Chapitre 8

Etalement d'une goutte par effet Josephson

Introduction

On a solid substrate, superfluid ^{4}He may form stratified liquid films (above a solid ^{4}He layer). Assuming the existence of these films, we analyse [10] the dynamics of wetting for a two-layer droplet which tends to spread and transform into a one-layer droplet (at low temperature: no vapour is present). The permeation between layers, which exists in classical liquids, is replaced here by a Josephson coupling. Spreading should occur *via* Josephson vortices that drift towards the centre of the droplet.

8.1 Version brève de l'article

Objet de l'étude

Certains liquides classiques développent une structuration en marches lors de l'étalement sur un substrat solide. L'épaisseur quantifiée de ces marches correspond alors à des dimensions moléculaires (références 1 à 3 de l'article [1] reproduit ci-dessous, section 8.2). La dynamique de l'étalement dans ces étapes finales est bien décrite par un modèle fondé sur la perméation des molécules entre les couches (due à un abaissement du potentiel chimique) et sur la friction inter-couches, en supposant que chaque couche se comporte comme un liquide incompressible (références 4 et 5 de l'article ci-dessous).

Il est apparu récemment que près d'une surface solide, l'hélium superfluide pourrait présenter des oscillations importantes de son paramètre d'ordre $\psi(z)$ (et donc de la fraction superfluide du liquide) en fonction de la distance à la paroi (référence 8). Ce résultat suggère qu'il puisse exister des gouttes stratifiées de ^{4}He en fin d'étalement. Nous supposons que les oscillations de densité superfluide conduisent dans ce cas à une interaction faible entre les couches, sous la forme d'une jonction Josephson [2].

Le but de l'étude est de donner une description qualitative de la dynamique de l'étalement dans ces conditions. On suppose cependant qu'aucune marche cristalline ne risque de constituer une autre jonction Josephson, qu'un film bidimensionnel de ^{4}He est stable en tant que liquide (qu'il ne s'évapore pas à deux dimensions), enfin que l'étalement du film est le seul processus de mouillage (pas de transfert par la phase vapeur).

Courant de Josephson dans la région à deux couches

On considère une goutte unidimensionnelle, à symétrie de translation selon l'un des axes horizontaux, y. W_i désigne l'énergie d'un atome d'hélium dans la couche i (avec $i = 1$ ou 2), μ_i son potentiel chimique, v_i la vitesse horizontale de la couche i, δ_i la dilataton locale, p_i la pression, ϕ_i la phase macroscopique de chaque couche et $\phi \equiv \phi_1 - \phi_2$ leur différence, J est le courant de Josephson, F l'énergie du fluide par unité de surface du substrat, Ω le volume d'un atome, M sa masse, a^2 la surface moyenne qu'il occupe sur le substrat, c la vitesse du son.

Les premières équations (1 à 7) donnent les relations classiques entre ces grandeurs. Dans l'expression du potentiel chimique (équation 7), on a négligé le terme cinétique $Mv_i^2/2$ en supposant que les vitesses en jeu sont suffisamment faibles.

Si l'on introduit une longueur caractéristique l (équation 9), les relations précédentes fournissent l'équation d'évolution de la différence de phase ϕ (équation 8). Elle a la structure d'une équation d'ondes.

[1] Les références et les équations mentionnées ici sont toujours celles de l'article [10] reproduit à la section 8.2.

[2] Notons cependant que cette jonction est continue (elle est bidimensionnelle), à la différence des jonctions Josephson envisagées habituellement.

Ondes progressives

En écrivant $u = x + Vt$ et $\tilde{l} = l\sqrt{1 - V^2/c^2}$, et en posant $\phi(u) \equiv \phi(x, t)$, on constate que l'équation 8 se met sous la forme de l'équation du mouvement d'un pendule pesant (équation 10) de longueur $R = g\tilde{l}^2/2$ et au repos en $\phi = \pi$ en fonction du temps u.

La constante Q de l'intégrale première détermine le type de mouvement: oscillant pour $Q < 1$, rotation pour $Q > 1$ (et soliton pour $Q \equiv 1$). En choisissant $Q > 1$, on obtient pour le superfluide une dynamique de vortex (figure 1b), tous orientés dans le même sens et séparés par une distance D (équation 15). L'extension horizontale du champ de vitesse de chaque vortex est de l'ordre de $\tilde{l}$. Si l'on avait choisi $Q < 1$, les vortex auraient eu alternativement un sens de rotation opposé; une telle solution ne peut pas être stable (les vortex s'attirent et s'annihilent).

Si v_s désigne la vitesse de la marche séparant le domaine à une couche et la région à deux couches (figure 2 de l'article) et si l'on tient compte de la conservation de la matière (équation 18) et d'une condition de pression nulle à la marche (équation 19a), on constate que la marche progresse à la vitesse des vortex, qui est donnée par l'équation 20. C'est le résultat principal: *la marche d'épaisseur est animée d'une vitesse de progression constante.*

La vitesse et la distance de séparation des vortex (équations 20 et 22) ne dépendent finalement que de l'énergie interfaciale W, mais l'extension spatiale $\tilde{l}$ de chaque vortex dépend aussi de la force Λ du couplage Josephson (équation 9).

Discussion

1. L'énergie interfaciale W devrait être faible par rapport à Mc^2; la vitesse V devrait donc être bien plus faible que la vitesse du son, peut-être de l'ordre de 10m/s.

2. Nous n'avons pas abordé les mécanismes possibles de dissipation au centre de la goutte dans une géométrie tridimensionnelle réelle (figure 1a).

3. Au cours de l'étalement que nous avons décrit, l'ensemble du fluide est en mouvement. Cette situation est très différente de celle des fluides classiques où seuls la région à une couche, ainsi qu'un voisinage de la marche, sont en mouvement.

4. La bicouche devrait toujours évoluer vers la solution des vortex, présentée ici.

8.2 Article: Josephson Droplets

JOSEPHSON DROPLETS.

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PACS. 68.45.Gd Wetting.
PACS. 67.40.Rp Film and weak link transport.
PACS. 67.40.Vs Vortices and turbulence.

Abstract.

On a solid substrate, superfluid ^{4}He may show stratified liquid films (above a solid ^{4}He layer). Assuming the existence of these films, we analyse the dynamics of wetting for a two-layer droplet which tends to spread and transform into a one-layer droplet (at low temperature: no vapor phase is present). The permeation process between layers, which exists in classical liquids, is replaced here by a Josephson coupling. Spreading should occur via Josephson vortices that drift towards the center of the droplet.

1. General aims.

Many classical liquids, when spontaneously spreading on a solid surface, show "Aztec pyramids", i.e. layers of quantized thickness; the bottom layers are spreading while the top layers shrink [1-3]. The dynamics of Aztec pyramids can be roughly understood using two concepts [4,5]:
(a) a mutual friction between layers with different horizontal velocities,
(b) a permeation process, according to which a molecule, e.g. from layer 2, can jump into layer 1 if this corresponds to a drop in chemical potential.

What happens if we substitute a superfluid — liquid ^{4}He — to the classical fluid? Recent calculations by J.W.Woods Halley and C.E.Campbell [6] show that near a solid substrate the density $|\psi^2(z)|$ and the superfluid order

parameter $\psi(z)$ display marked oscillations, when plotted as a function of the distance z from the solid part. This evokes two reactions:
(a) the $|\psi^2|$ oscillations suggest that we may have stratified droplets
(b) the minima in $|\psi^2|$ imply that exchanges between layers should occur via a relatively weak Josephson coupling.

Our aim in the present note is to get a qualitative picture of what might happen with a superfluid Aztec pyramid (Fig 1) when it tends to spread. Of course there are many provisos:
(1) we assume that the solid is ideally flat (no crystalline steps): a single step on a crystal substrate may be enough to generate a completely different type of Josephson coupling for a one-layer liquid film (the step edge is a region of weaker attraction between solid and ^{4}He: thus, it may act as a weak junction between the films on each side).
(2) we assume that ^{4}He can exist in the form of a dense one-layer film: i.e. , that the two-dimensional liquid does not spontaneously transform into a 2D vapor. For classical liquids, this condition turns out to be rather stringent.
(3) we also assume that ^{4}He vapor pressure above the films (in 3D) is negligible: we want to avoid wetting processes which would take place from a droplet to a dry surface via the gas phase.

2. Josephson flow in the two-layer region.

Our model is described on Fig 1b. Helium atoms in the i-th layer have an energy (due to Van der Waals interactions with the solid substrate) W_i and we assume $W_2 - W_1 = W > 0$. This corresponds to a condition of complete wetting. In each layer, we have a horizontal velocity

$$v_i = (\hbar / M)\partial_x \varphi_i, \qquad (1)$$

where φ_i is the superfluid phase, M being the ^{4}He mass.
For the moving vortices which will be of interest here, we cannot directly assume that the fluid is incompressible; thus, the conservation equation is of the form:

$$\partial_x v_1 - \partial_t \delta_1 = J = -\partial_x v_2 + \partial_t \delta_2, \qquad (2)$$

where δ_i is the local dilatation of the fluid in layer i, and $J = J_{2\to 1}$ is a Josephson current[7] from layer 2 to layer 1, measured as a fluid vertical velocity divided by the interlayer distance. It is also the number of atoms that pass through an atomic area in the horizontal plane per unit time. It can be expressed as:

$$J = (\Lambda / \hbar)\sin\varphi, \tag{3}$$

where Λ is a Josephson coupling (per ^{4}He atom) and $\varphi = \varphi_1 - \varphi_2$ is the phase difference between layers. These equations can be related to a simple energy structure

$$F = \frac{1}{a^2}\int dxdy\left\{\frac{\hbar^2}{2M}\left[(\partial_x\varphi_1)^2 + (\partial_x\varphi_2)^2\right] - \Lambda\cos\varphi + \frac{p_1^2 + p_2^2}{2Mc^2}\Omega^2\right\} \tag{4}$$

where a^2 is the area per ^{4}He (assumed to be the same in both layers), Ω ($\approx a^3$) the volume per ^{4}He atom and ρ the density. In eq(4), the first term is kinetic; the second describes the Josephson coupling; the third is an elastic energy with a sound velocity c. The pressures p_i are related to the dilatations δ_i via:

$$\Omega p_i = -Mc^2\delta_i \tag{5}$$

Ultimately, we need the rate equations for the phase:

$$\hbar\partial_t\varphi_i = -\mu_i, \tag{6}$$

where μ_i is the chemical potential of the i-th layer:

$$\mu_i = \mu_b + W_i + p_i\Omega, \tag{7}$$

μ_b being the bulk value. In fact, the complete chemical potential takes into account the molecule kinetic energy: $\mu_i = \mu_b + Mv_i^2/2 + W_i + p_i\Omega$, but here, we assume that fluid velocities are small enough for this extra term to be neglected. Combining eqs(1-3, 5-7) we arrive at the basic equation for the relative phase:

$$\partial_x^2\varphi - (1/c^2)\partial_t^2\varphi = (2/l^2)\sin\varphi \tag{8}$$

where we have introduced a characteristic length l related to the Josephson coupling Λ through:

$$\Lambda = \hbar^2 / Ml^2 \tag{9}$$

To solve equation (8) we shall consider that the droplet is infinite in size. Solving from one edge naturally leads to traveling wave solutions with a constant (atmospheric) pressure condition at the step; this is done in the next section.

3. Traveling waves.

We can search for solutions to eq(8) in the form of waves propagating towards the inside of the droplet with a certain velocity V: $\varphi(x,t) = \varphi(u)$, where $u = x + Vt$. We arrive at the standard soliton form

$$\partial_u^2\varphi = (2/\tilde{l}^2)\sin\varphi, \tag{10}$$

with $\tilde{l} = (1 - V^2/c^2)^{1/2}l$ (Eq(10) is the equation of movement of a pendulum of length $R = g\tilde{l}^2/2$ (angle $\varphi = \pi$ at rest) as a function of "time" u).

The constant Q in the first integral

$$(\partial_u \varphi)^2 = (4/\tilde{l}^2)(Q - \cos\varphi) \tag{11}$$

could be taken as smaller than unity (oscillating pendulum); this would correspond to a train of vortices, alternatively clockwise and anticlockwise, all moving to the left with constant velocity V; but such vortices would attract one another and eventually annihilate. Hence, we rather take $Q \geq 1$ (high-energy, rotating pendulum). Solutions of eq(10) now have the shape of Figure 2. The superfluid phase difference $\varphi(x,t)$ is now derived from the second integration:

$$\int_0^{\varphi} d\theta (Q - \cos\theta)^{-1/2} = 2(x + Vt - x_0)/\tilde{l} \tag{12}$$

where x_0 is an integration constant. Of course, this is directly related to elliptic functions as defined in ref [8]:

$$\varphi(x,t) = \pi + 2\mathrm{am}\left(\sqrt{Q+1}\,\frac{x + Vt - \tilde{x}_0}{\tilde{l}}\,\middle|\,\frac{2}{Q+1}\right) \tag{13}$$

where $\tilde{x}_0$ is a constant given by $\tilde{x}_0 = x_0 + D/2$, with the inter-vortex distance D given by eq(15). If we assume that fluid velocity is equal in both layers but opposite in sign, at any position and time, then the velocity field is readily derived from the first integral (eq 11):

$$\mathrm{v}_1(x,t) = -\mathrm{v}_2(x,t) = (\hbar/2M)\partial_x \varphi(x,t) = (\hbar/M\tilde{l})\sqrt{Q - \cos\varphi} \tag{14}$$

Velocity oscillates between minimum value $\hbar\sqrt{Q-1}/M\tilde{l}$ and maximum value $\hbar\sqrt{Q+1}/M\tilde{l}$. High-velocity regions are confined to a region whose extension is of order $\tilde{l}$: we call them vortices (see Figure 2). The inter-vortex distance D is given by $\varphi(u + D) = 2\pi + \varphi(u)$, i.e.:

$$D = \tilde{l}\int_0^{\pi} d\theta\,(Q - \cos\theta)^{-1/2} \tag{15}$$

For Q close to unity, D is large (soliton limit): $D \approx (\tilde{l}/\sqrt{2})\ln(32/Q - 1)$. For $Q \to \infty$, D is small: $D \approx \pi\tilde{l}/\sqrt{Q}$.

The mean fluid velocity in layer 1 is:

$$< \mathrm{v}_1 > = (\hbar/2M) < \partial_x \varphi > = \hbar\pi/MD \tag{16}$$

This corresponds to quantization of circulation. A long circuit (length L) enclosing L/D vortices, with a circulation $2\pi\hbar/M$ per vortex, gives a total circulation:

$$\oint \mathrm{v}_i ds = 2L < \mathrm{v}_1 > = (L/D)\,2\pi\hbar/M \tag{17}$$

The position $x_s(t)$ of the step is given by the condition that no fluid should go through the step:

$$-\mathrm{v}_s(t) \equiv \partial_t x_s(t) = \mathrm{v}_2(x_s(t),t) \tag{18}$$

The zero-pressure condition at the step also imposes constant step velocity v_s:

$$\begin{aligned} \hbar\partial_t \varphi(x_s(t),t) &= W - \Omega p(x_s(t),t) = W \\ &= \hbar V \partial_x \varphi(x_s(t),t) = 2MV\mathrm{v}_s(t) \end{aligned} \tag{19}$$

Constant phase derivative at the step also implies constant phase. Hence, the step velocity is equal to the vortex drift velocity: $v_s = V$. From eq(19), we finally get:

$$v_s = V = \sqrt{W / 2M} \tag{20}$$

This is in fact also the average fluid velocity $< v_1 >$:

$$< v_1 >= V \tag{21}$$

A compact derivation of this simply uses the relation between frequency ($W / \hbar$), wave vector ($2\pi / D$) and velocity (V) for a traveling wave:

$$(2\pi / D)\, V = W / \hbar \tag{22}$$

Comparing this with eqs(16, 20) yields eq(21).

The inter-vortex distance D can be readily derived from equations (20, 22):

$$D = \hbar\pi\sqrt{2 / MW} \tag{23}$$

Note from eqs(20, 23) that the drift velocity and the spacing of vortices depend only on the interfacial energy W; but the vortex aspect ratio $D / \tilde{l}$ also depends on the Josephson coupling energy Λ.

4. Discussion.

1) The main conclusion from our (tentative) analysis is that a two-layer droplet should spread at a constant velocity V. From eq(20), we see that $(V / c)^2 = W / 2Mc^2$. The factor Mc^2 is comparable to the binding energy of liquid ^{4}He. The interfacial energy W is due to Van der Waals interactions between the solid substrate and the fluid — mediated through one or two layers of solid ^{4}He. Even when the substrate is of high polarisability, we do not expect a large W, because the thickness of the solid ^{4}He layer should then increase. Thus we expect W / Mc^2 to be constantly small, and V may be in the range of 10 meters per second.

2) Interfacial energy is transformed into kinetic energy. But we do not discuss dissipation. For a three-dimensional case, where the ingoing vortices annihilate in the center of the droplet. At this point, they may create various types of excitations : phonons, evaporated ^{4}He atoms, etc.

3) In our picture, all the droplet is set into motion: this is very different from the behavior of classical fluids , where only a small annulus at the edge of the second layer carries the action.

4) Our picture describes a steady state contraction of the upper layer. We have not discussed how this state may be reached, starting with some prescribed initial conditions:

a) if, at $t = 0$, the pressure in all parts of the droplet was atmospheric pressure, then the chemical potential difference between layers would originally be equal to W and then oscillate with time, as the relative phase would start to precess at all points simultaneously: at later times, the influence from the step would propagate inwards and should probably lead to the vortex solution (eqs 13, 14).

b) if, at $t = 0$, the pressures had canceled the difference in chemical potential ($\Omega p_1 + W_1 = \Omega p_2 + W_2$), a pressure wave should start from the edge of layer 2, and/or a rarefaction wave from the edge of layer 1; they should propagate (roughly at velocity c) inwards: this tends to lead us back to the starting point (a). Hence, in any case, the system should probably evolve towards the vortex state.

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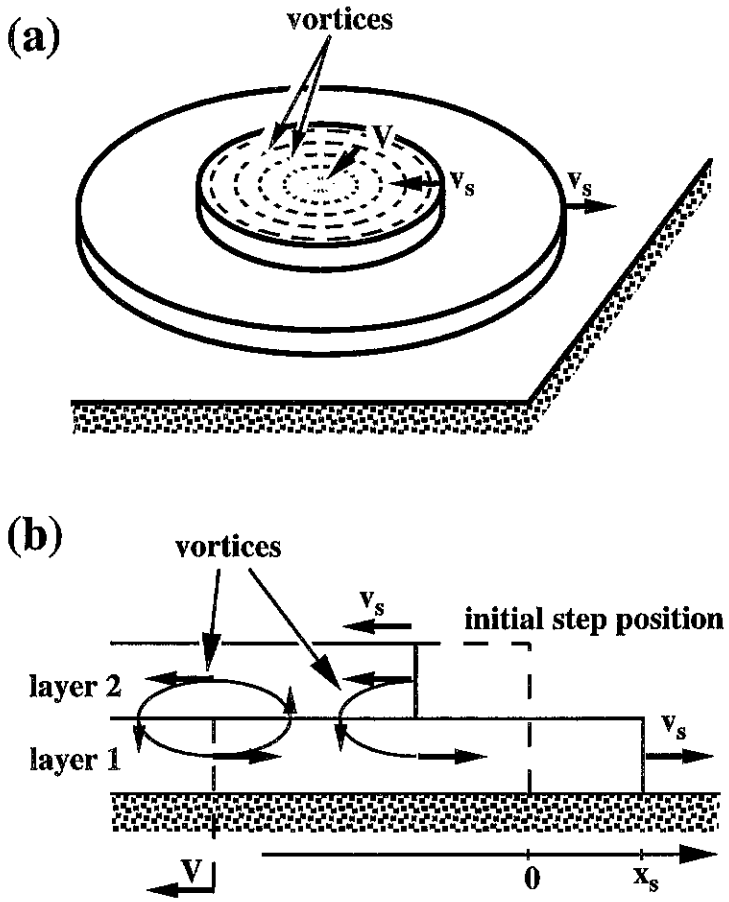

Figure 8.1: Spreading of a superfluid droplet made of two layers. (a) Circular droplet. In the course of spreading, vortices converge towards the center, and annihilate there. (b) Semi-infinite, 1D model of the droplet. Vortices drift towards the left side. At the starting time $t = 0$, the two layers made a double step at $x = 0$.

Figure 8.1: Etalement d'une goutte superfluide constitu'ee de deux couches. (a) Goutte circulaire. Au cours de son étalement, des vortex convergent vers le centre et s'y annihilent. (b) Modèle unidimensionnel et semi-infini de la goutte. Les vortex dérivent vers la gauche. A l'origine des temps, les deux couches forment une double marche en $x = 0$.

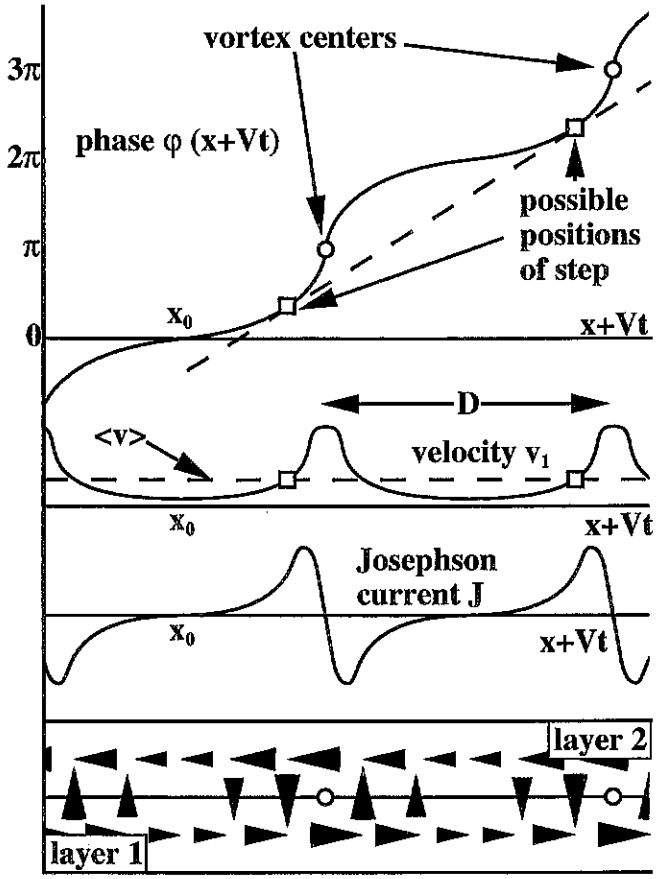

Figure 8.2: Indicative phase profile of the vortex train. The circles indicate the vortex centers. The points marked with a square are allowed positions for the step, following equation (20).

Figure 8.2: Profil indicatif de la phase dans le train de vortex. Les cercles indiquent le centre des vortex. Les points marqués d'un carré sont les positions possibles de la marche, d'après l'équation (20).