Probabilistic numerical methods: multi-scale and mean-field problems

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INTRODUCTION

1 Introduction Générale

Cette thèse s'inscrit dans le contexte de l'analyse numérique stochastique et son objectif principal est de proposer des algorithmes numériques pour résoudre efficacement quelques problèmes stochastiques nécessitant une approche particulière.

Aujourd'hui, grâce à leur capacité d'incorporer des termes complexes ou des incertitudes, les modèles stochastiques sont très présents dans tous les domaines qui utilisent de façon intensive des résultats quantitatifs (physique, biologie, ingénierie, finance, économie, etc.). L'adoption généralisée des modèles stochastiques trouve ses racines dans les premiers études du mouvement Brownien et ses applications en finance faites par Bachelier (1900) et, bien évidemment, dans les travaux d'Einstein (1905) sur le mouvement des grains de pollen sur l'eau observé par Brown. Puis, elle a été favorisé par le développement de l'analyse stochastique et ses outils associés à partir du début des années 1930.

Sans aucune doute, une bonne partie de la popularité des modèles stochastiques est expliquée par l'arrivée de l'ordinateur et l'apparition et le développement de l'analyse numérique stochastique. En effet, à l'exception de quelques exemples bien connus, la plupart des problèmes quantitatifs reliés au calcul stochastique (résoudre une EDS, un problème de contrôle stochastique, ...) ne peuvent pas être résolus analytiquement. Un praticien souhaitant utiliser des outils stochastiques doit alors choisir entre l'utilisation de modèles assez simples, qui peuvent être résolus analytiquement, ou de modèles plus riches, qui doivent être résolus numériquement sous certaines restrictions (précision, temps de calcul, utilisation de mémoire, etc).

Ces restrictions demandent l'utilisation de méthodes numériques plus adaptées et plus efficaces. Ainsi, la définition de nouveaux algorithmes doit profiter de la structure spécifique des problèmes stochastiques à résoudre. Dans ce travail, nous considérons deux problèmes indépendants pour lesquels nous proposons des schémas de résolution, et un troisième problème découlant de l'étude de complexité d'une des méthodes disponibles pour l'approximation faible des EDS.

Le premier problème que nous avons abordé est l'approximation numérique des modèles stochastiques fortement oscillants, c'est-à-dire, un système stochastique formé par deux groupes de

variables en interaction ayant des temps d'évolution très différents (un groupe de variables dites « rapides » et un autre de variables dites « lentes »). Si les variables d'intérêt sont les plus lentes, on peut vérifier que la séparation des échelles rend inefficace toute utilisation simple des algorithmes de discrétisation classiques pour la résolution du système (comme le schéma d'Euler). Nous proposons un algorithme pour résoudre un tel problème lorsque les variables rapides sont ergodiques. Sous cette hypothèse, on peut profiter de l'homogénéisation pour résoudre le système en utilisant un schéma d'Euler pour les variables d'échelle lente couplé à un estimateur à pas décroissant pour les moyennes ergodiques des variables rapides. Nous prouvons la convergence forte de l'algorithme ainsi qu'un résultat du type « théorème limite central » pour la distribution limite de l'erreur normalisée. Nous proposons également une version extrapolée de l'algorithme avec une complexité asymptotique inférieure et satisfaisant les mêmes propriétés que la version originale. Ce problème fait l'objet d'une prépublication en révision à Annals of Applied Probability [42] et il est présenté au Chapitre 1.

Ensuite, nous nous intéressons à l'approximation numérique d'une équation différentielle stochastique progressive-rétrograde découplée de type McKean-Vlasov (EDSPR-MKV). Les équations différentielles de type McKean-Vlasov sont des équations différentielles stochastiques ayant des coefficients qui dépendent de la loi de la diffusion elle-même. Elles apparaissent comme limite d'un système de particules avec des interactions de champ moyen lorsque le nombre de particules tend vers l'infini. De manière analogue, une EDSPR-MKV est une équation progressive-retrograde où les coefficients dépendent de la loi de la solution elle-même. Nous montrons que ce type d'équations permet de résoudre toute une famille de systèmes de contrôle. Puis, pour répondre à la question de l'approximation numérique, nous avons proposé un nouvel algorithme déterministe pour approcher faiblement la solution d'une EDS de type McKean-Vlasov. L'algorithme est basé sur la méthode de cubature sur l'espace de Wiener proposée par Lyons et Victoir (2004). Sa propriété la plus remarquable est qu'il peut être paramétré de manière à obtenir n'importe quel ordre donné de convergence. Pour conclure, en utilisant cette nouvelle méthode, nous construisons un algorithme pour résoudre l'EDSPR-MKV découplée. Nous donnons deux algorithmes et montrons qu'ils convergent respectivement aux ordres un et deux sous des conditions de régularité appropriées. Ce problème fait l'objet d'une prépublication soumise [23] en collaboration avec P. E. Chaudru de Raynal et il est présenté au Chapitre 2.

Bien qu'indépendantes, la définition du schéma de solution et l'analyse de l'erreur de ces deux problèmes reposent sur les mêmes raisonnements. L'idée de base est de combiner quelques résultats limites connus (sur l'échelle rapide pour le première problème ou sur le nombre de particules dans le deuxième) avec un découplage des termes rendant difficile l'approximation (les variables rapides dans le premier cas et les termes de loi dans le deuxième). Le système découplé étant plus simple, nous sommes alors en mesure de le résoudre en utilisant des outils plus classiques. Il est alors évident que le point le plus important dans la démonstration de la convergence du schéma général est de contrôler les erreurs de découplage et d'approximation. Dans les deux problèmes que nous considérons, nous estimons tout d'abord des erreurs en temps court et puis nous contrôlons leur accumulation. La régularité des variables stochastiques approchées sera très importante pour mener à bien ce projet.

Enfin, nous abordons un troisième problème découlant de l'étude sur la solution de l'EDSPR-MKV. En effet, le défaut le plus important de la méthode de cubature est le suivant : lorsqu'elle est appliquée directement, sa complexité présente une croissance exponentielle. Bien que l'algorithme donne de bons résultats en termes d'erreur (même avec peu de pas de discrétisation) il est important de réduire cette contrainte sur la précision maximale de l'algorithme. Ainsi, le troisième problème que nous avons étudié est la modification de l'algorithme de cubature permettant de contrôler sa complexité pour la résolution d'EDSR, sans sacrifier sa vitesse de convergence.

Dans l'Annexe A, nous présentons quelques résultats d'une étude en cours visant a résoudre le problème de complexité et basée sur la méthode de recombinaison de Litterer et Lyons [66].

Dans la suite, nous introduisons chacun des problèmes présentés ainsi que les algorithmes que nous avons proposé pour trouver leurs solutions. Afin de exposer clairement les idées à la base des méthodes présentées, nous introduisons également les principaux outils mathématiques utilisés.

2 Équations différentielles stochastiques fortement oscillantes

On considère l'EDS suivante

$$\begin{cases} X_t^\epsilon &= x_0 + \int_0^t f(X_s^\epsilon, Y_s^\epsilon)ds + \int_0^t g(X_s^\epsilon, Y_s^\epsilon)dB_s \\ Y_t^\epsilon &= y_0 + \epsilon^{-1}\int_0^t b(X_s^\epsilon, Y_s^\epsilon)ds + \epsilon^{-1/2}\int_0^t \sigma(X_s^\epsilon, Y_s^\epsilon)d\tilde{B}_s, \end{cases} \tag{1}$$

où X_t^ϵ est un processus stochastique de dimension d_x, Y_t^ϵ est un processus stochastique de dimension d_y, B et $\tilde{B}$ sont deux mouvements Browniens indépendants de dimensions respectives d_x et d_y. Ce type de dynamique modélise un système avec deux ensembles de variables évoluant à des échelles de temps différentes. L'écart entre les deux échelles de temps est défini par le paramètre ϵ. Notre étude porte sur le régime $\epsilon \ll 1$, c'est-à-dire, le cas où X^ϵ évolue lentement par rapport à Y^ϵ (nous dirons que X^ϵ est la variable lente et Y^ϵ la variable rapide).

Voici quelques exemples d'application de l'EDS (1) :

- Dans [74] (voir aussi les références incluses), les auteurs posent la question de simplification des modèles climatiques : ils supposent qu'il existe deux types de variables dans les systèmes climatiques : les variables climatiques et les variables non-résolues, et il supposent aussi que les variables climatiques évoluent très lentement par rapport aux variables non-résolues. Dans ce cadre, ils proposent un modèle climatique général donné par

$$\begin{aligned} dX_t =& \epsilon^{-1}\left[f_1(\epsilon^{-1}t) + L_{11}X_t + L_{22}Y_t + Q_{12}^1(X_t, Y_t) + Q_{22}^1(Y_t, X_t)\right]dt \\ &+ \left[F_1(t) + bX_t + Q_{11}^1(X_t, X_t)\right]dt \\ dY_t =& \epsilon^{-1}\left[f_2(\epsilon^{-1}t) + L_{21}X_t + L_{22}Y_t + Q_{11}^2(X_t, X_t) + Q_{12}^1(X_t, Y_t)\right]dt \\ &+ \epsilon^{-2}\Gamma Y_t dt + \epsilon^{-1}\sigma dB_t, \end{aligned} \tag{2}$$

  où $Q_{11}^1, Q_{12}^1, Q_{22}^1, Q_{12}^1, Q_{22}^1$ sont des opérateurs quadratiques. Par exemple, on retrouve le système (2) lorsqu'on considère la transformée de Fourier d'un système d'équations aux dérivées partielles décrivant un système climatique. Souvent, on est intéressé par la résolution du système en considérant uniquement l'échelle appelée « advective ». Nous considérons alors le système réduit

$$\begin{aligned} dX_t =& \left[F_1(t) + bX_t + Q_{11}^1(X_t, X_t)\right]dt \\ dY_t =& \epsilon^{-2}\Gamma Y_t dt + \epsilon^{-1}\sigma dB_t, \end{aligned}$$

  qui peut être compris comme un cas particulier de l'équation (1) avec une volatilité de l'échelle lente nulle.

- A partir de tests numériques effectués sur les données de l'indice S&P 500, Fouque, Papanicolaou et Sircar affirment dans [36] qu'il existe une composante rapide dans la volatilité de la série temporelle de l'indice. À partir de ces observations, les mêmes auteurs, en collaboration avec Solna, ont proposé et étudié dans [35] (puis dans [37]), un système similaire à (1) pour établir le prix de produits dérivés dans le cadre de modèles de volatilité stochastique. Leur modèle prévoit des prix d'actifs qui suivent une loi conditionnellement log-normale avec une volatilité exprimée comme une fonction positive d'un processus d'Ornstein Uhlenbeck. Plus précisément, leur modèle s'écrit

$$\begin{aligned} dX_t &= rX_t dt + f(Y_t)X_t dB_t \\ dY_t^\epsilon &= \left[\epsilon^{-1}(m - Y_t^\epsilon) - \epsilon^{-1/2}\nu\sqrt{2}\Lambda(Y_t^\epsilon)\right] dt + \epsilon^{-1/2}\nu\sqrt{2}dW_t, \end{aligned}$$

  où f est une fonction positive continue, B et W sont deux mouvements Browniens ayant comme corrélation instantanée $\rho \in (-1, 1)$ et

$$\Lambda(y) = \frac{\rho(\mu - r)}{f(y)} + \gamma(y)\sqrt{1 - \rho^2},$$

  correspond au prix de marché du risque combiné.

  On remarque que l'équation (1) exhibe les caractéristiques principales du modèle de Fouque et al. (à savoir, le retour à la moyenne de la volatilité et son caractère de variable rapide) tout en permettant l'ajout d'une dépendance explicite de la variable rapide par rapport à la variable lente. Nous sommes, cependant, limités au cas d'une corrélation nulle où nous avons négligé la contribution du terme Λ lorsque ϵ est très petit.

Dans les exemples présentés, la variable d'intérêt est la variable lente. Lorsque l'utilisation d'une approche numérique s'impose, on considère d'abord la possibilité d'appliquer directement un schéma d'approximation bien connu, comme le schéma d'Euler. Soit h la taille du pas de discrétisation d'un schéma d'Euler. Sous des hypothèses usuelles de régularité et croissance des coefficients (par exemple on suppose les coefficients étant uniformément Lipschitz continus avec une croissance polynomiale), on peut montrer, en utilisant un changement d'échelle et des résultats connus sur le schéma d'Euler (voir Théorème 10.2.2. dans [53]), que l'erreur forte d'une telle approximation est de l'ordre de $O(\epsilon^{-1/2}h^{1/2})$. Ainsi, afin d'approcher fortement l'équation (1) à une tolérance Δ près, il faut prendre le pas de discrétisation de l'ordre de $\epsilon\Delta^2$ ce qui implique effectuer environ $(d_x + d_y)\epsilon^{-1}\Delta^{-2}$ opérations. On en déduit que l'approche directe devient inefficace quand on considère une grande séparation des échelles.

Bien évidemment, la question que nous abordons n'est pas exclusive au cadre stochastique et elle apparaît également pour des systèmes purement déterministes. Dans ce cas, plusieurs méthodes numériques sont disponibles ; la plupart ont un trait commun : l'utilisation d'un premier algorithme pour résoudre la variable lente appelé « macro-solveur » qui reçoit comme paramètre des informations provenant d'un deuxième algorithme dit « micro-solveur » agissant sur la variable rapide (voir par exemple [31]).

Le cadre stochastique a été, à notre connaissance, moins étudié. Dans [32], les auteurs présentent un algorithme qui rappelle l'idée de base des algorithmes déterministes. Il est établi à partir de l'utilisation d'un schéma pour la variable lente (par exemple le schéma d'Euler) et, à chaque étape d'approximation de la variable lente, un autre schéma de discrétisation usuel est utilisé pour résoudre la contribution de la variable rapide. On trouve également dans [32] une analyse de l'erreur forte de l'algorithme quand on suppose que la volatilité de la variable lente est nulle (c'est-à-dire quand $g(x, y) = 0$ dans (1)), et une analyse de l'erreur faible dans le cas plus général. Récemment, cette méthode a été utilisée pour résoudre des EDPS à deux échelles

[13].

Dans le Chapitre 1, notre objectif est de proposer et d'étudier un algorithme alternatif pour approcher efficacement le problème à deux échelles lorsqu'on considère une EDS couplée fortement oscillante. La procédure se fait de la manière suivante : nous profitons de certains résultats d'approximation montrés par Pardoux et Veretennikov dans une série de papiers [83, 84, 85] qui permettent de déduire la possibilité d'approcher le système fortement oscillant par un système effectif plus simple. Ainsi nous avons des expressions explicites pour le système effectif : ses coefficients s'expriment comme espérances par rapport à la mesure ergodique de la variable rapide et sont donc, a priori, inconnus. Nous proposons de calculer ces espérances à l'aide de l'algorithme à pas décroissant proposé par Lamberton et Pagès dans [59]. L'utilisation de cet algorithme introduit de bonnes propriétés mathématiques pour le schéma, et constitue la principale différence par rapport au schéma [32].

2.1 Équation effective

Puisque, a priori, nous espérons que le système limite lorsque $\epsilon \to 0$ (étant supposé que celui-ci existe) soit très proche du système avec une petite valeur de ϵ, nous considérons le cas limite quand la séparation des échelles tend vers l'infini pour approcher le système fortement oscillant. En effet, supposons que la variable rapide soit ergodique. Intuitivement nous pouvons trouver un temps assez court de façon à avoir un très petit changement dans la variable lente et, en même temps, suffisamment grande pour permettre à la variable rapide d'atteindre son état stable. Cette image suggère que la variable lente peut être approchée par un autre processus stochastique où la dépendance par rapport à la variable rapide est remplacée par une moyenne ergodique, état stable d'une copie de la diffusion rapide avec dépendance par rapport à la variable lente gelée. Nous montrons que cette idée est juste, quitte à poser des bonnes hypothèses sur le système fortement oscillant.

La formalisation mathématique de l'idée présentée ci-dessus, demande l'application de techniques de perturbation singulière, comme celles présentées par Bensoussan, Lions et Papanicolaou dans [6] pour les cas déterministes ou par Papanicolaou, Stroock et Varadhan dans [80], Friedlin et Wentzell [40], Khasminskii [52] ou Pardoux et Veretennikov [84] (voir aussi [83], [85]) pour le cas stochastique. Nous considérons principalement l'approche de Pardoux et Veretennikov car le cadre de leur étude est le plus adapté à notre situation.

Le but est de montrer qu'il existe une limite faible pour la famille de diffusions X^ϵ, lorsque ϵ tend vers zéro. Pour cela, nous comparons l'action du générateur infinitésimal de X^ϵ sur un ensemble bien choisi de fonctions tests lorsque ϵ décroît à celle du générateur infinitésimal de notre candidat à devenir la diffusion limite (suggéré par l'argument intuitif que nous avons présenté). Afin d'illustrer les arguments de la preuve, montrons quelques calculs formels : soit $\mathcal{L}^\epsilon$ le générateur infinitésimal de (1), que l'on peut écrire comme

$$\mathcal{L}^\epsilon = \epsilon^{-1}\mathcal{L}_y^x + \mathcal{L}_x'^y$$

où $\mathcal{L}_x'^y$ est le générateur infinitésimal de la diffusion lente avec comme paramètre fixe y

$$\mathcal{L}_x'^y := \sum_{i,j=1}^{d_y} h_{ij}(x,y)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d_y} f_i(x,y)\frac{\partial}{\partial x_i}.$$

où $h(x,y) = gg^*(x,y)$; et $\mathcal{L}_y^x$ est le générateur infinitésimal de la diffusion rapide après change-

ment d'échelle $\{Y^\epsilon_{\epsilon t}\}_{t\geq 0}$ avec comme paramètre fixe x, c'est-à-dire,

$$\mathcal{L}^x_y := \sum_{i,j=1}^{d_y} a_{ij}(x,y)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{d_y} b_i(x,y)\frac{\partial}{\partial y_i}. \tag{3}$$

Dans la suite, nous notons μ^x la mesure invariante associée à $\{Y^\epsilon_{\epsilon t}\}_{t\geq 0}$ avec comme paramètre fixe x (l'existence et unicité de celle-ci est une conséquence de l'ergodicité de la diffusion rapide).

Nous allons maintenant comparer $\mathcal{L}^\epsilon$ avec un opérateur $\mathcal{L}'^y_x$ lorsque la variable rapide a déjà atteint sa limite stable. Ainsi, nous introduisons l'opérateur $\bar{\mathcal{L}}$ défini comme l'espérance de $\mathcal{L}'^y_x$ par rapport à μ^x,

$$\bar{\mathcal{L}} := \sum_{i,j=1}^{d_x} H_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d_x} F_i(x)\frac{\partial}{\partial x_i},$$

où

$$F(x) = \int f(x,y)\mu^x(dy) \qquad H(x) = \int h(x,y)\mu^x(dy). \tag{4}$$

Soit ψ une application régulière arbitraire définie entre $\mathbb{R}^{d_x}$ et $\mathbb{R}$. On peut trouver une application ψ_ϵ allant de $\mathbb{R}^{d_x}\times\mathbb{R}^{d_y}$ dans $\mathbb{R}$ telle que l'action de $\mathcal{L}^\epsilon$ sur ψ_ϵ soit proche de l'action de $\bar{\mathcal{L}}$ sur ψ. En effet, nous pouvons trouver un terme de correction ϕ_ψ (qui sera défini dans la suite) tel que

$$\psi_\epsilon(x,y) = \psi(x) + \epsilon\phi_\psi(x,y).$$

On en déduit que

$$\begin{aligned}\mathcal{L}^\epsilon\psi_\epsilon(x,y) &= (\epsilon^{-1}\mathcal{L}^x_y + \mathcal{L}'^y_x)(\psi(x) + \epsilon\phi_\psi(x,y)) \\ &= \mathcal{L}'^y_x\psi(x) + \mathcal{L}^x_y\phi_\psi(x,y) + \epsilon\mathcal{L}'^y_x\phi_\psi(x,y) \\ &= \bar{\mathcal{L}}\psi(x) + O(\epsilon),\end{aligned}$$

lorsque

$$\mathcal{L}^x_y\phi_\psi(x,y) = (\bar{\mathcal{L}} - \mathcal{L}'^y_x)\psi(x). \tag{5}$$

Cette dernière condition implique que ϕ_ψ est solution de l'équation de Poisson associée à $\mathcal{L}^x_y$ avec terme source $(\mathcal{L}'^y_x - \bar{\mathcal{L}})\psi(x)$. Nous remarquons que pour donner un sens au terme $O(\epsilon)$, il faut que ϕ_ψ satisfasse à des bonnes propriétés de régularité.

Ainsi, formellement, lorsque ϵ tend vers zéro, le terme de correction et l'erreur d'approximation tendent vers zéro, et nous en déduisons que $\mathcal{L}^\epsilon\psi_\epsilon$ converge vers $\bar{\mathcal{L}}\psi$.

Le développement formel que nous avons présenté permet d'illustrer le résultat rigoureux de Pardoux et Veretennikov, que nous énonçons ici dans une version adaptée à notre cadre d'application.

Nous notons par $C_b^{j,r+\alpha}$ la collection des fonctions ayant j dérivées bornées par rapport à x et r dérivées bornées par rapport à y, de telle sorte que les dérivées de tout ordre soient $\alpha-$Hölder continues par rapport à y uniformément en x.

Theorem 0.1 (Théorème 4 dans [84]). *Soient b,σ,f,g définis en (1) et $a=\sigma\sigma^*$. Supposons que $\lim_{|y|\to\infty} b(x,y)\cdot y = -\infty$ et que la matrice 'a' soit uniformément non-dégénérée. Supposons également que $a,b\in C_b^{2,1+\alpha}$, et f,g Lipschitz continues par rapport à x uniformément en y et ayant une croissance au plus polynomiale en y et linéaire en x.*

Alors pour tout $T > 0$, la famille de processus $\{X_t^\epsilon, 0 \leq t \leq T\}_{0<\epsilon\leq 1}$ est faiblement compacte dans $C([0,T];\mathbb{R}^l)$ et tout point d'accumulation X est une solution du problème de martingale associé à l'opérateur $\bar{\mathcal{L}}$.

De plus, lorsque le problème de martingale est bien posé, alors $X^\epsilon \Rightarrow X$, où X est l'unique (en distribution) diffusion ayant $\bar{\mathcal{L}}$ comme générateur.

Il est important de souligner que le cadre dans lequel Pardoux et Veretennikov montrent leur résultat permet aussi de considérer le cas avec un terme d'ordre ϵ^{-1} dans la variable lente. Ce terme additionnel est une source de difficulté dans la preuve. La démonstration est faite en deux parties : on montre d'abord la compacité faible dans $C([0,T],\mathbb{R}^d)$ puis, on identifie la limite. La première partie est démontrée en utilisant des résultats de tension classiques comme ceux de Billingsley [9], à savoir, montrer que :

- Pour tout $\delta > 0$, il existe $M > 0$, tel que

$$P\left(\sup_{0\leq t\leq T}|X_t^\epsilon| > M\right) \leq \delta \quad 0 < \epsilon < 1.$$

- Pour tout $\delta > 0$, $M > 0$, il existe ϵ_0 et γ, tels que

$$P\left(\sup_{0\leq t_0\leq T}\sup_{t\in[t_0,t_0+\gamma]}|X_t^\epsilon - X_{t_0}^\epsilon| \geq \delta;\ \sup_{0\leq t\leq T}|X_t^\epsilon| \leq M\right) < \delta.$$

L'effet de la variable rapide apparaît lorsqu'on s'intéresse au deuxième critère, et on doit alors faire un changement de temps bien choisi. Dans la deuxième partie, on s'occupe de l'identification de la limite. La preuve consiste à choisir une suite de fonctions étagées (par rapport à la variable lente) qui partagent la même limite que l'ensemble original, puis à profiter de la structure avec une variable lente gelée afin de passer à la limite dans l'échelle rapide, grâce au théorème ergodique.

Par conséquent, quand le problème est bien posé, nous pouvons exprimer le résultat de la façon suivante. Nous définissons *l'équation effective*

$$X_t = x_0 + \int_0^t F(X_s)ds + \int_0^t G(X_s)dB_s, \tag{6}$$

où $G(x) = \sqrt{H(x)}$ et F, H sont définies en (4). Notons que l'unique contrainte sur $G(x)$ est d'être une racine de la matrice semi-définie positive $H(x)$. En conséquence il y aurait a priori plusieurs valeurs admissibles pour $G(x)$. Par convenance, nous choisissons de définir $G(x)$ comme la décomposition de Cholesky de H. Ainsi, sous des hypothèses appropriées, $X^\epsilon \xrightarrow{\mathcal{L}} X$ lorsque $\epsilon \to 0$. Il est donc possible de continuer le programme présenté au départ : nous pouvons approcher la solution du système (1) via l'approximation du système (6).

Il n'est pas, cependant, facile de résoudre explicitement la mesure invariante ou les espérances ergodiques de (4), sauf pour quelques exemples particuliers. C'est pourquoi la méthode numérique que nous proposons doit aussi prévoir le calcul des ces termes inconnus.

2.2 Approximation de la mesure invariante

Voyons comment approcher les quantités de (4). Une première méthode proposée par Talay dans [92, 93], permet d'approcher les espérances par rapport à la loi invariante via le théorème

ergodique. Plus précisément, si X_t est un processus stochastique ergodique avec mesure invariante μ, alors

$$\int f(x)\mu(dx) = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(X_s)ds \approx \frac{1}{N}\sum_{k=0}^{N\Delta} f(\bar{X}_{k\Delta})ds,$$

où $0 < \Delta \ll 1$, N est un entier $N \gg 1$ et $\bar{X}_{k\Delta}$ est un schéma d'approximation pris aux temps $k\Delta$, pour $k = 1, \ldots, N$. Un tel schéma introduit deux types d'erreur :

- Une erreur de troncature, conséquence du remplacement de la limite à l'infini par une somme finie de N termes.

- Une erreur de changement de mesure invariante, puisque la limite ergodique est prise sur le schéma d'approximation et non sur la diffusion originale.

L'erreur de troncature peut être contrôlée à l'aide de résultats sur les chaînes de Markov ergodiques, grâce au théorème limite central et à des estimations sur la convergence du schéma discret vers sa limite (voir [79], [75]). Cette erreur est d'ordre de $N^{-1/2}$ plus un terme dépendant de hN (typiquement décroissant de façon exponentielle). L'erreur de discrétisation est ensuite étudiée en utilisant tout d'abord la structure de semi-groupe de l'opérateur d'espérance conditionnelle pour trouver un développement au première ordre (comme fait par Talay et Tubaro dans [94] pour le schéma d'Euler), puis, à l'aide des résultats de convergence vers la limite invariante (cette fois-ci de la diffusion continue). Voir par exemple [79] pour le cas de vitesse de convergence exponentielle ou [97] pour le cas polynomial.

Remarquons qu'un pas indispensable du programme consiste à démontrer que le schéma de discrétisation est ergodique. Il est possible que ce ne soit pas le cas, même si le processus à approcher est ergodique. Par exemple (voir [75]), on montre assez facilement que la diffusion

$$dX_t = -X_t^3 dt + dB_t$$

est ergodique et que $\mathbb{E}\left(|X_t|^2\right) \leq 3/2$ pour tout $t \in \mathbb{R}^+$, tandis que la discrétisation d'Euler correspondante

$$\bar{X}_{k+1} = \bar{X}_k - \bar{X}_k^3 \Delta_{T_{k+1}} + \Delta B_{k+1}$$

a des moments de deuxième ordre divergents lorsque k augmente.

Ainsi, il est indispensable de montrer l'ergodicité du schéma. Plusieurs arguments sont disponibles, ayant tous la même idée fondamentale : on cherche à montrer que la chaîne de Markov associée aux schémas est irréductible et récurrente positive. Par exemple, le schéma présenté dans [92] est construit à partir d'un schéma de Milstein et il est ergodique lorsqu'on utilise des incréments Gaussians pour les simulations de Monte Carlo : la propriété d'irréductibilité de la chaîne se déduit de l'uniforme ellipticité de la matrice de diffusion tandis que la propriété de récurrence positive est montrée à partir d'une hypothèse forte sur le terme de dérive de retour à la moyenne. Enfin, des résultats sur l'ergodicité de schémas utilisant des conditions plus générales sont présentés dans [75] ou [79].

2.2.a *Méthode à pas décroissant*

Une autre méthode d'approximation de la mesure invariante a été proposée par Lamberton et Pagès dans [59, 60], puis étudiée dans [65]. Nous présentons une version simplifiée de cette méthode, adaptée à notre cadre d'application.

Soit $\{\gamma_k\}_{k\in\mathbb{N}}$ une suite de pas décroissants avec $\gamma_k > 0$ pour tout k, $\lim_{n\to\infty} \gamma_k = 0$ et $\lim_{k\to\infty} \Gamma_k = \infty$ où

$$\Gamma_k := \sum_{j=0}^{k} \gamma_j.$$

Soit $\tilde{B}$ un mouvement Brownien. Fixons $\sqrt{\gamma_{k+1}}U_{k+1} := \tilde{B}_{\Gamma_{k+1}} - \tilde{B}_{\Gamma_k}$ de sorte que U_{k+1} soit un vecteur Gaussien standard. Soit $y_0 \in \mathbb{R}^{d_y}$. Nous définissons l'approximation d'Euler à pas décroissant d'une diffusion ergodique par

$$\begin{aligned} \tilde{Y}_0^x &= y_0 \\ \tilde{Y}_{k+1}^x &= \tilde{Y}_k^x + \gamma_{k+1} b(x, \tilde{Y}_k^x) + \sqrt{\gamma_{k+1}}\sigma(x, \tilde{Y}_k^x)U_{k+1}, \end{aligned} \tag{7}$$

et l'*estimateur à pas décroissant de la moyenne* par

$$\tilde{F}^k(x) = \frac{1}{\Gamma_k} \sum_{j=1}^{k} \gamma_j f(x, \tilde{Y}_{j-1}^x) \tag{8}$$

Les équations (7) et (8) présentent une version simplifiée de l'algorithme présenté dans [59]. La version plus générale permet l'utilisation de suites différentes pour les pas du schéma d'Euler dans (7) et pour définir les poids de la moyenne dans (8). Néanmoins, nous nous limitons à cette version réduite car elle simplifie la présentation des résultats sans pour autant limiter la vitesse de convergence asymptotique de l'algorithme complet.

La principale caractéristique de la méthode à pas décroissant est que, sous réserve de vérifier l'existence d'une fonction de Lyapunov satisfaisant quelques conditions de contrôle, non seulement la discrétisation est ergodique, mais l'estimateur à pas décroissant associé n'induit pas d'erreur de discrétisation. Autrement dit, l'estimateur converge presque sûrement vers la moyenne ergodique de la diffusion continue lorsque $k \to \infty$ (voir [59]). De plus, l'estimateur à pas décroissant peut être écrit récursivement comme

$$\tilde{F}^0(x) = 0; \quad \tilde{F}^k(x) = \tilde{F}^{k-1}(x) + \frac{\gamma_k}{\Gamma_k}\left(f(x, \tilde{Y}_{k-1}^x) - \tilde{F}^{k-1}(x)\right)$$

ce qui est très appréciable d'un point de vue pratique.

La preuve de la convergence presque sûre utilise l'hypothèse d'existence d'une fonction de Lyapunov et des techniques de martingale pour montrer la tension de la suite de mesures quand $k \to \infty$. Ensuite, on considère un problème de martingale afin de montrer que la loi limite peut être exprimée comme la loi d'une diffusion limite. Ceci est fait à l'aide du théorème d'Echeverría-Weiss. Enfin, des résultats de stabilité et de contrôle de martingale permettent l'identification de la limite.

Une autre caractéristique de l'algorithme est qu'il permet une certaine forme de « développement de l'erreur » (comme montré dans [65]) lorsqu'il est appliqué à une certaine famille de fonctions. Cette propriété s'avère très utile, permettant l'application d'une extrapolation de Romberg. Nous profitons de cette propriété pour proposer une version extrapolée de l'algorithme.

2.3 Algorithme proposé

La discussion précédente sur l'approximation de la diffusion et la mesure invariante motive notre choix de proposer un algorithme d'approximation pour le système fortement oscillant (1). Nous

appelons cette procédure *algorithme à pas décroissant multi-échelle (MsDS pour ses initiales en anglais)*. Il est défini comme une composition d'un schéma d'Euler pour l'échelle lente et des réalisations indépendantes de la méthode d'Euler à pas décroissant pour calculer les coefficients en fonction des moyennes ergodiques à chaque étape. Puisque nous envisageons une mise en œuvre avec des simulations de Monte Carlo, nous devons ajouter une procédure additionnelle à chaque étape, à savoir une décomposition de Cholesky pour trouver le coefficient de volatilité.

2.3.a *Hypothèses*

Les résultats que nous présentons sont valables sous certaines hypothèses sur les coefficients de la diffusion rapide (hypothèses $(\mathcal{H}_{f.s.})$), notamment une condition de non-dégénérescence, une condition de récurrence et l'existence de certaines bornes sur la croissance de la diffusion, qui garantissent l'existence d'une unique mesure invariante pour la diffusion rapide $\{Y^{\epsilon}_{\epsilon t}\}_{t\geq 0}$ avec paramètre d'échelle lente fixé x, pour tout $x \in \mathbb{R}^{d_x}$. La condition de non-dégénérescence est plus forte que nécessaire pour l'application de la méthode d'Euler à pas décroissant, mais va nous permettre d'utiliser les résultats de Pardoux et Veretennikov pour obtenir un développement de l'erreur du schéma d'approximation des moyennes ergodiques.

Nous imposons également des conditions sur les coefficients d'échelle lente (hypothèses $(\mathcal{H}_{s.s.})$), à savoir, une condition de régularité (Lipschitz par rapport à la variable x), un contrôle sur la croissance (polynomiale en y et linéaire en x) et nous supposons la matrice $h = gg^*$ uniformément non dégénérée ou identiquement nulle (une hypothèse nécessaire pour des raisons de stabilité par rapport à la décomposition de Cholesky). Une définition plus précise de l'ensemble des hypothèses requises est donnée dans la section 1 du chapitre 1.

2.3.b *Multi-scale decreasing step (MsDS)*

Pour une maturité T donnée, pour $n \in \mathbb{N}^*$ et en fixant $t_k = Tk/n$, l'algorithme MsDS est défini comme

$$\check{X}^n_{t_{k+1}} = \tilde{F}^{M(n)}(\check{X}^n_{t_k})\Delta t_{k+1} + \tilde{G}^{M(n)}(\check{X}^n_{t_k})\Delta B_{k+1},$$

où $\tilde{F}^M$ est défini en (8) et $\tilde{G}^M(x)$ est calculé en deux étapes : nous approchons d'abord $\tilde{H}^M(x)$ avec l'algorithme à pas décroissant (8) (rappelons que $h(x,y) = gg^*(x,y)$), puis nous faisons une décomposition de Cholesky pour trouver $\tilde{G}^M(x) = \sqrt{\tilde{H}^M(x)}$. Enfin, $M(n)$ est le nombre de pas à réaliser à chaque utilisation de la méthode à pas décroissant et il est donné en fonction du nombre total de pas pour le schéma d'Euler appliqué sur l'équation effective. Une expression pour cette dépendance est donnée dans le théorème principal.

Rappelons que nous avons choisi d'utiliser la racine donnée par la décomposition de Cholesky pour répondre à des considérations pratiques d'implémentation de l'algorithme et que ce choix, est d'un point de vue analytique arbitraire.

Nous avons également étudié une version extrapolée de l'algorithme (EMsDS pour les initiales en anglais) qui suit la même structure de base que l'algorithme MsDS mais pour laquelle nous utilisons comme estimateur une extrapolation Richardson-Romberg de l'estimateur de base : i.e. une combinaison linéaire de l'estimateur à pas décroissant calculé en utilisant deux ensembles de paramètres différents choisis pour réduire la complexité de l'algorithme (les détails sont présentés dans le chapitre 1).

Nous noterons la version extrapolée des estimateurs de F et H par $\hat{F}^{(n);\lambda}(x)$, $\tilde{H}^{\lambda,M(n)}(x)$ et, suivant la même idée que pour l'algorithme de base, nous fixons $\tilde{G}^{\lambda,M(n)}(x) = \sqrt{\tilde{H}^{\lambda,M(n)}(x)}$. L'algorithme extrapolé est alors défini comme l'algorithme MsDS de base, mais en remplaçant les estimateurs originaux par ses homologues extrapolés.

2.4 Résultats principaux

Tous nos résultats sont montrés pour une version continue de l'algorithme obtenue par simple interpolation. C'est-à-dire, si nous notons $\underline{t} = \lfloor nt \rfloor / n$, alors nous considérons

$$\tilde{X}_t^n = x_0 + \int_0^t \tilde{F}^{M(n)}\left(\tilde{X}_{\underline{s}}^n\right) ds + \int_0^t \tilde{G}^{M(n)}\left(\tilde{X}_{\underline{s}}^n\right) dB_s. \tag{9}$$

Le théorème 0.2 contient les principaux résultats sur l'algorithme MsDS : il converge fortement vers la solution exacte et satisfait de plus une version non-standard d'un théorème limite central (TCL), de sorte que la distribution des erreurs normalisées converge vers la solution d'une EDS.

Theorem 0.2. *Soient $0 < \theta < 1$, $\gamma_0 \in \mathbb{R}^+$ et $\gamma_k = \gamma_0 k^{-\theta}$. Soit M_1 une constante positive. Supposons que les hypothèses $(\mathcal{H}_{f.s.})$ et $(\mathcal{H}_{s.s.})$ (esquissées précédemment et précisées dans la section 1 du chapitre1) soient satisfaites. On définit $M(n)$ par*

$$M(n) = M_1 n^{\frac{1}{1-\theta}},$$

alors

i) (Convergence forte). Il existe une constante K telle que

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|X_t - \tilde{X}_t^n\right|^2\right) \le K n^{-[(1-\theta)\wedge\theta]/(1-\theta)}$$

ii) (T.L.C. généralisé pour le cas EDO : $g \equiv 0$). Supposons de plus que f est au moins 7 fois dérivable par rapport à y avec toutes ses dérivées polynomialement bornées. Supposons également $\theta \ge 1/2$. Alors,

$$n\left(X - \tilde{X}^n\right) =: \zeta^n \Rightarrow \zeta^\infty$$

(convergence en loi de la suite de processus) pour ζ^∞, solution d'une EDS dont les coefficients sont fonction de la loi invariante. L'EDS est donnée explicitement dans la proposition 1.23 dans le chapitre(1).

iii) (T.L.C. généralisé pour le cas EDS : $h = gg^*$ non-dégénérée). Supposons de plus que f, h sont au moins 7 fois dérivables par rapport à y avec toutes ses dérivées polynomialement bornées. Supposons également $\theta \ge 1/3$. Alors,

$$n^{1/2}\left(X - \tilde{X}^n\right) =: \zeta^n \Rightarrow \zeta^\infty$$

(convergence en loi de la suite de processus) pour ζ^∞, solution d'une EDS dont les coefficients sont fonction de la loi invariante. L'EDS est donnée explicitement dans la proposition 1.23 dans le chapitre(1).

La preuve se fait en plusieurs étapes : d'abord, nous trouvons un contrôle sur l'approximation de la moyenne ergodique à chaque étape. Nous montrons que ce contrôle est basé sur l'existence, la régularité et la vitesse de croissance de la solution de l'équation de Poisson associée à la diffusion à échelle rapide. Ces propriétés sont étudiées à partir des résultats de Pardoux et

Veretennikov [84]. Ensuite, nous contrôlons l'erreur obtenue après avoir effectué une décomposition de Cholesky. Les résultats précédents permettent de borner, de manière standard, l'erreur L_2 nous donnant la convergence forte de notre algorithme et nous donne un taux candidat pour la vitesse de convergence. De plus, en définissant une structure appropriée et en utilisant les estimations mentionnées auparavant, nous montrons, en utilisant les résultats de distribution limite de Jakubowski et Pagès [50], Jacod et Protter [49] et Kurtz and Protter [56], le résultat sur le TLC généralisé.

En outre, nous montrons dans la section 2.4.a que l'algorithme MsDS est plus efficace que l'utilisation directe d'un schéma d'Euler lorsque le paramètre de séparation des échelles ϵ est très petit.

La version extrapolée de l'algorithme, définie dans la section 2.1 du chapitre 1, possède les mêmes propriétés que la version de base.

Corollary 0.3. *Soient $0 < \theta < 1$, $\gamma_0 \in \mathbb{R}^+$ et $\gamma_k = \gamma_0 k^{-\theta}$. Soit M_1 une constante positive. Supposons que les hypothèses $(\mathcal{H}_{f.s.})$ et $(\mathcal{H}_{s.s.})$ (esquissées précédemment et précisées dans la section 1 du chapitre1) soient satisfaites. Soit $M(n)$ défini comme dans le Théorème 0.2. Soit $\hat{X}^n$ le résultat de l'utilisation de la version extrapolée de l'algorithme définie dans la section 2.1 du chapitre 1. Alors,*

i) (Convergence forte). Il existe une constante K telle que

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|X_t - \tilde{X}_t^n\right|^2\right) \leq K n^{-[(1-\theta)\wedge\theta]/(1-\theta)}$$

ii) (T.L.C. généralisé pour le cas EDO : $g \equiv 0$). Supposons de plus que f est au moins 8 fois dérivable par rapport à y avec toutes ses dérivées polynomialement bornées. Supposons également $\theta \geq 1/3$. Alors,

$$n\left(X - \tilde{X}^n\right) =: \zeta^n \Rightarrow \zeta^\infty$$

(convergence en loi de la suite de processus) pour ζ^∞, solution d'une EDS dont les coefficients sont fonction de la loi invariante. Les coefficients de l'EDS sont donnés explicitement dans le Corollaire 1.25 du chapitre (1).

iii) (T.L.C. généralisé pour le cas EDS : $h = gg^*$ non-dégénéré). Supposons de plus que f, h sont au moins 8 fois dérivables par rapport à y avec toutes ses dérivées polynomialement bornées. Supposons également $\theta \geq 1/5$. Alors,

$$n^{1/2}\left(X - \tilde{X}^n\right) =: \zeta^n \Rightarrow \zeta^\infty$$

(convergence en loi de la suite de processus) pour ζ^∞, solution d'une EDS dont les coefficients sont fonction de la loi invariante. Les coefficients de l'EDS sont donnés explicitement dans le Corollaire 1.25 du chapitre (1).

L'avantage d'utiliser l'estimateur extrapolé n'est pas évident à première vue, car la vitesse de convergence reste la même que pour l'algorithme original. En fait, l'utilisation de l'estimateur interpolé *permet de choisir une plus petite valeur de θ dans la définition de la suite $\gamma_k = \gamma_0 k^{-\theta}$*. Ce changement conduit à une réduction asymptotique de la complexité de l'algorithme, sans changer ses autres caractéristiques. Le prix à payer est la régularité, plus forte, imposée aux coefficients. La section suivante illustre clairement cette situation.

2.4.a *Analyse de la complexité*

Nous pouvons approcher le temps d'exécution des deux algorithmes (MsDS et EMsDS) en estimant le nombre total d'opérations nécessaires pour simuler un chemin d'approximation de l'équation effective (6). Notons que, comme les deux algorithmes partagent la même structure, la même analyse vaut pour tous les deux. Le coût total $\kappa(n)$ de l'algorithme avec n pas peut s'écrire comme

$$\kappa(n) = [\kappa_1(n, d_x, d_y) + \kappa_2(d_x)]\, n,$$

où κ_1 représente le coût d'estimation des coefficients à chaque pas de l'algorithme d'Euler décroissant et κ_2 est le coût de calcul de chaque pas du schéma d'Euler sur l'équation effective. Celui-ci est de l'ordre de $O(d_x)$ dans le cas EDO est $O(d_x^2)$ dans le cas EDS.

Considérons tout d'abord le terme κ_1. Les deux algorithmes doivent faire $M_1 n^{1/(1-\theta)}$ itérations pour approcher la diffusion $\tilde{Y}$ et calculer les estimateurs $\tilde{F}$ et $\tilde{G}$. Pour l'algorithme MsDS, chacune de ces itérations coûte $O(d_y d_x)$ opérations pour le cas EDO et $O(d_y d_x^2)$ pour le cas EDS. Par ailleurs, dans ce dernière cas il faut aussi calculer une décomposition de Cholesky avec un coût de $O(d_x^3)$ opérations. Ainsi,

$$\kappa_1^{MsDS}(n, d_x, d_y) = \begin{cases} O\left(d_y d_x n^{1/(1-\theta)}\right) & \text{pour le cas EDO} \\ O\left([d_y d_x^2 + d_x^3] n^{1/(1-\theta)}\right) & \text{pour le cas EDS} \end{cases}.$$

Pour l'algorithme EMsDS, on peut montrer que $\kappa_1^{EMsDS} \leq \lambda \kappa_1^{MsDS}$ (voir la définition dans la section2.1 du chapitre 1). Donc, les deux algorithmes partagent la même formule pour la complexité, sauf que *la version extrapolée de l'algorithme permet de choisir un θ plus petit.*

Il est en fait plus intéressant de comparer l'efficacité des deux algorithmes. Autrement dit, le temps (mesuré en nombre d'operations) τ investi pour satisfaire une tolérance Δ sur l'erreur. Le Théorème 0.2 et le Corollaire 0.3 montrent que $\Delta(n) := O(n^{-1})$ pour le cas EDO et $\Delta(n) := O(n^{-1/2})$ pour le cas EDS. Si l'on remplace la valeur minimale pour θ, on déduit les valeurs de la Table 1 :

	EDO	EDO (extrapol.)	EDS	EDS (extrapol.)
$\theta_{\min}$	1/2	1/3	1/3	1/5
$\tau_{\min}(\Delta)$	$O(d_y d_x \Delta^{-3})$	$O(d_x d_y \Delta^{-2.5})$	$O([d_x^2 d_y + d_x^3]\Delta^{-5})$	$O([d_x^2 d_y + d_x^3]\Delta^{-4.5})$

TABLE 1 – Efficacité minimale (mesurée en nombre d'opérations pour satisfaire une tolérance Δ) de l'algorithme de base et de la version extrapolée pour les cas EDO et EDS.

Est-il plus avantageux d'utiliser ces algorithmes plutôt que d'utiliser directement un schéma d'Euler sur le système fortement oscillant ? Pour le cas EDO, un tel schéma demandera $(dx + dy)\epsilon^{-1}\Delta^{-2}$ opérations. Ainsi, l'algorithme MsDS est plus efficace lorsque $\epsilon < \Delta(d_x \vee d_y)^{-1}$, et l'algorithme EMsDS si $\epsilon < \Delta^{1/2}(d_x \vee d_y)^{-1}$. Nous pouvons aussi nous comparer avec l'algorithme proposé dans [32]. Si les complexités des algorithmes MsDS et celui proposé dans [32] sont équivalentes, l'algorithme MsDS permet d'obtenir en outre le T.L.C. généralisé. Ce résultat est très intéressant pour estimer la variance de la solution lorsqu'on utilise des méthodes de Monte Carlo.

Enfin, pour le cas EDS, l'algorithme proposé est plus efficace qu'un Euler traditionnel quand $\epsilon < \Delta^3(d_x \vee d_y)^{-1}$ si l'on utilise l'algorithme MsDS, et quand $\epsilon < \Delta^{2.5}(d_x \vee d_y)^{-1}$ si l'on utilise le EMsDS. Ces bornes marquent le régime d'application de notre méthode. Nous remarquons que dans [32] il n'y a pas d'étude explicite de l'algorithme pour le cas EDS.

2.5 Remarques

Nous avons présenté un algorithme pour approcher fortement l'équation effective, qui approche elle même le système fortement oscillant. Nous avons étudié la convergence et sa vitesse de convergence optimale associée, à l'aide d'un résultat de type TLC généralisé.

Nous croyons qu'il y a encore des possibilités d'amélioration de nos résultats. Par exemple, il semble possible d'assouplir l'hypothèse de non-dégénérescence imposée sur les échelles rapides et lentes. Concernant la variable lente, ne plus poser l'hypothèse de non-dégénérescence nécessiterait de reprendre l'algorithme de Cholesky (car des singularités pourraient apparaître dans l'analyse de l'erreur) et modifier notre étude des résultats limites, en ajoutant quelques difficultés techniques supplémentaires. Assouplir l'hypothèse pour la variable rapide est, en revanche, plus difficile. Bien que la méthode à pas décroissant n'ait pas vraiment besoin de ce genre d'hypothèse, nous aurions besoin de modifier l'analyse de Pardoux et Veretennikov, basée sur des résultats sur les EDP uniformément elliptiques. Nous croyons que des résultats similaires peuvent être retrouvés sous une hypothèse plus faible sur la structure de la diffusion comme la condition Hörmander [41].

Dans notre analyse, nous avons mis en valeur la convergence forte de l'algorithme. L'analyse de la convergence faible peut être réalisée en suivant les techniques utilisées par Talay et Tubaro dans [94], en y ajoutant la présence de l'estimateur à pas décroissant. Nous ne donnerons pas les détails là, mais nous soulignons que le point crucial dans l'analyse est la disponibilité du développement explicite de l'erreur induite par l'estimateur à pas décroissant (voir la proposition 1.9). L'espérance de ce développement ne dépend pas des trajectoires prises pour l'approximation. La conclusion est que, sous les mêmes hypothèses du théorème 0.2, la vitesse de convergence au sens faible est de l'ordre de n^{-1}, soit un ordre analogue au schéma d'Euler. En outre, tout comme pour le schéma d'Euler , nous pouvons effectuer une extrapolation de Romberg pour eliminer l'erreur du premier ordre, mais cette fois *sur l' approximation de la diffusion effective* et pas seulement sur l'estimateur, comme nous l'avons fait pour le cas de l'approximation forte. En choisissant les bonnes suites $\{\gamma_k\}_{k\in\mathbb{N}}$, l'extrapolation conduira à une approximation d'ordre supérieur (sans pour autant atteindre l'ordre du cas classique, *i.e.* n^{-2}) .

Un autre aspect qui pourrait être considéré est de trouver des modifications amenant à accélérer davantage la convergence (tant pour la convergence faible comme pour la forte) . Par exemple, s'il est nécessaire de résoudre plusieurs problèmes liés au même système (par exemple lorsque on s'interésse à l'évaluation des options contingentes en finance), on peut former une grille sur l'échelle lente et résoudre le problème en projetant sur cette grille. La variable rapide est alors calculée uniquement pour les valeurs de la grille. Ainsi, on vise à optimiser le nombre de points pour lesquels la moyenne ergodique doit être estimée (étant donné que c'est la partie la plus coûteuse de l'algorithme). Nous avons effectué des études numériques en utilisant des grilles creuses, mais les résultats n'étaient pas concluants.

3 Approximation numérique d'une EDSPR découplée de type McKean-Vlasov

Nous considérons le système

$$\begin{cases} dX_t^x = \sum_{i=0}^{d} V_i(t, X_t^x, \mathbb{E}\varphi_i(X_t^x))dB_t^i \\ dY_t^x = -f(t, X_t^x, Y_t^x, Z_t^x, \mathbb{E}\varphi_f(X_t^x, Y_t^x))dt + Z_t^x dB_t^{1:d} \\ X_0^x = x, \quad Y_T^x = \phi(X_T^x) \end{cases} \tag{10}$$

où $B^{1:d}$ représente un mouvement Brownien de dimension d, où nous avons pris la convention $B_t^0 = t$, pour tout t en $[0, T]$, $T > 0$ donné. L'équation (10) est composé d'une variable progressive X avec condition initiale x_0 et d'une variable rétrograde Y avec condition finale fonction de X. En outre, puisque les coefficients des parties progressive et rétrograde dépendent de la loi des variables elles-mêmes et que la variable rétrograde n'apparaît pas dans la formulation de la variable progressive, nous disons que l'équation (10) est une équation différentielle stochastique progressive rétrograde découplée de type McKean-Vlasov (MKV-FBSDE pour les initiales en anglais).

Notre objectif au chapitre 2 est de présenter une méthode numérique pour résoudre l'équation (10). Ce problème est intéressant en soi, mais il est également utile pour résoudre une certaine classe de problèmes de contrôle sur un système d'un grand nombre de particules avec interaction de type champ moyen, inspirée de la théorie des jeux à champ moyen, mais conçue de telle manière que la dynamique du processus contrôlé n'a pas d'influence sur l'environnement du champ moyen. Pour une belle discussion sur le régime asymptotique des jeux différentiels stochastiques avec un nombre fini de joueurs lorsque le nombre de joueurs tend vers l'infini voir [20].

Voyons un exemple assez concret de ce genre de problème de contrôle. On considère la question d'optimiser la richesse d'un investisseur ayant un large portefeuille d'actifs de crédit. Ici, nous nous inspirons du cadre présenté dans [16]. Une des méthodes utilisées pour la modélisation de la dynamique d'actifs de crédit est le modèle dit structurel (voir [8] pour une revue des modèles de risque de crédit). Selon ce modèle, nous supposons que le défaut de crédit est déclenché lorsque la valeur de l'actif de crédit correspondant est inférieure à un certain seuil. Sous la configuration originale de ce problème, appelé aussi modèle de Merton, le défaut ne peut être déclenché qu'à un moment donné d'échéance fixe T. Un modèle plus réaliste consiste à considérer tout l'intervalle $[0, T]$ et permet le déclenchement du défaut de crédit dès la première fois que l'actif de crédit passe sous un seuil.

On suppose que tous les actifs de crédit dans le portefeuille sont homogènes (ce qui peut être juste si, par exemple, nous supposons qu'ils appartiennent au même secteur économique) de sorte que leur valeur est modélisée par des copies indépendantes de la même EDS (même volatilité et même dérive). Pour simplifier, nous allons considérer le modèle de Merton. Afin de tenir compte des effets de contagion par secteur économique, nous supposons qu'il existe une interdépendance entre les dynamiques des différents emprunteurs. Ainsi, on suppose que les coefficients de l'EDS sont fonction de la moyenne empirique de tous les emprunteurs. Nous supposons également que l'investisseur dispose, en plus des actifs de crédit, d'un portefeuille de marché utilisé, par exemple, pour sauvegarder le risque de crédit ou pour fournir des liquidités à sa division de crédit. Ensuite, la valeur de la position du créditeur est modélisée par une EDS avec des coefficients fonction de la contribution de tous les actifs de crédit. L' objectif du problème de contrôle est de maximiser la valeur de la position du créditeur.

Mathématiquement, on s'intéresse à contrôler une particule marquée $\hat{X}$ évoluant dans un environnement construit d'un grand nombre de particules indépendantes et identiques $X^1, \ldots, X^M$ avec des interactions du type moyenne empirique et où nous supposons que la particule marquée

«ressent» l'environnement, mais n'agit pas sur lui. Alors, la dynamique d'un tel système suit l'équation

$$
\begin{aligned}
dX_t^j &= b\left(t, X_t^j, \frac{1}{M}\sum_{i=1}^{M}\delta_{X_t^i}\right)dt + \sigma\left(t, X_t^j, \frac{1}{M}\sum_{i=1}^{M}\delta_{X_t^i}\right)dB_j(t) \quad \text{for } j = 1, \dots M;\\
d\hat{X}_t^{1:M;\alpha} &= b^0\left(t, \hat{X}_t^{1:M;\alpha}, \frac{1}{M}\sum_{i=1}^{M}\delta_{X_t^i}, \alpha_t\right)dt + \sigma^0\left(t, \hat{X}_t^{1:M;\alpha}, \frac{1}{M}\sum_{i=1}^{M}\delta_{X_t^i}, \alpha_t\right)dW_t\\
X_0^1 &= \dots X_0^M = x, \quad \hat{X}_0^{1:M;\alpha} = \tilde{x}
\end{aligned}
$$

où B_j $j = 1, \dots M$ sont des mouvement Browniens[1] de dimension d, W est un mouvement Brownien de dimension d', et α représente le contrôle que nous supposons appartient à un ensemble de contrôles admissibles $\mathcal{A}$. Nous supposons que la fonctionnelle objectif est donnée par

$$
J(t, x, \tilde{x}, \alpha) = \mathbb{E}\left[g\left(X_T^{1:M;\alpha;t,\tilde{x}}, \frac{1}{M}\sum_{i=1}^{M}\delta_{X_T^i}\right) + \int_0^T f\left(s, X_s^{1:M;\alpha;t,\tilde{x}}, \frac{1}{M}\sum_{i=1}^{M}\delta_{X_s^i}\right)ds\right]
$$

L'idée maintenant est d'approcher le problème présenté par un autre qui, nous espérons, est plus simple à résoudre. Nous nous concentrons alors sur la solution du système obtenu en passant à la limite sur le nombre de particules comme expliqué, dans [20], et résolvons le problème d'optimisation limite associé en espérant que la solution du problème limite ne soit pas trop éloignée de la solution du problème original. Dans les sections suivantes, nous expliquons que grâce à un phénomène de conservation du chaos le système asymptotique suit une équation de McKean-Vlasov

$$
\begin{cases}
dX_t = b(t, X_t, \mu_t)dt + \sigma(t, X_t, \mu_t)dB_t\\
d\tilde{X}_t^\alpha = b^0(t, X_t, \tilde{X}_t^\alpha, \mu_t, \alpha_t)dt + \sigma^0(t, X_t, \tilde{X}_t^\alpha, \mu_t)dW_t\\
X_0 = x, \quad \tilde{X}_0^\alpha = \tilde{x}
\end{cases}
$$

avec μ_t la loi de X_t, que nous supposons fixe dans la suite. Notons que, dans la version limite, nous permettons en plus une dépendance de la variable marquée par rapport à la variable d'état de l'environnement. Nous présentons dans la section 3.1 des conditions garantissant l'existence et l'unicité de la solution de ce système.

De façon similaire, la nouvelle fonctionnelle pour le système asymptotique s'écrit

$$
J(t, x, \tilde{x}, \alpha) = \mathbb{E}\left[g(X_T^{t,x}, \tilde{X}_T^{\alpha;t,\tilde{x}}, \mu_T) + \int_0^T f(s, X_s^{t,x}, \tilde{X}_s^{\alpha;t,\tilde{x}}, \mu_s)ds\right].
$$

Imaginons que nous soyons intéressés par la solution optimale

$$
u(t, x, \tilde{x}) = \sup\{J(t, x, \tilde{x}, \alpha),\ \alpha \in \mathcal{A}\}.
$$

Pourvu que toutes les équations soient bien définies, nous pouvons résoudre u en utilisant l'équation de Hamilton Jacobi Bellman (HJB) associée

$$
D_t u(t, x, \tilde{x}) + \frac{1}{2}\mathrm{Tr}(\bar{a}D^2_{x,\tilde{x}}u(t, x, \tilde{x})) + b(t, x, \mu_t)D_x u + H(t, x, \tilde{x}, D_x u, \mu_t) = 0,
$$

avec

$$
\bar{a} = \begin{bmatrix} \sigma\sigma^T & \sigma\rho(\sigma^0)^T \\ \sigma^0\rho^T\sigma^T & \sigma^0(\sigma^0)^T \end{bmatrix}, \qquad \rho = [B, W]
$$

[1] et non les entrées d'un Brownien que l'on note avec l'exposant !

où $[\cdot,\cdot]$ est le crochet et H est l'Hamiltonien

$$H(t,x,\tilde{x},z,\mu_t) = \sup_{\alpha\in\mathcal{A}} \left[b^0(t,x,\tilde{x},\mu_t,\alpha)z + f(t,x,\tilde{x},\mu_t)\right].$$

Nous ne discuterons pas ici de l'existence d'une solution à l'équation HJB (voir par exemple [34] ou [87] pour une revue sur le sujet). En supposant qu'elle est bien définie, l'équation HJB (2.21) peut être réécrite d'un point de vue probabiliste : $u(t,x,\tilde{x}) = Y_t^{t,x,\tilde{x}}$ où $Y^{t,x,\tilde{x}}$ est donné par la solution d'une EDSPR-MV.

$$\begin{cases} dX_s^{t,x,\tilde{x}} = b(s,X_s^{t,x,\tilde{x}},\mu_s)ds + \sigma(s,X_s^{t,x,\tilde{x}},\mu_s)dB_s \\ d\tilde{X}_s^{t,x,\tilde{x}} = \sigma^0(s,X_s^{t,x,\tilde{x}},\tilde{X}_s^{t,x,\tilde{x}},\mu_s)dW_s \\ -dY_s^{t,x,\tilde{x}} = H(s,X_s^{t,x,\tilde{x}},\tilde{X}_s^{t,x,\tilde{x}},\tilde{Z}_s^{t,x,\tilde{x}},\mu_s) - \tilde{Z}_s^{t,x,\tilde{x}}dW_s + Z^{t,x,\tilde{x}}dB_s \\ X_t^{t,x,\tilde{x}} = x, \quad \tilde{X}_t^{t,x,\tilde{x}} = \tilde{x}, \quad Y_0^{t,x,\tilde{x}} = g(X_T^{t,x,\tilde{x}},\tilde{X}_T^{t,x,\tilde{x}},\mu_T). \end{cases}$$

Ce qui montre que résoudre une EDSPR-MV est intéressant pour trouver des solutionnes optimales pour une classe de problèmes de contrôle.

Afin de donner un cadre approprié pour la définition et les propriétés des EDSPR-MVs et de mettre en évidence les difficultés éventuelles pour les résoudre, nous faisons une brève revue de quelques propriétés bien connues sur les processus de McKean-Vlasov et les EDSPRs. Nous présenterons également la méthode de cubature sur l'espace de Wiener, principal outil de notre algorithme. Ensuite, nous expliquons l'algorithme proposé et les résultats principaux associés.

3.1 Processus de McKean-Vlasov

Les équations de McKean-Vlasov sont des EDS dont les coefficients dépendent de la loi du processus lui même, c'est-à-dire

$$dX_t = b(t,X_t,\mu_t)dt + \sigma(t,X_t,\mu_t)dB_t, \quad 0\le t\le T; \quad \mu_0 = \bar{\mu}_0 \tag{11}$$

où μ_t est la loi de X au temps t. Elles ont été introduites par McKean dans [76], comme la contrepartie probabiliste d'EDPs figurant dans des problèmes de transport mécanique. Ces EDPs s'écrivent de façon faible :

$$\partial_t\langle\mu_t,\phi\rangle = \langle\mu_t,\mathcal{L}(\mu_t)\phi\rangle \tag{12}$$

où $\mathcal{L}$ est le générateur infinitésimal associé à (11).

Les équations de McKean-Vlasov apparaissent naturellement comme limite des systèmes de particules avec bruits indépendants et des interactions du type moyenne empirique lorsque le nombre de particules devient très grand. L'étude du comportement limite d'un tel système de particules utilise le concept de « propagation du chaos », introduit par Kac dans [51] et puis étudié entre autres par McKean [76, 77], Tanaka [96], Sznitman [90, 91] et Méléard [78]. Nous dirons qu'il y a de la propagation du chaos pour un système dynamique de N particules en interaction de type moyenne empirique si, étant donné que le système est initialisé avec des particules indépendantes et identiquement distribuées, la loi de $k < N$ particules tend vers la loi de k particules i.i.d. lorsque N tend vers l'infini. De plus, lorsque le système est échangeable, la propagation du chaos est équivalente à la convergence faible de la mesure empirique du système vers une loi commune unique quand la taille du système tend vers l'infini. Cette dernière propriété est derrière la caractérisation de l'équation de McKean-Vlasov comme modèle pour le comportement asymptotique d'un système de particules avec des interactions à champ moyen. Une excellente revue sur le sujet est celle de Sznitman [91]. Des résultats sur la vitesse de convergence d'une telle approximation sont présentées par exemple par Méléard dans [78].

Bien que les processus de McKean-Vlasov soient apparus initialement en mécanique statistique, ils sont maintenant utilisés dans de nombreux domaines en raison de la vaste gamme d'applications impliquant des grandes populations. Par exemple, ils sont utilisés en finance, pour modéliser la volatilité stochastique d'un modèle à facteur [7] ou dans des modèles à volatilité incertaine [45] ; en économie, dans la théorie des « jeux à champ moyen » récemment développée par J.M. Lasry et P.L. Lions dans une série de papiers [61, 62, 63, 64] (voir aussi [20, 19, 18] pour le point de vue probabiliste) et aussi en physique, neurosciences, biologie, etc.

La question de l'existence et de l'unicité de la solution d'une équation de McKean-Vlasov donnée peut être abordée sous des hypothèses de régularité appropriées pour les coefficients, incluant la régularité par rapport à la loi. Prenons l'espace $\mathcal{P}_2(C([0,T],\mathbb{R}^d))$ des distributions sur $C([0,T],\mathbb{R}^d)$ admettant des moments du second ordre. Dans cet espace, nous définissons la métrique de Wasserstein-2

$$D_T(m_1,m_2) = \left[\inf_{m\in\Gamma(m_1,m_2)} \int \left(\sup_{s\in[0,T]} \|x_t - y_t\|\right)^2 m(dx,dy)\right]^{1/2}$$

où $\Gamma(m_1,m_2)$ désigne la collection de toutes les mesures sur $\mathcal{P}_2(C([0,T],\mathbb{R}^d)) \times \mathcal{P}_2(C([0,T],\mathbb{R}^d))$ avec comme lois marginales m_1 et m_2 respectivement.

Proposition 0.4. *Supposons que les coefficients b et σ définis dans (11) soient uniformément Lipschitz continus en $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(C([0,T],\mathbb{R}^d))$ et que μ_0 a un second moment fini. Alors, il y a existence et unicité forte de (11).*

La proposition est démontrée dans [78]. L'idée est de procéder par un argument de point fixe : on définit une application de l'espace des fonctions de $\mathcal{P}_2([0,T],\mathbb{R}^d)$ dans lui-même, et on montre qu'elle est une contraction. Ensuite, l'unicité forte découle de la continuité Lipschitz par un argument classique.

Une fois résolus les problèmes d'existence et d'unicité de la solution de l'équation de McKean-Vlasov, la question des méthodes numériques pour la résoudre apparaît naturellement. En général, il est très difficile de résoudre l'EDP (12). La technique habituelle pour résoudre ce genre d'équation est basée sur l'équation de Fokker Planck. Il est aussi possible d'utiliser des méthodes probabilistes basées sur la représentation comme limite d'un système de particules en interaction (voir par exemple [2], [95] or [11] et les références incluses). L'idée, dans ce genre d'algorithme, est de simuler un grand nombre de particules en interaction avec des bruits indépendants, où la trajectoire de chaque particule est approchée en utilisant des techniques de discrétisation traditionnelles et la dépendance de la loi dans les coefficients est remplacée par la mesure empirique de l'ensemble des particules.

Dans le cadre de cette thèse, nous explorons une approche différente, basée sur la méthode de cubature sur l'espace de Wiener.

3.2 Cubature sur l'espace de Wiener

La cubature sur l'espace de Wiener (aussi connue comme méthode KLV pour Kusuoka-Lyons-Victoir) est une méthode inspirée des algorithmes d'intégration multi-dimensionnelles, qui cherche à approcher faiblement et de façon déterministe une EDS.

Expliquons comment la méthode se déroule. Soit $(\Omega',\mathcal{F}',\mathbb{P}')$ un espace de probabilité avec $\Omega' = C^0([0,1],\mathbb{R}^d)$ l'ensemble de fonctionnes continues de $[0,1]$ sur $\mathbb{R}^d$ ayant comme valeur

initiale 0. Soit $\mathcal{F}'$ la tribu de Borel associé à cet espace et $\mathbb{P}_x$ la mesure de Wiener, autrement dit, la mesure associé à un mouvement Brownien dans cet espace de probabilité.
Alors, si X est un processus stochastique de dimension d défini sur $(\Omega, \mathcal{F}, \mathbb{P})$ par

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s,$$

nous avons pour chaque f fonction mesurable,

$$\mathbb{E}[f(X_T)] = \int_{C^0([0,T],\mathbb{R}^d)} f(X'_T(\omega))\mathbb{P}(\omega). \tag{13}$$

où

$$X'_t(\omega') = x_0 + \int_0^t b(X'_s(\omega'))ds + \int_0^t \sigma(X'_s(\omega'))d\omega'_s. \tag{14}$$

Ainsi, nous pouvons approcher faiblement les lois marginales de X en approchant l'intégrale de dimension infinie à droite dans (13). L'idée principale de la méthode de cubature est de trouver une mesure finie sur Ω' qui « approche » la mesure $\mathbb{P}'$, rendant plus simple le calcul de l'intégrale. Ici, nous nous intéressons à une façon particulière de dire que deux mesures sont proches : la mesure discrète se rapproche de la dimension infinie si les espérances des intégrales itérées du mouvement Brownien sont conservées jusqu'à un certain ordre. Nous précisons ce concept en définissant une m-formule de cubature.

Definition 0.5. Soit m un entier positif est $t \in \mathbb{R}^+$. Une m-formule de cubature dans l'espace de Wiener $C^0([0,t],\mathbb{R}^d)$ est une mesure discrète de probabilité $\mathbb{Q}_t$ avec support fini dans $C^0_{\text{bv}}([0,t],\mathbb{R}^d)$ (fonctionnes continues et à variation bornée avec valeur initiale zéro) telle que les espérances des intégrales de Stratonovitch itérées d'ordre m sous la mesure de Wiener soient les mêmes que sous la mesure de cubature Q_t, c'est-à-dire, pour tout multi-indice $(i_1, \cdots, i_l) \in \{1, \cdots, d\}^l$, $l \leq m$

$$\begin{aligned}\mathbb{E}\int_{0<t_1<\cdots<t_l<t} \circ dB^{i_1}_{t_1} \cdots \circ dB^{i_l}_{t_l} &= \mathbb{E}_{\mathbb{Q}_t}\int_{0<t_1<\cdots<t_l<t} \circ dB^{i_1}_{t_1} \cdots \circ dB^{i_l}_{t_l} \\ &= \sum_{j=1}^{l} \lambda_j \int_{0<t_1<\cdots<t_l<t} d\omega^{i_1}_j(t_1) \cdots d\omega^{i_l}_j(t_l),\end{aligned}$$

où «$\circ$» signale l'integrale de Stratonovitch, et nous notons par ω^i_j la $i-$ème entré du $j-$ème chemin composant la mesure de cubature.

La mesure de cubature permet alors d'approcher faiblement la solution d'une EDS. Il suffit de regarder la mesure construite avec les solutions des EDO issue des équations (14) pour chaque élément du support de la formule de cubature, avec ses poids associés. Comme une conséquence directe du développement de Taylor-Stratonovitch appliqué à $f(X_t)$, une m-formule de cubature satisfait

$$|(\mathbb{E} - \mathbb{E}_{\mathbb{Q}_t})f(X_t)| \leq Ct^{(m+1)/2}||f||_{m+2,\infty}, \tag{15}$$

pour toute fonction f avec $m+2$ dérivés continues et bornées lorsque les coefficients b, σ sont réguliers et bornés.

Bien sûr, ce contrôle d'erreur n'est pas en général petit. Mais, en utilisant la propriété de Markov et un changement de échelle du mouvement Brownien, il est possible d'appliquer la méthode de cubature de façon répétée sur des petites subdivisions de l'intervalle $[0,t]$ pour lesquels le contrôle sera petit, de sorte à avoir un bon contrôle total.

En effet, imaginons que nous voulions approcher une EDS sur l'intervalle $[0,T]$. Considérons une formule de cubature $\mathbb{Q}_1$ d'ordre $m \in \mathbb{N}^*$ ayant un support égal à $\{\omega_1,\dots,\omega_\kappa\}$ et dont les poids associés sont $\{\lambda_1,\dots,\lambda_\kappa\}$. Pour tout $h>0$ et tout $t \in [0,T-h]$, nous pouvons construire une mesure de cubature $\mathbb{Q}_{t,t+h}$ d'ordre m avec un support fini $\{\tilde{\omega}_1,\cdots,\tilde{\omega}_\kappa\}$ où $\tilde{\omega}_j : s \in [t,t+h] \mapsto \tilde{\omega}(s) = \sqrt{h}\omega_j((s-t)/h)$ pour tout $1 \leq j \leq \kappa$; auxquels nous associons les poids originaux $\{\lambda_1,\dots,\lambda_\kappa\}$.

Ainsi, grâce à la propriété de Markov, une subdivision $0 = T_0 < T_1 < \ldots < T_N = T$ de l'intervalle $[0,T]$ conduit à la construction d'un arbre avec κ^k nœuds (le nombre de chemins différents) à la $k-$ème subdivision : en effet, il y a un seul nœud (la valeur initiale) au temps T_0 puis, pour chaque nœud à l'instant T_k, l'application de la cubature génère κ branches et donc κ nœuds au temps T_{k+1}. Les trajectoires sont notées $\tilde{\omega}_{(i_1,\dots,i_k)}$, où $(i_1,\dots,i_k)$ représente la trajectoire où l'on prendre d'abord le $i_1$ème branche, puis le $i_2$ème branche, etc. La trajectoire $\tilde{\omega}_{(i_1,\dots,i_k)}$ est associée aux poids cumulés $\Lambda_{(i_1,\dots,i_k)} = \prod_{j=1}^{k} \lambda_{i_j}$.

La Figure 1 illustre l'arbre de cubature associé à une équation d'Ornstein-Uhlenbeck en dimension un, où nous avons fixé $N=5$ et nous avons utilisé une 3-formule de cubature. Le poids de chaque particule est représenté par la taille de chaque nœud.

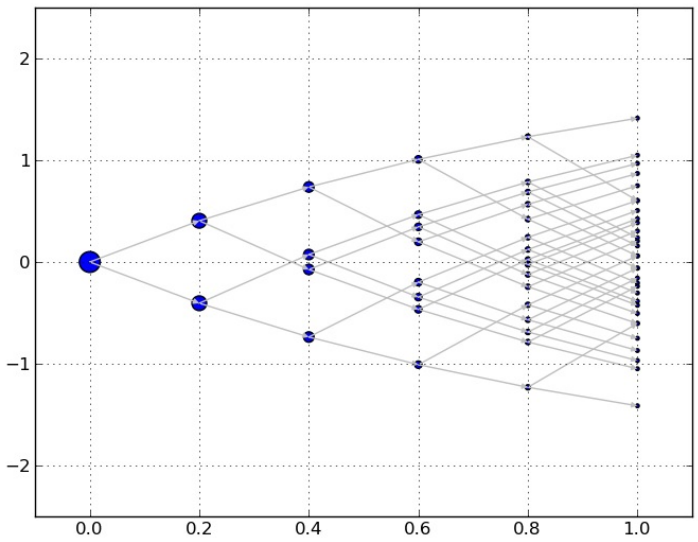


FIGURE 1 – Illustration d'un exemple d'une mesure de cubature en dimension un.

L'idée principale de la méthode de cubature sur l'espace de Wiener, c'est-à-dire, approcher une loi de probabilité en dimension infinie par une loi dont le support est fini, a été considéré auparavant en utilisant des outils mathématiques differents. Mentionnons, par exemple, le étude menée par Luschgy et Pagès dans [68] sur la quantification de processus Gaussiens. Le but de la quantification est de construire une grille spatiale et des poids associés pour approcher une loi de probabilité donnée de façon optimale par rapport à une distance L_p (où l'on prend usuellement $p=2$). Le lien entre ces deux méthodes est aussi present lorsqu'elles sont utilisées pour résoudre certains problèmes de probabilité numérique. Par exemple, nous montrons dans 3.3.b que les deux méthodes peuvent être employées à la résolution des EDSRs.

Concluons par quelques remarques. Notons que les hypothèses de régularité sur f peuvent être assouplies : le point clé étant de bénéficier de l'effet de régularisation induit par la diffusion sous certaines conditions de structure supplémentaires. On pourra consulter [41] qui montre cet aspect dans le cas uniformément elliptique et [57] pour une hypothèse plus faible sur la structure de diffusion appelée condition (UFG) [2]. Grâce à ces résultats, Lyons et Victoir analysent le contrôle de l'erreur pour f Lipschitz et pour la discrétisation de l'intervalle donnée dans [69] (nous utilisons cette même subdivision dans le chapitre 2). Remarquons, enfin, que dans [46] le lecteur pourra retrouver des résultats applicables lorsque f est continue par morceaux.

Avant tout chose, insistons sur le fait que la définition de la formule de cubature utilisée ici fonctionne avec l'intégrale de Stratonovitch et non d'Itô. Cela s'explique par la convergence des intégrales de Stratonovitch lorsque le pas de temps décroît, mais aussi par la structure algébrique typique des intégrales de Stratonovitch qui est extrêmement pratique pour la construction des formules de cubature. Ce dernier aspect a été étudié par Chen dans [24] et les autres papiers reliés ; Lyons ayant utilisé les résultats de Chen dans sa théorie sur les chemins rugueux en 1998 [70]. Plus concrètement, on retrouve dans [69] et [46] des exemples explicites de formules de cubature à différents ordres et différentes dimensions mettant en avant la structure particulière des intégrales de Stratonovitch.

3.3 Aperçu des EDSPRs

Les Équations Différentielles Stochastiques Retrogrades (EDSR) ont été introduites par Bismut en 1973 [10] pour jouer le rôle d'équation adjointe d'un problème d'optimisation stochastique linéaire, et ensuite généralisées par Pardoux et Peng [81] dans le cas non-linéaire. On les retrouve dans de nombreux domaines comme par exemple en finance où elles constituent un outil puissant pour l'évaluation des produits dérivés et de leur couverture particulièrement lorsque des contraintes apparaissent ou que le marché est incomplet. Le fait de les rencontrer dans de nombreux domaines s'explique en particulier par leur lien avec les EDP non-linéaires et les équations de contrôle stochastique. Le lecteur intéressé pourra consulter [73] pour plus de détails sur les EDSR ainsi que [33] pour le cas d'application en finance.

Considérons un espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P})$ de filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ ainsi qu'un mouvement Brownien standard de dimension m, $(B_t)_{0\leq t\leq T}$, adapté à cette filtration. La solution de l'EDSR est constituée de deux processus stochastiques $(Y_t)_{0\leq t\leq T}, (Z_t)_{0\leq t\leq T}$, adaptés à $(\mathcal{F}_t)_{0\leq t\leq T}$, vérifiant

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \tag{16}$$

où ξ est une variable aléatoire $\mathcal{F}_T$- mesurable qui représente la condition au bord et où f est une application mesurable de $\Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{m\times d}$ dans $\mathbb{R}^d$, communément appelé générateur. Malgré le fait que (16) ne fasse apparaître qu'une seule équation, nous pouvons supposer l'unicité de la solution sous certaines contraintes appropriées grâce à la condition d'adaptabilité imposée à Y et Z.

Proposition 0.6. *Pour toute fonction f uniformément Lipschitzienne tel que $f(.,0,0)$ soit L_2 dans $\Omega \times [0,T]$, il existe une unique solution de l'EDSR (16) .*

Nous rappelons les arguments principaux de la preuve de Pardoux et Peng [81] puisqu'ils permettent de comprendre le résultat analogue dans le cas McKean-Vlasov. Comme pour les EDS, l'unicité est obtenue par une méthode classique qui consiste à supposer l'existence de deux solutions (Y^1, Z^1) et (Y^2, Z^2). On conclut, en utilisant le lemme de Gronwall et les propriétés

[2] UFG : « uniformly finitely generated »

Lipschitz des coefficients. En ce qui concerne l'existence, on introduit une suite de processus Y^n, Z^n telle que

$$Y_t^{(n+1)} = \xi + \int_t^T f(s, Y_s^{(n)}, Z_s^{(n)})ds - \int_t^T Z_s^{(n+1)} dB_s,$$

ce qui permet de découpler l'équation d'origine. Pour terminer il suffit de montrer que Y^n, Z^n est une suite de Cauchy (à l'aide du lemme d'Itô est de la propriété Lipschitz).

Le système d'équation suivant :

$$\begin{aligned} X_t &= x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \\ Y_t &= \phi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)dt - \int_0^T Z_s dB_s \end{aligned} \tag{17}$$

est un exemple d'EDSR où la condition au bord ainsi que le générateur dépendent des valeurs marginales d'un autre processus stochastique. Si le problème progressif est bien posé, le résultat précédent nous permet d'obtenir l'existence et l'unicité de la solution. Notons que ce type d'équation est appelé Équation Différentielle Stochastique Progressive Retrograde (EDSPR) découplée (car les coefficients de l'équation progressive ne dependent pas des variables retrogrades).

Remarque. Dans la suite nous ne considérerons pas d'EDSPRs couplées, c'est-à-dire de la forme

$$\begin{aligned} X_t &= x_0 + \int_0^t b(t, X_s, Y_s, Z_s)ds + \int_0^t \sigma(t, X_s, Y_s)dB_s \\ Y_t &= g(X_T) + \int_t^T f(t, X_t, Y_t, Z_t)dt - \int_t^T Z_t dB_t. \end{aligned}$$

3.3.a *Champ découplant*

Considérons une version légèrement modifiée de l'équation (17) où on impose que la condition $X_t = x$, c'est-à-dire,

$$\begin{aligned} X_s^{t,x} &= x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r \\ Y_s^{t,x} &= \phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dB_r. \end{aligned} \tag{18}$$

On définit la fonction de découplage (déterministe) u comme :

$$u(t, x) = Y_t^{t,x} \ p.s.$$

L'appellation « champ découplant » vient du cas d'une EDSPR complètement couplée où faire apparaître la fonction u constitue l'étape clé du schéma « à quatre étapes» introduit par Ma, Protter et Yong [72]. Elle permet de découpler l'interaction entre la partie progressive et la partie rétrograde.

Intéressons nous à quelques propriétés de la fonction u. Pour commencer, il est possible de montrer à l'aide de la propriété de Markov que

$$Y_s^{t,x} = u(s, X_s^{t,x}).$$

Ensuite, il est important de rappeler que la fonction de découplage crée un lien entre les EDSPRs et les EDPs. En effet, il a été montré par Peng dans [86] que la solution adaptée à (18) peut être interprétée comme solution de viscosité de l'EDP :

$$D_t u(t, x) + \mathcal{L}u(t, x) = f(t, x, u(t, x), \sigma^* D_x u(t, x)), \qquad u(T, x) = g(x), \tag{19}$$

où $\mathcal{L}$ est le générateur infinitésimal de la variable progressive de l'équation (18). De plus en utilisant le calcul de Malliavin, Pardoux et Peng ont montré que, si ϕ, b, σ sont C_b^3 par rapport à x et si $f(0,.,.)$ appartient à C_b^3 par rapport à (x,y), alors u appartient à $C^{1,2}$ tout en étant une solution classique de (19). En complément, Crisan et Delarue ont étudié la régularité de la fonction u avec des hypothèses plus faibles sur g en ajoutant des hypothèses sur les propriétés de diffusion de la variable progressive [25] ; ce résultat peut être vu comme une extension du programme du Kusuoka et Stroock [58].

Réciproquement, d'après le lemme d'Itô, si une fonction u appartient à $C^{1,2}$ et qu'elle satisfait l'équation (19), alors le couple $Y_s^{t,x} := u(s, X_s^{t,x})$ et $Z_s^{t,x} := \sigma^* D_x u(s, X_s^{t,x})$ est solution de (18).

3.3.b *Solution numérique*

Plusieurs méthodes pour approcher numériquement la solution d'une EDSPR découplée sont disponibles. Ces méthodes peuvent être classifiées en deux grandes groupes. D'un côté, nous avons des algorithmes qui cherchent à résoudre directement la fonction de découplage u en utilisant l'EDP (19) (différences finies, éléments finis, etc.). De l'autre côté, nous avons des algorithmes qui utilisent le point de vue probabiliste. Dans cette section nous nous occupons de ce dernier groupe. L'idée principale est d'utiliser la fonction de découplage et des résultats sur sa régularité pour résoudre l'équation rétrograde en partant de la condition au bord et puis remontant en amont. Cette idée nécessite un schéma d'approximation de l'EDSPR et une méthode d'approximation des espérances conditionelles. En effet, soit $0 = T_0 < T_1 < \ldots < T_N = T$ une grille de l'intervalle $[0,T]$, et notons $\Delta_{T_k} = T_k - T_{k-1}$. Alors, nous pouvons approcher la solution de la fonction de découplage $u(t,x) = Y_t^{t,x}$ et la fonction $v(t,x) = Z_t^{t,x}$ par :

$$\begin{aligned} u(T_N, x) &= g(x) \\ v(T_k, x) &\approx \frac{1}{\Delta_{T_{k+1}}} \mathbb{E}\left(u(T_{k+1}, X_{T_{k+1}}^{T_k,x}) \Delta W_{k+1} \right) \\ u(T_k, x) &\approx \mathbb{E}\left(u(T_{k+1}, X_{T_{k+1}}^{T_k,x}) + \Delta_{T_{k+1}} f(\Theta_{k+1,x}) \right), \end{aligned} \tag{20}$$

où

$$\Theta_{k+1,x} := \left(T_{k+1}, X_{T_{k+1}}^{T_k,x}, u(T_{k+1}, X_{T_{k+1}}^{T_k,x}), v(T_{k+1}, X_{T_{k+1}}^{T_k,x}) \right). \tag{21}$$

Nous remarquons que la régularité du champ de découplage permettra d'utiliser plutôt une variante explicite de $\Theta_{k+1,x}$.

Afin de rendre le schéma utile en pratique, il faut choisir une méthode d'approximation de l'espérance conditionnelle dans (20). Selon le type d'approche choisi, nous pouvons regrouper les algorithmes probabilistes en trois grandes classes : algorithmes de régression en L_2, algorithmes basées sur le calcul de Malliavin et algorithmes de discrétisation spatiale.

Les algorithmes de régression redéfinissent le problème d'approximation : l'objectif devient alors celui de trouver les fonctions de découplage u, v à l'instant T_k, sachant leurs valeurs au temps T_{k+1}, en faisant une régression sur l'espace généré par un ensemble bien choisi de fonctions en L_2. Le choix de l'ensemble de fonctions génératrices met en compétition deux objectifs. D'un côté, nous privilégions les ensembles petits, afin de simplifier la régression, et de l'autre nous préférons les ensembles riches, afin de permettre une bonne précision d'approximation de la solution. Typiquement nous choisirons comme ensemble générateur une famille des fonctions constantes par morceaux ou plus généralement des polynômes locaux à degré choisi. La régression se fait en utilisant comme critère une approximation de l'erreur L_2 en T_k normalement obtenue à partir de simulations de Monte Carlo. Cette approche a été proposée d'abord par Longstaff-Schwarz [67] ; puis étudiée en profondeur dans plusieurs articles avec de conditions plus faibles

de convergence par exemple par Gobet, Lemor et Warin dans [44] et Gobet et Lemor dans [43].

Une alternative consiste à approcher l'opérateur espérance conditionnelle en utilisant le calcul de Malliavin. En effet, il est possible de calculer l'espérance conditionnelle d'une fonction en calculant l'espérance totale de la fonction ponderée par une fonction aléatoire qui peut être explicitement calculée grâce au calcul de Malliavin. Intuitivement, nous pouvons comprendre cette procédure comme un changement de numéraire. La fonction de pondération et l'espérance totale sont calculées en pratique à l'aide des simulations de Monte Carlo. Dans plusieurs cas, il est aussi important d'ajouter une fonction de troncature afin de contrôler l'existence de singularités de la fonction de pondération. Cette technique a été introduite par Fournié et al. dans [39] puis étudiée dans [38]. Le lecteur peu trouver aussi dans le papier de Bouchard et Touzi [12] une description complète d'un algorithme de solution de EDSR utilisant le calcul de Malliavin.

Finalement, nous avons les algorithmes qui cherchent à approcher la loi de la variable progressive en construisant une grille dans l'espace de solution et des poids associés à chaque point de la grille. Dans ce type d'approximation, l'algorithme se fait en deux étapes : d'abord l'on s'occupe de la construction de la grille, puis, un schéma comme (20) est utilisé pour trouver la solution à chaque point du réseau. En général cette deuxième étape est calculée très rapidement et la plus complexe des deux étapes est la première. De plus, les deux étapes sont indépendantes, ce qui nous permet de conclure qu'une même grille peut être utilisée pour résoudre plusieurs problèmes rétrogrades ayant des données différentes. La méthode la plus connue dans cette famille est la méthode de quantification proposée par Bally et Pagès dans [4] et [3]. La première étape de la méthode de quantification consiste à construire une grille formée par N points, avec ses poids associées, qui minimise l'erreur de quantification L_2 (ou plus généralement L_p). Pour cela, l'algorithme combine les procédures de construction de diagrammes de Voronoi, le gradient stochastique pour la minimisation[3] et des approximations empiriques des probabilités de transition. Dans [4] et [3], Bally et Pagès montrent que la méthode de quantification est efficace pour résoudre des EDSR multidimensionnelles (même dans le cadre plus général des équations réfléchies) sans terme Z dans le générateur. La méthode a été aussi appliqué pour résoudre des EDSPR complètement couplées par Delarue et Menozzi dans [29].

Plus récemment, une autre méthode de type grille spatiale a été proposé par Crisan et Manoralakis dans [28]. Les auteurs utilisent la méthode de cubature sur l'espace de Wiener (présentée dans la section 3.2). L'idée principale derrière leur algorithme est de profiter de la propriété de Markov pour approcher l'espérance conditionnelle.

Une propriété intéressante de l'algorithme de Crisan et Manoralakis est le fait qu'il peut être construit pour atteindre une vitesse de convergence polynomiale avec un exposant $r > 1$, à condition que la vraie solution soit suffisamment régulière. En effet, il suffit d'utiliser une formule de cubature d'ordre supérieur à trois et de travailler avec un schéma d'ordre supérieur à celui dans (20). Le lecteur peut trouver des exemples de discrétisation de plusieurs ordres dans les travaux de Crisan et Manoralakis [27], et Chassagneux et Crisan [22].

3.4 Équations EDSPR-MV découplées

Revenons à notre problème d'origine et considérons l'équation EDSPR-MV (10). Soit $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ un espace probabilisé, $B_t^{1:d}$ un mouvement Brownien d-dimensional adapté et $B_t^0 = t$. Nous supposons que les fonctions $V_i : (t, y, w) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \mapsto V_i(t, y, w)$; $\varphi_i : y \in$

[3] Dans la literature d'apprentissage, le cas L_2 est équivalent à une extension de l'Apprentissage Competitive par Quantification Vectorielle.

$\mathbb{R}^d \mapsto \varphi_i(y) \in \mathbb{R}$, $i = 0, \cdots, d$; $\varphi_f : (y, y') \in \mathbb{R}^d \times \mathbb{R} \mapsto \varphi_f(y, y')$ et $f : t, y, y', z, w \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \mapsto f(t, y, y', z, w) \in \mathbb{R}$ sont bornées et lisses avec de dérivées de tout ordre bornées. On suppose également que l'application $\phi : \mathbb{R}^d \to \mathbb{R}$ est au moins Lipschitz continue et nous préciserons sa régularité dans la suite.

Notons que dans (10) la forme de la dépendance par rapport à la loi est très spécifique, à savoir, les coefficients dépendent de la loi agissant sur les fonctions φ_i, $i = 1, \ldots, d$ et φ_f. Nous avons choisi ce cadre pour plusieurs raisons : premièrement, parce que c'est un cadre qui englobe une bonne partie des applications pratiques, deuxièmement parce que ce cadre a des bonnes propriétés mathématiques, notamment, ce type de dépendance est Lipschitz en distance Wasserstein-2 ; et finalement, parce qu'il simplifie la compréhension de l'algorithme.

Dans [15], Buckdahn Li et Peng ont démontré l'existence et l'unicité d'un système très similaire à (10). Ici, nous profitons de la structure découplée : nous avons déjà donné les conditions garantissant l'existence et l'unicité de l'équation progressive. Il ne reste qu'à considérer l'équation rétrograde. La preuve est similaire à celle d'une simple EDSR, quitte à prendre en compte le nouveau terme de loi (c'est-à-dire, nous pouvons définir une fonction définie sur l'espace des trajectoires continues de L_2 vers lui même, et montrer qu'il s'agit d'une contraction par rapport à une distance appropriée).

Passons maintenant à la question principale du chapitre (2) : la définition d'un algorithme permettant d'approcher l'équation (10). Notons que les parties progressives et rétrogrades sont découplées et que nous pouvons profiter de ce découplage pour résoudre l'équation en deux temps. D'abord la partie progressive puis la partie rétrograde. Une première idée est d'utiliser une méthode particulaire pour approcher le terme McKean-Vlasov et ensuite une des méthodes présentées pour la partie rétrograde. Cependant, les méthodes particulaires introduisent de nouvelles copies indépendantes du mouvement Brownien, augmentant la dimension du Brownien dans le problème rétrograde : autant de dimensions que de particules s'ajoutent pour l'approximation du terme de loi. Par conséquent, la complexité d'un tel algorithme est très importante, même si l'on pourrait espérer profiter des symétries intrinsèques à ce système.

Nous proposons une approche différente, construite à partir de la méthode de cubature sur l'espace de Wiener. L'idée de considérer la méthode de cubature est assez naturelle, car comme nous l'avons mentionné auparavant, elle sert à obtenir des bonnes approximations faibles d'une SDE et à résoudre des EDSPRs découplées. La principale difficulté du cas EDSPR-MV vient de la dépendance McKean-Vlasov, puisque la diffusion ainsi définie ne vérifie pas la propriété de Markov (sur $\mathbb{R}^d$) ce qui empêche l'utilisation directe de certains outils d'analyse classique. Pour gérer cette condition, nous allons découpler la dépendance de loi : *si l'on se donne la loi de la solution du système, (10) est une EDSPR classique, découplée et inhomogène en temps (car la loi agit comme un paramètre temporel).*

Soit $(\eta_t)_{0 \leq t \leq T}$ une famille de mesures de probabilités sur $\mathbb{R}^d$. Fixons la famille de lois dans les termes McKean-Vlasov de (10) égale à $(\eta_t)_{0 \leq t \leq T}$. Ce système modifié est une équation EDSPR classique, et donc nous pouvons l'approcher à l'aide de la méthode de cubature avec une dépendance en temps (qui est gérée par discrétisation). Ainsi, nous approchons le système découplé à loi gelée : d'abord nous construisons un arbre de cubature et puis nous remontons en temps en calculant pour chaque nœud à chaque étape la valeur des processus rétrogrades.

Évidemment, la précision de l'algorithme dépendra du choix de la loi utilisée à la place de la loi de la diffusion dans le terme McKean-Vlasov. Ainsi, cette loi doit être bien choisie pour espérer une convergence de la solution approchée vers la vraie solution. Par exemple, un « bon

choix » est celui de la loi de cubature obtenue à chaque pas de l'approximation comme la somme pondérée des mesures de Dirac prises aux valeurs de solution des EDO associées à chaque chemin de cubature. Nous montrons que si l'on utilise une $m-$ formule de cubature et N pas de discrétisation, ce choix d'approximation de la loi implique une erreur en norme infinie de l'ordre de $N^{-(m-1)/2}$ lorsque l'on approche l'espérance d'une fonction bornée $m+2$ fois dérivable avec toutes ses dérivées bornées[4]. On obtient également une erreur en norme infinie de l'ordre de N^{-1} pour la partie rétrograde du schéma. Des corrections sur l'algorithme que nous présentons postérieurement permettent l'obtention de vitesses de convergence supérieures.

Nous avons remarqué dans la section correspondant à la méthode de cubature, en suivant les analyses de [69] et [27], que si nous supposons l'uniforme ellipticité de la diffusion progressive dans (10) alors nous pouvons affaiblir l'hypothèse de régularité de la condition terminale ϕ de (10) (en fait, comme expliqué auparavant, la condition demandée dans les références données est plus faible ; il s'agit de la condition appelée UFG). L'explication vient des propriétés de régularisation des EDPs paraboliques et semi-linéaires (le lecteur intéressé peut voir [41] pour un aperçu du cas elliptique et [58] et [25] pour le cas UFG). Nous montrons que la possibilité d'affaiblir les conditions de régularité reste valable dans le cas McKean-Vlasov, et que dans ce cas les vitesses de convergence annoncées restent inchangées.

Le système conditionnel. Nous présentons tout d'abord brièvement l'argument derrière le découplage de la loi. Rappelons qu'il existe une unique solution $\{X_t^x, Y_t^x\}_{t\geq 0}$ au système (10). Rappelons également que d'après la discussion que nous avons présentée sur les EDSPRs classiques dans le cadre Markovien, $Y_t^x = u(t, X_t^x)$ où u est la fonction de découplage, et donc que la loi du couple (X^x, Y^x) est déterminée entièrement par la loi de $(X_t^x)_{0\leq t\leq T}$ i.e. $\mu = (\mu_t)_{0\leq t\leq T}$. Nous montrons que dans le cas McKean-Vlasov ce résultat reste valable (la démonstration est présentée dans la section 7), et qu'il existe une fonction déterministe $u(t,y)$, fonction aussi de μ_x, telle que pour tout $t \in \mathbb{R}_+$,

$$Y_t = u(t, X_t). \tag{22}$$

Nous étudions également la régularité de u et donnons des bornes sur ses dérivées sous des conditions appropriées. Ainsi, nous pouvons maintenant définir l'équation EDSPR-MV conditionnelle par

$$\begin{cases} dX_s^{t,y,\mu} = \sum_{i=0}^{d} V_i(s, X_s^{t,y,\mu}, \langle \mu_s, \varphi_i \rangle) dB_s^i \\ dY_s^{t,y,\mu} = -f(s, X_s^{t,y,\mu}, Y_s^{t,y,\mu}, Z_s^{t,y,\mu}, \langle \mu_s, \varphi_f(\cdot, u(s,\cdot)) \rangle) ds + Z_s^{t,y} dB_s^{1:d} \\ X_t^{t,y,\mu} = y, \quad Y_T^{t,y,\mu} = \phi(X_T^{t,y,\mu}). \end{cases} \tag{23}$$

On remarquera que nous avons découplé la dépendance par rapport a la loi de la variable rétrograde, à l'aide de la fonction de découplage.

Dans la suite, nous utilisons également la fonction $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ définie par

$$Z_t = v(t, X_t), \tag{24}$$

aussi déterministe et fonction de μ_x, et qui satisfait

$$v(t,x) = (\mathcal{V}^\mu u(t,x))^T,$$

si u est régulière.

[4] ceci correspond au cas lorsque ϕ est $m+2$ fois dérivable avec dérivées bornées et $f = 0$ en (10).

3.5 Algorithmes proposés

Nous donnons au chapitre 2, section 1, une description détaillée des algorithmes proposés. Ici nous présentons une description générale visant à développer davantage les idées principales de l'algorithme. On se donne une subdivision de l'intervalle $[0,T]$ en $N \in \mathbb{N}^*$ étapes $0 = T_0 < \cdots < T_N = T$.

- Algorithme pour la partie progressive (Algorithme 1 au chapitre 2 section 1.1.a) : L'algorithme a deux paramètres d'entrée : m, qui définit l'ordre des formules de cubature et q qui va définir l'ordre d'un développement limité.

  1. Démarrage ($k = 0$) : À l'étape 0, on fixe $\hat{X}_0 = x_0$ et $\hat{\mu}_0 = \delta_{x_0}$, c'est-à-dire une mesure de Dirac au point de départ.
  2. Propagation ($k = 1, \ldots, N$)
     (a) Geler la loi : Pour tout temps $t \in [T_k, T_{k+1})$ on fixe la loi dans le coefficients de l'équation MKV progressive égale à $\hat{\mu}_{k-1}$, puis on fait sur la dépendance McKean-Vlasov un développement limité d'ordre q, i.e. on change $\langle \mu_{T_k}, \varphi \rangle$ par

$$\sum_{p=0}^{q-1} \frac{1}{p!}(t - T_k)^p \langle \hat{\mu}_{T_{k+1}}, (\mathcal{L}^{\hat{\mu}})^p \varphi_i \rangle.$$

     L'EDS obtenue est une diffusion ordinaire.
     (b) Cubature : On utilise la méthode de cubature pour approcher l'EDS du pas précédent : autrement dit, on choisit une $m-$ formule de cubature et, pour chaque nœud $\hat{X}_{T_{k-1}}$ au temps T_{k-1} on résout l'EDO associé à chaque ω^j dans la formule de cubature. À la fin de cette procédure on obtient un nuage de particules déterministes prises au temps T_k
     (c) Mise à jour de la loi discrète. On définie $\hat{\mu}_k$ comme la mesure associée aux particules déterministes et leurs poids cumulés associés.

- Algorithme rétrograde (Algorithme 2 au chapitre 2 section 1.1.b) : Décrivons d'abord l'algorithme du première ordre :

  1. Démarrage ($k = N$) : On fixe $\hat{u}(T_N, \hat{X}_N) = \phi(\hat{X}_N)$, $\hat{v}(T_N, \hat{X}_N) = 0$.
  2. Propagation (Étapes $k = N-1, \ldots, 0$)
     (a) Geler la loi : Pour tout temps $t \in [T_k, T_{k+1})$ on fixe la loi dans le générateur de l'équation rétrograde égale à $(I \otimes \phi)(\hat{\mu}_{k+1})$. On obtient ainsi une EDSR classique.
     (b) Schéma rétrograde : On trouve $\hat{u}(T_k, X_k)$ et $\hat{v}(T_k, X_k)$ pour chaque nœud de l'arbre de cubature en suivant l'algorithme de Crisan et Manoralakis.

  L'algorithme rétrograde du second ordre (Algorithme 3 au chapitre 2 section 1.1.b), est similaire à celui décrit au-dessus, mais il utilise un schéma de discrétisation d'ordre supérieur. Pour éviter d'utiliser des schémas implicites (ce qui serais difficile à cause de la présence des termes de loi) et le calcul de dérivées , nous proposons un schéma prédicteur-correcteur, où le terme prédicteur est construit à partir du schéma d'ordre un.

3.6 Résultats principaux

Hypothèses. Le lecteur peut déduire des arguments que nous avons donné jusqu'à présent que l'analyse de l'erreur de l'algorithme proposé utilise les propriétés de régularité de la fonction $(t,x) \in [0,T] \times \mathbb{R}^d \mapsto \mathbb{E}\phi(X_T^{t,x,\mu})$ pour la partie progressive et de u, solution exacte de (22), pour la partie rétrograde. Ainsi, nous présentons deux types d'hypothèses qui conduisent aux

propriétés de régularité désirées.

La première option consiste à demander à la condition terminale et aux coefficients des deux équations (progressive et rétrograde) d'être infiniment réguliers. La seconde option consiste à considérer des équations satisfaisant des conditions de régularité plus faibles et une condition de uniforme éllipticité sur la matrice de diffusion V.

(SB) Nous dirons que l'hypothèse **(SB)** est satisfaite si la condition terminale ϕ définie en (10) est C_b^∞

(LB) Nous dirons que l'hypothèse **(LB)** est satisfaite si la condition terminale ϕ est uniformément Lipschitz continue et si la matrice VV^T est uniformément elliptique, c'est-à-dire, s'il existe $c > 0$ telle que

$$\forall (t, y, w) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}, \forall \varsigma \in \mathbb{R}^d, c^{-1}|\varsigma|^2 \leq VV^T(t, y, w)\varsigma \cdot \varsigma \leq c|\varsigma|^2.$$

On introduit une notation adaptée pour la présentation des résultats d'analyse de l'erreur. Pour $i = 1, 2$:

$$\mathcal{E}_u^i(k) := \max_{\pi \in \mathcal{S}_\kappa(k)} |u(T_k, \hat{X}_{T_k}^\pi) - \hat{u}^i(T_k, \hat{X}_{T_k}^\pi)|; \qquad \mathcal{E}_v^i(k) := \max_{\pi \in \mathcal{S}_\kappa(k)} |v(T_k, \hat{X}_{T_k}^\pi) - \hat{v}^i(T_k, \hat{X}_{T_k}^\pi)|, \quad (25)$$

où u, v sont définis en (22), (24) et $\hat{u}^1, \hat{v}^1, \hat{u}^2$ et $\hat{v}^2$ sont les solutions approchées, obtenues d'après les algorithmes rétrogrades du premier et du second ordres.

Principaux résultats sous les conditions (SB) .

Theorem 0.7. *Supposons que **(SB)** est satisfaite. Si nous appliquons l'algorithme 1 avec paramètres m et q en utilisant une grille uniforme avec N points, alors il existe une constante positive C, qui dépend uniquement de T, q, d, $||\varphi_{0:d}||_{2q+m+2,\infty}$ et $||\phi||_{m+2,\infty}$, telle que :*

$$\max_{k \in \{0,\dots,N\}} |\langle \mu_{T_k} - \hat{\mu}_{T_k}, \phi \rangle| \leq C \left(\frac{1}{N}\right)^{[(m-1)/2] \wedge q}. \quad (26)$$

où $\hat{\mu}$ est définie comme dans l'algorithme 1. Supposons en plus que $q \geq 1$ and $m \geq 3$. On déduit qu'il existe une constante positive C_1, qui dépend uniquement de T, q, d, $||\varphi_{0:d}||_{2q+m+2,\infty}$, $||\varphi_f||_{m+2,\infty}$ et $||\phi||_{m+3,\infty}$, telle que pour tout $k = 0, \dots, N$:

$$\mathcal{E}_u^1(k) + \Delta_{T_k}^{1/2} \mathcal{E}_v^1(k) \leq C_1 \left(\frac{1}{N}\right). \quad (27)$$

Supposons en plus que $q \geq 2$ et $m \geq 5$. On déduit qu'il existe une constante positive C_2, qui dépend uniquement de T, q, d, $||\varphi_{0:d}||_{2q+m+2,\infty}$, $||\varphi_f||_{m+2,\infty}$ et $||\phi||_{m+3,\infty}$, telle que pour tout $k = 0, \dots, N$:

$$\mathcal{E}_u^2(k) + \Delta_{T_k}^{1/2} \mathcal{E}_v^2(k) \leq C_2 \left(\frac{1}{N}\right)^2. \quad (28)$$

Le cœur de la démonstration de la convergence des algorithmes progressif et rétrograde est le contrôle des erreurs à chaque étape.

- Pour montrer le résultat de l'algorithme progressif, nous réécrivons l'erreur totale comme une somme télescopique des erreurs à chaque étape. Puis, l'erreur à chaque étape est exprimée en termes de :
  - Une erreur provenant du gel de la loi au démarrage de chaque pas de temps de l'algorithme. Cet erreur est majorée en utilisant un contrôle sur la croissance des coefficients de la variable progressive.

  - Une erreur provenant de l'utilisation de la cubature. L'analyse de cette erreur découle, comme nous l'avons montré dans la section 3.2, des résultats de régularité et majoration des dérivées de $(t,x) \in [0,T] \times \mathbb{R}^d \mapsto \mathbb{E}\phi(X_T^{t,x,\mu})$.
  - Une erreur de propagation provenant du fait que nous utilisons la loi de cubature plutôt que la vraie loi du processus.

  La proposition se déduit grâce à l'application du lemme de Gronwall discret sur les termes trouvés.

- La preuve de la convergence de l'algorithme rétrograde du premier ordre est semblable. Cette fois ci, l'erreur est exprimée en termes de
  - Une erreur de discrétisation résultant de l'utilisation du schéma (20) pour approcher les fonctions u et v aux temps de discrétisation.
  - Une erreur résultant du calcul des espérances conditionnelles. L'analyse de cette erreur demande l'étude de régularité de u et v.
  - Une erreur de propagation.

  Tout comme dans le cas de la composante progressive, nous contrôlons l'erreur totale à l'aide des termes précédents et du lemme de Gronwall.

- L'algorithme rétrograde du second ordre suit les mêmes idées, mais demande en plus la majoration du prédicteur à chaque étape de l'algorithme. Il est aussi important de se rappeler que cet algorithme utilise un schéma d'ordre supérieur et une formule de cubature d'ordre plus grand.

Vitesse de convergence sous le conditions (LB) .

Corollary 0.8. *Supposons que l'hypothèse* ***(LB)*** *soit satisfaite. Si nous appliquons l'algorithme 1 avec comme paramètres m et q en utilisant une grille à pas décroissant avec N points donnée par*

$$T_k = T\left[1 - \left(1 - \frac{k}{N}\right)^{\gamma}\right],$$

pour $\gamma > 0$, alors il existe une constante positive C, qui dépend uniquement de T, $||\varphi||_{2q+m+2,\infty}$ et $||\phi||_{1,\infty}$, telle que :

$$|\langle \mu_T - \hat{\mu}_T, \phi \rangle| \quad \leq \quad C\left(\left(\frac{1}{N}\right)^{(\gamma/2)\wedge q} \vee L(\gamma, m)\right). \tag{29}$$

où

$$L(\gamma,m) = \begin{cases} N^{-\gamma/2} & si \quad \gamma \in (0, m-1) \\ N^{-(m-1)/2}\ln(N) & si \quad \gamma = m-1 \\ N^{-(m-1)/2} & si \quad \gamma \in (m-1, +\infty) \end{cases} \tag{30}$$

De plus, si $\gamma > m-1$, les résultats de majoration de l'erreur de $(\hat{u}^1, \hat{v}^1)$ et $(\hat{u}^2, \hat{v}^2)$ respectivement donnés en (27) et (28) restent vrais.

Notons que la majoration (29) n'est valable qu'à l'instant $T_N = T$. Ceci est une conséquence du fait que le résultat utilise l'effet régularisant de la diffusion. La preuve du corollaire consiste d'abord à s'assurer des propriétés de régularité de $(t,x) \in [0,T] \times \mathbb{R}^d \mapsto \mathbb{E}\phi(X_T^{t,x,\mu})$. Ces propriétés permettent de majorer l'erreur à chaque temps de discrétisation comme dans le cas régulier. Ensuite, nous obtenons un contrôle sur l'erreur cumulée lorsque l'on applique le lemme de Gronwall aux termes majorés.

3.7 Extensions

Discutons maintenant de quelques généralisations du cadre d'application de l'algorithme présenté. D'abord, on peut assez facilement montrer que l'algorithme peut être appliqué lorsque la condition terminale $\phi : (y, w) \in \mathbb{R}^d \times \mathbb{R} \mapsto \phi(y, w)$ dépend aussi de la loi du processus $(X_t^x,\ 0 \leq t \leq T)$ à condition d'avoir des hypothèses de contrôle sur cette dépendance. Par exemple, l'on peut considérer le cas

$$\mathbb{E}\left[\phi(X_t^x, \mathbb{E}[\varphi_\phi(X_T^x)])\right]$$

pour une fonction $\varphi_\phi \in C_b^{m+2}$ et ϕ Lipschitz en w uniformément en y.

Ensuite, l'algorithme peut aussi résoudre des équations où la forme de la dépendance par rapport à la loi est différente. Par exemple, il peut être utilisé pour résoudre des équations EDSPR-MV ayant dépendances du type de celles explorées dans [15], i.e. :

$$V_i(t, y, \mu) = \langle \mu_t, V_i(t, y, \cdot)\rangle,\ i = 0, \cdots, d.$$

Les résultats du théorème 0.7 et le corollaire 0.8 restent valables. Bien évidemment, pour ce type de dépendance et le cas **(LB)** , l'hypothèse d'uniforme ellipticité doit être comprise sur la matrice $[\langle \eta_t, V(t, y, \cdot)\rangle]\,[\langle \eta_t, V(t, y, \cdot)\rangle]^*$ uniformément en y, t en $\mathbb{R}^d \times \mathbb{R}^+$ et η (famille de mesures de probabilité en $\mathbb{R}^d$).

Quelques généralisations additionnelles (plus techniques) sont aussi possibles et présentées dans la section 2 aux chapitre 2.

3.8 Remarques finales

Nous avons présenté un nouvel algorithme basé sur la cubature sur l'espace de Wiener pour résoudre les EDS de type McKean-Vlasov et deux algorithmes pour résoudre les équations EDSPRs de type McKean-Vlasov découplé. Nous avons étudié l'erreur d'approximation des algorithmes sous des hypothèses de régularité et majoration des fonctions V_i, φ_i et f de (10), et sous deux types d'hypothèses pour la condition terminale ϕ : infiniment régulière ou Lipschitzienne (plus uniforme ellipticité de l'équation progressive).

Naturellement, nous nous demandons s'il est possible d'affaiblir les hypothèses admises. La première condition que l'on pourrait affaiblir est la régularité des fonctions $\varphi_i, i = 0, \ldots, d$ dans la dépendance McKean-Vlasov. On peut imaginer qu'un effet régularisant rende les termes $\langle \mu_t^x, \varphi_i\rangle, i = 0, \ldots, d$ plus réguliers que les termes φ_i de façon analogue à ce qui se passe avec la condition terminale. Ainsi, on pourrait demander aux fonctions $\varphi_i, i = 0, \ldots, d$ de n'être que Lipschitz plutôt que lisses. Mais, cette fois-ci, la singularité se trouve au temps initial et la régularisation agirait en s'éloignant du départ. Ainsi, l'analyse n'est pas assurée car on a besoin d'étudier en temps petit la propagation de la singularité. En particulier, il faut trouver des majorations sur les dérivées de la fonction $(t, x) \to \mathbb{E}\left(\varphi_i(X_t^{0,x})\right)$. Si la régularisation reste vraie et si les majorations sont obtenues, il est possible d'utiliser l'algorithme à condition de modifier la grille de discrétisation pour faire face à la singularité de départ : nous utiliserions une discrétisation croissante près de zéro.

Une autre hypothèse pouvant être affaiblie est l'uniforme ellipticité pour le cas Lipschitz. Comme mentionné auparavant, lorsqu'on utilise la méthode de cubature pour approcher une EDSPR découplée sans le terme McKean-Vlasov et *homogène* en temps, on peut profiter de résultats de Kusuoka [58] et Crisan et Delarue [25] pour montrer la régularité demandée à u et v sous une condition plus faible que l'ellipticité (la condition UFG). Cependant, dans le cadre

que nous avons fixé, le terme McKean-Vlasov fait apparaître un terme inhomogène en temps dans les coefficients de la diffusion. Actuellement, nous n'avons pas des résultats semblables à la condition UFG pour le cadre inhomogène, à l'exception de demander que la variable temporelle « agisse comme les variables spatiales » car, dans ce cas, nous pouvons examiner une version homogène de l'équation. Un effort additionnel doit être dédié à l'interprétation de ces conditions dans le cadre McKean- Vlasov.

Considérons également les coefficients V_i et f. L'hypothèse qui demande aux coefficients $V_i; i = 0, \dots, d$ d'être bornés est technique et simplifie notre analyse, mais nous sommes convaincus que ce type de condition peut être affaiblie en une condition de majoration par des polynômes, sans modifier l'efficacité de l'algorithme. Le cas de f est plus compliqué, puisque tout changement sur la condition de croissance peut avoir un effet sur la stabilité de l'EDSR. Ainsi, l'algorithme devrait être modifié pour inclure des techniques de troncature. Enfin, nous remarquons que l'algorithme de cubature demande la régularité des fonctions V_i pour assurer les résultats d'approximation que nous avons présentés et donc, à notre connaissance, cette hypothèse ne peut pas être modifiée.

En ce qui concerne l'utilisation de l'algorithme dans des cadres plus généraux, au-delà des extensions déjà présentées, nous estimons que bien que l'algorithme ait été conçu pour résoudre des EDSPR-MVs découplées, il peut être utilisé au cœur d'un algorithme pour résoudre des EDSPR-MVs couplées. L'idée serait de combiner l'algorithme avec des procédures de point fixe. Cependant, ce programme exige des études additionnelles afin de comprendre les conditions garantissant l'existence d'une solution (qui n'est, en général, pas unique) du problème couplé et celles permettant à un algorithme de point fixe de converger vers une des solutions .

Nous pouvons aussi nous demander si l'algorithme pourrait être modifié pour calculer la solution du problème lorsque ϕ dépend de toute la trajectoire de la variable progressive et non de la condition terminale seulement. Bayer et Friz [5] ont démontré récemment que la méthode de cubature permet l'approche faible des fonctions des trajectoires des EDS. Néanmoins leur méthode ne permet pas l'obtention d'une vitesse de convergence explicite. Un résultat donnant des vitesses de convergence serait d'intérêt pour le cas sans ou avec la dépendance McKean-Vlasov.

Enfin, reste le problème de la complexité de l'algorithme utilisé, celle-ci étant exponentielle. Nous avons considéré ce problème avec plus de détails et nous le présentons dans la section suivante.

4 Annexe.- Implémentation d'une méthode de cubature pour résoudre des EDSR

Dans la section 3.3.b, nous avons présenté l'algorithme de Crisan et Manoralakis pour la solution des EDSR découplées. Malheureusement, l'algorithme hérite de la complexité exponentielle de la méthode de cubature. Ce défaut de l'algorithme été déjà identifié dans [28], où les auteurs proposent un algorithme TBBA[5], qui contrôle la complexité au prix d'introduire un bruit additionnel. L'algorithme TBBA préserve presque sûrement les vitesses de convergence de l'algorithme de base. Néanmoins, la vitesse précise de convergence presque sûre n'est pas disponible.

Dans [66], Litterer et Lyons ont proposé une autre solution au problème de complexité de la méthode de cubature. Il s'agit de la recombinaison d'ordre élevé que nous expliquons ci dessous.

[5] TBBA : Tree branched based algorithme

4.1 Recombinaison

Brièvement, la recombinaison d'ordre élevé est une procédure consistant à redistribuer la masse des particules dans l'arbre de cubature sans beaucoup modifier la loi marginale du processus. Son but est de préserver la vitesse de convergence de l'approximation faible d'une EDS mais avec une complexité polynomiale.

Plus précisément, la recombinaison est une procédure qui, à partir d'une mesure de probabilité discrète μ sur $\mathbb{R}^d$, produit une mesure de probabilité réduite $\tilde{\mu}$ dont le support est un sous-ensemble propre du support de μ et telle que pour tout élément h dans une famille de fonctions $\mathcal{H}$,

$$\langle \mu, h \rangle = \langle \tilde{\mu}, h \rangle.$$

Autrement dit, l'action de la mesure originale sur la famille de fonctions test $\mathcal{H}$ est préservé par la réduction.

L'obtention d'une mesure réduite est faite à l'aide d'un algorithme utilisant des techniques « diviser pour régner » et un schéma de réduction provenant de la preuve constructive du théorème de Carathéodory : supposons que $\{h^1, \ldots, h^l\}$ soit une base pour $\mathcal{H}$ et que μ soit une loi discrète sur $\mathbb{R}^d$ définie par

$$\mu = \sum_{i=1}^{N} \lambda_i \delta_{x_i}; \qquad \text{où } x_1, \ldots, x_N \in \mathbb{R}^d.$$

Soit $\hat{\mu}$ la mesure sur $\mathbb{R}^l$ définie par

$$\hat{\mu} = \sum_{i=1}^{N} \lambda_i \delta_{h_i}; \qquad \text{où } h_i = [h^1(x_i), \ldots, h^l(x_i)]^T.$$

Alors, l'algorithme se déroule de la façon suivante,

1. Trouver une solution non-triviale $c = [c_1, \ldots, c_N]$ du système

$$\sum_{i=1}^{N} c_i \left(h_i - \langle \hat{\mu}, h \rangle\right) = 0; \quad \sum_{i=1}^{N} c_i = 0$$

2. Soit $\theta = \min_{|c_i|>0} |\lambda_i c_i^{-1}|$. Soit $i^\star$ un point où le minimum est atteint. Soit $\tilde{\lambda} = \lambda - \text{sign}(c_{i^\star})\theta c$. On déduit que $\hat{\lambda}_{i^\star} = 0$, et que l'on peut donc définir la mesure réduite

$$\tilde{\mu} = \sum_{\{i=1,\ldots,N : \tilde{\lambda}_i > 0\}} \tilde{\lambda}_i \delta_{x_i},$$

dont le support est un sous-ensemble propre de $\{x_1, \ldots, x_N\}$ avec cardinal $\tilde{N} < N$. De plus, par construction, les moyennes de chaque fonction h^j par rapport à μ et par rapport à $\tilde{\mu}$ sont égales.

Notons que nous pouvons répéter la procédure tant que $\tilde{N} > l + 1$, car cette condition garantit l'existence d'une solution non-triviale c dans l'étape 1. Ainsi, nous assurons l'existence d'une mesure réduite dont le support a au plus $l + 1$ éléments.

Lorsque $N \gg l + 1$, l'algorithme précédent est couplé à une stratégie « diviser pour régner » :

- Diviser le support de la mesure μ en $2(l + 1)$ ensembles de tailles à peu près égales. On note ces ensembles $A_1, \ldots, A_{2(l+1)}$.

- Soient $\Lambda_k = \sum_{i=1}^{N} \lambda_i \mathbf{1}_{\{x_i \in A_k\}}$ et $g_k = (\Lambda_k)^{-1} \sum_{i=1}^{N} \lambda_i h_i \mathbf{1}_{\{x_i \in A_k\}}$.
- On applique l'algorithme de réduction à la mesure discrète sur $\mathbb{R}^l$ donnée par $\sum_{k=1}^{2l+1} \Lambda_k \delta_{g_k}$.

La procédure est répétée tant que $N \gg l+1$. Remarquons que cette procédure est plus efficace car, au lieu d'éliminer un point, nous éliminons un ensemble A_i à chaque itération de l'algorithme de réduction.

4.1.a *Utilisation de la méthode de cubature pour l'approximation faible des EDS*

Afin d'utiliser l'algorithme de recombinaison dans le cadre de la méthode de cubature, nous devons choisir un bon ensemble de fonctions tests et appliquer la recombinaison aux mesures marginales à chaque temps de discrétisation T_k. Par exemple, dans [66], la méthode est étudiée lorsque $\mathcal{H}$ est l'ensemble de polynômes locaux de dégré inférieur à m : plus précisément, pour tout temps de discrétisation T_k, définissons un recouvrement $\{O_j^k,\ j = 1, \dots, M^k\}$ du support de la mesure de cubature par des boules O_j^k de rayon r_k. Alors nous pouvons prendre

$$\mathcal{H}^k = \left\{ \sum_{j=1}^{l} a_i p_i^j \mathbb{1}_{O_j^k} : a_i \in \mathbb{R}; i = 1, \dots \binom{m+d}{d}; j = 1, \dots, M^k, \right\}.$$

L'avantage de choisir une telle famille de fonctions tests est la suivante : soit $f \in C_b^{m+1}$ i.e. une fonction bornée $m+1$ fois dérivable dont les dérivées sont bornées. Soit $\Pi_{\mathcal{H}^k}$ l'opérateur de projection sur $\mathcal{H}^k$. On déduit du théorème de Taylor et la définition de $\mathcal{H}^k$ que

$$|f(x) - \Pi_{\mathcal{H}^k} f(x)| \leq K(r_k)^{m+1},$$

ce qui implique

$$|\mathbb{E}_{\mathbb{Q}_{T_k}}[f(X_{T_k})] - \mathbb{E}_{\tilde{\mathbb{Q}}_{T_k}}[f(X_{T_k})]| \leq K(r_k)^{m+1},$$

où $\mathbb{Q}_{T_k}$ est la mesure de cubature et $\tilde{\mathbb{Q}}_{T_k}$ la mesure réduite.

Remarquons que, a priori, nous pouvons appliquer la procédure de réduction à chaque pas de temps k, mais cela ne peut être efficace que si le nombre de particules à réduire est grand par rapport au nombre minimal de particules après la réduction. De plus, puisque la recombinaison utilise la régularité de la fonction à moyenner, si l'on souhaite approcher l'espérance de fonctions seulement Lipschitziennes il faut éviter l'utilisation de la réduction au temps T_N.

La procédure de recombinaison est conçue pour préserver l'approximation faible des lois marginales d'un processus progressif, mais elle ne préserve pas la structure Markovienne dans la construction de la mesure de cubature. Nous illustrons cette situation dans la Figure 2 où nous montrons un arbre de cubature recombiné (le lecteur peut comparer cette image avec la Figure 1). En effet, la réduction crée des points dont la loi conditionnelle n'est pas approchée par le sous-arbre associé (points verts dans la Figure 2) et même des points sans sous-arbre associé (points rouges dans la Figure 2).

La conséquence de la perte de la structure Markovienne est l'impossibilité d'utiliser l'arbre réduit pour résoudre des EDSR en suivant le schéma de Crisan et Manoralakis.

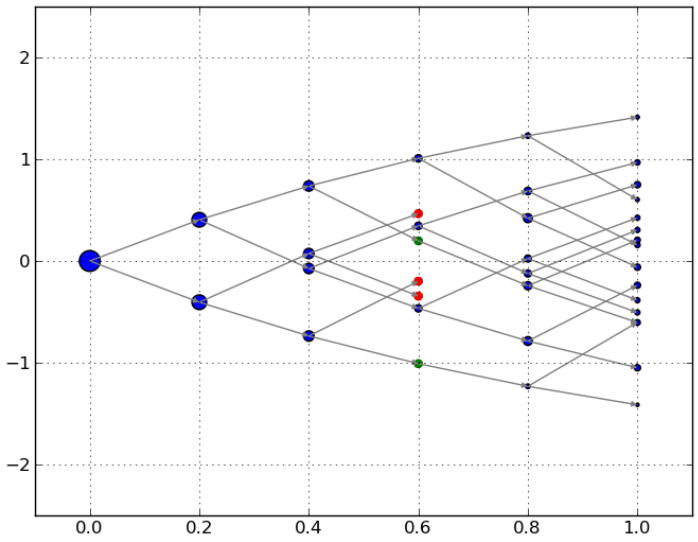


FIGURE 2 – Un exemple d'un arbre de cubature réduit. La cubature réduite perd la propriété de Markov car la structure partant des points verts est incomplète et celle partant des points rouge est inexistante.

4.2 Implémentation d'une méthode de cubature réduite pour la solution des EDSR

Notre objectif dans l'Annexe A est de présenter des résultats préliminaires sur un algorithme pour adapter la recombinaison afin de produire des arbres réduits pour résoudre des EDSR.

L'idée principale est de reconstruire, à une petite erreur près, l'arbre de cubature original à partir de l'arbre réduit. Nous proposons de passer d'une recombinaison produisant des mesures de probabilité à une réduction produisant des mesures signées préservant, comme avant, une famille de fonctions tests. Nous montrons que l'algorithme de résolution des EDSR utilisant la cubature réduite est stable et vérifie les mêmes vitesses de convergence que l'algorithme sans réduction sous certaines hypothèses contrôlant la norme l_1 des coefficients conditionnels de la réduction. Bien que nous n'ayons pas de résultats analytiques sur le ratio de réduction de la procédure que nous proposons, les tests numériques que nous avons effectué semblent montrer que l'algorithme recombiné présente une complexité polynomiale.

INTRODUCTION (ENGLISH)

1 General introduction

This Ph.D. thesis is inscribed in the numerical stochastic analysis domain and its main objective is to propose numerical algorithms to solve efficiently some stochastic approximation problems requiring an ad-hoc approach.

Stochastic models are widely used in almost any domain requiring quantitative results like physics, biology, engineering, finance, economics, etc., thanks to their capacity to represent complex terms or uncertainties. This widespread adoption started in with the first modeling of Brownian motion done by Bachelier and his proposed application in finance (1900) and of course with the work of Einstein (1905) on the motion of a pollen grain in water first observed by Brown; and was possible thanks to the development of stochastic analysis tools starting from the early 1930's when the theory was formalized.

But, without any doubt, the common use of stochastic models owes a lot to the development of numerical stochastic analysis and the arrival of the digital computer. Indeed, except for some notable and well known examples, most of stochastic related problems (from solving an SDE to optimizing a stochastic control problem) cannot be solved analytically. Practitioners must then choose between using simple well-known tools and using existing numerical methods to solve their interest problem subject to some pre-existing constraints (precision, time, memory,etc.).

As the complexity of the mathematical models increases, there is a rising need for coming up with adapted and efficient numerical algorithms. This frequently means designing algorithms profiting from the specific structure of the treated stochastic problem. In this work, we will consider two unrelated problems for which we propose a numerical solution scheme, and a third problem arising from the study of an existing numerical method of weak diffusion approximation.

The first problem we investigate is the numerical approximation of strongly oscillating stochastic models, i.e. a stochastic system composed by two interacting groups of variables with very different evolution time scales ('slow' variables and 'fast' variables). If the interest variables are the slow ones we can verify that the scale separation renders inefficient any straightforward use of classical discretization algorithms for solving the system (like the Euler scheme). We propose

an algorithm for solving such a problem when the fast variables are ergodic. In this case, we can profit from homogenization to propose the use of an Euler scheme for the slow scale variables coupled with a decreasing step estimator for the ergodic averages of the quick variables. We prove the strong convergence of the algorithm as well as a C.L.T. like limit result for the normalized error distribution. In addition, we propose an extrapolated version that has an asymptotically lower complexity and satisfies the same properties as the original version. This problem is the subject of a preprint submitted to Annals of Applied Probability (currently under minor revision) [42] and is presented in Chapter 1 .

Then, the second problem we consider is the numeric approximation of a decoupled McKean Vlasov forward-backward stochastic differential equation (MKVFBSDE). McKean Vlasov equations are stochastic differential equations with coefficients depending on the law of the diffusion itself, and appear as the asymptotic limit of a system of particles with mean field interactions when the number of particles tend to infinity. Some control problems defined on this kind of system are effectively approached by decoupled MKVFBSDE, i.e. decoupled forward-backward problems where the dynamics of the forward and backward parts are of the McKean Vlasov type. We propose a new deterministic algorithm to approximate weakly the solution of a McKean-Vlasov SDE. The algorithm is based on the cubature on Wiener spaces method of Lyons and Victoir [69] and has the notable property that it can be parametrized in order to obtain any given order of convergence. Then, using this new method, we construct implementable algorithms to solve the decoupled FBSDE of McKean-Vlasov type. We give two algorithms and show that they have convergence of orders one and two under appropriate regularity conditions. This problem is the subject of a submitted preprint [23] coauthored by P.E. Chaudru de Raynal and is presented in Chapter 2.

Although these two problems are of independent nature, we follow a similar reasoning to define the solution and analyze its convergence. The underlying idea is to combine existing limit result (for the fast scale in the first example or for the number of particles in the second) with an appropriate decoupling of the difficult terms in the approximation (the fast scale in the first problem and the law in the second one). The decoupled system being simpler, we might use known schemes to find an approximating solution to it. Hence, the main point when performing the error analysis will be to prove that we manage to keep errors controlled after the decoupling and approximation steps: this combines finding appropriate bounds on the small time errors and then controlling their cumulation, a step in which we will require regularity and estimation properties of the considered stochastic processes.

Finally, we consider a third problem emerging form our study of the cubature method when solving the McKean-Vlasov FBSDE. In fact, there is a possible drawback of using the cubature method to solve backward equations: if applied directly, the method shows an exponential complexity growth in terms of its defining parameter. Although the algorithm gives nice error results even with only a few discretization steps, during the extensive numeric tests we performed for solving the MKVFBSDE problem we became aware of the constraints imposed by the complexity growth. Hence, the third problem we study is how to modify the cubature algorithm in order to control the complexity growth of the cubature method when solving BSDEs without sacrificing the order of convergence of the algorithm. We present a procedure, based on the high order recombination method of Litterer and Lyons [66] to cope with this difficulty. This problem is presented in the Appendix A

In what follows, we will introduce properly each one of the enumerated problems and the algorithms we have proposed to solve them. To clearly explain the ideas behind our methods, we also introduce the main mathematical tools associated to the solution of the presented problems.

2 Strongly oscillating stochastic differential equations

Let us consider a SDE system given by

$$\begin{cases} X_t^\epsilon &= x_0 + \int_0^t f(X_s^\epsilon, Y_s^\epsilon)ds + \int_0^t g(X_s^\epsilon, Y_s^\epsilon)dB_s \\ Y_t^\epsilon &= y_0 + \epsilon^{-1}\int_0^t b(X_s^\epsilon, Y_s^\epsilon)ds + \epsilon^{-1/2}\int_0^t \sigma(X_s^\epsilon, Y_s^\epsilon)d\tilde{B}_s, \end{cases} \tag{31}$$

where X_t^ϵ is a d_x-dimensional process, Y_t^ϵ a d_y-dimensional process, B and $\tilde{B}$ are two independent Brownian motions of dimensions d_x and d_y. This type of system models the dynamics of two sets of interacting variables evolving in different time scales. The difference between time scales is controlled by the parameter ϵ. We are interested in considering the regime when $\epsilon \ll 1$, representing the case in which X^ϵ is evolving slowly with respect to Y^ϵ (for this reason we will denominate them as *slow scale* and *fast scale* variables respectively).

Such a stochastic system is useful to model several applications. Let us cite two examples of application of system (31)

- In [74] (see also references within), the authors consider the question of simplifying climate modeling. Their basic assumptions are that there are two main types of variables in climate systems, climate variables and unresolved variables, and that the climate variables evolve much slower than the unresolved ones. With this basic assumptions, they pose the general model

$$\begin{aligned} dX_t =& \epsilon^{-1}\left[f_1(\epsilon^{-1}t) + L_{11}X_t + L_{22}Y_t + Q_{12}^1(X_t, Y_t) + Q_{22}^1(Y_t, X_t)\right]dt \\ &+ \left[F_1(t) + bX_t + Q_{11}^1(X_t, X_t)\right]dt \\ dY_t =& \epsilon^{-1}\left[f_2(\epsilon^{-1}t) + L_{21}X_t + L_{22}Y_t + Q_{11}^2(X_t, X_t) + Q_{12}^1(X_t, Y_t)\right]dt \\ &+ \epsilon^{-2}\Gamma Y_t dt + \epsilon^{-1}\sigma dB_t \end{aligned}$$

  where $Q_{11}^1, Q_{12}^1, Q_{22}^1, Q_{12}^1, Q_{22}^1$ are quadratic operators. This system may appear, for example, when working in the Fourier space some partial differential equation relating climate and unresolved variables. Frequently, one is interested in the so called *advective* time scale, and the equation is reduced to

$$\begin{aligned} dX_t =& \left[F_1(t) + bX_t + Q_{11}^1(X_t, X_t)\right]dt \\ dY_t =& \epsilon^{-2}\Gamma Y_t dt + \epsilon^{-1}\sigma dB_t \end{aligned}$$

  which is a particular instance of equation (31) with no volatility in the slow scale.

- Based on numerical tests run on S&P 500 data, Fouque, Papanicolaou and Sircar argue in [36] on the presence of a fast scale volatility in real time series. Consequently, the same authors and Solna propose and study in [35] (in a framework extended and complemented

in [37]) a system similar to (31) for pricing derivatives in the context of stochastic volatility models. They assume a framework under which stock prices are conditionally lognormal with volatility given as a positive increasing function of an Ornstein Uhlenbeck process. More precisely, their model is written as

$$dX_t = rX_t dt + f(Y_t)X_t dB_t$$
$$dY_t^\epsilon = \left[\epsilon^{-1}(m - Y_t^\epsilon) - \epsilon^{-1/2}\nu\sqrt{2}\Lambda(Y_t^\epsilon)\right] dt + \epsilon^{-1/2}\nu\sqrt{2}dW_t$$

where f is a positive continuous function, the Brownian motions B and W have instantaneous correlation $\rho \in (-1, 1)$ and

$$\Lambda(y) = \frac{\rho(\mu - r)}{f(y)} + \gamma(y)\sqrt{1 - \rho^2}$$

is a combined market price of risk.

Note that equation (31) retains some of the main characteristics of the model of Fouque et al. (namely the fast scale mean reverting volatility variable) allowing in addition for explicit dependence of the fast scale variable on the slow scale one, for the case in which there is no noise correlation and where we neglect the contribution of Λ when ϵ is really small.

In the previous applications, we are concerned with solving the slow scale variables. If we need to find the solution by numerical means, we are tempted to use directly some classic approach like the Euler scheme. Letting h be the step size for the Euler scheme, and under the usual Lipschitz and growth assumptions, we get (see for example Theorem 10.2.2. in [53]) that the strong error is of order $O(\epsilon^{-1/2}h^{1/2})$, as a consequence of the presence of the ϵ^{-1} and $\epsilon^{-1/2}$ term in the drift and volatility coefficients of the fast variable. Assuming we would like to approximate strongly equation (31) with a tolerance Δ, this means we should take a step size proportional to $\epsilon\Delta^2$ or a total of $(d_x + d_y)\epsilon^{-1}\Delta^{-2}$ operations. Hence, the bigger the scale separation, the more inefficient the direct application of Euler scheme becomes.

Of course this question is not exclusive of the SDE case. We might consider the same question referring to a purely deterministic system. Several numerical methods to solve such a system in the deterministic case have been developed (see for example the review [31]). Most of them are based on choosing a macro-solver for the slow scale in which some information from the fast scale is added via parameters introduction to guarantee the correct approximation.

The literature with respect to numerical approximation of the general stochastic case we have presented here is, to our knowledge, much more restricted. In [32] the authors present an algorithm based on the use of an approximation scheme for the slow scale (for example Euler scheme) and at each step of the slow scale another scheme is used to solve for the fast scale contribution; it includes as well a weak and strong error analysis of the algorithm on the particular case of an ODE slow scale equation and a stochastic ergodic fast scale variable (i.e. when $g(x, y) = 0$ in (31)), and a weak error analysis for the full stochastic case. Recently, this method has been extended to solve SPDEs [13].

Our goal in Chapter 1 is to propose and study an alternative algorithm to approach efficiently the two scale problem in the fully coupled SDE case by taking advantage of the scale difference in the following way: we take benefit from some diffusion approximation results by Pardoux and Veretennikov in a series of papers [83, 84, 85] that show the strongly oscillating system may be approached by an effective simplified systems. This effective system has coefficients obtained as ergodic averages that we calculate by means of the invariant measure approximation algorithm

proposed by Lamberton and Pagès in [59].

2.1 Diffusion approximation

A way to approach the strongly oscillating system is to consider the limit situation when the scale separation tends to infinity, i.e. instead of looking at the system with a small ϵ, we study the limit of (31) as $\epsilon \to 0$ (if it exists) and we expect the error induced by this approximation to be small Suppose the fast scale is ergodic. Then, our intuition suggests that a very short time suffices to the fast variable to attain its stable limit while keeping the slow variable almost unchanged. Then, the slow variable should be approached by another stochastic process where the dependence with respect to the fast variable has been replaced by an ergodic average. This insight turns out to be accurate, provided that certain conditions are satisfied.

Mathematically, studying the limit when ϵ tends to zero requires some singular perturbation techniques as the ones presented for example by Bensoussan, Lions and Papanicolaou in [6] for deterministic models or in Papanicolaou, Stroock and Varadhan in [80], Friedlin and Wentzell [40], Khasminskii [52] or Pardoux and Veretennikov [84] (see also [83], [85]) for the stochastic case. We will focus on the approach of Pardoux and Veretennikov as the framework under which they study the limit approximation is adapted to our own strong oscillating problem.

The goal is to prove that the family of diffusions X^ϵ does have a weak limit as ϵ tends to zero and to identify that limit. For this, we look for an understanding of how the action of its infinitesimal generator over an appropriately chosen set of test functions evolves as ϵ decreases, and to compare it to the limit candidate suggested by the intuitive development we have given. Let us show some formal computations to illustrate how this works: Let $\mathcal{L}^\epsilon$ be the infinitesimal generator of the system (31). It can be written as

$$\mathcal{L}^\epsilon = \epsilon^{-1}\mathcal{L}_y^x + \mathcal{L}_x'^y$$

where $\mathcal{L}_x'^y$ is the generator of the slow diffusion with fixed parameter y

$$\mathcal{L}_x'^y := \sum_{i,j=1}^{d_y} h_{ij}(x,y)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d_y} f_i(x,y)\frac{\partial}{\partial x_i}.$$

with $h(x,y) = gg^*(x,y)$; and $\mathcal{L}_y^x$ is the generator of the time renormalized fast diffusion $\{Y_{\epsilon t}^\epsilon\}_{t\geq 0}$ with fixed x, i.e.

$$\mathcal{L}_y^x := \sum_{i,j=1}^{d_y} a_{ij}(x,y)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{d_y} b_i(x,y)\frac{\partial}{\partial y_i}. \tag{32}$$

Let μ^x be the invariant measure associated to $\{Y_{\epsilon t}^\epsilon\}_{t\geq 0}$ with fixed parameter x (uniqueness and existence follow as we have assumed the fast scale diffusion is ergodic).

Following the intuitive argument we presented, we want to compare $\mathcal{L}^\epsilon$ with an operator corresponding to $\mathcal{L}_x'^y$ when its dependence with respect to the variable y has attained its stable limit. Hence, we introduce the operator $\bar{\mathcal{L}}$ to be the averaged operator $\mathcal{L}_x'^y$ with respect to μ^x, i.e., let $\bar{\mathcal{L}}$ be given by

$$\bar{\mathcal{L}} := \sum_{i,j=1}^{d_x} H_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d_x} F_i(x)\frac{\partial}{\partial x_i},$$

where

$$F(x) = \int f(x,y)\mu^x(dy) \qquad H(x) = \int h(x,y)\mu^x(dy), \tag{33}$$

Now, let ψ be an arbitrary smooth application from $\mathbb{R}^{d_x}$ to $\mathbb{R}$. We can find an application ψ_ϵ from $\mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$ to $\mathbb{R}^m$ which is essentially ψ plus a small correction term, such that the action $\mathcal{L}^\epsilon$ over ψ_ϵ is close to the action of $\bar{\mathcal{L}}$ over ψ. Indeed, we can find a "corrector" function ϕ_ψ (that will be chosen appropriately in the following), so that fixing

$$\psi_\epsilon(x,y) = \psi(x) + \epsilon\phi_\psi(x,y),$$

gives,

$$\begin{aligned}\mathcal{L}^\epsilon\psi_\epsilon(x,y) &= (\epsilon^{-1}\mathcal{L}_y^x + \mathcal{L}_x'^y)(\psi(x) + \epsilon\phi_\psi(x,y)) \\ &= \mathcal{L}_x'^y\psi(x) + \mathcal{L}_y^x\phi_\psi(x,y) + \epsilon\mathcal{L}_x'^y\phi_\psi(x,y) \\ &= \bar{\mathcal{L}}\psi(x) + O(\epsilon)\end{aligned}$$

provided that

$$\mathcal{L}_y^x\phi_\psi(x,y) = (\bar{\mathcal{L}} - \mathcal{L}_x'^y)\psi(x), \tag{34}$$

which accounts for ϕ_ψ to be the solution of the Poisson equation associated to $\mathcal{L}_y^x$ and source term $(\mathcal{L}_x'^y - \bar{\mathcal{L}})\psi(x)$. Note that some regularity properties for ϕ_ψ are needed to give a sense to the $O(\epsilon)$ term.

Thus, formally, when ϵ tends to zero, both the corrector term and the error in the approximation tend to zero, and we would conclude that $\mathcal{L}^\epsilon\psi$ converges to $\bar{\mathcal{L}}\psi$.

Having presented this formal argument, let us state the rigorous diffusion approximation result of Pardoux and Veretennikov adapted to our context.

Note. We define $C_b^{j,r+\alpha}$ to be the collection of functions having j bounded derivatives in the x variable and r bounded derivatives in the y variable, where the derivatives of every order are $\alpha-$Hölder continuous with respect to y uniformly in x.

Theorem 0.9 (Theorem 4 in [84]). *Let b, σ, f, g be defined as in (31) and $a = \sigma\sigma^*$. Assume we have a recurrence condition of the type $\lim_{|y|\to\infty} b(x,y)\cdot y = -\infty$, and that the matrix 'a' is non-degenerate and uniformly elliptic. Assume that $a, b \in C_b^{2,1+\alpha}$, and that f, g are Lipschitz with respect to the x variable uniformly in y and have at most polynomial growth in y and linear growth in x.*

Then, for any $T > 0$, the family of processes $\{X_t^\epsilon, 0 \le t \le T\}_{0<\epsilon\le 1}$ is weakly relatively compact in $C([0,T];\mathbb{R}^l)$. Any accumulation point X is a solution of the martingale problem associated with the operator $\bar{\mathcal{L}}$.

If moreover, the martingale problem is well posed, then $X^\epsilon \Rightarrow X$, where X is the unique (in law) diffusion process with generator $\bar{\mathcal{L}}$.

It is worth mentioning that the actual framework of Pardoux and Veretennikov's statement includes the case in which there is an ϵ^{-1} order term in the slow variable, which complicates the proof. The main argument is divided in two parts: proving the weak compactness in $C([0,T],\mathbb{R}^d)$ and then identifying the result. The first part is achieved using classical tightness results as appearing in Billingsley [9]: namely to show that

- For all $\delta > 0$, there exists $M > 0$, such that

$$P\left(\sup_{0\le t\le T} |X_t^\epsilon| > M\right) \le \delta \quad 0 < \epsilon < 1;$$

- For all $\delta > 0, M > 0$, there exists ϵ_0 and γ, such that

$$P\left(\sup_{0\leq t_0\leq T}\sup_{t\in[t_0,t_0+\gamma]}|X_t^\epsilon - X_{t_0}^\epsilon| \geq \delta;\ \sup_{0\leq t\leq T}|X_t^\epsilon| \leq M\right) < \delta$$

The effect of the fast variable appears when studying the second criterion, which will require a well chosen change of time. To identify the limit, the basic idea is to choose a sequence of step functions sharing the same limit and profiting from the frozen slow scale structure to pass to the limit in the fast scale, using the ergodic theorem.

Hence, when the martingale problem is well posed, we restate the result as follows. Define the *effective equation*

$$X_t = x_0 + \int_0^t F(X_s)ds + \int_0^t G(X_s)dB_s, \tag{35}$$

where $G(x) = \sqrt{H(x)}$ and F, H are defined in (33). Note that $G(x)$ could be in principle any matrix with square given by H but we choose it to represent the Cholesky decomposition of the positive semi-definite matrix H. Then, under appropriate assumptions $X^\epsilon \xrightarrow{\mathcal{L}} X$ as $\epsilon \to 0$. Therefore, to follow our program, we could approach a solution to the original system (31) by approaching the limit system (35).

However, except for a few particular examples, it is not an easy task to find explicit expressions for the invariant measure or the averages in (33). Then, our numerical method must include a way to approach these invariant distribution averages.

2.2 Invariant distribution approximation

A first method presented by Talay in [92, 93], proposes to approach averages with respect to the invariant distribution of the ergodic SDE by means of the ergodic property. More precisely, if X_t is an ergodic stochastic process with a unique invariant law μ,

$$\int f(x)\mu(dx) = \lim_{t\to\infty}\frac{1}{t}\int_0^t f(X_s)ds \approx \frac{1}{N}\sum_{k=0}^{N\Delta} f(\bar{X}_{k\Delta})ds$$

for some small Δ and large N, where $\bar{X}_{k\Delta}$ is the outcome of a weak approximation scheme taken at time $k\Delta$. The scheme induces two types of approximation errors:

- A truncation error coming from the fact that we approach the limit to infinity by stopping the sum at some big N;

- A change in invariant measure error, which comes from the fact that the ergodic limit is taken on the approximating scheme and not on the original diffusion.

The truncation error might be controlled by using Markov chain ergodicity results, relying on the central limit theorem and some estimates on the convergence of the discrete scheme towards its limit, and give an error of order $N^{-1/2}$ plus some term depending on hN (typically exponentially decreasing). The discretization error might be studied using first error expansions issue of semi-group structure techniques, as explained by Talay and Tubaro for the Euler scheme in [94]; and then some results to find explicit rates of convergence towards the invariant limit (see for

example [79] for exponential rates or [97] for polynomial ones).

Note that we are assuming implicitly an important condition for the presented method to work: the weakly approximating sequence must also be ergodic. It might occur that a scheme is not ergodic even when the approached diffusion is ergodic. For example (see [75]), we can show that the diffusion

$$dX_t = -X_t^3 dt + dB_t$$

is ergodic with $\mathbb{E}\left(|X_t|^2\right) \leq 3/2$ for all $t \in \mathbb{R}^+$, while the corresponding Euler discretization

$$\bar{X}_{k+1} = \bar{X}_k - \bar{X}_k^3 \Delta_{T_{k+1}} + \Delta B_{k+1}$$

has diverging second moments for increasing k.

Hence, the ergodicity of the discrete scheme must be shown. Several approaches are available, the common goal is to show both the irreducibility and positive recurrence properties of the Markov chains associated to the scheme. For example, the scheme presented in [92] based on a Milstein scheme is ergodic provided that gaussian increments are used, and the irreducibility property is a consequence of a uniform ellipticity assumption while the positive recurrence comes from a strong mean reverting condition on the drift. Weaker conditions imposing the existence of a Lypunov function plus some extra conditions on the diffusion are used in [75] or [79].

2.2.a *The decreasing step method*

A similar but different approach was proposed by Lamberton and Pages in [59, 60], further studied in [65]. Let us present a simplified version of the method, adapted to the use we will make of it.

Let $\{\gamma_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of steps satisfying $\gamma_k > 0$ for all k, $\lim_{n\to\infty} \gamma_k = 0$ and $\lim_{k\to\infty} \Gamma_k = \infty$; where

$$\Gamma_k := \sum_{j=0}^{k} \gamma_j.$$

Let $\tilde{B}$ be a Brownian motion and define $\sqrt{\gamma_{k+1}} U_{k+1} := \tilde{B}_{\Gamma_{k+1}} - \tilde{B}_{\Gamma_k}$ implying that U_{k+1} is a standard Gaussian vector. Let $y_0 \in \mathbb{R}^{d_y}$. We define a decreasing step Euler approximation of the ergodic diffusion by

$$\begin{aligned} \tilde{Y}_0^x &= y_0 \\ \tilde{Y}_{k+1}^x &= \tilde{Y}_k^x + \gamma_{k+1} b(x, \tilde{Y}_k^x) + \sqrt{\gamma_{k+1}} \sigma(x, \tilde{Y}_k^x) U_{k+1}, \end{aligned} \tag{36}$$

and the *decreasing step average estimator* by

$$\tilde{F}^k(x) = \frac{1}{\Gamma_k} \sum_{j=1}^{k} \gamma_j f(x, \tilde{Y}_{j-1}^x) \tag{37}$$

We must remark that we have presented here a simplified version of the algorithm in [59] with minimal hypothesis: the most general version of the algorithm allows the use of different sequences for the Euler scheme step and for the weights in the average. We consider only the reduced version as it simplifies the presentation of our results without requiring some compromise in the asymptotic rate of convergence of the algorithm.

The main feature of the decreasing step method is that, provided the existence of Lyapunov function satisfying some control conditions, the scheme is not only ergodic, but also it eliminates the discretization error in the approximation, that is, the estimator converges when $k \to \infty$ almost surely to the ergodic average of the continuous diffusion (see [59]). Moreover, it has an easy recursive implementation as

$$\tilde{F}^0(x) = 0; \quad \tilde{F}^k(x) = \tilde{F}^{k-1}(x) + \frac{\gamma_k}{\Gamma_k}\left(f(x, \tilde{Y}^x_{k-1}) - \tilde{F}^{k-1}(x)\right)$$

The proof of the almost sure convergence towards the ergodic average uses the existence of a Lypaunov function and martingale techniques to prove the tightness of the sequence of laws as $k \to \infty$. A martingale problem then follows to prove that the limit law might actually be expressed in terms of a limit diffusion. This is done using the Echeverría-Weiss theorem. Finally, stability and martingale techniques are used to show that the limit result is indeed the invariant measure of the continuous diffusion.

An additional feature of the algorithm is that it allows for some kind of "error expansion" (as shown in [65]) when applied to a certain family of functions. This property is very useful when considering plugging this algorithm as part of a more general diffusion estimation.

2.3 Proposed algorithm

The previous discussion on diffusion approximation and invariant measure approximation should motivate our choice for proposing an approximation algorithm for the strongly oscillating system (31). We propose a *Multi-scale Decreasing Step (MsDS)* algorithm defined as a composition of an Euler scheme for the slow scale and independent realizations of the decreasing Euler step algorithm and estimator to calculate the coefficients depending on ergodic averages at each step. Since we are considering a Monte Carlo implementation of the algorithm we need to add a Cholesky decomposition at each step for finding the volatility coefficient.

2.3.a *Assumptions*

Our results follow under some sets of assumptions on the fast scale coefficients (hypothesis $(\mathcal{H}_{f.s.})$) (in essence, a non-degeneracy condition, a recurrence condition and bounds on the diffusion growth) that guarantee the existence of a unique invariant measure for the rescaled fast scale diffusion for any fixed x parameter. The non-degeneracy condition is stronger than needed by the decreasing Euler step method, but will allow us to use Pardoux and Veretennikov results to obtain an error development for the ergodic approximation.

We also impose some conditions on the slow scale coefficients (hypothesis $(\mathcal{H}_{s.s.})$) namely regularity condition (Lipschitz in the x variable), growth control (polynomial in y and linear in x) and for the matrix $h = gg^*$ either to be uniformly non-degenerate or equal to zero (a hypothesis mostly needed for stability purposes with respect to the Cholesky decomposition). The precise set of assumptions is given in Section 1 of Chapter 1.

2.3.b *Multi-scale decreasing step (MsDS)*

The MsDS algorithm is then defined as, assuming a time horizon T, for $n \in \mathbb{N}^*$ and putting $t_k = Tk/n$, as

$$\check{X}^n_{t_{k+1}} = \tilde{F}^{M(n)}(\check{X}^n_{t_k})\Delta t_{k+1} + \tilde{G}^{M(n)}(\check{X}^n_{t_k})\Delta B_{k+1},$$

where $\tilde{F}^M$ is defined in (37) and $\tilde{G}^M(x)$ is defined in two steps: First we find $\tilde{H}^M(x)$ using the decreasing step algorithm as in (37) (recall that $h(x,y) = g^* g(x,y)$) and then we perform a Cholesky decomposition on it to find $\tilde{G}^M(x) = \sqrt{\tilde{H}^M(x)}$. Finally, $M(n)$ is the number of steps in the decreasing Euler estimator expressed as a function of the number of steps in the Euler scheme for the slow scale. The form of $M(n)$ will be clear from the main theorems.

We remark that choosing as square root the result of a Cholesky decomposition is arbitrary, and aims simply to allow for Monte Carlo simulations hence giving a completely implementable algorithm.

We studied also an extrapolated version of the algorithm (EMsDS), that follows the same basic structure of the MsDS algorithm but in which we use as estimator a Richardson-Romberg extrapolation: a linear combination of the decreasing step average estimator with two different sets of parameters chosen to reduce the complexity of the algorithm (the details are presented in Chapter 1).

We denote the extrapolated approximation estimator of F and H by $\hat{F}^{(n);\lambda}(x)$, $\tilde{H}^{\lambda,M(n)}(x)$, and define as before $\tilde{G}^{\lambda,M(n)}(x) = \sqrt{\tilde{H}^{\lambda,M(n)}(x)}$. Then, we can apply the same idea as in the basic MsDS algorithm, but using the new estimator.

2.4 Main results

All of our results will be derived for a continuous version of the algorithm obtained by interpolation, i.e. if we denote $\underline{t} = \lfloor nt \rfloor / n$, we consider

$$\tilde{X}_t^n = x_0 + \int_0^t \tilde{F}^{M(n)}\left(\tilde{X}_{\underline{s}}^n\right) ds + \int_0^t \tilde{G}^{M(n)}\left(\tilde{X}_{\underline{s}}^n\right) dB_s. \tag{38}$$

The continuous MsDS algorithm strongly converges to the exact solution and it features a non-standard C.L.T. property in the sense that the normalized error distribution converges towards the solution of a SDE. These are the main results of Theorem 0.10.

Theorem 0.10. *Let $0 < \theta < 1$, $\gamma_0 \in \mathbb{R}^+$ and $\gamma_k = \gamma_0 k^{-\theta}$. Let M_1 be a positive constant. Assume $(\mathcal{H}_{f.s.})$ and $(\mathcal{H}_{s.s.})$ (roughly described before, but precisely defined in Section 1 of Chapter 1). Define $M(n)$ by*

$$M(n) = \left\lceil M_1 n^{\frac{1}{1-\theta}} \right\rceil,$$

then

i) (Strong convergence). There exists a constant K such that

$$\mathbb{E}\left(\sup_{0 \le t \le T} \left| X_t - \tilde{X}_t^n \right|^2 \right) \le K n^{-[(1-\theta)\wedge 2\theta]/(1-\theta)}$$

ii) (C.L.T like result for the ODE case: $g \equiv 0$). Assume in addition that f is at least 7 times differentiable in y with all derivatives being polynomially bounded. Assume as well $\theta \ge 1/2$. Then

$$n\left(X - \tilde{X}^n\right) =: \zeta^n \Rightarrow \zeta^\infty$$

(convergence in law of the sequence of processes) for ζ^∞ given as the solution of the system of SDEs with coefficients depending on the invariant law explicitly given in Proposition 1.23 in Chapter (1).

iii) (C.L.T like result for the SDE case $h = gg^*$ non degenerate). Assume in addition that f, h are at least 7 times differentiable in y with all derivatives being polynomially bounded. Assume as well $\theta \geq 1/3$. Then

$$n^{1/2}\left(X - \tilde{X}^n\right) =: \zeta^n \Rightarrow \zeta^\infty$$

(convergence in law of the sequence of processes) for ζ^∞ given as the solution of the system of SDEs with coefficients depending on the invariant law explicitly given in Proposition 1.23.

The proof is done in several steps: first, we find an estimate on the ergodic average approximation at each step. We show that this control is based on the existence, regularity and control of the solution of the Poisson equation associated to the fast scale diffusion, for which we use the results of Pardoux and Veretennikov ([84]). Then, we control the error obtained after performing a Cholesky decomposition. The previous results allow to control, in a standard way, the L_2 error, giving us the strong convergence of our algorithm and a candidate for the rate of convergence. Moreover, by defining an appropriate structure and using the mentioned estimates, we prove, using limit distribution results as the ones in Jakubowski and Pagès [50], Jacod and Protter [49] and Kurtz and Protter [56], the C.L.T.-like result.

In addition to the convergence results, we show in section 2.4.a that the MsDS algorithm is more efficient than a simple Euler scheme when the scale separation parameter ϵ is really small .

For the extrapolated version of the algorithm, defined in section 2.1 in Chapter 1, we have the same properties.

Corollary 0.11. *Let $0 < \theta < 1$, $\gamma_0 \in \mathbb{R}^+$ and $\gamma_k = \gamma_0 k^{-\theta}$. Assume $(\mathcal{H}_{f.s.})$ and $(\mathcal{H}_{s.s.})$. With $M(n)$ defined as in Theorem 0.10.*

Let $\hat{X}^n$ be the result of applying the extrapolated version of the algorithm defined in section 2.1 in Chapter 1. Then

i) (Strong convergence). There exists a constant K such that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|X_t - \tilde{X}_t^n\right|^2\right) \leq K n^{-[(1-\theta)\wedge 4\theta]/(1-\theta)}$$

ii) (ODE case: $g \equiv 0$). Assume in addition that f is at least 8 times differentiable in y with all derivatives being polynomially bounded. Assume as well $\theta \geq 1/3$, then $\hat{\zeta}^n := n\left(X - \hat{X}^n\right)$ satisfies the limit distribution result given in Theorem 1.1 ii) with new coefficients given in Corollary 1.25.

iii) (SDE case) Assume in addition that f, h are at least 8 times derivable in y with all differentiable being polynomially bounded. Assume as well and $\theta \geq 1/5$,. Then $\hat{\zeta}^n := n^{1/2}\left(X - \hat{X}^n\right)$ satisfies the limit distribution result given in Theorem 1.1 iii) with new coefficients given in Corollary 1.25.

The advantage of using the extrapolated estimator might not be apparent, as the convergence rate is the same as in the original algorithm. It is the following one : *it allows a lower value for θ in the definition of the sequence $\gamma_k = \gamma_0 k^{-\theta}$*. This change implies an asymptotic reduction in the complexity of the algorithm, without affecting its other features, at the price of demanding higher coefficient regularity. The next section shows clearly this situation.

2.4.a *Efficiency analysis*

We can approximate the execution time of both algorithms by estimating the total number of operations needed to perform one path approximation of the effective equation (35). Note that since both algorithms share the same structure, a similar analysis is valid for both of them: the total cost $\kappa(n)$ of the algorithm with n steps may be written as

$$\kappa(n) = [\kappa_1(n, d_x, d_y) + \kappa_2(d_x)]\, n,$$

where κ_1 stands for the cost coefficient estimation at each step of the decreasing Euler, and κ_2 for the cost of calculating the Euler iteration. The latter will be of order $O(d_x)$ in the ODE case and $O(d_x^2)$ for the SDE case.

Let us focus now on κ_1. Both algorithms perform $M_1 n^{1/(1-\theta)}$ iterations for approximating the diffusion $\tilde{Y}$ and the calculation of estimators $\tilde{F}, \tilde{G}$. For the MsDS algorithm, each one of this iterations has a cost of $O(d_y d_x)$ in the *ODE* case, or $O(d_y d_x^2)$ in the SDE case. In the latter, we need also to perform a Cholesky decomposition with a cost of $O(d_x^3)$ operations. Hence,

$$\kappa_1^{MsDS}(n, d_x, d_y) = \begin{cases} O\left(d_y d_x n^{1/(1-\theta)}\right) & \text{in ODE case} \\ O\left([d_y d_x^2 + d_x^3] n^{1/(1-\theta)}\right) & \text{in SDE case} \end{cases}.$$

On the other hand, from the definition of the EMsDs algorithm, we get $\kappa_1^{EMsDS} \leq \lambda \kappa_1^{MsDS}$, and thus both share the same order of complexity, with the only difference that θ *is allowed to be smaller in the extrapolated algorithm.*

It may be more interesting to compare the *efficiency* of both algorithms, that is the time spent to obtain a given error tolerance Δ, that we will denote by $\tau(\Delta)$. We have from Theorem 0.10 and Corollary 0.11 that $\Delta(n) := O(n^{-1})$ for the ODE, and $\Delta(n) := O(n^{-1/2})$ for the SDE case. Replacing the minimum possible θ values we obtain the complexity figures given in Table 2.

	ODE	ODE (extrapol.)	SDE	SDE (extrapol.)
$\theta_{\min}$	$1/2$	$1/3$	$1/3$	$1/5$
$\tau_{\min}(\Delta)$	$O(d_y d_x \Delta^{-3})$	$O(d_x d_y \Delta^{-2.5})$	$O([d_x^2 d_y + d_x^3]\Delta^{-5})$	$O([d_x^2 d_y + d_x^3]\Delta^{-4.5})$

Table 2: Minimal efficiency (operations for fixed error) of the basic and extrapolated algorithm for ODE and full SDE cases

How do these figures compare with a straightforward Euler scheme applied to the original system? For the ODE case, an Euler scheme implemented for the original system (31) would require a total of $(dx + dy)\epsilon^{-1}\Delta^{-2}$ operations. Then the MsDS algorithm is more efficient if $\epsilon < \Delta(d_x \vee d_y)^{-1}$, and the EMsDS if $\epsilon < \Delta^{1/2}(d_x \vee d_y)^{-1}$. With respect to the algorithm presented in [32], the efficiency is equivalent to the one obtained when using a weak scheme of order one for approximating the ergodic averages. The advantage of our method is that we have in addition to the rate of convergence an expression for a C.L.T. type result.

In the SDE case, on the other hand, the proposed algorithm will be advantageous in the case in which $\epsilon < \Delta^3(d_x \vee d_y)^{-1}$ for the MsDS version, and $\epsilon < \Delta^{2.5}(d_x \vee d_y)^{-1}$ for the EMsDS. In other words, our proposed algorithms will be more efficient in our regime of interest of a strong scale scale separation (i.e. when $\epsilon \to 0$). It should be remarked that the SDE case is not explicitly studied for the algorithm in [32].

2.5 Final remarks

We presented an algorithm to approach strongly the effective equation, approaching in turn the strongly oscillating system. We studied its convergence and gave a sharp order of convergence and a C.L.T. like result for it.

We believe there is still some room to improve our results. For example with respect to the assumed hypothesis, we believe that the non-degeneracy assumptions on both the fast and slow scale may be relaxed, particularly since the decreasing step method does not really require this kind of assumption. With respect to the slow variable, getting rid of the non-degeneracy assumptions implies taking care of the Cholesky algorithm, and modifying our study for the limit results, adding some additional technical difficulties. However, the bigger changes appear when considering the fast scale, as we would need to modify the analysis of Pardoux and Veretennikov, which is based on uniformly elliptic PDE results. We believe that similar results might be recovered under some weaker hypothesis on the structure of the diffusion like Hörmander condition.

In our analysis, we have focused on the strong convergence of the algorithm. The analysis of the weak convergence may be performed following the work of Talay and Tubaro [94], adding the presence of the additional decreasing step estimator. We will not give the details, but let us underline that the key point for the analysis is that we have an explicit expansion on the error done using the decreasing step estimator (see Proposition 1.9), whose expectation does not depend on the taken path. The bottom line is that under the same hypotheses of Theorem 0.10, the weak rate of convergence is n^{-1}, in line with the similar result for the Euler scheme. Moreover, just as for the Euler scheme, we can perform a Romberg extrapolation to get rid of the first order error, but this time *on the whole effective diffusion approximation* and not just on the estimator as we have done for the strong case. Up to choosing the right sequence $\{\gamma_k\}_{k\in\mathbb{N}}$, this will lead to a higher order scheme (but with convergence slower than n^{-2}).

Another aspect that might be considered is to find ways to further accelerate convergence (both for the weak and strong convergence). For instance, if there is a need to solve several problems associated to the same system (a possible case being contingent option pricing), one may try some kind of grid technique on the slow scale, with the constraint of optimizing the number of points for which the ergodic average is calculated (given that this is the most expensive part of the algorithm). We performed some numerical studies using sparse grids, but the results were not conclusive.

3 Numeric approximation of decoupled FBSDEs of the McKean Vlasov type

Let us consider the system

$$\begin{cases} dX_t^x = \sum_{i=0}^{d} V_i(t, X_t^x, \mathbb{E}\varphi_i(X_t^x))dB_t^i \\ dY_t^x = -f(t, X_t^x, Y_t^x, Z_t^x, \mathbb{E}\varphi_f(X_t^x, Y_t^x))dt + Z_t^x dB_t^{1:d} \\ X_0^x = x, \quad Y_T^x = \phi(X_T^x) \end{cases} \tag{39}$$

where $B^{1:d}$ denotes a d dimensional Brownian motion and with the convention that $B_t^0 = t$, for any t in $[0,T]$, $T>0$ be given. Equation (39) is composed by a forward variable X with initial condition x_0 and a backward variable with final condition depending on X. Moreover, since the

coefficients of the forward and backward parts depend on the law of the variables themselves and the backward variable does not appear in the forward formulation, we say that equation 39 is a decoupled McKean-Vlasov forward-backward equation (MKVFBSDE).

Our objective in chapter 2 is to present a numerical method to solve equation (39). This is not only interesting in itself, but is also useful to solve a certain class of control problems on a system of a large number of particles with mean field interaction, inspired from the theory of mean field games but designed in such a way that the dynamics of the controlled process have no influence on the mean field environment. For a nice discussion on the the asymptotic regime of stochastic differential games with a finite number of players as the number of players tends to the infinity see [20].

A rather concrete example of this kind of control problem appears when considering the question of optimizing the wealth of an issuer having a large portfolio of credit assets inspired in the framework presented in [16]. One of the methods used to model credit asset dynamics is the so called structural model (see [8] for a review on credit risk models). Under this model, we assume that a credit default is triggered when the value of the corresponding credit asset is below a certain threshold. In the original Merton setup, the default may only be triggered at a certain fixed maturity time T. In a rather more realistic view, the default is triggered the first time the credit asset is below the threshold.

Assuming all the credit assets in the basket are small and homogeneous (which may be justified if, for example, we suppose they belong to the same economic sector) so that their value is modeled by SDEs with the same volatility and drift function terms. To simplify, we will consider the simpler Merton model. Moreover, in order to account for sector-wise contagion effects, we suppose there is a mean field type dependence in the dynamics. In addition to the credit assets, we suppose the issuer has a market portfolio used by the issuer to backup the credit risk, for example to comply with credit risk regulations, or to provide liquidity to its credit branch. Then, the value of the position of the issuer is modeled by an SDE with coefficients depending on the contribution of all credit assets. The objective of the control problem is to maximize the value of the issuer position.

From a mathematical point of view, the described model belongs to a class control problems in which one is interested in controlling a marked particle $\hat{X}$ evolving in an environment of a large number of independent and identical particles $X^1, \dots, X^M$ with mean field interactions, and where we assume the marked particle "feels" the environment but does not act on it. The dynamics of such a system evolve as

$$
\begin{aligned}
dX_t^j &= b\left(t, X_t^j, \frac{1}{M}\sum_{i=1}^{M}\delta_{X_t^i}\right)dt + \sigma\left(t, X_t^j, \frac{1}{M}\sum_{i=1}^{M}\delta_{X_t^i}\right)dB_j(t) \quad \text{for } j = 1, \dots M;\\
d\hat{X}_t^{1:M;\alpha} &= b^0\left(t, \hat{X}_t^{1:M;\alpha}, \frac{1}{M}\sum_{i=1}^{M}\delta_{X_t^i}, \alpha_t\right)dt + \sigma^0\left(t, \hat{X}_t^{1:M;\alpha}, \frac{1}{M}\sum_{i=1}^{M}\delta_{X_t^i}, \alpha_t\right)dW_t\\
X_0^1 &= \dots X_0^M = x, \quad \hat{X}_0^{1:M;\alpha} = \tilde{x}
\end{aligned}
$$

where the B_j are independent $d-$dimensional Brownian motions[6], W is a d'- dimensional Brownian motion, and α denotes the control that we suppose belongs to a set of admissible controls

[6] and not the entries of a multi-dimensional Brownian motion that we denote with a super-script!

$\mathcal{A}$. Similarly, the target functional would be given by

$$J(t,x,\tilde{x},\alpha) = \mathbb{E}\left[g\left(X_T^{1:M;\alpha;t,\tilde{x}}, \frac{1}{M}\sum_{i=1}^{M}\delta_{X_T^i}\right) + \int_0^T f\left(s, X_s^{1:M;\alpha;t,\tilde{x}}, \frac{1}{M}\sum_{i=1}^{M}\delta_{X_s^i}\right) ds\right]$$

We approach the system by passing to the limit in the number of particles as done for example in [20]. As we will discuss in the following, the asymptotic dynamics will be given by the McKean Vlasov system

$$\begin{cases} dX_t = b(t,X_t,\mu_t)dt + \sigma(t,X_t,\mu_t)dB_t \\ d\tilde{X}_t^\alpha = b^0(t,X_t,\tilde{X}_t^\alpha,\mu_t,\alpha_t)dt + \sigma^0(t,X_t,\tilde{X}_t^\alpha,\mu_t)dW_t \\ X_0 = x, \quad \tilde{X}_0^\alpha = \tilde{x} \end{cases}$$

where μ_t is the law of X_t that we will assume in the following to be fixed. Note we have allowed for a further dependence of the marked variable on the state of the environmental variable. Conditions that guarantee a unique solution of such a system are given in Section 3.1.

Now, for the asymptotic approximation, the target function now reads

$$J(t,x,\tilde{x},\alpha) = \mathbb{E}\left[g(X_T^{t,x}, \tilde{X}_T^{\alpha;t,\tilde{x}}, \mu_T) + \int_0^T f(s, X_s^{t,x}, \tilde{X}_s^{\alpha;t,\tilde{x}}, \mu_s)ds\right].$$

Assume we want to solve for the optimal value function

$$u(t,x,\tilde{x}) = \sup\{J(t,x,\tilde{x},\alpha),\ \alpha\in\mathcal{A}\}.$$

Provided all our equations are well defined, we can the solve the associated Hamilton Jacobi Bellman (HJB) equation for u,

$$D_t u(t,x,\tilde{x}) + \frac{1}{2}\mathrm{Tr}(\bar{a}D^2_{x,\tilde{x}}u(t,x,\tilde{x})) + b(t,x,\mu_t)D_x u + H(t,x,\tilde{x},D_x u,\mu_t) = 0,$$

with

$$\bar{a} = \begin{bmatrix} \sigma\sigma^T & \sigma\rho(\sigma^0)^T \\ \sigma^0\rho^T\sigma^T & \sigma^0(\sigma^0)^T \end{bmatrix}, \qquad \rho = [B,W]$$

and where $[\cdot,\cdot]$ stands for the covariation and H is the Hamiltonian

$$H(t,x,\tilde{x},z,\mu_t) = \sup_{\alpha\in\mathcal{A}}\left[b^0(t,x,\tilde{x},\mu_t,\alpha)z + f(t,x,\tilde{x},\mu_t)\right].$$

We will not discuss here the resolvability of the HJB equation (see e.g. [34] or [87] for a partial revue). We can interpret (2.21) from a probabilistic point of view: we have that $u(t,x,\tilde{x}) = Y_t^{t,x,\tilde{x}}$ where $Y^{t,x,\tilde{x}}$ is given by the MKV-FBSDE

$$\begin{cases} dX_s^{t,x,\tilde{x}} = b(s,X_s^{t,x,\tilde{x}},\mu_s)ds + \sigma(s,X_s^{t,x,\tilde{x}},\mu_s)dB_s \\ d\tilde{X}_s^{t,x,\tilde{x}} = \sigma^0(s,X_s^{t,x,\tilde{x}},\tilde{X}_s^{t,x,\tilde{x}},\mu_s)dW_s \\ -dY_s^{t,x,\tilde{x}} = H(s,X_s^{t,x,\tilde{x}},\tilde{X}_s^{t,x,\tilde{x}},\tilde{Z}_s^{t,x,\tilde{x}},\mu_s) - \tilde{Z}_s^{t,x,\tilde{x}}dW_s + Z^{t,x,\tilde{x}}dB_s \\ X_t^{t,x,\tilde{x}} = x, \quad \tilde{X}_t^{t,x,\tilde{x}} = \tilde{x}, \quad Y_0^{t,x,\tilde{x}} = g(X_T^{t,x,\tilde{x}},\tilde{X}_T^{t,x,\tilde{x}},\mu_T). \end{cases}$$

cases, the truncated problem will then satisfy the needed assumptions and may be solved with the presented Algorithm 1, 2.

Which shows that solving MKV-FBSDEs may be instrumental to find optimal target problem values for a class of control problems, via the use of the HJB equation.

In order to give an appropriate context to the definition and properties of MKVFBSDE and to highlight the possible difficulties in solving them, we will briefly review some well known properties of each one of its composing concepts, that is McKean Vlasov processes and FBSDEs, and we will also present the cubature method. Then, we will explain the proposed algorithm and the associated main results.

3.1 McKean Vlasov processes

McKean Vlasov equations are stochastic differential equations where the coefficients in the equation depend on the law of the equation itself,

$$dX_t = b(t, X_t, \mu_t)dt + \sigma(t, X_t, \mu_t)dB_t,\ 0 \leq t \leq T; \mu_0 = \bar{\mu}_0 \tag{40}$$

where μ_t denotes the law of X at time t. They were introduced by McKean in [76], as a probabilistic counterpart of a family of PDEs originally appearing in mechanical transport problems, written in weak form as

$$\partial_t \langle \mu_t, \phi \rangle = \langle \mu_t, \mathcal{L}(\mu_t)\phi \rangle \tag{41}$$

where $\mathcal{L}$ is the associated infinitesimal generator of (40).

The McKean Vlasov equations appear naturally as the asymptotic limit in particle systems with mean field interactions and independent noise when the number of particles becomes very large. The study of the limit behavior of such system of particles uses the concept of "propagation of chaos", first introduced by Kac in [51] and further studied among others by McKean [76, 77], Tanaka [96], Sznitman [90, 91] and Méléard [78]. We will say that there is propagation of chaos for a system of N interacting particles with empirical mean interaction if, given that the system is initialized with i.i.d. particles, the law of $k < N$ particles tends to the law of k i.i.d. particles as N tends to infinity. Moreover, when the system is exchangeable, propagation of chaos is equivalent to the weak convergence of the empirical measure of the system to a unique common law when the size of the system grows to infinity. This latter property is behind the characterization of the McKean Vlasov equation as model for the asymptotic behavior in number of particles of mean field systems. For an excellent review see Sznitman [91]. Results on the rate of convergence of such an approximation may be found, for example, in Méléard [78].

Although McKean Vlasov processes appeared initially in statistical mechanics, they are now used in many fields because of the wide range of applications requiring large populations interactions. For example, they are used in finance, as factor stochastic volatility models [7] or uncertain volatility models [45]; in economics, in the theory of "mean field games" recently developed by J.M. Lasry and P.L. Lions in a series of papers [61, 62, 64] (see also [20, 19, 18] for the probabilistic counterpart) and also in physics, neuroscience, biology, etc.

The question of existence and uniqueness of the solution to a given McKean Vlasov equation can be addressed under appropriate regularity assumptions of the coefficients, including regularity with respect to the law dependence term. Let us consider the space $\mathcal{P}_2(C([0,T], \mathbb{R}^d))$ of distributions on $C([0,T], \mathbb{R}^d)$ with second order moments. In this space, we define the Wasserstein-2 metric as

$$D_T(m_1, m_2) = \left[\inf_{m \in \Gamma(m_1, m_2)} \int \left(\sup_{s \in [0,T]} \|x_t - y_t\| \right)^2 m(dx, dy) \right]^{1/2}$$

where $\Gamma(m_1, m_2)$ denotes the collection of all measures on $\mathcal{P}_2(C([0,T], \mathbb{R}^d)) \times \mathcal{P}_2(C([0,T], \mathbb{R}^d))$ with marginals m_1 and m_2 respectively.

Proposition 0.12. *Assume that the coefficients b and σ in (40) are uniformly Lipschitz continuous in $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(C([0,T] \times \mathbb{R}^d))$ and that μ_0 has finite second moment. Then, there is existence and uniqueness (pathwise and in law) of (40).*

The proof may be found in [78]. The idea is to proceed by a fixed point argument: one defines an application from the space of functions $\mathcal{P}_2([0,T] \times \mathbb{R}^d)$ into itself, and show that it

is in fact a contraction. Then, pathwise uniqueness follows from the Lipschitz continuity by a classic argument.

Having the existence and uniqueness results on the solution of the McKean Vlasov equation, the question of numerical methods to solve it appears naturally. Solving PDE (41) is, in general, quite difficult. The usual technique to do so uses the Fokker Plank equation. A probabilistic approach is also available, based on the representation as limit of an interacting particle system (see for example [2], [95] or [11] and references therein). The idea in this kind of algorithm is to simulate a large number of interacting particles with independent noises, where the trajectory of each particle is approached using traditional discretization techniques and the law dependence in the coefficients is replaced with the empirical measure of the whole set of particles.

As part if this Ph.D. thesis, we explore a different approach based on the cubature on Wiener spaces method.

3.2 Cubature on Wiener spaces

Inspired from multi-dimensional integration algorithms, the cubature on Wiener space method (also known as KLV for Kusuoka-Lyons-Victoir) is a deterministic algorithm to weakly approach a given stochastic differential equation.

Let us explain how the method works. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a probability space with $\Omega' = C^0([0,1], \mathbb{R}^d)$ the space of continuous functions from $[0,1]$ into $\mathbb{R}^d$ with initial value 0, $\mathcal{F}'$ its associated Borel sigma algebra and $\mathbb{P}_x$ the Wiener measure, i.e. the probability measure of the Brownian motion in this probability space.

Hence, if X is a d-dimensional stochastic process defined in $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s,$$

we have for every measurable f

$$\mathbb{E}[f(X_T)] = \int_{C^0([0,T],\mathbb{R}^d)} f(X'_T(\omega))\mathbb{P}(\omega). \tag{42}$$

where

$$X'_t(\omega') = x_0 + \int_0^t b(X'_s(\omega'))ds + \int_0^t \sigma(X'_s(\omega'))d\omega'_s.$$

Then, we can approach weakly the marginals of X by approaching the infinite dimensional integral at the right hand side of (42). The main idea of the cubature method is to find a finite measure on Ω' "approaching" $\mathbb{P}'$ making it easy to evaluate the integral. Here, we say that the the discrete measure approaches the infinite dimension if the expectation of iterated integrals of the Brownian motion are conserved up to a certain order. Let us precise this concept by defining an m-cubature formula.

Definition 0.13. Let m be a natural number and $t \in \mathbb{R}^+$. An m-cubature formula on the Wiener space $C^0([0,t], \mathbb{R}^d)$ is a discrete probability measure $\mathbb{Q}_t$ with finite support on $C^0_{\text{bv}}([0,t], \mathbb{R}^d)$ (continuous functions and bounded variation starting in 0) such that the expectation of the iterated Stratonovitch integrals of degree m under the Wiener measure and under the cubature

measure $\mathbb{Q}_t$ are the same, i.e., for all multi-index $(i_1, \cdots, i_l) \in \{1, \cdots, d\}^l$, $l \leq m$

$$\mathbb{E} \int_{0<t_1<\cdots<t_l<t} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_l}^{i_l} = \mathbb{E}_{\mathbb{Q}_t} \int_{0<t_1<\cdots<t_l<t} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_l}^{i_l}$$
$$= \sum_{j=1}^{l} \lambda_j \int_{0<t_1<\cdots<t_l<t} d\omega_j^{i_1}(t_1) \cdots d\omega_j^{i_l}(t_l),$$

where "$\circ$" stands for the Stratonovich operator and ω_j^i for the i^{th} coordinate of the j^{th} path.

Then, as a direct consequence of the Taylor-Stratonovitch expansion applied to $f(X_t)$, a cubature formula of degree m is such that:

$$|(\mathbb{E} - \mathbb{E}_{\mathbb{Q}_t}) f(X_t)| \leq C t^{(m+1)/2} ||f||_{m+2,\infty}, \tag{43}$$

for all bounded and $m+2$ times continuously differentiable function f with bounded derivatives, provided that b, σ are smooth and bounded. Of course this error control is not in general small, but the Markov and scaling properties of the Brownian motion can be used to apply the cubature method iteratively in small subdivisions of the interval $[0, t]$ for which we have a good error control.

Indeed suppose we want to approach an SDE on $[0, T]$. Consider a cubature formula $\mathbb{Q}_1$ of order $m \in \mathbb{N}^*$ with support $\{\omega_1, \ldots, \omega_\kappa\}$ and corresponding weights $\{\lambda_1, \ldots, \lambda_\kappa\}$. For all $h > 0$ and any $t \in [0, T-h]$, one can deduce a cubature measure $\mathbb{Q}_{t,t+h}$ of order m with finite support on $C_{\text{bv}}^0([t, t+h], \mathbb{R}^d)$ equal to $\{\tilde{\omega}_1, \cdots, \tilde{\omega}_\kappa\}$ with the same weights $\{\lambda_1, \ldots, \lambda_\kappa\}$ and where the paths are defined as $\tilde{\omega}_j : s \in [t, t+h] \mapsto \tilde{\omega}(s) = \sqrt{h}\omega_j((s-t)/h)$ for all $1 \leq j \leq \kappa$.

Then, by virtue of the Markov property, a subdivision of $[0, T]$ leads to the construction of a tree which has κ^k nodes (corresponding to the number of paths) at the k^{th} subdivision. Each path $\tilde{\omega}_{(i_1,\ldots,i_k)}$, where $(i_1, \ldots, i_k)$ stands for the trajectory of the path, has then a cumulate weight of the form $\Lambda_{(i_1,\ldots,i_k)} = \prod_{j=1}^{k} \lambda_{i_j}$.

Figure 3 shows an example of the cubature tree of order one associated to a particular instance of an Ornstein-Uhlenbeck process in dimension one, which the size of each node representing the associated wight in the discrete measure. For this particular cubature measure, at each stage, each node in the tree is divided in two equal nodes placed at appropriate places.

The basic underlying idea behind the cubature method, that is, to replace a probability law in infinite dimensions by a law with finite support, has been explored by using different techniques. We can mention, for example, the work of Luschgy and Pagès [68] on quantization of Gaussian processes. The objective of the quantization method is to construct finite spacial grids and associated weights to approach a given probability law optimally in some L_p space (the most common case being $p = 2$) . The relation between both methods extends to further applications in numerical probability. For example, both methods can be used to solve BSDEs of different type, as we present in 3.3.b.

Some final remarks are in order. It is worth noticing that the regularity assumptions of f may be relaxed: the key point is to profit from a regularization effect induced by the diffusion under some additional structure conditions. Results in this direction are found in [41] when uniform ellipticity is assumed and [57] when a weaker assumption on the diffusively structure (the so called uniformly finitely generated (UFG) condition) is assumed. Using this results, Lyons and Victoir show the error control for f Lipschitz, and a well chosen time discretization in [69] (we

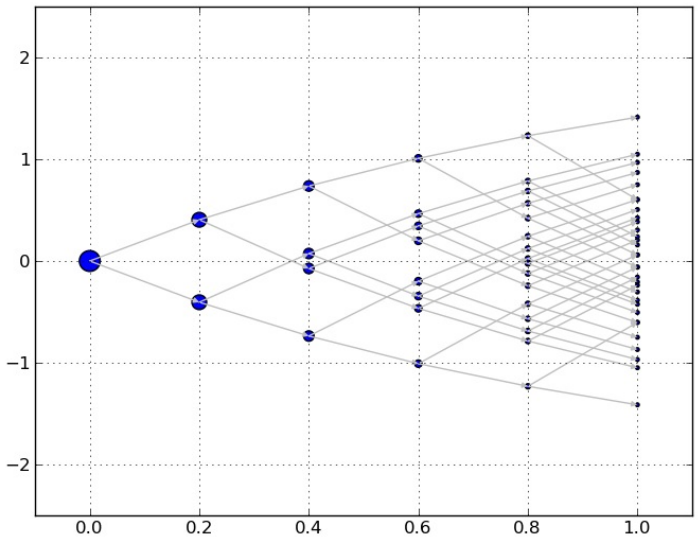


Figure 3: An example of a cubature tree of order three in dimension one.

will use this structure whens solving the MKV-FBSDE problem in Chapter 2). In [46] some results are also available when f is piecewise continuous.

We emphasize that at the definition of cubature formulas, we work with Stratonovich (and not Itô) integrals. From the probabilistic view, this is explained as we will have convergence towards Stratonovich integrals when decreasing the step size, but it is also justified by an algebraic structure satisfied by Stratonovich integrals that is extremely useful for the construction of cubature formulas. This is studied in Chen ([24]) and related papers, and used in the rough path theory of Lyons [70]. Applications of this structure along with explicit cubature formulas of different orders and different dimensions are presented in [69] and [46].

3.3 A short review on decoupled FBSDEs

Introduced first by Bismut [10] in the linear case, when establishing an adjoint equation for stochastic optimization, and then extended by Pardoux and Peng [81] in the general non-linear case, backward stochastic differential equations (BSDEs) are found in many applications thanks to their links with non-linear PDEs and stochastic control. For instance, in finance, they are a powerful tool for contingent claim valuation and hedging, specially when constraints appear, or when considering incomplete markets. See [73] for a review on BSDEs and [33] for a review of BSDEs in finance.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$, and a standard $m-$ dimensional Brownian motion $B_{0\leq t\leq T}$ adapted to the filtration, the solution to a BSDE is a couple of $\mathcal{F}_t$-adapted stochastic processes Y, Z satisfying a certain dynamic and boundary condition, namely

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \tag{44}$$

where ξ, the boundary condition, is an $\mathcal{F}_T$ measurable random variable and f, called the driver, is a measurable mapping from $\Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{m\times d}$ into $\mathbb{R}^d$. Note that although only one

equation is defined, the adaptedness condition imposes an additional constraint allowing us to expect uniqueness of the solution under appropriate conditions.

Proposition 0.14. *Assuming f is uniformly Lipschitz continuous and that $f(.,0,0)$ is square integrable on $\Omega \times [0,T]$, then there exists a unique solution of BSDE (44) .*

We recall the main arguments of the proof of Pardoux and Peng in [81] as they help to understand the analogous result in the McKean Vlasov case. First, we interest ourselves in uniqueness. The programs the same as for the classical SDE uniqueness result: assuming there exist two solutions (Y^1, Z^1) and (Y^2, Z^2) of the BSDE, we use the Lipschitz property of the coefficients, maximal inequalities and Gronwall's lemma to conclude that their must coincide. To prove existence, the equation is decoupled by introducing a sequence of processes Y^n, Z^n with

$$Y_t^{(n+1)} = \xi + \int_t^T f(s, Y_s^{(n)}, Z_s^{(n)})ds - \int_t^T Z_s^{(n+1)} dB_s,$$

and then, we show Y_n, Z_n is a Cauchy sequence by using Itô's Lemma and Lipschitz continuity.

A particular example of BSDE appear when we choose the boundary condition and driver to depend upon the marginal values of another stochastic process, i.e. when we deal with a system like

$$\begin{aligned} X_t &= x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \\ Y_t &= \phi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)dt - \int_0^T Z_s dB_s. \end{aligned}$$

that are called decoupled FBSDEs. Evidently existence and uniqueness follow from Proposition 0.14, provided the forward equation is well posed.

Remark. In our development we will not consider coupled FBSDEs i.e. FBSDEs of the type

$$\begin{aligned} X_t &= x_0 + \int_0^t b(t, X_s, Y_s, Z_s)ds + \int_0^t \sigma(t, X_s, Y_s)dB_s \\ Y_t &= g(X_T) + \int_t^T f(t, X_t, Y_t, Z_t)dt - \int_t^T Z_t dB_t. \end{aligned}$$

3.3.a *Decoupling field*

Consider a slightly modified version of our decoupled FBSDE where we condition the forward variable to be at time t at point x, i.e.

$$\begin{aligned} X_s^{t,x} &= x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r \\ Y_s^{t,x} &= \phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dB_r. \end{aligned} \tag{45}$$

We define the (deterministic) *"decoupling" function* u as

$$u(t,x) = Y_t^{t,x} \ a.s.$$

The terminology of decoupling field comes from the fully coupled FBSDE case, since this function is the key step in the *four-step scheme* of Ma, Protter and Yong [72] to decouple the interaction

between the forward and backward parts, but here we focus on the decoupled case only.

We are interested in some of the properties of the decoupling function. For starters, one shows using the Markov property that

$$Y_s^{t,x} = u(s, X_s^{t,x}).$$

Then, it should be recalled that the decoupling function creates a link between FBSDEs and PDEs. Indeed, as shown by Peng in [86], the adapted solution to (45) may be interpreted as the probabilistic version of a viscosity solution to the PDE

$$D_t u(t,x) + \mathcal{L}u(t,x) = f(t,x,u(t,x),\sigma^* D_x u(t,x)), \qquad u(T,x) = g(x), \tag{46}$$

where $\mathcal{L}$ is the infinitesimal generator of the forward variable in (45). Moreover, in [82], Pardoux and Peng showed using Malliavin calculus techniques, that if ϕ, b, σ are in C_b^3 with respect to x and $f(0,.,.)$ are in C_b^3 with respect to (x,y), then u is in $C^{1,2}$ and it will be a classical solution to (46). Crisan and Delarue, studied in [25] the regularity of the function u under weaker assumptions on g (but adding a hypothesis on the diffusivity of the forward variable). Their result may be seen as an extension of the program of Kusuoka and Stroock [58].

Conversely, it follows from Ito's lemma that if a function u is $C^{1,2}$ and satisfies equation (46), then $Y_s^{t,x} := u(s, X_s^{t,x})$ and $Z_s^{t,x} := \sigma^* D_x u(s, X_s^{t,x})$ solve (45).

3.3.b *Numeric solution*

There are numerous available methods to approach decoupled FBSDEs that we can grossly classify in two big families. We have PDE based methods, like the finite differences and finite elements methods, that aim to solve directly for the decoupling function using PDE (46). On the other hand, we have probabilistic based algorithms. We focus on the latter. The main idea in probabilistic type algorithms is to use the decoupling function and results on their regularity, to propose backward iterations in small time needing the calculation of conditional expectations. Let $0 = T_0 < T_1 < \ldots < T_N = T$ be a grid on $[0,T]$. Denote $\Delta_{T_k} = T_k - T_{k-1}$. Then, we can approach numerically the decoupled field $u(t,x) = Y_t^{t,x}$ and the function $v(t,x) = Z_t^{t,x}$ by using the scheme

$$\begin{aligned} u(T_N, x) &= g(x) \\ v(T_k, x) &\approx \frac{1}{\Delta_{T_{k+1}}} \mathbb{E}\left(u(T_{k+1}, X_{T_{k+1}}^{T_k,x}) \Delta W_{k+1} \right) \\ u(T_k, x) &\approx \mathbb{E}\left(u(T_{k+1}, X_{T_{k+1}}^{T_k,x}) + \Delta_{T_{k+1}} f(\Theta_{k+1,x}) \right) \end{aligned} \tag{47}$$

Where we denote,

$$\Theta_{k+1,x} := \left(T_{k+1}, X_{T_{k+1}}^{T_k,x}, u(T_{k+1}, X_{T_{k+1}}^{T_k,x}), v(T_{k+1}, X_{T_{k+1}}^{T_k,x}) \right). \tag{48}$$

Note that depending on the regularity of the decoupling field, it is possible to use instead a semi-implicit or completely explicit version of $\Theta_{k+1,x}$.

Now, to render the scheme implementable, we need to give an explicit way to approach the conditional expectations in (48). We can classify the available approaches in roughly three groups: regression type, Malliavin type and grid type algorithms.

Regression algorithms propose to reinterpret the problem of finding the unknown values of the decoupling functions u, v at time T_k knowing their values at time T_{k+1} as a regression problem

The regression is performed over a set generated by some well-chosen set of functions: in general one would like the generating set to be small enough as to simplify the regression solution but rich enough to allow for a good approximation precision (for example a set of local polynomials of certain degree). The error to be minimized for the regression is the L_2 error in T_k, usually calculated using Monte Carlo simulations. This approach was first proposed by Longstaff-Schwarz [67]; and it was further studied under weakest regularity conditions for example by Gobet, Lemor and Warin in [44] and Gobet and Lemor in [43].

Another alternative is to approach the conditional expectation operator y using Malliavin calculus. This approximating operator essentially transforms conditional expectations in total expectations by multiplying by some random weight function that is explicitly calculated, and that intuitively acts as a *numéraire* change. The weight function and the total expectation are implemented in practice using Monte Carlo simulations and, frequently, truncation functions to control possible instabilities in the random weight function. The Malliavin calculus approach to calculate conditional expectations was first introduced by Fournié et al. in [39] and further explored in [38]. We can refer to Bouchard and Touzi [12] for a complete description of an algorithm solving general BSDEs based on this approach.

Finally, we have the algorithms based on the construction of a grid that attempt to give a faithful representation of the law of the forward variable. In general, this type of approach runs in two steps: first, the effort is focused on constructing a well approximating grid. Then, a second algorithm is used to obtain the solution of the backward problem at each point in the grid. In general both parts are independent. Moreover, from the complexity point of view, the first part is much more expensive to accomplish than the second one. This characterizes the main difference with respect to the previous two methods: once a grid is defined, several backward problems with different data (i.e. with different boundary condition and driver) can be solved at almost no cost. The best known method of this type is the quantization method first proposed by Bally and Pagès in [4] and [3]. In the first stage, a grid of points and associated weights having at most N elements and minimizing the L^2 (or in general L^p) quantization error is constructed by combining a Voronoi tesselation and stochastic gradient optimization[7] with some empirical approximation of transition probabilities. In [4] and [3], the quantization method was shown to effectively solve BSDEs in the multidimensional case (even considering additional reflection terms), for the case where the driver does not depend on Z. It has also been successfully applied to solve fully coupled equations by Delarue and Menozzi [29].

Recently, another method belonging to the grid type family was introduced by Crisan and Manoralakis in [28] based on the cubature on Wiener spaces algorithm. The basic idea is to use the fact that the subtree spanning from a given node on the cubature tree, thanks to the Markovian property, approaches the conditional law.

In addition to the already mentioned possibility to approach several BSDE problems with the same cubature tree, the cubature approach, provided that the true solution is regular enough, allows for arbitrary orders of convergence in the conditional expectation approximation. Then, we could combine higher order discretization generalizing the scheme (47) and the cubature method to obtain higher order methods to approach BSDEs. Higher order schemes have recently been studied by Crisan and Manoralakis [27] and by Chassagneux and Crisan [22].

[7] In the machine learning terminology, the quadratic case is equivalent to a extended Competitive Learning Vector Quantization.

3.4 Decoupled MKV-FBSDEs

Let us go back to our original problem and consider equation (39). We place ourselves in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$, with $B_t^{1:d}$ a d-dimensional adapted Brownian motion and $B_t^0 = t$. We take $V_i : (t, y, w) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \mapsto V_i(t, y, w)$; functions $\varphi_i : y \in \mathbb{R}^d \mapsto \varphi_i(y) \in \mathbb{R}$, $i = 0, \cdots, d$ and $\varphi_f : (y, y') \in \mathbb{R}^d \times \mathbb{R} \mapsto \varphi_f(y, y')$ and the mapping $f : t, y, y', z, w \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \mapsto f(t, y, y', z, w) \in \mathbb{R}$ to be infinitely differentiable with bounded derivatives. The mapping ϕ is an at least Lipschitz function from $\mathbb{R}^d$ to $\mathbb{R}$ whose precise regularity is given below.

Note, that we have assumed a particular form for the law dependence of the coefficients in equation (39), as we have assumed that the law appears acting over the functions φ_i, $i = 1, \ldots, d$ and φ_f. We have chosen to work with this particular dependence for several reasons: first, because it is a very natural way of having a law dependence and it covers a large number of practical examples; then because it has nice mathematical properties, notably this dependence is Lipschitz with respect to the Wasserstein-2 norm we have defined before, and finally because it simplifies the comprehension of the algorithm.

Existence and uniqueness of the system (39) in a slightly modified version is presented in Buckdahn et al. [15]. The forward equation is simply a McKean Vlasov equation and we have already commented on the arguments leading to existence and uniqueness for it. The proof of the backward part resembles that of a simple BSDE (i.e. is done by defining an appropriate function and showing it is a contraction), except the additional law term must be considered.

Now, as announced, we are interested in finding numerical methods to solve equation (39). The forward and backward parts being decoupled, it seems reasonable to try to solve first the forward McKean Vlasov equation, and then solve the backward part. A first idea would be to use directly a particular method to approach the McKean equation and then applying one of the presented numerical methods to deal with the backward equation. However, particular methods introduce as many dimensions in the Brownian motion as particles. Since the dimension of the noise in the backward equation is the same, this has the effect of increasing the complexity of solving numerically the backward equation, even if some dimension reduction might be possible thanks to symmetry.

Instead we follow a different approach by considering the cubature on Wiener space algorithm. The method is a natural candidate as we showed it is effective for weak SDE approximation (and hence to approximate our assumed kind of law dependence) and solving decoupled FBSDEs. Let us explain the basic idea. The main issue in the case of a MKV-FBSDE is the McKean-Vlasov dependence that appears in the coefficients. This dependence breaks the Markov property (considered only on $\mathbb{R}^d$) of the process so that it is not possible to apply, a priori, many classical analysis tools. In order to handle this problem, the idea consists in decoupling the law dependence : *given the law of the solution of the system,* (39) *is a classical time inhomogeneous decoupled FBSDE* (the law just acts as a time dependent parameter).

Let $(\eta_t)_{0\leq t\leq T}$ be a family of probability measures on $\mathbb{R}^d$, and let us fix the law in the McKean-Vlasov terms of (39) to be $(\eta_t)_{0\leq t\leq T}$. For this modified system, we may apply a classical cubature scheme for the FBSDE (the time dependence of the coefficients being handled by time freezing). The trick consists in taking advantage of the decoupled setting: we first build a cubature tree (depending on the order of the cubature) and then go back along the nodes of the tree by computing the current value of the backward process as a conditional expectation at each node.

Obviously, at each step of the scheme, we pay the price of using an arbitrary probability measure as parameter for the coefficients instead of the law of the process. Therefore this law has to be chosen carefully in order to keep a good control on the error and achieve convergence. An example of a "good choice" is to take at each step of the cubature tree the discrete marginal law given by the solution of the ODEs along the cubature paths and corresponding weights. We show that for a cubature of order m and a number N of discretization steps, this choice of approximation law leads to a $N^{-(m-1)/2}$ order approximation of the expectation of any $m+2$ times continuously differentiable functional of the forward component, when all the derivatives are bounded[8], and to a first order approximation scheme of the backward component, where the given orders stand for the supremum norm error. Higher orders of approximation are also obtained by correcting some terms in the algorithm.

As it is pointed out in [69] and [27], the regularity of the terminal condition ϕ in (39) may be relaxed to Lipschitz and the approximation convergence rate preserved, provided that the vector fields are uniformly non-degenerate (in fact, the condition in the given references is weaker, since the vector fields are supposed to satisfy an UFG condition, see [58]). This relies on the regularization properties of parabolic and semi-linear parabolic PDEs (see [41] for an overview in the elliptic case and respectively [58] and [25] for the UFG case). We show that this remains true in the McKean-Vlasov case and that the convergence rate still holds when the function ϕ is Lipschitz only and when the vector fields are uniformly elliptic.

The conditional system. Let us shortly develop how to perform the law decoupling. Recall we have argued that there exists a unique solution $\{X_t^x, Y_t^x\}_{t\geq 0}$ to the system (39). As we have explained in the review, for a classical decoupled FBSDE in a Markovian setting, the law of this couple is entirely determined by the law $\mu = (\mu_t)_{0\leq t\leq T}$ of the forward process $(X_t)_{0\leq t\leq T}$ thanks to the decoupling function u. In our case, one can show that this remains true (see Section 7 below for a proof) so that there exists a deterministic $u(t,y)$ such that for all t,

$$Y_t = u(t, X_t). \tag{49}$$

We prove that under appropriate assumptions u is regular and we will have estimates on it and its derivatives. The conditional MKV-FBSDE system may be then defined as

$$\begin{cases} dX_s^{t,y,\mu} = \sum_{i=0}^{d} V_i(s, X_s^{t,y,\mu}, \langle \mu_s, \varphi_i \rangle) dB_s^i \\ dY_s^{t,y,\mu} = -f(s, X_s^{t,y,\mu}, Y_s^{t,y,\mu}, Z_s^{t,y,\mu}, \langle \mu_s, \varphi_f(\cdot, u(s,\cdot)) \rangle) ds + Z_s^{t,y} dB_s^{1:d} \\ X_t^{t,y,\mu} = y, \quad Y_T^{t,y,\mu} = \phi(X_T^{t,y,\mu}). \end{cases} \tag{50}$$

Note that in this system, we have decoupled the law dependence with respect to the backward variable, leaving only the dependence with respect to the law of X.

Let us remark that in this setting, we can also define the deterministic mapping $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$Z_t = v(t, X_t). \tag{51}$$

As in the classical BSDE theory, under appropriate regularity conditions,

$$v(t,x) = (\mathcal{V}^\mu u(t,x))^T.$$

[8] this is the special case when ϕ is $m+2$ times continuously differentiable with bounded derivatives and $f = 0$ in (39).

3.5 Proposed algorithms

Detailed descriptions of the algorithm are available in Chapter 2 section 1. We present here a brief description to develop the intuition behind our proposed algorithm. We assume we are given a subdivision of the time interval $[0,T]$ into $N \in \mathbb{N}^*$ steps $0 = T_0 < \cdots < T_N = T$.

- Forward approximation (Algorithm 1 in Chapter 2 section 1.1.a): The algorithm receives two parameters: m defining the order of the cubature formula and q that will appear in a Taylor expansion.

  1. Initialization (Step $k = 0$): At step 0, fix $\hat{X}_0 = x_0$ and $\hat{\mu}_0 = \delta_{x_0}$, i.e. the Dirac mass in the starting point.
  2. Propagation (Steps $k = 1, \ldots, N$)
     (a) Law freezing: Between times T_{k-1}, T_k, take the law appearing in the coefficients of the MKV equation equal to $\hat{\mu}_{k-1}$, after performing an Ito-Taylor expansion of order q on the McKean dependence, i.e., we change $\langle \mu_{T_k}, \varphi \rangle$ by

     $$\sum_{p=0}^{q-1} \frac{1}{p!} (t - T_k)^p \langle \hat{\mu}_{T_{k+1}}, (\mathcal{L}^{\hat{\mu}})^p \varphi_i \rangle.$$

     The SDE is now an ordinary diffusion.
     (b) Cubature: Apply the cubature method to the modified ordinary SDE, i.e., choose a cubature formula of order m and for each particle $\hat{X}_{T_{k-1}}$ at time T_{k-1} solve the ODEs appearing for each ω^j of the cubature formula. This will produce a cloud of deterministic particles at time T_k given by the solution of the resulting ODEs.
     (c) Update the discrete law: Fix $\hat{\mu}_k$ to be the discrete measure coming from the obtained cloud of particles and their associated cumulative weights.

- Backward approximation (Algorithm 2 in Chapter 2 section 1.1.b): Let us describe the first order backward algorithm.

  1. Initialization (Step $k = N$): At step 0, fix $\hat{u}(T_N, \hat{X}_N) = \phi(\hat{X}_N)$, $\hat{v}(T_N, \hat{X}_N) = 0$.
  2. Propagation (Steps $k = N-1, \ldots, 0$)
     (a) Law freezing: Between times T_k, T_{k+1}, take the law appearing in the coefficients of the backward equation equal to $(I \otimes \phi)(\hat{\mu}_{k+1})$ to obtain an ordinary BSDE.
     (b) Backward scheme plus cubature: Find $\hat{u}(T_k, X_k)$ and $\hat{v}(T_k, X_k)$ for each node in the cubature tree. This is done as in the Crisan and Manoralakis scheme, combining the discretization in (47) and the cubature to take conditional expectations.

  The second order backward algorithm (Algorithm 3 in Chapter 2 section 1.1.b), follows the same lines, but uses a higher order scheme for approximating the decoupling function. To avoid implicit terms (difficult to calculate because of the law term) or explicit derivatives, we use a predictor-corrector approach, where the prediction step is based on the first order algorithm.

3.6 Main results

Assumptions. As the reader might guess from the previous discussion, the error analysis of the proposed algorithm uses extensively the regularity of $(t,x) \in [0,T] \times \mathbb{R}^d \mapsto \mathbb{E}\phi(X_T^{t,x,\mu})$ for the forward part and of the solution u of (49) for the backward part. Therefore, we present two types of hypotheses that guarantee that the required regularity is attained.

The first option is to require smoothness on the boundary condition and all the coefficient functions, from where we will deduce the necessary regularity. However, it is also interesting to consider boundary conditions with less regularity. In this case, we need to compensate the regularity loss by imposing stronger diffusion conditions on the forward variable, namely asking for uniform ellipticity of the diffusion matrix V.

(SB) We say that assumption **(SB)** holds if the mapping ϕ in (39) is C_b^∞.

(LB) We say that assumption **(LB)** holds if the mapping ϕ in (39) is uniformly Lipschitz continuous and if the matrix VV^T is uniformly elliptic i.e., there exists $c > 0$ such that

$$\forall (t, y, w) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}, \forall \varsigma \in \mathbb{R}^d, c^{-1}|\varsigma|^2 \leq VV^T(t, y, w)\varsigma \cdot \varsigma \leq c|\varsigma|^2.$$

In order to make the exposition of our results clear, let us define, for $i = 1, 2$:

$$\mathcal{E}_u^i(k) := \max_{\pi \in \mathcal{S}_\kappa(k)} |u(T_k, \hat{X}_{T_k}^\pi) - \hat{u}^i(T_k, \hat{X}_{T_k}^\pi)|; \qquad \mathcal{E}_v^i(k) := \max_{\pi \in \mathcal{S}_\kappa(k)} |v(T_k, \hat{X}_{T_k}^\pi) - \hat{v}^i(T_k, \hat{X}_{T_k}^\pi)|, \quad (52)$$

$\hat{u}^1, \hat{u}^2, \hat{v}^1$ and $\hat{v}^2$ being the approximated solutions using the first and and second order algorithms where u, v are defined in (49), (51).

Main result in a smooth setting. We have that

Theorem 0.15. *Suppose that assumption **(SB)** holds. Apply Algorithm 1 with parameters m and q on a uniform time grid with N points. Then, there exists a positive constant C, depending only on T, q, d, $||\varphi_{0:d}||_{2q+m+2,\infty}$, $||\phi||_{m+2,\infty}$, such that:*

$$\max_{k \in \{0,\dots,N\}} |\langle \mu_{T_k} - \hat{\mu}_{T_k}, \phi \rangle| \leq C \left(\frac{1}{N}\right)^{[(m-1)/2] \wedge q}, \quad (53)$$

with $\hat{\mu}$ as defined in Algorithm 1. and where we used the duality notation $\langle \mu, \varphi_i \rangle$ for $\int \varphi_i d\mu$. Suppose in addition that $q \geq 1$ and $m \geq 3$. Then, there exists a positive constants C_1, depending only on T, q, d, $||\varphi_{0:d}||_{2q+m+2,\infty}$, $||\varphi_f||_{m+2,\infty}$,$||\phi||_{m+3,\infty}$, such that for all $k = 0, \dots, N$:

$$\mathcal{E}_u^1(k) + \Delta_{T_k}^{1/2} \mathcal{E}_v^1(k) \leq C_1 \left(\frac{1}{N}\right), \quad (54)$$

Moreover, suppose in addition that $q \geq 2$ and $m \geq 5$. Then, there exists a positive constants C_2, depending only on T, q, d, $||\varphi_{0:d}||_{2q+m+2,\infty}$, $||\varphi_f||_{m+2,\infty}$,$||\phi||_{m+3,\infty}$, such that for all $k = 0, \dots, N$:

$$\mathcal{E}_u^2(k) + \Delta_{T_k}^{1/2} \mathcal{E}_v^2(k) \leq C_2 \left(\frac{1}{N}\right)^2. \quad (55)$$

The proof of both the forward and the backward algorithms are based on controlling one step errors:

- To prove the result for the forward algorithm, we rewrite the total error as a telescopic sum of one step errors. Then, each one step error is decomposed in turn as:
  - An error coming from freezing both the law and the time variable at the initial time of the step. This is controlled using estimates on the coefficients of the forward diffusion.
  - An error coming from the cubature approximation. Regularity and estimates on $(t, x) \in [0, T] \times \mathbb{R}^d \mapsto \mathbb{E}\phi(X_T^{t,x,\mu})$ are therefore needed.

– A propagation error coming from the fact that we use an empirical law instead of the exact law of the process.

The claim follows then by using a discrete Gronwall's lemma on the resulting expressions.

- The first order backward algorithm is proved in a similar manner. This time, each one step error is decomposed in

  – A scheme error coming from the scheme used to approach the functions u and v at the discretization times.

  – An error coming from the cubature approximation to find conditional expectations. Regularity and estimates on u, v are needed.

  – A propagation error.

  As in the forward case, the estimates allow for the use of Gronwall's lemma to finish the proof.

- The second order backward algorithm follows the same idea but demands taking care of the additional prediction terms and the new higher order scheme.

Convergence order for a Lipschitz boundary condition.

Corollary 0.16. *Suppose that assumption **(LB)** holds. Let m be a given cubature order and q a given non-negative integer (marking the Ito-Taylor expansion of the McKean term), parameters for Algorithm 1 applied on a decreasing time grid with N points, with*

$$T_k = T\left[1 - \left(1 - \frac{k}{N}\right)^{\gamma}\right],$$

for some $\gamma > 0$. Then, there exists a positive constant C depending only on T, $||\varphi||_{2q+m+2,\infty}$, $||\phi||_{1,\infty}$, such that:

$$|\langle \mu_T - \hat{\mu}_T, \phi \rangle| \quad \leq \quad C\left(\left(\frac{1}{N}\right)^{(\gamma/2)\wedge q} \vee L(\gamma, m)\right). \tag{56}$$

where

$$L(\gamma, m) = \begin{cases} N^{-\gamma/2} & \text{if} \quad \gamma \in (0, m-1) \\ N^{-(m-1)/2}\ln(N) & \text{if} \quad \gamma = m-1 \\ N^{-(m-1)/2} & \text{if} \quad \gamma \in (m-1, +\infty) \end{cases} \tag{57}$$

Moreover, if $\gamma > m-1$, the results on the error control of $(\hat{u}^1, \hat{v}^1)$ and $(\hat{u}^2, \hat{v}^2)$ respectively given by (54) and (55) remain valid.

Note that, in (56), the control holds only at time T although it holds at each step in (53): this is because the boundary condition is Lipschitz only so that we have to wait for the smoothing effect to take place. The proof of this corollary is basically a consequence of guaranteeing the needed regularity of the functions $(t, x) \in [0, T] \times \mathbb{R}^d \mapsto \mathbb{E}\phi(X_T^{t,x,\mu})$ for the forward part and of the solution u, v for the backward part.

3.7 Extensions

It is easily shown that the algorithm will work seamlessly if we let the boundary condition $\phi : (y, w) \in \mathbb{R}^d \times \mathbb{R} \mapsto \phi(y, w)$ depend also on the law of the process $(X_t^x,\ 0 \leq t \leq T)$, as long as this dependence behaves nicely. For example one can consider

$$\mathbb{E}\left[\phi(X_t^x, \mathbb{E}[\varphi_\phi(X_T^x)])\right]$$

for a given $\varphi_\phi \in C_b^{m+2}$ and where ϕ is Lipschitz in w uniformly in y.

Similarly, the algorithm can be easily adapted to different forms of law dependence. For example, it can be used with the particular dependence explored in [15]:

$$V_i(t, y, \mu) = \langle \mu_t, V_i(t, y, \cdot)\rangle,\ i = 0, \cdots, d,$$

and the result of Theorem 0.15 and Corollary 0.16 remain valid. Note that in that case the uniform ellipticity **(LB)** has to be understood for the matrix $[\langle \eta_t, V(t, y, \cdot)\rangle]\,[\langle \eta_t, V(t, y, \cdot)\rangle]^*$ uniformly in y, t in $\mathbb{R}^d \times \mathbb{R}^+$ and η family of probability measures on $\mathbb{R}^d$.

Some additional (and more technical) generalizations are also explored in section 2 in Chapter 2.

3.8 Final remarks

We have presented a new cubature based method to approach McKean Vlasov SDEs and constructed based on it two algorithms to approach decoupled FBSDEs. We have presented our results assuming smoothness and boundedness of the functions V_i, φ_i and f in (39), under two types of regularity for the boundary condition ϕ: smoothness and Lipschitz continuity plus uniform ellipticity of the forward equation.

Naturally we should examine the question of weakening the assumed conditions. The first condition we could relax is the one related to the functions appearing in the McKean dependence term $\varphi_i, i = 0, \ldots, d$. Indeed, as the coefficients depend on averages of φ_i and not on the φ_i themselves, one could argue using the smoothing effect, as we have done for the boundary condition, that the analysis will hold assuming less regularity on those functions (for example, only Lipschitz regularity). In order to do this, we need some careful study to determine how is the smoothing affected when the McKean term is present, and particularly to determine estimates on the derivatives of the function $(t, x) \to \mathbb{E}\left(\varphi_i(X_t^{0,x})\right)$. We would expect to encounter a situation analogous to the case of Lipschitz boundary condition: the derivative bounds should be singular close to $t = 0$ and progressively smooth. From the point of view of the algorithm formulation, this would force us to change the discretization steps: just as we used a a decreasing step discretization to profit form the smoothing effect for the Lipschitz boundary condition, we would need an *increasing* step in the case of Lipschitz functions in the McKean term. Hence we would need an increasing then decreasing time step to manage simultaneously both situations.

Another condition that could eventually be relaxed is the ellipticity hypothesis needed for the Lipschitz case. When considering a *homogeneous* decoupled FBSDE without the McKean term, the smoothing results and estimates of Kusuoka [58] and Crisan and Delarue [25] follow under the so called UFG condition, which is a condition weaker than the Hörmander condition, and, of course than uniform ellipticity. In our setup, we have asked for ellipticity because the McKean Vlasov term puts us in an inhomogeneous case, and we have not found equivalent results in the literature under that setup. A first idea is to recover the homogeneous case by imposing conditions on the time variable to "behave like a space variable", but in this case some work is

needed to understand how to translate those conditions to the McKean Vlasov setup.

Let us consider as well the conditions on the coefficients V_i and f. The boundedness of the coefficients V_i is mainly a technical assumption that simplifies our analysis, and we believe that this type of condition could be exchanged for some polynomial growth without changing the effectiveness of the algorithm. The case for f is more complicated, as changes in its growth condition may affect the stability of the BSDE. Then, the algorithm should be complemented with truncation techniques to be applied in this kind of situations. Finally the cubature algorithm needs a certain regularity of the functions V_i, f to work properly and to our knowledge this hypothesis cannot be relaxed.

Regarding extended uses of the algorithm, we believe that although it has been designed to work for decoupled MKV-FBSDEs, this solver may also be considered as a building block if one is interested in approaching the fully coupled case (when the forward coefficients depend on the backward variable), for example via fixed point procedures. Nevertheless, a lot of work is required to define the precise conditions and setup required for proving the existence of a solution and those assuring that the proposed fixed point algorithm would converge to the (or at least a) solution of the fully coupled problem.

Another interesting question is if the algorithm may be used when the boundary condition ϕ depends on the whole trajectory of the forward variable X and not only on its final value. Bayer and Friz have recently shown that the cubature method allows for weak convergence of general path-dependent functions [5], although their method does not allow to obtain an explicit convergence rate. A result in this direction would be interesting even in the non-McKean Vlasov case.

Finally, there is the question on the complexity of the curator method, which is in fat exponential. We have considerers this problem in more detail, and we present it in the next section.

4 Appendix.- Implementation of a reduced cubature method to solve BSDEs

In Section 3.3.b, we presented Crisan and Manoralakis' algorithm to solve decoupled FBSDEs using the cubature on Wiener scheme method. Unfortunately, the algorithm inherits the exponential complexity growth cubature methods with its obvious consequences. For many applications, this will not be a problem as small errors may be obtained even with few discretization steps, but some applications may require a higher precision.

The growth limitation was already identified in [28] where a Tree Branch Based Algorithm (TBBA) was proposed to control the growth of the number of particles needed in KLV approach. The TBBA algorithm succeeds in its complexity growth control, at the cost of introducing an additional randomness. The TBBA shows almost sure convergence, recovering the whole power of the cubature for a large enough number of steps. Currently there are no estimates on the rate of this almost sure convergence in the general case.

Another possible solution to the complexity growth problem comes from the recombination procedure proposed by Litterer and Lyons in [66]. Let us present this method.

4.1 High order recombination

High order recombination is a procedure that, when applied to any discrete measure μ in $\mathbb{R}^d$, aims to produce a reduced measure $\tilde{\mu}$ whose support is a subset of the support of the original discrete measure, and such that for every element h in a given family of functions $\mathcal{H}$ we have

$$\langle \mu, h \rangle = \langle \tilde{\mu}, h \rangle,$$

i.e. the action of the original measure over a set $\mathcal{H}$ of test function is preserved after the reduction. Hence, if the set of test functions is "well chosen", we can expect to reduce the number of elements in the support of the discrete measure while keeping its essential properties.

The actual implementation to obtain the reduced measure is a combination of divide and conquer techniques with a reduction scheme known in constructive proofs of Carathéodory's theorem. Suppose that $\{h^1, \dots, h^l\}$ is a basis for $\mathcal{H}$, and that the discrete measure is given by

$$\mu = \sum_{i=1}^{N} \lambda_i \delta_{x_i}; \qquad \text{for some } x_1, \dots, x_N \in \mathbb{R}^d.$$

Instead of considering the discrete measure μ in $\mathbb{R}^d$, we work with the related measure in $\mathbb{R}^l$

$$\hat{\mu} = \sum_{i=1}^{N} \lambda_i \delta_{h_i}; \qquad \text{where } h_i = [h^1(x_i), \dots, h^l(x_i)]^T,$$

and the reduction algorithm is as follows,

1. Find a non-trivial solution $c = [c_1, \dots, c_N]$ of

$$\sum_{i=1}^{N} c_i \left(h_i - \langle \hat{\mu}, h \rangle\right) = 0; \quad \sum_{i=1}^{N} c_i = 0$$

2. Fix $\theta = \min_{|c_i|>0} |\lambda_i c_i^{-1}|$ attained at least in one point $i^\star$, and set $\tilde{\lambda} = \lambda - \text{sign}(c_{i^\star})\theta c$. Note that at least one value of $\tilde{\lambda}_i$ is equal to zero (namely $\hat{\lambda}_{i^\star}$). Hence, we can define the reduced law

$$\tilde{\mu} = \sum_{\{i=1,\dots,N : \tilde{\lambda}_i > 0\}} \tilde{\lambda}_i \delta_{x_i}$$

which has support of cardinality $\tilde{N}$ subset of $\{x_1, \dots, x_N\}$ with $\tilde{N} < N$ and, by construction, preserves the average of each h^j.

Note that thanks to basic linear algebra properties, we can repeat this process as long as $\tilde{N} > l+1$, because we can always find a non-trivial c in step 1. Therefore we assure the existence of a reduced measure with cardinality of its support less than $l+1$.

When $N \gg l+1$, Litterer and Lyons propose to combine the previous algorithm with a divide and conquer strategy:

- Divide the support of μ in $2(l+1)$ sets of roughly the same size (say $A_1, \dots, A_{2(l+1)}$.
- Define $\Lambda_k = \sum_{i=1}^{N} \lambda_i \mathbf{1}_{\{x_i \in A_k\}}$ and $g_k = (\Lambda_k)^{-1} \sum_{i=1}^{N} \lambda_i h_i \mathbf{1}_{\{x_i \in A_k\}}$, and consider the associated discrete measure in $\mathbb{R}^l$, $\sum_{k=1}^{2l+1} \Lambda_k \delta_{g_k}$.

- Apply the reduction algorithm on the associated discrete law.

Here, the idea is that at each call of the reduction algorithm, an entire subset A_i is eliminated (i.e. roughly $\tilde{N}/(2(l+1))$ points), accelerating the reduction process.

4.1.a *Application to the cubature method for weak SDE approximation*

To apply the recombination algorithm to the cubature method, we choose a suitable set of test functions, and apply the recombination to the marginal cubature measure at the discretization times T_k. For example, in [66], the method is studied when considering $\mathcal{H}$ as the set of local polynomials of degree at most m: more specifically, for a given discretization time T_k, we take O_j^k, $j = 1, \dots, M^k$ a finite collection of open balls of radius r_k covering the support of the cubature tree. Then if $\{p_1^j, \dots, p_l^j\}$, again with $j = 1, \dots, M^k$, are basis functions for the set of polynomials of degree m with d variables, we take

$$\mathcal{H}^k = \left\{ \sum_{j=1}^{l} a_i p_i^j \mathbb{1}_{O_j^k} : a_i \in \mathbb{R}; i = 1, \dots \binom{m+d}{d}; j = 1, \dots, M^k, \right\}.$$

The advantage of choosing such a family of test functions is the following one: let f be in C_b^{m+1} (a $m+1$ differentiable function with bounded derivatives up to order $m+1$), and let $\Pi_{\mathcal{H}^k}$ be the projection operator into $\mathcal{H}^k$. Then, Taylor's theorem and the definition of $\mathcal{H}^k$ imply

$$|f(x) - \Pi_{\mathcal{H}^k} f(x)| \leq K(r_k)^{m+1}.$$

from where we deduce, by the definition of the recombined cubature measure that,

$$|\mathbb{E}_{\mathbb{Q}_{T_k}}[f(X_{T_k})] - \mathbb{E}_{\tilde{\mathbb{Q}}_{T_k}}[f(X_{T_k})]| \leq K(r_k)^{m+1}.$$

where $\mathbb{Q}_{T_k}$ is the original cubature measure and $\tilde{\mathbb{Q}}_{T_k}$ is the recombined one.

Note that we can, in principle, apply the reduction procedure at each desired time k. In practice, however, this might not be the best approach due to the complexity of the reduction algorithm. In addition, some care must be taken when using the reduced cubature method to weakly approach the average of Lipschitz regular functions: the reduction step should not be applied on the final step as there is not enough regularity to apply the Taylor theorem we invoked in the approximation result. Any other step will benefit form the regularization effects we mentioned in the cubature presentation, and is thus subject to the use of recombination.

The recombination procedure is designed to preserve the weak approximation of the marginals of the forward process, at the cost of losing the Markovian structure the cubature measure featured. This is better illustrated in Figure 4.1.a where it is shown an example of a recombined cubature tree (compare with Figure 3). Observe that as a consequence of the reduction, there are points for which we do not have a subtree effectively approaching the conditional law, as shown by the red and green dots in Figure 4.1.a.

As a consequence, a recombined cubature tree cannot be used to solve FBSDEs following the scheme of Crisan and Manoralakis. Indeed, referring once again to Figure 4.1.a, we see that Crisan and Manoralakis' scheme will have some problems when arriving to the colored nodes: there is no information available to find the solution at the red points, and incomplete solution leading to possibly wrong values for the green points. Evidently, this situation propagates to previous discretization times.

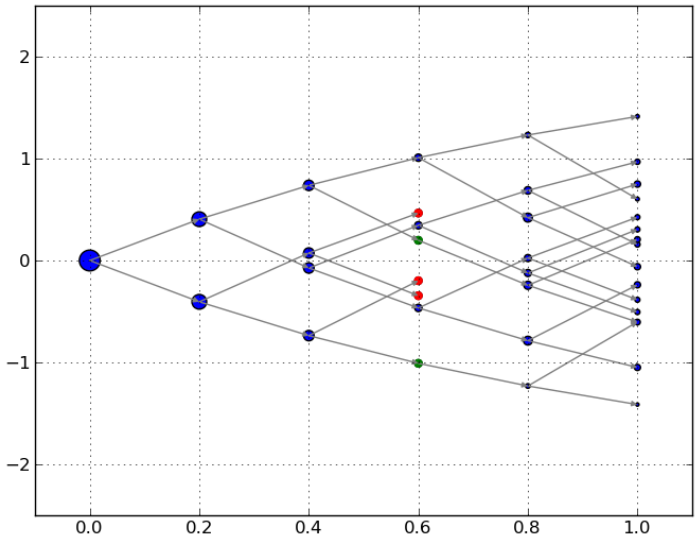


Figure 4: An example of a reduced cubature tree. The reduced cubature tree losses its Markovian property, as the structure departing from the green dots is incomplete and the one of red dots is inexistent.

4.2 Implementation of a recombined cubature method to solve BSDEs

Our objective in Appendix A is to explore a procedure to adapt the recombination algorithm in order to work for the solution of BSDEs.

The main idea aims to reconstruct, up to a controlled error, the original cubature tree from the reduced one. Although there may be several ways to achieve this goal, we propose to change the objective of obtaining a reduced probability measure after the recombination procedure and aim instead to obtain a bounded *signed* measure. The structure of the algorithm will allow us to reconstruct the values at the missing points and to use the structure of the original cubature tree to find conditional expectations. Under appropriate restrictions on the reduced signed measure, we show the algorithm is stable and presents the same orders of convergence of the original cubature method up to a constant. Although we do not have analytic results on the reduction ratio, namely the ratio between the number of nodes in the reduced cubature and those in the original cubature, the numerical tests seem to suggest that the method is effective to change the exponential growth of the algorithm for a polynomial one.

Chapter 1

A Decreasing Step Method for Strongly Oscillating Stochastic Models

In this chapter, we will focus on obtaining a numerical scheme to efficiently approach the following system

$$\begin{cases} X_t^\epsilon &= x_0 + \int_0^t f(X_s^\epsilon, Y_s^\epsilon)ds + \int_0^t g(X_s^\epsilon, Y_s^\epsilon)dW_s \\ Y_t^\epsilon &= y_0 + \epsilon^{-1}\int_0^t b(X_s^\epsilon, Y_s^\epsilon)ds + \epsilon^{-1/2}\int_0^t \sigma(X_s^\epsilon, Y_s^\epsilon)d\tilde{W}_s, \end{cases} \tag{1.1}$$

where X_t^ϵ is a d_x-dimensional process, Y_t^ϵ a d_y-dimensional process, W and $\tilde{W}$ are two independent Brownian motions of dimensions d_x and d_y, and the functions b, σ, f and g have the right dimensions.

As mentioned in the introduction, to approach the strong oscillating system we profit from the mentioned result (Theorem 0.9) of Pardoux and Veretennikov (presented in [84]), stating that under appropriate assumptions (that we will recall in the following) $X^\epsilon \xrightarrow{\mathcal{L}} X$ as $\epsilon \to 0$, where X is the solution to the *effective equation*

$$X_t = x_0 + \int_0^t F(X_s)ds + \int_0^t G(X_s)dW_s, \tag{1.2}$$

with coefficients given by

$$F(x) = \int f(x,y)\mu^x(dy) \quad G(x) = \sqrt{H(x)} \quad H(x) = \int h(x,y)\mu^x(dy), \tag{1.3}$$

where $h(x,y) = gg^*(x,y)$, and $G(x)$ could be in principle any matrix with square given by H, but we choose it to represent the Cholesky decomposition of the positive semi-definite matrix H ; provided that

$$Y_t^x = y_0 + \int_0^t b(x, Y_s^x)ds + \int_0^t \sigma(x, Y_s^x)d\tilde{W}_s, \tag{1.4}$$

is an ergodic diffusion with unique invariant measure μ^x.

Then, we focus our efforts in approaching the effective equation (1.2). As announced, we propose two algorithms. A first algorithm, that we will call multi-scale decreasing step algorithm (MsDS) algorithm and we show it strongly converges to the exact solution and proves to be more efficient than a simple Euler scheme for highly oscillating problems. Moreover, it features a non-standard C.L.T. property in the sense that the normalized error distribution converges towards the solution of a SDE. The coefficients appearing in this normalized error SDE depend on the invariant measure and hence are, in general, unknown. Nevertheless, the available explicit expression for them is valuable for the estimation of confidence intervals and eases the task of parameter tuning for actual implementation of the algorithm. We study as well an *extrapolated MsDS (EMsDS)* version of the algorithm, differing from the original one in that it uses as estimator of the ergodic average a Richardson-Romberg extrapolation of the decreasing step estimator (i.e. a well chosen linear combination of the decreasing step Euler estimator with appropriate parameters). As the MsDS, the EMsDS also features a non-standard C.L.T. property and shares the same rate of convergence. However, the extrapolated version has lower asymptotic complexity and hence higher asymptotic efficiency than the original one.

In order to control the total error approximation of our proposed algorithms we need to take into account four effects. First, we need an estimate on the ergodic average approximation at each step. We show that this control is based on the existence, regularity and control of the solution of the Poisson equation associated to the fast scale diffusion

$$\mathcal{L}_y^x \phi_\psi(x,y) = \psi(x,y) \tag{1.5}$$

where

$$\mathcal{L}_y^x := \sum_{i,j=1}^{d_y} a_{ij}(x,y) \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{d_y} b_i(x,y) \frac{\partial}{\partial y_i} \tag{1.6}$$

with $a := \sigma\sigma^*$, when considering as sources (i.e. the right hand side functions) the coefficients F and H centered with respect to their respective invariant measures. Second, we need to control the error obtained after performing a Cholesky decomposition. Third, we have to account for discretization errors. Finally, we need to control the error propagation which will be possible under some growth control on the coefficients of the effective equation.

The organization of the chapter is as follows: in section 1 we present our general framework (probability space and main hypothesis). We describe the algorithm in its two versions (with and without extrapolation); the main results (strong convergence, limit distribution): this will be the plan for section 2. Then we proceed to develop the proof of the main theorems in three steps: first, on section 3, we remind some known regularity properties of the effective equation and available results on the decreasing Euler estimation algorithm; then, on section 4, we prove the convergence and limit distribution results for the MsDs algorithm; finally on section 5 we extend those results to the EMsDS algorithm. We close this chapter with an illustration of the presented results by performing some numerical tests in section 6 and with the proof of a technical result on convergence of a tuple used in the main proof in section 7.

1 Probability space and standing hypothesis

We introduce a probability space with a particular structure that will be appropriate for studying such an algorithm. Consider the space $(\Omega, \mathcal{F}, \mathbb{P})$ with an adapted Brownian motion W. Suppose

as well we are given an independent probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a family of independent Brownian motions $\tilde{W}^q, q \in \mathbb{Q}$ with an associated filtration $\tilde{\mathcal{F}}_t^q := \sigma\{\tilde{W}_s^q, s \leq t\}$. Define the extended space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}})$ by

$$\bar{\Omega} := \Omega \times \tilde{\Omega}, \qquad \bar{\mathbb{P}}(d\omega, d\bar{\omega}) = \mathbb{P}(d\omega)\tilde{\mathbb{P}}(d\bar{\omega}),$$
$$\bar{\mathcal{F}} := \mathcal{F} \otimes \tilde{\mathcal{F}}, \qquad \tilde{\mathcal{F}}_t^q := \bigvee_{q \in \mathbb{Q}; q \leq t} \tilde{F}_\infty^q, \qquad \bar{F}_t := \mathcal{F}_t \vee \tilde{\mathcal{F}}_t^q.$$

Such extended space will be useful for treating independently the noise coming from the Brownian in the effective diffusion and the one related to the ergodic diffusion averages. Hence, we may look at $\mathcal{F}$ as the slow scale information filtration while the family $\tilde{\mathcal{F}}^q$ accounts for several independent Brownian realizations needed to obtain independent realizations of the decreasing Euler estimator.

1.1 Standing hypothesis

Let us introduce the set of hypothesis under which we obtain our results.

Hypothesis $(\mathcal{H}_{s.s.})$ (*On the slow-scale coefficients*).

i) Lipschitz in x: There exist constants K, m such that for all $x, x' \in \mathbb{R}^{d_x}$ and $y \in \mathbb{R}^{d_y}$,

$$|f(x,y) - f(x',y)| + |g(x,y) - g(x',y)| \leq K|y|^m|x - x'|;$$

ii) Regularity: f, h belong to $C_{b,p}^{2,r^y}$ for some $r^y > 3$, where the subindex b, p means the derivatives $\partial_x^i \partial_y^j$ for $0 \leq i \leq 2$ and $0 \leq j \leq r^y - i$ are bounded in x and polynomially bounded in y;

iii) Degeneracy: Either h is identically zero, or it is uniformly non degenerate, that is, there exists λ'_- such that $\lambda'_- I \leq h(x,y)$.

Before giving the standing hypothesis on the fast scale equation, recall that we have defined the matrix $a(x,y) = \sigma\sigma^*(x,y)$.

Hypothesis $(\mathcal{H}_{f.s.})$ (*On the fast-scale coefficients*).

i) $a, b \in C_{b,l}^{2,0}$, i.e. they are continuous and linearly bounded in y, and C^2 and bounded in x.

ii) The matrix a is uniformly continuous and uniformly non-degenerate and bounded, i.e. there exist $\lambda_-, \lambda_+ \in \mathbb{R}_*^+$ such that

$$\lambda_- I \leq a(x,y) \leq \lambda_+ I;$$

iii) $\sup_x b(x,y) \cdot y \leq -c_1|y|^2 + c_2$, for some $c_1 \in \mathbb{R}_+^*, c_2 \in \mathbb{R}$

The regularity and growth assumptions contained in $(\mathcal{H}_{s.s.})$ are assumed to control the error propagation. On the other hand, $(\mathcal{H}_{f.s.})$ is quite restrictive as it requires uniform bounds on x. Note in particular that $(\mathcal{H}_{f.s.})$ (i) implies $\sup_x |b(x,y)| = O(|y|)$ and $(\mathcal{H}_{f.s.})$ (iii) deduces $\lim_{|y| \to \infty} \sup_x b(x,y) \cdot y = -\infty$, meaning that the drift has at most linear growth in y and is mean

reverting uniformly in x. These conditions on the fast scale diffusion appear as we are aiming to have an invariant limit for any possible fixed value of x and a uniform control on its averages. In turn, the ellipticity and non-degeneracy assumption ($\mathcal{H}_{f.s.}$) (ii) is helpful to deduce the uniqueness of the invariant measure.

2 The algorithm

We recall the definition of the decreasing step algorithm and estimator given in the introduction, but modified to add an additional parameter to signal different realizations in accordance to the structure of the probability space we have introduced.

Let $q \in \mathbb{Q}$, and let $\sqrt{\gamma_{k+1}}U^q_{k+1} := \tilde{W}^q_{\Gamma_{k+1}} - \tilde{W}^q_{\Gamma_k}$ implying that U_{k+1} is a standard Gaussian vector. Let $y_0 \in \mathbb{R}^{d_y}$. We define a decreasing step Euler approximation of the ergodic diffusion by

$$\begin{aligned} \tilde{Y}^{x,q}_0 &= y_0 \\ \tilde{Y}^{x,q}_{k+1} &= \tilde{Y}^{x,y_0,q}_k + \gamma_{k+1} b(x, \tilde{Y}^{x,q}_k) + \sqrt{\gamma_{k+1}}\sigma(x, \tilde{Y}^{x,q}_k)U^q_{k+1}, \end{aligned} \tag{1.7}$$

where $\{\gamma_k\}_{k\in\mathbb{N}}$ is a decreasing sequence of steps satisfying

Hypothesis ($\mathcal{H}_\gamma$) (*On the sequence of steps for the average estimation algorithm*).

i) $\gamma_k > 0$ for all k;

ii) γ_k is a sequence of decreasing steps with $\lim_{n\to\infty} \gamma_k = 0$;

iii) $\lim_{k\to\infty} \Gamma_k = \infty$; where $\Gamma_k := \sum_{j=0}^{k} \gamma_j$;

iv) $\sum_{k=1}^{\infty} \left(\frac{\gamma_k^2}{\Gamma_k}\right) < +\infty$.

Then, we define the *decreasing step average estimator* by

$$\tilde{F}^k(x,q) = \frac{1}{\Gamma_k} \sum_{j=1}^{k} \gamma_j f(x, \tilde{Y}^{x,q}_{j-1}) \tag{1.8}$$

Note. In what follows, we will denote by $\Gamma^{[r]}_M = \sum_{k=1}^{M} (\gamma_k)^r$, so in particular, $\Gamma^{[1]}_M = \Gamma_M$.

2.1 The MsDS and EMsDS algorithms

We define an Euler scheme to approach our effective diffusion. Assuming a time horizon T, for $n \in \mathbb{N}^*$ we put $t_k = Tk/n$, so that the Euler scheme will be given by

$$\check{X}^n_{t_{k+1}} = \check{X}^n_{t_k} + \tilde{F}^{M(n)}(\check{X}^n_{t_k}, t_k)\Delta t_{k+1} + \tilde{G}^{M(n)}(\check{X}^n_{t_k}, t_k)\Delta W_{k+1},$$

where $\tilde{F}^M$ is defined in (1.8) and $\tilde{G}^M(x,q)$ is defined in two steps: First we find $\tilde{H}^M(x,q)$ using the decreasing step algorithm as in (1.8) (recall that $h(x,y) = g^* g\,(x,y)$) and then we perform a Cholesky decomposition on it to find $\tilde{G}^M(x,q) = \sqrt{\tilde{H}^M(x,q)}$. Finally, $M(n)$ is the number

of steps in the decreasing Euler estimator expressed as a function of the number of steps in the Euler scheme for the slow scale. The form of $M(n)$ will be clear from the main theorems.

It will be easier to work mathematically with a continuous version of the Euler approximation. Let us denote by $\underline{t}(n) = \lfloor nt \rfloor / n$ and we will usually omit the explicit dependence on n and write $\underline{t}$ when clear from the context. We define the continuous Euler approximation by

$$\tilde{X}_t^n = x_0 + \int_0^t \tilde{F}^{M(n)}\left(\tilde{X}_{\underline{s}}^n, \underline{s}\right) ds + \int_0^t \tilde{G}^{M(n)}\left(\tilde{X}_{\underline{s}}^n, \underline{s}\right) dW_s \tag{1.9}$$

Note that at times t_k the continuous Euler coincide with the Euler algorithm. All our results will be derived for this version of the algorithm.

Remark. For implementation purposes, it could be more convenient to calculate the estimator $\tilde{F}^n$ (similarly $\tilde{H}^n$) recursively as

$$\tilde{F}^0(x,q) = 0; \quad \tilde{F}^k(x,q) = \tilde{F}^{k-1}(x,q) + \frac{\gamma_k}{\Gamma_k}\left(f(x, \tilde{Y}_{k-1}^{x,q}) - \tilde{F}^{k-1}(x,q)\right)$$

The extrapolated version of the algorithm follows the same basic structure of the MsDS algorithm but we use as estimator a Richardson-Romberg extrapolation: a linear combination of the estimator with two different set of parameters chosen to reduce the complexity of the algorithm.

Indeed, let λ be a positive real. If $\{\gamma_k\}$ is a sequence of steps satisfying $(\mathcal{H}_\gamma)$, then the sequence $\gamma_k^\lambda := \dfrac{\gamma_k}{\lambda}$ will also satisfy $(\mathcal{H}_\lambda)$. We will denote Γ_M^λ and $\Gamma_M^{\lambda,[r]}$ the sum of the γ_k^λ and their r-power as before.

Let us denote by $\tilde{F}^{\lambda,M}(x,q)$ the ergodic average estimator as defined in (1.8) when the coefficients $\{\gamma_k^\lambda\}$ are used.

We define an extrapolated approximation estimator as

$$\hat{F}^{(n);\lambda}(x,q) = \frac{1}{\lambda - 1}\left(\lambda \tilde{F}^{\lambda,M(n)}(x,q) - \tilde{F}^{M(n)}(x,q)\right). \tag{1.10}$$

In analogy to the MsDS case, we define $\tilde{G}^{\lambda,M(n)}(x,q) = \sqrt{\tilde{H}^{\lambda,M(n)}(x,q)}$ for $\tilde{H}^{\lambda,M(n)}(x,q)$ defined applying (1.10) to h. We denote as well $\hat{X}$ the continuous approximation (1.9) with coefficients given by the estimators just defined.

Let us state our main Theorems on the MsDS algorithm. Its proof is found in section 4.

Theorem 1.1. *Let $0 < \theta < 1$, $\gamma_0 \in \mathbb{R}^+$ and $\gamma_k = \gamma_0 k^{-\theta}$. Let M_1 be a positive constant. Assume $(\mathcal{H}_{f.s.})$ and $(\mathcal{H}_{s.s.})$. Define $M(n)$ by*

$$M(n) = \left\lceil M_1 n^{\frac{1}{1-\theta}} \right\rceil,$$

then

i) (Strong convergence). $\lim_{n\to\infty} \mathbb{E}\left(\sup_{0\le t\le T}\left|X_t - \tilde{X}_t^n\right|^2\right) \to 0$

ii) (C.L.T like result for the ODE case: $G(x) \equiv 0$). Assume in addition that $r^y \ge 7$ and $1/2 \le \theta$. Then

$$n\left(X - \tilde{X}^n\right) =: \zeta^n \Rightarrow \zeta^\infty$$

for ζ^∞ given as the solution of the system of SDEs

$$\zeta_t^{\infty,i} = \sum_{j=1}^{d_x}\left(\int_0^t \partial_{x^j} F^i(X_s)\zeta_s^{\infty,j} ds + \frac{1}{2}\int_0^t \partial_{x^j} F^i(X_s) F^j(X_s) ds\right) + \sum_{j=1}^{d_x}\int_0^t \varphi_F^{i,j}(X_s) dB_s^{1;j}$$

where B^1 is a d_x-dimensional Brownian motion independent of W; $F^i, G^{i,j}, \zeta^i$ are the entries of F, G, ζ respectively; and $\varphi_F^{i,j}$ are functions depending only on the coefficients b, σ, the constants M_1, γ_0, θ and F and its derivatives up to order 4, and are explicitly given in Proposition 1.23.

iii) (C.L.T like result for the SDE case $G(x) \neq 0$) Assume in addition that $r^y \geq 7$ and $1/3 \leq \theta$. Then

$$n^{1/2}\left(X - \tilde{X}^n\right) =: \zeta^n \Rightarrow \zeta^\infty$$

for ζ^∞ given as the solution of the system of SDEs,

$$\begin{aligned}\zeta_t^{\infty,i} =& \sum_j \left(\int_0^t \partial_{x^j} F^i(X_s)\zeta_s^{\infty,j} ds + \int_0^t R_G^{i,j}(X_s) dW_s^j\right) \\ &+ \sum_{j,l=1}^{d_x} \int_0^t \partial_{x^j} G^{i,l}(X_s)\zeta_s^j dW_s^l + \sum_{j,l,k=1}^{d_x} \int_0^t \varphi_G^{i,j,l,k}(X_s) dB_s^{3;l,k,j} \\ &+ \frac{1}{\sqrt{2}} \sum_{l,j,k=1}^{d_x} \int_0^t \partial_{x^j} G^{i,l}(X_s) G^{j,k}(X_s) dB_s^{2;k,l}\end{aligned}$$

where B^2 is a d_x^2 and B^3 is a $d_x^2 \times d_x$ dimensional Brownian motions independent between them and independent from W; $R_G^{i,j}, \varphi_G^{i,j,l,k}$ are functions depending only on b, σ, the constants M_1, γ_0, θ, F and its first derivatives, G and its derivatives up to order 4, and are explicitly given in Proposition 1.23.

As will be clear from Proposition 1.23, the coefficients $R_G^{i,j}, \varphi_F^{i,j}$ and $\varphi_G^{i,j,l,k}$ are explicitly given in terms of the invariant law of the ergodic diffusion and are consequently unknown. The key point is that, being explicit, they might be estimated numerically for practical purposes.

With respect to the extrapolated version, Corollary 1.2 shows the advantage of using the EMsDS algorithm: assuming higher regularity, all the properties of the MsDS algorithm are conserved but the extrapolated version *allows a lower value for θ in the definition of the sequence* $\gamma_k = \gamma_0 k^{-\theta}$. More precisely we pass from 1/2 to 1/3 in the ODE case and from 1/3 to 1/5 in the SDE case as minimal θ values. As a consequence of this reduction, the complexity of the modified version is in general asymptotically lower than that of the non extrapolated version.

Corollary 1.2. *Let $0 < \theta < 1$, $\gamma_0 \in \mathbb{R}^+$ and $\gamma_k = \gamma_0 k^{-\theta}$. Assume $(\mathcal{H}_{f.s.})$ and $(\mathcal{H}_{s.s.})$. With $M(n)$ defined as in Theorem 1.1.*

Let $\hat{X}^n$ be the approximated diffusion where we replace the ergodic estimator (1.8) by (1.10). Then

i) (Strong convergence). $\lim_{n\to\infty} \mathbb{E}\left(\sup_{0\leq t\leq T} \left|X_t - \hat{X}_t^n\right|^2\right) \to 0$

ii) (ODE case: $G(x) \equiv 0$). Assume in addition that $r^y \geq 8$ and $1/3 \leq \theta$, then $\hat{\zeta}^n := n\left(X_t - \hat{X}^n\right)$ satisfies the limit distribution result given in Theorem 1.1 ii) with new coefficients $\hat{\varphi}^F$ that will be given in Corollary 1.25.

iii) (SDE case) Assume in addition that $r^y \geq 8$ and $1/5 \leq \theta$,. Then $\hat{\zeta}^n := n^{1/2}\left(X_t - \hat{X}^n\right)$ satisfies the limit distribution result given in Theorem 1.1 iii) with new coefficients $\hat{R}^F, \hat{R}^G, \hat{\varphi}^F$ and $\hat{\varphi}^G$ that will be given in Corollary 1.25.

The proof of Corollary 1.2 is given in section 5.

3 Preliminaries

To prove our main results we will need to use different techniques and results related to the interacting steps of the defined algorithms. In this section we present briefly some results from the available literature (with some slight modifications) that will be helpful for our task.

3.1 A priori estimates

An important result is related to some a priori estimates valid for general SDEs. Since they are quite standard, we will state the result without giving the details of the proof.

Proposition 1.3. *Let*

$$\vartheta_t = \vartheta_0 + \int_0^t V_1(\vartheta_s, s)ds + \int_0^t V_2(\vartheta_s, s)dW_s, \tag{1.11}$$

where V_1, V_2 are adapted random functions.

i) For all $\alpha \geq 2$,

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|\vartheta_t|^\alpha\right) \leq K_\alpha \mathbb{E}\left(|\vartheta_0|^\alpha\right) + K(\alpha, T)\int_0^T \left(\mathbb{E}\left(|V_1(\vartheta_s, s)|^\alpha\right) + \mathbb{E}\left(|V_2(\vartheta_s, s)|^\alpha\right)\right) ds$$

and consequently

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|\vartheta_t|^\alpha\right) \leq K_\alpha \mathbb{E}\left(|\vartheta_0|^\alpha\right) + K(\alpha, T)\left(\sup_{0\leq t\leq T}\mathbb{E}\left(|V_1(\vartheta_t, t)|^\alpha\right) + \sup_{0\leq t\leq T}\mathbb{E}\left(|V_2(\vartheta_t, t)|^\alpha\right)\right).$$

ii) Assume that $\forall \alpha \geq 2$,

$$\mathbb{E}\left(|V_1(\vartheta_t, t)|^\alpha\right) + \mathbb{E}\left(|V_2(\vartheta_t, t)|^\alpha\right) \leq K\left(1 + \mathbb{E}\left(|\vartheta_t|\right)^\alpha\right).$$

Then,

a) For $t \in [0, T]$ and $\alpha \geq 2$, $\mathbb{E}\left(|\vartheta_t|^\alpha\right) \leq K(\alpha, T)$

b) For $\alpha \geq 2$, $\mathbb{E}\left(\sup_{0\leq s\leq t}|\vartheta_s|^\alpha\right) \leq K(\alpha, T)$

c) Let $\tau_r = \inf\{t : \vartheta_t \geq r\}$. Then

$$\mathbb{P}\left(\sup_{0\leq s\leq t}\tau_r \leq t\right) \leq \frac{K'(\alpha, t)}{r^\alpha}.$$

3.2 Cholesky decomposition

The Cholesky decomposition of a positive definite matrix consists in expressing this matrix as the product of a lower triangular matrix and its conjugate transpose. A stability analysis of this procedure is a key point in our analysis for the SDE case behavior of our algorithm.

Recall that we denote by $|\cdot|$ the induced operator norm. Let us denote by $\|\cdot\|_F$ the Frobenius norm. Recall that if H is a $d \times d$ matrix

$$|H| \leq \|H\|_F \leq \sqrt{d}|H|. \tag{1.12}$$

Theorem 1.4 (Theorem 1.1 in [89]). *Let H be a $d \times d$ positive definite matrix with Cholesky factorization $H = GG^*$. If ΔH is a $d \times d$ symmetrical matrix satisfying*

$$|H^{-1}|\|\Delta H\|_F < 1/2$$

then there is a unique Cholesky factorization $H + \Delta H = (G + \Delta G)(G + \Delta G)^*$ and

$$\frac{\|\Delta G\|_F}{|G|} \leq \sqrt{2}\frac{\kappa_2(H)\kappa}{1 + \sqrt{1 - 2\kappa_2(H)\kappa}}, \tag{1.13}$$

where $\kappa_2(H) = |H||H^{-1}|$ and

$$\kappa = \frac{\|\Delta H\|_F}{|H|}. \tag{1.14}$$

Theorem 1.4 gives a control on the sensitivity of the Cholesky procedure. We might study the propagation effect at each stage of the Cholesky factorization to say a little bit more on the particular form of the error.

Lemma 1.5. *Suppose the hypothesis of Theorem 1.4 hold. Then,*

$$\Delta G_{i,i} = \frac{\Delta H_{i,i} - 2\sum_{k=1}^{i-1} \Delta G_{i,k} G_{i,k}}{2G_{i,i}} + O(|\Delta H|^2), \tag{1.15}$$

$$\Delta G_{i,j} = \frac{\Delta H_{i,j} - G_{i,j}\Delta G_{j,j} - \sum_{k=1}^{j-1} (\Delta G_{j,k} G_{i,k} + \Delta G_{i,k} G_{j,k})}{G_{j,j}} + O(|\Delta H|^2) \tag{1.16}$$

for $i > j$.

Proof. Since $G + \Delta G$ is the lower triangular factor of $H + \Delta H$, we have

$$(G_{i,i} + \Delta G_{i,i})^2 = H_{i,i} + \Delta H_{i,i} - \sum_{k=1}^{i-1} (G_{i,k} + \Delta G_{i,k})^2.$$

By algebraic manipulation and the fact that G is the Cholesky decomposition of H we get,

$$\Delta G_{i,i} = \frac{\Delta H_{i,i} - 2\sum_{k=1}^{i-1} \Delta G_{i,k} G_{i,k}}{2G_{i,i}} - \left((\Delta G_{i,i})^2 + \sum_{k=1}^{i-1} (\Delta G_{i,k})^2 \right).$$

The first claim follows by controlling the last term by induction in i, Theorem 1.4 and norm equivalence given by (1.12). The case $i > j$ is proved in the same way. □

Lemma 1.5 gives a first order approximation of the error matrix ΔG knowing the perturbation matrix ΔH. Moreover, we can deduce on the regularity of the Cholesky approximation.

Corollary 1.6. *Let $H : \mathbb{R}^d \to M^{d\times d}$ be C_b^2 and non-degenerate (in the sense given in hypothesis $(\mathcal{H}_{s.s.})$. Then G is also C_b^2 and non-degenerate.*

Proof. The non-degenerate and boundedness part follows since from the definition, $|G|^2 = |GG^*| = |H|$, so that $\sqrt{\lambda_-}$ and $\sqrt{\lambda_+}$ are respectively upper and lower bounds for $|G|$. The regularity claim is proven from the limit definition of the derivatives using the result in Lemma 1.5. □

3.3 Decreasing step Euler algorithm

In this section we present some control and error expansion results valid for the Decreasing Step Euler algorithm. The results here presented are found in [59] or in the PhD thesis [65].

A first interesting property is that the sequence of estimators defined in (1.8) converges almost surely to the ergodic average for any fixed x.

Proposition 1.7. *Assume $(\mathcal{H}_{f.s.})$, and let $\psi : \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \to \mathbb{R}^{d'}$ for some $d' \in \mathbb{N}^*$, and suppose that $\psi(x,y) \le C(x)(1+|y|^\pi)$. Let $\tilde{\Psi}^M(x,q)$ be defined as in (1.8). Then, for any $x \in \mathbb{R}^{d_1}, q \in \mathbb{Q}$,*

$$\tilde{\Psi}^M(x,q) \xrightarrow{a.s.} \int \psi(x,y)\mu^x(dy), \text{ as } M \to \infty$$

where μ^x is the invariant measure of (1.4).

Proof. $(\mathcal{H}_{f.s.})$ imply that $V(y) := 1 + |y|^2$ is a uniformly in x function satisfying the hypothesis of Theorem 1 in [59], from which the claim follows. □

We have as well a control on the moments of any order of $\tilde{Y}_k^{x,q}$.

Proposition 1.8. *Let $\pi > 0$ and let $\tilde{Y}_k^{x,q}$ be given by (1.7). Then there exists a constant K_π given only by π, λ_-, λ_+ and γ_0 such that for all $x \in \mathbb{R}^{d_x}$ and $q \in \mathbb{Q}$*

$$\sup_{i\in\mathbb{N}} \mathbb{E}\left(|\tilde{Y}_i^{x,q}|^\pi\right) < K_\pi.$$

Moreover, for every $\pi > 1$,

$$\sup_{M\in\mathbb{N}} \left(\frac{1}{\Gamma_M}\sum_{i=1}^{M} \gamma_i |\tilde{Y}_i^{x,q}|^\pi\right) < +\infty$$

Proof. By Lemma 2 in [59] given that U_k^q has moments of any order and $V(y) = |y|^2 + 1$ satisfies the needed hypothesis uniformly in x, we get that for any $\pi \ge 1$ and $q \in \mathbb{Q}$,

$$\sup_{i\in\mathbb{N}} \mathbb{E}\left(|\tilde{Y}_i^{x,q}|^{2\pi}\right) \le \sup_{i\in\mathbb{N}} \mathbb{E}\left(V(\tilde{Y}_i^{x,q})^\pi\right) < K_\pi.$$

The extension to all $\pi > 0$ is straightforward.

The second claim is a consequence of Theorem 3 in [59]. □

Proposition 1.9 is an adaptation of an expansion result appearing in the PhD thesis ([65]). The proof comes from performing a Taylor expansion and reordering the terms in a proper way.

Remark. Proposition 1.9, introduces in addition to the sequence $\gamma_{\{k\ge0\}}$ a new sequence that we we will denote by $\eta_{\{k\ge0\}}$ (that, of course, may be taken equal to the former). This added flexibility will be useful in the following, in particular to prove Proposition (1.10). We may interpret Proposition 1.9 as an error expansion result. Indeed if we fix $\eta_k = \gamma_k$ satisfying $(\mathcal{H}_\gamma)$ then we will have an explicit expression for the approximation error of the decreasing Euler algorithm.

Proposition 1.9. *Let $\psi : \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \to \mathbb{R}$. Under the assumptions of Proposition 1.7, suppose that for each $x \in \mathbb{R}^{d_x}$ there exists ϕ_ψ^x solution of the centered Poisson equation*

$$\mathcal{L}_y^x \phi_\psi^x(y) = \psi(x,y) - \int \psi(x,z)\mu^x(dz) \tag{1.17}$$

Suppose as well for $r \in \mathbb{N}$, $r \geq 2$, that ϕ_ψ^x is C^r in the y-variable uniformly in x, and $D^r\phi_\psi$ is Lipschitz in y uniformly in x. Let γ_k and η_k be two decreasing sequences with $\gamma_k \to 0$, $\eta_k \to 0$, $\Gamma_k = \sum_{1\leq j\leq k} \gamma_k$, $\mathcal{H}_k = \sum_{1\leq j\leq k} \eta_k$. Let $\tilde{Y}_k^{x,q}$ be defined as in (1.7) (with step sequence γ_k). Then,

$$\sum_{k=1}^M \eta_k \left(\psi(x, \tilde{Y}_{k-1}^{x,q}) - \int \psi(x,z)\mu^x(dz) \right) = A_{\psi,M}^0 - N_{\psi,M} - \sum_{i=2}^r A_{\psi,M}^i - Z_{\psi,M}^r$$

where

$$A_{\psi,M}^0(x,q) := \sum_{k=1}^M \frac{\eta_k}{\gamma_k} \left[\phi_\psi^x(\tilde{Y}_k^{x,q}) - \phi_\psi^x(\tilde{Y}_{k-1}^{x,q}) \right] \tag{1.18}$$

$$N_{\psi,M}(x,q) := \sum_{k=1}^M \frac{\eta_k}{\sqrt{\gamma_k}} \langle D_y\phi_\psi^x(\tilde{Y}_{k-1}^{x,q}), \sigma(x, \tilde{Y}_{k-1}^{x,q}) U_k^q \rangle \tag{1.19}$$

$$\begin{aligned} A_{\psi,M}^2(x,q) := \frac{1}{2} \sum_{k=1}^M \eta_k \Big[& D^2\phi_\psi^x(\tilde{Y}_{k-1}^{x,q}) \cdot (\sigma(x, \tilde{Y}_{k-1}^{x,q}) U_k^q)^{\otimes 2} \\ & - Tr\left(D^2\phi_\psi^x(\tilde{Y}_{k-1}^{x,q})(\sigma^*\sigma(x, \tilde{Y}_{k-1}^{x,q})) \right) \Big] \end{aligned} \tag{1.20}$$

$$A_{\psi,M}^i(x,q) := \sum_{k=1}^M \eta_k \gamma_k^{i/2-1} v_\psi^{i,r}(x, \tilde{Y}_{k-1}^{x,q}, U_k^q) \tag{1.21}$$

for $i = 3, \ldots, r$ with

$$v_\psi^{i,r}(x,y,z) = \sum_{j\geq i/2}^{i\wedge r} \binom{j}{i-j} \frac{1}{j!} D_y^j \phi_\psi^x(y) \cdot \left\langle b(x,y)^{\otimes(i-j)}, (\sigma(x,y)z)^{\otimes(2j-i)} \right\rangle;$$

and

$$|Z_{\psi,M}^r|(x,q) \leq K \sum_{k=1}^M \eta_k \gamma_k^{\frac{r-1}{2}} (1 + |\tilde{Y}_{k-1}^{x,q}|^{r+1})(1 + |U_k^q|)^{r+1}. \tag{1.22}$$

The average of each expansion term will play an important role in our analysis, so that we will present a special notation for them. Indeed, let

$$\begin{aligned} \bar{v}_\psi^{i,r}(x,y) &:= \mathbb{E}\left(v_\psi^{i,r}(x,y,U_1^0) \right) \\ &= \sum_{j\geq i/2}^{i\wedge r} \binom{j}{i-j} \frac{1}{j!} D_y^j \phi_\psi^x(y) \mathbb{E}\left(\left\langle b(x,y)^{\otimes(i-j)}, (\sigma(x,y)U_1^0)^{\otimes(2j-i)} \right\rangle \right) \\ &= \sum_{j\geq i/2}^{i\wedge r} \binom{j}{i-j} \frac{1}{j!} D_y^j \phi_\psi^x(y) \mathbb{E}\left(\left\langle b(x,y)^{\otimes(i-j)}, (\sigma(x,y)U_k^q)^{\otimes(2j-i)} \right\rangle | \tilde{F}_{\Gamma_{k-1}} \right) \end{aligned} \tag{1.23}$$

Remark. Consider $A_{\psi,M}^{2i+1}$ for $i \leq \lfloor (r-1)/2 \rfloor$. As $2j - 2i - 1$ is odd for any j integer and given the fact that the odd powers of a centered Gaussian are centered we deduce $\bar{v}_\psi^{2i+1,r} = 0$. Of course this property transfers to $A_{\psi,M}^{2i+1}$ so that $\mathbb{E}\left(A_{\psi,M}^{2i+1}\right) = 0$, implying in turn that the terms with an odd index are centered.

Under some additional hypothesis, Proposition 1.9 may be used to obtain a L_2 control on the error of the approximation.

Proposition 1.10. *Under the assumptions of Proposition 1.9, let $\alpha \geq 1$. Assume $\{\gamma_k\}$ satisfies $(\mathcal{H}_\gamma)$, and that $\Gamma_M^{[\alpha]} \to \infty$. Assume as well that the solution of the centered Poisson equation ϕ_ψ is in $C_{b,p}^{2,r}$ for $r > 3$. Let $\bar{\Psi} := \int \psi(x,z)\mu^x(dz)$, then*

$$\mathbb{E}\left(\left|\frac{1}{\Gamma_M^{[\alpha]}}\sum_{k=1}^{M}\gamma_k^\alpha\left(\psi(x,\tilde{Y}_{k-1}^{x,q}) - \bar{\Psi}(x)\right)\right|^2\right) \leq K\frac{1+\Gamma_M^{[2\alpha-1]}+\Gamma_M^{[2\alpha]}+(\Gamma_M^{[\alpha+1]})^2}{(\Gamma_M^{[\alpha]})^2}$$

Proof. We recall first some martingale inequalities. Let $\{a_k\}$ be any sequence of random tensors. By Cauchy-Schwarz inequality we have that

$$\mathbb{E}\left(\left|\sum_{k=1}^{M}\gamma_k^p a_k\right|^2\right) \leq \mathbb{E}\left(\Gamma_M^{[p]}\sum_{k=1}^{M}\gamma_k^p|a_k|^2\right) = \Gamma_M^{[p]}\sum_{k=1}^{M}\gamma_k^p\mathbb{E}\left(|a_k|^2\right). \tag{1.24}$$

Let $\{b_k\}$ be also a sequence of tensors. If $s_0 < s_1 < \ldots < s_k < \ldots$, the $\{a_k\},\{b_k\}$ are $\tilde{\mathcal{F}}_{s_k}^q$ adapted and for all k, $\mathbb{E}\left(a_k|\tilde{\mathcal{F}}_{s_k}^q\right) = \mathbb{E}\left(b_k|\tilde{\mathcal{F}}_{s_k}^q\right) = 0$, we have by martingale properties that

$$\mathbb{E}\left(\left\langle\sum_{k=1}^{M}\gamma_k^p a_k, \sum_{k=1}^{M}\gamma_k^p b_k\right\rangle\right) = \sum_{k=1}^{M}\gamma_k^{2p}\mathbb{E}\left(\langle a_k, b_k\rangle\right); \tag{1.25}$$

and in particular

$$\mathbb{E}\left(\left|\sum_{k=1}^{M}\gamma_k^p a_k\right|^2\right) = \sum_{k=1}^{M}\gamma_k^{2p}\mathbb{E}\left(|a_k|^2\right). \tag{1.26}$$

Now, take the error expansion in Proposition 1.9 with $r = 3$ and Let $\eta_k = \gamma_k^\alpha$. By Abel's transformation, using convexity, the estimate (1.24), the regularity properties of ϕ_ψ and Proposition 1.8, we get

$$\begin{aligned}
\mathbb{E}\left(|A_{\psi,M}^0(x,q))|^2\right) &= \mathbb{E}\left(\left|\sum_{k=1}^{M}\gamma_k^{\alpha-1}\left[\phi_\psi^x(\tilde{Y}_k^{x,q}) - \phi_\psi^x(\tilde{Y}_{k-1}^{x,q})\right]\right|^2\right) \\
&= \mathbb{E}\left(\left|\gamma_M^{\alpha-1}\phi_\psi^x(\tilde{Y}_M^{x,q}) - \gamma_0^{\alpha-1}\phi_\psi^x(\tilde{Y}_0^{x,q}) + \sum_{k=1}^{M-1}\left[(\gamma_k^{\alpha-1}-\gamma_{k+1}^{\alpha-1})\phi_\psi^x(\tilde{Y}_k^{x,q})\right]\right|^2\right) \\
&\leq 3\mathbb{E}\left(\left|\gamma_M^{\alpha-1}\phi_\psi^x(\tilde{Y}_M^{x,q})\right|^2\right) + 3\mathbb{E}\left(\left|\gamma_0^{\alpha-1}\phi_\psi^x(\tilde{Y}_0^{x,q})\right|^2\right) \\
&\quad + 3\mathbb{E}\left(\left|\sum_{k=1}^{M-1}\left[(\gamma_k^{\alpha-1}-\gamma_{k+1}^{\alpha-1})\phi_\psi^x(\tilde{Y}_k^{x,q})\right]\right|^2\right) \\
&\leq K\left[(\gamma_M^{\alpha-1})^2 + 1 + \left(\sum_{k=1}^{M-1}(\gamma_k^{\alpha-1}-\gamma_{k+1}^{\alpha-1})\right)^2\right] \leq K
\end{aligned} \tag{1.27}$$

Moreover using the fact that the terms are centered from Remark 3.3, equation (1.26) and

the finite moments of the Brownian increments imply

$$\mathbb{E}\left(|N_{\psi,M}(x,q)|^2\right) = \sum_{k=1}^{M} \gamma_k^{2\alpha-1} \mathbb{E}\left(|\langle \sigma^* D_y \phi_\psi(x, \tilde{Y}_{k-1}^{x,q}), U_k^q\rangle|^2\right) \leq K\Gamma_M^{2\alpha-1} \tag{1.28}$$

$$\begin{aligned}\mathbb{E}\left(|A_{\psi,M}^2(x,q)|^2\right) &\leq \frac{1}{4}\sum_{k=1}^{M} \gamma_k^{2\alpha} \mathbb{E}\left(|D_y^2\phi(x, \tilde{Y}_{k-1}^{x,q}) \cdot (\sigma(x, \tilde{Y}_{k-1}^{x,q}) U_k^q)^{\otimes 2})|^2\right) \\ &\leq K\Gamma_M^{[2\alpha]}\end{aligned} \tag{1.29}$$

More generally, estimate (1.26) leads to

$$\mathbb{E}\left(|A_{\psi,M}^3(x,q)|^2\right) = \sum_{k=1}^{M} \gamma_k^{2\alpha+1} \mathbb{E}\left(|v_{F^i}^{3,3}(x, \tilde{Y}_{k-1}^{x,q}, U_k^q)|^2\right) \leq K\Gamma_M^{[2\alpha+1]}, \tag{1.30}$$

while by virtue of (1.24), we find as estimate

$$\mathbb{E}\left(|Z_{\psi,M}^3(x,q)|^2\right) \leq K\mathbb{E}\left(\left|\sum_{k=1}^{M} \gamma_k^{\alpha+1}(1+|\tilde{Y}_{k-1}^{x,q}|^4)(1+|U_k^q|)^4\right|^2\right) \leq K(\Gamma_M^{[\alpha+1]})^2 \tag{1.31}$$

On the other hand, from $(\mathcal{H}_\gamma)$ and given that $\Gamma_M^{[\alpha]} \to \infty$, we have for M large enough that if $i > j$

$$\frac{\Gamma_M^{[i]}}{\Gamma_M^{[\alpha]}} \leq \frac{\Gamma_M^{[j]}}{\Gamma_M^{[\alpha]}}.$$

The claim follows from Proposition 1.9 and equations (1.27),(1.28),(1.29),(1.30),(1.31). □

3.4 Ergodic average and Poisson equation

Being basic to our analysis, we introduce in this section some known properties of the exact averages and the effective diffusion. These results appear in Pardoux and Veretennikov [83] and [84]. They complement the main limit approximation result mentioned in the introduction, Theorem 0.9.

Let us start with some control results. In [97] it is shown that for any given α, we have a uniform bound for the moment of order α of the marginals of Y^x solution of (1.4).

Proposition 1.11. *Let $\alpha > 0$ and let Y_t^x be the solution of (1.4) with deterministic initial condition y_0 and coefficients satisfying $(\mathcal{H}_{f.s.})$.*

Then there exists a constant K given only by α, λ_-, λ_+ such that for all $t \geq 0$ and $x \in \mathbb{R}^{d_1}$

$$\mathbb{E}\left(|Y_t^x|^\alpha\right) < K(1+|y_0|^{\alpha+2}).$$

This proposition has a natural corollary

Corollary 1.12. *Under the same hypothesis of the theorem, for any $\alpha > 0$ and all $x \in \mathbb{R}^{d_1}$*

$$\int |y|^\alpha \mu^x(dy) < K.$$

Lemma 1.13. *Let $\psi(x,y)$ be a function satisfying the regularity and growth conditions in $(\mathcal{H}_{s.s.})$, and let $\Psi(x) = \int \psi(x,y)\mu^x(dy)$, then $\Psi(x)$ is C_b^2.*

Proof. The claim follows from adapting Theorems 3 and 5 in [98] to the linear growth case: the needed equivalent results of convergence in total variation and control of expectations may be found in [79]. □

Proposition 1.9 shows that the solution of the centered Poisson equation (1.17) plays a special role in understanding the error expansion of the decreasing Euler algorithm. Proposition 1.14, which is an adaptation of Theorem 1 in [83] and [98], states some sufficient conditions for having the solution of such an equation when f belongs to a certain family of functions.

Proposition 1.14. *Consider a function $\psi(x,y)$ satisfying the regularity and growth conditions in $(\mathcal{H}_{s.s.})$ i), ii); and such that*

$$\int \psi(x,y)\mu^x(dy) = 0, \forall x.$$

Assume $(\mathcal{H}_{f.s.})$. Then, there exists a function $\phi_\psi(x,y)$, continuous in y and belonging to the class $\bigcap_{p>1} W^2_{p,loc}$ in y, such that for every $x \in \mathbb{R}^{d_x}$,

i) $\mathcal{L}^x_y \phi_\psi(x,y) = \psi(x,y)$,

ii) $\int \phi_\psi(x,y)\mu^x(dy) = 0$,

iii) $\phi_\psi \in C^{2,r^y}_{b,p}$.

This function is the unique solution up to an additive constant of the Poisson equation on the class of continuous and $\bigcap_{p>1} W^2_{p,loc}$ functions in y which are locally bounded and grow at most polynomially in $|y|$ as $|y| \to \infty$. Moreover, it has the representation

$$\phi_\psi(x,y) = -\int_0^\infty \mathbb{E}_{x,y}(\psi(x, Y^x_t))dt$$

4 Convergence results for the MsDS algorithm

We focus now on the study of the MsDS algorithm. First, we show that the proposed approximated coefficients (by means of Decreasing Euler step and Cholesky procedures) satisfy a growth control and error control properties. As a consequence, we will conclude on some regularity property of the approximated diffusion (1.9) and show its strong convergence towards (1.2). Then, we will study the limit error distribution property.

4.1 Existence, uniqueness, continuity

It follows from Hypothesis $(\mathcal{H}_{s.s.})$, $(\mathcal{H}_{f.s.})$, Proposition 1.11 and Proposition 1.3 that there exists a unique solution to equation 1.2, and that it has a continuous modification. We show the defined approximation has the same properties.

Proposition 1.15 uses the results of section 3 to show that, under the standing hypothesis, the coefficients of the approximated diffusion have finite moments of any order, and that its error with respect to the exact coefficients decrease as a power of the number of steps n.

Proposition 1.15. *Assume $(\mathcal{H}_{s.s.})$, $(\mathcal{H}_{f.s.})$ and $(\mathcal{H}_\gamma)$. Let $\beta_0 > 0$, and define $M(n)$ implicitly by $\Gamma_{M(n)} = C_0 n^{2\beta_0}$, where C_0 is some constant.*

i) There exist ϕ_f and ϕ_h solutions of the centered Poisson equations

$$\mathcal{L}_y^x \phi_f(x,y) = f(x,y) - \int f(x,y')\mu^x(dy'),$$
$$\mathcal{L}_y^x \phi_h(x,y) = h(x,y) - \int h(x,y')\mu^x(dy'),$$

ii) Let

$$\varsigma := \min_{l\geq 4, i=1,\dots d}(\bar{v}_{F^i}^{l,r^y} \neq 0) \wedge \min_{l\geq 4, i,j=1,\dots d}(\bar{v}_{H^{i,j}}^{l,r^y} \neq 0) \wedge (r^y+1) \tag{1.32}$$

(with the convention that $\min(\emptyset) = \infty$) and $\bar{v}_{F^i}^{l,r}, \bar{v}_{H^{i,j}}^{l,r^y}$ defined as in (1.23) applied to

$$F^1, \dots, F^{d_x}, H^{1,1}, \dots, H^{d_x,d_x}.$$

Assume that we have the asymptotic expansion

$$\frac{\Gamma_M^{[\varsigma/2]}}{\Gamma_M} = C_1 n^{-\beta_1} + o(n^{-\beta_1}) \tag{1.33}$$

for some $\beta_1 > 0$ and some constant C_1. Let

$$\beta := \beta_0 \wedge \beta_1. \tag{1.34}$$

Then $\tilde{F}^n$ (and respectively $\tilde{H}^n, \tilde{G}^n := \sqrt{\tilde{H}^n}$) satisfies for any $\alpha \in \mathbb{R}^+$ and $k = 0, \dots, n$

$$\begin{cases} \mathbb{E}\left(|\tilde{F}^n(x,t_k)|^\alpha\right) \leq K \\ \mathbb{E}\left(|\tilde{F}^n(x,t_k) - F(x)|^2\right) \leq K n^{-2\beta}. \end{cases}$$

Remark. We should understand ς as marking the first non-zero value in the error expansion of either $\tilde{F}^n$ or $\tilde{H}^n$, depends exclusively on the coefficients of the effective and ergodic diffusion (in particular it does not depend on n).

Remark. Proposition 1.15 means that we have a rate of convergence in norm L_2 for the coefficient estimators of order $O(n^{-\beta})$. Since we *choose* β_0 by taking $M(n)$ as needed, the actual limit to β comes from β_1. But of course, increasing β_0 implies growing M faster as a function of n, increasing the algorithm's cost.

Proof of Proposition 1.15. Note first that *i)* follows from $(\mathcal{H}_{s.s.})$ and Proposition 1.14.

We prove *ii)*. By Jensen inequality and Proposition 1.8 we have for every $\alpha \geq 1$ and n big enough,

$$\mathbb{E}\left(|\tilde{F}^n(x,q)|^\alpha\right) = \mathbb{E}\left(\left|\frac{1}{\Gamma_M}\sum_{k=1}^M \gamma_k f(x, \tilde{Y}_{k-1}^{x,q})\right|^\alpha\right) \leq \mathbb{E}\left(\frac{1}{\Gamma_M}\sum_{k=1}^M \gamma_k |f(x, \tilde{Y}_{k-1}^{x,q})|^\alpha\right) \leq K,$$

and similarly for every $\alpha \geq 2$

$$\mathbb{E}\left(|\tilde{G}^n(x,q)|^\alpha\right) = \mathbb{E}\left(|\tilde{H}^n(x,q)|^{\alpha/2}\right) \leq K,$$

since $|G|^2 = |H|$. The result extends trivially to every $\alpha > 0$.

It remains to prove the error control. We obtain an expansion of order r^y in Proposition 1.9. We can bound the first terms as we did in Proposition 1.10 by taking $\gamma_k = \eta_k$ for all $k = 1, \dots, M$

(i.e. taking $\alpha = 1$ in the statement of Proposition 1.10). More generally, from the definition of ς in (1.32), we have that for every $l < \varsigma$ or l odd $\bar{v}^{l,r^y}_{F^i}(x,y) = 0$, hence (1.26) leads to

$$\mathbb{E}\left(|A^l_{F^i,M}(x,q)|^2\right) = \sum_{k=1}^{M} \gamma_k^l \mathbb{E}\left(|v^{l,r^y}_{F^i}(x, \tilde{Y}^{x,q}_{k-1}, U^q_k)|^2\right) \leq K\Gamma^{[l]}_M, \tag{1.35}$$

while for even l with $l \geq \varsigma$, by virtue of (1.24), we find as estimate

$$\mathbb{E}\left(|A^l_{F^i,M}(x,q)|^2\right) \leq \Gamma_M^{l/2} \sum_{k=1}^{M} \gamma_k^{l/2} \mathbb{E}\left(|v^{l,r^y}_{F^i}(x, \tilde{Y}^{x,q}_{k-1}, U^q_k)|^2\right) \leq K(\Gamma^{[l/2]}_M)^2, \tag{1.36}$$

Likewise,

$$\begin{aligned}\mathbb{E}\left(|Z^{r^y}_{F^i,M}(x,q)|^2\right) &\leq K\mathbb{E}\left(\left|\sum_{k=1}^{M} \gamma_k^{r^y+1/2}(1+|\tilde{Y}^{x,q}_{k-1}|^{r^y+1})(1+|U^q_k|)^{r^y+1}\right|^2\right) \\ &\leq K(\Gamma^{[r^y+1/2]}_M)^2\end{aligned} \tag{1.37}$$

Note that estimates (1.35) and (1.36) are uniform in x. On the other hand, from $(\mathcal{H}_\gamma)$, we have for M big enough and $l \leq r^y$ that

$$1 \geq \frac{\Gamma^{[2]}_M}{\Gamma_M} \geq \frac{\Gamma^{[3]}_M}{\Gamma_M} \geq \dots \frac{\Gamma^{[l]}_M}{\Gamma_M};$$

Hence, from Proposition 1.9 and equations (1.27),(1.28),(1.29),(1.35),(1.36),

$$\mathbb{E}\left(|\tilde{F}^{i;n}(x,q) - F^i(x,q)|^2\right) \leq \frac{K(\Gamma^{[\varsigma/2]}_M)^2}{(\Gamma_M)^2} + \frac{K}{\Gamma_M} \leq K' n^{-2(\beta_0 \wedge \beta_1)}.$$

implying our claim for F, $\tilde{F}^n$. Since H satisfies the same properties than F, the claim follows for $H, \tilde{H}^n$.

As a final step, we prove the error control for $\tilde{G}^n$. Let us denote $\Delta H^n(x,q) := H(x) - \tilde{H}^n(x,q)$. Let $E = \left\{|\Delta H^n(x,q)| \geq |2H^{-1}|^{-1}\right\}$. Markov inequality gives us the control

$$\mathbb{P}(E) \leq 4|H^{-1}(x)|^2 \mathbb{E}\left(|\Delta H^n(x,q)|^2\right) \leq K n^{-2\min(\beta_0,\beta_1)}$$

from this control and Theorem 1.4, we get

$$\begin{aligned}&\mathbb{E}\left(|G(x) - \tilde{G}^n(x,q)|^2\right) \\ &= \mathbb{E}\left(|G(x) - \tilde{G}^n(x,q)|^2 \mathbf{1}_E\right) + \mathbb{E}\left(|G(x) - \tilde{G}^n(x,q)|^2 \mathbf{1}_{E^\complement}\right) \\ &\leq K' n^{-2\min(\beta_0,\beta_1)} + \mathbb{E}\left(|G(x) - \tilde{G}^n(x,q)|^2 \mathbf{1}_{E^\complement}\right) \\ &\leq K' n^{-2\min(\beta_0,\beta_1)} + K n^{-2\min(\beta_0,\beta_1)} = K'' n^{-2\min(\beta_0,\beta_1)}.\end{aligned}$$

□

We can deduce from Proposition 1.15 and the assumed structure the following a priori estimates.

Corollary 1.16. *Under the hypothesis and notation of Proposition 1.15, for any $0 \leq s \leq T$,*

$$\mathbb{E}\left(|\tilde{F}^n(\tilde{X}^n_{\underline{s}}, \underline{s})|^\alpha\right) \leq K, \tag{1.38}$$

and

$$\mathbb{E}\left(|\tilde{F}^n(\tilde{X}^n_s, \underline{s}) - F(\tilde{X}^n_s)|^2\right) \leq K n^{-2\beta}. \tag{1.39}$$

The same bounds hold with $\tilde{F}^n, F$ replaced by $\tilde{H}^n, H$ and $\tilde{G}^n, G$.

Proof. Define

$$\bar{\mathcal{F}}_{t,t^-} := \left(\mathcal{F}_t \vee \bigvee_{q\in\mathbb{Q},q<t} \tilde{\mathcal{F}}^q_\infty \right) \tag{1.40}$$

by construction, $\tilde{X}_{\underline{s}}$ is $\bar{\mathcal{F}}_{\underline{s},\underline{s}^-}$ measurable and for any deterministic x, $\tilde{F}^n(x,\underline{s}) \perp\!\!\!\perp \bar{\mathcal{F}}_{\underline{s},\underline{s}^-}$ so that we get from Proposition 1.15

$$\mathbb{E}\left(|\tilde{F}^n(\tilde{X}_{\underline{s}},\underline{s})|^\alpha\right) = \mathbb{E}\left(\mathbb{E}\left(|\tilde{F}^n(\tilde{X}_{\underline{s}},\underline{s})|^\alpha \middle| \bar{\mathcal{F}}_{\underline{s},\underline{s}^-}\right)\right) \leq \mathbb{E}\left(K\right) = K.$$

A similar argument leads to (1.39), and to the claims for $\tilde{H}^n, H$ and $\tilde{G}^n, G$. □

Corollary 1.16 should be understood as an a priori control on the approximated process. From this control, we can deduce, using Proposition (1.3) as in the case of the effective equation, the existence and strong uniqueness of the solution of the approximated diffusion (1.9). In addition, Proposition 1.17 states that the approximation (1.9) has a continuous modification. The result follows from Proposition 1.11, the estimates in Corollary 1.16 and Kolmogorov's criterion.

Proposition 1.17. *Under the hypothesis and notation of Proposition 1.15, for every* $\alpha \geq 2$

$$\mathbb{E}\left(|\tilde{X}^n_t - \tilde{X}^n_s|^\alpha\right) \leq K_{\alpha,T}(t-s)^{\alpha/2}((t-s)^{\alpha/2}+1).$$

Moreover, the solution of (1.9) *has a continuous modification.*

4.2 Strong convergence

In what follows, we choose $\tilde{X}$ to be continuous in time. We can proceed to show the strong convergence of $\tilde{X}^n$ towards X.

Theorem 1.18. *Under* $(\mathcal{H}_{s.s.})$, $(\mathcal{H}_{f.s.})$ *and* $(\mathcal{H}_\gamma)$, *let* X *be defined by* (1.2) *and* $\tilde{X}^n$ *by* (1.9). *Let* β *be defined as in* (1.34). *Then,*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|X_t - \tilde{X}^n_t|^2\right) \leq Kn^{-(1\wedge 2\beta)}.$$

Proof. By definition

$$X_t - \tilde{X}^n_t = \int_0^t \left[F(X_s) - \tilde{F}^n(\tilde{X}^n_{\underline{s}},\underline{s})\right] ds + \int_0^t \left[G(X_s) - \tilde{G}^n(\tilde{X}^n_{\underline{s}},\underline{s})\right] dW_s.$$

Our plan is to use Proposition 1.3 *i*). By convexity

$$\begin{aligned}\left|F(X_s) - \tilde{F}^n\left(\tilde{X}^n_{\underline{s}},\underline{s}\right)\right|^2 &\\ \leq 3\left|F(X_s) - F\left(\tilde{X}^n_s\right)\right|^2 &+ 3\left|F\left(\tilde{X}^n_s\right) - F\left(\tilde{X}^n_{\underline{s}}\right)\right|^2 \\ + 3\left|F\left(\tilde{X}^n_{\underline{s}},\underline{s}\right) - \tilde{F}^n\left(\tilde{X}^n_{\underline{s}},\underline{s}\right)\right|^2&.\end{aligned}$$

By Lipschitz assumption in $(\mathcal{H}_{s.s.})$,

$$\begin{aligned}\mathbb{E}\left(\left|F(X_s) - F\left(\tilde{X}^n_s\right)\right|^2\right) &\leq K\mathbb{E}\left(\left|X_s - \tilde{X}^n_s\right|^2\right) \\ \mathbb{E}\left(\left|F\left(\tilde{X}^n_s\right) - F\left(\tilde{X}^n_{\underline{s}}\right)\right|^2\right) &\leq K\mathbb{E}\left(\left|\tilde{X}^n_s - \tilde{X}^n_{\underline{s}}\right|^2\right) \leq Kn^{-1},\end{aligned}$$

the last inequality being possible for n large enough thanks to Proposition 1.17. Also, by Corollary 1.16 , we get

$$\mathbb{E}\left(\left|F\left(\tilde{X}^n_{\underline{s}},\underline{s}\right)-\tilde{F}^n\left(\tilde{X}^n_{\underline{s}},\underline{s}\right)\right|^2\right)\leq Kn^{-2\beta}.$$

Therefore

$$\mathbb{E}\left(\left|F(X_s)-\tilde{F}^n\left(\tilde{X}^n_{\underline{s}},\underline{s}\right)\right|^2\right)\leq K\left(n^{-(1\wedge\beta)}+\mathbb{E}\left(|X_s-\tilde{X}^n_s|^2\right)\right), \tag{1.41}$$

Since we may obtain similar bounds for the terms with G, we also have

$$\mathbb{E}\left(\left|G(X_s)-\tilde{G}^n\left(\tilde{X}^n_{\underline{s}},\underline{s}\right)\right|^2\right)\leq K\left(n^{-(1\wedge\beta)}+\mathbb{E}\left(|X_s-\tilde{X}^n_s|^2\right)\right). \tag{1.42}$$

Now, Proposition 1.3 *i*) shows

$$\mathbb{E}\left(|X_t-\tilde{X}^n_t|^2\right)\leq K\int_0^T\left(n^{-(1\wedge\beta)}+\mathbb{E}\left(|X_s-\tilde{X}^n_s|^2\right)\right)ds,$$

Therefore, by Gronwall's lemma,

$$\sup_{0\leq t\leq T}\mathbb{E}\left(|X_t-\tilde{X}^n_t|^2\right)\leq Kn^{-(1\wedge 2\beta)}.$$

Replacing on (1.41) and (1.42) we get

$$\sup_{0\leq t\leq T}\left(\mathbb{E}\left(\left|F(X_s)-\tilde{F}^n\left(\tilde{X}^n_{\underline{s}},\underline{s}\right)\right|^2\right)+\mathbb{E}\left(\left|G(X_s)-\tilde{G}^n\left(\tilde{X}^n_{\underline{s}},\underline{s}\right)\right|^2\right)\right)\leq Kn^{-(1\wedge 2\beta)}.$$

So that by Proposition 1.3 *i*),

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|X_t-\tilde{X}^n_t|^2\right)\leq Kn^{-(1\wedge 2\beta)}.$$

□

4.3 Limit distribution

In this section we show that, under slightly stronger regularity assumptions on the coefficients of the diffusion, we have convergence in the weak (uniform topology) sense towards a limit distribution given as the solution of a particular SDE.

Our plan to prove the limit distribution result is to look at the rescaled error and its associated stochastic differential equation. We prove the joint weak convergence of the terms appearing in that SDE and use the fact that under certain hypothesis the joint convergence of the terms suffices to deduce the weak convergence of the solution of the equation. The reader may find most of the needed material on weak convergence of stochastic integrals and stochastic SDEs in [50], [54] and [56].

Definition 1.19. Let X^n be a sequence of $\mathbb{R}^d$-valued semimartingales and let $A^n(\delta)$ be the predictable process with finite variation null at zero and $M^n(\delta)$ the local martingale null at zero appearing in the representation of X^n as

$$X^n_t=X^n_0+A^n_t(\delta)+M^n_t(\delta)+\sum_{s\leq t}\Delta X^n_s\mathbf{1}_{\{|\Delta X^n_s|>\delta\}}.$$

We say that the sequence X^n satisfies property (*) if for some $\delta>0$

$$\langle M^n(\delta),M^n(\delta)\rangle_T+\int_0^T|dA^n(\delta)_s|+\sum_{s\leq T}|\Delta X^n_s|\mathbf{1}_{\{|\Delta X^n_s|>\delta\}} \tag{*}$$

is tight. (The notation $\int_0^T|dA|$ denotes the total variation of A on $[0,T]$)

The importance of property (*) is shown by the following theorem (see [50], [49] and [56]).

Theorem 1.20. *Let X^n be a sequence of $\mathbb{R}^d-$ valued semimartingales relative to the filtration $\mathcal{F}_t$. Suppose that X^n weakly converges in the Skorokhod topology $D_{\mathbb{R}^{d_x}}$. Then (*) is necessary and sufficient for goodness: for any sequence H^n of $(\mathcal{F}_t)-$ adapted càdlàg processes such that $(H^n, X^n) \Rightarrow (H, X)$ in the Skorokhod topology $D_{M^{d_x\times d_x}\times\mathbb{R}^{d_x}}$, then X is a semimartingale w.r.t the filtration generated by (H, X) and $(H^n, X^n, \int H^n dX^n) \Rightarrow (H, X, \int HdX)$ in the Skorokhod topology $D_{M^{d_x\times d_x}\times\mathbb{R}^{d_x}\times\mathbb{R}^{d_x}}$.*

Goodness gives us a direct way to show the convergence of sequences of stochastic integrals, and will play a key role for the convergence of sequences of SDEs.

Before proceeding to the main propositions of this section, we cite another useful result concerning weak convergence of sequences of solutions of SDEs, allowing to compare the limit of two sequences with converging coefficients.

Theorem 1.21 (Theorem 2.5 (b) [49]). *Consider a sequence of linear SDEs*

$$\vartheta_t^n = P_t^n + \int_0^t \vartheta_{s-}^n Q_t^n dJ_t \tag{1.43}$$

where the P_t^n are stochastic processes in $\mathbb{R}^d$, Q_t^n are stochastic processes in $\mathbb{R}^{d\times d'}$ and the J_t^n are semimartingales in $\mathbb{R}^{d'}$, and all processes are in same the filtered probability space. Suppose that we have another sequence of equations like (1.43) with solution ϑ'^n and coefficients P'^n and Q'^n. If the sequences $\sup_{0\le s\le T}\|P_s^n\|$ and $\sup_{0\le s\le T}\|Q_s^n\|$ are tight and if

$$\sup_{0\le s\le T}\|P_s^n - P_s'^n\| \xrightarrow{P} 0 \qquad \sup_{0\le s\le T}\|Q_s^n - Q_s'^n\| \xrightarrow{P} 0$$

then,

$$\sup_{0\le s\le T}\|\vartheta_s^n - \vartheta_s'^n\| \xrightarrow{P} 0$$

Proposition 1.22 shows the weak convergence of some tuples appearing in the rescaled error SDE.

Proposition 1.22. *Let $\mathcal{I}$ be a set of indices, and consider a family of independent standard Gaussian variables $\{\nu_{t_k}^{i;n}\}_{n\in\mathbb{N}^*;0\le k\le n;i\in\mathcal{I}}$ where for any n, i we have $\nu_{t_k}^{i;n}$ is $\bar{\mathcal{F}}_{t_k}$ measurable.*

Consider the sequence of random processes $A^{0;n}$ (dimension 1), $A^{1;n}, B^{0;n}$ (dimension d_x), $B^{2;n}$ (dimension $d_x\times d_x$), $B^{1;n}$ (dimension $|\mathcal{I}|$) and $B^{3;n}$ (dimension $|\mathcal{I}|\times d_x$) defined componentwise by

$$B_t^{0;j;n} := \int_0^t (s-\underline{s})dW_s^j; \qquad A_t^{0;n} := 2\int_0^t (s-\underline{s})ds; \tag{1.44}$$

$$B_t^{2;l,j;n} := \int_0^t \sqrt{2}(W_s^l - W_{\underline{s}}^l)dW_s^j; \qquad A_t^{1;j;n} := \int_0^t (W_s^j - W_{\underline{s}}^j)ds; \tag{1.45}$$

$$B_t^{3;i,j;n} := \int_0^t \nu_{\underline{s}}^{i;n} dW_s^j; \qquad B_t^{1;i;n} := \int_0^t \nu_{\underline{s}}^{i;n} ds; \tag{1.46}$$

Then, we have the following limit results

$$(X, \tilde{X}^n, W, nA^{0;n}, \sqrt{n}B^{1;n}) \Rightarrow (X, X, W, A^0, B^1) \tag{1.47}$$

$$(X, \tilde{X}^n, W, n^{\frac{1}{2}}A^{0;n}, n^{\frac{1}{2}}B^{0;n}, n^{\frac{1}{2}}A^{1;n}, n^{\frac{1}{2}}B_s^{2;n}, B^{1;n}, B^{3;n}) \Rightarrow (X, X, W, 0, 0, 0, B^2, 0, B^3) \tag{1.48}$$

where $A_t^0 = t$; B^0, B^1, B^2 and B^3 are standard Brownian motions defined on an extension of the space W, with dimensions d_x, d_x^2, $|\mathcal{I}| \times d_x$ and $|\mathcal{I}|$ respectively.

Moreover, we have $\{B^0, B^2, B^3, W\}$ are independent; $\{B^0, B^2, B^1, W\}$ are independent, and $B^{1;n}$, $\sqrt{n}B^{2;n}$ and $B^{3;n}$ are "good" in the sense of Theorem 1.20.

The proof of Proposition 1.22 will be given in section 7.1.

Proposition 1.23. *Under the assumptions and notation of Proposition 1.15, assume that $r^y > \varsigma + 3$ in $(\mathcal{H}_{s.s.})$, and that there is $\beta_2 \geq 0$ such that we have the asymptotic expansion*

$$\frac{\Gamma_M^{[\varsigma/2+1]}}{\Gamma_M^{[\varsigma/2]}} = C_2 n^{-\beta_2} + o(n^{-\beta_2}) \tag{1.49}$$

where ς is defined in (1.32). Let

$$\rho = \mathbf{1}_{\{\beta_0 > \beta_1\}}(\beta_2 \wedge (\beta_0 - \beta_1)) + \mathbf{1}_{\{\beta_0 < \beta_1\}}(\beta_0 \wedge (\beta_1 - \beta_0)). \tag{1.50}$$

i) Let Φ_F be the $d_x \times d_x$ matrix defined component-wise as

$$\Phi_F^{i,j}(x) := C_0^{-1} \int \langle \sigma^* D_y \phi_{F^i}(x,y), \sigma^* D_y \phi_{F^j}(x,y) \rangle \mu^x(dy),$$

where ϕ_{F^i} is the solutions of the Poisson equation (1.17) with source F^i. Let

$$\varphi_F(x) := \mathbf{1}_{\{\beta_1 \geq \beta_0\}} \sqrt{\Phi_F(x)}; \qquad R_{F^i}(x) := -\mathbf{1}_{\{\beta_0 \geq \beta_1\}} C_1 \int \bar{v}_{F^i}^{\varsigma, r^y}(x,y) \mu^x(dy),$$

with the square root meaning the Cholesky root. Then, there exist a family of independent standard Gaussian variables $\{\nu_k^{i;n}\}_{n \in \mathbb{N}^*; 0 \leq k \leq n; 1 \leq i \leq d_x}$, such that each $\nu_k^{i;n}$ is $\bar{\mathcal{F}}_{t_k}$ measurable and

$$\mathbb{E}\left(\left| n^\beta \left(F^i(x) - \tilde{F}^{i;n}(x, t_k) \right) - \sum_{j=1}^{d_x} \varphi_F^{i,j}(x) \nu_k^{j;n} - R_F^i(x) \right|^2 \right) = O(n^{-2\rho}),$$

for all $x \in \mathbb{R}^{d_x}$.

ii) In the SDE case ($h \neq 0$), define in a similar way a d_x^2 dimensional random function R_H and a $d_x^2 \times d_x^2$ dimensional random function Φ_H, with

$$\Phi_H^{i,j,i',j'}(x) := C_0^{-1} \int \langle \sigma^* D_y \phi_{H^{i,j}}(x,y), \sigma^* D_y \phi_{H^{i',j'}}(x,y) \rangle \mu^x(dy),$$

$$\varphi_H(x) := \mathbf{1}_{\{\beta_1 \geq \beta_0\}} \sqrt{\Phi_H(x)}; \qquad R_H^{i,j}(x) := -\mathbf{1}_{\{\beta_0 \geq \beta_1\}} C_1 \int \bar{v}_{H^{i,j}}^{\varsigma, r^y}(x,y) \mu^x(dy),$$

Then, there exist a family of independent standard Gaussian variables $\{\nu_k^{i,j;n}\}_{n \in \mathbb{N}^*; 0 \leq k \leq n; 0 \leq i,j \leq d_x}$, such that each $\nu_k^{i,j;n}$ is $\bar{\mathcal{F}}_{t_k}$ measurable and

$$\mathbb{E}\left(\left| n^\beta \left(H^{i,j}(x) - \tilde{H}^{i,j;n}(x, t_k) \right) - \sum_{i',j'=1}^{d_x} \varphi_H^{i,j,i',j'}(x) \nu_k^{i',j';n} - R_H^{i,j}(x) \right|^2 \right) = O(n^{-2\rho}),$$

for all $x \in \mathbb{R}^{d_x}$. *Moreover, letting* R_G, φ_G *be defined component-wise for* $0 \le i', j' \le d_x$ *as*

$$R_G^{i,i} = \frac{R_H^{i,i} - 2\sum_{k=1}^{i-1} R_G^{i,k} G^{i,k}}{2G^{i,i}}, \qquad \varphi_G^{i,i,i',j'} = \frac{\varphi_H^{i,i,i',j'} - 2\sum_{j=1}^{i-1} \varphi_G^{i,j,i',j'} G^{i,j}}{2G^{i,i}};$$

and for $i > j$

$$R_G^{i,j} = \frac{R_H^{i,j} - R_G^{j,j} G^{i,j} - \sum_{l=1}^{j-1} (R_G^{j,l} G^{i,l} + R_G^{i,l} G^{j,l})}{G^{j,j}}$$
$$\varphi_G^{i,j,i',j'} = \frac{\varphi_H^{i,j,i',j'} - \varphi_G^{j,j,i',j'} G^{i,j} - \sum_{l=1}^{j-1} \left[\varphi_G^{j,l,i',j'} G^{i,l} + \varphi_G^{i,l,i',j'} G^{j,l}\right]}{G^{j,j}}.$$

Then,

$$\mathbb{E}\left(\left| n^\beta \left(G^{i,j}(x) - \tilde{G}^{i,j;n}(x,t_k)\right) - \sum_{i',j'=1}^{d_x} \varphi_G^{i,j,i',j'}(x) \nu_k^{i',j';n} - R_G^{i,j}(x)\right|^2\right) = O(n^{-2\rho}).$$

Proof. i) We prove the first claim. We use the expansion of Proposition 1.9 as in Proposition 1.15, and estimates (1.27), (1.28), (1.29), (1.35), (1.36); to get for any x that

$$\begin{aligned}\mathbb{E}\left(\left|(F^i(x) - \tilde{F}^{i;n}(x,q)) - \frac{1}{\Gamma_M}\left(N_{F^i,M}(x,q) + A^{(\varsigma)}_{F^i,M}(x,q)\right)\right|^2\right) \\ = O\left(\max\left\{\frac{K}{(\Gamma_M)^2}, \left(\frac{\Gamma_M^{[\varsigma/2+1]}}{\Gamma_M}\right)^2\right\}\right)\end{aligned} \tag{1.51}$$

Let us examine separately three cases depending on the relation between β_0 and β_1:

- If $\beta_0 > \beta_1$:

$$\frac{\Gamma_M^{[\varsigma/2]}}{\sqrt{\Gamma_M}} = \frac{\Gamma_M^{[\varsigma/2]}\sqrt{\Gamma_M}}{\Gamma_M} = \frac{C_1 n^{\beta_0-\beta_1}}{C_0} + o(n^{\beta_0-\beta_1}). \tag{1.52}$$

We can rescale (1.51) to get

$$\begin{aligned}\mathbb{E}\left(\left|\frac{\Gamma_M}{\Gamma_M^{[\varsigma/2]}}(F^i(x) - \tilde{F}^{i;n}(x,q)) - \frac{1}{\Gamma_M^{[\varsigma/2]}} N_{F^i,M}(x,q) + \frac{1}{\Gamma_M^{[\varsigma/2]}} A^{\varsigma}_{F^i,M}(x,q)\right|^2\right) \\ = O\left(\left(\frac{\Gamma_M^{[\varsigma/2+1]}}{\Gamma_M^{[\varsigma/2]}}\right)^2 + \left(\frac{1}{\Gamma_M^{[\varsigma/2]}}\right)^2\right)\end{aligned} \tag{1.53}$$

so that the difference converges uniformly towards 0 in L_2. And from (1.28) we have

$$\mathbb{E}\left(\left|\frac{1}{\Gamma_M^{[\varsigma/2]}} N_{F^i,M}(x,q)\right|^2\right) = O\left(\frac{\Gamma_M}{(\Gamma_M^{[\varsigma/2]})^2}\right). \tag{1.54}$$

Let us define

$$\bar{A}^{\varsigma}_{F^i,M}(x,q) := \sum_{k=1}^{M} \gamma_k^{\varsigma/2} \bar{v}_{F^i}^{\varsigma,r^y}(x, \bar{Y}_{k-1}^{x,q}) \tag{1.55}$$

for $\bar{v}_{F^i}^{\varsigma,r^y}$ defined in (1.23). We can compare $(\Gamma_M^{[\varsigma/2]})^{-1}A^{\varsigma}_{F^i,M}$ and $(\Gamma_M^{[\varsigma/2]})^{-1}\bar{A}^{\varsigma}_{F^i,M}$ in L_2 by (1.26). Indeed, thanks to controls (1.35) and (1.36), we have that

$$\begin{aligned}\mathbb{E}\left(\left|(\Gamma_M^{[\varsigma/2]})^{-1}\left(A^{\varsigma}_{F^i,M}(x,q)-\bar{A}^{\varsigma}_{F^i,M}(x,q)\right)\right|^2\right)&\\ =\mathbb{E}\left(\left|(\Gamma_M^{[\varsigma/2]})^{-1}\sum_{k=1}^{M}\gamma_k^{\varsigma/2}\left(v_{F^i}^{\varsigma,r^y}(x,\bar{Y}^{x,q}_{k-1},U^q_k)-\bar{v}_{F^i}^{\varsigma,r^y}(x,\bar{Y}^{x,q}_{k-1})\right)\right|^2\right)&\\ \leq\frac{\Gamma_M^{[\varsigma]}}{(\Gamma^{[\varsigma/2]})^2}=O\left(\frac{\Gamma_M}{(\Gamma_M^{[\varsigma/2]})^2}\right).&\quad(1.56)\end{aligned}$$

Substitute (1.56) and (1.54) in (1.53) to get

$$\begin{aligned}\mathbb{E}\left(\left|\frac{\Gamma_M}{\Gamma_M^{[\varsigma/2]}}(F^i(x)-\tilde{F}^{i;n}(x,q))+\frac{1}{\Gamma_M^{[\varsigma/2]}}\bar{A}^{\varsigma}_{F^i,M}(x,q)\right|^2\right)&\\ =O\left(\left(\frac{\Gamma_M^{[\varsigma/2+1]}}{\Gamma_M^{[\varsigma/2]}}\right)^2+\frac{\Gamma_M}{\left(\Gamma_M^{[\varsigma/2]}\right)^2}\right)=O(n^{-2\beta_2}+n^{2(\beta_1-\beta_0)}).&\end{aligned}$$

Hence, to prove our claim, it suffices to show that

$$\mathbb{E}\left(\left|\frac{\bar{A}^{\varsigma}_{F^i,M}(x,q)}{\Gamma_M^{[\varsigma/2]}}+R^i_F(x)\right|^2\right)=O(n^{-2\rho}).$$

Now, (1.52) and $\Gamma_M\to\infty$ imply $\Gamma_M^{[\varsigma/2]}\to\infty$. Moreover, from the assumed regularity hypothesis, $\bar{v}_{F^i}^{\varsigma,r^y}(x,y)$ is $C^{2,r^y-\varsigma}_{p,b}$. Therefore, Proposition 1.14 guarantees the existence of a solution to the centered Poisson equation with source $\bar{A}_{F^i}^{\varsigma,r^y}(x,y)$ of the same regularity, and thus 1.10 shows that $\bar{A}^{\varsigma}_{F^i,M}(x,q)$ converges uniformly with respect to x in L^2 to $-R^i_F(x)$ with rate $(\beta_0-\beta_1)\wedge\beta_2\geq\rho$ since

$$\begin{aligned}K\frac{1+\Gamma_M^{[\varsigma]}+\Gamma_M^{[\varsigma-1]}+(\Gamma_M^{[\varsigma/2+1]})^2}{(\Gamma_M^{[\varsigma/2]})^2}&\leq K\frac{\Gamma_M+(\Gamma_M^{[\varsigma/2+1]})^2}{(\Gamma_M^{\varsigma/2})^2}\\ &=O(n^{-2((\beta_0-\beta_1)\wedge\beta_2)}).\end{aligned}$$

- If $\beta_1>\beta_0$: In this case we get from (1.52) that $\frac{\Gamma_M^{[\varsigma/2]}}{\sqrt{\Gamma_M}}\to 0$. Moreover by rescaling (1.51) we get

$$\begin{aligned}\mathbb{E}\left(\left|\sqrt{\Gamma_M}(F^i(x)-\tilde{F}^{i;n}(x,q))-\frac{1}{\sqrt{\Gamma_M}}N_{F^i,M}(x,q)+\frac{1}{\sqrt{\Gamma_M}}A^{\varsigma}_{F^i,M}(x,q)\right|^2\right)&\\ =O\left(\frac{\left(\Gamma_M^{[\varsigma/2+1]}\right)^2}{\Gamma_M}+\frac{1}{\Gamma_M}\right)&\end{aligned}$$

But from (1.36)

$$\mathbb{E}\left(\left|\frac{1}{\sqrt{\Gamma_M}}A^{\varsigma}_{F^i,M}(x,q)\right|^2\right)=O\left(\frac{(\Gamma_M^{[\varsigma/2]})^2}{\Gamma_M}\right).$$

Hence,

$$\mathbb{E}\left(\left|\sqrt{\Gamma_M}(F^i(x)-\tilde{F}^{i;n}(x,q))+\frac{1}{\sqrt{\Gamma_M}}A^{\varsigma}_{F^i,M}(x,q)\right|^2\right)$$
$$=O\left(\frac{\left(\Gamma_M^{[\varsigma/2]}\right)^2}{\Gamma_M}+\frac{1}{\Gamma_M}\right)=O(n^{-2(\beta_1-\beta_0)}+n^{-2\beta_0})$$

So it remains to consider the N_M term. Note that since the U_k^q are independent standard Gaussian vectors, $(C_0\sqrt{\Gamma_M})^{-1}N_{F^i,M}(x,q)$ when i ranges $1,\dots,d_x$ is a Gaussian vector.

Let us study its covariance matrix Φ_F^n. Using (1.25) we get for $i,j=1,\dots,n$

$$\Phi_F^{i,j;n}(x,q):=\mathbb{E}\left(\frac{1}{\Gamma_M}N_{F^i,M}(x,q)N_{F^j,M}(x,q')\right)$$
$$=\mathbf{1}_{\{q=q'\}}\sum_{k,k'=1}^{M}\gamma_k\langle\sigma^*(\cdot)D_y\phi_{F^i}(\cdot),\sigma^*(\cdot)D_y\phi_{F^j}(\cdot)\rangle(x,\tilde{Y}^{x,q}_{k-1}).$$

Define $\varphi_F^n=\sqrt{\Phi_F^n}$ (the Cholesky decomposition). Then, there exists a family of independent gaussian variables $\nu_{t_k}^{i,j;n}$, $\bar{\mathcal{F}}_{t_k}$-measurable such that $(\Gamma_M)^{-1}N_{F^i,M}(x,q)=\sum_{j=1}^{d_x}\varphi^{i,j;n}\nu_{t_k}^{i,j;n}$. Moreover from Proposition 1.7 and Proposition 1.15 we have that $\Phi_F^n(x,q)$ converges uniformly in x in L^2 to $\Phi_F(x)$ as defined in the claim with rate $O(n^{-\beta})$. By Theorem 1.4 we get the same uniform convergence for φ_F^n. The claim follows in this case.

- The case $\beta_0=\beta_1$ is straightforward from what has been proven in the previous cases.

ii) Since $H,\tilde{H}^n$ satisfy the same properties as $F,\tilde{F}^n$, we get the claim for R_H, φ_H and $\nu_k^{i,j;n}$ by analogous arguments. Replacing this result in the sensitivity of the Cholesky procedure given in Lemma 1.5, and taking into account the independence of the Gaussian entries, we get the claim for R_G and φ_G.

□

Let $\{\upsilon_n\}$ be a sequence of increasing positive numbers and let us consider the sequence of rescaled error processes ζ^n, defined by

$$\zeta_t^n:=\upsilon_n(X_t-\tilde{X}_t^n).$$

We can show that this sequence of processes converges in distribution in the uniform convergence topology to a process ζ defined as the solution to a certain stochastic differential equation. We divide the analysis in two main cases: a first one in which $G(x)\equiv 0$, i.e. when X is the solution to an ordinary differential equation; and the case when $G(x)$ is non degenerate. Just as in the asymptotic error obtained for the usual stochastic Euler method given in [49], we will obtain different rates and different components in the equation for both cases.

Theorem 1.24 (Limit distribution). *Under the assumptions and notation of Proposition 1.23, let $\rho,R_F,\varphi_F,R_G,\varphi_G$ be defined as in Proposition 1.23 and β defined in (1.34).*

i) (ODE case- $G(x)\equiv 0$.) Let B^1 be the Brownian process given in Proposition 1.22. Let $r=\min\{1,1/2+\beta\}$, and suppose $\rho\geq r-\beta$. Let

$$\zeta_t^n:=n^r(X_t-\tilde{X}_t^n).$$

Then $\zeta^n \Rightarrow \zeta^\infty$ in the uniform convergence sense, where ζ^∞ is solution of the system

$$\zeta_t^{\infty,i} = \sum_{j=1}^{d_x} \left(\int_0^t \partial_{x^j} F^i(X_s) \zeta_s^{\infty,j} ds + \mathbf{1}_{\{\beta \geq 1/2\}} \frac{1}{2} \int_0^t \partial_{x^j} F^i(X_s) F^j(X_s) ds \right) + \mathbf{1}_{\{\beta \leq 1/2\}} \left(\int_0^t R_F^i(X_s) ds + \sum_{l=1}^{d_x} \int_0^t \varphi_F^{i,l}(X_s) dB_s^{1;l} \right). \quad (1.57)$$

ii) (SDE case - $G(x) \neq 0$) Let B^2 and B^3 be the independent Brownian processes given in Proposition 1.22. Let $r = \min\{1/2, \beta\}$ and

$$\zeta_t^n := n^r (X_t - \tilde{X}_t^n).$$

Then $\zeta^n \Rightarrow \zeta^\infty$, where ζ^∞ is solution of the system for $i = 1, \ldots, d_x$ of

$$\begin{aligned} \zeta_t^{\infty,i} = & \sum_j \left(\int_0^t \partial_{x^j} F^i(X_s) \zeta_s^{\infty,j} ds + \int_0^t R_G^{i,j}(X_s) dW_s^j \right) \\ & + \mathbf{1}_{\{\beta \leq 1/2\}} \sum_{j,k,l=1}^{d_x} \int_0^t \varphi_G^{i,j,l,k}(X_s) dB_s^{3;l,k,j} \\ & + \mathbf{1}_{\{\beta \leq 1/2\}} \sum_{j,l=1}^{d_x} \int_0^t \partial_{x^j} G^{i,l}(X_s) \zeta_s^{\infty,j} dW_s^l \\ & + \mathbf{1}_{\{\beta \geq 1/2\}} \frac{1}{\sqrt{2}} \sum_{j,k,l=1}^{d_x} \int_0^t \partial_{x^j} G^{i,l}(X_s) G^{j,k}(X_s) dB_s^{2;k,l}. \end{aligned} \quad (1.58)$$

Proof. a) Assume we are in the SDE case. We have from the definition of ζ^n that

$$\zeta_t^n = \int_0^t n^r \left(F(X_s) - \tilde{F}^n(\tilde{X}_{\underline{s}}^n, \underline{s}) \right) ds + \int_0^t n^r \left(G(X_s) - \tilde{G}^n(\tilde{X}_{\underline{s}}^n, \underline{s}) \right) dW_s. \quad (1.59)$$

Let us examine each one of these terms separately. Denoting by x^i the $i-th$ component of x, let $x, y \in \mathbb{R}^{d_x}$. We define the set of vectors $\Delta^j(x,y)$

$$\Delta^j(x,y) := \begin{cases} x & \text{for } j = 0 \\ (y_1, y_2, \ldots, y_j, x_{j+1}, x_{j+2}, \ldots, x_{d_x})^* & \text{for } 1 \leq j \leq d_x \end{cases}$$

and

$$\Delta^j F^i(x,y) := \mathbf{1}_{\{x^j \neq y^j\}} \left(\frac{F^i(\Delta^{j-1}(x,y)) - F^i(\Delta^j(x,y))}{x^j - y^j} \right) + \mathbf{1}_{\{x^j = y^j\}} \partial_{x^j} F^i(x),$$

and recalling that

$$\tilde{X}_s^{j,n} - \tilde{X}_{\underline{s}}^{j,n} = F^j(\tilde{X}_{\underline{s}}^n)(s - \underline{s}) + \sum_{l=1}^{d_x} G^{j,l}(\tilde{X}_{\underline{s}}^n)(W_s^l - W_{\underline{s}}^l),$$

we have

$$\begin{aligned}
&\int_0^t n^r \left[F^i(X_s) - \tilde{F}^{i;n}(\tilde{X}^n_{\underline{s}}, \underline{s})\right] ds \\
&\quad = \int_0^t n^r (F^i(X_s) - F^i(\tilde{X}^n_s)) ds + \int_0^t n^r \left(F^i(\tilde{X}^n_s) - F^i(\tilde{X}^n_{\underline{s}})\right) ds + \\
&\qquad + \int_0^t n^r \left(F^i(\tilde{X}^n_{\underline{s}}) - \tilde{F}^{i;n}(\tilde{X}^n_{\underline{s}}, \underline{s})\right) ds \\
&\quad = \int_0^t \sum_j \Bigg[n^r \Delta^j F^i(X_s, \tilde{X}^n_s)(X^j_s - \tilde{X}^{j;n}_s) \\
&\qquad + n^r \Delta^j F^i(\tilde{X}^n_s, \tilde{X}^n_{\underline{s}}) F^j(\tilde{X}^n_{\underline{s}})(s - \underline{s}) \\
&\qquad + \sum_{l=1}^{d_x} n^r \Delta^j F^i(\tilde{X}^n_s, \tilde{X}^n_{\underline{s}}) G^{j,l}(\tilde{X}^n_{\underline{s}})(W^l_s - W^l_{\underline{s}}) \Bigg] ds \\
&\qquad + n^{r-\beta} \int_0^t n^\beta \left(F^i(\tilde{X}^n_{\underline{s}}) - \tilde{F}^{i;n}(\tilde{X}^n_{\underline{s}}, \underline{s})\right) ds.
\end{aligned}$$

Following the same approach we obtain for each $l = 1 \ldots, d_x$

$$\begin{aligned}
&\int_0^t n^r \left[G^{i,l}(X_s) - \tilde{G}^{i,l;n}(\tilde{X}^n_{\underline{s}}, \underline{s})\right] dW^l_s. \\
&\quad = \int_0^t \sum_j \Bigg[n^r \Delta^j G^{i,l}(X_s, \tilde{X}^n_s)(X^j_s - \tilde{X}^{j;n}_s) \\
&\qquad + n^r \Delta^j G^{i,l}(\tilde{X}^n_s, \tilde{X}^n_{\underline{s}}) F^j(\tilde{X}^n_{\underline{s}})(s - \underline{s}) \\
&\qquad + \sum_{k=1}^{d_x} n^r \Delta^j G^{i,l}(\tilde{X}^n_s, \tilde{X}^n_{\underline{s}}) G^{j,k}(\tilde{X}^n_{\underline{s}})(W^k_s - W^k_{\underline{s}}) \Bigg] dW^l_s \\
&\qquad + n^{r-\beta} \int_0^t n^\beta \left(G^{i,l}(\tilde{X}^n_{\underline{s}}) - \tilde{G}^{i,l;n}(\tilde{X}^n_{\underline{s}}, \underline{s})\right) dW^l_s.
\end{aligned}$$

By identifying terms in the obvious way, we write

$$\zeta^{i,n}_t = (P^{i,n}_1(t) + P^{i,n}_2(t)) + \int_0^t \langle Q^{i;n}_1(s), \zeta^n_s \rangle ds + \sum_{l=1}^{d_x} \int_0^t \langle Q^{i,l;n}_2(s), \zeta^n_s \rangle dW^l_s.$$

where Q^i_1, $Q^{i,l}_2$ are d_x dimensional random processes with components

$$Q^{j,i;n}_1(s) = \Delta^j F^i(X_s, \tilde{X}^n_s) \qquad Q^{j,i,l;n}_2(s) = \Delta^j G^{i,l}(X_s, \tilde{X}^n_s)$$

and

$$\begin{aligned}
P_1^{i;n}(s) = \int_0^t \sum_j \Bigg[& n^r \Delta^j F^i(\tilde{X}_s^n, \tilde{X}_{\underline{s}}^n) F^j(\tilde{X}_{\underline{s}}^n)(s-\underline{s}) \\
& + \sum_{l=1}^{d_x} n^r \Delta^j F^i(\tilde{X}_s^n, \tilde{X}_{\underline{s}}^n) G^{j,l}(\tilde{X}_{\underline{s}}^n)(W_s^l - W_{\underline{s}}^l) \Bigg] ds \\
& + \int_0^t \sum_{j,l=1}^{d_x} \Bigg[n^r \Delta^j G^{i,l}(\tilde{X}_s^n, \tilde{X}_{\underline{s}}^n) F^j(\tilde{X}_{\underline{s}}^n)(s-\underline{s}) dW_s^l \\
& + \sum_{k=1}^{d_x} n^r \Delta^j G^{i,l}(\tilde{X}_s^n, \tilde{X}_{\underline{s}}^n) G^{j,k}(\tilde{X}_{\underline{s}}^n)(W_s^k - W_{\underline{s}}^k) \Bigg] dW_s^l \\
P_2^{i;n}(s) = & n^{r-\beta} \int_0^t n^\beta \left(F^i(\tilde{X}_{\underline{s}}^n) - \tilde{F}^{i;n}(\tilde{X}_{\underline{s}}^n, \underline{s}) \right) ds \\
& + n^{r-\beta} \int_0^t n^\beta \left(G^{i,l}(\tilde{X}_{\underline{s}}^n) - \tilde{G}^{i,l;n}(\tilde{X}_{\underline{s}}^n, \underline{s}) \right) dW_s^l.
\end{aligned}$$

b) In this step, we propose another diffusion with nicer coefficients and study its convergence. We claim that both diffusions share the same limit distribution. Let

$$\check{\zeta}_t^{i,n} = (\check{P}_1^{i,n}(t) + \check{P}_2^{i,n}(t)) + \int_0^t \langle \check{Q}_1^{i;n}(s), \check{\zeta}_s^n \rangle ds + \sum_{l=1}^{d_x} \int_0^t \langle \check{Q}_2^{i,l;n}(s), \check{\zeta}_s^n \rangle dW_s^l.$$

where

$$\check{Q}_1^{i;n}(s) = \nabla F^i(X_s) \qquad\qquad \check{Q}_2^{i,l;n}(s) = \nabla G^{i,l}(X_s)$$

$$\begin{aligned}
\check{P}_1^{i;n}(s) = & \frac{1}{2} \int_0^t n^r \left\langle \nabla F^i(X_s), F(X_s) \right\rangle dA^{0;n} \\
& + \sum_{l=1}^{d_x} \int_0^t n^r \langle \nabla F^i(X_s), G^{\cdot,l}(X_s) \rangle dA_s^{1;l;n} \\
& + \int_0^t n^r \langle \nabla G^{i,l}(X_s), F(X_s) \rangle dB_s^{0;l,n} \\
& + \sum_{k,l=1}^{d_x} \frac{1}{\sqrt{2}} \int_0^t n^r \langle \nabla G^{i,l}(X_s) G^{\cdot,k}(X_s) \rangle dB_s^{2;k,l,n},
\end{aligned}$$

$$\begin{aligned}
\check{P}_2^{i;n}(s) = & n^{r-\beta} \int_0^t \sum_{j,k,l=1}^{d_x} \varphi_G^{i,j,l,k}(X_s) dB_s^{3;l,k,j,n} + n^{r-\beta} \int_0^t \sum_{j=1}^{d_x} R_G^{i,j}(X_s) dW_s^j \\
& + n^{r-\beta} \int_0^t \sum_{j=1}^{d_x} \varphi_F^{i,j}(X_s) dB_s^{1;j,n} + n^{r-\beta} \int_0^t R_F^i(X_s) ds,
\end{aligned}$$

for $R_F, R_G, \varphi_F, \varphi_G$ defined in Proposition 1.23. By $(\mathcal{H}_{s.s.})$, F, G are bounded; by Lemma 1.13, ∇F and ∇G are well defined, bounded, and have bounded derivatives; and from the definition of $R_F, R_G, \varphi_F, \varphi_G$ are C_b^1.

Note that (1.48) in Proposition 1.22 gives us goodness and convergence of the tuple $(n^r A^{0,n}, n^r A^{1,n}, n^r B^{0,n}, B^{1,n}, n^r B^{2,n}, B^{3,n})$. Hence, by virtue of Theorem 5.4 in [54] $\check{\zeta}^n(\cdot \wedge \tau_K^n)$ is tight and any limit point will satisfy (1.58) on the interval $[0, \tau_K]$ where $\tau_K = (\inf\{t : |\zeta(t)| > K\} \wedge T)$. Moreover

$$\sup_{0\le s\le \tau_K} \|\check{P}_s^n\|, \qquad \sup_{0\le s\le \tau_K} \|\check{Q}_1^n(s)\|, \qquad \sup_{0\le s\le \tau_K} \|\check{Q}_2^n(s)\|;$$

are tight.

c) We prove now that both ζ^n and $\check{\zeta}^n$ have the same limit on the interval $[0, \tau_K]$. By Theorem 1.21, it suffices to prove that sup norm of the difference of the coefficients converge in probability. By Theorem 1.18 the regularity properties of F and the mean value theorem we have

$$\mathbb{E}\left(\sup_{0\le t\le \tau_K} |Q_1^{i,n}(t) - \check{Q}_1^{i,n}(t)|\right) \le \mathbb{E}\left(\sup_x |D^2 F(x)| \sup_{0\le t\le \tau_K} |X_t - \tilde{X}_t^n|\right) \to 0.$$

The terms of Q_2^n, P_1^n are treated in the same way. On the other hand, we get from Corollary 1.16, Proposition 1.3, and Burckholder-Davis-Gundy inequality that

$$\sup_{0\le t\le T}\left| n^{r-\beta}\int_0^t \left[n^\beta\left(F^i(\cdot) - \tilde{F}^{i;n}(\cdot,\underline{s})\right) - \sum_{j=1}^{d_x} \varphi_F^{i,j}(\cdot)\nu_{\underline{s}}^{j;n} - R_F^i(\cdot)\right](\tilde{X}_{\underline{s}}^n)ds,\right|$$

$$\sup_{0\le t\le T}\left| n^{r-\beta}\int_0^t \left[n^\beta\left(G^{i,j}(\cdot) - \tilde{G}^{i,j;n}(\cdot,\underline{s})\right) - \sum_{j,k=1}^{d_x} \varphi_G^{i,j,l,k}(\cdot)\nu_{\underline{s}}^{l,k;n} - R_G^{i,j}(\cdot)\right](\tilde{X}_{\underline{s}}^n)dW_s^l\right|$$

are tight and converge to zero.

Thus, by Theorem 1.21 we will have that $\zeta^{i;n}$ and $\check{\zeta}^{i;n}$ will converge to the same limit.

d) Finally, note that $\tau_K^n \to \infty$ and $\tau_K \to \infty$, proving our claim in the SDE case.

e) To prove *i*) it suffices to follow the same approach. We obtain an equivalent development for the ODE case (replacing by zero the "G-term"). The rest of the proof proceeds as before, this time using (1.47) for the weak convergence of the tuple.

□

Remark. If $\beta > 1/2$ in Theorem 1.24, the error of the Euler scheme dominates: we recover the limit distribution error for an Euler scheme with exact coefficients given in [55] or [49]. By contrast, if $\beta < 1/2$, it is the decreasing Euler estimate error that becomes dominant. Since a higher β is general only achieved by paying a higher price in the required number of steps for the decreasing Euler step, the optimal choice implies fixing $\beta = 1/2$.

Proof of Theorem 1.1. The result is obtained, from Theorems 1.18 and 1.24, by verifying the conditions imposed in those theorems.

The sequence defined as $\gamma_k = \gamma_0 k^{-\theta}$ for $0 < \theta < 1$ satisfies Hypothesis $(\mathcal{H}_\gamma)$. Recall that we fixed $M(n) = M_1 n^{1/(1-\theta)}$, and we have for n large enough

$$\Gamma_M \approx \frac{\gamma_0 M_1^{1-\theta} n}{1-\theta}, \qquad \frac{\Gamma_M^{[\varsigma/2]}}{\Gamma_M} \approx \frac{(1-\theta)M_1^{-(\varsigma/2-1)\theta} n^{-\frac{(\varsigma/2-1)\theta}{1-\theta}}}{1-\varsigma\theta/2},$$

$$\frac{\Gamma_M^{[\varsigma/2+1]}}{\Gamma_M^{[\varsigma/2]}} \approx \frac{(1-\varsigma\theta/2)M_1^{-\theta} n^{-\frac{\theta}{1-\theta}}}{1-(\varsigma/2+1)\theta};$$

So that we get from Proposition 1.23, that $\beta_0 = 1/2$ and

$$\beta_1 = \frac{(\varsigma/2-1)\theta}{1-\theta}, \qquad \beta_2 = \frac{\theta}{1-\theta}, \qquad C_0 \approx \frac{\gamma_0 M_1^{1-\theta}}{1-\theta}, \qquad C_1 \approx \frac{(1-\theta)M_1^{-\theta}}{1-2\theta}.$$

Recall that ς is defined in (1.32) and stands for the first non-zero term in the error expansion of the decreasing Euler estimator. Let us assume we are in the more general case when it attains its minimal value $\varsigma = 4$. Hence

$$\beta_1 = \frac{\theta}{1-\theta}, \qquad C_1 = \frac{(1-\theta)M_1^{-\theta}}{1-2\theta}.$$

Then, from the conditions of Theorem 1.24 we have for the ODE case

$$r = 1 \wedge \left(\frac{1}{2} + \beta\right) = \frac{1}{2} + \left(\frac{1}{2} \wedge \beta\right) = \frac{1}{2} + \left(\frac{1}{2} \wedge (\beta_0 \wedge \beta_1)\right) = \frac{1}{2} + \beta$$

since we should have $\rho \geq r - \beta = 1/2$, we must verify

$$|\beta_0 - \beta_1| = \left|\frac{1}{2} - \frac{\theta}{1-\theta}\right| \geq \rho \geq \frac{1}{2},$$

which is the case if $\theta \in [1/2, 1)$. Moreover, since in this case $\beta_1 \geq 1 > \beta_0 = 1/2$, we get $r = 1/2$, and the R_F term disappears.

For the SDE case, we have $r = \beta = \min(1/2, \theta/(1-\theta))$ the only restriction comes from imposing $\beta = 1/2$. This is obtained for $1/3 \leq \theta < 1$. Note that the R_G term is different from zero only if $\theta = 1/3$.

Let us consider now the case $\varsigma > 4$. From the constraints $\theta \in [1/2, 1)$ in the ODE case and $\theta \in [1/3, 1)$ in the SDE case, we deduce $\beta_1 > \beta_0 = 1/2$, $\beta_1 > \beta_2$, and in the ODE case $\beta_1 > \beta_0 + 1/2$, and the result follows. Note that in both cases the R_G term is zero. □

Clearly, from the proof of Theorem 1.1 shows that knowing a priori that $\varsigma > 4$, would allow for lower inferior bound for θ in the theorem. Since in general we do not know ς, we stay with the sometimes sub-optimal limits.

5 Convergence results for the EMsDS algorithm

Given the error expansion for the decreasing step algorithm presented in Proposition 1.9, it seems natural to explore if a Richardson-Romberg extrapolation may be used to obtain the approximation with the same convergence properties we have proven. The idea of such a procedure is to decrease the complexity by performing a linear combination of two (or more) realizations of the algorithm with carefully chosen parameters. We borrow here the procedure as defined in [65].

Let λ be a positive real. If $\{\gamma_k\}$ is a sequence of steps satisfying $\mathcal{H}_\lambda$, the sequence $\gamma_k^\lambda := \dfrac{\gamma_k}{\lambda}$ will also satisfy $(\mathcal{H}_\lambda)$. We will denote Γ_M^λ and $\Gamma_M^{\lambda,[r]}$ the sum of the γ_k^λ and its power as before.

Let us denote by $\tilde{F}^{\lambda,M}(x,q)$ the approximation as defined in (1.8) when the coefficients $\{\gamma_k^\lambda\}$ are used.

Let ς defined in (1.32). We define an extrapolated approximation estimator as

$$\hat{F}^{\lambda;M(n)}(x,q) = \frac{1}{\lambda^{\varsigma/2-1}-1}\left(\lambda^{\varsigma/2-1}\tilde{F}^{\lambda,M(n)}(x,q) - \tilde{F}^{M(n)}(x,q)\right) \tag{1.60}$$

The first question we might ask is if estimator (1.60) does converge to the actual ergodic average, and what type of properties it inherits. To clarify the situation consider an extension of (1.4). Let $\vec{Y} = (Y^1, Y^2)^*$ with

$$\begin{aligned} Y_t^{1;x} &= y_0^1 + \int_0^t \frac{b(x, Y_s^{1;x})}{\lambda} ds + \int_0^t \frac{\sigma(x, Y_s^{1;x})}{\sqrt{\lambda}} d\hat{W}_s^1, \\ Y_t^{2;x} &= y_0^2 + \int_0^t b(x, Y_s^{2;x}) ds + \int_0^t \sigma(x, Y_s^{2;x}) d\hat{W}_s^2. \end{aligned} \tag{1.61}$$

If $\hat{W}^1$ and $\hat{W}^2$ are independent, then this system satisfies $(\mathcal{H}_{f.s.})$ with a unique invariant measure defined by $\vec{\mu}^x(d\vec{y}) = \mu^x(dy^1)\mu^x(dy^2)$. If we define

$$\vec{f}(x, \vec{y}) := \frac{1}{\lambda^{\varsigma/2-1} - 1} \left(\lambda^{\varsigma/2-1} f(x, y^1) - f(x, y^2) \right), \tag{1.62}$$

and defining in an analogous way $\vec{h}$, then it can be seen that $\vec{f}, \vec{g}, \vec{h} := \vec{g}^* \vec{g}$ satisfy $(\mathcal{H}_{s.s.})$. Moreover if we apply the decreasing step algorithm to $\vec{f}$ (respectively $\vec{h}$) in the extended framework, we obtain the expression (1.60). *Hence, we conclude that the EMsDS algorithm is equivalent to the MsDS algorithm applied to an extended system.*

Let us denote by $\hat{X}^n$ the approximation of the diffusion X using the extrapolated version of the algorithm. In view of the discussion we presented before, the following result is just a re-statement of Theorems 1.18 and 1.24.

Corollary 1.25 (of Theorems 1.18 and 1.24)**.** *Under the hypothesis and notation of Theorems 1.18 and 1.24.*

- $\hat{X}^n$ *converges strongly to* X.
- *Suppose that* $r^y > \varsigma + 4$ *and*

$$\frac{\Gamma_M^{[\varsigma/2+1]}}{\Gamma_M} = \hat{C}_1 n^{-\hat{\beta}_1} + o(n^{-\hat{\beta}_1}) \; ; \qquad \frac{\Gamma_M^{[\varsigma/2+2]}}{\Gamma_M^{[\varsigma/2+1]}} = \hat{C}_2 n^{-\hat{\beta}_2} + o(n^{-\hat{\beta}_2}) \tag{1.63}$$

Define

$$\hat{C}_\varphi = \frac{(\lambda^{\varsigma-1} + 1)^{1/2}}{\lambda^{\varsigma/2-1} - 1}. \tag{1.64}$$

Then, the error process $\hat{\zeta}^n := \hat{X}^n - X$ *satisfies the C.L.T like result presented in Theorem 1.24 with* $\hat{r}, \hat{\rho}, \hat{R}_F, \hat{R}_H, \hat{\varphi}_F, \hat{\varphi}_H$ *given by*

$$\begin{aligned} \hat{r} &= \min(\beta_0, \hat{\beta}_1); \\ \hat{\rho} &= \mathbf{1}_{\{\beta_0 > \hat{\beta}_1\}} (\hat{\beta}_2 \wedge (\beta_0 - \hat{\beta}_1)) + \mathbf{1}_{\{\beta_0 < \hat{\beta}_1\}} (\beta_0 \wedge (\hat{\beta}_1 - \beta_0)); \\ \hat{\varphi}_F(x) &:= \mathbf{1}_{\{\beta_0 \geq \hat{\beta}_1\}} \hat{C}_\varphi \sqrt{\hat{\Phi}_F(x)}; \quad \hat{\varphi}_G(x) := \mathbf{1}_{\{\beta_0 \geq \hat{\beta}_1\}} \hat{C}_\varphi \sqrt{\hat{\Phi}_G(x)}; \\ \hat{R}_F^i(x) &:= \mathbf{1}_{\{\hat{\beta}_1 \geq \beta_0\}} \hat{C}_1 (1 - \lambda^{-1}) \int \bar{v}_{F^i}^{\varsigma+2, r^y}(x, y) \mu^x(dy); \\ \hat{R}_H^{i,j}(x) &:= \mathbf{1}_{\{\hat{\beta}_1 \geq \beta_0\}} \hat{C}_1 (1 - \lambda^{-1}) \int \bar{v}_{H^{i,j}}^{\varsigma+2, r^y}(x, y) \mu^x(dy). \end{aligned}$$

Proof. As was highlighted in the discussion preceding the Corollary, the EMsDS algorithm is the MsDS algorithm applied to an extended system, and hence the strong convergence and limit distribution properties are a consequence from Theorems 1.18 and 1.24: it remains just to express

the values of the functions and constants appearing in Propositions 1.15 and 1.23 in terms of the original system. Indeed, recall that

$$\vec{b}(x,\vec{y}) = \begin{pmatrix} \lambda^{-1} b(x,y^1) \\ b(x,y^2) \end{pmatrix}; \qquad \vec{\sigma}(x,\vec{y}) = \begin{pmatrix} \lambda^{-1/2}\sigma(x,y^1) & 0 \\ 0 & \sigma(x,y^2) \end{pmatrix}; \tag{1.65}$$

By i) in Proposition 1.15 applied to the extended problem (i.e for the system (1.61) and $\vec{f}$ defined in (1.62)), we have a solution for the extended centered Poisson equation given by

$$\vec{\phi}_{F^i}(x,\vec{y}) = \frac{1}{\lambda^{\varsigma/2-1}-1}\left(\lambda^{\varsigma/2}\phi_{F^i}(x,y^1) - \phi_{F^i}(x,y^2)\right),$$

i.e., the solution of equation (1.17) with function F^i under the extended set-up is a linear combination of the solution in the original set-up. Thus, for any $j>0$,

$$D_y^j\vec{\phi}_{F^i}(x,\vec{y}) = \frac{1}{\lambda^{\varsigma/2-1}-1}\begin{pmatrix} \lambda^{\varsigma/2} D_y^j\phi_{F^i}(x,y^1) \\ -D_y^j\phi_{F^i}(x,y^2) \end{pmatrix}. \tag{1.66}$$

It follows that

$$\begin{aligned} &D_y^j\vec{\phi}_{F^i}(x,\vec{y})\mathbb{E}\left(\left\langle \vec{b}(x,\vec{y})^{\otimes(l-j)}, (\vec{\sigma}(x,\vec{y})U_1^0)^{\otimes(2j-l)}\right\rangle\right) \\ &= \frac{\lambda^{\varsigma/2}}{\lambda^{\varsigma/2-1}-1} D_y^j\phi_{F^i}(x,y^1)\mathbb{E}\left(\left\langle \left(\frac{b(x,y^1)}{\lambda}\right)^{\otimes(l-j)}, \left(\frac{\sigma(x,y^1)}{\sqrt{\lambda}}U_1^0\right)^{\otimes(2j-l)}\right\rangle\right) \\ &\quad - \frac{1}{\lambda^{\varsigma/2-1}-1} D_y^j\phi_{F^i}(x,y^2)\mathbb{E}\left(\left\langle \vec{b}(x,y^2)^{\otimes(l-j)}, (\sigma(x,y^2)U_1^0)^{\otimes(2j-l)}\right\rangle\right). \end{aligned}$$

Therefore

$$\vec{\bar{v}}_{F^i}^{l,r^y} = \left(\frac{\lambda^{(\varsigma-l)/2}-1}{\lambda^{\varsigma/2-1}-1}\right)\bar{v}_{F^i}^{l,r^y}, \tag{1.67}$$

and we deduce that the terms of the error expansion will be zero for $l \le \varsigma+1$.

Let $\vec{\varsigma}$, be defined by (1.32) under the extended setup. From (1.67) we conclude that $\vec{\varsigma}=\varsigma+2$. Hence, we deduce that $\hat{\beta}_1$, $\hat{\beta}_2$ defined in (1.63) are the corresponding ones in Proposition 1.15, and that $\hat{R}_F^i$ is the functions appearing in Proposition 1.23. Similar developments for H allow to extend the conclusion to $\hat{R}_H^{i,j}$.

On the other hand, looking at the definition of φ_F and Φ_F from Proposition 1.23 and (1.66) we get that

$$\begin{aligned} \hat{\Phi}_F^{i,j}(x) = \frac{C_0^{-1}}{(\lambda^{\varsigma/2-1}-1)^2}\Big(&\lambda^{\varsigma-2}\int\langle\sigma^*D_y\phi_{F^i},\sigma^*D_y\phi_{F^j}\rangle(x,y^1)\mu^x(dy^1) \\ &+\int\langle\sigma^*D_y\phi_{F^i},\sigma^*D_y\phi_{F^j}\rangle(x,y^2)\mu^x(dy^2)\Big); \end{aligned}$$

i.e. $\hat{\Phi}_F(x) = (\lambda^{\varsigma-2}+1)(\lambda^{\varsigma/2-1}-1)^{-2}\Phi_F(x)$. We get a similar result or $\hat{\Phi}_G$. We obtain the value $\hat{C}_\varphi$ given in (1.64). The claim follows. □

Proof of Corollary 1.2. We apply Corollary 1.25 to the presented particular framework. As in the proof of 1.1, we deal with the most general case $\varsigma=4$, $(\mathcal{H}_\gamma)$ is satisfied and the powers and constants we need to determine are given by:

$$\Gamma_M \approx \frac{\gamma_0 M_1^{1-\theta} n}{1-\theta}, \qquad \frac{\Gamma_M^{[3]}}{\Gamma_M} \approx \frac{\gamma_0^2(1-\theta)M_1^{-2\theta}n^{-\frac{2\theta}{1-\theta}}}{1-3\theta}, \qquad \frac{\Gamma_M^{[4]}}{\Gamma_M^{[3]}} \approx \frac{\gamma_0(1-3\theta)M_1^{-\theta}n^{-\frac{\theta}{1-\theta}}}{1-4\theta};$$

hence

$$\beta_0 = \frac{1}{2}, \qquad \hat{\beta}_1 = \frac{2\theta}{1-\theta} \qquad \beta_2 = \frac{\theta}{1-\theta},$$
$$\hat{C}_\varphi \approx \frac{(\lambda^3+1)^{1/2}}{\lambda - 1}, \qquad \hat{C}_1 \approx \frac{\gamma_0^2(1-\theta)M_1^{-2\theta}}{1-3\theta}.$$

For the ODE case, Theorem 1.24 asks for

$$\rho \geq r - \beta = \min\left(\frac{1}{2}, 1 - \min\left(\frac{1}{2}, \frac{2\theta}{1-\theta}\right)\right) = \frac{1}{2}$$

because $r = \min(1, 1/2 + \min(1/2, 2\theta/(1-\theta)))$. Hence,

$$|\beta_0 - \beta_1| = \left|\frac{1}{2} - \frac{2\theta}{1-\theta}\right| \geq \rho \geq \frac{1}{2}$$

which is the case if $\theta \in [1/3, 1)$, and implying $\beta_1 \geq 1 > \beta_0$. Note that in this interval we have $r = 1/2$ as wanted.

For the SDE case, we have $r = \beta = \min(1/2, 2\theta/(1-\theta))$, so that to have $r = 1/2$, we ask for $1/5 < \theta < 1$. □

Remark. We remark that $\hat{C}_\varphi$ is a constant multiplying the uncertainty coming from the decreasing step estimator. Since we would like this quantity as small as possible, having an explicit value for $\hat{C}_\varphi$ is very useful from a numerical point of view: we can choose λ to minimize $\hat{C}_\varphi$. In the general case $\varsigma = 4$ we get

$$\lambda_* = 1 + (\sqrt{3}+1)^{1/3} + (\sqrt{3}+1)^{-1/3} \approx 3.196$$

inducing $\hat{C}_\varphi \approx 2.64$. This is the initial additional cost that has to be paid for the extrapolation, making the EMsDS algorithm useful for large n, where the reduction in complexity of the EMsDS is enough to compensate for the higher error, as seen in the numerical tests of section 6.

6 Numerical tests

To illustrate the main features of the algorithm, we present in this section its numerical behavior when used for solving a toy system for which we are able to obtain an exact solution. Consider

$$dY_t^x = \left(\sqrt{\frac{1}{|x|^2+1}} - Y_t^x\right) + \sqrt{2}d\tilde{W}_t,$$

which is an Ornstein-Uhlenbeck system having a unique invariant measure with distribution $\mathcal{N}\left((|x|^2+1)^{-1/2}, 1\right)$; and define the SDE system

$$dX_t = F(X_t)dt + G(X_t)dW_t,$$

with

$$f(x,y) := \begin{pmatrix} 1 + y - (|x|^2+1)^{-1/2} \\ 1 \end{pmatrix}; \qquad g(x,y) := \sqrt{\frac{|x|^2+1}{2|x|^2+3}(y^2+1)} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

with F, G defined as before and where $\tilde{W}$ is a real Brownian motion independent of the planar Brownian motion W. The form of the assumed coefficients is chosen to satisfy the regularity

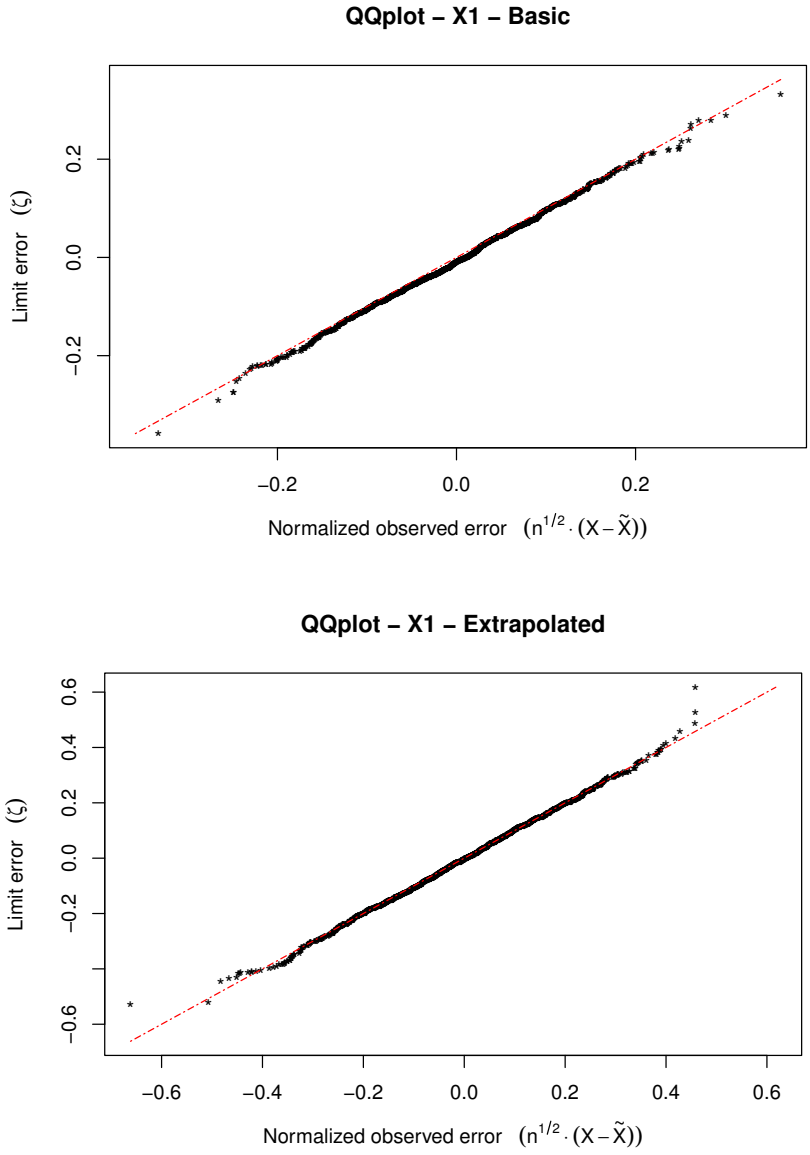


Figure 1.1: QQ-plot comparing the rescaled errors (first variable) in the simulation with $n = 510$ and the theoretical limit distribution (the reference line represents a perfect match). Top : SDE decreasing step. Down: SDE interpolated.

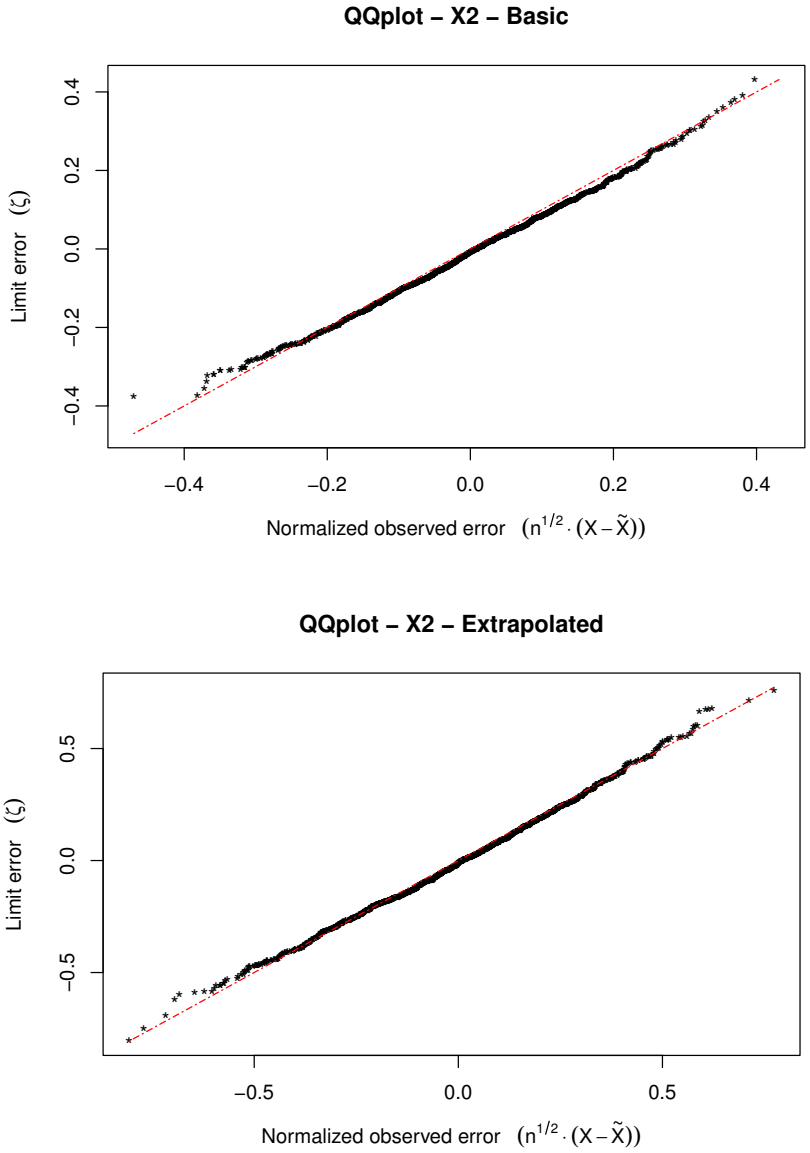


Figure 1.2: QQ-plot comparing the rescaled errors in the simulation (second variable) with $n = 510$ and the theoretical limit distribution (the reference line represents a perfect match). Top : SDE decreasing step. Down: SDE interpolated.

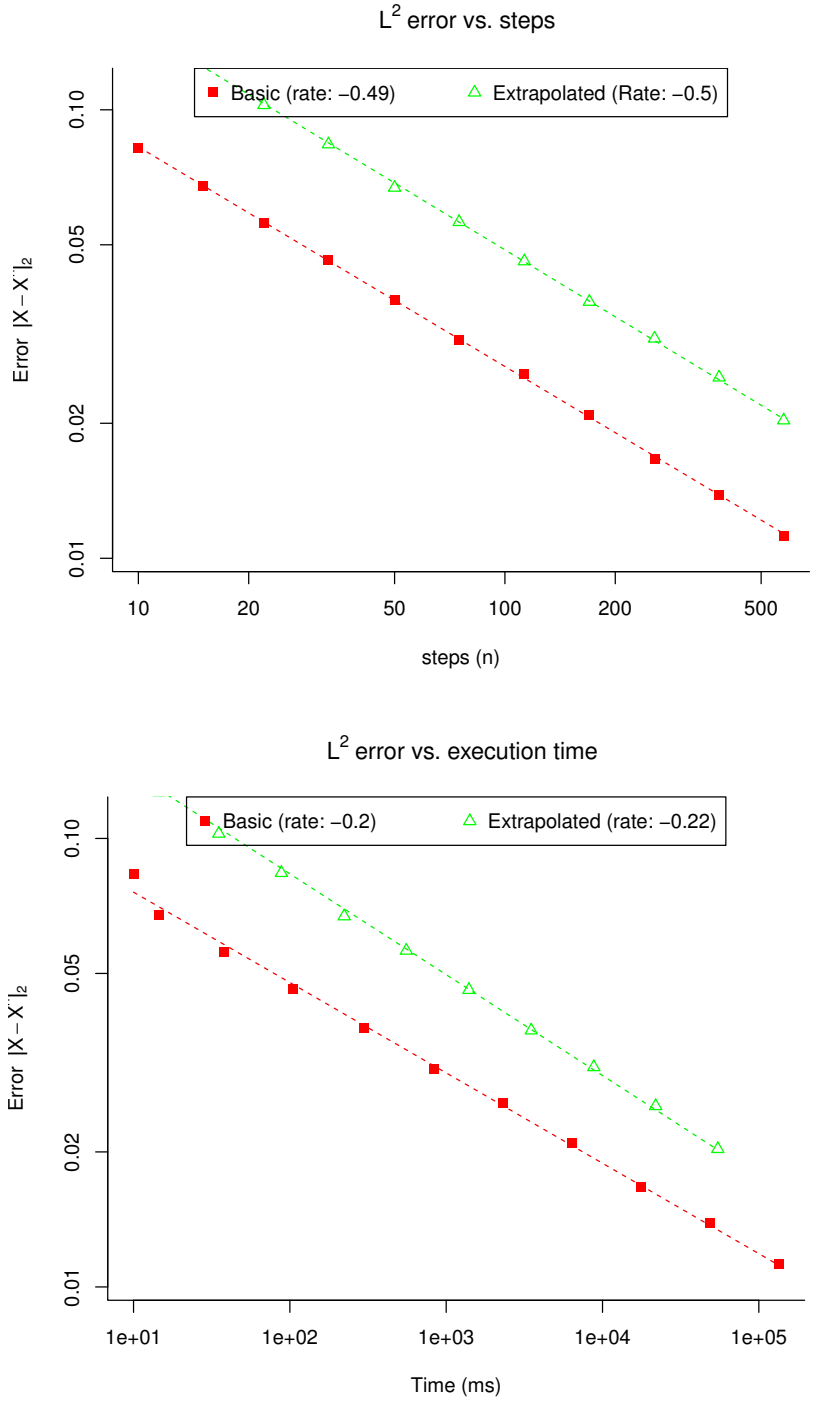


Figure 1.3: Left: L_2 error as a function of steps for the SDE case (log-log scale). Note that the estimated values for the slopes verify the rate of convergence for the algorithm in both implementations. Right: L_2 error as a function of execution time for the SDE case (log-log scale). Although a higher price must be payed for a small step number, the slope difference signals a change in the asymptotic order of convergence.

and uniform bound hypothesis in $(\mathcal{H}_{s.s.})$ and $(\mathcal{H}_{f.s.})$ and to give a simple effective equation expression. In fact, it is easily verified that the exact effective equation is

$$X_s = \begin{pmatrix} x_0^1 + s + W_s^1 \\ x_0^2 + s + W_s^1 + W_s^2 \end{pmatrix}$$

We will look at the numerical results of applying the decreasing step with sequence $\gamma_k = k^{-1/3}$ and the EMsDS version with sequence $\gamma_k = k^{-1/5}$ and $\lambda = 3$. Let us examine the distribution of the error at a fixed time $T = 1$ (i.e. $\zeta = \tilde{X}_1 - X_1$). Figures 1.1 and 1.2 show the Q-Q plots of the rescaled simulated errors $\sqrt{n}\zeta$ and the limit distribution error in the studied cases. As shown, the empirical distributions obtained after 1600 simulations with $n = 510$ verify the expected limit behavior.

Figure 1.3 Left plots in a log-log scale the evolution of the L_2 error

$$\zeta_{L_2} = \left(\mathbb{E} \left(\sup_{0 \le t \le T} |\tilde{X}_t - X_t|^2 \right) \right)^{1/2}$$

in function of the number of steps n, comparing both versions of the algorithm. The empirically obtained slope (close to -0.5 in both cases) represents the power of the approximation and is the one expected from the convergence theorems.

We show as well in Figure 1.3 Right a comparison in the efficiency of both methods (measured as the error in terms of the execution time) of each one of the algorithms. The effect of the extrapolation in the cost of the algorithm is evidenced in the difference in slope of the empirical plot for both algorithms. Recall the efficiency Table 2. Note that solving for Δ in this table we get $\Delta_{MsDS} = O(\tau^{-0.2})$ and $\Delta_{EMsDS} \approx O(\tau^{-0.222})$, values that are retrieved in the numerical experiment. It is worth observing the difference in the intercept of both lines, showing that the higher slope comes with a cost in the initial error. The conclusion drawn from the toy example may well be generalized: the user should consider implementing the extrapolated version only when requiring a very high precision on the approximation results.

7 Technical results

7.1 Weak convergence of tuples

Proof of Proposition 1.22. a) Let us start by proving (1.47). Note that the approximations defined by (1.9) are defined in the same sample space of the effective equation (1.2) and that we have thanks to Theorem 1.18 that

$$\sup_{0 \le s \le t} |\tilde{X}_s^n - X_s| \xrightarrow{P} 0.$$

Hence,

$$(X, \tilde{X}^n, W) \Rightarrow (X, X, W). \tag{1.68}$$

Now, $nA^{0;n}$ is deterministic, continuous and

$$\lim_{n\to\infty} nA_t^{0;n} = \lim_{n\to\infty} 2n \left(\frac{t^2}{2} - \frac{\lfloor nt \rfloor (\lfloor nt \rfloor - 1)}{2n^2} \right) = t.$$

an the convergence is uniform in t.

On the other hand, we can easily verify that for any t, and $1 \le i \le d_x$

$$\sqrt{n} B_t^{1;i,n} = \sqrt{n} \int_0^t \nu_{\underline{s}}^{i;n} ds = \sum_{i=0}^{\lfloor nt \rfloor} \frac{1}{\sqrt{n}} \nu_{\underline{s}}^{i;n} + \frac{nt - \lfloor nt \rfloor}{n^{3/2}} \nu_{\underline{t}}^{i;n}$$

but by Cauchy-Schwarz inequality we have

$$\mathbb{E}\left(\left|\sup_{0\le t\le T}\left(\frac{nt-\lfloor nt\rfloor}{n^{3/2}}\nu^{i;n}_{\underline{t}}\right)\right|^2\right)\le\mathbb{E}\left(\left|\sum_{k=1}^{n}\frac{1}{n^{3/2}}|\nu^{i;n}_{t_k}|\right|^2\right)$$
$$\le\mathbb{E}\left(\frac{n}{n^3}\sum|\nu^{i;n}_{t_k}|^2\right)\to 0.$$

then, it suffices to study the convergence of the Gaussian martingale $\sum_{i=0}^{\lfloor nt\rfloor} n^{-1/2}\nu^{i;n}_{\underline{s}}$. Let $0\le j\le d_x$, then the independence properties and an application of a multi-dimensional C.L.T. gives us that

$$\left\langle\sum_{i=0}^{\lfloor nt\rfloor}\frac{1}{\sqrt{n}}\nu^{i;n}_{\underline{s}},\sum_{i=0}^{\lfloor nt\rfloor}\frac{1}{\sqrt{n}}\nu^{j;n}_{\underline{s}}\right\rangle=\frac{1}{n}\sum_{i=0}^{\lfloor nt\rfloor}\nu^{i;n}_{\underline{s}}\nu^{j;n}_{\underline{s}}\xrightarrow{P}\delta_{i=j}.$$

We conclude that B^1 is (up to a modification) a Brownian motion independent from W and X, by remarking its Gaussian nature with independent increments property and covariance matrix as the one of the standard Brownian and hence (1.47) follows. Note that we have as well shown property (*) and consequently goodness of the sequence.

b) To prove (1.48) note first that $\sqrt{n}B^{2;i,j;n}$ is a continuous martingale. In view of the results in [48], we examine the component-wise quadratic variation. By standard techniques we find

$$\langle\sqrt{n}B^{2;i,j;n},\sqrt{n}B^{2;i',j;n}\rangle_t=2n\int_0^t(W^i_s-W^i_{\underline{s}})(W^{i'}_s-W^{i'}_{\underline{s}})ds$$
$$\xrightarrow{P}\mathbf{1}_{\{i=i'\}}t,$$

and due to independence we find as well that, taking, $j\neq j'$,

$$\langle\sqrt{n}B^{2;i,j;n},W^j\rangle_t\xrightarrow{P}0$$
$$\langle\sqrt{n}B^{2;i,j;n},\sqrt{n}B^{2;i,j';n}\rangle_t=0\ ;\qquad\langle\sqrt{n}B^{2;k;n},W^j\rangle_t=0.$$

By Theorem 4-1 in [48], $B^{2;n}$ convergences stably in law towards B^2 a standard Brownian Motion independent from W (for the definition of this type of convergence see [1] or [48]). Since all the processes are continuous, convergence stably in law implies joint convergence. Therefore considering (1.68), we have

$$(X,\tilde{X}^n,W,B^{2;n})\Rightarrow(X,X,W,B^2).$$

Note that we proved tightness of the quadratic variation of the martingale $\sqrt{n}B^{2;n}$, so that it has property (*), and therefore it is good.

Now, $B^{3;n}$ is also a continuous Gaussian martingale, so that we can make use again of Theorem 4-1 in [48]. Let us check the convergence in probability of its quadratic variation towards tI and of its quadratic covariation with respect to the other martingales. Indeed, it is straightforward that if $j\neq j'$, $\langle B^{3;i,j;n},B^{3;i',j';n}\rangle_t=0$, while we deduce from the multidimensional C.L.T $\langle B^{3;i,j;n},B^{3;i',j;n}\rangle_t\xrightarrow{P}\mathbf{1}_{\{i=i'\}}t$. As before this also shows goodness of $B^{3;n}$. Using the same techniques we prove for any i,j,l that

$$\langle B^{3;l,j;n},\sqrt{n}B^{2;i;n}\rangle_t=0,\ \text{and}\ \langle B^{3;l,j;n},W\rangle=0.$$

Hence

$$(X,\tilde{X}^{(n)},W,B^{2;n},B^{3;n})\Rightarrow(X,X,W,B^2,B^3).$$

We prove now the convergence in probability towards zero of the remaining terms in the left side tuple in (1.48).

Since $n^{-1/2} \to 0$ and $nA^{0,n} \Rightarrow A^0$, we have $n^{1/2}A^{0,n} \Rightarrow 0$ and thus $n^{1/2}A^{0,n} \xrightarrow{P} 0$.

On the other hand, for any $t \geq 0$ and k,

$$\mathbb{E}\left(\left\langle \sqrt{n}B^{0;k;n}, \sqrt{n}B^{0;k;n}\right\rangle_t\right) = n\int_0^t (s-\underline{s})^2 ds$$
$$= \sum_{i=0}^{\lfloor nt\rfloor} n\int_0^{1/n} r^2 dr + n\int_0^{t-\lfloor nt\rfloor/n} r^2 dr \leq \sum_{i=0}^{\lfloor nt\rfloor+1} \frac{1}{3n^2} = O(n^{-1}),$$

So that by Burckholder-Davis-Gundy inequality, $\mathbb{E}\left(\sup_{0\leq t\leq T} |\sqrt{n}B^{0;n}|^2\right)$ tends to zero as $n \to \infty$, implying $\sqrt{n}B^{0;n} \xrightarrow{P} 0$. In addition, it can be readily seen that

$$\mathbb{E}\left(|nA^{1;k;n}A^{1;j;n}|\right) = 0$$

for $j \neq k$, so that we have by using convex and Cauchy-Schwarz inequalities,

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |\sqrt{n}A_t^{1;n}|^2\right) \leq nT\sum_{j=1}^{d_x}\int_0^T \mathbb{E}\left((W_s^j - W_{\underline{t}s}^j)\right) \leq \frac{d_x T}{2}$$

hence, by the law of large numbers, $\sqrt{n}A^{1;n} \xrightarrow{P} \mathbb{E}\left(\sqrt{n}A^{1;n}\right) = 0$.

Finally, as $\sqrt{n}B^{1;n}$ converges in law to a Brownian, $B^{3;n} \xrightarrow{P} 0$. Therefore (1.48) is proved. □

Chapter 2

Numeric approximation of decoupled FBSDEs of the McKean Vlasov type

Consider the decoupled McKean-Vlasov forward backward stochastic differential equation (MKV-FBSDE) system:

$$\begin{cases} dX_t^x = \sum_{i=0}^{d} V_i(t, X_t^x, \mathbb{E}\varphi_i(X_t^x))dB_t^i \\ dY_t^x = -f(t, X_t^x, Y_t^x, Z_t^x, \mathbb{E}\varphi_f(X_t^x, Y_t^x))dt + Z_t^x dB_t^{1:d} \\ X_0^x = x, \quad Y_T^x = \phi(X_T^x) \end{cases} \tag{2.1}$$

for any t in $[0,T]$, $T>0$ be given. We place ourselves in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$, with $B_t^{1:d}$ a d-dimensional adapted Brownian motion and $B_t^0 = t$. We take $V_i : (t,y,w) \in [0,T]\times\mathbb{R}^d\times\mathbb{R} \mapsto V_i(t,y,w)$; functions $\varphi_i : y \in \mathbb{R}^d \mapsto \varphi_i(y) \in \mathbb{R}$, $i = 0, \cdots, d$ and $\varphi_f : (y,y') \in \mathbb{R}^d\times\mathbb{R} \mapsto \varphi_f(y,y')$ and the mapping $f : t,y,y',z,w \in [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R} \mapsto f(t,y,y',z,w) \in \mathbb{R}$ to be bounded and infinitely differentiable with bounded derivatives. The mapping ϕ is an at least Lipschitz function from $\mathbb{R}^d$ to $\mathbb{R}$ whose precise regularity is given below.

Recall that our program is to solve the system in two steps. First we solve the forward component by decoupling the law dependence. We choose some family of probability measures on $\mathbb{R}^d$ $(\eta_t)_{0\leq t\leq T}$ and fix the law in the McKean-Vlasov terms of (2.1) to be $(\eta_t)_{0\leq t\leq T}$. For this modified system, we may apply a classical cubature scheme for the forward component (the time dependence of the coefficients being handled as an additional dimension). Then, to solve the backward part, we take advantage of the decoupled setting going back along the nodes of the cubature tree by computing the current value of the backward process as a conditional expectation at each node, as in Crisan and Manoralakis' scheme as proposed in [28] and presented in Section 3.2 of the Introduction.

Our choice for the decoupling law is to take at each step of the cubature tree the discrete marginal law given by the solution of the ODEs along the cubature paths and corresponding weights. We show that for a cubature of order m and a number N of discretization steps, this choice of approximation law leads to a $N^{-(m-1)/2}$ order approximation of the expectation of any $m+2$ times continuously differentiable functional of the forward component, when all the derivatives are bounded[1], and to a first order approximation scheme of the backward component, where the given orders stand for the supremum norm error. Higher orders of approximation are also obtained by correcting some terms in the algorithm.

Usually, forward MKV-SDEs are solved by using particle algorithms (see for example [2, 95] or [11] and references therein) in which the McKean term is approached with the empirical measure of a large number of interacting particles with independent noise. Adapting such algorithms to the forward-backward problem is not obvious as the high dimension of the involved Brownian motion (given by the number of particles) induces, a priori, a high dimension backward problem with the obvious consequences for the numerical implementation. In comparison, our proposed algorithm gives a deterministic approximation of the McKean term, and since it does not induce any additional noise, it does not increase the dimension of the backward problem.

Although our algorithm works for decoupled MKV-FBSDEs, we believe this solver may also be considered as a building block if one is interested in approaching the fully coupled case (when the forward coefficients depend on the backward variable), for example via fixed point procedures. Nevertheless, a lot of work is required to define the precise conditions and setup in which such algorithm would converge to the (or at least a) solution of the fully coupled problem.

The conditional system. Let us shortly develop what we mean with the sentence "*given the law of the solution of the system,* (2.1) *is a classical time inhomogeneous FBSDE*". Working with a non-linear problem, such as MKV-FBSDE, could be tricky. In our case, the main object to work with is the *conditional system*. This is the formulation that allows to get rid of the dependence on the law and to replace it by a time dependent parameter.

Following the same line of arguments presented in Buckdahn et al. [15], it is well seen that there exists a unique solution $\{X_t^x, Y_t^x\}_{t\geq 0}$ to the system (2.1). In a Markovian setting, the law of this couple is entirely determined by the law $\mu = (\mu_t)_{0\leq t\leq T}$ of the forward process $(X_t)_{0\leq t\leq T}$ and a given deterministic function $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$. In our case, one can show that this remains true (see Section 7 below for a proof) so that there exists a deterministic $u(t,y)$ such that for all t,

$$Y_t = u(t, X_t). \tag{2.2}$$

We prove that under appropriate assumptions u is regular and satisfies the parametrized non-local semi linear PDE:

$$\begin{cases} D_t u(t,y) + \mathcal{L}^\mu u(t,y) = f\left(t, y, u(t,y), (\mathcal{V}^\mu u(t,y))^T, \langle \mu_t, \varphi_f(\cdot, u(t,\cdot))\rangle\right) \\ u(T,y) = \phi(y) \end{cases}, \tag{2.3}$$

where $\mathcal{V}^\mu u$ stands for the row vector $(\nabla u \cdot V_1, \dots, \nabla u \cdot V_d)$, $(\mathcal{V}^\mu u)^T$ is the transpose of $\mathcal{V}^\mu u$ and $\mathcal{L}^\mu$ is the generator of the forward component in (2.5) below and given by:

$$\mathcal{L}^\mu := V_0(\cdot,\cdot,\langle\mu_\cdot,\varphi_0\rangle)\cdot D_y + \frac{1}{2}\mathrm{Tr}[VV^T(\cdot,\cdot,\langle\mu_\cdot,\varphi_i\rangle)D_y^2]. \tag{2.4}$$

Here, we used the duality notation $\langle\mu,\varphi_i\rangle$ for $\int \varphi_i d\mu$. Likewise, the superscript μ means that the vector fields are taken at the point $(\cdot,\cdot,\langle\mu_\cdot,\varphi_i\rangle)$ (where the $i\in\{0,\cdots,d\}$ is taken with respect

[1] this is the special case when ϕ is $m+2$ times continuously differentiable with bounded derivatives and $f=0$ in (2.1).

to the corresponding vector field), V is the matrix $[V_1, \cdots, V_d]$, "$\cdot$" stands for the euclidean scalar product on $\mathbb{R}^d$ and "Tr" for the trace.

The conditional MKV-FBSDE system is then defined as

$$\begin{cases} dX_s^{t,y,\mu} = \sum_{i=0}^{d} V_i(s, X_s^{t,y,\mu}, \langle \mu_s, \varphi_i \rangle) dB_s^i \\ dY_s^{t,y,\mu} = -f(s, X_s^{t,y,\mu}, Y_s^{t,y,\mu}, Z_s^{t,y,\mu}, \langle \mu_s, \varphi_f(\cdot, u(s,\cdot)) \rangle) ds + Z_s^{t,y} dB_s^{1:d} \\ X_t^{t,y,\mu} = y, \quad Y_T^{t,y,\mu} = \phi(X_T^{t,y,\mu}). \end{cases} \tag{2.5}$$

Let us remark that in this setting, we can also define the deterministic mapping $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$Z_t = v(t, X_t). \tag{2.6}$$

As in the classical BSDE theory, under appropriate regularity conditions,

$$v(t,x) = (\mathcal{V}^\mu u(t,x))^T.$$

Assumptions. From the previous discussion, the error analysis of the proposed algorithm uses extensively the regularity of $(t,x) \in [0,T] \times \mathbb{R}^d \mapsto \mathbb{E}\phi(X_T^{t,x,\mu})$ for the forward part and of the solution u of (2.3) for the backward part. Therefore, we present two types of hypotheses that guarantee that the required regularity is attained.

(SB) We say that assumption **(SB)** holds if the mapping ϕ in (2.1) is C_b^∞.

(LB) We say that assumption **(LB)** holds if the mapping ϕ in (2.1) is uniformly Lipschitz continuous and if the matrix VV^T is uniformly elliptic i.e., there exists $c > 0$ such that

$$\forall (t,y,w) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}, \forall \varsigma \in \mathbb{R}^d, c^{-1}|\varsigma|^2 \leq VV^T(t,y,w)\varsigma \cdot \varsigma \leq c|\varsigma|^2.$$

As discussed in the introduction, the regularity loss under the Lipschitz condition is compensated by imposing stronger diffusion conditions on the forward variable, namely asking for uniform ellipticity of the diffusion matrix V.

Remark. A reader familiarized with the cubature method might wonder why we assume uniform ellipticity instead of the weaker UFG condition usually needed for applying the method with Lipschitz boundary conditions. The reason is that the smoothing results of Kusuoka and Stroock [58] hold for space dependent vector fields only, and therefore do not apply directly to our framework with a time dependence coming from the McKean term. There is some extension in the time inhomogeneous case that do not include derivatives in the V_0 direction (see for example [21] and references therein), but, to the best of our knowledge, there is no result that could be applied to our framework.

Towards a more general class of coefficients. We have chosen to work with the assumed explicit dependence of the coefficients with respect to the law, as it is very natural in practice. In fact as a consequence of our analysis, our algorithm works for a more general class of MKV-FBSDE, i.e. for a system written as

$$\begin{cases} dX_t^x = \sum_{i=0}^{d} V_i(t, X_t^x, \mu_t) dB_t^i \\ dY_t^x = -f(t, X_t^x, Y_t^x, Z_t^x, \mu_t^{X,Y}) dt + Z_t^x dB_t^{1:d} \\ X_0^x = x, \quad Y_T^x = \phi(X_T^x), \end{cases} \tag{2.7}$$

where $\mu^{X,Y} = (\mu_t^{X,Y})_{0\leq t\leq T}$ denotes the joint law of $(X_t, Y_t)_{0\leq t\leq T}$, the coefficients V_i, $0 \leq i \leq d$ (and f) are Lipschitz continuous with respect to an appropriately defined distance in the space of probability measures on $\mathbb{R}^d$ (respectively $\mathbb{R}^d \times \mathbb{R}$) which integrates the square of the usual Euclidean norm. The distance we consider is defined by duality: let $\mathcal{F}$ be a (sufficiently rich[2]) class of functions (that will be detailed in the following). Then we define the distance $d_{\mathcal{F}}$ between two probability measures on $\mathbb{R}^n$ by

$$d_{\mathcal{F}}(\mu, \nu) = \sup_{\varphi\in\mathcal{F}} |\langle \varphi, \mu - \nu\rangle|. \tag{2.8}$$

In this decoupled case the Lipschitz property of the coefficients with respect to $d_{\mathcal{F}}$ ensures the existence of a unique solution of (2.7) [3]. Then, we are able to analyze the convergence of our procedure in two different cases. When $\mathcal{F}$ is the class of 1-Lipschitz functions, i.e. when the distance is the so-called Wasserstein-1 distance, and when the vector fields are uniformly elliptic our algorithm leads to an $N^{-1/2}$ order approximation[4]. When $\mathcal{F}$ is the class of C_b^∞ functions we obtain an N^{-1} order approximation without any ellipticity assumption on the diffusion matrix.

Objectives and organization of the chapter. As a corollary of the discussion on the conditional system we can resume our objective as the approximation of $\mathbb{E}\phi(X_T^x)$, where $(X_t^x)_{0\leq t\leq T}$ is the solution of (2.1) and of u satisfying (2.2).

This chapter is organized as follows: Section 1 states the algorithm, while the convergence rate of the forward and backward approximations is stated in Section 2. Then, we give a numerical example for each set of hypotheses **(SB)** and **(LB)** in Section 3. A class of control problems is introduced in Section 4. The remainder of the chapter is dedicated to the proof of the convergence. For the sake of simplicity, we first recall some definitions, basic facts and notations in Section 5. The forward and backward convergence rates for regular boundary conditions are successively proved in Section 6 and the common mathematical tools are given in Section 7. Section 8 presents the extension to the Lipschitz boundary condition case and the announced generalization of the law dependence of the McKean terms.

Notations. As we are treating with objects exhibiting different dependences, the notation can become a bit heavy. For the sake of simplicity, we adopt the following conventions. We denote by φ the function $\varphi = [\varphi_1, \cdots, \varphi_d]$. For two positive integers $i < j$, the notation "$i : j$" means "from index i to j". For all $\varsigma \in \mathbb{R}^n$, $n \in \mathbb{N}$ the partial derivative $[\partial/\partial\varsigma]$ is denoted by ∂_ς. Let $g : y \in \mathbb{R}^d \mapsto g(y) \in \mathbb{R}$ be a p-continuously differentiable function. We set $||g||_{\infty,p} := \max_{j\leq p} ||\partial_y^j g||_\infty$. We say that a function g from $[0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ is C_b^p, $p \in \mathbb{N}^*$ if it is bounded and p-times continuously differentiable with bounded derivatives. We usually denote by η (eventually with an exponent) a family of probability measures $(\eta_t)_{0\leq t\leq T}$ on $\mathbb{R}^d$. For such a family, we set $\mathcal{L}^\eta$ to be the second order operator of the form (2.4) with η instead of μ. In general, we will work with the vector fields taken at the point $(\cdot, \cdot, \langle \mu_\cdot, \varphi_i\rangle)$ (where the $i \in \{0, \cdots, d\}$ signals the corresponding vector field) and in general we will omit the explicit dependence on μ in the notation. In any case, we will mark the law dependence explicitly when needed, in particular when a dependence with respect to a different law appears.

Multi-index. Multi-indices allow to easily manage differentiation and integration in several dimensions. Let

$$\mathcal{M} = \{\emptyset\} \cup \bigcup_{l\in\mathbb{N}^*} \{0, 1, \ldots, d\}^l, \tag{2.9}$$

[2] It should contain the space of 1-Lipschitz functions.

[3] by definition of $\mathcal{F}$, the distance $d_{\mathcal{F}}$ is less than or equal to the Wasserstein 2 distance. Then, one uses the same kind of arguments as in [91].

[4] Recall that N denotes the number of discretization steps.

denotes the set of multi-indices where $\emptyset$ refers, for the sake of completeness, to the zero-length multi-index. We define "$*$" to be the concatenation operator such that if $\beta^1 = (\beta_1^1, \dots, \beta_l^1)$ and $\beta^2 = (\beta_1^2, \dots, \beta_n^2)$ then $\beta^1 * \beta^2 = (\beta_1^1, \dots, \beta_l^1, \beta_1^2, \dots, \beta_n^2)$.

1 Algorithm

1.1 Algorithms

We proceed to give a precise description of each of the two main parts of our proposed algorithm. Since the cubature involves Stratanovitch integrals, we set:

$$\bar{V}_0^k = V_0^k - \frac{1}{2}\sum_{i,j=1}^{d} V_j^i \frac{\partial}{\partial x_j} V_k^j, \tag{2.10}$$

for all $k \in \{1, \cdots, d\}$ and rewrite the system (2.1) as:

$$\begin{cases} dX_t^x = \bar{V}_0(t, X_t^x, \mathbb{E}\varphi_0(X_t^x))dt + \sum_{i=1}^{d} V_i(t, X_t^x, \mathbb{E}\varphi_i(X_t^x)) \circ dB_t^i \\ dY_t^x = -f(t, X_t^x, Y_t^x, Z_t^x, \mathbb{E}\varphi_f(X_t^x, Y_t^x))dt + Z_t^x dB_t^{1:d} \\ X_0^x = x, \quad Y_T^x = \phi(X_T^x). \end{cases} \tag{2.11}$$

In order to make the description of the algorithm as clear as possible, for any k, κ in $\mathbb{N}$, we set $\mathcal{S}_\kappa(k) = \{\text{multi-index } (j_1, \cdots, j_k) \in \{1, \cdots, \kappa\}^k\}$, i.e., $\mathcal{S}_\kappa(k)$ is the set of multi-indices with entries between $1, \dots, \kappa$ of length (exactly) k.

1.1.a *Building the tree* $\mathcal{T}(\gamma, q, m)$

The subdivision. Let $\gamma > 0$, $N \in \mathbb{N}^*$, let $0 = T_0 < \ldots < T_N = T$ be a discretization of the time interval $[0, T]$ given as

$$T_k = T\left[1 - \left(1 - \frac{k}{N}\right)^\gamma\right] \tag{2.12}$$

and let $\Delta_{T_k} = T_k - T_{k-1}$.

Remark. When the boundary condition is not smooth, we take a non-uniform subdivision in order to refine the discretization step close to the boundary as proposed by Kusuoka in [57]. If, on the contrary, the boundary condition is smooth, we may use a classical uniform discretization. For this reason, in the following we assume that $\gamma = 1$ if **(SB)** holds, and that $\gamma > m - 1$ under **(LB)**.

Let γ be given as explained above, q and m be two given integers, and $\big\{\{\omega_1, \cdots, \omega_\kappa\}, \{\lambda_1, \cdots, \lambda_\kappa\}\big\}$ be a m order cubature (the number κ of paths and weights depends on m). Recall that $\omega_j : t \in [0,1] \mapsto (\omega_j^1(t), \dots, \omega_j^d(t)) \in \mathbb{R}^d$ is some continuous function with bounded variation and for all t in $[0, T]$, we set $\omega_0(t) = t$. Examples of cubature formulas of order $3, 5, 7, 9, 11$ can be found in [69] or [46].

Algorithm 1 Cubature Tree $\mathcal{T}(\gamma, q, m)$

Set $(X^{\emptyset}, \hat{\mu}_{T_0}, \Lambda_0) = (x, \delta_x, 1)$
for $0 \leq i \leq d$ **do**
    Set $F_i(t, \hat{\mu}_{T_0}) = \sum_{p=0}^{q-1} \frac{1}{p!}(t - T_0)^p \langle \delta_x, (\mathcal{L}^{\delta_x})^p \varphi_i \rangle$
end for
for $1 \leq k \leq N-1$ **do**
    for $\pi \in \mathcal{S}_\kappa(k)$ **do**
        for $1 \leq j \leq \kappa$ **do**
            Define $\hat{X}^{\pi * j}_{T_{k+1}}$ as the solution of the ODE:

$$d\hat{X}^{\pi*j}_t = \sum_{i=0}^{d} V_i(t, \hat{X}^{\pi*j}_t, F_i(t, \hat{\mu}_{T_k}))\sqrt{\Delta_{T_{k+1}}} d\omega^i_j((t - T_k)/\Delta_{T_{k+1}}),$$
$$\hat{X}^{\pi*j}_{T_k} = \hat{X}^{\pi}_{T_k}$$

            Set the associated weight: $\Lambda_{\pi*j} = \Lambda_\pi \lambda_j$
        end for
    end for
    Set $\hat{\mu}_{T_{k+1}} = \sum_{\pi \in \mathcal{S}_\kappa(k+1)} \Lambda_\pi \delta_{\hat{X}^{\pi}_{T_{k+1}}}$
    for $0 \leq i \leq d$ **do**
        Set $F_i(t, \hat{\mu}_{T_{k+1}}) = \sum_{p=0}^{q-1} \frac{1}{p!}(t - T_k)^p \langle \hat{\mu}_{T_{k+1}}, (\mathcal{L}^{\hat{\mu}})^p \varphi_i \rangle$
    end for
end for

1.1.b *Backward scheme*

Algorithm 2 First order backward scheme

for $\pi \in \mathcal{S}_\kappa(N)$ **do**
    Set $\hat{u}^1(T_N, \hat{X}^{\pi}_{T_N}) = \phi(\hat{X}^{\pi}_{T_N})$
    Set $\hat{v}^1(T_N, \hat{X}^{\pi}_{T_N}) = 0$
end for
for $N - 1 \geq k \geq 1$ **do**
    for $\pi \in \mathcal{S}_\kappa(k)$ **do**
        $\hat{v}^1(T_k, \hat{X}^{\pi}_{T_k}) = \frac{1}{\Delta_{T_{k+1}}} \sum_{j=1}^{\kappa} \lambda_j \hat{u}^1(T_{k+1}, X^{\pi*j}_{T_{k+1}}) \sqrt{\Delta_{T_{k+1}}} \omega_j(1)$
        for $1 \leq j \leq \kappa$ **do**
            $\hat{\Theta}^{\pi,1}_{k+1,k}(j) = \left(T_{k+1}, \hat{X}^{\pi*j}_{T_{k+1}}, \hat{u}^1(T_{k+1}, \hat{X}^{\pi*j}_{T_{k+1}}), \hat{v}^1(T_k, \hat{X}^{\pi}_{T_k}), F^1(T_{k+1}, \hat{\mu}_{T_{k+1}})\right)$
        end for
        $\hat{u}^1(T_k, \hat{X}^{\pi}_{T_k}) = \sum_{j=1}^{\kappa} \lambda_j \left(\hat{u}^1(T_{k+1}, X^{\pi*j}_{T_{k+1}}) + \Delta_{T_{k+1}} f(\hat{\Theta}_{k+1,k}(j))\right)$
    end for
    Set $F^1(T_{k+1}, \hat{\mu}_{T_{k+1}}) = \langle \hat{\mu}_{T_{k+1}}, \varphi_f(\cdot, \hat{u}^1(T_{k+1}, \cdot)) \rangle$
end for

Algorithm 3 Second order backward scheme

for $\pi \in \mathcal{S}_\kappa(N)$ **do**
  Set $\hat{u}^2(T_N, \hat{X}^\pi_{T_N}) = \phi(\hat{X}^\pi_{T_N})$
  Set $\hat{v}^2(T_N, \hat{X}^\pi_{T_N}) = 0$
end for
for $\pi \in \mathcal{S}_\kappa(N-1)$ **do**
  Set $\hat{u}^2(T_{N-1}, \hat{X}^\pi_{T_{N-1}}) = \hat{u}^1(T_{N-1}, \hat{X}^\pi_{T_{N-1}})$ and $\hat{v}^2(T_{N-1}, \hat{X}^\pi_{T_{N-1}}) = \hat{v}^1(T_{N-1}, \hat{X}^\pi_{T_{N-1}})$
  Set $F^2(T_{N-1}, \hat{\mu}_{T_{N-1}}) = \langle \hat{\mu}_{T_{N-1}}, \varphi_f(\cdot, \hat{u}^2(T_{N-1}, \cdot))\rangle$
end for
for $N-2 \geq k \geq 1$ **do**
  for $\pi \in \mathcal{S}_\kappa(k)$ **do**
    for $1 \leq j \leq \kappa$ **do**
      $\hat{\Theta}^{\pi,2}_{k+1}(j) = \left(T_{k+1}, \hat{X}^{\pi * j}_{T_{k+1}}, \hat{u}^2(T_{k+1}, \hat{X}^{\pi * j}_{T_{k+1}}), \hat{v}^2(T_{k+1}, \hat{X}^{\pi * j}_{T_{k+1}}), F^2(T_{k+1}, \hat{\mu}_{T_{k+1}})\right)$
      $\hat{\zeta}^{\pi * j}_{k+1} := 4\frac{1}{\Delta_{T_{k+1}}}\sqrt{\Delta_{T_{k+1}}}\omega_j(1) - 6\frac{1}{\Delta^2_{T_{k+1}}}\int_{T_k}^{T_{k+1}} (s - T_k)\sqrt{\Delta_{T_{k+1}}} d\omega_j((s-T_k)/\Delta_{T_{k+1}})$
    end for
    Set $\hat{v}^2(T_k, \hat{X}^\pi_{T_k}) = \sum_{j=1}^{\kappa} \lambda_j \left(\hat{u}^2(T_{k+1}, X^{\pi * j}_{T_{k+1}}) + \Delta_{T_{k+1}} f(\hat{\Theta}^{\pi * j, 2}_{k+1})\right) \hat{\zeta}^{\pi * j}_{k+1}$
    [*Predictor*] $\tilde{u}(T_k, \hat{X}^\pi_{T_k}) = \sum_{j=1}^{\kappa} \lambda_j \left(\hat{u}^2(T_{k+1}, X^{\pi * j}_{T_{k+1}}) + \Delta_{T_{k+1}} f(\hat{\Theta}^{\pi * j, 2}_{k+1})\right)$
    Set $\tilde{F}(T_k, \hat{\mu}_{T_k}) = \langle \hat{\mu}_{T_k}, \varphi_f(\cdot, \tilde{u}(T_k, \cdot))\rangle$
    [*Corrector*] $\tilde{\Theta}^\pi_k = \left(T_k, \hat{X}^\pi_{T_k}, \tilde{u}(T_k, \hat{X}^\pi_{T_k}), \hat{v}^2(T_k, \hat{X}^\pi_{T_k}), \tilde{F}(T_k, \hat{\mu}_{T_k})\right)$
    $\hat{u}^2(T_k, \hat{X}^\pi_{T_k}) = \sum_{j=1}^{\kappa} \lambda_j \left(\hat{u}^2(T_{k+1}, X^{\pi * j}_{T_{k+1}}) + \frac{1}{2}\Delta_{T_{k+1}} \left(f(\hat{\Theta}^{\pi * j, 2}_{k+1}) + f(\tilde{\Theta}^\pi_k)\right)\right)$
    Set $F^2(T_k, \hat{\mu}_{T_k}) = \langle \hat{\mu}_{T_k}, \varphi_f(\cdot, \hat{u}^2(T_k, \cdot))\rangle$
  end for
end for

Remark. The initialization value for v at the boundary, that we have fixed in 0, is arbitrary, given that the first steps in both algorithms does not use this value.

However, if the algorithm is used under **(SB)** and the values of $D_x u$ can be easily calculated on the boundary, we have a natural initialization value for v. In this case, we may initialize the backward algorithms of order one and two to reflect this additional information, by setting $v(t,x) = (\mathcal{V}^\mu u(t,x))^T$.

This modification will have no effect at all for the first order scheme, and is interesting only from the point of view of consistence. On the other hand, on the second order algorithm, the natural initialization of v allows to skip the first order step. This change does not affect the overall rate of convergence of the algorithm but will induce a reduction in the error constant whence of the total approximation error.

It is worth noticing that the given algorithm is particularly effective for treating the McKean dependence of the backward component. Indeed, note that the expectation of any regular enough function of $\hat{u}$ is readily available given that the support of the approximating measure $\hat{\mu}$ coincides with the points where $\hat{u}$ is available. Of course the situation is quite different when a different approach, like a particle method, is used.

Example: one dimensional cubature of order $m = 3$. In this case, we may use a cubature formula with $\kappa = 2$ paths given by $\{+t, -t\}$, and associated weights: $\{\lambda_1 = 1/2, \lambda_2 = 1/2\}$. Let us explain the idea behind the algorithm with an example for 2 steps, as shown in Figure 2.1. We initialize the tree at a given point x, and the law at T_0 as δ_x. Then, we find two descendants given as the solution of an ODE that uses the position X, the two cubature paths, and an approximated law using the information at time 0. Each descendant will have a weight equal to the product of the weight of its parent times the weight given to the corresponding cubature path. Once all nodes at time T_1 are calculated, we obtain the discrete measure $\hat{\mu}_{T_1}$, the law approximation at time T_1. The process is then repeated for each node at time T_1 to reach the final time $T = T_2$.

Figure 2.1 right illustrates the idea behind the backward approximation: the approximated function $\hat{u}$ is defined first at the leaves of the constructed tree, and then back-propagates using the approximated law to obtain $\hat{u}$ at previous times. The back-propagation is made by conditional expectation: average with respect to the weight of each cubature path.

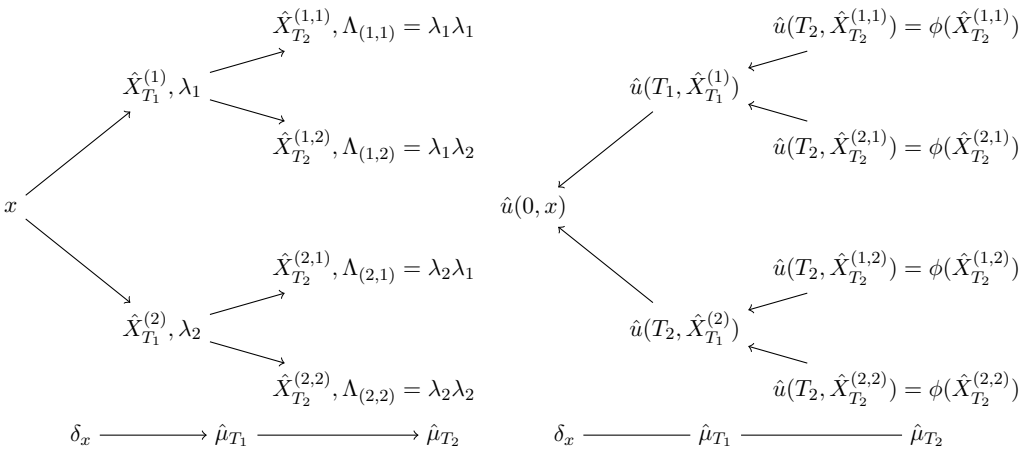


Figure 2.1: Left: Cubature tree. Right: Backward scheme.

2 Main Results

In this section, we first give the rate of convergence of our algorithms 1, 2 and 3 when both the coefficients and terminal condition in (2.1) are smooth. This is given in Theorem 2.1 below. Then, we give the rate when the boundary condition is Lipschitz and when the diffusion part of (2.1) is uniformly non-degenerate. This does not really affect the convergence order, provided the subdivision is taken appropriately. The result is summarized in Corollary 2.2 below. Finally, we give the convergence of a version of our algorithm applied to equation (2.7): when the dependence of the coefficients with respect to the law is general. This is given in Corollary 2.3.

In order to make the exposition of our results clear, let us define, for $i = 1, 2$:

$$\mathcal{E}_u^i(k) := \max_{\pi \in \mathcal{S}_\kappa(k)} |u(T_k, \hat{X}_{T_k}^\pi) - \hat{u}^i(T_k, \hat{X}_{T_k}^\pi)|; \qquad \mathcal{E}_v^i(k) := \max_{\pi \in \mathcal{S}_\kappa(k)} |v(T_k, \hat{X}_{T_k}^\pi) - \hat{v}^i(T_k, \hat{X}_{T_k}^\pi)|, \tag{2.13}$$

with $\hat{u}^1, \hat{u}^2, \hat{v}^1$ and $\hat{v}^2$ as defined by the algorithms 2 and 3 and where u, v are defined in (2.2), (2.6).

Main result in a smooth setting. We have that

Theorem 2.1. Suppose that assumption **(SB)** holds. Let m be a given cubature order, q a given non-negative integer and $\mathcal{T}(1, q, m)$ the cubature tree defined by Algorithm 1. Then, there exists a positive constant C, depending only on T, q, d, $||\varphi_{0:d}||_{2q+m+2,\infty}$, $||\phi||_{m+2,\infty}$, such that:

$$\max_{k\in\{0,\dots,N\}} |\langle \mu_{T_k} - \hat{\mu}_{T_k}, \phi\rangle| \leq C \left(\frac{1}{N}\right)^{[(m-1)/2]\wedge q}, \tag{2.14}$$

with $\hat{\mu}$ as defined in Algorithm 1.
Suppose in addition that $q \geq 1$ and $m \geq 3$. Then, there exists a positive constants C_1, depending only on T, q, d, $||\varphi_{0:d}||_{2q+m+2,\infty}$, $||\varphi_f||_{m+2,\infty}$,$||\phi||_{m+3,\infty}$, such that for all $k = 0, \dots, N$:

$$\mathcal{E}_u^1(k) + \Delta_{T_k}^{1/2} \mathcal{E}_v^1(k) \leq C_1 \left(\frac{1}{N}\right), \tag{2.15}$$

Moreover, suppose in addition that $q \geq 2$ and $m \geq 7$. Then, there exists a positive constant C_2, depending only on T, q, d, $||\varphi_{0:d}||_{2q+m+2,\infty}$, $||\varphi_f||_{m+2,\infty}$,$||\phi||_{m+4,\infty}$, such that for all $k = 0, \dots, N$:

$$\mathcal{E}_u^2(k) + \Delta_{T_k}^{1/2} \mathcal{E}_v^2(k) \leq C_2 \left(\frac{1}{N}\right)^2. \tag{2.16}$$

Convergence order for a Lipschitz boundary condition.

Corollary 2.2. Suppose that assumption **(LB)** holds. Let m be a given cubature order, q a given non-negative integer, γ a non negative real and $\mathcal{T}(\gamma, q, m)$ the cubature tree defined by the algorithm 1. Then, there exists a positive constant C depending only on T, $||\varphi||_{2q+m+2,\infty}$, $||\phi||_{1,\infty}$, such that:

$$|\langle \mu_T - \hat{\mu}_T, \phi\rangle| \quad \leq \quad C \left(\left(\frac{1}{N}\right)^{(\gamma/2)\wedge q} \vee L(\gamma, m) \right). \tag{2.17}$$

where

$$L(\gamma, m) = \begin{cases} N^{-\gamma/2} & \text{if} \quad \gamma \in (0, m-1) \\ N^{-(m-1)/2} \ln(N) & \text{if} \quad \gamma = m-1 \\ N^{-(m-1)/2} & \text{if} \quad \gamma \in (m-1, +\infty) \end{cases} \tag{2.18}$$

Moreover, if $\gamma > m-1$, the results on the error control of $\hat{u}^1, \hat{v}^1; \hat{u}^2$, and $\hat{v}^2$ respectively given by (2.15) and (2.16) remain valid (with a constant C_2' depending only on T, q, d, $||\varphi_{0:d}||_{2q+m+2,\infty}$, $||\varphi_f||_{m+2,\infty}$,$||\phi||_{1,\infty}$).

Note that, in (2.17), the control holds only at time T although it holds at each step in (2.14): this is because the boundary condition is Lipschitz only so that we have to wait for the smoothing effect to take place.

We emphasize that the result still applies if we let the boundary condition $\phi : (y, w) \in \mathbb{R}^d \times \mathbb{R} \mapsto \phi(y, w)$ depend also on the law of the process $(X_t^x,\ 0 \leq t \leq T)$. For example one can consider

$$\mathbb{E}\left[\phi(X_t^x, \mathbb{E}[\varphi_\phi(X_T^x)])\right]$$

for a given $\varphi_\phi \in C_b^{m+2}$ and where ϕ is Lipschitz in w uniformly in y.

The algorithm can be easily adapted to the case of the particular dependence explored in [15]:

$$V_i(t, y, \mu) = \langle \mu_t, V_i(t, y, \cdot)\rangle,\ i = 0, \cdots, d,$$

and the result of Theorem 2.1 and Corollary 2.2 remain valid. Note that in that case the uniform ellipticity **(LB)** has to be understood for the matrix $[\langle \eta_t, V(t,y,\cdot)\rangle]\,[\langle \eta_t, V(t,y,\cdot)\rangle]^*$ uniformly in y, t in $\mathbb{R}^d \times \mathbb{R}^+$ and η family of probability measures on $\mathbb{R}^d$.

Results under a more general law dependence.

As mentioned in the introduction, the algorithm may be modified to solve problems in a naturally extended framework. Let us precise the framework of this extension.

Let $\mathcal{F}$ and $\mathcal{F}'$ be two classes of functions, dense in the space of continuous functions that are zero at infinity. Let $d_{\mathcal{F}}$ $d_{\mathcal{F}'}$ be two distances as defined in (2.8). Recall that we suppose the vector fields V_i $0 \leq i \leq d$ that appear in (2.7) to be Lipschitz continuous with respect to $d_{\mathcal{F}}$ and the driver f to be Lipschitz continuous with respect to $d_{\mathcal{F}'}$. Furthermore, let us suppose that there exists a unique solution X_t^x, Y_t^x, Z_t^x to such a system and, as before, denote by u the decoupling function defined as (2.2) given its existence.

Clearly, we need to modify Algorithm 1 in the natural way to be used in this framework, that is, at each discretization time, we plug directly in the coefficients the cubature based law.

In order to retrieve higher orders of convergence, we need to expand the McKean term that appears in the coefficients. In this extended case, we have to be careful when considering the forward algorithm with $q > 1$: indeed, we must give sense to the expansion proposed at the definition of the functions F_i $0 \leq i \leq d$ in Algorithm 1. A good notion may be the one proposed in Section 7 of [17]. To avoid further technicalities, we will consider here only the case with $q = 1$, i.e., when no expansion is performed.

With this definitions and observations in mind, we give the main result under a more general law dependence.

Corollary 2.3. Let μ_T be the marginal law of the forward process in (2.7) at time T. Let $m \geq 3$ be a given cubature order and $\hat{\mu}_T$ be the discrete measure given by the cubature tree $\mathcal{T}(1,1,m)$ defined by the algorithm 1. Then, there exist two positive constants C_1 and C_2, depending only on T, d such that:

- If **(SB)** holds and $\mathcal{F}$ (resp. $\mathcal{F}'$) is the class of functions φ in $C_b^\infty(\mathbb{R}^d, \mathbb{R})$ (resp. $C_b^\infty(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$) such that $||\varphi||_{\infty,\infty} \leq 1$, then

$$d_{\mathcal{F}}(\mu_T, \hat{\mu}_T) + \mathcal{E}_u^1(k) + \Delta_{T_k}^{1/2}\mathcal{E}_v^1(k) \leq C_1 N^{-1}. \tag{2.19}$$

- If **(LB)** holds and $\mathcal{F}$ (resp. $\mathcal{F}'$) is the class of functions φ in $C_b^1(\mathbb{R}^d, \mathbb{R})$ (resp. $C_b^1(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$) such that $||\varphi||_{1,\infty} \leq 1$, then

$$d_{\mathcal{F}}(\mu_T, \hat{\mu}_T) + \mathcal{E}_u^1(k) + \Delta_{T_k}^{1/2}\mathcal{E}_v^1(k) \leq C_2 N^{-1/2}. \tag{2.20}$$

We emphasize that, when $\mathcal{F} = \{\varphi \in C_b^1(\mathbb{R}^d, \mathbb{R}),\ s.t.\ ||\varphi||_{1,\infty} \leq 1\}$, thanks to the Monge-Kantorovitch duality theorem, the distance $d_{\mathcal{F}}$ is the so-called Wasserstein-1 distance.

3 Numerical examples

In this section, we illustrate the algorithm behavior by applying it to a toy model for which the exact solution is available.

Consider the $d-$dimensional MKV-FBSDE on the interval $[0,1]$ with dynamics given by

$$\begin{aligned} dX_t &= \mathbb{E}\left(\sin(X_t)\right) dt + dB_t \\ -dY_t &= \left(\frac{\mathbf{1}\cdot\cos(X_t)}{2} + \mathbb{E}\left((\mathbf{1}\cdot\sin(X_t))\exp(-Y_t^2)\right)\right) dt - Z_t \cdot dB_t, \end{aligned}$$

where $(B_t)_{0\leq t\leq 1}$ is a $d-$dimensional Brownian motion, $\mathbf{1}$ is a $d-$dimensional vector having each entry equal to one and the sin and cos functions are applied entry-wise. Moreover, suppose that $X_0 = \mathbf{0}$. It is easily verified that a solution for the forward variable is $X = B$, and thanks to the uniqueness result this is the unique solution for the forward variable.

With respect to the backward part, take two different boundary conditions corresponding to the two considered set of assumptions **(SB)** and **(LB)**.

(SB) For $x \in \mathbb{R}^d$, we fix $\phi(x) = \mathbf{1}\cdot\cos(x)$. In this case, the solution to the backward part of the problem is

$$u(t,x) = \mathbf{1}\cdot\cos(x); \text{ and } v(t,x) = -\sin(x),$$

which clearly implies $Y_t = \mathbf{1}\cdot\cos(X_t)$ and $Z_t = -\sin(X_t)$.

(LB) We fix the boundary condition to be $\phi(x) := \phi'(d^{-1/2}(\mathbf{1}\cdot x))$ where ϕ' is the triangular function defined for all $y \in \mathbb{R}$ as

$$\phi'(y) = \begin{cases} y + K & \text{if } y \in (-K, 0] \\ -y + K & \text{if } y \in (0, K] \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the solution is given by

$$u(t,x) = \mathbb{E}\left[\phi(X_T^x) + \int_t^T \left(\frac{\mathbf{1}\cdot\cos(X_t)}{2} + \mathbb{E}\left((\mathbf{1}\cdot\sin(X_t))\exp(-Y_t^2)\right)\right) ds\right].$$

Basic properties of the Brownian motion imply that

$$u(t,x) = U(t, d^{-1/2}(\mathbf{1}\cdot x)) + (\mathbf{1}\cdot\cos(x))\left[\exp\left(\frac{t-1}{2}\right) - 1\right]$$

where

$$\begin{aligned} U(t,y) =& \sqrt{\frac{1-t}{2\pi}}\left[\exp\left(\frac{-(K+y)^2}{2(1-t)}\right) + \exp\left(\frac{-(K-y)^2}{2(1-t)}\right) - 2\exp\left(\frac{-y^2}{2(1-t)}\right)\right] \\ &+ (K+y)\left[F\left(\frac{-y}{\sqrt{1-t}}\right) - F\left(\frac{-K-y}{\sqrt{1-t}}\right)\right] + (K-y)\left[F\left(\frac{K-y}{\sqrt{1-t}}\right) - F\left(\frac{-y}{\sqrt{1-t}}\right)\right] \end{aligned}$$

and F is the cumulative distribution function of the standard normal distribution. Evidently $v(t,x) = D_x u(t,x)$, and is defined for $t < 1$.

3.1 Tests in dimension one

Given that the law dependence already increases the dimension of the problem, we start by presenting some results when we fix $d = 1$.

3.1.a *Forward component*

To implement the forward variable, we use the cubature formulae of order 3 and 5 presented in [69], which have paths support of size $\kappa = 2$ and $\kappa = 3$ respectively. Given the simple structure of the forward variable dynamics and the piecewise linear definition of the cubatures, we are able to solve explicitly the ODEs appearing during the tree construction. Hence there is no need to use any ODE solver.

In our first test we evaluate the weak approximation error of X using the function ϕ as test function. Indeed we plot as error for the **(SB)** case

$$\max_{k=1,\dots,N} |\langle \hat{\mu}_{T_k} - \mu_{T_k}, \cos \rangle| ,$$

while for the **(LB)** case, we plot

$$|\langle \hat{\mu}_{T_k} - \mu_{T_k}, \phi \rangle| ,$$

where ϕ is the defined triangular function with $K = 0.6$. As was pointed out before, the difference between the kind of error we are observing for each case is justified as a smoothing effect is needed for the approximation to be valid under **(LB)** .

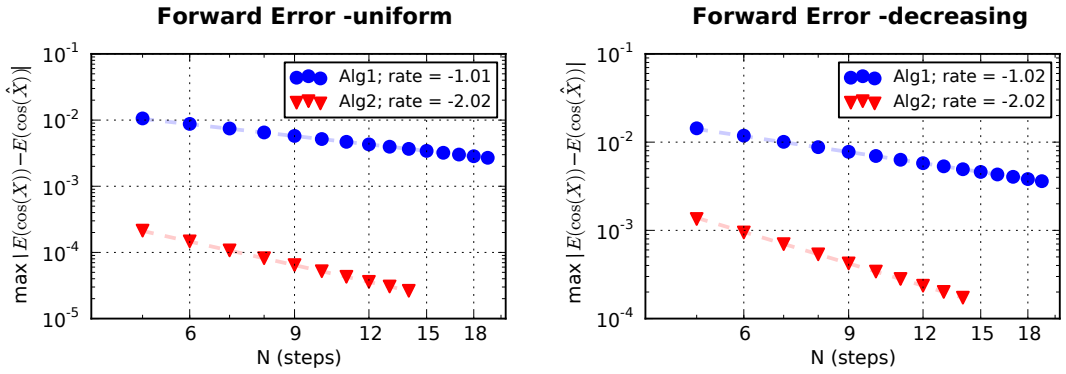


Figure 2.2: Weak approximation of the forward variable: The calculated rates are the slope of a linear regression on the last 8 points.

Figure 2.2 shows the obtained rate of convergence where we have used the uniform discretization in the **(SB)** case and the discretization with $\gamma = 2$ for the **(LB)** case. With the exception of the rate of convergence for the second order algorithm under **(LB)** (which is actually better than the predicted one), the expected rates of convergence are verified in both cases.

Moreover, under the smooth case, the benefit of using the higher order scheme is not only evident from a quickest convergence, but the error constant itself is smaller. This is an effect that depends on the particular example, but we remark it as it is interesting to notice that a higher order of convergence does not imply necessarily a higher initial constant.

3.1.b *Backward component*

Let us check now the approximation of the backward variable. We evaluate numerically

$$\max_{0\le k\le N-2;\ \pi\in\mathcal{S}_\kappa(k)} \left|\hat{u}^1(T_k, \hat{X}^\pi_{T_k}) - u(T_k \hat{X}^\pi_{T_k})\right| \quad \text{and} \quad \max_{0\le k\le N-2;\ \pi\in\mathcal{S}_\kappa(k)} \left|\hat{u}^2(T_k, \hat{X}^\pi_{T_k}) - u(T_k \hat{X}^\pi_{T_k})\right| ;$$

for both the (SB) and (LB) cases, where we fix $K = 0.6$ for the latter.

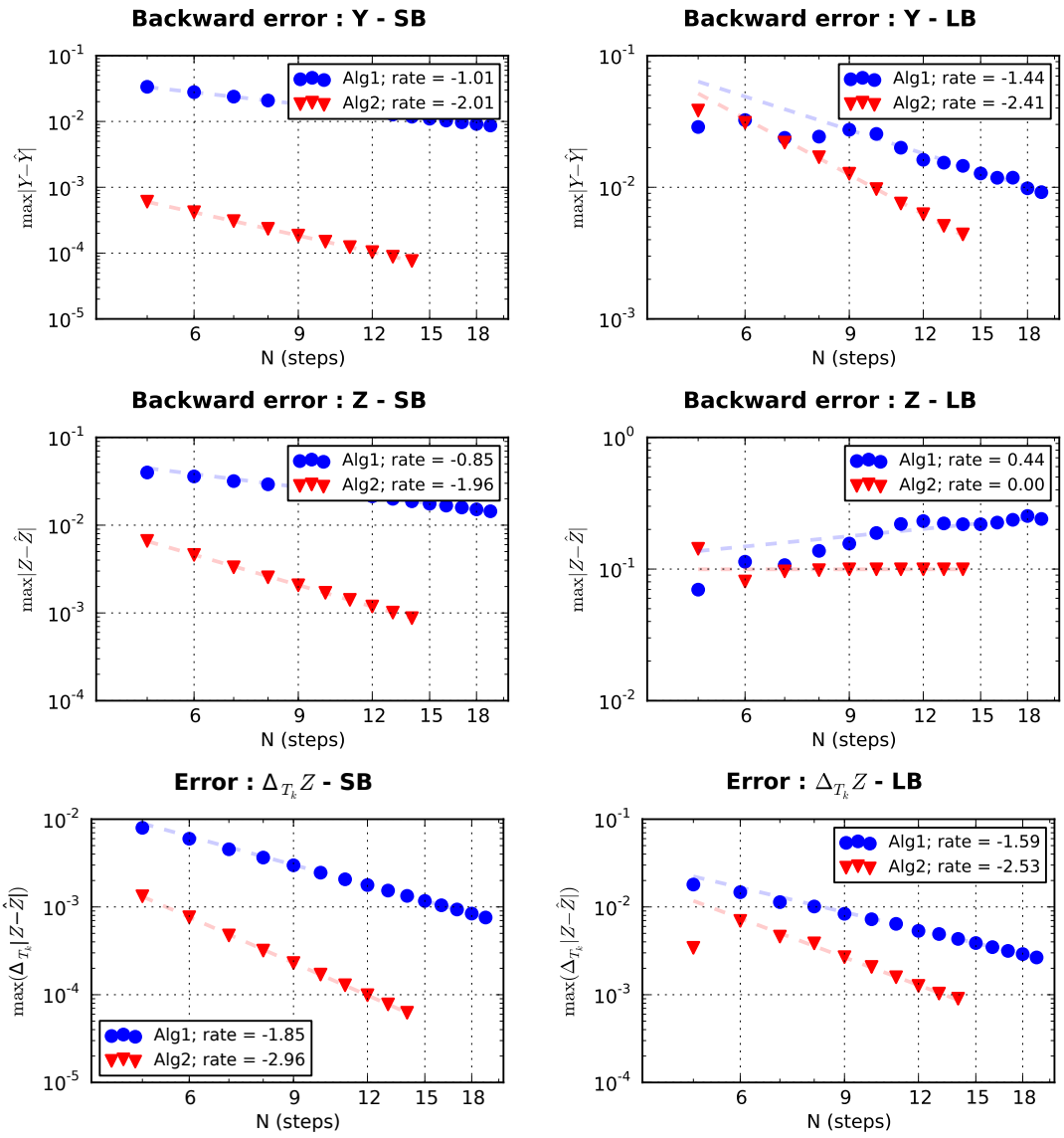


Figure 2.3: Weak approximation of the backward variable: The calculated rates are the slope of a linear regression on the last 8 points.

The specific structure of our examples allows us to obtain a second order convergence scheme with a cubature of order only 5. Indeed, in such a case, the terms in front of the leading rate of convergence on the cubature error estimate (cf Claim 2.13) are identically 0. Given that the order 5 cubature induces a lower complexity, it is simpler to carry out simulations for a larger number of steps.

As can be appreciated from the two uppermost plots in Figure 2.3, the expected rates of convergence for both algorithms are verified under the smooth and Lipschitz conditions. Just as we remarked in the forward approximation, solving the backward variable in the smooth case with the higher order scheme has the double benefit of better rate of convergence and smaller constant. As one would expect, due to the use of higher order derivatives, this is no longer true for the Lipschitz case.

It is interesting to look at the behavior of the other backward variable, Z. We look first at

an error of the type

$$\max_{0\leq k\leq N;\ \pi\in\mathcal{S}_\kappa(k)}\left|\hat{v}^1(T_k,\hat{X}^\pi_{T_k})-v(T_k\hat{X}^\pi_{T_k})\right| \quad \text{and} \quad \max_{0\leq k\leq N;\ \pi\in\mathcal{S}_\kappa(k)}\left|\hat{v}^2(T_k,\hat{X}^\pi_{T_k})-v(T_k\hat{X}^\pi_{T_k})\right|.$$

The two plots in the middle of Figure 2.3 are concerned with these errors. Although nice convergence is obtained in the smooth case, this is no longer true for the **(LB)** case, where the error stagnates. As will be clear from the analysis, this is a consequence of the singularity appearing at the boundary on the control of derivatives in this case. Hence, a more adequate error analysis considers errors given by

$$\max_{0\leq k\leq N;\ \pi\in\mathcal{S}_\kappa(k)}\Delta_{T_k}^{1/2}\left|\hat{v}^1(T_k,\hat{X}^\pi_{T_k})-v(T_k\hat{X}^\pi_{T_k})\right| \quad \text{and} \quad \max_{0\leq k\leq N;\ \pi\in\mathcal{S}_\kappa(k)}\Delta_{T_k}^{1/2}\left|\hat{v}^2(T_k,\hat{X}^\pi_{T_k})-v(T_k\hat{X}^\pi_{T_k})\right|.$$

The expected rate of convergence of this type of error is, respectively for $\hat{v}^1$ and $\hat{v}^2$, of the same order of the order of the error of $\hat{u}^1,\hat{u}^2$ with respect to u. As shown in the bottommost plots in Figure 2.3, the numerical tests for the **(SB)** and **(LB)** cases reflect the expected rates.

3.2 Tests in higher dimensions

We evaluate as well the algorithm using our test models **(SB)** ,**(LB)** with dimensions $d=2$ and $d=4$. For these tests, we evaluate only the first order schemes and use the 3-cubature formulae presented in [46] which have supports of size $\kappa=4$ and $\kappa=6$ respectively.

Figure 2.4 shows that, just as is in the one-dimensional case, the announced rates of convergence for the forward and backward variables are verified. Note that for the particular chosen examples, the error value changes just slightly with dimension.

The case of dimensions 2 and 4 show one of the current limitations of the method: its complexity grows, in general, exponentially both in terms of the number of iterations and the dimension of the problem. Indeed, considering once again the 3-cubature formula, we have that in general the number of nodes in the tree is

$$\sharp(nodes)=\frac{(2d)^{n-1}-1}{2d-1},$$

with the obvious effects on memory management and execution time. We remark that for some particular cases, the complexity can be radically lower. For instance, under the case of constant drift and diffusion coefficients and smooth boundary conditions, using symmetric cubature formulae (as we did here) leads to a kind of "auto-pruning" of the cubature tree leading to complexity grow of the form

$$\sharp(nodes)=\sum_{i=1}^{n} i^d,$$

i.e. polynomial in n with the order of the polynomial depending on the dimension d.

4 A class of control problems in a mean field environment

In this section, we show that equation (2.1) appears when solving a class of control problems inspired from the theory of mean field games but designed in such a way that the dynamics of the controlled process have no influence on the mean field environment.

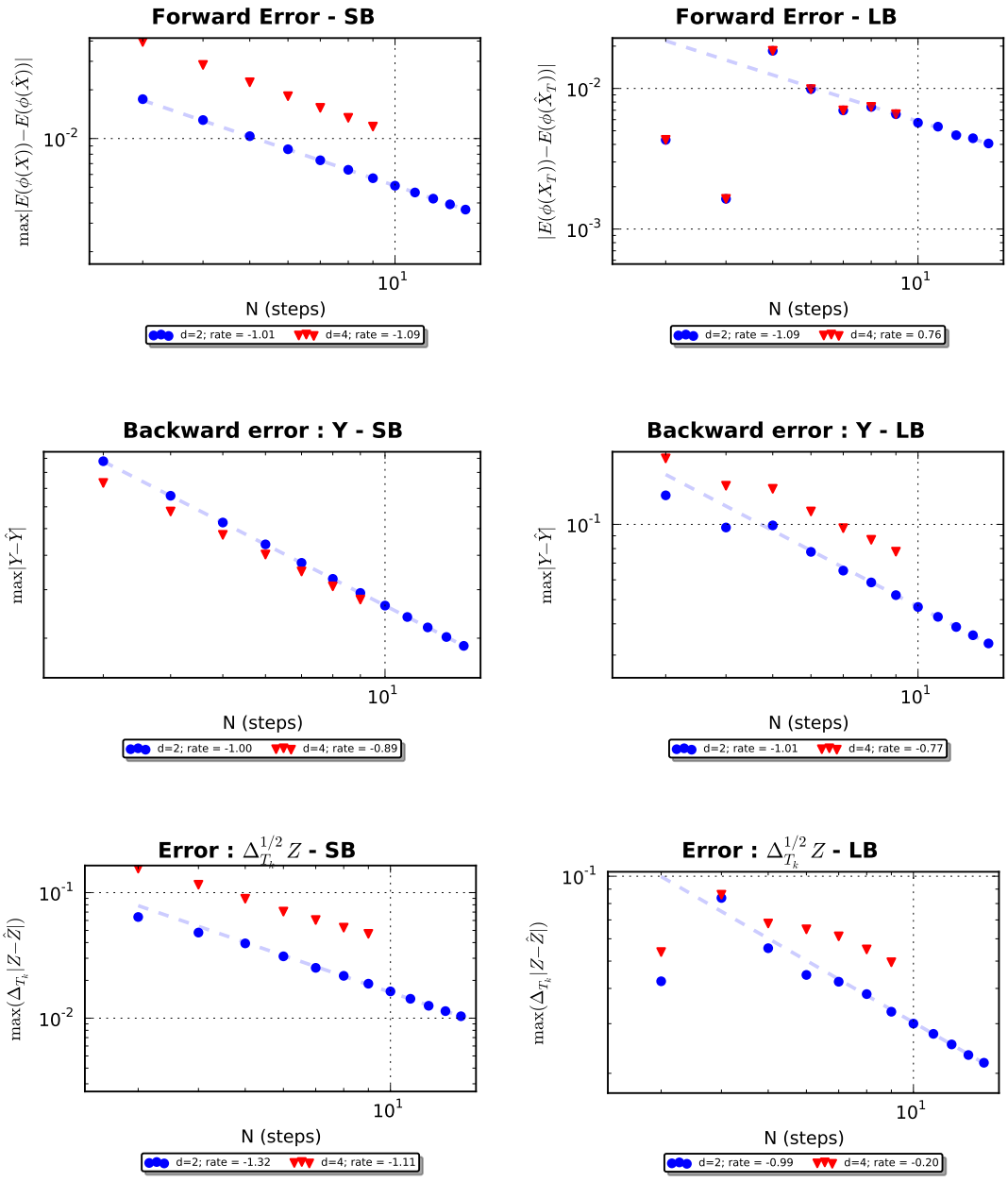


Figure 2.4: Results in dimension 2 and 4.

For the sake of illustration, consider for instance the problem of optimization of an issuer having a large portfolio of credit assets inspired in the framework presented in [16]. One of the methods used to model credit asset dynamics is the so called structural model (see [8] for a review on credit risk models). Under this model, we assume that a credit default is triggered when the value of the corresponding credit asset is below a certain threshold. In the original Merton setup, the default may only be triggered at a certain fixed maturity time T. In a rather more realistic view, the default is triggered the first time the credit asset is below the threshold.

We assume that the credit assets in the basket are small and homogeneous (for example, we suppose they belong to the same economic sector) so that their value is modeled by SDEs with the

same volatility and drift function terms. To simplify, we will consider the simpler Merton model. Moreover, in order to account for sector-wise contagion effects, we suppose there is a mean field type dependence in the dynamics. In addition to the credit assets, we suppose the issuer has a market portfolio used by the issuer to backup the credit risk, for example to comply with credit risk regulations, or to provide liquidity to its credit branch. Then, the value of the position of the issuer position is modeled by an SDE with coefficients depending on the contribution of all credit assets. The objective of the control problem is to maximize the value of the issuer position.

We will formalize mathematically a generalized version of the presented example. For this, we introduce a system in which a *marked* particle (the issuer in our example) with a controlled state variable X^α is immersed in an *environment* of M interacting particles (the credit assets in our example) with state variables $X^1, \dots, X^M$, and which dynamics are given by

$$\begin{cases} dX_t^1 = b(t, X_t^1, \frac{1}{M}\sum_{i=1}^M \delta_{X_t^i})dt + \sigma(t, X_t^1, \frac{1}{M}\sum_{i=1}^M \delta_{X_t^i})dB_t^1 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ dX_t^M = b(t, X_t^M, \frac{1}{M}\sum_{i=1}^M \delta_{X_t^i})dt + \sigma(t, X_t^M, \frac{1}{M}\sum_{i=1}^M \delta_{X_t^i})dB_t^M \\ dX_t^{1:M;\alpha} = b^0(t, X_t^1, X_t^{1:M;\alpha}, \frac{1}{M}\sum_{i=1}^M \delta_{X_t^i}, \mu_t, \alpha_t)dt + \sigma^0(t, X_t^1, X_t^{1:M;\alpha}, \frac{1}{M}\sum_{i=1}^M \delta_{X_t^i}, \mu_t, \alpha_t)dW_t \\ X_0^1 = \dots X_0^M = x, \quad X_0^{1:M;\alpha} = \bar{x} \end{cases}$$

where $(\alpha_t,\ t \geq 0)$ is a progressively measurable process with image in $A \subset \mathbb{R}$, $\{(B_t^1, \dots, B_t^M)^T = \bar{B}_t,\ t \geq 0\}$ is an M-dimensional Brownian motion and $(W_t,\ t \geq 0)$ is a d-dimensional Brownian motion possibly correlated with $\bar{B}$. Note that, in this framework, the marked player does not influence the dynamics of the other players.

For a large number of environment players, the system is approached by the McKean Vlasov system

$$\begin{cases} dX_t = b(t, X_t, \mu_t)dt + \sigma(t, X_t, \mu_t)dB_t \\ d\bar{X}_t^\alpha = b^0(t, X_t, \bar{X}_t^\alpha, \mu_t, \alpha_t)dt + \sigma^0(t, X_t, \bar{X}_t^\alpha, \mu_t)dW_t \\ X_0 = x, \quad \bar{X}_0^\alpha = \bar{x} \end{cases}$$

where μ_t is the law of X_t that we will assume in the following to be fixed. Assume that the marked player is interested in minimizing the cost functional

$$J(t, x, \bar{x}, \alpha) = \mathbb{E}\left[g(X_T^{t,x}, \bar{X}_T^{\alpha;t,\bar{x}}, \mu_T) + \int_0^T f(s, X_s^{t,x}, \bar{X}_s^{\alpha;t,\bar{x}}, \mu_s)ds\right],$$

for $\alpha \in \mathcal{A}$, the set of all progressively measurable process $\alpha = (\alpha_t,\ t \geq 0)$ valued in A (the maximization case is available up to a change of sign). We want to solve for the optimal value function $u(t, x, \bar{x}) = \inf\{J(t, x, \bar{x}, \alpha),\ \alpha \in \mathcal{A}\}$. Then, the associated Hamilton Jacobi Bellman equation for u reads

$$\begin{aligned} 0 =& D_t u(t, x, \bar{x}) + \frac{1}{2}\mathrm{Tr}(\bar{a} D^2_{x,\bar{x}} u(t, x, \bar{x})) + b(t, x, \mu_t) D_x u \\ &+ H(t, x, \bar{x}, D_x u, \mu_t), \end{aligned} \tag{2.21}$$

with

$$\bar{a} = \begin{bmatrix} \sigma\sigma^T & \sigma\rho(\sigma^0)^T \\ \sigma^0\rho^T\sigma^T & \sigma^0(\sigma^0)^T \end{bmatrix}, \qquad \rho = [B, W]$$

and where $[\cdot,\cdot]$ stands for the quadratic variation and H is the Hamiltonian

$$H(t,x,\bar{x},z,\mu_t) = \inf_{\alpha\in\mathcal{A}} \left[b^0(t,x,\bar{x},\mu_t,\alpha)z + f(t,x,\bar{x},\mu_t)\right].$$

We will not discuss here the resolvability of the HJB equation (see e.g. [34] or [87] for a partial revue). We can interpret (2.21) from a probabilistic point of view: we have that $u(t,x,\bar{x}) = Y_t^{t,x,\bar{x}}$ where $Y^{t,x,\bar{x}}$ is given by the MKV-FBSDE

$$\begin{cases} dX_s^{t,x,\bar{x}} = b(s,X_s^{t,x,\bar{x}},\mu_s)ds + \sigma(s,X_s^{t,x,\bar{x}},\mu_s)dB_s \\ d\bar{X}_s^{t,x,\bar{x}} = \sigma^0(s,X_s^{t,x,\bar{x}},\bar{X}_s^{t,x,\bar{x}},\mu_s)dW_s \\ -dY_s^{t,x,\bar{x}} = H(s,X_s^{t,x,\bar{x}},\bar{X}_s^{t,x,\bar{x}},\bar{Z}_s^{t,x,\bar{x}},\mu_s) - \bar{Z}_s^{t,x,\bar{x}}dW_s + Z^{t,x,\bar{x}}dB_s \\ X_t^{t,x,\bar{x}} = x, \quad \bar{X}_t^{t,x,\bar{x}} = \bar{x}, \quad Y_0^{t,x,\bar{x}} = g(X_T^{t,x,\bar{x}},\bar{X}_T^{t,x,\bar{x}},\mu_T). \end{cases}$$

The reader may object that the Hamiltonian H does not satisfy the boundedness condition we have assumed for the analysis of the algorithm (bounded with bounded derivatives w.r.t. the variable z). However, some relatively mild assumptions guarantee that the first derivative term $\bar{Z}$ will be bounded. This is almost direct when the boundary condition g is bounded and smooth and proved in [25] when g is Lipschitz and the diffusion matrix uniformly elliptic. Hence, given an estimate on this quantity, one may introduce a modified system in which we replace in the function $(Z,\bar{Z})$ by $(\psi(Z),\bar{\psi}(\bar{Z}))$, where $\psi,\bar{\psi}$ are truncation functions used to make the value of $Z,\bar{Z}$ satisfy its known estimates, as in [88] (if the estimate is not explicitly known, a sequence of functions approximating the identity may be used as in [47], but some additional work would be needed to account for the truncation error). In both cases, the truncated problem will then satisfy the needed assumptions and may be solved with the presented Algorithm 1, 2.

5 Preliminaries

In the following we set a subdivision $T_0 = 0 < \cdots < T_N = T$ of $[0,T]$.

Artificial dynamics. We denote by $\underline{s}$ the mapping $s \mapsto \underline{s} = T_k$ if $s \in [T_k, T_{k+1})$, $k \in \{0,\cdots,N-1\}$.

For any family of probability measures η^1 and η^2, one denotes by P^{η^1} and $\tilde{P}^{\eta^2}$ the operators such that, for all $t < s$ in $[0,T]$, for all measurable function g from $\mathbb{R}^d$ to $\mathbb{R}$ and for all y in $\mathbb{R}^d$:

$$P_{t,s}^{\eta^1} g(y) = \mathbb{E}[g(X_s^{t,y,\eta^1})] \text{ and } \tilde{P}_{t,s}^{\eta^2} g(y) = \mathbb{E}[g(\tilde{X}_s^{t,y,\eta^2})]$$

and $(\mathcal{L}_s^{\eta^1})_{t\le s\le T}$ and $(\tilde{\mathcal{L}}_s^{\eta^2})_{t\le s\le T}$ their infinitesimal generator, where for all g in $C^2(\mathbb{R}^d,\mathbb{R})$

$$\mathcal{L}_s^{\eta^1} g(y) := V_0(s,y,\langle\eta_s^1,\varphi_0\rangle)\cdot D_y g(y) + \frac{1}{2}\mathrm{Tr}[VV^T(s,y,\langle\eta_s^1,\varphi\rangle)D_y^2 g(y)] \tag{2.22}$$

and by definition $\tilde{\mathcal{L}}_s^{\eta^2} = \mathcal{L}_{\underline{s}}^{\eta^2}$. Here X^{t,y,η^1} and $\tilde{X}^{t,y,\eta^2}$ are the respective solutions of

$$dX_s^{t,y,\eta^1} = \sum_{i=0}^d V_i\left(s,X_s^{t,y,\eta^1},\langle\eta_s^1,\varphi_i\rangle\right)dB_s^i, \quad X_t^{t,y,\eta^1} = y, \tag{2.23}$$

$$d\tilde{X}_s^{t,y,\eta^2} = \sum_{i=0}^d V_i\left(s,\tilde{X}_s^{t,y,\eta^2},\sum_{p=0}^{q-1}[(t-\underline{t})^p/p!]\langle\eta_{\underline{s}}^2,(\tilde{\mathcal{L}}^{\eta^2})^p\varphi_i\rangle\right)dB_t^i, \quad \tilde{X}_t^{t,y,\eta^2} = y. \tag{2.24}$$

Finally, let us define the operator associated to the cubature measure, $Q^{\hat{\mu}}$, as

$$Q^{\hat{\mu}}_{t,s}g(y) = \mathbb{E}_{\mathbb{Q}_{t,s}}[g(\tilde{X}^{t,y,\hat{\mu}}_s)] \tag{2.25}$$

for all $t < s$ in $[0,T]$ for all y in $\mathbb{R}^d$ and for all measurable function g from $\mathbb{R}^d$ to $\mathbb{R}$. Note that for all k in $\{1,\dots,N\}$:

$$Q^{\hat{\mu}}_{0,T_k}g(x) = \langle \hat{\mu}_{T_k}, g\rangle.$$

Multi-index (2). Let $\mathcal{M}$ be defined by (2.9). Let $\beta \in \mathcal{M}$. We define $|\beta| = l$ if $\beta = (\beta_1,\dots,\beta_l)$, $|\beta|_0 := \text{card}\{i : \beta_i = 0\}$ and $\|\beta\| := |\beta| + |\beta|_0$. Naturally $|\emptyset| = |\emptyset|_0 = \|\emptyset\| = 0$. For every $\beta \neq \emptyset$, we set $-\beta := (\beta_2,\dots,\beta_l)$ and $\beta- := (\beta_1,\dots,\beta_{l-1})$. We set β^+ the multi-index obtained by deleting the zero components of β.

We will frequently refer to the set of multi-indices of degree at most l denoted by $\mathcal{A}_l := \{\beta \in \mathcal{M} : \|\beta\| \leq l\}$. We define as well its *frontier set* $\partial\mathcal{A} := \{\beta \in \mathcal{M} \setminus \mathcal{A} : -\beta \in \mathcal{A}\}$. We can easily check that $\partial\mathcal{A}_l \subset \mathcal{A}_{l+2} \setminus \mathcal{A}_l$.

Directional derivatives. For notational convenience, let us define the second order operator

$$\mathcal{V}_{(0)} := \partial_t + \mathcal{L},$$

and for $j = \{1,\dots,d\}$ the operator

$$\mathcal{V}_{(j)} := V_j \frac{\partial}{\partial x_j}.$$

where, as announced in the notation section, we do not mark explicitly the time, space and law dependence. For every $\|\beta\| \leq l$ let us define recursively

$$\mathcal{V}_\beta g := \begin{cases} g & \text{if } |\beta| = 0 \\ V_{\beta_1}\mathcal{V}_{-\beta}g & \text{if } |\beta| > 0, \end{cases} \tag{2.26}$$

provided that $g : [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}$ is smooth enough. Hence, for $n \in \mathbb{N}$ we denote by $\mathcal{D}^n_b$, the space of such functions g for which $\mathcal{V}_\beta g$ exists and is bounded for every $\beta \in \mathcal{A}_n$. For any function g in $\mathcal{D}^n_b$, we set for all $\beta \in \mathcal{A}_n$,

$$D_\beta g := \frac{\partial}{\partial y_{\beta_1}} \cdots \frac{\partial}{\partial y_{\beta_{|\beta|}}} g,$$

where $[\partial/\partial y_0]$ must be understood as $[\partial/\partial t]$.

Iterated integrals. For any multi-index β and adapted process g we define for all $t < s \in [0,T]$ the *multiple Itô integral* $I^{t,s}_\beta[g]$ recursively by

$$I^{t,s}_\beta(g) = \begin{cases} g(\tau) & \text{if } |\beta| = 0 \\ \displaystyle\int_t^s I^{\rho,r}_{\beta-}(g)dr & \text{if } |\beta| > 0 \text{ and } \beta_l = 0 \\ \displaystyle\int_t^s I^{t,r}_{\beta-}(g)dB^{\beta_l}_r & \text{if } |\beta| > 0 \text{ and } \beta_l > 0 \end{cases}$$

We will write $I^{t,s}_\beta := I^{t,s}_\beta(1)$.

The previous notation is very convenient to introduce an Itô-Taylor expansion, that is an analogue of Taylor formula when dealing with Itô processes. The proof follows simply by repeated iteration of Itô's lemma, and may be found (without the law dependence) in [53].

Lemma 2.4. Let $t < s \in [0,T]$ and $y \in \mathbb{R}^d$. Let $n \in \mathbb{N}^*$ and let g in D_b^n Then, for each family of probability measures η on $\mathbb{R}^d$, we have have an *Itô-Taylor expansion of order* n, that is

$$g(s, X_s^{t,y,\eta}) = g(t,y) + \sum_{\beta \in \mathcal{A}_n} \mathcal{V}_\beta g(t,y) I_\beta^{t,s} + \sum_{\beta \in \partial\mathcal{A}_n} I_\beta^{t,s}[\mathcal{V}_\beta g(.,X_.^{t,y,\eta})]$$

where $(X_s^{t,y,\eta},\ t \le s \le T)$ is the solution of (2.24).

The following lemma is a particular case of a result in [53]. It follows from integration by parts formula and expectation properties.

Lemma 2.5. Let $\beta \in \mathcal{M}$, and let $t_1 < t_2 \in [0,T]$. Then for any bounded and measurable functions g_1 and g_2 in $[t_1,t_2]$ there exists a constant depending only on β and i such that

$$\mathbb{E}\left(I_\beta^{t_1,t_2}(g_1) I_{(i)}^{t_1,t_2}(g_2) \middle| \mathcal{F}_{t_1}\right) \le \mathbf{1}_{\{(\beta)^+ = i\}} C(\beta,i)(t_2-t_1)^{(||\beta||+1)/2} \sup_{t_1 \le s \le t_2} |g_1(s)| \sup_{t_1 \le s \le t_2} |g_2(s)|,$$

$$\mathbb{E}\left(I_\beta^{t_1,t_2}(g_1) I_{(0,i)}^{t_1,t_2}(g_2) \middle| \mathcal{F}_{t_1}\right) \le \mathbf{1}_{\{(\beta)^+ = i\}} C'(\beta,i)(t_2-t_1)^{(||\beta||+3)/2} \sup_{t_1 \le s \le t_2} |g_1(s)| \sup_{t_1 \le s \le t_2} |g_2(s)|.$$

6 Proof of Theorem 2.1 under (SB)

6.1 Rate of convergence of the forward approximation: proof of (2.14)

Here we prove the approximation order of the forward component. Let η be a given family of probability measures on $\mathbb{R}^d$. We have the following decomposition of the error:

$$\begin{aligned}
(P_{T_0,T_N} - Q^{\hat\mu}_{T_0,T_N})\phi(x) &= (P_{T_0,T_N} - P^\eta_{T_0,T_N})\phi(x) + Q^{\hat\mu}_{T_0,T_{N-1}}(P^\eta_{T_{N-1},T_N} - Q^{\hat\mu}_{T_{N-1},T_N})\phi(x) \\
&\quad + Q^{\hat\mu}_{T_0,T_{N-2}} P^\eta_{T_{N-2},T_N}\phi(x) - Q^{\hat\mu}_{T_0,T_{N-1}} P^\eta_{T_{N-1},T_N}\phi(x) \\
&\quad + Q^{\hat\mu}_{T_0,T_{N-3}} P^\eta_{T_{N-3},T_N}\phi(x) - Q^{\hat\mu}_{T_0,T_{N-2}} P^\eta_{T_{N-2},T_N}\phi(x) \\
&\quad \vdots \\
&\quad + P^\eta_{T_0,T_N}\phi(x) - Q^{\hat\mu}_{T_0,T_1} P^\eta_{T_1,T_N}\phi(x) \\
&= (P_{T_0,T_N} - P^\eta_{T_0,T_N})\phi(x) + (P^\eta_{T_0,T_1} - Q^{\hat\mu}_{T_0,T_1}) P^\eta_{T_1,T_N}\phi(x) \\
&\quad + \sum_{j=1}^{N-1} Q^{\hat\mu}_{T_0,T_j}\left[(P^\eta_{T_j,T_{j+1}} - Q^{\hat\mu}_{T_j,T_{j+1}}) P^\eta_{T_{j+1},T_N}\phi(x)\right],
\end{aligned}$$

so that:

$$\begin{aligned}
(P_{T_0,T_N} - Q^{\hat\mu}_{T_0,T_N})\phi(x) &= (P_{T_0,T_N} - P^\eta_{T_0,T_N})\phi(x) + (P^\eta_{T_0,T_1} - Q^{\hat\mu}_{T_0,T_1}) P^\eta_{T_1,T_N}\phi(x) \\
&\quad + \sum_{j=1}^{N-2} Q^{\hat\mu}_{T_0,T_j}\left[(P^\eta_{T_j,T_{j+1}} - Q^{\hat\mu}_{T_j,T_{j+1}}) P^\eta_{T_{j+1},T_N}\phi(x)\right] \\
&\quad + Q^{\hat\mu}_{T_0,T_{N-1}}\left[(P^\eta_{T_{N-1},T_N} - Q^{\hat\mu}_{T_{N-1},T_N})\phi(x)\right]
\end{aligned}$$

That is, the global error is decomposed as a sum of local errors (i.e., as the sum of errors on each interval). These local errors can be also split. Let us define the function

$$\psi(T_k, x) := P^\eta_{T_k,T_N}\phi(x). \tag{2.27}$$

We emphasize that for all t in $[0,T)$, $y \mapsto \psi(t,y)$ is C_b^∞. Indeed, this function can be seen as the solution of the PDE

$$\begin{cases} \partial_t \psi(t,y) + \mathcal{L}^\eta \psi(t,y) = 0, \text{ on } [0,T] \times \mathbb{R}^d \\ \psi(T,y) = \phi(y) \end{cases} \tag{2.28}$$

taken at time T_k, where $\mathcal{L}^\eta$ is defined by (2.22). The claim follows from Lemma 2.14. On each interval Δ_{T_k}, $k = 1, \cdots, N-1$, the local error $(P^\eta_{T_k,T_{k+1}} - Q^{\hat{\mu}}_{T_k,T_{k+1}})\psi(T_{k+1},x)$ is expressed as

$$\begin{aligned} (P^\eta_{T_k,T_{k+1}} - Q^{\hat{\mu}}_{T_k,T_{k+1}})\psi(T_{k+1},x) &= (P^\eta_{T_k,T_{k+1}} - \tilde{P}^{\hat{\mu}}_{T_k,T_{k+1}})\psi(T_{k+1},x) && (2.29) \\ &\quad + (\tilde{P}^{\hat{\mu}}_{T_k,T_{k+1}} - Q^{\hat{\mu}}_{T_k,T_{k+1}})\psi(T_{k+1},x). && (2.30) \end{aligned}$$

Error (2.29) can be identified as a frozen (in time) error (and so, a sort of weak Euler error) plus an approximation error, in the sense that in step k, the measure μ_{T_k} is approximated by the discrete law $\hat{\mu}_{T_k}$. Then, (2.30) is a (purely) cubature error on one step, and we have:

$$\begin{aligned} &(P_{T_0,T_N} - Q^{\hat{\mu}}_{T_0,T_N})\phi(x) \\ &= (P_{T_0,T_N} - P^\eta_{T_0,T_N})\phi(x) \\ &\quad + (P^\eta_{T_0,T_1} - \tilde{P}^{\hat{\mu}}_{T_0,T_1})\psi(T_1,x) + \sum_{j=1}^{N-2} Q^{\hat{\mu}}_{T_0,T_j}\left[(P^\eta_{T_j,T_{j+1}} - \tilde{P}^{\hat{\mu}}_{T_j,T_{j+1}})\psi(T_{j+1},x)\right] \\ &\quad + (\tilde{P}^{\hat{\mu}}_{T_0,T_1} - Q^{\hat{\mu}}_{T_0,T_1})\psi(T_1,x) + \sum_{j=1}^{N-2} Q^{\hat{\mu}}_{T_0,T_j}\left[(\tilde{P}^{\hat{\mu}}_{T_j,T_{j+1}} - Q^{\hat{\mu}}_{T_j,T_{j+1}})\psi(T_{j+1},x)\right] \\ &\quad + Q^{\hat{\mu}}_{T_0,T_{N-1}}\left[(P^\eta_{T_{N-1},T_N} - Q^{\hat{\mu}}_{T_{N-1},T_N})\phi(x)\right] \end{aligned}$$

We have the two following Claims:

Claim 2.6. There exists a positive constant $C(T,V_{0:d})$ depending on the regularity of the $V_{0:d}$ and on T such that, for all y in $\mathbb{R}^d$, for all family of probability measures η, for all $k \in \{1,\cdots,N-1\}$, one has:

$$\begin{aligned} &\left|(P^\eta_{T_k,T_{k+1}} - \tilde{P}^{\hat{\mu}}_{T_k,T_{k+1}})\psi(T_{k+1},y)\right| \\ &\leq C(T,V_{0:d})||\psi(T_{k+1},.)||_{2,\infty}\sum_{i=0}^{d}\int_{T_k}^{T_{k+1}}\left|\langle\eta_t,\varphi_i\rangle - \sum_{p=0}^{q-1}[(t-T_k)^p/p!]\langle\eta_{T_k},(\mathcal{L}^\eta)^p\varphi_i\rangle\right|dt \\ &\quad + C(T,V_{0:d})||\psi(T_{k+1},.)||_{2,\infty}\sum_{i=0}^{d}\sum_{p=0}^{q-1}[\Delta^p_{T_{k+1}}/p!]\left|\langle\eta_{T_k},(\mathcal{L}^\eta)^p\varphi_i\rangle - \langle\hat{\mu}_{T_k},(\mathcal{L}^{\hat{\mu}})^p\varphi_i\rangle\right|. \end{aligned}$$

Proof. We deduce the claim by applying Lemma 2.16 to $y \mapsto \psi(T_k,y)$ for each $k \in \{0,\cdots,N-2\}$. □

Claim 2.7. There exists a positive constant $C(T,V_{0:d},d,m)$ depending on the regularity of the $V_{0:d}$ on the dimension d and the cubature order m such that: for all y in $\mathbb{R}^d$, for all family of probability measure η, for all $k \in \{0,\cdots,N-1\}$, one has:

$$\left|(\tilde{P}^{\hat{\mu}}_{T_k,T_{k+1}} - Q^{\hat{\mu}}_{T_k,T_{k+1}})\psi(T_{k+1},y)\right| \leq C(T,V_{0:d},d,m)\sum_{l=m+1}^{m+2}||\psi(T_{k+1},\cdot)||_{l,\infty}\Delta^{\frac{l}{2}}_{T_{k+1}}.$$

Proof. The claim follows, by applying Lemma 2.17 with $\hat{\mu}$ to the function $y \in \mathbb{R}^d \mapsto \psi(T_k, y)$ for each $k \in \{1, \cdots, N-1\}$. □

Then, by plugging estimates of Claims 2.6 and 2.7 in the error expansion we deduce that:

$$\begin{aligned}
&\left|(P_{T_0,T_N} - Q^{\hat{\mu}}_{T_0,T_N})\phi(x)\right| \qquad (2.31)\\
&\leq C(T, V_{0:d}) \sum_{j=0}^{N-1} ||\psi(T_{j+1}, \cdot)||_{2,\infty} \int_{T_j}^{T_{j+1}} \sum_{i=0}^{d} \left| \langle \eta_t, \varphi_i \rangle - \sum_{p=0}^{q-1} [(t-T_k)^p/p!] \langle \eta_{T_j}, (\mathcal{L}^\eta)^p \varphi_i \rangle \right| dt \\
&\quad + C(T, V_{0:d}) \sum_{j=0}^{N-1} ||\psi(T_{j+1}, \cdot)||_{2,\infty} \sum_{p=0}^{q-1} \Delta_{T_{j+1}}^{p+1} \sum_{i=0}^{d} \left| \langle \eta_{T_j}, (\mathcal{L}^\eta)^p \varphi_i \rangle - \langle \hat{\mu}_{T_j}, (\mathcal{L}^{\hat{\mu}})^p \varphi_i \rangle \right| \\
&\quad + C(T, V_{0:d}, d, m) \sum_{j=0}^{N-1} \sum_{l=m+1}^{m+2} ||\psi(T_{j+1}, \cdot)||_{l,\infty} \Delta_{T_{j+1}}^{\frac{l}{2}} + \left| (P_{T_0,T_N} - P^\eta_{T_0,T_N})\phi(x) \right|
\end{aligned}$$

Up to now, the analysis holds for any family of probability measures η. *The key point in the proof is to note that we can actually choose* $\eta = \mu$, that is, the law of the solution of the forward component in (2.1). In that case for all measurable function g:

$$\langle \eta_\cdot, g \rangle = \langle \mu_\cdot, g \rangle = \mathbb{E}[g(X^x_\cdot)].$$

Then, (2.31) becomes:

$$\begin{aligned}
&\left|(P_{T_0,T_N} - Q^{\hat{\mu}}_{T_0,T_N})\phi(x)\right| \qquad (2.32)\\
&\leq C(T, V_{0:d}) \sum_{j=0}^{N-1} ||\psi(T_{j+1}, \cdot)||_{2,\infty} \int_{T_j}^{T_{j+1}} \sum_{i=0}^{d} \left| \mathbb{E}\left[\varphi_i(X_t^{x,\mu})\right] - \sum_{p=0}^{q-1} [(t-T_k)^p/p!] \mathbb{E}\left[(\mathcal{L}^\mu)^p \varphi_i(X_{T_j}^{x,\mu})\right] \right| dt \\
&\quad + C(T, V_{0:d}) \sum_{j=0}^{N-1} ||\psi(T_{j+1}, \cdot)||_{2,\infty} \sum_{p=0}^{q-1} \Delta_{T_{j+1}}^{p+1} \sum_{i=0}^{d} \left| \mathbb{E}\left[(\mathcal{L}^\mu)^p \varphi_i(X_{T_j}^{x,\mu})\right] - \mathbb{E}\left[(\mathcal{L}^{\hat{\mu}})^p \varphi_i(\hat{X}_{T_j}^{x,\hat{\mu}})\right] \right| \\
&\quad + C(T, V_{0:d}, d, m) \sum_{j=0}^{N-1} \sum_{l=m+1}^{m+2} ||\psi(T_{j+1}, \cdot)||_{l,\infty} \Delta_{T_{j+1}}^{\frac{l}{2}}
\end{aligned}$$

since $P_{s,t} = P^\mu_{s,t}$ for all $s < t \in [0, T]$, by definition. Now we have:

Claim 2.8. For any $k \in \{0, \cdots, N-1\}$, and for all t in $[T_k; T_{k+1})$ there exists a positive constant $C(d, V_{0:d})$ such that:

$$\int_{T_k}^{T_{k+1}} \sum_{i=0}^{d} \left| \mathbb{E}\left[\varphi_i(X_t^{x,\mu})\right] - \sum_{p=0}^{q-1} [(t-T_k)^p/p!] \mathbb{E}\left[(\mathcal{L}^\mu)^p \varphi_i(X_{T_k}^{x,\mu})\right] \right| dt \leq C(d, V_{0:d}) \|\varphi\|_{2q,\infty} \Delta_{T_{k+1}}^{q+1}$$

Proof. This follows by Itô-Taylor expansion of order q of $\varphi_i(X^{X_{T_k},\mu})$ for each $i = 0, \cdots, d$ and for any k in $\{0, \cdots, N-1\}$. □

Therefore,

$$\begin{aligned}
&\left|(P_{T_0,T_N} - Q^{\hat{\mu}}_{T_0,T_N})\phi(x)\right| \qquad (2.33)\\
&\leq C(T, V_{0:d}) \sum_{j=0}^{N-1} ||\psi(T_{j+1},\cdot)||_{2,\infty} \sum_{p=0}^{q-1} \Delta_{T_{j+1}}^{p+1} \sum_{i=0}^{d} \left|(P_{T_0,T_N} - Q^{\hat{\mu}}_{T_0,T_N})\mathcal{L}^p\varphi_i(x)\right|\\
&\quad + C(T, V_{0:d}, d)\|\varphi\|_{2q,\infty} \sum_{j=0}^{N-1} ||\psi(T_{j+1},\cdot)||_{2,\infty} \Delta_{T_{j+1}}^{q+1}\\
&\quad + C(T, V_{0:d}, d, m) \sum_{j=0}^{N-1} \sum_{l=m+1}^{m+2} ||\psi(T_{j+1},\cdot)||_{l,\infty} \Delta_{T_{j+1}}^{\frac{l}{2}}.
\end{aligned}$$

Thanks to estimate (2.72) in Lemma 2.14, for all n in $\mathbb{N}$, we have the following bound on the supremum norm of the derivatives of ψ up to order n:

$$||\nabla_y^n \psi(t,\cdot)||_\infty \leq C(T, V_{0:d})||\phi||_{n,\infty}. \qquad (2.34)$$

By plugging this bound in (2.33) we get

$$\begin{aligned}
&\left|(P_{T_0,T_N} - Q^{\hat{\mu}}_{T_0,T_N})\phi(x)\right|\\
&\leq C(T, V_{0:d})||\phi||_{2,\infty} \sum_{j=0}^{N-1} \sum_{p=0}^{q-1} \Delta_{T_{j+1}}^{p+1} \sum_{i=0}^{d} \left|(P_{T_0,T_j} - Q^{\hat{\mu}}_{T_0,T_j})\mathcal{L}^p\varphi_i(x)\right|\\
&\quad + C(T, V_{0:d}, d, m)(||\phi||_{2m+2,\infty} + ||\varphi||_{2q,\infty}) \left(\frac{1}{N}\right)^{q\wedge(m-1)/2}.
\end{aligned}$$

It should be remarked that the term on the right hand side is controlled in terms of the approximation error itself, acting on the functions $\mathcal{L}^p\varphi_i,\ i = 1,\cdots,d,\ p = 0,\cdots,q-1$, from step 0 to j for any j in $\{1,\cdots,N-1\}$. To proceed, the argument is the following one: since these bounds hold for all (at least) ϕ smooth enough, one can let $\phi = \varphi_0$, and use the discrete Gronwall Lemma to get the following bound on φ_0:

$$\begin{aligned}
\left|(P_{T_0,T_N} - Q^{\hat{\mu}}_{T_0,T_N})\varphi_0(x)\right| \quad \leq \quad & C(T, V_{0:d}, m, ||\phi||_{m+2,\infty}, ||\varphi||_{2q,\infty}, ||\varphi_0||_{2q+m,\infty}) \left(\frac{1}{N}\right)^{q\wedge[(m-1)/2]}\\
&\times \Bigg\{ \sum_{j=0}^{N-1} \sum_{p=0}^{q-1} \Delta_{T_{j+1}}^{p+1} \sum_{i=1}^{d} \left|(P_{T_0,T_j} - Q^{\hat{\mu}}_{T_0,T_j})\mathcal{L}^p\varphi_i(x)\right|\\
&\quad + \sum_{j=0}^{N-1} \sum_{p=1}^{q-1} \Delta_{T_{j+1}}^{p+1} \left|(P_{T_0,T_j} - Q^{\hat{\mu}}_{T_0,T_j})\mathcal{L}^p\varphi_0(x)\right| \Bigg\}
\end{aligned}$$

It is clear that, by iterating this argument (i.e., by letting $\phi = \varphi_1$ and then $\phi = \varphi_2, \ldots$, $\phi = \mathcal{L}\varphi_0$, etc...) we obtain:

$$\left|(P_{T_0,T_N} - Q^{\hat{\mu}}_{T_0,T_N})\phi(x)\right| \quad \leq \quad C(T, V, d, q, m, ||\phi||_{m+2,\infty}, ||\varphi||_{2q+m,\infty}) \left(\frac{1}{N}\right)^{q\wedge[(m-1)/2]}.$$

This concludes the proof (2.14) at time T. From these arguments, we easily deduce that the estimate holds for any T_k, $k = 1, \ldots, N$. □

6.2 Rate of convergence for the backward approximation: proof of (2.15) and (2.16)

Here we prove the approximation order of the backward component. Before presenting the proof, we introduce some notations. Let us define the Brownian counterparts of $\hat{\Theta}_{k+1,k}, \hat{\Theta}_k$ and $\hat{\zeta}_k$ given in step 9 in Algorithm 2 and steps 12, 18 and 13 in Algorithm 3. For all family of probability measures η we set

$$\begin{aligned}\Theta_k^{\eta}(y) &:= (T_l, y, u(T_k, y), v(T_k, y), \langle \eta_{T_k}, \varphi_f[\cdot, u(T_k, \cdot)]\rangle)\,, \\ \bar{\Theta}_{k+1,k}^{\eta^1,\eta^2}(y) &:= \left(T_{k+1}, X_{T_{k+1}}^{T_k,y,\eta^1}, u(T_{k+1}, X_{T_{k+1}}^{T_k,y,\eta^1}), v(T_k, y), \langle \eta_{T_k}^2, \varphi_f[\cdot, u(T_k, \cdot)]\rangle\right),\end{aligned} \tag{2.35}$$

and

$$\zeta_k = 4\frac{B_{T_{k+1}} - B_{T_k}}{\Delta_{T_{k+1}}} - 6\frac{\int_{T_k}^{T_{k+1}}(s - T_k)dB_s}{\Delta_{T_{k+1}}^2}.$$

The proof uses extensively the regularity of the function u. From Lemma 2.15, for all $t \in [0, T)$, the function $y \in \mathbb{R}^d \mapsto u(t, y)$ is C_b^∞ with uniform bounds in time. In the elliptic case the same situation holds, although the bounds depend on time and blow up in the boundary. Hence, we keep track of the explicit dependence of each error term on u and its derivatives in such a way that the proof is simplified for the elliptic case.

Moreover, we will expand and bound terms of the form $y \mapsto f(\cdot, y, u(\cdot, y), \mathcal{V}u(\cdot, y), \cdot)$. When differentiating such a term, the bounds involve the product of the derivatives of u with respect to the space variable. Namely, the r^{th} differentiation of f involves a product of at most $r + 1$ derivatives of u. To keep track of the order of the derivatives that appear in the bound, we introduce the set of positive integers for which their sum is less than or equal to r: $\mathcal{I}(l, r) = \{I = (I_1, \dots, I_l) \in \{1, \dots, r\}^l : \sum_j I_j \leq r\}$, and define the quantity:

$$M_u(r, s) := \sum_{l=1}^{r} \sum_{I \in \mathcal{I}(l,r)} \prod_{j=1}^{l} ||u(s, .)||_{I_j,\infty}. \tag{2.36}$$

(1) Proof of the order of convergence for the first order algorithm (Algorithm 2).
Let $k \in \{1, \dots, N-1\}$. We first break the error between u and $\hat{u}^1$ as follows:

$$\begin{aligned} u(T_k, \hat{X}_{T_k}^\pi) - \hat{u}^1(T_k, \hat{X}_{T_k}^\pi) &= u(T_k, \hat{X}_{T_k}^\pi) - \mathbb{E}\left(u(T_{k+1}, X_{T_{k+1}}^{T_k, \hat{X}_{T_k}^\pi, \mu}) + \Delta_{T_{k+1}} f(\bar{\Theta}_{k+1,k}^{\mu,\mu}(\hat{X}_{T_k}^\pi))\right) && (2.37)\\ &\quad + \mathbb{E}\left[u(T_{k+1}, X_{T_{k+1}}^{T_k, \hat{X}_{T_k}^\pi, \mu}) + \Delta_{T_{k+1}} f(\bar{\Theta}_{k+1,k}^{\mu,\mu}(\hat{X}_{T_k}^\pi))\right] && (2.38)\\ &\quad - \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[u(T_{k+1}, X_{T_{k+1}}^{T_k, \hat{X}_{T_k}^\pi, \hat{\mu}}) + \Delta_{T_{k+1}} f(\bar{\Theta}_{k+1,k}^{\hat{\mu},\mu}(\hat{X}_{T_k}^\pi))\right] \\ &\quad + \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\Big[u(T_{k+1}, X_{T_{k+1}}^{T_k, \hat{X}_{T_k}^\pi, \hat{\mu}}) - \hat{u}^1(T_{k+1}, X_{T_{k+1}}^{T_k, \hat{X}_{T_k}^\pi, \hat{\mu}}) && (2.39)\\ &\quad + \Delta_{T_{k+1}}\left(f(\bar{\Theta}_{k+1,k}^{\hat{\mu},\mu}(\hat{X}_{T_k}^\pi)) - f(\hat{\Theta}_{k+1,k}^{\pi,1})\right)\Big]. \end{aligned}$$

Similarly, we can expand the error between v and $\hat{v}^1$ as:

$$\begin{aligned}
&\Delta_{T_{k+1}}\left[v(T_k,\hat{X}^\pi_{T_k})-\hat{v}^1(T_k,\hat{X}^\pi_{T_k})\right]\\
&=\Delta_{T_{k+1}}v(T_k,\hat{X}^\pi_{T_k})-\mathbb{E}\left(u\left(T_{k+1},X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}}\right)\Delta B_{T_{k+1}}\right) &(2.40)\\
&+\mathbb{E}\left(u\left(T_{k+1},X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}}\right)\Delta B_{T_{k+1}}\right)-\mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[u\left(T_{k+1},X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}}\right)\Delta B_{T_{k+1}}\right] &(2.41)\\
&+\mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left(\left[u\left(T_{k+1},X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}}\right)-\hat{u}^1\left(T_{k+1},X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}}\right)\right]\Delta B_{T_{k+1}}\right). &(2.42)
\end{aligned}$$

Then, at each step, the approximation error on the backward variables can be expanded as: a first term (2.37) and (2.40), corresponding to scheme errors; a second term, (2.38) and (2.41), corresponding to generalized cubature errors and can be viewed as one step versions of the forward error (2.14) in Theorem 2.1 ; and a third term, (2.39) and (2.42), which are propagation errors.

Let us explain how the proof works. We will bound separately each error: the scheme, cubature and propagation errors. Each bound is summarized in a Claim (respectively Claims 2.9, 2.10 and 2.10 below). Then, we will deduce the dynamics of the error at step k, $\mathcal{E}^1_u(k)$ defined as (2.13) and conclude with a Gronwall argument.

The first claim below gives the bounds on the scheme errors.

Claim 2.9. There exists a constant C depending on the regularity of $V_{0:d}$ and f (and not on k) such that the scheme errors (2.37) and (2.40) are bounded by:

$$\left|u(T_k,\hat{X}^\pi_{T_k})-\mathbb{E}\left(u(T_{k+1},X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}})+\Delta_{T_{k+1}}f(\bar{\Theta}^{\mu,\mu}_{k+1,k}(\hat{X}^\pi_{T_k}))\right)\right|\leq C\sup_{s\in[T_k,T_{k+1}]}||u(s,\cdot)||_{4,\infty}\Delta^2_{T_{k+1}}$$

$$\left|\Delta_{T_{k+1}}v(T_k,\hat{X}^\pi_{T_k})-\mathbb{E}\left(u\left(T_{k+1},X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}}\right)\Delta B_{T_{k+1}}\right)\right|\leq C\sup_{s\in[T_k,T_{k+1}]}||u(s,\cdot)||_{3,\infty}\Delta^2_{T_{k+1}}$$

Proof. The proof of the first estimate follows from a second order Itô-Taylor expansion. Applying Lemma 2.4 with $n=2$ to u and taking the expectation leads to:

$$\mathbb{E}\left(u(T_{k+1},X^{T_k,y,\mu}_{T_{k+1}})\right)=u(T_k,y)+\Delta_{T_k}\mathcal{V}_{(0)}u(T_k,y)+\sum_{\beta\in\partial\mathcal{A}_2}\mathbb{E}\left(I^{T_k,T_{k+1}}_\beta[\mathcal{V}_\beta u(\cdot,X^{T_k,y,\mu}_\cdot)]\right) \quad (2.43)$$

and applying again Lemma 2.4 with $n=1$ to $\mathcal{V}_{(0)}u$ and taking the expectation gives:

$$\mathbb{E}\left(\mathcal{V}_{(0)}u(T_{k+1},X^{T_k,y,\mu}_{T_{k+1}})\right)=\mathcal{V}_{(0)}u(T_k,y)+\sum_{\beta\in\partial\mathcal{A}_1}\mathbb{E}\left(I^{T_k,T_{k+1}}_\beta[\mathcal{V}_\beta\mathcal{V}_{(0)}u(\cdot,X^{T_k,y,\mu}_\cdot)]\right). \quad (2.44)$$

Now, note that since u is the solution of PDE (2.3) we have $f=-\mathcal{V}_{(0)}u$. So that, by combining (2.43), (2.44) and estimate of Lemma 2.5, we obtain

$$\begin{aligned}
&\left|u(T_k,\hat{X}^\pi_{T_k})-\mathbb{E}\left(u(T_{k+1},X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}})+\Delta_{T_{k+1}}f(\bar{\Theta}^{\mu,\mu}_{k+1,k}(\hat{X}^\pi_{T_k}))\right)\right|\\
&\leq 2\sum_{\beta\in\partial\mathcal{A}_2}\mathbb{E}\left(I^{T_k,T_{k+1}}_\beta[\mathcal{V}_\beta u(\cdot,X^{T_k,y,\mu}_\cdot)]\right)\\
&\leq C(T,V_{0:d},f)\sup_{s\in[T_k,T_{k+1}]}||u(s,\cdot)||_{4,\infty}\Delta^2_{T_{k+1}}.
\end{aligned}$$

This concludes the proof of the first estimate. The proof of the second estimate is similar. We first apply an Itô-Taylor expansion of Lemma 2.4 with $n=1$ on u. Then, by noticing that

$\Delta B_{T_{k+1}} = (I_{(1)}^{T_k,T_{k+1}}, \dots, I_{(d)}^{T_k,T_{k+1}})^T$ and by multiplying by $I_{(j)}^{T_k,T_{k+1}}$ the previous expansion of u, and taking the expectation gives, thanks to Itô's Formula

$$\mathbb{E}\left(u(T_{k+1}, X_{T_{k+1}}^{T_k,y,\mu}) I_{(j)}^{T_k,T_{k+1}}\right) = \mathcal{V}_{(j)}u(T_k,y)\Delta_{T_{k+1}} + \sum_{\beta\in\partial\mathcal{A}_1} \mathbb{E}\left(I_\beta^{T_k,T_{k+1}}[\mathcal{V}_\beta u(\cdot, X_\cdot^{T_k,y,\mu})] I_{(j)}^{T_k,T_{k+1}}\right),$$

for $j = 1, \dots, d$ and where the first term in the right hand side is the bracket between the stochastic integrals. The last term is controlled by using Lemma 2.5. Recalling that the j-th component of the function v is given by $\mathcal{V}_{(j)}u$ and reordering the terms we obtain the second inequality. □

We now turn to bound the cubature like error terms (2.38) and (2.41). This is summarized by:

Claim 2.10. There exist two constants C, depending on d, q, T, m, and the regularity of $V_{0:d}$ and $\varphi_{0:d}$ (and not on k), and C', depending in addition on the regularity of f, such that:

$$\begin{aligned}
&\left|\mathbb{E}\left[u(T_{k+1}, X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\mu}) + \Delta_{T_{k+1}} f(\bar{\Theta}_{k+1,k}^{\mu,\mu}(\hat{X}_{T_k}^\pi))\right] - \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[u(T_{k+1}, X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\hat{\mu}}) + \Delta_{T_{k+1}} f(\bar{\Theta}_{k+1,k}^{\hat{\mu},\mu}(\hat{X}_{T_k}^\pi)\right.\right. \\
&\quad \le C\left(||u(T_{k+1},\cdot)||_{2,\infty}\left[\Delta_{T_{k+1}}^{q+1} + \Delta_{T_{k+1}} N^{-[(m-1)\wedge 2q]/2}\right] + ||u(T_{k+1},\cdot)||_{m+1,\infty}\Delta_{T_{k+1}}^{(m+1)/2}\right. \\
&\qquad \left. + ||u(T_{k+1},\cdot)||_{m+2,\infty}\Delta_{T_{k+1}}^{(m+2)/2}\right) + C'\left(M_u(2,T_{k+1})\left[\Delta_{T_{k+1}}^{q+1} + \Delta_{T_{k+1}} N^{-[(m-1)\wedge 2q]/2}\right]\right. \\
&\qquad \left. + M_u(m+1,T_{k+1})\Delta_{T_{k+1}}^{(m+1)/2} + M_u(m+2,T_{k+1})\Delta_{T_{k+1}}^{(m+2)/2}\right) \\
&\left|\mathbb{E}\left(u\left(T_{k+1}, X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\mu}\right)\Delta B_{T_{k+1}}\right) - \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[u\left(T_{k+1}, X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\hat{\mu}}\right)\Delta B_{T_{k+1}}\right]\right| \\
&\quad \le C\left(||u(T_{k+1},\cdot)||_{3,\infty}\left[\Delta_{T_{k+1}}^{q+1} + \Delta_{T_{k+1}}^2 N^{-[(m-1)\wedge 2q]/2}\right] + ||u(T_{k+1},\cdot)||_{m,\infty}\Delta_{T_{k+1}}^{(m+1)/2}\right. \\
&\qquad \left. + ||u(T_{k+1},\cdot)||_{m+1,\infty}\Delta_{T_{k+1}}^{(m+2)/2}\right)
\end{aligned}$$

Proof. Note that the r^{th} derivative of the function $y \mapsto f(\cdot, y, u(\cdot,y), \cdot, \cdot)$ is bounded by $C'M_u(r,\cdot)$ defined by (2.36). Then, the proof of the first assertion follows from (2.84) in Lemma 2.18 applied to u and f and the second assertion from (2.85) in Lemma 2.18 applied to u. □

Finally, an estimate on the propagation error (2.39) is given by:

Claim 2.11. There exists a constant C depending on d, q, T, m, and the regularity of $V_{0:d}$ and $\varphi_{0:d}$ (and not on k) such that:

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[u(T_{k+1}, X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\hat{\mu}}) - \hat{u}^1(T_{k+1}, X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\hat{\mu}}) + \Delta_{T_{k+1}}\left(f(\bar{\Theta}_{k+1,k}^{\hat{\mu},\mu}(\hat{X}_{T_k}^\pi)) - f(\hat{\Theta}_{k+1,k}^{\pi,1})\right)\right]\right| \\
&\quad \le (1 + C\Delta_{T_{k+1}})\mathcal{E}_u^1(k+1) + C\left(||u(T_{k+1},\cdot)||_{m+2,\infty}\Delta_{T_{k+1}} N^{-[(m-1)/2]\wedge q}\right. \\
&\qquad + ||u(T_{k+1},\cdot)||_{3,\infty}\left[\Delta_{T_{k+1}}^2 + \Delta_{T_{k+1}}^{q+1} + \Delta_{T_{k+1}}^2 N^{-[(m-1)\wedge 2q]/2}\right] \\
&\qquad \left. + ||u(T_{k+1},\cdot)||_{m+1,\infty}\Delta_{T_{k+1}}^{(m+1)/2} + ||u(T_{k+1},\cdot)||_{m+2,\infty}\Delta_{T_{k+1}}^{(m+2)/2}\right)
\end{aligned}$$

Proof. Let us start by expanding the f term. We get from the mean value theorem that there exist three random variable Ψ_1, Ψ_2, Ψ_3, respectively bounded by $\|\partial_{y'} f\|_\infty$, $\|\partial_z f\|_\infty$ and $\|\partial_w f\|_\infty$

almost surely, depending on each argument of $\hat{\Theta}^{\pi,1}_{k+1,k}$ and $\bar{\Theta}^{\hat{\mu},\mu}_{k+1,k}$ such that:

$$\begin{aligned}
&\Delta_{T_{k+1}}\left(f(\bar{\Theta}^{\hat{\mu},\mu}_{k+1,k}(\hat{X}^{\pi}_{T_k}))-f(\hat{\Theta}^{\pi,1}_{k+1,k})\right)\\
&=\Delta_{T_{k+1}}\Psi_1\left(u(T_{k+1},X^{T_k,\hat{X}^{\pi}_{T_k},\hat{\mu}}_{T_{k+1}})-\hat{u}^1(T_{k+1},X^{T_k,\hat{X}^{\pi}_{T_k},\hat{\mu}}_{T_{k+1}})\right)+\Delta_{T_{k+1}}\Psi_2\left(v(T_k,\hat{X}^{\pi}_{T_k})-\hat{v}^1(T_k,\hat{X}^{\pi}_{T_k})\right)\\
&\quad+\Delta_{T_{k+1}}\Psi_3\left(\langle\mu_{T_{k+1}},\varphi_f[\cdot,u(T_{k+1},\cdot)]\rangle-\langle\hat{\mu}_{T_{k+1}},\varphi_f[\cdot,\hat{u}^1(T_{k+1},\cdot)]\rangle\right).
\end{aligned}$$

Now, we can use the error expansion (2.40), (2.41) and (2.42) of $\Delta_{T_{k+1}}\left(v(T_k,\hat{X}^{\pi}_{T_k})-\hat{v}^1(T_k,\hat{X}^{\pi}_{T_k})\right)$ together with the second assertion of Claims 2.9 and 2.10 to get

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[u(T_{k+1},X^{T_k,\hat{X}^{\pi}_{T_k},\hat{\mu}}_{T_{k+1}})-\hat{u}^1(T_{k+1},X^{T_k,\hat{X}^{\pi}_{T_k},\hat{\mu}}_{T_{k+1}})+\Delta_{T_{k+1}}\left(f(\bar{\Theta}^{\hat{\mu},\mu}_{k+1,k}(\hat{X}^{\pi}_{T_k}))-f(\hat{\Theta}^{\pi,1}_{k+1,k})\right)\right]\right|\\
&\leq\left|\mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[\left(u(T_{k+1},X^{T_k,\hat{X}^{\pi}_{T_k},\hat{\mu}}_{T_{k+1}})-\hat{u}^1(T_{k+1},X^{T_k,\hat{X}^{\pi}_{T_k},\hat{\mu}}_{T_{k+1}})\right)(1+\Psi_1\Delta_{T_{k+1}}+\Psi_2\Delta B_{T_{k+1}})\right]\right|\\
&\quad+C||u(T_{k+1},\cdot)||_{3,\infty}\left[\Delta^2_{T_{k+1}}+\Delta^{q+1}_{T_{k+1}}+\Delta^2_{T_{k+1}}N^{-[(m-1)\wedge 2q]/2}\right]+C||u(T_{k+1},\cdot)||_{m+1,\infty}\Delta^{(m+1)/2}_{T_{k+1}}\\
&\quad+C||u(T_{k+1},\cdot)||_{m+2,\infty}\Delta^{(m+2)/2}_{T_{k+1}}+\Delta_{T_{k+1}}\|\partial_w f\|_\infty\left|\langle\mu-\hat{\mu}_{T_{k+1}},\varphi_f[\cdot,u(T_{k+1},\cdot)]\rangle\right|\\
&\quad+\Delta_{T_{k+1}}\|\partial_w f\|_\infty\left|\langle\hat{\mu}_{T_{k+1}},\varphi_f[\cdot,u(T_{k+1},\cdot)]-\varphi_f[\cdot,\hat{u}^1(T_{k+1},\cdot)]\rangle\right|. \qquad (2.45)
\end{aligned}$$

Note that from the forward result (2.14) in Theorem 2.1, we have

$$\Delta_{T_{k+1}}\|\partial_w f\|_\infty\left|\langle\mu-\hat{\mu}_{T_{k+1}},\varphi_f[\cdot,u(T_{k+1},\cdot)]\rangle\right|\leq C(||u(T_{k+1},\cdot)||_{m+2,\infty})\Delta_{T_{k+1}}N^{-((m-1)\wedge 2q)/2}, \qquad (2.46)$$

while using the regularity of φ_f and the definition of $\hat{\mu}$ gives,

$$\Delta_{T_{k+1}}\|\partial_w f\|_\infty\left|\langle\hat{\mu}_{T_{k+1}},\varphi_f[\cdot,u(T_{k+1},\cdot)]-\varphi_f[\cdot,\hat{u}^1(T_{k+1},\cdot)]\rangle\right|\leq C'\Delta_{T_{k+1}}\mathcal{E}^1_u(k+1). \qquad (2.47)$$

The Claim follows by applying the Cauchy-Schwartz inequality on the first term in the right hand side of (2.45) and plugging (2.46) and (2.47) in (2.45). □

We can now analyze the local error at step k. By plugging the estimates from Claims 2.9, 2.10 and 2.11 in the expansion (2.37), (2.38) and (2.39) of $u-\hat{u}$ we obtain that:

$$\mathcal{E}^1_u(k)\leq\left(1+C\Delta_{T_{k+1}}\right)\mathcal{E}^1_u(k+1)+\bar{\epsilon}(k+1), \qquad (2.48)$$

with

$$\begin{aligned}
\bar{\epsilon}(k+1)\;=\;&C\Bigg(\sup_{s\in[T_k;T_{k+1}]}||u(s,\cdot)||_{2,\infty}\Delta^4_{T_{k+1}}+||u(T_{k+1},\cdot)||_{2,\infty}\Delta_{T_{k+1}}N^{-[(m-1)\wedge 2q]/2}\\
&+||u(T_{k+1},\cdot)||_{m+1,\infty}\Delta^{(m+1)/2}_{T_{k+1}}+||u(T_{k+1},\cdot)||_{m+2,\infty}\Delta^{(m+2)/2}_{T_{k+1}}\\
&+||u(T_{k+1},\cdot)||_{3,\infty}\left[\Delta^2_{T_{k+1}}+\Delta^{q+1}_{T_{k+1}}+\Delta^2_{T_{k+1}}N^{-[(m-1)\wedge 2q]/2}\right]\Bigg) \qquad (2.49)\\
&+C'\Bigg(M_u(m+1,T_{k+1})\Delta^{(m+1)/2}_{T_{k+1}}+M_u(m+2,T_{k+1})\Delta^{(m+2)/2}_{T_{k+1}}\\
&+M_u(2,T_{k+1})\left[\Delta^{q+1}_{T_{k+1}}+\Delta_{T_{k+1}}N^{-[(m-1)\wedge 2q]/2}\right]\Bigg).
\end{aligned}$$

Under **(SB)** , we have from Lemma 2.15 that for all $n\in\mathbb{N}^*$ there exists a constant K, depending on the regularity of $V_{0:d}$ and ϕ, such that for all $k\in\{0,\cdots,N-1\}$,

$$M_u(n,T_{k+1})+\sup_{s\in[T_k,T_{k+1}]}||u(s,\cdot)||_{n,\infty}\leq K.$$

Therefore, Gronwall's Lemma applied to (2.48) and the definition of Δ_{T_k} implies

$$\mathcal{E}_u^1 \leq CN^{-1}. \tag{2.50}$$

Moreover, Claims 2.9 and 2.10 and expansion (2.40), (2.41) and (2.42) show that:

$$\begin{aligned}\Delta_{T_{k+1}}\mathcal{E}_v^1(k+1) \leq C\Bigg(&\sup_{s\in[T_k,T_{k+1}]}||u(s,\cdot)||\Delta_{T_{k+1}}^2 + ||u(T_{k+1},\cdot)||_{3,\infty}\left[\Delta_{T_{k+1}}^{q+1} + \Delta_{T_{k+1}}^2 N^{-[(m-1)\wedge 2q]/2}\right] \\ &+ ||u(T_{k+1},\cdot)||_{m+1,\infty}\Delta_{T_{k+1}}^{(m+1)/2} + C||u(T_{k+1},\cdot)||_{m+2,\infty}\Delta_{T_{k+1}}^{(m+2)/2}\Bigg) + \Delta_{T_{k+1}}^{1/2}\mathcal{E}_u^1(k+1),\end{aligned}$$

which together with (2.50) imply

$$\Delta_{T_k}^{1/2}\mathcal{E}_v^1 \leq CN^{-1},$$

and the result holds.

(2) Proof of the order of convergence for the second order algorithm (Algorithm 3).

Let $k \in \{0,\dots,N-2\}$. We can expand as before the errors on the u and v approximations as:

$$\begin{aligned}u(T_k,\hat{X}_{T_k}^\pi) - \hat{u}^2(T_k,\hat{X}_{T_k}^\pi) = &\mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[u(T_{k+1},X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\hat{\mu}}) - \hat{u}^2(T_{k+1},X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\hat{\mu}})\right] \\ &+ \frac{1}{2}\Delta_{T_{k+1}}\mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[f(\Theta_{k+1}^\mu(X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\mu})) - f(\hat{\Theta}_{k+1}^{\pi,2})\right] \qquad (2.51) \\ &+ \frac{1}{2}\Delta_{T_{k+1}}[f(\Theta_k^\mu(\hat{X}_{T_k}^\pi)) - f(\tilde{\Theta}_k^\pi)] \qquad (2.52) \\ &+ \epsilon_{\hat{u}^2,k}^s(\pi) + \epsilon_{\hat{u}^2,k}^c(\pi),\end{aligned}$$

where

$$\epsilon_{\hat{u}^2,k}^s(\pi) = u(T_k,\hat{X}_{T_k}^\pi) - \mathbb{E}\left(u(T_{k+1},X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\mu}) + \frac{\Delta_{T_{k+1}}}{2}\left(f(\Theta_k^\mu(\hat{X}_{T_k}^\pi)) + f(\Theta_{k+1}^\mu(X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\mu}))\right)\right) \tag{2.53}$$

$$\begin{aligned}\epsilon_{\hat{u}^2,k}^c(\pi) = &\mathbb{E}\left(u(T_{k+1},X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\mu})\right) - \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[u(T_{k+1},X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\hat{\mu}})\right] \\ &+ \frac{\Delta_{T_{k+1}}}{2}\left(\mathbb{E}\left(f(\Theta_{k+1}^\mu(X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\mu}))\right) - \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[f(\Theta_{k+1}^\mu(X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\mu}))\right]\right). \qquad (2.54)\end{aligned}$$

Similarly, we have

$$\begin{aligned}v(T_k,\hat{X}_{T_k}^\pi) - \hat{v}^2(T_k,\hat{X}_{T_k}^\pi) = &\mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left(\left[u\left(T_{k+1},X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\hat{\mu}}\right) - \hat{u}^2\left(T_{k+1},X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\hat{\mu}}\right)\right]\zeta_{k+1}\right) \\ &+ \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[\left(f[\Theta_{k+1}^\mu(X_{T_{k+1}}^{T_k,\hat{X}_{T_k}^\pi,\mu})] - f(\hat{\Theta}_{k+1}^{\pi,2})\right)\Delta_{T_{k+1}}\zeta_{k+1}\right] \qquad (2.55) \\ &+ \epsilon_{\hat{v}^2,k}^s(\pi) + \epsilon_{\hat{v}^2,k}^c(\pi),\end{aligned}$$

with

$$\epsilon^s_{\hat{v}^2,k}(\pi) = v(T_k, \hat{X}^\pi_{T_k}) - \mathbb{E}\left(\left[u(T_{k+1}, X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}}) + \Delta_{T_{k+1}} f[\Theta^\mu_{k+1}(X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}})]\right]\zeta_{k+1}\right) \tag{2.56}$$

$$\begin{aligned}\epsilon^c_{\hat{v}^2,k}(\pi) = \mathbb{E}\left(u(T_{k+1}, X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}})\zeta_{k+1}\right) - \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[u(T_{k+1}, X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}})\zeta_{k+1}\right] \\ + \mathbb{E}\left(f[\Theta^\mu_{k+1}(X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}})]\Delta_{T_{k+1}}\zeta_{k+1}\right) - \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[f[\Theta^\mu_{k+1}(X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}})]\Delta_{T_{k+1}}\zeta_{k+1}\right].\end{aligned} \tag{2.57}$$

We identify, as for the first order expansion, some error terms corresponding to the scheme error (2.53) and (2.56), generalized cubature errors (2.54) and (2.57) and propagation errors (2.51), (2.55). Some important changes are clear from the expansion: we have in addition a prediction error term (2.52) reflecting the fact that we perform a new intermediate step, and we have some f term in (2.55), adding to the propagation error.

The proof for the second order approximation is then similar to its first order equivalent, but we will have to consider the mentioned additional terms. In particular, the fact that the second order approximation $\hat{v}^2(\hat{X}^\pi_{T_k})$ includes the term $f(\hat{\Theta}^{\pi,2}_{T_{k+1}})$, adds an additional coupling effect. With this in mind, and in order to simplify the analysis, we introduce the following quantity

$$\mathcal{E}_f(k) = \max_{\pi\in\mathcal{S}(k)} |f(\Theta^\mu_k(\hat{X}^\pi_{T_k})) - f(\hat{\Theta}^{\pi,2}_k)|$$

and we will analyze the dynamics of the sum of errors at step k, $\mathcal{E}^2_u(k) + \Delta_{T_k}\mathcal{E}_f(k)$.

To this aim, we will bound separately the scheme and cubature errors. Each bound is summarized in a Claim (respectively Claims 2.12, 2.13 below). Then, we will conclude with a Gronwall argument.

The scheme error terms (2.53) and (2.56) and the generalized cubature errors (2.54) and (2.57) are treated similarly as in the first order scheme. We show this in Claims 2.12 and 2.13.

Claim 2.12. There exists a constant C, depending on the regularity of $V_{0:d}$ and f (and not on k) such that:

$$\begin{aligned}\left|\epsilon^s_{\hat{u}^2,k}(\pi)\right| &\leq C \sup_{s\in[T_k,T_{k+1}]} ||u(s,\cdot)||_{6,\infty}\Delta^3_{T_{k+1}} \\ \left|\epsilon^s_{\hat{v}^2,k}(\pi)\right| &\leq C \sup_{s\in[T_k,T_{k+1}]} ||u(s,\cdot)||_{5,\infty}\Delta^2_{T_{k+1}}.\end{aligned}$$

where $\epsilon^s_{\hat{u}^2,k}(\pi), \epsilon^s_{\hat{v}^2,k}(\pi)$ are defined in (2.53) and (2.56).

Proof. The proof follows in the same way as the one of Claim 2.9, by performing a Taylor expansion to one additional order. The choice of ζ_{k+1} is the one needed to match the lower order terms (recall that the i^{th} component of ζ_k is expressed as $\zeta^i_k = 4\Delta^{-1}_{T_k} I^{T_k,T_{k+1}}_{(i)} - 6\Delta^{-2}_{T_k} I^{T_k,T_{k+1}}_{(0,i)}$).

Applying Lemma 2.4 with $n = 4$ to u and with $n = 2$ to $\mathcal{V}_{(0)}u$ implies, after taking the expectation, that

$$\mathbb{E}\left(u(T_{k+1}, X^{T_k,y,\mu}_{T_{k+1}})\right) = u(T_k,y) + \Delta_{T_k}\mathcal{V}_{(0)}u(T_k,y) + \frac{1}{2}\mathcal{V}_{(0,0)}u(T_k,y)\Delta^2_{T_{k+1}} + \sum_{\beta\in\partial\mathcal{A}_4}\mathbb{E}\left(I^{T_k,T_{k+1}}_\beta[\mathcal{V}_\beta u(.,X^{T_k,y,\mu}_.)]\right)$$

and

$$\mathbb{E}\left(\mathcal{V}_{(0)}u(T_{k+1}, X^{T_k,y,\mu}_{T_{k+1}})\right) = \mathcal{V}_{(0)}u(T_k,y) + \mathcal{V}_{(0,0)}u(T_k,y)\Delta_{T_{k+1}} + \sum_{\beta\in\partial\mathcal{A}_2}\mathbb{E}\left(I^{T_k,T_{k+1}}_\beta[\mathcal{V}_{(\beta*0)}u(.,X^{T_k,y,\mu}_.)]\right).$$

Then, the estimate of Lemma 2.5 gives

$$\begin{aligned}
&\mathbb{E}\left(u(T_{k+1},X_{T_{k+1}}^{T_k,y,\mu})-\frac{\Delta_{T_k}}{2}\left(\mathcal{V}_{(0)}u(T_k,y)+\mathcal{V}_{(0)}u(T_{k+1},X_{T_{k+1}}^{T_k,y,\mu})\right)\right)-u(T_k,y)\\
&\quad=\sum_{\beta\in\partial\mathcal{A}_4}\mathbb{E}\left(I_\beta^{T_k,T_{k+1}}[\mathcal{V}_{(\beta)}u(.,X_.^{T_k,y,\mu})]\right)-\frac{\Delta_{T_k}}{2}\sum_{\beta\in\partial\mathcal{A}_2}\mathbb{E}\left(I_\beta^{T_k,T_{k+1}}[\mathcal{V}_{(\beta*0)}u(.,X_.^{T_k,y,\mu})]\right)\\
&\quad\le C(T,V_{0:d},f)\Delta_{T_{k+1}}^3\sup_{s\in[T_k,T_{k+1}]}||u(s,.)||_{6,\infty},
\end{aligned}$$

from where we deduce the first inequality.

Similarly, by using Lemma 2.4 with $n=3$ on u and with $n=1$ on $\mathcal{V}_{(0)}u$ and taking the expectation, using the fact that $\zeta_k^i=4\Delta_{T_k}^{-1}I_{(i)}^{T_k,T_{k+1}}-6\Delta_{T_k}^{-2}I_{(0,i)}^{T_k,T_{k+1}}$, it follows

$$\begin{aligned}
&\mathbb{E}\left(\left[u(T_{k+1},X_{T_{k+1}}^{T_k,y,\mu})-\Delta_{T_{k+1}}\mathcal{V}_{(0)}u(T_{k+1},X_{T_{k+1}}^{T_k,y,\mu})\right]\Delta_{T_{k+1}}\zeta_k^j\right)\\
&=\mathbb{E}\Bigg[\left[\mathcal{V}_{(j)}u(T_k,y)I_{(j)}^{T_k,T_{k+1}}+\mathcal{V}_{(0,j)}u(T_k,y)I_{(0,j)}^{T_k,T_{k+1}}+\mathcal{V}_{(j,0)}u(T_k,y)I_{(j,0)}^{T_k,T_{k+1}}\right.\\
&\qquad\left.-\Delta_{T_{k+1}}\mathcal{V}_{(j,0)}u(T_k,y)I_{(j)}^{T_k,T_{k+1}}\right]\left(4I_{(j)}^{T_k,T_{k+1}}-6\frac{I_{(0,j)}^{T_k,T_{k+1}}}{\Delta_{T_{k+1}}}\right)\Bigg]+\mathcal{R}(k,j)\\
&=\Delta_{T_{k+1}}\mathcal{V}_{(j)}u(T_k,y)+\mathcal{R}(k,j)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}(k,j)=\Delta_{T_{k+1}}&\sum_{\beta\in\partial\mathcal{A}_3}\mathbb{E}\left(I_\beta^{T_k,T_{k+1}}[\mathcal{V}_{(\beta)}u(.,X_.^{T_k,y,\mu})]\zeta_k^j\right)\\
&-\Delta_{T_{k+1}}^2\sum_{\beta\in\partial\mathcal{A}_1}\mathbb{E}\left(I_\beta^{T_k,T_{k+1}}[\mathcal{V}_{(\beta*0)}u(.,X_.^{T_k,y,\mu})]\zeta_k^j\right).
\end{aligned}$$

Using Lemma 2.5 we bound the residual term $\mathcal{R}(k,j)$ and obtain

$$\begin{aligned}
&\Delta_{T_{k+1}}\sum_{\beta\in\partial\mathcal{A}_3}\mathbb{E}\left(I_\beta^{T_k,T_{k+1}}[\mathcal{V}_{(\beta)}u(.,X_.^{T_k,y,\mu})]\zeta_k^j\right)-\Delta_{T_{k+1}}^2\sum_{\beta\in\partial\mathcal{A}_1}\mathbb{E}\left(I_\beta^{T_k,T_{k+1}}[\mathcal{V}_{(\beta*0)}u(.,X_.^{T_k,y,\mu})]\zeta_k^j\right)\\
&\quad\le\Delta_{T_k}^3\sup_{s\in[T_k,T_{k+1}]}||u(s,.)||_{5,\infty}+\Delta_{T_k}^{7/2}\sup_{s\in[T_k,T_{k+1}]}||u(s,.)||_{5,\infty},
\end{aligned}$$

recalling that $v(T_k,y)=\mathcal{V}_{(j)}u(T_k,y)$ we deduce the second inequality. □

Claim 2.13. There exist two constants C, depending on d, q, T, m, the regularity of $V_{0:d}$ and $\varphi_{0:d}$ (and not on k), and C', depending in addition on the regularity of f, such that:

$$\begin{aligned}
\left|\epsilon_{\hat{u}^2,k}^c(\pi)\right|\le C&\Bigg(||u(T_{k+1},\cdot)||_{2,\infty}\left[\Delta_{T_{k+1}}^{q+1}+\Delta_{T_{k+1}}N^{-[(m-1)\wedge 2q]/2}\right]+||u(T_{k+1},\cdot)||_{m+1,\infty}\Delta_{T_{k+1}}^{(m+1)/2}\\
&+||u(T_{k+1},\cdot)||_{m+2,\infty}\Delta_{T_{k+1}}^{(m+2)/2}\Bigg)\\
+C'&\Bigg(M_u(3,T_{k+1})\left[\Delta_{T_{k+1}}^{q+2}+\Delta_{T_{k+1}}^2N^{-[(m-1)\wedge 2q]/2}\right]+M_u(m+2,T_{k+1})\Delta_{T_{k+1}}^{(m+3)/2}\\
&+M_u(m+3,T_{k+1})\Delta_{T_{k+1}}^{(m+4)/2}\Bigg),
\end{aligned}$$

$$\begin{aligned}
\left|\epsilon_{\hat{v}^2,k}^c(\pi)\right|\le C&\Bigg(||u(T_{k+1},\cdot)||_{3,\infty}\left[\Delta_{T_{k+1}}^{q}+\Delta_{T_{k+1}}N^{-[(m-1)\wedge 2q]/2}\right]+\sum_{i=m-2}^{m+1}||u(T_{k+1},\cdot)||_{i,\infty}\Delta_{T_{k+1}}^{(i-1)/2}\Bigg)\\
+C'&\Bigg(M_u(4,T_{k+1})\left[\Delta_{T_{k+1}}^{q+1}+\Delta_{T_{k+1}}^2N^{-[(m-1)\wedge 2q]/2}\right]+\sum_{i=m-1}^{m+2}M_u(i,T_{k+1})\Delta_{T_{k+1}}^{i/2}\Bigg).
\end{aligned}$$

where $\epsilon^c_{\hat{u}^2,k}(\pi), \epsilon^c_{\hat{v}^2,k}(\pi)$ are defined in (2.54) and (2.57).

Remark. Although the rates of convergence have a leading term of order $\Delta^{(m-1)}_{T_{k+1}}$ that is worst than the one in the first order scheme result, (Claim 2.10), here we assume that m is bigger, and thus they are suitable for a second order scheme.

Proof. Note first that the r^{th} derivative of the function $y \mapsto f(\cdot, y, u(\cdot, y), \mathcal{V}u(\cdot, y), \cdot)$ is bounded by $M_u(r+1, \cdot)$ defined by (2.36). This estimate goes up to $r+1$ and not just r as in Claim 2.10, because the differentiation of $y \in \mathbb{R}^d \mapsto f(\Theta^\mu_{k+1}(y))$ involves the additional dependence on $\mathcal{V}u$.

Then, the first assertion follows from applying (2.84) in Lemma 2.18 to u and $f(\Theta^\mu_{k+1})$. For the second assertion, recall that the i^{th} component of ζ_k is expressed as $\zeta^i_k = 4\Delta^{-1}_{T_k} I^{T_k,T_{k+1}}_{(i)} - 6\Delta^{-2}_{T_k} I^{T_k,T_{k+1}}_{(0,i)}$. Then, applying (2.85) and (2.86) with $n = m$ in Lemma 2.18 to u and $f(\Theta^\mu_{k+1})$, we conclude on the second assertion. □

It will be handy to have an expansion on the prediction error, by recalling that $\tilde{u}$ is essentially an application of the first order scheme, it follows that

$$(u-\tilde{u})(T_k, \hat{X}^\pi_{T_k}) = \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[(u-\hat{u}^2)(T_{k+1}, X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}}) + \Delta_{T_{k+1}}\left(f(\Theta^{\hat{\mu}}_{k+1}(X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}}) - f(\hat{\Theta}^{\pi,2}_{k+1})\right)\right] \tag{2.58}$$
$$+ \epsilon^s_{\tilde{u},k}(\pi) + \epsilon^c_{\tilde{u},k}(\pi)$$

where

$$\epsilon^c_{\tilde{u},k}(\pi) := \mathbb{E}\left[u(T_{k+1}, X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}}) + \Delta_{T_{k+1}} f(\Theta^\mu_{k+1}(X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}})\right]$$
$$- \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[u(T_{k+1}, X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}}) + \Delta_{T_{k+1}} f(\Theta^{\hat{\mu}}_{k+1}(X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}})\right] \tag{2.59}$$

$$\epsilon^s_{\tilde{u},k}(\pi) := u(T_k, \hat{X}^\pi_{T_k}) - \mathbb{E}\left(u(T_{k+1}, X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}}) + \Delta_{T_{k+1}} f[\Theta^\mu_{k+1}(X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}})]\right). \tag{2.60}$$

Note that $\epsilon^s_{\tilde{u},k}(\pi)$ and $\epsilon^c_{\tilde{u},k}(\pi)$ may be bounded respectively as in Claims (2.9) and (2.10) . The fact that we are using here Θ^μ_{k+1} instead of $\bar{\Theta}^{\mu,\mu}_{k+1,k}$ in those claims, is not really problematic. For the scheme error in Claim (2.9), the difference can be controlled by an additional application of Ito's theorem, but we skip the details. For the cubature error in Claim (2.10) this difference plays no role at all.

Let us now focus on the errors when approaching the driver. Using the mean value theorem, we know that there exist Ψ_1, Ψ_2, Ψ_3 respectively bounded by $\|\partial_{y'} f\|_\infty$, $\|\partial_z f\|_\infty$ and $\|\partial_w f\|_\infty$, such that

$$f(\Theta^\mu_k(\hat{X}^\pi_{T_k})) - f(\hat{\Theta}^{\pi,2}_k) = \Psi_1\left[u(T_k, \hat{X}^\pi_{T_k}) - \hat{u}^2(T_k, \hat{X}^\pi_{T_k})\right] + \Psi_2\left[v(T_k, \hat{X}^\pi_{T_k}) - \hat{v}^2(T_k, \hat{X}^\pi_{T_k})\right] \tag{2.61}$$
$$+ \Psi_3\left(\langle\hat{\mu}_{T_k}, \varphi_f[\cdot, u(T_k, \cdot)]\rangle - \langle\hat{\mu}_{T_k}, \varphi_f[\cdot, \hat{u}^2(T_k, \cdot), \cdot)]\rangle\right) \tag{2.62}$$
$$+ \Psi_3\left(\langle\mu_{T_k}, \varphi_f[\cdot, u(T_k, \cdot)]\rangle - \langle\hat{\mu}_{T_k}, \varphi_f[\cdot, u(T_k, \cdot), \cdot)]\rangle\right), \tag{2.63}$$

Similarly, there exist random variables and $\Psi'_1, \Psi'_2, \Psi'_3$ respectively bounded by $\|\partial_{y'} f\|_\infty$, $\|\partial_z f\|_\infty$ and $\|\partial_w f\|_\infty$, such that

$$f(\Theta^\mu_k(\hat{X}^\pi_{T_k})) - f(\tilde{\Theta}^\pi_k) = \Psi'_1\left[u(T_k, \hat{X}^\pi_{T_k}) - \tilde{u}(T_k, \hat{X}^\pi_{T_k})\right] + \Psi'_2\left[v(T_k, \hat{X}^\pi_{T_k}) - \hat{v}^2(T_k, \hat{X}^\pi_{T_k})\right] \tag{2.64}$$
$$+ \Psi'_3\left(\langle\hat{\mu}_{T_k}, \varphi_f[\cdot, u(T_k, \cdot)]\rangle - \langle\hat{\mu}_{T_k}, \varphi_f[\cdot, \tilde{u}(T_k, \cdot), \cdot)]\rangle\right) \tag{2.65}$$
$$+ \Psi'_3\left(\langle\mu_{T_k}, \varphi_f[\cdot, u(T_k, \cdot)]\rangle - \langle\hat{\mu}_{T_k}, \varphi_f[\cdot, u(T_k, \cdot), \cdot)]\rangle\right) .. \tag{2.66}$$

Then, using the error development for $(u-\hat{u}^2)(T_k, \hat{X}^\pi_{T_k})$ given in (2.53),(2.54), (2.51),(2.52) the error and the ones for $f(\Theta^\mu_k(\hat{X}^\pi_{T_k})) - f(\hat{\Theta}^{\pi,2}_k)$ in (2.61),(2.62) and (2.63) and $f(\Theta^\mu_k(\hat{X}^\pi_{T_k})) - f(\tilde{\Theta}^\pi_k)$ in (2.64), (2.65) and (2.66) , it follows

$$
\begin{aligned}
(u-\hat{u}^2)(T_k, \hat{X}^\pi_{T_k}) &+ \frac{1}{2}\Delta_{T_k}\left(f(\Theta^\mu_k(\hat{X}^\pi_{T_k})) - f(\hat{\Theta}^{\pi,2}_k)\right) \\
&= (1+\Delta_{T_k}\Psi_1)\mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\Big[u(T_{k+1}, X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}}) - \hat{u}^2(T_{k+1}, X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}}) \\
&\qquad + \frac{1}{2}\Delta_{T_{k+1}}\left(f(\Theta^\mu_{k+1}(X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}})) - f(\hat{\Theta}^{\pi,2}_{k+1})\right)\Big] \\
&+ (1+\Delta_{T_k}\Psi_1)(\epsilon^s_{\hat{u}^2,k}(\pi) + \epsilon^c_{\hat{u}^2,k})(\pi) \\
&+ \frac{1}{2}[\Psi_2\Delta_{T_k} + \Psi'_2\Delta_{T_{k+1}}]\big[v(T_k, \hat{X}^\pi_{T_k}) - \hat{v}^2(T_k, \hat{X}^\pi_{T_k})\big] \\
&+ \Delta_{T_{k+1}}\frac{\Psi'_1}{2}\big[u(T_k, \hat{X}^\pi_{T_k}) - \tilde{u}(T_k, \hat{X}^\pi_{T_k})\big] \\
&+ \frac{1}{2}[\Psi_3\Delta_{T_k} + \Psi'_3\Delta_{T_{k+1}}]\left(\langle\mu_{T_k}, \varphi_f[\cdot, u(T_k,\cdot)]\rangle - \langle\hat{\mu}_{T_k}, \varphi_f[\cdot, u(T_k,\cdot),\cdot)]\rangle\right), \\
&+ \Delta_{T_k}\frac{\Psi_3}{2}\left(\langle\hat{\mu}_{T_k}, \varphi_f[\cdot, u(T_k,\cdot)]\rangle - \langle\hat{\mu}_{T_k}, \varphi_f[\cdot, \hat{u}^2(T_k,\cdot),\cdot)]\rangle\right) \\
&+ \Delta_{T_{k+1}}\frac{\Psi'_3}{2}\left(\langle\hat{\mu}_{T_k}, \varphi_f[\cdot, u(T_k,\cdot)]\rangle - \langle\hat{\mu}_{T_k}, \varphi_f[\cdot, \tilde{u}(T_k,\cdot),\cdot)]\rangle\right)
\end{aligned}
$$

Then, replacing the expansion for the error of $\hat{v}^2$ in terms of (2.56),(2.57), (2.55) and the one for $(u-\tilde{u})$ in terms of (2.58), (2.59), (2.60), and up to rescaling some of the Ψ random variables, one obtains

$$
\begin{aligned}
(u-\hat{u}^2)(T_k, \hat{X}^\pi_{T_k}) &+ \frac{1}{2}\Delta_{T_k}\left(f(\Theta^\mu_k(\hat{X}^\pi_{T_k})) - f(\hat{\Theta}^{\pi,2}_k)\right) \qquad (2.67)\\
&= \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\Bigg((1+\Delta_{T_{k+1}}\Psi''_1 + \Delta_{T_k}\zeta_{k+1}\Psi''_2)\Big[(u-\hat{u}^2)(T_{k+1}, X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}}) \\
&\qquad + \frac{1}{2}\Delta_{T_{k+1}}\left(f(\Theta^\mu_{k+1}(X^{T_k,\hat{X}^\pi_{T_k},\mu}_{T_{k+1}})) - f(\hat{\Theta}^{\pi,2}_{k+1})\right)\Big]\Bigg) \\
&- \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left(\left[(u-\hat{u}^2)(T_{k+1}, X^{T_k,\hat{X}^\pi_{T_k},\hat{\mu}}_{T_{k+1}})\right](\Delta_{T_{k+1}}\Psi''_1 + \Delta_{T_{k+1}}\zeta_{k+1}\Psi''_2)\right) \\
&+ (1+\Delta_{T_k}\Psi_1)(\epsilon^s_{\hat{u}^2,k}(\pi) + \epsilon^c_{\hat{u}^2,k}(\pi)) + \Delta_{T_{k+1}}\Psi''_3(\epsilon^s_{\tilde{u},k}(\pi) + \epsilon^c_{\tilde{u},k}(\pi)) \\
&+ \Delta_{T_{k+1}}\Psi''_4(\epsilon^s_{\hat{v}^2,k}(\pi) + \epsilon^c_{\hat{v}^2,k}(\pi)) \\
&+ \frac{1}{2}[\Psi_3\Delta_{T_k} + \Psi'_3\Delta_{T_{k+1}}]\left(\langle\mu_{T_k}, \varphi_f[\cdot, u(T_k,\cdot)]\rangle - \langle\hat{\mu}_{T_k}, \varphi_f[\cdot, u(T_k,\cdot),\cdot)]\rangle\right), \\
&+ \Delta_{T_k}\frac{\Psi_3}{2}\left(\langle\hat{\mu}_{T_k}, \varphi_f[\cdot, u(T_k,\cdot)]\rangle - \langle\hat{\mu}_{T_k}, \varphi_f[\cdot, \hat{u}^2(T_k,\cdot),\cdot)]\rangle\right) \\
&+ \Delta_{T_{k+1}}\frac{\Psi'_3}{2}\left(\langle\hat{\mu}_{T_k}, \varphi_f[\cdot, u(T_k,\cdot)]\rangle - \langle\hat{\mu}_{T_k}, \varphi_f[\cdot, \tilde{u}(T_k,\cdot),\cdot)]\rangle\right).
\end{aligned}
$$

We are now in position to give the dynamics of the sum of the maximal errors $\mathcal{E}^2_u(k)+\Delta_{T_k}\mathcal{E}_f(k)$. We use Cauchy-Schwartz inequality on the first two terms of (2.67), bound the last two terms in (2.67) using the Lipschitz property of φ_f, as in the first order error analysis, we can deduce that for some constants C, C',

$$
\mathcal{E}^2_u(k)+\Delta_{T_{k+1}}\mathcal{E}_f(k) \le (1+C\Delta_{T_{k+1}})\left[\mathcal{E}^2_u(k+1) + \Delta_{T_k}\mathcal{E}_f(k+1)\right] + C'(\bar{\epsilon}_2(k)+\Delta_{T_{k+1}}N^{-[(m-1)\wedge q]/2}) \qquad (2.68)
$$

with

$$\bar{\epsilon}_2(k) = \sup_{\pi\in\mathcal{S}_k} |\epsilon^s_{\hat{u}^2,k}(\pi)+\Delta_{T_{k+1}}\epsilon^s_{\hat{v}^2,k}(\pi)|+\sup_{\pi\in\mathcal{S}_k} |\epsilon^c_{\hat{u}^2,k}(\pi)+\Delta_{T_{k+1}}\epsilon^c_{\hat{v}^2,k}(\pi)|+\Delta_{T_{k+1}}(\sup_{\pi\in\mathcal{S}_k} |\epsilon^s_{\tilde{u},k}(\pi)+\epsilon^c_{\tilde{u},k}(\pi)|). \tag{2.69}$$

We can bound the two first terms of $\bar{\epsilon}_2(k)$ using respectively Claims 2.12, 2.13, while the last term may be bounded using Claims 2.9 and 2.10 as we have discussed before, so that there is a constant C''

$$\begin{aligned}\bar{\epsilon}_2(k) \leq & C''\Bigg(\sup_{s\in[T_k,T_{k+1}]} ||u(s,\cdot)||_{4,\infty}\Delta^3_{T_{k+1}} \\ & + [||u(T_{k+1},\cdot)||_{3,\infty} + M_u(4,T_{k+1})]\left[\Delta^{q+1}_{T_{k+1}} + \Delta^2_{T_{k+1}}N^{-[(m-1)\wedge 2q]/2}\right] \\ & + \sum_{i=m-2}^{m+1}[||u(T_{k+1},\cdot)||_{i,\infty} + M_u(i,T_{k+1}))]\Delta^{(i+1)/2}_{T_{k+1}} + \sum_{i=m+2}^{m+3} M_u(i,T_{k+1})\Delta^{(i+1)/2}_{T_{k+1}}\Bigg).\end{aligned} \tag{2.70}$$

Given that the initialization step is the first order scheme, Claims 2.9, 2.10 and 2.11 imply $\mathcal{E}^2_u(N-1) \leq KN^{-2}$ and $\Delta_{T_{N-1}}\mathcal{E}^2_f(N-1) \leq KN^{-2}$. Moreover, under **(SB)** , we have from Lemma 2.15 that for all $n \in \mathbb{N}^*$ there exists a constant K, depending on the regularity of $V_{0:d}$ and ϕ, such that for all $k \in \{0,\cdots,N-1\}$,

$$M_u(n,T_{k+1}) + \sup_{s\in[T_k,T_{k+1}]} ||u(s,\cdot)||_{n,\infty} \leq K.$$

An application of the discrete Gronwall lemma on the sum $\mathcal{E}^2_u(k) + \Delta_{T_{k+1}}\mathcal{E}_f(k)$, gives

$$\sup_{k\leq N-1} \mathcal{E}^2_u(k) + \Delta_{T_k}\mathcal{E}_f(k) \leq C\left(\mathcal{E}^2_u(N-1) + \Delta_{T_N}\mathcal{E}_f(N-1) + N^{-[(m-1)\wedge q]/2} + \sum_{i=0}^{N-2}\bar{\epsilon}_2(k)\right).$$

Using (2.70) we deduce that, if $m \geq 7$,where the bound is a consequence of the second assertion of Claim 2.13, we have

$$\sup_{k\leq N-1} \mathcal{E}^2_u(k) + \Delta_{T_k}\mathcal{E}_f(k) \leq CN^{-2}.$$

As for the first order case, the previous result together with the expansion of $v-\hat{v}^2$ given (2.56), (2.57), (2.55) and Cauchy-Schwartz inequality, implies

$$\Delta^{1/2}_{T_k}\mathcal{E}^2_v(k) \leq CN^{-2}.$$

This concludes the proof of assertions (2.15) and (2.16) in Theorem 2.1. □

7 Mathematical tools

Here we will intensively use the notions defined in section 5.

7.1 The conditional linear PDE

Lemma 2.14. Let η be a given family of probability measures on $\mathbb{R}^d$ and consider the PDE

$$\begin{cases} \partial_t\psi(t,y) + \mathcal{L}^\eta\psi(t,y) = 0, \text{ on } [0,T]\times\mathbb{R}^d \\ \psi(T,y) = \phi(y) \end{cases} \tag{2.71}$$

where $\mathcal{L}^\eta$ is defined by (2.22). Suppose that assumption **(SB)** holds, then, this PDE admits a unique infinitely differentiable solution ψ and for every multi-index $\beta \in \mathcal{M}$ there exists a positive constant C depending on the regularity of $V_{0:d}$, $\varphi_{0:d}$ and T such that, :

$$||D_\beta \psi(t,.)||_\infty \leq C||\phi||_{||\beta||,\infty} \tag{2.72}$$

Proof. Since the law that appears in the coefficients of the SDE is fixed, this is an obvious consequence of the regularity of the coefficients and the terminal condition.

□

7.2 The conditional semi-linear PDE

Lemma 2.15. Under **(SB)** there exists a function u from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}$ such that

$$Y_t^y = u(t, X_t^y)$$

where Y_t^y is defined in (2.1). This function is in $C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$ and is the unique solution of the semi-linear PDE:

$$\begin{cases} \partial_t u(t,y) + \mathcal{L}^\mu u(t,y) = f\left(t, y, u(t,y), (\mathcal{V}^\mu u(t,y))^T, \langle \mu_t, \varphi_f(\cdot, u(t,\cdot))\rangle\right), \text{ on } [0,T] \times \mathbb{R}^d \\ u(T,y) = \phi(y) \end{cases} \tag{2.73}$$

where $\mathcal{L}^\mu$ is defined as in (2.4).

Moreover, u is infinitely differentiable, and for every multi-index $\beta \in \mathcal{M}$ there exists a positive constant C depending on the regularity of $V_{0:d}, f, \varphi_{0:d}, \phi$ and T such that,

$$||D_\beta u||_\infty \leq C \tag{2.74}$$

Proof of Lemma 2.15.

(i) *Existence and PDE solution.* Consider the conditional BSDE:

$$\begin{cases} dX_s^{t,y,x} = \sum_{i=0}^{d} V_i(s, X_s^{t,y,x}, \mathbb{E}\left[\varphi_i(X_s^x)\right]) dB_s^i \\ d\bar{Y}_s^{t,y,x} = -f(s, X_s^{t,y,x}, \bar{Y}_s^{t,y,x}, \bar{Z}_s^{t,y,x}, \mathbb{E}\left[\varphi_f(X_s^x, Y_s^x)\right])ds + \bar{Z}_s^{t,y,x} dB_s^{1:d} \\ X_t^{t,y,x} = y, \quad \bar{Y}_T^{t,y,x} = \phi(X_T^{t,y,x}), \end{cases} \tag{2.75}$$

for s in $[t,T]$. Note that the McKean term in (2.75), $\mathbb{E}\varphi_f(X_s^x, Y_s^x)$, does not depend on y. In fact, if the solution of (2.1) is found, one can consider the term $\mathbb{E}\varphi_f(X_s^x, Y_s^x)$ simply as a term depending on time, so that conditionally to knowing the joint law of $(X_t^x, Y_t^x,\ 0 \leq t \leq T)$, equation (2.75) is classical and Markov. As pointed out before, the existence of a unique solution to (2.1) follows the lines of the results in [15].

It is clear from the previous discussion that we might apply classical results on BSDE to analyze equation (2.75). In particular, we have from the results of Pardoux and Peng in [82] given the regularity in space of f and ϕ, that the mapping $(t,y) \mapsto \bar{u}(t,y) = \bar{Y}_t^{t,y,x}$ the solution of (2.75) is differentiable, once in time and twice in space with bounded derivatives and satisfies the PDE:

$$\begin{cases} \partial_t \bar{u}(t,y) + \mathcal{L}^\mu \bar{u}(t,y) = f\left(t, y, \bar{u}(t,y), (\mathcal{V}^\mu \bar{u}(t,y))^T, \mathbb{E}[\varphi_f(X_t^x, Y_t^x)]\right) \\ \bar{u}(T,y) = \phi(y) \end{cases},$$

Besides, thanks to the bounds on f and ϕ it follows from Proposition 3.2 in [14] that

$$||\bar{u}(t,\cdot)||_\infty \leq C(T, V_{0:d}, f)(1 + ||\phi||_\infty^p)^{1/p} \leq C(T, V_{0:d}, f, \phi).$$

Moreover, as shown in [82] the derivative of $\nabla_y \bar{Y}_s^{t,y,x}$ is given as the solution of the BSDE:

$$\begin{aligned}\nabla_y \bar{Y}_s^{t,y,x} =& \nabla_y \phi(X_T^{t,y,x}) \nabla_y X_T^{t,y,x} - \int_s^T \nabla_y \bar{Z}_r^{t,y,x} dB_r \\ &+ \int_s^T \Big[\partial_y f(r, X_r^{t,y,x}, \bar{Y}_r^{t,y,x}, \bar{Z}_r^{t,y,x}, \mathbb{E}[\varphi(X_r^x, Y_r^x,)]) \nabla_y X_r^{t,y,x} \\ &\quad + \partial_{y'} f(r, X_r^{t,y,x}, \bar{Y}_r^{t,y,x}, \bar{Z}_r^{t,y,x}, \mathbb{E}[\varphi(X_r^x, Y_r^x)]) \nabla_y \bar{Y}_r^{t,y,x} \\ &\quad + \partial_z f(r, X_r^{t,y,x}, \bar{Y}_r^{t,y,x}, \bar{Z}_r^{t,y,x}, \mathbb{E}[\varphi(X_r^x, Y_r^x)]) \nabla_y \bar{Z}_r^{t,y,x} \Big] dr,\end{aligned} \tag{2.76}$$

and $\nabla_y \bar{u}(t,y) = \nabla_y \bar{Y}_t^{t,y,x}$. Once again using the bounds on the derivatives in space of ϕ and f and the results in [14] we deduce

$$||\nabla_y \bar{u}(t,\cdot)||_\infty \leq C(T, V_{0:d}, f)(1 + ||\partial_y \phi||_\infty^p)^{1/p} \leq C(T, V_{0:d}, f, \phi). \tag{2.77}$$

Finally, one can show that $\bar{u}$ solves (2.73). To see this, notice that due to the uniqueness of the solutions to (2.1) and (2.75),

$$\bar{u}(t, X_t^x) = \bar{u}(t, X_t^{0,x,x}) = \bar{Y}_t^{t, X_t^{0,x}, x} = \bar{Y}_t^{0,x,x} = Y_t^x.$$

This equality implies

$$\mathbb{E}[\varphi_f(X_s^x, \bar{u}(s, X_s^x))] = \mathbb{E}[\varphi_f(X_s^x, Y_s^x)]$$

for all s in $[0,T]$. Therefore we set $u := \bar{u}$. This concludes the proof of the first assertion.

(ii) *Control on the derivatives.*

To prove the regularity of u and the the bound on its derivatives, we consider first the case involving only space derivatives. In this case the whole argument of Pardoux and Peng may be iterated reasoning on the BSDE for the first derivative, as long as the hypotheses remain valid, to obtain a BSDE for higher order derivatives in space. We turn the reader to the paper of Crisan and Delarue [25] where this is done in detail (taking into account the additional law dependence that must be considered in our framework).

It remains to consider the case of general derivatives including time derivatives. As we have said before, iterative applications of the Pardoux and Peng argument lead to PDEs similar to (2.73). Then, we can argue that we are able to differentiate once in time for every two derivatives in space. It is also clear that the control on the space derivatives plus the regularity properties of the coefficients imply the control for time derivatives.

7.3 One-step errors

Let η be a given family of probability measures and $X^{t,y,\eta}$, $\tilde{X}^{t,y,\eta}$ defined as in (2.23) and (2.24). We recall that $\hat{\mu}$ denotes the discrete probability measure defined by Algorithm 1 and μ the law of the forward part of (2.1).

Lemma 2.16. Let g be a C_b^2 function from $\mathbb{R}^d$ to $\mathbb{R}$. Then, there exists a constant C depending only on $V_{0:d}$ and T such that for all $k = 1, \cdots, N-1$:

$$\begin{aligned}&\left| (P_{T_k, T_{k+1}}^{\eta} - \tilde{P}_{T_k, T_{k+1}}^{\hat{\mu}}) g(y) \right| \\ &\leq C||g||_{2,\infty} \sum_{i=0}^{d} \int_{T_k}^{T_{k+1}} \left| \langle \eta_t, \varphi_i \rangle - \sum_{p=0}^{q-1} [(t-T_k)^p / p!] \langle \eta_{T_k}, (\mathcal{L}^\eta)^p \varphi_i \rangle \right| dt \\ &+ C||g||_{2,\infty} \sum_{i=0}^{d} \sum_{p=0}^{q-1} \Delta_{T_{k+1}}^{p+1} \left| \langle \eta_{T_k}, (\mathcal{L}^\eta)^p \varphi_i \rangle - \langle \hat{\mu}_{T_k}, (\mathcal{L}^{\hat{\mu}})^p \varphi_i \rangle \right|.\end{aligned} \tag{2.78}$$

Moreover, if g is C_b^3, there exists a constant C depending only on $V_{0:d}, d$ such that for all $l = 1, \cdots, d$ and $k = 1, \cdots, N-1$:

$$\begin{aligned}
&\left|\mathbb{E}\left(\left[g\left(X_{T_{k+1}}^{T_k,y,\eta}\right) - g\left(\tilde{X}_{T_{k+1}}^{T_k,y,\hat{\mu}}\right)\right] I_{(l)}^{T_k,T_{k+1}}\right)\right| \qquad (2.79)\\
&\leq C||g||_{3,\infty} \sum_{i=0}^{d} \int_{T_k}^{T_{k+1}} \int_{T_k}^{t} \left|\langle \eta_s, \varphi_i\rangle - \sum_{p=0}^{q-1} [(t-T_k)^p/p!] \langle \eta_{T_k}, (\mathcal{L}^\eta)^p \varphi_i\rangle\right| ds dt\\
&+C||g||_{3,\infty} \sum_{i=0}^{d} \sum_{p=0}^{q-1} \Delta_{T_{k+1}}^{p+1} \left|\langle \eta_{T_k}, (\mathcal{L}^\eta)^p \varphi_i\rangle - \langle \hat{\mu}_{T_k}, (\mathcal{L}^{\hat{\mu}})^p \varphi_i\rangle\right|
\end{aligned}$$

and

$$\begin{aligned}
&\left|\mathbb{E}\left(\left[g\left(X_{T_{k+1}}^{T_k,y,\eta}\right) - g\left(\tilde{X}_{T_{k+1}}^{T_k,y,\hat{\mu}}\right)\right] I_{(0,l)}^{T_k,T_{k+1}}\right)\right| \qquad (2.80)\\
&\leq C||g||_{3,\infty} \Delta_{T_{k+1}} \sum_{i=0}^{d} \int_{T_k}^{T_{k+1}} \int_{T_k}^{t} \left|\langle \eta_s, \varphi_i\rangle - \sum_{p=0}^{q-1} [(t-T_k)^p/p!] \langle \eta_{T_k}, (\mathcal{L}^\eta)^p \varphi_i\rangle\right| ds dt\\
&+C||g||_{3,\infty} \sum_{i=0}^{d} \sum_{p=0}^{q-1} \Delta_{T_{k+1}}^{p+2} \left|\langle \eta_{T_k}, (\mathcal{L}^\eta)^p \varphi_i\rangle - \langle \hat{\mu}_{T_k}, (\mathcal{L}^{\hat{\mu}})^p \varphi_i\rangle\right|.
\end{aligned}$$

Lemma 2.17. Let $n \geq 1$, g be a C_b^{n+2} function from $\mathbb{R}^d$ to $\mathbb{R}$ and $\mathbb{Q}$ be a cubature measure of order n. Then, there exist constants C, C' depending only on $V_{0:d}, d, n$ such that for all $i = 1, \ldots, d$, for all $k = 1, \cdots, N-1$:

$$\left|(\tilde{P}_{T_k,T_{k+1}}^{\eta} - \tilde{Q}_{T_k,T_{k+1}}^{\eta}) g(y)\right| = \left|(\mathbb{E} - \mathbb{E}_{\mathbb{Q}})\left[g(\tilde{X}_{T_{k+1}}^{T_k,y,\eta})\right]\right| \leq C \sum_{l=n+1}^{n+2} ||g||_{l,\infty} \Delta_{T_{k+1}}^{(l)/2} \qquad (2.81)$$

$$\left|(\mathbb{E} - \mathbb{E}_{\mathbb{Q}})\left[g(\tilde{X}_{T_{k+1}}^{T_k,y,\eta}) I_{(i)}^{T_k,T_{k+1}}\right]\right| \leq C \sum_{l=n}^{n+1} ||g||_{l,\infty} \Delta_{T_{k+1}}^{(l+1)/2} \qquad (2.82)$$

$$\left|(\mathbb{E} - \mathbb{E}_{\mathbb{Q}})\left[g(\tilde{X}_{T_{k+1}}^{T_k,y,\eta}) I_{(0,i)}^{T_k,T_{k+1}}\right]\right| \leq C' \sum_{l=n-2}^{n-1} ||g||_{l,\infty} \Delta_{T_{k+1}}^{(l+3)/2}, \qquad (2.83)$$

for all $y \in \mathbb{R}^d$.

Lemma 2.18. Let $n \geq 1$ and g be a C_b^{n+2} function from $\mathbb{R}^d$ to $\mathbb{R}$. Let $\mathbb{Q}$ be a cubature measure of order n, and $\hat{X}$ be the associated cubature tree. Then, there exists a constant C depending only on d, q, $V_{0:d}, n, T, ||\varphi_{0:d}||_{2q+n+2,\infty}$ such that, for all $k = 1, \cdots, N-1$:

$$\begin{aligned}
\left|\mathbb{E}\left[g\left(X_{T_{k+1}}^{T_k,y,\mu}\right)\right] - \mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[g\left(X_{T_{k+1}}^{T_k,y,\hat{\mu}}\right)\right]\right| &\leq C||g||_{2,\infty}\left[\Delta_{T_{k+1}}^{q+1} + \Delta_{T_{k+1}} N^{-[(n-1)\wedge 2q]/2}\right]\\
&+C||g||_{n+1,\infty}\Delta_{T_{k+1}}^{(n+1)/2} + C||g||_{n+2,\infty}\Delta_{T_{k+1}}^{(n+2)/2} \qquad (2.84)
\end{aligned}$$

$$\begin{aligned}
\left|\mathbb{E}\left[\left(g\left(X_{T_{k+1}}^{T_k,y,\mu}\right) - g\left(\hat{X}_{T_{k+1}}^{T_k,y,\hat{\mu}}\right)\right) I_{(l)}^{T_k,T_{k+1}}\right]\right| &\leq C||g||_{3,\infty}\left[\Delta_{T_{k+1}}^{q+1} + \Delta_{T_{k+1}}^{2} N^{-[(n-1)\wedge 2q]/2}\right]\\
&+C||g||_{n,\infty}\Delta_{T_{k+1}}^{(n+1)/2} + C||g||_{n+1,\infty}\Delta_{T_{k+1}}^{(n+2)/2} \qquad (2.85)
\end{aligned}$$

and

$$\begin{aligned}
\left|\mathbb{E}\left[\left(g\left(X_{T_{k+1}}^{T_k,y,\mu}\right) - g\left(\hat{X}_{T_{k+1}}^{T_k,y,\hat{\mu}}\right)\right) I_{(0,l)}^{T_k,T_{k+1}}\right]\right| &\leq C||g||_{3,\infty}\left[\Delta_{T_{k+1}}^{q+2} + \Delta_{T_{k+1}}^{3} N^{-[(n-1)\wedge 2q]/2}\right]\\
&+C||g||_{n-2,\infty}\Delta_{T_{k+1}}^{(n+1)/2} + C||g||_{n-1,\infty}\Delta_{T_{k+1}}^{(n+2)/2} \qquad (2.86)
\end{aligned}$$

7.4 Proofs of Lemmas 2.16, 2.17 and 2.18

7.4.a *Proof of Lemma 2.16*

Let $k \in \{0, \cdots, N-1\}$, consider the PDE (2.14) with g as boundary condition. It is clear that this PDE admits a unique solution $\tilde{u}$ and that there exists a positive constant $C(T, V_{0:d})$ such that for every multi-index of space derivatives with $\beta \in \mathcal{A}_3$

$$||\tilde{u}||_\infty + ||D_\beta \tilde{u}||_\infty \leq C(T, V_{0:d})||g||_{||\beta||,\infty}. \tag{2.87}$$

Let us write:

$$\left|(P^\eta_{T_k,T_{T_{k+1}}} - \tilde{P}^{\hat{\mu}}_{T_k,T_{T_{k+1}}})g(y)\right| = \left|\mathbb{E}\left[g\left(X^{T_k,y,\eta}_{T_{k+1}}\right) - g\left(\tilde{X}^{T_k,y,\hat{\mu}}_{T_{k+1}}\right)\right]\right|.$$

Now, we have

$$\begin{aligned} g\left(X^{T_k,y,\eta}_{T_{k+1}}\right) &= \tilde{u}(T_k,y) + \int_{T_k}^{T_{k+1}} \mathcal{V}^\eta_{(0)}\tilde{u}(s, X^{T_k,y,\eta}_s)ds + \sum_{j=1}^d \int_{T_k}^{T_{k+1}} \mathcal{V}^\eta_{(j)}\tilde{u}(s, X^{T_k,y,\eta}_s)dB^j_s \\ &= \tilde{u}(T_k,y) + \sum_{j=1}^d \int_{T_k}^{T_{k+1}} \mathcal{V}^\eta_{(j)}\tilde{u}(s, X^{T_k,y,\eta}_s)dB^j_s, \end{aligned} \tag{2.88}$$

using the fact that $\tilde{u}$ is the solution of (2.14), and

$$\begin{aligned} g\left(\tilde{X}^{T_k,y,\hat{\mu}}_{T_{k+1}}\right) &= \tilde{u}(T_k,y) + \int_{T_k}^{T_{k+1}} \mathcal{V}^{\hat{\mu}}_{(0)}\tilde{u}(s, \tilde{X}^{T_k,y,\hat{\mu}}_s)ds + \sum_{j=0}^d \int_{T_k}^{T_{k+1}} \mathcal{V}^{\hat{\mu}}_{(j)}\tilde{u}(s, X^{T_k,y,\hat{\mu}}_s)dB^j_s \\ &= \tilde{u}(T_k,y) + \int_{T_k}^{T_{k+1}} (\mathcal{L}^\eta - \tilde{\mathcal{L}}^{\hat{\mu}})\tilde{u}(s, \tilde{X}^{T_k,y,\hat{\mu}}_s)ds \\ &\quad + \sum_{j=1}^d \int_{T_k}^{T_{k+1}} \mathcal{V}^{\hat{\mu}}_{(j)}\tilde{u}(s, \tilde{X}^{T_k,y,\hat{\mu}}_s)dB^j_s. \end{aligned} \tag{2.89}$$

Therefore,

$$\mathbb{E}\left[g\left(X^{T_k,y,\eta}_{T_{k+1}}\right) - g\left(\tilde{X}^{T_k,y,\hat{\mu}}_{T_{k+1}}\right)\right] = \mathbb{E}\int_{T_k}^{T_{k+1}} \left(\mathcal{L}^\eta - \tilde{\mathcal{L}}^{\hat{\mu}}\right)\tilde{u}(t, \tilde{X}^{T_k,y,\hat{\mu}}_t)dt, \tag{2.90}$$

since $\partial_t \tilde{u} = -\mathcal{L}^\eta \tilde{u}$. As (2.87) implies that $\tilde{u}$ and its two first derivatives are bounded, we may control the term above by the difference between the two generators, then by the difference between the frozen (in space of probability measure) $(\eta_t)_{T_K \leq t \leq T_{k+1}}$ and the approximate and frozen (in time) measure $\hat{\mu}_{T_k}$. Hence, taking into account the particular dependence on the measure in our framework, we deduce:

$$\begin{aligned} &\left|\mathbb{E}\left[g\left(X^{T_k,y,\eta}_{T_{k+1}}\right) - g\left(\tilde{X}^{T_k,y,\hat{\mu}}_{T_{k+1}}\right)\right]\right| \\ &\leq C||g||_{2,\infty}\int_{T_k}^{T_{k+1}} \left[\sum_{i=0}^d \left|\langle \eta_t, \varphi_i\rangle - \sum_{p=0}^{q-1}[(t-T_k)^p/p!]\langle \eta_{T_k}, (\mathcal{L}^\eta)^p\varphi_i\rangle\right| \right. \\ &\qquad \left. + \sum_{i=0}^d \sum_{p=0}^{q-1} \Delta^p_{T_{k+1}} \left|\langle \eta_{T_k}, (\mathcal{L}^\eta)^p\varphi_i\rangle - \langle \hat{\mu}_{T_k}, (\mathcal{L}^{\hat{\mu}})^p\varphi_i\rangle\right|\right] dt \\ &\leq C(T, V_{0:d})||g||_{2,\infty}\int_{T_k}^{T_{k+1}} \sum_{i=0}^d \left|\langle \eta_t, \varphi_i\rangle - \sum_{p=0}^{q-1}[(t-T_k)^p/p!]\langle \eta_{T_k}, (\mathcal{L}^\eta)^p\varphi_i\rangle\right| dt \\ &\quad + C(T, V_{0:d})||g||_{2,\infty} \sum_{i=0}^d \sum_{p=0}^{q-1} \Delta^p_{T_{k+1}}\Delta_{T_{k+1}} \left|\langle \eta_{T_k}, (\mathcal{L}^\eta)^p\varphi_i\rangle - \langle \hat{\mu}_{T_k}, (\mathcal{L}^{\hat{\mu}})^p\varphi_i\rangle\right| \end{aligned}$$

This concludes the proof of the first assertion. Now, we deduce from (2.88) and (2.89) and integration by parts, that:

$$
\begin{aligned}
&\mathbb{E}\left(\left[g\left(X_{T_{k+1}}^{T_k,y,\eta}\right)-g\left(\tilde{X}_{T_{k+1}}^{T_k,y,\hat{\mu}}\right)\right]I_{(l)}^{T_k,T_{k+1}}\right) \qquad (2.91)\\
&=\mathbb{E}\int_{T_k}^{T_{k+1}}\left(\mathcal{L}^{\eta}-\tilde{\mathcal{L}}^{\hat{\mu}}\right)\tilde{u}(t,\tilde{X}_t^{T_k,y,\hat{\mu}})I_{(l)}^{T_k,t}dt\\
&\quad+\mathbb{E}\int_{T_k}^{T_{k+1}}\left[\mathcal{V}_{(l)}(t,X_t^{T_k,y,\eta},\langle\eta_t,\varphi_l\rangle)-\mathcal{V}_{(l)}(t,X_t^{T_k,y,\eta},\langle\hat{\mu}_{T_k},\varphi_l\rangle)\right]\tilde{u}\left(t,X_t^{T_k,y,\eta}\right)dt\\
&\quad+\frac{1}{2}\mathbb{E}\int_{T_k}^{T_{k+1}}\left[\mathcal{V}_{(l)}(t,X_t^{T_k,y,\eta},\langle\hat{\mu}_{T_k},\varphi_l\rangle)\tilde{u}(t,X_t^{T_k,y,\eta})-\mathcal{V}_{(l)}(t,\tilde{X}_t^{T_k,y,\hat{\mu}},\langle\hat{\mu}_{T_k},\varphi_l\rangle)\tilde{u}(t,\tilde{X}_t^{T_k,y,\hat{\mu}})\right]dt.
\end{aligned}
$$

On a first hand, note that the first two terms in the right hand side above may be controlled by the difference between the coefficients times the supremum norm of the first and second order derivatives of $\tilde{u}$ times the order of the integrals, as we did for (2.90). On the other hand, note that the first assertion of Lemma 2.16 can be applied to the function $\mathcal{V}_{(l)}(t,.,\langle\hat{\mu}_{T_k},\varphi_l\rangle)\tilde{u}(t,.)$ in the last term on the right hand side above. These arguments, together with the bound (2.87) lead to:

$$
\begin{aligned}
&\left|\mathbb{E}\left(\left[g\left(X_{T_{k+1}}^{T_k,y,\eta}\right)-g\left(\tilde{X}_{T_{k+1}}^{T_k,y,\hat{\mu}}\right)\right]I_{(l)}^{T_k,T_{k+1}}\right)\right|\\
&\leq C(T,V_{0:d})\int_{T_k}^{T_{k+1}}(||g||_{1,\infty}+||g||_{2,\infty}(t-T_k)^{1/2})\sum_{i=0}^{d}\left|\langle\eta_t,\varphi_i\rangle-\sum_{p=0}^{q-1}[(t-T_k)^p/p!]\langle\eta_{T_k},(\mathcal{L}^{\eta})^p\varphi_i\rangle\right|dt\\
&\quad+C(T,V_{0:d})\sum_{i=0}^{d}\sum_{p=0}^{q-1}\Delta_{T_{k+1}}^{p}\Delta_{T_{k+1}}(||g||_{1,\infty}+||g||_{2,\infty}\Delta_{T_{k+1}}^{1/2})\left|\langle\eta_{T_k},(\mathcal{L}^{\eta})^p\varphi_i\rangle-\langle\hat{\mu}_{T_k},(\mathcal{L}^{\hat{\mu}})^p\varphi_i\rangle\right|\\
&\quad+C(T,V_{0:d})||g||_{3,\infty}\left\{\int_{T_k}^{T_{k+1}}\int_{T_k}^{t}\sum_{i=0}^{d}\left|\langle\eta_s,\varphi_i\rangle-\sum_{p=0}^{q-1}[(s-T_k)^p/p!]\langle\eta_{T_k},(\mathcal{L}^{\eta})^p\varphi_i\rangle\right|dsdt\right.\\
&\qquad\qquad\left.+\sum_{i=0}^{d}\sum_{p=0}^{q-1}\Delta_{T_{k+1}}^{p}\Delta_{T_{k+1}}^{2}\left|\langle\eta_{T_k},(\mathcal{L}^{\eta})^p\varphi_i\rangle-\langle\hat{\mu}_{T_k},(\mathcal{L}^{\hat{\mu}})^p\varphi_i\rangle\right|\right\},
\end{aligned}
$$

and this concludes the proof of the second assertion. Finally, (2.88) and (2.89) and integration by parts, give:

$$
\begin{aligned}
&\mathbb{E}\left(\left[g\left(X_{T_{k+1}}^{T_k,y,\eta}\right)-g\left(\tilde{X}_{T_{k+1}}^{T_k,y,\hat{\mu}}\right)\right]I_{(0,l)}^{T_k,T_{k+1}}\right)\\
&=\mathbb{E}\int_{T_k}^{T_{k+1}}\left(\mathcal{L}^{\eta}-\tilde{\mathcal{L}}^{\hat{\mu}}\right)\tilde{u}(t,\tilde{X}_t^{T_k,y,\eta})I_{(0,l)}^{T_k,t}dt\\
&\quad+\mathbb{E}\int_{T_k}^{T_{k+1}}\left[\mathcal{V}_{(l)}(t,X_t^{T_k,y,\eta},\langle\eta_t,\varphi_l\rangle)-\mathcal{V}_{(l)}(t,X_t^{T_k,y,\eta},\langle\hat{\mu}_{T_k},\varphi_l\rangle)\right]\tilde{u}\left(t,X_t^{T_k,y,\eta}\right)I_{(0)}^{T_k,t}dt\\
&\quad+\frac{1}{2}\mathbb{E}\int_{T_k}^{T_{k+1}}\left[\mathcal{V}_{(l)}(t,X_t^{T_k,y,\eta},\langle\hat{\mu}_{T_k},\varphi_l\rangle)\tilde{u}(t,X_t^{T_k,y,\eta})-\mathcal{V}_{(l)}(t,\tilde{X}_t^{T_k,y,\hat{\mu}},\langle\hat{\mu}_{T_k},\varphi_l\rangle)\tilde{u}(t,\tilde{X}_t^{T_k,y,\hat{\mu}})\right]I_{(0)}^{T_k,t}dt.
\end{aligned}
$$

Note the similarity with (2.91). So that a similar development gives

$$
\begin{aligned}
&\left|\mathbb{E}\left(\left[g\left(X_{T_{k+1}}^{T_k,y,\eta}\right)-g\left(\tilde{X}_{T_{k+1}}^{T_k,y,\hat{\mu}}\right)\right]I_{(0,l)}^{T_k,T_{k+1}}\right)\right| \\
&\leq C(T,V_{0:d})\int_{T_k}^{T_{k+1}}(||g||_{1,\infty}(t-T_k)+||g||_{2,\infty}(t-T_k)^{3/2})\sum_{i=0}^{d}\left|\langle\eta_t,\varphi_i\rangle-\sum_{p=0}^{q-1}[(t-T_k)^p/p!]\langle\eta_{T_k},(\mathcal{L}^{\eta})^p\varphi_i\rangle\right|dt \\
&\quad+C(T,V_{0:d})\Delta_{T_{k+1}}\sum_{i=0}^{d}\sum_{p=0}^{q-1}\Delta_{T_{k+1}}^{p}\Delta_{T_{k+1}}(||g||_{1,\infty}+||g||_{2,\infty}\Delta_{T_{k+1}}^{1/2})\left|\langle\eta_{T_k},(\mathcal{L}^{\eta})^p\varphi_i\rangle-\langle\hat{\mu}_{T_k},(\mathcal{L}^{\hat{\mu}})^p\varphi_i\rangle\right| \\
&\quad+C(T,V_{0:d})\Delta_{T_{k+1}}||g||_{3,\infty}\Bigg\{\int_{T_k}^{T_{k+1}}\int_{T_k}^{t}\sum_{i=0}^{d}\left|\langle\eta_s,\varphi_i\rangle-\sum_{p=0}^{q-1}[(s-T_k)^p/p!]\langle\eta_{T_k},(\mathcal{L}^{\eta})^p\varphi_i\rangle\right|dsdt \\
&\qquad+\sum_{i=0}^{d}\sum_{p=0}^{q-1}\Delta_{T_{k+1}}^{p}\Delta_{T_{k+1}}^{2}\left|\langle\eta_{T_k},(\mathcal{L}^{\eta})^p\varphi_i\rangle-\langle\hat{\mu}_{T_k},(\mathcal{L}^{\hat{\mu}})^p\varphi_i\rangle\right|\Bigg\}
\end{aligned}
$$

from where the last claim is deduced. □

7.4.b *Proof of Lemma 2.17*

Let $k\in\{0,\cdots,N-1\}$, once again, we consider the unique infinitely differentiable solution $\tilde{u}$ of PDE (2.14) with g as boundary condition. Recall that for every $\beta\in\mathcal{M}$ there exists a positive constant $C(T,V_{0:d})$ such that:

$$||\tilde{u}||_{\infty}+||D_{\beta}\tilde{u}||_{\infty}\leq C(T,V_{0:d})||g||_{||\beta||,\infty}. \tag{2.92}$$

The result then follows from Stratonovich-Taylor expansion of $(t,y)\mapsto\tilde{u}(t,y)$ around (T_k,X_{T_k}) by Theorem 5.6.1 in [53] and bounding the remainder as in Proposition 2.1 of [69]. □

7.4.c *Proof of Lemma 2.18*

Note that

$$
\begin{aligned}
\left|\mathbb{E}\left[g\left(X_{T_{k+1}}^{T_k,y,\mu}\right)\right]-\mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[g\left(X_{T_{k+1}}^{T_k,y,\hat{\mu}}\right)\right]\right| &= \left|\mathbb{E}\left[g\left(X_{T_{k+1}}^{T_k,y,\mu}\right)\right]-\mathbb{E}\left[g\left(\tilde{X}_{T_{k+1}}^{T_k,y,\hat{\mu}}\right)\right]\right| \\
&\quad+\left|\mathbb{E}\left[g\left(\tilde{X}_{T_{k+1}}^{T_k,y,\mu}\right)\right]-\mathbb{E}_{\mathbb{Q}_{T_k,T_{k+1}}}\left[g\left(X_{T_{k+1}}^{T_k,y,\hat{\mu}}\right)\right]\right|.
\end{aligned} \tag{2.93}
$$

Combining estimate (2.78) of Lemma 2.16 with Claim 2.8 and (2.14) in Theorem 2.1, we get that the first term in the right hand side is bounded by:

$$C||g||_{2,\infty}\left[\Delta_{T_{k+1}}^{q+1}+\Delta_{T_{k+1}}N^{-[(n-1)\wedge 2q]/2}\right].$$

The second term in the right hand side of (2.93) can be estimated by combining this bound with the estimate (2.81) in 2.17 (when choosing $\eta=\mu$).

The other assertion follows from the same procedure, substituting (2.79) (resp. (2.80)) to (2.78) and (2.82) (resp. (2.83)) to (2.81).

8 Proofs of Corollary 2.2 and 2.3

8.1 Proof of Corollary 2.2

Many practical applications, particularly in finance, require the algorithm to be able to solve problems in which the boundary condition ϕ is less regular, e.g. when ϕ is just Lipschitz. In this section, we prove how the results obtained in the regular case extend to the case when assumption **(LB)** holds as Corollary 2.2 state.

A preliminary result. We use in addition an auxiliary result shown in the proof of Theorem 8 in [26]:

Lemma 2.19. There exists a positive constant C such that:

$$\sum_{j=0}^{N-2} \Delta_{T_{j+1}}^{(m+1)/2}(T-T_j)^{-m/2} \leq CL(\gamma,m),$$

where L is defined in (2.18).

We are now ready to examine the error convergence for the forward and backward components of the algorithm.

Proof of the forward approximation in Corollary 2.2

The regularity of the solution of the linear associated linear PDE is essential to our analysis. We start by stating a result in this sense under **(LB)**. This is summarized by

Claim 2.20. Under **(LB)**, there exists a unique solution ψ to the PDE (2.71) and for every multi-index $\beta \in \mathcal{M}$ there exists a constant C such that:

$$||D_\beta \psi(t,\cdot)||_\infty \leq C(T-t)^{-(||\beta||-1)/2} \tag{2.94}$$

Proof. This follows from classical results of parabolic equations with parameter, see Chapter 9, Section 3 of [41].

□

Thanks to the uniform ellipticity assumption, even if the terminal condition is not differentiable, we know that the solution of the PDE (2.28) is smooth except at the boundary. Precisely, the gradient bounds (2.34) are now given by

$$||\nabla_y^n \psi(t,\cdot)||_\infty \leq C(T,V_{0:d})||\phi||_{1,\infty}(T-t)^{(1-n)/2}, \tag{2.95}$$

where ψ is defined in (2.27). With this in hand, we can follow the proof exactly as the one of the corresponding forward part in Theorem 2.1 up to estimate (2.33) but where we separate the error on the last step, since there is no smoothing effect there. Then, plugging estimate (2.95) in (2.33) instead of (2.34), we get:

$$\begin{aligned}
&\left|(P_{T_0,T_N} - Q^{\hat{\mu}}_{T_0,T_N})\phi(x)\right| \\
&\leq C(T,V_{0:d})||\phi||_{1,\infty} \sum_{j=0}^{N-2}\sum_{p=0}^{q-1} \Delta_{T_{j+1}}^{p+1}(T-T_{j+1})^{-1/2} \sum_{i=0}^{d} \left|(P_{T_0,T_N} - Q^{\hat{\mu}}_{T_0,T_N})\varphi_i(x)\right| \\
&\quad + C(T,V_{0:d},d)||\phi||_{1,\infty}\|\varphi\|_{2q,\infty} \sum_{j=0}^{N-2} \Delta_{T_{j+1}}^{q+1}(T-T_{j+1})^{-1/2} \\
&\quad + C(V_{0:d},d,m)||\phi||_{1,\infty} \sum_{j=0}^{N-2}\sum_{l=m+1}^{m+2} \Delta_{T_{j+1}}^{\frac{l}{2}}(T-T_{j+1})^{(1-l)/2} \\
&\quad + \left|(P_{T_0,T_N} - P^{\eta}_{T_0,T_N})\phi(x)\right|.
\end{aligned}$$

We conclude the proof by using Lemma 2.19 on the sums and by combining Lipschitz property of ϕ and adapted time-step on the last step error. □

Proof of the backward approximation in Corollary 2.2

Just as in the forward case, our analysis relies on the regularization properties of the associated non-linear PDE under **(LB)**. We have

Claim 2.21. Under **(LB)**, there exists a unique solution u of (2.73), for all $(t,y) \in [0,T] \times \mathbb{R}^d$, it is given by

$$u(t,y) = Y_t^{t,y,\mu},$$

where $Y_t^{t,y,\mu}$ is defined in (2.5). Moreover, for ϕ Lipschitz and bounded, for every multi-index $\beta \in \mathcal{M}$ there exists a positive constant C depending on the regularity of $V_{0:d}, f, \varphi$ and T such that:

$$||D_\beta u(t,\cdot)||_\infty \leq C||\phi||_{1,\infty}^{|\beta|}(T-t)^{-(||\beta||-1)/2} \tag{2.96}$$

Proof. To prove Claim 2.21, we follow the same arguments given for Lemma 2.15. First, due to the regularity properties of the diffusion under the elliptic case, we have similar properties as those used in the paper of Crisan and Delarue [25], even in the non-homogeneous case (notably, the integration by parts property as shown in [71]). Hence, we get the control on derivatives result for space derivatives, and extend it, as before, to time derivatives. □

Armed with the regularity of the function u, we can repeat the proof of the backward approximation in Theorem (2.1). We recover (2.48) for the first order scheme and (2.68) for the second order scheme, i.e.

$$\mathcal{E}_u^1(k) \leq \left(1 + C\Delta_{T_{k+1}}\right)\mathcal{E}_u^1(k+1) + \bar{\epsilon}(k+1), \tag{2.97}$$

$$\mathcal{E}_u^2(k) + \Delta_{T_{k+1}}\mathcal{E}_f(k) \leq (1 + C\Delta_{T_{k+1}})\left[\mathcal{E}_u^2(k+1) + \Delta_{T_k}\mathcal{E}_f(k+1)\right] + C'(\bar{\epsilon}_2(k) + \Delta_{T_{k+1}}N^{-[(m-1)\wedge q]/2}), \tag{2.98}$$

where $\bar{\epsilon}, \bar{\epsilon}_2$ are respectively defined in (2.49), (2.69). Now, if we show that

$$\sum_{k=0}^{N-1}\bar{\epsilon}(k) \leq N^{-1}; \quad \text{and} \quad \sum_{k=0}^{N-2}\bar{\epsilon}_2(k) \leq N^{-2}; \tag{2.99}$$

then, as in the smooth setting, we can apply Gronwall lemma on (2.97) and (2.98) and conclude on the desired rates of convergence for the approximation of u. The arguments for the rate of the approximation of v are exactly as in the smooth setting thus completing the proof of the claimed result.

Therefore, we only need to prove (2.99). But, Claim 2.21 and the definition of M_u given in (2.36) imply

$$M_u(n, T_{k+1}) \leq (T - T_{k+1})^{(1-n)/2}||\phi||_{1,\infty}^n,$$

which together with Claim 2.21 show that under the Lipschitz boundary setup,

$$\begin{aligned}
\bar{\epsilon}(k+1) &\leq C\Big((T-T_k)^{-1/2}\Delta_{T_{k+1}}\left[\Delta_{T_{k+1}}^3 + \Delta_{T_{k+1}}^q + N^{-[(m-1)\wedge 2q]/2}\right] \\
&\quad + (T-T_k)^{-m/2}\Delta_{T_{k+1}}^{(m+1)/2} + (T-T_k)^{-(m+1)/2}\Delta_{T_{k+1}}^{(m+2)/2} \\
&\quad + (T-T_k)^{-1}\Delta_{T_{k+1}}^{3/2}\left[\Delta_{T_{k+1}}^{1/2} + \Delta_{T_{k+1}}^{q-1/2} + \Delta_{T_{k+1}}^{1/2}N^{-[(m-1)\wedge 2q]/2}\right]\Big). \\
&\leq C\Big((T-T_k)^{-1/2}\Delta_{T_{k+1}}N^{-1} + (T-T_k)^{-m/2}\Delta_{T_{k+1}}^{(m+1)/2} \\
&\quad + (T-T_k)^{-(m+1)/2}\Delta_{T_{k+1}}^{(m+2)/2} + (T-T_k)^{-1}\Delta_{T_{k+1}}^{3/2}N^{-1/2}\Big);
\end{aligned} \tag{2.100}$$

where we have used the fact that $\Delta_{T_k} \leq CN^{-1}$ even on the decreasing discretization. We can proceed similarly for $\bar{\epsilon}_2$ from inequality (2.70) , to get

$$\bar{\epsilon}_2(k) \leq C''\Bigg((T-T_k)^{-3/2}\Delta^3_{T_{k+1}} + (T-T_k)^{-3/2}\left[\Delta^{q+1}_{T_{k+1}} + \Delta^2_{T_{k+1}} N^{-[(m-1)\wedge 2q]/2}\right] + \sum_{i=m-2}^{m+3} (T-T_k)^{-(i-1)/2}\Delta^{(i+1)/2}_{T_{k+1}}\Bigg). \tag{2.101}$$

Then, (2.99) follows by applying Lemma 2.19 to (2.100) and (2.101) □

8.2 Proof of Corollary 2.3

On a first hand, by following the proof of (2.14) in Theorem 2.1 we get (2.32), where the difference between the integral of the φ_i, $i = 1, \ldots, d$ against the measures in the right hand side are replaced by the distance $d_{\mathcal{F}}$. Since for all $T_k < t < T$ $d_{\mathcal{F}}(\mu_t, \mu_{T_k}) \leq C(t - T_k)$ (resp. $(t - T_k)^{1/2}$) in the case (2.19) (resp. (2.20)), the result follows from Gronwall's Lemma. This gives the rate of approximation of the law of the forward process.

On a second hand, the backward errors are then obtained by the same arguments already developed in the proof of (2.15) in Theorem 2.1, using the new forward approximation (2.19) (resp. (2.20)) instead of (2.14) (resp. (2.17)) in the proofs of Claims 2.10 and 2.11.

Appendix A

Implementation of a reduced cubature method to solve BSDEs

In the Introduction of this thesis we remarked how the cubature method suffers from the exponentially growth of its complexity. We mentioned that there exists some techniques that allow to control this complexity problem without sacrificing the rate of convergence of the cubature algorithm, notably the TBBA method and the high order recombination procedure. Focusing on the latter, we showed how the procedure reduces the number of needed nodes at the expense of sacrificing the Markov structure in the cubature tree construction. As a consequence, the procedure cannot be used directly to solve BSDEs.

In this appendix, we present some early results to design a new recombination procedure leading to reduced cubatures to solve BSDEs. The key point is to take care at the reduction step of the dependencies between nodes at one time and their descendants: we will compensate the loss of information due to the elimination of some nodes at the recombination step by adding extra dependencies with respect to some of the surviving ones.

More precisely, we propose an approach in which we add and *subtract* additional trajectories, sharing the history of the eliminated ones, but passing at time T_k only trough non eliminated nodes. The fact that we subtract some trajectories means passing from a structure of a node and its descendants characterized by a probability measure to another characterized by a *signed measure*.

The appendix starts by introducing conditional reduction of signed measures in section 1. Here, we will explain our choice of working with signed measures instead of probability measures, and show a link with this procedure and interpolation. Then, in section 2, we use the properties of the defined reduction to show that using conditional reduction as a black-box we can construct a reduced cubature measure to solve FBSDE. In this section we present the main result on the error for the FBSDE approximation in terms of the parameters introduced for the reduced cubature tree construction. Section 3 presents a proposed method of implementation of the conditional reduction for the unidimensional case with potential to be extended to the

multidimensional case. We discuss as well some insights with respect to the parameter choice for the reduced cubature. Finally, we illustrate our results by solving a toy problem in section 4.

1 Conditional reduction of signed measures

We introduce a concept that modifies the reduction definition behind the recombination procedure: it will deal with a class of normalized and bounded probability measures extending probability measures and impose some additional structure.

Let Ω_1, Ω_2 be two sets, and let μ be a discrete measure defined on $\Omega_1 \times \Omega_2$ given by

$$\mu = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \mu_{i,j} \delta_{x_i} \delta_{y_j},$$

for some $x_1, \dots, x_{m_1} \in \Omega_1$, and $y_1, \dots, y_{m_2} \in \Omega_2$. We will not restrict ourselves to probability measures (i.e. the coefficients might be negative), but to an extension of them: we will deal with signed measures such that

$$\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \mu_{i,j} = 1; \qquad \max_{i,j} |\mu_{i,j}| \leq 1. \tag{A.1}$$

i.e. we deal with normalized signed measures with bounded coefficients. Of course probability measures are included in this class.

Finally, we denote by μ^1 and μ^2 the corresponding marginal measures over Ω_1 and Ω_2 respectively, and let

$$\mu_i^1 = \sum_{j=1}^{m_2} \mu_{i,j} \quad \text{and} \quad \mu_j^2 = \sum_{i=1}^{m_1} \mu_{i,j}$$

Definition A.1. Let $\mathcal{H}$ be a finite set of test functions with support on Ω_2, integrable with respect to μ^2. We say that $\tilde{\mu}$ is a $K_{max}-$ *bounded conditionally reduced (c-reduced) measure of* μ *with respect to* $\mathcal{H}$ if it satisfies the following conditions

1. $\tilde{\mu}^1 = \mu^1$ where $\tilde{\mu}^1$ is the marginal measure of $\tilde{\mu}$ on Ω_1 ;
2. $\text{supp}(\tilde{\mu}^2) \subsetneq \text{supp}(\mu^2)$ where $\tilde{\mu}^2$ is the marginal measure of $\tilde{\mu}$ on Ω_2 ;
3. $\tilde{\mu}$ is also normalized with bounded coefficients,

$$\sum_{i=1}^{m_1} \sum_{j=1}^{\tilde{m}_2} \tilde{\mu}_{i,j} = 1; \qquad \max_{i,j} |\tilde{\mu}_{i,j}| \leq 1.$$

4. $\tilde{\mu}$ satisfies

$$\sum_{j=1}^{\tilde{m}_2} |\tilde{\mu}_{i,j}| \leq K_{max} |\tilde{\mu}_i^1|, \quad \text{for } i = 1, \dots, m_1.$$

In other words the reduced conditional measures with respect to each node in Ω_1 have l_1 norm bounded by K_{max}.

5. For all $i = 1, \dots, m_1$ and $h \in \mathcal{H}$,

$$\sum_{j=1}^{\tilde{m}_2} \tilde{\mu}_{i,j} h(\hat{y}_j) = \sum_{j=0}^{m_2} \mu_{i,j} h(y_j)$$

where $\text{supp}(\tilde{\mu}^2) = \{\hat{y}_1, \dots, \hat{y}_{\tilde{m}_2}\}$.

The main motivation behind our definition of a c-reduced measure comes from our idea of preserving a kind of Markov property when constructing a measure obtained by using iteratively the reduction algorithm: we want to be able to approach conditional expectations as well as total expectations, controlling at the same time the stability of the iteration (the reason for introducing the constraint K_{max}).

Evidently, we would rather work with reduced *probability measures* because we would have all the probability machinery available. Note that in our framework this would account to fix $K_{max} \equiv 1$. We believe this would be a very strong condition that might prevent effective reduction in many cases. What we expect from allowing some negative weights is to add enough flexibility for the reduction to take place keeping at the same time the stability controlled.

To illustrate this point, consider the following simple example: let $\Omega_1 = \Omega_2$ be a finite subset of $[-1, 1]$ with $\{-1, 0, 1\} \subset \Omega_2$. Let $\mu_{i,j} = |\text{supp}(\Omega_2)|^{-1} \delta_{i,j}$. If $\mathcal{H}$ is the set of polynomials up to order 2, we can show that, if $K_{max} = 1$ (the probability measure case), the minimal size set measure satisfying the properties in Definition (A.1) is μ itself.

To see this, assume by contradiction that there exists a c-reduced measure $\tilde{\mu}_{i,j}$, and let $S = \{\hat{y}_1, \dots, \hat{y}_l\}$ be the support of its second marginal. As the support of $\tilde{\mu}$ should be smaller than Ω_2, there exists j^* such that $x_{j^*} \in \Omega_2$ and $x_{j^*} \notin S$. But since $\tilde{\mu}_{i,j}$ is a c-reduced measure, it follows that $[x_{j^*}, x_{j^*}^2]^T$ can be expressed as a convex combination of the elements of the set $\{[\hat{y}_1, \hat{y}_1^2]^T, \dots, [\hat{y}_l, \hat{y}_l^2]^T\}$ (otherwise the conservation property of $\mathcal{H}$ does not hold). But the space of convex combinations of $\{[\hat{y}_1, \hat{y}_1^2]^T, \dots, [\hat{y}_l, \hat{y}_l^2]^T\}$ is just the convex hull, which intersects the set $\{[x, x^2]^T, x \in [-1, 1]\}$ only at the points in S due to the strong convexity of the square function (see Figure A.1 for an illustration). This would imply $x_{j^*} \notin S$ which is the desired contradiction.

In comparison, if we take $K_{max} = 1.25$ one can construct a measure $\tilde{\mu}$ having as support for its second marginal the set $\{-1, 0, 1\}$, and weights given by

$$\tilde{\mu}_{x,0} = \frac{(x+1)(1-x)}{|\text{supp}(\Omega_2)|} \quad \tilde{\mu}_{x,-1} = \frac{x(x-1)}{2|\text{supp}(\Omega_2)|} \quad \tilde{\mu}_{x,1} = \frac{x(x+1)}{2|\text{supp}(\Omega_2)|},$$

where, for the sake of clarity, we abused of our notation to denote the points in the support of the measure. Indeed, note that

$$\begin{aligned}
\tilde{\mu}_{x,0} + \tilde{\mu}_{x,-1} + \tilde{\mu}_{x,1} &= \frac{1}{|\text{supp}(\Omega_2)|} = \mu_{x,x} \\
\tilde{\mu}_{x,0}(0) + \tilde{\mu}_{x,-1}(-1) + \tilde{\mu}_{x,1}(1) &= \frac{x}{|\text{supp}(\Omega_2)|} = x\mu_{x,x} \\
\tilde{\mu}_{x,0}(0) + \tilde{\mu}_{x,-1}(-1)^2 + \tilde{\mu}_{x,1}(1)^2 &= \frac{x^2}{|\text{supp}(\Omega_2)|} = x^2\mu_{x,x}.
\end{aligned}$$

Finally, note that both $\tilde{\mu}_{x,-1}, \tilde{\mu}_{x,1}$ can be negative, but not at the same time and moreover $|\tilde{\mu}_{x,-1}|, |\tilde{\mu}_{x,1}| \leq (1/8)|\text{supp}(\Omega_2)|^{-1}$ for any point $x \in [-1, 1]$. Hence

$$|\tilde{\mu}_{x,0}| + |\tilde{\mu}_{x,1}| + |\tilde{\mu}_{x,-1}| \leq \tilde{\mu}_{x,0} + \tilde{\mu}_{x,1} + \tilde{\mu}_{x,-1} + \frac{2}{8} \leq 1.25|\text{supp}(\Omega_2)|^{-1}.$$

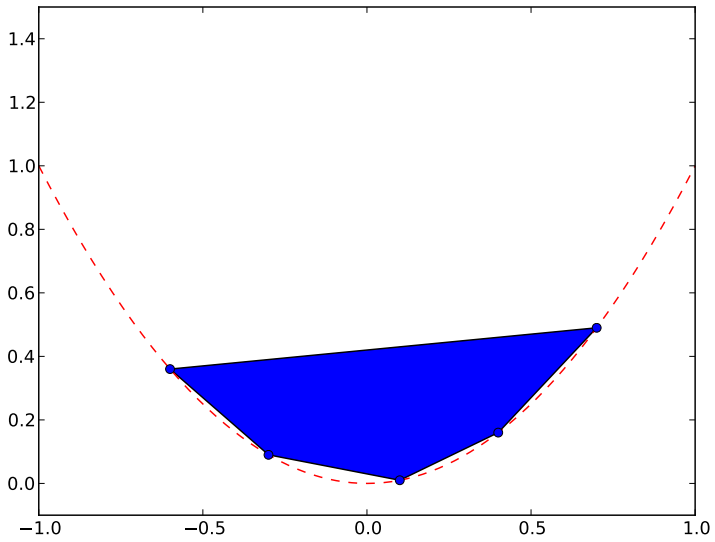


Figure A.1: The strict convexity implies that the convex hull of the set of points S intersects the curve $[x, x^2]$ only at the points S.

Hence, we passed from not being able to find a c-recombination to give a c-reduced measure with only 3 points in the support of the second marginal regardless of the size of Ω_2. In fact, the coefficients we proposed for this reduction come from a polynomial interpolation. We develop the link between reduction and interpolation in Lemma A.2.

Having illustrated our motivation to work in the signed measure framework, let us go back to discussing c-reduction. Although the marginal $\tilde{\mu}^2$ is a reduction (in the sense of Litterer and Lyons) of the marginal μ^2, the conditional reduction procedure is not a reduction taken on the space $\Omega_1 \times \Omega_2$ (i.e. in the space of couples) because new points in this space appear in the support, combination of points in the support of each marginal that were not initially coupled together.

We can give a probability interpretation to the class of measures satisfying a property as the one in (A.1). Suppose $\mathbb{P}$ is a probability measure defined over a set Ω, and A_1, A_2 are two non mutually exclusive events such that $\Omega = A_1 \cup A_2$, then

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2),$$

and $\mathbb{P}(A_1), \mathbb{P}(A_2), \mathbb{P}(A_1 \cap A_2)$ are all in $[0, 1]$.

Similarly, we can think about the negative entries in any measure satisfying property (A.1), as compensating for redundant information given by positive entries. This image has sense particularly in the framework of reduction as in Definition (A.1), when we suppose μ is a probability measure, and we interpret $\tilde{\mu}$ as composed by the information carried by some events (positive entries) and some terms we have to subtract to compensate for counting certain effects more than once (negative entries).

From the definition, we can see that the conditional reduction of a measure depends on the masses and elements in the support of the joint measure to be reduced. However, assuming the

initial measure is a probability measure, we will be able to give a sufficient condition to find a c-reduced measure depending only on the elements of the support of μ^2, the second marginal of μ.

Lemma A.2. *With the same notation as before, assume that μ is a probability measure. If there exists $S = \{\hat{y}_1, \dots, \hat{y}_{\tilde{m}_2}\} \subsetneq \Omega_2$ and $\alpha_j^k \in \mathbb{R}$ for $j = 1, \dots m_2; k = 1, \dots \tilde{m}_2$ with $|\alpha_j^k| < 1$ such that for every $h \in \mathcal{H}$ and $j = 1, \dots, m_2$*

$$h(y_j) = \sum_{k=1}^{\tilde{m}_2} \alpha_j^k h(\hat{y}_k) \quad ; \quad \sum_{k=1}^{\tilde{m}_2} \alpha_j^k = 1 \quad \text{and} \quad \sum_{k=1}^{\tilde{m}_2} |\alpha_j^k| \leq K_{max},$$

then

$$\bar{\mu}_{i,k} = \sum_{j=1}^{m_2} \alpha_j^k \mu_{i,j}, \qquad \text{for } k = 1, \dots \tilde{m}_2 \tag{A.2}$$

is a K_{max} bounded c-reduced measure of μ with respect to $\mathcal{H}$.

Lemma A.2 shows that the problem of finding a reduced c-measure for a probability measure can be solved by a constrained interpolation problem: we look for a subset of nodes that interpolate the functions $h \in \mathcal{H}$ in a compact domain containing all the nodes in the support of μ, such that the l_1 norm of the interpolating coefficients evaluated at the points in the support of the original measure is bounded by K_{max}. This motivates the following definition.

Definition A.3. We will say that $(\bar{S}, (\alpha_j^k)_{1 \leq j \leq m_2; 1 \leq k \leq \tilde{m}_2})$ is a K_{max} bounded interpolation of Ω_2 with respect to $\mathcal{H}$, if it satisfies the conditions of Lemma A.2. Likewise, we will say that $\tilde{\mu}$ is a K_{max} bounded c-reduction of μ with respect to $\mathcal{H}$ *coming from interpolation* if it can be written as in (A.2).

Proof of Lemma A.2 . The proof proceeds by straightforward verification: we show that $\bar{\mu}$ is a c-reduced measure as defined in Definition A.1. Trivially, $\text{supp}(\bar{\mu}^2) \subsetneq \text{supp}(\mu^2)$, and for any $i = 1, \dots, m_1$

$$\bar{\mu}_i^1 = \sum_{k=1}^{\tilde{m}_2} \bar{\mu}_{i,k} = \sum_{j=1}^{m_2} \left(\sum_{k=1}^{\tilde{m}_2} \alpha_j^k \right) \mu_{i,j} = \sum_{j=1}^{m_2} \mu_{i,j} = \mu_i^1,$$

Moreover,

$$\max_{i=1,\dots,m_1, k=1,\dots \tilde{m}_2} |\bar{\mu}_{i,k}| = \max_{i=1,\dots,m_1, k=1,\dots \tilde{m}_2} \left| \sum_{j=1}^{m_2} \alpha_j^k \mu_{i,j} \right| \leq \max_{i=1,\dots,m_1} \left(\max_{k=1,\dots \tilde{m}_2} |\alpha_i^k| \sum_{j=1}^{m_2} \mu_{i,j} \right) \leq 1$$

where we use the fact that the initial measure is a probability measure.

Reorganizing the sum, the assumed l_1 bound gives

$$\sum_{k=1}^{\tilde{m}_2} |\bar{\mu}_{i,k}| \leq \sum_{j=1}^{m_2} \sum_{k=1}^{\tilde{m}_2} |\alpha_j^k| \mu_{i,j} \leq \sup_{j=1\dots,m_2} \left[\sum_{k=1}^{\tilde{m}_2} |\alpha_j^k| \right] \sum_{j=1}^{m_2} \mu_{i,j} \leq K_{max} \mu_i^1.$$

Finally, by a similar argument,

$$\sum_{k=1}^{\tilde{m}_2} \bar{\mu}_i^k h(\bar{y}_k) = \sum_{j=1}^{m_2} \mu_{i,j} \left(\sum_{k=1}^{\tilde{m}_2} \alpha_j^k h(\bar{y}_k) \right) = \sum_{j=1}^{m_2} \mu_{i,j} h(y_j)$$

□

2 Conditionally reduced cubature measure

In this section we present how to apply the presented procedure to the cubature on Wiener space method. Let $(\Omega, \mathbb{P}, \mathcal{F})$ be a filtered probability space. For $t \in [0,T]$ and $x \in \mathbb{R}^d$, consider the SDE written in Stratonovitch notation given by

$$X_t = x + \sum_{i=0}^{d} \int_0^t V^i(X_s) \circ dB_s^i \tag{A.3}$$

for any t in $[0,T]$, $T > 0$ be given, with $B_t^{1:d}$ a d-dimensional adapted Brownian motion and the convention $B_t^0 = t$. We take all functions $V_i : (t,y) \in [0,T] \times \mathbb{R}^d \mapsto V_i(t,y)$ to be infinitely differentiable with bounded derivatives.

For $\omega \in C_{0,bv}([0,t], \mathbb{R}^d)$, and using the convention $\omega^0(t) = t$, let $\Phi_{t,x}(\omega)$ be the solution of the ODE

$$\Phi_{t,x}(\omega) = x_0 + \sum_{i=0}^{d} \int_0^t V^i(\Phi_{s,x}(\omega)) d\omega^i(s).$$

Let m be a positive integer. Assume we are given an m-cubature formula (for the time interval $[0,1]$), defined by the couples $(\lambda_1, \omega_1), \ldots, (\lambda_\kappa, \omega_\kappa)$, were $\omega_i \in C_{0,bv}([0,1], \mathbb{R}^d)$. Recall from the introduction that the λ_i and $\omega_{i;t}(s) := \sqrt{t}\omega_i(s/t)$ define a cubature measure Q_t in $[0,t]$.

Let us first introduce some useful mappings between certain discrete measure spaces. Following [66], let $KLV(.;s)$ be a mapping taking discrete measures in $\mathbb{R}^d$ to discrete measures in $\mathbb{R}^d$, such that if $\mu = \sum_{j=1}^{l} \mu_j \delta_{x_j}$,

$$KLV(\mu, t) = \sum_{j=1}^{l} \sum_{i=1}^{\kappa} \mu_j \lambda_i \delta_{\Phi_{t,x_j}(\omega_{i;t})}. \tag{A.4}$$

This transition mapping may be seen as the building block of the cubature tree construction: if $0 = T_0 < T_1 < \ldots < T_N = T$ is the discretization grid, and $\Delta_{T_k} := T_k - T_{k-1}$ for $k = 1, \ldots, N$, then we define cubature measures approaching X at each discretization times iteratively as

$$Q_{T_0} := \delta_{x_0}; \quad Q_{T_k} := KLV(Q_{T_{k-1}}, \Delta_{T_k}).$$

It will be useful as well to introduce the enlarged mapping $\overline{KLV}$ taking discrete measures in $\mathbb{R}^d$ to discrete measures in $\mathbb{R}^d \times \mathbb{R}^d$, such that if $\mu = \sum_{j=1}^{l} \mu_j \delta_{x_j}$,

$$\overline{KLV}(\mu, t) = \sum_{j=1}^{l} \sum_{i=1}^{\kappa} \mu_j \lambda_i \delta_{\left(\Phi_{t,x_j}(\omega_{i;t}), \omega_{i;t}\right)},$$

which is like KLV but carries the information to form a cubature measure approaching jointly X and B.

Finally, we introduce the application $jKLV(.;s)$, a mapping taking discrete measures in $\mathbb{R}^d$ to discrete measures in $\mathbb{R}^d \times (\mathbb{R}^d \times \mathbb{R}^d)$, such that if $\mu = \sum_{j=1}^{l} \mu_j \delta_{x_j}$,

$$jKLV(\mu, t) = \sum_{j=1}^{l} \sum_{i=1}^{\kappa} \mu_j \lambda_i \delta_{\left(x_j, \left(\Phi_{t,x_j}(\omega_{i;t}), \omega_{i;t}\right)\right)},$$

so that resulting image is a joint law having as first marginal μ and as second marginal the $\overline{KLV}(\mu, t)$ measure.

Then, the c-reduced cubature measure $\tilde{Q}$ is defined as follows.

Definition A.4 (c-reduced cubature measure $\tilde{Q}$)**.** Let $(r_k)_{k=1,\dots,N-1}$, $(\eta_k)_{k=1,\dots,N-1}$ be two sequences of positive real numbers, and let $(m'_k)_{k=1,\dots,N-1}$ a sequence in $\mathbb{N}$ that will be used as parameters of the algorithm.

- For $k = 0$, fix $\tilde{Q}_0 = \delta_{x_0}$, and let $\mathcal{S}_0 = \{x_0\}$.
- For $k = 1, \dots, N-1$:
  - Set the auxiliary measure $\check{Q}_{T_{k-1},T_k} := jKLV(\tilde{Q}_{T_{k-1}}, \Delta_{T_k})$ and let $\check{Q}_{T_k}$ be the first submarginal of its second marginal (i.e. $\check{Q}_{T_k} = KLV(\tilde{Q}_{T_{k-1}}, \Delta_{T_k})$).
  - Let O_j^k, $j = 1, \dots, M^k$ be a finite collection of disjoint open balls of radius r_k covering the support of $\check{Q}_{T_k}$, and $\{p_1, \dots, p_l\}$, with $l = \binom{m'_k + d}{d}$ be a basis for the set of polynomials in $\mathbb{R}^d$ of degree at most m'_k in $\mathbb{R}^d$.
  - Let $\mathcal{S}_k = \{\hat{x}^1_{T_k}, \dots, \hat{x}^{n_k}_{T_k}\}$ and α_j^k form an η_k bounded interpolation of the support of $\check{Q}_{T_k}$ with respect to the set of test functions $\mathcal{H}_k$ of local polynomials of degree at most m'_k i.e.
  $$\mathcal{H}_k = \{p_i \mathbf{1}_{O_k^j} : i = 1, \dots, l; j = 1, \dots M^k\}. \tag{A.5}$$
  Set $\tilde{Q}_{T_{k-1},T_k}$ be the associated c-reduced law (as in Lemma A.2) and $\tilde{Q}_{T_k}$ be its second marginal.
- For $k = N$, set $\tilde{Q}_{T_{N-1},T_N} := jKLV(\tilde{Q}_{T_{N-1}}, \Delta_{T_N})$, let $\tilde{Q}_{T_N}$ be its corresponding second marginal, the latter having support $\mathcal{S}_N = \{\hat{x}^1_{T_N}, \dots, \hat{x}^{n_N}_{T_N}\}$.

Let us comment on the c-reduced cubature measure. First of all, the marginal measures $\tilde{Q}_{T_k}$ for $k = 0, \dots, N$ are probability measures and form a reduced cubature measure in the sense of the definition of Litterer and Lyons. But in comparison with the reduced cubature tree, we have an additional structure to approach conditional expectations between successive times. This will be an essential feature to be able to solve backward problems.

Remark that we have extended the reduced measures to contain also the information regarding Brownian increments. In practice, this means that we should keep in memory the product of the weight coefficients and the increment of each trajectory in the cubature formula. This extra memory requirement will allow us in return the possibility to find conditional expectations involving Brownian increments .

2.1 Solution of BSDEs: General Algorithm

Consider now for, $t \in [0,T]$ and $x \in \mathbb{R}^d$, a decoupled forward backward stochastic differential equation (BSDE) given by

$$\begin{aligned} X_t &= x_0 + \sum_{i=0}^{d} \int_0^t V^i(X_s) \circ dB_s^i \\ Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)dt - \int_t^T Z_s dB_s, \end{aligned} \tag{A.6}$$

for any t in $[0,T]$, $T > 0$ be given, with $B_t^{1:d}$ a d-dimensional adapted Brownian motion and the convention $B_t^0 = t$. We take all functions $V_i : (t,y) \in [0,T] \times \mathbb{R}^d \mapsto V_i(t,y)$ and the mapping $f : t,x,y,z \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \mapsto f(t,x,y,z) \in \mathbb{R}$ to be infinitely differentiable with bounded derivatives, and g a smooth and bounded function from $\mathbb{R}^d$ to $\mathbb{R}$. Note that as our objective is to work with the cubature method, we have directly expressed the forward equation in Stratonovich notation (see section 3.2 of the introduction for a review on the cubature method).

From now on, let $0 \leq T_0, \ldots \leq T_N = T$ be a discretization of the interval $[0,T]$. In the smooth setting we are allowed to take a uniform discretization or the decreasing step discretization we introduced before,

$$T_k = T\left[1 - \left(1 - \frac{k}{N}\right)^{\gamma}\right]. \tag{A.7}$$

As before, let $\Delta_{T_k} = T_k - T_{k-1}$. Then, we can use the c-reduced cubature tree construction for the forward part, coupled with the backward algorithm as proposed by Crisan and Manoralakis [28] and that coincides with the one we presented in Section 1 in Chapter 2 when the McKean term is zero. In short, it is composed of

- C-reduced cubature tree construction as defined in Definiton A.4
- Backward propagation:
  1. At $k = N$ (i.e. $t = T$), we can approximate directly the value of $u(T,x)$ for $x \in \mathcal{S}_N$ as follows:

$$\begin{cases} \hat{u}(T_N, \hat{X}^j_{T_N}) & := g\left(\hat{X}^j_{T_N}\right); \\ \hat{v}(T_N, \hat{X}^j_{T_N}) & := 0 \end{cases}$$

  2. Let us suppose that for $k \leq N$, we have found $\hat{u}\left(T_{k+1}, \hat{X}^i_{T_{k+1}}\right)$ and $\hat{v}\left(T_{k+1}, \hat{X}^i_{T_{k+1}}\right)$ for all $i = 1, \ldots, n_{k+1}$.
  3. We calculate the approximation scheme for each $x \in \mathcal{S}_k$

$$\begin{cases} \hat{v}(T_k, x) & = \dfrac{1}{\Delta_{T_{k+1}}} \mathbb{E}_{\tilde{Q}}\left[\hat{u}(T_{k+1}, X^{T_k,x}_{T_{k+1}}) \Delta B_{k+1}\right] \\ \hat{u}(T_k, x) & = \mathbb{E}_{\tilde{Q}}\left[\hat{u}(T_{k+1}, X^{T_k,x}_{T_{k+1}}) + \Delta_{T_{k+1}} f(\hat{\Theta}_{k+1,x})\right] \end{cases} \tag{A.8}$$

  Where we denote

$$\hat{\Theta}_{k+1,x} := \left(T_{k+1}, X^{T_k,x}_{T_{k+1}}, \hat{u}(T_{k+1}, X^{T_k,x}_{T_{k+1}}), \hat{v}(T_k, x)\right). \tag{A.9}$$

Remark. Just as in Chapter 2, we can relax the smoothness condition on g and ask for uniformly Lipschitz, by imposing an additional diffusivity condition (this time we can use the uniformly finitely generated (UFG) condition see [25]). However, even under this alternative hypothesis, the effect of using the c-recombined measure appears only when considering the conditional expectation of the already regularized function as the analysis is treated differently on the last step. Hence, in essence it suffices to study the mechanism under smooth assumptions

2.2 Solution of BSDEs: Error Analysis

We follow closely the proof for the first order backward algorithm 2 in the **(SB)** case, in Chapter (2).

Let us first present some simple control property for the c-reduced measure.

Lemma A.5. *Let ψ be a bounded function from $\mathbb{R}^d$ to $\mathbb{R}$, and K a given real. Then, for every $x \in \mathcal{S}_k$*

$$\left|\mathbb{E}_{\tilde{Q}}\left[\psi(X_{T_{k+1}}^{T_k,x})\right]\right| \leq \eta_{k+1}||\psi||$$

and

$$\left|\mathbb{E}_{\tilde{Q}}\left[\psi(X_{T_{k+1}}^{T_k,x})\left(K+\Delta B_{T_{k+1}}\right)\right]\right| \leq \eta_{k+1}||\psi||_\infty (K+\Delta_{T_{k+1}})^{1/2}$$

Proof. Let $\mu_{x,.}^{k+1} = KLV(\delta_x, \Delta_{T_{k+1}})$ be the cubature measure result of applying the KLV operator defined in (A.4). Note that this already corresponds to a conditional measure.

Consider $\tilde{Q}_{[T_k,T_{k+1}]}$ and $S_{k+1} = \{\hat{x}_1^{k+1}, \dots, \hat{x}_{\tilde{m}}^{k+1}\}$ the reduced cubatures nodes at T_{k+1} as in Definition (A.4). By construction, there exist α_j^i for $j = 1, \dots, |\text{supp}(\mu)|$ and $i = 1, \dots, \tilde{m} := |\text{supp}(\tilde{Q}_{[T_k,T_{k+1}]})|$ such that

$$\mathbb{E}_{\tilde{Q}}\left[\psi(X_{T_{k+1}}^{T_k,x})\right] = \sum_{i=1}^{\tilde{m}} \sum_{j=1}^{|\text{supp}(\mu)|} \alpha_j^i \mu_{x,j}^{k+1} \psi(\hat{x}_i^{k+1}) = \sum_{j=1}^{|\text{supp}(\mu)|} \mu_{x,j}^{k+1} \sum_{i=1}^{\tilde{m}} \alpha_j^i \psi(\hat{x}_i^{k+1})$$

from where the first claim follows. Similarly,

$$\begin{aligned}\mathbb{E}_{\tilde{Q}}\left[\psi(X_{T_{k+1}}^{T_k,x})\left(K+\Delta B_{T_{k+1}}\right)\right] &= \sum_{i=1}^{\tilde{m}} \sum_{j=1}^{|\text{supp}(\mu)|} \alpha_j^i \mu_{x,j}^{k+1} \left(\psi(\hat{x}_i^{k+1})\left(K+\Delta\omega^j\right)\right) \\ &= \sum_{j=1}^{|\text{supp}(\mu)|} \mu_{x,j}^{k+1} \left(\left(K+\Delta\omega^j\right)\right) \sum_{i=1}^{\tilde{m}} \alpha_j^i \psi(\hat{x}_i^{k+1})\end{aligned}$$

So that the second claim follows from Cauchy Schwarz inequality. □

It will be important to study one step error approximations. Let us recall first some one step error results applying to the cubature method, that we have already presented and discussed on the introduction and in a slightly different framework in Lemma 2.17.

Lemma A.6. *Let ψ be a C_b^{m+2} function from $\mathbb{R}^d$ to $\mathbb{R}$. Then, there exist constants C, C' depending only on $V_{0:d}, d, m$ such that, for all $i = 1, \dots, d$, for all $k = 0, \cdots, N-1$ and for all $x \in \text{supp}(\mathbb{Q}_{T_k})$:*

$$\left|(\mathbb{E}-\mathbb{E}_{\mathbb{Q}})\left[\psi(X_{T_{k+1}}^{T_k,x})\right]\right| \leq C \sum_{l=m+1}^{m+2} ||\psi||_{l,\infty} \Delta_{T_{k+1}}^{(l)/2}$$

$$\left|(\mathbb{E}-\mathbb{E}_{\mathbb{Q}})\left[\psi(X_{T_{k+1}}^{T_k,x})\Delta B_{k+1}\right]\right| \leq C' \sum_{l=m}^{m+1} ||\psi||_{l,\infty} \Delta_{T_{k+1}}^{(l+1)/2}$$

where $\Delta B_{k+1} = B_{T_{k+1}} - B_{T_k}$.

The c-reduced cubature measure has a very similar property.

Lemma A.7. *Let $(r_k)_{k=0,\dots,N}$, $(m'_k)_{k=0,\dots,N}$ and $(\eta_k)_{k=0,\dots,N}$ be respectively the sequence of radii of the covering balls, the sequence of maximal powers of the approximating polynomials and the sequence of l_1 bounds as defined in Definition A.4.*

Let ψ be a $C_b^{(m+2)\vee(m'_{k+1}+1)}$ function from $\mathbb{R}^d$ to $\mathbb{R}$. Then, there exist constants C, C' depending only on $V_{0:d}, d, m$ and $(m'_k)_{0\le k\le N}$ such that for all $k = 0, \cdots, N-1$, for all $i, j = 1, \dots d$, and for all $z \in \mathcal{S}_k$:

$$\left|(\mathbb{E} - \mathbb{E}_{\tilde{Q}})\left[\psi(X_{T_{k+1}}^{T_k,x})\right]\right| \le C\eta_{k+1}||\psi||_{m'_{k+1}+1,\infty} r_{k+1}^{m'_{k+1}+1} + C\sum_{l=m+1}^{m+2} ||\psi||_{l,\infty}\Delta_{T_{k+1}}^{(l)/2} \tag{A.10}$$

$$\left|(\mathbb{E} - \mathbb{E}_{\tilde{Q}})\left[\psi(X_{T_{k+1}}^{T_k,x})\Delta B_{k+1}^i\right]\right| \le C'\eta_{k+1}||\psi||_{m'_{k+1}+1,\infty} r_{k+1}^{m'_{k+1}+1}\Delta_{T_{k+1}}^{1/2} + C'\sum_{l=m}^{m+1} ||\psi||_{l,\infty}\Delta_{T_{k+1}}^{(l+1)/2} \tag{A.11}$$

Proof. Let us decompose the error as

$$\begin{aligned}\left|(\mathbb{E} - \mathbb{E}_{\tilde{Q}})\left[\psi(X_{T_{k+1}}^{T_k,x})\right]\right| \le &\left|(\mathbb{E} - \mathbb{E}_{\mathbb{Q}})\left[\psi(X_{T_{k+1}}^{T_k,x})\right]\right| + \left|(\mathbb{E}_{\mathbb{Q}} - \mathbb{E}_{\tilde{Q}})\left[\Pi_{\mathcal{H}^k}\psi(X_{T_{k+1}}^{T_k,x})\right]\right| \\ &+ \left|(\mathbb{E}_{\mathbb{Q}} - \mathbb{E}_{\tilde{Q}})\left[\psi(X_{T_{k+1}}^{T_k,x}) - \Pi_{\mathcal{H}^k}\psi(X_{T_{k+1}}^{T_k,x})\right]\right|\end{aligned} \tag{A.12}$$

The first term in the right hand side of (A.12) is a cubature error and is controlled using the results of Lemma 2.17. The second term is null by definition of the conditionally reduced measure $\tilde{Q}$. The last term is controlled as follows. Let $O_{k+1}^1, \dots, O_{k+1}^{M^{k+1}}$ be the collection of disjoint open balls of radius r_{k+1} used for the recombination algorithm. Let $\Pi_{\mathcal{H}^k}$ be the projection operator into the set of local polynomials $\mathcal{H}_k$ defined in (A.5). Thanks to the regularity of ψ and the Taylor theorem it follows that for all $y \in \text{supp}(\tilde{Q}_{T_{k+1}})$

$$|\psi(y) - \Pi_{\mathcal{H}^k}\psi(y)| \le \frac{\|\psi\|_{m'_{k+1}+1,\infty}}{(m'_{k+1}+1)!} r_{k+1}^{m'_{k+1}+1}.$$

Hence, given that $\mathbb{Q}$ is a probability measure and using Lemma A.6, it follows

$$\left|(\mathbb{E}_{\mathbb{Q}} - \mathbb{E}_{\tilde{Q}})\left[\psi(X_{T_{k+1}}^{T_k,x}) - \Pi_{\mathcal{H}^k}\psi(X_{T_{k+1}}^{T_k,x})\right]\right| \le \eta_{k+1}\frac{\|\psi\|_{m'_{k+1}+1,\infty}}{(m'_{k+1}+1)!} r_k^{m'_{k+1}+1},$$

from where we deduce (A.10). Similar arguments lead to (A.11). □

We are ready to prove our main theorem on error approximation of the BSDE scheme using the c-reduced cubature method. Recall that when $k = N$, by definition, for any $x \in \mathcal{S}_N$ $\hat{u}(T_N, x) = u(T_N, x)$, hence, we focus only on the L^∞ error between 0 and $N-1$.

Theorem A.8. *In addition to the standing assumptions, suppose that $m \ge 3$ and there exist positive constants K, K' such that*

$$r_{k+1}^{m'_{k+1}+1} \le K\Delta_{T_{k+1}}^{(m+1)/2} \text{ for all } k = 0, \dots, N$$

and

$$\sum_{k=1}^{N-1}(\eta_k - 1) \le K'.$$

Let

$$\mathcal{E}_u(k) := \max_{x\in\mathcal{S}_k} |u(T_k, x) - \hat{u}(T_k, x)|; \qquad \mathcal{E}_v(k) := \max_{x\in\mathcal{S}_k} |v(T_k, x) - \hat{v}(T_k, x)|.$$

Then,

$$\max_{0\le k\le N-1}(\mathcal{E}_u(k) + \Delta_{T_{k+1}}^{1/2}\mathcal{E}_v(k)) \le CN^{-1}$$

Proof. The proof is a modification of the first order backward algorithm 2 in the **(SB)** case, in Chapter (2). Here we will particularly emphasize the points in the argument where the use of the c-reduced measure intervenes.

Let $k \in \{0, \dots, N-1\}$. For any $x \in \mathcal{S}_k$, we break the error between u and $\hat{u}$ as follows:

$$u(T_k,x) - \hat{u}(T_k,x) = u(T_k,x) - \mathbb{E}\left(u(T_{k+1}, X_{T_{k+1}}^{T_k,x}) + \Delta_{T_{k+1}} f(\bar{\Theta}_{k+1,k}(x))\right) \tag{A.13}$$

$$+ (\mathbb{E} - \mathbb{E}_{\tilde{Q}})\left[u(T_{k+1}, X_{T_{k+1}}^{T_k,x}) + \Delta_{T_{k+1}} f(\bar{\Theta}_{k+1,k}(x))\right] \tag{A.14}$$

$$+ \mathbb{E}_{\tilde{Q}}\Big[u(T_{k+1}, X_{T_{k+1}}^{T_k,x}) - \hat{u}(T_{k+1}, X_{T_{k+1}}^{T_k,x}) \tag{A.15}$$

$$+ \Delta_{T_{k+1}}\left(f(\bar{\Theta}_{k+1,k}(x)) - f(\hat{\Theta}_{k+1,x})\right)\Big].$$

with

$$\bar{\Theta}_{k+1,k}(x) := \left(T_{k+1}, X_{T_{k+1}}^{T_k,x}, u(T_{k+1}, X_{T_{k+1}}^{T_k,x}), v(T_k,x)\right).$$

Similarly, we can expand the error between v and $\hat{v}$ as:

$$\Delta_{T_{k+1}}\left[v(T_k,x) - \hat{v}(T_k,x)\right] = \Delta_{T_{k+1}} v(T_k,x) - \mathbb{E}\left(u\left(T_{k+1}, X_{T_{k+1}}^{T_k,x}\right)\Delta B_{T_{k+1}}\right) \tag{A.16}$$

$$+ (\mathbb{E} - \mathbb{E}_{\tilde{Q}})\left[u\left(T_{k+1}, X_{T_{k+1}}^{T_k,x}\right)\Delta B_{T_{k+1}}\right] \tag{A.17}$$

$$+ \mathbb{E}_{\tilde{Q}}\left(\left[u\left(T_{k+1}, X_{T_{k+1}}^{T_k,x}\right) - \hat{u}\left(T_{k+1}, X_{T_{k+1}}^{T_k,x}\right)\right]\Delta B_{T_{k+1}}\right). \tag{A.18}$$

We deal first with the control of the propagation errors (A.15) and (A.18). From the mean value theorem, there exist random variables Ψ_1, Ψ_2, respectively bounded by $\|\partial_y f\|_\infty$, $\|\partial_z f\|_\infty$ almost surely, such that:

$$\Delta_{T_{k+1}}\left(f(\bar{\Theta}_{k+1,k}(x)) - f(\hat{\Theta}_{k+1,x})\right) = \Delta_{T_{k+1}}\Psi_1\left(u(T_{k+1}, X_{T_{k+1}}^{T_k,x}) - \hat{u}(T_{k+1}, X_{T_{k+1}}^{T_k,x})\right) \tag{A.19}$$

$$+ \Delta_{T_{k+1}}\Psi_2\left(v(T_k,x) - \hat{v}(T_k,x)\right).$$

Replacing (A.16), (A.17) and (A.18) in (A.19) and then in (A.15), we get

$$\begin{aligned}
|(\text{A.15}) + (\text{A.18})| &= \Big|\mathbb{E}_{\tilde{Q}}\Big[\left(u(T_{k+1}, X_{T_{k+1}}^{T_k,x}) - \hat{u}(T_{k+1}, X_{T_{k+1}}^{T_k,x})\right)\left(1 + \Delta B_{T_{k+1}}\right) \\
&\qquad + \Delta_{T_{k+1}}\left(f(\bar{\Theta}_{k+1,k}(x)) - f(\hat{\Theta}_{k+1,x})\right)\Big]\Big| \\
&\leq \left|\mathbb{E}_{\tilde{Q}}\left[\left(u(T_{k+1}, X_{T_{k+1}}^{T_k,x}) - \hat{u}(T_{k+1}, X_{T_{k+1}}^{T_k,x})\right)\left(1 + \Psi_1\Delta_{T_{k+1}} + (1+\Psi_2)\Delta B_{T_{k+1}}\right)\right]\right| \\
&\leq \eta_{k+1}\mathcal{E}_u(k+1)(1 + C\Delta_{T_{k+1}})^{1/2}.
\end{aligned}$$

thanks to Lemma A.5.

Having found a control for the propagation errors (A.15) and (A.18), we consider the other terms in the error expansion. The scheme errors (A.13) and (A.16) can be controlled as in Claim 2.9 in Chapter 2 by putting the McKean term equal to zero. Likewise, the conditional cubature errors (A.14) and (A.17) are controlled using Lemma (A.7). Hence, we have from the expansion of u in (A.13), (A.14), (A.15) that

$$\mathcal{E}_u(k) \leq \eta_{k+1}\left(1 + C\Delta_{T_{k+1}}\right)\mathcal{E}_u(k+1) + \bar{\epsilon}(k+1), \tag{A.20}$$

with

$$\bar{\epsilon}(k+1) = C\Big(\sup_{s\in[T_k;T_{k+1}]} ||u(s,\cdot)||_{2,\infty}\Delta_{T_{k+1}}^4 + \eta_{k+1}||\psi||_{m'_{k+1}+1,\infty} r_{k+1}^{m'_{k+1}+1} + \sum_{l=m+1}^{m+2} ||\psi||_{l,\infty}\Delta_{T_{k+1}}^{(l)/2}\Big)$$
$$\leq C\Big(\sup_{s\in[T_k;T_{k+1}]} ||u(s,\cdot)||_{2,\infty}\Delta_{T_{k+1}}^4 + K(K'+1)||\psi||_{m'_{k+1}+1,\infty}\Delta_{T_{k+1}}^{(m+1)/2} + \sum_{l=m+1}^{m+2} ||\psi||_{l,\infty}\Delta_{T_{k+1}}^{(l)/2}\Big),$$

where the last inequality uses the condition on $(r_k)_{0\leq k\leq N}$ and $(m'_k)_{0\leq k\leq N}$. Therefore, Gronwall's Lemma applied to (A.20), using the condition on the sequence $(\eta_k)_{0\leq k\leq N}$ and the definition of Δ_{T_k} implies

$$\mathcal{E}_u(i) \leq \sum_{k=1}^{N-1} C' \exp(K')\bar{\epsilon}(k+1) \leq CN^{-1}.$$

The result for $\mathcal{E}_v$ follows from the bound for $\mathcal{E}_u$ and the expansion (A.16), (A.17), (A.18). □

3 Implementation (dimension 1)

In the previous sections, we explained a possible way to extend the cubature algorithm to approach BSDEs that depend, in essence, on the possibility to find a bounded interpolation for a given finite set of points Ω_2 and a family of test functions as in Lemma (A.2).

The output of a reduction procedure is satisfactory if we manage to find a small interpolation set with interpolation coefficients satisfying our assumptions. Our problem is then written as an optimization problem as follows: minimize the size of a set S such that there exist coefficients α making (S,α) a K_{max} bounded interpolation of Ω_2 with respect to $\mathcal{H}$. In what follows, we take $\mathcal{H}$ to be the set of polynomials up to order m' (although the algorithm can be well applied for a different set of test functions).

Recall that the reason why we keep $S \subset \Omega_2$ comes from the cubature measure: the idea is not to explore directions that are not visited by the diffusion to be approximated. Of course in dimension one this limitation does not really appear, as there is only one direction for noise propagation. However, although we are limiting ourselves here to the case of dimension one, we are aiming to propose an scalable solution.

Note that the proposed optimization problem is of a combinatoric type, so it is quite complex to study and to solve. In general we do not expect for it to have a unique solution. Written in the optimization form, the c-reduction problem is related to that of choosing the "sparsest" basis to represent a set of points, a question that appears in different fields (most notably in signal treatment where it is known as compressive sensing). The most common technique to solve such a problem passes by transforming the l_0 problem of minimizing the number of elements in the basis set (for us the interpolation nodes) to a problem of minimizing their l_1 norm (see for example [30]). Note however that we cannot use this sort of technique: the minimal l_1 norm in our case would be obtained by taking the entire set Ω_2 as support of the reduced measure, as any combination increases the l_1 norm. We have to consider another approach.

Having in mind the complexity of finding a numerical solution, the two notable exceptions to this sentence being the cases $m' = \{0,1\}$ for which there are simple formulas available (S being any singleton in the first case or the extreme points of the interpolation interval in the

second), we propose to follow an approach based on the classical interpolation and interior point optimization techniques and the recombination method. We denote it *top-down*, as it initializes with $S = \Omega_2$ and then progressively eliminates from S the nodes that can be represented as linear combination of the remaining ones without trespassing the limits imposed by conditions

Let us introduce the algorithm:

Algorithm A.9 (Top-down reduction algorithm)**.** *The algorithm receives as parameters the l_1 bound for the interpolation coefficients K_{max}, the maximal degree of interpolating polynomials m' and the set $\Omega_2 = \{x_1, \dots, x_{m_2}\}$ of points to be interpolated.*

At each iteration, $S = \{\bar{x}_1, \dots, \bar{x}_{\tilde{m}_2}\}$ is the set of interpolation nodes and A is a $m_2 \times \tilde{m}_2$ matrix such that the $i-th$ row of A contains the interpolation coefficients of the point x_i in terms of the interpolation points.

1. *Initialization: Set $S = \Omega_2$, $\tilde{m}_2 = m_2$. Let $A = Id(m_2)$ be the identity $m_2 \times m_2$ matrix.*

2. *Size verification: We verify if the minimal number of interpolation points is attained. If*

$$\tilde{m}_2 \leq \binom{m' + d}{d},$$

   end the algorithm (go to step 7).

3. *Kernel calculation: Let $\bar{V}_S$ be the Vandermonde matrix of the set S,*

$$\bar{V}_S = \begin{pmatrix} 1 & \bar{x}_1 & \bar{x}_1^2, & \dots & \bar{x}_1^{m'} \\ 1 & \bar{x}_2 & \bar{x}_2^2 & \dots & \bar{x}_2^{m'} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \bar{x}_{\tilde{m}_2} & \bar{x}_{\tilde{m}_2}^2 & \dots & \bar{x}_{\tilde{m}_2}^{m'} \end{pmatrix}.$$

   Find a basis $\mathcal{C} = \{c^1, \dots, c^{l'}\} \in \mathbb{R}^{\tilde{m}_2}$ for the kernel of $\bar{V}_S^T$ (for example using singular value decomposition).

4. *Choosing the node to be reduced: For each $i = 1, \dots, l'$ let $j^*(i) = \text{argmin}_{j=1,\dots,m_2} |c_j^i|$. Then let i^*, j^* be defined by*

$$i^* = \text{argmin}_{i=1,\dots,l'} \sum_{j=1}^{m_2} \left| \frac{c_j^i}{c_{j^*(i)}^i} \right| \quad \text{and } j^* = j^*(i^*).$$

   Define $\bar{c}$ by $\bar{c}_j := -c_j^{i^*} / c_{j^*}^{i^*}$

   Here, the idea is the following: note that for each $c^i \in \mathcal{C}$ and $k = 0, \dots, m'$, we have

$$\sum_{j=1}^{\tilde{m}_2} c_j^i \bar{x}_j^k = 0 \Rightarrow c_{j^*(i)}^i x_{j^*(i)}^k = \sum_{j=1; j \neq j^*(i)}^{\tilde{m}_2} -c_j^i \bar{x}_j^k,$$

   so that the interpolation coefficients are defined like in $\bar{c}$. Note that we choose $j^*(i)$ to guarantee the coefficients are less than one, and we choose i^* to represent node having interpolation coefficients in terms of the other ones with lower total l_1 norm.

5. *Reduction: Let α be an updating representation matrix. It will have dimension $(\tilde{m}_2) \times (\tilde{m}_2 - 1)$, and be given by*

$$\alpha = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ \bar{c}_1 & \bar{c}_2 & \dots & \bar{c}_{j^*-1} & \bar{c}_{j^*+1} & \bar{c}_{j^*+2} & \dots & \bar{c}_{\tilde{m}_2} \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

6. *Bound verification : Calculate the new representation matrix: $A' := A \times \alpha$. If*

$$\sum_{k=1}^{\tilde{m}_2 - 1} |A'_{j,k}| \leq K_{max}, \forall j = 1, \dots, m_2, \tag{A.21}$$

i.e., if the l_1 condition is verified, make the update: set $A := A'$, $S := S \backslash \{\bar{x}_{j^*}\}$, $\tilde{m}_2 := \tilde{m}_2 - 1$ and go to step 2.

Otherwise, make $\mathcal{C} := \mathcal{C} \setminus \{c^{i^*}\}$. If $\mathcal{C} \neq \emptyset$, try another reduction: go to step 4.

7. *End: The nodes S and the weights given in A are a K_{max} bounded interpolation of the set Ω_2 with respect to the polynomials up to order m'.*

Thus, the algorithm starts by considering all nodes as part of the basis set. It then looks for candidates to be removed from this set. To do so, it looks at the null space which gives alternative ways of writing a given value in terms of the remaining ones. The criteria we have proposed to choose a candidate node to be let out of the interpolation nodes, guarantees the condition that each interpolation coefficient must be smaller than one, and is based on the idea of retiring nodes that would add a small l_1 weight in terms of the remaining ones. Nevertheless, we remark that we pay the price of considering only combinations lying in the direction of the chosen basis of the null space, but it would be very expensive to look at all possible combinations.

The top-down algorithm is a heuristic method. Up to now, we do not have convergence results for it. We cannot give a complexity figure for the algorithm either, as it depends on the size of the minimal interpolation set and the number of effective iterations. We know, however, the cost of one iteration of the algorithm. The matrix multiplication in step 6 is in fact a vector matrix multiplication. Hence, taking into account that the SVD in step 3 has in general a complexity of $\tilde{m}_2(r+1)^2$ (i.e. linear in $\tilde{m}_2$), we conclude the most expensive part of one iteration is the matrix to vector operation in step 6 and choosing the minimal point in step 4 that account for a complexity of $(\tilde{m}_2)^2$.

3.1 Parameter choice

The actual implementation of the interpolation procedure to be applied in the framework of c-reducing a cubature measure, demands the definition of the main parameters of the scheme as defined in Definition A.4, namely the sequences $(r_k)_{k=0,\dots,N}$, $(m'_k)_{k=0,\dots,N}$ and $(\eta_k)_{k=0,\dots,N}$, that should satisfy the conditions

$$r_{k+1}^{m'_{k+1}+1} \leq K \Delta_{T_{k+1}}^{(m+1)/2} \text{ for all } k = 0, \dots, N \tag{A.22}$$

$$\sum_{k=1}^{N-1} (\eta_k - 1) \leq K'. \tag{A.23}$$

The user has to deal with some compromises when satisfying both these conditions. From the first condition, we deduce that

$$r_{k+1} \propto \Delta_{T_{k+1}}^{\frac{(m+1)}{2(m'_{k+1}+1)}}.$$

Hence, from a numerical point of view, it seems reasonable to avoid small values of m'_{k+1} when dealing with very small $\Delta_{T_{k+1}}$, for example at the final steps of a decreasing step discretization as defined in (A.7). Note as well that condition (A.23) and the size of the Gronwall constant in the error analysis, impose a small value for η_k, as N grows, for most k in $\{0, \dots, N\}$: this implies taking small values of m'_{k+1}. But on the contrary, taking very small values of $\eta_k - 1$ would, as argued before, severely limit the possibility to effectively reduce the cubature measure.

From the previous discussion, a good compromise might be obtained as follows: let $(a_k)_{0 \leq k}$ be sequence with associated convergent series (not necessarily with very fast convergence). For example, take $a_k = k^{-2}$). For some positive integer M, set $\eta_k = 1 + Ma_{N-k}$. Finally, set $\theta > 0$ be a reasonable threshold for interpolation. Then, fix $m'_k = m$ if $\eta_k > \theta$ or $m'_k = 1$ otherwise. The idea of such a choice is to profit from the use of interpolation of higher order when dealing with more points and smaller intervals. The threshold serves to a double purpose: decide cheaply whether or not to perform a c-reduction, and cap the number of items of the sum, effectively reducing the Gronwall constant.

3.2 Main difficulties to extend the algorithm to the multidimensional case

We have proposed a scheme to treat the FBSDE problem in dimension one. A priori, most of the presented steps might be generalized to higher dimensions, but there are some issues we must consider.

First, there is the problem (that also applies to high order recombination) of implementing numerically the open covering used for the reduction. In dimension one it is quite simple to subdivide an interval containing the support of the considered measure. In dimension bigger than one, there is a real issue in choosing a good way to do this division, as it is quite expensive to define the groups of points that belong to the same ball (even when simple norms, from the numerical point of view, as the l_1 or l_∞ norms are used).

Then, an extra difficulty comes from the fact that reduction is not guaranteed except if $\tilde{m} = 0$. Indeed, the case $\tilde{m} = 1$, that was easily solved in dimension one, consists now in finding the convex hull of the support of the original measure, and it is not only guarantee to induce some reduction.

The previous issues add a lot to the complexity of the algorithm in the multidimensional case, but we are convinced, that they can be addressed in order make the algorithm efficient also in this case.

4 Numerical tests

In this section, we illustrate the algorithm behavior by applying it to a toy model for which the exact solution is available. For $t \in [0, 1]$, consider the equation given by

$$
\begin{aligned}
&dX_t = dt + dB_t \\
&-dY_t = \left(\cos(X_t) - \frac{1}{2}Y_t - Z_t\right) dt - Z_t \cdot dB_t, \\
&X_0 = 0; \qquad Y_1 = \cos(X_t)
\end{aligned}
$$

which has exact solution given by $Y_t = \cos(X_t)$, $Z_t = -\sin(X_t)$. We solve the system with a straightforward approximation of the cubature method and then with an implementation of the c-reduced cubature with parameters given by

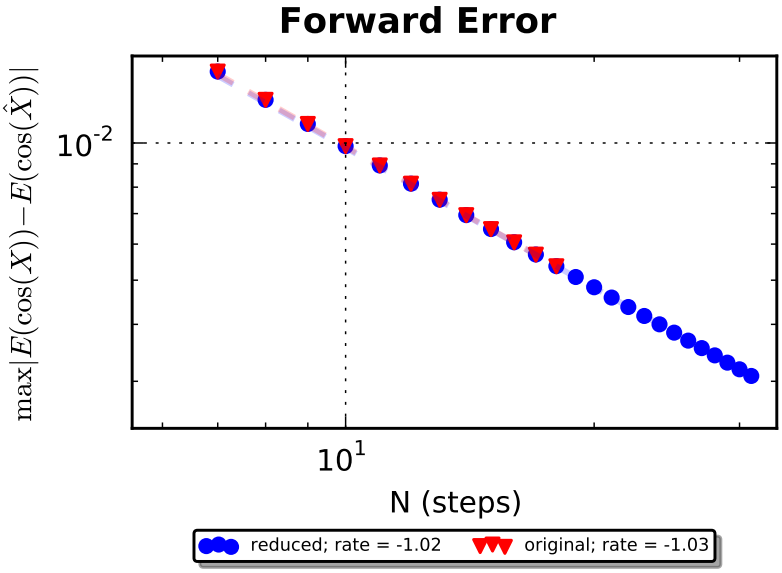
- m=3 (binary cubature)
- $\eta_k = 1 + 15\Delta_{T_{N-k}}$
- Threshold (as defined in section 3.1): $\theta = 1.1$
- $m'_k = m$ above the threshold, and $m'_k = 1$ below.
- $r_k = \Delta_{T_k}^{1/2}$ above the threshold, and $r_k = \Delta_{T_k}^{2/3}$ below.

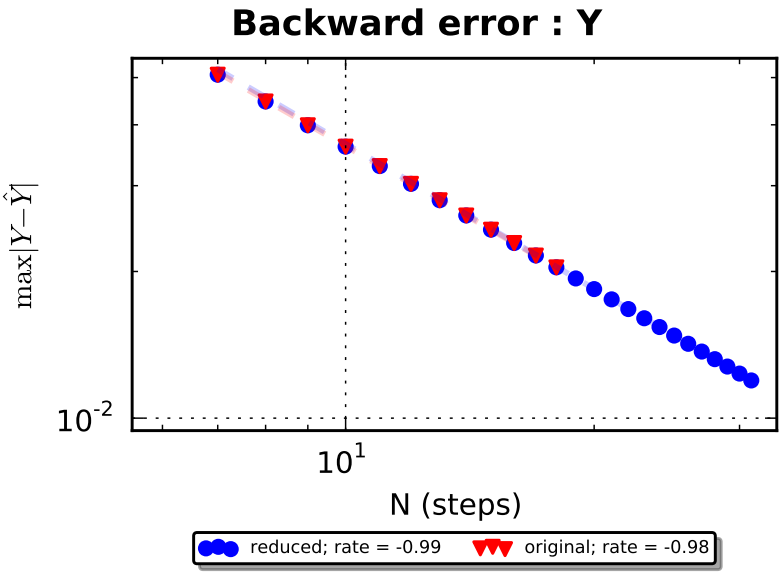
Here, it is important to remark that a cubature used on a uniform discretization of the interval $[0, 1]$ for the binary cubature is "auto-reducing", in the sense that each node shares descendants with its neighbors (hence, the cubature tree grows only linearly in this case). For our testing purposes, we block this auto-reducing feature by using instead a decreasing step as the one presented in (A.7).

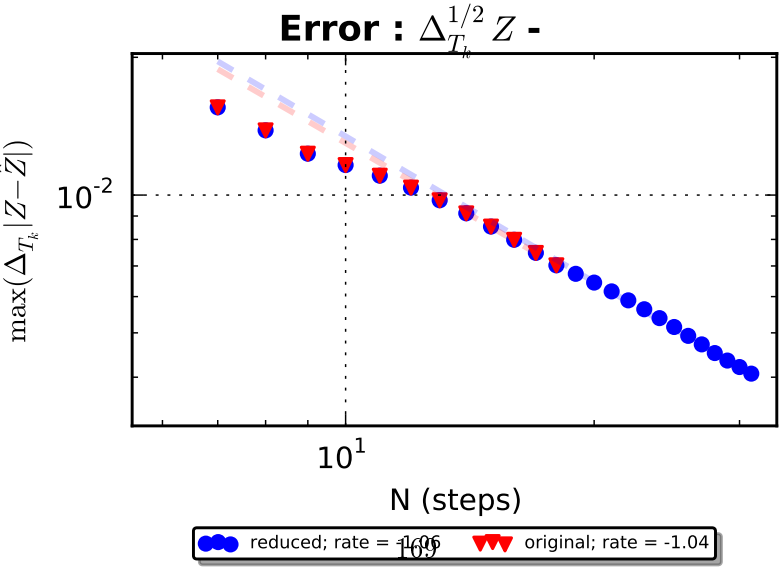
Let us look first at the forward and backward approximation results. Figure A.2 shows not only that the predicted rates of convergence of the c-reduced cubature are attained, but also that we have managed to tune the parameters to our particular example in such a way as to show almost no degradation in the approximating powers, either for the forward or backward variables, of the cubature method after applying the c-reduction algorithm.

Moreover, as expected, there is a sensible reduction in the complexity of the algorithm. Figure A.3 left shows the evolution of the total number of nodes as a function of the number of steps in the approximation, showing the important savings in necessary memory to run the algorithm after using the reduction. We observe a polynomial growth with an degree less than 2.5, which seems reasonable for scaling.

Figure A.3 right gives us even more information, as it shows explicitly the additional cost of the reduction procedure, changing both the constant and the rate of growth of the complexity: indeed, the main cost of the algorithm is now the calculation of the reduction procedure.


Forward Error
$\max|E(\cos(X))-E(\cos(\hat{X}))|$
10^{-2}
10^{1}
N (steps)
reduced; rate = -1.02
original; rate = -1.03


Backward error : Y
$\max|Y-\hat{Y}|$
10^{-2}
10^{1}
N (steps)
reduced; rate = -0.99
original; rate = -0.98


Error : $\Delta_{T_k}^{1/2} Z$ -
$\max(\Delta_{T_k}|Z-\hat{Z}|)$
10^{-2}
10^{1}
N (steps)
reduced; rate = -1.06
original; rate = -1.04

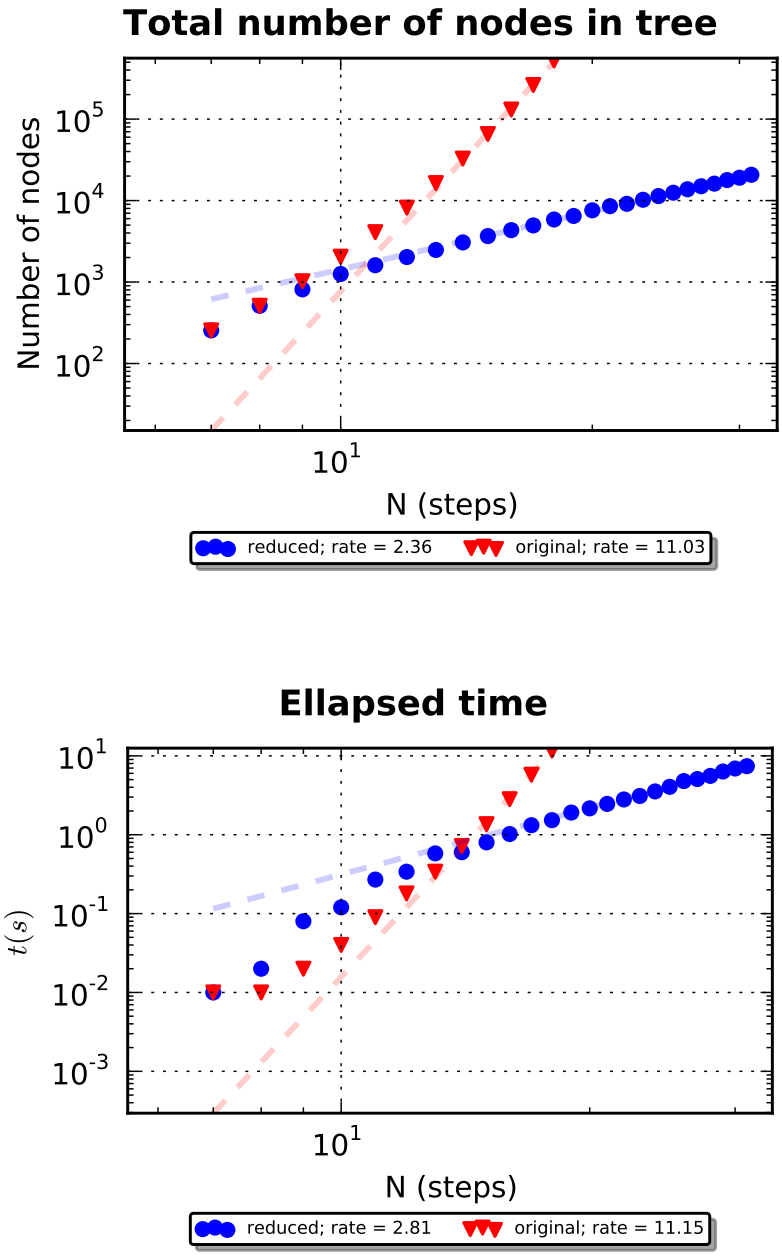


Figure A.3: Two measures of complexity growth for the algorithm: number of total nodes and total execution time as a function of the number of steps (log-log). The c-reduction version shows a sensible improvement with respect to the straightforward cubature.

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Méthodes numériques probabilistes: problèmes multi-échelles et problèmes de champ moyen

Résumé. Cette thèse traite de la solution numérique de deux types de problèmes stochastiques : les équations différentielles stochastiques (EDS) fortement oscillantes, c'est-à-dire, les systèmes composés de variables ergodiques évoluant rapidement par rapport aux autres ; et les équations différentielles stochastiques progressives-rétrogrades découplées de type McKean-Vlasov (EDSPR-MKV) associées à la solution de certains problèmes de contrôle sous un environnement formé d'un grand nombre de particules ayant des interactions du type champ-moyen.

Premièrement, nous nous intéressons aux EDS fortement oscillantes. Nous proposons un algorithme basé sur des résultats d'homogénéisation. Il est défini par un schéma d'Euler appliqué aux variables lentes couplé avec un estimateur à pas décroissant pour approcher la limite ergodique des variables rapides. Nous prouvons la convergence forte de l'algorithme et montrons que son erreur normalisée satisfait un résultat du type théorème limite centrale généralisé. Nous proposons également une version extrapolée de l'algorithme ayant une meilleure complexité asymptotique en satisfaisant les mêmes propriétés que la version originale.

Ensuite, nous étudions la solution des EDSPR-MKV. D'abord, nous présentons un nouvel algorithme, basé sur la méthode de cubature sur l'espace de Wiener, pour approcher faiblement la solution d'une EDS du type McKean-Vlasov. Il est déterministe et peut être paramétré pour atteindre tout ordre de convergence souhaité. Puis, en utilisant ce nouvel algorithme, nous construisons deux schémas pour résoudre les EDSPR-MKV découplées et nous montrons que ces schémas ont des convergences d'ordres un et deux.

Enfin, nous considérons le problème de réduction de la complexité de la méthode de cubature lorsqu'elle est utilisée pour résoudre des équations rétrogrades. Nous étudions le cas de la dimension un, où l'on peut profiter des résultats sur l'interpolation polynomiale, et nous présentons un algorithme afin de recombiner les poids de la mesure de cubature en respectant l'approximation des espérances conditionnelles.

Mots-clés : *méthodes numériques, systèmes multi-échelles, systèmes fortement oscillants, équations de McKean Vlasov, EDSPR, cubature, recombinaison*

Probabilistic numerical methods: multi-scale and mean-field problems

Abstract. This Ph.D. thesis deals with the numerical solution of two types of stochastic problems : strongly oscillating SDEs, i.e. systems in which some ergodic state variables evolve quickly with respect to the remaining ones, and decoupled Forward Backward Stochastic Differential equations of McKean-Vlasov type (MKV-FBSDE) which appear in some stochastic control problems in an environment of a large number of particles with mean field interactions.

First, we investigate the strongly oscillating SDEs solution. We propose an algorithm that uses homogenization results and consists of an Euler scheme for the slow scale variables coupled with a decreasing step estimator for the ergodic averages of the fast variables. We prove the strong convergence of the algorithm as well as a generalized central limit theorem result for the normalized error distribution. In addition, we propose an extrapolated version applicable under stronger regularity assumptions and which satisfies the same properties of the original algorithm with lower asymptotic complexity.

Then, we treat the problem of solving MKV-FBSDEs. As a first step, we propose a new algorithm, based on the cubature method on Wiener spaces, to weakly approach the solution of a McKean-Vlasov SDE. It is deterministic and can be parametrized to obtain any given order of convergence. Using this first forward approximation algorithm, we construct two procedures to solve the decoupled MKV-FBSDE and show that they converge with orders one and two under appropriate regularity conditions.

Finally, we consider the problem of reducing the complexity of the cubature method when used to solve a backward equation. We focus on the case on dimension one, where we can profit from polynomial interpolation results, and we present a recombination procedure to recombine the cubature measure weights while preserving the approximation of conditional expectations.

Keywords : *numerical methods, multi-scale system, strongly oscillating systems, McKean Vlasov equations, FBSDE, cubature, recombination*