{\ell^r}\\ &\leq \varepsilon\|\nabla\psi\|_{\widetilde{L}^1_t(B^{s+1}_{p,r})}+C(\varepsilon)\int_0^t \|\nabla\varphi\|^{q'}_{B^{d/p-2/q}_{p,\infty}}\|\nabla\psi\|_{B^{s-1}_{p,r}}\,dt'.\end{aligned}$$

Put the five parts $\mathcal{R}^i$, $i=1,\cdots,5$ together and (A.30) issues. The proof for (A.31) is exactly the same, which is omitted here. We remark here that as in the proof of Proposition A.4, among the quantities $\mathcal{R}^i$, $i=1,\cdots,5$, only the term $(\partial_k\Delta_{j'}\widetilde{\varphi}\cdot\Delta_j S_{j'+1}\nabla\psi)$ in $\mathcal{R}^2$ is not included in the ball with radius of size 2^j. □

Annexe B

A priori estimates

B.1 Parabolic equations

In this section we are interested in the following linear parabolic equation :

$$\left\{\begin{array}{rcl}\partial_t\varrho + u\cdot\nabla\varrho - \operatorname{div}(\kappa\nabla\varrho) & = & f,\\ \varrho|_{t=0} & = & \varrho_0.\end{array}\right. \tag{B.1}$$

Here the divergence-free vector field u, the positive coefficient κ, the source term f and the initial data ϱ_0 are smooth enough and decay sufficiently at infinity, such that $\kappa(x,t)\geq\kappa_*>0$ holds true for all $x\in\mathbb{R}^d$, $t\in\mathbb{R}^+$. The key to get a priori estimates for such a linear equation is applying the commutator estimates presented in Section §A.2 to the following two commutators :

$$\mathcal{R}_j^1 = [u,\Delta_j]\cdot\nabla\varrho,\quad \mathcal{R}_j^2 = \operatorname{div}([\kappa,\Delta_j]\nabla\varrho). \tag{B.2}$$

In fact, a standard process : we apply the operator Δ_j to the equation, we integrate first in space and then in time ; then we use the Bernstein's inequality and also the commutator estimates ; at last we perform Gronwall's Inequality. This gives us the estimate.

The a priori estimate concerning low regularity (which extends the corresponding one in [29]) reads :

Proposition B.1. *Let ϱ satisfy* (B.1) *on $[0,T]\times\mathbb{R}^d$. Let $(p_1,p_2)\in(1,\infty)^2$ and $s\in\mathbb{R}$ fulfill*

$$-1-d\min\Big\{\frac{1}{p_1},\frac{1}{p_1'},\frac{1}{p_2}\Big\} < s \leq d\min\Big\{\frac{1}{p_1},\frac{1}{p_2}+\frac{1}{d}\Big\}. \tag{B.3}$$

Then there exist constants $c_1(d,p_1,\kappa_*)$, $C_1(d,p_1,p_2,s,\kappa_*)$, $\widetilde{C}_1(d,p_1,s,\kappa_*)$ such that the solution to (B.1) *satisfies for all $t\in[0,T]$:*

$$\|\varrho\|_{\widetilde{L}_t^\infty(B^s_{p_1,1})} + c_1\|\varrho\|_{L_t^1(B^{s+2}_{p_1,1})} \leq e^{C_1\mathcal{K}_L(t)}\times\Big(\|\varrho_0\|_{B^s_{p_1,1}} + \widetilde{C}_1\|\Delta_{-1}\varrho\|_{L_t^1(L^{p_1})} + \|f\|_{L_t^1(B^s_{p_1,1})}\Big), \tag{B.4}$$

where

$$\mathcal{K}_L(t) := \|\nabla u\|_{L_t^1(B^{d/p_2}_{p_2,1})} + \|\nabla\kappa\|^2_{L_t^2(B^{d/p_1}_{p_1,1})}. \tag{B.5}$$

Proof. In the following it is understood that $\|(c_j)_j\|_{\ell^1}=1$.
As a warm up, we focus on the special case $p_1=p_2=2$ and $s\in(-d/2,d/2]$ which may be achieved by classical energy arguments.

Applying Δ_j to (B.1) yields for all $j\geq -1$,

$$\partial_t\varrho_j + u\cdot\nabla\varrho_j - \operatorname{div}(\kappa\nabla\varrho_j) = f_j + \mathcal{R}_j^1 - \mathcal{R}_j^2, \tag{B.6}$$

where $\varrho_j = \Delta_j \varrho$, $f_j = \Delta_j f$.

Taking the L^2 inner product of (B.6) with ϱ_j and integrating by parts (recall that $\operatorname{div} u = 0$), we get

$$\frac{1}{2}\frac{d}{dt}\|\varrho_j\|_{L^2}^2 + \int \kappa |\nabla \varrho_j|^2 \leq \|\varrho_j\|_{L^2}(\|f_j\|_{L^2} + \|\mathcal{R}_j^1\|_{L^2} + \|\mathcal{R}_j^2\|_{L^2}).$$

Notice that we have $\|\nabla \varrho_j\|_{L^2} \approx 2^j \|\varrho_j\|_{L^2}$ for $j \geq 0$. Hence, dividing formally both sides of the inequality by $\|\varrho_j\|_{L^2}$ and integrating with respect to the time variable, we get for some constant c_1 depending only on d, κ_*,

$$\begin{aligned}\|\varrho_j\|_{L_t^\infty(L^2)} + c_1 2^{2j}\|\varrho_j\|_{L_t^1(L^2)} \leq &\|(\varrho_0)_j\|_{L^2} + \delta_j^{-1} c_1 2^{2j}\|\Delta_{-1}\varrho\|_{L_t^1(L^2)} \\ &+ \|f_j\|_{L_t^1(L^2)} + \|\mathcal{R}_j^1\|_{L_t^1(L^2)} + \|\mathcal{R}_j^2\|_{L_t^1(L^2)},\end{aligned}$$

where

$$\delta_j^{-1} = 1 \text{ if } j = -1 \quad \text{and} \quad \delta_j^{-1} = 0 \text{ if } j \neq -1.$$

Applying Inequality (A.11) to $\mathcal{R}_j^1$ and Inequality (A.12) with regularity index $s+1$ and $\nu = 1$ to $\mathcal{R}_j^2$ yields

$$\begin{aligned}\|\mathcal{R}_j^1\|_{L_t^1(L^2)} &\lesssim 2^{-js} c_j \int_0^t \|\nabla u\|_{B_{2,1}^{d/2}} \|\nabla \varrho\|_{B_{2,1}^{s-1}}\, dt' \quad \text{if} \quad -d/2 < s \leq d/2+1, \\ \|\mathcal{R}_j^2\|_{L_t^1(L^2)} &\lesssim 2^{-js} c_j \int_0^t \|\nabla \kappa\|_{B_{2,1}^{d/2}} \|\nabla \varrho\|_{B_{2,1}^{s}}\, dt' \quad \text{if} \quad -d/2-1 < s \leq d/2.\end{aligned}$$

Now multiplying both sides by 2^{js}, summing up over j and taking advantage of the interpolation inequality $\|\cdot\|_{B_{2,1}^{s+1}} \lesssim \|\cdot\|_{B_{2,1}^s}^{1/2} \|\cdot\|_{B_{2,1}^{s+2}}^{1/2}$ in Proposition 0.1 yields

$$\begin{aligned}\|\varrho\|_{\widetilde{L}_t^\infty(B_{2,1}^s)} + c_1\|\varrho\|_{L_t^1(B_{2,1}^{s+2})} \leq &\|\varrho_0\|_{B_{2,1}^s} + \widetilde{C}_1\|\Delta_{-1}\varrho\|_{L_t^1(L^2)} + \|f\|_{L_t^1(B_{2,1}^s)} \\ &+ C_1 \int_0^t (\|\nabla u\|_{B_{2,1}^{d/2}} + \|\nabla \kappa\|_{B_{2,1}^{d/2}}^2)\|\varrho\|_{B_{2,1}^s}\, dt'.\end{aligned}$$

Then applying Gronwall's inequality leads to Inequality (B.4).

To treat the general case $1 < p_1 < \infty$ we multiply (B.6) by $|\varrho_j|^{p_1-2}\varrho_j$. We arrive at

$$\frac{1}{p_1}\frac{d}{dt}\int |\varrho_j|^{p_1}\, dx + (p_1 - 1)\int \kappa |\varrho_j|^{p_1-2}|\nabla \varrho_j|^2\, dx \leq \|\varrho_j\|_{L^{p_1}}^{p_1-1}(\|f_j\|_{L^{p_1}} + \|\mathcal{R}_j^1\|_{L^{p_1}} + \|\mathcal{R}_j^2\|_{L^{p_1}}).$$

Next, we use (bearing in mind that $1 < p_1 < \infty$) the following Bernstein type inequality (see Inequality (0.41)) :

$$\int |\varrho_j|^{p_1-2}|\nabla \varrho_j|^2 \gtrsim 2^{2j}\int |\varrho_j|^{p_1}\, dx \quad \text{for} \quad j \geq 0. \tag{B.7}$$

Hence we get for $j \geq 0$,

$$\frac{d}{dt}\|\varrho_j\|_{L^{p_1}}^{p_1} + 2^{2j}\|\varrho_j\|_{L^{p_1}}^{p_1} \lesssim \|\varrho_j\|_{L^{p_1}}^{p_1-1}(\|f_j\|_{L^{p_1}} + \|\mathcal{R}_j^1\|_{L^{p_1}} + \|\mathcal{R}_j^2\|_{L^{p_1}}). \tag{B.8}$$

Therefore dividing both sides by $\|\varrho_j\|_{L^{p_1}}^{p_1-1}$ and using that, according to (A.11) and (A.12),

$$\begin{aligned}\|\mathcal{R}_j^1\|_{L^{p_1}} &\lesssim 2^{-js} c_j \|\nabla u\|_{B_{p_2,1}^{d/p_2}} \|\nabla \varrho\|_{B_{p_1,1}^{s-1}} \quad \text{if} \quad -d\min\left(\frac{1}{p_1'}, \frac{1}{p_2}\right) < s \leq 1 + d\min\left(\frac{1}{p_1}, \frac{1}{p_2}\right), \\ \|\mathcal{R}_j^2\|_{L^{p_1}} &\lesssim 2^{-js} c_j \|\nabla \kappa\|_{B_{p_1,1}^{d/p_1}} \|\nabla \varrho\|_{B_{p_1,1}^{s}} \quad \text{if} \quad -d\min\left(\frac{1}{p_1'}\frac{1}{p_1}\right) < s+1 \leq 1 + \frac{d}{p_1},\end{aligned}$$

integrating in time, multiplying both sides with 2^{js}, summing up over $j \geq -1$ and performing an interpolation inequality, one arrives at

$$\|\varrho\|_{\widetilde{L}^\infty_t(B^s_{p_1,1})} + c_1\|\varrho\|_{L^1_t(B^{s+2}_{p_1,1})} \leq \|\varrho_0\|_{B^s_{p_1,1}} + c_1 2^{-(s+2)}\|\Delta_{-1}\varrho\|_{L^1_t(L^{p_1})} + \|f\|_{L^1_t(B^s_{p_1,1})} + C_1\int_0^t (\|\nabla u\|_{B^{d/p_2}_{p_2,1}} + \|\nabla\kappa\|^2_{B^{d/p_1}_{p_1,1}})\|\varrho\|_{B^s_{p_1,1}}\,dt',$$

which yields (B.4) by Gronwall inequality, except in the case where s is too negative.

To improve the condition over s for s negative, it suffices to use the fact that, owing to $\operatorname{div} u = 0$, one has

$$\mathcal{R}^1_j = \operatorname{div}([u,\Delta_j]\varrho).$$

Then one may apply Inequality (A.12) to $\operatorname{div}([u,\Delta_j]\varrho)$ with $s+1$ instead of s. The details are left to the reader. □

Remark B.1. *Let us further remark that*

$$\mathcal{R}^2_j = [\nabla\kappa,\Delta_j]\cdot\nabla\varrho + [\kappa,\Delta_j]\Delta\varrho.$$

This decomposition allows to improve the condition (B.3) *for positive s : if we only assume that $s \leq 1 + d\min(1/p_1, 1/p_2)$ then Inequality* (B.4) *holds true with the additional term $\|\nabla^2\kappa\|_{L^1_t(B^{d/p_1}_{p_1,1})}$ in $\mathcal{K}_L$. As only the case $s = d/p_1$ is needed for proving Theorem 1.2, we do not provide more details here.*

Note also that, for $s = d/p_1$, Condition (B.3) *holds if and only if $1/p_1 \leq 1/p_2 + 1/d$.*

Remark B.2. *From the proof we can see that the quantity $\widetilde{C}_1\|\Delta_{-1}\varrho\|_{L^1_t(L^{p_1})}$ in the a priori estimates* (B.4) *can be absorbed if the time t is small. Indeed, for instance, one has for any $s \in \mathbb{R}$,*

$$\|\Delta_{-1}\varrho\|_{L^1_t(L^{p_1})} \lesssim \|\varrho\|_{L^1_t(B^s_{p_1,1})} \leq t\|\varrho\|_{L^\infty_t(B^s_{p_1,1})},$$

hence $\widetilde{C}_1\|\Delta_{-1}\varrho\|_{L^1_t(L^{p_1})}$ can be absorbed by the left-hand side if t is small.

Remark B.3. *Another natural way to deal with the transport term $u\cdot\nabla\varrho$ is to view it as a source term and we apply Proposition A.1 to it :*

$$\|u\cdot\nabla\varrho\|_{B^s_{p_1,1}} \lesssim \|u\|_{B^{\frac{d}{p_2}}_{p_2,1}}\|\nabla\varrho\|_{B^s_{p_1,1}} + \|u\|_{B^{\frac{d}{p_2}+\delta}_{p_2,1}}\|\nabla\varrho\|_{B^{s-\delta}_{p_1,1}}$$

$$\leq C_\varepsilon\Big(\|u\|^2_{B^{\frac{d}{p_2}}_{p_2,1}} + \|u\|^{2/(1+\delta)}_{B^{\frac{d}{p_2}+\delta}_{p_2,1}}\Big)\|\varrho\|_{B^s_{p_1,1}} + \varepsilon\|\varrho\|_{B^{s+2}_{p_1,1}},\ -\min\{\frac{d}{p_1'},\frac{d}{p_2}\} < s \leq \frac{d}{p_2}+\delta,\ \delta\in(-1,1].$$

The following a priori estimate concerning high regularity holds true :

Proposition B.2. *Let ϱ satisfy* (B.1) *and the triplet (s,p,r) verify*

$$s \geq \frac{d}{p},\quad p \in]1,+\infty[,\quad r\in[1,+\infty],\quad \text{with } r = 1 \text{ if } s = 1+\frac{d}{p} \text{ or } \frac{d}{p}. \tag{B.9}$$

Then there exists a positive constant C such that the following estimate holds true :

$$\|\varrho\|_{\widetilde{L}^\infty_t(B^s_{p,r})} + \|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})} \leq C\,e^{C\mathcal{K}_H(t)}\Big(\|\varrho_0\|_{B^s_{p,r}} + \|f\|_{\widetilde{L}^1_t(B^s_{p,r})}\Big), \tag{B.10}$$

where we have defined $\mathcal{K}_H(0) = 0$ and

$$\mathcal{K}'_H(t) := 1 + \|\nabla u\|_{B^{\frac{d}{p}}_{p,1}\cap B^{s-1}_{p,r}} + \|\nabla\kappa\|_{B^s_{p,r}} + \|\nabla\kappa\|^2_{L^\infty}. \tag{B.11}$$

Proof. We follow the proof of Proposition B.1.

It is easy to get (B.8) with p instead of p_1, for all $j \geq 0$. Now, we divide both members of the previous inequality by $\|\varrho_j\|_{L^p}^{p-1}$ and we integrate in time; then we multiply by 2^{js} and take the ℓ^r norm with respect to j. Hence, for all t we get :

$$\|\varrho\|_{\widetilde{L}_t^\infty(B^s_{p,r})} + \|\varrho\|_{\widetilde{L}_t^1(B^{s+2}_{p,r})} \leq C\bigg(\|\varrho_0\|_{B^s_{p,r}} + 2^{-(s+2)} \|\Delta_{-1}\varrho\|_{L^1_t(L^p)} + \|f\|_{\widetilde{L}^1_t(B^s_{p,r})} + \\ + \left\| 2^{js} \int_0^t \left\|\mathcal{R}^1_j(\tau)\right\|_{L^p} d\tau \right\|_{\ell^r} + \left\| 2^{js} \int_0^t \left\|\mathcal{R}^2_j(\tau)\right\|_{L^p} d\tau \right\|_{\ell^r} \bigg). \tag{B.12}$$

The low-frequency term $\Delta_{-1}\varrho$ can be easily bounded. As a matter of fact, by definition we immediately have, for all t,

$$2^{-(s+2)} \|\Delta_{-1}\varrho\|_{L^1_t(L^p)} \leq C \int_0^t \|\varrho\|_{B^s_{p,r}} d\tau . \tag{B.13}$$

For the first commutator term, by Proposition 0.4, we get

$$\int_0^t \left\| 2^{js} \left\|\mathcal{R}^1_j\right\|_{L^p} \right\|_{\ell^r} d\tau \lesssim \begin{cases} \int_0^t \|\nabla u\|_{B^{\frac{d}{p}}_{p,1}} \|\varrho\|_{B^s_{p,r}} & \text{if } s \in (-d\min\{\frac{d}{p}, \frac{d}{p'}\}, 1+\frac{d}{p}), \\ \int_0^t \|\nabla u\|_{B^{s-1}_{p,r}} \|\varrho\|_{B^s_{p,r}} & \text{if } s \geq 1+\frac{d}{p}, \text{ and } r=1 \text{ if } s = 1+\frac{d}{p}. \end{cases} \tag{B.14}$$

For the second commutator term, instead, we apply (A.24) with $s_1 = s$, $s_2 = s+2$ and $s_3 = s+1$, and we get, for any small $\varepsilon > 0$ and some corresponding constant C_ε,

$$\left\| 2^{js} \int_0^t \left\|R^2_j\right\|_{L^p} \right\|_{\ell^r} d\tau \leq \varepsilon \|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})} + C_\varepsilon \int_0^t \left(\|\nabla\kappa\|_{B^s_{p,r}} + \|\nabla\kappa\|^2_{L^\infty} \right) \|\varrho\|_{B^s_{p,r}} d\tau \tag{B.15}$$

for all $s \geq d/p$, with $r = 1$ if $s = d/p$, such that embedding $B^{d/p}_{p,r} \hookrightarrow L^\infty$ holds true.

We put (B.13), (B.14) and (B.15) into (B.12), choose sufficiently small ε and apply Gronwall's lemma to arrive at (B.10). $\square$

Remark B.4. *According to* (B.11)*, we need only* $\nabla\kappa \in L^1_t(B^s_{p,r})$*, instead of* $\nabla\kappa \in L^2_t(B^s_{p,r})$*. And it is easy to see that*

$$\mathcal{K}_H(t) = \int_0^t \mathcal{K}'_H \lesssim t + \|\nabla u\|_{L^1_t(B^{d/p}_{p,1} \cap B^{s-1}_{p,r})} + \|\nabla\kappa\|^2_{L^2_t(B^s_{p,r})}.$$

Remark B.5. *Just as mentioned in Remark B.3, we can replace* $\|\nabla u\|_{L^1_t(B^{d/p}_{p,1} \cap B^{s-1}_{p,r})}$ *by* $\|u\|^2_{L^2_t(B^s_{p,r})}$ *in* $\mathcal{K}_H$*, without assuming the divergence-free condition on* u*.*

B.2 Parabolic equations in spaces with infinite Lebesgue exponent

The aim of this section is to show an estimate for parabolic equations in Besov spaces of type $B^s_{\infty,r}$, which is needed to prove Theorem 3.6.

Proposition B.3. *Let* ρ *be such that* $\rho - 1 \in \mathcal{S}$ *solves the following linear parabolic-type system*

$$\begin{cases} \partial_t\rho - \operatorname{div}(\kappa\nabla\rho) = f, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{B.16}$$

with $\kappa, f, \rho_0 \in \mathcal{S}$ *and*

$$0 < \rho_* \leq \rho_0 \leq \rho^*, \quad 0 < \kappa_* \leq \kappa(t,x) \leq \kappa^*.$$

If $s > 0$, *then the following estimate holds true :*

$$\|\rho\|_{\widetilde{L}_t^\infty(B^s_{\infty,r}) \cap \widetilde{L}_t^1(B^{s+2}_{\infty,r})} \leq Ce^{C\mathcal{K}_\infty(t)} \times \left(\|\rho_0\|_{B^s_{\infty,r}} + \|f\|_{\widetilde{L}_t^1(B^s_{\infty,r})} \right), \tag{B.17}$$

where C *is a constant depending on* $d, s, r, \rho_*, \rho^*, \kappa_*, \kappa^*$ *and* $\|\kappa\|_{L_t^\infty(\dot{C}^\epsilon)}$ *for some* $\epsilon \in (0,1)$, *and*

$$\mathcal{K}_\infty(t) := \int_0^t \left(1 + \|\nabla\kappa\|^2_{L^\infty} + \|\nabla\kappa\|^{\max\{2/(1+s),1\}}_{B^s_{\infty,r}} \right) d\tau \,.$$

Remark B.6. *Let us point out that this proposition provides us with estimates analogous to those found in Proposition B.2. Moreover, althrough Equation* (B.16) *involves no transport term* $u \cdot \nabla\rho$, *we can view it as a source term and apply product estimate to it. Finally by Gronwall's inequality we still have* (B.10) *for the case*

$$s \geq 1, \ \text{with } r = 1 \text{ if } s = 1,$$

with $\mathcal{K}'_H(t)$ *replaced by*

$$(\mathcal{K}^u_\infty)' = 1 + \|u\|^2_{L^\infty} + \|u\|_{B^s_{\infty,r}} + \|\nabla\kappa\|^2_{L^\infty} + \|\nabla\kappa\|_{B^s_{\infty,r}}.$$

We will first consider the case $s \in (0,1)$ and $r = \infty$, i.e. we take solutions in time-dependent-Hölder's spaces $\widetilde{L}_T^\infty(C^\epsilon) \cap \widetilde{L}_T^1(C^{2+\epsilon})$, $\epsilon \in (0,1)$. Since Maximum Principle applied to parabolic equations has already given us the control on low frequencies of the solution

$$\|\rho\|_{L_t^\infty(L^\infty)} \leq \|\rho_0\|_{L^\infty} + \int_0^t \|f\|_{L^\infty}, \tag{B.18}$$

and, besides, the classical a priori estimates for heat equations are simpler (at least formally, see Proposition 0.7) in homogeneous Besov spaces, we only have to focus on

$$\dot{E}^\epsilon \triangleq \widetilde{L}_T^\infty(\dot{C}^\epsilon) \cap \widetilde{L}_T^1(\dot{C}^{2+\epsilon}), \quad \epsilon \in (0,1).$$

We can localize the function ρ into countable many functions ϱ_n, each of which is supported on some ball $B(x_n, \delta)$, with the small radius $\delta \in (0,1)$ to be determined in the proof. It is useful to use another description of time-dependent Hölder's spaces stated in Proposition B.5 : roughly speaking, up to a constant, $\dot{E}^\epsilon$-norm can be determined locally, which reduces to bound $\{\varrho_n\}$ instead of the whole ρ. The systems for $\{\varrho_n\}$ (see System (B.23) below) derive from (B.16) by mutiplying by some partition of unity and thus the coefficient κ can be viewed as a small perturbation of a function which depends only on time t, because ϱ_n is supported on a ball with sufficiently small radius. Consequently changing the time variable and making use of estimates of heat equations entail an a priori estimate for ϱ_n in $\dot{E}^\epsilon$, with the "source" terms being either small or of lower regularity, and hence easy to control. At last we will show how the Hölder case will yield the general one.

We agree here that in this section $\{\varrho_n(t,x)\}$ always denote localized functions of $\rho(t,x)$ in x-space, while ρ_j as usual, denotes $\{\Delta_j\rho\}$ (localization in the phase space).

Before going on, Proposition 0.7 ensures that for any $s \in \mathbb{R}$, and $f_0, f, F \in \mathcal{S}(\mathbb{R}^d)$ such that

$$F(t,x) \triangleq e^{t\Delta} f_0 + \int_0^t e^{(t-\tau)\Delta} f(\tau)\, d\tau,$$

there exists a constant C_0 such that

$$\|F\|_{L^\infty_T(\dot{C}^s)\cap\widetilde{L}^1_T(\dot{C}^{2+s})} \leq C_0(\|f_0\|_{\dot{C}^s} + \|f\|_{\widetilde{L}^1_T(\dot{C}^s)}). \tag{B.19}$$

It is also convenient to show an a priori estimate for the paraproduct $\dot{T}_v u$ in the space $\widetilde{L}^1_T(\dot{C}^s)$:

Proposition B.4. *For any $s > 0$, $\varepsilon > 0$, $a, b > 0$, there exists a constant $C_\varepsilon \sim \varepsilon^{-a/b}$ such that*

$$\|\dot{T}_v u\|_{\widetilde{L}^1_T(\dot{C}^s)} \leq C_\varepsilon \int_0^T \|u(t)\|_{\dot{C}^s}^{\frac{a+b}{b}} \|v\|_{\dot{C}^{-a}} + \varepsilon\|v\|_{\widetilde{L}^1_T(\dot{C}^b)}. \tag{B.20}$$

Proof. First, let us notice that,

$$\|\dot{T}_v u\|_{\widetilde{L}^1_T(\dot{C}^s)} \leq \int_0^T \sum_{j\in\mathbb{Z}} \|u\|_{\dot{C}^s}\|\dot{\Delta}_j v\|_{L^\infty}.$$

For any $\varepsilon > 0$, $a, b > 0$ and $t \in (0, T)$, we fix an integer

$$N_t = \left[\frac{1}{b}\log_2(\varepsilon^{-1}\|u(t)\|_{\dot{C}^s})\right] + 1,$$

then noticing that $\varepsilon^{-1}\|u(t)\|_{\dot{C}^s} \sim 2^{N_t b}$, we have

$$\begin{aligned}
\int_0^T \sum_{j\in\mathbb{Z}} \|u\|_{\dot{C}^s}\|\dot{\Delta}_j v\|_{L^\infty} &\leq \int_0^T \sum_{j\leq N_t} 2^{ja}\|u(t)\|_{\dot{C}^s}2^{-ja}\|\dot{\Delta}_j v\|_{L^\infty} + \sum_{j\geq N_t+1} 2^{-jb}\|u(t)\|_{\dot{C}^s}2^{jb}\|\dot{\Delta}_j v\|_{L^\infty} \\
&\lesssim \int_0^T 2^{N_t a}\|u(t)\|_{\dot{C}^s}\|v\|_{\dot{C}^{-a}} + \sum_{j\geq N_t+1} 2^{-(j-N_t)b}\,\varepsilon\,2^{jb}\|\dot{\Delta}_j v\|_{L^\infty} \\
&\lesssim \int_0^T \varepsilon^{-a/b}\|u(t)\|_{\dot{C}^s}^{(a+b)/b}\|v\|_{\dot{C}^{-a}} + \varepsilon\sum_{j\geq 1} 2^{-jb}\,2^{(j+N_t)b}\|\dot{\Delta}_{j+N_t} v\|_{L^\infty} \\
&\lesssim \varepsilon^{-a/b}\int_0^T \|u(t)\|_{\dot{C}^s}^{(a+b)/b}\|v\|_{\dot{C}^{-a}} + \varepsilon\sup_j \int_0^T 2^{jb}\|\dot{\Delta}_j v\|_{L^\infty}.
\end{aligned}$$

Thus the result follows. □

There is also an equivalent description for Hölder's spaces :

Proposition B.5. $\forall\epsilon \in (0, 1)$*, there exists a constant C such that for all $u \in \mathcal{S}$,*

$$C^{-1}\|u\|_{\dot{C}^\epsilon} \leq \left\|\frac{\|u(x+y) - u(x)\|_{L^\infty_x}}{|y|^\epsilon}\right\|_{L^\infty_y} \leq C\|u\|_{\dot{C}^\epsilon}, \tag{B.21}$$

and

$$C^{-1}\|u\|_{\widetilde{L}^1_t(\dot{C}^\epsilon)} \leq \left\|\int_0^t \frac{\|u(x+y) - u(x)\|_{L^\infty_x}}{|y|^\epsilon}\right\|_{L^\infty_y} \leq C\|u\|_{\widetilde{L}^1_t(\dot{C}^\epsilon)}. \tag{B.22}$$

Proof. The proof of (B.21) is very classic. Let us just show the left-hand inequality of (B.22). The inverse inequality follows immediately after similar changes with respect to time in the classical proof. Since

$$\dot{\Delta}_j u(t, x) = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)(u(t, x-y) - u(t, x))\,dy,$$

then

$$\int_0^t 2^{j\epsilon}\|\dot{\Delta}_j u(\tau,\cdot)\|_{L^\infty_x} \le 2^{jd}\int_{\mathbb{R}^d} 2^{j\epsilon}|y|^\epsilon |h(2^j y)| \int_0^t \frac{\|u(\tau,x-y)-u(\tau,x)\|_{L^\infty_x}}{|y|^\epsilon}\,d\tau\,dy$$
$$\le C\left\|\int_0^t \frac{\|u(x+y)-u(x)\|_{L^\infty_x}}{|y|^\epsilon}\right\|_{L^\infty_y},$$

which ensures the left -hand inequality of (B.22). □

Next we will prove Proposition B.3 in three steps. In Step 1 we will mainly deal with the localized solutions $\{\varrho_n\}$ or equivalently, after a one-to-one transformation in time variables, with $\{\widetilde{\varrho}_n\}$, which solve heat equations. Thus Estimate (B.19) ensures an a priori estimate for $\widetilde{\varrho}_n$ and Proposition B.4 provides the control on "source" terms. Thanks to Proposition B.5, we can carry the results from $\{\varrho_n\}$ to ρ : this will be done in Step 2. In order to handle general Besov spaces of form $B^s_{\infty,r}$, we again localize the system, but in Fourier variables, in Step 3. There we apply the result in Step 2 to ρ_j and a tricky calculation on commutator term will yield the thesis.

Step 1 Estimate for ϱ_n in $\dot{E}^\epsilon$

Let us take first a partition of unity $\{\psi_n\}_{n\in\mathbb{N}}$ of class C^3 subordinated to a locally finite covering of $\mathbb{R}^d$. We suppose also that the ψ_n's satisfy the following conditions :

(i) $\operatorname{Supp}\psi_n \subset B(x_n,\delta) \triangleq B_n$, $\forall\, n\in\mathbb{N}$, with $\delta<1$ to be determined later ;

(ii) $\sum_n \psi_n \equiv 1$;

(iii) $0\le\psi_n\le 1$, $\forall n\in\mathbb{N}$ with $\psi_n\equiv 1$ on $B(x_n,\delta/2)$;

(iv) $\|\nabla^\eta\psi_n\|_{L^\infty}\le C|\delta|^{-|\eta|}$, $\forall n\in\mathbb{N}$, for $|\eta|\le 3$;

(v) for each $x\in\mathbb{R}^d$, there are at most 3 elements in $\{\psi_n\}_{n\in\mathbb{N}}$ such that $x\in\operatorname{Supp}\psi_n$.

Now by multiplying ψ_n to Equation (B.16), we get the equation for compactly supported function $\varrho_n \triangleq \rho\psi_n$ which is supported on B_n :

$$\begin{cases} \partial_t\varrho_n - \bar{\kappa}_n\Delta\varrho_n = (\kappa-\bar{\kappa}_n)\Delta\varrho_n + \nabla\kappa\cdot\nabla\varrho_n + g_n, \\ \varrho_n|_{t=0} = \varrho_{0,n} = \psi_n\rho_0, \end{cases} \tag{B.23}$$

where

$$\bar{\kappa}_n(t) \triangleq \frac{1}{\mathrm{vol}(B_n)}\int_{B_n}\kappa(t,y)\,dy \quad \text{is a function depending only on } t,$$

and

$$g_n = -2\kappa\nabla\psi_n\cdot\nabla\rho - (\kappa\Delta\psi_n + \nabla\kappa\cdot\nabla\psi_n)\rho + f\psi_n. \tag{B.24}$$

For convenience we suppose that there exists a positive constant $C_\kappa \sim \|\kappa\|_{L^\infty_{t_0}(\dot{C}^\epsilon)}$ such that

$$|\kappa(t,x)-\kappa(t,y)| \le C_\kappa|x-y|^\epsilon, \quad \forall\, x,y\in\mathbb{R}^d,\ t\in[0,t_0]. \tag{B.25}$$

Noticing that, by (B.25), we have $\bar{\kappa}_n\ge\kappa_*>0$, this ensures that, for all $t\in[0,t_0]$,

$$\|\kappa/\bar{\kappa}_n - 1\|_{L^\infty(B_n)} \le \kappa_*^{-1}\left\|\frac{1}{\mathrm{vol}(B_n)}\int_{B_n}\kappa(t,x)-\kappa(t,y)\,dy\right\|_{L^\infty(B_n)} \le C_\kappa\kappa_*^{-1}\delta^\epsilon. \tag{B.26}$$

In order to get rid of the variable coefficient $\bar{\kappa}_n(t)$, let us make the one-to-one change in time variable

$$\tau \triangleq S(t) = \int_0^t \bar{\kappa}_n(t')\,dt'. \tag{B.27}$$

Therefore, the new unknown

$$\widetilde{\varrho}_n(\tau, x) \triangleq \rho_n(t, x),$$

satisfies (noticing that $\frac{d\tau}{dt} = \bar{\kappa}_n(t)$)

$$\begin{cases} \partial_\tau \widetilde{\varrho}_n - \Delta \widetilde{\varrho}_n = \left(\frac{\widetilde{\kappa}(\tau)}{\widetilde{\kappa}_n(\tau)} - 1\right) \Delta \widetilde{\varrho}_n + \frac{\nabla \widetilde{\kappa}(\tau)}{\widetilde{\kappa}_n(\tau)} \cdot \nabla \widetilde{\varrho}_n + \frac{\widetilde{g}_n(\tau)}{\widetilde{\kappa}_n(\tau)}, \\ \widetilde{\varrho}_n|_{\tau=0} = \varrho_{0,n}, \end{cases} \tag{B.28}$$

where $\widetilde{\kappa}(\tau, x) = \kappa(t, x)$, $\widetilde{\kappa}_n(\tau) = \bar{\kappa}_n(t)$, $\widetilde{\rho}(\tau, x) = \rho(t, x)$, $\widetilde{g}_n(\tau, x) = g_n(t, x)$.

This is a heat equation and hence according to (B.19), it rests to bound the "source" terms. Estimate (B.20) and the following estimate,

$$\|\dot{T}_u v + \dot{R}(u, v)\|_{\widetilde{L}^1_T(\dot{C}^\epsilon)} \leq C\|u\|_{L^\infty_T(L^\infty)} \|v\|_{\widetilde{L}^1_T(\dot{C}^\epsilon)},$$

imply that the first source term of Equation (B.28) can be controlled by

$$\begin{aligned} \left\| \left(\frac{\widetilde{\kappa}(\tau, \cdot)}{\widetilde{\kappa}_n(\tau)} - 1 \right) \Delta \widetilde{\varrho}_n(\tau, \cdot) \right\|_{\widetilde{L}^1_\tau(\dot{C}^\epsilon)} &\leq C \, \|\widetilde{\kappa}/\widetilde{\kappa}_n - 1\|_{L^\infty_\tau(L^\infty(B_n))} \, \|\Delta \widetilde{\varrho}_n\|_{\widetilde{L}^1_\tau(\dot{C}^\epsilon)} \\ &\quad + C_{\eta_1} \int_0^\tau \|\widetilde{\kappa}/\widetilde{\kappa}_n - 1\|_{\dot{C}^\epsilon}^{\frac{2}{\epsilon}} \, \|\Delta \widetilde{\varrho}_n\|_{\dot{C}^{\epsilon-2}} + \eta_1 \|\Delta \widetilde{\varrho}_n\|_{\widetilde{L}^1_\tau(\dot{C}^\epsilon)}, \end{aligned}$$

for any $\eta_1 \in (0, 1)$ with $C_{\eta_1} \sim \eta_1^{\frac{\epsilon-2}{\epsilon}}$. Besides, Inequality (B.26) ensures that for all $\tau \in [0, \tau_0]$, with $\tau_0 = S(t_0)$,

$$\|\widetilde{\kappa}/\widetilde{\kappa}_n - 1\|_{L^\infty_\tau(B_n)} \leq C_\kappa \kappa_*^{-1} \delta^\epsilon,$$

which implies, for $\tau \in [0, \tau_0]$,

$$\left\| \left(\frac{\widetilde{\kappa}(\tau, \cdot)}{\widetilde{\kappa}_n(\tau)} - 1 \right) \Delta \widetilde{\varrho}_n(\tau, \cdot) \right\|_{\widetilde{L}^1_\tau(\dot{C}^\epsilon)} \leq C_{\eta_1} \int_0^\tau \|\widetilde{\kappa}\|_{\dot{C}^\epsilon}^{\frac{2}{\epsilon}} \, \|\widetilde{\varrho}_n\|_{\dot{C}^\epsilon} + (C C_\kappa \kappa_*^{-1} \delta^\epsilon + \eta_1) \|\widetilde{\varrho}_n\|_{\widetilde{L}^1_\tau(\dot{C}^{2+\epsilon})}. \tag{B.29}$$

By Proposition A.3 (more precisely, the proof of Proposition A.3), for any $\eta > 0$, there exists $C_\eta \sim \eta^{-1}$ such that

$$\|\dot{T}_u v + \dot{R}(u, v)\|_{\widetilde{L}^1_\tau(\dot{C}^\epsilon)} \leq C_\eta \int_0^\tau \|u\|_{L^\infty}^2 \|v\|_{\dot{C}^{\epsilon-1}} + \eta \|v\|_{\widetilde{L}^1_\tau(\dot{C}^{\epsilon+1})}.$$

Thus, also by use of Proposition B.4 with $a = 1 - \epsilon$ and $b = 1 + \epsilon$, for any $\eta_2 \in (0, 1)$ with $C_{\eta_2} \sim \eta_2^{-1}$ we have the following :

$$\left\| \frac{\nabla \widetilde{\kappa}(\tau, \cdot)}{\widetilde{\kappa}_n(\tau)} \cdot \nabla \widetilde{\varrho}_n(\tau, \cdot) \right\|_{\widetilde{L}^1_\tau(\dot{C}^\epsilon)} \leq C_{\eta_2} \int_0^\tau \left(\|\nabla \widetilde{\kappa}\|_{L^\infty}^2 + \|\nabla \widetilde{\kappa}\|_{\dot{C}^\epsilon}^{\frac{2}{1+\epsilon}} \right) \|\widetilde{\varrho}_n\|_{\dot{C}^\epsilon} + \eta_2 \|\widetilde{\varrho}_n\|_{\widetilde{L}^1_\tau(\dot{C}^{2+\epsilon})}. \tag{B.30}$$

Now let us choose δ, η_1, η_2 such that

$$C_0 C C_\kappa \kappa_*^{-1} \delta^\epsilon, \; C_0 \eta_1, \; C_0 \eta_2 \leq 1/6, \tag{B.31}$$

with the same C_0 in (B.19). Then, from (B.19) and estimates (B.29), (B.30), for any $\tau \in [0, \tau_0]$ we get, for some "harmless" constant still denoted by C,

$$\begin{aligned} &\|\widetilde{\varrho}_n\|_{L^\infty_\tau(\dot{C}^\epsilon) \cap \widetilde{L}^1_\tau(\dot{C}^{2+\epsilon})} \\ &\quad \leq C \left(\|\varrho_{0,n}\|_{\dot{C}^\epsilon} + \int_0^\tau \left(\|\widetilde{\kappa}\|_{\dot{C}^\epsilon}^{\frac{2}{\epsilon}} + \|\nabla \widetilde{\kappa}\|_{L^\infty}^2 + \|\nabla \widetilde{\kappa}\|_{\dot{C}^\epsilon}^{\frac{2}{1+\epsilon}} \right) \|\widetilde{\varrho}_n\|_{\dot{C}^\epsilon} + \|\widetilde{g}_n\|_{\widetilde{L}^1_\tau(\dot{C}^\epsilon)} \right). \end{aligned}$$

Since $\kappa_* \le \bar{\kappa}_n \le \kappa^*$, we arrive at from transformation in time (B.27) that

$$\|\varrho_n\|_{L_t^\infty(\dot{C}^\epsilon)\cap\widetilde{L}_t^1(\dot{C}^{2+\epsilon})} \le C\left(\|\varrho_{0,n}\|_{\dot{C}^\epsilon} + \int_0^t \mathcal{K}_1\|\varrho_n\|_{\dot{C}^\epsilon} + \|g_n\|_{\widetilde{L}_t^1(\dot{C}^\epsilon)}\right), \quad \forall t \in [0, t_0], \quad \text{(B.32)}$$

with

$$\mathcal{K}_1 = \|\kappa\|_{\dot{C}^\epsilon}^{\frac{2}{\epsilon}} + \|\nabla\kappa\|_{L^\infty}^2 + \|\nabla\kappa\|_{\dot{C}^\epsilon}^{\frac{2}{1+\epsilon}},$$

provided

$$\delta^{-\epsilon} = C_\delta C_\kappa \le C_\delta\|\kappa\|_{L_{t_0}^\infty(\dot{C}^\epsilon)} \quad \text{for some constant } C_\delta \text{ depending only on } d, \epsilon. \qquad \text{(B.33)}$$

Step 2 Estimate for ρ

Now we return to consider $\rho = \sum_n \varrho_n$. It is easy to see that, by assumptions on the partition of unity $\{\psi_n\}$, for any x, there exist at most 4 balls of our covering which cover the small ball $B(x, \delta/4)$. Therefore, from Inequality (B.21) we have

$$\begin{aligned}\|\rho\|_{\widetilde{L}_t^1(\dot{C}^\epsilon)} &\le C\left\|\int_0^t \frac{\|\rho(x+y)-\rho(x)\|_{L_x^\infty}}{|y|^\epsilon}\right\|_{L_y^\infty}\\ &\le C\sup_{|y|>\delta/4}\int_0^t \frac{\|\rho(x+y)-\rho(x)\|_{L_x^\infty}}{|y|^\epsilon} + C\sup_{|y|\le\delta/4}\int_0^t \frac{\|\rho(x+y)-\rho(x)\|_{L_x^\infty}}{|y|^\epsilon},\end{aligned}$$

whose second term can be controlled by

$$4C\sup_{|y-z|\le\delta/4}\int_0^t \frac{\sup_n\|\varrho_n(x+y)-\varrho_n(x+z)\|_{L_x^\infty}}{|y-z|^\epsilon}.$$

Thus we find

$$\|\rho\|_{\widetilde{L}_t^1(\dot{C}^\epsilon)} \le C\delta^{-\epsilon}\int_0^t \|\rho\|_{L^\infty} + 4C\sup_n\|\varrho_n\|_{\widetilde{L}_t^1(\dot{C}^\epsilon)}.$$

Similarly, we have

$$\|\rho\|_{\widetilde{L}_t^\infty(\dot{C}^\epsilon)} \le C\delta^{-\epsilon}\|\rho\|_{L_t^\infty(L^\infty)} + C\sup_n\|\varrho_n\|_{\widetilde{L}_t^\infty(\dot{C}^\epsilon)}.$$

Since $\nabla^2\rho = \sum_n(\nabla^2\varrho_n)$, from the same arguments as before we infer

$$\|\rho\|_{\widetilde{L}_t^1(\dot{C}^{2+\epsilon})} \le C\|\nabla^2\rho\|_{\widetilde{L}_t^1(\dot{C}^\epsilon)} \le C\delta^{-\epsilon}\int_0^t \|\nabla^2\rho\|_{L^\infty} + C\sup_n\|\varrho_n\|_{\widetilde{L}_t^1(\dot{C}^{2+\epsilon})}.$$

Therefore, to sum up, for all $t \in [0, t_0]$,

$$\begin{aligned}\|\rho\|_{\widetilde{L}_t^\infty(\dot{C}^\epsilon)\cap\widetilde{L}_t^1(\dot{C}^{2+\epsilon})} &\le C\delta^{-\epsilon}\left(\|\rho\|_{L_t^\infty(L^\infty)} + \int_0^t \|\nabla^2\rho\|_{L^\infty}\right) + C\sup_n\|\varrho_n\|_{\widetilde{L}_t^\infty(\dot{C}^\epsilon)\cap\widetilde{L}_t^1(\dot{C}^{2+\epsilon})}\\ &\le C\delta^{-\epsilon}\left(\|\rho\|_{L_t^\infty(L^\infty)} + \int_0^t \|\nabla^2\rho\|_{L^\infty}\right)\\ &\quad + C\sup_n\left(\|\varrho_{0,n}\|_{\dot{C}^\epsilon} + \int_0^t \mathcal{K}_1\|\varrho_n\|_{\dot{C}^\epsilon} + \|g_n\|_{\widetilde{L}_t^1(\dot{C}^\epsilon)}\right),\end{aligned}$$

with the second inequality deriving from Estimate (B.32). Noticing (B.18) and

$$\|\varrho_n\|_{C^\epsilon} = \|\rho\psi_n\|_{C^\epsilon} \le C\|\rho\|_{C^\epsilon}\|\psi_n\|_{C^\epsilon} \le C\delta^{-\epsilon}\|\rho\|_{C^\epsilon},$$

we thus have

$$\begin{aligned}\|\rho\|_{\widetilde{L}^\infty_t(C^\epsilon)\cap\widetilde{L}^1_t(C^{2+\epsilon})} &\le C\|\rho\|_{L^\infty_t(L^\infty)} + C\int_0^t \|\rho\|_{L^\infty} + \|\rho\|_{\widetilde{L}^\infty_t(\dot C^\epsilon)\cap\widetilde{L}^1_t(\dot C^{2+\epsilon})}\\ &\le C\delta^{-\epsilon}\left(\|\rho_0\|_{C^\epsilon} + \int_0^t (\|\nabla^2\rho\|_{L^\infty} + \|\rho\|_{L^\infty} + \|f\|_{L^\infty}) + \int_0^t \mathcal{K}_1\|\rho\|_{C^\epsilon}\right)\\ &\quad + C\sup_n \|g_n\|_{\widetilde{L}^1_t(\dot C^\epsilon)},\end{aligned}$$

It rests us to bound g_n uniformly. In fact, by the definition of g_n (B.24), we follow the same method to get (B.29) and (B.30) to arrive at

$$\begin{aligned}\|g_n\|_{\widetilde{L}^1_t(\dot C^\epsilon)} &\le C\int_0^t \left(\eta^{-1}(\|\kappa\nabla\psi_n\|^2_{L^\infty} + \|\kappa\nabla\psi_n\|^{\frac{2}{1+\epsilon}}_{\dot C^\epsilon}) + \|\kappa\Delta\psi_n + \nabla\kappa\cdot\nabla\psi_n\|_{L^\infty}\right)\|\rho\|_{\dot C^\epsilon}\\ &\quad + \eta\|\rho\|_{\widetilde{L}^1_t(\dot C^{2+\epsilon})} + C\int_0^t \|\kappa\Delta\psi_n + \nabla\kappa\cdot\nabla\psi_n\|_{\dot C^\epsilon}\|\rho\|_{L^\infty} + C\delta^{-\epsilon}\int_0^t \|f\|_{L^\infty} + C\|f\|_{\widetilde{L}^1_t(\dot C^\epsilon)}\\ &\le C_\eta\delta^{-2}\int_0^t \left(1 + \|\kappa\|^{\frac{2}{1+\epsilon}}_{C^\epsilon} + \|\kappa\|_{C^{1+\epsilon}}\right)\|\rho\|_{C^\epsilon} + \eta\|\rho\|_{\widetilde{L}^1_t(\dot C^{2+\epsilon})} + C\delta^{-\epsilon}\|f\|_{\widetilde{L}^1_t(C^\epsilon)},\end{aligned}$$

where we have used $\|f\|_{L^1_t(L^\infty)\cap\widetilde{L}^1_t(\dot C^\epsilon)} \le C\|f\|_{\widetilde{L}^1_t(C^\epsilon)}$.

We finally get a priori estimate for ρ :

$$\|\rho\|_{\widetilde{L}^\infty_t(C^\epsilon)\cap\widetilde{L}^1_t(C^{2+\epsilon})} \le C\delta^{-\epsilon}\left(\|\rho_0\|_{C^\epsilon} + \int_0^t \|\rho\|_{C^2} + \|f\|_{\widetilde{L}^1_t(C^\epsilon)}\right) + C\delta^{-2}\int_0^t \mathcal{K}_2\|\rho\|_{C^\epsilon},$$

with

$$\mathcal{K}_2 = 1 + \|\kappa\|^{\frac{2}{1+\epsilon}}_{C^{1+\epsilon}} \ge C\left(\mathcal{K}_1 + 1 + \|\kappa\|^{\frac{2}{1+\epsilon}}_{C^\epsilon} + \|\kappa\|_{C^{1+\epsilon}}\right).$$

Thus by a direct interpolation inequality, that is to say

$$\delta^{-\epsilon}\|\rho\|_{L^1_t(C^2)} \le C_\eta\delta^{-2}\int_0^t \|\rho\|_{C^\epsilon} + \eta\|\rho\|_{\widetilde{L}^1_t(C^{2+\epsilon})},$$

Gronwall's Inequality tells us

$$\|\rho\|_{\widetilde{L}^\infty_t(C^\epsilon)\cap\widetilde{L}^1_t(C^{2+\epsilon})} \le C\delta^{-\epsilon}\exp\left(C\delta^{-2}\int_0^t \mathcal{K}_2\right)\left(\|\rho_0\|_{C^\epsilon} + \|f\|_{\widetilde{L}^1_t(C^\epsilon)}\right), \tag{B.34}$$

which is just Estimate (B.17) when $s = \epsilon$ and $r = \infty$.

Step 3 General Case $B^s_{\infty,r}$

Now we want to deal with the general case $B^s_{\infty,r}$. Let us apply $\widetilde{\Delta_j} = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$, $j \ge 0$, to System (B.16), yielding

$$\begin{cases} \partial_t\overline{\rho}_j - \operatorname{div}(\kappa\nabla\overline{\rho}_j) = \overline{f}_j - \overline{R}_j,\\ \overline{\rho}_j|_{t=0} = \overline{\rho}_{0,j}, \end{cases} \tag{B.35}$$

with

$$\overline{\rho}_j = \widetilde{\Delta_j}\rho, \quad \overline{f}_j = \widetilde{\Delta_j}f, \quad \overline{R}_j = \operatorname{div}([\kappa, \widetilde{\Delta_j}]\nabla\rho), \quad \overline{\rho}_{0,j} = \widetilde{\Delta_j}\rho_0.$$

We apply the a priori estimate (B.34) to the solution $\overline{\rho}_j$ of System (B.35), for some $\epsilon < s$, entailing

$$\|\overline{\rho}_j\|_{\widetilde{L}^\infty_t(C^\epsilon)\cap\widetilde{L}^1_t(C^{2+\epsilon})} \le C\delta^{-\epsilon}\exp\left(C\delta^{-2}\int_0^t \mathcal{K}_2\right)\left(\|\overline{\rho}_{0,j}\|_{C^\epsilon} + \|\overline{f}_j - \overline{R}_j\|_{\widetilde{L}^1_t(C^\epsilon)}\right).$$

Let us notice that for $j \geq 0$, denoted by $\rho_j = \Delta_j \rho$ and $\rho_q = \Delta_q \rho$ as usual, then we have

$$\Delta_j \overline{\rho}_j = \rho_j \quad \text{and} \quad \Delta_q \overline{\rho}_j \equiv 0 \text{ if } |q-j| \geq 2.$$

Hence the above inequality gives

$$2^{j\epsilon} \|\rho_j\|_{L_t^\infty(L^\infty)} + 2^{j(2+\epsilon)} \int_0^t \|\rho_j\|_{L^\infty} \\ \leq C\delta^{-\epsilon} \exp\left(C\delta^{-2} \int_0^t \mathcal{K}_2\right) \left(2^{j\epsilon} \sum_{|j-q|\leq 1} \left(\|\rho_{0,q}\|_{L^\infty} + \int_0^t \|f_q\|_{L^\infty}\right) + \|\overline{R}_j\|_{\widetilde{L}_t^1(C^\epsilon)}\right).$$

Let us focus on $\|\overline{R}_j\|_{\widetilde{L}_t^1(C^\epsilon)}$ for a while. As usual, we decompose $\overline{R}_j$ into four parts :

$$\overline{R}_j^1 = \sum_{|j-q|\leq 3} \operatorname{div}\left([S_{q-1}\widetilde{\kappa}, \widetilde{\Delta_j}]\nabla \Delta_q \rho\right), \quad \overline{R}_j^2 = \sum_{q \geq j-3} \operatorname{div} \widetilde{\Delta_j}(\Delta_q \widetilde{\kappa}\, S_{q+1} \nabla \rho),$$
$$\overline{R}_j^3 = \sum_{q\geq j-3} \operatorname{div}\left(\Delta_q \widetilde{\kappa}\, \widetilde{\Delta_j} S_{q+1} \nabla \rho\right), \quad \overline{R}_j^4 = \operatorname{div}\left([\Delta_{-1}\kappa, \widetilde{\Delta_j}]\, \nabla \rho\right), \quad \text{with} \quad \widetilde{\kappa} = \kappa - \Delta_{-1}\kappa, .$$

It is easy to see that the Fourier transform of the terms $\overline{R}_j^1$ and $\overline{R}_j^2$ is supported near the ring $2^j\mathcal{C}$; thus, we get for some sequence $(c_j)_j \in \ell^r$,

$$\|\overline{R}_j^1\|_{\widetilde{L}_t^1(C^\epsilon)} \leq C 2^{j(1+\epsilon)} \sum_{|j-q|\leq 3} \int_0^t \|[S_{q-1}\widetilde{\kappa}, \widetilde{\Delta_j}]\nabla \Delta_q \rho\|_{L^\infty} \\ \leq C 2^{j(\epsilon - s)}\, c_j \int_0^t \|\nabla \kappa\|_{L^\infty} \|\nabla \rho\|_{B^s_{\infty,r}}$$

and, for some $s > -1$ and $(c_j)_j \in \ell^r$

$$\|\overline{R}_j^2\|_{\widetilde{L}_t^1(C^\epsilon)} \leq C 2^{j(1+\epsilon)} \sum_{q\geq j-3} \int_0^t \|\Delta_q \widetilde{\kappa}\|_{L^\infty} \|S_{q+1}\nabla\rho\|_{L^\infty} \\ \leq C 2^{j(\epsilon-s)} \sum_{q\geq j-3} 2^{(j-q)(1+s)} c_q \int_0^t \|\nabla\kappa\|_{B^s_{\infty,r}} \|\nabla \rho\|_{L^\infty}, \\ \leq C 2^{j(\epsilon-s)} c_j \int_0^t \|\nabla \kappa\|_{B^s_{\infty,r}} \|\nabla\rho\|_{L^\infty}.$$

Let us now consider $\overline{R}_j^3$ carefully. Each $\Delta_q \widetilde{\kappa}\, \widetilde{\Delta_j} S_{q+1} \nabla \rho$ has Fourier transform supported near a ring of small radius proportional to $2^q - 2^j$, and big radius proportional to $2^q + 2^j \sim 2^q$. Therefore, arguing as above, we find, for some $(c_j)_j \in \ell^r$

$$\|\overline{R}_j^3\|_{\widetilde{L}_t^1(C^\epsilon)} \leq \sup_{j' \geq j-2} 2^{j'\epsilon} \sum_{q\geq j-3, q\sim j'} \int_0^t \|\Delta_{j'} \operatorname{div}\left(\Delta_q \widetilde{\kappa} \widetilde{\Delta_j} S_{q+1}\nabla\rho\right)\|_{L^\infty} \\ \leq \sup_{j'\geq j-2} 2^{j'(\epsilon+1)} \sum_{q\geq j-3, q\sim j'} \int_0^t 2^{-q} \|\Delta_q \nabla \widetilde{\kappa}\|_{L^\infty} \|\Delta_j \nabla \rho\|_{L^\infty} \\ \leq C 2^{j(\epsilon-s)} c_j \int_0^t \|\nabla\kappa\|_{B^s_{\infty,r}} \|\nabla \rho\|_{L^\infty}.$$

Since $\overline{R}_j^4$'s Fourier transform is always supported near a ring $2^j\mathcal{C}$, arguing for instance as in the proof of Proposition A.3, we can get

$$\|\overline{R}_j^4\|_{\widetilde{L}_t^1(C^\epsilon)} \leq 2^{j(\epsilon-s)}c_j \int_0^t \|\nabla\kappa\|_{L^\infty} \|\nabla\rho\|_{B^s_{\infty,r}}.$$

To conclude, we have got a priori estimate for ρ_j :

$$\|\rho_j\|_{L_t^\infty(L^\infty)} + 2^{2j}\int_0^t \|\rho_j\|_{L^\infty} \leq C\delta^{-\epsilon}\exp\left(C\delta^{-2}\int_0^t \mathcal{K}_2\right)$$
$$\times\left(\sum_{|j-q|\leq 2}\left(\|\rho_{0,q}\|_{L^\infty} + \int_0^t \|f_q\|_{L^\infty}\right) + 2^{-js}c_j\int_0^t \|\nabla\kappa\|_{B^s_{\infty,r}}\|\nabla\rho\|_{L^\infty} + \int_0^t \|\nabla\kappa\|_{L^\infty}\|\nabla\rho\|_{B^s_{\infty,r}}\right).$$

Therefore, we multiply both sides by 2^{js} and then take ℓ^r norm, to arrive at for $s > -1$,

$$\|\rho\|_{\widetilde{L}_t^\infty(B^s_{\infty,r})\cap\widetilde{L}_t^1(B^{s+2}_{\infty,r})} \leq C\delta^{-\epsilon}\exp\left(C\delta^{-2}\int_0^t \mathcal{K}_2\right)$$
$$\times\left(\|\rho_0\|_{B^s_{\infty,r}} + \|f\|_{\widetilde{L}_t^1(B^s_{\infty,r})} + \int_0^t \|\nabla\kappa\|_{L^\infty}\|\nabla\rho\|_{B^s_{\infty,r}} + \|\nabla\kappa\|_{B^s_{\infty,r}}\|\nabla\rho\|_{L^\infty}\right).$$

The direct application of interpolation inequalities and embedding results reduces the above estimate to (B.17).

B.3 Linearized Euler equation

In this section we consider the following linearized Euler equations for the unknowns u and q, with variable coefficient λ before the pressure ∇q :

$$\begin{cases} \partial_t u + v\cdot\nabla u + \lambda\nabla q = h, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{B.36}$$

As usual, we suppose the given functions v, λ, h, u_0 are smooth and decay sufficiently at infinity. But pay attention here that the transport velocity v is not assumed to be solenoidal. The following a priori estimate holds :

Proposition B.6. *Let*

$$s > \frac{d}{p} - \frac{d}{4}, \quad p\in[2,4], \quad r\in[1,+\infty], \qquad \textit{with} \qquad r = 1 \;\; \textit{if} \;\; s = \frac{d}{p} \;\; \textit{or} \;\; 1+\frac{d}{p}. \tag{B.37}$$

Suppose $0 < \lambda_* \leq \lambda(t,x)$ *for all* $t\geq 0$ *and* $x\in\mathbb{R}^d$.

Then for any smooth solution u *of* (B.36)*, there exists a positive constant* C *such that the following estimates hold true :*

$$\|u\|_{\widetilde{L}_t^\infty(B^s_{p,r})} \leq Ce^{CV(t)}\Big(\|u_0\|_{B^s_{p,r}} + \|h\|_{\widetilde{L}_t^1(B^s_{p,r})\cap L_t^1(L^2)}\Big), \tag{B.38}$$

$$\|\nabla q\|_{\widetilde{L}_t^1(B^s_{p,r})\cap L_t^1(L^2)} \leq C\Big(\|h\|_{\widetilde{L}_t^1(B^s_{p,r})\cap L_t^1(L^2)} + V(t)\|u\|_{\widetilde{L}_t^\infty(B^s_{p,r})}\Big), \tag{B.39}$$

where $V(t)$ *is defined by* (0.53) *and the constant* C *depends on* $d, p, s, r, \lambda_*, \Lambda$ *with*

$$\Lambda(t) := \|\lambda\|_{L_t^\infty(L^\infty)} + \begin{cases} \|\nabla\lambda\|_{\widetilde{L}_t^\infty(B^{\frac{d}{p}}_{p,1})} & \textit{if } s\in(\frac{d}{p}-\frac{d}{4}, 1+\frac{d}{p}], \\ \|\nabla\lambda\|_{\widetilde{L}_t^\infty(B^{s-1}_{p,r})} & \textit{if } s > 1+\frac{d}{p}. \end{cases} \tag{B.40}$$

Proof. From Proposition 0.6 we easily get the following estimate for u with $V(t)$ defined by (0.53) :

$$\|u(t)\|_{\widetilde{L}_t^\infty(B^s_{p,r})} \le e^{C(d,p,s)V(t)}\Big(\|u_0\|_{B^s_{p,r}} + \|h-\lambda\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})}\Big). \tag{B.41}$$

Just as in last section §B.4, we decompose $\lambda\nabla q$ into $(\Delta_{-1}\lambda)\nabla q + ((Id-\Delta_{-1})\lambda)\nabla q$. Then by product estimates given in Proposition 0.3, we have

$$\|\lambda\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} \le C\Lambda(t)\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})}, \text{ if } s > -\min\{\frac{d}{p},\frac{d}{p'}\}, \text{ with } r=1 \text{ if } s=\frac{d}{p}, \tag{B.42}$$

with Λ defined by (B.40). Hence it is sufficient to estimate $\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})}$, in order to obtain Estimate (B.38).

According to Equality $(\mathrm{B.36})_2$, applying the operator "div" to the first equation of (B.36) yields the elliptic equation for q :

$$\mathrm{div}\,(\lambda\nabla q) = \mathrm{div}\,(h - v\cdot\nabla u) = \mathrm{div}\,(h - u\cdot\nabla v + u\,\mathrm{div}\,v). \tag{B.43}$$

We apply the localization operator Δ_j to (B.43), we multiply $|q_j|^{p-2}q_j$, then we integrate with respect to space variable and we use the Bernstein type Inequality (0.41) and Hölder's Inequality, and we finally find

$$\lambda_* 2^{2j}\|q_j\|_{L^p} \lesssim \|\Delta_j \mathrm{div}\,(h-v\cdot\nabla u)\|_{L^p} + \|\mathrm{div}\,[\lambda,\Delta_j]\nabla q\|_{L^p},\ \forall j\ge 0.$$

On the other hand, for $p\ge 2$, classic Bernstein's inequalities entail

$$\|\Delta_{-1}\nabla q\|_{L^p} \le C\|\nabla q\|_{L^2}, \quad 2^j\|q_j\| \sim \|\nabla q_j\|,\ \forall j\ge 0,$$

which ensure that

$$\begin{aligned}\lambda_*\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} \lesssim\ & \|\nabla q\|_{L^1_t(L^2)} + \|\mathrm{div}\,(h-v\cdot\nabla u)\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})} \\ & + \|2^{j(s-1)}\|\mathrm{div}\,[\lambda,\Delta_j]\nabla q\|_{L^1_t(L^p)}\|_{\ell^r}.\end{aligned} \tag{B.44}$$

Let us focus on the above commutator term for a while. Proposition 0.4 implies that it can be controlled by

$$\|2^{j(s-1)}\|\mathrm{div}\,[\lambda,\Delta_j]\nabla q\|_{L^1_t(L^p)}\|_{\ell^r} \lesssim \Lambda\|\nabla q\|_{L^1_t(B^{s-1}_{p,r})}, \text{ if } s > -\min\{\frac{d}{p},\frac{d}{p'}\}.$$

Motivatived by the embeddings (3.26) and $L^2 \hookrightarrow B^{\frac{d}{p}-\frac{d}{2}}_{p,\infty}$, $\forall p\ge 2$, we apply interpolation inequalities between Besov spaces, obtaining

$$\|\nabla q\|_{L^1_t(B^{s-1}_{p,r})} \lesssim \|\nabla q\|_{\widetilde{L}^1_t(B^{s+\epsilon-1}_{p,r})} \lesssim \|\nabla q\|^{1-\delta}_{\widetilde{L}^1_t(B^{\frac{d}{p}-\frac{d}{2}}_{p,\infty})}\|\nabla q\|^\delta_{\widetilde{L}^1_t(B^s_{p,r})} \lesssim \|\nabla q\|^{1-\delta}_{L^1_t(L^2)}\|\nabla q\|^\delta_{\widetilde{L}^1_t(B^s_{p,r})},$$

for some $\epsilon\in(0,1)$, such that

$$\delta = \frac{s+\epsilon-1-(d/p-d/2)}{s-(d/p-d/2)} \in (0,1), \quad \text{i.e.} \quad s > \frac{d}{p}-\frac{d}{2}+1-\epsilon.$$

Therefore, one can use directly Young's Inequality on the above bound for commutator term

$$\|2^{j(s-1)}\|\mathrm{div}\,[\lambda,\Delta_j]\nabla q\|_{L^1_t(L^p)}\|_{\ell^r} \le C(d,s,p,r,\varepsilon,\epsilon)\Lambda^{1/(1-\delta)}\|\nabla q\|_{L^1_t(L^2)} + \varepsilon\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})},$$

such that for some small enough $\varepsilon > 0$, Estimate (B.44) becomes

$$\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} \leq C(d,s,p,r,\varepsilon,\epsilon,\lambda_*)\Lambda^{1/(1-\delta)}\|\nabla q\|_{L^1_t(L^2)} + C\|\mathrm{div}\,(h - v\cdot\nabla u)\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})}. \quad \text{(B.45)}$$

Thanks to Proposition 0.9, we have already got by Equation (B.43),

$$\|\nabla q\|_{\widetilde{L}^1_t(B^s_{p,r})} \leq C(d,s,p,r,\lambda_*,\Lambda)\Big(\|h - u\cdot\nabla v + u\,\mathrm{div}\,v\|_{L^1_t(L^2)} + \|\mathrm{div}\,(h - v\cdot\nabla u)\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})}\Big).$$

It rests to dealing with

$$\|u\cdot\nabla v\|_{L^1_t(L^2)}, \quad \|u\,\mathrm{div}\,v\|_{L^1_t(L^2)} \quad \text{and} \quad \|\mathrm{div}\,(v\cdot\nabla u)\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})}.$$

We can easily find, by Hölder's Inequality and embedding results, that

$$\|u\cdot\nabla v\|_{L^1_t(L^2)} \leq \int_0^t \|u\|_{L^4}\|\nabla v\|_{L^4} \lesssim \int_0^t \|u\|_{B^{s_1}_{4,\infty}}\|\nabla v\|_{B^{s_2}_{4,\infty}}\,d\tau\,, \quad \forall s_1, s_2 > 0\,.$$

Hence for $p \leq 4$, $s > \dfrac{d}{p} - \dfrac{d}{4}$, we have

$$\|u\cdot\nabla v\|_{L^1_t(L^2)} \lesssim \int_0^t \|u\|_{B^s_{p,r}}\|\nabla v\|_{B^{\frac{d}{p}}_{p,\infty}}\,d\tau\,. \quad \text{(B.46)}$$

The term $u\mathrm{div}\,v$ is actually analogous to the previous one.

On the other hand, recalling the divergence-free condition over u, it is easy to decompose $\|\mathrm{div}\,(v\cdot\nabla u)\|_{B^{s-1}_{p,r}}$ into

$$\|T_{\partial_i v^j}\partial_j u^i + T_{\partial_j u^i}\partial_i v^j + \mathrm{div}\,\big(R(v^j,\partial_j u)\big)\|_{B^{s-1}_{p,r}}, \quad \text{(B.47)}$$

which can be controlled, according to Proposition 0.2, by

$$V'(t)\|\nabla u\|_{B^{s-1}_{p,r}}, \quad \text{if} \quad s > -\min\{\frac{d}{p}, \frac{d}{p'}\},$$

with $V'(t)$ defined by (0.53).

To conclude, in the case $p \in [2,4]$, $s > \frac{d}{p} - \frac{d}{4}$, Estimate (B.39) holds and hence estimate (B.38) also holds true, according to Estimate (B.41) and Gronwall's Inequality.

□

B.4 Linearized Stokes equation

In this section we will deal with the following linearized Stokes equation with variable coefficients :

$$\begin{cases} \partial_t u + v\cdot\nabla u - \mathrm{div}\,(\eta\nabla u) + \lambda\nabla Q &= h, \\ \mathrm{div}\,u &= 0, \\ u|_{t=0} &= u_0, \end{cases} \quad \text{(B.48)}$$

where λ, η, v, h, u_0 be smooth and decay sufficiently at infinity with $\mathrm{div}\,v = 0$. Let us first consider the case $p_1 = p_2 = 2$ which may be handled by standard energy arguments.

Proposition B.7. *Let $(u,\nabla Q)$ satisfy* (B.48) *on $[0,T]\times\mathbb{R}^d$. Let $s \in [0, d/2]$. Suppose that, for some positive constants λ_* and Λ^*, we have*

$$\|\nabla\lambda\|_{L^\infty_T(B^{d/2-1}_{2,1})} + \|\lambda\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq \Lambda^* \quad \textit{and} \quad \min(\lambda,\eta) \geq \lambda_* > 0. \quad \text{(B.49)}$$

There exists a constant $c_\lambda = c_\lambda(d,s)$ such that, if for some integer N, one has

$$\inf_{x\in\mathbb{R}^d, t\in[0,T]} S_N\lambda(t,x) \geq \lambda_*/2, \quad \|\lambda - S_N\lambda\|_{L^\infty_T(B^{d/2}_{2,1})} \leq c_\lambda\lambda_*, \tag{B.50}$$

then there exist constants $c_2(d,\lambda_*)$, $C_2(d,s,\lambda_*,\Lambda^*,N)$, $\widetilde{C}_2(d,s,\lambda_*)$, $C_3(d,s,\lambda_*,\Lambda^*,N)$, such that for all $t\in[0,T]$,

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_t(B^s_{2,1})} + c_2\|u\|_{L^1_t(B^{s+2}_{2,1})} \leq & e^{C_2\left(\|\nabla v\|_{L^1_t(B^{d/2}_{2,1})}+\|\nabla\eta\|^2_{L^2_t(B^{d/2}_{2,1})}\right)} \\ & \times\left(\|u_0\|_{B^s_{2,1}} + \widetilde{C}_2\|\Delta_{-1}u\|_{L^1_t(L^2)} + C_2\|h\|_{L^1_t(B^s_{2,1})}\right),\end{aligned} \tag{B.51}$$

$$\|\nabla Q\|_{L^1_t(B^s_{2,1})} \leq C_3\int_0^t\left(\|h\|_{B^s_{2,1}} + \|\nabla v\|_{B^{d/2}_{2,1}}\|u\|_{B^s_{2,1}} + \|\nabla\eta\|_{B^{d/2}_{2,1}}\|\nabla u\|_{B^s_{2,1}}\right)d\tau. \tag{B.52}$$

Proof. Following the proof of Proposition B.1, we apply Δ_j to (1.37). This yields

$$\partial_t u_j + v\cdot\nabla u_j - \operatorname{div}(\eta\nabla u_j) = h_j + \mathcal{R}^3_j - \mathcal{R}^4_j - \Delta_j(\lambda\nabla Q),$$

where

$$u_j := \Delta_j u, \quad h_j := \Delta_j h, \quad \mathcal{R}^3_j := [v,\Delta_j]\cdot\nabla u, \quad \mathcal{R}^4_j := \operatorname{div}([\eta,\Delta_j]\cdot\nabla u).$$

As in the proof of Proposition B.1, we apply Inequalities (A.11) and (A.12) to the two commutators $\mathcal{R}^3_j$ and $\mathcal{R}^4_j$, getting if $-d/2 < s \leq d/2$,

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_t(B^s_{2,1})} + c_2(d)\lambda_*\|u\|_{L^1_t(B^{s+2}_{2,1})} \leq \|u_0\|_{B^s_{2,1}} + \widetilde{C}_2(c_2,s)\|\Delta_{-1}u\|_{L^1_t(L^2)} + \|h\|_{L^1_t(B^s_{2,1})} \\ + C(d,s,\lambda_*)\int_0^t\left(\|\nabla v\|_{B^{d/2}_{2,1}}\|u\|_{B^s_{2,1}} + \|\nabla\eta\|^2_{B^{d/2}_{2,1}}\|\nabla u\|_{B^s_{2,1}}\right)dt' + \|\lambda\nabla Q\|_{L^1_t(B^s_{2,1})}.\end{aligned} \tag{B.53}$$

We now have to bound ∇Q. We follow the analysis for the pressure ∇q in the proof of Proposition B.6. Applying the divergence operator to the first equation of (B.48), we then arrive at the following elliptic equation *with variable coefficients* (similar to (B.43)) [1] :

$$\operatorname{div}(\lambda\nabla Q) = \operatorname{div} L \quad \text{with} \quad L := -u\cdot\nabla v + \nabla\eta\cdot(\nabla u)^T + h. \tag{B.54}$$

Then (B.44) holds for ∇Q, with $p=2$ and $h - v\cdot\nabla u$ replaced by L. Applying the commutator estimate (A.9) on $\|[\lambda,\Delta_j]\nabla Q\|_{L^2}$, we easily get if $-d/2 < \nu \leq 1$ and $-d/2 < s \leq \nu + d/2$,

$$\lambda_*\|\nabla Q\|_{L^1_t(B^s_{2,1})} \leq \|L\|_{L^1_t(B^s_{2,1})} + C_Q(d,s,\nu)\|\nabla\lambda\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})}\|\nabla Q\|_{L^1_t(B^{s-\nu}_{2,1})}. \tag{B.55}$$

Now we consider two cases :

– Case $0 < s \leq d/2$. Let us first assume that $\nabla\lambda$ has some extra regularity : suppose for instance that it belongs to $L^\infty_T(B^{d/2+\nu-1}_{2,1})$ for some ν such that $\nu + d/2 \geq s > \nu > 0$. As $\|\cdot\|_{B^0_{2,2}} = \|\cdot\|_{L^2}$ we arrive (by interpolation) at

$$\|\nabla Q\|_{B^{s-\nu}_{2,1}} \lesssim \|\nabla Q\|^{\nu/s}_{L^2}\|\nabla Q\|^{1-\nu/s}_{B^s_{2,1}} \lesssim \|L\|^{\nu/s}_{L^2}\|\nabla Q\|^{1-\nu/s}_{B^s_{2,1}}. \tag{B.56}$$

Hence (B.55) implies that

$$\|\nabla Q\|_{L^1_t(B^s_{2,1})} \leq C(d,s,\nu,\lambda_*)(1 + \|\nabla\lambda\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})})^{s/\nu}\|L\|_{L^1_t(B^s_{2,1})}. \tag{B.57}$$

1. Here we use that $\operatorname{div}(v\cdot\nabla u) = \operatorname{div}(u\cdot\nabla v)$ and $\operatorname{div}\big(\operatorname{div}(\eta\nabla u)\big) = \operatorname{div}\big(\nabla u\cdot\nabla\eta\big)$ owing to $\operatorname{div} u = \operatorname{div} v = 0$.

Now, if λ satisfies only Conditions (B.49) and (B.50) then we decompose it into

$$\lambda = \lambda_N + (\lambda - \lambda_N) \quad \text{with } \lambda_N := S_N\lambda.$$

Note that $\nabla\lambda_N \in H^\infty$ and that the equation for Q recasts in

$$\operatorname{div}(\lambda_N \nabla Q) = \operatorname{div}(L + E_N), \quad \text{where} \quad E_N = (\lambda_N - \lambda)\cdot\nabla Q.$$

Therefore, following the procedure leading to (B.55) and bearing the first part of Condition (B.50) in mind, yields

$$\frac{\lambda_*}{2}\|\nabla Q\|_{L^1_t(B^s_{2,1})} \le \|L+E_N\|_{L^1_t(B^s_{2,1})} + C_Q(d,s,\nu)\|\nabla\lambda_N\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})}\|\nabla Q\|_{L^1_t(B^{s-\nu}_{2,1})}. \tag{B.58}$$

We notice that for $-d/2 < s \le d/2$,

$$\begin{aligned}
\|\nabla\lambda_N\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})} &\le C_\lambda(d)2^{N\nu}\|\nabla\lambda\|_{L^\infty_t(B^{d/2-1}_{2,1})},\\
\|E_N\|_{L^1_t(B^s_{2,1})} &\le C_\lambda(d,s)\|\lambda_N-\lambda\|_{L^\infty_t(B^{d/2}_{2,1})}\|\nabla Q\|_{L^1_t(B^s_{2,1})},
\end{aligned}$$

hence the term pertaining to E_N may be absorbed by the left-hand side of (B.58) if c_λ is small enough in (B.50). Then using the same interpolation argument as above, we end up with

$$\lambda_*\|\nabla Q\|_{L^1_t(B^s_{2,1})} \le C_Q(d,s,\nu)2^{N\nu}\big(1+\|\nabla\lambda\|_{L^\infty_t(B^{d/2-1}_{2,1})}\big)^{s/\nu}\|L\|_{L^1_t(B^s_{2,1})}. \tag{B.59}$$

Now, in order to complete the proof of Inequality (B.52), it is only a matter of using the product estimates (ii) stated in Proposition A.2 for bounding L, which implies that

$$\|L\|_{B^s_{2,1}} \lesssim \|\nabla v\|_{B^{\frac d2}_{2,1}}\|u\|_{B^s_{2,1}} + \|\nabla\eta\|_{B^{\frac d2}_{2,1}}\|\nabla u\|_{B^s_{2,1}} + \|h\|_{B^s_{2,1}} \quad \text{if} \quad -d/2 < s \le d/2.$$

– Case $s=0$: in this case, the interpolation inequality (B.56) fails, so that we have to modify the proof accordingly. First we apply Inequality (B.55) for some $0<\nu<1$, and Inequality (A.9), to get :

$$\begin{aligned}
\|[\lambda,\Delta_j]\nabla Q\|_{L^1_t(B^0_{2,1})} &\lesssim c_j\|\nabla\lambda\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})}\|\nabla Q\|_{L^1_t(B^{0-\nu}_{2,1})}\\
&\lesssim c_j\|\nabla\lambda\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})}\|L\|_{L^1_t(L^2)},
\end{aligned}$$

hence

$$\begin{aligned}
\|\nabla Q\|_{L^1_t(B^0_{2,1})} &\lesssim \|L\|_{L^1_t(B^0_{2,1})} + \|\nabla\lambda\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})}\|L\|_{L^1_t(L^2)}\\
&\le C(d,\nu,m)(1+\|\nabla\lambda\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})})\|L\|_{L^1_t(B^0_{2,1})},
\end{aligned} \tag{B.60}$$

which is quite similar as (B.57) and hence the same procedure implies also (B.59).

In order to prove (B.51), it suffices to plug the above estimate for the pressure in (B.53). The main point is that, if $-d/2 < s \le d/2$ then we have (similar as (B.42))

$$\|\lambda\nabla Q\|_{B^s_{2,1}} \lesssim \big(\|\lambda\|_{L^\infty} + \|\nabla\lambda\|_{B^{\frac d2-1}_{2,1}}\big)\|\nabla Q\|_{B^s_{2,1}}.$$

Then Gronwall lemma leads to Inequality (B.51). □

Remark B.7. *In Proposition B.7 we need the assumption $s \geq 0$ to get the necessary L^2 estimate for ∇Q. However, some negative indices may be achieved by duality arguments. As the corresponding estimates are not needed in this thesis, we here do not give more details on that issue.*

Remark B.8. *Similar as pointed out in Remark B.2, the quantity $\widetilde{C}_2\|\Delta_{-1}u\|_{L^1_t(L^2)}$ in the a priori estimate* (B.51) *can be absorbed if the time t is small.*
Similar as mentioned in Remark B.3, without Condition $\mathrm{div}\, v = 0$*, the quantity $\|\nabla v\|_{L^1_t(B^{d/2}_{2,1})}$ can be replaced by $\|v\|^2_{L^2_t(B^{d/2}_{2,1})}$ in the exponential of Estimate* (B.51)*, according to Identity* $\mathrm{div}\,(v\cdot\nabla u) \equiv \mathrm{div}\,(u\cdot\nabla v + u\,\mathrm{div}\, v)$ *because of* $\mathrm{div}\, u = 0$.

Remark B.9. *Compared to Λ defined by* (B.40)*, Λ^* here (see* (B.49)*) requires less regularity, thanks to a decomposition of λ into a big smooth part $S_N\lambda$ and a small remainder $\lambda - S_N\lambda$, given by* (B.50)*.*

We now want to extend Proposition B.7 to more general Besov spaces which are not directly related to the energy space. To simplify the presentation, we focus on the regularity exponent $s = d/p_2 - 1$ which is the only one that we will have to consider in the proof of Theorem 1.2 :

Proposition B.8. *Let $T > 0$ and $(u, \nabla Q)$ be a solution to* (B.48) *on $[0,T]\times\mathbb{R}^d$. Let p_1 be in $[1,\infty)$ and $p_2 \in [2,4]$ satisfy*

$$p_2 \leq \frac{2p_1}{p_1-2} \quad \text{if} \quad p_1 > 2, \qquad \frac{1}{p_2} \leq \frac{1}{p_1} + \frac{1}{d}, \qquad \text{and} \quad (p_1,p_2) \neq (4,4) \quad \text{if} \quad d=2. \tag{B.61}$$

Assume that there exist two constants λ_*, Λ^*, a small enough constant c_λ which depends only on d, p_1, p_2 and an integer $N \in \mathbb{N}$ such that

$$\begin{aligned} &\min(\eta,\lambda) \geq \lambda_*, \quad \|\lambda\|_{L^\infty([0,T]\times\mathbb{R}^d)} + \|\nabla\lambda\|_{L^\infty_T(B^{d/p_1-1}_{p_1,1})} \leq \Lambda^*, \\ &\inf_{x\in\mathbb{R}^d, t\in[0,T]} S_N\lambda(t,x) \geq \lambda_*/2, \quad \|\lambda - S_N\lambda\|_{L^\infty_T(B^{d/p_1}_{p_1,1})} \leq c_\lambda\lambda_*. \end{aligned} \tag{B.62}$$

Then there exist constants $c_2(d,p_2,\lambda_*)$, $C_2(d,p_1,p_2,\lambda_*,\Lambda^*,N)$, $\widetilde{C}_2(d,p_2,\lambda_*)$, $C_3(d,p_1,p_2,\lambda_*,\Lambda^*,N)$ such that the following a priori estimates hold :

$$\begin{aligned} \|u\|_{\widetilde{L}^\infty_t(B^{d/p_2-1}_{p_2,1})} &+ c_2\|u\|_{L^1_t(B^{d/p_2+1}_{p_2,1})} \\ &\leq e^{C_2\left(\|\nabla v\|_{L^1_t(B^{d/p_2}_{p_2,1})} + \|\nabla v\|^{4/3}_{L^{4/3}_t(B^{d/p_2-1/2}_{p_2,1})} + \|\nabla\eta\|^2_{L^2_t(B^{d/p_1}_{p_1,1})} + \|\nabla\eta\|^{2/(1-\delta)}_{L^{2/(1-\delta)}_t(B^{d/p_1-\delta}_{p_1,1})}\right)} \\ &\times\Big(\|u_0\|_{B^{d/p_2}_{p_2,1}} + \widetilde{C}_2\|\Delta_{-1}u\|_{L^1_t(L^{p_2})} + C_2\|h\|_{L^1_t(L^2\cap B^{d/p_2-1}_{p_2,1})}\Big), \end{aligned} \tag{B.63}$$

with $\delta = \min(1/2, d/p_1)$, and

$$\begin{aligned} \|\nabla Q\|_{L^1_t(B^{\frac{d}{p_2}-1}_{p_2,1}\cap L^2)} \leq C_3\int_0^t \Big(&\|h\|_{L^2\cap B^{d/p_2-1}_{p_2,1}} + \|\nabla v\|_{B^{d/p_2}_{p_2,1}}\|u\|_{B^{d/p_2-1}_{p_2,1}} + \|u\cdot\nabla v\|_{L^2} \\ &+ \|\nabla\eta\|_{B^{d/p_1}_{p_1,1}}\|\nabla u\|_{B^{d/p_2-1}_{p_2,1}} + \|\nabla u\cdot\nabla\eta\|_{L^2}\Big)d\tau. \end{aligned} \tag{B.64}$$

Proof. With the notation in the proof of Proposition B.7, we have

$$\|(2^{j(d/p_2-1)}\|\mathcal{R}_j^3\|_{L^{p_2}})_{j\in\mathbb{Z}}\|_{\ell^1} \lesssim \|\nabla v\|_{B^{d/p_2}_{p_2,1}}\|u\|_{B^{d/p_2-1}_{p_2,1}}, \tag{B.65}$$

$$\|(2^{j(d/p_2-1)}\|\mathcal{R}_j^4\|_{L^{p_2}})_{j\in\mathbb{Z}}\|_{\ell^1} \lesssim \|\nabla \eta\|_{B^{d/p_1}_{p_1,1}}\|\nabla u\|_{B^{d/p_2-1}_{p_2,1}}, \tag{B.66}$$

$$\|\lambda\nabla Q\|_{B^{d/p_2-1}_{p_2,1}} \lesssim \big(\|\lambda\|_{L^\infty}+\|\nabla\lambda\|_{B^{d/p_1-1}_{p_1,1}}\big)\|\nabla Q\|_{B^{d/p_2-1}_{p_2,1}}. \tag{B.67}$$

Indeed [2] Inequality (B.65) follows from (A.9) with "p_1" = "p_2" = p_2, "ν" = 1, "s" = d/p_2-1 (note that the condition $p_2<2d$ is not required for $\operatorname{div} v=0$, a consequence of (A.12) with "s" = d/p_2 and "ν" = 1) while (B.66) stems from (A.12) with "p_1" = p_2, "p_2" = p_1, "ν" = 1, "s" = d/p_2 (here we need that $1/p_2\le 1/p_1+1/d$) ; and (B.67) is a consequence of (A.6) with "s_1" = d/p_1, "s_2" = d/p_2-1 (which requires that $1/p_2\le 1/p_1+1/d$ and $1/p_1+1/p_2>1/d$).

Now, granted with the above inequalities, the same procedure as in Proposition B.1 yields

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_t(B^{d/p_2-1}_{p_2,1})}+c_2\|u\|_{L^1_t(B^{d/p_2+1}_{p_2,1})} &\le \|u_0\|_{B^{d/p_2-1}_{p_2,1}}+c_2 2^{-(d/p_2+1)}\|\Delta_{-1}u\|_{L^1_t(L^{p_2})}\\ &+C(d,p_1,p_2,\lambda_*)\int_0^t(\|\nabla v\|_{B^{d/p_2}_{p_2,1}}+\|\nabla\eta\|^2_{B^{d/p_1}_{p_1,1}})\|u\|_{B^{d/p_2-1}_{p_2,1}}\,dt'\\ &+\|h\|_{L^1_t(B^{d/p_2-1}_{p_2,1})}+\big(\|\lambda\|_{L^\infty_t(L^\infty)}+\|\nabla\lambda\|_{L^\infty_t(B^{d/p_1-1}_{p_1,1})}\big)\|\nabla Q\|_{L^1_t(B^{d/p_2-1}_{p_2,1})}.\end{aligned} \tag{B.68}$$

So bounding ∇Q is our next task. First of all, using the fact that Q satisfies the elliptic equation (B.54), we still have

$$\lambda_*\|\nabla Q\|_{L^2}\le\|L\|_{L^2}\quad\text{with}\quad L=h+u\cdot\nabla v+\nabla u\cdot\nabla\eta.$$

Thus (B.44) holds for ∇Q with $p=p_2$, $s=d/p_2-1$ and $h-v\cdot\nabla u$ replaced by L.

Applying Inequality (A.9) with

$$\text{“}\varphi\text{”}=\lambda,\quad \text{“}\psi\text{”}=\nabla Q,\quad \text{“}p_1\text{”}=p_2,\quad \text{“}p_2\text{”}=p_1,\quad \text{“}\nu\text{”}=1/4,\quad \text{“}s\text{”}=d/p_2-1,$$

(which is possible provided $1/p_1+1/p_2>1/d$ and $1/p_2\le 1/p_1+5/(4d)$), we get

$$\|[\lambda,\Delta_j]\nabla Q\|_{L^{p_2}}\lesssim 2^{-j(d/p_2-1)}c_j\|\nabla\lambda\|_{B^{d/p_1-3/4}_{p_1,1}}\|\nabla Q\|_{B^{d/p_2-5/4}_{p_2,1}}\quad\text{with}\quad\sum_j c_j=1. \tag{B.69}$$

In the case $d\ge3$, arguing by interpolation, we get

$$\|\nabla Q\|_{B^{d/p_2-5/4}_{p_2,1}}\lesssim\|\nabla Q\|^{\frac{2d-5}{2d-4}}_{B^{d/p_2-1}_{p_2,1}}\|\nabla Q\|^{\frac{1}{2d-4}}_{B^{d/p_2-d/2}_{p_2,2}}. \tag{B.70}$$

Therefore, together with (B.44), this implies that

$$\lambda_*\|\nabla Q\|_{B^{d/p_2-1}_{p_2,1}}\lesssim\|L\|_{B^{d/p_2-1}_{p_2,1}}+\big(1+\|\nabla\lambda\|_{B^{d/p_1-3/4}_{p_1,1}}\big)^{2d-4}\|L\|_{L^2}. \tag{B.71}$$

In the case $d=2$, the interpolation inequality (B.70) fails. However, from (B.69) we directly get for p_1,p_2 satisfying (B.61),

$$\begin{aligned}\lambda_*\|\nabla Q\|_{B^{2/p_2-1}_{p_2,1}}&\lesssim\|L\|_{B^{2/p_2-1}_{p_2,1}}+\|\nabla\lambda\|_{B^{2/p_1-3/4}_{p_1,1}}\|\nabla Q\|_{B^{2/p_2-5/4}_{p_2,1}}\\&\lesssim\|L\|_{B^{2/p_2-1}_{p_2,1}}+(1+\|\nabla\lambda\|_{B^{2/p_1-3/4}_{p_1,1}})\|L\|_{L^2}.\end{aligned} \tag{B.72}$$

2. Here it is understood that the quote marks designate the indices in the original inequalities (A.6), (A.9) and (A.12).

Therefore, in any dimension $d \geq 2$, we have

$$\lambda_* \|\nabla Q\|_{B^{d/p_2-1}_{p_2,1}\cap L^2} \leq C_{Q,d,p_1,p_2}\big(1+\|\nabla\lambda\|_{B^{d/p_1-3/4}_{p_1,1}}\big)^{\max(1,2d-4)}\|L\|_{B^{d/p_2-1}_{p_2,1}\cap L^2}. \tag{B.73}$$

In order to treat the case where $\nabla\lambda$ is only in $L^\infty_T(B^{d/p_1-1}_{p_1,1})$, we proceed exactly as in the proof of Proposition B.7, decomposing λ into two parts, the smooth large part $S_N\lambda$ and the small rough part $\lambda - S_N\lambda$. Under the same assumptions as in (B.67), we find that

$$\|E_N\|_{B^{d/p_2-1}_{p_2,1}} \leq C_{\lambda,p_1,p_2}\|\lambda-\lambda_N\|_{B^{d/p_1}_{p_1,1}}\|\nabla Q\|_{B^{d/p_2-1}_{p_2,1}}.$$

Therefore, if c_λ is small enough in (B.62), we get

$$\|\nabla Q\|_{L^1_t(B^{d/p_2-1}_{p_2,1}\cap L^2)} \leq C_{Q,d,p_1,p_2}2^{N/4}\big(1+\|\nabla\lambda\|_{L^\infty_T(B^{d/p_1-1}_{p_1,1})}\big)^{\max(1,2d-4)}\|L\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}. \tag{B.74}$$

In order to complete the proof of (B.64), we now have to bound L. First, we notice that, applying (A.6) with "p_1" = "p_2" = p_2, "s_1" = d/p_2 and "s_2" = d/p_2-1 yields, if $p_2 < 2d$,

$$\|u\cdot\nabla v\|_{B^{d/p_2-1}_{p_2,1}} \lesssim \|\nabla v\|_{B^{d/p_2}_{p_2,1}}\|u\|_{B^{d/p_2-1}_{p_2,1}}. \tag{B.75}$$

If $2d \leq p_2 < \infty$ then, owing to $\operatorname{div} u = 0$, the same inequality is true. Indeed applying Bony's decomposition, we discover that

$$u\cdot\nabla v = T_u\nabla v + T_{\nabla v}u + \operatorname{div} R(u, v-\Delta_{-1}v) + R(u,\Delta_{-1}\nabla v). \tag{B.76}$$

The first two terms may be bounded as in (B.75). For the third one, one has (because $\mathcal{F}(v-\Delta_{-1}v)$ is supported away from the origin)

$$\begin{aligned}\|\operatorname{div} R(u,v-\Delta_{-1}v)\|_{B^{d/p_2-1}_{p_2,1}} &\lesssim \|R(u,v-\Delta_{-1}v)\|_{B^{d/p_2}_{p_2,1}}\\ &\lesssim \|u\|_{B^{d/p_2-1}_{p_2,1}}\|v-\Delta_{-1}v\|_{B^{d/p_2+1}_{p_2,1}} \lesssim \|u\|_{B^{d/p_2-1}_{p_2,1}}\|\nabla v\|_{B^{d/p_2}_{p_2,1}}.\end{aligned}$$

And finally, we have

$$R(u,\Delta_{-1}\nabla v) = \sum_{-1\leq j\leq 0}\Delta_j\Delta_{-1}\nabla v\,(\Delta_{j-1}+\Delta_j+\Delta_{j+1})u,$$

so it is clear that we have

$$\|R(u,\Delta_{-1}\nabla v)\|_{B^{d/p_2-1}_{p_2,1}} \lesssim \|R(u,\Delta_{-1}\nabla v)\|_{L^{p_2}} \lesssim \|\nabla v\|_{L^\infty}\|S_2 u\|_{L^{p_2}} \lesssim \|\nabla v\|_{B^{d/p_2}_{p_2,1}}\|u\|_{B^{d/p_2-1}_{p_2,1}}.$$

Next, just as in (B.67), under the conditions $1/p_2 \leq 1/p_1 + 1/d$ and $1/p_1 + 1/p_2 > 1/d$, we have

$$\|\nabla u\cdot\nabla\eta\|_{B^{d/p_2-1}_{p_2,1}} \lesssim \|\nabla\eta\|_{B^{d/p_1}_{p_1,1}}\|\nabla u\|_{B^{d/p_2-1}_{p_2,1}}. \tag{B.77}$$

This gives (B.64).

In order to complete the proof of the lemma, we still have to bound $u\cdot\nabla v$ and $\nabla u\cdot\nabla\eta$ in L^2. To handle the former term, we just use the fact that

$$B^{d/p_2-1/2}_{p_2,1} \hookrightarrow L^4 \quad \text{if} \quad p_2 \leq 4.$$

Hence, by virtue of Hölder's inequality,

$$\|u\cdot\nabla v\|_{L^2} \lesssim \|u\|_{B^{d/p_2-1/2}_{p_2,1}}\|\nabla v\|_{B^{d/p_2-1/2}_{p_2,1}}. \tag{B.78}$$

Concerning the latter term, if both p_1 and p_2 are less than or equal to 4 then one may merely use the embedding

$$B^{d/p_1-1/2}_{p_1,1} \hookrightarrow L^4 \quad \text{and} \quad B^{d/p_2-1/2}_{p_2,1} \hookrightarrow L^4,$$

hence

$$\|\nabla u \cdot \nabla \eta\|_{L^2} \lesssim \|\nabla \eta\|_{B^{d/p_1-1/2}_{p_1,1}} \|\nabla u\|_{B^{d/p_2-1/2}_{p_2,1}}.$$

Now, if $p_1 > 4$ then we first write

$$\|\nabla u \cdot \nabla \eta\|_{L^2} \leq \|\nabla \eta\|_{L^{p_1}} \|\nabla u\|_{L^{\widetilde{p}_1}} \quad \text{with} \;\; \widetilde{p}_1 = \frac{2p_1}{p_1 - 2}\cdot$$

Let $\delta = \min(1/2, d/p_1)$. Then we notice that if $p_2 \leq \widetilde{p}_1$ then

$$B^{d/p_1-\delta}_{p_1,1} \hookrightarrow L^{p_1} \quad \text{and} \quad B^{d/p_2-1+\delta}_{p_2,1} \hookrightarrow L^{\widetilde{p}_1}.$$

Therefore

$$\|\nabla u \cdot \nabla \eta\|_{L^2} \lesssim \|\nabla \eta\|_{B^{d/p_1-\delta}_{p_1,1}} \|\nabla u\|_{B^{d/p_2-1+\delta}_{p_2,1}}.$$

Together with (B.68), interpolation inequalities and Gronwall lemma, this enables us to complete the proof of (B.63). □

Bibliographie

[1] H. Abidi and M. Paicu. Existence globale pour un fluide inhomogène. *Ann. Inst. Fourier (Grenoble)*, 57(3) :883–917, 2007.

[2] T. Alazard. Low Mach number limit of the full Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 180(1) :1–73, 2006.

[3] S.N. Antontsev, A.V. Kazhikhov, and V.N. Monakhov. *Boundary value problems in mechanics of nonhomogeneous fluids*, volume 22 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1990. Translated from the Russian.

[4] V. Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1) :319–361, 1966.

[5] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[6] C. Bardos, F. Golse, and D. Levermore. Sur les limites asymptotiques de la théorie cinétique conduisant à la dynamique des fluides incompressibles. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(11) :727–732, 1989.

[7] B.J. Bayly, C.D. Levermore, and T. Passot. Density variations in weakly compressible flows. *Phys. Fluids A*, 4(5) :945–954, 1992.

[8] J. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94(1) :61–66, 1984.

[9] H. Beirão da Veiga. Diffusion on viscous fluids. Existence and asymptotic properties of solutions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 10(2) :341–355, 1983.

[10] H. Beirão da Veiga. Singular limits in fluid dynamics. *Rend. Sem. Mat. Univ. Padova*, 94 :55–69, 1995.

[11] H. Beirão da Veiga, R. Serapioni, and A. Valli. On the motion of nonhomogeneous fluids in the presence of diffusion. *J. Math. Anal. Appl.*, 85(1) :179–191, 1982.

[12] H. Beirão da Veiga and A. Valli. Existence of C^∞ solutions of the Euler equations for nonhomogeneous fluids. *Comm. Partial Differential Equations*, 5(2) :95–107, 1980.

[13] H. Beirão da Veiga and A. Valli. On the Euler equations for nonhomogeneous fluids. I. *Rend. Sem. Mat. Univ. Padova*, 63 :151–168, 1980.

[14] H. Beirão da Veiga and A. Valli. On the Euler equations for nonhomogeneous fluids. II. *J. Math. Anal. Appl.*, 73(2) :338–350, 1980.

[15] J.-M. Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)*, 14(2) :209–246, 1981.

[16] D. Bresch and B. Desjardins. On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids. *J. Math. Pures Appl. (9)*, 87(1) :57–90, 2007.

[17] D. Bresch, El H. Essoufi, and M. Sy. Effect of density dependent viscosities on multiphasic incompressible fluid models. *J. Math. Fluid Mech.*, 9(3) :377–397, 2007.

[18] X. Cai, L. Liao, and Y. Sun. Global regularity for the initial value problem of a 2-D Kazhikhov-Smagulov type model. *Nonlinear Anal.*, 75(15) :5975–5983, 2012.

[19] C. Calgaro, E. Creusé, and T. Goudon. Simulation of mixture flows : pollution spreading and avalanches. *Submitted*.

[20] M. Cannone. A generalization of a theorem by Kato on Navier-Stokes equations. *Rev. Mat. Iberoamericana*, 13(3) :515–541, 1997.

[21] J.-Y. Chemin. Sur le mouvement des particules d'un fluide parfait incompressible bidimensionnel. *Invent. Math.*, 103(3) :599–629, 1991.

[22] J.-Y. Chemin. *Perfect incompressible fluids*, volume 14 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.

[23] Y. Cho and H. Kim. Unique solvability for the density-dependent Navier-Stokes equations. *Nonlinear Anal.*, 59(4) :465–489, 2004.

[24] Y. Cho and H. Kim. Existence results for viscous polytropic fluids with vacuum. *J. Differential Equations*, 228(2) :377–411, 2006.

[25] Y. Cho and H. Kim. On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities. *Manuscripta Math.*, 120(1) :91–129, 2006.

[26] P. Constantin and C. Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[27] R. Danchin. Global existence in critical spaces for compressible Navier-Stokes equations. *Invent. Math.*, 141(3) :579–614, 2000.

[28] R. Danchin. Global existence in critical spaces for flows of compressible viscous and heat-conductive gases. *Arch. Ration. Mech. Anal.*, 160(1) :1–39, 2001.

[29] R. Danchin. Local theory in critical spaces for compressible viscous and heat-conductive gases. *Comm. Partial Differential Equations*, 26(7-8) :1183–1233, 2001.

[30] R. Danchin. Zero mach number limit for compressible flows with periodic boundary conditions. *Amer. J. Math.*, 124(6) :1153–1219, 2002.

[31] R. Danchin. Zero mach number limit in critical spaces for compressible Navier-Stokes equations. *Ann. Sci. École Norm. Sup. (4)*, 35(1) :27–75, 2002.

[32] R. Danchin. Density-dependent incompressible visous fluids in critical spaces. *Proc. Roy. Soc. Edinburgh Sect. A*, 133(6) :1311–1334, 2003.

[33] R. Danchin. Local and global well-posedness results for flows of inhomogeneous viscous fluids. *Advances in differential equations*, 9(3-4) :353–386, 2004.

[34] R. Danchin. Uniform estimates for transport-diffusion equations. *J. Hyperbolic Differ. Equ.*, 4(1) :1–17, 2007.

[35] R. Danchin. On the well-posedness of the incompressible density-dependent euler equations in the L^p framework. *J. Differential Equations*, 248(8) :2130–2170, 2010.

[36] R. Danchin and F. Fanelli. The well-posedness issue for the density-dependent Euler equations in endpoint Besov spaces. *J. Math. Pures Appl.*, 96(3) :253 – 278, 2011.

[37] R. Danchin and X. Liao. On the well-posedness of the full low-Mach number limit system in general critical Besov spaces. *Commun. Contemp. Math.*, 14(3) :1250022, 2012.

[38] J.-M. Delort. Existence de nappes de tourbillon en dimension deux. *J. Amer. Math. Soc.*, 4(3) :553–586, 1991.

[39] B. Desjardins. Global existence results for the incompressible density-dependent Navier-Stokes equations in the whole space. *Differential Integral Equations*, 10(3) :587–598, 1997.

[40] B. Desjardins and E. Grenier. Low Mach number limit of viscous compressible flows in the whole space. *R. Sco. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 455(1986) :2271–2279, 1999.

[41] B. Desjardins, E. Grenier, P.-L. Lions, and N. Masmoudi. Incompressible limit for solutions of the isentropic Navier-Stokes equations with Dirichlet boundary conditions. *J. Math. Pures Appl. (9)*, 78(5) :461–471, 1999.

[42] D.G. Ebin. The motion of slightly compressible fluids viewed as a motion with strong constraining force. *Ann. of Math. (2)*, 105(1) :141–200, 1977.

[43] D.G. Ebin and J. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. of Math. (2)*, 92 :102–163, 1970.

[44] P. Embid. Well-posedness of the nonlinear equations for zero Mach number combustion. *Comm. Partial Differential Equations*, 12(11) :1227–1283, 1987.

[45] P. Embid. On the reactive and nondiffusive equations for zero Mach number flow. *Comm. Partial Differential Equations*, 14(8-9) :1249–1281, 1989.

[46] L.C. Evans and S. Müller. Hardy spaces and the two-dimensional Euler equations with nonnegative vorticity. *J. Amer. Math. Soc.*, 7(1) :199–219, 1994.

[47] F. Fanelli and X. Liao. The well-posedness issue in endpoint spaces for an inviscid low-Mach number limit system. *Submitted.*

[48] E. Feireisl. On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not square integrable. *Comment. Math. Univ. Carolin.*, 43(1) :83–98, 2001.

[49] E. Feireisl. *Dynamics of viscous compressible fluids*, volume 26 of *Oxford Lectures Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[50] E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids*. Advances in Mathematical Fluid Mechanics. Birkhäuser Verlag, Basel, 2009.

[51] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.*, 3(4) :358–392, 2001.

[52] D.A. Frank-Kamenetskii. *Diffusion and Heat Transfer in Chemical Kinetics*. Plenum, New York/London, 1969.

[53] H. Fujita and T. Kato. On the Navier-Stokes initial value problem. I. *Arch. Rational Mech. Anal.*, 16 :269–315, 1964.

[54] F. Guillén-González and J.V. Gutiérrez-Santacreu. Conditional stability and convergence of a fully discrete scheme for three-dimensional Navier-Stokes equations with mass diffusion. *SIAM J. Numer. Anal.*, 46(5) :2276–2308, 2008.

[55] F. Guillén-González and J.V. Gutiérrez-Santacreu. Unconditional stability and convergence of fully discrete schemes for 2D viscous fluids models with mass diffusion. *Math. Comp.*, 77(263) :1495–1524, 2008.

[56] T. Hagstrom and J. Lorenz. All-time existence of classical solutions for slightly compressible flows. *SIAM J. Math. Anal.*, 29(3) :652–672 (electronic), 1998.

[57] T. Hmidi and S. Keraani. Incompressible viscous flows in borderline Besov spaces. *Arch. Ration. Mech. Anal.*, 189(2) :283–300, 2008.

[58] D. Hoff. Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. *Trans. Amer. Math. Soc.*, 303(1) :169–181, 1987.

[59] D. Hoff. Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. *J. Differential Equations*, 120(1) :215–254, 1995.

[60] D. Hoff. Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids. *Arch. Rational Mech. Anal.*, 139(4) :303–354, 1997.

[61] D. Hoff. The zero-Mach limit of compressible flows. *Comm. Math. Phys.*, 192(3) :543–554, 1998.

[62] H. Isozaki. Singular limits for the compressible Euler equation in an exterior domain. *J. Reine Angew. Math.*, 381 :1–36, 1987.

[63] H. Isozaki. Singular limits for the compressible Euler equation in an exterior domain. ii. bodies in a uniform flow. *Osaka J. Math.*, 26(2) :399–410, 1989.

[64] N. Itaya. On the Cauchy problem for the system of fundamental equations describing the movement of compressible viscous fluid. *Kōdai Math. Sem. Rep.*, 23 :60–120, 1971.

[65] S. Itoh and A. Tani. Solvability of nonstationary problems for nonhomogeneous incompressible fluids and the convergence with vanishing viscosity. *Tokyo J. Math.*, 22(1) :17–42, 1999.

[66] V.I. Judovič. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 3 :1032–1066, 1963.

[67] T. Kato. Nonstationary flows of viscous and ideal fluids in $\mathbf{R}^3$. *J. Functional Analysis*, 9 :296–305, 1972.

[68] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(7) :891–907, 1988.

[69] A.V. Kazhikhov and V.V. Shelukhin. Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. *Prikl. Mat. Meh.*, 41(2) :282–291, 1977.

[70] A.V. Kazhikhov and Sh. Smagulov. The correctness of boundary value problems in a certain diffusion model of an inhomogeneous fluid. *Dokl. Akad. Nauk SSSR*, 234(2) :330–332, 1977.

[71] A.V. Kazhikhov and V.A. Vaĭgant. On the existence of global solutions of two-dimensional Navier-Stokes equations of a compressible viscous fluid. *Sibirsk. Mat. Zh.*, 36(6) :1283–1316, ii, 1995.

[72] S. Klainerman and A. Majda. Compressible and incompressible fluids. *Comm. Pure Appl. Math.*, 35(5) :629–651, 1982.

[73] H. Koch and D. Tataru. Well-posedness for the Navier-Stokes equations. *Adv. Math.*, 157(1) :22–35, 2001.

[74] H. Kreiss. Problems with different time scales for partial differential equations. *Comm. Pure Appl. Math.*, 33(3) :399–439, 1980.

[75] O.A. Ladyzhenskaya, V.A. Solonnikov, and H. True. Résolution des équations de Stokes et Navier-Stokes dans des tuyaux infinis. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(4) :251–254, 1981.

[76] J. Leray. Etude de diverses équations intégrales non linéaires et quelques problèmes que pose l'hydrodynamique. *J. Math. Pures Appl.*, 9(12) :1–82, 1933.

[77] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63(1) :193–248, 1934.

[78] X. Liao. A global existence result for a zero Mach number system. *Submitted.*

[79] L. Lichtenstein. Über einige Existenzprobleme der Hydrodynamik homogener, unzusammendrückbarer, reibungsloser Flüssigkeiten und die Helmholtzschen Wirbelsätze. *Math. Z.*, 23(1) :89–154, 1925.

[80] J.-L. Lions and G. Prodi. Un théorème d'existence et unicité dans les équations de Navier-Stokes en dimension 2. *C. R. Acad. Sci. Paris*, 248 :3519–3521, 1959.

[81] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol.1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.

[82] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 2*, volume 10 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998. Compressible models, Oxford Science Publications.

[83] P.-L. Lions and N. Masmoudi. Incompressible limit for a viscous compressible fluid. *J. Math. Pures Appl. (9)*, 77(6) :585–627, 1998.

[84] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1984.

[85] A. Majda. *The interaction of nonlinear analysis and modern applied mathematics*. ICM-90. Mathematical Society of Japan, Tokyo, 1990. A plenary address presented at the International Congress of Mathematicians held in Kyoto, August 1990.

[86] A. Majda. Remarks on weak solutions for vortex sheets with a distinguished sign. *Indiana Univ. Math. J.*, 42(3) :921–939, 1993.

[87] A. Matsumura and T. Nishida. Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. *Comm. Math. Phys.*, 89(4) :445–464, 1983.

[88] W.H. Matthaeus and M.R. Brown. Nearly incompressible magnetohydrodynamics at low Mach number. *Phys. Fluids*, 31(12) :3634–3644, 1988.

[89] G. Métivier and S. Schochet. The incompressible limit of the non-isentropic Euler equations. *Arch. Ration. Mech. Anal.*, 158(1) :61–90, 2001.

[90] J. Nash. Le problème de Cauchy pour les équations différentielles d'un fluide général. *Bull. Soc. Math. France*, 90 :487–497, 1962.

[91] F. Planchon. Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbf{R}^3$. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 13(3) :319–336, 1996.

[92] S. Schochet. The compressible Euler equations in a bounded domain : existence of solutions and the incompressible limit. *Comm. Math. Phys.*, 104(1) :49–75, 1986.

[93] P. Secchi. On the initial value problem for the equations of motion of viscous incompressible fluids in the presence of diffusion. *Boll. Un. Mat. Ital. B (6)*, 1(3) :1117–1130, 1982.

[94] P. Secchi. On the motion of viscous fluids in the presence of diffusion. *SIAM J. Math. Anal.*, 19(1) :22–31, 1988.

[95] P. Secchi. On the singular incompressible limit of inviscid compressible fluids. *J. Math. Fluid Mech.*, 2(2) :107–125, 2000.

[96] D. Serre. Solutions faibles globales des équations de Navier-Stokes pour un fluide compressible. *C. R. Acad. Sci. Paris Sér. I Math.*, 303(13) :639–642, 1986.

[97] J. Serrin. On the uniqueness of compressible fluid motions. *Arch. Rational Mech. Anal.*, 3 :271–288, 1959.

[98] J. Simon. Écoulement d'un fluide non homogène avec une densité initiale s'annulant. *C. R. Acad. Sci. Paris Sér. A-B*, 287(15) :A1009–A1012, 1978.

[99] M. Sy. A remark on the Kazhikhov-Smagulov type model : the vanishing initial density. *Appl. Math. Lett.*, 18(12) :1351–1358, 2005.

[100] R. Temam. *Navier-Stokes equations and nonlinear functional analysis*, volume 41 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1983.

[101] S. Ukai. The incompressible limit and the initial layer of the compressible Euler equation. *J. Math. Kyoto Univ.*, 26(2) :323–331, 1986.

[102] M. Vishik. Hydrodynamics in Besov spaces. *Arch. Ration. Mech. Anal.*, 145(3) :197–214, 1998.

[103] W. Wolibner. Un theorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long. *Math. Z.*, 37(1) :698–726, 1933.

[104] Z. Xin. Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density. *Comm. Pure Appl. Math.*, 51(3) :229–240, 1998.

[105] G.P. Zank and W.H. Matthaeus. The equations of nearly incompressible fluids. I. Hydrodynamics, turbulence, and waves. *Phys. Fluids A*, 3(1) :69–82, 1991.

[106] R.Kh. Zeytounian. *Theory and applications of viscous fluid flows.* Springer-Verlag, Berlin, 2004.