Weakly enriched categories over a symmetric monoidal category

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Catégories faiblement enrichies sur une catégorie monoïdale symétrique

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Résumé

Dans cette thèse nous développons une théorie de *catégories faiblement enrichies*. Par 'faiblement' on comprendra ici une catégorie dont la composition de morphismes est associative à homotopie près; à l'inverse d'une catégorie enrichie classique où la composition est strictement associative. Il s'agit donc de notions qui apparaissent dans un contexte homotopique. Nous donnons une notion de *catégorie enrichie de Segal* et une notion de *catégorie enrichie co-Segal*; chacune de ces notions donnant lieu à une structure de catégorie supérieure. L'une des motivations de ce travail était de fournir une théorie de catégories linéaires supérieures, connues pour leur importance dans des différents domaines des mathématiques, notamment dans les géométries algébriques commutative et non-commutative.

La première partie de la thèse est consacrée à la notion de catégorie enrichie de Segal. Nous définissons une telle catégorie enrichie comme morphisme (*colax*) de 2-catégories satisfaisant certaines conditions dites *conditions de Segal*. Le fil rouge de notre démarche est la définition de *monoïde à homotopie près* donnée par Leinster. Les monoïdes de Leinster correspondent précisément aux catégories enrichies de Segal avec un seul objet; ici on suit la coutume en théorie des catégories qui consiste à identifier un monoïde avec l'espace des endomorphismes d'un objet. Notre contribution ici est donc une généralisation des travaux de Leinster. Nous montrons comment notre formalisme couvre le cas des catégories de Segal classique, les monoïdes de Leinster et surtout apporte une définition de DG-catégorie de Segal. Les catégories enrichies 'classiques' sont des catégorie enrichies sur une catégorie monoïdale. L'École australienne a étudié la notion plus générale de catégorie enrichie lorsqu'on remplace 'monoïdale' par '2-catégorie'. Notre formalisme généralise de manière naturelle le *cas australien* en ajoutant de l'homotopie dans la 2-catégorie sur laquelle on enrichit.

Les principaux résultats de la thèse sont dans la deuxième partie qui porte sur les catégories enrichies co-Segal. Nous avons introduit ces nouvelles structures lorsqu'on s'est aperçu que les catégories enrichies de Segal ne sont pas faciles à manipuler pour faire une théorie de l'homotopie. En effet il semble devoir imposer une condition supplémentaire qui est trop restrictive dans beaucoup de cas. Ces nouvelles catégories s'obtiennent en 'renversant' la situation du cas Segal, d'où le préfixe 'co' dans 'co-Segal'. Nous définissons une catégorie co-Segal comme morphisme (lax) de 2-catégories satisfaisant des *conditions co-Segal*. Ces structures se révèlent plus souples à manipuler et notamment pour faire de l'homotopie. Notre résultat principal est l'existence d'une structure de modèles au sens de Quillen sur la catégorie des précatégories co-Segal; avec comme particularité que les objets fibrants sont des catégories co-Segal. Cette structure de modèle s'obtient comme localisation de Bousfield et repose sur des méthodes initialement développées par Jardine et Joyal.

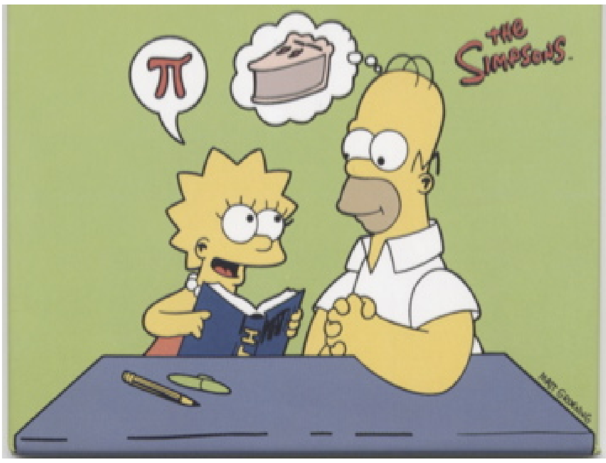
Abstract

In this thesis we develop a theory of *weakly enriched categories*. By 'weakly' we mean an enriched category where the composition is not strictly associative but associative up-to-homotopy. We introduce the notion of *Segal enriched categories* and of *co-Segal categories*. The two notions give rise to higher categorical structures. One of the motivations of this work was to provide an alternative notion of higher linear categories, which are known by the experts to be important in both commutative and noncommutative algebraic geometry.

The first part of the thesis is about Segal enriched categories. We define such an enriched category as a (colax) morphism of 2-categories satisfying the so called *Segal conditions*. Our definition is deeply inspired by the notion of *up-to-homotopy monoid* introduced by Leinster. These weak monoids correspond precisely to Segal enriched categories having a single object. Our work here was to generalize Leinster's work by giving the many object form of his definition. We show that our formalism cover the definition of classical Segal categories and generalizes Leinster's definition. Furthermore we give a definition of Segal DG-category. The theory of enriched categories started with enrichment over a monoidal category. Then the theory was generalized to enrichment over a 2-category, notably by the Australian school. Our formalism generalizes naturally this idea of enrichment over a 2-category by bringing homotopy enrichment at this level.

The main results of this work are in the second part of the thesis which is about co-Segal categories. The origin of this notion comes from the fact that Segal enriched categories are not easy to manipulate for homotopy theory purposes. In fact when trying to have a model structure on them, it seems important to require an extra hypothesis that can be too restrictive. We define a co-Segal category as a (lax) morphism of 2-categories satisfying the *co-Segal conditions*. The idea was to 'reverse' everything of the Segal case i.e from colax to lax, hence the terminology 'co-Segal'. These new structures are much easier to study and to have a homotopy theory of them. The main theorem is the existence of a Quillen model structure on the category of co-Segal precategories; with the property that fibrant objects are co-Segal categories. This model structure is a Bousfield localisation of a preexisting one and lies on techniques which go back to Jardine and Joyal.

Remerciements



«Tout est intéressant !»

Cette phrase, Carlos Simpson n'a eu cesse de me la répéter, sans doute par sa grande vision de la science et des mathématiques en particulier. Il a toujours pris le temps d'écouter mes suggestions et mes idées; et même quand celles-ci étaient dénuées de sens, il a su en tirer quelque chose de très profond; me donnant l'impression que je soulevais un bon point. J'ai eu la chance de découvrir la Recherche auprès de lui avec un sujet riche et passionnant. Sa façon de concevoir les objets mathématiques et son intuition m'ont toujours impressionné. Et le plus surprenant c'est ce mélange d'excellence, humilité, humanité et générosité. Quelques personnes seulement savent à quel point j'ai été enthousiaste et fier de l'avoir eu comme directeur de thèse. Je ferai simple et te dis, Carlos: *Merci*.

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L'histoire sans fin... Je ne pouvais écrire ces mots sans dire comment ça a commencé.

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« Le diable se cache dans les détails... »
Carlos Simpson

Je dédie cette thèse **à l'enfant** de Grothendieck

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CHAPTER 1

Introduction

Version française

Aperçu général

Soit $\mathscr{M} = (\underline{\mathrm{M}}, \otimes, I)$ une catégorie monoïdale. On définit une catégorie $\mathcal{C}$ *enrichie sur* $\mathscr{M}$ en se donnant:

- des objets $A,\ B,\ C, ...$ de $\mathcal{C}$;
- des *hom-objets* $\mathcal{C}(A, B) \in \mathrm{Ob}(\underline{\mathrm{M}})$, pour chaque pair d'objets;
- un morphisme *identité* $I_A : I \longrightarrow \mathcal{C}(A, A)$ pour chaque objet;
- une composition: $c_{ABC} : \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$, pour chaque triplet (A, B, C);

avec les axiomes naturels sur la composition qui sont l'associativité et l'invariance par les identités. On dira que $\mathcal{C}$ est une '$\mathscr{M}$-catégorie'. Lorsque $\mathscr{M}$ est $(\mathbf{Set}, \times), (\mathbf{Ab}, \otimes_{\mathbb{Z}}), (\mathbf{Top}, \times), (\mathbf{Cat}, \times)$,..., une $\mathscr{M}$-catégorie est, respectivement, une catégorie au sens classique [1], une catégorie *pré-additive*, une catégorie *pré-topologique*, une 2-catégorie, etc. On appelle $\mathscr{M}$ "la base d'enrichissement" ou simplement 'la base'.

On a une notion de foncteur entre $\mathscr{M}$-catégories (les $\mathscr{M}$-foncteurs), de transformations naturelles entre $\mathscr{M}$-foncteurs, qui généralisent les notions usuelles pour les petites catégories. Les catégories enrichies sur une base $\mathscr{M}$ forment une catégorie notée $\mathscr{M}$-$\mathbf{Cat}$ qui est en fait une 2-catégorie. Le lecteur trouvera dans le livre de Kelly [49] une exposition de la théorie des catégories enrichies sur une catégorie monoïdale.

Dans son article fondateur, Bénabou [10] introduit les *bicatégories* ainsi que les différents types de morphisme entre elles. Il a remarqué qu'une bicatégorie ayant un seul objet est la même chose qu'une catégorie monoïdale. Cette généralisation des catégories monoïdales en bicatégories a donné lieu à un élargissement naturel de la théorie des catégories enrichies en une théorie où la base $\mathscr{M}$ est une bicatégorie. Il y a dans la littérature de nombreuses références à ce sujet; on peut par exemple citer [52], [86], où l'on trouve plusieurs aspects de cette théorie et d'autres références.

Street [86] a observé que les catégories enrichies sur une bicatégorie $\mathscr{M}$ apparaissaient déjà dans l'article de Bénabou [10] sous la terminologie *polyade*. Bénabou a défini une polyade dans une bicatégorie $\mathscr{M}$ comme étant la version "plusieurs objets" de *monade*. Pour un ensemble X, il définit une X-*polyade* comme étant un morphisme *lax* de bicatégories de $\overline{X}$ vers $\mathscr{M}$; où $\overline{X}$ est la catégorie *indiscrète* [2] associée à X. Dans ce contexte, pour une polyade $F : \overline{X} \longrightarrow \mathscr{M}$, si on note $\mathscr{M}_F^X$ la $\mathscr{M}$-catégorie correspondante; on peut baptiser F comme étant le *le nerf* de $\mathscr{M}_F^X$ et identifier F et $\mathscr{M}_F^X$, tout comme Grothendieck a caractérisé le nerf d'une petite catégorie.

L'approche de Grothendieck d'identifier et définir une catégorie par son nerf, donc par un morphisme, est utilisée dans la théorie des *catégories de Segal*. Rappelons qu'une catégorie de

[1] Ce qu'on appelle couramment une petite ou localement petite catégorie

[2] On l'appelle également catégorie 'grossière' ou 'chaotique' associée à X

Segal est un *objet simplicial* d'une catégorie monoïdale $\mathscr{M} = (\underline{M}, \times)$, qui satisfait les dites *conditions de Segal*. Les origines de la notion de catégorie de Segal sont dans l'article de Segal [78] dans lequel il donne un critère de reconnaissance des espaces de lacets en topologie (*'the delooping problem'*). L'idée d'utiliser les méthodes de Segal pour définir des catégories (faibles) apparaît dans les travaux de Dwyer-Kan-Smith [31] et Schwänzl-Vogt [76]. La théorie des catégories de Segal en général (les n-catégories de Segal) est traitée par Hirschowitz et Simpson [41]. Le lien avec les espaces de lacets et les catégories de Segal est simplement le fait qu'un espace de lacets a naturellement une structure de 1-catégorie de Segal avec un seul objet (monoïde de Segal).

La définition de n-catégories d'Hirschowitz et Simpson est inspirée par les travaux de Tamsamani [87] et Dunn [29], qui à leur tour ont suivi les idées de Segal [78]. Ils définissent une n-catégorie de Segal par son nerf qui est un foncteur défini sur Δ^{op} à valeurs dans une catégorie $\mathscr{M}$ et qui satisfait les conditions de Segal. Ici $\mathscr{M}$ est une catégorie possédant une classe de morphismes appelés *equivalence faibles*; le plus souvent $\mathscr{M}$ est une catégorie de modèles possédant des *objets discrets*. Les objets discrets de $\mathscr{M}$ servent à apporter "l'ensemble d'objets" de la n-catégorie de Segal qu'on définit.

On peut interpréter cette définition comme un enrichissement sur $\mathscr{M}$ même si il est préferable de dire "objet en catégorie interne". Ces idées ont été reprises dans la thèse de Pellissier [71] pour définir les catégories enrichies faibles mais toujours dans le cas où $\mathscr{M}$ a un produit cartésien et possède des objets discrets. L'une des motivations de cette thèse était d'étendre les travaux de Pellissier dans un contexte où $\mathscr{M}$ n' a pas d'objets discrets et a un produit $\otimes$ différent du produit cartésien.

Parallèlement Rezk [73] s'est aussi inspiré des idées de Segal et a introduit les "*Espaces complets de Segal*" [3] comme étant des catégories faiblement enrichies sur $(\mathbf{Top}, \times)$ et $(\mathbf{SSet}, \times)$. Les 1-catégories de Segal et les espaces complets de Segal sont tous deux des modèles de ce qu'on appelle $(\infty, 1)$-catégories. Nous renvoyons le lecteur à l'article de Bergner [15] où l'on trouve une comparaison entre catégories de Segal, epaces complets de Segal, quasicatégories, $(\infty, 1)$-catégories et catégories simpliciales.

Afin d'enlever l'hypothèse 'présence d'objets discrets dans $\mathscr{M}$', Lurie [66] a utilisé la catégorie Δ_X associée à un ensemble X, dont l'origine remonte à Bergner [17, 18]. Δ_X est une version colorée de la catégorie usuelle [4] des simplexes Δ; lorsque X n'a qu'un élément alors Δ_X est isomorphe à Δ. Récemment, Simpson [79] a utilisé cette catégorie Δ_X pour définir les catégories de Segal comme un "vrai" enrichissement sur $\mathscr{M}$. L'adjectif "vrai" ici signifie simplement que l'ensemble d'objets X n'est pas dans $\mathscr{M}$.

Dans cette thèse nous introduisons deux types de catégories faiblement enrichies : les **catégories enrichies de Segal** et les **catégories enrichies co-Segal**. Les deux notions découlent de la philosophie des catégories de Segal classiques et généralisent la notion usuelle de catégories (strictement) enrichies donnée au début de cette introduction. Chacun de ces deux type de 'catégories' se définit comme un morphisme à valeurs dans $\mathscr{M}$ qu'on interpretera alors comme étant le nerf.

La notion de catégories enrichies de Segal complète en quelque sorte la théorie classique qui est développée jusqu'ici dans un contexte où $\mathscr{M} = (\underline{M}, \times)$ est monoïdale pour le produit cartésien. Nous donnons une définition adaptée au cas où $\mathscr{M} = (\underline{M}, \otimes)$ a un produit qui n'est pas le produit cartésien, mais qui s'applique aussi bien lorsque $\mathscr{M} = (\underline{M}, \times)$. Notre definition est étroitement

[3] 'complete Segal spaces' en anglais

[4] Par usuelle nous entendons ici la catégorie Δ qui ne contient pas le vide

guidée par la notion de *monoïde à homotopie près* de Leinster [61] qui correspond exactement à une catégorie enrichie de Segal avec un seul objet. Notre travail ici se positionne donc comme une généralisation des travaux de Leinster (passer d'un seul objet à plusieurs objets).

Pour définir ces monoïdes faibles, Leinster utilise la catégorie monoïdale $(\Delta^+, +, \mathbf{0})$ dont les objets sont les ensembles finis totalement ordonnés $\mathbf{n} = \{0, ..., n-1\}$, en prenant $\mathbf{0}$ pour l'ensemble vide. Nous attirons l'attention du lecteur sur le fait que Δ et Δ^+ sont deux catégories différentes. Les morphismes de Δ^+ sont les fonctions croissantes et $+$ représente l'addition des ordinaux. Cette catégorie $(\Delta^+, +, \mathbf{0})$ est fondamentale dans la théorie des monoïdes classiques (strictes) car elle joue de rôle de 'catégorie classifiante' comme nous l'apprend Mac Lane [68, p. 175]. Plus précisément Mac Lane a montré qu'avoir un monoïde $\mathbf{c}$ dans une catégorie monoïdale $\mathscr{M}$ est équivalent à avoir un foncteur monoïdal $\mathcal{C} : (\Delta^+, +, \mathbf{0}) \longrightarrow \mathscr{M}$ avec $\mathcal{C}(\mathbf{1}) = \mathbf{c}$; en ce sens $(\Delta^+, +, \mathbf{0})$ contient le monoïde universel qui correspond à l'objet $\mathbf{1}$. Par exemple la multiplication $\mu : \mathbf{c} \otimes \mathbf{c} \longrightarrow \mathbf{c}$ est encodée dans le diagramme ci-dessous:

$$\begin{tikzcd} & \mathcal{C}(\mathbf{1}) \\ \mathcal{C}(\mathbf{1}) \otimes \mathcal{C}(\mathbf{1}) \arrow[ur, dotted] \arrow[r, equal, "\sim"'] & \mathcal{C}(\mathbf{2}) \arrow[u, "\mathcal{C}(\mathbf{2} \longrightarrow \mathbf{1})"'] \end{tikzcd}$$

En suivant cette philosophie, Leinster considère la notion analogue d'objet simplicial à valeur dans une catégorie monoïdale $\mathscr{M} = (\underline{M}, \otimes)$; il s'agit de la notion de foncteur *colax monoïdal* de $(\Delta^+, +, \mathbf{0})$ vers $\mathscr{M}$. Sommairement, un foncteur colax monoïdal $\mathcal{C} : (\Delta^+, +, \mathbf{0}) \longrightarrow \mathscr{M}$ consiste en la donnée d'un foncteur $\mathcal{C} : \Delta^+ \longrightarrow \mathscr{M}$, avec des *morphismes de colaxité* $\varphi_{n,m}$:

$$\varphi_{n,m} : \mathcal{C}(\mathbf{n} + \mathbf{m}) \longrightarrow \mathcal{C}(\mathbf{n}) \otimes \mathcal{C}(\mathbf{m})$$

qui doivent satisfaire des conditions de cohérence. Leinster montre que si $\mathscr{M} = (\underline{M}, \times)$ est monoïdale pour le produit cartésien alors il y a une équivalence de catégories entre foncteurs colax monoïdaux $\mathcal{C} : (\Delta^+, +, \mathbf{0}) \longrightarrow \mathscr{M}$ et objets simpliciaux de $\mathscr{M}$ (voir [60, Prop. 3.1.7]).

Avec ces deux ingrédients, Leinster définit un monoïde à homotopie près dans une catégorie monoïdale $\mathscr{M}$ comme étant un foncteur colax monoïdal $\mathcal{C} : (\Delta^+, +, \mathbf{0}) \longrightarrow \mathscr{M}$ satisfaisant les conditions de Segal, c'est à dire, tous les morphismes de colaxité $\varphi_{n,m}$, et $\varphi_0 : \mathcal{C}(\mathbf{0}) \longrightarrow I$ sont des équivalences faibles:

$$\varphi_{n,m} : \mathcal{C}(\mathbf{n} + \mathbf{m}) \xrightarrow{\text{equiv. faible}} \mathcal{C}(\mathbf{n}) \otimes \mathcal{C}(\mathbf{m}).$$

Comme dans le cas stricte, la structure de monoïde (faible) est sur l'objet $\mathcal{C}(\mathbf{1})$ sauf qu'il n'y a plus de multiplication explicite; on peut en avoir une avec n'importe quel inverse *faible* du morphisme de colaxité $\varphi_{1,1}$:

$$\begin{tikzcd} & \mathcal{C}(\mathbf{1}) \\ \mathcal{C}(\mathbf{1}) \otimes \mathcal{C}(\mathbf{1}) \arrow[ur, dotted] & \mathcal{C}(\mathbf{2}) \arrow[u, "\text{canonique}"'] \arrow[l, "\varphi_{1,1}"', "\sim"] \end{tikzcd}$$

En prenant $\mathscr{M}$ égale à $(\mathbf{Top}, \times)$ ou $(\mathbf{SSet}, \times)$, on obtient une 1-catégorie de Segal classique ayant un seul objet.

Pour définir une $\mathscr{M}$-catégorie de Segal $\mathcal{C}$ ayant comme ensemble d'objets X, nous imitons la définition de Leinster et définissons $\mathcal{C}$ comme un foncteur colax de 2-catégories $\mathcal{C} : \mathscr{P}_{\overline{X}} \longrightarrow \mathscr{M}$ qui satisfait les conditions de Segal. Ici $\mathscr{P}_{\overline{X}}$ est une 2-catégorie qui est le substitut de Δ_X dans notre situation. Les objets de $\mathscr{P}_{\overline{X}}$ sont les éléments de X, un 1-morphisme de A vers B est une suite d'éléments $(A_0, ..., A_n)$ avec $A_0 = A$ et$A_n = B$. Les 2-morphismes sont paramétrés par les morphismes de Δ^+ et se traduisent par les répétitions ou suppressions de lettres pour passer d'une suite à une autre tout en gardant A et B fixés. On a par exemple un 2-morphisme $(A, E, B) \longrightarrow (A, B)$ paramétré par l'unique morphisme $\mathbf{2} \longrightarrow \mathbf{1}$ de Δ^+. La composition dans $\mathscr{P}_{\overline{X}}$ est la concaténation de suites. Lorsque X a un seul élément on a un isomorphisme $\mathscr{P}_{\overline{X}} \cong (\Delta^+, +, \mathbf{0})$ et on peut donc dire que $\mathscr{P}_{\overline{X}}$ est un 'gros' $(\Delta^+, +, \mathbf{0})$.

Un tel morphisme colax $\mathcal{C}$ donne lieu à des diagrammes dans $\mathscr{M}$ du type:

$$\begin{tikzcd}
 & \mathcal{C}(A,C) \\
\mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \arrow[ur, dotted] & \mathcal{C}(A,B,C) \arrow[l, "\varphi_{ABC}"'] \arrow[u, "\text{canonique}"']
\end{tikzcd}$$

Les conditions de Segal reviennent à demander que tous les morphismes de colaxités (= les morphismes de Segal) soient des équivalences faibles:

$$\varphi : \mathcal{C}(A_0, ..., A_i, ..., A_k) \xrightarrow{\text{equiv. faible}} \mathcal{C}(A_0, ..., A_i) \otimes \mathcal{C}(A_i, ..., A_k).$$

En particulier le morphisme $\varphi_{ABC} : \mathcal{C}(A, B, C) \longrightarrow \mathcal{C}(A, B) \otimes \mathcal{C}(B, C)$ dans le diagramme précédent est une équivalence faible; ceci permet de choisir un inverse faible et avoir une composition associative à homotopie près.

Après avoir trouvé le bon formalisme, il reste à développer une théorie d'homotopie de ces structures, c'est à dire avoir une structure de catégorie de modèle sur les $\mathscr{M}$-catégories de Segal pour $\mathscr{M} = (\underline{M}, \otimes, I)$ une catégorie monoïdale de modèles avec un produit $\otimes$ non cartésien. Lorsque $\otimes$ est le produit cartésien la théorie est largement traitée dans le livre de Simpson [79]. Pour les catégories de Segal classiques, on met la structure de modèles sur la catégories des precatégories de Segal qui sont simplement tous les diagrammes (colax) sans imposer les conditions de Segal. Pour avoir la structure de modèles on a besoin d'un procédé de "Segalification" **Seg** qui a pour but d'associer à toute précategorie une catégorie de Segal qui est minimal dans un sens homotopique. Simpson traite ce procédé dans son livre, en termes de *projection monadique faible*.

La structure de modèles sur les précatégories de Segal est inspirée par les travaux de Jardine [44], Joyal [46] sur les (pré)-faisceaux simpliciaux, et est comprise comme étant un cas spécial de localisation de Bousfield (voir [79] ainsi que les références mentionnés dedans). Ces techniques sont désormais standards pour les experts et ont été beaucoup utilisées dans la littérature.

Dans notre cas, on a voulu suivre la même philosophie; c'est à dire de construire d'abord un foncteur de Segalification **Seg** qui associe à tout diagramme colax $\mathcal{C} : \mathscr{P}_{\overline{X}} \longrightarrow \mathscr{M}$, un diagramme $\mathbf{Seg}(\mathcal{C})$ qui satisfait les conditions de Segal. Mais nous sommes confrontés à deux obstacles:

- Premièrement les morphismes de Segal $\varphi : \mathcal{C}(A, B, C) \longrightarrow \mathcal{C}(A, B) \otimes \mathcal{C}(B, C)$ ne sont pas 'naturels', dans le sens où ce sont des morphismes qui vont vers un produit tensoriel, où il n'y a *à priori* pas de possibilité de récupérer une projection sur chaque facteur. De fait lorsqu'on a un produit $\otimes$ qui n'est pas le produit cartésien, le produit de deux objets est habituellement une source d'un morphisme plutôt que le but.

– L'autre obstacle vient du fait que pour construire un tel foncteur **Seg** l'idée est de factoriser les morphismes de Segal φ comme la composée d'une cofibration suivie d'une fribration triviale:

$$\varphi : \mathcal{C}(A,B,C) \xhookrightarrow{i} Q(A,B,C) \xrightarrow[\sim]{j} \mathcal{C}(A,B) \otimes \mathcal{C}(B,C).$$

Seulement dans le diagramme colax $\mathcal{C}$ il y a des produits de morphismes de Segal; et le produit des factorisations vont induire des produits de fibrations triviales $j \otimes j$ dans $\mathscr{M}$ que l'on demanderait d'être une fibration triviale.

Le comportement des fibrations triviales par rapport au produit tensoriel n'a pas été vraiment considéré par les spécialistes. Et il est même certain qu'en général les fibrations (triviales) ne sont pas stables par produit tensoriel. Cependant il existe des cas où les fibrations sont stables par produit. Un des exemples les plus intéressants de cette situation c'est lorsque $\mathscr{M} = \mathbf{Ch}(\mathbf{R})$, la catégorie des complexes de (co)-chaines sur un anneau commutatif $\mathbf{R}$; les fibrations sont juste les morphismes surjectifs degré par degré (voir [42, Prop. 4.2.13]).

Ces deux contraintes sont apparues pendant une discussion avec Lurie et nous ont conduit à changer de direction pour le reste de la thèse. Cela dit on peut quand même conjecturer que si les fibrations sont stables par produit, alors on devrait pouvoir construire un foncteur **Seg**. Ceci est un problem ouvert que nous avons laissé pour des travaux futurs.

Dans la deuxième partie de cette thèse, nous avons introduit la notion de *$\mathscr{M}$-catégories co-Segal*; celles-ci sont encore des catégories homotopiquement enrichies. Cette fois-ci les diagrammes qui donnent lieu à une composition faiblement associative sont de la forme:

$$\begin{tikzcd} & \mathcal{C}(A,C) \arrow[d, "\wr"', "\text{equiv. faible}"] \\ \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \arrow[r, "\varphi_{ABC}"] & \mathcal{C}(A,B,C) \end{tikzcd}$$

Comme on peut le remarquer, ce diagramme s'obtient à partir du diagramme du cas Segal en inversant le sens des flèches; d'où la terminologie 'co-Segal'. Les conditions 'co-Segal' ici consistent à demander que le morphisme verticale soit une équivalence faible à chaque fois que nous avons un diagramme de ce type. L'autre différence importante avec le cas Segal vient du fait que pour $f \in \mathcal{C}(A,B)$ et $g \in \mathcal{C}(B,C)$ il y a déjà une *précomposée* $\varphi(f \otimes g)$ dans "objet-tampon" $\mathcal{C}(A,B,C)$.

Nous définissons une $\mathscr{M}$-catégorie co-Segal ayant un ensemble d'objets X comme étant un foncteur lax $\mathcal{C} : (\mathbb{S}_{\overline{X}})^{\text{2-op}} \longrightarrow \mathscr{M}$ satisfaisant les conditions co-Segal. Ici on a $\mathbb{S}_{\overline{X}} \subset \mathscr{P}_{\overline{X}}$ et si X a un seul élément on a un isomorphisme $\mathbb{S}_{\overline{X}} \cong (\Delta^{+}_{\text{epi}}, +, \mathbf{0})$.

Ces nouvelles structures sont faciles à étudier et le reste de la thèse était de développer une théorie d'homotopie pour elles. Comme pour le cas Segal, nous mettons une structure de modèles sur la catégories de tous les diagrammes lax dans $\mathscr{M}$ indexés par $(\mathbb{S}_{\overline{X}})^{\text{2-op}}$ quand X parcourt la catégorie **Set** des ensembles. Nous baptisons ces diagrammes comme étant les *précatégories co-Segal* et notons par $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ la catégorie qu'ils forment. Le résultat principal ici est le:

Théorème. *Soit $\mathscr{M}$ une catégorie monoïdale symétrique de modèles qui est cofibrement engendrée et telle que tous les objets sont cofibrants. Alors on a*

1. *La catégorie $\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})$ des précatégories co-Segal admet une structure de modèles cofibrement engendrée;*

2. *les objets fibrants sont des $\mathscr{M}$-catégories co-Segal.*

3. Si $\mathscr{M}$ est combinatorielle alors $\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})$ est aussi combinatorielle.

Cette structure de modèle s'obtient comme localisation de Bousfield d'une structure préexistante; nous utilisons un "foncteur de co-Segalification" et suivons la démarche de Simpson qui a proposé une *méthode de localisation directe* (voir [79, Chap. 9]).

Perspectives de recherche

Les deux types de catégories faiblement enrichies considérées dans cette thèse donnent lieu à des structure de catégories supérieures. La théorie des catégories supérieures constitue à elle seule un vaste domaine de recherche et nous espérons que ce travail contribuera à lever un coin du voile sur certains aspects de ces structures mystérieuses et fascinantes.

Les idées amorcées ici peuvent être poursuivies dans plusieurs directions; tant sur le plan fondationnel que sur les applications dans les différents domaines des mathématiques. Nous indiquons brièvement dans cette section, quelques une des différentes pistes qui seront considérées dans un avenir proche.

L'homotopie des $\mathscr{M}$-catégories de Segal

Comme nous l'avons dit plus haut, nous avons changé de direction au cours de la thèse car le cas des catégories de Segal semble nécessiter l'hypothèse suivante sur $\mathscr{M} = (\underline{M}, \otimes, I)$ (avec $\otimes \neq \times$).

Hypothèse (Sketch)**.** La classe des fibrations (triviales) est stable par produit tensoriel, c'est à dire: $f \otimes g$ est une fibration (triviale) si f et g sont tous deux des fibrations (triviales).

Nous savons que cette hypothèse est vraie dans la catégorie $\mathbf{Ch}(\mathbf{R})$ des complexes de (co)-chaines sur un anneau commutatif $\mathbf{R}$; et en particulier lorsque $\mathbf{R}$ est un corps. Dans ce cas les catégories enrichies de Segal qu'on obtient sont ce qu'on appellera DG-catégories de Segal dont la définition a été la motivation première d'avoir le formalisme Segal dans un contexte avec produit $\otimes$ non cartésien. Cette idée d'avoir une notion de DG-catégorie faible remonte au projet de Toën [93] d'avoir une théorie de catégories linéaires supérieures pour faire de la dualité de Tannaka supérieure. On imagine une théorie itérative *à la* Simpson-Tamsamani; c'est à dire qu'il devrait y avoir une catégorie monoïdale de modèles de toutes les $\mathscr{M}$-(pré)-catégories de Segal avec un produit $\boxtimes$ qu'on utiliserait comme base d'enrichissement.

Lurie a dégagé la notion de $(\infty, 1)$-*catégorie stable* comme exemple de catégories linéaire supérieure. Cette notion est déjà utilisée par beaucoup pour faire de la dualité de Tannaka supérieure. Le lecteur trouvera une exposition de cette théorie dans la thèse de Wallbridge dont le sujet porte sur la dualité de Tannaka supérieure.

Sous l'hypothèse précédente on conjecture que

Conjecture.

– *Il existe un foncteur de Segalification* $\mathbf{Seg}$ *de PreSeg($\mathscr{M}$) vers elle même qui à tout diagramme colax* $\mathcal{C}$ *associe une catégorie de Segal* $\mathbf{Seg}(\mathcal{C})$.

– *Le foncteur* $\mathbf{Seg}$ *est muni d'une transformation naturelle* $\eta : \mathrm{Id} \longrightarrow \mathbf{Seg}$*, qui est une cofibration niveau par niveau; et tel que si* $\mathcal{C}$ *satisfait les condition de Segal alors* $\eta : \mathcal{C} \longrightarrow \mathbf{Seg}(\mathcal{C})$ *est une équivalence faible dans PreSeg($\mathscr{M}$) (= niveau par niveau).*

– *Il y a deux structures de modèles $PreSeg(\mathscr{M})_{proj}$ et $PreSeg(\mathscr{M})_{Reedy}$ sur la catégorie $PreSeg(\mathscr{M})$; on les appellera respectivement la structure projective et la structure Reedy. Dans chacun de ces deux cas, les trois classes, cofibrations, fibrations et équivalences faibles seront définies comme pour le cas où $\otimes = \times$ (voir [79]).*

Unités faibles pour les $\mathscr{M}$-catégories co-Segal

La définition d'une $\mathscr{M}$-catégorie co-Segal donne lieu à une structure de semi-catégorie faiblement enrichie, ce qui veut dire qu'il n'y a pas de morphismes identité. Il y a, cela dit, une notion naturelle d'unité faible que nous allons brièvement décrire dans un instant. Nous avons la même situation que pour les A_∞-catégories qui sont apparues dans un premier temps sans morphisme identité (voir [35, 36]). Si $\mathcal{C}$ est une $\mathscr{M}$-catégorie co-Segal, nous désignerons par $[\mathcal{C}]$ la $\mathbf{ho}(\mathscr{M})$-catégorie co-Segal qu'on obtient par le changement de base d'enrichissement $\mathrm{L} : \mathscr{M} \longrightarrow \mathbf{ho}(\mathscr{M})$. $[\mathcal{C}]$ est une $\mathbf{ho}(\mathscr{M})$-catégorie co-Segal stricte, ce qui signifie que c'est une semi-catégorie enrichie sur $\mathbf{ho}(\mathscr{M})$.

Définition (Sketch)**.** Nous dirons qu'une $\mathscr{M}$-catégorie co-Segal $\mathcal{C}$ a des identités faibles si $[\mathcal{C}]$ est une catégorie enrichie sur $\mathbf{ho}(\mathscr{M})$ (=avec des morphismes identités).

Il est naturel de se demander si on ne peut pas avoir directement les identités i.e sans utiliser le changement de base $\mathrm{L} : \mathscr{M} \longrightarrow \mathbf{ho}(\mathscr{M})$. Pour l'heure nous ne savons pas si une telle démarche est naturelle mais nous indiquons ci-dessous quelques directions possibles.

Directions possibles

1. La première tentative est demander que pour chaque objet A de $\mathcal{C}$, on a un morphisme $I_A : I \longrightarrow \mathcal{C}(A, A)$ tel que pour tout B, le diagramme suivant commute (à homotopie près ?):

$$\begin{tikzcd} I \otimes \mathcal{C}(A,B) \arrow[r, "\cong"] \arrow[d, "I_A \otimes \mathrm{Id}"] & \mathcal{C}(A,B) \arrow[d, "\wr"', "\text{equiv. faible}"] \\ \mathcal{C}(A,A) \otimes \mathcal{C}(A,B) \arrow[r, "\varphi_{AAB}"] & \mathcal{C}(A,A,B) \end{tikzcd}$$

   Ce diagramme donnera l'invariance à gauche par composition avec I_A; et le même type de diagramme donnera l'invariance à droite. Il est important de remarquer que nous ne considérons là que l'invariance vis à vis des '1-simplexes' $\mathcal{C}(A, B)$ de $\mathcal{C}$ et non par rapport à tous les $\mathcal{C}(A_0, ...A_n)$ avec $n > 1$.

   Il y a au moins deux raisons qui motivent cette limitation. La première vient du fait que pour définir les A_∞-catégories avec unité, on se limite à l'invariance vis à vis de la multiplication binaire 'm_2' (voir par exemple Kontsevich-Soibelman [54, Sec. 4.2], Lyubashenko [67, Def. 7.3]).

   L'autre raison est le fait que $\mathcal{C}(A, B)$ et $\mathcal{C}(A, ..., A_i, ..., B)$ ont le même type d'homotopie (grâce aux conditions co-Segal); par conséquent si $\mathcal{C}(A, B)$ est faiblement invariant par I_A, il en sera de même pour $\mathcal{C}(A, ..., A_i, ..., B)$.

   Enfin dans le schéma général de l'algèbre, il est évident que l'ajout de conditions supplémentaires réduit la classe d'objets qui sont soumis à celles-ci.

2. Une autre alternative serait de suivre la démarche de Kock [53] pour avoir des unités faibles dans les catégories supérieures. Kock utilise une catégorie **'Δ^+ épaissie'** pour produire des unités faibles. Puisque les 2-catégories $\mathbb{S}_{\overline{X}}$ et $\mathscr{P}_{\overline{X}}$ sont des généralisations de Δ^+, il est naturel de se demander s'il y a une version 'épaissie' de $\mathbb{S}_{\overline{X}}$ ou $\mathscr{P}_{\overline{X}}$.

L'homotopie des catégories co-Segal unitaires

Avec la notion de catégories co-Segal avec les unités (unitaires), on peut donner la *bonne* définition d'équivalence faible entre catégories co-Segal. Jusqu'ici un morphisme $\sigma : \mathcal{C} \longrightarrow \mathcal{D}$ de catégories co-Segal est une équivalence faible si la fonction $\mathrm{Ob}(\sigma) : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{D})$ est un isomorphisme, et si c'est une équivalence niveau par niveau. Ceci n'est guère surprenant puisqu'on ne peut pas exprimée le fait d'être "essentiellement surjectif" sans unités (faibles). Nous adopterons donc la définition suivante:

Définition. Un morphisme $\sigma : \mathcal{C} \longrightarrow \mathcal{D}$ de catégories co-Segal unitaires est *essentiellement surjectif* si le morphisme induit $[\sigma] : [\mathcal{C}] \longrightarrow [\mathcal{D}]$ est essentiellement surjectif comme foncteur de catégories strictement enrichies sur $\mathbf{ho}(\mathscr{M})$.

Nouvelle structure de modèles Cette nouvelle définition d'équivalences faibles changera la structure de modèles actuelle sur $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$; il faudra donc construire une autre structure de modèles qui prendra en compte la présence d'unités faibles.

Une catégorie monoïdale de modèles La structure de modèle sur $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ donnée ici ne tient pas compte du produit tensoriel sur $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$. L'une des taches importantes qui restent sera d'inclure le produit tensoriel dans la structure de modèles.

Segal, co-Segal et strictification

Il est coutume de comparer chaque nouvelle notion ou généralisation avec celles qui existent déjà. Ce principe est d'autant plus important dans la théorie des catégories supérieures, qui est un domaine réputé pour la multitude de définitions des objets qu'on étudie. À l'heure actuelle plusieurs groupes de chercheurs se penchent sur la comparaison des différentes approches et définitions qui existent pour les catégories supérieures. Il y a déjà plusieurs résultats à ce sujet dont les travaux de Barwick et Schommer-Pries [8], Bergner [15], Bergner-Rezk [16], Cheng [21], Lurie [66], Simpson [79], Toën [92] et beaucoup d'autres.

1. Nous nous intéresserons au problème de la comparaison de l'homotopie des $\mathscr{M}$-catégories de Segal, $\mathscr{M}$-catégories co-Segal et $\mathscr{M}$-catégories strictes.
2. La méthode utilisée par Bergner [17] qui repose sur des techniques développées par Badzioch [5] semble s'appliquer au cas co-Segal; on imagine une équivalence de Quillen entre $\mathscr{M}$-$\mathbf{Cat}$ et 'co-Segal $\mathscr{M}$-$\mathbf{Cat}$' (donc une *strictification*).
3. Lorsque $\mathscr{M}$ a un produit cartésien, la comparaison entre $\mathscr{M}$-catégories de Segal $\mathscr{M}$-catégories strictes a déjà été étudiée (voir [15], [79] et les références qu'on y trouve).

Comparaison avec les A_∞-catégories et les $(\infty, 1)$-catégories stables Lorsqu'on considère une $\mathscr{M}$-catégorie (Co)-Segal pour $\mathscr{M} = (\mathbf{Ch}(\mathbf{R}), \otimes_{\mathbf{R}}, \mathbf{R})$, on obtient une DG-catégorie faible qu'on interprète comme étant un exemple de catégorie supérieure linéaire. Comme indiqué plus haut la définition de structures linéaires supérieures était l'une des motivations principales de cette thèse. Il y a actuellement deux notions de structures linéaires supérieures qui sont utilisées:

- la notion de A_∞-catégorie introduite par Fukaya et Kontsevich (voir [34],[37], [55],[54],[48], [67]) qui remonte à la notion de A_∞-algèbre de Stasheff [82].
- la notion de ∞-catégorie stable introduite par Lurie [65].

1. Nous essaierons donc d'avoir une compréhension profonde des connections qu'il y a entre les catégories co-Segal linéaires, les A_∞-catégories et les ∞-catégories stables, d'un point de vue homotopique.

2. Une possible équivalence entre les catégories co-Segal linéaires et et les A_∞-catégories permettrait d'avoir une approche simpliciale et moins combinatorielle de la géométrie algébrique non commutative et les sujets connexes. On peut aussi espérer simplifier certaines constructions du ' monde A_∞'.

3. La plupart des notions des catégories enrichies strictes se transportent naturellement aux cas (Co)-Segal. Pour $\mathscr{M} = (\mathbf{Ch}(\mathbf{R}), \otimes_{\mathbf{R}}, \mathbf{R})$ les notions de DG-module et A_∞-module jouent un rôle central en géométrie algébrique non commutative. Dans la théorie des catégories enrichies les DG-modules correspondent aux préfaisceaux à valeur dans $\mathscr{M}$ et sont un cas particulier de la notion plus générale de *bimodules*, qu'on appelle également *distributeurs* (enrichis) ou *profoncteurs*.

La géométrie algébrique avec des 'outils co-Segal'

Suivre les suggestions de Deligne

Deligne [26] a dégagé beaucoup d'idées pour faire de la géométrie algébrique dans une catégorie tannakienne en analysant ce qui se passe pour la catégorie tannakienne des représentations linéaires Rep(G) d'un $\mathbf{k}$-schéma en groupes affine G (pour un corps $\mathbf{k}$).

Deligne définit un anneau commutatif avec unité dans une catégorie tannakienne $\mathcal{T}$ comme étant un objet $R \in \mathcal{T}$ avec une structure de monoïde commutatif et une unité. Il définit en suite la catégories des $\mathcal{T}$-*schémas affines* comme d'habitude, c'est à dire, comme la catégorie opposée des anneaux commutatifs avec unité dans $Ind\mathcal{T}$ [5]. Pour $\mathcal{T} = \text{Rep}(G)$ les $\mathcal{T}$-schémas affines correspondent aux $\mathbf{k}$-schémas affines classiques munies d'une action à droite de G (les G-modules). Les $\mathbf{k}$-schémas affines 'tout court' sont vus comme ayant l'action triviale de G. De même un morphisme de $\mathbf{k}$-schémas devient de manière tautologique un morphisme de $\mathcal{T}$-schémas affines. Deligne indique que les notions usuelles de la géométrie algébrique se transportent aussi bien pour les $\mathcal{T}$-schémas affines: produits fibrés, schémas en groupes, platitude, module (quasi-cohérent), torseurs, etc.

L'un des avantages quand on est dans une catégorie tannakienne c'est qu'il y a un <u>Hom</u> *interne*; ce qui fait de $\mathcal{T}$ une catégorie *monoïdale fermée* et en particulier une catégorie enrichie sur elle même. Pour des $\mathbf{k}$-schémas affines classiques, on sait grâce à l'anti-équivalence de Grothendieck qu'un morphisme de k-schémas est induit par un morphisme de k-algèbres commutatives (qui va dans l'autre sens); seulement les morphismes de k-algèbres ne forment qu'un ensemble et on perd toute la structure vectoriel des morphismes entre $\mathbf{k}$-espaces vectoriels sous-jacents (aux $\mathbf{k}$-algèbres).

Pour $\mathcal{T} = \text{Rep}(G)$, le fait de voir un $\mathbf{k}$-schéma affine classique comme un $\mathcal{T}$-schéma (avec l'action triviale), permet de garder une 'trace' de la structure de morphismes entres $\mathbf{k}$-espaces vectoriels sous-jacents tout en manipulant des objets géométriques. Nous voyons en cela un enrichissement de la catégorie des $\mathbf{k}$-schémas affines sur $\mathcal{T}$.

Pour le peu que je comprends de la philosophie des motifs, on cherche à 'héberger' la catégorie des variétés algébriques dans une catégorie (supérieure) linéaire qui est (homotopiquement) minimale. Ici par 'héberger' j'entends simplement trouver un enrichissement, donc des <u>Hom</u> à

[5] $Ind\mathcal{T}$ représente le complété par petites limites inductives de $\mathcal{T}$; les objets de $Ind\mathcal{T}$ sont les limites inductives de représentables dans $\widehat{\mathcal{T}} = \text{Hom}(\mathcal{T}^{\text{op}}, \mathbf{Set})$

coefficient linéaires qui contiennent les 'vieux' morphismes comme sous-objet 'simple'. Cette tentative de mettre une structure sur les morphismes de schémas apparaît également dans les travaux de Voevodsky qui a considéré la catégorie 'Cor' des correspondances entre schémas dans sa définition des dits motifs de Voevodsky (voir [69]).

Récemment il y a eu des notions d'homotopie en géométrie algébrique qui se sont dégagées telles que la $\mathbf{A}^1$-homotopie ou l'Homotopie Étale. Le prolongement des idées de Deligne dans ce contexte homotopique fera intervenir des catégories linéaires supérieures comme le dit par exemple Toën [93]; et on espère que les notions que nous développons ici contribueront à révéler certains aspect de l'idée de motifs imaginée par Grothendieck.

Pour concrétiser cette démarche nous aurons besoin des notions suivantes.

1. Une notion correcte de catégorie monoïdale symétrique co-Segal.
2. Une définition de catégorie co-Segal tannakienne (neutre)
3. Par ailleurs il semble naturel de penser qu'une DG-algèbre commutative co-Segal devrait être équivalent à un E_∞-anneau. Les E_∞-anneaux et les anneaux simpliciaux sont au cœur de la *Géométrie algébrique Homotopique et Dérivée.*

Organisation du manuscrit

Nous avons essayé de rendre les chapitres aussi indépendants que possible. Chaque chapitre commence par une brève introduction qui donne un panorama du contenu.

- ◇ Le chapitre 2 contient essentiellement la prépublication de l'auteur [3] dans lequel nous exposons la définition des catégories enrichies de Segal. Les définitions sont données dans le langage général des catégories enrichies sur une 2-catégorie $\mathscr{M}$. Nous montrons comment notre définition généralise naturellement le travail de Leinster [60] et que, comme on s'y attend, le cas Segal stricte donne bien une $\mathscr{M}$-catégorie classique.

- ◇ Les principaux résultats de cette thèse sont dans le chapitre 3 qui parle des catégories co-Segal. Afin d'avoir la structure de modèles sur $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ énoncée plus haut, nous avons d'abord traité le cas $\mathscr{M}_{\mathbb{S}}(X)$ où l'on fixe l'ensemble d'objets X. Ensuite on a propagé la structure de modèles en utilisant un résultat de Roig-Stanculescu qui permet d'avoir une structure de modèles de la catégorie totale $\mathscr{E}$ d'une fibration de Grothendieck $\mathbf{p} : \mathscr{E} \longrightarrow \mathscr{B}$.

- ◇ Enfin le chapitre 4 contient les différents appendices.

English version

Overview

Let $\mathscr{M} = (\underline{\mathrm{M}}, \otimes, I)$ be a monoidal category. An *enriched* category $\mathcal{C}$ over $\mathscr{M}$, shortly called 'an $\mathscr{M}$-category', consists roughly speaking of :

- objects $A,\ B,\ C, \cdots$
- *hom-objects* $\mathcal{C}(A, B) \in \mathrm{Ob}(\underline{\mathrm{M}})$,
- a unit map $I_A : I \longrightarrow \mathcal{C}(A, A)$ for each object A,
- a composition law : $c_{ABC} : \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$, for each triple of objects (A, B, C),

satisfying the obvious axioms, associativity and identity, suitably adapted to the situation. Taking $\mathscr{M}$ equal to $(\mathbf{Set}, \times), (\mathbf{Ab}, \otimes_{\mathbb{Z}}), (\mathbf{Top}, \times), (\mathbf{Cat}, \times)$,..., an $\mathscr{M}$-category is, respectively, an ordinary[6] category, a *pre-additive* category, a *pre-topological* category, a 2-category, etc. The category $\mathscr{M}$ is called the *base* as "base of enrichment".

Just like for **Set**-categories, we have a notion of $\mathscr{M}$-functor, $\mathscr{M}$-natural transformation, etc. The reader can find an exposition of the theory of enriched categories over a monoidal category in the book of Kelly [49]. For a base $\mathscr{M}$, we commonly denote by $\mathscr{M}$-Cat the category of $\mathscr{M}$-categories.

Bénabou defined *bicategories*, and morphisms between them (see [10]). He pointed out that a bicategory with one object was the same thing as a monoidal category. This gave rise to a general theory of enriched categories where the base $\mathscr{M}$ is a bicategory. We refer the reader to [52], [86] and references therein for *enrichment* over a bicategory.

Street noticed in [86] that for a set X, an X-*polyad* [7] of Bénabou in a bicategory $\mathscr{M}$ was the same thing as a category enriched over $\mathscr{M}$ whose set of objects is X. Here an X-polyad means a *lax* morphism of bicategories from $\overline{X}$ to $\mathscr{M}$, where $\overline{X}$ is the *coarse*[8] category associated to X. Then given a polyad $F : \overline{X} \longrightarrow \mathscr{M}$, if we denote by $\mathscr{M}_F^X$ the corresponding $\mathscr{M}$-category, one can interpret F as *the nerve* of $\mathscr{M}_F^X$ and identify F with $\mathscr{M}_F^X$, like Grothendieck's characterization of the nerve of a category.

Recall that a Segal category is a *simplicial object* of a cartesian monoidal category $\mathscr{M}$, satisfying the so called *Segal conditions*. The theory of Segal categories has its roots in the paper of Segal [78] in which he proposed a solution of the *delooping problem*. The general theory starts with the works of Dwyer-Kan-Smith [31] and Schwänzl-Vogt [76]. The major development of Segal n-categories was given by Hirschowitz and Simpson [41].

Hirschowitz and Simpson used the same philosophy as Tamsamani [87] and Dunn [29], who in turn followed the ideas of Segal [78]. A Segal n-category is defined by its nerve which is an $\mathscr{M}$-valued functor satisfying the suitable Segal conditions. The target category $\mathscr{M}$ needs to have a class of maps called *weak* or *homotopy* equivalences. Moreover they required the presence of *discrete objects* in $\mathscr{M}$ which will play the role of 'set of objects'. We can interpret their approach as an enrichment over $\mathscr{M}$, even though it's better to say "internal weak-category-object of $\mathscr{M}$". The same approach was used by Pellissier [71].

[6]By ordinary category we mean small category
[7]Bénabou called *polyad* the 'many objects' case of *monad*. 'X-polyad' means here "polyad associated to X"
[8]Some authors call it the 'chaotic' or 'indiscrete' category associated to X

Independently Rezk [73] followed also the ideas of Segal to define *complete Segal spaces* as weakly enriched categories over $(\mathbf{Top}, \times)$ and $(\mathbf{SSet}, \times)$. We refer the reader to the paper of Bergner [15] for an exposition of the interactions between Segal categories, complete Segal spaces, quasicategories, $(\infty, 1)$-categories, etc.

To avoid the use of discrete objects, Lurie [66] used a category Δ_X introduced by Bergner [17, 18] which is a 'colored version' of the usual[9] category of simplices Δ. Simpson [79] used this Δ_X to define Segal categories as a "proper" enrichment over $\mathscr{M}$. Here by "proper" we simply mean that the set X which will be the set of objects is taken 'outside' $\mathscr{M}$.

In this thesis we introduce two types of weakly enriched categories. The first ones are called **Segal enriched categories** and the second ones are called **co-Segal categories**. The two notions derive from the philosophy of classical Segal categories; and have the expected behavior that is: strict Segal and co-Segal $\mathscr{M}$-categories are just the usual $\mathscr{M}$-categories. Furthermore they are both defined as a morphism that we shall also consider to be the corresponding nerve.

Segal enriched categories complete somehow the theory of classical Segal categories which were defined so far only when the base $\mathscr{M} = (\underline{M}, \times)$ is monoidal for the cartesian product. In fact the main motivation here was to push forward the Segal formalism to have a definition of $\mathscr{M}$-categories when $\mathscr{M}$ has a tensor product $\otimes$ different from the cartesian product. Their definition is a generalization of the notion of up-to-homotopy monoid introduced by Leinster [61] which is precisely a Segal enriched category with one object.

To define these weak monoids, Leinster used the monoidal category $(\Delta^+, +, \mathbf{0})$ whose objects are the finite totally ordered sets $\mathbf{n} = \{0, ..., n-1\}$ with $\mathbf{0}$ being the empty set. We would like to warn the reader that Δ and Δ^+ are different categories. The morphisms of Δ^+ are the nondecreasing functions and $+$ is the ordinal addition. As pointed out by Mac Lane [68, p. 175], $(\Delta^+, +, \mathbf{0})$ contains the universal monoid which corresponds to the object $\mathbf{1}$; the universality here is to be understood that for any monoid $\mathbf{c}$ in a monoidal category $\mathscr{M}$ there is a unique monoidal functor $\mathcal{C} : (\Delta^+, +, \mathbf{0}) \longrightarrow \mathscr{M}$ with $\mathcal{C}(\mathbf{1}) = \mathbf{c}$. For example the multiplication $\mu : \mathbf{c} \otimes \mathbf{c} \longrightarrow \mathbf{c}$ is encoded in the diagram below:

$$\begin{tikzcd} & \mathcal{C}(\mathbf{1}) \\ \mathcal{C}(\mathbf{1}) \otimes \mathcal{C}(\mathbf{1}) \arrow[ur, dotted] \arrow[r, equal, "\sim"'] & \mathcal{C}(\mathbf{2}) \arrow[u, "\mathcal{C}(\mathbf{2} \longrightarrow \mathbf{1})"'] \end{tikzcd}$$

Following this philosophy Leinster considered the appropriate notion of simplicial object which is given by the notion of *colax monoidal* functor from $(\Delta^+, +, \mathbf{0})$ to $\mathscr{M}$. A colax monoidal functor $\mathcal{C} : (\Delta^+, +, \mathbf{0}) \longrightarrow \mathscr{M}$ consists roughly speaking of a functor $\mathcal{C} : \Delta^+ \longrightarrow \mathscr{M}$, together with *colaxity maps* $\varphi_{n,m}$:

$$\varphi_{n,m} : \mathcal{C}(\mathbf{n} + \mathbf{m}) \longrightarrow \mathcal{C}(\mathbf{n}) \otimes \mathcal{C}(\mathbf{m})$$

which must satisfy a coherence condition. He showed that if $\mathscr{M}$ is monoidal with the cartesian product then there is an equivalence of categories between colax morphisms $\mathcal{C} : (\Delta^+, +, \mathbf{0}) \longrightarrow \mathscr{M}$ and simplicial objects of $\mathscr{M}$ (see [60, Prop. 3.1.7]).

With these two ingredients Leinster defined an up-to-homotopy monoid $\mathcal{C}$ in $\mathscr{M}$ to be a colax monoidal functor, $\mathcal{C} : (\Delta^+, +, \mathbf{0}) \longrightarrow \mathscr{M}$ satisfying the Segal conditions, that is all the colaxity

[9] By usual 'Δ' we mean the topological one which doesn't contain the empty set

maps $\varphi_{n,m}$, and $\varphi_0 : \mathcal{C}(\mathbf{0}) \longrightarrow I$ are weak equivalences:

$$\varphi_{n,m} : \mathcal{C}(\mathbf{n}+\mathbf{m}) \xrightarrow{\text{weak. equiv}} \mathcal{C}(\mathbf{n}) \otimes \mathcal{C}(\mathbf{m}).$$

Just like in the strict case, the monoid structure is on the object $\mathcal{C}(\mathbf{1})$ but there is no specification of a mutliplication; one gets a multiplication using any weak inverse of the colaxity map $\varphi_{1,1}$:

$$\begin{tikzcd} & \mathcal{C}(\mathbf{1}) \\ \mathcal{C}(\mathbf{1}) \otimes \mathcal{C}(\mathbf{1}) \arrow[ur, dotted] & \mathcal{C}(\mathbf{2}) \arrow[l, "\varphi_{1,1}", "\sim"'] \arrow[u, "\text{canonical}"'] \end{tikzcd}$$

When $\mathscr{M}$ is $(\mathbf{Top}, \times)$ or $(\mathbf{SSet}, \times)$, we get a usual Segal 1-category with one object.

To define a Segal $\mathscr{M}$-category $\mathcal{C}$ having a set of objects X, we mimic Leinster's definition and define $\mathcal{C}$ as a colax morphism of 2-categories $\mathcal{C} : \mathscr{P}_{\overline{X}} \longrightarrow \mathscr{M}$ satisfying the suitable Segal conditions. Here $\mathscr{P}_{\overline{X}}$ is a 2-category which is the substitute of Δ_X in our context. The objects of $\mathscr{P}_{\overline{X}}$ are the elements of X, a 1-morphism from A to B is a sequence of elements $(A_0, ..., A_n)$ with $A_0 = A$ and $A_n = B$. The 2-morphism are parametrized by the morphisms of Δ^+ and consists of repeating of deleting some letters keeping A and B fixed. We have for example a 2-morphism $(A, E, B) \longrightarrow (A, B)$ which is parametrized by the unique map $\mathbf{2} \longrightarrow \mathbf{1}$ of Δ^+. The composition in $\mathscr{P}_{\overline{X}}$ is the concatenation of sequences side by side. When X has one element then $\mathscr{P}_{\overline{X}} \cong (\Delta^+, +, \mathbf{0})$ so we can consider $\mathscr{P}_{\overline{X}}$ as a big $(\Delta^+, +, \mathbf{0})$.

Such a colax morphism $\mathcal{C}$ comprises the following type of diagram in $\mathscr{M}$:

$$\begin{tikzcd} & \mathcal{C}(A,C) \\ \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \arrow[ur, dotted] & \mathcal{C}(A,B,C) \arrow[l, "\varphi_{ABC}"'] \arrow[u, "\text{canonical}"'] \end{tikzcd}$$

The Segal conditions require that all the colaxity maps (= the Segal maps) are weak equivalences:

$$\varphi : \mathcal{C}(A_0, ..., A_i, ..., A_k) \xrightarrow{\text{weak. equiv}} \mathcal{C}(A_0, ..., A_i) \otimes \mathcal{C}(A_i, ..., A_k).$$

In particular the map $\varphi_{ABC} : \mathcal{C}(A, B, C) \longrightarrow \mathcal{C}(A, B) \otimes \mathcal{C}(B, C)$ in the above diagram has to be a weak equivalence; this will allow us to consider a weak inverse and get a weak composition.

Once we've established the formalism it remains to develop a homotopy theory i.e put a model structure for Segal $\mathscr{M}$-categories when $\mathscr{M} = (\underline{M}, \otimes, I)$ is a monoidal model category with a non-cartesian product $\otimes$. When $\otimes$ is the cartesian product the theory is widely treated in the book of Simpson [79]. For classical Segal categories, one puts a model structure on the category of Segal precategories which are simply all the (colax) diagrams without demanding the Segal conditions. The model structure uses a "Segalification functor" **Seg** which assigns to every Segal precategory a Segal category which is homotopically minimal in some sense. Simpson formalized this in term of weak monadic projection.

The model structure on Segal precategories is inspired by earlier works of Jardine [44], Joyal [46] on simplicial (pre)-sheaves and has been understood to be a special case of Bousfield localization (see [79] and the many references therein). These techniques are now 'standard' for homotopy theorists and have been widely used in the literature.

In our case we would like to follow the same philosophy, that is to construct first a functor **Seg** assigning to every colax diagram $\mathcal{C} : \mathscr{P}_{\overline{X}} \longrightarrow \mathscr{M}$ a colax diagram $\mathbf{Seg}(\mathcal{C})$ satisfying the Segal conditions. But we are confronted with the following constraints.

- First the Segal maps $\varphi : \mathcal{C}(A,B,C) \longrightarrow \mathcal{C}(A,B) \otimes \mathcal{C}(B,C)$ are not 'natural', in the sense that they are maps going into a product where there is no *a priori* a way to have a projection on each factor. In fact for a noncartesian product $\otimes$, the product of two objects gives much more information when it's a a source of morphisms rather than a target.
- The other problem is that to construct **Seg**, the idea is to factorize each Segal map φ as a composite of a cofibration followed by a trivial fibration:

$$\varphi : \mathcal{C}(A,B,C) \overset{i}{\hookrightarrow} Q(A,B,C) \xrightarrow[\sim]{j} \mathcal{C}(A,B) \otimes \mathcal{C}(B,C).$$

  But since in the colax diagram $\mathcal{C}$ we have also a product of Segal maps, the above factorization leads us to consider a tensor product $j \otimes j$ of trivial fibrations in the model category $\mathscr{M}$; and demand that the result is again a trivial fibration.

  The behavior of the (trivial) fibrations with respect to the tensor product is not something that has really been studied by homotopy theorists. In fact it's surely not true in general that (trivial) fibrations are closed under tensor product. However there are some monoidal model categories in which it's true. An interesting example is the category $\mathbf{Ch}(\mathbf{R})$ of chain complex over a commutative ring $\mathbf{R}$ where the fibrations are just the degree-wise surjective morphisms (see [42, Prop. 4.2.13]).

These two constraints emerged during a discussion with Lurie and led us to change the direction of this thesis. We can conjecture though that if the fibrations are closed by tensor product, one should be able to construct such a functor **Seg**. This is an open problem and is reserved for the future.

In the second part of this thesis we set up another type of 'homotopical enrichment' with the notion of *co-Segal $\mathscr{M}$-categories*. This time the diagrams which provide a weak composition are of the form:

$$\begin{tikzcd} & \mathcal{C}(A,C) \arrow[d, "\wr"', "\text{weak.equiv}"] \\ \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \arrow[r, "\varphi_{ABC}"] & \mathcal{C}(A,B,C) \end{tikzcd}$$

As one can see, this diagram is obtained by reversing the morphisms in the Segal situation, hence the terminology 'co-Segal'. The 'co-Segal conditions demand that the vertical map is a weak equivalence whenever we have this type of diagram. There is another difference with the Segal formalism which is the fact that given $f \in \mathcal{C}(A,B)$ and $g \in \mathcal{C}(B,C)$ there is already a *precomposite* $\varphi(f \otimes g)$ in the "buffer-object" $\mathcal{C}(A,B,C)$.

A co-Segal $\mathscr{M}$-category having a set of objects X is defined as a <u>lax</u> morphism $\mathcal{C} : (\mathbb{S}_{\overline{X}})^{\text{2-op}} \longrightarrow \mathscr{M}$ satisfying the co-Segal conditions. Here we have $\mathbb{S}_{\overline{X}} \subset \mathscr{P}_{\overline{X}}$ and when X has one element then $\mathbb{S}_{\overline{X}} \cong (\Delta^+_{\text{epi}}, +, \mathbf{0})$.

These new structures are much easier to study and the rest of the thesis was to set up a homotopy theory of them. Just like for classical Segal categories, we put a model structure on the category of all lax diagrams in $\mathscr{M}$ indexed by $(\mathbb{S}_{\overline{X}})^{2\text{-op}}$ as X runs through **Set**. We called these diagrams *co-Segal precategories* and denoted by $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ the category they form. The main result here is the following:

Theorem. *Let $\mathscr{M}$ be a symmetric monoidal model category which is cofibrantly generated and such that all the objects are cofibrant. Then the following holds.*

1. *the category $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$, of co-Segal precategories admits a model structure which is cofibrantly generated,*
2. *fibrant objects are co-Segal categories,*
3. *If $\mathscr{M}$ is combinatorial then so is $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$.*

The above model structure is a Bousfield localization of a preexisting model structure; we use a "co-Segalification functor" and follow the same method as in the book of Simpson who proposed a *Direct localization method* (see [79, Chap. 9]).

Research perspectives

The two types of weakly enriched categories considered in this thesis give rise to higher categorical structures. Higher category theory constitutes on its own a large research area and we hope that this work will help to enlighten some facets of these fascinating structures. The ideas developed here can be pursued in different ways; from the foundational level to the applications in other areas of mathematics. We outline very briefly some of the directions that will be considered in the near future.

Homotopy theory of Segal $\mathscr{M}$-categories

We will consider the Segal case which was left aside for a moment. As said before, it seems that one needs the following extra hypothesis on the (symmetric) monoidal model category $\mathscr{M} = (\underline{M}, \otimes, I)$ (when $\otimes \neq \times$).

Hypothesis (Sketch)**.** The class of fibrations is closed under tensor product that is: $f \otimes g$ is a (trivial) fibration if f and g are simultaneously (trivial) fibrations.

As we mentioned earlier, this hypothesis holds in the category $\mathbf{Ch}(\mathbf{R})$ of chain complexes over a commutative ring $\mathbf{R}$; and in particular when $\mathbf{R}$ is a field. In this case we have the corresponding notion of Segal DG-category which was one of the motivation of bringing the Segal formalism in this context. The idea goes back to Toën's considerations in [93] to have a Segal like theory of higher linear categories which will be used in his approach to higher Tannaka duality. The theory should be iterative *à la* Simpson-Tamsamani; in particular one should have a new monoidal model category of Segal $\mathscr{M}$-categories with an appropriate tensor product $\boxtimes$, that can be used to enrich over.

Lurie introduced the notion of *stable* $(\infty, 1)$*-category* as an example of higher linear category. The reader can find in Wallbridge's PhD thesis [94] and the references therein, an account of higher Tannaka duality using these higher categorical structures.

Under the above hypothesis we can conjecture that

Conjecture. – *There exists a Segalification functor* **Seg** *from PreSeg($\mathscr{M}$) to itself which assigns to any colax diagram $\mathcal{C}$ a colax diagram* $\mathbf{Seg}(\mathcal{C})$ *satisfying the Segal conditions.*

– *The functor* **Seg** *comes equipped with a natural transformation* $\eta : \mathrm{Id} \longrightarrow \mathbf{Seg}$, *which is a level-wise cofibration; and such that if* $\mathcal{C}$ *satisfies the Segal conditions then* $\eta : \mathcal{C} \longrightarrow \mathbf{Seg}(\mathcal{C})$ *is a weak equivalence in PreSeg*($\mathscr{M}$) *(= level-wise weak equivalence).*

– *There are two model structures PreSeg*$(\mathscr{M})_{proj}$ *and PreSeg*$(\mathscr{M})_{Reedy}$ *on the category PreSeg*($\mathscr{M}$)*; these are called respectively projective and Reedy model structures. In each case the three classes of cofibrations, fibrations and weak equivalences are defined as in the case where* $\otimes = \times$ *(see [79]).*

Weak unity in co-Segal $\mathscr{M}$-categories

The definition of a co-Segal $\mathscr{M}$-category gives rise to a weakly enriched semi-category, which means that there is no identity morphism. But there is a natural notion of weak unity we are going to explain very briefly. This is the same situation as for A_∞-categories which arose with weak identity morphisms (see [35, 36]). If $\mathcal{C}$ is a co-Segal $\mathscr{M}$-category denote by $[\mathcal{C}]$ the co-Segal $\mathbf{ho}(\mathscr{M})$-category we get by the change of enrichment (=base change) $\mathrm{L} : \mathscr{M} \longrightarrow \mathbf{ho}(\mathscr{M})$. $[\mathcal{C}]$ is a strict co-Segal category which means that it's a semi-enriched $\mathbf{ho}(\mathscr{M})$-category. Then we can define

Definition (Sketch)**.** Say that a co-Segal $\mathscr{M}$-category $\mathcal{C}$ has weak identity morphisms if $[\mathcal{C}]$ is a classical enriched category (=with identity morphisms) over $\mathbf{ho}(\mathscr{M})$.

There is a natural question which is to find out whether or not it's relevant to consider a "direct" identity morphism i.e without using the base change $\mathrm{L} : \mathscr{M} \longrightarrow \mathbf{ho}(\mathscr{M})$. At this level we don't know for the moment if such consideration is 'natural'. Below we list some possible directions to address this question.

Possible directions

1. A first attempt will be to demand that for any object A of $\mathcal{C}$ there is a map $I_A : I \longrightarrow \mathcal{C}(A, A)$ such that for any object B the following commutes (up-to-homotopy ?):

$$\begin{tikzcd} I \otimes \mathcal{C}(A,B) \arrow[rr, "\cong"] \arrow[d, "I_A \otimes \mathrm{Id}"] & & \mathcal{C}(A,B) \arrow[d, "\wr"', "\text{weak.equiv}"] \\ \mathcal{C}(A,A) \otimes \mathcal{C}(A,B) \arrow[rr, "\varphi_{AAB}"] & & \mathcal{C}(A,A,B) \end{tikzcd}$$

   This will give the left invariance of I_A; the same type of diagram will give the right invariance. Note that we've limited the invariance to the '1-simplices' $\mathcal{C}(A, B)$ of $\mathcal{C}$ i.e we do not require such a diagram with $\mathcal{C}(A_0, ...A_n)$ with $n > 1$. There are two reasons that suggest this limitation. The first one comes from the fact that for unital A_∞-categories, the unity condition is only required for the binary multiplication 'm_2' (see for example Kontsevich-Soibelman [54, Sec. 4.2], Lyubashenko [67, Def. 7.3]).

   The other reason is that $\mathcal{C}(A, B)$ and $\mathcal{C}(A, ..., A_i, ..., B)$ have the same homotopy type (the co-Segal conditions); thus if $\mathcal{C}(A, B)$ is weakly invariant under I_A we should have the same thing for $\mathcal{C}(A, ..., A_i, ..., B)$. Finally we should mention that in the grand scheme of algebra, imposing further conditions reduces the class of objects.

2. Another alternative is to follow the ideas developed by Kock [53] of having weak units in higher category. Kock considered a category **'fat Δ^+'** to produce weak units. Since $\mathbb{S}_{\overline{X}}$ and $\mathscr{P}_{\overline{X}}$ are big versions of Δ^+ it's natural to find out if there is a corresponding notion of **'fat $\mathbb{S}_{\overline{X}}$ or fat $\mathscr{P}_{\overline{X}}$'**.

Homotopy theory for unital co-Segal categories

Having at hand the notion of weak identity arrows, we can define the "correct" notion of weak equivalence between co-Segal categories. So far a morphism $\sigma : \mathcal{C} \longrightarrow \mathcal{D}$ of co-Segal categories can be a weak equivalence only if the underlying function $\mathrm{Ob}(\sigma) : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{D})$ is an isomorphism of sets. This is not surprising since we cannot define a notion of being "essentially surjective" without (weak) identity maps. For unital co-Segal categories we will say that

Definition. A morphism $\sigma : \mathcal{C} \longrightarrow \mathcal{D}$ if *essentially surjective* if the induced map $[\sigma] : [\mathcal{C}] \longrightarrow [\mathcal{D}]$ is essentially surjective as a functor of strict enriched category over $\mathbf{ho}(\mathscr{M})$.

New model structure These new weak equivalences will change the model structure on $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$, so we will have to construct a new one which will take account of the identity morphisms.

A monoidal model category The model structure on $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ does not take account of the monoidal structure on $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$. One of the remaining pieces of work to do will be to include the monoidal structure in the model structure.

Segal, co-Segal and Strictification

It's an ancestral task to compare the 'new notions' with the existing ones. This principle is even more important in higher category theory, which is an area where the objects of study have different definitions and 'shadows'. The comparison of these theories and variant definitions is under active consideration by many people; results on this problem include the works of Barwick and Schommer-Pries [8], Bergner [15], Bergner-Rezk [16], Cheng [21], Lurie [66], Simpson [79], Toën [92] and others.

1. We will try then to compare the homotopy theory of Segal $\mathscr{M}$-categories, co-Segal $\mathscr{M}$-categories and strict $\mathscr{M}$-categories.
2. The method used by Bergner [17] which goes back to Badzioch [5] seems to apply directly to co-Segal categories; this will give a Quillen equivalence between $\mathscr{M}$-**Cat** and co-Segal $\mathscr{M}$-**Cat** (a *strictification*). However it's not clear that one should proceed in the same manner for the Segal case.
3. When $\mathscr{M}$ has a cartesian product, the comparison between Segal $\mathscr{M}$-categories and strict $\mathscr{M}$-categories is already treated (see [15], [79] and references therein).

Comparison with A_∞-categories, Stable $(\infty,1)$-categories When we consider a (Co)-Segal $\mathscr{M}$-category for $\mathscr{M} = (\mathbf{Ch}(\mathbf{R}), \otimes_{\mathbf{R}}, \mathbf{R})$, the category of chain complexes over a commutative ring $\mathbf{R}$, we get a weak differential graded (=DG) category which we think as a higher linear category. As we said before the definition of higher linear structures is one of the main motivation of this thesis. The two notions which are now used as weak linear categories are:

- the notion of A_∞-category introduced by Fukaya and Kontsevich (see [34],[37], [55],[54],[48], [67]) which goes back to Stasheff's notion of A_∞-algebra [82].
- Lurie's notion of *stable* ∞-category (see [65]).

1. We will try to have a deep understanding of the connections between linear co-Segal categories, A_∞-categories and stable ∞-categories on the homotopic theoretical point of view.

2. Having a hypothetical equivalence between linear co-Segal categories and A_∞-categories will give a less combinatorial approach of non commutative algebraic geometry and the many related fields. We can also hope that it will provide some simplification of many constructions in the 'A_∞-world'.

3. Almost all the notions of classical enriched categories can be defined for both co-Segal and Segal $\mathscr{M}$-categories. For $\mathscr{M} = (\mathbf{Ch}(\mathbf{R}), \otimes_{\mathbf{R}}, \mathbf{R})$ the notion of DG-module and A_∞-module play a central role in the program of non commutative algebraic geometry. In enriched category theory, DG-modules are the equivalent notion of $\mathscr{M}$-valued presheaf and are special cases of the notions of *bimodules* also known as (enriched) *distributors* or *profunctors*.

   The considerations of these objects in the co-Segal world will lead immediately to some obvious questions such as:
   How shall we interpret a co-Segal category being equivalent to a smooth dg-manifold in the sense of Kontsevich-Soibelman [54] ?

Algebraic Geometry in co-Segal settings

Pursuing Deligne's idea of algebraic geometry in monoidal categories

In [26] Deligne proposed a formal categorical framework for doing algebraic geometry in a tannakian categories; the guiding example is the category $\mathrm{Rep}(G)$ of linear representations of an affine $\mathbf{k}$-group scheme G, for a field $\mathbf{k}$.

Deligne defines a commutative ring object R in a tannakian category $\mathcal{T}$, to be a commutative monoid with unit. He then defines the category of *affine $\mathcal{T}$-schemes* to be the opposite category of commutative ring objects with unit in $\mathcal{T}$, like in the classical case. For $\mathcal{T} = \mathrm{Rep}(G)$, affine $\mathcal{T}$-schemes correspond to classical affine $\mathbf{k}$-schemes equipped with right action of G (G-modules). In this setting any affine $\mathbf{k}$-scheme is an affine $\mathcal{T}$-scheme because we can take the trivial action of G. And any map between two affine $\mathbf{k}$-schemes induces in a tautological way a map of affine $\mathcal{T}$-scheme. Deligne points out that many of the classical notion and constructions can be given for $\mathcal{T}$-schemes: fiber products, group schemes, flatness, (quasi-coherent) modules, torsors, etc.

In a Tannakian category there is an *internal* $\underline{\mathrm{Hom}}$ which makes $\mathcal{T}$ an enriched category over itself. By the Grothendieck anti-equivalence, we know that a morphism of classical affine $\mathbf{k}$-schemes is induced by a morphism of commutative $\mathbf{k}$-algebras (which goes in the opposite sense). But morphisms of $\mathbf{k}$-algebras form only a set and have no extra structure; we loose for example the vector space structure of the morphisms between the underlying $\mathbf{k}$-vector spaces (of the $\mathbf{k}$-algebras).

For $\mathcal{T} = \mathrm{Rep}(G)$, when we view a classical affine $\mathbf{k}$-schemes as affine $\mathcal{T}$-schemes, we keep track of their geometric nature while 'hosting' the morphisms between them into a richer object (the linear maps between the underlying vector spaces). We view this situation as an enrichment of the category of affine $\mathbf{k}$-schemes over $\mathcal{T}$.

As far as I understand the philosophy of motives, we try to 'host' the category of algebraic variety in a (higher) linear category which is somehow (homotopically) minimal. Here by 'host' I mean find a enrichment, thus new $\underline{\mathrm{Hom}}$ in coefficient in a linear category, that contains the 'old' morphisms as a 'simple' sub-object. This attempt of putting an extra structure on morphism between schemes can be seen in the work of Voevodsky who considered the category 'Cor' of correspondences between schemes in his definition of Voevodsky motives (see [69]).

There are different type of homotopy theory in algebraic geometry that are now studied, such as $\mathbf{A}^1$-homotopy theory or Etale Homotopy theory. The consideration of Deligne's ideas in these theories will undoubtedly involves higher linear structures as pointed out, for example, by Toën [93]. We hope that the notions developed here will help to understand some facets of Grothendieck's dream about *motives*.

Such consideration will require the following notions:

1. A good definition of a symmetric monoidal co-Segal category. Usually one would define it as a Γ-object in the monoidal category of unital co-Segal categories;

2. A definition of a (neutral) Tannakian co-Segal category.

3. It seems that a commutative co-Segal DG-algebra should be equivalent to an E_∞-ring. E_∞-rings and simplicial rings are in the center of *Homotopical and Derived Algebraic Geometry*.

Organization of the manuscript

All the chapters are meant to be readable independently. Each chapter begins with a short introduction which gives an overview of the discussion.

- Chapter 2 contains essentially the author's preprint [3] in which we set up the definition of Segal enriched categories. All the definitions are given in the setting of enrichment over a 2-category $\mathscr{M}$. We show how the definition generalizes the work of Leinster [60] and that the strict case correspond to the classical $\mathscr{M}$-categories.

- The main results of the thesis appear in chapter 3 which is about co-Segal categories. To get the model structure on $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$, we treat first the case $\mathscr{M}_{\mathbb{S}}(X)$ consisting of co-Segal precategories having a fixed set of objects X; then we extend the model structure using a Roig-Stanculescu theorem of having a model structure on the total category $\mathscr{E}$ of a Grothendieck bifibred category $\mathbf{p} : \mathscr{E} \longrightarrow \mathscr{B}$.

- Chapter 4 contains the different appendices.

Warning: Δ and Δ^+

In this thesis two different categories "**Delta**" are used. We include this short notice to warn the reader about the potential confusion. We outline very briefly some known facts about these two "Delta".

- Δ is the category of finite ordinals $\underline{n} = \{0, ..., n\}$, **without the empty set**. The morphisms are the nondecreasing functions.
- Δ^+ is the category of all finite ordinals $\mathbf{n} = \{0, ..., n-1\}$, **with the empty set** $(= \mathbf{0})$. The morphisms are also the nondecreasing functions.

So the 'underlined n', $\underline{n} = \{0, ..., n\}$, represents an object of Δ corresponding to the first $(n+1)$ natural numbers; while the 'bold n', $\mathbf{n} = \{0, ..., n-1\}$, represents an object of Δ^+ corresponding to the first n natural numbers. In both $\underline{n}$ and $\mathbf{n}$ we consider the natural order.

From Δ to Δ^+

If $\underline{n} = \{0, ..., n\}$ and $\underline{m} = \{0, ..., m\}$ are two objects of Δ, say that $f : \underline{n} \longrightarrow \underline{m}$ *preserves the extremities* if:

$$f(0) = 0 \quad \text{and} \quad f(n) = m.$$

Let $\Omega \subset \Delta$ be the subcategory having the same objects as Δ and whose morphisms are the ones that preserve the extremities. Then we claim that:

Claim. There is an isomorphism of categories between Ω^{op} and Δ^+.

We will not give a detailed proof of the claim but we will give the main idea. To show that the claim holds we explicitly construct an isomorphism $T : \Delta^+ \longrightarrow \Omega^{op}$.

On the objects, T maps $\mathbf{n} = \{0, ..., n-1\}$ to $\underline{n} = \{0, ..., n\}$. To see what T does on morphisms we need to go back to Mac Lane's description of the category Δ^+ [68, p.172]. The category Δ^+ has a monoidal structure given by the the ordinal addition $+$. Mac Lane showed that the arrows in Δ^+ are generated by addition and composition from $\mu : \mathbf{2} \longrightarrow \mathbf{1}$ and $\eta : \mathbf{0} \longrightarrow \mathbf{1}$. Therefore in order to define T we simply have to give $T(\mu)$ and $T(\eta)$.

The maps $T(\mu)$ and $T(\eta)$ are, respectively, the opposite of the following maps of Ω:

- $T(\mu)^{op} : \{0, 1\} \longrightarrow \{0, 1, 2\}$, the unique map that takes 0 to 0 and 1 to 2.
- $T(\eta)^{op} : \{0, 1\} \longrightarrow \{0\}$, the unique constant map.

We leave the reader to check that the functor T we get is an isomorphism.

Chapter 2

Segal Enriched Categories

2.1 Introduction

In this chapter we present the theory of Segal enriched categories and provide some situations where they appear naturally. We give our definitions in the context of enrichment over a bicategory (2-category). The theory of enriched categories over a 2-category generalizes the classical theory of enrichment over a monoidal category and also gives rise to various points of view in many classical situations. Walters [96] showed for example that a sheaf on a Grothendieck site $\mathcal{C}$ was the same thing as *Cauchy-complete* enriched category over a bicategory Rel($\mathcal{C}$) built from $\mathcal{C}$. Later Street [86] extended this result to describe *stacks* as enriched categories with extra properties and gave an application to nonabelian cohomology.

Both Street and Walters used the notion of *bimodule* (also called *distributor, profunctor or module*) between enriched categories. The notion of *Cauchy completeness* introduced by Lawvere [58] plays a central role in their respective work. In fact 'Cauchy completeness' is a property of *representability* and is used there to have the restriction of sections and to express the *descent conditions.*

This characterization of stacks as enriched categories is close to the definition of a stack as fibered category satisfying the descent conditions. One can obviously adapt their result with the formalism we develop here. We can consider a Segal version of their results using the notion of *Segal site* of Toën-Vezzosi [91] or Lurie's notion of ∞-site [66].

By the Giraud characterization theorem [38] we know that a sheaf is an object of a Grothendieck topos. Then the results of Walters and Street say that a Grothendieck (higher) topos is equivalent to a subcategory of $\mathscr{M}$-Cat for a suitable base $\mathscr{M}$. A *Segal topos* of Toën-Vezzosi should be a subcategory of the category of Segal-enriched categories over a base $\mathscr{M}$. Street [85] has already provided a characterization theorem of the bicategory of stacks on a site $\mathcal{C}$, then a *bitopos.* Here again one may propose a characterization theorem for Segal topoi of Toën-Vezzosi by suitably adapting the results of Street.

More generally we can extend the ideas of Jardine [45], Thomason [88] followed by, Dugger [28], Hirschowitz-Simpson [41], Morel-Voevodsky [70], Toën-Vezzosi [89, 90] and others, who developed a homotopy theory in situations, e.g in algebraic geometry, where the notion of homotopy was not *natural.* The main ingredients in these theories are essentially the use of *simplicial presheaves* with their (higher) generalizations, and the *functor of points* initiated by Grothendieck.

Enriched categories appear naturally there because, for example, the category of simplicial presheaves is a *simplicially enriched* category i.e an **SSet**-category. An interesting task will be, for example to 'linearize' the work of Toën-Vezzosi and develop a Morita theory in 'Segal settings'. This will be discussed in future work.

Plan

Finding a 'big' Δ^+

We start by introducing the new tool which generalizes the monoidal category $(\Delta^+, +, 0)$. The reason for this approach is the fact that this category $(\Delta^+, +, 0)$ is known to contain the universal monoid which is the object 1. More precisely, Mac Lane [68] showed that a monoid V in a monoidal category $\mathscr{M}$ can be obtained as the image of 1 by a **monoidal functor** $\mathscr{N}(V) : (\Delta^+, +, 0) \longrightarrow \mathscr{M}$. And as mentioned previously a monoid is viewed as an $\mathscr{M}$-category with one object, so we can consider the functor $\mathscr{N}(V)$ as the **nerve** of the [1] corresponding category whose hom-object is V.

From this observation it becomes natural to find a big tool which will be used to 'depict' many monoids and bimodules in $\mathscr{M}$ to form a general $\mathscr{M}$-category. This led us to the notion of: **2-path-category associated to a 1-category** $\mathcal{C}$ (see Proposition-Definition 2.2.1).

We construct from any 1-category $\mathcal{C}$ a 2-category $\mathscr{P}_{\mathcal{C}}$ which is characterized by the following universal property: for any 2-category $\mathscr{M}$ there is an isomorphism of sets between:

- the set of *lax morphisms*, in the sense of Bénabou [10], from $\mathcal{C}$ to $\mathscr{M}$
- the set of *strict homomorphisms* from $\mathscr{P}_{\mathcal{C}}$ to $\mathscr{M}$.

Furthermore for the unit category $\mathbb{1}$ we have a monoidal isomorphisms $\mathscr{P}_{\mathbb{1}} \cong (\Delta^+, +, 0)$.

Similar constructions have been considered by Dawson, Paré and Pronk for double categories (see [25]). One can compare the *Example 1.2 and Remark 1.3* of their paper with the fact that here we have: $\mathscr{P}_{\mathbb{1}}$ 'is' $(\Delta^+, +, 0)$.

As mentioned above the idea of enrichment will be to consider special types of morphisms from $\mathscr{P}_{\mathcal{C}}$ to other bicategories. We will see that when $\mathcal{C}$ is $\overline{X}$, $\mathscr{P}_{\overline{X}}$ will replace Lurie's Δ_X and will be used to define Segal enriched categories. This will generalize the definition of up-to-homotopy monoid in the sense of Leinster which may be called up-to-homotopy *monad* in the language of bicategories.

The fact that $\mathcal{C}$ is an arbitrary small category allows us to consider geometric situations when $\mathcal{C}$ is a Grothendieck site and in this way we can 'transport' geometry in enriched category context.

The environment

Before giving the definition of enrichment, we describe the type of category $\mathscr{M}$ which will contain the hom-objects $\mathcal{C}(A, B)$ (see Definition 2.2.4).

We will work with a bicategory $\mathscr{M}$ equipped with a class $\mathscr{W}$ of 2-cells satisfying the following properties.

- Every invertible 2-cell of $\mathscr{M}$ is in $\mathscr{W}$, in particular 2-identities are in $\mathscr{W}$,
- $\mathscr{W}$ is stable under horizontal composition,
- $\mathscr{W}$ has the vertical '3 out of 2' property.

[1] the category is unique up to isomorphism.

Such a pair $(\mathscr{M}, \mathscr{W})$ will be called *base* as 'base of enrichment'. When $\mathscr{M}$ has one object, therefore a monoidal category, we get the same environment given by Leinster [61].

Since we work with bicategories, $\mathscr{M}$ can also be :

- any 1-category viewed as a bicategory with all the 2-morphisms being identities,
- the '2-level part' of a strict ∞-category.

Note. To define Segal enriched categories, $\mathscr{W}$ will be a class of 2-morphisms called **homotopy 2-equivalences**. In this case, following the terminology of Dwyer, Hirschhorn, Kan and Smith [33] one may call $\mathscr{M}$ together with $\mathscr{W}$ 'a homotopical bicategory'.

Relative enrichment

With the previous materials we give the definition of **relative enrichment** in terms of **path-objects** (*Definition* 2.2.5). One can compare the following definition with *Definition* 2.3.2.

Definition.

Let $(\mathscr{M}, \mathscr{W})$ be a base of enrichment. A *path-object* of $(\mathscr{M}, \mathscr{W})$ is a pair $(\mathcal{C}, F)$, where $\mathcal{C}$ is a small category and $F = (F, \varphi)$ a **colax morphism** of Bénabou:

$$F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$$

such that for any objects A, B, C of $\mathcal{C}$ and any pair (t, s) in $\mathscr{P}_{\mathcal{C}}(B, C) \times \mathscr{P}_{\mathcal{C}}(A, B)$, all the 2-cells

$$F_{AC}(t \otimes s) \xrightarrow{\varphi(A,B,C)(t,s)} F_{BC}(t) \otimes F_{AB}(s)$$

$$F_{AA}([0, A]) \xrightarrow{\varphi_A} I'_{FA}$$

are in $\mathscr{W}$. Such a colax morphism will be called a $\mathscr{W}$**-colax morphism** and $\varphi(A, B, C)$ will be called **'colaxity maps'**.

- When $\mathscr{W}$ is a class of homotopy 2-equivalences, then $(\mathcal{C}, F)$ will be called **a Segal path-object** of $\mathscr{M}$ and $F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ will give a relative enrichment of $\mathcal{C}$ over $(\mathscr{M}, \mathscr{W})$.
- We will say for short that $(\mathcal{C}, F)$ is a **'$\mathcal{C}$-point'** or a **'$\mathcal{C}$-module'** of $\mathscr{M}$. In [4] a duality theory of enrichment is developed and we will prefer the terminology $\mathcal{C}$-module.
- When $\mathcal{C} = \overline{X}$ a Segal $\overline{X}$-point of $(\mathscr{M}, \mathscr{W})$ is called a Segal $\mathscr{M}_{\mathscr{W}}$-category.

The reason we consider **colax** morphisms, is the fact that 'colax' is the appropriate replacement of **simplicial** (see *Proposition* 2.3.4) when working with general monoidal categories, in particular for noncartesian monoidal ones. In fact when $\mathscr{M}$ is monoidal for the cartesian product, the *colaxity maps* will give *all* the *face maps* and we can consider a path-object of $(\mathscr{M}, \mathscr{W})$ as a 'super-simplicial object' of $(\mathscr{M}, \mathscr{W})$ 'colored by' $\mathcal{C}$. In the Segal case we may consider F as a '$\mathcal{C}$-homotopy coherent nerve'

But there are also other interpretations that arise when we consider special bases $(\mathscr{M}, \mathscr{W})$. Some times in the Segal case we can consider F as a *homotopic $\mathscr{M}$-representation* of $\mathcal{C}$ (see 2.3.5).

One of the advantages of having enrichment as a morphism is the fact that classical operations such as *base change* will follow immediately. In addition to that, we can use the notions of **transformations** and **modifications** to have a first categorical structure of the 'moduli space' of relative enrichments of $\mathcal{C}$ over $\mathscr{M}$. This is discussed in *sections* 2.4 and 2.4.4.

Examples

In *section* 2.3 we show that the formalism we've adopted covers the following situations.

Category theory

- Up-to-homotopy monoid in the sense of Leinster (*Proposition* 2.3.5).

- Simplicial object (*Proposition* 2.3.5).

- Classical enriched categories (*Proposition* 2.3.7).

- Segal categories in the sense of Hirschowitz-Simpson (*Proposition* 2.3.9).

- Linear Segal categories are defined in Definition 2.3.10

Nonabelian cohomology

- In section 2.3.5 we remarked that for a group G in $(\mathbf{Set}, \times)$, a G-torsor, e.g EG, is the same thing as a "full" G-category. The cocyclicity property of torsors reflect a 'degenerated' composition i.e the composition maps c_{ABC} are identities. We recover the classification role of BG because to define a G-category we take a path-object of BG.

  This remark can be extended to the general case of a group-object using the functor of points.

- For a nonempty set X, the coarse category $\overline{X}$ is the 'EG' of some group G (see Remark 2.2.5). And as we shall see we will take $\overline{X}$-point of $(\mathscr{M}, \mathscr{W})$ to define enriched categories. When $\mathscr{M}$ is a 1-category e.g **Vect** the category of vector space, an EG-point of **Vect** will give in some case a representation of G (all elements of G, which correspond to the objects of EG, are sent to the same object of **Vect**).

- In section 2.3.5 we give an example of 1-functor considered as a (free) path-object. We want to consider a *parallel transport functor* as a path-object. In the Segal case we will have *homotopic holonomy.*

- Finally we've introduced some material for the future with the notion of *quasi-presheaf* (see 2.3.6). We define a *quasi-presheaf* on $\mathcal{C}$ to be a Segal $\mathcal{C}^{op}$-point of $(\mathscr{M}, \mathscr{W})$. This is not simply a 'generalization-nonsense' of classical presheaves. We want to consider, for example, the Grothendieck anti-equivalence between affine schemes and commutative rings as a 'co-enrichment' having a *reconstruction property*, then a *good* enrichment (see Example 2.3.15). A detailed account will appear in [4].

  Another example is to consider any cohomology theory on $\mathcal{C}$ as a family of 'free' $\mathcal{C}^{op}$-points i.e relative enrichments of $\mathcal{C}^{op}$. We hope that using the *machine* of 'Segal enrichment': base changes of $\mathcal{C}^{op}$-points, enriched Kan extension, Segal categories etc, together with model categories, we can understand some facets of *motivic cohomology.*

  These considerations will require an appropriate *descent theory* of relative enrichment which will be discussed in [4].

Morphisms, Bimodules and Reduction

In section 2.4 we've revisited some classical notions adapted to our formalism. We've tried as much as possible to express these notions in terms of morphisms of bicategories. The idea is to have everything 'at once' using path-objects.

– Given two path-objects $F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ and $G : \mathscr{P}_{\mathcal{D}} \longrightarrow \mathscr{M}$, we define first an $\mathscr{M}$-premorphism to be a pair $\Sigma = (\Sigma, \sigma)$ consisting of a functor $\Sigma : \mathcal{C} \longrightarrow \mathcal{D}$ together with a transformation of (colax) morphisms of bicategories $\sigma : F \longrightarrow G \circ \mathscr{P}_{\Sigma}$ (Definition 2.4.2). An $\mathscr{M}$-morphism is a special type of an $\mathscr{M}$-premorphism.

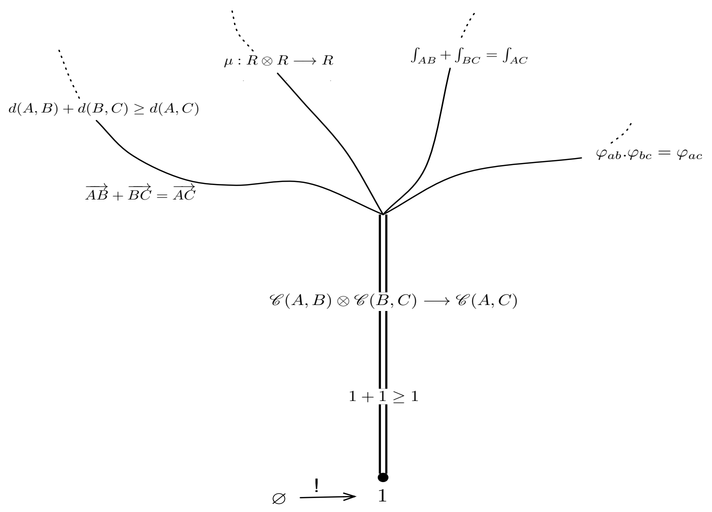
– We define bimodules (also called "distributors", "profunctors" or "modules") in terms of path-object (Definition 2.4.3).

– Finally in Proposition 2.4.10, we introduce a bicategory $\mathscr{W}^{-1}\mathscr{M}$ which is rougly speaking the 'secondary' Gabriel-Zisman localization of a base $(\mathscr{M}, \mathscr{W})$ with respect to $\mathscr{W}$. With this bicategory $\mathscr{W}^{-1}\mathscr{M}$ we can *reduce* any Segal point to its homotopic part.

Yoga of enrichment

" In mathematics, there are not only theorems. There are, what we call, 'philosophies' or 'yogas', which remain vague. Sometimes we can guess the flavor of what should be true but cannot make a precise statement. When I want to understand a problem, I first need to have a panorama of what is around it. A philosophy creates a panorama where you can put things in place and understand that if you do something here, you can make progress somewhere else. That is how things begin to fit together. "

Pierre Deligne, in *Mathematicians*, Mariana Cook, PUP, 2009, p156.



A Big Bang : From the *empty set* to other structures

In the following we present some facets of enrichment and give some interpretations [2] around it. We hope that it will help a reader who is unfamiliar with Segal categories to have a panoramic understanding of the subject.

The general picture

In a classical enriched category $\mathcal{C}$, we have :

- compositions $c_{ABC} : \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \longrightarrow \mathcal{C}(A,C)$: thought as *partial multiplications*
- an identity map $I_A : I \longrightarrow \mathcal{C}(A,A)$; then the pair think $[\mathcal{C}(A,A), I_A]$ is like a *pointed space with multiplication* e.g $\pi_1(x,X)$, $\Omega_x X$, etc.

In a Segal category **there is no prescription**, in general, of the previous data but we have the following diagrams.

1)

$$\begin{tikzcd} & \mathcal{C}(A,B,C) \arrow[dl, "\text{weak equiv.}"'] \arrow[dr, "\text{canonical}"] & \\ \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) & & \mathcal{C}(A,C) \end{tikzcd}$$

2)

$$\begin{tikzcd} & \mathcal{C}_{[0,A]} \arrow[dl, "\text{weak equiv.}"'] \arrow[dr, "\text{canonical}"] & \\ I & & \mathcal{C}(A,A) \end{tikzcd}$$

The two types of maps

$$\mathcal{C}(A,B,C) \longrightarrow \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \quad \text{and} \quad \mathcal{C}_{[0,A]} \longrightarrow I$$

are called 'Segal maps' and they are required to be *weak equivalences.*

The idea is that when these maps are isomorphisms (strong equivalences) then using their respective inverses we can run these diagrams from the left to the right and we will have the data of a classical category.

But when the Segal maps are not isomorphisms but only weak equivalences then we can think that each weak inverse of the previous maps will give a 'quasi-composition' and a 'quasi-identity map'. It turns out that Segal categories are more general than classical categories and appear to be a good tool for homotopy theory purposes.

Note. In this paper the Segal maps will be the 'colaxity maps'.

[2] or philosophies

Why *relative* enrichment ?

For a given small category $\mathcal{C}$ and a bicategory $\mathscr{M}$ we define a relative enrichment of $\mathcal{C}$ over $\mathscr{M}$ to be a morphism of bicategories $F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ satisfying some extra conditions which can be interpreted as 'generalized Segal conditions'.

To understand the meaning of 'relative' it suffices to consider the trivial case where $\mathscr{M}$ is a 1-category viewed as a bicategory with identity 2-morphisms. In this case the morphism F is determined by a 1-functor $F_{|\mathcal{C}} : \mathcal{C} \longrightarrow \mathscr{M}$ (see Observations 2.2.1).

And the idea is to observe that given any functor $G : \mathcal{C} \longrightarrow \mathscr{M}$ then we can form the category $G[\mathcal{C}]$ described as follows.

- $\mathrm{Ob}(G[\mathcal{C}]) = \mathrm{Ob}(\mathcal{C})$
- For each pair of objects (A, B) we take the morphism to be the image [3] of the function

$$G_{AB} : \mathcal{C}(A, B) \longrightarrow \mathscr{M}(GA, GB)$$

- The composition is defined in the obvious way.

In this situation we will consider $G[\mathcal{C}]$ as a relative enrichment of $\mathcal{C}$ over $\mathscr{M}$. One can interpret $G[\mathcal{C}]$ as *a copy of* $\mathcal{C}$ *of type* $\mathscr{M}$.

As usual we have the following philosophical questions.

- What is the 'best copy' of $\mathcal{C}$ of type $\mathscr{M}$?
- Does such a 'motivic copy' of $\mathcal{C}$ exist for a given $\mathscr{M}$?
- Which $\mathscr{M}$ shall we consider to have much informations about $\mathcal{C}$?

The *machine* of enriched categories allows us to do base changes i.e move $\mathscr{M}$, and we hope that the ideas of *Segal-like enrichment* can guide us, using homotopy theory, to find an answer to those questions.

In the previous example we can see that we have the usual factorization of the functor G as

$$\mathcal{C} \xrightarrow{j} G[\mathcal{C}] \xrightarrow{i} \mathscr{M}$$

where j is full and i is faithful.

If the functor is an equivalence then we may say that $G[\mathcal{C}]$ is a 'good copy' of $\mathcal{C}$ of type $\mathscr{M}$ and if G reflects isomorphisms then $G[\mathcal{C}]$ will provide a good copy of some subcategory of the *interior of* $\mathcal{C}$, etc. We can also form a category whose set of objects is the image of the function $G : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathscr{M})$; the morphisms are those 'colored' or coming from $\mathcal{C}$. In this way we will form a category which *lives* in $\mathscr{M}$.

With this point of view, we can also consider that enrichment over $\mathscr{M}$ is a process which *enlarges* $\mathscr{M}$. In fact it's well known that some properties of $\mathscr{M}$ are *transferred* to $\mathscr{M}$-Cat and $\mathscr{M}$-Dist.

The recent work of Lurie [64] on cobordism hypothesis (framed version) says roughly speaking that a *copy of* $\mathbf{Bord}_n^{\mathrm{fr}}$ which respects the monoidal structure and the symmetry i.e a symmetric

[3] As G may not be faithful we need to take the image to have a set

monoidal functor, is determined by the copy of the point which must be a *fully dualizable object*. This reflect the fact that $\mathbf{Bord}_n^{\mathrm{fr}}$ is in some sense *built* from the point using bordisms and disjoint unions.

Remark 2.1.1. To have the classical theory of enriched categories we will consider the case where $\mathcal{C}$ is of the form $\overline{X}$ (see 2.2.5) for a nonempty set X and $\mathscr{M}$ a bicategory with one object, hence a monoidal category.

A 'Big Bang'

> " Will mathematics merely become more sophisticated and specialized, or will we find ways to drastically simplify and unify the subject? Will we only build on existing foundations, or will we also reexamine basic concepts and seek new starting-points? Surely there is no shortage of complicated and interesting things to do in mathematics. But it makes sense to spend at least a little time going back and thinking about *simple* things."
>
> John C. Baez, James Dolan, *From Finite Sets to Feynman Diagrams* [6] p2.

If we look closely the composition '$\mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \longrightarrow \mathcal{C}(A,C)$' in any category, we can see the similarity with the other classical formulas such as:

- $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$: basic geometry
- $d(A,B) + d(B,C) \geq d(A,C)$: triangle inequality
- $\varphi_{AB} \cdot \varphi_{BC} = \varphi_{AC}$: cocyclicity of transition functions for a vector bundle, etc.

In fact all of these formulas can be described in terms of enriched categories and base changes. For example Lawvere [58] remarked that the *triangle inequality* is the composition in a metric space when considered as an enriched category over $(\overline{\mathbb{R}}_+, +, 0, \geq)$.

We've used the terminology 'Big bang' because many structures are encoded in this way by enriched categories. But enriched categories are defined using a big version of the category Δ. And Δ is itself built from 1, which in turn can be taking to be $\{\varnothing\}$.

It appears that almost *everything* comes from the *empty* ...

2.2 Path-Objects in Bicategories

2.2.1 The 2-path-category

We follow the notation of Deligne [27], and denote here by Δ^+ the "augmented" category of <u>all</u> finite totally ordered sets, including the empty set. We will denote by Δ the"topological" one, which does not contain the empty set and which is commonly used to define simplicial objects.

Recall that the objects of Δ^+ are ordinal numbers $n = \{0, ..., n-1\}$ and the arrows are nondecreasing functions $f : n \longrightarrow m$. Δ^+ is a monoidal category for the *ordinal addition*, has an initial object 0 and a terminal object 1. The object 1 is a "universal" monoid in the sense that any monoid in a monoidal category $\mathscr{M}$ is the image of 1 under a *monoidal functor* from Δ to $\mathscr{M}$. The reader can find this result and a complete description of Δ^+ in [68].

Warning. The category Δ^+ we consider here is denoted by Leinster [61] and MacLane [68] as Δ.

Proposition-Definition 2.2.1. [2-Path-category] *Let $\mathcal{C}$ be a small category.*

1. *There exists a 2-category $\mathscr{P}_{\mathcal{C}}$ having the following properties:*

   – *the objects of $\mathscr{P}_{\mathcal{C}}$ are the objects of $\mathcal{C}$,*

   – *for every pair (A, B) of objects, $\mathscr{P}_{\mathcal{C}}(A, B)$ is posetal and is a category over Δ i.e we have a functor called **length** or **degree***

$$\mathscr{L}_{AB} : \mathscr{P}_{\mathcal{C}}(A, B) \longrightarrow \Delta^+$$

   – *0 is in the image of $\mathscr{L}_{AB}$ if and only if $A = B$; and in this case $\mathscr{L}_{AA}$ becomes a* <u>*monoidal*</u> *functor with the composition.*

2. *if $\mathcal{C} \cong \mathbf{1}$, say $Ob(\mathcal{C}) = \{\boldsymbol{O}\}$ and $\mathcal{C}(\boldsymbol{O}, \boldsymbol{O}) = \{\mathrm{Id}_{\boldsymbol{O}}\}$, we have* <u>*monoidal*</u> *isomorphism:*

$$\mathscr{P}_{\mathcal{C}}(\boldsymbol{O}, \boldsymbol{O}) \xrightarrow{\sim} (\Delta^+, +, 0)$$

3. *the operation $\mathcal{C} \mapsto \mathscr{P}_{\mathcal{C}}$ is functorial in $\mathcal{C}$:*

$$\mathscr{P}_{[-]} : \mathbf{Cat}_{\leq 1} \longrightarrow \textbf{2-Cat}$$
$$\mathcal{C} \xrightarrow{F} \mathcal{D} \longmapsto \mathscr{P}_{\mathcal{C}} \xrightarrow{\mathscr{P}_F} \mathscr{P}_{\mathcal{D}}$$

where $\mathbf{Cat}_{\leq 1}$ and **2-Cat** *are respectively the 1-category of small categories and the category of 2-categories (and homomorphisms).*

Proof. The construction of $\mathscr{P}_{\mathcal{C}}$ is given in Definition 4.7.2. Below we give a short description.

– The objects of $\mathscr{P}_{\mathcal{C}}$ are the objects of $\mathcal{C}$;

– a 1-morphism from A to B is a chain of composable morphisms $A \longrightarrow A_1 \longrightarrow \cdots \longrightarrow B$; we shall denote such chain as $[n, A \longrightarrow A_1 \longrightarrow \cdots \longrightarrow B]$ where n represents the number of arrow in the chain;

– the composition is the concatenation of chains;

– the identity of A is the chain $(A) = [0, A]$;

– the 2-morphisms are parametrized by the morphism of Δ. For example we have the following 2-morphisms which somehow generate all the other ones:

$$\begin{tikzcd} & X \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow"] & \\ A \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & {} & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\mathrm{Id}_A"] \arrow[d, phantom, "\Uparrow"', shift left=4ex] & A \\ (A) & \end{tikzcd}$$

In the above diagrams the left one is a 2-morphism $(f, g) \longrightarrow (g \circ f)$ which is parametrized by the unique map $\sigma_0 : \mathbf{2} \longrightarrow \mathbf{1}$ of Δ; and the one on the right is a 2-morphism $(A) \longrightarrow (\mathrm{Id}_A)$ parametrized by the unique map $\mathbf{0} \longrightarrow 1$ of Δ.

■

From the construction of $\mathscr{P}_{\mathcal{C}}$ one can easily show that (see [22] and references therein for a general statement).

Corollary 2.2.2. *For any* 2*-category* $\mathscr{M}$ *we have an isomorphism of sets functorial in* $\mathcal{C}$*:*

$$\text{Lax}(\mathcal{C}, \mathscr{M}) \cong \text{Hom}(\mathscr{P}_{\mathcal{C}}, \mathscr{M})$$

where the left hand side is the set of lax morphisms from $\mathcal{C}$ *to* $\mathscr{M}$ *while the right hand side is the set of strict homomorphisms in the sense of Bénabout [10].*

Remark 2.2.1.

1. The 'path-functor' as presented above doesn't extend immediately to a 2-functor because natural transformations in Cat are not sent to transformation in Bicat.
2. We can fix the problem either by using co-spans in $\mathscr{P}_{\mathcal{C}}(A, B)$ or by localizing each $\mathscr{P}_{\mathcal{C}}(A, B)$ with respect to the class of maps which correspond to compositions in $\mathcal{C}$, then 'reversing the composition'.
3. Another solution could be to work in the area of Leinster's fc-multicategories instead of staying in Bicat, but we won't that do here. In fact $\mathscr{P}_{\mathcal{C}}$ carries a good enough combinatoric for our first purpose which is to have a Segal version of enriched categories.

Observations 2.2.1. For a small category $\mathcal{C}$, the following properties follow directly from the construction of $\mathscr{P}_{\mathcal{C}}$.

1. Since $\mathbf{1}$ is terminal in Cat, we have by functoriality a homomorphism (strict 2-functor): $\mathscr{P}_{\mathcal{C}} \to \mathscr{P}_{\mathbf{1}}$. We may call it the 'skeleton-morphism' and we will **view $\mathscr{P}_{\mathcal{C}}$ to be over $\mathscr{P}_{\mathbf{1}}$**.
2. We have $(\mathscr{P}_{\mathcal{C}})^{\text{1-op}} \cong \mathscr{P}_{\mathcal{C}^{op}}$, where $(\mathscr{P}_{\mathcal{C}})^{\text{1-op}}$ is the 1-opposite of $\mathscr{P}_{\mathcal{C}}$, which is characterized by:
$$(\mathscr{P}_{\mathcal{C}})^{\text{1-op}}(A, B) := \mathscr{P}_{\mathcal{C}}(B, A).$$

2.2.2 Basic properties

In the following we give some basic properties of the path-functor.
It is clear that $\mathscr{P}_{[-]}$ preserves equivalence and is an isomorphism reflecting in the sense that if $\mathscr{P}_F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{P}_{\mathcal{D}}$ is an isomorphism (of 2-categories) then so is $F : \mathcal{C} \longrightarrow \mathcal{D}$.
From the definition one has immediately that $\mathscr{P}_{\mathcal{C} \coprod \mathcal{D}} \cong \mathscr{P}_{\mathcal{C}} \coprod \mathscr{P}_{\mathcal{D}}$. For the product we need to be careful.

Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories and $\mathcal{C} \times \mathcal{D}$ their cartesian product. The projections from $\mathcal{C} \times \mathcal{D}$ to each factor induce two maps

$$\mathscr{P}_{\mathcal{C}\times\mathcal{D}} \longrightarrow \mathscr{P}_{\mathcal{C}}, \quad \mathscr{P}_{\mathcal{C}\times\mathcal{D}} \longrightarrow \mathscr{P}_{\mathcal{D}}$$

which in turn give a map

$$\mathscr{P}_{\mathcal{C}\times\mathcal{D}} \to (\mathscr{P}_{\mathcal{C}} \times \mathscr{P}_{\mathcal{D}})$$

by universal property of the cartesian product.

This map has no section, which means that it cannot be a biequivalence. We can see it from the fact that we have a canonical map $\mathscr{P}_{\mathcal{C}\times\mathcal{D}} \to \mathscr{P}_{\mathbf{1}}$ while such map doesn't exist with $\mathscr{P}_{\mathcal{C}} \times \mathscr{P}_{\mathcal{D}}$. This is related to the fact that $(\Delta, +, 0)$, which is $\mathscr{P}_{\mathbf{1}}$, **is not symmetric monoidal**.

But everything is not lost since we have.

Proposition 2.2.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories and $\mathscr{P}_{\mathcal{C}} \to \mathscr{P}_{\mathbf{1}}$, $\mathscr{P}_{\mathcal{D}} \to \mathscr{P}_{\mathbf{1}}$ the corresponding 2-path-categories. Then we have an isomorphism of 2-categories*

$$\mathscr{P}_{\mathcal{C}\times\mathcal{D}} \xrightarrow{\sim} (\mathscr{P}_{\mathcal{C}} \times_{\mathscr{P}_{\mathbf{1}}} \mathscr{P}_{\mathcal{D}}).$$

Sketch of proof. It suffices to write the definition of $\mathscr{P}_{\mathcal{C}\times\mathcal{D}}$. A chain $[n, s]$ in $\mathscr{P}_{\mathcal{C}\times\mathcal{D}}$ is by definition the same thing as a pair of chains $([n, s_{\mathcal{C}}], [n, s_{\mathcal{D}}])$. And a morphism of chains is $\mathscr{P}_{\mathcal{C}\times\mathcal{D}}$ is by definition a morphism of Δ which is 'simultaneously' the same in both $\mathscr{P}_{\mathcal{C}}$ and $\mathscr{P}_{\mathcal{D}}$ which means that it's a morphism of the fiber product $\mathscr{P}_{\mathcal{C}} \times_{\mathscr{P}_{\mathbf{1}}} \mathscr{P}_{\mathcal{D}}$.

Here $\mathscr{P}_{\mathcal{C}} \times_{\mathscr{P}_{\mathbf{1}}} \mathscr{P}_{\mathcal{D}}$ is given by :

- Objects : $\mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$
- Morphisms : Consider two pairs (A, X), (B, Y), with A, B objects of $\mathcal{C}$ and X, Y objects of $\mathcal{D}$. From the length functors :

$$\mathscr{L}_{AB} : \mathscr{P}_{\mathcal{C}}(A, B) \longrightarrow \Delta, \qquad \mathscr{L}_{XY} : \mathscr{P}_{\mathcal{D}}(X, Y) \longrightarrow \Delta$$

  we define

$$(\mathscr{P}_{\mathcal{C}} \times_{\mathscr{P}_{\mathbf{1}}} \mathscr{P}_{\mathcal{D}})[(A, X), (B, Y)] := \mathscr{P}_{\mathcal{C}}(A, B) \times_{\Delta} \mathscr{P}_{\mathcal{D}}(X, Y).$$

- The composition is given by the concatenation of chains *factor-wise.*

■

Remark 2.2.2.

1. If we use cospans in each $\mathscr{P}_{\mathcal{C}}(A, B)$ and extend $\mathscr{P}_{[-]}$ to a 2-functor, then one can compute the general limits and colimits with respect to $\mathscr{P}_{[-]}$, but we won't do it here.
2. If we apply the 2-functor to a monoidal category $\mathcal{D}$, then $\mathscr{P}_{\mathcal{D}}$ will be a monoidal 2-category with a suitable tensor product.

2.2.3 Base of enrichment

Let $\mathscr{M}$ be a bicategory and $\mathscr{W}$ a class of 2-cells of $\mathscr{M}$.

Definition 2.2.4. *The pair $(\mathscr{M}, \mathscr{W})$ is said to be a* ***base of enrichment*** *if $\mathscr{W}$ has the following properties:*

1. *Every invertible 2-cell of $\mathscr{M}$ is in $\mathscr{W}$, in particular 2-identities are in $\mathscr{W}$,*
2. *$\mathscr{W}$ has the vertical '3 out of 2' property, that is :*

$$\begin{tikzcd}
U \arrow[r, bend left=70, "f"] \arrow[r] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, shift left=3ex, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3ex, "\beta\Downarrow"] & V \arrow[r, rightsquigarrow] & U \arrow[r, bend left=40, "f"] \arrow[r, bend right=40, "h"'] \arrow[r, phantom, "\beta\star\alpha\Downarrow"] & V
\end{tikzcd}$$

   if 2 of α, β, $\beta \star \alpha$ are in $\mathscr{W}$ then so is the third,

3. $\mathscr{W}$ *is stable under* horizontal *composition, that is :*

$$\begin{tikzcd} W & V \arrow[l, bend right, "g"'] \arrow[l, bend left, "g'"] \arrow[l, phantom, "\beta\Downarrow"] & U \arrow[l, bend right, "f"'] \arrow[l, bend left, "f'"] \arrow[l, phantom, "\alpha\Downarrow"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} W & & U \arrow[ll, bend right, "g\otimes f"'] \arrow[ll, bend left, "g'\otimes f'"] \arrow[ll, phantom, "\beta\otimes\alpha\Downarrow"] \end{tikzcd}$$

if α *and* β *are both in* $\mathscr{W}$ *then so is* $\beta \otimes \alpha$*.*

Observations 2.2.2.
This definition is simply a generalization of the environment required by Leinster to define the notion of *up-to-homotopy monoid* in [61], when $\mathscr{M}$ is a monoidal category, hence a bicategory with one object.

Remark 2.2.3. The reader may observe that for any bicategory $\mathscr{M}$, the class $\mathscr{W}$ = 2-Iso consisting of all invertible 2-cells of $\mathscr{M}$ satisfies the previous properties. In this way we can say that the pair ($\mathscr{M}$, 2-Iso) is the smallest base of enrichment since by definition every base ($\mathscr{M}, \mathscr{W}$) contains ($\mathscr{M}$, 2-Iso).

Note that if we take $\mathscr{W}$ to be the class 2-Mor($\mathscr{M}$) of all of 2-cells we get the largest base ($\mathscr{M}$, 2-Mor($\mathscr{M}$))

2.2.4 Path-object

Definition 2.2.5. *Let* ($\mathscr{M}, \mathscr{W}$) *be a base of enrichment. A* Path-object *of* ($\mathscr{M}, \mathscr{W}$) *is a pair* $(\mathcal{C}, F)$*, where* $\mathcal{C}$ *is a small category and* $F = (F, \varphi)$ *a* <u>colax</u> *morphism of Bénabou:*

$$F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$$

such that for any objects A*,* B*,* C *of* $\mathcal{C}$ *and any pair* (t, s) *in* $\mathscr{P}_{\mathcal{C}}(B, C) \times \mathscr{P}_{\mathcal{C}}(A, B)$*, all the* 2*-cells*

$$F_{AC}(t \otimes s) \xrightarrow{\varphi(A,B,C)(t,s)} F_{BC}(t) \otimes F_{AB}(s)$$

$$F_{AA}([0, A]) \xrightarrow{\varphi_A} I'_{FA}$$

<u>*are in*</u> $\mathscr{W}$*. Such a colax morphism will be called a* $\mathscr{W}$***-colax morphism****.*

Terminology.

1. If $\mathscr{W}$ is a class of 2-cells called 2-*homotopy equivalences* then $(\mathcal{C}, F)$ will be called a *Segal path-object*. The maps $\varphi(A, B, C)$ and φ_A will be called *Segal maps.* If F is a strict homomorphism (respectively nonstrict homomorphism) we will say that that $(\mathcal{C}, F)$ is a <u>strict</u> Segal path-object (respectively *pseudo-strict* Segal path-object).

2. For U in Ob($\mathscr{M}$), an object *over* U is an object A of $\mathcal{C}$ such that $FA = U$ (See Figure 1 below). Here we've followed the geometric picture in enrichment over bicategories as in [86], [95], [96]. Sometimes it's also worthy to thinkof it as an object **connected to** U because we're going to use the combinatoric of $\mathscr{P}_{\mathcal{C}}$ to 'extract' from $\mathscr{M}$, 'the skeleton' of a category.

3. If $\mathscr{W} = 2\text{-}Mor(\mathscr{M})$, we will not mention $\mathscr{W}$ and call $(\mathcal{C}, F)$ a path-object of $\mathscr{M}$.

4. Since a path-object is a sort of morphism from $\mathscr{P}_{\mathcal{C}}$ to $\mathscr{M}$ we will call it a '$\mathscr{P}_{\mathcal{C}}$-point' or a '$\mathscr{P}_{\mathcal{C}}$-module' of $\mathscr{M}$. And for short we will simply say **$\mathcal{C}$-point** or **$\mathcal{C}$-module** of $\mathscr{M}$. We will therefore say Segal $\mathcal{C}$-point (or $\mathcal{C}$-module) for a Segal path-object $(\mathcal{C}, F)$.

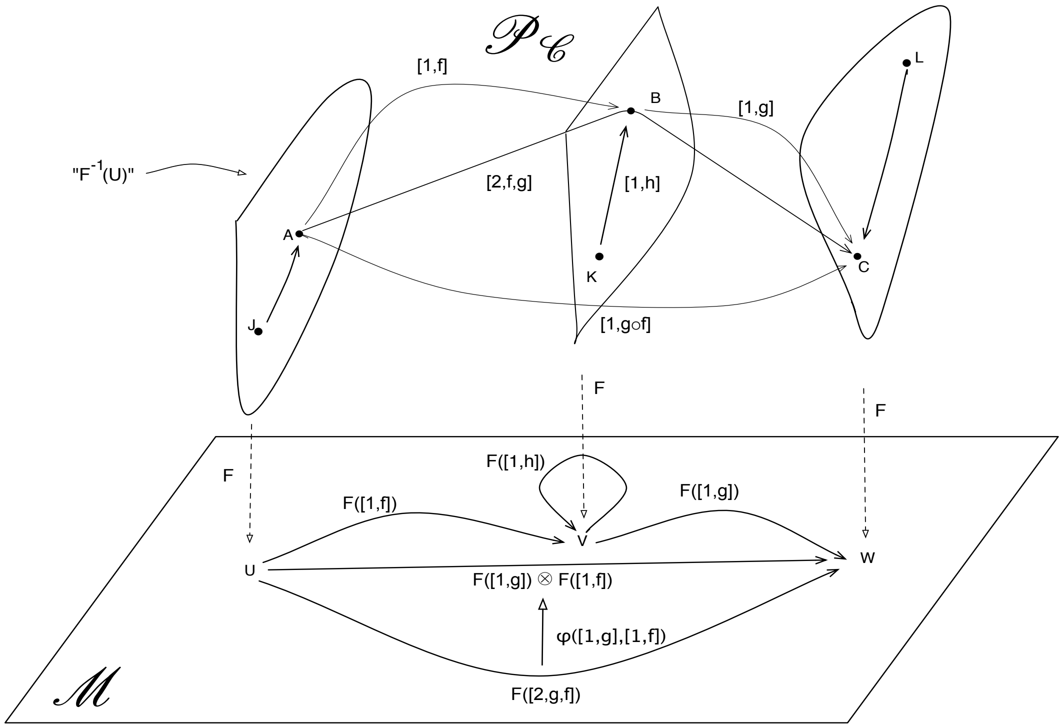


Figure 1

Observations 2.2.3.

In Figure 1, we took $t = [1, B \xrightarrow{g} C]$ and $s = [1, A \xrightarrow{f} B]$.
We have $t \otimes s = [2, A \xrightarrow{f} B \xrightarrow{g} C]$ and a canonical 2-cell in $\mathscr{P}_{\mathcal{C}}(A, C)$

$$\begin{tikzcd} A \arrow[r, bend left, "{[2,A\xrightarrow{f}B\xrightarrow{g}C]}"] \arrow[r, bend right, "{[1,A\xrightarrow{g\circ f}C]}"'] \arrow[r, phantom, "2\xrightarrow{!}1\Downarrow"] & C \end{tikzcd}$$

given by the composition in $\mathcal{C}$ and 'parametrized' by the (unique) arrow $2 \xrightarrow{!} 1$ of Δ. The image by F of this 2-cell is a 2-cell of $\mathscr{M}$

$$\begin{tikzcd} U \arrow[r, bend left, "{F([2,A\xrightarrow{f}B\xrightarrow{g}C])}"] \arrow[r, bend right, "{F([1,A\xrightarrow{g\circ f}C])}"'] \arrow[r, phantom, "\Downarrow"] & W \end{tikzcd} .$$

Now if we combine this with the colaxity map $\varphi(A, B, C)(t, s)$ we have the following span in $\mathscr{M}(U, W)$:

$$\begin{tikzcd} & F([2, A \xrightarrow{f} B \xrightarrow{g} C]) \arrow[dl, "{\varphi(A,B,C)(t,s)}"'] \arrow[dr, "{F(`2\xrightarrow{!}1')}"] & \\ F([1, B \xrightarrow{g} C]) \otimes F([1, A \xrightarrow{f} B]) & & F([1, A \xrightarrow{g\circ f} C]) \end{tikzcd}$$

If $\varphi(A,B,C)(t,s)$ is a *weak equivalence* (*e.g* a Segal map) therefore is *weakly* invertible, any choice of a weak inverse of $\varphi(A,B,C)(t,s)$ will give a map :

$$F([1, B \xrightarrow{g} C]) \otimes F([1, A \xrightarrow{f} B]) \longrightarrow F([1, A \xrightarrow{g\circ f} C])$$

by running the span from the left to the right.

In that situation if we want this construction to be consistent, we have to assume that all the weak inverses of $\varphi(A,B,C)(t,s)$ must be *homotopy equivalent* in some sense. This way the 'space' of the maps

$$F([1, B \xrightarrow{g} C]) \otimes F([1, A \xrightarrow{f} B]) \longrightarrow F([1, A \xrightarrow{g\circ f} C])$$

obtained for each weak inverse, will be *contractible*.

One of the interesting situations is when $F([1, B \xrightarrow{g} C])$ and $F([1, A \xrightarrow{f} B])$ stand for hom-objects of some category-like structure. The maps

$$F([1, B \xrightarrow{g} C]) \otimes F([1, A \xrightarrow{f} B]) \longrightarrow F([1, A \xrightarrow{g\circ f} C])$$

will be a sort of composition up-to *homotopy*, like for classical Segal categories.

Remark 2.2.4.

1. For every object U of $\mathscr{M}$, denote by $F^{-1}(U)$ the set of objects of $\mathcal{C}$ over U *via* F. For each U $F^{-1}(U)$ we have a full subcategory $\mathcal{C}_U$ of $\mathcal{C}$, corresponding to the "restriction" of $\mathcal{C}$ to $F^{-1}(U)$. Then F gives a **'foliation' of** $\mathcal{C}$ of 'leaves' $\mathcal{C}_U$. We get by functoriality a canonical injection $\mathscr{P}_{\mathcal{C}_U} \hookrightarrow \mathscr{P}_{\mathcal{C}}$ and the composition by F gives a $\mathcal{C}_U$-point of $(\mathscr{M}_{UU}, \mathscr{W}_{UU})$.

2. We see that a $\mathcal{C}$-point of a bicategory $(\mathscr{M}, \mathscr{W})$ is a *'moduli'*[4] of 'bimodules' between $\mathcal{C}_i$-points of some monoidal bases of enrichment $(\mathscr{M}_i, \mathscr{W}_i)$. As one can see if we start with a monoidal category $(\mathscr{M}, \mathscr{W})$, all object of $\mathcal{C}$ will be over the same object, say $*$, with $\operatorname{Hom}(*,*) = \mathscr{M}$.

3. We've used the terminology of foliation theory because each $\mathcal{C}_U$ can be an algebraic leaf of a foliated manifold $\mathcal{C}$. In that case $\mathcal{C}_U$ is determined by a collection of rings satisfying a (co)-descent condition. We will have a path-object of the bicategory **Bim** of rings, bimodules and morphism of bimodules (see [10]). Here again we will need a descent theory for path-objects.

4. In **Bim** we have both commutative and noncommutative rings so it appears to be a good place where both commutative and noncommutative geometry meet. Then the study of path-objects of **Bim** (and its higher versions) needs to be considered seriously.

Observations 2.2.4.

The collection of $\mathcal{C}$-points of $(\mathscr{M}, \mathscr{W})$ forms naturally a bicategory $\mathscr{M}^+_{\mathscr{W}}(\mathcal{C}) = \operatorname{Bicat}[\mathscr{W}](\mathscr{P}_{\mathcal{C}}, \mathscr{M})$, of $\mathscr{W}$-colax morphisms, transformations and modifications. In fact, in Bicat one has an *internal colax*-Hom between any two bicategories. In particular we have a 'colax-Yoneda' functor (of points)[5] $\operatorname{Bicat}_{colax}(-, \mathscr{M})$. We recall briefly this bicategorical structure on $\mathscr{M}^+_{\mathscr{W}}(\mathcal{C})$ as follows.

1. $\operatorname{Ob}(\mathscr{M}^+_{\mathscr{W}}(\mathcal{C})) = \{F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}\}$, the collection of $\mathscr{W}$-colax morphisms of Bénabou between $\mathscr{P}_{\mathcal{C}}$ and $\mathscr{M}$,

2. For every pair (F, G) of $\mathscr{W}$-colax morphisms, a 1-cell $\sigma : F \longrightarrow G$ is a <u>transformation</u> of morphisms of bicategories

$$\begin{tikzcd} \mathscr{P}_{\mathcal{C}} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathscr{M} \arrow[from=U, to=D, Rightarrow, "\sigma"'] \end{tikzcd} .$$

[4] or is "generated" by $\mathcal{C}_i$

[5] This justifies our terminology of '$\mathcal{C}$-points'

3. For every pair (σ_1, σ_2) of 1-cells, a 2-cell $\Gamma : \sigma_1 \longrightarrow \sigma_2$ is a modification of transformations :

$$\begin{tikzcd} \mathscr{P}_{\mathcal{C}} \arrow[rr, bend left=60, "F"] \arrow[rr, bend right=60, "G"'] \arrow[rr, phantom, "\sigma_1 \Downarrow \overset{\Gamma}{\Rrightarrow} \Downarrow \sigma_2"] & & \mathscr{M} \end{tikzcd} .$$

The definitions of transformations and modifications are recalled in section 2.4.

2.2.5 The coarse or indiscrete category

Recall that the 'object functor' Ob : **Cat** $\longrightarrow$ **Set** that takes a category $\mathcal{B}$ to it set of objects $\mathrm{Ob}(\mathcal{B})$, has a left adjoint **disc** : **Set** $\longrightarrow$ **Cat** 'the discrete functor'. It turns out that this functor has also a right adjoint **indisc** : **Set** $\longrightarrow$ **Cat**. We will denote by $\overline{X} := \mathbf{indisc}(X)$. By definition for any category $\mathcal{B}$ and any set X we have an isomorphism of sets:

$$\mathrm{Hom}(\mathcal{B}, \overline{X}) \cong \mathrm{Hom}(\mathrm{Ob}(\mathcal{B}), X)$$

functorial in X and $\mathcal{B}$; where the left-hand side is the set of functors from $\mathcal{B}$ to $\overline{X}$ while the right-hand side is the set of functions from $\mathrm{Ob}(\mathcal{B})$ to X. Below we give a brief description of $\overline{X}$.

Brief description of $\overline{X}$ The category $\overline{X}$ is the terminal connected groupoid having X as the set of objects. There is precisely a unique morphism between any pair of elements:

$$\overline{X}(a, b) = \mathrm{Hom}_{\overline{X}}(a, b) := \{(a, b)\} \cong 1.$$

The composition is obvious: it's the bijection $1 \times 1 \cong 1$. Given a function $g : \mathrm{Ob}(\mathcal{B}) \longrightarrow X$, the associated functor $\overline{g} : \mathcal{B} \longrightarrow \overline{X}$ is given by the (unique) constant functions

$$\overline{g}_{UV} : \mathcal{B}(U, V) \longrightarrow \overline{X}(g(U), g(V)) \cong 1.$$

Remark 2.2.5.
1. If X has two elements then $\overline{X}$ is the "**walking-isomorphism category**" in the sense that any isomorphim in a category $\mathcal{B}$ is the same thing as a functor $\overline{X} \longrightarrow \mathcal{B}$.
2. One may observe that $\overline{X}$ looks like EG for some group G. As we shall see in a moment EG is a G-category.
3. When X has only one element, say $X = \{A\}$, $\overline{X}$ consist of the object A with the identity 1_A, hence $\overline{X} \cong \mathbf{1}$. The by Proposition 2.2.1 we have a monoidal isomorphism between $\mathscr{P}_{\overline{X}}(A, A)$ and $(\Delta^+, +, 0)$.

Terminology. For a set X we will simply say **$\overline{X}$-point** or **$\overline{X}$-module** of $(\mathscr{M}, \mathscr{W})$ for a path-object $(\overline{X}, F)$ of $(\mathscr{M}, \mathscr{W})$.

We will write $\mathscr{M}^+_{\mathscr{W}}(X) := \mathrm{Bicat}[\mathscr{W}](\mathscr{P}_{\overline{X}}, \mathscr{M})$, for the bicategory of $\mathscr{W}$-colax morphisms from $\mathscr{P}_{\overline{X}}$ to $\mathscr{M}$.

Some natural path-objects For a set X, we've considered the coarse category $\overline{X}$ which is a groupoid, but one may consider any *preorder*. Recall that by preorder we mean a category in which there is at most one morphism between any two objects. For preorders R, The R-points of $(\mathscr{M}, \mathscr{W})$ are important because in some sense they 'generate' the general $\mathcal{C}$-points for arbitrary small categories $\mathcal{C}$. This comes from the *nerve* construction of a small category.

Recall that for a category $\mathcal{C}$ one defines the nerve of $\mathcal{C}$ to be the following functor :

$$\mathscr{N}(\mathcal{C}) : \Delta^{op} \longrightarrow Set$$

$$n \mapsto \mathrm{Hom}([n], \mathcal{C})$$

where $[n]$ is the preorder with n objects. Explicitly $[n]$ is the category defined as follows.
Take $\mathrm{Ob}([n]) = \{0, 1, \cdots, n\}$ the set of the first $n+1$ natural numbers and

$$[n](i,j) = \begin{cases} \{(i,j)\} & \text{if } i < j \\ \{\mathrm{Id}_i = (i,i)\} & \text{if } i = j \\ \varnothing & \text{if } i > j \end{cases}$$

The composition is the obvious one.

The set $\mathscr{N}(\mathcal{C})_n = \mathrm{Hom}([n], \mathcal{C})$ is the set of n composable arrows of $\mathcal{C}$ through $(n+1)$ objects $A_0, ..., A_n$:

$$A_0 \xrightarrow{f_1} \cdots A_{i-1} \xrightarrow{f_i} A_i \longrightarrow \cdots \xrightarrow{f_n} A_n.$$

Each element of $\mathscr{N}(\mathcal{C})_n$ is called a *n-simplex* of $\mathcal{C}$. The n-simplices of $\mathcal{C}$ from A_0 to A_n are exactly the 1-cells (of length n) in $\mathscr{P}_{\mathcal{C}}(A_0, A_n)$.
It's important to notice that for every n-simplex r of $\mathcal{C}$, *i.e* a functor $r : [n] \longrightarrow \mathcal{C}$, we have a pullback-category $[r]$ described as follows.
$\mathrm{Ob}([r]) = \mathrm{Ob}([n]) = \{0, 1, \cdots, n\}$ and

$$[r](i,j) = \begin{cases} \{f_j \circ \cdots \circ f_i\} & \text{if } i < j \\ \{\mathrm{Id}_{r(i)}\} & \text{if } i = j \\ \varnothing & \text{if } i > j \end{cases}$$

Now if all the arrows $f_0, \cdots, f_n$ are invertible we can extend $[r]$ to a coarse category by adding the inverse of each f_i or by formally adding the inverses of the f_i. This will be the case where $\mathcal{C}$ is a groupoid or by localizing $\mathcal{C}$ with respect to some class of morphisms $\mathcal{S}$.

Since the construction of the path-category is functorial it follows that for every n-simplex r of $\mathcal{C}$, *i.e* a functor $r : [n] \longrightarrow \mathcal{C}$, we have a strict homomorphism $\mathscr{P}_r : \mathscr{P}_{[n]} \longrightarrow \mathscr{P}_{\mathcal{C}}$. Therefore any $\mathcal{C}$-point $F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ gives by pullback an $[n]$-point. The reason we've considered these constructions is that:

Proposition 2.2.6. *A colax morphism $F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ satisfies the Segal conditions if and only if for very n-simplex $r : [n] \longrightarrow \mathcal{C}$ of $\mathcal{C}$ the pullback $\mathscr{P}_r^{\star}(F) : \mathscr{P}_{[n]} \longrightarrow \mathscr{M}$ satisfies the Segal conditions.*

Proof. Obvious. ∎

2.3 Examples of path-objects

Our goal in this section is to outline how the language of path-objects covers some classical situations, from enriched category theory to other areas.

2.3.1 Up-to-homotopy monoids and Simplicial objects

Definition 2.3.1. *Let $\mathcal{N}$, $\mathcal{N}'$ be two monoidal categories. A* colax *monoidal functor $\mathcal{N}' \longrightarrow \mathcal{N}$ consists of a functor $Y : \mathcal{N}' \longrightarrow \mathcal{N}$ together with maps*

$$\xi_{AB} : Y(A \otimes B) \longrightarrow Y(A) \otimes Y(B)$$

$$\xi_0 : Y(I) \longrightarrow I$$

$(A, B \in \mathcal{N}')$, satisfying naturality and coherence axioms.

Here $\otimes$ and I denote the tensor operation and unit object in both monoidal categories $\mathcal{N}'$ and $\mathcal{N}$. We refer the reader to [60] for the coherence axioms of colax functors.

The following definition is due to Leinster. The generalization of this definition was the motivation of this work.

Definition 2.3.2. *Let $\mathscr{M}$ be a monoidal category equipped with a class of* homotopy equivalences *$\mathscr{W}$ such that the pair $(\mathscr{M}, \mathscr{W})$ is a base of enrichment. A* homotopy monoid *in $\mathscr{M}$ is colax monoidal functor*

$$(Y, \xi) : (\Delta^+, +, 0) \longrightarrow \mathscr{M}$$

for which the maps ξ_0, ξ_{mn} are in $\mathscr{W}$ for every m, n in Δ^+.

Definition 2.3.3. *Let $\mathscr{M}$ be a category. A simplicial object of $\mathscr{M}$ is a functor $Y : \Delta^{op} \longrightarrow \mathscr{M}$.*

Remark 2.3.1. In this definition $\mathscr{M}$ may be a **higher category**. For example in [41], Simpson and Hirschowitz use simplicial objects of some higher (model) category to define inductively Segal n-categories.

A special case of simplicial object which is relevant to our path-objects comes when the ambient category $\mathscr{M}$ is a category with finite products with an empty-product object 1, which is terminal by definition. In this case $\mathscr{M}$ turns to be a **cartesian** monoidal category $(\mathscr{M}, \times, 1)$.

The following proposition is due to Leinster [60].

Proposition 2.3.4. *Let $(\mathscr{M}, \times)$ be a category with finite products. Then there is an isomorphism of categories*

$$Colax((\Delta^+, +, 0), (\mathscr{M}, \times, 1)) \cong \mathrm{Hom}[\Delta^{op}, \mathscr{M}].$$

Remark 2.3.2. $Colax((\Delta^+, +, 0), (\mathscr{M}, \times, 1))$ represents the category of colax monoidal functors.

In what follows we are going to rephrase the proposition and the definitions given above in terms of points of $\mathscr{M}$. We will use the following notation.
$\mathbb{1} = \{\mathbf{O}, \mathbf{O} \xrightarrow{Id_{\mathbf{O}}} \mathbf{O}\}$ = the unit category.
$Iso(\mathscr{M})$ = the class of all invertible morphisms of $\mathscr{M}$.
$Mor(\mathscr{M})$ = the class of all morphisms of $\mathscr{M}$.
As usual, since Bénabou, we will identify $\mathscr{M}$ with a bicategory with one object (See Example 4.6.2 of the Appendix A).

Proposition 2.3.5. *Let $(\mathscr{M}, \otimes, I)$ be a monoidal category.*

1. *We have an equivalence between the following data:*
   - *a $\mathbb{1}$-point of $(\mathscr{M}, Iso(\mathscr{M}))$ i.e an object of $\mathscr{M}^+_{Iso(\mathscr{M})}(\mathbb{1})$,*
   - *a monoid of $\mathscr{M}$.*

2. *Assume that $\mathscr{M}$ is equipped with a class $\mathscr{W}$ of morphisms called homotopy equivalences, such that $(\mathscr{M}, \mathscr{W})$ is a base of enrichment. Then we have an equivalence between the following data*

   – *a $\mathbb{1}$-point of $(\mathscr{M}, \mathscr{W})$ i.e an object of $\mathscr{M}^{+}_{\mathscr{W}}(\mathbb{1})$,*

   – *an up-to homotopy monoid in the sense of Leinster [61].*

3. *If $\mathscr{M}$ has finite products, and is considered to be monoidal for the cartesian product, then we have an equivalence between*

   – *a $\mathbb{1}$-point of $(\mathscr{M}, Mor(\mathscr{M}))$ i.e an object of $\mathscr{M}^{+}_{Mor(\mathscr{M})}(\mathbb{1})$,*

   – *a simplicial object of $\mathscr{M}$.*

Each of the above equivalences will be automatically an equivalence of categories with the appropriate notions of morphism of $\mathcal{C}$-points.

Observations 2.3.1.

1. The assertion (1) is simply a particular case of (2) when $\mathscr{W} = Iso(\mathscr{M})$. The main motivation to consider general classes of *homotopy equivalences* $\mathscr{W}$ other than $Iso(\mathscr{M})$ is to have a Segal version of enriched categories over monoidal categories. The idea is to view a monoid of $\mathscr{M}$ as a category enriched over $\mathscr{M}$ with one object and to view an up-to-homotopy monoid of Leinster as the Segal version of it.

   The 'several objects' case is considered in the upcoming examples. We will call them **Segal enriched categories**.

2. As pointed out earlier, in the assertion (3), $\mathscr{M}$ may be a higher category having finite product. This suggests to extend the definition of $\mathcal{C}$-points of $(\mathscr{M}, \mathscr{W})$ (Definition 2.2.5) to a general one where $\mathcal{C}$ and $\mathscr{M}$ are ∞-categories. A first attempt would consist to use **Postnikov systems** (see [7]) to **"go down"** to the current situation. We will come back to this later.

3. Again in the assertion (3), the category $\mathscr{M}$ may have **discrete objects** and we can give a definition of Segal categories or *(weak) internal category object* in $\mathscr{M}$ (see [41]) in terms of $\mathbb{1}$-point of $(\mathscr{M}, \mathscr{W})$.
   An immediate step is to ask what we will have with general $\overline{X}$-points. This is discussed later.

Proof of Proposition 2.3.5

The proof of the proposition is based on the following two facts:

– the path-bicategory of $\mathbf{1}$ 'is' Δ^{+} (see Proposition 2.2.1).

– bicategories with one object and morphisms between them are identified with monoidal categories and the suitable functors of monoidal categories, and vice versa.

Let F be $\mathbb{1}$-point of $(\mathscr{M}, \mathscr{W})$. By definition F is a $\mathscr{W}$-colax morphism of bicategories $F : \mathscr{P}_{\mathbb{1}} \longrightarrow \mathscr{M}$.
As $\mathscr{P}_{\mathbb{1}}$ is a one-object bicategory, F is entirely determined by the following data:

1. a functor $F_{\mathbf{OO}} : \mathscr{P}_{\mathbb{1}}(\mathbf{O}, \mathbf{O}) \longrightarrow \mathscr{M}$ which is the only component of F

2. arrows $F_{\mathbf{OO}}(t \otimes s) \xrightarrow{\varphi(\mathbf{O},\mathbf{O},\mathbf{O})(t,s)} F_{\mathbf{OO}}(t) \otimes F_{\mathbf{OO}}(s)$ in $\mathscr{W}$, for every pair (t, s) in $\mathscr{P}_{\mathbb{1}}(\mathbf{O}, \mathbf{O})$,

3. an arrow $F_{\mathbf{OO}}([0,\mathbf{O}]) \xrightarrow{\varphi_{\mathbf{O}}} I$ in $\mathscr{W}$,

4. coherences on $\varphi(\mathbf{O},\mathbf{O},\mathbf{O})(t,s)$ and $\varphi_{\mathbf{O}}$.

But one can check that these data say exactly that $F_{\mathbf{OO}}$ is a **colax monoidal functor** from $(\mathscr{P}_{\mathbb{1}}(\mathbf{O},\mathbf{O}), c(\mathbf{O},\mathbf{O},\mathbf{O}), [0,\mathbf{O}])$ to $(\mathscr{M},\otimes,I)$.

As remarked previously we have an isomorphism of monoidal categories

$$(\Delta^+,+,0) \cong (\mathscr{P}_{\mathbb{1}}(\mathbf{O},\mathbf{O}), c(\mathbf{O},\mathbf{O},\mathbf{O}), [0,\mathbf{O}]).$$

We recall that this isomorphism is determined by the following identifications.
$0 \longleftrightarrow [0,\mathbf{O}]$.
$n \longleftrightarrow [n, \underbrace{\mathbf{O} \xrightarrow{Id_{\mathbf{O}}} \mathbf{O} \cdots \mathbf{O} \xrightarrow{Id_{\mathbf{O}}} \mathbf{O}}_{n \quad identities}] = s.$
$m \longleftrightarrow [m, \underbrace{\mathbf{O} \xrightarrow{Id_{\mathbf{O}}} \mathbf{O} \cdots \mathbf{O} \xrightarrow{Id_{\mathbf{O}}} \mathbf{O}}_{m \quad identities}] = t.$
$(n+m) \longleftrightarrow [n+m, \underbrace{\mathbf{O} \xrightarrow{Id_{\mathbf{O}}} \cdots \mathbf{O}}_{n \quad identities} \underbrace{\xrightarrow{Id_{\mathbf{O}}} \cdots \mathbf{O}}_{m \quad identities}] = c(\mathbf{O},\mathbf{O},\mathbf{O})(t,s) = t \otimes s.$
$\{$Coface maps in Δ $\} \longleftrightarrow \{$Replacing consecutive arrows by their composite $\}$.
$\{$Codegeneracy maps in Δ $\} \longleftrightarrow \{$ Adding identities $\}$ (see Appendix B).

Summing up the above discussion we see that F is equivalent to a colax monoidal functor from $(\Delta^+,+,0)$ to $(\mathscr{M},\otimes,I)$:

- $\widetilde{F} : \Delta^+ \longrightarrow \mathscr{M}$,
- $\varphi_{mn} : \widetilde{F}(m+n) \longrightarrow \widetilde{F}(m) \otimes \widetilde{F}(n) \in \mathscr{W}$
- $\varphi_0 : \widetilde{F}(0) \longrightarrow I \in \mathscr{W}$.

If $\mathscr{W}$ is a class of homotopy equivalences, we recover the definition of a homotopy monoid given by Leinster in [61], which proves the assertion (2).

If $\mathscr{W} = Mor(\mathscr{M})$ and $\mathscr{M}$ is cartesian monoidal we get an object of $Colax((\Delta^+,+,0),(\mathscr{M},\times,1))$ and the assertion (3) follows from the Proposition 7 above. ■

2.3.2 Classical enriched categories

The theory of enriched categories over a monoidal category $\mathscr{M}$ has a natural extension when $\mathscr{M}$ is a 2-category (see [86]). For completeness we recall hereafter the definition of an $\mathscr{M}$-category for a 2-category $\mathscr{M}$.

Definition 2.3.6. *Let $\mathscr{M}$ be a 2-category. An $\mathscr{M}$-category $\mathcal{C}$ consists of the following data :*

- *for each object U of $\mathscr{M}$, a set $\mathcal{C}_U$ of objects* over *U;*
- *for objects A, B over U, V, respectively an arrow $\mathcal{C}(A,B) : U \longrightarrow V$ in $\mathscr{M}$;*

– *for each object A over U, a 2-cell $I_A : \mathrm{Id}_U \Longrightarrow \mathcal{C}(A,A)$:*

$$\begin{tikzcd} U & & U \arrow[ll, bend right, "\mathrm{Id}_U"'] \arrow[ll, bend left, "\mathcal{C}(A,A)"] \arrow[ll, phantom, "\Downarrow I_A"] \end{tikzcd}$$

in $\mathscr{M}$;

– *for object A, B, C over U, V, W, respectively, a 2-cell $c_{ABC} : \mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \Longrightarrow \mathcal{C}(A,C)$:*

$$\begin{tikzcd} & V \arrow[dl, bend right, "\mathcal{C}(B,C)"'] & \\ W & & U \arrow[ul, bend right, "\mathcal{C}(A,B)"'] \arrow[ll, bend right=40, "\mathcal{C}(B,C)\otimes\mathcal{C}(A,B)"'] \arrow[ll, bend left, "\mathcal{C}(A,C)"] \arrow[ll, phantom, "\Downarrow c_{ABC}"] \end{tikzcd}$$

in $\mathscr{M}$;

satisfying the obvious three axioms of left and right identities and associativity.

The reader can immediately check that if $\mathscr{M}$ has one object we recover the enrichment over a monoidal category as in [49]. We've followed here the terminology *'object over'* of Street [86]. This provides a geometric vision in the theory of enriched categories which is very useful in some situations.

Remark 2.3.3. We will assume that all the sets $\mathcal{C}_U$ are nonempty, otherwise we replace $\mathscr{M}$ by its "restriction" to the set of U such that $\mathcal{C}_U$ is nonempty.

Proposition 2.3.7. *Let $\mathscr{M}$ be a bicategory and $\mathscr{W}$ be the class of invertible 2-cells of $\mathscr{M}$. We have an equivalence between the following data*

1. *an $\overline{X}$-point of $(\mathscr{M}, \mathscr{W})$*

2. *an enriched category over $\mathscr{M}$ having X as set of objects i.e a polyad of Bénabou.*

Note. Here again, the above equivalence is an equivalence of categories with the appropriate notions of morphisms of X-points.

Conventions.

– If $(f_1, \cdots, f_n)$ is a n-tuple of composable 1-morphisms of $\mathscr{M}$ we will write $f_1 \otimes \cdots \otimes f_n$ for the horizontal composition of $f_1, \cdots, f_n$ with all pairs of parentheses starting in front,

– Similarly if $(\alpha_1, \cdots, \alpha_n)$ is a n-tuple of composable 2-morphisms we will write $\alpha_1 \otimes \cdots \otimes \alpha_n$ for the horizontal composition of the $\alpha_1, \cdots, \alpha_n$ with all pairs of parentheses starting in front,

– For every 1-morphism $f : U \longrightarrow V$ of $\mathscr{M}$ we will write r_f (resp. l_f) for the right identity (resp. left identity) isomorphism $f \otimes \mathrm{Id}_U \xrightarrow{\sim} f$ (resp. $\mathrm{Id}_V \otimes f \xrightarrow{\sim} f$). We will write Id_f for identity 2-morphism.

Proof. We start with the direction (2) $\Rightarrow$ (1).

Let $\mathcal{C}$ be an $\mathscr{M}$-category. Denote by X the set $\bigsqcup \mathcal{C}_U$. Our goal is to construct a (colax) homomorphism denoted again $\mathcal{C} : \mathscr{P}_{\overline{X}} \longrightarrow \mathscr{M}$. Before doing this we recall some basics fact about the path-category.

By definition for each pair (A, B) of elements of X, $\mathscr{P}_{\overline{X}}(A, B)$ is the category of elements of some functor from Δ^+ to Set [6]. It follows that the morphisms of $\mathscr{P}_{\overline{X}}(A, B)$ are parametrized by the morphisms of Δ.

Since the morphisms of Δ are generated by the *cofaces* $d^i : n+1 \longrightarrow n$, and the *codegeneracies* $s^i : n \longrightarrow n+1$ (see [68]), one has that the morphisms of $\mathscr{P}_{\overline{X}}(A, B)$ are generated by the following two types of morphisms:

$$(*) \qquad \begin{array}{c} [n+1, A \longrightarrow \cdots A_{i-1} \xrightarrow{(A_{i-1},A_i)} A_i \xrightarrow{(A_i,A_{i+1})} A_{i+1} \cdots \longrightarrow B] \\ \Big\downarrow d^i \\ [n, A \longrightarrow \cdots A_{i-1} \xrightarrow{(A_{i-1},A_{i+1})} A_{i+1} \cdots \longrightarrow B] \end{array}$$

and

$$(**) \qquad \begin{array}{c} [n, A \longrightarrow \cdots A_{i} \xrightarrow{(A_{i},A_{i+1})} A_{i+1} \cdots \longrightarrow B] \\ \Big\downarrow s^i \\ [n+1, A \longrightarrow \cdots A_{i} \xrightarrow{(A_i,A_i)} A_i \xrightarrow{(A_i,A_{i+1})} A_{i+1} \cdots \longrightarrow B] \cdot \end{array}$$

The morphisms of type $(*)$ correspond to composition and those of type $(**)$ correspond to add the identity of the ith object. With these observations we define the component $\mathcal{C}_{AB} : \mathscr{P}_{\overline{X}}(A, B) \longrightarrow \mathscr{M}$ as follows.

1. The image of $[n, A \longrightarrow \cdots A_i \xrightarrow{(A_i,A_{i+1})} A_{i+1} \cdots \longrightarrow B]$ by $\mathcal{C}_{AB}$ is the 1-morphism of $\mathscr{M}$:
$$\mathcal{C}(A_{n-1}, B) \otimes \cdots \otimes \mathcal{C}(A_i, A_{i+1}) \otimes \cdots \otimes \mathcal{C}(A, A_1).$$

2. When $A = B \in \mathcal{C}_U$, then the image of $[0, A]$ is Id_U, the unity of $\mathscr{M}(U, U)$.

3. The image of a morphism of type $(*)$ by $\mathcal{C}_{AB}$ is the composite :
$$\begin{array}{c} \mathcal{C}(A_n, B) \otimes \cdots \otimes \mathcal{C}(A_i, A_{i+1}) \otimes \mathcal{C}(A_{i-1}, A_i) \otimes \cdots \otimes \mathcal{C}(A, A_1) \\ \Big\downarrow \wr \\ \mathcal{C}(A_n, B) \otimes \cdots \otimes [\mathcal{C}(A_i, A_{i+1}) \otimes \mathcal{C}(A_{i-1}, A_i)] \otimes \cdots \otimes \mathcal{C}(A, A_1) \\ \Big\downarrow \mathrm{Id}_{\mathcal{C}(A_n,B)} \otimes \cdots \otimes c_{A_{i-1}A_iA_{i+1}} \otimes \cdots \otimes \mathrm{Id}_{\mathcal{C}(A,A_1)} \\ \mathcal{C}(A_n, B) \otimes \cdots \otimes \mathcal{C}(A_{i-1}, A_{i+1}) \otimes \cdots \otimes \mathcal{C}(A, A_1) \end{array} .$$

[6] from Δ to Set if $A = B$

4. Similarly the image of a morphism of type $(**)$ is the composite:

$$\begin{array}{c}
\mathcal{C}(A_{n-1},B)\otimes\cdots\otimes\mathcal{C}(A_i,A_{i+1})\otimes\cdots\otimes\mathcal{C}(A,A_1)\\
\wr\big\downarrow \mathrm{Id}_{\mathcal{C}(A_{n-1},B)}\otimes\cdots\otimes l^{-1}_{\mathcal{C}(A_i,A_{i+1})}\otimes\cdots\otimes\mathrm{Id}_{\mathcal{C}(A,A_1)}\\
\mathcal{C}(A_{n-1},B)\otimes\cdots\otimes[I\otimes\mathcal{C}(A_i,A_{i+1})]\otimes\cdots\otimes\mathcal{C}(A,A_1)\\
\big\downarrow \mathrm{Id}_{\mathcal{C}(A_n,B)}\otimes\cdots\otimes[I_{A_i}\otimes\mathrm{Id}_{\mathcal{C}(A_i,A_{i+1})}]\otimes\cdots\otimes\mathrm{Id}_{\mathcal{C}(A,A_1)}\\
\mathcal{C}(A_{n-1},B)\otimes\cdots\otimes[\mathcal{C}(A_i,A_i)\otimes\mathcal{C}(A_i,A_{i+1})]\otimes\cdots\otimes\mathcal{C}(A,A_1)\\
\wr\big\downarrow\\
\mathcal{C}(A_{n-1},B)\otimes\cdots\otimes\mathcal{C}(A_i,A_i)\otimes\mathcal{C}(A_i,A_{i+1})\otimes\cdots\otimes\mathcal{C}(A,A_1)
\end{array}.$$

5. Using the bifunctoriality of the composition $\otimes$ in $\mathscr{M}$, its associativity and the fact that morphisms of type $(*)$ and $(**)$ generate all the morphisms of $\mathscr{P}_{\overline{X}}(A,B)$, one extends the above formula to a functor $\mathcal{C}_{AB}:\mathscr{P}_{\overline{X}}(A,B)\longrightarrow\mathscr{M}$.

The construction of the homomorphism is not complete until we say what are the colaxity maps $\varphi(A,B,C)(t,s):\mathcal{C}_{AC}(t\otimes s)\longrightarrow\mathcal{C}_{BC}(t)\otimes\mathcal{C}_{AB}(s)$.

But if $s=[n,A\cdots A_i\xrightarrow{(A_i,A_{i+1})}A_{i+1}\cdots\longrightarrow B]$, and $t=[m,B\cdots B_j\xrightarrow{(B_j,B_{j+1})}B_{j+1}\cdots\longrightarrow C]$, we have:

- $t\otimes s=[n+m,A\longrightarrow\cdots B\longrightarrow\cdots C]$,
- $\mathcal{C}_{AC}(t\otimes s)=\mathcal{C}(B_{m-1},C)\otimes\cdots\otimes\mathcal{C}(B,B_1)\otimes\mathcal{C}(A_{n-1},B)\otimes\cdots\otimes\mathcal{C}(A,A_1)$,
- $\mathcal{C}_{BC}(t)\otimes\mathcal{C}_{AB}(s)=[\mathcal{C}(B_{m-1},C)\otimes\cdots\otimes\mathcal{C}(B,B_1)]\otimes[\mathcal{C}(A_{n-1},B)\otimes\cdots\otimes\mathcal{C}(A,A_1)]$.

Then the map $\varphi(A,B,C)(t,s)$ is the unique isomorphism from $\mathcal{C}_{AC}(t\otimes s)$ to $\mathcal{C}_{BC}(t)\otimes\mathcal{C}_{AB}(s)$ given by the associativity of the composition $\otimes$. This map consists to move the parentheses from the front to the desired places. Clearly $\varphi(A,B,C)(t,s)$ is functorial in t and s.
We leave the reader to check that the functors $\mathcal{C}_{AB}$ together with the maps $\varphi(A,B,C)(t,s)$ and $\varphi_A=\mathrm{Id}_I$, satisfy the coherence axioms of morphism of bicategories. There $\mathcal{C}$ is a (colax) unitary [7] homomorphism from $\mathscr{P}_{\overline{X}}$ to $\mathscr{M}$ as desired.

The direction (1) $\Rightarrow$ (2) has a short proof. Let $\mathcal{C}$ be an $\overline{X}$-point of $(\mathscr{M},\mathscr{W})$. We construct an $\mathscr{M}$-category denoted again $\mathcal{C}$ as follows.

1. Put $\mathrm{Ob}(\mathcal{C})=X$.
2. For every pair (A,B) of elements of X, the hom-object is $\mathcal{C}(A,B):=\mathcal{C}_{AB}([1,(A,B)])$.
3. If $\mathcal{C}$ takes $A\in X$ to $U\in\mathrm{Ob}(\mathscr{M})$, then the component $\mathcal{C}_{AA}:\mathscr{P}_{\overline{X}}(A,A)\longrightarrow\mathscr{M}(U,U)$ is a (colax) strict monoidal functor; in particular $\varphi_A:\mathcal{C}([0,A])\longrightarrow\mathrm{Id}_U$ is invertible.
Furthermore we have a canonical 2-cell $[0,A]\xrightarrow{!}[1,(A,A)]$ in $\mathscr{P}_{\overline{X}}(A,A)$ whose image under $\mathcal{C}$ is a 2-morphism $\mathcal{C}([0,A])\xrightarrow{\mathcal{C}_{AA}(!)}\mathcal{C}([1,(A,A)])$. One takes the unit map Id_A to be the composite:
$$\mathrm{Id}_U\xrightarrow{\varphi_A{}^{-1}}\mathcal{C}([0,A])\longrightarrow\mathcal{C}([1,(A,A)])=\mathcal{C}(A,A).$$

[7] unitary means φ_A is the identity for every object A.

4. For every triple (A, B, C) of elements of X we construct the composition as follows. Consider the following span

$$\begin{tikzcd} & \mathcal{C}([2,(A,B,C)]) \arrow[dl, "\varphi(A,B,C)(t,s)"'] \arrow[dr, "\mathcal{C}\{[2,(A,B,C)]\xrightarrow{!}[1,(A,C)]\}"] & \\ \mathcal{C}([1,(B,C)]) \otimes \mathcal{C}([1,(A,B)]) \arrow[rr, dotted, "c_{ABC}"] & & \mathcal{C}([1,(A,C)]) \end{tikzcd}$$

where the left leg is the colaxity map while the right leg is an evaluation of the component $\mathcal{C}_{AC}$. By hypothesis the map φ is invertible, and we take the composition to be:

$$c_{ABC} = \mathcal{C}\{[2,(A,B,C)] \xrightarrow{!} [1,(A,C)]\} \star \varphi(A,B,C)(t,s)^{-1}$$

5. Finally one can easily check that the associativity and unity axioms required in $\mathcal{C}$ follow directly from the coherence axiom required in the definition of the morphism $\mathcal{C}$.

It's clear that the above data give an $\mathscr{M}$-category. ■

Remark 2.3.4. The above proof can be shortened if we use the isomorphism mentioned before in Corollary 2.2.2:

$$\mathrm{Lax}(\overline{X}, \mathscr{M}) \cong \mathrm{Hom}(\mathscr{P}_{\overline{X}}, \mathscr{M}).$$

$\mathrm{Lax}(\overline{X}, \mathscr{M})$ is precisely the category of $\mathscr{M}$-category having as set of objects X. The reason we did not present that proof in the first place is the fact that we wanted to outline the combinatorics that $\mathscr{P}_{\overline{X}}$ carries.

2.3.3 Segal categories

We recall that Δ is the "topologists's category of simplices" (the empty set has been removed from Δ^+). We mentioned previously that for a small category $\mathcal{C}$ we can associate functorially a simplicial set $\mathscr{N}(\mathcal{C}) : \Delta^{op} \longrightarrow \mathbf{Set}$, called *nerve* of $\mathcal{C}$. The natural maps, called *Segal maps*

$$\mathscr{N}(\mathcal{C})_k \longrightarrow \mathscr{N}(\mathcal{C})_1 \times_{\mathscr{N}(\mathcal{C})_0} \cdots \times_{\mathscr{N}(\mathcal{C})_0} \mathscr{N}(\mathcal{C})_1$$

are isomorphisms.
Simpson and Hirschowitz [41] generalized this process to define inductively Segal n-categories. They defined first a 'category' nSePC of *Segal n-précat* as follows.

- A Segal 0-précat is a simplicial set, hence a $\mathbf{1}$-point of $(\mathbf{Set}, \times, 1)$ in our terminology.
- For $n \geq 1$, a Segal n-précat is a functor :

$$\mathscr{A} : \Delta^{op} \longrightarrow (n-1)\mathrm{SePC}$$

  such that $\mathscr{A}_0 = \mathscr{A}(0)$ is a discrete object of $(n-1)$SePC. Since $(n-1)$SePC has finite products, we can formulate it again in terms of $\mathbf{1}$-point of $(n-1)$SePC.

- A *morphism* of Segal n-précat is a natural transformation of functors.

These data define the category nSePC. They gave a notion of *equivalence* in nSePC and model structure on it.

Finally they define a Segal n-category to be a Segal n-précat $\mathscr{A} : \Delta^{op} \longrightarrow (n-1)$SePC such that :

- for every k, $\mathscr{A}_k$ is a Segal $(n-1)$-category ,
- for every $k \geq 1$, the canonical maps

$$\mathscr{A}_k \longrightarrow \mathscr{A}_1 \times_{\mathscr{A}_0} \cdots \times_{\mathscr{A}_0} \mathscr{A}_1$$

are equivalences of Segal $(n-1)$-précats.

Remark 2.3.5. These definitions involved the use of **discrete** objects. A discrete object in [41], is by definition an object in the image of some fully faithful functor from *Set* to $(n-1)$SePC. For a Segal n-category $\mathscr{A}$ the discrete object $\mathscr{A}_0$ plays the role of "set of objects". We can see the analogy with the nerve of a small category.
It's important to notice that in the above definitions, one needs a notion of **fiber product** to define the Segal maps

$$\mathscr{A}_k \longrightarrow \mathscr{A}_1 \times_{\mathscr{A}_0} \cdots \times_{\mathscr{A}_0} \mathscr{A}_1.$$

In fact $(n-1)$SePC is a cartesian monoidal category.

One could interpret $\mathscr{A}$ as a **generalized nerve** of a category enriched over $((n-1)\text{SePC}, \times, 1)$ with an 'internal set' of object $\mathscr{A}_0$.

If we do not have a notion of *discrete object* and a *fiber product* we need to change the construction a little bit to define generalized Segal categories. For this purpose, one needs a category $\mathscr{M}$ together with a class of *homotopy equivalences* such that $(\mathscr{M}, \mathscr{W})$ form a base of enrichment . We take the set of objects 'outside' $\mathscr{M}$, to avoid the use of discrete objects, by introducing the set X.

The following definition is on the "level 2" when $\mathscr{M}$ is bicategory. We will extend it later to the case where $\mathscr{M}$ is an ∞-category .

Definition 2.3.8. *For $(\mathscr{M}, \mathscr{W})$ a base of enrichment with $\mathscr{W}$ a class of homotopy equivalences. For any set X, an $\overline{X}$-point of $(\mathscr{M}, \mathscr{W})$ will be called a* Segal *$\mathscr{M}$-category.*

Proposition 2.3.9. *Let $\mathscr{M} = (n-1)$SePC and $\mathscr{W}$ be the equivalences of Simpson and Hirschowitz. For a set X we have an equivalence of categories between*

– *a Segal n-category $\mathscr{A}$ in the sense of Simpson-Hirschowitz, with $\mathscr{A}_0 = X$*

– *an $\overline{X}$-point F of $(\mathscr{M}, \mathscr{W})$, satisfying the induction hypothesis:*
$F[p, (x_0, \cdots, x_p)]$ is a Segal $(n-1)$-category.

Proof. Obvious. ■

2.3.4 Linear Segal categories

We fix $\mathscr{M} = (\mathbf{ChMod}_{\mathbf{R}}, \otimes_{\mathbf{R}}, \mathbf{R})$ the monoidal category of (co)-chain complexes of $\mathbf{R}$-modules for a commutative ring $\mathbf{R}$.

Choice of the class of maps $\mathscr{W}$

1. When working with a general commutative ring $\mathbf{R}$ then we will take $\mathscr{W}$ to be the class of **chain homotopy equivalences**.
2. But if $\mathbf{R}$ is a field we can take $\mathscr{W}$ to be the class of **quasi-isomorphisms**.

Warning. We do not require a priori $\mathscr{M}$ to be a model category where $\mathscr{W}$ is the class of weak equivalences. However the examples that are relevant for our purposes are where $\mathscr{M} = (\mathbf{ChMod_R}, \otimes_\mathbf{R}, \mathbf{R})$ is equipped with the monoidal model structure (see [42, Prop. 4.2.13]) with $\mathscr{W}$ the class of quasi-isomorphisms.

Remark 2.3.6. Leinster [60] pointed out that for a general commutative ring $\mathbf{R}$, the quasi-isomorphisms may not be stable under tensor product because of the Künneth formula.

Definition 2.3.10. *Let X be a set and $\mathscr{M} = (\mathbf{ChMod_R}, \otimes_\mathbf{R}, \mathbf{R})$ together with $\mathscr{W}$ the suitable class of weak equivalences. A Segal DG-category is an $\overline{X}$-point of $(\mathscr{M}, \mathscr{W})$, that is a $\mathscr{W}$-colax morphism*

$$F : \mathscr{P}_{\overline{X}} \longrightarrow \mathscr{M}$$

Remark 2.3.7.

- As one can see a strict Segal point of $(\mathscr{M}, \mathscr{W})$ is a classical DG-category.
- As usual we can use the iterative process *à la* Simpson-Tamsamani by defining enrichment over $\mathscr{M}$-Cat with the suitable weak equivalences. This way we can define also higher linear Segal categories. But in order to iterate the process one needs a definition of a weak equivalence between Segal DG-categories (see Definition 2.4.12).

2.3.5 Nonabelian cohomology

G-categories

Bénabou [10] pointed out that we can use polyads to 'pick up' a coherent family of isomorphisms satisfying **cocyclicity**. But polyads are enriched categories and correspond to strict Segal $\overline{X}$-points in our language, it appear that **the cocyclicity conditions of torsors reflect a composition operation**. For example EG for a group G in $(\mathbf{Set}, \times)$ is a G-category in an obvious manner. The reader can find in [47] an account on torsors.

We denote by BG the usual category having one object say, $\star$, and $\mathrm{Hom}(\star, \star) = G$.

Definition 2.3.11. *Let X be a set. An $\overline{X}$-point $F : \mathscr{P}_{\overline{X}} \longrightarrow BG$ is called a G-category.*

If we denote by $\mathscr{M}_F^X$ the corresponding category then we have for every pair (a, b) of elements of X, an element $\mathscr{M}_F^X(a, b)$ of G. The composition is the identity and gives a cocyclicity condition $\mathscr{M}_F^X(a, b) \centerdot \mathscr{M}_F^X(b, c) = \mathscr{M}_F^X(a, c)$ and $\mathscr{M}_F^X(a, a) = e$, where e is the unit in G.

Observations 2.3.2.

- We've considered a group in the category of sets but we can generalize it to any group object using the functor of points. This will be an iterative process of enrichment, that is enrichment over the categories of G-Cat when G is group in $(\mathbf{Set}, \times)$.
- It follows immediately that any group homomorphism from G to H will take a G-category to an H-category.
- The geometric picture behind a G-category is the notion of G-bundle. Roughly speaking we want to consider each element of X as an open set of some space and to consider $\mathscr{M}_F^X(a, b)$ as a transition function. We can then consider F a **generic trivialization**. When all the $\mathscr{M}_F^X(a, b)$ are equal to e, then our vector bundle (or local system) is trivial.

Terminology. Let $\mathscr{A}$ be a small category. Following the terminology of Simpson, we will call **interior of** $\mathscr{A}$ and denote by $\mathrm{Int}(\mathscr{A})$ the biggest groupoid contained in $\mathscr{A}$. For a base $(\mathscr{M}, \mathscr{W})$, we take the interior $\mathrm{Int}[(\mathscr{M}, \mathscr{W})]$ to be the sub-bicategory whose underlying 1-category is the interior of $\mathscr{M}_{\leq 1}$.

Definition 2.3.12. *Let $\mathcal{C}$ be a small category. A $\mathcal{C}$-generic cohomological class in coefficient in $\mathscr{M}$ is a Segal $\mathcal{C}$-point of* $\mathrm{Int}[(\mathscr{M}, \mathscr{W})]$.

Remark 2.3.8. When $\mathcal{C} = \overline{X}$ an $\overline{X}$-generic cohomological class is precisely a Segal $\mathscr{M}_{\mathscr{W}}$-category having X as set of object and such that each $\mathrm{Hom}(A, B)$ is invertible. We can require also that each $\mathrm{Hom}(A, A)$ is *contractible.*

Parallel transport

In the following we give an example of 1-functor which is viewed as an enrichment. We refer the reader to Schreiber-Waldorf [75] and references therein for an account on *parallel transport* with a guidance toward higher categories.

Warning. For a smooth manifold M we will consider below the **Path-groupoid** $\mathcal{P}_1(M)$ as defined in [75] which is different from the fundamental groupoid $\Pi_1(M)$.

Definition 2.3.13. *Let x, y be two points in M. A path γ from x to y is a smooth map $\gamma : [0, 1] \longrightarrow M$ such that there is a positive number ϵ, $0 < \epsilon < 1$ with*

- $\gamma_{|[0,\epsilon]} = x$;
- $\gamma_{|[1-\epsilon,1]} = y$.

The terminology can be confusing since the paths we consider here are not the usual paths for a topological space. There is a notion of **thin homotopy equivalence** of paths (see [75, Definition 2.2]). Parallel transport along these special paths is invariant by thin homotopy equivalence even if we have a non flat connection. We can form the path groupoid $\mathcal{P}_1(M)$ where the objects are the points of M and the morphisms are the class of thin-homotopy classes of smooth paths in M.

Let M be a smooth manifold and $\mathscr{E} \longrightarrow M$ a vector bundle equipped with a connection ∇. The connection induces a functor

$$\mathrm{Tra}\nabla : \mathcal{P}_1(M) \longrightarrow \mathbf{Vect}$$

called 'parallel transport functor' where **Vect** is the category of vector spaces.
The functor sends each point x of M to its fiber $\mathscr{E}_x$, and each path $f : x \longrightarrow y$, to the *parallel transport* $\mathrm{Tra}\nabla(f) : \mathscr{E}_x \longrightarrow \mathscr{E}_y$ induced by the connection along the path.

The relation with enriched categories comes when we view each point x of $\mathcal{P}_1(M)$ to be **over its fiber** $\mathscr{E}_x$.

In fact if we consider **Vect** as a bicategory, and even a strict 2-category, with all the 2-cells being identities (or *degenerate*) we can "lift" the functor

$$\mathrm{Tra}\nabla : \mathcal{P}_1(M) \longrightarrow \mathbf{Vect}$$

to a strict homomorphism from the 2-path-bicategory of $\mathcal{P}_1(M)$ to **Vect** (see Observations 2.2.1). In our terminology this will be strict 'free' $\mathcal{P}_1(M)$-point of **Vect** but we may prefer the terminology $\mathcal{P}_1(M)$-module in this situation.

The corresponding $\mathcal{P}_1(M)$-module will be denoted $\mathscr{E}^{-1}$ and is described as follows.

1. For every x in $\mathcal{P}_1(M)$, $\mathscr{E}^{-1}(x) = \mathscr{E}_x$.

2. For every pair (x, y) , the component $\mathscr{E}_{xy}^{-1} : \mathscr{P}_{\mathcal{P}_1(M)}(x, y) \longrightarrow \mathbf{Vect}$ is given by :
   if $s = [n, x \longrightarrow \cdots x_i \xrightarrow{f_i} x_{i+1} \cdots \longrightarrow y]$ with each $f_i : x_i \longrightarrow x_{i+1}$ a morphism in $\mathcal{P}_1(M)$ then we set
   $$\mathscr{E}^{-1}(s) := \mathrm{Tra}\nabla(f_{n-1}) \circ \cdots \circ \mathrm{Tra}\nabla(f_i) \circ \cdots \circ \mathrm{Tra}\nabla(f_0).$$
   We see that $\mathscr{E}^{-1}(s)$ is a linear map from $\mathscr{E}_x$ to $\mathscr{E}_y$.

3. For $x = y$, we have $\mathscr{E}^{-1}([0, x]) = \mathrm{Id}_{\mathscr{E}_x}$.

4. for every s, s' in $\mathscr{P}_{\mathcal{P}_1(M)}(x, y)$, and any morphism $u : s \longrightarrow s'$ then we define $\mathscr{E}_{xy}^{-1}(u) = \mathrm{Id}_{\mathscr{E}^{-1}(s)}$. This definition is well defined because we know that morphisms in $\mathscr{P}_{\mathcal{P}_1(M)}(x, y)$ are generated by the morphisms of type $(*)$ and $(**)$ as we saw in the proof of Proposition 2.3.7. And one easily see that the image of a morphism of type $(*)$ or $(**)$ is the identity, therefore $\mathscr{E}^{-1}(s) = \mathscr{E}^{-1}(s')$.

5. Finally for every triple (x, y, z) and every (t, s) in $\mathscr{P}_{\mathcal{P}_1(M)}(y, z) \times \mathscr{P}_{\mathcal{P}_1(M)}(x, y)$ it's easy to see that
   $$\mathscr{E}^{-1}(t \otimes s) = \mathscr{E}^{-1}(t) \circ \mathscr{E}^{-1}(s).$$

These data satisfy the coherences axioms and $\mathscr{E}^{-1}$ is a *strict* $\mathcal{P}_1(M)$-module (or $\mathcal{P}_1(M)$-point) of $(\mathbf{Vect}, \mathrm{Id}_{\mathbf{Vect}})$.

Observations 2.3.3. The following observations are also suggestions that one may want to consider.

1. Since $\mathcal{P}_1(M)$ is a groupoid, every morphism $f : x \longrightarrow y$ is invertible therefore the induced map
   $\mathrm{Tra}\nabla(f) : \mathscr{E}_x \longrightarrow \mathscr{E}_y$ is invertible in **Vect**. Taking $x = y$ we see that $\mathscr{E}_{xx}^{-1}$ is a **representation** of the (smooth) fundamental group $\pi_1(M, x)$. Therefore studying $\mathcal{C}$-point with $\mathcal{C}$ a groupoid becomes important to understand the homotopy of *generalized spaces* M.

2. It's well known that if we consider flat connection ∇, then the functor $\mathrm{Tra}\nabla$ factorizes through $\mathcal{P}_1(M)$, the fundamental groupoid of X. And we can still work in enriched category context.

3. The idea of thinking a vector bundle on M as an enriched category extend our intuition which consists to 'view' a category as a topological space (the classifying space). We can consider a vector bundle with a connection as **a linear copy** [8] of our space M. A point x is identify with the corresponding fiber $\mathscr{E}_x$ and every path from a point x to a point y gives a linear map by parallel transport.

4. Grothendieck defined the fundamental group in algebraic geometry as the group automorphism of a fiber functor (see [39]). This suggests to identify a point x of a generalized space M with it's *"motivic" fiber functor* $\mathrm{Mot}(\omega_x)$ (to be defined). In our terminology we will view x as being over (or taking as 'copy') $\mathrm{Mot}(\omega_x)$. We will then have an enrichment over the "category of fiber functors". Enrichment in this situation can be thought as *giving a copy of* $\mathcal{C}$ 'of type $\mathscr{M}$'.

[8] this terminology matches with the expression 'linear representation'

5. We see through out this example how enriched category theory appears in geometry and homotopy context. We saw that if we take **Vect** as our base of enrichment we have a "linearization" of the 1-homotopy type of M. Now if want more information on the higher homotopy, we need to replace $\Pi_1(M)$ by $\Pi_\infty(M)$ and **Vect** by another base which contains more information, then doing **a base changes and base extensions**.

   One can take for example $\mathbf{SVect}, \mathbf{nVect}, \mathbf{ChVect}, \mathbf{Perf}$, which are ,respectively, the category of simplicial vector spaces, n-vector spaces, complex of vector spaces, perfect complexes. In these categories there is a notion of *weak equivalence* , and we can consider Segal $\mathcal{C}$-points (or $\mathcal{C}$-module). It appears that having a theory of Segal enriched categories becomes important.

6. A further step will be to consider the notion of *gluing* Segal $\mathcal{C}_i$-points of $\mathscr{M}$ where $\mathcal{C}_i$ is a *covering* of $\mathcal{C}$. This will be part of [4].

2.3.6 Quasi-presheaf

Definition 2.3.14. *Let $\mathcal{C}$ be a small category and $(\mathscr{M}, \mathscr{W})$ a base of enrichment with $\mathscr{W}$ a class of homotopy 2-equivalences. A Segal $\mathscr{M}_{\mathscr{W}}$-presheaf in values in $\mathscr{M}$ is a Segal $\mathcal{C}^{op}$-point of $(\mathscr{M}, \mathscr{W})$, that is a $\mathscr{W}$-colax morphism*

$$\mathscr{F} : \mathscr{P}_{\mathcal{C}^{op}} \longrightarrow \mathscr{M}.$$

Example 2.3.15. Let's consider the Grothendieck anti-equivalence given by the 'global section functor' :

$$\text{Aff}^{\ op} \xrightarrow{\Gamma} \text{ComRing}$$

where Aff is the category of affine schemes and ComRing is the category of commutative rings; Γ is the global section functor of the structure sheaf.
We want to consider this functor as a quasi-presheaf which is a real presheaf taking its values in **Bim**. Recall that **Bim** is the bicategory described as follows.

- Objects are rings : R, S,...
- a 1-morphism from R to S is a bimodule ${}_SM_R$,
- a 2-morphism from ${}_SM_R$ to ${}_SN_R$ is a morphism of bimodule,
- The composition is given by the obvious tensor product.

The reader can find a detailed description of **Bim** in the paper of Bénabou [10].

Then the presheaf consists roughly speaking to send

- each $(Spec(R), \mathcal{O}_{Spec(R)})$ to R
- each morphism of schemes $f : Spec(R) \longrightarrow Spec(S)$ to the (S, R)-bimodule $\varphi^\star : S \nrightarrow R$, where φ is the corresponding ring homomorphism given by the anti-equivalence.

Note. In a more compact way we obtain the presheaf using the 'embedding' described in [10] from ComRing to **Bim**.

2.4 Morphisms of path-objects

2.4.1 Transformation

We recall briefly the notion of transformation between colax morphisms.

Definition 2.4.1. [Transformation]
Let $\mathscr{B}$ and $\mathscr{M}$ be two bicategories and $F = (F, \varphi)$, $G = (G, \psi)$ be two colax morphisms from $\mathscr{B}$ to $\mathscr{M}$. A transformation $\sigma : F \longrightarrow G$

$$\begin{tikzcd} \mathscr{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\sigma\Downarrow"] & \mathscr{M} \end{tikzcd} .$$

is given by the following data and axioms.

Data :

- 1-*cells* $\sigma_A : FA \longrightarrow GA$ *in* $\mathscr{M}$
- *Natural transformations*

$$\begin{tikzcd} \mathscr{B}(A,B) \arrow[r, "G_{AB}"] \arrow[d, "F_{AB}"'] \arrow[dr, bend left] \arrow[dr, bend right] \arrow[dr, phantom, "\Nearrow \sigma_{AB}"] & \mathscr{M}(GA,GB) \arrow[d, "-\otimes\sigma_A"] \\ \mathscr{M}(FA,FB) \arrow[r, "\sigma_B\otimes-"'] & \mathscr{M}(FA,GB) \end{tikzcd}$$

thus 2-*cells of* $\mathscr{M}$, $\sigma_t : \sigma_B \otimes Ft \longrightarrow Gt \otimes \sigma_A$, *for each* t *in* $\mathscr{B}(A, B)$.

Axioms :

The following commute :

$$\begin{tikzcd} \sigma_C \otimes F(t\otimes s) \arrow[rrr, "\sigma_{t\otimes s}"] \arrow[d, "\mathrm{Id}\otimes\varphi"'] & & & G(t\otimes s)\otimes\sigma_A \arrow[d, "\psi\otimes\mathrm{Id}"] \\ \sigma_C \otimes (Ft\otimes Fs) \arrow[rrr, dotted] \arrow[d, "a^{-1}"'] & & & (Gt\otimes Gs)\otimes\sigma_A \\ (\sigma_C\otimes Ft)\otimes Fs \arrow[dr, "\sigma_t\otimes\mathrm{Id}"'] & & & Gt\otimes(Gs\otimes\sigma_A) \arrow[u, "a^{-1}"'] \\ & (Gt\otimes\sigma_B)\otimes Fs \arrow[r, "a"] & Gt\otimes(\sigma_B\otimes Fs) \arrow[ur, "\mathrm{Id}\otimes\sigma_s"'] & \end{tikzcd}$$

$$\begin{tikzcd} \sigma_A\otimes FI_A \arrow[rr, "\sigma_{I_A}"] \arrow[d, "\mathrm{Id}\otimes\varphi_A"'] & & GI_A\otimes\sigma_A \arrow[d, "\psi_A\otimes\mathrm{Id}"] \\ \sigma_A\otimes I_{FA} \arrow[r, "\sim", "r"'] & \sigma_A \arrow[r, "\sim", "l^{-1}"'] & I_{GA}\otimes\sigma_A \end{tikzcd}$$

Remark 2.4.1. When all the 1-cells $\sigma_A : FA \longrightarrow GA$ are identities we will not represent them in the diagrams.

2.4.2 Morphism of path-objects

In this section we're going to define what is a morphism between $\mathcal{C}$-point and $\mathcal{D}$-point of $(\mathscr{M}, \mathscr{W})$, for $\mathcal{C}$ and $\mathcal{D}$ two small categories, we call them **pre-morphisms**. We will see in a moment that the morphisms of points of $(\mathscr{M}, \mathscr{W})$ which are relevant to enrichment behave exactly as morphisms of vector bundle over $\mathscr{M}$, which means *fiber wise* compatible. This is not surprising because it only makes sense to speak about 'morphism' between enriched categories having the same 'type of enrichment'. When $\mathscr{M}$ has one object then this condition will be fulfilled but the morphisms we consider are more general than a classical morphisms between enriched categories. The $\mathscr{M}$-morphisms correspond to the notion of **icons** in the sense of Lack [57].

Recall that for any category $\mathcal{C}$, by construction of $\mathscr{P}_{\mathcal{C}}$ we have $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathscr{P}_{\mathcal{C}})$. Moreover any functor
$\Sigma : \mathcal{C} \longrightarrow \mathcal{D}$ extends to a strict homomorphism $\mathscr{P}_{\Sigma} : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{P}_{\mathcal{D}}$.

Definition 2.4.2. *Let $F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ and $G : \mathscr{P}_{\mathcal{D}} \longrightarrow \mathscr{M}$ be respectively two path-objects of $(\mathscr{M}, \mathscr{W})$. An **$\mathscr{M}$-premorphism** from F to G, is a pair $\Sigma = (\Sigma, \sigma)$ consisting of a functor $\Sigma : \mathcal{C} \longrightarrow \mathcal{D}$ together with a transformation of morphisms of bicategories $\sigma : F \longrightarrow G \circ \mathscr{P}_{\Sigma}$*

$$\begin{tikzcd}
\mathscr{P}_{\mathcal{C}} \arrow[rr, "\mathscr{P}_{\Sigma}"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\sigma}{\Longrightarrow}"] & \mathscr{P}_{\mathcal{D}} \arrow[dl, "G"] \\
& \mathscr{M} &
\end{tikzcd}$$

An $\mathscr{M}$-premorphism is called an $\mathscr{M}$-morphism if all the 1*-cells σ_A are identities. In particular if A is over $U \in Ob(\mathscr{M})$ then so is ΣA (see* Figure 4 *below).*

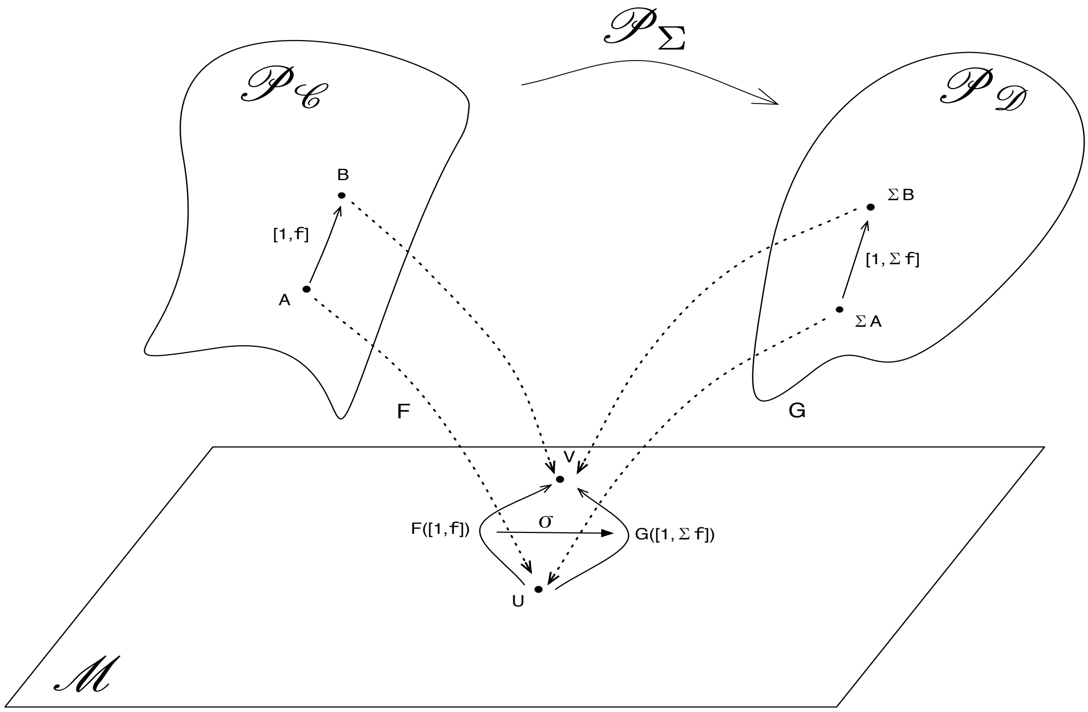


Figure 4

Observations 2.4.1. Given two sets X and Y, if F and G are respectively strict $\overline{X}$-point and $\overline{Y}$-point of $\mathscr{M}$, it's easy to check that an $\mathscr{M}$-morphism is exactly an $\mathscr{M}$-functor from $\mathscr{M}_F^X$ to $\mathscr{M}_G^Y$ in the classical sense.

2.4.3 Bimodules

Warning. We remind the reader that the composition in an $\mathscr{M}$-category is presented here in this order :

$$\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C).$$

Then each $\mathcal{C}(A,B)$ is a $(\mathcal{C}(B,B),\mathcal{C}(A,A))$-bimodule with $\mathcal{C}(B,B)$ **acting on the left** and $\mathcal{C}(A,A)$ **on the right**. But as one can see if the composition was presented as : $\mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \longrightarrow \mathcal{C}(A,C)$, then the action of $\mathcal{C}(B,B)$ would have been on the right.

We saw that a monoid T (or monad) in $(\mathscr{M},\mathscr{W})$ is given by a strict **1**-point that is a homomorphism:

$$T : \mathscr{P}_{\mathbf{1}} \longrightarrow \mathscr{M}.$$

Let **2** be the posetal category described as follows.
$\mathrm{Ob}(\mathbf{2}) = \{0,1\}$ and

$$\mathbf{2}(i,j) = \begin{cases} \{(i,j)\} & \text{if } i < j \\ \{\mathrm{Id}_i = (i,i)\} & \text{if } i = j \\ \varnothing & \text{if } i > j \end{cases}$$

The composition is the obvious one.

We have two functors : $\mathbf{1} \xrightarrow{i_0} \mathbf{2}$ and $\mathbf{1} \xrightarrow{i_1} \mathbf{2}$. These functors induce by functoriality two functors $\mathscr{P}_{i_0}$ and $\mathscr{P}_{i_0}$ from $\mathscr{P}_{\mathbf{1}}$ to $\mathscr{P}_{\mathbf{2}}$.

Definition 2.4.3. *Let T_0, T_1 be two Segal* **1***-points of $(\mathscr{M},\mathscr{W})$. A bimodule from T_0 to T_1 is a Segal path-object*

$$\Psi : \mathscr{P}_{\mathbf{2}} \longrightarrow \mathscr{M}$$

such that $\Psi \circ \mathscr{P}_{i_0} = T_0$ and $\Psi \circ \mathscr{P}_{i_1} = T_1$.

This definition has a natural generalization for every $\mathcal{C}$-point and $\mathcal{D}$-point of $(\mathscr{M},\mathscr{W})$.

The general case All through this work we've always identified monoids with enriched categories with one object. Now for bimodules in $\mathscr{M}$, e.g $\mathcal{C}(A,B)$, we want to identify them with **oriented enriched categories** having two objects. Here by 'oriented' we mean that there may not be a hom-object between some pair of objects.

We saw previously, that in some cases, given a $\mathcal{C}$-point $F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ we want to identify F with a generalized $\mathscr{M}$-category $\mathscr{M}_F^{\mathcal{C}}$. In the following we're going to express the classical notion of bimodule (also called distributor, profunctor or module) using path-objects. We will express everything in terms of morphisms of path-object but one should keep in mind that these definitions generalize the classical ones.

Our idea to define a bimodule in general between a $\mathcal{C}$-point and a $\mathcal{D}$-point is to consider an $\mathscr{E}$-point, where $\mathscr{E}$ contains both $\mathcal{C}$ and $\mathcal{D}$ together with an 'order' in $\mathscr{E}$ between the objects of $\mathcal{C}$ and $\mathcal{D}$. This lead us to the following.

Definition 2.4.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories. A **bridge** from $\mathcal{C}$ to $\mathcal{D}$ (resp. $\mathcal{D}$ to $\mathcal{C}$) is a category $\mathscr{E}$ equipped with two embedding [9] functors*

$$\mathscr{E}_{|\mathcal{C}} : \mathcal{C} \longrightarrow \mathscr{E}, \quad \mathscr{E}_{|\mathcal{D}} : \mathcal{D} \longrightarrow \mathscr{E}$$

such that for every A in $Ob(\mathcal{C})$ and B in $Ob(\mathcal{D})$ we have $\mathscr{E}(B, A) = \varnothing$ (resp. $\mathscr{E}(A, B) = \varnothing$). [10]

A morphism of bridges is a functor $\beta : \mathscr{E} \longrightarrow \mathscr{G}$ such that : $\beta \circ \mathscr{E}_{|\mathcal{C}} = \mathscr{G}_{|\mathcal{C}}$ and $\beta \circ \mathscr{E}_{|\mathcal{D}} = \mathscr{G}_{|\mathcal{D}}$.

*A bridge $\mathscr{E}$ is said to be **rigid** if $Ob(\mathscr{E}) \cong Ob(\mathcal{C}) \coprod Ob(\mathcal{D})$.*

Warning. We make no claim of inventing or introducing these notions. The theory of distributors (or profunctors or modules) is widely treated in the literature; the reader can find an account on distributors in [11],[20], [58], [86]. It was acknowledged to the author that what we call 'rigid bridge' appear in Street's paper [84], as a special case of *collage.*

Example 2.4.5.

1. The first example is the previous category **2** which is a bridge from **1** to **1**.

2. In the following we're going to construct the 'thin' bridge between any small categories.

   Let's denote by $\mathcal{C} \prec \mathcal{D}$ the small category described as follows.

   We take $\mathrm{Ob}(\mathcal{C} \prec \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \coprod \mathrm{Ob}(\mathcal{D})$.

$$[\mathcal{C} \prec \mathcal{D}](A, B) = \begin{cases} \mathcal{C}(A, B) & \text{if } (A, B) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}) \\ \mathcal{D}(A, B) & \text{if } (A, B) \in \mathrm{Ob}(\mathcal{D}) \times \mathrm{Ob}(\mathcal{D}) \\ \{(A, B)\} \cong 1 & \text{if } (A, B) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D}) \\ \varnothing & \text{if } (A, B) \in \mathrm{Ob}(\mathcal{D}) \times \mathrm{Ob}(\mathcal{C}) \end{cases}$$

   The composition is given by the following rules.

   - For A in $\mathrm{Ob}(\mathcal{C})$ and B in $\mathrm{Ob}(\mathcal{D})$, if O is an object of $\mathcal{C}$ then the composition c_{OAB} is the constant (unique) function which sends every pair $[f, (A, B)]$ to (O, B).
   - Similarly if P is an object of $\mathcal{D}$, then the composition c_{ABP} is the constant function which sends every pair $[(A, B), g]$ to (A, P).
   - The restriction of the composition to $\mathcal{C}$ (resp. to $\mathcal{D}$) is the original one.

   One easily check that $\mathcal{C} \prec \mathcal{D}$ is a category and we have two canonical embeddings : $i_{\mathcal{C}} : \mathcal{C} \longrightarrow (\mathcal{C} \prec \mathcal{D})$ and $i_{\mathcal{D}} : \mathcal{D} \longrightarrow (\mathcal{C} \prec \mathcal{D})$.

Remark 2.4.2. It's easy to see that $(\mathcal{C} \prec \mathcal{D})$ is the terminal rigid bridge. In some cases depending of the base $\mathscr{M}$ we will only consider this terminal rigid bridge.

Notation 2.4.1. We will denote by $\mathscr{P}_{\mathcal{C} \hookrightarrow \mathscr{E}}$ and $\mathscr{P}_{\mathcal{D} \hookrightarrow \mathscr{E}}$ the induced embeddings on the 2-path-categories.

[9] By 'embedding' we mean injective on object and fully faithful

[10] We've identified A with $\mathscr{E}_{|\mathcal{C}}(A)$ and B with $\mathscr{E}_{|\mathcal{D}}(B)$

Bridges and classical bimodules (or distributors) are connected by the following proposition that we attribute to Street [83, Theorem 6.13]. Since our context is very simple, we give a direct proof using less technology than in Street's paper.

Proposition 2.4.6. *We have an equivalence categories between the following data.*

1. *A distributor $\mathscr{X} : \mathcal{D} \nrightarrow \mathcal{C}$ (equivalently a functor $\mathscr{X} : \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$);*
2. *A rigid bridge $\mathscr{E}$ from $\mathcal{C}$ to $\mathcal{D}$*

This equivalence is an equivalence of categories.

Sketch of proof. Given a bridge $\mathscr{E}$ from $\mathcal{C}$ to $\mathcal{D}$ one defines the associated distributor $\mathscr{X}(\mathscr{E}) : \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$ by the functor of points $\mathscr{X}(\mathscr{E})(D) := \mathrm{Hom}(\mathscr{E}_{|\mathcal{C}}(-), D)$ and similarly on morphisms $\mathscr{X}(\mathscr{E})(f) := \mathrm{Hom}(\mathscr{E}_{|\mathcal{C}}(-), f)$.

Conversely given a distributor $\mathscr{X} : \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$, we define the associated bridge $\mathscr{E}(\mathscr{X})$ as follows.

Set $\mathrm{Ob}(\mathscr{E}(\mathscr{X})) = \mathrm{Ob}(\mathcal{C}) \coprod \mathrm{Ob}(\mathcal{D})$.

The restriction of $\mathscr{E}$ to $\mathrm{Ob}(\mathcal{C})$ (resp. $\mathrm{Ob}(\mathcal{D})$) is equal to $\mathcal{C}$ (resp. $\mathcal{D}$) and for A in $\mathrm{Ob}(\mathcal{C})$ and D in $\mathrm{Ob}(\mathcal{D})$ we take $\mathscr{E}(A, D) := \mathscr{X}(D)(A)$.

We define the composition in the following manner.

– For a triple of object (A, A', D) with A, A' in $\mathrm{Ob}(\mathcal{C})$ and D in $\mathrm{Ob}(\mathcal{D})$, the composition function is given by

$$c_{AA'D} : \mathscr{X}(D)(A') \times \mathcal{C}(A, A') \longrightarrow \mathscr{X}(D)(A)$$

which sends each element (a, f) of $\mathscr{X}(D)(A') \times \mathcal{C}(A, A')$ to $\mathscr{X}(D)(f)a$.

– Similarly given D, D' two objects of $\mathcal{D}$ and A an object of $\mathcal{C}$ then

$$c_{ADD'} : \mathcal{D}(D, D') \times \mathscr{X}(D)(A) \longrightarrow \mathscr{X}(D')(A)$$

sends an element (g, b) of $\mathcal{D}(D, D') \times \mathscr{X}(D)(A)$ to $\mathscr{X}(g)_A(b)$, where $\mathscr{X}(g)_A$ is the component at A of the natural transformation $\mathscr{X}(g) : \mathscr{X}(D) \longrightarrow \mathscr{X}(D')$.

■

Definition 2.4.7. *Let $F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ and $G : \mathscr{P}_{\mathcal{D}} \longrightarrow \mathscr{M}$ be respectively two Segal $\mathcal{C}$-point and $\mathcal{D}$-point of $(\mathscr{M}, \mathscr{W})$ and $\mathscr{E}$ a rigid bridge from $\mathcal{C}$ to $\mathcal{D}$,*

– *An $\mathscr{E}$-(G, F)-bimodule $\Psi : G \nrightarrow F$ is a Segal $\mathscr{E}$-point of $(\mathscr{M}, \mathscr{W})$*

$$\Psi : \mathscr{P}_{\mathscr{E}} \longrightarrow \mathscr{M}$$

satisfying the 'boundary conditions': $\Psi \circ \mathscr{P}_{\mathcal{C} \hookrightarrow \mathscr{E}} = F$ and $\Psi \circ \mathscr{P}_{\mathcal{D} \hookrightarrow \mathscr{E}} = G$

– *Given Ψ_1, Ψ_2 two $\mathscr{E}$-(G, F)-bimodules, a morphism of bimodules from Ψ_1 to Ψ_2 is an $\mathscr{M}$-morphism $(\mathrm{Id}_{\mathscr{E}}, \Theta)$ which induces the identity on both F and G.*

- *More generally, let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two rigid bridges from $\mathcal{C}$ to $\mathcal{D}$ and Ψ_1 (resp. Ψ_2) be an $\mathscr{E}_1$-(G,F)-bimodule (resp. $\mathscr{E}_2$-(G,F)-bimodule).*

  A morphism of (G,F)-bimodules from Ψ_1 to Ψ_2 is an $\mathscr{M}$-morphism

$$\Sigma = (\Sigma, \sigma) : \Psi_1 \longrightarrow \Psi_2$$

  such that the induced morphism from Ψ_1 to $\Sigma^\star\Psi_2$ is a morphism of $\mathscr{E}_1$-(G,F)-bimodules. Here $\Sigma^\star\Psi_2$ is the obvious pullback of Ψ_2 along Σ.

Observations 2.4.2.

1. To understand what's really happening in this definition it suffices to write it when $\mathcal{C} = \overline{X}$, $\mathcal{D} = \overline{Y}$, $\mathscr{E} = (\overline{X} \prec \overline{Y})$ and F, G and Ψ are respectively strict $\overline{X}$-point, $\overline{Y}$-point and $(\overline{X} \prec \overline{Y})$-point of $\mathscr{M}$.

   Let $\Psi =_G \Psi_F$ be an $(\overline{X} \prec \overline{Y})$-strict point of $\mathscr{M}$. Given a pair (P,Q) of objects of $\overline{X}$ and an object R of $\overline{Y}$, we have by definition of Ψ the following span of the same type as the ones which give the composition in the categories F and F.

$$\begin{tikzcd}
 & \Psi([2,(P,Q,R)]) \arrow[dl, "\varphi"'] \arrow[dr, "\Psi\{[2,(P,Q,R)] \xrightarrow{!} [1,(P,R)]\}"] & \\
\Psi([1,(Q,R)]) \otimes \Psi([1,(P,Q)]) \arrow[rr, dotted, "c_{PQR}"] & & \Psi([1,(P,R)])
\end{tikzcd}$$

   And the condition $\Psi \circ \mathscr{P}_{i_{\mathcal{C}}} = F$ says that $\Psi([1,(P,Q)]) = F_{PQ}([1,(P,Q)]) = F(P,Q)$ and we have a map

$$c_{PQR} : \Psi([1,(Q,R)]) \otimes F(P,Q) \longrightarrow \Psi([1,(P,R)]).$$

   Similarly if we take one object Q in $\overline{X}$ and two objects R, S in $\overline{Y}$, we will have a map

$$c_{QRS} : \Psi([1,(Q,R)]) \otimes G(R,S) \longrightarrow \Psi_{QS}([1,(Q,S)]).$$

   It's clear that these data together with unity and the associativity coherences contained in the definition of Ψ give a bimodule (also called distributor, profunctor or module) from F to G in the classical sense.

2. We have the classical fact any $\mathscr{M}$-morphism $\Sigma = (\Sigma, \sigma)$ from F to G induces two bimodules : $\Sigma^\star : F \nrightarrow G$ and $\Sigma_\star : G \nrightarrow F$, see [11], [20] [58] [86], for a description.

Remark 2.4.3.

1. We can define the classical operations such as composite or 'tensor product' of a $\mathscr{E}$-(G,F)-bimodule by another $\mathscr{E}'$-(F,D)-bimodule but the existence of such (G,D)-bimodule will involve some cocompleteness conditions on the hom-categories in $\mathscr{M}$. The idea consists to consider the 'composite bridge' which is given by the composite of the corresponding distributors and define a path-object satisfying the 'boundary conditions'.

2. With this composite we can define a category of 'enriched distributors' in a suitable manner. We will come back to this when we will give a model structure in [4].

For the moment we will assume that $\mathscr{M}$ is 'big and good' enough to have all these operations. We will denote by $\mathscr{M}$-Dist the bicategory described as follows.

- Objects are Segal path-objects $(\mathcal{C}, F)$
- Morphisms are Bimodules
- 2-morphisms are morphism of bimodules.

Presheaves on path-object

For a given Segal path-object $F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$, denote by $\mathscr{M}_F^{\mathcal{C}}$ the corresponding generalized Segal $\mathscr{M}$-category. In the following we give the definition of the analogue of a presheaf on $\mathscr{M}_F^{\mathcal{C}}$, that is functor from $(\mathscr{M}_F^{\mathcal{C}})^{op}$ to $\mathscr{M}$. When $\mathscr{M}$ is $(\mathbf{ChVect}, \otimes_{\mathbf{k}}, \mathbf{k})$, then $\mathscr{M}_F^{\mathcal{C}}$ will be a generalized DG-category, and a functor $\mathscr{A} : (\mathscr{M}_F^{\mathcal{C}})^{op} \longrightarrow \mathscr{M}$ is sometimes called '$\mathscr{A}$-DG-module' or simply $\mathscr{A}$-module. So in general we may call such a functor an $\mathscr{M}$-module, like in [86].

Notation 2.4.2.

1. We will denote by η the 'generic object' of $\mathscr{M}$, which consist to select an object U of $\mathscr{M}$ with it's identity arrow Id_U. We have
$$\eta_U : \mathscr{P}_{\mathbf{1}} \xrightarrow{[U, \mathrm{Id}_U]} \mathscr{M}$$
which express Id_U as the trivial monoid. We will identify η_U with U. When $\mathscr{M}$ has one object, hence a monoidal category there is only one generic object. η_U is sometimes denoted simply U or $\hat{U}$.

2. For an object A of $\mathcal{C}$, we have the canonical distributor $\mathcal{Y}_A : \mathbf{1} \longrightarrow \widehat{\mathcal{C}}$ which consists to select the functor of points $\mathcal{C}(-, A)$. We will denote by $\mathscr{E}(\mathcal{C})_A$ the associated bridge from $\mathcal{C}$ to $\mathbf{1}$ given by Proposition 2.4.3.

   Taking $\mathbf{1} = \{\star, \mathrm{Id}_\star\}$, $\mathscr{E}(\mathcal{C})_A$ is described as follows.

   - $\mathrm{Ob}(\mathscr{E}(\mathcal{C})_A) = \mathrm{Ob}(\mathcal{C}) \coprod \{\star\}$
   - For every B in $\mathrm{Ob}(\mathcal{C})$ we define $\mathscr{E}(\mathcal{C})_A(B, \star) := \mathcal{C}(B, A)$ and , $\mathscr{E}(\mathcal{C})_A(\star, B) := \varnothing$
   - We take $\mathscr{E}(\mathcal{C})_A(\star, \star) = \{\mathrm{Id}_\star\}$

   The composition is the obvious one and we check easily that $\mathscr{E}(\mathcal{C})_A$ is a rigid bridge from $\mathcal{C}$ to $\mathbf{1}$.

3. In general a distributor $\mathbf{1} \longrightarrow \widehat{\mathcal{C}}$ consists precisely to select an object of $\widehat{\mathcal{C}}$, say $\mathcal{Z}$, and we will denote by $\mathscr{E}_{\mathcal{Z}}$ the corresponding bridge from $\mathcal{C}$ to $\mathbf{1}$.

Definition 2.4.8. *Let $F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ be a Segal path-object. We denote by $\mathcal{P}F$ the category described as follows.*

1. *Objects are (η_U, F)-bimodules i.e Segal path-object $\Psi : \mathscr{P}_{\mathscr{E}(\eta_U)} \longrightarrow \mathscr{M}$ with $\mathscr{E}(\eta_U)$ a rigid bridge from $\mathcal{C}$ to $\mathbf{1}$, such that the 'boundary conditions' are satisfied:*
   $\Psi \circ \mathscr{P}_{\mathcal{C} \hookrightarrow \mathscr{E}(\eta_U)} = F$ and $\Psi \circ \mathscr{P}_{\mathbf{1} \hookrightarrow \mathscr{E}(\eta_U)} = \eta_U$.

2. *For Ψ_U, Ψ_V respectively in $\mathscr{M}$-Dist(η_U, F), $\mathscr{M}$-Dist(η_V, F), a morphism $\alpha : \Psi_U \longrightarrow \Psi_V$ when it exists, is the object of $\mathscr{M}$-Dist$(\eta_U, \eta_V) = \mathscr{M}(U, V)$ who represents the functor*

$$\mathscr{M}\text{-}Dist[\Psi_V \otimes -, \Psi_U] : \mathscr{M}\text{-}Dist(\eta_U, \eta_V) \longrightarrow \mathscr{M}\text{-}Dist(\eta_U, F)$$

Relative Yoneda functor We have a relative F-Yoneda functor $\mathcal{Y}_F : \mathcal{C} \longrightarrow \mathcal{P}F$ who sends an object A of $\mathcal{C}$ to a path-object $\mathcal{Y}_{F,A} : \mathscr{P}_{\mathscr{E}(\mathcal{C})_A} \longrightarrow \mathscr{M} \in \mathscr{M}$-Dist$(\eta_{FA}, F)$, described as follows.

Recall that here $\mathscr{E}(\mathcal{C})_A$ is the rigid bridge from $\mathcal{C}$ to $\mathbf{1}$ obtained by the distributor $\mathcal{Y}_A : \mathbf{1} \longrightarrow \widehat{\mathcal{C}}$. To define $\mathcal{Y}_{F,A}$ we need to specify the image of a chain $[1, P \xrightarrow{\gamma} \star]$ which generated the other chains of $\mathscr{E}(\mathcal{C})_A$ ending at $\star$. But for this it suffices to specify only for $P = A$, because the morphisms in $\mathscr{E}(\mathcal{C})_A$ between P and $\star$ are generated by $\mathcal{C}(P, A)$ and the morphism between A and $\star$. But in some sense we can think $\star$ 'as' a copy of A, which means that A has a *'multiplicity'*.

So to define the path-object $\mathcal{Y}_{F,A}$ we need to remove the discrepancy between the actions of A and $\star$. We do it by sending every chain $[1, A \xrightarrow{\gamma} \star]$ to the identity arrow Id_{FA}. More generally for a chain $[n, s]$ ending at $\star$, we take the image of $[n, s]$ to be the image of $[n', s']$, where s' is the 'longest' chain ending at A contained in s.

When $\mathcal{C} = \overline{X}$ and F is a strict path-object, then $\mathcal{Y}_{F,A}$ is just the classical Yoneda functor, see for example [86].

2.4.4 Base Change and Reduction

Definition 2.4.9. *Given two bases of enrichment $(\mathscr{M}_1, \mathscr{W}_1)$, $(\mathscr{M}_2, \mathscr{W}_2)$, a morphism of bases is a homomorphism of bicategories $L : \mathscr{M}_1 \longrightarrow \mathscr{M}_2$ such that $L(\mathscr{W}_1) \subseteq \mathscr{W}_2$.*
Then if $(\mathcal{C}, F)$ is a point of $(\mathscr{M}_1, \mathscr{W}_1)$ it follows immediately that $(\mathcal{C}, L \circ F)$ is a point of $(\mathscr{M}_2, \mathscr{W}_2)$. This operation is called ***base change along L****.*

Proposition 2.4.10. *Let $(\mathscr{M}, \mathscr{W})$ be a base of enrichment. There exists a bicategory $\mathscr{W}^{-1}\mathscr{M}$ together with a homomorphism $L_{\mathscr{W}} : \mathscr{M} \longrightarrow \mathscr{W}^{-1}\mathscr{M}$ such that:*

1. *$L_{\mathscr{W}}$ makes $\mathscr{W}$ invertible,*

2. *any homomorphism $\Phi : \mathscr{M} \longrightarrow \mathscr{B}$ which makes $\mathscr{W}$ invertible factorizes as $\Phi = \overline{\Phi} \circ L_{\mathscr{W}}$ with*

$$\overline{\Phi} : \mathscr{W}^{-1}\mathscr{M} \longrightarrow \mathscr{B}$$

   a homomorphism.

3. *$\mathscr{W}^{-1}\mathscr{M}$ is unique up to isomorphism.*

Remark 2.4.4. The above result is restricted to the underlying 1-category of the 2-category of bicategories, homomorphisms and icons. The reader who might be interested can generalize the above result and make $\mathscr{W}^{-1}\mathscr{M}$ weakly universal i.e universal in the 2-category.

Proof. See Appendix 4.9. ■

Definition 2.4.11. *Let $(\mathscr{M}, \mathscr{W})$ be a base of enrichment and $L_{\mathscr{W}} : \mathscr{M} \longrightarrow \mathscr{W}^{-1}\mathscr{M}$ a localization. For any Segal point $(\mathcal{C}, F)$ of $(\mathscr{M}, \mathscr{W})$ the pair $(\mathcal{C}, L_{\mathscr{W}} \circ F)$ is called* a reduction *of $(\mathcal{C}, F)$. It's a strict Segal point of $\mathscr{W}^{-1}\mathscr{M}$. We will denote for short $[F] = (\mathcal{C}, L_{\mathscr{W}} \circ F)$.*

Definition 2.4.12. *A morphism of Segal $\mathscr{M}$-categories $\sigma : F \longrightarrow G$ is a weak equivalence of Segal $\mathscr{M}$-categories if the induced map $[\sigma] : [F] \longrightarrow [G]$ is a classical equivalence of $\mathscr{W}^{-1}\mathscr{M}$-categories.*

The above definition concerns only morphisms between Segal $\mathscr{M}$-categories, which is not sufficient since we need to work in the largest category of Segal path-objects (=Segal $\mathscr{M}$-precategories). The natural notion of weak equivalences therein needs a **"Segalification functor" Seg** that takes a Segal $\mathscr{M}$-precategory F to a Segal $\mathscr{M}$-category $\mathbf{Seg}(F)$.

Having at hand the functor **Seg**, we can define a morphism $\sigma : F \longrightarrow G$ of Segal $\mathscr{M}$-precategories to be a weak equivalences if $\mathbf{Seg}(\sigma)$ is a weak equivalence of Segal $\mathscr{M}$-category in the sense of Definition 2.4.12. This new definition involving **Seg** agrees with old one for a morphism of Segal $\mathscr{M}$-categories; this is because **Seg** preserves the homotopy type and $\mathscr{W}$ has the 3-for-2 property.

Lax diagrams and Enrichment: co-Segal categories

3.1 Introduction

We pursue here the idea initiated in the previous chapter of having a theory of weakly enriched categories over a symmetric monoidal model category $\mathscr{M} = (\underline{M}, \otimes, I)$. We introduce the notion of co-Segal $\mathscr{M}$-category. The main idea is this. Rather than specifying a composition law

$$\mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \longrightarrow \mathcal{C}(A,C)$$

we have instead the following diagram:

$$\begin{tikzcd} & \mathcal{C}(A,C) \arrow[d, "\wr"'] \\ \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \arrow[r] & \mathcal{C}(A,B,C) \end{tikzcd}$$

where the vertical map $\mathcal{C}(A,C) \longrightarrow \mathcal{C}(A,B,C)$ is required to be a *weak equivalence*. As one can see when this weak equivalence is an isomorphism or an identity (the strict case) then we will have a classical composition and everything is as usual. In the non-strict case, one gets a weak composition given by any choice of a weak inverse of that vertical map.

The previous diagram is obtained by 'reversing the morphisms' in the Segal situation, hence the terminology 'co-Segal'. The diagrams below outline this idea:

$$\begin{tikzcd} & \mathcal{C}(A,C) \\ \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) & \mathcal{C}(A,B,C) \arrow[l, "\sim"'] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{C}(A,C) \arrow[d, "\wr"'] \\ \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \arrow[r] & \mathcal{C}(A,B,C) \end{tikzcd}$$

In a Segal category **In a co-Segal category**

If the tensor product $\otimes$ of the category $\mathscr{M} = (\underline{M}, \otimes, I)$ is different from the cartesian product $\times$ e.g $\mathscr{M}$ is a Tannakian category, the so called *Segal map* $\mathcal{C}(A,B,C) \longrightarrow \mathcal{C}(A,B) \otimes \mathcal{C}(B,C)$ appearing in the Segal situation is not 'natural'; it's a map going into a product where there is no *a priori* a way to have a projection on each factor. The co-Segal formalism was introduced precisely to bypass this problem.

In [3], following an idea introduced by Leinster [61], we define a Segal enriched category $\mathcal{C}$ having a set of objects X, as a <u>colax</u> morphism of 2-categories

$$\mathcal{C} : \mathscr{P}_{\overline{X}} \longrightarrow \mathscr{M}$$

satisfying the usual Segal conditions. As we shall see a co-Segal category is defined as a <u>lax</u> morphism of 2-categories

$$\mathcal{C} : (\mathbb{S}_{\overline{X}})^{\text{2-op}} \longrightarrow \mathscr{M}$$

satisfying the co-Segal conditions (Definition 3.4.7). Here $\mathscr{P}_{\overline{X}}$ is a 2-category over Δ^+ while $\mathbb{S}_{\overline{X}} \subset \mathscr{P}_{\overline{X}}$ is over Δ^+_{epi}. These 2-categories are probably examples of what we called a *locally Reedy 2-category*, that is a 2-category such that each category of 1-morphisms is a Reedy category and the composition is coherent with the Reedy structures.

To develop a homotopy theory of these co-Segal categories we follow the same philosophy as for Segal categories, that is we consider the more general objects consisting of lax morphisms $\mathcal{C} : (\mathbb{S}_{\overline{X}})^{2\text{-op}} \longrightarrow \mathscr{M}$ without demanding the co-Segal conditions yet; these are called *co-Segal precategories.*

As X runs through **Set** we have a category $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ of all *co-Segal precategories* with morphisms between them. We have a natural Grothendieck bifibration $\text{Ob} : \mathscr{M}_{\mathbb{S}}(\mathbf{Set}) \longrightarrow \mathbf{Set}$.

The main result in this chapter is the following

Theorem. *Let $\mathscr{M}$ be a symmetric monoidal model category which is cofibrantly generated and such that all the objects are cofibrant. Then the following holds.*

1. *the category $\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})$, of co-Segal precategories admits a model structure which is cofibrantly generated,*
2. *fibrant objects are co-Segal categories,*
3. *If $\mathscr{M}$ is combinatorial then so is $\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})$.*

Plan

We begin this chapter with a review the definition of a lax diagram in a 2-category $\mathscr{M}$, which are simply lax functors of 2-category in the sense of Bénabou [10]. We point out that $\mathscr{M}$-categories are special cases of lax diagrams as earlier observed by Street [86].

Then in section 3.3 we recall some basic definitions about multisorted operads or colored operads. The idea is to use the powerful language of operads to treat 2-categories and lax morphisms in terms of $\mathcal{O}$-algebras and lax morphisms of $\mathcal{O}$-algebras for some suitable operad. The operads we're working with are the ones enriched in **Cat**.

Given two $\mathcal{O}$-algebras $\mathcal{C}_{\bullet}$ and $\mathcal{M}_{\bullet}$ there is a category $\text{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_{\bullet}, \mathcal{M}_{\bullet})$ of lax morphisms and morphism of lax morphisms. After setting up some definitions we prove that:

- for a locally presentable $\mathcal{O}$-algebra $\mathcal{M}_{\bullet}$ the category $\text{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_{\bullet}, \mathcal{M}_{\bullet})$ is also locally presentable (Theorem 3.3.8);
- If $\mathcal{M}_{\bullet}$ is a special Quillen $\mathcal{O}$-algebra (Definition 3.3.9) and under some hypothesis, the category $\text{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_{\bullet}, \mathcal{M}_{\bullet})$ carries a model structure (Theorem 3.3.12).

In section 3.4 we introduce the language of co-Segal categories starting with an overview of the one-object case. We've tried as much as possible to make this section independent from the previous ones. We only use the language of lax functor between 2-categories rather than lax morphisms of $\mathcal{O}$-algebras. We introduce first the notion of an $\mathbb{S}$-diagram in $\mathscr{M}$ which correspond to *co-Segal precategories* (Definition 3.4.6). Then we define a co-Segal category to be an $\mathbb{S}$-diagram satisfying the **co-Segal conditions** (Definition 3.4.8). After giving some definitions we show that

- A strict co-Segal $\mathscr{M}$-category is the same thing as a strict (semi) $\mathscr{M}$-category (Proposition 3.4.9);
- The co-Segal conditions are stable under weak equivalences (Proposition 3.4.12).

In section 3.5 we show that the category $\mathscr{M}_{\mathbb{S}}(X)$ of co-Segal precategories with a fixed set of objects X is:

- is cocomplete if $\mathscr{M}$ is so (Theorem 3.5.2) ; and
- locally presentable if $\mathscr{M}$ is so (Theorem 3.5.1).

For both of these two theorems, we've presented a 'direct proof' i.e which doesn't make use of the language of operads; the idea is to make the content accessible for a reader who is not familiar with operads.

In section 3.6 we consider the notion of locally Reedy 2-category. The main idea is to provide a *direct* model structure on the category $\mathscr{M}_{\mathbb{S}}(X)$ (Corollary 3.6.15).

In section 3.7 we give two type of model structures on $\mathscr{M}_{\mathbb{S}}(X)$, using a different method. These model structures play an important role in the later sections. We show precisely that if $\mathscr{M}$ is a symmetric monoidal model category, which is cofibrantly generated and such that all the objects are cofibrant, then we have:

- a *projective* model structure on $\mathscr{M}_{\mathbb{S}}(X)$ denoted $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ (Theorem 3.7.6);
- an *injective* model structure on $\mathscr{M}_{\mathbb{S}}(X)$ denoted $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ (Theorem 3.7.7);
- the identity functor $\text{Id} : \mathscr{M}_{\mathbb{S}}(X)_{\text{proj}} \rightleftarrows \mathscr{M}_{\mathbb{S}}(X)_{\text{inj}} : \text{Id}$ is a Quillen equivalence (Corollary 3.7.8);

These model structures are both cofibrantly generated (and combinatorial if $\mathscr{M}$ is so). The projective model structure is the same as the one given by Corollary 3.6.15.

The section 3.8 is dedicated to study of the category $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ of all co-Segal precategories. We show that:

- $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ inherits the cocompleteness and local presentability of $\mathscr{M}$ (Theorem 3.8.2); and
- that $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ carries a *fibered projective* model structure which is cofibrantly generated. And if $\mathscr{M}$ is combinatorial then so is $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ (Theorem 3.8.8 and Corollary 3.8.13).

In section 3.9, we begin by constructing for each set X, an endofunctor $\mathscr{S} : \mathscr{M}_{\mathbb{S}}(X) \longrightarrow \mathscr{M}_{\mathbb{S}}(X)$, called *'co-Segalifcation'* which takes any co-Segal precategory to a co-Segal category (Proposition 3.9.7). Assuming that $\mathscr{M}_{\mathbb{S}}(X)$ is left proper if $\mathscr{M}$ is so (Hypothesis 3.9.1) we prove that:

- There exists a *new injective* model structure on $\mathscr{M}_{\mathbb{S}}(X)$ denoted $\mathscr{M}_{\mathbb{S}}(X)^{+}_{\text{inj}}$ which is combinatorial and such that the fibrant objects are co-Segal categories. $\mathscr{M}_{\mathbb{S}}(X)^{+}_{\text{inj}}$ is the left Bousfield localization of $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ with respect to some set of maps $\mathbf{K}_{\text{inj}}$ (Theorem 3.9.12).
- There is also a *new projective* model structure on $\mathscr{M}_{\mathbb{S}}(X)$ denoted $\mathscr{M}_{\mathbb{S}}(X)^{+}_{\text{proj}}$ which is combinatorial and such that the fibrant objects are co-Segal categories. The model category $\mathscr{M}_{\mathbb{S}}(X)^{+}_{\text{proj}}$ is the left Bousfield localization of $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ with respect to some set of maps $\mathbf{K}_{\text{proj}}$ (Theorem 3.9.21).
- We have also a *new fibered projective* model structure on $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ denoted $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}$ which is combinatorial and such that the fibrant objects are co-Segal categories (Theorem 3.9.24).

In section 3.10 we reviewed the basics about $\mathscr{M}$-categories for a 2-category $\mathscr{M}$. For a fixed set of objects X, we show that if $\mathscr{M}$ is locally a model category (Definition 3.10.1) and all the objects are cofibrant, then the category $\mathscr{M}$- $\mathbf{Cat}(X)$ has a model structure which is cofibrantly generated and combinatorial if $\mathscr{M}$ is so. We leave the reader who might be interested to give a fibered model structure on $\mathscr{M}$-**Cat** and even the 'canonical model structure' in the sense of Berger-Moerdijk [13].

It seems clear that all the previous results on co-Segal categories should hold if we replace the monoidal model category $\mathscr{M}$ by a 2-category which is locally a model category.

3.2 Lax Diagrams

Warning. By "2-category" we mean the same thing as bicategory; that is the composition is associative up-to natural isomorphisms and the identities are invariant up-to natural isomorphisms. By "strict 2-category" we mean a 2-category where the composition is strictly associative and the identities are strictly invariant.

In the following we fix a 2-category $\mathscr{M}$. For a sufficiently large universe $\mathbb{V}$ we will assume that all the 2-categories we will consider (including $\mathscr{M}$) have a $\mathbb{V}$-small set of 2-morphisms.

Definition 3.2.1. *A* ***lax diagram*** *in $\mathscr{M}$ is a lax morphism $F : \mathcal{D} \longrightarrow \mathscr{M}$, where $\mathcal{D}$ is a strict 2-category.*

For each $\mathcal{D}$ we will consider $\text{Lax}(\mathcal{D}, \mathscr{M})$ the 1-category of lax morphisms from $\mathcal{D}$ to $\mathscr{M}$ and **icons** in the sense of Lack [57].

- The objects of $\text{Lax}(\mathcal{D}, \mathscr{M})$ are lax morphisms,
- the morphisms are *icons* (see [57]) .

Icons are what we call later **simple transformations** (Definition 3.4.10). The reader can find for example in [57, 59] these definitions.

Warning. Note that in general there is only a 2-category (which is not a strict 2-category) of lax morphisms. This 2-category is described as follows

- The objects of are lax morphisms,
- the 1-morphisms are transformations of lax morphisms,
- the 2-morphisms are modifications of transformations.

By definition of a lax morphism $F : \mathcal{D} \longrightarrow \mathscr{M}$ we have a function between the corresponding set of objects

$$\text{Ob}(F) : \text{Ob}(\mathcal{D}) \longrightarrow \text{Ob}(\mathscr{M}).$$

This defines a function $\text{Ob} : \text{Ob}[\text{Lax}(\mathcal{D}, \mathscr{M})] \longrightarrow \text{Hom}[\text{Ob}(\mathcal{D}), \text{Ob}(\mathscr{M})]$ which sends F to $\text{Ob}(F)$.

Given a function ϕ from $\text{Ob}(\mathcal{D})$ to $\text{Ob}(\mathscr{M})$ we will say that $F \in \text{Lax}(\mathcal{D}, \mathscr{M})$ is *over* ϕ if $\text{Ob}(F) = \phi$. We will denote by $\text{Lax}_{/\phi}(\mathcal{D}, \mathscr{M})$ be the full subcategory of $\text{Lax}(\mathcal{D}, \mathscr{M})$ consisting of objects over ϕ and transformations of lax morphisms.

$\mathscr{M}$-categories are lax morphisms Given an $\mathscr{M}$-category $\mathcal{C}$ having a set of objects X, then we can define a lax morphism denoted again $\mathcal{C} : \overline{X} \longrightarrow \mathscr{M}$. The category $\overline{X}$ is defined in the previous chapter (see 2.2.5); we refer to it as the *indiscrete* or *coarse* category associated to X. In this context one interprets the lax morphism as the nerve of the enriched category.

This identification of $\mathscr{M}$-categories as lax morphisms goes back to Bénabou [10] as pointed out by Street [86]. Bénabou defined them as *polyads* as the plural form of monad.

We pursue the spirit of this identification which is somehow the 'universal lax situation'.

3.3 Operads and Lax morphisms

In the following we use the language of *multisorted operads* also called 'colored operads' to treat the theory of 2-categories and lax functors as $\mathcal{O}$-algebras and morphism of $\mathcal{O}$-algebras of a certain multisorted operad $\mathcal{O}$. When there is no confusion we will simply say operads to mean multisorted operads.

Although the results of this section will be stated for a general operad $\mathcal{O}$, one should keep in mind the special case where $\mathcal{O}$ is the operad we will see in the Example 3.3.1 below.

We recall briefly hereafter the definition of the type of operad we will consider. For a detailed definition of these one can see, for example, [14] or [62]. In the later reference multisorted operads are called *multicategories*.

Let C be a set (thought as a set of colors or sorts).

A C-multisorted operad $\mathcal{O}$ in **Cat**, or a **Cat**-operad, consists of the following data.

1. For all $n \geq 0$ and each $(n+1)$-tuple $(i_1, ..., i_n; j)$ of elements of C there is a category $\mathcal{O}(i_1, ..., i_n; j)$.
2. For each $i \in C$, we have an identity operation expressed as a functor $\mathbf{1} \xrightarrow{1_i} \mathcal{O}(i; i)$, where $\mathbf{1}$ is the unit category.
3. There is a composition operation:
$$\mathcal{O}(i_1, ..., i_n; j) \times \mathcal{O}(h_{1,1}, ..., h_{1,k_1}; i_1) \times \cdots \times \mathcal{O}(h_{n,1}, ..., h_{n,k_n}; i_n) \longrightarrow \mathcal{O}(h_{1,1}, ..., h_{n,k_n}; j)$$
$$(\theta, \theta_1, ..., \theta_n) \mapsto \theta \circ (\theta_1, ..., \theta_n).$$
4. The composition satisfies associativity and unity conditions. The reader can find all the details in [62, Chap.2] or in [56, Part I].

When the set C has only one element (one color) we recover the definition of an operad.

Remark 3.3.1. In the condition (1) above, when $n = 0$ we have no color in 'input', so we will denote by $\mathcal{O}(0, i)$ this category. Here the '0' means zero input.

This category $\mathcal{O}(0, i)$ allows us to have an 'identity' or 'unity object' when we want to have the notion of *unital* $\mathcal{O}$-algebra.

For a fixed set of colors C, we have a category of C-multisorted operads in **Cat** with the obvious notion of morphism. The reader can find a definition in [14]. We follow the same notation as in [14] and will denote by $\mathrm{Oper}_C(\mathbf{Cat})$the category of C-multisorted operads in **Cat**. Similarly if $\mathscr{E}$ is a monoidal category, we will denote by $\mathrm{Oper}_C(\mathscr{E})$ the category of C-multisorted operads in $\mathscr{E}$.

Below we give an example of a multi-sorted operad which will play an important role in the upcoming sections. This is the multi-sorted operad whose algebras are 2-categories i.e enriched categories over **Cat**. The construction we present here is equivalent to the one given in [14, Section 1.5.4].

Example 3.3.1. Let X be a set and $\overline{X}$ be the associated indiscrete or coarse category. Recall that $\overline{X}$ is the category with X as the set objects and such that there is exactly one morphism between any pair of elements.

Let $C = X \times X$ be the set of pairs of elements of X. There is a one-to-one correspondence between C and the set of morphisms of $\overline{X}$. We will denote by $\mathscr{N}(\overline{X})$ the nerve of $\overline{X}$ and by $\mathscr{N}(\overline{X})_n$ its set of n-simplices.

We define a C-multisorted operad $\mathcal{O}_X$ as follows.

- for $n > 0$ we take

$$\mathcal{O}_X(i_1, ..., i_n; j) = \begin{cases} \mathbf{1} = \text{the unit category} & \text{if } (i_1, ..., i_n) \in \mathscr{N}(\overline{X})_n \text{ and } j = i_n \circ \cdots \circ i_1 \\ \varnothing = \text{the empty category} & \text{if not} \end{cases}$$

- For $n = 0$ we set

$$\mathcal{O}_X(0, i) = \begin{cases} \mathbf{1} & \text{if } i = \mathrm{Id}_A \text{ in } \overline{X} \text{ i.e } i = (A, A) \text{ for some } A \in X \\ \varnothing & \text{if not} \end{cases}$$

- The 'identity-operation' functor $\mathbf{1} \longrightarrow \mathcal{O}_X(i, i)$ is the identity $\mathrm{Id}_{\mathbf{1}}$.
- The composition:

$$\mathcal{O}_X(i_1, ..., i_n; j) \times \mathcal{O}_X(h_{1,1}, ..., h_{1,k_1}; i_1) \times \cdots \times \mathcal{O}_X(h_{n,1}, ..., h_{n,k_n}; i_n) \longrightarrow \mathcal{O}_X(h_{1,1}, ..., h_{n,k_n}; j)$$

  is either one of the (unique) functors:

$$\begin{cases} \mathbf{1} \times \cdots \times \mathbf{1} \xrightarrow{\cong} \mathbf{1} \\ \varnothing \xrightarrow{\mathrm{Id}} \varnothing \\ \varnothing \longrightarrow \mathbf{1} \end{cases}$$

- The associativity and unity axioms are straightforward.

We will see in a moment that an $\mathcal{O}_X$-algebra in **Cat** is equivalent to a 2-category having X as its set of objects.

Claim. Given a category $\mathcal{B}$, If we replace everywhere $\overline{X}$ by $\mathcal{B}$ in the construction above, one gets a multisorted operad $\mathcal{O}_{\mathcal{B}}$ in **Cat** where the set of colors C is the set $\mathrm{Arr}(\mathcal{B})$ of all morphisms of $\mathcal{B}$. An $\mathcal{O}_{\mathcal{B}}$-algebra is the same thing as a lax morphism from $\mathcal{B}$ to $(\mathbf{Cat}, \times, \mathbf{1})$.
And more generally given a symmetric monoidal category $\mathscr{M} = (\underline{M}, \otimes, I)$ having an initial object $\mathbf{0}$, we can construct a multisorted $\mathscr{M}$-operad $\mathcal{O}_{\mathcal{B}}$, replacing $\mathbf{1}$ and $\varnothing$ respectively by I and $\mathbf{0}$. As in the previous case an $\mathcal{O}_{\mathcal{B}}$-algebra in $\mathscr{M}$ will be the same thing as a lax morphism from $\mathcal{B}$ to $\mathscr{M}$.

Definition 3.3.2. *Let $\mathcal{O}$ be a C-multisorted operad in* **Cat**.

*An $\mathcal{O}$-**algebra** $\mathcal{M}$. is given by the following data.*

- *For each $i \in C$ we have a category $\mathcal{M}_i$.*
- $\mathcal{M}_0 = \mathbf{1}$.

- *For each $(n+1)$-tuple $(i_1, ..., i_n; j)$ of elements of C there is a functor:*

$$\mathcal{O}(i_1, ..., i_n; j) \times \mathcal{M}_{i_1} \times \cdots \times \mathcal{M}_{i_n} \xrightarrow{\theta_{i_\bullet|j}} \mathcal{M}_j$$

- *We have also a functor $\mathcal{O}(0, i) \times \mathcal{M}_0 \longrightarrow \mathcal{M}_i$.*
- *These functors are compatible with the associativity and unity of the composition of $\mathcal{O}$.*

Notation 3.3.1. Given $(x, m_1, ..., m_n) \in \mathcal{O}(i_1, ..., i_n; j) \times \mathcal{M}_{i_1} \times \cdots \times \mathcal{M}_{i_n}$ we will use the suggestive notation $\otimes_x(m_1, ..., m_n) = \theta_{i_\bullet|j}(x, m_1, ..., m_n)$. The idea is to think each functor $\theta_{i_\bullet|j}(x, -)$ as a general tensor product.

The following proposition shows us how the theory of lax functors and operads are related within the theory of enriched categories.

Proposition 3.3.3. *Let X be a set. We have an equivalence between the following data.*

i) An $\mathcal{O}_X$-algebra in **Cat***,*

ii) A 2-category with X as the set of objects.

iii) A lax morphism $F : \overline{X} \longrightarrow (\mathbf{Cat}, \times, \mathbf{1})$

Remark 3.3.2. We can also include a fourth equivalence between the strict homomorphism from $\mathscr{P}_{\overline{X}}$ to $(\mathbf{Cat}, \times, \mathbf{1})$, where $\mathscr{P}_{\overline{X}}$ is the 2-path category associated to $\overline{X}$ (see [3]). And as claimed above, one can replace everywhere $\overline{X}$ by an arbitrary category $\mathcal{B}$. The fourth equivalence will be a homomorphism from $\mathscr{P}_{\mathcal{B}}$ to $(\mathbf{Cat}, \times, \mathbf{1})$.

Sketch of proof. The equivalence between ii) and iii) is well known and is left to the reader. We simply show how we get a 2-category from an $\mathcal{O}_X$-algebra. The implication ii) $\Rightarrow$ i) will follow immediately by 'reading backwards' the argumentation we present hereafter.

Let $\mathcal{M}_\bullet$ be an $\mathcal{O}_X$-algebra in **Cat**. We construct a 2-category $\mathcal{M}$ as follows.

1. $\mathrm{Ob}(\mathcal{M}) = X$

2. Given a pair $(A, B) \in X^2 = C$, we have a category $\mathcal{M}_{(A,B)}$ and we set $\mathcal{M}(A, B) = \mathcal{M}_{(A,B)}$.

3. Given A, B, C in X, if we set $i_1 = (A, B), i_2 = (B, C)$ and $j = (A, C)$ we have $\mathcal{O}(i_1, i_2; j) = \mathbf{1}$ and the functor $\mathcal{O}_X(i_1, i_2; j) \times \mathcal{M}_{i_1} \times \mathcal{M}_{i_2} \longrightarrow \mathcal{M}_j$ gives the composition:

$$\mathcal{M}(A, B) \times \mathcal{M}(B, C) \xrightarrow[\text{canonical}]{\cong} \mathbf{1} \times \mathcal{M}(A, B) \times \mathcal{M}(B, C) \longrightarrow \mathcal{M}(A, C).$$

4. Each $\mathcal{O}_X(i, i)$ acts trivially on $\mathcal{M}_i$ i.e the map $\mathcal{O}_X(i, i) \times \mathcal{M}_i \longrightarrow \mathcal{M}_i$ is the canonical isomorphism $\mathbf{1} \times \mathcal{M}_i \xrightarrow{\cong} \mathcal{M}_i$.

5. One gets the associativity of the composition in $\mathcal{M}$ using the fact the following functors are invertible and have the same codomain:

   - $\mathcal{O}_X(i_1 \circ i_2, i_3; j) \times \mathcal{O}_X(i_1, i_2; i_1 \circ i_2) \times \mathcal{O}_X(i_3; i_3) \xrightarrow{\cong} \mathcal{O}_X(i_1, i_2, i_3; j)$

- $\mathcal{O}_X(i_1, i_2 \circ i_3; j) \times \mathcal{O}_X(i_1, i_1) \times \mathcal{O}_X(i_2, i_3; i_2 \circ i_3) \xrightarrow{\cong} \mathcal{O}_X(i_1, i_2, i_3; j)$

with $i_1 = (A, B), i_2 = (B, C), i_3 = (C, D)$ and $i_1 \circ i_2 = (A, C)$, $i_2 \circ i_3 = (B, D)$. This provides a natural isomorphism between the domains of the two functors. Putting these together with the fact that the action of $\mathcal{O}_X$ on $\mathcal{M}_\bullet$ is compatible with the composition of $\mathcal{O}_X$, we get the desired natural isomorphism expressing the associativity of the composition in $\mathcal{M}$.

6. For each i of the form (A, A) we have $\mathcal{O}_X(0, i) = \mathbf{1}$ and the unity condition of the algebra provides a morphism $\mathbf{1} \longrightarrow \mathcal{M}(A, A)$ which satisfies the desired conditions of an identity morphism in a 2-category.

■

The functor Ob : **Cat** $\longrightarrow$ **Set** which sends a category to its set of objects, commutes with the cartesian product, so that it's actually a (strict) monoidal functor. As a consequence we get a functor

$$\mathrm{Ob} : \mathrm{Oper}_C(\mathbf{Cat}) \longrightarrow \mathrm{Oper}_C(\mathbf{Set}).$$

For $\mathcal{R} \in \mathrm{Oper}_C(\mathbf{Set})$ and $\mathcal{O} \in \mathrm{Oper}_C(\mathbf{Cat})$ we will say that $\mathcal{O}$ *is over* $\mathcal{R}$ if $\mathrm{Ob}(\mathcal{O}) = \mathcal{R}$.

Remark 3.3.3. It's not hard to see that since the functor Ob is monoidal, for any $\mathcal{O}$-algebra $\mathcal{M}_\bullet$ then $\mathrm{Ob}(\mathcal{M}_\bullet)$ is automatically an $\mathrm{Ob}(\mathcal{O})$-algebra.

3.3.1 Lax morphism of $\mathcal{O}$-algebra

We now consider the type of morphism of $\mathcal{O}$-algebras we are going to work with. Our definition is different than the standard definition of morphism of algebras. The idea is to recover the definition of lax functor between 2-categories when $\mathcal{O}$ is of the form $\mathcal{O}_X$.

Definition 3.3.4. *Let $\mathcal{O}$ be an object of Oper$_C$(**Cat**) and $\mathcal{C}_\bullet$, $\mathcal{M}_\bullet$ be two $\mathcal{O}$-algebras.*

*A **lax** morphism $\mathcal{F}_\bullet : \mathcal{C}_\bullet \longrightarrow \mathcal{M}_\bullet$ of $\mathcal{O}$-algebras, or simply a lax $\mathcal{O}$-morphism,is given by the following data and axioms.*

Data:

- *A family of functors $\{\mathcal{F}_i : \mathcal{C}_i \longrightarrow \mathcal{M}_i\}_{i \in C}$.*
- *For each $(n+1)$-tuple $(i_1, ..., i_n; j)$, a family of natural transformations $\{\varphi = \varphi(i_\bullet; j)\}$:*

$$\begin{tikzcd}
\mathcal{O}(i_1, ..., i_n; j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n} \arrow[r, "\theta"] \arrow[d, "\mathrm{Id} \times \mathcal{F}_{i_1} \times \cdots \times \mathcal{F}_{i_n}"'] & \mathcal{C}_j \arrow[d, "\mathcal{F}_j"] \\
\mathcal{O}(i_1, ..., i_n; j) \times \mathcal{M}_{i_1} \times \cdots \times \mathcal{M}_{i_n} \arrow[r, "\rho"'] \arrow[ur, phantom, "\Nearrow \varphi"] & \mathcal{M}_j
\end{tikzcd}$$

$$\otimes_x(\mathcal{F}_{i_1} c_1, ..., \mathcal{F}_{i_n} c_n) \xrightarrow{\varphi(x, c_1, ... c_n)} \mathcal{F}_j[\otimes_x(c_1, ..., c_n)]$$

Axioms: *The natural transformations $\varphi_{i_\bullet|j}(-)$ satisfy the following coherence conditions, which are the ' 2-dimensional' analogue of those satisfied by $\theta_{i_\bullet|j}(-)$ and $\rho_{i_\bullet|j}(-)$:*

$$\varphi_{i_\bullet|j} \otimes_\theta \{\mathrm{Id}_{\mathrm{Id}_{\mathcal{O}(i_\bullet|j)}} \times [(\prod_i \varphi_{h_{i,\bullet}|i}) \otimes \mathrm{Id}_{shuffle}]\} = \varphi_{h_{\bullet,\bullet}|j} \otimes \{\mathrm{Id}_{\gamma_{h_{\bullet,\bullet}|i_\bullet|j}} \times \mathrm{Id}_{\mathrm{Id}_{\prod \mathcal{M}_{h_{\bullet,\bullet}}}}\}.$$

More explicitly, given

- $(x, x_1, ..., x_n) \in \mathcal{O}(i_\bullet|j) \times \mathcal{O}(h_{1,\bullet}|i_1) \times \cdots \times \mathcal{O}(h_{n,\bullet}|i_n)$
- $(d_{1,1}, ..., d_{1,k_1}, ..., d_{n,1}, ..., d_{n,k_n}) \in \mathcal{M}_{h_{1,1}} \times \cdots \times \mathcal{M}_{h_{1,k_1}} \times \cdots \times \mathcal{M}_{h_{n,k_n}}$
- $\otimes_{\gamma(x,x_1,...,x_n)}(d_{1,1}, ..., d_{n,k_n}) = c$
- $\otimes_{x_i}(d_{i,1}, ..., d_{i,k_i}) = c_i$, $i \in \{1, ..., n\}$,
- $\otimes_x(c_1, ..., c_n) = c$
- $\varphi_i = \varphi(x_i, d_{i,1}, ..., d_{i,k_i}) : \otimes_{x_i}(\mathcal{F}d_{i,1}, ..., \mathcal{F}d_{i,k_i}) \longrightarrow \mathcal{F}[\otimes_{x_i}(d_{i,1}, ..., d_{i,k_i})] = \mathcal{F}c_i$

  we require the equality :

$$\varphi(\gamma(x, x_1, ..., x_n), d_{1,1}, ..., d_{n,k_n}) = \varphi(x, c_1, ...c_n) \circ [\otimes_x(\varphi_1, ..., \varphi_n)].$$

In the next paragraph we make some comments about the coherence conditions in the previous definition.

Coherences

The previous coherence can be easily understood when we think that the family of functors $\{\mathcal{F}_i : \mathcal{C}_i \longrightarrow \mathcal{M}_i\}_{i\in C}$ equipped with the family of natural transformations $\{\varphi_{i_\bullet|j}(x)\}_{x\in\mathrm{Ob}(\mathcal{O}(i_1,...,i_n;j))}$, is an $\mathcal{O}$-algebra of some arrow-category we are about to describe.

Let's consider $\mathrm{Arr}(\mathbf{Cat})_+$ the double category given by the following data.

- The objects are the arrows of **Cat** i.e functors F
- A morphism from F to G consists of a triple (α, β, φ) where α and β are functors and φ a natural transformation as shown in the following diagram:

$$\begin{tikzcd}
\cdot \arrow[r, "\alpha"] \arrow[d, "F"'] & \cdot \arrow[d, "G"] \\
\cdot \arrow[r, "\beta"'] \arrow[ur, phantom, "\varphi\,\Nearrow"] & \cdot
\end{tikzcd}$$

  We will represent such morphism as a column or a row: $\begin{pmatrix} \alpha \\ \beta \\ \varphi \end{pmatrix}$, $(\alpha; \beta; \varphi)$.

- The horizontal composition $\otimes_h$ and vertical composition $\otimes_v$ in $\mathrm{Arr}(\mathbf{Cat})_+$ are given as follows

$$\begin{pmatrix}\alpha'\\ \beta'\\ \varphi'\end{pmatrix}_{G\longrightarrow H} \otimes_h \begin{pmatrix}\alpha\\ \beta\\ \varphi\end{pmatrix}_{F\longrightarrow G} = \begin{pmatrix}\alpha'\circ\alpha\\ \beta'\circ\beta\\ \varphi'_{\alpha x}\circ\beta'(\varphi_x)\end{pmatrix}_{F\longrightarrow H}$$

$$\begin{pmatrix}\alpha\\ \beta\\ \varphi'\end{pmatrix}_{K\longrightarrow L} \otimes_v \begin{pmatrix}\beta\\ \gamma\\ \varphi\end{pmatrix}_{F\longrightarrow G} = \begin{pmatrix}\alpha\\ \gamma\\ L(\varphi_x)\circ\varphi'_{Fx}\end{pmatrix}_{KF\longrightarrow LG}$$

It's not hard to see that $\mathrm{Arr}(\mathbf{Cat})_+$ carries a monoidal structure with the cartesian product of functors where the unity is the identity functor $\mathrm{Id}_{\mathbf{1}}$. The product of two morphisms (α,β,φ) and $(\alpha',\beta',\varphi')$ is given by:

$$\begin{pmatrix}\alpha\\ \beta\\ \varphi\end{pmatrix} \times \begin{pmatrix}\alpha'\\ \beta'\\ \varphi'\end{pmatrix} = \begin{pmatrix}\alpha\times\alpha'\\ \beta\times\beta'\\ \varphi\times\varphi'\end{pmatrix}$$

Remark 3.3.4. We have a functor $\mathbf{Cat} \hookrightarrow \mathrm{Arr}(\mathbf{Cat})_+$ sending a natural transformation $\sigma : F \longrightarrow G$ to $(\mathrm{Id}_s;\mathrm{Id}_t;\sigma)$ where s and t are the source and target of both F and G.

Given an object F of $\mathrm{Arr}(\mathbf{Cat})_+$, we will use the notation $\mathcal{O}\odot F := \mathrm{Id}_{\mathcal{O}}\times F$. With the monoidal category $(\mathrm{Arr}(\mathbf{Cat})_+,\times,\mathrm{Id}_{\mathbf{1}})$ we can say that the coherence conditions on $\varphi_{i_\bullet|j}$ are equivalent to saying that the family $\{\mathcal{F}_i\}_{i\in C}$ is an $\mathcal{O}$-algebra in $(\mathrm{Arr}(\mathbf{Cat})_+,\times,\mathrm{Id}_{\mathbf{1}})$ where the maps

$$\mathcal{O}(i_1,...,i_n;j)\odot[\mathcal{F}_{i_1}\times\cdots\times\mathcal{F}_{i_n}] \longrightarrow \mathcal{F}_j$$

are given by the family of triples $(\theta_{i_\bullet|j},\rho_{i_\bullet|j},\varphi_{i_\bullet|j})$.

Composition of lax $\mathcal{O}$-morphisms

Let $\mathcal{C}_\bullet,\mathcal{M}_\bullet,\mathcal{N}_\bullet$ be three $\mathcal{O}$-algebras and $(\mathcal{F}_\bullet,\varphi_{i_\bullet|j}) : \mathcal{C}_\bullet \longrightarrow \mathcal{M}_\bullet$, $(\mathcal{G}_\bullet,\psi_{i_\bullet|j}) : \mathcal{M}_\bullet \longrightarrow \mathcal{N}_\bullet$ be two lax $\mathcal{O}$-morphisms.

We define the composite $\mathcal{G}_\bullet\circ\mathcal{F}_\bullet$ to be the lax $\mathcal{O}$-morphism given by the following data.

- The family of functors $\{\mathcal{G}_i\circ\mathcal{F}_i : \mathcal{C}_i \longrightarrow \mathcal{N}_i\}_{i\in C}$.
- For each $(n+1)$-tuple $(i_1,...,i_n;j)$, the family of natural transformations $\{\chi_{i_\bullet|j}(x)\}_{x\in\mathrm{Ob}(\mathcal{O}(i_1,...,i_n;j))}$ where:

$$\chi_{i_\bullet|j}(x) = G_j[\varphi_{i_\bullet|j}(x)]\circ\psi_{i_\bullet|j}(x)_{\prod\mathcal{F}_i(-)}.$$

- More precisely the component of $\chi_{i_\bullet|j}(x)$ at $(c_1,...,c_n)\in\mathcal{C}_{i_1}\times\cdots\times\mathcal{C}_{i_n}$ is the morphism:

$$\chi_{i_\bullet|j}(x)_{(c_1,...,c_n)} = G_j[\varphi_{i_\bullet|j}(x)_{(c_1,...,c_n)}]\circ\psi_{i_\bullet|j}(x)_{(\mathcal{F}_{i_1}c_1,...,\mathcal{F}_{i_n}c_n)}.$$

We leave the reader to check that these data satisfy the coherence conditions of the Definition 3.3.4.

Remark 3.3.5.
The identity $\mathcal{O}$-morphism of an algebra $(\mathcal{M}_\bullet,\theta_{i_\bullet|j})$ is given by the family of functors $\{\mathrm{Id}_{\mathcal{M}_i}\}_{i\in C}$ and natural transformations $\{\mathrm{Id}_{\theta_{i_\bullet|j}(x)}\}_{x\in\mathrm{Ob}(\mathcal{O}(i_1,...,i_n;j))}$.

3.3.2 Morphisms of lax $\mathcal{O}$-morphisms

$\mathcal{O}$-algebras and lax $\mathcal{O}$-morphisms form naturally a category. But there is an obvious notion of 2-morphism we now describe. A 2-morphism is the analogue of the transformations of lax functors.

Definition 3.3.5. *Let $(\mathcal{F}_\bullet, \varphi_{i_\bullet|j})$ and $(\mathcal{F}_\bullet', \varphi'_{i_\bullet|j})$ be two lax $\mathcal{O}$-morphisms from $\mathcal{C}_\bullet$ to $\mathcal{M}_\bullet$.*

A 2-morphism $\sigma_\bullet : \mathcal{F}_\bullet \longrightarrow \mathcal{F}_\bullet'$ is given by the following data and axioms.

Data: *A family of natural transformations $\{\sigma_i : \mathcal{F}_i \longrightarrow \mathcal{F}_i'\}_{i \in C}$.*

Axioms: *For any $x \in \mathcal{O}(i_1, ..., i_n; j)$, and any $(c_1, ..., c_n) \in \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n}$, the following commutes :*

$$\begin{tikzcd}
\otimes_x(\mathcal{F}_{i_1}c_1, ..., \mathcal{F}_{i_n}c_n) \arrow[r, "\varphi(x,c_1,...,c_n)"] \arrow[d, "\otimes_x(\sigma_{i_1,c_1},...,\sigma_{i_n,c_n})"'] & \mathcal{F}_j[\otimes_x(c_1, ..., c_n)] \arrow[d, "\sigma_{j,\otimes_x(c_\bullet)}"] \\
\otimes_x(\mathcal{F}'_{i_1}c_1, ..., \mathcal{F}'_{i_n}c_n) \arrow[r, "\varphi'(x,c_1,...,c_n)"'] & \mathcal{F}'_j[\otimes_x(c_1, ..., c_n)]
\end{tikzcd}$$

The composition of 2-morphisms is the obvious one i.e component-wise. We will denote by $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ the category of lax $\mathcal{O}$-morphisms between two $\mathcal{O}$-algebras $\mathcal{C}_\bullet$ and $\mathcal{M}_\bullet$.

3.3.3 Locally presentable $\mathcal{O}$-algebras

Below we extend the notion of locally presentable category $\mathscr{M}$ to $\mathcal{O}$-algebras for an operad $\mathcal{O} \in \mathrm{Oper}_C(\mathbf{Cat})$.

Definition 3.3.6. *Let $(\mathcal{M}_\bullet, \theta_{i_\bullet|j})$ be an $\mathcal{O}$-algebra. We say that $\mathcal{M}_\bullet$ is a locally presentable $\mathcal{O}$-algebra if the following conditions holds.*

- *For every $i \in C$ the category $\mathcal{M}_i$ is a locally presentable category in the usual sense.*
- *For every $(i_1, ..., i_n; j)$ the functor $\theta_{i_\bullet|j}$ preserves the colimits on each factor 'i_k' $(1 \leq k \leq n)$ that is for every $(m_l)_{l \neq k} \in \prod_{l, l \neq k} \mathcal{M}_{i_l}$ and every $x \in \mathcal{O}(i_1, ..., i_n; j)$ the functor*

  $$\theta_{i_\bullet|j}(x; (m_l)) := \theta_{i_\bullet|j}(x; ..., m_l, ...m_{k-1}, -, m_{k+1}, ...) : \mathcal{M}_{i_k} \longrightarrow \mathcal{M}_j$$

  preserves all colimits.

Example 3.3.7.

1. If $\mathcal{O}$ is the operad of enriched categories, then any symmetric closed monoidal category $\mathscr{M}$ which is locally presentable is automatically a locally presentable $\mathcal{O}$-algebra. The second condition of the definition follows from the fact that being closed monoidal implies that the tensor product of $\mathscr{M}$ (which is a left adjoint) preserves colimits on each factor.
2. More generally any biclosed monoidal category $\mathscr{M}$ (see [49, 1.5]), not necessarily symmetric, which is locally presentable is a locally presentable $\mathcal{O}$-algebra.
3. Any 2-category (or bicategory) such that the composition preserves the colimits on each factor and all the category of morphisms are locally presentable, is a locally presentable $\mathcal{O}_X$-algebra for the operad $\mathcal{O}_X$ of the Example 3.3.1.

Remark 3.3.6. In the same way we will say that $\mathcal{M}_\bullet$ is a cocomplete $\mathcal{O}$-algebra if all the $\mathcal{M}_i$ are cocomplete and if the second condition of the previous definition holds.

The main result in this section is the following.

Theorem 3.3.8. *Let $\mathcal{M}_\bullet$ be a locally presentable $\mathcal{O}$-algebra. For any $\mathcal{O}$-algebra $\mathcal{C}_\bullet$ the category of lax $\mathcal{O}$-morphisms $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ is locally presentable.*

Proof. See Appendix 4.4 ■

3.3.4 Special Quillen $\mathcal{O}$-algebra

In the following we consider an *ad-hoc* notion of Quillen $\mathcal{O}$-algebra.

Definition 3.3.9. *Let $(\mathcal{M}_\bullet, \theta_{i_\bullet|j})$ be an $\mathcal{O}$-algebra. We say that $\mathcal{M}_\bullet$ is a **special Quillen $\mathcal{O}$-algebra** if the following conditions holds.*

1. *$\mathcal{M}_\bullet$ is complete and cocomplete,*
2. *For every $i \in C$ the category $\mathcal{M}_i$ is a Quillen closed model category in the usual sense.*
3. *For every $x \in \mathcal{O}(i_1, ..., i_n; j)$, the functor $\otimes_x$ preserves (trivial) cofibrations with cofibrant domain. This means that for every n-tuple of morphisms $(g_k)_k$ in $\mathcal{M}_{i_1} \times \cdots \times \mathcal{M}_{i_n}$, such that each g_k has a cofibrant domain, then $\otimes_x(g_1, ..., g_n)$ is a (trivial) cofibration in $\mathcal{M}_j$ if all $g_1, ..., g_n$ are (trivial) cofibrations.*

Say that $\mathcal{M}_\bullet$ is cofibrantly generated if all the $\mathcal{M}_i$ are cofibrantly generated. Similarly if each $\mathcal{M}_i$ is combinatorial we will say that $\mathcal{M}_\bullet$ is combinatorial.

Example 3.3.10.

- Any model category is obviously a special Quillen $\mathcal{O}$-algebra with the tautological operad (no operations except the 1-ary identity operation).
- Another example of special Quillen algebra is a symmetric monoidal model category. In fact using the pushout-product axiom one has that (trivial) cofibrations with cofibrant domain are closed by tensor product.

Remark 3.3.7. Note that in our definition we did not include a generalized pushout product axiom; it doesn't seem relevant, for our purposes, to impose this axioms in general. But if one is interested in having such axiom, a first approximation will be of course to mimic the monoidal situation. Below we give a sketchy one.

<u>Axiom</u>: Say that $\mathcal{M}_\bullet$ is pushout-product compatible if:

- for every $x \in \mathcal{O}(i_1, ..., i_n; j)$
- for every cofibrations $f : a_k \longrightarrow b_k \in \mathcal{M}_{i_k}$, $g : a_l \longrightarrow b_l \in \mathcal{M}_{i_l}$,
- for every $(n-2)$-tuple of cofibrant objects $(c_r)_{r \neq l, r \neq l}$

then the map

$$\delta : \otimes_x(-, a_k, -, b_l, -) \cup_{\otimes_x(-,a_k,-,a_l,-)} \otimes_x(-, b_k, -, a_l, -) \longrightarrow \otimes_x(-, b_k, -, b_l, -)$$

is a cofibration which is moreover a trivial cofibration if either f of g is.

$$\begin{tikzcd}
\otimes_x(c_1,...,a_k,-,a_l,...,c_n) \arrow[rr, "{\otimes_x(\mathrm{Id},...,\mathrm{Id},-,g,...,\mathrm{Id})}"] \arrow[dd, "{\otimes_x(\mathrm{Id},...,f,-,\mathrm{Id},...,\mathrm{Id})}"'] & & \otimes_x(c_1,...,a_k,-,b_l,...,c_n) \arrow[dd, "{\otimes_x(\mathrm{Id},...,f,-,\mathrm{Id},...,\mathrm{Id})}"] \arrow[dl, dotted] \\
& \cdot \arrow[dr, "\delta"] & \\
\otimes_x(c_1,...,b_k,-,a_l,...,c_n) \arrow[rr, "{\otimes_x(\mathrm{Id},...,\mathrm{Id},-,g,...,\mathrm{Id})}"'] \arrow[ur, dotted] & & \otimes_x(c_1,...,b_k,-,b_l,...,c_n)
\end{tikzcd}$$

The main result in this section is to say that under some hypothesis on the triple $(\mathcal{O}, \mathcal{C}_{\bullet}, \mathcal{M}_{\bullet})$ then there is a model structure on $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_{\bullet}, \mathcal{M}_{\bullet})$. We don't know for the moment if we have the same result without any restriction. We will denote by $\mathcal{K}_{\mathcal{C}_{\bullet}} = \prod_i \mathrm{Hom}(\mathcal{C}_i, \mathcal{M}_i)$.

Definition 3.3.11. *Let $(\mathcal{C}_{\bullet}, \rho)$ and $(\mathcal{M}_{\bullet}, \theta)$ be two $\mathcal{O}$-algebras.*

1. *Say that $\mathcal{C}_{\bullet}$ is $\mathcal{O}$-**well-presented**, or $\mathcal{O}$-**identity-reflecting** (henceforth* **ir**-*$\mathcal{O}$-algebra) if for every $n+1$-tuple $(i_1, ..., i_n; j)$ the following functor reflects identities*

$$\rho : \mathcal{O}(i_1, ..., i_n; j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n} \longrightarrow \mathcal{C}_j.$$

   This means that the image of $(u, f_1, ..., f_n) \in \mathcal{O}(i_1, ..., i_n; j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n}$ is an identity morphism in $\mathcal{C}_j$ (if and) only if all $u, f_1, ..., f_n$ are simultaneously identities.

2. *Say that $(\mathcal{C}_{\bullet}, \mathcal{M}_{\bullet})$ is an $\mathcal{O}$-**homotopy-compatible pair** if $\mathbf{F} : \mathcal{K}_{\mathcal{C}_{\bullet}} \longrightarrow \mathcal{K}_{\mathcal{C}_{\bullet}}$ preserves level-wise trivial cofibrations, where $\mathcal{K}_{\mathcal{C}_{\bullet}}$ is endowed with the injective model structure. Here $\mathbf{F}$ is the left adjoint of the functor $\mathcal{U}$ which forgets the laxity maps (see Appendix 4.4.1).*

The motivation of these definitions is explained in the Appendix 4.5.2.
With the previous material we have

Theorem 3.3.12. *For an* **ir**-*$\mathcal{O}$-algebra $\mathcal{C}_{\bullet}$, and a special Quillen $\mathcal{O}$-algebra $\mathcal{M}_{\bullet}$ assume that*

- *$(\mathcal{C}_{\bullet}, \mathcal{M}_{\bullet})$ is an $\mathcal{O}$-homotopy compatible pair,*
- *all objects of $\mathcal{M}_{\bullet}$ are cofibrant,*
- *$\mathcal{M}$ is cofibrantly generated with $\mathbf{I}_{\bullet}$ (resp. $\mathbf{J}_{\bullet}$) the generating set of (trivial) cofibrations*

then there is a model structure on $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_{\bullet}, \mathcal{M}_{\bullet})$ which is cofibrantly generated. A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is

- *a weak equivalence if $\mathcal{U}\sigma$ is a weak equivalence in $\mathcal{K}_{\mathcal{C}_{\bullet}}$,*
- *a fibration if $\mathcal{U}\sigma$ is a fibration in $\mathcal{K}_{\mathcal{C}_{\bullet}}$,*
- *a cofibration if it has the LLP with respect to all maps which are both fibrations and weak equivalences,*
- *a trivial cofibration if it has the LLP with respect to all fibrations.*
- *the set $\mathbf{F}(\mathbf{I}_{\bullet})$ and $\mathbf{F}(\mathbf{J}_{\bullet})$ constitute respectively the set of generating cofibrations and trivial cofibrations in $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_{\bullet}, \mathcal{M}_{\bullet})$.*

The pair

$$\mathcal{U} : \mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_{\bullet}, \mathcal{M}_{\bullet}) \leftrightarrows \prod_i \mathrm{Hom}(\mathcal{C}_i, \mathcal{M}_i) : \mathbf{F}$$

is a Quillen pair, where $\mathbf{F}$ is left Quillen and $\mathcal{U}$ right Quillen.

Proof. The idea is to transfer the (product) model structure on $\mathcal{K}_{\mathcal{C}_\bullet} = \prod_i \mathrm{Hom}(\mathcal{C}_i, \mathcal{M}_i)$ through the monadic adjunction $\mathbf{F} \dashv \mathcal{U}$ using a lemma of Schwede-Shipley [77]. In fact $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ is equivalent to $\mathcal{T}$-alg for the monad $\mathcal{T} = \mathcal{U}\mathbf{F}$. The method is exactly the same as in the proof of theorem 3.7.6.
All we have to check is that the pushout of $\mathbf{F}\sigma$ is a weak equivalence for every generating trivial cofibration σ in $\mathcal{K}_{\mathcal{C}_\bullet}$. This is exposed in the Appendix 4.5.2. ■

An alternative description of $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$

In the following we fix a multi-sorted operad $\mathcal{O}$ and consider two $\mathcal{O}$-algebra $\mathcal{C}_\bullet$ and $\mathcal{M}_\bullet$. Our goal is to describe the category $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ as subcategory of $\mathrm{Lax}_{\mathcal{O}'\text{-alg}}(\mathbf{1}_\bullet, \mathcal{M}_\bullet)$ for some operad $\mathcal{O}' = \mathcal{O}_{\mathcal{C}_\bullet} = \int_{\mathcal{C}_\bullet}$; here $\mathbf{1}_\bullet$ is the terminal algebra. This will simplify many constructions such as pushouts and colimit in general.

Definition of $\mathcal{O}_{\mathcal{C}_\bullet}$

By definition of $\mathcal{C}_\bullet$, for each $(n+1)$-tuple we have an action of $\mathcal{O}$ given by a functor

$$\theta_{i_\bullet|j} : \mathcal{O}(i_1, ..., i_n; j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n} \longrightarrow \mathcal{C}_j.$$

When there is no confusion we will omit the subscript and will write simply θ.

The set D of colors or sorts of $\mathcal{O}_{\mathcal{C}_\bullet}$ is the set of object of $\mathcal{C}_\bullet$, that is $D = \coprod_{i \in C} \mathrm{Ob}(\mathcal{C}_i)$.
Given an $(n+1)$-tuple $(c_1, ..., c_n, c_j) \in \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n} \times \mathcal{C}_j$, we define the category of operations $\mathcal{O}_{\mathcal{C}_\bullet}(c_1, ..., c_n, c_j)$ as follows:

- the objects are pairs (x, h), with $x \in \mathcal{O}(i_1, ..., i_n; j)$ and $h : \theta(x, c_1, ..., c_n) \longrightarrow c_j$ a morphism in $\mathcal{C}_j$
- a morphism from (x, h) to (y, k) is a morphism $u : x \longrightarrow y$ in $\mathcal{O}(i_1, ..., i_n; j)$ such that $h = k \circ \theta(u, c_1, ..., c_n)$; or equivalently $\theta(u, c_\bullet)$ is a morphism from h to k in the slice category $\mathcal{C}_j/c_j$.

If γ is the multiplication or substitution of $\mathcal{O}$, then we define the associated multiplication $\gamma_{\mathcal{C}_\bullet}$ in the natural way to be a mixture of γ and θ.

Given $[(x_i, h_i)]_{1 \le i \le n}$ with $(x_i, h_i) \in \mathcal{O}_{\mathcal{C}_\bullet}(d_{i,1}, ..., d_{1,k_i}, c_i)$ then we set

$$\gamma_{\mathcal{C}_\bullet}[(x_1, h_1), ..., (x_n, h_n)] := [\gamma(x_1, ..., x_n), \theta(d_{1,1}, ..., d_{n,k_n})].$$

Proposition 3.3.13. *The data $\mathcal{O}_{\mathcal{C}_\bullet}(c_1, ..., c_n, c_j)$ with $\gamma_{\mathcal{C}_\bullet}$ constitute a D-multisorted* **Cat***-operad.*

Proof. The associativity of $\gamma_{\mathcal{C}_\bullet}$ follows from the associativty of γ and θ. ■

Remark 3.3.8. Note that there is a function $p : D \longrightarrow C$ between the set of colours which is just the subscript-reading operation: for $c_i \in \mathrm{Ob}(\mathcal{C}_i)$ $p(c_i) = i$. Pulling back $\mathcal{O}$ along p we get a D multisorted operad $p^\star \mathcal{O}$.
We have then that for each $(c_1, ..., c_n, c_j) \in D^{n+1}$, $\mathrm{p}^\star\, \mathcal{O}(c_1, ..., c_n, c_j) = \mathcal{O}(i_1, ..., i_n; j)$.
The projection on the first factor is a functor $\pi : \mathcal{O}_{\mathcal{C}_\bullet}(c_1, ..., c_n, c_j) \longrightarrow \mathcal{O}(i_1, ..., i_n; j)$ ($\pi(x, h) = x$) and it's not hard to see that these functors π fit coherently to form a morphism of D-multisorted operads denoted again $\pi : \mathcal{O}_{\mathcal{C}_\bullet} \longrightarrow \mathrm{p}^\star\, \mathcal{O}$.

For an $\mathcal{O}$-algebra $\mathcal{M}$, by p we have an $\mathrm{p}^\star\, \mathcal{O}$-algebra and by π we have an $\mathcal{O}_{\mathcal{C}_\bullet}$-algebra $\pi^\star[\mathrm{p}^\star\, \mathcal{M}_\bullet]$. When there is no confusion we will simply write $\pi^\star\, \mathcal{M}_\bullet$.

Definition 3.3.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be two small* 1*-categories. A* ***prefunctor*** $F : \mathcal{C} \longrightarrow \mathcal{M}$ *is an object given by the same data and axioms of a functor except the preservation of identities, that is we do not require to have* $F(\mathrm{Id}_A) = \mathrm{Id}_{FA}$ *for* $A \in \mathcal{C}$*.*

In other terms a prefunctor is the same thing as a morphism between the underlying graphs which is compatible with the composition on both sides.
The compatibility of the composition forces each $F(\mathrm{Id}_A)$ to be an idempotent in $\mathcal{M}$. Obviously any functor is a prefunctor.

In the same way given two $\mathcal{O}$-algebras $\mathcal{C}_{\bullet}$ and $\mathcal{M}_{\bullet}$ a **prelax $\mathcal{O}$-morphism** $\mathcal{F}_{\bullet} : \mathcal{C}_{\bullet} \longrightarrow \mathcal{M}_{\bullet}$ is the same thing as a lax $\mathcal{O}$-morphism except that each $\mathcal{F}_i : \mathcal{C}_i \longrightarrow \mathcal{M}_i$ is a prefunctor.

Proposition 3.3.15. *Let $\mathcal{C}_{\bullet}$ and $\mathcal{M}_{\bullet}$ be two $\mathcal{O}$-algebras. We have an equivalence between the following data:*

1. *a* ***prelax $\mathcal{O}$-morphism*** *from $\mathcal{C}_{\bullet}$ to $\mathcal{M}_{\bullet}$,*
2. *a* ***lax $\mathcal{O}_{\mathcal{C}_{\bullet}}$-morphism*** *from* $\mathbf{1}_{\bullet}$ *to* $\pi^{\star}\mathcal{M}_{\bullet}$*.*

Proof. Simply write the definition of each object. ■

3.4 co-Segal Categories

3.4.1 The one-object case

Conventions.

- By **semi-monoidal category** we mean the same structure as a monoidal category except that no unit object is required. Obviously any monoidal category has an underlying semi-monoidal category.

- A **lax functor** between semi-monoidal categories is the same thing as a lax functor between monoidal categories without the data involving the units. A strict functor will be call as well 'monoidal functor'.

- More generally we will say **semi-bicategory** (resp. semi-2-category) to be the same thing as bicategory (resp. 2-category) except that we don't require the identity 1-morphisms.

- We have also the notion of lax morphism, transformation of lax morphisms, between semi-bicategories in the natural way.

- For a semi-bicategory $\mathscr{A}$ and a bicategory $\mathscr{B}$, a lax morphism from $\mathscr{A}$ to $\mathscr{B}$ will be a morphism from $\mathscr{A}$ to the underlying semi-bicategory of $\mathscr{B}$ which will be denoted again $\mathscr{B}$.

In the following we fix $\mathscr{M} = (\underline{\mathrm{M}}, \otimes, I)$ a monoidal category.

3.4.2 Overview

As we identify $\mathscr{M}$-categories with one object and monoids of $\mathscr{M}$, we shall expect that a co-Segal category with one object will be a kind of *homotopical semi-monoid*[1] of $\mathscr{M}$. We will call them *co-Segal semi-monoids*.

To define a co-Segal category $\mathcal{C}$ with one object A, we need a sequence of objects of $\mathscr{M}$

$$\begin{cases} \mathcal{C}(A,A) \rightsquigarrow \mathcal{C}(\mathbf{1}) & : \text{ the 'hom-space' of } A \\ \mathcal{C}(A,A,A) \rightsquigarrow \mathcal{C}(\mathbf{2}) & \\ \quad \cdots & \\ \mathcal{C}(\mathbf{n} * A) = \mathcal{C}(\underbrace{A, ..., A}_{(n+1)\text{-A}}) \rightsquigarrow \mathcal{C}(\mathbf{n}) & n \geq 1 \end{cases}$$

together with the following data.

1. A diagram expressing a 'quasi-multiplication'

$$\begin{tikzcd} & \mathcal{C}(A,A) \arrow[d, "\gamma_0"', "\text{weak.equiv}"] \\ \mathcal{C}(A,A) \otimes \mathcal{C}(A,A) \arrow[ur, dotted, "\text{generic lifting}"] \arrow[r, "\mu_{1,1}"] & \mathcal{C}(A,A,A) \end{tikzcd}$$

[1] In the standard terminology we would have said 'up-to-homotopy' monoid but this terminology is already used for another notion of weak monoid (see [3]).

2. Some other semi-multiplications: $\mathcal{C}(\mathbf{n} * A) \otimes \mathcal{C}(\mathbf{m} * A) \xrightarrow{\mu_{n,m}} \mathcal{C}((\mathbf{n}+\mathbf{m}) * A)$

3. In the 'semi-cubical' diagram below each face is commutative and the weak equivalences γ_i must satisfy : $\gamma_1 \circ \gamma_0 = \gamma_2 \circ \gamma_0$ to have an associativity up-to homotopy.

$$\begin{tikzcd}
 & \mathcal{C}(A,A) \otimes \mathcal{C}(A,A) \otimes \mathcal{C}(A,A) \arrow[dl, "\mu_{1,1} \otimes \mathrm{Id}"'] \arrow[rr, "\mathrm{Id} \otimes \mu_{1,1}"] & & \mathcal{C}(A,A) \otimes \mathcal{C}(A,A,A) \arrow[dl, "\mu_{1,2}"] \\
\mathcal{C}(A,A,A) \otimes \mathcal{C}(A,A) \arrow[rr, "\mu_{2,1}"'] & & \mathcal{C}(A,A,A,A) & \\
 & & & \mathcal{C}(A,A) \otimes \mathcal{C}(A,A) \arrow[uu, "\mathrm{Id} \otimes \gamma_0"'] \arrow[dl, "\mu_{1,1}"] \\
\mathcal{C}(A,A) \otimes \mathcal{C}(A,A) \arrow[uu, "\gamma_0 \otimes \mathrm{Id}"] \arrow[rr, "\mu_{1,1}"'] & & \mathcal{C}(A,A,A) \arrow[uu, bend left, "\gamma_1"] \arrow[uu, bend right, "\gamma_2"'] & \\
 & & \mathcal{C}(A,A) \arrow[u, "\gamma_0"] &
\end{tikzcd}$$

4. We have other commutative diagrams of the same type as above which give the coherences of this weak associativity of the quasi-multiplication etc.

As one can see when all of the maps 'γ_i' are isomorphisms we will have the data of semi-category with one object i.e a semi-monoid of $\mathscr{M}$. In this case we know from Mac Lane [68] that a semi-monoid in $\mathscr{M}$ is given by a monoidal functor:

$$\mathscr{N}(\mathcal{C}) : (\Delta^+_{\mathrm{epi}}, +, \mathbf{0}) \longrightarrow \mathscr{M}$$

which we interpret as the nerve of the semi-enriched category $\mathcal{C}$ with one object.

Remark 3.4.1. The object $\mathbf{0}$ doesn't play any role here since there is no morphism from any another object to it. So we can restrict this functor to the underlyinng semi-monoidal categories (see Definition 3.4.2 below).

3.4.3 Definitions

Definition 3.4.1. *We will denote by* $(\Delta^+_{epi}, +)$ *be the semi-monoidal subcategory of* $(\Delta^+, +, \boldsymbol{0})$ *described as follows.*

- *The set of objects is:* $\mathrm{Ob}(\Delta^+) - \{\boldsymbol{0}\}$.

- *The morphisms are surjective maps $f : \boldsymbol{m} \longrightarrow \boldsymbol{n}$ of Δ^+*

Remark 3.4.2.

1. For a morphism $f : \mathbf{m} \longrightarrow \mathbf{n}$ in $(\Delta^+_{\text{epi}}, +)$, by definition f is surjective and nondecreasing then it follows that f preserves the 'endpoints' i.e $f(0) = 0$ and $f(m-1) = n-1$.

2. For $\mathbf{n} \geq \mathbf{1}$ we denote by σ_i^n the unique map of $(\Delta^+_{\text{epi}}, +)$ from $\mathbf{n+1}$ to $\mathbf{n}$ such that $\sigma_i(i) = \sigma_i(i+1)$ for $i \in \mathbf{n} = \{0, \cdots, n-1\}$. The maps σ_i^n generate all the maps in $(\Delta^+_{\text{epi}}, +)$ (see [68]) and satisfies the simplicial identities:
$$\sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1}, \qquad\qquad i \leq j.$$

3. Mac Lane [68] pointed out that just like $(\Delta^+, +, 0)$, $(\Delta^+_{\text{epi}}, +)$ contains the universal semi-monoid which still corresponds to the object $\mathbf{1}$ together the (unique) map $\sigma_0^1 : \mathbf{2} \longrightarrow \mathbf{1}$.

Now we can take as definition.

Definition 3.4.2. *Let $\mathscr{M} = (\underline{M}, \otimes, I)$ be a monoidal category. A semi-monoid of $\mathscr{M}$ is a **monoidal functor***
$$F : (\Delta^+_{epi}, +) \longrightarrow \mathscr{M}.$$

We now assume that $\mathscr{M}$ is equipped with a class of map called homotopy or weak equivalences. We refer the reader to [3] for the definition of *base of enrichment.*

Definition 3.4.3. *Let $(\mathscr{M}, \mathscr{W})$ be abase of enrichment. A **co-Segal semi-monoid** of $(\mathscr{M}, \mathscr{W})$ is a **lax monoidal** functor*
$$F : ({\Delta^+_{epi}}^{op}, +) \longrightarrow \mathscr{M}$$
satisfying the co-Segal conditions:
for every $f \in (\Delta^+_{epi}, +)(\boldsymbol{m}, \boldsymbol{n})$ the morphism $F(f) : F(\boldsymbol{n}) \longrightarrow F(\boldsymbol{m})$ is a weak equivalence i.e $F(f) \in \mathscr{W}$.

Remark 3.4.3.

1. It's important to notice that in the first definition we use $(\Delta^+_{\text{epi}}, +)$ while in the second we use $({\Delta^+_{\text{epi}}}^{op}, +)$.

2. Here as usual, the underlying semi-monoid is the object $F(\mathbf{1})$.

3. Finally it's important to notice that since the morphism of $(\Delta^+_{\text{epi}}, +)$ are generated by the maps σ_i^n and because $\mathscr{W}$ is stable under composition, it suffices to require the co-Segal conditions only for the maps $F(\sigma_i^n)$.

To understand the definition one needs to see the data that F carries.

Observations 3.4.1.

1. By definition of a lax morphism for every $\mathbf{n}, \mathbf{m}$ we have a 'laxity map'
$$F_{n,m} : F(\mathbf{n}) \otimes F(\mathbf{m}) \longrightarrow F(\mathbf{n} + \mathbf{m}).$$
In particular for $\mathbf{m} = \mathbf{n} = \mathbf{1}$ we have a map $F_{1,1} : F(\mathbf{1}) \otimes F(\mathbf{1}) \longrightarrow F(\mathbf{2})$.

2. The co-Segal condition for $f = \sigma_0^1 : \mathbf{2} \longrightarrow \mathbf{1}$ says that the map $F(\sigma_0^1) : F(\mathbf{1}) \longrightarrow F(\mathbf{2})$ is a weak equivalence. If we combine this map with the previous laxity map we will have a quasi-multiplication as we described earlier:

$$\begin{tikzcd} & F(\mathbf{1}) \arrow[d, "F(\sigma_0^1)"', "\text{weak.equiv}"] \\ F(\mathbf{1}) \otimes F(\mathbf{1}) \arrow[r, "F_{1,1}"] & F(\mathbf{2}) \end{tikzcd}$$

3. For every $f : \mathbf{n} \longrightarrow \mathbf{n'}$ and $g : \mathbf{m} \longrightarrow \mathbf{m'}$ the following diagram commutes

$$\begin{tikzcd} F(\mathbf{n'}) \otimes F(\mathbf{m'}) \arrow[r, "F_{n',m'}"] \arrow[d, "F(f)\otimes F(g)"'] & F(\mathbf{n'} + \mathbf{m'}) \arrow[d, "F(f+g)"] \\ F(\mathbf{n}) \otimes F(\mathbf{m}) \arrow[r, "F_{n,m}"] & F(\mathbf{n} + \mathbf{m}) \end{tikzcd}$$

4. For every triple of objects $(\mathbf{m}, \mathbf{n}, \mathbf{p})$ using the laxity maps and the maps '$F(f)$' we have some semi-cubical commutative diagrams as before, which will give the associativity up-to-homotopy and the suitable coherences.

Terminology.

- When all the maps $F(f)$ are isomorphisms then we will say F is a strict co-Segal semi-monoid or a co-Segal semigroup.
- Without the co-Segal conditions in the Definition 3.4.3 we will say that F is a pre-semi-monoid.

Proposition 3.4.4. *We have an equivalence between the following data:*

- *a classical semi-monoid or semigroup of $\mathscr{M}$*
- *a strict co-Segal semi-monoid of $\mathscr{M}$.*

When we will define the morphisms between co-Segal semi-monoids, this equivalence will automatically be an equivalence of categories.

Sketch of proof.

a) Let $F : (\Delta^+_{\text{epi}}, +) \longrightarrow \mathscr{M}$ be a semi-monoid. We define the corresponding co-Segal semi-monoid $\widetilde{F}$ as follows.

  - We set $\widetilde{F}(\mathbf{n}) = \widetilde{F}(\mathbf{1}) := F(\mathbf{1})$ for every $\mathbf{n}$, and for every $f : \mathbf{m} \longrightarrow \mathbf{n}$ we set $\widetilde{F}(f) := \mathrm{Id}_{F(\mathbf{1})}$.

- Finally the laxity maps correspond to the multiplication of the semi-monoid $F(\mathbf{1})$ i.e $\widetilde{F}_{n,m}$ is the composite:

$$F(\mathbf{1}) \otimes F(\mathbf{1}) \xrightarrow{\mathrm{Id}} F(\mathbf{2}) \xrightarrow{F(\sigma_0^1)} F(\mathbf{1})$$

for $\mathbf{m} \geq \mathbf{1}, \mathbf{n} \geq \mathbf{1}$.

b) Conversely let $G : (\Delta_{\text{epi}}^{+\,op}, +) \longrightarrow \mathscr{M}$ be a strict co-Segal semi-monoid. We get a semi-monoid $[G]$ in the following manner.

- $[G](\mathbf{1}) = G(\mathbf{1})$,
- $[G](\mathbf{n}) = G(\mathbf{1})^{\otimes n} = \underbrace{G(\mathbf{1}) \otimes \cdots \otimes G(\mathbf{1})}_{\text{n-times}}$,
- We have a multiplication $\mu : [G](\mathbf{1}) \otimes [G](\mathbf{1}) \longrightarrow [G](\mathbf{1})$ which is the map $G(\sigma_0^1)^{-1} \circ G_{1,1}$ obtained from the diagram below:

$$\begin{tikzcd} & G(\mathbf{1}) \arrow[d, "G(\sigma_0^1)"', "\cong"] \\ G(\mathbf{1}) \otimes G(\mathbf{1}) \arrow[ur, dashed, "\mu"] \arrow[r, "G_{1,1}"] & G(\mathbf{2}) \end{tikzcd}$$

- On morphism, we define $[G]$ on the generators by $[G](\sigma_i^n) := \mathrm{Id}_{G(\mathbf{1})^{\otimes i}} \otimes \mu \otimes \mathrm{Id}_{G(\mathbf{1})^{\otimes n-i-1}}$
- Finally one gets the associativity from the semi-cubical diagram mentioned before.

■

3.4.4 The General case: co-Segal Categories

3.4.5 $\mathbb{S}$-Diagrams

In addition to the notations of the previous section, we will also use the following ones.

Notation 3.4.1.
$\mathrm{Cat}_{\leq 1}$ = the 1-category of small categories with functors.
$\mathbf{Bicat_2}$= the 2-category of bicategories, lax morphisms and icons ([57, Thm 3.2]). [2]
$\frac{1}{2}\,\mathbf{Bicat_2}$ = the category of semi-bicategories, lax morphisms and icons.
$\mathscr{P}_{\mathcal{C}}$ = the 2-path-category associated to a small category $\mathcal{C}$ (see [3]) .
$\mathbb{1} = \{\mathbf{O}, \mathbf{O} \xrightarrow{Id_{\mathbf{O}}} \mathbf{O}\}$ = the unit category.
$\overline{X}$ = the coarse category associated to a set X.
$(\mathscr{B})^{\text{2-op}}$ = the 2-opposite (semi) bicategory of $\mathscr{B}$. We keep the same 1-cells but reverse the 2-cells i.e

$$(\mathscr{B})^{\text{2-op}}(A, B) := \mathscr{B}(A, B)^{\text{op}}.$$

[2]Note that $\mathbf{Bicat_2}$ is not the standard one which includes all transformation. The standard one is **not** a 2-category.

$(\mathscr{M}, \mathscr{W})$ = a base of enrichment, with $\mathscr{M}$ a general bicategory.
2-Iso($\mathscr{M}$) = the class of invertible 2-morphisms of $\mathscr{M}$. Recall that ($\mathscr{M}$, 2-Iso) is the smallest base of enrichment.

Note. We will freely identify bicategories and 2-categories. And as usual monoidal categories will be identified with bicategories with one object.

Recall that the 2-path category $\mathscr{P}_{\mathcal{C}}$ is a generalized version of the monoidal category $(\Delta^+, +, \mathbf{0})$ in the sense that when $\mathcal{C} \cong \mathbb{1}$ then $\mathscr{P}_{\mathcal{C}} \cong (\Delta^+, +, \mathbf{0})$. It has been shown in [3] that a classical enriched category with a small set of objects was the same thing as a homomorphism in the sense of Bénabou from $\mathscr{P}_{\overline{X}}$ to $\mathscr{M}$, for some set X.

In what follows we introduce a generalized version of the semi-monoidal category $(\Delta^+_{\text{epi}}, +)$ just like we did for $(\Delta^+, +, \mathbf{0})$. For $\mathcal{C}$ a small category, we consider $\mathbb{S}_{\mathcal{C}}$, a semi-2-category contained in $\mathscr{P}_{\mathcal{C}}$, such that $\mathbb{S}_{\mathbb{1}}$ 'is' $(\Delta^+_{\text{epi}}, +)$.

Proposition-Definition 3.4.5. *Let $\mathcal{C}$ be a small category.*
There exists a strict semi-2-category $\mathbb{S}_{\mathcal{C}}$ having the following properties.

- *the objects of $\mathbb{S}_{\mathcal{C}}$ are the objects of $\mathcal{C}$,*
- *for every pair (A, B) of objects, $\mathbb{S}_{\mathcal{C}}(A, B)$ is a category over Δ^+_{epi} i.e we have a functor called* ***length*** *or* ***degree***
$$\mathscr{L}_{AB} : \mathbb{S}_{\mathcal{C}}(A, B) \longrightarrow \Delta^+_{epi}$$
- *$\mathscr{L}_{AA}$ becomes naturally a monoidal functor when we consider the composition on $\mathbb{S}_{\mathcal{C}}(A, A)$,*
- *if $\mathcal{C} \cong \mathbb{1}$, say $ob(\mathcal{C}) = \{\boldsymbol{O}\}$ and $\mathcal{C}(\boldsymbol{O}, \boldsymbol{O}) = \{\mathrm{Id}_{\boldsymbol{O}}\}$, we have an isomorphism of semi-monoidal categories:*
$$\mathbb{S}_{\mathcal{C}}(\boldsymbol{O}, \boldsymbol{O}) \xrightarrow{\sim} (\Delta^+_{epi}, +)$$
- *the operation $\mathcal{C} \mapsto \mathbb{S}_{\mathcal{C}}$ is functorial in $\mathcal{C}$:*

$$\mathbb{S}_{[-]} : \mathbf{Cat}_{\leq 1} \longrightarrow \tfrac{1}{2}\,\mathbf{Bicat_2}$$
$$\mathcal{C} \xrightarrow{F} \mathcal{D} \longmapsto \mathbb{S}_{\mathcal{C}} \xrightarrow{\mathbb{S}_F} \mathbb{S}_{\mathcal{D}}$$

Proof. $\mathbb{S}_{\mathcal{C}}$ is the object obtained from the genuine fibred product of semi-2-categories:

$$\begin{tikzcd}
\mathbb{S}_{\mathcal{C}} \arrow[r, hook, "i"] \arrow[d, dotted, "\mathscr{L}"'] & \mathscr{P}_{\mathcal{C}} \arrow[d, "\mathscr{L}"] \\
(\Delta^+_{\text{epi}}, +) \arrow[r, hook, "i"'] & (\Delta^+, +, \mathbf{0})
\end{tikzcd}$$

■

Note. We will be interested in particular to the cases where $\mathcal{C}$ is of the form $\overline{X}$, the *indiscrete* or *coarse category* associated to a set X. In that case an object of $\mathbb{S}_{\overline{X}}(A,B)$ can be identified with an $(n+1)$-tuple $(E_0,\cdots,E_n)$ of elements of X for some n, with $E_0=A$ and $E_n=B$. For simplicity we will use small letters: r,s,t,..., to represent such chains $(E_0,\cdots,E_n)$.

A morphism $u:t\longrightarrow s$ of $\mathbb{S}_{\overline{X}}(A,B)$ can be viewed as an operation which deletes some letters of t to get s, keeping A and B fixed.

In the upcoming definitions we consider a 2-category $\mathscr{M}$ which is also a special Quillen $\mathcal{O}$-algebra for the operad '$\mathcal{O}_X$' of 2-categories. This situation covered also the special case of a 2-category which is locally a model category (Definition 3.10.1).

Definition 3.4.6. *Let $\mathscr{M}$ be a 2-category which is a special Quillen algebra.*
An $\mathbb{S}$***-diagram*** *of $\mathscr{M}$ is a lax morphism $F:(\mathbb{S}_{\mathcal{C}})^{2\text{-}op}\longrightarrow\mathscr{M}$ for some $\mathcal{C}$. We will say for short that F is an $\mathbb{S}_{\mathcal{C}}$-diagram of $\mathscr{M}$.*

One can observe that this definition is the generalization of Definition 3.4.3 without the co-Segal conditions.

Definition 3.4.7. *Let $\mathscr{M}$ be a 2-category which is a special Quillen algebra.*
A ***co-Segal*** $\mathbb{S}$***-diagram*** *is an $\mathbb{S}$-diagram*

$$F:(\mathbb{S}_{\mathcal{C}})^{2\text{-}op}\longrightarrow\mathscr{M}$$

satisfying the ***co-Segal conditions****: for every pair (A,B) of object of $\mathcal{C}$, the component*

$$F_{AB}:\mathbb{S}_{\mathcal{C}}(A,B)^{op}\longrightarrow\mathscr{M}(FA,FB)$$

takes its values in the subcategory of weak equivalences. This means that for every $u:s\longrightarrow s'$ in $\mathbb{S}_{\mathcal{C}}(A,B)$, the 2-morphism

$$F_{AB}(u):F_{AB}(s')\longrightarrow F_{AB}(s)$$

is a weak equivalence in the model category $\mathscr{M}(FA,FB)$.

Terminology. When all the maps $F_{AB}(u)$ are 2-isomorphisms , then we will say that F is a strict co-Segal $\mathbb{S}_{\mathcal{C}}$-diagram of $\mathscr{M}$.

Observations 3.4.2. By construction of $\mathbb{S}_{\mathcal{C}}$, for every pair of objects (A,B) and for every $t\in\mathbb{S}_{\mathcal{C}}(A,B)$ we have a unique element $f\in\mathcal{C}(A,B)$ and a unique morphism $u_t:t\longrightarrow[1,f]$.
Concretely t is a chain of composable morphisms such that the composite is f, or equivalently t is a 'presentation' (or factorization) of f with respect to the composition. It follows that for any morphism $v:t\longrightarrow s$ we have that $u_t=u_s\circ v$.

Since in each $\mathscr{M}(FA,FB)$ the weak equivalences have the 3 for 2 property and are closed under composition, it's easy to see that F satisfies the co-Segal conditions if and only if $F(u_t)$ is a weak equivalence for all t and all pairs (A,B).

Definition 3.4.8. *A* ***co-Segal $\mathscr{M}$-category*** *is a co-Segal $\mathbb{S}_{\overline{X}}$-diagram for some set X.*

The classical examples

In the following discussion we will use the following conventions.

- By **semi-enriched category** we mean a structure given by the same data and axioms of an enriched category without the identities . We will say as well **$\mathscr{M}$-semi-category** to mention the base $\mathscr{M}$ which contains the 'Hom'. This is the generalized version of semi-monoids.

– As for $\mathscr{M}$-categories, we have morphisms between $\mathscr{M}$-semi-categories by simply ignoring the data involving the identities.

– Our $\mathscr{M}$-categories and $\mathscr{M}$-semi-categories will always have a small set of objects.

The following proposition is the generalized version of Proposition 3.4.4.

Proposition 3.4.9. *We have an equivalence between the following data:*

1. *an $\mathscr{M}$-semi-category*
2. *a strict co-Segal $\mathbb{S}_{\overline{X}}$-diagram of $\mathscr{M}$.*

The proof is very similar and is straightforward. We give hereafter an outline for the case where $\mathscr{M}$ is a monoidal category.

Sketch of proof. Let $\mathscr{A}$ be an $\mathscr{M}$-semi-category with $X = Ob(\mathscr{A})$. We define the corresponding strict co-Segal $\mathbb{S}_{\overline{X}}$-diagram $F = (F, \varphi)$ as follows:

1. each component $F_{AB} : \mathbb{S}_{\overline{X}}(A, B) \longrightarrow \mathscr{M}$ is a constant functor :
$$\begin{cases} F_{AB}([\mathbf{n}, s]) = F_{AB}([\mathbf{1}, (A, B)]) := \mathscr{A}(A, B) & \text{for all } [\mathbf{n}, s] \\ F_{AB}(f) := \mathrm{Id}_{\mathscr{A}(A,B)} & \text{for all } f : [\mathbf{n}, s] \longrightarrow [\mathbf{n}', s'] \text{ in } \mathbb{S}_{\overline{X}}(A, B) \end{cases}$$
2. the laxity maps are given by the composition:
$$\varphi_{s,t} := c_{ABC} : \mathscr{A}(B, C) \otimes \mathscr{A}(A, B) \longrightarrow \mathscr{A}(A, C)$$

Conversely let $F : (\mathbb{S}_{\overline{X}})^{2\text{-op}} \longrightarrow \mathscr{M}$ be a <u>strict</u> co-Segal $\mathbb{S}_{\overline{X}}$-diagram. We simply show how we get the composition of the $\mathscr{M}$-semi-category which is denoted by $\mathscr{M}_F^X$.

1. First we have $\mathrm{Ob}(\mathscr{M}_F^X) = X$.
2. We take $\mathscr{M}_F^X(A, B) := F_{AB}([\mathbf{1}, (A, B)]$, for every $A, B \in X$.
3. The laxity map $\varphi_{s,t}$ for $s = [\mathbf{1}, (A, B)]$, $t = [\mathbf{1}, (B, C)]$ is a map of $\mathscr{M}$
$$\varphi_{s,t} : \mathscr{M}_F^X(B, C) \otimes \mathscr{M}_F^X(A, B) \longrightarrow \mathscr{M}_F^X(A, B, C)$$
where $\mathscr{M}_F^X(A, B, C) := F_{AC}([\mathbf{2}, (A, B, C)])$.
4. Now in $\mathbb{S}_{\overline{X}}(A, C)$ we have a unique map $[\mathbf{2}, (A, B, C)] \xrightarrow{\sigma_0^1} [\mathbf{1}, (A, C)]$ parametrized by the map $\sigma_0^1 : \mathbf{2} \longrightarrow \mathbf{1}$ of $(\Delta_{\text{epi}}^+, +)$. The image of this map by F_{AC} is a map
$$F(\sigma_0^1) : \mathscr{M}_F^X(A, C) \longrightarrow \mathscr{M}_F^X(A, B, C)$$
which is invertible by hypothesis.
5. And we take the composition $c_{ABC} = F(\sigma_0^1)^{-1} \circ \varphi_{s,t}$ as illustrated in the the diagram below:
$$\begin{tikzcd} & \mathscr{M}_F^X(A, C) \arrow[d, "F(\sigma_0^1)"', "\cong"] \\ \mathscr{M}_F^X(B, C) \otimes \mathscr{M}_F^X(A, B) \arrow[r, "\varphi_{s,t}"] \arrow[ur, dashed, "c_{ABC}"] & \mathscr{M}_F^X(A, B, C) \end{tikzcd}$$

■

Remark 3.4.4. The previous equivalence will turn to be an equivalence of categories when we will have the morphisms of $\mathbb{S}$-diagrams.

3.4.6 Morphisms of $\mathbb{S}$-Diagrams

As our $\mathbb{S}$-diagrams are lax morphisms of semi-bicategories, one can guess that a morphism of $\mathbb{S}$-diagrams will be a *transformation of lax morphisms* in the sense of Bénabou. This is the same approach as in [3] where the morphism of *path-objects* were defined as transformations of colax morphisms.

But just like in [3] not every transformation will give a morphism of semi-enriched categories. In [3], a general transformation is called '$\mathscr{M}$-premorphism' and an $\mathscr{M}$-morphism was defined as special $\mathscr{M}$-premorphism.

Warning. In the following, we will only consider the transformations which will give the classical notion of morphism between semi-enriched categories. We decide not to mention '$\mathscr{M}$-premorphisms' between $\mathbb{S}$-diagrams.

We recall hereafter the definition of the transformations of morphisms of semi-bicategories we are going to work with. The following definition is slightly different from the standard one, even though in the monoidal case, it is the standard one.

Definition 3.4.10. *Let $\mathscr{B}$ and $\mathscr{M}$ be two semi-bicategories and $F = (F, \varphi)$, $G = (G, \psi)$ be two lax morphisms from $\mathscr{B}$ to $\mathscr{M}$ such that $FA = GA$ for every object A of $\mathscr{B}$.*

A ***simple transformation*** $\sigma : F \longrightarrow G$

$$\begin{tikzcd} \mathscr{B} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathscr{M} \arrow[Rightarrow, from=U, to=D, "\sigma"'] \end{tikzcd} .$$

is given by the following data and axioms.

Data: *A natural transformation for each pair of objects (A, B) of $\mathscr{B}$:*

$$\begin{tikzcd} \mathscr{B}(A,B) \arrow[r, bend left, "F_{AB}"{name=U}] \arrow[r, bend right, "G_{AB}"'{name=D}] & \mathscr{M}(FA, FB) \arrow[from=U, to=D, "\sigma"'] \end{tikzcd} .$$

hence a 2-morphism of $\mathscr{M}$, $\sigma_t : Ft \longrightarrow Gt$, for each t in $\mathscr{B}(A, B)$, natural in t.

Axioms: *The following commutes :*

$$\begin{tikzcd} Fs \otimes Ft \arrow[r, "\varphi_{s,t}"] \arrow[d, "\sigma_s \otimes \sigma_t"'] & F(s \otimes t) \arrow[d, "\sigma_{s \otimes t}"] \\ Gs \otimes Gt \arrow[r, "\psi_{s,t}"] & G(s \otimes t) \end{tikzcd}$$

With this definition we can now give the definition of morphism of $\mathbb{S}$-diagrams.

Definition 3.4.11. *Let F and G be respectively an $\mathbb{S}_{\mathcal{C}}$-diagram and an $\mathbb{S}_{\mathcal{D}}$-diagram of $\mathscr{M}$. A morphism of $\mathbb{S}$-diagrams from F to G is a pair (Σ, σ) where:*

1. $\Sigma : \mathcal{C} \longrightarrow \mathcal{D}$ *is a functor such that for every* $A \in Ob(\mathcal{C})$ *we have* $FA = G(\Sigma A)$*,*

2. $\sigma : F \longrightarrow G \circ \mathbb{S}_\Sigma$ *is a simple tranformation of lax morphisms:*

$$\begin{tikzcd}
(\mathbb{S}_{\mathcal{C}})^{2\text{-}op} \arrow[rr, "(\mathbb{S}_\Sigma)^{2\text{-}op}"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\sigma}{\Longrightarrow}"] & (\mathbb{S}_{\mathcal{D}})^{2\text{-}op} \arrow[dl, "G"] \\
 & \mathscr{M} &
\end{tikzcd}$$

When all the components 'σ_t*' of* σ *are weak equivalences we will say that* (Σ, σ) *is a level-wise weak equivalences.*

Notation 3.4.2.

1. For a small category $\mathcal{C}$, we will denote by $\mathrm{Lax}^*[(\mathbb{S}_{\mathcal{C}})^{2\text{-op}}, \mathscr{M}]$ the category of $\mathbb{S}_{\mathcal{C}}$-diagrams with morphisms of $\mathbb{S}_{\mathcal{C}}$-diagrams.
2. We will denote by $\mathscr{M}_{\mathbb{S}}(\mathcal{C})$ the subcategory of $\mathrm{Lax}^*[(\mathbb{S}_{\mathcal{C}})^{2\text{-op}}, \mathscr{M}]$ with morphisms of the form $(\mathrm{Id}_{\mathcal{C}}, \sigma)$. It follows that the morphisms in $\mathscr{M}_{\mathbb{S}}(\mathcal{C})$ are simply determined by the simple transformations 'σ'.
3. For $\mathcal{C} = \overline{X}$, we will write $\mathscr{M}_{\mathbb{S}}(X)$ to mean $\mathscr{M}_{\mathbb{S}}(\overline{X})$.

Proposition 3.4.12. *Let* $\mathscr{M}$ *be a 2-category which is a base of enrichment or a special Quillen algebra, and* $F : (\mathbb{S}_{\mathcal{C}})^{2\text{-}op} \longrightarrow \mathscr{M}$*,* $G : (\mathbb{S}_{\mathcal{C}})^{2\text{-}op} \longrightarrow \mathscr{M}$ *be two* $\mathbb{S}$*-diagrams in* $\mathscr{M}$*. For a level-wise weak equivalence* $(\Sigma, \sigma) : F \longrightarrow G$ *we have:*

1. If G *is a co-Segal* $\mathbb{S}$*-diagram then so is* F*,*

2. If F *is a co-Segal* $\mathbb{S}$*-diagram and if* Σ *is surjective on objects and full then* G *is also a co-Segal* $\mathbb{S}$*-diagram.*

Remark 3.4.5. In the category $\mathscr{M}_{\mathbb{S}}(\mathcal{C})$ the condition required in (2) is automatically fulfilled because the morphisms in $\mathscr{M}_{\mathbb{S}}(\mathcal{C})$ are of the form $(\mathrm{Id}_{\mathcal{C}}, \sigma)$.

Sketch of proof. The key of the proof is to use the '3-out-of-2' property of weak equivalences in $\mathscr{M}$. This says that whenever we have a composable pair of morphisms (f, g), then if 2 members of the set $\{f, g, g \circ f\}$ are weak equivalences then so is the third.

For assertion (1), we need to show that for every $u : s \longrightarrow s'$ in $\mathbb{S}_{\mathcal{C}}(A, B)$, we have $F_{AB}(u) : F_{AB}(s') \longrightarrow F_{AB}(s)$ is a weak equivalence in $\mathscr{M}$. To simplify the notations we will not mention the subscript 'AB' on the components of F and G.

By definition of (Σ, σ) for every $u : s \longrightarrow s'$ in $\mathbb{S}_{\mathcal{C}}(A, B)$, the following diagram commutes:

$$\begin{tikzcd}
F(s') \arrow[r, "\sigma_{s'}", "\sim"'] \arrow[d, "F(u)"'] & G[\mathbb{S}_\Sigma(s')] \arrow[d, "G[\mathbb{S}_\Sigma(u)]", "\wr"'] \\
F(s) \arrow[r, "\sigma_s", "\sim"'] & G[\mathbb{S}_\Sigma(s)]
\end{tikzcd}$$

Since all three maps are weak equivalences by hypothesis, we deduce by 3 for 2 that $F(u)$ is also a weak equivalence, which gives (1).

For assertion (2) we proceed as follows. The assumptions on Σ imply that for any morphism $v : t \longrightarrow t'$ in $\mathbb{S}_{\mathcal{D}}(U, V)$ there exists a pair of objects (A, B) of $\mathcal{C}$ and s', s in $\mathbb{S}_{\mathcal{C}}(A, B)$ together with a maps $u : s \longrightarrow s'$ such that:
$\Sigma A = U$, $\Sigma B = V$,
$\mathbb{S}_{\Sigma}(s) = t$, $\mathbb{S}_{\Sigma}(s') = t'$,
$\mathbb{S}_{\Sigma}(u) = v$.

And we have the same type of commutative diagram:

$$\begin{tikzcd}
F(s') \arrow[r, "\sigma_{s'}", "\sim"'] \arrow[d, "F(u)"', "\wr"] & G(t') \arrow[d, "G(v)"] \\
F(s) \arrow[r, "\sigma_s", "\sim"'] & G(t)
\end{tikzcd}$$

Just like in the previous case we have by 3 for 2 that $G(v)$ is also a weak equivalence.

■

3.5 Properties of $\mathscr{M}_{\mathbb{S}}(\mathcal{C})$

This section is devoted to the study of the properties that $\mathscr{M}_{\mathbb{S}}(X)$ inherits from $\mathscr{M}$ e.g (co)-completeness, accessibility, etc. For simplicity we consider here only the cases $\mathcal{C} = \overline{X}$, for some set X. The methods are the same for an arbitrary $\mathcal{C}$.

Environment: We assume that $\mathscr{M} = (\underline{M}, \otimes, I)$ is a **symmetric closed** monoidal category (see [49] for a definition). One of the consequences of this hypothesis is the fact that for every object A of $\mathscr{M}$ the two functors

$$- \otimes A : \underline{M} \longrightarrow \underline{M} \qquad \text{and (hence)} \qquad A \otimes - : \underline{M} \longrightarrow \underline{M}$$

preserve the colimits. Note that these conditions turn $\mathscr{M}$ into a special Quillen algebra.

Warning.

1. When we say that '$\mathscr{M}$ is complete/cocomplete' we mean of course that the underlying category $\underline{M}$ is complete/cocmplete. And by colimits and limits in $\mathscr{M}$ we mean colimits and limits in $\underline{M}$.

2. We will say as well that $\mathscr{M}$ is locally presentable, accessible when $\underline{M}$ is so.

3. Some results in this section are presented without proof since they are easy and are sometime considered as 'folklore' in category theory.

3.5.1 $\mathscr{M}_{\mathbb{S}}(X)$ is locally presentable if $\mathscr{M}$ is so

Our goal here is to prove the following

Theorem 3.5.1. *Let $\mathscr{M}$ be a symmetric closed monoidal category which is locally presentable. Then for every set X the category $\mathscr{M}_{\mathbb{S}}(X)$ is locally presentable.*

To prove this, we proceed in the same way as in the paper of Kelly and Lack [50] where they established that $\mathscr{M}$-**Cat** is locally presentable if $\mathscr{M}$ is so.

The idea is to use the fact that given a locally presentable category $\mathcal{K}$ and a monad $\mathcal{T}$ on $\mathcal{K}$, then if $\mathcal{T}$ preserve the directed colimits then the category of algebras of $\mathcal{T}$ (called the Eilenberg-Moore category of $\mathcal{T}$) is also locally presentable (see [1, Remark 2.78]).

In our case we will have:

- $\mathcal{K}$ is the category $\prod_{(A,B)\in X^2} \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$. We will write $\mathcal{K}_X$ to emphasize that it depends of the set X
- There is a forgetful functor $\mathcal{U} : \mathscr{M}_{\mathbb{S}}(X) \longrightarrow \mathcal{K}_X$ which is faithful and injective on object, therefore we can consider $\mathscr{M}_{\mathbb{S}}(X)$ is a subcategory of $\mathcal{K}_X$.
- There is left adjoint Γ of $\mathcal{U}$ inducing the monad $\mathcal{T} = U\Gamma$.
- The category of algebra of $\mathcal{T}$ is precisely $\mathscr{M}_{\mathbb{S}}(X)$.

Remark 3.5.1. The theory of locally presentable categories tells us that any (small) diagram category of a locally presentable category, is locally presentable (see [1, Corollary 1.54]). It follows that each $\mathrm{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$ is locally presentable if $\mathscr{M}$ is so. Finally $\mathcal{K}_X$ is locally presentable since it's a (small) product of locally presentable categories.

But before proving the Theorem 3.5.1 we must first show that $\mathscr{M}_{\mathbb{S}}(X)$ is co-complete to be able to consider (filtered) colimits. This is given by the following

Theorem 3.5.2. *Given a co-complete symmetric monoidal category $\mathscr{M}$, for any set X the category $\mathscr{M}_{\mathbb{S}}(X)$ is co-complete.*

Proof of Theorem 3.5.2. See Appendix 4.3 ■

Proposition 3.5.3. *The monad $\mathcal{T} = \mathcal{U}\Gamma : \mathcal{K}_X \longrightarrow \mathcal{K}_X$ is finitary, that is, it preserves filtered colimits.*

Proof of the proposition. Filtered and directed colimits are essentially the same and it's known that a functor preserves filtered colimits if and only if it preserves directed colimits (see [1, Chap. 1, Thm 1.5 and Cor]). This allows us to reduce the proof to directed colimits.

Recall that colimits in $\mathcal{K}_X = \prod_{(A,B)\in X^2} \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$ are computed factor-wise.

For $\mathcal{F} = (\mathcal{F}_{AB})$ recall that $\mathcal{T}\mathcal{F} = ([\Gamma\mathcal{F}]_{AB})$ where Γ is the left adjoint of $\mathcal{U}$ (see Appendix 4.2.1).[3] To simplify the notation we will not mention the subscript 'AB'. For each pair (A,B) the AB-component of $\Gamma\mathcal{F}$ is $\Gamma\mathcal{F} : \mathbb{S}_{\overline{X}}(A,B)^{op} \longrightarrow \mathscr{M}$ the functor given by:

- for $t \in \mathbb{S}_{\overline{X}}(A,B)$ we have

$$\Gamma\mathcal{F}(t) = \coprod_{(t_0,\cdots,t_l)\in \mathrm{Dec}(t)} \mathcal{F}(t_0)\otimes\cdots\otimes\mathcal{F}(t_l).$$

[3] Note that actually $\mathcal{T}\mathcal{F} = ([\mathcal{U}\Gamma\mathcal{F}]_{AB})$ but since $\mathcal{U}$ consists to forget the laxity maps it's not necessary to mention it.

- for $u : t \longrightarrow t'$, we have

$$\Gamma\mathcal{F}(u) = \coprod_{(u_0,\cdots,u_l)\in \mathrm{Dec}(u)} \mathcal{F}(u_0)\otimes\cdots\otimes\mathcal{F}(u_l)$$

  where $u_i : t_i \longrightarrow t'_i$.

Let $\lambda < \kappa$ be an ordinal and $(\mathcal{F}^k)_{k\in\lambda}$ be a λ-directed diagram in $\mathcal{K}_X$ whose colimit is denoted by $\mathcal{F}^\infty$.

For any l the diagonal functor $d : \lambda \longrightarrow \prod_{i=0\dots l}\lambda$ is cofinal therefore the following colimits are the same

$$\begin{cases} \mathrm{colim}_{(k_0,\dots,k_l)\in\lambda^{l+1}}\{\mathcal{F}^{k_0}(t_0)\otimes\cdots\otimes\mathcal{F}^{k_l}(t_l)\} \\ \\ \mathrm{colim}_{k\in\lambda}\{\mathcal{F}^k(t_0)\otimes\cdots\otimes\mathcal{F}^k(t_l)\} \end{cases}$$

The first colimit is easy to compute as $\mathscr{M}$ is symmetric closed and we have

$$\mathrm{colim}_{(k_0,\dots,k_l)\in\lambda^{l+1}}\{\mathcal{F}^{k_0}(t_0)\otimes\cdots\otimes\mathcal{F}^{k_n}_{t_l}\} = \mathcal{F}^\infty(t_0)\otimes\cdots\otimes\mathcal{F}^\infty_{t_l}.$$

Consequently $\mathrm{colim}_{k\in\lambda}\{\mathcal{F}^k(t_0)\otimes\cdots\otimes\mathcal{F}^k_{t_l}\} = \mathcal{F}^\infty(t_0)\otimes\cdots\otimes\mathcal{F}^\infty(t_l)$.

From this we deduce successively that:

$$\begin{aligned} \mathrm{colim}_{k\in\lambda}\Gamma\mathcal{F}^k(t) &= \mathrm{colim}_{k\in\lambda}\{\coprod_{(t_0,\cdots,t_l)\in\mathrm{Dec}(t)} \mathcal{F}^k(t_0)\otimes\cdots\otimes\mathcal{F}^k(t_l)\} \\ &= \coprod_{(t_0,\cdots,t_l)\in\mathrm{Dec}(t)} \mathrm{colim}_{k\in\lambda}\{\mathcal{F}^k(t_0)\otimes\cdots\otimes\mathcal{F}^k(t_l)\} \\ &= \coprod_{(t_0,\cdots,t_l)\in\mathrm{Dec}(t)} \mathcal{F}^\infty(t_0)\otimes\cdots\otimes\mathcal{F}^\infty(t_l) \\ &= \Gamma\mathcal{F}^\infty(t) \end{aligned}$$

which shows that $\mathcal{T} = \mathcal{U}\Gamma$ preserves directed colimits as desired. ■

Now we can give the proof of Theorem 3.5.1 as follows.

Proof of Theorem 3.5.1. Thanks to Theorem 4.3.9 we know that $\mathcal{U} : \mathscr{M}_\mathbb{S}(X) \longrightarrow \mathcal{K}_X$ is monadic therefore $\mathscr{M}_\mathbb{S}(X)$ is equivalent to the category $\mathcal{T}$-alg of $\mathcal{T}$-algebras. Now since $\mathcal{T}$ is a finitary monad on the locally presentable category $\mathcal{K}_X$, we know from a classical result that $\mathcal{T}$-alg (hence $\mathscr{M}_\mathbb{S}(X)$) is also locally presentable (see [1, Remark 2.78]). ■

3.6 Locally Reedy 2-categories

In the following we give an *ad hoc* definition of a locally Reedy 2-category. One can generalize this notion to $\mathcal{O}$-algebra but we will not go through that here. The horizontal composition in 2-categories will be denoted by $\otimes$.

Definition 3.6.1. *A small* 2*-category* $\mathcal{C}$ *is called a* ***locally Reedy* 2*-category*** *if the following holds.*

1. *For each pair* (A, B) *of objects, the category* $\mathcal{C}(A, B)$ *is a classical Reedy* 1*-category;*
2. *The composition* $\otimes : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C)$ *is a functor of Reedy categories i.e takes direct (resp. inverse) morphisms to direct (resp. inverse) morphism.*

Example 3.6.2.

1. The examples that motivated the above definition are of course the 2-categories, $\mathscr{P}_{\mathcal{D}}$ and $\mathbb{S}_{\mathcal{D}}$, and their respective 1-opposite and 2-opposite 2-categories: $(\mathscr{P}_{\mathcal{D}})^{\text{op}}$, $(\mathbb{S}_{\mathcal{D}})^{2\text{-op}}$, etc. In particular $(\Delta, +, 0)$ is a monoidal Reedy category (a locally Reedy 2-category with one object).
2. Any classical Reedy 1-category $\mathcal{D}$ can be considered as a 2-category with two objects 0 and 1 with $\text{Hom}(0, 1) = \mathcal{D}$, $\text{Hom}(1, 0) = \varnothing$; $\text{Hom}(0, 0) = \text{Hom}(1, 1) = \mathbb{1}$ (the unit category); the composition is the obvious one (left and right isomorphism of the cartesian product). It follow that any classical category is also a locally Reedy 2-category.
3. As any set is a (discrete) Reedy category, it follows that any 1-category viewed as a 2-category with only identity 2-morphisms is a locally Reedy 2-category. In that case the linear extension are constant functors.

Warning. It's important to notice that we've chosen to say 'locally Reedy 2-category' rather than 'Reedy 2-category'. The reason is that the later terminology may refer to the notion of 'Reedy $\mathcal{M}$-category' (=enriched Reedy category) introduced by Angeltveit [2] when $\mathcal{M} = (\mathbf{Cat}, \times, \mathbb{1})$.

In our definition we've implicitly used the fact that if $\mathcal{A}$ and $\mathcal{B}$ are two classical Reedy categories, then there is a natural Reedy structure on the cartesian product $\mathcal{A} \times \mathcal{B}$ (see [40, Prop. 15.1.6]). This way we form a monoidal category of Reedy categories and morphisms of Reedy categories with the cartesian product; the unit is the same i.e $\mathbb{1}$. We will denote by $\mathbf{Cat}^{\times}_{\text{Reedy}}$ this monoidal category.

Our definition is equivalent to

Definition 3.6.3. *A locally Reedy* 2*-category is a category enriched over* $\mathbf{Cat}^{\times}_{Reedy}$.

Remark 3.6.1.

1. This definition can be generalized to locally Reedy n-categories, but we won't consider it, since the spirit of this work is to use lower dimensional objects to define higher dimensional ones.
2. One can replace $\mathbf{Cat}^{\times}_{\text{Reedy}}$ by a suitable monoidal category $\mathscr{M}\text{-}\mathbf{Cat}_{\text{Reedy}}{}^{\otimes}$ of Reedy $\mathscr{M}$-categories in the sense of Angeltveit [2]; but we don't know how relevant this would be.
3. We can also enrich over the category of generalized Reedy categories in the sense of Berger-Moerdijk [12] and in the sense of Cisinski [23, Chap. 8].
4. We can also push the definition far by considering not only Reedy $\mathcal{O}$-algebras, but defining first a Reedy multisorted operad as being multicategory enriched over $\mathbf{Cat}^{\times}_{\text{Reedy}}$,$\mathscr{M}$-$\mathbf{Cat}_{\text{Reedy}}{}^{\otimes}$, etc.

3.6.1 (Co)lax-latching and (Co)lax-matching objects

Let $\mathcal{C}$ be a locally Reedy 2-category (henceforth **lr**-category). Given a lax morphism or colax morphism $\mathcal{F} : \mathcal{C} \longrightarrow \mathscr{M}$ we would like to define the corresponding latching and matching objects of $\mathcal{F}$ at a 1-morphism z of $\mathcal{C}$. Our definitions are restricted to the case where $\mathcal{C}$ is equipped with a **global linear extension** which respects the composition. This means that we have an ordinal λ such that the linear extension $\mathbf{deg} : \mathcal{C}(A,B) \longrightarrow \lambda$ satisfies $\mathbf{deg}(g \otimes f) = \mathbf{deg}(g) + \mathbf{deg}(f)$.

Note. We don't know many examples other than the 2-categories that motivated this consideration, but we choose to have a common language for both $\mathscr{P}_{\overline{X}}$, $\mathbb{S}_{\overline{X}}$ and the others 2-categories we can construct out of them. However it's clear to see that for a classical Reedy 1-category $\mathcal{D}$, if we view $\mathcal{D}$ as an **lr**-category with two objects (see Example 3.6.2) and if we declare both $\mathbf{deg}(\mathrm{Id}_0) = \mathbf{deg}(\mathrm{Id}_1) = 0$ then $\mathcal{D}$ has this property.

We will consider lax morphism $\mathcal{F} : \mathcal{C} \longrightarrow \mathscr{M}$ which are unitary in the sense of Bénabou i.e such that $\mathcal{F}(\mathrm{Id}) = \mathrm{Id}$ and the laxity maps $\mathcal{F}\,\mathrm{Id} \otimes \mathcal{F}f \longrightarrow \mathcal{F}(\mathrm{Id} \otimes f)$ are the natural left and right isomorphisms.

Let λ be an infinite ordinal containing ω. We can make λ into a monoidal category with the addition; and we can consider it a usual as a 2-category with one object having as hom-category λ.

Definition 3.6.4. *A locally Reedy 2-category $\mathcal{C}$ is **simple** if there exist an infinite ordinal λ such that the linear extension form a strict 2-functor $\boldsymbol{deg} : \mathcal{C} \longrightarrow \lambda$. Here on object $\boldsymbol{deg} : Ob(\mathcal{C}) \longrightarrow \{*\}$*

From this definition we have the following consequences:

- First if $\mathcal{C}$ is a simple **lr**-category then composition reflects the identities; thus $\mathcal{C}$ is an **ir**-$\mathcal{O}$-algebra for the operad of 2-categories.
- For any object $A \in \mathcal{C}$, we have $\mathbf{deg}(\mathrm{Id}_A) = \mathbf{0}$ since $\mathbf{deg}(\mathrm{Id}_A) = \mathbf{deg}(\mathrm{Id}_A \otimes \mathrm{Id}_A) = \mathbf{deg}(\mathrm{Id}_A) + \mathbf{deg}(\mathrm{Id}_A)$.
- Another important consequence is that a 1-morphism z cannot appear in the set
$$\otimes^{-1}(z) = \coprod_{l>1} \{(s_1, ..., s_l); \otimes(s_1, ..., s_l) = z\}$$
if $\mathbf{deg}(s_i) > \mathbf{0}$ for all i.

Using the Grothendieck construction For each pair (A, B) we have a composition diagram which is organized into a functor $\mathbf{c} : \Delta^+_{\mathrm{epi}} \longrightarrow \mathbf{Cat}$ and represented as:

$$\mathcal{C}(A,B) \longleftarrow \coprod \mathcal{C}(A,A_1) \times \mathcal{C}(A_1,B) \Lleftarrow \coprod \mathcal{C}(A,A_1) \times \mathcal{C}(A_1,A_2) \times \mathcal{C}(A_2,B) \cdots$$

More precisely one defines:

- $\mathbf{c}(\mathbf{1}) = \mathcal{C}(A,B)$,
- $\mathbf{c}(\mathbf{n}) = \coprod_{(A,\ldots,B)} \mathcal{C}(A,A_1) \times \mathcal{C}(A,A_1) \times \cdots \times \mathcal{C}(A_{n-1},B)$.

Note that the morphisms in Δ^+_{epi} are generated by the maps $\sigma_i^n : \mathbf{n+1} \longrightarrow \mathbf{n}$ which are characterized by $\sigma_i^n(i) = \sigma_i^n(i+1)$ for $i \in \mathbf{n} = \{0, \cdots, n-1\}$ (see [68, p.177]). Then the functor $\mathbf{c}(\sigma_i^n) : \mathbf{c}(\mathbf{n}+1) \longrightarrow \mathbf{c}(\mathbf{n})$ is the functor which consists to compose at the vertex A_{i+1}.

Let $\int \mathbf{c}$ be the category we obtained by the Grothendieck construction:

- the objects are pairs $(\mathbf{n}, a)$ with $a \in \mathrm{Ob}(\mathbf{c}(\mathbf{n}))$. Such object $(\mathbf{n}, a)$ can be identified with an n-tuple of 1-morphisms $(s_1, ..., s_n)$ with $s_i \in \mathcal{C}(A_{i-1}, A_i)$.
- a morphism $\gamma : (\mathbf{n}, a) \longrightarrow (\mathbf{m}, b)$ is a pair $\gamma = (f, u)$ where $f : \mathbf{n} \longrightarrow \mathbf{m}$ is a morphism of Δ^+_{epi} and $u : \mathbf{c}(f)a \longrightarrow b$ is a morphism in $\mathbf{c}(\mathbf{m})$. Here $\mathbf{c}(f) : \mathbf{c}(\mathbf{n}) \longrightarrow \mathbf{c}(\mathbf{m})$ is a functor (image of f by $\mathbf{c}$).
- the composite of $\gamma = (f, u)$ and $\gamma' = (g, v)$ is $\gamma' \circ \gamma := (g \circ f, v \circ \mathbf{c}(g)u)$.

One can easily check that these data define a category. Note that for each $\mathbf{n} \in \Delta^+_{\text{epi}}$ the subcategory of objects over $\mathbf{n}$ is isomorphic to $\mathbf{c}(\mathbf{n})$.

Claim. For a simple **lr**-category $\mathcal{C}$ and for each pair (A, B) there is a natural Reedy structure on $\int \mathbf{c}$.

In fact one has a linear extension by setting $\mathbf{deg}(\mathbf{n}, a) = \mathbf{deg}(a) = \mathbf{deg}(s_1) + \cdots + \mathbf{deg}(s_n)$ for $a = (s_1, ..., s_n)$. A morphism $\gamma = (f, u) : (\mathbf{n}, a) \longrightarrow (\mathbf{m}, b)$ is said to be a direct (resp. inverse) if u is a direct (resp. inverse) morphism in $\mathbf{c}(\mathbf{m})$. The factorization axiom follows from the fact that $\mathbf{c}(\mathbf{m})$ is a Reedy category.

Remark 3.6.2.

1. There is a general statement for the category $\int F$ associated to any functor $F : \mathcal{D} \longrightarrow \mathbf{Cat}^{\times}_{\text{Reedy}}$.
2. For any morphism of $h : x \longrightarrow y$ of $\mathcal{D}$ and any morphism $u : a \longrightarrow b$ of $F(x)$, the following commutes in $\int F$:

$$\begin{tikzcd}
(x, a) \arrow[r, "(h, \mathrm{Id}_{F(h)a})"] \arrow[d, "(\mathrm{Id}_x, u)"'] & (y, F(h)a) \arrow[d, "(\mathrm{Id}_y, F(h)u)"] \\
(x, b) \arrow[r, "(h, \mathrm{Id}_{F(h)b})"'] & (y, F(h)b)
\end{tikzcd}$$

Let $\int \overrightarrow{\mathbf{c}} \subset \int \mathbf{c}$ be the direct category. We will denote by $\int \overrightarrow{\mathbf{c}} \downarrow (\mathbf{n}, a)$ the slice category at $(\mathbf{n}, a)$.

Definition 3.6.5. *Let $z \in \mathcal{C}(A, B)$ be a 1-morphism. Define the **generalized latching category at** z, $\partial^{\bullet}_{\mathcal{C}/z}$, to be the subcategory of $\int \overrightarrow{\mathbf{c}} \downarrow (\mathit{1}, z)$ described as follows.*

- *the objects are <u>direct</u> morphisms $\gamma : (\boldsymbol{n}, a) \longrightarrow (\boldsymbol{1}, z)$ such that z doesn't appear in a; or equivalently we have $a = (s_1, ..., s_n)$ with $\boldsymbol{deg}(s_i) < \boldsymbol{deg}(z)$ for all i.*
- *the morphisms are the usual morphisms of the comma category $\int \overrightarrow{\mathbf{c}} \downarrow (\boldsymbol{1}, z)$:*

$$\begin{tikzcd}
(\boldsymbol{n}, a) \arrow[r, "\gamma"] \arrow[d, "\delta"'] & (\boldsymbol{1}, z) \\
(\boldsymbol{m}, b) \arrow[ur, "\gamma'"'] &
\end{tikzcd}$$

- *the composition is the one in $\int \overrightarrow{\mathbf{c}} \downarrow (\boldsymbol{1}, z)$.*

Lax functor and diagram on $\int \mathbf{c}$ Given a lax functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathscr{M}$ we can define a natural functor denoted again $\mathcal{F}$ on $\int \mathbf{c}$ as follows.

1. $\mathcal{F}(\mathbf{n}, a) = \otimes(\mathcal{F}s_1, ...\mathcal{F}s_n)$ for $a = (s_1, ..., s_n)$;
2. To define $\mathcal{F}$ on morphisms it suffices to define the image of morphisms $\gamma = (\sigma_i^n, u)$ since they generated all the other morphisms. Such morphism $\gamma : (\mathbf{n+1}, a) \longrightarrow (\mathbf{n}, b)$ for $a = (s_1, ..., s_{n+1})$ and $b = (t_1, ..., t_n)$ corresponds to a n direct morphisms $\{\alpha_l : s_l \longrightarrow t_l\}_{l \neq i, l \neq i+1} \cup \{\alpha_i : s_i \otimes s_{i+1} \longrightarrow t_i\}$. With these notations one defines $\mathcal{F}(\gamma) : \mathcal{F}(\mathbf{n}+1, a) \longrightarrow \mathcal{F}(\mathbf{n}, b)$ to be the composite:
$$\otimes(\mathcal{F}s_1, ..., \mathcal{F}s_{n+1}) \xrightarrow{\mathrm{Id} \otimes ... \otimes \varphi(s_i, s_{i+1}) \otimes ... \mathrm{Id}} \otimes(\mathcal{F}s_1, ..., \mathcal{F}(s_i \otimes s_{i+1}), ...\mathcal{F}s_n) \xrightarrow{\otimes \mathcal{F}(\alpha_l)} \otimes(\mathcal{F}t_1, ..., \mathcal{F}t_n)$$
where $\varphi(s_i, s_{i+1}) : \mathcal{F}s_i \otimes \mathcal{F}s_{i+1} \longrightarrow \mathcal{F}(s_i \otimes s_{i+1})$ is the laxity map.

These data won't define a functor until we show that $\mathcal{F}(\gamma \circ \gamma') = \mathcal{F}(\gamma) \circ \mathcal{F}(\gamma')$. But this is given by the coherence axioms for the lax functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathscr{M}$ as we are going to explain. First of all we will denote by φ_{σ_i} the above map $\mathrm{Id} \otimes \cdots \otimes \varphi \otimes \cdots \otimes \mathrm{Id}$ which uses the laxity map of the ith and $(i+1)$th terms. For any map $f : \mathbf{n} \longrightarrow \mathbf{m}$ of Δ^+_{epi} and any object $a \in \mathbf{c}(\mathbf{n})$ there is a canonical map $f_a : (\mathbf{n}, a) \longrightarrow (\mathbf{m}, \mathbf{c}(f)a)$ given by $f_a = (f, \mathrm{Id}_{\mathbf{c}(f)a})$. As pointed out by Mac Lane [68, p.177], each morphism $f : \mathbf{n} \longrightarrow \mathbf{m}$ of Δ^+_{epi} has a *unique presentation* $f = \sigma_{j_1} \circ \cdots \circ \sigma_{j_{n-m}}$ where the string of subscripts j satisfy:
$$0 \leq j_1 < \cdots < j_{m-n} < n - 1.$$

With the previous notations we can define $\varphi_f = \mathcal{F}(f_a) := \varphi_{\sigma_{j_1}} \circ \cdots \circ \varphi_{\sigma_{j_{n-m}}}$ to be the **laxity map governed by** f. Here we omit a in φ_f for simplicity.

Proposition 3.6.6. *Given $f : \mathbf{n} \longrightarrow \mathbf{m}$ and $g : \mathbf{m} \longrightarrow \mathbf{m'}$ then $\mathcal{F}(g \circ f_a) = \mathcal{F}(g_{\mathbf{c}(f)a}) \circ \mathcal{F}(f_a)$ i.e $\varphi_{g \circ f} = \varphi_g \circ \varphi_f$.*

The proposition will follow from the

Lemma 3.6.7. *The maps φ_{σ_i} respect the simplicial identities $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$ $(i \leq j)$. This means that we have $\varphi_{\sigma_j} \circ \varphi_{\sigma_i} = \varphi_{\sigma_i} \circ \varphi_{\sigma_{j+1}}$.*

Sketch of proof. If $i < j$ then the assertion follows from the bifunctoriality of $\otimes$. In fact given two morphisms u, v of $\mathscr{M}$ then $u \otimes v = (\mathrm{Id} \otimes v) \circ (u \otimes \mathrm{Id}) = (u \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes v)$. So the only point which needs to be clarified is when $i = j$. In that case the equality is given by the coherence condition which says that the following commutes:

$$\begin{tikzcd}
\mathcal{F}s_i \otimes \mathcal{F}s_{i+1} \otimes \mathcal{F}s_{i+2} \arrow[r, "\sigma_{i+1}"] \arrow[d, "\sigma_i"'] & \mathcal{F}s_i \otimes \mathcal{F}(s_{i+1} \otimes s_{i+2}) \arrow[d, "\sigma_i"] \\
\mathcal{F}(s_i \otimes s_{i+1}) \otimes \mathcal{F}s_{i+2} \arrow[r, "\sigma_i"'] & \mathcal{F}(s_i \otimes s_{i+1} \otimes s_{i+2})
\end{tikzcd}$$

■

To prove the proposition one needs to see how we build a presentation of $g \circ f$ out of the presentation of $f = \sigma_{j_1} \circ \cdots \circ \sigma_{j_{n-m}}$ and $g = \sigma_{l_1} \circ \cdots \circ \sigma_{l_{m-m'}}$ where $0 \leq l_1 < \cdots < l_{m-m'} < m - 1$ and $0 \leq j_1 < \cdots < j_{m-n} < n - 1$. By induction one reduces to the case where $g = \sigma_l$. Then $g \circ f = \sigma_l \circ \sigma_{j_1} \cdots \sigma_{j_{n-m}}$, and we proceed as follows.

1. if $l \geq j_1$ then we use the simplicial identities to replace $\sigma_l \circ \sigma_{j_1}$ by $\sigma_{j_1} \circ \sigma_{l+1}$; if not we're done.

2. then $g \circ f = \sigma_{j_1} \circ (\sigma_{l+1}\sigma_{j_2} \cdots \sigma_{j_{n-m}})$ and we apply the first step with $g' = \sigma_{l+1}$ and $f' = \sigma_{j_2} \cdots \sigma_{j_{n-m}}$.

3. after a finite number of steps one has the presentation of $g \circ f = \sigma_{k_1} \circ \cdots \circ \sigma_{k_{n-m+1}}$ with $0 \leq k_1 < \cdots < k_{m-n+1} < n-1$.

Proof of Proposition 3.6.6. By definition $\mathcal{F}(g \circ f) = \varphi_{\sigma_{k_1}} \circ \cdots \circ \varphi_{\sigma_{k_{n-m'}}}$ and we have

$$\mathcal{F}(g_{\mathbf{c}(f)a}) \circ \mathcal{F}(f_a) = \varphi_{\sigma_{l_1}} \cdots \varphi_{\sigma_{l_{m-m'}}} \circ \varphi_{\sigma_{j_1}} \cdots \varphi_{\sigma_{j_{n-m}}}.$$

From the last expression we apply the lemma to each step we've followed to get the presentation of $g \circ f$; after a finite number of steps we end up with the expression $\varphi_{\sigma_{k_1}} \circ \cdots \circ \varphi_{\sigma_{k_{n-m'}}}$ which completes the proof. ■

As we already said, each morphism $f : \mathbf{n} \longrightarrow \mathbf{m}$ induces a functor $\mathbf{c}(f) : \mathbf{c}(\mathbf{n}) \longrightarrow \mathbf{c}(\mathbf{m})$. A morphism u in $\mathbf{c}(\mathbf{n})$ is a n-tuple of morphism $u = (u_1, ..., u_n)$. When there is no confusion we will write $\otimes(\mathcal{F}(u))$ to mean $\otimes(\mathcal{F}u_1, ..., \mathcal{F}u_n)$; the image of u by f will be denoted by fu instead of $\mathbf{c}(f)u$ and we may consider $\otimes(\mathcal{F}(fu))$.

According to these notations we can define shortly $\mathcal{F}$ on the morphisms of $\int \mathbf{c}$ by:

$$\mathcal{F}(\gamma) = \otimes(\mathcal{F}u) \circ \varphi_f \qquad \text{with} \quad \gamma = (f, u)$$

Lemma 3.6.8. *For every morphism $f : \boldsymbol{n} \longrightarrow \boldsymbol{m}$ of Δ^+_{epi} and every morphism $u = (u_1, ..., u_n)$ of $\mathbf{c}(\boldsymbol{n})$ we have an equality:*

$$\otimes[\mathcal{F}(fu)] \circ \varphi_f = \varphi_f \circ \otimes(\mathcal{F}u).$$

This means that the following commutes:

$$\begin{tikzcd}
\otimes(\mathcal{F}s_i) \arrow[r, "\varphi_f"] \arrow[d, "\otimes(\mathcal{F}u)"'] & \otimes[\mathcal{F}f(s_i)] \arrow[d, "\otimes[\mathcal{F}(fu)]"] \\
\otimes(\mathcal{F}t_i) \arrow[r, "\varphi_f"'] & \otimes[\mathcal{F}f(t_i)]
\end{tikzcd}$$

Sketch of proof. By standard arguments (repeating the process), one reduces the assertion to the case where $f = \sigma_i$. In this case the result follows from the functoriality of the coherence for the laxity maps, that is that all the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}s_i \otimes \mathcal{F}s_{i+1} \arrow[r, "\varphi_{\sigma_i}"] \arrow[d, "\mathcal{F}u_i \otimes \mathcal{F}u_{i+1}"'] & \mathcal{F}(s_i \otimes s_{i+1}) \arrow[d, "\mathcal{F}(u_i \otimes u_{i+1})"] \\
\mathcal{F}t_i \otimes \mathcal{F}t_{i+1} \arrow[r, "\varphi_{\sigma_i}"'] & \mathcal{F}(t_i \otimes t_{i+1})
\end{tikzcd}$$

■

With the previous lemma at hand we conclude that

Proposition 3.6.9. *Given two composable morphisms γ and γ' then we have $\mathcal{F}(\gamma' \circ \gamma) = \mathcal{F}(\gamma') \circ \mathcal{F}(\gamma)$ i.e $\mathcal{F}$ is a functor on $\int \mathbf{c}$.*

Proof. Let $\gamma = (f, u)$ and $\gamma' = (g, v)$. Then by definition $\gamma' \circ \gamma = [(g \circ f), v \circ g(u)]$.

In one hand we have by definition $\mathcal{F}(\gamma' \circ \gamma) = \otimes[\mathcal{F}(v \circ g(u))] \circ \varphi_{g \circ f}$. On the other hand we have $\mathcal{F}(\gamma) = \otimes(\mathcal{F}u) \circ \varphi_f$ and $\mathcal{F}(\gamma') = \otimes(\mathcal{F}v) \circ \varphi_g$. Using the functoriality of $\mathcal{F}$ and $\otimes$ together with the fact that $\varphi_{g \circ f} = \varphi_g \circ \varphi_f$ we establish that

$$\begin{aligned}
\mathcal{F}(\gamma' \circ \gamma) &= \otimes[\underbrace{\mathcal{F}(v \circ g(u))}_{\mathcal{F}v \circ \mathcal{F}g(u)}] \circ \underbrace{\varphi_{g \circ f}}_{=\varphi_g \circ \varphi_f} \\
&= \otimes(\mathcal{F}v) \circ \underbrace{\otimes[\mathcal{F}g(u)] \circ \varphi_g}_{=\varphi_g \circ \otimes(\mathcal{F}u)} \circ \varphi_f \\
&= \underbrace{\otimes(\mathcal{F}v) \circ \varphi_g}_{=\mathcal{F}(\gamma')} \circ \underbrace{\otimes(\mathcal{F}u) \circ \varphi_f}_{=\mathcal{F}(\gamma)} \\
&= \mathcal{F}(\gamma') \circ \mathcal{F}(\gamma)
\end{aligned}$$

■

Observations 3.6.1. As we pointed out earlier, for each $\mathbf{n}$ the category of objects in $\int \mathbf{c}$ over $\mathbf{n}$ is isomorphic to $\mathbf{c}(\mathbf{n})$ and we have an embedding $\mathbf{c}(\mathbf{n}) \xhookrightarrow{\iota_n} \int \mathbf{c}$. By the universal property of the coproduct we get a functor

$$\iota : \coprod \mathbf{c}(\mathbf{n}) \longrightarrow \int \mathbf{c}\,.$$

Given a family of functors $\{\mathcal{F}_{AB} : \mathcal{C}(A, B) \longrightarrow \mathscr{M}\}$ we can define a functor for each $\mathbf{n}$, $\mathcal{F}_n : \mathbf{c}(\mathbf{n}) \longrightarrow \mathscr{M}$ by the above formula $\mathcal{F}(s_1, ..., s_n) := \otimes(\mathcal{F}s_1, ..., \mathcal{F}s_n)$. These functors define in turn a functor

$$\coprod \mathcal{F}_n : \coprod \mathbf{c}(\mathbf{n}) \longrightarrow \mathscr{M}.$$

Then the left Kan extension of $\coprod \mathcal{F}_n$ along $\iota : \coprod \mathbf{c}(\mathbf{n}) \longrightarrow \int \mathbf{c}$ **creates laxity maps**. This is the same idea we use to define the 'free lax-morphism' generated by the family $\{\mathcal{F}_{AB} : \mathcal{C}(A, B) \longrightarrow \mathscr{M}\}$ (see Appendix 4.4.1).

Denote by U the canonical forgetful functor $\mathrm{U} : \partial^{\bullet}_{\mathcal{C}/z} \longrightarrow \int \mathbf{c}$.

Definition 3.6.10. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathscr{M}$ be a lax functor.*

1. *The **lax-latching** object of $\mathcal{F}$ at z is the colimit*

$$\mathbf{Latch}_{\mathrm{lax}}(\mathcal{F}, z) := colim_{\partial^{\bullet}_{\mathcal{C}/z}} U_{\star}\mathcal{F}.$$

2. *Define the **lax-matching** object of $\mathcal{F}$ at z to be the usual matching object of the component $\mathcal{F}_{AB} : \mathcal{C}(A, B) \longrightarrow \mathscr{M}(\mathcal{F}A, \mathcal{F}B)$ at $z \in \mathcal{C}(A, B)$.*

Remark 3.6.3.

1. By the universal property of the colimit, there is a unique map $\varepsilon : \mathbf{Latch}_{\mathrm{lax}}(\mathcal{F}, z) \longrightarrow \mathcal{F}z$.

2. There is a canonical map $\eta : \mathbf{Latch}(\mathcal{F}, z) \longrightarrow \mathbf{Latch}_{\mathrm{lax}}(\mathcal{F}, z)$, from the classical latching object to the lax-latching object and we have an equality

$$\mathbf{Latch}(\mathcal{F}, z) \longrightarrow \mathcal{F}z = \varepsilon \circ \eta.$$

Locally direct categories

We focus our study to lax diagrams indexed by locally directed category $\mathcal{C}$ which are also simple in the sense of Definition 3.6.4. This case is precisely what motivated our considerations. Indeed the 2-category $(\mathbb{S}_{\overline{\mathrm{X}}})^{2\text{-op}}$ has the property that each $\mathbb{S}_{\overline{\mathrm{X}}}(A,B)^{\mathrm{op}}$ is a direct category (like $\Delta_{\mathrm{epi}}^{+\ op}$).

Given a lax morphism $\mathcal{F} : \mathcal{C} \longrightarrow \mathscr{M}$ and an element $m \in \lambda$, we would like to consider a truncation '$\mathcal{F}^{\leq m}$' just like for simplicial sets. The problem is that to define such a morphism we need to change $\mathbb{S}_{\overline{\mathrm{X}}}$ into '$\mathbb{S}_{\overline{\mathrm{X}}}^{\leq m}$' and consider the corresponding notion of lax functor.
An obvious attempt is to define '$\mathbb{S}_{\overline{\mathrm{X}}}^{\leq m}$' as follows.

- The objects are the same i.e $\mathrm{Ob}(\mathbb{S}_{\overline{\mathrm{X}}}^{\leq m}) = X$;
- the category of morphisms between from A to B is $\mathbb{S}_{\overline{\mathrm{X}}}^{\leq m}(A,B) := \mathbb{S}_{\overline{\mathrm{X}}}(A,B)_{\leq m}$, the full subcategory of objects of degree $\leq m$.

But these data don't define a 2-category since the horizontal composition $z \otimes z'$ is only defined if $\mathbf{deg}(z) + \mathbf{deg}(z') \leq m$. This is the same situation where $(\Delta, +, \mathbf{0})$ is a monoidal category but any truncation $(\Delta_{\leq \mathbf{n}}, +, 0)$ fails to be stable by addition.

By the above observation we need to enlarge a bit our 2-categories and consider a more general notion of **2-groupement** *à la* Bonnin [19]. The notion of *groupement* was introduced by Bonnin [19] as a generalization of a category. The concept of groupement covers the idea of a category without a set of objects in the following sense. For a small 1-category $\mathcal{D}$ denote by $\mathrm{Arr}(\mathcal{D})$ the 'set' of all morphisms on $\mathcal{D}$. We can embed the set of objects $\mathrm{Ob}(\mathcal{D})$ in $\mathrm{Arr}(\mathcal{D})$ using the identity morphism and the composition gives a partial multiplication on $\mathrm{Arr}(\mathcal{D})$. This way the category structure is transferred on $\mathrm{Arr}(\mathcal{D})$ and we no longer mention a set of objects.

Warning. We will not provide an explicit definition of 2-groupement but will use the terminology to refer a sort of 2-category where the horizontal composition is partially defined. Our discussion will be limited to $\mathcal{C}^{\leq m}$ for locally directed category $\mathcal{C}$ which is simple.

From now $\mathcal{C}^{\leq m}$ is the 2-groupement or the 'almost 2-category' having the same objects as $\mathcal{C}$ and all 1-morphisms of degree $\leq m$; the 2-morphisms are the same.

Definition 3.6.11. *A lax* $\mathbf{g}$*-morphism* $\mathcal{G} : \mathcal{C}^{\leq m} \longrightarrow \mathscr{M}$ *consists of:*

1. *A family of functors* $\mathcal{G}_{AB} : \mathcal{C}(A,B)^{\leq m} \longrightarrow \mathscr{M}(\mathcal{G}A, \mathcal{G}B)$*;*
2. *laxity maps* $\varphi : \mathcal{G}s \otimes \mathcal{G}t \longrightarrow \mathcal{G}(s \otimes t)$ *if* $\boldsymbol{deg}(s) + \boldsymbol{deg}(t) \leq m$*;*
3. *the laxity maps respect the functoriality i.e the following commutes when all the laxity maps exist*

$$\begin{tikzcd}
\mathcal{G}s \otimes \mathcal{G}t \arrow[r, "\varphi"] \arrow[d, "\mathcal{G}u\otimes\mathcal{G}v"'] & \mathcal{G}(s \otimes t) \arrow[d, "\mathcal{G}(u\otimes v)"] \\
\mathcal{G}s' \otimes \mathcal{G}t' \arrow[r, "\varphi"'] & \mathcal{G}(s' \otimes t')
\end{tikzcd}$$

4. *a coherence condition which say that the following commutes if all the laxity maps are defined*

$$\begin{tikzcd}
\mathcal{G}r \otimes \mathcal{G}s \otimes \mathcal{G}t \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & \mathcal{G}r \otimes \mathcal{G}(s \otimes t) \arrow[d, "\varphi"] \\
\mathcal{G}(r \otimes s) \otimes \mathcal{G}t \arrow[r, "\varphi"'] & \mathcal{G}(r \otimes s \otimes t)
\end{tikzcd}$$

There is an obvious notion of transformation of lax $\mathbf{g}$-morphism given by the usual data except that we limit everything to the 1-morphisms of degree $\leq m$. We will denote by $\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m}, \mathscr{M})$ the category of lax $\mathbf{g}$-morphisms and transformations (the $\mathbf{g}$ here stands for groupement).
We leave the reader to check that any lax functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathscr{M}$ induces a lax $\mathbf{g}$-morphisms $\mathcal{F}^{\leq m} : \mathcal{C}^{\leq m} \longrightarrow \mathscr{M}$, functorially in $\mathcal{F}$. Thus we have a truncation functor

$$\tau_m : \mathrm{Lax}(\mathcal{C}, \mathscr{M}) \longrightarrow \mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m}, \mathscr{M}).$$

It's natural to ask if this functor has a left adjoint. This is the same situation with simplicial sets. In the next paragraph we will see that there is an affirmative answer to that question.

Lax Left Kan extensions Our problem can be interpreted as an existence of a *lax left Kan extension*. Proceeding by induction on m we reduces our original question to the existence of a left adjoint of the truncation functor

$$\tau_m : \mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m+1}, \mathscr{M}) \longrightarrow \mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m}, \mathscr{M}).$$

Proposition 3.6.12. *For every $m \in \lambda$ there is a left adjoint to τ_m*

$$\mathbf{sk}_m : \mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m}, \mathscr{M}) \longrightarrow \mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m+1}, \mathscr{M})$$

Sketch of proof. Given a lax functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathscr{M}$ and a 1-morphism z, we defined previously the lax-latching object of $\mathcal{F}$ at z to be

$$\mathbf{Latch}_{\mathrm{lax}}(\mathcal{F}, z) := \mathrm{colim}_{\partial^{\bullet}_{\mathcal{C}/z}} \otimes (\mathcal{F}s_1, ..., \mathcal{F}s_n)$$

where the colimit is taking over the sub-comma category of (direct) morphisms $\gamma : (n, s_1, ..., s_n) \longrightarrow (1, z)$ such that $\mathbf{deg}(s_j) < \mathbf{deg}(z)$ for all j.
Then given a lax $\mathbf{g}$-morphism $\mathcal{G} : \mathcal{C}^{\leq m} \longrightarrow \mathscr{M}$ and a 1-morphism z of degree $m+1$, all the values $\mathcal{G}s_j$ are defined for $\mathbf{deg}(s_j) < m+1$. Using the coherence of the lax $\mathbf{g}$-morphism and the functoriality of its components, one can show like in Proposition 3.6.9, that we have a functor $L_z : \partial^{\bullet}_{\mathcal{C}/z} \longrightarrow \mathscr{M}$ by the formula $L_z(\gamma) = \otimes(\mathcal{G}s_1, ..., \mathcal{G}s_n)$. Since $\mathscr{M}$ is (locally) complete, then we can define

$$\mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z) = \mathrm{colim}_{\partial^{\bullet}_{\mathcal{C}/z}} \otimes (\mathcal{G}s_1, ..., \mathcal{G}s_n) = \mathrm{colim}_{\partial^{\bullet}_{\mathcal{C}/z}} L_z.$$

For each γ we have a canonical map $\iota_\gamma : \otimes(\mathcal{G}s_1, ..., \mathcal{G}s_n) \longrightarrow \mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z)$. If $n = 1$ then γ is just a 2-morphism $s \longrightarrow z$ of $\mathcal{C}(A, B)$ with $z \in \mathcal{C}(A, B)$; and ι_γ is a *structure map* $\mathcal{G}s \longrightarrow \mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z)$. If $n > 1$ then ι_γ is a laxity map (eventually composed with a structure map); in particular when $(s_1, s_2) \in \otimes^{-1}(z)$ and $\gamma = \mathrm{Id}$, we have a *pure laxity map*

$$\iota_\gamma : \mathcal{G}s_1 \otimes \mathcal{G}s_2 \longrightarrow \mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z).$$

So for every 1-morphism z of degree $m+1$ the object $\mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z)$ comes equipped with structure maps and laxity map which are compatible with the old ones. If we assemble these data for all z of degree $m+1$ we can define $\mathbf{sk}_m\, \mathcal{G} : \mathcal{C}^{\leq m+1} \longrightarrow \mathscr{M}$ as follows

– $(\mathbf{sk}_m\, \mathcal{G})z := \mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z)$;

– $(\mathbf{sk}_m\, \mathcal{G})_{|\mathcal{C}^{\leq m}} = \mathcal{G}$

– the structure maps $\mathcal{G}s \longrightarrow \mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z)$ give the components

$$(\mathbf{sk}_m\, \mathcal{G})_{AB} : \mathcal{C}(A, B)^{\leq m+1} \longrightarrow \mathscr{M}(\mathcal{G}A, \mathcal{G}B)$$

– the laxity maps are the obvious ones.

– the coherences come with the definition of each $\mathbf{Latch}_{\text{lax}}(\mathcal{G}, z)$

We leave the reader to check that these data define a lax $\mathbf{g}$-morphism $\mathbf{sk}_m\, \mathcal{G} : \mathcal{C}^{\leq m+1} \longrightarrow \mathscr{M}$ and that $\mathbf{sk}_m$ is indeed a left adjoint to τ_m. ■

Remark 3.6.4. 1. According to the description of $\mathbf{sk}_m$, given a transformation $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ in $\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m}, \mathscr{M})$, one defines $\mathbf{sk}_m(\alpha)$ as the transformation given the maps $\alpha_s : \mathcal{F}s \longrightarrow \mathcal{G}s$ ($\mathbf{deg}(s) < m$) together with the maps $\mathbf{Latch}_{\text{lax}}(\mathcal{F}, z) \longrightarrow \mathbf{Latch}_{\text{lax}}(\mathcal{G}, z)$ ($\mathbf{deg}(z) = m+1$) induced by the universal property of the colimits. In particular for each pair (A, B) we have a natural transformation $(\mathbf{sk}_m\, \alpha)_{AB} : (\mathbf{sk}_m\, \mathcal{F})_{AB} \longrightarrow (\mathbf{sk}_m\, \mathcal{G})_{AB}$ extending $\alpha_{AB} : \mathcal{F}_{AB} \longrightarrow \mathcal{G}_{AB}$.

2. It turns out that a map $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ in $\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m+1}, \mathscr{M})$ is determined by its restriction $\alpha^{\leq m}$ together with the following commutative squares for all z of degree $m+1$:

$$\begin{tikzcd}
\mathcal{F}z \arrow[r] & \mathcal{G}z \\
\mathbf{Latch}_{\text{lax}}(\mathcal{F}, z) \arrow[u] \arrow[r] & \mathbf{Latch}_{\text{lax}}(\mathcal{G}, z) \arrow[u]
\end{tikzcd}$$

Colimits and Factorization system

Let $\mathscr{M}$ be a 2-category which is locally complete and such that each $\mathscr{M}(U, V)$ has a factorization system. For simplicity we will reduce our study to the case where $\mathscr{M}$ is a monoidal category having a factorization system (L, R). Let $\mathcal{C}$ be as above and consider:

– $\mathscr{R}$ = the class of lax morphisms $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ such that for all z, the map

$$g_z : \mathcal{F}z \cup_{\mathbf{Latch}_{\text{lax}}(\mathcal{F},z)} \mathbf{Latch}_{\text{lax}}(\mathcal{G}, z) \longrightarrow \mathcal{G}z$$

is in R;

– $\mathscr{L}$ = the class of lax morphisms $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ such that for all z the map $\alpha_z : \mathcal{F}z \longrightarrow \mathcal{G}_z$ is in L.

Similarly for each $m \in \lambda$ there are two classes $\mathscr{L}_m$ and $\mathscr{R}_m$ in $\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m}, \mathscr{M})$.

Lemma 3.6.13. *With the above notations the following holds.*

1. *The functor* $\tau_m : \mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m+1}, \mathscr{M}) \longrightarrow \mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m}, \mathscr{M})$ *creates colimits.*

2. *Let* $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ *be an object* $\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m+1}, \mathscr{M})$ *such that* $\tau_m\alpha$ *has a factorization of type* $(\mathscr{L}_m, \mathscr{R}_m)$:
$$\tau_m\mathcal{F} \xrightarrow{i} \mathcal{K} \xrightarrow{p} \tau_m\mathcal{G}.$$
Then there is a factorization of α *of type* $(\mathscr{L}_{m+1}, \mathscr{R}_{m+1})$ *in* $\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m+1}, \mathscr{M})$.

3. *Let* $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ *be in* $\mathscr{L}_{m+1}$ *(resp.* $\mathscr{R}_{m+1}$*). If* $\tau_m\alpha$ *has the LLP (resp. RLP) with respect to all maps in* $\mathscr{R}_m$ *(resp.* $\mathscr{L}_m$*) then* α *has the LLP (resp. RLP) with respect to all maps in* $\mathscr{L}_{m+1}$ *(resp.* $\mathscr{R}_{m+1}$*).*

We dedicate the next paragraph for the proof of the lemma.

Proof of Lemma 3.6.13

Proof of (1) Let $\mathcal{X} : \mathscr{I} \longrightarrow \mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m+1}, \mathscr{M})$ be a diagram such that $\tau_m \mathcal{X}$ has a colimit $\mathcal{E}$ in $\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m}, \mathscr{M})$. For $i \in \mathscr{I}$ we have a canonical map $e_i : \tau_m \mathcal{X}_i \longrightarrow \mathcal{E}$.
Let z be a 1-morphism of degree $m+1$ in $\mathcal{C}^{\leq m+1}$. By the universal property of the colimit there is canonical map

$$\pi_i : \mathbf{Latch}_{\mathrm{lax}}(\tau_m \mathcal{X}_i, z) \longrightarrow \mathbf{Latch}_{\mathrm{lax}}(\mathcal{E}, z).$$

Note that $\mathbf{Latch}_{\mathrm{lax}}(\tau_m \mathcal{X}_i, z) \cong \mathbf{Latch}_{\mathrm{lax}}(\mathcal{X}_i, z)$ so we can drop the τ_m here. Furthermore the maps π_i are functorial in i, that is we have an obvious functor $\pi : \mathscr{I} \longrightarrow (\mathscr{M} \downarrow \mathbf{Latch}_{\mathrm{lax}}(\mathcal{E}, z))$.

Let Λ_z be the diagram in $\mathscr{M}$ made of the following spans (= pushout data) which are connected in the obvious manner:

$$\begin{tikzcd}
 & \mathcal{X}_i z \arrow[rr, "\mathcal{X}(i\longrightarrow j)"] & & \mathcal{X}_j z \\
 & \mathbf{Latch}_{\mathrm{lax}}(\mathcal{X}_i, z) \arrow[u, "\varepsilon_i"'] \arrow[rr] \arrow[dl, "\pi_i"'] & & \mathbf{Latch}_{\mathrm{lax}}(\mathcal{X}_j, z) \arrow[u, "\varepsilon_j"'] \arrow[dlll, "\pi_j"] \\
\mathbf{Latch}_{\mathrm{lax}}(\mathcal{E}, z) & & &
\end{tikzcd}$$

Denote by $\widetilde{\mathcal{E}}z$ the colimit of Λ_z. The are several ways to compute this colimit. One can proceed as follows.

- Introduce $\mathcal{X}_\infty z = \mathrm{colim}_{\mathscr{I}} \mathcal{X}_i z = \mathrm{colim}_{\mathscr{I}} \mathrm{Ev}_z \circ \mathcal{X}$; we have a canonical map $\delta_i : \mathcal{X}_i z \longrightarrow \mathcal{X}_\infty z$.
- Let $\mathcal{O}_i(z) = \mathcal{X}_\infty z \cup_{\mathbf{Latch}_{\mathrm{lax}}(\mathcal{X}_i, z)} \mathbf{Latch}_{\mathrm{lax}}(\mathcal{E}, z)$ be the object obtained by the pushout

$$\begin{tikzcd}
\mathbf{Latch}_{\mathrm{lax}}(\mathcal{X}_i, z) \arrow[r, "\delta_i \circ \varepsilon_i"] \arrow[d, "\pi_i"'] & \mathcal{X}_\infty z \arrow[d] \\
\mathbf{Latch}_{\mathrm{lax}}(\mathcal{E}, z) \arrow[r] & \mathcal{O}_i(z)
\end{tikzcd}$$

- The objects $\mathcal{O}_i(z)$ are functorial in i, that is we have a functor $\mathcal{O}(z) : \mathscr{I} \longrightarrow \mathscr{M}$ that takes i to $\mathcal{O}_i(z)$.
- Then it's easy to see that $\widetilde{\mathcal{E}}z \cong \mathrm{colim}_{\mathscr{I}} \mathcal{O}(z)$.

So for each i and each z of degree $m+1$ we have a canonical map $\iota_i : \mathcal{X}_i z \longrightarrow \widetilde{\mathcal{E}}z$ and the following commutes

$$\begin{tikzcd}
\mathcal{X}_i z \arrow[r, "\iota_i"] & \widetilde{\mathcal{E}}z \\
\mathbf{Latch}_{\mathrm{lax}}(\mathcal{X}_i, z) \arrow[u, "\varepsilon_i"] \arrow[r, "\mathrm{can}"'] & \mathbf{Latch}_{\mathrm{lax}}(\mathcal{E}, z) \arrow[u, "\mathrm{can}"']
\end{tikzcd}$$

The objects $\widetilde{\mathcal{E}}z$ together with the obvious maps defined a unique lax $\mathbf{g}$-morphism $\widetilde{\mathcal{E}} : \mathcal{C}^{\leq m+1} \longrightarrow \mathscr{M}$ such that $\tau_m(\widetilde{\mathcal{E}}) = \mathcal{E}$. We leave the reader to check that $\widetilde{\mathcal{E}}$ equipped with the natural cocone satisfies the universal property of the colimit in $\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m+1}, \mathscr{M})$. This proves the assertion (1).

Proof of (2) Let $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ be in $\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m+1}, \mathscr{M})$ and z be of degree $m+1$. By hypothesis the following commutes

$$\begin{tikzcd}
\mathcal{F}_i z \arrow[rr, "\alpha"] & & \mathcal{G}z \\
\mathbf{Latch}_{\mathrm{lax}}(\mathcal{F}, z) \arrow[u, "\varepsilon"] \arrow[r, "i"'] & \mathbf{Latch}_{\mathrm{lax}}(\mathcal{K}, z) \arrow[r, "p"'] & \mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z) \arrow[u, "\varepsilon"']
\end{tikzcd}$$

So we have a unique map $\mathcal{F}z \cup_{\mathbf{Latch}_{\mathrm{lax}}(\mathcal{F},z)} \mathbf{Latch}_{\mathrm{lax}}(\mathcal{K}, z) \longrightarrow \mathcal{G}z$. We use the factorization in $\mathscr{M}$ to factorize this map as

$$\mathcal{F}z \cup_{\mathbf{Latch}_{\mathrm{lax}}(\mathcal{F},z)} \mathbf{Latch}_{\mathrm{lax}}(\mathcal{K}, z) \xrightarrow{i'} \mathcal{K}'z \xrightarrow{p'} \mathcal{G}z$$

where $i' \in L$ and $p' \in R$. Write $p_z = p'$ and i_z for the composite

$$\mathcal{F}z \longrightarrow \mathcal{F}z \cup_{\mathbf{Latch}_{\mathrm{lax}}(\mathcal{F},z)} \mathbf{Latch}_{\mathrm{lax}}(\mathcal{K}, z) \xrightarrow{i'} \mathcal{K}'z.$$

If we assemble these data for all z of degree $m+1$, we have an object $\mathcal{K}' \in \mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m+1}, \mathscr{M})$ such that $\tau_m\mathcal{K}' = \mathcal{K}$ with maps $i : \mathcal{F} \longrightarrow \mathcal{K}' \in \mathscr{L}_{m+1}$ and $p : \mathcal{K}' \longrightarrow \mathcal{G} \in \mathscr{R}_{m+1}$ such that $\alpha = p \circ i$. And the assertion (2) follows.

Proof of (3) Consider a lifting problem in $\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m+1}, \mathscr{M})$ defined by $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ and $p : \mathcal{X} \longrightarrow \mathcal{Y}$:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "\alpha"'] & \mathcal{X} \arrow[d, "p"] \\
\mathcal{G} \arrow[r] & \mathcal{Y}
\end{tikzcd}$$

By hypothesis, in the two cases, there is a solution $h : \tau_m\mathcal{G} \longrightarrow \tau_m\mathcal{X}$ for the truncated problem in $\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m}, \mathscr{M})$. The idea is to extend h into a lax $\mathbf{g}$-morphism $h' : \mathcal{G} \longrightarrow \mathcal{X}$. As usual we reduce the problem to find h'_z for z of degree $m+1$. For each z of degree $m+1$, we have by h a canonical map $\mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z) \longrightarrow \mathbf{Latch}_{\mathrm{lax}}(\mathcal{X}, z)$; if we compose with ε we get a map

$$\mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z) \longrightarrow \mathbf{Latch}_{\mathrm{lax}}(\mathcal{X}, z) \xrightarrow{\varepsilon} \mathcal{X}z.$$

By the universal property of the pushout we get a unique map $\mathcal{F}z \cup_{\mathbf{Latch}_{\mathrm{lax}}(\mathcal{F},z)} \mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z) \longrightarrow \mathcal{X}z$ and the following commutes:

$$\begin{tikzcd}
\mathcal{F}z \cup_{\mathbf{Latch}_{\mathrm{lax}}(\mathcal{F},z)} \mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z) \arrow[r] \arrow[d, "g_z"'] & \mathcal{X}z \arrow[d, "p_z"] \\
\mathcal{G}z \arrow[r] & \mathcal{Y}z
\end{tikzcd}$$

So if either $g_z \in L$ or $p_z \in R$ we can find a lift $h'_z : \mathcal{G}z \longrightarrow \mathcal{X}z$ making everything commutative. In particular the following commutes:

$$\begin{tikzcd}
\mathcal{G}z \arrow[r, "h'_z"] & \mathcal{X}z \\
\mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z) \arrow[u, "\varepsilon"] \arrow[r, "h"'] & \mathbf{Latch}_{\mathrm{lax}}(\mathcal{X}, z) \arrow[u, "\varepsilon"']
\end{tikzcd}$$

Thus the collection of h together with the maps h'_z constitutes a lax $\mathbf{g}$-morphism $\mathcal{G} \longrightarrow \mathcal{X}$ which is obviously a solution to the original problem. ■

3.6.2 Application: a model structure

We apply the previous material to establish the following theorem.

Theorem 3.6.14. *Let $\mathscr{M}$ be a 2-category which is a Quillen algebra, and $\mathcal{C}$ be a locally direct category which is simple and such that the degree function $\boldsymbol{deg} : \mathcal{C} \longrightarrow \lambda$ has a minimal value m_0 for non identity 1-morphisms.*

Then there is a model structure on the category $\mathrm{Lax}(\mathcal{C}, \mathscr{M})_u$ of all normal lax morphisms; where a morphism $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is:

1. *a* weak equivalence *if for all 1-morphism z, $\alpha_z : \mathcal{F}z \longrightarrow \mathcal{G}z$ is a weak equivalence;*
2. *a* fibration *if for all 1-morphism z, $\alpha_z : \mathcal{F}z \longrightarrow \mathcal{G}z$ is a fibration;*
3. *a* cofibration *if for all z the canonical map*

$$g_z : \mathcal{F}z \cup_{\mathbf{Latch}_{\mathrm{lax}}(\mathcal{F}, z)} \mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z) \longrightarrow \mathcal{G}z$$

is a cofibration.

Corollary 3.6.15. *For a monoidal model category $\mathscr{M}$ such that all objects are cofibrant, the category $\mathscr{M}_{\mathbb{S}}(X)$ has a model structure, called the* Reedy model structure*, with the above three classes of weak equivalence, fibration, cofibration. We will denote it by $\mathscr{M}_{\mathbb{S}}(X)_{Reedy}$.*

Proof of Theorem 3.6.14. The proof is very similar to the one given by Hovey [42, Thm 5.1.3] for classical diagrams indexed by direct categories.
An easy exercise shows that the above three classes of maps are closed under retracts. Following Hovey, we will say that α is a *good trivial cofibration* if for all z

$$g_z : \mathcal{F}z \cup_{\mathbf{Latch}_{\mathrm{lax}}(\mathcal{F}, z)} \mathbf{Latch}_{\mathrm{lax}}(\mathcal{G}, z) \longrightarrow \mathcal{G}z$$

is a trivial cofibration.

It's not hard to see that in the almost-2-category $\mathcal{C}^{\leq m_0}$, the only 1-morphisms of degree $\leq m_0$ are identities, which are of degree 0; and the ones of degree m_0. This is a consequence of m_0 being a minimal value. In addition to that, there are no nontrivial 2-morphisms between 1-morphisms of degree m_0; indeed, the factorization axiom in the Reedy 1-categories $\mathcal{C}(A, B)$ will contradict the minimality of m_0.

Furthermore since we assumed that the composition in $\mathcal{C}$ adds the degrees i.e, $\mathbf{deg}(x \otimes y) = \mathbf{deg}(x) + \mathbf{deg}(y)$, it's easy to see that for z such that $\mathbf{deg}(z) = m_0$, the only pairs (x, y) such that $x \otimes y = z$ are: (Id, z) and (z, Id) (as m_0 is minimal). If we put these observations together, we see that objects of the category $\mathrm{Lax}_{\mathbf{g}}[\mathcal{C}^{\leq m_0}, \mathscr{M}]_n$ have no *pure laxity maps* i.e, the only laxity maps are the isomorphisms $\mathrm{Id} \otimes \mathcal{F}(z) \xrightarrow{\cong} \mathcal{F}(z)$ and $\mathcal{F}(z) \otimes \mathrm{Id} \xrightarrow{\cong} \mathcal{F}(z)$. Consequently it's not hard to see that the functor that forgets the laxity map induces an equivalence of categories:

$$\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m_0}, \mathscr{M})_u \longrightarrow \coprod_{\mathrm{Ob}(C) \xrightarrow{\phi} \mathrm{Ob}(\mathscr{M})} \prod_{(A,B) \in \mathrm{Ob}(\mathcal{C})^2} \mathrm{Hom}[\mathcal{C}^{\leq m_0}(A,B), \mathscr{M}(\phi A, \phi B)].$$

On the right hand side, each summand is a product of diagram categories, where each factor is indexed by the Reedy 1-category $\mathcal{C}^{\leq m_0}(A,B)$. From the classical theory of Reedy diagrams, we know that each factor in each summand carries a model structure, in particular is complete and cocomplete. Therefore the whole right hand side carries the obvious model structure and by equivalence, we deduce that $\mathrm{Lax}_{\mathbf{g}}(\mathcal{C}^{\leq m_0}, \mathscr{M})_u$ carries also a model structure; and in particular is cocomplete and has the obvious two factorization systems. Now thanks to Lemma 3.6.13 we establish (by induction) that:

- $\mathrm{Lax}(\mathcal{C}, \mathscr{M})_u$ is cocomplete; it's also complete since limits are computed level-wise.
- Any map α can be factorized as a cofibration followed by a trivial fibration.
- Any map α can be factorized as a good trivial cofibration followed by a fibration.
- Good trivial cofibrations have the LLP with respect to all fibrations; and trivial fibrations have the RLP with respect to all cofibrations.

Finally following the same method as Hovey one shows using a retract argument that every map which is both a weak equivalence and a cofibration is a good trivial cofibration. ■

Remark 3.6.5. For a classical Reedy 1-category $\mathcal{D}$, if we view it as an **lr**-category which is simple, then the previous theorem gives the same model structure for diagrams in $\mathscr{M}$ indexed by $\mathcal{D}$ (see [42, Thm 5.1.3]).

3.7 A model structure on $\mathscr{M}_{\mathbb{S}}(X)$

In this section we want to show, with a different method that for a fixed set X, the category $\mathbb{S}_X$-diagrams whose objects are called *co-Segal precategories* has a model structure when $\mathscr{M}$ is monoidal model category. In the first case we will assume that $\mathscr{M}$ is cofibrantly generated model category with a set $\mathbf{I}$ (resp. $\mathbf{J}$) of generating cofibrations (resp. acyclic cofibrations).

The model structure will be obtained by transfering the model structure on the category $\mathcal{K}_X = \prod_{(A,B) \in X^2} \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$ along the monadic adjunction $\mathscr{M}_{\mathbb{S}}(X) \leftrightarrows \mathcal{K}_X$.

On $\mathcal{K}_X$ we will consider for our purposes the *projective* and *injective* model structures. Each of these model structures is the product of the one on each factor $\mathcal{K}_{X,AB} = \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$. Since each $\mathbb{S}_{\overline{X}}(A,B)$ is an inverse category (like Δ_{epi}) the projective and Reedy model structure on the presheaf category $\mathcal{K}_{X,AB}$ are the same. In fact the identity is an isomorphism of model categories between $(\mathcal{K}_{X,AB})_{\text{proj}}$ and $(\mathcal{K}_{X,AB})_{\text{Reedy}}$ see [9, 3.17], [42, Ch. 5]. In the last reference one views $\mathcal{K}_{X,AB}$ as a functor category where the source is the directed category $\mathbb{S}_{\overline{X}}(A,B)^{op}$. The reader can find in [9, Prop 3.3], [40, Ch. 11.6; Ch.15],[42, Ch. 5], [66, A.3.3] [79, Ch. 7.6.2], a description of these model structures on diagram categories.

Denote by $\mathcal{K}_{X\text{-proj}}$ (resp. $\mathcal{K}_{X\text{-inj}}$) the projective (resp. injective) model structure on $\mathcal{K}_X$. These are cofibrantly generated model categories as (small) product of such model categories. The generating cofibrations and acyclic cofibration are respectively $\mathbf{I}_{\bullet} = \prod_{(A,B) \in X^2} \mathbf{I}_{AB}$ and

$\mathbf{J}_\bullet = \prod_{(A,B)\in X^2} \mathbf{J}_{AB}$, where $\mathbf{I}_{AB}$ (resp. $\mathbf{J}_{AB}$) is the corresponding set of cofibration (resp. acyclic cofibrations) in $\mathrm{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$. For $\mathcal{K}_{X\text{-proj}}$ one can actually tell more about the sets $\mathbf{I}_{AB}$ and $\mathbf{J}_{AB}$; the reader can find a nice description in the above references.

In contrast to the projective model structure, there is not an explicit characterization in $\mathcal{K}_{X\text{-inj}}$ for the generating set of (trivial) cofibrations. The generating cofibrations are known so far to be (trivial) cofibrations between presentable objects see [9], [66], [79] and references therein.

Extra hypothesis on $\mathscr{M}$ For the moment we will assume that all objects of $\mathscr{M}$ are cofibrant.

The following lemma due to Schwede-Shipley [77] is the key step for the transfer of the model structure on $\mathcal{K}_X$ to $\mathscr{M}_{\mathbb{S}}(X)$ through the monadic adjunction

$$\mathcal{U} : \mathscr{M}_{\mathbb{S}}(X) \rightleftarrows \mathcal{K}_X : \Gamma$$

Lemma 3.7.1. *Let $\mathcal{T}$ be a monad on a cofibrantly generated model category $\mathcal{K}$, whose underlying functor preserves directed colimits. Let $\mathbf{I}$ be the set of generating cofibrations and $\mathbf{J}$ be the set of generating acyclic cofibrations for $\mathcal{K}$. Let $\mathbf{I}_{\mathcal{T}}$ and $\mathbf{J}_{\mathcal{T}}$ be the images of these sets under the free $\mathcal{T}$-algebra functor. Assume that the domains of $\mathbf{I}_{\mathcal{T}}$ and $\mathbf{J}_{\mathcal{T}}$ are small relative to $\mathbf{I}_{\mathcal{T}}$-cell and $\mathbf{J}_{\mathcal{T}}$-cell respectively. Suppose that*

1. *every regular $\mathbf{J}_{\mathcal{T}}$-cofibration is a weak equivalence, or*
2. *every object of $\mathcal{K}$ is fibrant and every $\mathcal{T}$-algebra has a path object.*

Then the category of $\mathcal{T}$-algebras is a cofibranty generated model category with $\mathbf{I}_{\mathcal{T}}$ a generating set of cofibrations and $\mathbf{J}_{\mathcal{T}}$ a generating set of acyclic cofibrations.

In our case we will need only to show that the condition (1) holds. We do this in the next paragraph.

Note. In the formulation of Schwede-Shipley [77], $\mathbf{I}_{\mathcal{T}}$-cell and $\mathbf{J}_{\mathcal{T}}$-cell are respectively denoted by $\mathbf{I}_{\mathcal{T}}$-$\mathrm{cof}_{\mathrm{reg}}$ and $\mathbf{J}_{\mathcal{T}}$-$\mathrm{cof}_{\mathrm{reg}}$.

3.7.1 Pushouts in $\mathscr{M}_{\mathbb{S}}(X)$

Our goal here is to understand the pushout in $\mathscr{M}_{\mathbb{S}}(X)$ of $\Gamma\alpha$ where $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ is a (trivial) cofibration in $\mathcal{K}_{X\text{-inj}}$ or $\mathcal{K}_{X\text{-proj}}$.

By construction Γ preserves level-wise cofibrations and weak equivalences in $\mathcal{K}_X$ so $\Gamma\alpha$ is clearly a level-wise (trivial) cofibration if α is a (trivial) cofibration.
For an object $\mathcal{F}$ of $\mathscr{M}_{\mathbb{S}}(X)$ we want to analyze the pushout of $\Gamma\mathcal{B} \xleftarrow{\Gamma\alpha} \Gamma\mathcal{A} \longrightarrow \mathcal{F}$. Before going to this task we start below with a constant case; we consider three objects with lax morphisms which are coherent. The goal is to outline how one builds laxity maps when we move each of the three objects.

Analysis of the constant case Let $m_1, m_2, m_3, m_{12}, m_{23}$ and m be objects of $\mathscr{M}$ with maps:

- $\varphi : m_1 \otimes m_2 \otimes m_3 \longrightarrow m$,
- $\varphi_{1,2} : m_1 \otimes m_2 \longrightarrow m_{12}$,
- $\varphi_{2,3} : m_2 \otimes m_3 \longrightarrow m_{23}$,

– $\varphi_{1,23} : m_1 \otimes m_{23} \longrightarrow m$,

– $\varphi_{12,3} : m_{12} \otimes m_3 \longrightarrow m$,

Assume moreover that the following 'associativity condition' holds:

$$\varphi_{12,3} \circ (\varphi_{1,2} \otimes \mathrm{Id}_{m_3}) = \varphi_{1,23} \circ (\mathrm{Id}_{m_1} \otimes \varphi_{2,3}) = \varphi.$$

These equalities are pieces of the coherence conditions required for the laxity maps: think $F(s) = m_1, F(t) = m_2, F(u) = m_3, F(s \otimes t) = m_{12}$, $\varphi_{t,s} = \varphi_{1,2}$, etc. We've considered only three generic objects because the coherences for lax morphisms involves three terms.

Terminology. We will say that the objects m_i, m_{ij} together with the maps φ satisfying the previous equality form a **3-ary coherent system**. There is also a notion of n-ary-coherent system when we consider n objects $m_1, ..., m_n$ with compatible laxity maps. These are the 'constant data' of lax morphism between $\mathcal{O}$-algebras.

Given three maps $\alpha_i : m_i \longrightarrow m'_i$ ($i \in \{1, 2, 3\}$), we consider successively:

– $\alpha_{12} : m_{12} \longrightarrow \mathcal{R}_{12}$ the pushout of $\alpha_1 \otimes \alpha_2$ along $\varphi_{1,2}$. $\mathcal{R}_{12} = m'_1 \otimes m'_2 \cup_{m_1 \otimes m_2} m_{12}$.

– $\alpha_{23} : m_{23} \longrightarrow \mathcal{R}_{23}$ the pushout of $\alpha_2 \otimes \alpha_3$ along $\varphi_{2,3}$.

These pushouts come with canonical maps: $\widetilde{\varphi}_{i,i+1} : m'_i \otimes m'_{i+1} \longrightarrow \mathcal{R}_{i,i+1}$ ($i \in \{1, 2\}$).

Definition 3.7.2. *Define the* ***coherent object*** m' *to be the colimit of the diagram below:*

$$\begin{tikzcd}
 & & m_{12} \otimes m_3 \arrow[dr, "\varphi"] \arrow[ddd] & \\
m_1 \otimes m_2 \otimes m_3 \arrow[urr, "\varphi\otimes\mathrm{Id}"] \arrow[dr, "\mathrm{Id}\otimes\varphi"'] \arrow[ddd, "\otimes\alpha_i"'] & & & m \arrow[ddd, dotted, "\beta"] \\
 & m_1 \otimes m_{23} \arrow[urr, "\varphi"'] \arrow[ddd] & & \\
 & & \mathcal{R}_{12} \otimes m'_3 \arrow[dr, dotted] & \\
m'_1 \otimes m'_2 \otimes m'_3 \arrow[urr, "\widetilde{\varphi}\otimes\mathrm{Id}"] \arrow[dr, "\mathrm{Id}\otimes\widetilde{\varphi}"'] & & & m' \\
 & m'_1 \otimes \mathcal{R}_{23} \arrow[urr, dotted] & &
\end{tikzcd}$$

Proposition 3.7.3. *With the above notation, assume that all objects of the ambient category* $\mathscr{M}$ *are cofibrant. Then if each* $\alpha_i : m_i \longrightarrow m'_i$ *is a (trivial) cofibration, then the canonical map* $\beta : m \longrightarrow m'$ *is a (trivial) cofibration as well.*

Remark 3.7.1. The reason we demand the objects to be cofibrant is the fact that tensoring with a cofibrant object preserves (trivial) cofibrations. This is a consequence of the pushout-product axiom.

The strategy to prove the proposition is to 'divide then conquer'; we will use the following lemma which treats the case where one of the faces in the original semi-cube is a pushout square. This is a classical Reedy-style lemma (see for example Lemma 7.2.15 in[40]).

Lemma 3.7.4. *Let Q be a semi-cube in a model category $\mathscr{M}$ whose colimit is an object m':*

$$
\begin{tikzcd}
 & & \cdot \arrow[dr] \arrow[ddd, dotted, "\delta_1"] & \\
\cdot \arrow[urr] \arrow[dr] \arrow[ddd] & & & m \\
 & \cdot \arrow[urr] \arrow[ddd, "\delta_2"] & & \\
 & & \cdot & \\
\cdot \arrow[urr, dotted] \arrow[dr] & & & \\
 & \cdot & &
\end{tikzcd}
$$

Assume that the face containing δ_1 is a pushout square. Then if δ_2 is a (trivial) cofibration, then the canonical map $\beta : m \longrightarrow m'$ is also a (trivial) cofibration.

In practice we will use the lemma when all the vertical map are (trivial) cofibrations.

Proof of the lemma . We simply treat the case where δ_2 is a trivial cofibration; the method is the same when δ_2 is just a cofibration. β will be a trivial cofibration if we show that it has the LLP with respect to all fibrations.
Consider a lifting problem defined by β and a fibration $p : x \twoheadrightarrow y$:

$$
\begin{tikzcd}
m \arrow[r, "i"] \arrow[d, "\beta"'] & x \arrow[d, "p"] \\
m' \arrow[r, "j"] & y
\end{tikzcd}
$$

A solution to this problem is a map out of m', $h : m' \longrightarrow x$, satisfying the obvious equalities. Since m' is a colimit-object, we simply have to show that we can complete in a suitable way the semi-cube Q into a commutative cube ending at x; the map h will be then induced by universal property of the colimit.

If we join the lifting problem to the universal cube we get a commutative diagram displayed below

$$
\begin{tikzcd}
 & & \cdot \arrow[dr] \arrow[ddd, "\delta_1"] & & \\
\cdot \arrow[urr] \arrow[dr] \arrow[ddd] & & & m \arrow[r, "i"] \arrow[ddd, dotted, "\beta"'] & x \arrow[ddd, "p"] \\
 & \cdot \arrow[urr] \arrow[ddd, hook, "\delta_2"', "\wr"] & & & \\
 & & \cdot \arrow[dr, dotted] \arrow[uurr, dotted, "g"] & & \\
\cdot \arrow[urr] \arrow[dr] & & & m' \arrow[r, "j"] \arrow[uuur, "h"] & y \\
 & \cdot \arrow[urr, dotted] \arrow[uuuurrr, "f"] & & &
\end{tikzcd}
$$

Since δ_2 is a trivial cofibration there is a solution f to the lifting problem defined by δ_2 and p. With the map f we get a commutative square starting from the horn defining the pushout square in the back (the one containing δ_1) and ending at x; so by universal property of the pushout, there is a unique map g making the obvious diagram commutative.

With the maps f and g we have a commutative cube ending at x, so by universal property of the colimit we have a unique map $h : m' \longrightarrow x$ making everything commutative. In particular h satisfies the equality $i = h \circ \beta$.

By construction the two cubes ending at y obtained with the maps j and $p \circ h$ are the same, so by uniqueness of the map out of the colimit we have $j = p \circ h$. Consequently h is a solution to the original lifting problem and β is a trivial cofibration as desired. ■

Remark 3.7.2. The statement of the lemma remains valid if we consider a more general situation where the pushout square containing δ_1 is replaced by another commutative square in which the morphism ε out of the pushout is a (trivial) cofibration:

$$\begin{tikzcd}
\cdot \arrow[rr] \arrow[dd] & & \cdot \arrow[dl, dotted] \arrow[dd, "\delta_1"] \\
& \cdot \arrow[dr, hook, "\varepsilon"'] & \\
\cdot \arrow[ur, dotted] \arrow[rr] & & \cdot
\end{tikzcd}$$

Proof of Proposition 3.7.3

To prove the proposition, we will present the cube defining m' as a concatenation of other universal cubes where each of them satisfies the condition of the previous lemma. The proof is organized as follows.

- First we treat the case where only m_1 moves that is $\alpha_2 = \mathrm{Id}_{m_2}$ and $\alpha_3 = \mathrm{Id}_{m_3}$. We will denote by z_1 the coherent object defined with these data and denote by Q_1 the induced universal cube. Denote by $\beta_1 : m \longrightarrow z_1$ the canonical map.
- The lower face of the cube Q_1 is a coherent system ending at z_1. We construct z_2 to be the coherent object with respect to that associative system, where only m_2 moves i.e $\alpha_1 = \mathrm{Id}_{m'_1}$ and $\alpha_3 = \mathrm{Id}_{m_3}$. We will denote by Q_2 the new universal cube. There is a canonical map $\beta_2 : z_1 \longrightarrow z_2$.
- Finally with the lower face of Q_2, we treat the case where only m_3 moves, which is similar to the first case. We have a coherent object z_3 with a new cube Q_3; there is also a canonical map $\beta_3 : z_2 \longrightarrow z_3$.
- By universal property we have $z_3 \cong m'$, thus we can take $\beta = \beta_3 \circ \beta_2 \circ \beta_1$.
- Each cube Q_i is constructed from a semi-cube satisfying the conditions of the previous lemma, thus each β_i will be a (trivial) cofibration and the result will follow.

We need some pieces of notation for the rest of the proof.

Notation 3.7.1.

1. Let O_{12} and P_{12} be the objects obtained from the pushout squares:

$$S_1 = \begin{tikzcd}
m_1 \otimes m_2 \arrow[r, "\varphi"] \arrow[d, "\alpha_1 \otimes \mathrm{Id}"'] & m_{12} \arrow[d, "h_{12}"] \\
m'_1 \otimes m_2 \arrow[r, "\widetilde{\varphi}"] & O_{12}
\end{tikzcd}
\qquad
S_2 = \begin{tikzcd}
m'_1 \otimes m_2 \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\mathrm{Id} \otimes \alpha_2"'] & O_{12} \arrow[d, "k_{12}"] \\
m'_1 \otimes m'_2 \arrow[r, "\widetilde{\varphi}'"] & P_{12}
\end{tikzcd}$$

From lemma 4.1.2 we know that the 'vertical' concatenation '$\frac{S_1}{S_2}$' of these pushout squares is 'the' pushout square defining $\mathcal{R}_{12}$; it follows that $P_{12} \cong \mathcal{R}_{12}$.

Now since colimits distribute over the tensor product, tensoring S_1 and S_2 with m_3 gives two pushout squares $S_1 \otimes m_3$ and $S_2 \otimes m_3$. The concatenation of the latter squares is the pushout square

$$D = \begin{tikzcd}
m_1 \otimes m_2 \otimes m_3 \arrow[r, "\varphi \otimes \mathrm{Id}"] \arrow[d, "\alpha_1 \otimes \alpha_2 \otimes \mathrm{Id}"'] & m_{12} \otimes m_3 \arrow[d, "p_{12} \otimes \mathrm{Id}"] \\
m'_1 \otimes m'_2 \otimes m_3 \arrow[r, "\widetilde{\varphi} \otimes \mathrm{Id}"] & \mathcal{R}_{12} \otimes m_3
\end{tikzcd}$$

2. Let K_{23} and L_{23} be the objects obtained from the the pushout squares:

$$T_1 = \begin{tikzcd} m_2 \otimes m_3 \arrow[r, "\varphi"] \arrow[d, "\alpha_2 \otimes \mathrm{Id}"'] & m_{23} \arrow[d] \\ m_2' \otimes m_3 \arrow[r, "\widetilde{\varphi}"] & K_{23} \end{tikzcd} \qquad T_2 = \begin{tikzcd} m_2' \otimes m_3 \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\mathrm{Id} \otimes \alpha_3"'] & K_{23} \arrow[d] \\ m_2' \otimes m_3' \arrow[r, "\widetilde{\varphi}'"] & L_{23} \end{tikzcd}$$

As usual the concatenation of T_1 and T_2 is the pushout square defining $\mathcal{R}_{23}$ so we can take $L_{23} = \mathcal{R}_{23}$. And if we tensor everywhere by m_1' we still have pushout square $m_1' \otimes T_1$ and $m_1' \otimes T_2$ and their concatenation is the pushout square:

$$E = \begin{tikzcd} m_1' \otimes m_2 \otimes m_3 \arrow[r, "\mathrm{Id} \otimes \varphi"] \arrow[d, "\mathrm{Id} \otimes \alpha_2 \otimes \alpha_3"'] & m_1' \otimes m_{23} \arrow[d] \\ m_1' \otimes m_2' \otimes m_3' \arrow[r, "\mathrm{Id} \otimes \widetilde{\varphi}"] & m_1' \otimes \mathcal{R}_{23} \end{tikzcd}$$

Step 1: Moving m_1
In this case we consider the following semi-cube whose colimit is z_1:

$$\begin{tikzcd}
& m_{12} \otimes m_3 \arrow[dr] \arrow[ddd, dotted] & \\
m_1 \otimes m_2 \otimes m_3 \arrow[ur] \arrow[dr] \arrow[ddd] & & m \\
& m_1 \otimes m_{23} \arrow[ur] \arrow[ddd] & \\
& O_{12} \otimes m_3 & \\
m_1' \otimes m_2 \otimes m_3 \arrow[ur] \arrow[dr] & & \\
& m_1' \otimes m_{23} &
\end{tikzcd}$$

The face in the back is precisely the pushout square $S_1 \otimes m_3$ and the map $\delta_2 = \alpha_1 \otimes \mathrm{Id}_{m_{23}}$ is a (trivial) cofibration since α_1 is so (Remark 3.7.1); then by Lemma 3.7.4 we know that the canonical map $\beta_1 : m \longrightarrow z_1$ is also a (trivial) cofibration.

Step 2: Moving m_2
Introduce the following semi-cube whose colimit is z_2:

$$\begin{tikzcd}
& O_{12} \otimes m_3 \arrow[dr] \arrow[ddd, dotted] & \\
m_1' \otimes m_2 \otimes m_3 \arrow[ur] \arrow[dr] \arrow[ddd] & & z_1 \\
& m_1' \otimes m_{23} \arrow[ur] \arrow[ddd] & \\
& \mathcal{R}_{12} \otimes m_3 & \\
m_1' \otimes m_2' \otimes m_3 \arrow[ur] \arrow[dr] & & \\
& m_1' \otimes N_{23} &
\end{tikzcd}$$

The two faces not containing z_1 are pushout squares; the one in the back is $S_2 \otimes m_3$ and the other one is $m_1' \otimes T_1$. All the vertical maps appearing there are (trivial) cofibrations since α_2 is so, therefore by Lemma 3.7.4 the canonical map $\beta_2 : z_1 \longrightarrow z_2$ is also a (trivial) cofibration.

Step 3: Moving m_3

This time we consider the semi-cube below whose colimit is denoted by z_3:

$$
\begin{tikzcd}
& \mathcal{R}_{12}\otimes m_3 \arrow[rr] \arrow[dd, dotted] & & z_2 \\
m'_1\otimes m'_2\otimes m_3 \arrow[ur] \arrow[r] \arrow[dd] & m'_1\otimes N_{23} \arrow[urr] \arrow[dd] & & \\
& \mathcal{R}_{12}\otimes m'_3 & & \\
m'_1\otimes m'_2\otimes m'_3 \arrow[ur] \arrow[r] & m'_1\otimes \mathcal{R}_{23} & &
\end{tikzcd}
$$

The face on the left is a pushout square and corresponds to $m'_1 \otimes T_2$. The map $\delta_2 = \mathrm{Id}_{\mathcal{R}_{12}} \otimes \alpha_3$ in the face on the back is a (trivial) cofibration since α_3 is so; applying lemma 3.7.4 again we deduce that the canonical map $\beta_3 : z_2 \longrightarrow z_3$ is also a (trivial) cofibration.

One can easily see that the (vertical) concatenation of the previous universal cubes constitutes a universal cube for the original semi-cube defining m'. By uniqueness of the colimit we can take $m' = z_3$ and $\beta = \beta_3 \circ \beta_2 \circ \beta_1$. Since each β_i is a (trivial) cofibration, by composition β is a (trivial) cofibration as well, which is just we wanted to prove. ■

Remark 3.7.3. The proposition remains valid if we allow the objects m_{12} and m_{23} to move by (trivial) cofibrations. This time we will have to use the more general version of Lemma 3.7.4 pointed out in Remark 3.7.2.

The main lemma

In the following our goal is to establish that

Lemma 3.7.5. *Given a diagram $\Gamma\mathcal{B} \xleftarrow{\Gamma\alpha} \Gamma\mathcal{A} \longrightarrow \mathcal{F}$, consider the pushout in $\mathscr{M}_{\mathbb{S}}(X)$:*

$$
\begin{tikzcd}
\Gamma\mathcal{A} \arrow[r, "\sigma"] \arrow[d, hook, "\Gamma\alpha"'] & \mathcal{F} \arrow[d, dotted, "H_\alpha"] \\
\Gamma\mathcal{B} \arrow[r, dotted] & \mathcal{G}
\end{tikzcd}
$$

Then if α is a level-wise trivial cofibration in $\mathcal{K}_X$ then $\mathcal{U}H_\alpha$ is a level-wise trivial cofibration; in particular H_α is a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)$.

Proof. This is a special case of Lemma 4.5 in the Appendix. In fact $(\mathbb{S}_{\overline{X}})^{2\text{-op}}$ is an $\mathcal{O}$-algebra where $\mathcal{O}$ is the multisorted operad for (nonunital) 2-categories; $\mathscr{M}$ is a special Quillen $\mathcal{O}$-algebra with all the objects cofibrant. Furthermore:

1. $(\mathbb{S}_{\overline{X}})^{2\text{-op}}$ is an **ir**-$\mathcal{O}$-algebra in the sense of Definition 4.5.1. This follows from the fact that the composition in $\mathbb{S}_{\overline{X}}$ is a concatenation of chains and the 2-morphisms are parametrized by the morphisms in Δ^+_{epi}. In fact the composition of 2-morphisms is simply a generalization of the ordinal addition of morphisms in $(\Delta^+_{\text{epi}}, +, \mathbf{0})$; consequently the concatenation of 2-morphisms cannot be the identity unless all of them are identities.
2. The pair $((\mathbb{S}_{\overline{X}})^{2\text{-op}}, \mathscr{M})$ is an $\mathcal{O}$-**hc**-pair in the sense of Definition 4.5.1, since the left adjoint Γ preserves the level-wise trivial cofibrations (see Remark 4.2.3).

We have $\mathscr{M}_{\mathbb{S}}(X) = \mathrm{Lax}_{\mathcal{O}\text{-alg}}((\mathbb{S}_{\overline{X}})^{2\text{-op}}, \mathscr{M})$. ■

Remark 3.7.4. It's important to notice that in the lemma we've considered a level-wise cofibration α in $\mathcal{K}_X$; these are precisely the injective cofibrations therein. But this situation covers also the projective case, since projective cofibrations are also injective ones.
So in either $\mathcal{K}_{X\text{-proj}}$ or $\mathcal{K}_{X\text{-proj}}$, the pushout of $\Gamma\alpha$ is a level-wise weak equivalence and the condition (1) of lemma 3.7.1 will hold.

3.7.2 The projective model structure

According to a well known result on diagram categories in cofibrantly generated model categories, see [40, Theorem 11.6.1], each diagram category $\text{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$ has a cofibrantly generated model structure which is known to be the projective model structure.

In these settings a morphism $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is:

- A weak equivalence in $\text{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$ if it is a level-wise equivalence: for every w the component $\sigma_w : \mathcal{F}w \longrightarrow \mathcal{G}w$ is a weak equivalence in $\mathscr{M}$,
- A fibration in $\text{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$ if it is a level-wise fibration: $\sigma_w : \mathcal{F}w \longrightarrow \mathcal{G}w$ is a fibration in $\mathscr{M}$.
- A trivial fibration is a map which is both a fibration and a weak equivalence.

Left adjoint of evaluations For any object $w \in \mathbb{S}_{\overline{X}}(A,B)^{op}$ the evaluation functor at w : $\text{Ev}_w : \text{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}] \longrightarrow \mathscr{M}$ has a left adjoint

$$\mathbf{F}^w_- : \mathscr{M} \longrightarrow \text{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$$

One defines $\mathbf{F}^w_-$ by 'the body' of the Yoneda functor $\mathcal{Y}_w$ (see [40, Section 11.5.21]):

$$\mathbf{F}^w_m = m \otimes \mathcal{Y}_w = \coprod_{\text{Hom}(w,-)} m, \qquad \text{for } m \in \mathscr{M}.$$

This means that for $v \in \mathbb{S}_{\overline{X}}(A,B)^{op}$, $\mathbf{F}^w_m(v)$ is the coproduct of copies of m indexed by the set $\text{Hom}_{\mathbb{S}_{\overline{X}}(A,B)^{op}}(w,v)$. The fact that $\mathbf{F}^w_-$ has the desired properties follows from the Yoneda lemma.

With the functor $\mathbf{F}$ we have that the set of generating cofibrations is:

$$\mathbf{I}_{AB} = \coprod_{w \in \mathbb{S}_{\overline{X}}(A,B)^{op}} \mathbf{F}^w_{\mathbf{I}} = \coprod_{w \in \mathbb{S}_{\overline{X}}(A,B)^{op}} \{\mathbf{F}^w_m \xrightarrow{\mathbf{F}^w_\alpha} \mathbf{F}^w_{m'}\}_{(m \xrightarrow{\alpha} m') \in \mathbf{I}}$$

Similarly the set of generating acyclic cofibrations is:

$$\mathbf{J}_{AB} = \coprod_{w \in \mathbb{S}_{\overline{X}}(A,B)^{op}} \mathbf{F}^w_{\mathbf{J}}$$

Consider the product model structure on $\mathcal{K}_{X\text{-proj}} = \prod_{(A,B) \in X^2} \text{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]_{\text{proj}}$ where the three class of maps, cofibrations, fibrations, weak equivalences, are the natural ones i.e, factor-wise cofibrations, fibrations, and weak equivalences (see [42, Example 1.16]).

The main theorem

Theorem 3.7.6. *The category $\mathscr{M}_{\mathbb{S}}(X)$ has a combinatorial model structure where:*

- *a weak equivalence is a map σ such that $\mathcal{U}(\sigma)$ is a weak equivalence in $\mathcal{K}_{X\text{-}proj}$,*
- *a fibration is a map σ such that $\mathcal{U}(\sigma)$ is a fibration in $\mathcal{K}_{X\text{-}proj}$,*
- *a cofibration is a map having the left lifting property (LLP) with respect to all trivial fibrations,*
- *the set of generating cofibrations is $\Gamma(\mathbf{I})$,*
- *the set of generating acyclic cofibrations is $\Gamma(\mathbf{J})$.*

We will refer this model structure as the 'projective' model structure on $\mathscr{M}_{\mathbb{S}}(X)$ and denote it by $\mathscr{M}_{\mathbb{S}}(X)_{proj}$.

Proof. Thanks to our lemma 3.7.5, the condition (1) of lemma 3.7.1 holds. It follows from the lemma that $\mathscr{M}_{\mathbb{S}}(X)$ is a cofibrantly generated model category with the corresponding set of generating (trivial) cofibrations. And from Theorem 3.5.1) we know that $\mathscr{M}_{\mathbb{S}}(X)$ is locally presentable. ■

3.7.3 Lifting the injective model structure on $\mathcal{K}_X$

By the same argument as in the projective case we establish the following.

Theorem 3.7.7. *The category $\mathscr{M}_{\mathbb{S}}(X)$ has a combinatorial model structure where:*

- *a weak equivalence is a map σ such that $\mathcal{U}(\sigma)$ is a weak equivalence in $\mathcal{K}_{X\text{-}inj}$,*
- *a fibration is a map σ such that $\mathcal{U}(\sigma)$ is a fibration in $\mathcal{K}_{X\text{-}inj}$,*
- *a cofibration is a map having the left lifting property (LLP) with respect to all trivial fibrations,*
- *the set of generating cofibrations is $\Gamma(\mathbf{I})$,*
- *the set of generating acyclic cofibrations is $\Gamma(\mathbf{J})$.*

We will refer this model structure as the 'injective' model structure on $\mathscr{M}_{\mathbb{S}}(X)$ and denote it by $\mathscr{M}_{\mathbb{S}}(X)_{inj}$.

Proof. The same as for the previous theorem. ■

Corollary 3.7.8. *The identity functor* $\mathrm{Id} : \mathscr{M}_{\mathbb{S}}(X)_{proj} \rightleftarrows \mathscr{M}_{\mathbb{S}}(X)_{inj} : \mathrm{Id}$ *is a Quillen equivalence.*

Proof. The weak equivalences are the same and a projective (trivial) cofibration is also an injective (trivial) cofibration. ■

Remark 3.7.5. Note that in both $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ and $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ the fibrations and weak equivalences are the underlying ones in $\mathcal{K}_{X\text{-inj}}$ and $\mathcal{K}_{X\text{-proj}}$ respectively. Since limits in $\mathscr{M}_{\mathbb{S}}(X)$ are computed level-wise, it's easy to see that both $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ and $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ are *right proper* if $\mathscr{M}$ is so. In fact one establishes first that $\mathcal{K}_{X\text{-proj}}$ and $\mathcal{K}_{X\text{-inj}}$ are also right proper. For left properness the situation is a bit complicated, we will discuss it later.

3.8 Variation of the set of objects

Let **Set** be the category of sets of some universe $\mathbb{U} \subsetneq \mathbb{U}'$. So far we've considered the category $\mathscr{M}_{\mathbb{S}}(X)$ for a fixed set $X \in \mathbb{U}$. In this section we are going to vary X.

Since the construction of $\mathbb{S}_{\overline{X}}$ is functorial in X, any function $f : X \longrightarrow Y$ induces a strict 2-functor $\mathbb{S}_f : \mathbb{S}_{\overline{X}} \longrightarrow \mathbb{S}_{\overline{Y}}$. We then have a functor $f^\star : \mathscr{M}_{\mathbb{S}}(X) \longrightarrow \mathscr{M}_{\mathbb{S}}(Y)$. Below we will see that there is a left adjoint $f_!$ of $f^\star$. When no confusion arises we will simply write again f to mean $\mathbb{S}_f$.

Let $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ be the category described as follows.

1. The objects are pairs $(X, \mathcal{F})$ with $X \in \mathbf{Set}$ and $\mathcal{F} \in \mathscr{M}_{\mathbb{S}}(X)$,

2. The morphisms from $(X, \mathcal{F})$ to $(Y, \mathcal{G})$ are pairs (f, σ) with $f \in \mathbf{Set}(X, Y)$ and $\sigma \in \mathscr{M}_{\mathbb{S}}(X)(\mathcal{F}, f^{\star}\mathcal{G})$.

In the same way we have a category $\mathcal{K}_{\mathbf{Set}}$ and a forgetful functor $\mathcal{U} : \mathscr{M}_{\mathbb{S}}(\mathbf{Set}) \longrightarrow \mathcal{K}_{\mathbf{Set}}$.

Lemma 3.8.1. *If $\mathscr{M}$ is a symmetric closed monoidal category which is cocomplete then:*

1. $\mathcal{U}$ is monadic

2. The monad induced by $\mathcal{U}$ preserves directed colimits.

Proof. Assertion (1) is easy and is treated in the same way as in the fixed set case. For assertion (2) we simply need to see how one computes colimits in $\mathcal{K}_{\mathbf{Set}}$. Each function $f : X \longrightarrow Y$ induces an adjoint pair: $f_! : \mathcal{K}_X \rightleftarrows \mathcal{K}_Y : f^{\star}$.

Every diagram $\mathcal{J} : \mathcal{D} \longrightarrow \mathcal{K}_{\mathbf{Set}}$, induces on the set of objects, a diagram $\mathbf{pr}_1(\mathcal{J}) : \mathcal{D} \longrightarrow \mathbf{Set}$ and we can take the colimit $X_\infty = \operatorname{colim} \mathbf{pr}_1(\mathcal{J})$. For each $d \in \mathcal{D}$ the canonical map $i_d : X_d \longrightarrow X_\infty$ induces an object $i_{d!}\mathcal{J}d$ in $\mathcal{K}_{X_\infty}$. It's not hard to see that $\mathcal{J}$ induces a diagram $i_!\mathcal{J} : \mathcal{D} \longrightarrow \mathcal{K}_{X_\infty}$ where the morphisms connecting the different $i_{d!}\mathcal{J}d$ are induced by the universal property of the adjoint.

The colimit of $\mathcal{J}$ is the colimit of the pushforward diagram $i_!\mathcal{J}$. Given a directed diagram $\mathcal{J}$, one has to show that the pushforward of the colimit of $\mathcal{J}$ is the colimit of the pushforward diagram. One proceeds exactly in the same manner as Kelly and Lack [50, Lemma 3.2, Thm 3.3] who treated the case for $\mathscr{M}$-categories. ■

Theorem 3.8.2. *Let $\mathscr{M}$ be a symmetric monoidal closed category.*

1. If $\mathscr{M}$ is cocomplete then so is $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$,

2. If $\mathscr{M}$ is locally presentable then so is $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$.

Proof. All is proved in the same way as for $\mathscr{M}_{\mathbb{S}}(X)$. ■

3.8.1 Some model structures on $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$

$f^{\star}$ has a left adjoint

Let $f : X \longrightarrow Y$ be a function. As pointed out above we have an adjunction $f_! : \mathcal{K}_X \rightleftarrows \mathcal{K}_Y : f^{\star}$ which is just the product adjunction for each pair (A, B):

$$f_{!AB} : \operatorname{Hom}[\mathbb{S}_{\overline{X}}(A, B)^{op}, \mathscr{M}] \rightleftarrows \operatorname{Hom}[\mathbb{S}_{\overline{Y}}(A, B)^{op}, \mathscr{M}] : f^{\star}_{AB}.$$

The last adjunction is a Quillen adjunction between the projective model structure: this is Proposition 3.6 in [9]. It follows that $f_! : \mathcal{K}_X \rightleftarrows \mathcal{K}_Y : f^{\star}$ is also a Quillen adjunction between the respective projective model structure.

In what follows we will show that we have also a Quillen adjunction between the projective model structures on $\mathscr{M}_{\mathbb{S}}(X)$ and $\mathscr{M}_{\mathbb{S}}(Y)$. We will denote again by $f^{\star} : \mathscr{M}_{\mathbb{S}}(Y) \longrightarrow \mathscr{M}_{\mathbb{S}}(X)$ the pullback functor. By definition $f^{\star}$ preserves everything which is level-wise: (trivial) fibrations , weak equivalences, limits in $\mathscr{M}_{\mathbb{S}}(Y)$ (limits are computed level-wise). To show that we have a Quillen adjunction it suffices to show that $f^{\star}$ has a left adjoint since it already preserves fibrations and trivial fibrations (see [42, lemma 1.3.4]). We will use the adjoint functor theorem for locally presentable categories since $\mathscr{M}_{\mathbb{S}}(X)$ and $\mathscr{M}_{\mathbb{S}}(Y)$ are such categories.

Theorem 3.8.3. *A functor between locally presentable categories is a right adjoint if and only if it preserves limits and λ-directed colimits for some regular cardinal λ.*

Proof. See [1, 1.66] ■

Proposition 3.8.4. *For a symmetric monoidal model category $\mathscr{M}$ which is locally presentable, and a function $f : X \longrightarrow Y$ the following hold.*

1. *The functor $f^\star$ has a left adjoint $f_! : \mathscr{M}_{\mathbb{S}}(X) \longrightarrow \mathscr{M}_{\mathbb{S}}(Y)$.*
2. *The adjunction $f_! : \mathscr{M}_{\mathbb{S}}(X) \rightleftarrows \mathscr{M}_{\mathbb{S}}(Y) : f^\star$ is a Quillen adjunction.*
3. *We have a square of Quillen adjunctions*

$$\begin{tikzcd} \mathscr{M}_{\mathbb{S}}(X) \arrow[r, shift right, "f_!"'] \arrow[d, shift left, "\mathcal{U}"] & \mathscr{M}_{\mathbb{S}}(Y) \arrow[l, shift right, "f^\star"'] \arrow[d, shift left, "\mathcal{U}"] \\ \mathcal{K}_X \arrow[u, shift left, "\Gamma"] \arrow[r, shift right, "f_!"'] & \mathcal{K}_Y \arrow[u, shift left, "\Gamma"] \arrow[l, shift right, "f^\star"'] \end{tikzcd}$$

in which only two squares are commutative:

- $\mathcal{U} \circ f^\star = f^\star \circ \mathcal{U}$ *and*
- $\Gamma \circ f_! = f_! \circ \Gamma$.

Proof. Since $f^\star$ preserves limits and thanks to the adjoint functor theorem, it suffices to show that it also preserves directed colimits. But as the functor $\mathcal{U} : \mathscr{M}_{\mathbb{S}}(Y) \longrightarrow \mathcal{K}_Y$ preserves filtered colimits (Proposition 3.5.3) , it follows that filtered colimits in $\mathscr{M}_{\mathbb{S}}(Y)$ are computed level-wise and since $f^\star : \mathscr{M}_{\mathbb{S}}(Y) \longrightarrow \mathscr{M}_{\mathbb{S}}(X)$ preserves every level-wise property it certainly preserves them and assertion (1) follows.

Assertion (2) is a consequence of [42, lemma 1.3.4]: from (1) we know that $f^\star$ is a right adjoint functor but as it preserves (trivial) fibrations, the adjunction $f_! \dashv f^\star$ is automatically a Quillen adjunction. Assertion (3) is clear. ■

Recall that for $\mathcal{F} \in \mathscr{M}_{\mathbb{S}}(X)$, $\mathcal{G} \in \mathscr{M}_{\mathbb{S}}(Y)$ a morphism $\sigma \in \mathrm{Hom}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}(\mathcal{F}, \mathcal{G})$ is a pair $\sigma = (f, \sigma)$ where $f \in \mathbf{Set}(X, Y)$ and $\sigma \in \mathrm{Hom}_{\mathscr{M}_{\mathbb{S}}(X)}(\mathcal{F}, f^\star\mathcal{G})$. An easy exercise shows that:

Proposition 3.8.5. *The canonical functor* $\mathrm{P} : \mathscr{M}_{\mathbb{S}}(\boldsymbol{Set}) \longrightarrow \boldsymbol{Set}$ *is a Grothendieck fibration, the fiber category over $X \in \boldsymbol{Set}$ being $\mathscr{M}_{\mathbb{S}}(X)$ and the inverse image functor is $f^\star$ for $f \in \boldsymbol{Set}(X, Y)$.*

Remark 3.8.1. Note that for $f \in \mathbf{Set}(X, Y)$ and $\mathcal{G} \in \mathscr{M}_{\mathbb{S}}(Y)$, $f^\star\mathcal{G}$ is the composite

$$(\mathbb{S}_{\overline{X}})^{2\text{-op}} \xrightarrow{(\mathbb{S}_f)^{2\text{-op}}} (\mathbb{S}_{\overline{Y}})^{2\text{-op}} \xrightarrow{\mathcal{G}} \mathscr{M}.$$

The identity $\mathrm{Id}_{\mathcal{G}(f(s))}$ gives, in a tautological way, a canonical cartesian lifting of f, therefore P has a cleavage (or is cloven).

As we saw previously the inverse image functor has a left adjoint $f_!$ so we deduce that

Proposition 3.8.6. *The canonical functor* $\mathrm{P} : \mathscr{M}_{\mathbb{S}}(\boldsymbol{Set}) \longrightarrow \boldsymbol{Set}$ *is a bifibration, that is,*

$$\mathrm{P}^{op} : \mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})^{op} \longrightarrow \boldsymbol{Set}^{op}$$

is also a Grothendieck fibration (or $\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})$ is cofibered).

Proof. Apply lemma 9.1.2 in [43]. ■

Remark 3.8.2. From the adjunction $f_! \dashv f^\star$, it's not hard to see that P^{op} has a cleavage; thus P is a cloven bifibration.

The fibred model structure on $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$

In what follows we give a first model structure on $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ using the previous bifibration $\mathrm{P} : \mathscr{M}_{\mathbb{S}}(\mathbf{Set}) \longrightarrow \mathbf{Set}$. The key ingredient is to use Roig's work [74] on Quillen model structure on the 'total space' of a Grothendieck bifibration. As pointed out by Stanculescu [81], there is a gap in Roig's theorem. A reformulation was given by Stanculescu in *loc. cit* and is recalled hereafter.

Theorem 3.8.7 (Roig-Stanculescu)**.** *Let* $\mathrm{P} : \mathbf{E} \longrightarrow \mathbf{B}$ *be a cloven Grothendieck bifibration. Assume that*

i. $\mathbf{E}$ *is complete and cocomplete,*

ii. the base category $\mathbf{B}$ *as a model structure* $(\mathbf{cof}, \mathbf{we}, \mathbf{fib})$

iii. for each object $X \in \mathbf{B}$ *the fiber category* $\mathbf{E}_X$ *admits a model structure* $(\mathbf{cof}_X, \mathbf{we}_X, \mathbf{fib}_X)$*,*

iv. for every morphism $f : X \longrightarrow Y$ *of* $\mathbf{B}$*, the adjoint pair is* $(f_!, f^\star)$ *is a Quillen pair,*

v. for $f = \mathrm{P}(\sigma)$ *a weak equivalence in* $\mathbf{B}$*, the functor* $f^\star$ *preserves and reflects weak equivalences,*

vi. for $f = \mathrm{P}(\sigma)$ *a trivial cofibration in* $\mathbf{B}$*, the unit of the adjoint pair* $(f_!, f^\star)$ *is a weak equivalence.*

Then there is a model structure on $\mathbf{E}$ *where a map* $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ *in* $\mathbf{E}$ *is*

- *a weak equivalence if* $f = \mathrm{P}(\sigma) \in \mathbf{we}$ *and* $\sigma^f : \mathcal{F} \longrightarrow f^\star\mathcal{G} \in \mathbf{we}_X$*,*
- *a cofibration if* $f = \mathrm{P}(\sigma) \in \mathbf{cof}$ *and* $\sigma_f : f_!\mathcal{F} \longrightarrow \mathcal{G} \in \mathbf{cof}_Y$*,*
- *a fibration if* $f = \mathrm{P}(\sigma) \in \mathbf{fib}$ *and* $\sigma^f : \mathcal{F} \longrightarrow f^\star\mathcal{G} \in \mathbf{fib}_X$*.*

Let $\mathbf{Set}_{min}$ be the category of $\mathbf{Set}$ with the minimal model structure: weak equivalences are isomorphisms, cofibration and fibrations are all morphisms. In particular trivial cofibrations and fibrations are simply isomorphisms. Recall that if $f : X \longrightarrow Y$ an isomorphism then $(\mathbb{S}_f)^{2\text{-op}}$ is also an isomorphism, and we can take $f_! = (f^{-1})^\star$; one clearly sees that conditions (v) and (vi) of the theorem hold on the nose.

Let's fix the projective model structure on each $\mathscr{M}_{\mathbb{S}}(X)$ as X runs through $\mathbf{Set}$. By virtue of the previous theorem we deduce that

Theorem 3.8.8. *For a symmetric closed monoidal model category* $\mathscr{M}$*, the category* $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ *has a Quillen model structure where a map* $\sigma = (f, \sigma) : \mathcal{F} \longrightarrow \mathcal{G}$ *is*

1. *a weak equivalence if* $f : X \longrightarrow Y$ *is an isomorphism of sets and* $\sigma : \mathcal{F} \longrightarrow f^\star\mathcal{G}$ *is a weak equivalence in* $\mathscr{M}_{\mathbb{S}}(X)$*,*
2. *a cofibration if the adjoint map* $\widetilde{\sigma} : f_!\mathcal{F} \longrightarrow \mathcal{G}$ *is a cofibration in* $\mathscr{M}_{\mathbb{S}}(Y)$*,*
3. *a fibration if* $\sigma : \mathcal{F} \longrightarrow f^\star\mathcal{G}$ *is a fibration in* $\mathscr{M}_{\mathbb{S}}(X)$*.*

We will denote $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ *endowed with this model structure by* $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})_{fib}$*.*

Proof. $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ is complete and cocomplete as any locally presentable category. The other conditions of Theorem 3.8.7 are clearly fulfilled. ■

Remark 3.8.3. If we replace everywhere $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ by $\mathcal{K}_{\mathbf{Set}}$ in the previous theorem we will get as well a fibered model structure on $\mathcal{K}_{\mathbf{Set}}$. The adjunction $\mathcal{U} : \mathscr{M}_{\mathbb{S}}(\mathbf{Set}) \rightleftarrows \mathcal{K}_{\mathbf{Set}} : \Gamma$ is a Quillen adjunction.

3.8.2 The canonical model structure

In the following we use the fact that we have a fibred model structure to construct a new model structure on $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$, called the *canonical model structure* (following the terminology of [13]). In the new model structure we are about to construct, the fibrations are the same but the weak equivalences are no longer required to induce an isomorphism on the set of objects.

Recall that we assume that $\mathscr{M}$ is cofibrantly generated with a set $\mathbf{I}$ (resp. $\mathbf{J}$) of generating cofibrations (resp. trivial cofibrations).

Some natural $\mathbb{S}$-diagrams The discussion we present here follows closely Simpson's considerations in [79, 13.2].

Let $[\mathbf{n}]$ be the indiscrete category associated to the set $\{0, ..., n\}$. In the 2-category $\mathbb{S}_{[\mathbf{n}]}$, there is a special 1-morphism from 0 to n corresponding to the $n+1$-tuple $(0, ..., n)$. It is the maximal nondegenerate simplex in the nerve of $[\mathbf{n}]$. We will denote this 1-morphism by s_n. Let $\mathbf{F}^{s_n}_{-} : \mathscr{M} \longrightarrow \mathrm{Hom}[\mathbb{S}_{[\mathbf{n}]}(0,n)^{op}, \mathscr{M}]$ be the left adjoint of the evaluation at s_n.

We have as usual the categories $\mathscr{M}_{\mathbb{S}}([\mathbf{n}])$ and $\mathcal{K}_{[\mathbf{n}]}$ with the monadic adjunction

$$\mathcal{U} : \mathscr{M}_{\mathbb{S}}([\mathbf{n}]) \leftrightarrows \mathcal{K}_{[\mathbf{n}]} : \Gamma.$$

This adjunction is moreover a Quillen adjunction. For the record $\mathscr{M}_{\mathbb{S}}([\mathbf{n}])$ is the category of normal lax morphisms from $\mathbb{S}_{[\mathbf{n}]}^{2\text{-op}}$ to $\mathscr{M}$ and $\mathcal{K}_{[\mathbf{n}]} = \prod_{(i,j) \in \mathrm{Ob}([\mathbf{n}])^2} \mathrm{Hom}[\mathbb{S}_{[\mathbf{n}]}(i,j)^{\mathrm{op}}, \mathscr{M}]$.

For $B \in \mathrm{Ob}(\mathscr{M})$ we will denote by $\delta(s_n, B)$ the object of $\mathcal{K}_{[\mathbf{n}]}$ given by:

$$\delta(s_n, B)_{ij} = \begin{cases} \mathbf{F}^{s_n}_B & \text{if } i = 0, j = n \\ (\varnothing, \mathrm{Id}_{\varnothing}) & \text{the constant functor otherwise.} \end{cases}$$

For $B \in \mathrm{Ob}(\mathscr{M})$ define $h([\mathbf{n}]; B) \in \mathscr{M}_{\mathbb{S}}([\mathbf{n}])$ to be $\Gamma\delta(s_n, B)$.

Lemma 3.8.9. *For any $B \in Ob(\mathscr{M})$ and $\mathcal{F} \in \mathscr{M}_{\mathbb{S}}(Y)$ the following are equivalent.*

1. *A morphism $\sigma : h([\mathbf{n}]; B) \longrightarrow \mathcal{F}$ in $\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})$.*
2. *A sequence of elements $(y_0, ..., y_n)$ of Y together with a morphism $B \longrightarrow \mathcal{F}(y_0, ..., y_n)$ in $\mathscr{M}$.*

Sketch of proof. A morphism $\sigma = (f, \sigma) : h([\mathbf{n}]; B) \longrightarrow \mathcal{F}$ is by definition a function

$$f : \{0, ..., n\} \longrightarrow Y$$

together with a morphism $\sigma : h([\mathbf{n}]; B) \longrightarrow f^{\star}\mathcal{F}$ in $\mathscr{M}_{\mathbb{S}}([\mathbf{n}])$. Setting $y_i = f(i)$ we get $fs_n = (y_0, ..., y_n)$ and by adjunction we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}[(h([\mathbf{n}]; B), \mathcal{F}] &= \mathrm{Hom}_{\mathscr{M}_{\mathbb{S}}([\mathbf{n}])}[(h([\mathbf{n}]; B), f^{\star}\mathcal{F}] \\
&= \mathrm{Hom}_{\mathscr{M}_{\mathbb{S}}([\mathbf{n}])}[\Gamma\delta(s_n, B), f^{\star}\mathcal{F}] \\
&\cong \mathrm{Hom}_{\mathcal{K}_{[\mathbf{n}]}}[\delta(s_n, B), \mathcal{U}(f^{\star}\mathcal{F})] \\
&\cong \mathrm{Hom}[\mathbf{F}^{s_n}_B, f^{\star}\mathcal{F}_{y_0 y_n}] \\
&\cong \mathrm{Hom}[B, \mathcal{F}_{y_0 y_n}(fs_n)] \\
&= \mathrm{Hom}[B, \mathcal{F}(y_0, ..., y_n)].
\end{aligned}$$

∎

Definition 3.8.10. *Say that a map $\sigma = (f, \sigma) : \mathcal{F} \longrightarrow \mathcal{G}$ is:*

1. *a* local fibration *if it's a fibration in the fibred model structure i.e if for all s, the map*
$$\mathcal{F}s \xrightarrow{\sigma_s} \mathcal{G}f(s)$$
is a fibration;

2. *a* local weak equivalence *if for all s, the map*
$$\mathcal{F}s \xrightarrow{\sigma_s} \mathcal{G}f(s)$$
is a weak equivalence.

Notation 3.8.1. We will use the following notation.

1. $\mathscr{W}_{loc}$ = the class of local weak equivalences.
2. $\mathbf{I}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})} = \coprod_{n\geq 1}\{h([\mathbf{n}]; q) : h([\mathbf{n}]; A) \longrightarrow h([\mathbf{n}]; B)\}_{q:A\longrightarrow B\in \mathbf{I}}$.
3. $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})} = \coprod_{n\geq 1}\{h([\mathbf{n}]; q) : h([\mathbf{n}]; A) \longrightarrow h([\mathbf{n}]; B)\}_{q:A\longrightarrow B\in \mathbf{J}}$.
4. $\mathbf{I}_{\mathcal{K}_{\mathbf{Set}}} = \coprod_{n\geq 1}\{\delta(s_n, q) : \delta(s_n, A) \longrightarrow \delta(s_n, B)\}_{q:A\longrightarrow B\in \mathbf{I}}$.
5. $\mathbf{J}_{\mathcal{K}_{\mathbf{Set}}} = \coprod_{n\geq 1}\{\delta(s_n, q) : \delta(s_n, A) \longrightarrow \delta(s_n, B)\}_{q:A\longrightarrow B\in \mathbf{J}}$.

Since projective (trivial) fibrations are the object-wise (trivial) fibrations, it follows that a map $\sigma = (f, \sigma) : \mathcal{F} \longrightarrow \mathcal{G}$ is a (trivial) fibration if for all s the map $\mathcal{F}s \xrightarrow{\sigma_s} \mathcal{G}f(s)$ has the right lifting property (RLP) with respect to all maps in ($\mathbf{I}$) $\mathbf{J}$.

If we combine this observation and Lemma 3.8.9 we deduce that

Proposition 3.8.11. *With the previous notation the following hold.*

1. *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ has the RLP with respect to maps in $\mathbf{I}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$ if and only if it is a local fibration and is in $\mathscr{W}_{loc}$.*
2. *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ has the RLP with respect to maps in $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$ if and only if it is a local fibration. In particular $\mathscr{W}_{loc} \cap \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$-inj $\subseteq \mathbf{I}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$-inj.*
3. *We have $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$-cell $\subseteq \mathscr{W}_{loc} \cap \mathbf{I}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$-cof.*

Proof. We prove (1) and (2) by adjointness. Let $\mathcal{F} \in \mathscr{M}_{\mathbb{S}}(X)$, $\mathcal{G} \in \mathscr{M}_{\mathbb{S}}(Y)$, $f : X \longrightarrow Y$ and $\sigma : \mathcal{F} \longrightarrow f^\star\mathcal{G}$ a morphism in $\mathscr{M}_{\mathbb{S}}(X)$. Let $s = (x_0, ..., x_n)$ be a generic 1-morphism of $\mathbb{S}_{\overline{X}}$; $s = \mathbb{S}_f(s_n)$ where $f : \{0, ..., n\} \longrightarrow X$ is the function that takes i to x_i. Let $q : A \longrightarrow B \in \text{Arr}(\mathscr{M})$ be a generic morphism of $\mathscr{M}$. From Lemma 3.8.9, it's easy to see that we have functorial isomorphisms in the respective arrow-categories:

$$\text{Hom}_{\text{Arr}[\mathscr{M}_{\mathbb{S}}(\mathbf{Set})]}[h([\mathbf{n}]; q), \sigma] \cong \text{Hom}_{\text{Arr}[\mathcal{K}_{\mathbf{Set}}]}[\delta(s_n; q), \mathcal{U}\sigma] \cong \text{Hom}_{\text{Arr}(\mathscr{M})}[q, \sigma_s].$$

And as q runs through $\mathbf{I}$ (resp. $\mathbf{J}$), we get that σ_s has the RLP with respect to $\mathbf{I}$ (resp. $\mathbf{J}$) if and only if $\mathcal{U}\sigma$ has the RLP with respect to $\mathbf{I}_{\mathcal{K}_{\mathbf{Set}}}$ (resp. $\mathbf{J}_{\mathcal{K}_{\mathbf{Set}}}$). Finally $\mathcal{U}\sigma$ has the RLP with respect to $\mathbf{I}_{\mathcal{K}_{\mathbf{Set}}}$ (resp. $\mathbf{J}_{\mathcal{K}_{\mathbf{Set}}}$) if and only if σ has the RLP with respect to $\Gamma\mathbf{I}_{\mathcal{K}_{\mathbf{Set}}} = \mathbf{I}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$ (resp. to $\Gamma\mathbf{J}_{\mathcal{K}_{\mathbf{Set}}} = \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$); and the two assertion follows.

For Assertion (3) we proceed as follows. First observe that in $\mathscr{M}$, we have $\mathbf{J}$-cell $\subseteq$ $\mathbf{I}$-cof, therefore by adjointness we get the inclusion $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$-cell $\subseteq \mathbf{I}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$-cof. Finally observe that maps in $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$ are trivial cofibrations in the fibred model structure, thus all maps in $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$-cell are also trivial cofibration in the fibred model structure; in particular they are weak equivalences therein. But a weak equivalence in the fibred model structure is also a local weak equivalence, and we have the inclusion $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$-cell $\subseteq \mathscr{W}_{loc}$ which gives the assertion. ■

Remark 3.8.4. Since the functors h and δ are left Quillen functors they preserve (trivial) cofibrations. In particular if A is cofibrant then so are $h([\mathbf{n}]; A)$ and $\delta(s_n, A)$. It follows that if the domain of maps in $\mathbf{I}$ are cofibrant in $\mathscr{M}$ then so are the domain of maps in $\mathbf{I}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}$ and $\mathbf{I}_{\mathcal{K}_{\mathbf{Set}}}$. This is useful when we want to preserve *tractability*.

In order to establish our main theorem we need to use the recognition theorem for cofibrantly generated model categories. We recall it hereafter as stated in [42].

Theorem 3.8.12. *Suppose $\mathcal{C}$ is a category with all small colimits and limits. Suppose $\mathscr{W}$ is a subcategory of $\mathcal{C}$, and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and $\mathscr{W}$ as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

1. *The subcategory $\mathscr{W}$ has the two out of three property and is closed under retracts.*
2. *The domains of I are small relative to I-cell.*
3. *The domains of J are small relative to J-cell.*
4. *J-cell $\subseteq \mathscr{W} \cap I$-cof.*
5. *I-inj $\subseteq \mathscr{W} \cap J$-inj.*
6. *Either $\mathscr{W} \cap I$-cof $\subseteq J$-cof or $\mathscr{W} \cap J$-inj $\subseteq I$-inj.*

By virtue of the previous theorem we can establish that:

Theorem 3.8.13. *There is a cofibrantly generated model structure on $\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})$ with $\mathbf{I}_{\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})}$ as the set of generating cofibrations, $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})}$ as the set of generating trivial cofibrations, and $\mathscr{W}_{\mathrm{loc}}$ as the subcategory of weak equivalences. The fibrations are the local fibrations. If $\mathscr{M}$ is combinatorial then so is $\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})$.*

We will denote this model structure by $\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})_{proj}$ and will call it the canonical or projective model structure. *The identity functor* $\mathrm{Id} : \mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})_{fib} \longrightarrow \mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})_{proj}$ *is a right Quillen functor.*

Proof. Combine Proposition 3.8.11, Theorem 3.8.12 and Theorem 3.8.2. ■

3.9 co-Segalification of $\mathbb{S}$-diagrams

Environment: In this section $(\mathscr{M}, \mathscr{W})$ is a **symmetric monoidal model category** where $\mathscr{W}$ represents the class of weak equivalences. We refer the reader to [42] for the definition of (symmetric) monoidal model categories.

For simplicity we consider in this section only $\mathbb{S}_{\overline{X}}$-diagrams of $(\mathscr{M}, \mathscr{W})$. For a general category $\mathcal{C}$ the methods we will use will be the same.

Notation 3.9.1.
A cofibration of $\mathscr{M}$ will be represented by an arrow of the form: $\hookrightarrow$.
A fibration will be represented by: $\twoheadrightarrow$
A weak equivalence will be represented by an arrow: $\xrightarrow{\sim}$.
An isomorphism will be represented by: $\xrightarrow{\cong}$.
$\aleph_0$ = the first countable cardinal. $\aleph_0$ is identified with the ordinal $\omega = (\mathbb{N}, <)$.
κ = a regular uncountable cardinal.
$\mathrm{End}[\mathscr{M}_{\mathbb{S}}(X)]$ = the category of endofunctors of $\mathscr{M}_{\mathbb{S}}(X)$.
$\mathbf{I}$ = the class of cofibrations of $\mathscr{M}$.
$\mathbf{I}$-inj = the class of $\mathbf{I}$-injective maps.
$\mathscr{M}^{[\mathbf{1}]} = \mathrm{Hom}([\mathbf{1}], \mathscr{M})$ = the category of arrows of $\mathscr{M}$ (here $[\mathbf{1}]$ is the interval category).
$\emptyset$ = the initial object of $\mathscr{M}$.

All along our discussion X is a fixed set of cardinality $< \kappa$.
The purpose of this section is to build a process which associates to any $\mathbb{S}_{\overline{X}}$-diagram F a co-Segal $\mathbb{S}_{\overline{X}}$-diagram $\mathscr{S}(F)$. This process will be needed in the upcoming sections when we localize the previous model structures on the category $\mathscr{M}_{\mathbb{S}}(X)$.

We are going to construct a functor $\mathscr{S} : \mathscr{M}_{\mathbb{S}}(X) \longrightarrow \mathscr{M}_{\mathbb{S}}(X)$ equipped with a natural transformation

$$\eta_{\mathscr{S}} : \mathrm{Id}_{\mathscr{M}_{\mathbb{S}}(X)} \hookrightarrow \mathscr{S}$$

whose component at each F, $\eta_{\mathscr{S},F} : F \hookrightarrow \mathscr{S}(F)$, will be a *cofibration* in $\mathscr{M}_{\mathbb{S}}(X)$.

The natural transformation $\eta_{\mathscr{S}}$ will arise automatically from the construction of the functor $\mathscr{S}$.

The functor $\mathscr{S}$ will be obtained as a colimit of a κ-sequence of cofibrations in $\mathscr{M}_{\mathbb{S}}(X)$:

$$\mathrm{Id}_{\mathscr{M}_{\mathbb{S}}(X)} = \mathscr{S}_0 \hookrightarrow \mathscr{S}^1 \hookrightarrow \mathscr{S}^2 \cdots \hookrightarrow \mathscr{S}^{n-1} \hookrightarrow \mathscr{S}^n \hookrightarrow \cdots$$

3.9.1 co-Segalification by Generators and Relations

Recall that an $\mathbb{S}_{\overline{X}}$-diagram $\mathcal{F}$ is given by a family of functors $\{\mathcal{F}_{AB}\}_{(A,B)\in X^2}$ together with some laxity maps $\{\varphi_{s,t}\}$ and suitable coherences.
Here each $\mathcal{F}_{AB}$ is a classical functor $\mathcal{F}_{AB} : \mathbb{S}_{\overline{X}}(A,B)^{op} \longrightarrow \mathscr{M}$, with $\mathbb{S}_{\overline{X}}(A,B)$ a category over Δ_{epi}.

Such an $\mathcal{F}$ is said to be a *co-Segal $\mathbb{S}_{\overline{X}}$-diagram* if for every pair (A,B) and any morphism $u : t \longrightarrow s$ of $\mathbb{S}_{\overline{X}}(A,B)$, the morphism $\mathcal{F}(u) : \mathcal{F}(s) \longrightarrow \mathcal{F}(t)$ is a *weak equivalence* in $\mathscr{M}$. Following Observation 3.4.2 we know that it suffices to have these conditions for $u = u_t$ for all t, where $u_t : t \longrightarrow (A,B)$ is the unique map from t to (A,B).

The functor $\mathscr{S}$ we are about to construct will have the property that $\mathscr{S}(\mathcal{F})(u_t)$ will be a *trivial fibration* in $\mathscr{M}$ for all t. But since $\mathscr{M}$ is a model category [4] $\mathscr{S}(\mathcal{F})(u_t)$ is a trivial fibration if and only if $\mathscr{S}(\mathcal{F})(u_t) \in \mathbf{I}$-inj i.e it has the *right lifting property* (RLP) with respect to the class $\mathbf{I}$ of all cofibrations (see [42, Lemma 1.1.10], [72, Ch.5]). This lifting property amounts to saying that whenever we have a commutative diagram in $\mathscr{M}$

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, hook, "h"'] & \mathscr{S}[\mathcal{F}](A,B) \arrow[d, "{\mathscr{S}[\mathcal{F}](u_t)}"] \\
V \arrow[r, "g"] \arrow[ur, dotted, "k"] & \mathscr{S}[\mathcal{F}](t)
\end{tikzcd}$$

with $h \in \mathbf{I}$ then we can find a lifting i.e there exists $k : V \longrightarrow \mathscr{S}[\mathcal{F}](A,B)$ such that $k \circ h = f$ and $\mathscr{S}[\mathcal{F}](u_t) \circ k = g$.

If we consider separately in $\mathscr{M}$ the map $\mathcal{F}(u_t)$, the classical trick to produce $\mathscr{S}[\mathcal{F}](u_t)$ is to use the *small object argument* which gives, up to some hypothesis on $\mathbf{I}$, a functorial factorization $\mathcal{F}(u_t) = \beta_t(\mathcal{F}) \circ \alpha_t(\mathcal{F})$ with:

$$\begin{cases} \alpha_t(\mathcal{F}) : \mathcal{F}(A,B) \longrightarrow D & \text{an } \mathbf{I}\text{-cell complex} \\ \beta_t(\mathcal{F}) : D \longrightarrow \mathcal{F}(t) & \text{an element of } \mathbf{I}\text{-inj} \\ \text{for some } D \in \mathrm{Ob}(\mathscr{M}). \end{cases}$$

The map $\alpha_t(\mathcal{F})$ is obtained as a transfinite composition of pushouts of a coproduct of the maps in $\mathbf{I}$. The smallness or *compactness* of D is used to show that $\beta_t(\mathcal{F})$ has the RLP with respect to $\mathbf{I}$. The reader can find an exposition of the small object argument for example in [32, Section 7.12], [42, Theorem 2.1.14].

In this situation we can set $\mathscr{S}[\mathcal{F}](A,B) = D$, $\mathscr{S}[\mathcal{F}](t) = F(t)$, $\mathscr{S}[\mathcal{F}](u_t) = \beta_t(\mathcal{F})$ and the natural transformation $\eta_{\mathscr{S},\mathcal{F}}$ will be given by $\alpha_t(\mathcal{F}) : \mathcal{F}(A,B) \longrightarrow \mathscr{S}[\mathcal{F}](A,B)$ and $\mathrm{Id}_{\mathcal{F}(t)} : \mathcal{F}(t) \longrightarrow \mathscr{S}[\mathcal{F}](t)$.

In our case we want to use the same trick i.e using a transfinite composition of pushouts of maps of some class $\mathbf{I}_{\mathscr{M}_{\mathbb{S}}(X)} \subset \mathrm{Arr}(\mathscr{M})$, but we want these pushouts as well as the other operations to take place in $\mathscr{M}_{\mathbb{S}}(X)$.

An important adjunction

Let t be a 1-morphism of $\mathbb{S}_{\overline{X}}$ of length > 1 i.e $t \in \mathrm{Ob}(\mathbb{S}_{\overline{X}}(A,B)^{\mathrm{op}})$ for some pair of elements (A,B) of X. Recall that t corresponds to a sequence $(A_0, A_1, ..., A_n)$ with $A_0 = A$ and $A_n = B$.

Let $\mathcal{P}_t : \mathscr{M}_{\mathbb{S}}(X) \longrightarrow \mathscr{M}^{[\mathbf{1}]}$ be the evaluation functor at $u_t : (A,B) \longrightarrow t$:

- For $\mathcal{F} \in \mathscr{M}_{\mathbb{S}}(X)$ we have $\mathcal{P}_t(\mathcal{F}) = \mathcal{F}(u_t)$,
- For $\sigma \in \mathrm{Hom}_{\mathscr{M}_{\mathbb{S}}(X)}[\mathcal{F}, \mathcal{G}]$, we have $\mathcal{P}_t(\sigma) = (\sigma_{(A,B)}, \sigma_t)$ which corresponds to the commutative

[4]Here we adopt the modern language and simply say 'model category' to mean what Quillen [72, Ch.5] called 'closed model category'.

square:

$$\begin{tikzcd}
\mathcal{F}(A,B) \arrow[r, "\sigma_{(A,B)}"] \arrow[d, "\mathcal{F}(u_t)"'] & \mathcal{G}(A,B) \arrow[d, "\mathcal{G}(u_t)"] \\
\mathcal{F}(t) \arrow[r, "\sigma_t"] & \mathcal{G}(t)
\end{tikzcd}$$

Proposition 3.9.1. *For every object t of length > 1 the following holds.*

1. *The functor $\mathcal{P}_t$ has a left adjoint, that is, there exists a functor*

$$\mathcal{P}_{t!} : \mathscr{M}^{[\mathbf{1}]} \longrightarrow \mathscr{M}_{\mathbb{S}}(X)$$

such that for every $\mathcal{F} \in \mathscr{M}_S(X)$ and every $h \in \mathscr{M}^{[\mathbf{1}]}$ we have an isomorphism of sets:

$$\mathrm{Hom}_{\mathscr{M}_{\mathbb{S}}(X)}[\mathcal{P}_{t!}h, \mathcal{F}] \cong \mathrm{Hom}_{\mathscr{M}^{[\mathbf{1}]}}[h, \mathcal{F}(u_t)]$$

which is natural in both h and $\mathcal{F}$.

2. *$\mathcal{P}_{t!}$ is a left Quillen functor.*

Sketch of proof. For assertion (1), we write $\mathcal{P}_t$ as the composite of the following functors:

$$\mathscr{M}_{\mathbb{S}}(X) \xrightarrow{\mathcal{U}} \prod_{(A',B')\in X^2} \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A',B')^{op}, \mathscr{M}] \xrightarrow{\mathbf{pr}_{AB}} \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}] \xrightarrow{\mathrm{Ev}_{u_t}} \mathscr{M}^{[\mathbf{1}]}$$

where:

- $\mathcal{U}$ is the functor which forgets the laxity maps,
- $\mathbf{pr}_{AB}$ is the functor which gives the component at (A,B),
- Ev_{u_t} is the evaluation at u_t.

Thanks to Lemma 4.2.1 in the Appendix, $\mathcal{U}$ has a left adjoint Γ. Ev_{u_t} has a left adjoint $\mathbf{F}^{u_t}$ (see Appendix 4.2.2). Finally $\mathbf{pr}_{AB}$ has clearly a left adjoint δ_{AB} as explained below. The composite of these left adjoints gives a left adjoint of $\mathcal{P}_t$.
The functor δ_{AB} is simply the 'Dirac extension'. For $\mathcal{F} \in \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$ we define $\delta(\mathcal{F}) \in \mathcal{K}_X$ by

$$\delta(\mathcal{F})_{A'B'} = \begin{cases} \mathcal{F} & \text{if } (A',B') = (A,B) \\ (\varnothing, \mathrm{Id}_{\varnothing}) & \text{the constant functor otherwise.} \end{cases}$$

One can easily see that δ is a functor and that we have indeed an isomorphism of sets:

$$\mathrm{Hom}[\mathcal{F}, \mathcal{G}_{AB}] \cong \mathrm{Hom}[\delta(\mathcal{F}), \mathcal{G}]$$

which is natural in both $\mathcal{F}$ and $\mathcal{G}$; this completes the proof of assertion (1).

Assertion (2) follows from the fact that all the three functors Γ, δ and $\mathbf{F}^{u_t}$ are left Quillen functors. In fact Γ is a left Quillen functor by construction of the model structure on $\mathscr{M}_{\mathbb{S}}(X)$ (injective or projective). δ is clearly a left Quillen functor. For $\mathbf{F}^{u_t}$ see Corollary 4.2.3 and Corollary 4.2.5. It follows that $\mathcal{P}_{t!}$ is a composite of left Quillen functors therefore it's a left Quillen functor. ∎

For any map $h : U \longrightarrow V$ of $\mathscr{M}$, we have a tautological commutative square:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "h"'] & V \arrow[d, "\mathrm{Id}_V"] \\ V \arrow[r, "\mathrm{Id}_V"] & V \end{tikzcd}$$

which says that (h, Id_V) is, in a natural way, a morphism in $\mathscr{M}^{[\mathbf{1}]}$ from h to Id_V. We will denote by $h_{/V}$ this morphism.[5]

Lemma 3.9.2. *For a symmetric monoidal model category $\mathscr{M}$ which is also tractable, for any pushout square in either $\mathscr{M}_{\mathbb{S}}(X)_{inj}$ or $\mathscr{M}_{\mathbb{S}}(X)_{proj}$:*

$$\begin{tikzcd} \mathcal{P}_{t!}(h) \arrow[r, "\sigma"] \arrow[d, hook, "\mathcal{P}_{t!}(h_{/V})"'] & \mathcal{F} \arrow[d, dashed, "H"] \\ \mathcal{P}_{t!}\,\mathrm{Id}_V \arrow[r, dashed, "\overline{\sigma}"] & \mathcal{G} \end{tikzcd}$$

the following holds.

1. *If $h : U \longrightarrow V$ is a cofibration in $\mathscr{M}$ then H is a cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{inj}$.*
2. *If moreover $h : U \longrightarrow V$ is a trivial cofibration in $\mathscr{M}$ then H is a weak equivalence in both $\mathscr{M}_{\mathbb{S}}(X)_{inj}$ and $\mathscr{M}_{\mathbb{S}}(X)_{proj}$.*

Proof. The map $h_{/V}$ is an injective (trivial) cofibration in $\mathscr{M}^{[\mathbf{1}]}$ and since $\mathcal{P}_{t!}$ is a left Quillen functor, we know that $\mathcal{P}_{t!}(h_{/V})$ is a (trivial) cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{inj}}$. Applying Lemma 3.7.5 we deduce that H is a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{inj}}$ but weak equivalences in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{inj}}$ and $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{proj}}$ are the same. ■

The local 'co-Segalification' process

Let t be a fixed object in $\mathbb{S}_X(A, B)$ and $\mathcal{F}$ be an object of $\mathscr{M}_{\mathbb{S}}(X)$.
As $\mathscr{M}$ is a model category we can factorize the map $\mathcal{F}(u_t)$ as: $\mathcal{F}(u_t) = j \circ h$ where $h : \mathcal{F}(A, B) \hookrightarrow U$ is a cofibration and $j : U \twoheadrightarrow \mathcal{F}t$ is a trivial fibration.
The pair $(\mathrm{Id}_{\mathcal{F}(A,B)}, j)$ defines a morphism $S(j, h) \in \mathrm{Hom}_{\mathscr{M}^{[\mathbf{1}]}}[h, \mathcal{F}(u_t)]$ in a tautological way:

$$\begin{tikzcd} \mathcal{F}(A, B) \arrow[r, equal, "\mathrm{Id}"] \arrow[d, "h"'] & \mathcal{F}(A, B) \arrow[d, "\mathcal{F}(u_t)"] \\ U \arrow[r, "j"'] & \mathcal{F}t \end{tikzcd}$$

When necessary we will write $h = h(\mathcal{F}, t)$ and $j = j(\mathcal{F}, t)$ to mention that we working with the factorization of $\mathcal{F}(u_t)$.

[5] The notation '$h_{/V}$' is inspired from the fact that the commutative square above is the (unique) canonical map from h to Id_V in the slice category $\mathscr{M}_{/V}$. We recall that Id_V is final in $\mathscr{M}_{/V}$.

By adjunction we have a unique map $T(h,j,\mathcal{F},t) \in \operatorname{Hom}_{\mathscr{M}_{\mathbb{S}}(X)}[\mathcal{P}_{t!}(h), \mathcal{F}]$ 'lifting' $S(j,h)$. Denote by $\mathscr{S}_t^1(\mathcal{F})$ the object of $\mathscr{M}_{\mathbb{S}}(X)$ obtained by the pushout diagram [6]:

$$\begin{tikzcd}
\mathcal{P}_{t!}(h) \arrow[r, "T(h,j,\mathcal{F},t)"] \arrow[d, hook, "\mathcal{P}_{t!}(h_{/U})"'] & \mathcal{F} \arrow[d, dashed, "H_1"] \\
\mathcal{P}_{t!}\,\mathrm{Id}_U \arrow[r, dashed, "\alpha"] & \mathscr{S}_t^1(\mathcal{F})
\end{tikzcd}$$

Proposition 3.9.3. *With the above notation the following holds.*

1. *For every such factorization $(h,j) \in (\mathbf{cof}, \mathbf{we} \cap \mathbf{fib})$ of $\mathcal{F}(u_t)$ the map $H_1 : \mathcal{F} \longrightarrow \mathscr{S}_t^1(\mathcal{F})$ is an injective cofibration in $\mathscr{M}_{\mathbb{S}}(X)$.*

2. *If $\mathcal{F}(u_t)$ is a weak equivalence in $\mathscr{M}$, then H_1 is an injective trivial cofibration. In particular H_1 is a weak equivalence in both $\mathscr{M}_{\mathbb{S}}(X)_{inj}$ and $\mathscr{M}_{\mathbb{S}}(X)_{proj}$ and the map $[\mathscr{S}_t^1\mathcal{F}]u_t$ is a weak equivalence in $\mathscr{M}$.*

3. *If the factorization axioms in $\mathscr{M}$ is functorial then the operation $\mathcal{F} \mapsto \mathscr{S}_t^1\mathcal{F}$ is a functor.*

Sketch of proof. As h is a cofibration in $\mathscr{M}$, the assertion (1) follows immediately from Lemma 3.9.2 1.

If $\mathcal{F}(u_t) = j \circ h$ is a weak equivalence and as j is a weak equivalence by hypothesis, then by the 3 for 2 property we deduce that h is also a weak equivalence; therefore h is a trivial cofibration and half of assertion (2) follows also from Lemma 3.9.2 2.
By definition of map in $\mathscr{M}_{\mathbb{S}}(X)$, we know that the pair $(H_{1,AB}, H_{1,t})$ defines a map in $\mathscr{M}^{[\mathbf{1}]}$ from $\mathcal{F}(u_t)$ to $[\mathscr{S}_t^1\mathcal{F}](u_t)$. In particular we have an equality:

$$[\mathscr{S}_t^1\mathcal{F}]u_t \circ H_{1,AB} = H_{1,t} \circ \mathcal{F}(u_t).$$

Since H_1 is a weak equivalence, then both $H_{1,AB}$, $H_{1,t}$ are weak equivalences in $\mathscr{M}$; it follows that $H_{1,t} \circ \mathcal{F}(u_t)$ is a weak equivalence if $\mathcal{F}(u_t)$ is so. Now by the 3 for 2 property we deduce that $[\mathscr{S}_t^1\mathcal{F}](u_t)$ is also a weak equivalence. This complete the proof of (2).

Assertion (3) is clear and is left to the reader. ■

Remark 3.9.1. By adjoint transpose we have the following commutative square in $\mathscr{M}^{[\mathbf{1}]}$:

$$\begin{tikzcd}
h \arrow[r, "S(j,h)"] \arrow[d, hook, "h_{/U}"'] & \mathcal{F}(u_t) \arrow[d, dashed, "\mathcal{P}_t(H_1)"] \\
\mathrm{Id}_U \arrow[r, "\overline{\alpha}"] & \mathscr{S}_t^1(\mathcal{F})(u_t)
\end{tikzcd}$$

To simplify the notation in diagrams we will write $\mathcal{F}^1$ for $\mathscr{S}_t^1(\mathcal{F})$. The above diagram is

[6] "the *Gluing Construction*" in [32]

displayed as a commutative cube in $\mathscr{M}$:

$$\begin{tikzcd}
& \mathcal{F}(A,B) \arrow[rr, "\mathrm{Id}"] \arrow[dl, "h"'] \arrow[dd, dotted] & & \mathcal{F}(A,B) \arrow[dl, "H_{1,AB}"'] \arrow[dd, "\mathcal{F}(u_t)"] \\
U \arrow[rr, "\overline{\alpha}_{AB}", near start] \arrow[dd, "\mathrm{Id}"'] & & \mathcal{F}^1(A,B) \arrow[dd, "\mathcal{F}^1 u_t", near start] & \\
& U \arrow[rr, dotted, "j", near start] \arrow[dl, "\mathrm{Id}"'] & & \mathcal{F}t \arrow[dl, "H_{1,t}"] \\
U \arrow[rr, "\overline{\alpha}_t"] & & \mathcal{F}^1 t &
\end{tikzcd}$$

From the upper and bottom faces of that cube we deduce that $H_{1,AB} = \overline{\alpha}_{AB} \circ h$ and $\overline{\alpha}_t = H_{1,t} \circ j$; from the front face we have that $\overline{\alpha}_t = \mathcal{F}^1 u_t \circ \overline{\alpha}_{AB}$. If we put these together we see that in the diagram below everything is commutative (triangles and squares):

$$\begin{tikzcd}
\mathcal{F}(A,B) \arrow[rr, "H_{1,AB}"] \arrow[dr, hook, "h"'] \arrow[dd, "\mathcal{F}u_t"'] & & \mathcal{F}^1(A,B) \arrow[dd, "\mathcal{F}^1 u_t"] \\
& U \arrow[ur, "\overline{\alpha}_{AB}"] \arrow[dl, "j", "\wr"'] & \\
\mathcal{F}t \arrow[rr, "H_{1,t}"] & & \mathcal{F}^1 t
\end{tikzcd}$$

Warning. For the rest of the discussion we assume that factorization axioms, in the model category $\mathscr{M}$, are functorial.

For $k > 1$ we define inductively objects $\mathscr{S}_t^k(\mathcal{F})$ of $\mathscr{M}_{\mathbb{S}}(X)$ by setting $\mathscr{S}_t^k(\mathcal{F}) := \mathscr{S}_t^1[\mathscr{S}_t^{k-1}(\mathcal{F})]$ with $\mathscr{S}_t^0(\mathcal{F}) = \mathcal{F}$. One uses a (functorial) factorization $(h_k, j_k) \in (\mathbf{cof}, \mathbf{we} \cap \mathbf{fib})$ of the map $\mathscr{S}_t^{k-1}(\mathcal{F})u_t$ and apply the previous construction.

We have a canonical map $H_k : \mathscr{S}_t^{k-1}(\mathcal{F}) \longrightarrow \mathscr{S}_t^k(\mathcal{F})$ which is a cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{inj}}$. We have a κ-sequence in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{inj}}$:

$$\mathcal{F} = \mathscr{S}_t^0(\mathcal{F}) \xhookrightarrow{H_1} \mathscr{S}_t^1(\mathcal{F}) \hookrightarrow \cdots \hookrightarrow \mathscr{S}_t^{k-1}(\mathcal{F}) \xhookrightarrow{H_k} \mathscr{S}_t^k(\mathcal{F}) \hookrightarrow \cdots$$

Define $\mathscr{S}_t^\infty(\mathcal{F}) = \mathrm{colim}_k \mathscr{S}_t^k(\mathcal{F})$ and denote by $\eta_t : \mathcal{F} \longrightarrow \mathscr{S}_t^\infty(\mathcal{F})$ the canonical map.

Proposition 3.9.4. *For every $\mathcal{F} \in \mathscr{M}_{\mathbb{S}}(X)$ then:*

1. *The map $\mathscr{S}_t^\infty(\mathcal{F})(u_t)$ has the RLP with respect to all cofibrations in $\mathscr{M}$ i.e it's a trivial fibration in $\mathscr{M}$*
2. *The map η_t is a cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{inj}$.*
3. *If $\mathcal{F}(u_t)$ is a weak equivalence in $\mathscr{M}$, then η_t is trivial cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{inj}$, in particular a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)$.*

Proof. For notational convenience we will write in this proof $\mathcal{F}^k = \mathscr{S}_t^k(\mathcal{F})$ and $\mathcal{F}^\infty = \mathscr{S}_t^\infty(\mathcal{F})$.

Assertions (2) and (3) are straightforward: if $\mathcal{F}(u_t)$ is a weak equivalence then applying inductively Lemma 3.9.2, we get that all H_k are either cofibrations in case (2) or trivial cofibrations in case (3). In both cases η_t is a transfinite composition of such morphisms so it's also either a cofibration or a trivial cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$.

To prove the assertion (1) we use the small object argument in the locally presentable category $\mathscr{M}_{\mathbb{S}}(X)$. Choosing κ big enough we can assume that $\mathcal{P}_{t!}(g)$ is small for every cofibration g of $\mathscr{M}$.

Let g be a cofibration in $\mathscr{M}$ and consider a lifting problem defined by g and $\mathcal{F}^{\infty}(u_t)$:

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, hook, "g"'] & \mathcal{F}^{\infty}(A,B) \arrow[d, "\mathcal{F}^{\infty} u_t"] \\
Q \arrow[r, "l"] & \mathcal{F}^{\infty} t
\end{tikzcd}$$

Such a lifting problem is defined by a morphism $\theta \in \operatorname{Hom}_{\mathscr{M}^{[\mathbf{1}]}}[g, \mathcal{F}^{\infty}(u_t)]$. By adjunction θ corresponds to a unique morphism $\widetilde{\theta} \in \operatorname{Hom}_{\mathscr{M}_{\mathbb{S}}(X)}[\mathcal{P}_{t!}g, \mathcal{F}^{\infty}]$. Since $\mathcal{P}_{t!}g$ is κ-small, $\widetilde{\theta}$ factorizes through one of the $\mathcal{F}^{k}$, say $\mathcal{F}^{k_0}$: there is a map $\widetilde{\theta}_0 : \mathcal{P}_{t!}(h) \longrightarrow \mathcal{F}^{k_0}$ such that $\widetilde{\theta} = \iota_{k_0} \circ \widetilde{\theta}_0$, where $\iota_{k_0} : \mathcal{F}^{k_0} \longrightarrow \mathcal{F}^{\infty}$ is the canonical map.

By construction $\iota_{k_0} = \iota_{k_0+1} \circ H_{k_0}$ and from the adjunction we have an equality:

$$\theta = \mathcal{P}_t(\iota_{k_0}) \circ \theta_0 = \mathcal{P}_t(\iota_{k_0+1}) \circ \mathcal{P}_t(H_{k_0}) \circ \theta_0$$

where $\theta_0 = (f_0, l_0)$ is the adjoint transpose of $\widetilde{\theta}_0$.
It follows that our original lifting problem can be factorized as:

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, hook, "g"'] & \mathcal{F}^{\infty}(A,B) \arrow[d, "\mathcal{F}^{\infty} u_t"] \\
Q \arrow[r, "l"] & \mathcal{F}^{\infty} t
\end{tikzcd}
=
\begin{tikzcd}
P \arrow[r, "f_0"] \arrow[dd, hook, "g"'] & \mathcal{F}^{k_0}(A,B) \arrow[rr, "H_{k_0,AB}"] \arrow[dr, "h_{k_0}"] \arrow[dd, "\mathcal{F}^{k_0} u_t"'] & & \mathcal{F}^{k_0+1}(A,B) \arrow[r, "\iota"] \arrow[dd, "\mathcal{F}^{k_0+1} u_t"] & \mathcal{F}^{\infty}(A,B) \arrow[dd, "\mathcal{F}^{\infty} u_t"] \\
& & U_{k_0} \arrow[ur, "\overline{\alpha}_{k_0}"'] \arrow[dl, "j_{k_0}"] & & \\
Q \arrow[r, "l_0"'] \arrow[urr, dashed] & \mathcal{F}^{k_0} t \arrow[rr, "H_{k_0,t}"] & & \mathcal{F}^{k_0+1} t \arrow[r, "\iota"] & \mathcal{F}^{\infty} t
\end{tikzcd}$$

The pair (h_{k_0}, j_{k_0}) is 'the' factorization cofibration-trivial fibration used to construct H_{k_0} and $\overline{\alpha}_{k_0}$ is the obvious map (see Remark 3.9.1). As j_{k_0} is a trivial fibration, the induced lifting problem by g and j_{k_0} has a solution: there is a map $\beta_0 : Q \longrightarrow U$ satisfying the obvious equalities. We leave the reader to check that the composite

$$\iota \circ \overline{\alpha}_{k_0} \circ \beta_0 : Q \longrightarrow \mathcal{F}^{\infty}(A,B)$$

is a solution to the original lifting problem.

It follows that $\mathcal{F}^{\infty}(u_t)$ has the RLP with respect to all cofibrations of $\mathscr{M}$, thus it's a trivial fibration as desired. ■

$\mathscr{S}_t^{\infty}$ is homotopically minimal Let $\text{ho}(\mathscr{M}_{\mathbb{S}}(X))$ be the homotopy category associated to both $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ and $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$. Given a map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ in $\mathscr{M}_{\mathbb{S}}(X)$, we will denote by $[\sigma]$ the class of σ in $\text{ho}(\mathscr{M}_{\mathbb{S}}(X))$.

Denote by $\mathscr{R} \subset \mathscr{M}_{\mathbb{S}}(X)$ the subcategory consisting of co-Segal categories; these are objects $\mathcal{F}$ such that for every t, $\mathcal{F}(u_t)$ is a weak equivalence. For a fixed t denote by $\mathscr{R}_t \subset \mathscr{M}_{\mathbb{S}}(X)$ the subcategory of object $\mathcal{F}$ such that $\mathcal{F}(u_t)$ is a weak equivalence. We have $\mathscr{R} \subset \mathscr{R}_t \subset \mathscr{M}_{\mathbb{S}}(X)$.

Thanks to Proposition 3.9.4, for any $\mathcal{F} \in \mathscr{M}_{\mathbb{S}}(X)$ we have $\mathscr{S}_t^{\infty}\mathcal{F} \in \mathscr{R}_t$. In what follows we show that among all objects of $\mathscr{R}_t$, $\mathscr{S}_t^{\infty}\mathcal{F}$ is the 'homotopic-nearest object' to $\mathcal{F}$.

Definition 3.9.5. *Let $\mathcal{F}$ be an object of $\mathscr{M}_{\mathbb{S}}(X)$ and $\mathcal{G}$ be an object of $\mathscr{R}_t$. A map $\sigma_0 : \mathcal{F} \longrightarrow \mathcal{G}$ is* homotopically minimal *with respect to $\mathscr{R}_t$ if for any $\mathcal{Q} \in \mathscr{R}_t$ and any morphism $\sigma : \mathcal{F} \longrightarrow \mathcal{Q}$ there exist a morphism $\gamma : [\mathcal{G}] \longrightarrow [\mathcal{Q}]$ in $\mathrm{ho}(\mathscr{M}_{\mathbb{S}}(X))$ such that $[\sigma] = \gamma \circ [\sigma_0]$.*

Diagrammatically this is displayed in $\mathrm{ho}(\mathscr{M}_{\mathbb{S}}(X))$ as

$$\begin{tikzcd}
{[\mathcal{F}]} \arrow[r, "{[\sigma]}"] \arrow[d, "{[\sigma_0]}"'] & {[\mathcal{Q}]} \\
{[\mathcal{G}]} \arrow[ur, dotted, "\gamma"'] &
\end{tikzcd}$$

Proposition 3.9.6. *For every $\mathcal{F} \in \mathscr{M}_{\mathbb{S}}(X)$ the map $\eta_t : \mathcal{F} \longrightarrow \mathscr{S}_t^{\infty}\mathcal{F}$ is homotopically minimal with respect to $\mathscr{R}_t$.*

Proof. For a map $\sigma : \mathcal{F} \longrightarrow \mathcal{Q}$ with $\mathcal{Q} \in \mathscr{R}_t$, by functoriality we have an induced map

$$\mathscr{S}_t^{\infty}(\sigma) : \mathscr{S}_t^{\infty}\mathcal{F} \longrightarrow \mathscr{S}_t^{\infty}\mathcal{Q}$$

and the following commutes:

$$\begin{tikzcd}
{[\mathcal{F}]} \arrow[r, "\sigma"] \arrow[d, "(\eta_t)_{\mathcal{F}}"'] & {[\mathcal{Q}]} \arrow[d, "(\eta_t)_{\mathcal{Q}}"'] \\
{[\mathscr{S}_t^{\infty}\mathcal{F}]} \arrow[r, dotted, "\mathscr{S}_t^{\infty}\sigma"'] & {[\mathscr{S}_t^{\infty}\mathcal{Q}]}
\end{tikzcd}$$

Note that the map $\mathscr{S}_t^{\infty}\sigma$ is induced by universal property of the pushout (inductively), so it's a universal morphism. Since $\mathcal{Q} \in \mathscr{R}_t$ we have from Proposition 3.9.4 (3) that $(\eta_t)_{\mathcal{Q}}$ is a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)$, thus $[(\eta_t)_{\mathcal{Q}}]$ is an isomorphism in $\mathrm{ho}(\mathscr{M}_{\mathbb{S}}(X))$. Take $\gamma = [(\eta_t)_{\mathcal{Q}}]^{-1} \circ [\mathscr{S}_t^{\infty}\sigma]$. ■

The global co-Segalification process

In what follows we use the previous functors $\mathscr{S}_t^{\infty}$ to construct the desired functor $\mathscr{S}$ such that for any t and any $\mathcal{F}$, $\mathscr{S}(\mathcal{F})u_t$ is a weak equivalence, that is, $\mathscr{S}(\mathcal{F})$ is an object of $\mathscr{R}$.

Denote by $\mathrm{Mor}(\mathbb{S}_{\overline{X}})$ the set of all 1-morphisms t of degree > 1 in $\mathbb{S}_{\overline{X}}$.
Define $\mathscr{S}^1\mathcal{F}$ to be the object obtained from the generalized pushout diagram formed by all the morphisms $\eta_t : \mathcal{F} \longrightarrow \mathscr{S}_t^{\infty}\mathcal{F}$ as t runs through the set of all 1-morphisms of degree > 1:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook] \arrow[d, hook] \arrow[dr, hook] & \mathscr{S}_t^{\infty}\mathcal{F} \\
\mathscr{S}_{t'}^{\infty}\mathcal{F} & \mathscr{S}_{t''}^{\infty}\mathcal{F}
\end{tikzcd}$$

$$\mathscr{S}^1\mathcal{F} := \mathrm{colim}_{t,\mathbf{deg}(t)>1}\{\eta_t : \mathcal{F} \longrightarrow \mathscr{S}_t^{\infty}\mathcal{F}\}.$$

Let $\eta^1 : \mathcal{F} \longrightarrow \mathscr{S}^1\mathcal{F}$ and $\iota_t^1 : \mathscr{S}_t^{\infty}\mathcal{F} \longrightarrow \mathscr{S}^1\mathcal{F}$ be the canonical maps. It follows that for all t we have $\eta^1 = \iota_t^1 \circ \eta_t$. Like every $\mathscr{S}_t^{\infty}$, $\mathscr{S}^1$ is functorial in $\mathcal{F}$.

Remark 3.9.2. We leave the reader to check that we have the following properties.

1. For every $\mathcal{F}$ the map η^1 is a cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ (see Lemma 4.1.6).

2. If $\mathcal{F} \in \mathscr{R}$ then $\mathscr{S}^1\mathcal{F} \in \mathscr{R}$. Equivalently $\mathscr{S}^1$ induces an endofunctor on $\mathscr{R}$. Moreover η^1 is a trivial cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$; in particular a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)$.

Define inductively a sequence of functors $\mathscr{S}^k$ by $\mathscr{S}^k(\mathcal{F}) := \mathscr{S}^1[\mathscr{S}^{k-1}(\mathcal{F})]$:

$$\mathcal{F} = \mathscr{S}^0(\mathcal{F}) \xhookrightarrow{\eta_1} \mathscr{S}^1(\mathcal{F}) \hookrightarrow \cdots \hookrightarrow \mathscr{S}^{k-1}(\mathcal{F}) \xhookrightarrow{\eta_k} \mathscr{S}^k_t(\mathcal{F}) \hookrightarrow \cdots$$

Set $\mathscr{S}(\mathcal{F}) := \operatorname{colim}_k \mathscr{S}^k(\mathcal{F})$; denote by $\eta_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathscr{S}(\mathcal{F})$ the canonical map.

Proposition 3.9.7. *For every $\mathcal{F} \in \mathscr{M}_{\mathbb{S}}(X)$, the following hold.*

1. *For all 1-morphism t, $\mathscr{S}(\mathcal{F})(u_t)$ is a trivial fibration in $\mathscr{M}$, in particular a weak equivalence, thus $\mathscr{S}(\mathcal{F}) \in \mathscr{R}$ i.e satisfies the co-Segal conditions.*

2. *The canonical map $\eta : \mathcal{F} \longrightarrow \mathscr{S}(\mathcal{F})$ is a cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{inj}$.*

3. *If $\mathcal{F} \in \mathscr{R}$ then $\eta : \mathcal{F} \longrightarrow \mathscr{S}(\mathcal{F})$ is a trivial cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{inj}$, in particular a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)$.*

Sketch of proof. Assertions (2) and (3) are clear and are left to the reader.
To prove (1) one proceeds exactly in same way as in the proof of Proposition 3.9.4. We will adopt the notation $\mathcal{F}^k = \mathscr{S}^k(\mathcal{F})$ for simplicity.

For any cofibration g of $\mathscr{M}$, using suitably the small object argument and the adjunction $\mathcal{P}_{t!} \dashv \operatorname{Ev}_{u_t}$, any lifting problem defined by g and $(\mathscr{S}\mathcal{F})u_t$ can be factorized, for some k_0, as:

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, hook, "g"'] & (\mathscr{S}\mathcal{F})(A,B) \arrow[d, "(\mathscr{S}\mathcal{F})u_t"] \\
Q \arrow[r, "l"'] & (\mathscr{S}\mathcal{F})t
\end{tikzcd}
=
\begin{tikzcd}
P \arrow[r, "f_0"] \arrow[dddd, hook, "g"'] & \mathcal{F}^{k_0}(A,B) \arrow[rr] \arrow[dr] \arrow[dddd, "\mathcal{F}^{k_0}u_t"] & & \mathcal{F}^{k_0+1}(A,B) \arrow[r, "\text{can}"] \arrow[dddd, "\mathcal{F}^{k_0+1}u_t"] & (\mathscr{S}\mathcal{F})(A,B) \arrow[dddd, "(\mathscr{S}\mathcal{F})u_t"] \\
& & (\mathscr{S}_t^{\infty}\mathcal{F}^{k_0})(A,B) \arrow[ur, "\iota_t"] \arrow[dd, "\wr", "(\mathscr{S}_t^{\infty}\mathcal{F}^{k_0})u_t"'] & & \\
& & & & \\
& & (\mathscr{S}_t^{\infty}\mathcal{F}^{k_0})t \arrow[dr, "\iota_t"] & & \\
Q \arrow[r, "l_0"'] \arrow[uuurr, dashed] & \mathcal{F}^{k_0}t \arrow[ur] \arrow[rr, "\iota_t"] & & \mathcal{F}^{k_0+1}t \arrow[r, "\text{can}"] & (\mathscr{S}\mathcal{F})t
\end{tikzcd}$$

In the above diagram, everything is commutative (squares and triangles), and since $(\mathscr{S}_t^{\infty}\mathcal{F}^{k_0})u_t$ has the RLP with respect to all cofibration (Proposition 3.9.4(1)) there is a solution $\beta : Q \longrightarrow (\mathscr{S}_t^{\infty}\mathcal{F}^{k_0})(A,B)$ to the lifting problem induced by g and $(\mathscr{S}_t^{\infty}\mathcal{F}^{k_0})u_t$. Clearly the composite

$$\text{can} \circ \iota_t^{k_0} \circ \beta : Q \longrightarrow \mathscr{S}(\mathcal{F})(A,B)$$

is a solution to the original lifting problem. Here of course 'can' is the canonical map going to the colimit.

Consequently $(\mathscr{S}\mathcal{F})u_t$ has the RLP with respect to any cofibration g in $\mathscr{M}$, thus it's a trivial fibration as desired. ■

Note. Since weak equivalences in $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ and $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ are the same, if we choose a functorial factorization in $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ of the map $\eta_{\mathcal{F}}$ as:

$$\eta_{\mathcal{F}} : \mathcal{F} \xhookrightarrow{\widetilde{\eta}_{\mathcal{F}}} \mathcal{Z} \underset{\sim}{\overset{q}{\twoheadrightarrow}} \mathscr{S}(\mathcal{F})$$

where $\widetilde{\eta}_{\mathcal{F}}$ is cofibration and q is a trivial fibration, then we can set $\mathscr{S}(\mathcal{F}) := \mathcal{Z}$ when working in the projective model structure. This new functor has the same properties as the previous one.

3.9.2 Localization by weak monadic projections

Weak monadic projection

Let $\mathbf{M}$ be a model category and $\mathcal{R} \subset \mathbf{M}$ be a subcategory stable under weak equivalences. We recall very briefly the definition of weak monadic projection as stated in [79, 9.2.2].

A *weak monadic projection* from $\mathbf{M}$ to $\mathcal{R}$ is a functor $F : \mathbf{M} \longrightarrow \mathbf{M}$ together with a natural transformation $\eta_A : A \longrightarrow F(A)$ such that:

1. $F(A) \in \mathcal{R}$ for all $A \in \mathbf{M}$;
2. for any $A \in \mathcal{R}$, η_A is a weak equivalence;
3. for any $A \in \mathbf{M}$ the map $F(\eta_A) : F(A) \longrightarrow F(F(A))$ is a weak equivalence;
4. If $f : A \longrightarrow B$ is a weak equivalence between cofibrant objects then $F(f) : F(A) \longrightarrow F(B)$ is a weak equivalence; and
5. $F(A)$ is cofibrant for any cofibrant $A \in \mathbf{M}$.

Remark 3.9.3. If F is a monadic projection from $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ to $\mathcal{R}$ then we can extract a monadic projection $\widetilde{F}$ from $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ to $\mathcal{R}$. In fact one uses the (functorial) factorization in $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$:

$$\eta_A : A \xhookrightarrow{\widetilde{\eta}_A} \widetilde{F}(A) \xrightarrow[\sim]{p_A} F(A)$$

where $\widetilde{\eta}_A$ is a projective cofibration and p_A a trivial fibration.

(WPr1) holds because $p : \widetilde{F}(A) \longrightarrow F(A)$ is a weak equivalence and $\mathcal{R}$ is stable under weak equivalences;
(WPr2) follows by the 3 for 2 property of weak equivalences: p_A is already a weak equivalence, consequently if in addition η_A is a weak equivalence then $\widetilde{\eta}_A$ is also a weak equivalence ;
(WPr3) also follows from the 3 for 2 property: from the functoriality of the factorization in $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ one has that the following commutes:

$$\begin{tikzcd}
\widetilde{F}(A) \arrow[r, "\widetilde{F}(\widetilde{\eta}_A)"] \arrow[d, "p"', "\wr"] & \widetilde{F}(\widetilde{F}(A)) \arrow[d, "p"', "\wr"] \\
F(A) \arrow[r, "\sim", "F(\eta_A)"'] & F(F(A))
\end{tikzcd}$$

and all the other maps are weak equivalences.
For (WPr4) we use the fact that projective cofibrations are also injective cofibrations. Therefore if A is cofibrant in $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$, then it's also cofibrant in $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$. It follows that if $f : A \longrightarrow B$ is a weak equivalence between cofibrant objects in $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$, then $F(f) : F(A) \longrightarrow F(B)$ is a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)$. The functoriality of $\widetilde{F}$ gives a commutative square where all the other maps are weak equivalences:

$$\begin{tikzcd}
\widetilde{F}(A) \arrow[r, "\widetilde{F}(f)"] \arrow[d, "p_A"', "\wr"] & \widetilde{F}(B) \arrow[d, "p_B"', "\wr"] \\
F(A) \arrow[r, "\sim", "F(\eta_A)"'] & F(B)
\end{tikzcd}$$

and $\widetilde{F}(f)$ is a weak equivalence by 3 for 2;
(WPr5) holds 'on the nose' since $\widetilde{\eta}_A : A \longrightarrow \widetilde{F}(A)$ is a projective cofibration: if $\varnothing \longrightarrow A$ is a cofibration, by composition $\varnothing \longrightarrow \widetilde{F}(A)$ is also a cofibration.

In our case We would like to show that the functor $\mathscr{S}$ constructed previously is a weak monadic projection from $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ or $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ to $\mathscr{R}$. The only nontrivial condition in our case is the condition (WPr3), namely that the map induced by universal property $\mathscr{S}(\eta_{\mathcal{F}}) : \mathscr{S}(\mathcal{F}) \longrightarrow \mathscr{S}(\mathscr{S}\mathcal{F}))$ is a weak equivalence.

But rather than verifying step by step that $\mathscr{S}$ is a weak monadic projection, we will use the more general approach of Simpson [79, Chap. 9] who used **Direct localizing systems** to produce weak monadic projections.

Direct localizing system

The present discussion follows closely [79, Chap. 9].

Let $(\mathbf{M}, I, J)$ be a tractable left proper cofibrantly generated model category which is moreover locally presentable. Recall that *tractable* means that the domains of maps in I and J are cofibrant. Suppose we are given a subclass of objects considered as a full subcategory $\mathcal{R} \subset \mathbf{M}$, and a subset $\mathrm{K} \subset \mathrm{Arr}(\mathbf{M})$. We assume that:

1. K is a small set;
2. $J \subset \mathrm{K}$;
3. $\mathrm{K} \subset \mathbf{cof}(I)$ and the domain of arrows in K are cofibrant;
4. If $A \in \mathcal{R}$ and if $A \cong B$ in $\mathbf{ho}(\mathbf{M})$ then $B \in \mathcal{R}$; and
5. $\mathbf{inj}(\mathrm{K}) \subset \mathcal{R}$.
   Say that $(\mathcal{R}, \mathrm{K})$ is *direct localizing* if in addition to the above conditions:
6. for all $A \in \mathcal{R}$ such that A is fibrant, and any $A \longrightarrow B$ which is a pushout along an element of K, there exists $B \longrightarrow C$ in *cell*(K) such that $A \longrightarrow C$ is a weak equivalence.

Note. In our case $(\mathbf{M}, I, J)$ will be $(\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}, \mathbf{I}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}}, \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}})$ and $\mathcal{R}$ will be $\mathscr{R}$, the subcategory of co-Segal categories.

Notation 3.9.2.

1. We remind the reader that $h_{/V} : h \longrightarrow \mathrm{Id}_V$ is the map represented by the commutative square:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "h"'] & V \arrow[d, "\mathrm{Id}_V"] \\ V \arrow[r, "\mathrm{Id}_V"] & V \end{tikzcd}$$

2. Let $\mathbf{K}_{\text{inj}}$ be the set

$$\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}} \cup \coprod_{t \in \mathbb{S}_{\overline{X}}, \mathbf{deg}(t) > 1} \{\mathcal{P}_{t!}(h_{/V})\}_{h \in \mathbf{I}}.$$

Remark 3.9.4. Thanks to a theorem of Lurie [66, Prop. A.1.5.12] we can assume that every cofibration of $\mathscr{M}$ is an in $cell(\mathbf{I})$. It follows that for any cofibration $i : E \longrightarrow Q$ the map $\mathcal{P}_{t!}(i_{/Q})$ is in $cell(\mathbf{K}_{\text{inj}})$.

As we shall see in a moment the maps $h_{/V}$ allow us to transport in a tautological way, a lifting problem defined in $\mathscr{M}$ into a extension (or horn filling) problem in $\mathscr{M}^{[\mathbf{1}]}$. And thanks to the adjunction $\mathcal{P}_{t!} \dashv \mathrm{Ev}_{u_t}$, we will be able to test if $\mathcal{F}(u_t)$ is a trivial fibration or not in terms of being injective with respect to the maps $\mathcal{P}_{t!}(h_{/V})$.

The main result in this section is the following

Theorem 3.9.8. *With the above notation the pair $(\mathscr{R}, \mathbf{K}_{inj})$ is direct localizing in $(\mathscr{M}_{\mathbb{S}}(X)_{inj}, \mathbf{I}_{\mathscr{M}_{\mathbb{S}}(X)_{inj}}, \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{inj}})$.*

Proof of Theorem 3.9.8 To prove the theorem we will verify that all the conditions (A1),..., (A6) hold.

Conditions (A1) and (A2) are clear. Since we assumed that all objects of $\mathscr{M}$ are cofibrant, it follows that all objects in $\mathcal{K}_{X\text{-inj}}$ are cofibrant as well; therefore the elements of $\mathbf{I}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}} = \Gamma\mathbf{I}_{\mathcal{K}_{X\text{-inj}}}$ have cofibrant domain by definition of the model structure on $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$.

By construction $\mathcal{P}_{t!} : \mathscr{M}^{[\mathbf{1}]}{}_{\text{inj}} \longrightarrow \mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ is a left Quillen functor and since $h_{/V}$ is clearly a cofibration in $\mathscr{M}^{[\mathbf{1}]}{}_{\text{inj}}$ when $h \in \mathbf{I}$, we deduce that $\mathcal{P}_{t!}(h_{/V})$ is cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ (with cofibrant domain). Putting these together one has (A3).

Condition (A4) follows from the stability of $\mathscr{R}$ under weak equivalence (Proposition 3.4.12). We treat (A5) and (A6) in the next paragraphs.

The condition (A5) holds To prove this we begin by observing that

Proposition 3.9.9. *For a commutative square in $\mathscr{M}$*

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "p"] \\
V \arrow[r, "g"] & Y
\end{tikzcd}$$

considered as a morphism $\alpha = (f, g) : h \longrightarrow p$ in $\mathscr{M}^{[\mathbf{1}]}$ the following are equivalent.

- *There is a lifting in the commutative square above i.e there exists $k : V \longrightarrow X$ such that: $k \circ h = f$, $p \circ h = g$.*
- *We can fill the following* 'horn' *of $\mathscr{M}^{[\mathbf{1}]}$:*

$$\begin{tikzcd}
h \arrow[r, "\alpha"] \arrow[d, "h_{/V}"'] & p \\
\mathrm{Id}_V \arrow[ur, dotted] &
\end{tikzcd}$$

that is, there exists $\beta = (k, l) : \mathrm{Id}_V \longrightarrow p$ such that $\beta \circ h_{/V} = \alpha$.

Proof. Obvious. ■

Let $\mathcal{F} \in \mathscr{M}_{\mathbb{S}}(X)$ be an object in $\mathbf{inj}(\mathbf{K}_{\text{inj}})$. As $\mathcal{F}$ is $\mathbf{K}_{\text{inj}}$-injective, it has the left lifting property with respect to all maps in $\mathbf{K}_{\text{inj}}$, so in particular for any generating cofibration $h \in \mathbf{I}$ and any $t \in \mathbb{S}_{\overline{\mathrm{X}}}$, there is a solution to any lifting problem of the following form:

$$\begin{tikzcd}
\mathcal{P}_{t!}(h) \arrow[r, "a"] \arrow[d, hook, "\mathcal{P}_{t!}(h_{/V})"'] & \mathcal{F} \arrow[d, "!"] \\
\mathcal{P}_{t!}\,\mathrm{Id}_V \arrow[r, "!"] \arrow[ur, dotted] & *
\end{tikzcd}$$

where $*$ is the terminal object in $\mathscr{M}_{\mathbb{S}}(X)$. But such a lifting problem is equivalent to the extension or horn filling problem:

$$\begin{tikzcd}
\mathcal{P}_{t!}(h) \arrow[r, "a"] \arrow[d, hook, "\mathcal{P}_{t!}(h_{/V})"'] & \mathcal{F} \\
\mathcal{P}_{t!}\,\mathrm{Id}_V \arrow[ur, dotted] &
\end{tikzcd}$$

It follows by adjunction that $\mathcal{F}(u_t)$ has the extension property with respect to all $h_{/V}$, as h runs through $\mathbf{I}$. Thanks to the previous proposition, $\mathcal{F}(u_t)$ has the RLP with respect to any generating cofibration of $h \in \mathbf{I}$; therefore $\mathcal{F}(u_t)$ is a trivial fibration and in particular a weak equivalence. Assembling this for all t we get that $\mathcal{F}$ is a co-Segal category i.e an object of $\mathscr{R}$, and (A5) follows. ■

The condition (A6) holds The condition is given by the following:

Lemma 3.9.10. *Let $\mathcal{F}$ be a co-Segal category i.e an object of $\mathscr{R}$. For a pushout square in $\mathscr{M}_{\mathbb{S}}(X)_{inj}$*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \mathcal{F} \arrow[d, "q"] \\
\mathcal{B} \arrow[r] & \mathcal{Z}
\end{tikzcd}$$

if $\alpha \in \mathbf{K}_{inj}$ then there exists a map $\varepsilon : \mathcal{Z} \longrightarrow \mathcal{E}$ which is a pushout along an element $\gamma \in cell(\mathbf{K}_{inj})$ such that the composite $\varepsilon \circ q : \mathcal{F} \longrightarrow \mathcal{E}$ is a weak equivalence.

Proof. The assertion is clear if $\alpha \in \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}}$, just take $\mathcal{E} = \mathcal{Z}$ and $\varepsilon = \mathrm{Id}_{\mathcal{Z}}$; $\varepsilon \in \mathbf{K}_{\text{inj}}$ is the pushout of itself along itself and q is a trivial cofibration so in particular a weak equivalence.

Assume that $\alpha = \mathcal{P}_{t!}(h_{/V}) : \mathcal{P}_{t!}(h) \longrightarrow \mathcal{P}_{t!}(\mathrm{Id}_V)$; then α is clearly a cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$. Our map $\beta : \mathcal{P}_{t!}(h) \longrightarrow \mathcal{F}$ corresponds by adjunction to a map $(a_1, a_2) : h \longrightarrow \mathcal{F}(u_t)$ in $\mathscr{M}^{[\mathbf{1}]}$. Denote by E the object we get from the pushout of h along a_1.

$$\begin{tikzcd}
U \arrow[rr, "a_1"] \arrow[dd, hook, "h"'] & & \mathcal{F}(A,B) \arrow[dd, "\mathcal{F}(u_t)"] \arrow[dl, dotted, "f"'] \\
& E \arrow[dr, "r"'] & \\
V \arrow[rr, "a_2"'] \arrow[ur, dotted] & & \mathcal{F}t
\end{tikzcd}$$

This gives a factorization of $\mathcal{F}(u_t) : \mathcal{F}(A,B) \overset{f}{\hookrightarrow} E \overset{r}{\to} \mathcal{F}t$.

If we analyze our original pushout square at t like in Remark 3.9.1 we get a diagram in which every triangle and square is commutative.

$$\begin{tikzcd}
U \arrow[r, "a_1"] \arrow[d, hook, "h"'] & \mathcal{F}(A,B) \arrow[r, "q_{AB}"] \arrow[d] & \mathcal{Z}(A,B) \arrow[d, "\mathcal{Z}(u_t)"] \\
V \arrow[r, "a_2"'] \arrow[urr, dashed] & \mathcal{F}t \arrow[r, "q_t"'] & \mathcal{Z}t
\end{tikzcd}$$

By the universal property of the pushout of h along a_1, there exists a unique map $\delta : E \longrightarrow \mathcal{Z}(A,B)$ making everything commutative. The uniqueness of the map out of the pushout implies the commutativity of:

$$\begin{tikzcd}
E \arrow[r, "\delta"] \arrow[d, "r"'] & \mathcal{Z}(A,B) \arrow[d, "\mathcal{Z}(u_t)"] \\
\mathcal{F}t \arrow[r, "q_t"'] & \mathcal{Z}t
\end{tikzcd}$$

Choose a factorization of r: $E \overset{i}{\hookrightarrow} Q \underset{\sim}{\overset{j}{\twoheadrightarrow}} \mathcal{F}t$; this yields a factorization of $\mathcal{F}(u_t)$ by cofibration followed by a trivial fibration:

$$\mathcal{F}(u_t) = \mathcal{F}(A,B) \overset{i \circ f}{\hookrightarrow} Q \underset{\sim}{\overset{j}{\twoheadrightarrow}} \mathcal{F}t.$$

This factorization is like the one we used to construct the functor $\mathscr{S}_t^1$. Since j is already a weak equivalence, if $\mathcal{F}$ is in $\mathscr{R}$ then $\mathcal{F}(u_t)$ is a weak equivalence and by 3 for 2, $i \circ f$ is a weak equivalence and hence a trivial cofibration.

Let us set $h' = i \circ f : \mathcal{F}(A,B) \longrightarrow Q$ the previous trivial cofibration and denote as usual $h'_{/Q} = (h', \mathrm{Id}_Q) \in \mathrm{Hom}_{\mathscr{M}^{[1]}}(h', \mathrm{Id}_Q)$ the obvious map. The morphism $\mathcal{P}_{t!}(h'_{/Q})$ is an element of $\mathbf{K}_{\mathrm{inj}}$ and since h' is trivial cofibration then $\mathcal{P}_{t!}(h'_{/Q})$ is also a trivial cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{inj}}$.

Introduce as before $S(h', j) \in \mathrm{Hom}_{\mathscr{M}^{[1]}}(h', \mathcal{F}(u_t))$ the commutative square:

$$\begin{tikzcd}
\mathcal{F}(A,B) \arrow[r, equal, "\mathrm{Id}"] \arrow[d, "h'"'] & \mathcal{F}(A,B) \arrow[d, "\mathcal{F}(u_t)"] \\
Q \arrow[r, "j"'] & \mathcal{F}t
\end{tikzcd}$$

and denote by $T(h', j, \mathcal{F}, t) \in \mathrm{Hom}_{\mathscr{M}_{\mathbb{S}}(X)}[\mathcal{P}_{t!}(h'), \mathcal{F}]$ its adjoint transpose. Denote by H'_1 the pushout of $\mathcal{P}_{t!}(h'_{/Q})$ along $T(h', j, \mathcal{F}, t)$. By stability of cofibrations under pushout we know that

H'_1 is a trivial cofibration, so in particular a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)$:

$$\begin{tikzcd}
\mathcal{P}_{t!}(h') \arrow[r, "T(h',j,\mathcal{F},t)"] \arrow[d, hook, "\mathcal{P}_{t!}(h'_{/Q})"', "\wr"] & \mathcal{F} \arrow[d, hook, "H'_1", "\wr"'] \\
\mathcal{P}_{t!}\,\mathrm{Id}_Q \arrow[r] & \mathcal{E}
\end{tikzcd} \tag{3.9.2.1}$$

__Goal__: The rest of the proof will be to show that we can factorize H'_1 as:

$$\mathcal{F} \xrightarrow{q} \mathcal{Z} \xrightarrow{\varepsilon} \mathcal{E}$$

where ε is a pushout of a map $\gamma \in \mathbf{K}_{\text{inj}}$. This will complete the proof of the Lemma since H'_1 is a weak equivalence.

Claim. The map γ is $\mathcal{P}_{t!}(i_{/Q}) : \mathcal{P}_{t!}(i) \longrightarrow \mathcal{P}_{t!}(\mathrm{Id}_Q)$, where $i : E \hookrightarrow Q$ is the previous cofibration appearing in the factorization of the morphism $E \xrightarrow{r} \mathcal{F}t$. This map is in $\mathit{cell}(\mathbf{K}_{\text{inj}})$ (see Remark 3.9.4).

Let $R \in \mathrm{Hom}_{\mathscr{M}^{[\mathbf{1}]}}(h, h')$ be the morphism represented by the commutative square:

$$\begin{tikzcd}
U \arrow[rr, "a_1"] \arrow[dd, "h"'] & & \mathcal{F}(A,B) \arrow[dl, dotted] \arrow[dd, "h'"] \\
& E \arrow[dr, dotted] & \\
V \arrow[ur, dotted] \arrow[rr, "g"'] & & Q
\end{tikzcd}$$

The composite $S(h', j) \circ R : h \longrightarrow \mathcal{F}(u_t)$ is the map (a_1, a_2) whose adjoint transpose is $\beta : \mathcal{P}_{t!}(h) \longrightarrow \mathcal{F}$ of the original pushout square.

One can easily check that the morphism $\theta = (f, \mathrm{Id}_Q) : h' \longrightarrow i$ is the pushout in $\mathscr{M}^{[\mathbf{1}]}$ of $h_{/V}$ along R and that the following commutes:

$$\begin{tikzcd}
h \arrow[rr, "R"] \arrow[dd, hook, "h_{/V}"'] & & h' \arrow[dl, dotted, "\theta"'] \arrow[dd, hook, "h'_{/Q}", "\wr"'] \\
& i \arrow[dr, hook, "i_{/Q}"] & \\
\mathrm{Id}_V \arrow[ur, dotted] \arrow[rr] & & \mathrm{Id}_Q
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& U \arrow[rrr, "a_1"] \arrow[ddl, "h"'] \arrow[ddd, dotted] & & & \mathcal{F}(A,B) \arrow[dll, dotted] \arrow[ddd, "h'"] \arrow[ddl, "h'"'] \\
& & E \arrow[dr, "i"] \arrow[ddd, "i"] & & \\
V \arrow[urr, dotted] \arrow[rrr] \arrow[ddd, "\mathrm{Id}"'] & & & Q \arrow[ddd] & \\
& V \arrow[rrr, dotted] \arrow[ddl, "\mathrm{Id}"'] & & & Q \arrow[dll, dotted] \arrow[ddl, "\mathrm{Id}"'] \\
& & Q \arrow[dr] & & \\
V \arrow[urr, dotted] \arrow[rrr] & & & Q &
\end{tikzcd}$$

The functor $\mathcal{P}_{t!}$ is a left adjoint, therefore preserves colimits in general, in particular it preserves pushout squares. It follows that $\mathcal{P}_{t!}(\theta)$ is the pushout of $\mathcal{P}_{t!}(h_{/V})$ along $\mathcal{P}_{t!}(R)$. If we apply $\mathcal{P}_{t!}$ in the above square and join the pushout square (3.9.2.1) we get the following commutative diagram in $\mathscr{M}_{\mathbb{S}}(X)$:

$$\begin{tikzcd}
\mathcal{P}_{t!}(h) \arrow[rr, "\mathcal{P}_{t!}(R)"] \arrow[dd, hook, "\mathcal{P}_{t!}(h_{/V})"'] & & \mathcal{P}_{t!}(h') \arrow[r, dotted, "T(h',j,\mathcal{F},t)"] \arrow[dl, dotted, "\mathcal{P}_{t!}(\theta)"'] \arrow[dd, "\mathcal{P}_{t!}(h'_{/Q})"] & \mathcal{F} \arrow[dd, "H_1"] \\
& \mathcal{P}_{t!}(i) \arrow[dr, hook, "\mathcal{P}_{t!}(i_{/Q})"] & & \\
\mathcal{P}_{t!}(\mathrm{Id}_V) \arrow[ur, dotted] \arrow[rr] & & \mathcal{P}_{t!}(\mathrm{Id}_Q) \arrow[r] & \mathcal{E}
\end{tikzcd}$$

The naturality of the adjunction implies that $T(h', j, \mathcal{F}, t) \circ \mathcal{P}_{t!}(R) = \beta : \mathcal{P}_{t!}(h) \longrightarrow \mathcal{F}$. Now introduce the pushout of $\mathcal{P}_{t!}(\theta)$ along $T(h', j, \mathcal{F}, t)$:

$$D_1 = \begin{tikzcd} \mathcal{P}_{t!}(h) \arrow[r, "T(h',j,\mathcal{F},t)"] \arrow[d, hook, "\mathcal{P}_{t!}(\theta)"'] & \mathcal{F} \arrow[d, hook, "g"] \\ \mathcal{P}_{t!}(i) \arrow[r, "\xi"] & \mathcal{M} \end{tikzcd}$$

On one hand by Lemma 4.1.2, we know that 'a pushout of a pushout is a pushout', thus the concatenation of the two squares below is a pushout of $\mathcal{P}_{t!}(h_{/V})$ along β (which is the pushout of the Lemma):

$$\begin{tikzcd} \mathcal{P}_{t!}(h) \arrow[r, "\mathcal{P}_{t!}(R)"] \arrow[d, hook, "\mathcal{P}_{t!}(h_{/V})"'] & \mathcal{P}_{t!}(h') \arrow[r, dotted, "T(h',j,\mathcal{F},t)"] \arrow[d, hook, "\mathcal{P}_{t!}(\theta)"] & \mathcal{F} \arrow[d, "g"] \\ \mathcal{P}_{t!}(\mathrm{Id}_V) \arrow[r, dotted] & \mathcal{P}_{t!}(i) \arrow[r, "\xi"'] & \mathcal{M} \end{tikzcd}$$

By uniqueness of the pushout, we can assume (up to a unique isomorphism) that $\mathcal{M} = \mathcal{Z}$ and that $g = q$. On the other hand if we consider D_2 the pushout square of $\gamma = \mathcal{P}_{t!}(i_{/Q})$ along the previous map ξ:

$$D_2 = \begin{tikzcd} \mathcal{P}_{t!}(i) \arrow[r, "\xi"] \arrow[d, hook, "\mathcal{P}_{t!}(i_{/Q})"'] & \mathcal{Z} \arrow[d, hook, "\varepsilon"] \\ \mathcal{P}_{t!}(\mathrm{Id}_Q) \arrow[r, "\zeta"] & \mathcal{N} \end{tikzcd}$$

then the vertical concatenation $\frac{D_1}{D_2}$ is a pushout of $\mathcal{P}_{t!}(h'_{/Q})$ along $T(h', j, \mathcal{F}, t)$. By uniqueness of the pushout we can assume (up-to a unique isomorphism) that $\mathcal{N} = \mathcal{E}$. Consequently we have $H'_1 = \varepsilon \circ q$ and by construction ε is the pushout of $\mathcal{P}_{t!}(i_{/Q}) \in \mathbf{K}_{\text{inj}}$. This completes the proof of the Lemma. ■

Localization of the injective model structure

We now go back to the functor $\mathscr{S} : \mathscr{M}_{\mathbb{S}}(X)_{\text{inj}} \longrightarrow \mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ constructed before. Recall that $\mathscr{S}$ has the following properties:

1. for every $\mathcal{F} \in \mathscr{M}_{\mathbb{S}}(X)$, $\mathscr{S}(\mathcal{F}) \in \mathscr{R}$;
2. we have a natural transformation $\eta_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathscr{S}(\mathcal{F})$ which is a cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$.
3. if $\mathcal{F} \in \mathscr{R}$ then $\eta_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathscr{S}(\mathcal{F})$ is a trivial cofibration therein.

In order to apply the material developed by Simpson in [79], we need some other properties.

The first thing we need, that will not be proved for the moment is the

Hypothesis 3.9.1. We will assume from now that if $\mathscr{M}$ is left proper then $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ is also left proper.

Remark 3.9.5. We are not sure for the moment that this hypothesis is valid in all cases. But if it's not, there are many reasons to believe that we can have a structure of a **catégorie dérivable** in the sense of Cisinski [24]

Warning. We modify $\mathscr{S}$ by another functor denoted $\mathscr{S}_{\text{inj}}$ which is a $\mathbf{K}_{\text{inj}}$-injective replacement functor. $\mathscr{S}_{\text{inj}}$ is constructed by the *Gluing construction* (see [32, Prop. 7.17]) and the small object argument in the locally presentable category $\mathscr{M}_{\mathbb{S}}(X)$.

With the above modifications and hypothesis, and thanks to Theorem 3.9.8 we have

Proposition 3.9.11. *The pair $(\mathscr{S}_{inj}, \eta)$ is a weak monadic projection from $\mathscr{M}_{\mathbb{S}}(X)_{inj}$ to $\mathscr{R}$.*

Proof. This is Lemma 9.3.1 in [79]. ■

New homotopical data Let P be a cofibrant replacement functor on $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$.

Define a map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ to be

- a **new weak equivalence** if the map $\mathscr{S}_{\text{inj}}(\mathrm{P}\,\sigma) : \mathscr{S}_{\text{inj}}(\mathrm{P}\,\mathcal{F}) \longrightarrow \mathscr{S}_{\text{inj}}(\mathrm{P}\,\mathcal{G})$ is a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)$;
- a **new cofibration** if σ is a cofibration;
- a **new trivial cofibration** if σ is a cofibration and a new weak equivalence;
- a **new fibration** if it has the RLP with respect to all new trivial cofibrations; and
- a **new trivial fibration** if it's a new fibration and a new weak equivalence.

With the above definitions we have:

Theorem 3.9.12. *The classes of original cofibrations, new weak equivalences, and new fibrations defined above provide $\mathscr{M}_{\mathbb{S}}(X)$ with a structure of closed model category, cofibrantly generated and combinatorial. It is left proper. This structure is the left Bousfield localization of $\mathscr{M}_{\mathbb{S}}(X)_{inj}$ by the original set of maps $\mathbf{K}_{inj}$.*

The fibrant objects are the $\mathbf{K}_{inj}$-injective objects, in particular they are co-Segal categories; and a morphism $\mathcal{F} \longrightarrow \mathcal{G}$ to a fibrant object is a fibration if and only if it is in $\mathbf{inj}(\mathbf{K}_{inj})$. *We will denote by $\mathscr{M}_{\mathbb{S}}(X)^{+}_{inj}$ this new model structure on $\mathscr{M}_{\mathbb{S}}(X)$.*

Proof. Follows from Theorem 9.7.1 in [79]. The fact that fibrant objects are co-Segal categories follows from property $(A5)$. ■

3.9.3 Localization of the projective model structure

The Classical localization

In the fist place we begin by using the classical localization method to localize the projective model structure $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$.

We will use the following theorem, due to Smith [80], as stated by Barwick [9, Thm 4.7].

Theorem 3.9.13. *If $\mathbf{M}$ is left proper and $\mathbb{U}$-combinatorial, and $\mathbf{K}$ is an $\mathbb{U}$-small set of homotopy classes of morphisms of $\mathbf{M}$, the left Bousfield localization $\mathbf{L_K}\,\mathbf{M}$ of $\mathbf{M}$ along any set representing $\mathbf{K}$ exists and satisfies the following conditions.*

1. *The model category $\mathbf{L_K}\,\mathbf{M}$ is left proper and $\mathbb{U}$-combinatorial.*

2. *As a category,* $\mathbf{L_K}\,\mathbf{M}$ *is simply* $\mathbf{M}$.

3. *The cofibrations of* $\mathbf{L_K}\,\mathbf{M}$ *are exactly those of* $\mathbf{M}$.

4. *The fibrant objects of* $\mathbf{L_K}\,\mathbf{M}$ *are the fibrant* $\mathbf{K}$*-local objects* Z *of* $\mathbf{M}$.

5. *The weak equivalences of* $\mathbf{L_K}\,\mathbf{M}$ *are the* $\mathbf{K}$*-local equivalences.*

We introduce some pieces of notation.

Given a cofibration h, the map $h_{/V}$ considered previously is an injective cofibration in $\mathscr{M}^{[\mathbf{1}]}$ but not in general a projective cofibration. We will then replace $h_{/V}$ by a slight modification $\zeta(h)$ which is a projective cofibration.
If we consider $h_{/V}$ as a commutative diagram, by universal property of the pushout of h along itself, there is a unique map $k : V \cup_U V \longrightarrow V$ making everything commutative:

$$\begin{tikzcd}
U \arrow[rr, hook, "h"] \arrow[dd, hook, "h"'] & & V \arrow[dl, dotted, "i_1"'] \arrow[dd, "\mathrm{Id}"] \\
& V \cup_U V \arrow[dr, hook, "k"] & \\
V \arrow[ur, dotted, "i_0"] \arrow[rr, "\mathrm{Id}"'] & & V
\end{tikzcd}$$

Choose a factorization 'cofibration-trivial fibration' of the map $k : V \cup_U V \longrightarrow V$:

$$r = V \cup_U V \overset{a}{\hookrightarrow} Z \xrightarrow[\sim]{q} V.$$

Such factorization is a *relative cylinder object* for the cofibration $h : U \longrightarrow V$.

For each cofibration $h \in \mathbf{I}$, define $\zeta(h) = (h, ai_0) \in \mathrm{Hom}_{\mathscr{M}^{[\mathbf{1}]}}(h, ai_1)$ to be the induced map represented by the commutative square:

$$\begin{tikzcd}
U \arrow[rr, "h"] \arrow[dd, hook, "h"'] & & V \arrow[dl, dotted] \arrow[dd, "ai_1"] \\
& V \cup_U V \arrow[dr, hook, "a"] & \\
V \arrow[ur, dotted] \arrow[rr, hook, "ai_0"'] & & Z
\end{tikzcd}$$

By construction we have $j \circ (ai_0) = \mathrm{Id}_V$ and since j and Id_V are weak equivalences, it follows by 3 for 2 that ai_0 is a weak equivalence, hence a trivial cofibration.

Remark 3.9.6.

1. It's clear that $\zeta(h)$ is automatically a projective (= Reedy) cofibration in $\mathscr{M}^{[\mathbf{1}]}$; and if h is a trivial cofibration then so is $\zeta(h)$.

2. Since $\mathcal{P}_{t!} \dashv \mathrm{Ev}_{u_t}$ is a Quillen adjunction with the corresponding projective model structures, then $\mathcal{P}_{t!}\zeta(h)$ is a (trivial) cofibration in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{proj}}$ is h is so.

The map $\zeta(h)$ plays the role of $h_{/V}$, that is, we can detect if $\mathcal{F}(u_t)$ is a weak equivalence in $\mathscr{M}$ in terms of horn filling properties against all the map $\zeta(h)$ (using a homotopy lifting Lemma).

Definition 3.9.14. *Let $\mathscr{M}$ be a model category. Say that $g : A \longrightarrow B$ has the* ***right homotopy lifting property*** *(RHLP) with respect to $h : U \longrightarrow V$ if for any commutative square:*

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "h"'] & A \arrow[d, "g"] \\ V \arrow[r, "v"'] \arrow[ur, dotted, "r"] & B \end{tikzcd}$$

there exists a map $r : V \longrightarrow A$ such that:

- *$rh = u$ i.e the upper triangle commutes;*
- *gr and v are* ***homotopic relative to*** *U, that is, we have a relative cylinder object*

$$V \cup_U V \overset{a}{\hookrightarrow} Z \xrightarrow[\sim]{q} V$$

together with a map $f : Z \longrightarrow B$ restricting to $gu = vh$ on U; and inducing the equalities $fai_0 = v$, $fai_1 = gr$.

Remark 3.9.7. It's important to observe that this definition is well defined in the sense that it doesn't depend on the choice of the relative cylinder object for h. Indeed if $r' = V \cup_U V \overset{a'}{\hookrightarrow} Z' \xrightarrow[\sim]{q'} V$ is another cylinder, then by the lifting axiom we can find a map $k : Z' \longrightarrow Z$ such that $a = ka'$ and $q' = qk$. Therefore if $f : Z \longrightarrow B$ is a homotopy lifting with respect to r then automatically $f' = fk$ is a homotopy lifting with respect to r'.

Proposition 3.9.15. *For a cofibration $h : U \longrightarrow V$ and a map $g : A \longrightarrow B$ in a model category $\mathscr{M}$, the following are equivalent.*

1. *g has the RHLP with respect to h.*
2. *g is $\{\zeta(h)\}$-injective.*

Proof. We simply show how we get (1) from (2). The converse follows by 'reversing' the argumentation since we can assume that the relative cylinder chosen in (1) is the one used to construct $\zeta(h)$ thanks to Remark 3.9.7.

Assume that g is $\{\zeta(h)\}$-injective. A lifting problem defined by g and h

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "h"'] & A \arrow[d, "g"] \\ V \arrow[r, "v"'] & B \end{tikzcd}$$

corresponds to a map $\alpha = (u, v) \in \mathrm{Hom}_{\mathscr{M}^{[\mathbf{1}]}}(h, g)$; since g is $\{\zeta(h)\}$-injective we can fill the following horn in $\mathscr{M}^{[\mathbf{1}]}$

$$\begin{tikzcd} h \arrow[r, "\alpha"] \arrow[d, "\zeta(h)"'] & g \\ ai_1 \arrow[ur, dotted, "\beta"'] & \end{tikzcd}$$

with a map $\beta = (r, f)$. Displaying this horn in $\mathscr{M}$ we end up with with a commutative diagram where every triangle and square commutes:

$$\begin{tikzcd}
 & U \arrow[r, two heads, "u"] \arrow[dl, "h"'] \arrow[dd, "h"] & A \arrow[dd, "g"] \\
V \arrow[urr, dotted, "r"] \arrow[dd, "ai_1"'] & & \\
 & V \arrow[r, two heads, "v"'] \arrow[dl, "ai_0"'] & B \\
Z \arrow[urr, dotted, "f"'] & &
\end{tikzcd}$$

One clearly has that (r, f) gives the desired relative homotopy lifting. ■

The following is a classical result in model categories.

Lemma 3.9.16. *In a tractable left proper model category $\mathscr{M}$, if a map $g : A \longrightarrow B$ between fibrant objects has the RHLP with respect to any generating cofibration h, then g is a weak equivalence.*

Proof. This is Lemma 7.5.1 in [79]. ■

Notation 3.9.3.

1. We will denote for short $\zeta(\mathbf{I}) = \{\zeta(h)\}_{U \xrightarrow{h} V \in \mathbf{I}}$.
2. For $t \in \mathbb{S}_{\overline{X}}$ we will write $\mathcal{P}_{t!}\zeta(\mathbf{I})$ the image of $\zeta(I)$ by $\mathcal{P}_{t!}$.
3. Let $\mathbf{K}_{\text{proj}}$ be the set $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}} \cup (\coprod_{t \in \mathbb{S}_{\overline{X}}, \mathbf{deg}(t)>1} \mathcal{P}_{t!}\zeta(\mathbf{I}))$.

Recall that $\mathscr{R}$ is the subcategory of co-Segal categories. Under Hypothesis 3.9.1, we have by virtue of Theorem 3.9.13 that:

Theorem 3.9.17. *There exists a combinatorial model structure on $\mathscr{M}_{\mathbb{S}}(X)_{proj}$ such that the fibrant objects are co-Segal categories i.e, objects of $\mathscr{R}$. This model structure is moreover left proper and is the left Bousfield localization with respect to $\mathbf{K}_{proj}$.*

Proof. Take $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ with the Bousfield localization with respect to $\mathbf{K}_{\text{proj}}$ which exists by Theorem 3.9.13. The maps in $\mathbf{K}_{\text{proj}}$ are weak equivalences in the Bousfield localization and since they are old cofibrations, they become trivial cofibrations. It follows that if $\mathcal{F}$ is fibrant, it is then $\mathbf{K}_{\text{proj}}$-injective.

Let $\mathcal{F}$ be a $\mathbf{K}_{\text{proj}}$-injective object. On the one hand, as $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)\text{-proj}} \subset \mathbf{K}_{\text{proj}}$, we have that $\mathcal{F}$ is fibrant in the old model structure, which means that $\mathcal{F}$ is level-wise fibrant. On the other hand, since $\mathcal{F}$ is $\{\mathcal{P}_{t!}\zeta(\mathbf{I})\}$-injective for all t, it follows by the adjunction $\mathcal{P}_{t!} \dashv \mathrm{Ev}_{u_t}$, that $\mathcal{F}(u_t)$ is $\zeta(\mathbf{I})$-injective. Combining Proposition 3.9.15 and Lemma 3.9.16, we deduce that $\mathcal{F}(u_t)$ is a weak equivalence, thus $\mathcal{F} \in \mathscr{R}$. ■

Direct localizing the projective model structure.

Let's consider the injective localized model structure $\mathscr{M}_{\mathbb{S}}(X)^{+}_{\text{inj}}$ constructed with the direct localizing system $(\mathscr{R}, \mathbf{K}_{\text{inj}})$. As we mentioned before it's the left Bousfield localization of the original model structure with respect to $\mathbf{K}_{\text{inj}}$; in other words, it's the same Bousfield localization we will have using Theorem 3.9.17.

Since every map in $\mathbf{K}_{\text{inj}}$ is a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)^+_{\text{inj}}$, we have in particular that for all $h \in \mathbf{I}$, the map $\mathcal{P}_{t!}(h_{/V}) : \mathcal{P}_{t!}(h) \longrightarrow \mathcal{P}_{t!}(\text{Id}_V)$ is a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)^+_{\text{inj}}$. Recall that each map $\zeta(h)$ is constructed out of $h_{/V}$ and we have a factorization of $h_{/V}$:

$$h \xhookrightarrow{\zeta(h)} ai_1 \xrightarrow{\ell} \text{Id}_V .$$

The factorization is displayed below as:

$$\begin{tikzcd}
U \arrow[r, "h"] \arrow[d, hook, "h"'] & V \arrow[r, "\text{Id}"] \arrow[d, hook, "ai_1", "\wr"'] & V \arrow[d, "\text{Id}"] \\
V \arrow[r, hook, "ai_0"'] & Z \arrow[r, two heads, "\sim", "q"'] & V
\end{tikzcd}$$

As q is a trivial fibration, the map $\ell : ai_1 \longrightarrow \text{Id}$ is a level-wise trivial fibration in $\mathscr{M}^{[\mathbf{1}]}$, in particular, a weak equivalence therein. Since we've assumed that all the objects of $\mathscr{M}$ are cofibrant, then both the source and target of ℓ are cofibrant in $\mathscr{M}^2_{\text{proj}}$. From Ken Brown Lemma $\mathcal{P}_{t!}$ preserves weak equivalences between cofibrant objects (as any left Quillen functor); thus $\mathcal{P}_{t!}(\ell)$ is an old weak equivalence, hence a new weak equivalence (= a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)^+_{\text{inj}}$).

From the equality $\mathcal{P}_{t!}(h_{/V}) = \mathcal{P}_{t!}(\ell) \circ \mathcal{P}_{t!}\zeta(h)$ we deduce by 3 for 2 in the model category $\mathscr{M}_{\mathbb{S}}(X)^+_{\text{inj}}$, that $\mathcal{P}_{t!}\zeta(h)$ is a weak equivalence therein; moreover since projective cofibrations are also injective cofibration, then $\mathcal{P}_{t!}\zeta(h)$ is an old cofibration, hence a new cofibration. Putting these together one has that every $\mathcal{P}_{t!}\zeta(h)$ is a trivial cofibration in $\mathscr{M}_{\mathbb{S}}(X)^+_{\text{inj}}$.

New projective data Recall that weak equivalences in $\mathscr{M}_{\mathbb{S}}(X)^+_{\text{inj}}$ are those maps σ such that $\mathscr{S}_{\text{inj}}(\mathrm{P}\,\sigma)$ is a weak equivalence in the original model structure $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$. Here $\mathscr{S}_{\text{inj}}$ is a weak monadic projection from $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ to $\mathscr{R}$ and P is a cofibrant replacement functor in $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$. As pointed out in [79, Sec 9.3], the notion of new weak equivalence depends only on $\mathscr{R}$ and doesn't depend neither on $\mathbf{K}_{\text{inj}}$ nor on P.

Using again the fact that projective cofibrations are injective ones, and since old weak equivalences in $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ and $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ are the same, we clearly have that a cofibrant replacement functor in $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ is also a cofibrant replacement for $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$. Therefore we can assume that P is a projective cofibrant replacement functor.

As mentioned in Remark 3.9.3 we can extract from the weak monadic projection $\mathscr{S}_{\text{inj}}$ a weak monadic projection $\mathscr{S}_{\text{proj}}$ from $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ to $\mathscr{R}$. This is obtained by applying the functorial factorization of the type $(\mathbf{cof}, \mathbf{fib} \cap \mathbf{we})$ to the natural transformation $\eta : \text{Id}_{\mathscr{M}_{\mathbb{S}}(X)} \longrightarrow \mathscr{S}_{\text{inj}}$ in the model category $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$. In particular there is trivial projective fibration $\mathscr{S}_{\text{proj}} \longrightarrow \mathscr{S}_{\text{inj}}$.

Warning. In the upcoming paragraphs we will write 'new injective' or 'new projective' to avoid confusion when saying 'new weak equivalence'. We will remove this distinction later since the new weak equivalences will be the same.

Let P be cofibrant replacement functor on $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$.

Define a map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ to be:

- a **new projective weak equivalence** if the map $\mathscr{S}_{\text{proj}}(\mathrm{P}\,\sigma) : \mathscr{S}_{\text{proj}}(\mathrm{P}\,\mathcal{F}) \longrightarrow \mathscr{S}_{\text{proj}}(\mathrm{P}\,\mathcal{G})$ is a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)$;
- a **new cofibration** if σ is a cofibration;
- a **new trivial cofibration** if σ is a cofibration and a new (projective) weak equivalence;
- a **new fibration** if it has the RLP with respect to all new trivial cofibrations; and
- a **new trivial fibration** if it's a new fibration and a new weak equivalence.

Proposition 3.9.18. *The classes of new projective weak equivalences and and new injective weak equivalences coincide.*

Proof. For any morphism $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ we have by construction the commutativity of:

$$\begin{tikzcd}
\mathscr{S}_{\text{proj}}(\mathrm{P}\,\mathcal{F}) \arrow[r, "\mathscr{S}_{\text{proj}}(\mathrm{P}\,\sigma)"] \arrow[d, "\wr"] & \mathscr{S}_{\text{proj}}(\mathrm{P}\,\mathcal{G}) \arrow[d, "\wr"] \\
\mathscr{S}_{\text{inj}}(\mathrm{P}\,\mathcal{F}) \arrow[r, "\mathscr{S}_{\text{inj}}(\mathrm{P}\,\sigma)"'] & \mathscr{S}_{\text{inj}}(\mathrm{P}\,\mathcal{G})
\end{tikzcd}$$

with all the above vertical maps being weak equivalences in $\mathscr{M}_{\mathbb{S}}(X)$. Therefore if one of two maps $\mathscr{S}_{\text{proj}}(\mathrm{P}\,\sigma)$, $\mathscr{S}_{\text{inj}}(\mathrm{P}\,\sigma)$ is a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)$ then by 3 for 2 of weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)$ the other one is also a weak equivalence. ■

Remark 3.9.8. 1. It follows from the proposition that the new projective weak equivalences are closed under retracts, composition and satisfy the 3 for 2 property. We leave the reader to check that the class of new projective (trivial) cofibrations and (trivial) fibrations are also so closed under retract and composition.

2. Since the old projective cofibrations are also old injective cofibrations, by the proposition we deduce that the new projective trivial cofibrations are also a new injective trivial cofibrations.

We remind the reader that the functor $\mathscr{S}_{\text{inj}}$ is a $\mathbf{K}_{\text{inj}}$-injective replacement functor. The following proposition tells us that:

Proposition 3.9.19. *The functor $\mathscr{S}_{proj}$ is a $\mathbf{K}_{proj}$-injective replacement functor, that is, there is a lift to any diagram:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d, "\alpha"'] & \mathscr{S}_{proj}(\mathcal{F}) \arrow[d, "!"] \\
\mathcal{B} \arrow[r, "!"] \arrow[ur, dotted] & *
\end{tikzcd}$$

for any morphism $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathbf{K}_{proj}$.

Sketch of proof. Such lifting problem is simply an extension problem:

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathscr{S}_{\text{proj}}(\mathcal{F}) \\
\mathcal{B} \arrow[ur, dotted, "\exists?"] &
\end{tikzcd}$$

If $\alpha \in \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}} \subset \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}} \subset \mathbf{K}_{\text{inj}}$, by extending the above diagram using the projective trivial fibration $p : \mathscr{S}_{\text{proj}}(\mathcal{F}) \overset{\sim}{\twoheadrightarrow} \mathscr{S}_{\text{inj}}(\mathcal{F})$, we find a lift ① in the diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d, "\alpha"'] & \mathscr{S}_{\text{proj}}(\mathcal{F}) \arrow[r, two heads, "p", "\sim"'] & \mathscr{S}_{\text{inj}}(\mathcal{F}) \\
\mathcal{B} \arrow[ur, dotted, "②"] \arrow[urr, dotted, "①"'] & &
\end{tikzcd}$$

With the map ① we have a commutative square which gives a lifting problem defined by α and p; by the lifting axiom in the model category $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ we can find a lift $\mathcal{B} \overset{②}{\longrightarrow} \mathscr{S}_{\text{proj}}(\mathcal{F})$ making everything commutative, in particular we have the desired lift of the original extension problem.

Assume now that $\alpha = \mathcal{P}_{t!}\zeta(h)$, then by extending the diagram with the map p and the map $\mathcal{P}_{t!}(\ell)$ we end up with the following diagram

$$\begin{tikzcd}
\mathcal{P}_{t!}h \arrow[r] \arrow[d, "\mathcal{P}_{t!}\zeta(h)"] \arrow[dd, bend right, "\mathcal{P}_{t!}h_{/V}"'] & \mathscr{S}_{\text{proj}}(\mathcal{F}) \arrow[r, two heads, "p", "\sim"'] & \mathscr{S}_{\text{inj}}(\mathcal{F}) \\
ai_1 \arrow[d, "\mathcal{P}_{t!}\ell"] \arrow[ur, dotted, "②"'] & & \\
\mathrm{Id}_V \arrow[uurr, dotted, "①"'] & &
\end{tikzcd}$$

Since $\mathcal{P}_{t!}h_{/V} \in \mathbf{K}_{\text{inj}}$, there is a lift $\mathrm{Id}_V \overset{①}{\longrightarrow} \mathscr{S}_{\text{inj}}(\mathcal{F})$; the commutative square we get is a lifting problem defined by the projective cofibration $\mathcal{P}_{t!}\zeta(h)$ and the projective trivial fribration p. By the lifting axiom in $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ there is a lift $ai_1 \overset{②}{\longrightarrow} \mathscr{S}_{\text{proj}}(\mathcal{F})$ making everything commutative; in particular we have a lift to the original extension problem. ■

Pushout along new projective trivial cofibrations We need a last ingredient in order to apply Smith recognition theorem for combinatorial model categories as stated for example by Barwick [9, Proposition 2.2].

Lemma 3.9.20. *For a pushout square*

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d, "f"'] & \mathcal{F} \arrow[d, "g"] \\
\mathcal{B} \arrow[r] & \mathcal{G}
\end{tikzcd}$$

in $\mathscr{M}_{\mathbb{S}}(X)_{proj}$, if f is a new projective trivial cofibration then g is a new projective trivial cofibration.

Proof. Since g is already an old projective cofibration it suffices to show that g is a new projective weak equivalence. By Proposition 3.9.18, it's the same thing as being a new injective weak equivalence.

As pointed out above f is also a new injective trivial cofibration, and since (trivial) cofibrations are closed under pushout in any model category, it follows that g is also a new injective trivial cofibration in $\mathscr{M}_{\mathbb{S}}(X)^{+}_{\text{inj}}$. In particular g is a new injective weak equivalence, thus a new projective weak equivalence. ■

The new weak equivalences form an accessible subcategory of $\operatorname{Arr}(\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}})$ because they are the weak equivalences of the combinatorial model category $\mathscr{M}_{\mathbb{S}}(X)^+_{\text{inj}}$. A map in $\mathbf{inj}(\mathbf{I}_{\mathscr{M}_{\mathbb{S}}(X)})$ is a trivial fibration, in particular an old weak equivalence. But old weak equivalences are also new weak equivalences, therefore any map in $\mathbf{inj}(\mathbf{I}_{\mathscr{M}_{\mathbb{S}}(X)})$ is a new weak equivalence.

By virtue of Smith's theorem we have that:

Theorem 3.9.21. *The classes of original cofibrations, new weak equivalences, and new fibrations defined above provide $\mathscr{M}_{\mathbb{S}}(X)$ with a structure of closed model category, cofibrantly generated and combinatorial. It is indeed left proper.*

This model structure is the left Bousfield localization of $\mathscr{M}_{\mathbb{S}}(X)_{proj}$ with respect to $\mathbf{K}_{proj}$. Fibrant objects are co-Segal categories.

We will denote by $\mathscr{M}_{\mathbb{S}}(X)^+_{proj}$ this new model structure on $\mathscr{M}_{\mathbb{S}}(X)$.

Proof. The model structure is guaranteed by Smith's theorem. A fibrant object $\mathcal{F}$ is by definition an α-injective injective object for every new trivial cofibration α.

From the previous observations the maps $\mathcal{P}_{t!}\zeta(h)$ become new trivial cofibrations, and since the maps in $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}}$ are already new trivial cofibrations, all the elements of $\mathbf{K}_{\text{proj}}$ are new trivial cofibrations. Consequently if $\mathcal{F}$ is fibrant, it's in particular $\mathbf{K}_{\text{proj}}$-injective and one proceeds as in the proof of Theorem 3.9.17 to conclude that $\mathcal{F}$ is a co-Segal category which is fibrant in the old model structure (= level-wise fibrant).

It remains to prove that this model structure is the left Bousfield localization of $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ with respect to $\mathbf{K}_{\text{proj}}$. To prove this we will simply show that the model structure $\mathscr{M}_{\mathbb{S}}(X)^+_{\text{proj}}$ and the one of Theorem 3.9.17 are the same; that is, we have the same classes of cofibrations and fibrations on the underlying category $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$.

For the notation we will denote by $\mathscr{M}_{\mathbb{S}}(X)^{\mathbf{c}}_{\text{proj}}$ the 'classical' localized model structure of Theorem 3.9.17. In both $\mathscr{M}_{\mathbb{S}}(X)^+_{\text{proj}}$ and $\mathscr{M}_{\mathbb{S}}(X)^{\mathbf{c}}_{\text{proj}}$ the cofibrations are the old cofibrations in $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ so we only have to show that we have the same fibrations.

In the two model structures the fibrations are defined to be the maps having the RLP with respect to all maps which are both cofibration and new weak equivalences. It follows that the fibrations will be the same as soon as we show that we have the same weak equivalences. Thanks to the Lemma below we know that the weak equivalences are indeed the same. ■

Lemma 3.9.22. *Given a map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ in $\mathscr{M}_{\mathbb{S}}(X)$ the following are equivalent:*

1. *σ is a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)^{\mathbf{c}}_{proj}$ i.e a $\mathbf{K}_{proj}$-local equivalence;*
2. *σ is a weak in $\mathscr{M}_{\mathbb{S}}(X)^+_{proj}$, that is, $\mathscr{S}_{proj}(P\sigma)$ is an old weak equivalence;*
3. *σ is a weak in $\mathscr{M}_{\mathbb{S}}(X)^+_{inj}$, that is, $\mathscr{S}_{inj}(P\sigma)$ is an old weak equivalence or equivalently a $\mathbf{K}_{inj}$-local equivalence.*

Proof of the Lemma. The equivalence between (2) and (3) is clear; we mentioned it there just for a reminder. We will show that (3) is equivalent to (1).

The general picture is that $\mathscr{M}_{\mathbb{S}}(X)^{\mathbf{c}}_{\text{proj}}$ is obtained by turning the maps $\mathcal{P}_{t!}\zeta(h)$ to weak equivalences while $\mathscr{M}_{\mathbb{S}}(X)^+_{\text{inj}}$ is obtained by turning the maps $\mathcal{P}_{t!}h_{/V}$ into weak equivalences. The two type of maps are related by the equality $\mathcal{P}_{t!}(h_{/V}) = \mathcal{P}_{t!}(\ell) \circ \mathcal{P}_{t!}\zeta(h)$ where $\mathcal{P}_{t!}(\ell)$ is already a weak equivalence; thus if we turn one of them into a weak equivalence then the other one also become a weak equivalence by 3 for 2. This is the general philosophy; we shall now present a proof.

In the following we use the language of classical Bousfield localization of a model category with respect to a set of maps. This requires the notion of *local object* and *local equivalence*. These definitions can be found for example in [9], [33], [30], [40], [42] , [66]. We will denote by $\mathrm{Map}(-,-)$ a homotopy function complex on $\mathscr{M}_{\mathbb{S}}(X)$ (see the previous references for the definition of function complex). The homotopy type of $\mathrm{Map}(-,-)$ depends only on the weak equivalences, so we can use the same for $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{inj}}$ and $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{proj}}$.

We start with the direction (3) $\Rightarrow$ (1). Let $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ be a $\mathbf{K}_{\mathrm{inj}}$-local equivalence, that is, a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{inj}}^{+}$. Then σ will be a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{proj}}^{\mathbf{c}}$, by definition, if for all $\mathbf{K}_{\mathrm{proj}}$-local object $\mathcal{E}$ the induced map of simplicial sets $\sigma^{\star} : \mathrm{Map}(\mathcal{G}, \mathcal{E}) \longrightarrow \mathrm{Map}(\mathcal{F}, \mathcal{E})$ is weak equivalence.

Let $\mathcal{E}$ be a $\mathbf{K}_{\mathrm{proj}}$-local object. By definition $\mathcal{E}$ is fibrant in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{proj}}$ and for any $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathbf{K}_{\mathrm{proj}}$ then $\alpha^{\star} : \mathrm{Map}(\mathcal{B}, \mathcal{E}) \longrightarrow \mathrm{Map}(\mathcal{A}, \mathcal{E})$ is a weak equivalence. As $\mathcal{E}$ is not fibrant in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{inj}}$ we have to introduce a fibrant replacement $\widetilde{\mathcal{E}}$ in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{inj}}$; we have a weak equivalence $q : \mathcal{E} \longrightarrow \widetilde{\mathcal{E}}$. But fibrant objects in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{inj}}$ are also fibrant in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{proj}}$, therefore $\widetilde{\mathcal{E}}$ is fibrant in $\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{proj}}$.

Claim. $\widetilde{\mathcal{E}}$ is also $\mathbf{K}_{\mathrm{proj}}$-local and $\mathbf{K}_{\mathrm{inj}}$-local.

The fact that $\widetilde{\mathcal{E}}$ is $\mathbf{K}_{\mathrm{proj}}$-local is classical: $\mathbf{K}_{\mathrm{proj}}$-locality is invariant under weak equivalences of fibrant objects. In fact for any $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathbf{K}_{\mathrm{proj}}$, the following commutes:

$$\begin{tikzcd} \mathrm{Map}(\mathcal{B}, \mathcal{E}) \arrow[r, "\alpha^{\star}", "\sim"'] \arrow[d, "q^{\star}"', "\wr"] & \mathrm{Map}(\mathcal{A}, \mathcal{E}) \arrow[d, "q^{\star}", "\wr"'] \\ \mathrm{Map}(\mathcal{B}, \widetilde{\mathcal{E}}) \arrow[r, "\alpha^{\star}"'] & \mathrm{Map}(\mathcal{A}, \widetilde{\mathcal{E}}) \end{tikzcd}$$

The maps $\alpha^{\star} : \mathrm{Map}(\mathcal{B}, \mathcal{E}) \longrightarrow \mathrm{Map}(\mathcal{A}, \mathcal{E})$ is a weak equivalences of simplicial sets by hypothesis; the two vertical maps $q^{\star}$ are also weak equivalences (see [40, Thm 17.7.7]). Therefore by 3 for 2 in the model category of simplicial sets the other map $\alpha^{\star} : \mathrm{Map}(\mathcal{B}, \widetilde{\mathcal{E}}) \longrightarrow \mathrm{Map}(\mathcal{A}, \widetilde{\mathcal{E}})$ is also a weak equivalence, which proves that $\widetilde{\mathcal{E}}$ is $\mathbf{K}_{\mathrm{proj}}$-local.

To prove that $\widetilde{\mathcal{E}}$ is $\mathbf{K}_{\mathrm{inj}}$-local we have to show that for all $\alpha \in \mathbf{K}_{\mathrm{inj}}$ then

$$\alpha^{\star} : \mathrm{Map}(\mathcal{B}, \widetilde{\mathcal{E}}) \longrightarrow \mathrm{Map}(\mathcal{A}, \widetilde{\mathcal{E}})$$

is a weak equivalence i.e $\widetilde{\mathcal{E}}$ is α-local. If $\alpha \in \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\mathrm{inj}}}$ then α is in particular an old weak equivalence so $\widetilde{\mathcal{E}}$ is α-local by [40, Thm 17.7.7] (any object is local with respect to any weak equivalence). It remains the case where $\alpha = \mathcal{P}_{t!} h_{/V}$.

On the one hand since $\mathcal{P}_{t!}(\ell)$ is a weak equivalence, we have by [40, Thm 17.7.7] again that $\widetilde{\mathcal{E}}$ is $\mathcal{P}_{t!}(\ell)$-local. On the other hand since $\widetilde{\mathcal{E}}$ is $\mathbf{K}_{\mathrm{proj}}$-local, it's in particular $\mathcal{P}_{t!}\zeta(h)$-local. It follows that $\widetilde{\mathcal{E}}$ is $\{\mathcal{P}_{t!}(\ell) \circ \mathcal{P}_{t!}\zeta(h)\}$-local (as weak equivalences of simplicial sets are closed under composition); but $\mathcal{P}_{t!}(\ell) \circ \mathcal{P}_{t!}\zeta(h) = \mathcal{P}_{t!} h_{/V}$, thus $\widetilde{\mathcal{E}}$ is $\mathcal{P}_{t!} h_{/V}$-local. Summing up this, one has that $\widetilde{\mathcal{E}}$ is also $\mathbf{K}_{\mathrm{inj}}$-local.

Claim. A $\mathbf{K}_{\mathrm{inj}}$-local equivalence is also a $\mathbf{K}_{\mathrm{proj}}$-local equivalence.

By the above if σ is a $\mathbf{K}_{\text{inj}}$-local equivalence then $\sigma^\star : \mathrm{Map}(\mathcal{G}, \widetilde{\mathcal{E}}) \longrightarrow \mathrm{Map}(\mathcal{F}, \widetilde{\mathcal{E}})$ is a weak equivalence of simplicial sets. If we put this in the commutative diagram below:

$$\begin{tikzcd} \mathrm{Map}(\mathcal{G}, \mathcal{E}) \arrow[r, "\sigma^\star"] \arrow[d, "q^\star"', "\wr"] & \mathrm{Map}(\mathcal{F}, \mathcal{E}) \arrow[d, "q^\star", "\wr"'] \\ \mathrm{Map}(\mathcal{G}, \widetilde{\mathcal{E}}) \arrow[r, "\sim", "\sigma^\star"'] & \mathrm{Map}(\mathcal{F}, \widetilde{\mathcal{E}}) \end{tikzcd}$$

then all three maps are weak equivalences, therefore by 3 for 2 the map $\sigma^\star : \mathrm{Map}(\mathcal{G}, \mathcal{E}) \longrightarrow \mathrm{Map}(\mathcal{F}, \mathcal{E})$ is also a weak equivalence. Consequently σ is also a $\mathbf{K}_{\text{proj}}$-local equivalence, as claimed.

For the direction (1) $\Rightarrow$ (3) the proof is the same. Let $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ be a $\mathbf{K}_{\text{proj}}$-local equivalence. By definition for any $\mathbf{K}_{\text{proj}}$-local object $\mathcal{E}$, $\sigma^\star : \mathrm{Map}(\mathcal{G}, \mathcal{E}) \longrightarrow \mathrm{Map}(\mathcal{F}, \mathcal{E})$ is a weak equivalence.

Claim. If $\mathcal{E}$ is a $\mathbf{K}_{\text{inj}}$-local object then it is also a $\mathbf{K}_{\text{proj}}$-local object.

In fact if $\mathcal{E}$ is $\mathbf{K}_{\text{inj}}$-local, $\mathcal{E}$ is fibrant in $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ (hence in $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$) and the map

$$\alpha^\star : \mathrm{Map}(\mathcal{B}, \widetilde{\mathcal{E}}) \longrightarrow \mathrm{Map}(\mathcal{A}, \widetilde{\mathcal{E}})$$

is a weak equivalence for all $\alpha \in \mathbf{K}_{\text{inj}}$.

Recall that $\mathbf{K}_{\text{proj}} = \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}} \cup (\coprod_{t \in \mathbb{S}_{\overline{X}}, \mathbf{deg}(t)>1} \mathcal{P}_{t!}\zeta(\mathbf{I}))$.

If $\alpha \in \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}}$ then α is in particular an old weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ and $\mathscr{M}_{\mathbb{S}}(X)_{\text{inj}}$ so $\mathcal{E}$ is automatically α-local by [40, Thm 17.7.7].

Assume that $\alpha = \mathcal{P}_{t!}\zeta(h)$. By applying $\mathrm{Map}(-, \mathcal{E})$ to the equality $\mathcal{P}_{t!}(\ell) \circ \mathcal{P}_{t!}\zeta(h) = \mathcal{P}_{t!}h_{/V}$ we get $\mathcal{P}_{t!}\zeta(h)^\star \circ \mathcal{P}_{t!}(\ell)^\star = \mathcal{P}_{t!}h^\star_{/V}$; and since $\mathcal{E}$ is $\mathbf{K}_{\text{inj}}$-local, it is in particular $\mathcal{P}_{t!}\zeta(h)$-local, thus $\mathcal{P}_{t!}\zeta(h)^\star$ is a weak equivalence.

As $\mathcal{P}_{t!}(\ell)$ is an old weak equivalence then $\mathcal{E}$ is $\mathcal{P}_{t!}(\ell)$-local by [40, Thm 17.7.7] which means that $\mathcal{P}_{t!}(\ell)^\star$ is a weak equivalence; putting these together we conclude by 3 for 2 that $\mathcal{P}_{t!}\zeta(h)^\star$ is a weak equivalence and $\mathcal{E}$ is α-local.

Now if σ is a $\mathbf{K}_{\text{proj}}$-local equivalence, by the above for any $\mathbf{K}_{\text{inj}}$-local object $\mathcal{E}$ the map

$$\sigma^\star : \mathrm{Map}(\mathcal{G}, \mathcal{E}) \longrightarrow \mathrm{Map}(\mathcal{F}, \mathcal{E})$$

is also a weak equivalence which means precisely that σ is a $\mathbf{K}_{\text{inj}}$-local equivalence. ■

3.9.4 A new fibered model structure on $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$

In the following we want to vary the set X when $\mathscr{M}_{\mathbb{S}}(X)$ is equipped with the Bousfield localization with respect to $\mathbf{K}_{\text{proj}}$ constructed previously.

We will use the following notation.

Notation 3.9.4.
From now we will specify by $\mathbf{K}(X)_{\text{proj}}$, $\mathbf{K}(Y)_{\text{proj}}$ the corresponding sets.
$\mathscr{M}_{\mathbb{S}}(X)^+_{\text{proj}}$ = the Bousfield localization of $\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}$ with respect to $\mathbf{K}(X)_{\text{proj}}$, for a set X.
$\mathcal{L}_X : \mathscr{M}_{\mathbb{S}}(X)_{\text{proj}} \longrightarrow \mathscr{M}_{\mathbb{S}}(X)^+_{\text{proj}}$ = the canonical left Quillen functor.

Recall that for any function $f : X \longrightarrow Y$ there is a Quillen adjunction

$$f_! : \mathscr{M}_{\mathbb{S}}(X)_{\text{proj}} \rightleftarrows \mathscr{M}_{\mathbb{S}}(Y)_{\text{proj}} : f^\star$$

where $f_!$ is left Quillen and $f^\star$ is right Quillen.

Proposition 3.9.23. *For any sets X, Y and any function $f : X \longrightarrow Y$ there is an induced Quillen adjunction, denoted again $f_! \dashv f^\star$:*

$$f_! : \mathscr{M}_{\mathbb{S}}(X)^+_{proj} \leftrightarrows \mathscr{M}_{\mathbb{S}}(Y)^+_{proj} : f^\star$$

and the following commutes:

$$\begin{tikzcd}
\mathscr{M}_{\mathbb{S}}(X)_{proj} \arrow[r, shift right, "f_!"'] \arrow[d, "\mathcal{L}_X"] & \mathscr{M}_{\mathbb{S}}(Y)_{proj} \arrow[l, shift right, "f^\star"'] \arrow[d, "\mathcal{L}_Y"] \\
\mathscr{M}_{\mathbb{S}}(X)^+_{proj} \arrow[r, shift right, "f_!"'] & \mathscr{M}_{\mathbb{S}}(Y)^+_{proj} \arrow[l, shift right, "f^\star"']
\end{tikzcd}$$

Sketch of proof. We will show that $f_!(\mathbf{K}(X)_{\text{proj}}) \subset \mathbf{K}(Y)_{\text{proj}}$. The new '$f_!$' will be induced by universal property of the Bousfield localization.

Consider α in $\mathbf{K}(X)_{\text{proj}} = \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}} \cup (\coprod_{t \in \mathbb{S}_{\overline{X}}, \mathbf{deg}(t)>1} \mathcal{P}_{t!}\zeta(\mathbf{I}))$.

Claim. If $\alpha \in \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}}$ then $f_!(\alpha) \in \mathbf{J}_{\mathscr{M}_{\mathbb{S}}(Y)_{\text{proj}}} \subset \mathbf{K}(Y)_{\text{proj}}$.

To see this first recall that $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(X)_{\text{proj}}} = \Gamma \mathbf{J}_{\mathcal{K}_{X\text{-proj}}}$, where $\mathcal{K}_X = \prod_{(A,B)\in X^2} \text{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$. Furthermore there is also a Quillen adjunction $f_! : \mathcal{K}_{X\text{-proj}} \leftrightarrows \mathcal{K}_{Y\text{-proj}} : f^\star$; and by definition we have:

$$\mathbf{J}_{\mathcal{K}_{X\text{-proj}}} = \coprod_{t \in \mathbb{S}_{\overline{X}}} \{\mathbf{F}^t_h, h \in \mathbf{J}\},$$

where $\mathbf{F}^t$ is the left adjoint of the evaluation at t. It suffices then to show that $f_!(\mathbf{J}_{\mathcal{K}_{X\text{-proj}}}) \subset \mathbf{J}_{\mathcal{K}_{Y\text{-proj}}}$.

Using the adjunction one establishes that for any $m \in \mathscr{M}$ and any $\mathcal{G} \in \mathcal{K}_Y$:

$$\begin{aligned}
\text{Hom}(f_!\mathbf{F}^t_m, \mathcal{G}) &\cong \text{Hom}(\mathbf{F}^t_m, f^\star \mathcal{G}) \\
&\cong \text{Hom}(m, (f^\star \mathcal{G})(t)) \\
&= \text{Hom}(m, \mathcal{G} f(t)) \\
&\cong \text{Hom}(\mathbf{F}^{f(t)}_m, \mathcal{G})
\end{aligned}$$

Consequently $f_!\mathbf{F}^t_m \cong \mathbf{F}^{f(t)}_m$; similarly $f_!\mathbf{F}^t_\alpha \cong \mathbf{F}^{f(t)}_\alpha$ (we can actually assume that we have an equality). One clearly has that $f_!(\mathbf{J}_{\mathcal{K}_{X\text{-proj}}}) \subset \mathbf{J}_{\mathcal{K}_{Y\text{-proj}}}$ and the claim holds.

Claim. For every t and every $h \in \mathbf{I}$ then $f_!\mathcal{P}_{t!}\zeta(h) \cong \mathcal{P}_{f(t)!}\zeta(h)$

This also holds by the adjunction. For any h and any $\mathcal{G} \in \mathscr{M}_{\mathbb{S}}(Y)$ one has:

$$\begin{aligned}
\text{Hom}(f_!\mathcal{P}_{t!}\zeta(h), \mathcal{G}) &\cong \text{Hom}(\mathcal{P}_{t!}\zeta(h), f^\star \mathcal{G}) \\
&\cong \text{Hom}(\zeta(h), (f^\star \mathcal{G})(u_t)) \\
&= \text{Hom}(\zeta(h), \mathcal{G}\underbrace{f(u_t)}_{=u_{f(t)}}) \\
&= \text{Hom}(\zeta(h), \mathcal{G}u_{f(t)}) \\
&\cong \text{Hom}(\mathcal{P}_{f(t)!}\zeta(h), \mathcal{G})
\end{aligned}$$

and the claim follows. If we combine the two claims we have the desired result ■

By virtue of the Roig-Stanculescu result (Theorem 3.8.7) and under Hypothesis (3.9.1) we have

Theorem 3.9.24. *For a symmetric closed monoidal model category $\mathscr{M}$ whose objects are all cofibrant, the category $\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})$ has a Quillen model structure where a map $\sigma = (f, \sigma) : \mathcal{F} \longrightarrow \mathcal{G}$ is*

1. *a weak equivalence if $f : X \longrightarrow Y$ is an isomorphism of sets and $\sigma : \mathcal{F} \longrightarrow f^{\star}\mathcal{G}$ is a weak equivalence in $\mathscr{M}_{\mathbb{S}}(X)^{+}_{proj}$,*
2. *a cofibration if the adjoint map $\widetilde{\sigma} : f_{!}\mathcal{F} \longrightarrow \mathcal{G}$ is a cofibration in $\mathscr{M}_{\mathbb{S}}(Y)^{+}_{proj}$,*
3. *a fibration if $\sigma : \mathcal{F} \longrightarrow f^{\star}\mathcal{G}$ is a fibration in $\mathscr{M}_{\mathbb{S}}(X)^{+}_{proj}$.*
4. *fibrant objects are co-Segal categories*

We will denote by $\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})^{+}_{proj}$ the new model structure on $\mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})$. There is a canonical left Quillen functor

$$\mathcal{L} : \mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})_{proj} \longrightarrow \mathscr{M}_{\mathbb{S}}(\boldsymbol{Set})^{+}_{proj}$$

whose component over X is $\mathcal{L}_X : \mathscr{M}_{\mathbb{S}}(X)_{proj} \longrightarrow \mathscr{M}_{\mathbb{S}}(X)^{+}_{proj}$.

Proof. The proof is the same as for Theorem 3.8.8. On **Set** we take the minimal model structure: cofibrations and fibrations are all maps, weak equivalences are isomorphisms. One can check easily that all the conditions of Theorem 3.8.7 are fulfilled; this gives the model structure described above.

For $\mathcal{F} \in \mathscr{M}_{\mathbb{S}}(X)^{+}_{\text{proj}}$, $\mathcal{F}$ is fibrant if the canonical map $\jmath : \mathcal{F} \longrightarrow *$ is a fibration $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}$, where $*$ is the terminal object therein. By definition this is equivalent to $\jmath : \mathcal{F} \longrightarrow \jmath^{\star}*$ being a fibration in $\mathscr{M}_{\mathbb{S}}(X)^{+}_{\text{proj}}$; and it's not hard to see that $\jmath^{\star}$ is the terminal object in $\mathscr{M}_{\mathbb{S}}(X)$. Moreover for every X the terminal object in $\mathscr{M}_{\mathbb{S}}(X)$ is automatically fibrant in $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}$.

Summing this up, one has that $\mathcal{F}$ is fibrant in $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}$ if and only if it is fibrant in $\mathscr{M}_{\mathbb{S}}(X)^{+}_{\text{proj}}$, therefore $\mathcal{F}$ is a co-Segal category by Theorem 3.9.21. ■

Cofibrantly generated

Since the cofibrations in $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})_{\text{proj}}$ and $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}$ are the same, the set $\mathbf{I}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})_{\text{proj}}}$ constitutes also a generating set for the cofibrations in $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}$. Using the fact that $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}$ is already a model category which is locally presentable, one can easily check that the set $\mathbf{I}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})_{\text{proj}}}$ and the class of weak equivalences of $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}$ satisfy the conditions of Smith's recognition theorem ([9, Proposition 2.2]).

By Smith's theorem we have a combinatorial model structure on $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})_{\text{proj}}$ with the same cofibrations and weak equivalences of $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}$. The set $\mathbf{I}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})_{\text{proj}}}$ constitutes a generating set for the cofibrations and there exists a set $\mathbf{J}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}}$ which is a set of generating trivial cofibrations.

But since this new model category has the same cofibrations and weak equivalences (hence the same fibrations) as $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}$ we deduce that this new model structure is in fact isomorphic to $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}$; thus $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})^{+}_{\text{proj}}$ is combinatorial and in particular cofibrantly generated.

The monoidal category $(\mathscr{M}_{\mathbb{S}}(\mathbf{Set}), \otimes_{\mathbb{S}}, \mathbb{I})$

Given a small category $\mathcal{C}$, by construction there is a degree (or length) strict 2-functor $\mathbf{deg}: \mathbb{S}_{\mathcal{C}} \longrightarrow \mathbb{S}_{\mathbf{1}}$ where $\mathbf{1}$ is the unit category and $\mathbb{S}_{\mathbf{1}} \cong (\Delta^+_{\text{epi}}, +, \mathbf{0})$. If $\mathcal{D}$ is another category we can form the genuine fiber product of 2-categories $\mathbb{S}_{\mathcal{C}} \times_{\mathbb{S}_{\mathbf{1}}} \mathbb{S}_{\mathcal{D}}$.

Proposition 3.9.25. *There is an isomorphism of* 2*-categories:* $\mathbb{S}_{\mathcal{C}\times\mathcal{D}} \cong \mathbb{S}_{\mathcal{C}} \times_{\mathbb{S}_{\mathbf{1}}} \mathbb{S}_{\mathcal{D}}$.

Proof. Obvious. ∎

Tensor product of $\mathbb{S}$-diagrams

Let $\mathscr{M} = (\underline{M}, \otimes, I)$ be a symmetric monoidal category. Let $\mathbb{I} : (\Delta^+_{\text{epi}}, +)^{op} \longrightarrow \mathscr{M}$ be the constant lax functor of value I and Id_I; the laxity maps are the obvious natural isomorphism $I \otimes I \cong I$. The co-Segal diagram $\mathbb{I}$ exhibits I in a tautological way as a (semi) monoid.

Given $\mathcal{F} : (\mathbb{S}_{\mathcal{C}})^{2\text{-op}} \longrightarrow \mathscr{M}$ and $\mathcal{G} : (\mathbb{S}_{\mathcal{D}})^{2\text{-op}} \longrightarrow \mathscr{M}$ we define $\mathcal{F} \otimes_{\mathbb{S}} \mathcal{G} : (\mathbb{S}_{\mathcal{C}\times\mathcal{D}})^{2\text{-op}} \longrightarrow \mathscr{M}$ to be the lax functor given as follows.

1. For 1-morphisms $(s, s') \in (\mathbb{S}_{\mathcal{C}\times\mathcal{D}})$ we set $(\mathcal{F} \otimes_{\mathbb{S}} \mathcal{G})(s, s') := \mathcal{F}(s) \otimes \mathcal{G}(s')$,
2. The laxity map $\varphi_{\mathcal{F}\otimes_{\mathbb{S}}\mathcal{G}} : (\mathcal{F} \otimes_{\mathbb{S}} \mathcal{G})(s, s') \otimes (\mathcal{F} \otimes_{\mathbb{S}} \mathcal{G})(t, t') \longrightarrow (\mathcal{F} \otimes_{\mathbb{S}} \mathcal{G})(s \otimes t, s' \otimes t')$ is obtained as the composite:
$$\mathcal{F}(s) \otimes \mathcal{G}(s') \otimes \mathcal{F}(t) \otimes \mathcal{G}(t') \xrightarrow{\mathrm{Id} \otimes sym \otimes \mathrm{Id}} \mathcal{F}(s) \otimes \mathcal{F}(t) \otimes \mathcal{G}(s') \otimes \mathcal{G}(t') \xrightarrow{\varphi_{\mathcal{F}} \otimes \varphi_{\mathcal{G}}} \mathcal{F}(s \otimes t) \otimes \mathcal{G}(s' \otimes t')$$
where sym is the symmetry isomorphism in $\mathscr{M}$ (we have $sym : \mathcal{G}(s') \otimes \mathcal{F}(t) \xrightarrow{\cong} \mathcal{F}(t) \otimes \mathcal{G}(s')$).
3. One easily sees that if $f : \mathcal{C}' \longrightarrow \mathcal{C}$ and $g : \mathcal{D}' \longrightarrow \mathcal{D}$ then $(f \times g)^{\star}\mathcal{F} \otimes_{\mathbb{S}} \mathcal{G} \cong f^{\star}\mathcal{F} \otimes_{\mathbb{S}} g^{\star}\mathcal{G}$.
4. If $\sigma = (\sigma, f) \in \mathrm{Hom}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}(\mathcal{F}, \mathcal{G})$ and $\gamma = (\gamma, g) \in \mathrm{Hom}(\mathcal{F}', g^{\star}\mathcal{G}')$ we define
$$\sigma \otimes_{\mathbb{S}} \gamma = (\sigma \otimes \gamma, f \times g) \in \mathrm{Hom}_{\mathscr{M}_{\mathbb{S}}(\mathbf{Set})}[\mathcal{F} \otimes_{\mathbb{S}} \mathcal{G}, \mathcal{F}' \otimes_{\mathbb{S}} \mathcal{G}']$$
to be the morphism whose component at (s, s') is $\sigma_s \otimes \sigma_{s'}$.

We leave the reader to check that:

1. $\otimes_{\mathbb{S}}$ is a bifunctor and is associative,
2. we have a canonical symmetry: $\mathcal{F} \otimes_{\mathbb{S}} \mathcal{G} \cong \mathcal{G} \otimes_{\mathbb{S}} \mathcal{F}$,
3. for any $\mathcal{F}$ we have a natural isomorphism $\mathcal{F} \otimes_{\mathbb{S}} \mathbb{I} \cong \mathcal{F}$.

3.10 A model structure for $\mathscr{M}$-Cat for a 2-category $\mathscr{M}$

In the following $\mathscr{M}$ is a 2-category. We will use capital letters U, V, W for the objects of $\mathscr{M}$ and f, g, h, k for 1-morphisms and α, β, γ for 2-morphisms. For $U, V \in \mathrm{Ob}(\mathscr{M})$ we will write $\mathscr{M}_{UV}$ the category of morphisms from U to V; when $U = V$ we will simply write $\mathscr{M}_U$. If f, g are composable 1-morphisms, we will denote by $g \otimes f$ the horizontal composite. Similarly for 2-morphisms α, β we will write $\beta \otimes \alpha$ the horizontal composite while $\beta \circ \alpha$ will represents the vertical composite.

Definition 3.10.1. *Let $\mathscr{M}$ be* 2*-category. We will say that $\mathscr{M}$ is* ***locally a model category*** *if the following conditions hold:*

1. *Each $\mathscr{M}_{UV}$ is a model category in the usual sense.*

2. *$\mathscr{M}$ is a biclosed that is: for every 1-morphism f the following functors have right adjoints:*

$$f \otimes - \qquad\qquad - \otimes f$$

3. *The pushout-product axiom holds: given two cofibrations $\alpha : f \longrightarrow g \in Arr(\mathscr{M}_{UV})$ and $\beta : k \longrightarrow h \in Arr(\mathscr{M}_{VW})$ then the induced map $\beta \Box \alpha : h \otimes f \cup_{k \otimes f} k \otimes g \longrightarrow h \otimes g$ is a cofibration in $\mathscr{M}_{UW}$ which is moreover a trivial cofibration if either α or β is .*

4. *For every U and any 1-morphism $f \in \mathscr{M}_{UV}$, if $Q(\mathrm{Id}_U) \longrightarrow \mathrm{Id}_U$ is a cofibrant replacement then the induced map $Q(\mathrm{Id}_U) \otimes f \longrightarrow f$ is a weak equivalence in $\mathscr{M}_{UV}$.*

As one can see this is a straightforward generalization of a monoidal model category considered as a 2-category with one object. Condition (2) allows us to distribute colimits with respect to the composition on each factor.

We recall briefly below the definition of a category enriched over a 2-category $\mathscr{M}$.

An $\mathscr{M}$-category $\mathscr{X}$ consists roughly speaking of:

1. for each object U of $\mathscr{M}$, a set $\mathscr{X}_U$ of objects *over* U;
2. for objects A, B over U, V respectively, an arrow $\mathscr{X}(A, B) : U \longrightarrow V \in \mathscr{M}_{UV}$;
3. for each object A over U, a 2-cell $I_A : \mathrm{Id}_U \Longrightarrow \mathscr{X}(A, A) \in \mathscr{M}_U$;
4. for object A, B, C over U, V, W, respectively, a 2-cell $c_{ABC} : \mathscr{X}(B, C) \otimes \mathscr{X}(A, B) \Longrightarrow \mathscr{X}(A, C) \in \mathscr{M}_{UW}$ satisfying the obvious three axioms of left and right identities and associativity.

Equivalently $\mathscr{X}$ can be defined as a lax morphism $\mathscr{X} : \overline{X} \longrightarrow \mathscr{M}$ or a strict homomorphism $\mathscr{X} : \mathscr{P}_{\overline{X}} \longrightarrow \mathscr{M}$ (see [3]). Note that for each U, we have a category $\mathscr{X}_{|U}$ enriched over the monoidal category $\mathscr{M}_U$; the set of objects of $\mathscr{X}_{|U}$ is $\mathscr{X}_U$.

Given two $\mathscr{M}$-categories $\mathscr{X}$ and $\mathscr{Y}$ an $\mathscr{M}$-functor is given by the following data:

1. a function $\Phi : \mathrm{Ob}(\mathscr{X}) \longrightarrow \mathrm{Ob}(\mathscr{Y})$, $\Phi = \coprod_U \Phi_U$ with $\Phi_U : \mathscr{X}_U \longrightarrow \mathscr{Y}_U$;
2. for A, B in $\mathscr{X}$ over U, V, respectively we have a morphism $\Phi_{AB} : \mathscr{X}(A, B) \longrightarrow \mathscr{Y}(\Phi A, \Phi B)$ in $\mathscr{M}_{UV}$
3. for each object A over U we have $I_{\Phi A} = \Phi_{AA} \circ I_A$; satisfying the obvious compatibility with respect to the composition on both sides.

$\mathscr{M}$-categories with $\mathscr{M}$-functors form the category $\mathscr{M}$-**Cat**. There is an obvious category $\mathscr{M}$-**Graph** whose objects are $\mathscr{M}$-graphs and morphisms are just the natural ones.

We have a forgetful functor just like in the monoidal case $\mathcal{U} : \mathscr{M}\text{-}\mathbf{Cat} \longrightarrow \mathscr{M}\text{-}\mathbf{Graph}$.

Remark 3.10.1. As usual there is a restriction $\Phi_{|U} : \mathscr{X}_{|U} \longrightarrow \mathscr{Y}_{|U}$ of Φ, which is a $\mathscr{M}_U$-functor. And any $\mathscr{M}$-functor has an underlying morphism between the corresponding $\mathscr{M}$-graphs.

Proposition 3.10.2. *Let $\mathscr{M}$ be a biclosed 2-category which is locally* ***locally-presentable*** *that is: each $\mathscr{M}_{UV}$ is locally presentable in the usual sense. Then $\mathscr{M}$-***Cat*** is locally presentable.*

Proof. All is proved in the same manner as for a monoidal category $\mathscr{M}$. Below we list the different steps:

1. First one shows that $\mathscr{M}$-**Graph** is cocomplete. This is easy: just apply the same method as Wolff [97].

2. $\mathcal{U}$ is monadic: construct a left adjoint of $\mathcal{U}$ with the same formula given in [97]. Then show that $\mathscr{M}$-**Cat** has coequalizers of parallel $\mathcal{U}$-split pairs, again using the same idea in *loc. cit.* As $\mathcal{U}$ clearly reflects isomorphisms, it follows by Beck monadicity theorem that $\mathcal{U}$ is monadic.

3. Linton's result [63, Corollary 2] applies and we have: $\mathscr{M}$-**Cat** is cocomplete as well.

4. Following the same method as Kelly and Lack [50] one has that the monad induced by $\mathcal{U}$ preserves filtered colimits and $\mathscr{M}$-**Graph** is locally presentable. From this we apply [1, Remark 2.78] to establish that $\mathscr{M}$-**Cat** is also locally presentable.

Note that being biclosed is essential in order to permute (filtered) colimits and $\otimes$. ■

Terminology. Let W be a class of 2-morphisms in $\mathscr{M}$. An $\mathscr{M}$-functor $\Phi : \mathscr{X} \longrightarrow \mathscr{Y}$ is said to be locally in W if for every pair of objects A, B of $\mathscr{X}$ the 2-morphism $\Phi_{AB} : \mathscr{X}(A,B) \longrightarrow \mathscr{Y}(\Phi A, \Phi B)$ is in W. We will say the same thing for a morphism between $\mathscr{M}$-graphs.

The category $\mathscr{M}$- $\mathbf{Cat}(X)$

For each $U \in \mathscr{M}$ let's fix a set X_U of objects over U and consider $X = \coprod_U X_U$. Denote by $\mathscr{M}$- $\mathbf{Cat}(X)$ the category of $\mathscr{M}$-categories with fixed set of objects X and $\mathscr{M}$-functors which fix X. Similarly there is a category $\mathscr{M}$- $\mathbf{Graph}(X)$ of $\mathscr{M}$-graphs with vertices X and morphisms fixing X.

Just like in the case where $\mathscr{M}$ is a monoidal category there is a tensor product in $\mathscr{M}$- $\mathbf{Graph}(X)$ defined as follows. If $\mathscr{X}, \mathscr{Y} \in \mathscr{M}$- $\mathbf{Graph}(X)$ one defines $\mathscr{X} \otimes_X \mathscr{Y}$ by:

$$(\mathscr{X} \otimes_X \mathscr{Y})(A,B) = \coprod_{Z \in X} \mathscr{X}(A,Z) \otimes \mathscr{Y}(Z,B).$$

The unit of this product is the $\mathscr{M}$-graph $\mathscr{I}$ given by

$$\mathscr{I}(A,B) = \begin{cases} \mathrm{Id}_U & \text{if } A \text{ and } B \text{ are over the same object } U \\ \varnothing = \text{initial object in } \mathscr{M}_{UV} & \text{If } A \text{ over } U\text{, and } B \text{ over } V \text{ with } U \neq V \end{cases}$$

As usual it's not hard to see that $\mathscr{M}$- $\mathbf{Cat}(X)$ is the category of monoids of ($\mathscr{M}$- $\mathbf{Graph}(X), \otimes_X, \mathscr{I}$). We have an obvious isomorphism of categories:

$$\mathscr{M}\text{- }\mathbf{Graph}(X) \cong \prod_{(U,V) \in \mathrm{Ob}(\mathscr{M})^2} \mathscr{M}_{UV}^{(X_U \times X_V)}$$

where $\mathscr{M}_{UV}^{(X_U \times X_V)} = \mathrm{Hom}(X_U \times X_V, \mathscr{M}_{UV})$. From this we can endow $\mathscr{M}$- $\mathbf{Graph}(X)$ with the product model structure. In this model structure, fibrations, cofibrations and weak equivalences are simply component wise such morphism.

Theorem 3.10.3. *Let $\mathscr{M}$ be a 2-category which is locally a model category and locally cofibrantly generated. Assume moreover that all the objects of $\mathscr{M}$ are cofibrant. Then we have:*

1. *the category $\mathscr{M}$-$\mathbf{Cat}(X)$ admits a model structure which is cofibrantly generated.*

2. *if $\mathscr{M}$ is combinatorial, then so is $\mathscr{M}$-$\mathbf{Cat}(X)$*

Proof. This is a special case of Theorem 3.3.12 where $\mathcal{O} = \mathcal{O}_X$, $\mathcal{C}_\bullet = \overline{X}$ and $\mathcal{M}_\bullet = \mathscr{M}$ ■

Remark 3.10.2. One can remove the hypothesis 'all the objects of $\mathscr{M}$ are cofibrant' by using an analogue of the monoid axiom of [77]. In fact one can use the method in *loc.cit* to establish the theorem. Lurie [66] also presented a nice description of the model structure for the case where $\mathscr{M}$ is the monoidal category of simplicial sets. It seems obvious that we can adapt his method to calculate the pushout of interest in our case.

Appendices

4.1 Some classical lemmas

The first lemma we present is a classical result of category theory concerning the universal property of a pushout diagram. We include this part for completeness.

Warning. In this discussion κ is a regular uncountable cardinal and all 'sets' that will be considered in the sequel are assumed to have a cardinality $< \kappa$: this is what we mean by being a *κ-small* set.

Definition 4.1.1. *Let $\mathcal{C}$ be a small category and $f : A \longrightarrow B$, $g : A \longrightarrow C$ be two morphisms of $\mathcal{C}$ with the same source A.*

A pushout *of $\langle f, g\rangle$ is a commutative square:* $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "u"] \\ C \arrow[r, "v"] & R \end{tikzcd}$ *such that for any commutative square* $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & S \end{tikzcd}$ *there exists a unique morphism $t : R \longrightarrow S$ such that $h = t \circ u$ and $k = t \circ v$.*

Notation 4.1.1. To stress the fact a commutative square is a pushout square we will put the symbol '$\lrcorner$' at the center of the diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "u"] \\ C \arrow[r, "v"] & R \end{tikzcd}$$

Observations 4.1.1. It follows from the universal property of the pushout that if a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & S \end{tikzcd}$$

is also a pushout of $\langle f, g\rangle$ then the unique map $t : R \longrightarrow S$ we get from the definition is an isomorphism.

In this situation, up to composition by the morphism t, we can assume that $R = S$, $h = u$ and $k = v$.

Terminology. The map $v : C \longrightarrow R$ is said to be 'the pushout of f along g' and by symmetry v is the pushout of g along f.

Using again the universal property of the pushout, we get the following lemma which says that '*a pushout of a pushout is a pushout*'.

Lemma 4.1.2. *Let $\mathcal{C}$ be a small category. Given two commutative squares in $\mathcal{C}$:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "u"] \\ C \arrow[r, "v"] & R \end{tikzcd} \quad \begin{tikzcd} B \arrow[r, "p"] \arrow[d, "u"'] & D \arrow[d, "w"] \\ R \arrow[r, "q"] & U \end{tikzcd}$$

If the two are pushout squares then the 'composite square':

$$\begin{tikzcd} A \arrow[r, "p\circ f"] \arrow[d, "g"'] & D \arrow[d, "w"] \\ C \arrow[r, "q\circ v"] & U \end{tikzcd}$$

is also a pushout square.

Proof. Elementary. ∎

Remark 4.1.1. If the set $\mathrm{Mor}(\mathcal{C})$ of all morphisms of $\mathcal{C}$ is κ-small, then the previous lemma can be applied for any set of consecutive pushout squares indexed by an ordinal β with $\beta < \kappa$.

Definition 4.1.3. *Let K be set of cardinality $|K| < \kappa$ and $\mathbb{1} = \{\boldsymbol{O}, \boldsymbol{O} \xrightarrow{Id_{\boldsymbol{O}}} \boldsymbol{O}\}$ be the terminal category. We identify K with the discrete category whose set of objects is K.*
*The **cone** associated to the set K is the category $\epsilon(K)$ described as follows.*
$Ob(\epsilon(K)) = K \sqcup \{\boldsymbol{O}\}$ and for $x, y \in Ob(\epsilon(K))$ we have

$$\epsilon(K)(x,y) = \begin{cases} \{(\boldsymbol{O}, y)\} & \text{if } x = \boldsymbol{O} \text{ and } y \in K \\ \{\mathrm{Id}_x\} & \text{if } x = y \\ \varnothing & \text{otherwise} \end{cases}$$

The composition is the unique one.

Remark 4.1.2. Our notation '$\epsilon(K)$' is inspired by the category $\epsilon(n)$ used by Simpson (see [79] 13.1). In fact if K is a set of cardinality n then $\epsilon(K)$ is isomorphic to $\epsilon(n)$.
Following the terminology in [3], $\epsilon(K)$ is the <u>thin</u> bridge from the $\mathbb{1}$ to K which was denoted therein by '$\mathbb{1} < K$'.

With the category $\epsilon(K)$ we can give a general definition.

Definition 4.1.4. *Let $\mathscr{M}$ be a category and K be a set of cardinality $|K| < \kappa$. A cone of $\mathscr{M}$ indexed by K is a functor $\tau : \epsilon(K) \longrightarrow \mathscr{M}$.*

The ordinal $|K|$ is said to be the size *of τ.*

Concretely a cone of $\mathscr{M}$ corresponds to a K-indexed family $\{A \longrightarrow B_k\}_{k \in K}$ of morphisms of $\mathscr{M}$ having the same domain. We can write

$$\tau = \{A \longrightarrow B_k\}_{k \in K}$$

with $A = \tau(\mathbf{O})$, $B_k = \tau(k)$ and $\tau[(\mathbf{O}, k)] = A \longrightarrow B_k$.

Terminology.

- If $\mathscr{M}$ is a model category then a cone $\tau = \{A \longrightarrow B_k\}_{k \in K}$ is said to be a **cone of cofibrations** if every morphism $A \longrightarrow B_k$ is a cofibration.

- More generally given a class of maps $\mathbf{I}$ of a category $\mathscr{M}$, a cone $\tau = \{A \longrightarrow B_k\}_{k \in K}$ is said to be a cone of $\mathbf{I}$ if every map $A \longrightarrow B_k$ is a member of $\mathbf{I}$.

- A cone $\tau : \epsilon(K) \longrightarrow \mathscr{M}$ is said to be small if the index set K is such that $|K| < \kappa$ for some regular cardinal κ.

Definition 4.1.5. *Let $\mathscr{M}$ be a category. A* ***generalized pushout diagram*** *in $\mathscr{M}$ is a colimit of a cone τ of $\mathscr{M}$. Here the colimit is the colimit of the functor τ.*

One can check that for a cone τ associated to a set K of cardinality 2, then the colimit of the diagram τ is given by a classical pushout square.

Using the fact that in a model category, the pushout of a cofibration is again a cofibration we have the following lemma.

Lemma 4.1.6. *Let κ be regular cardinal. For any κ-small model category $\mathscr{M}$ the following hold.*

1. *Every small cone τ of $\mathscr{M}$ has a colimit.*
2. *If $\tau = \{A \hookrightarrow B_k\}_{k \in K}$ if a cone of (trivial) cofibrations then all the canonical maps:*

$$B_k \longrightarrow colim(\tau)$$

$$A \longrightarrow colim(\tau)$$

are also (trivial) cofibrations.

Sketch of proof. Assertion (1) follows from the fact that $\mathscr{M}$ has all small colimits by definition of a model category.

For the assertion (2) it suffices to prove that the canonical map $B_k \longrightarrow \mathrm{colim}(\tau)$ is a (trivial) cofibration for every k. The map $A \longrightarrow \mathrm{colim}(\tau)$ which is the composite of $A \hookrightarrow B_k$ and $B_k \longrightarrow \mathrm{colim}(\tau)$, will be automatically a (trivial) cofibration since (trivial) cofibrations are closed under composition. For the rest of the proof we will simply treat the trivial cofibration case; the other case is implicitly proved by the same method.

We proceed by induction on the cardinality of K.

- If $|K| = 1$, there is nothing to prove.
- If $|K| = 2$, $K = \{k_1, k_2\}$, the colimit of τ is a pushout diagram and the result is well known.
- Let K be an arbitrary κ-small set and assume that the assertion is true for any subset $J \subset K$ with $|J| < |K|$.

Let's now choose $k_0 \in K$ and set $J = K - \{k_0\}$ and $\tau' := \tau_{|\epsilon(J)}$.
As $|J| < |K|$ the assertion is true for τ' and we have that the canonical maps

$$\begin{cases} A \longrightarrow \operatorname{colim}(\tau') \\ B_k \longrightarrow \operatorname{colim}(\tau') \quad \text{for all } k \in J \end{cases}$$

are trivial cofibrations.

Consider in $\mathscr{M}$ the following pushout square:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & B_{k_0} \arrow[d] \\ \operatorname{colim}(\tau') \arrow[r] & S \end{tikzcd}$$

Since trivial cofibrations are closed under pushout, we know that the canonical maps

$$\begin{cases} \operatorname{colim}(\tau') \longrightarrow S \\ B_{k_0} \hookrightarrow S \end{cases}$$

are trivial cofibrations. We deduce that the following maps are also trivial cofibrations:

$$\begin{cases} B_k \hookrightarrow S = [\operatorname{colim}(\tau') \hookrightarrow S] \circ [B_k \hookrightarrow \operatorname{colim}(\tau')] \quad k \in J \\ A \hookrightarrow S. \end{cases}$$

Finally one can easily verify that the object S equipped with the morphisms:

$$\begin{cases} A \hookrightarrow S \\ B_{k_0} \hookrightarrow S \\ B_k \hookrightarrow S \quad \text{for all } k \in J. \end{cases}$$

is a colimit of the functor τ, that is, it satisfies the universal property of 'the' colimit of τ. So we can actually take $\operatorname{colim}(\tau) = S$, and the assertion follows.

■

Remark 4.1.3. If $\mathbf{I}$ is the set of cofibrations of $\mathscr{M}$ then what we've just showed can be rephrased in term of *relative* $\mathbf{I}$-*cell complex.* We refer the reader to [42, Ch. 2.1.2] or [79, Ch. 8.7] and references therein for the definition of relative cell complex.

In this terminology we've just showed that each map $B_k \longrightarrow \operatorname{colim}(\tau)$ is a *relative* $\mathbf{I}$-*cell complex.* Now it's well known that a relative $\mathbf{I}$-cell complex is an element of some set $\mathbf{I}$-cof (see [42, Lemma 2.1.10]). In general for an arbitrary class of maps $\mathbf{I}$ we have an inclusion $\mathbf{I} \subset \mathbf{I}$-cof; but in a model category in which $\mathbf{I}$ is the set of cofibrations of $\mathscr{M}$ we have an equality $\mathbf{I}\text{-cof} = \mathbf{I}$ (see [42, Ch. 2.1.2]).

4.2 Adjunction Lemma

4.2.1 Lemma 1

In the following we fix $\mathscr{M}$ a cocomplete a symmetric closed monoidal category.
We remind the reader that being symmetric monoidal closed implies that the tensor product commutes with colimits on both sides. In particular for every (small) diagram $D : J \longrightarrow \mathscr{M}$, with J a discrete category i.e a set, then we have:

$$[\coprod_{j\in J} D(j)] \otimes P \cong \coprod_{j\in J}[D(j) \otimes P]$$

$$P \otimes [\coprod_{j\in J} D(j)] \cong \coprod_{j\in J}[P \otimes D(j)].$$

Let X be a κ-small set and $\mathcal{U} : \mathscr{M}_{\mathbb{S}}(X) \longrightarrow \prod_{(A,B)\in X^2} \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$ be the functor defined as follows.

$$\begin{cases} \mathcal{U}(F) = \{F_{AB}\}_{(A,B)\in X^2} & \text{for } F = \{F_{AB}, \varphi_{s,t}\}_{(A,B)\in X^2} \in \mathscr{M}_{\mathbb{S}}(X) \\ \mathcal{U}(\sigma) = \{\sigma_{AB} : F_{AB} \longrightarrow G_{AB}\}_{(A,B)\in X^2} & \text{for } F \xrightarrow{\sigma} G \end{cases}$$

So concretely the functor $\mathcal{U}$ forgets the laxity maps '$\varphi_{s,t}$'.

Our goal is to prove the following lemma.

Lemma 4.2.1. *The functor $\mathcal{U}$ has a left adjoint, that is there exists a functor*

$$\Gamma : \prod_{(A,B)\in X^2} \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}] \longrightarrow \mathscr{M}_{\mathbb{S}}(X)$$

such that for all $F \in \mathscr{M}_{\mathbb{S}}(X)$ and all $\mathscr{X} \in \prod_{(A,B)\in X^2} \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A,B)^{op}, \mathscr{M}]$, we have an isomorphism of sets:

$$\mathrm{Hom}[\Gamma[\mathscr{X}], F] \cong \mathrm{Hom}[\mathscr{X}, \mathcal{U}(F)]$$

which is natural in F and $\mathscr{X}$.

We will adopt the following conventions.

Conventions.

- If $(U_1, \cdots, U_n)$ is a n-tuple of objects of $\mathscr{M}$ we will write $U_1 \otimes \cdots \otimes U_n$ for the tensor product of $U_1, \cdots, U_n$ with all pairs of parentheses starting in front.
- For a set J and J_1, J_2 two subsets of J such that $J_1 \coprod J_2 = J$ then for every family $(U_j)_{j\in J}$ of objects of $\mathscr{M}$ we will freely identify the two objects $\coprod_{j\in J} U_j$ and $(\coprod_{j\in J_1} U_j) \coprod (\coprod_{j\in J_2} U_j)$ and we will call it "the" coproduct of the U_j.

  In particular for each $k \in J_1$, the three canonical maps

  $$\begin{cases} i_k : U_k \longrightarrow \coprod_{j\in J} U_j \\ i_{J_1} : \coprod_{j\in J_1} U_j \longrightarrow \coprod_{j\in J} U_j \\ i_{k,J_1} : U_k \longrightarrow \coprod_{j\in J_1} U_j \end{cases}$$

  are linked by the equality: $i_k = i_{J_1} \circ i_{k,J_1}$.

Before giving the proof we make some observations.

Observations 4.2.1. Let (A, B) be a pair of elements of X and $t \in \mathbb{S}_{\overline{X}}(A, B)$. Denote by $\mathbf{d}$ the degree (or length) of t.

Consider the set $\mathrm{Dec}(t)$ of all decomposition or 'presentations' of t given by:

$$\mathrm{Dec}(t) = \coprod_{0 \leq l \leq \mathbf{d-1}} \{(t_0, \cdots, t_l), \text{with } t_0 \otimes \cdots \otimes t_l = t\}$$

where for $l = 0$ we have $t_0 = t_l = t$.

Given $t' \in \mathbb{S}_{\overline{X}}(A, B)$ of length $\mathbf{d}'$ and a morphism $u : t \longrightarrow t'$ (hence $\mathbf{d}' \leq \mathbf{d}$) then for any $(t'_0, \cdots, t'_l) \in \mathrm{Dec}(t')$, there exists a unique $(t_0, \cdots, t_l) \in \mathrm{Dec}(t)$ together with a unique $(l+1)$-tuple of morphisms $(u_0, \cdots, u_l)$ with $u_i : t_i \longrightarrow t'_i$ such that:

$$u = u_0 \otimes \cdots \otimes u_l.$$

This follows from the fact in $\mathbb{S}_{\overline{X}}$ the composition is a concatenation of chains 'side by side' which is a generalization of the ordinal addition in $(\Delta^+_{\text{epi}}, +, 0)$. In fact by construction each $\mathbb{S}_{\overline{X}}(A, B)$ is a category of elements of a functor from Δ^+_{epi} to the category of sets. In particular the morphisms in $\mathbb{S}_{\overline{X}}(A, B)$ are parametrized by the morphism of Δ^+_{epi} and we clearly have this property of decomposition of morphisms in $(\Delta^+_{\text{epi}}, +)$.

It follows that any map $u : t \longrightarrow t'$ of $\mathbb{S}_{\overline{X}}(A, B)$ determines a unique function $\mathrm{Dec}(u) : \mathrm{Dec}(t') \longrightarrow \mathrm{Dec}(t)$. Moreover it's not hard to see that if we have two composable maps $u : t \longrightarrow t'$, $u' : t' \longrightarrow t''$ then $\mathrm{Dec}(u' \circ u) = \mathrm{Dec}(u') \circ \mathrm{Dec}(u)$.

Remark 4.2.1. One can observe that for $t \in \mathbb{S}_{\overline{X}}(A, B)$ and $s \in \mathbb{S}_{\overline{X}}(B, C)$ we have a canonical map of sets

$$\mathrm{Dec}(s) \times \mathrm{Dec}(t) \longrightarrow \mathrm{Dec}(s \otimes t)$$

which is injective, so that we can view $\mathrm{Dec}(s) \times \mathrm{Dec}(t)$ as a subset of $\mathrm{Dec}(s \otimes t)$.
And more generally for each $(t_0, \cdots, t_l) \in \mathrm{Dec}(t)$ with $l > 0$ we can identify $\mathrm{Dec}(t_0) \times \cdots \times \mathrm{Dec}(t_l)$ with a subset of $\mathrm{Dec}(t)$.

Proof of Lemma 4.2.1

Let $\mathscr{X} = (\mathscr{X}_{AB})$ be an object of $\prod_{(A,B) \in X^2} \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A, B)^{op}, \mathscr{M}]$.
To prove the lemma we will proceed as follows.

- First we give the construction of the components $\Gamma[\mathscr{X}]_{AB}$.
- Then we define the laxity maps $\xi_{s,t}$.
- Finally we check that we have the universal property i.e that the functor $F \mapsto \mathrm{Hom}[\mathscr{X}, \mathcal{U}(F)]$ is co-represented by $\Gamma[\mathscr{X}]$.

The components $\Gamma[\mathscr{X}]_{AB}$

1. We define $\Gamma[\mathscr{X}](t)$ by induction on the degree of t by:

   - $\Gamma[\mathscr{X}](t) = \mathscr{X}(t)$ if $deg(t) = \mathbf{1}$, i.e $t = (A, B)$ for some $(A, B) \in X^2$

   - And if $deg(t) > \mathbf{1}$ then we set

   $$\Gamma[\mathscr{X}](t) = \mathscr{X}(t) \coprod_{l>0,(t_0,\cdots,t_l)\in \mathrm{Dec}(t)} \Gamma[\mathscr{X}](t_0) \otimes \cdots \otimes \Gamma[\mathscr{X}](t_l).$$

   This formula is well defined since for every $(t_i)_{0\leq i\leq l} \in \mathrm{Dec}(t)$ with $l > 0$ we have $deg(t_i) < deg(t)$ and therefore each $\Gamma[\mathscr{X}](t_i)$ is already defined by the induction hypothesis.

2. Note that by construction we have the following canonical maps:

$$\begin{cases} \Gamma[\mathscr{X}](t_0) \otimes \cdots \otimes \Gamma[\mathscr{X}](t_l) \xrightarrow{\xi_{(t_0,\cdots,t_l)}} \Gamma[\mathscr{X}](t) \quad \text{with } l > 0 \\ \mathscr{X}(t) \xrightarrow{\eta_t} \Gamma[\mathscr{X}](t) \end{cases} \tag{4.2.1.1}$$

3. Given a map $u : t \longrightarrow t'$ of $\mathbb{S}_{\overline{X}}(A, B)$, we also define the map $\Gamma[\mathscr{X}](u) : \Gamma[\mathscr{X}](t') \longrightarrow \Gamma[\mathscr{X}](t)$ by induction.

   - If t is of degree $\mathbf{1}$ we take $\Gamma[\mathscr{X}](u) = \mathscr{X}(u)$.

   - If the degree of t is $> \mathbf{1}$, for each $(t_0, \cdots, t'_l) \in \mathrm{Dec}(t')$ we have a unique $(t_0, \cdots, t_l) \in \mathrm{Dec}(t)$ together with maps $u_i : t_i \longrightarrow t'_i$ such that $u = u_0 \otimes \cdots \otimes u_l$.

   By the induction hypothesis all the maps $\Gamma[\mathscr{X}](u_i) : \Gamma[\mathscr{X}](t'_i) \longrightarrow \Gamma[\mathscr{X}](t_i)$ are defined, and we can consider the maps:

$$\begin{cases} \Gamma[\mathscr{X}](t'_0) \otimes \cdots \otimes \Gamma[\mathscr{X}](t'_l) \xrightarrow{\otimes \Gamma[\mathscr{X}](u_i)} \Gamma[\mathscr{X}](t_0) \otimes \cdots \otimes \Gamma[\mathscr{X}](t_l) \quad \text{with } l > 0 \\ \mathscr{X}(u) : \mathscr{X}(t') \longrightarrow \mathscr{X}(t) \end{cases} \tag{4.2.1.2}$$

   The composite of the maps in (4.2.1.2) followed by the maps in (4.2.1.1) gives the following

$$\begin{cases} \Gamma[\mathscr{X}](t'_0) \otimes \cdots \otimes \Gamma[\mathscr{X}](t'_l) \xrightarrow{\xi_{(t_0,\cdots,t_l)} \circ (\otimes \Gamma[\mathscr{X}](u_i))} \Gamma[\mathscr{X}](t) \quad \text{with } l > 0 \\ \mathscr{X}(t') \xrightarrow{\eta_t \circ \mathscr{X}(u)} \Gamma[\mathscr{X}](t) \end{cases} \tag{4.2.1.3}$$

   Finally using the universal property of the coproduct, we know that the maps in (4.2.1.3) determines a unique map:

   $$\Gamma[\mathscr{X}]_{AB}(u) : \Gamma[\mathscr{X}]_{AB}(t') \longrightarrow \Gamma[\mathscr{X}]_{AB}(t).$$

4. It's not hard to check that these data determine a functor $\Gamma[\mathscr{X}]_{AB} : \mathbb{S}_{\overline{X}}(A, B)^{op} \longrightarrow \mathscr{M}$.

5. We leave the reader to check that all the maps in (4.2.1.1) are natural in all variables t_i including t. In particular the maps η_t determine a natural transformation

$$\eta_{AB} : \mathscr{X}_{AB} \longrightarrow \Gamma[\mathscr{X}]_{AB}$$

These natural transformations $\{\eta_{AB}\}_{(A,B)\in X^2}$ will constitute the unit of the adjunction.

Remark 4.2.2. We can define alternatively $\Gamma[\mathscr{X}]$ without using induction by the formula:

$$\Gamma[\mathscr{X}](t) = \coprod_{(t_0,\cdots,t_l)\in \mathrm{Dec}(t)} \mathscr{X}(t_0) \otimes \cdots \otimes \mathscr{X}(t_l).$$

where we include also the case $l = 0$ to have $\mathscr{X}(t)$ in the coproduct.

It's not hard to check that the $\Gamma[\mathscr{X}]$ we get by this formula and the previous one are naturally isomorphic. But for simplicity we will work with the definition by induction.

The laxity maps The advantage of the definition by induction is that we have 'on the nose' the laxity maps which correspond to the canonical maps :

$$\Gamma[\mathscr{X}](s) \otimes \Gamma[\mathscr{X}](t) \xrightarrow[(s,t)\in \mathrm{Dec}(s\otimes t)]{\xi_{s,t}} \Gamma[\mathscr{X}](s \otimes t)$$

for all pair of composable morphisms (s, t). And one can check that these laxity maps satisfy the coherence axioms of a lax morphism, so that $\Gamma[\mathscr{X}]$ is indeed an $\mathbb{S}_{\overline{X}}$-diagram.

Given two objects $\mathscr{X}$, $\mathscr{X}'$ with a morphism $\delta : \mathscr{X} \longrightarrow \mathscr{X}'$, one defines $\Gamma(\delta) = \{\Gamma(\delta)_t\}$ with:

$$\Gamma(\delta)_t = \delta_t \coprod_{(t_0,\cdots,t_l)\in \mathrm{Dec}(t), l>0} \Gamma(\delta)_{t_0} \otimes \cdots \otimes \Gamma(\delta)_{t_l}.$$

We leave the reader check that Γ is a functor.

Remark 4.2.3. If $\mathscr{M}$ is a symmetric monoidal model category such that all the objects are cofibrant, then by the pushout-product axiom one has that the class of (trivial) cofibrations is closed under tensor product. It's also well known that (trivial) cofibrations are also closed under coproduct. If we combine these two facts one clearly sees that if $\delta : \mathscr{X} \longrightarrow \mathscr{X}'$ is a level-wise (trivial) cofibration, then $\Gamma\delta$ is also a level-wise (trivial) cofibrations.

The level-wise (trivial) cofibrations are precisely the (trivial) cofibration on $\mathcal{K}_{X\text{-inj}}$ ($= \mathcal{K}_X$ equipped with the *injective model structure*). We have also by Ken Brown lemma (see [42, Lemma 1.1.12]) that both $\otimes$ and $\coprod$ preserve the weak equivalences between cofibrant objects; thus if δ is a level-wise weak equivalence then so is $\Gamma\delta$.

The universal property It remains to show that we have indeed an isomorphism of sets:

$$\mathrm{Hom}[\Gamma[\mathscr{X}], F] \cong \mathrm{Hom}[\mathscr{X}, \mathcal{U}(F)]$$

Recall that the functor $\mathcal{U}$ forgets only the laxity maps, so for any $F, G \in \mathscr{M}_{\mathbb{S}}(X)$ we have:

- $(\mathcal{U}F)_{AB} = F_{AB}$,

- For any $\sigma \in \mathrm{Hom}_{\mathscr{M}_{\mathbb{S}}(X)}(F,G)$ then $\mathcal{U}[\sigma] = \sigma$.

So when there is no confusion, we will write F instead of $\mathcal{U}(F)$ and σ instead of $\mathcal{U}(\sigma)$.

Consider $\eta : \mathscr{X} \longrightarrow \Gamma[\mathscr{X}]$, the canonical map appearing in the construction of $\Gamma[\mathscr{X}]$. Actually η is a map from $\mathscr{X}$ to $\mathcal{U}(\Gamma[\mathscr{X}])$.

Let $\theta : \mathrm{Hom}[\Gamma[\mathscr{X}], F] \longrightarrow \mathrm{Hom}[\mathscr{X}, \mathcal{U}(F)]$ be the function defined by:

$$\theta(\sigma) = \mathcal{U}(\sigma) \circ \eta$$

with $\theta(\sigma)_t = \sigma_t \circ \eta_t$. By abuse of notation we will write $\theta(\sigma) = \sigma \circ \eta$.

Claim. θ is one-to-one and onto, hence an isomorphism of sets

Injectivity of θ

Suppose we have $\sigma, \sigma' \in \mathrm{Hom}[\Gamma[\mathscr{X}], F]$ such that $\theta(\sigma) = \theta(\sigma')$. We proceed by induction on the degree of t to show that for all t, $\sigma_t = \sigma'_t$.

• For t of degree $\mathbf{1}$ we have $\Gamma[\mathscr{X}](t) = \mathscr{X}(t)$ and $\eta_t = \mathrm{Id}_{\Gamma[\mathscr{X}](t)}$ therefore $\theta(\sigma)_t = \sigma_t$ and $\theta(\sigma')_t = \sigma'_t$. The assumption $\theta(\sigma)_t = \theta(\sigma')_t$ gives $\sigma_t = \sigma'_t$.

• Let t be of degree $\mathbf{d} > \mathbf{1}$ and assume that $\sigma_w = \sigma'_w$ for all w of degree $\leq \mathbf{d} - \mathbf{1}$. We will denote for short by $\mathrm{Dec}(t)^*$ the set $\mathrm{Dec}(t) - \{t\}$ which is exactly the set of all $(w_0, \cdots, w_p)$ in $\mathrm{Dec}(t)$ with $p > 0$.

For each $(w_0, \cdots, w_p) \in \mathrm{Dec}(t)^*$ we have two canonical maps:

$$\begin{cases} \Gamma[\mathscr{X}](w_0) \otimes \cdots \otimes \Gamma[\mathscr{X}](w_p) \xrightarrow{\xi_{(w_0,\cdots,w_p)}} \Gamma[\mathscr{X}](w_0 \otimes \cdots \otimes w_p) = \Gamma[\mathscr{X}](t) \\ F(w_0) \otimes \cdots \otimes F(w_p) \xrightarrow{\varphi_{(w_0,\cdots,w_p)}} F(w_0 \otimes \cdots \otimes w_p) = F(t) \end{cases} \tag{4.2.1.4}$$

The map $\xi_{(w_0,\cdots,w_p)}$ is the one in (4.2.1.1) and the map $\varphi_{(w_0,\cdots,w_p)}$ is uniquely determined by laxity maps of F and their coherences, together with the bifunctoriality of the product $\otimes$ and it associativity. We remind the reader that the choice and order of composition of these maps (laxity, associativity, identities) we use to build $\varphi_{(w_0,\cdots,w_p)}$ doesn't matter (Mac Lane coherence theorem [68, Ch. 7]).

Now by definition of a morphism of $\mathbb{S}_{\overline{X}}$-diagrams, for any (w_0, w_1) with $w_0 \otimes w_1 = t$ the following diagram commutes:

$$\begin{tikzcd} \Gamma[\mathscr{X}](w_0) \otimes \Gamma[\mathscr{X}](w_1) \arrow[r, "\xi_{w_0,w_1}"] \arrow[d, "\sigma_{w_0} \otimes \sigma_{w_1}"'] & \Gamma[\mathscr{X}](t) \arrow[d, "\sigma_t"] \\ F(w_0) \otimes F(w_1) \arrow[r, "\varphi_{w_0,w_1}"] & F(t) \end{tikzcd}$$

And using again a 'Mac Lane coherence style' argument we have a general commutative diagram for each $(w_0, \cdots, w_p) \in \mathrm{Dec}(t)$:

$$
\begin{tikzcd}
\Gamma[\mathscr{X}](w_0) \otimes \cdots \otimes \Gamma[\mathscr{X}](w_p) \arrow[r, "\xi_{(w_0,\cdots,w_p)}"] \arrow[d, "\bigotimes \sigma_{w_i}"'] & \Gamma[\mathscr{X}](t) \arrow[d, "\sigma_t"] \\
F(w_0) \otimes \cdots \otimes F(w_p) \arrow[r, "\varphi_{(w_0,\cdots,w_p)}"] & F(t)
\end{tikzcd}
\tag{4.2.1.5}
$$

If we replace σ by σ' everywhere we get a commutative diagram of the same type.

Let's denote by $\mathrm{Diag}_t(\sigma)$ and $\mathrm{Diag}_t(\sigma')$ the set of maps:

$$\mathrm{Diag}_t(\sigma) = \left\{\varphi_{(w_0,\cdots,w_p)} \circ (\otimes\sigma_{w_i}), (w_0, \cdots, w_p) \in \mathrm{Dec}(t)^*\right\} \sqcup \{\theta(\sigma)_t\}$$
$$\mathrm{Diag}_t(\sigma') = \left\{\varphi_{(w_0,\cdots,w_p)} \circ (\otimes\sigma'_{w_i}), (w_0, \cdots, w_p) \in \mathrm{Dec}(t)^*\right\} \sqcup \left\{\theta(\sigma')_t\right\}.$$

Using the induction hypothesis $\sigma_{w_i} = \sigma'_{w_i}$ and the fact that $\theta(\sigma) = \theta(\sigma')$ we have:

$$\begin{cases} \varphi_{(w_0,\cdots,w_p)} \circ (\otimes\sigma_{w_i}) = \varphi_{(w_0,\cdots,w_p)} \circ (\otimes\sigma'_{w_i}) & \text{for all } (w_0, \cdots, w_p) \in \mathrm{Dec}(t)^* \\ \theta(\sigma)_t = \theta(\sigma')_t \end{cases}$$

so that $\mathrm{Diag}_t(\sigma) = \mathrm{Diag}_t(\sigma')$.

The universal property of the coproduct says that there exists a **unique** map $\delta_t : \Gamma[\mathscr{X}](t) \longrightarrow F(t)$ such that:

$$\begin{cases} \varphi_{(w_0,\cdots,w_p)} \circ (\otimes\sigma_{w_i}) = \delta_t \circ \xi_{(w_0,\cdots,w_p)} \\ \varphi_{(w_0,\cdots,w_p)} \circ (\otimes\sigma'_{w_i}) = \delta_t \circ \xi_{(w_0,\cdots,w_p)} \\ \theta(\sigma)_t = \delta_t \circ \eta_t \\ \theta(\sigma')_t = \delta_t \circ \eta_t \end{cases}$$

But we know from the commutative diagrams (4.2.1.5) that both σ_t and σ'_t satisfy these relations, so by uniqueness we have $\sigma_t = \delta_t = \sigma'_t$.

By induction we have the equality $\sigma_t = \sigma'_t$ for all t which means that $\sigma = \sigma'$ and θ is injective.

Remark 4.2.4. The set of maps

$$\begin{cases} \varphi_{(w_0,\cdots,w_p)} : F(w_0) \otimes \cdots \otimes F(w_p) \longrightarrow F(t) & \text{for all } (w_0, \cdots, w_p) \in \mathrm{Dec}(t)^* \\ \mathrm{Id}_{F(t)} : F(t) \longrightarrow F(t) \end{cases}$$

determines by the universal property of the coproduct a unique map:

$$\varepsilon'_t : \coprod_{\mathrm{Dec}(t)} F(w_0) \otimes \cdots \otimes F(w_p) \longrightarrow F(t).$$

Note that the source of that map is a coproduct taken on $\mathrm{Dec}(t)$ i.e including t. We have the obvious factorizations of $\varphi_{(w_0,\cdots,w_p)}$ and $\mathrm{Id}_{F(t)}$ through ε'_t. From the Remark 4.2.2 we have

$$\Gamma[\mathcal{U}F](t) \cong \coprod_{\mathrm{Dec}(t)} F(w_0) \otimes \cdots \otimes F(w_p)$$

and ε'_t gives a map $\varepsilon_t : \Gamma[\mathcal{U}F](t) \longrightarrow F(t)$. That map ε_t will constitute the counit of the adjunction.

Surjectivity of θ
Let $\pi : \mathscr{X} \longrightarrow \mathcal{U}(F)$ be an element of $\text{Hom}[\mathscr{X}, \mathcal{U}(F)]$. In the following we construct by induction a morphism of $\mathbb{S}_{\overline{X}}$-diagrams $\sigma : \Gamma[\mathscr{X}] \longrightarrow F$ such that $\theta(\sigma) = \pi$.
Let t be a morphism in $\mathbb{S}_{\overline{X}}$ of degree $\mathbf{d}$.

• For $\mathbf{d} = \mathbf{1}$ since $\Gamma[\mathscr{X}](t) = \mathscr{X}(t)$ and $\eta_t = \text{Id}_{\Gamma[\mathscr{X}](t)}$, we set $\sigma_t = \pi_t$. We have:

$$\theta(\sigma)_t = \sigma_t \circ \eta_t = \pi_t \circ \text{Id}_{\Gamma[\mathscr{X}](t)} = \pi_t.$$

• For $\mathbf{d} > \mathbf{1}$, let's assume that we've constructed $\sigma_{w_i} : \Gamma[\mathscr{X}](w_i) \longrightarrow F(w_i)$ for all w_i of degree $\leq \mathbf{d} - \mathbf{1}$ such that:

$$\theta(\sigma)_{w_i} = \sigma_{w_i} \circ \eta_{w_i} = \pi_{w_i}.$$

Using the the universal property of the coproduct with respect to the following set of maps:

$$\begin{cases} \varphi_{(w_0,\cdots,w_p)} \circ (\otimes \sigma_{w_i}) : \Gamma[\mathscr{X}](w_0) \otimes \cdots \otimes \Gamma[\mathscr{X}](w_p) \longrightarrow F(t) & \text{for all } (w_0, \cdots, w_p) \in \text{Dec}(t)^* \\ \pi_t : \mathscr{X}(t) \longrightarrow F(t) \end{cases}$$

we have a unique map $\sigma_t : \Gamma[\mathscr{X}](t) \longrightarrow F(t)$ such that the following factorizations hold:

$$\begin{cases} \varphi_{(w_0,\cdots,w_p)} \circ (\otimes \sigma_{w_i}) = \sigma_t \circ \xi_{(w_0,\cdots,w_p)} & \text{for all } (w_0, \cdots, w_p) \in \text{Dec}(t)^* \\ \pi_t = \sigma_t \circ \eta_t. \end{cases}$$

These factorizations imply that all the diagrams of type (4.2.1.5) are commutative. So by induction for all t we have these commutative diagrams which gives the required axioms for transformations of lax-morphisms (see Definition 3.4.11).

Moreover we clearly have by construction that $\theta(\sigma)_t = \sigma_t \circ \eta_t = \pi_t$ for all t, that is $\theta(\sigma) = \pi$ and θ is surjective.

We leave the reader to check that all the constructions considered in the previous paragraphs are natural in both $\mathscr{X}$ and F. ■

4.2.2 Evaluations on morphism have left adjoint

In the following $\mathscr{M}$ represents a cocomplete category. Given a small category $\mathcal{C}$ and a morphism α of $\mathcal{C}$ we will denote by $\text{Ev}_\alpha : \text{Hom}(\mathcal{C}, \mathscr{M}) \longrightarrow \mathscr{M}^{[\mathbf{1}]}$ the functor that takes $\mathcal{F} \in \text{Hom}(\mathcal{C}, \mathscr{M})$ to $\mathcal{F}(\alpha)$.

Recall that $[\mathbf{1}] = (0 \longrightarrow 1)$ is the interval category. A morphism α of $\mathcal{C}$ can be identified with a functor denoted again $\alpha : [\mathbf{1}] \longrightarrow \mathcal{C}$, that takes 0 to the source of α and 1 to the target.
Then we can identify Ev_α with the pullback functor $\alpha^\star : \text{Hom}(\mathcal{C}, \mathcal{M}) \xrightarrow{\alpha \circ} \text{Hom}([\mathbf{1}], \mathscr{M}) = \mathscr{M}^2$. With this identification one easily establishes that:

Lemma 4.2.2. *For a small category $\mathcal{C}$ and any morphism $\alpha \in \mathcal{C}$ the functor Ev_α has a left adjoint $\mathbf{F}^\alpha : \mathscr{M}^{[\mathbf{1}]} \longrightarrow \text{Hom}(\mathcal{C}, \mathscr{M})$.*

Proof. This is a special case of a more general situation where we have a functor $\alpha : \mathcal{B} \longrightarrow \mathcal{C}$; the left adjoint of $\alpha^\star$ is $\alpha_!$ which corresponds to the left Kan extension functor along α. The reader can find a detailed proof for example in [66, A.2.8.7], [79, Ch. 7.6.1], [40, Ch. 11.6]. ■

Corollary 4.2.3. $\mathbf{F}^\alpha$ *is a left Quillen functor between the corresponding projective model structures.*

Proof. A left adjoint is a left Quillen functor if and only if its right adjoint is a right Quillen functor (see lemma [42, 1.3.4]); consequently $\mathbf{F}^\alpha$ is a left Quillen functor if and only if Ev_α is a right Quillen functor.

But in the respective projective model structure on $\mathrm{Hom}(\mathcal{C}, \mathscr{M})$ and $\mathscr{M}^{[\mathbf{1}]} = \mathrm{Hom}([\mathbf{1}], \mathscr{M})$, the (trivial) fibrations are the level-wise ones, so Ev_α clearly preserves them. Thus Ev_α is a right Quillen functor and the result follows. ■

Remark 4.2.5. From Ken Brown's lemma ([42, Lemma 1.1.12]), $\mathbf{F}^\alpha$ preserves weak equivalences between cofibrant objects. Recall that a cofibrant object in $\mathscr{M}^2_{\text{proj}}$ is a cofibration in $\mathscr{M}$ with a cofibrant domain. In particular if all the objects of $\mathscr{M}$ are cofibrant, then the cofibrant objects in $\mathscr{M}^2_{\text{proj}}$ are simply all the cofibrations of $\mathscr{M}$. Therefore if all the objects of $\mathscr{M}$ are cofibrant then $\mathbf{F}^\alpha$ preserves weak equivalence between any two cofibrations.

Our main interest in the above lemma is when $\mathcal{C} = \mathbb{S}_{\overline{X}}(A, B)^{op}$ and $\alpha = u_t$, the unique morphism in $\mathbb{S}_{\overline{X}}(A, B)^{op}$ from (A, B) to t (see Observation **??**).

Each category $\mathbb{S}_{\overline{X}}(A, B)^{op}$ is an example of *direct category*, that is, a category $\mathcal{C}$ equipped with a *linear extension functor* $\mathbf{deg} : \mathcal{C} \longrightarrow \lambda$, where λ is an ordinal (see [42]). One requires furthermore that $\mathbf{deg}$ takes nonidentity maps to nonidentity maps; this way any nonidentity map raises the degree. Note that $\mathbb{S}_{\overline{X}}(A, B)^{op}$ has an initial object e which corresponds to (A, B).

For such categories $\mathcal{C}$ one has the following

Lemma 4.2.4. *Let $\mathcal{C}$ be a direct category. Then for any model category $\mathscr{M}$, the adjunction*

$$\mathbf{F}^\alpha : \mathscr{M}^{[\mathbf{1}]} \rightleftarrows \mathrm{Hom}(\mathcal{C}, \mathscr{M}) : Ev_\alpha$$

is also a Quillen adjunction with the injective model structures on each side.

Corollary 4.2.5. *For any $t \in \mathbb{S}_{\overline{X}}(A, B)$ the functor*

$$\mathbf{F}^{u_t} : \mathscr{M}^{[\mathbf{1}]}_{inj} \longrightarrow \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A, B)^{op}, \mathscr{M}]_{inj}$$

is a left Quillen functor.

Proof of Lemma 4.2.4. We assume that α is an nonidentity map since the assertion is obvious when α is an identity. Let $c_0 = \alpha(0)$ and $c_1 = \alpha(1)$ be respectively the source and target of α, and $h : U \longrightarrow V$ be an object in $\mathscr{M}^{[\mathbf{1}]} = \mathrm{Hom}([\mathbf{1}], \mathscr{M})$. Note that since α is an nonidentity map we have $\mathbf{deg}(c_0) < \mathbf{deg}(c_1)$, thus $\mathrm{Hom}(c_1, c_0) = \varnothing$.

By definition of the left Kan extension along α one defines $\mathbf{F}^\alpha_h(c)$ as:

$$\mathbf{F}^\alpha_h(c) = \operatorname{colim}_{(\alpha(i) \xrightarrow{k} c) \in \alpha_{/c}} \alpha(i)$$

where $\alpha_{/c}$ represents the over category (see [68], [79, 7.6.1]). Let $D_\alpha(c_0, c) \subset \mathrm{Hom}(c_0, c)$ be the subset of morphisms that factorizes through α. One can check that the previous colimit is the following coproduct:

$$\mathbf{F}^\alpha_h(c) = (\coprod_{f \in D_\alpha(c_0,c)} V) \bigsqcup (\coprod_{f \notin D_\alpha(c_0,c)} U) \bigsqcup (\coprod_{f \in \mathrm{Hom}(c_1,c)} V).$$

In the above expression, when the set indexing the coproduct is empty, then the coproduct is the initial object of $\mathscr{M}$.

Given $g : c \longrightarrow c'$ the structure map $\mathbf{F}^{\alpha}_{h}(g) : \mathbf{F}^{\alpha}_{h}(c) \longrightarrow \mathbf{F}^{\alpha}_{h}(c')$ is given as follows:

1. On $\coprod_{f \in D_\alpha(c_0,c)} V$, one sends the V-summand corresponding to $f : c_0 \longrightarrow c$ to the V-summand corresponding to $gf : c_0 \longrightarrow c'$ by the identity Id_V. Note that this is well defined since gf factorizes also through α.
2. On $\coprod_{f \notin D_\alpha(c_0,c)} U$, one sends the U-summand corresponding to $f : c_0 \longrightarrow c$ to:
   - the V-summand corresponding to gf by the morphism $h : U \longrightarrow V$, if $gf \in D_\alpha(c_0, c')$.
   - the U-summand corresponding to gf, if $gf \notin D_\alpha(c_0, c')$ by the morphism Id_U.
3. On $\coprod_{f \in \mathrm{Hom}(c_1,c)} V$ we send the V-summand indexed by f to the V-summand in $\mathbf{F}^{\alpha}_{h}(c')$ corresponding to gf by the morphism Id_V.
4. If one of the coproducts is to the initial object $\varnothing$ of $\mathscr{M}$ then one uses simply the unique map out of it.

It follows that given an injective (trivial) cofibration $\theta = (a, b) : h \longrightarrow h'$:

$$\begin{tikzcd}
U \arrow[r, hook, "a"] \arrow[d, "h"'] & U' \arrow[d, "h'"] \\
V \arrow[r, hook, "b"'] & V'
\end{tikzcd}$$

the c-component of $\mathbf{F}^{\alpha}_{\theta}$ is the coproduct

$$\mathbf{F}^{\alpha}_{\theta,c} = (\coprod_{f \in D_\alpha(c_0,c)} b) \coprod (\coprod_{f \notin D_\alpha(c_0,c)} a) \coprod (\coprod_{f \in \mathrm{Hom}(c_1,c)} b).$$

Since (trivial) cofibrations are closed under coproduct we deduce that $\mathbf{F}^{\alpha}_{\theta,c}$ is a (trivial) cofibration if θ is so, thus $\mathbf{F}^{\alpha}$ is a left Quillen functor as desired. ∎

4.3 $\mathscr{M}_{\mathbb{S}}(X)$ is cocomplete if $\mathscr{M}$ is

In this section we want to prove the following:

Theorem 4.3.1. *Given a cocomplete symmetric monoidal category $\mathscr{M}$, for any set X the category $\mathscr{M}_{\mathbb{S}}(X)$ is cocomplete.*

The proof of this theorem follows exactly the same ideas as the proof of the co-completeness of $\mathscr{M}$-**Cat** given by Wolff [97].

We will proceed as follows.

- We will show first that $\mathscr{M}_{\mathbb{S}}(X)$ is monadic over $\mathcal{K}_X = \prod_{(A,B) \in X^2} \mathrm{Hom}[\mathbb{S}_{\overline{X}}(A, B)^{op}, \mathscr{M}]$ using the Beck monadicity theorem.
- As the adjunction is monadic we know that $\mathscr{M}_{\mathbb{S}}(X)$ has coequalizers of $\mathcal{U}$-split pairs of morphisms and $\mathcal{U}$ preserves them.
- Since $\mathcal{K}_X$ is cocomplete by a result of Linton [63, Corollary 2] the category of algebras of the monad, which is equivalent to $\mathscr{M}_{\mathbb{S}}(X)$, is cocomplete.

4.3.1 $\mathscr{M}_{\mathbb{S}}(X)$ has coequalizers of reflexive pairs

The question of existence of coequalizers in $\mathscr{M}_{\mathbb{S}}(X)$ is similar to that of coequalizers in $\mathscr{M}$-**Cat** which was treated by Wolff [97]. For our needs we only treat the question of coequalizer of reflexive pairs.

Given a parallel pair of morphisms in $\mathscr{M}_{\mathbb{S}}(X)$ $(\sigma_1, \sigma_2) : D \rightrightarrows F$ one can view it as defining a 'relation' '$\mathcal{R} = \mathrm{Im}(\sigma_1 \times \sigma_2) \subset F \times F$' on F . We will call such relation 'precongruence'. In this situation the question is to find out when a quotient object '$E = F/\mathcal{R}$' ('the coequalizer') exists in $\mathscr{M}_{\mathbb{S}}(X)$. We will proceed in the same manner as Wolff; below we outline the different steps before going to details.

1. We will start by giving a criterion which says under which conditions the coequalizer computed in $\mathcal{K}_X$ lifts to a coequalizer in $\mathscr{M}_{\mathbb{S}}(X)$.
2. When a parallel pair of morphisms is a reflexive pair (=the analogue of the relation $\mathcal{R}$ to be reflexive) we will show that the conditions of the criterion are fulfilled and the result will follow.

Lifting of coequalizer

Definition 4.3.2. *Let F be an object of $\mathscr{M}_{\mathbb{S}}(X)$.*

1. *A **precongruence** in F is a pair of parallel morphisms in $\mathcal{K}_X$*

$$A \underset{\sigma_1}{\overset{\sigma_2}{\rightrightarrows}} \mathcal{U}F$$

for some object $A \in \mathcal{K}_X$.

2. *Let E be a coequalizer in $\mathcal{K}_X$ of (σ_1, σ_2), with $L : \mathcal{U}F \longrightarrow E$ the canonical map. We say that the precongruence is a **congruence** if:*
   - *$E = \mathcal{U}(\widetilde{E})$ for some $\widetilde{E} \in \mathscr{M}_{\mathbb{S}}(X)$ and*
   - *$L = \mathcal{U}(\widetilde{L})$ for a (unique) morphism $\widetilde{L} : F \longrightarrow \widetilde{E}$ in $\mathscr{M}_{\mathbb{S}}(X)$.*

When there is no confusion we simply write E for $\widetilde{E}$ and L for $\widetilde{L}$.

Lemma 4.3.3. *Let F be an object of $\mathscr{M}_{\mathbb{S}}(X)$ and consider a precongruence:*

$$A \underset{\sigma_1}{\overset{\sigma_2}{\rightrightarrows}} \mathcal{U}F \xrightarrow{L} E$$

Denote by $\varphi_{s,t} : F(s) \otimes F(t) \longrightarrow F(s \otimes t)$ be the laxity maps of F.

Then the precongruence is a congruence if and only if for any pair (s,t) of composable 1-morphisms in $\mathbb{S}_{\overline{X}}$ the following equalities hold:

$$L_{s\otimes t} \circ [\varphi_{s,t} \circ (\mathrm{Id}_{F(s)} \otimes \sigma_1(t))] = L_{s\otimes t} \circ [\varphi_{s,t} \circ (\mathrm{Id}_{F(s)} \otimes \sigma_2(t))]$$

$$L_{s\otimes t} \circ [\varphi_{s,t} \circ (\sigma_1(s) \otimes \mathrm{Id}_{F(t)})] = L_{s\otimes t} \circ [\varphi_{s,t} \circ (\sigma_2(s) \otimes \mathrm{Id}_{F(t)})].$$

In this case the structure of $\mathbb{S}_{\overline{X}}$-diagram on E is unique.

Remark 4.3.1. If F was (the nerve of) an $\mathscr{M}$-category, taking $s = (A, B)$, and $t = (B, C)$, we have $s \otimes t = (B, C)$ and the laxity maps correspond to the composition: $c_{ABC} : F(A, B) \otimes F(B, C) \longrightarrow F(A, C)$. One can check that the previous conditions are the same as in [97, Lemma 2.7].

Sketch of proof. The fact that having a congruence implies the equalities is easy and follows from the fact that L is a morphism in $\mathscr{M}_{\mathbb{S}}(X)$ and that L is a coequalizer of σ_1 and σ_2. We will then only prove that the equalities force a congruence.

To prove the statement we need to provide the laxity maps for E: $\phi_{s,t} : E(s) \otimes E(t) \longrightarrow E(s \otimes t)$.
By definition of E, for any 1-morphism $s \in \mathbb{S}_{\overline{X}}$, $E(s)$ is a coequalizer of $(\sigma_1(s), \sigma_2(s))$, which is a particular case of colimit in $\mathscr{M}$. Since $\mathscr{M}$ is a symmetric monoidal closed, colimits commute on each factor with the tensor product $\otimes$ of $\mathscr{M}$. It follows that given a pair (s, t) of composable morphisms then $E(s) \otimes E(t)$ is the colimit of the 'diagram':

$$\epsilon(s,t) = \{A(s) \otimes A(t) \xrightarrow{\sigma_i(s) \otimes \sigma_j(t)} F(s) \otimes F(t)\}_{i,j \in \{1,2\}}.$$

Now we claim that all the composite $L_{s\otimes t} \circ \varphi_{s,t} \circ [\sigma_i(s) \otimes \sigma_j(t)] : A(s) \otimes A(t) \longrightarrow E(s \otimes t)$ are equal.
This equivalent to say that the diagram

$$\epsilon'(s,t) = L_{s\otimes t} \circ \varphi_{s,t} \circ \epsilon(s,t) := \{L_{s\otimes t} \circ \varphi_{s,t} \circ [\sigma_i(s) \otimes \sigma_j(t)]\}_{i,j \in \{1,2\}}$$

is a compatible cocone. Before telling why this is true let's see how we get the laxity maps for E. For that it suffices to observe that since $\epsilon'(s,t) := L_{s\otimes t} \circ \varphi_{s,t} \circ \epsilon(s,t)$ is a compatible cocone, by the universal property of the colimit of $\epsilon(s,t)$ there exists a unique map $\psi_{s,t} : E(s) \otimes E(t) \longrightarrow E(s \otimes t)$ making the obvious diagrams commutative. In particular for any s, t the following is commutative:

$$\begin{tikzcd}
F(s) \otimes F(t) \arrow[r, "\varphi_{s,t}"] \arrow[d, "L_s \otimes L_t"'] & F(s \otimes t) \arrow[d, "L_{s\otimes t}"] \\
E(s) \otimes E(t) \arrow[r, dashed, "\psi_{s,t}"] & E(s \otimes t)
\end{tikzcd}$$

The fact that the morphism $\psi_{s,t}$ fit coherently is left to the reader as it's straightforward: it suffices to introduce a cocone $\epsilon(s,t,u)$ whose colimit 'is' $E(s) \otimes E(t) \otimes E(u)$ and use the universal property of the colimit. This shows that $(E, \psi_{s,t})$ is an object of $\mathscr{M}_{\mathbb{S}}(X)$ and that L extends to a morphism in $\mathscr{M}_{\mathbb{S}}(X)$.

Now with some easy but tedious computations one gets successively the desired equalities:

$$
\begin{aligned}
L_{s\otimes t}\circ\varphi_{s,t}\circ[\sigma_1(s)\otimes\sigma_2(t)] &= L_{s\otimes t}\circ\varphi_{s,t}\circ[(\mathrm{Id}_{F(s)}\otimes\sigma_2(t))\circ(\sigma_1(s)\otimes\mathrm{Id}_{A(t)})]\\
&= \underbrace{L_{s\otimes t}\circ[\varphi_{s,t}\circ(\mathrm{Id}_{F(s)}\otimes\sigma_2(t))]}_{=L_{s\otimes t}\circ[\varphi_{s,t}\circ(\mathrm{Id}_{F(s)}\otimes\sigma_1(t))]}\circ(\sigma_1(s)\otimes\mathrm{Id}_{A(t)})\\
(1) &= L_{s\otimes t}\circ[\varphi_{s,t}\circ(\mathrm{Id}_{F(s)}\otimes\sigma_1(t))]\circ(\sigma_1(s)\otimes\mathrm{Id}_{A(t)})\\
&= \underline{L_{s\otimes t}\circ\varphi_{s,t}\circ[\sigma_1(s)\otimes\sigma_1(t)]}\\
\text{From (1)} \rightsquigarrow &= L_{s\otimes t}\circ\varphi_{s,t}\circ[(\mathrm{Id}_{F(s)}\otimes\sigma_1(t))\circ(\sigma_1(s)\otimes\mathrm{Id}_{A(t)})]\\
&= L_{s\otimes t}\circ\varphi_{s,t}\circ[(\sigma_1(s)\otimes\mathrm{Id}_{F(t)})\circ(\mathrm{Id}_{A(s)}\otimes\sigma_1(t))]\\
&= \underbrace{L_{s\otimes t}\circ[\varphi_{s,t}\circ(\sigma_1(s)\otimes\mathrm{Id}_{F(t)})]}_{=L_{s\otimes t}\circ[\varphi_{s,t}\circ(\sigma_2(s)\otimes\mathrm{Id}_{F(t)})]}\circ(\mathrm{Id}_{A(s)}\otimes\sigma_1(t))\\
(2) &= L_{s\otimes t}\circ[\varphi_{s,t}\circ(\sigma_2(s)\otimes\mathrm{Id}_{F(t)})]\circ(\mathrm{Id}_{A(s)}\otimes\sigma_1(t))\\
&= \underline{L_{s\otimes t}\circ\varphi_{s,t}\circ[\sigma_2(s)\otimes\sigma_1(t)]}\\
\text{From (2)} \rightsquigarrow &= L_{s\otimes t}\circ[\varphi_{s,t}\circ(\sigma_2(s)\otimes\mathrm{Id}_{F(t)})]\circ(\mathrm{Id}_{A(s)}\otimes\sigma_1(t))\\
&= L_{s\otimes t}\circ\varphi_{s,t}\circ[(\mathrm{Id}_{F(s)}\otimes\sigma_1(t))\circ(\sigma_2(s)\otimes\mathrm{Id}_{A(t)})]\\
&= \underbrace{L_{s\otimes t}\circ[\varphi_{s,t}\circ(\mathrm{Id}_{F(s)}\otimes\sigma_1(t))}_{=L_{s\otimes t}\circ[\varphi_{s,t}\circ(\mathrm{Id}_{F(s)}\otimes\sigma_2(t))}\circ(\sigma_2(s)\otimes\mathrm{Id}_{A(t)})\\
&= L_{s\otimes t}\circ[\varphi_{s,t}\circ(\mathrm{Id}_{F(s)}\otimes\sigma_2(t))\circ(\sigma_2(s)\otimes\mathrm{Id}_{A(t)})\\
&= \underline{L_{s\otimes t}\circ\varphi_{s,t}\circ[\sigma_2(s)\otimes\sigma_2(t)]}
\end{aligned}
$$

■

Following Linton [63] we introduce the

Definition 4.3.4. *Let $\sigma_1, \sigma_2 : A \rightrightarrows \mathcal{U}F$ be a parallel pair of morphisms in $\mathcal{K}_X$ i.e a precongruence in F. We will say that a morphism $L : F \longrightarrow E$ in $\mathscr{M}_{\mathbb{S}}(X)$ is a* ***coequalizer relative to*** $\mathcal{U}$ *if:*

1. *If $\mathcal{U}L\circ\sigma_1 = \mathcal{U}L\circ\sigma_2$ and*
2. *if for any morphism $L' : F \longrightarrow Z$ in $\mathscr{M}_{\mathbb{S}}(X)$ which satisfies $\mathcal{U}L'\circ\sigma_1 = \mathcal{U}L'\circ\sigma_2$ then there exists a unique morphism $H : E \longrightarrow Z$ in $\mathscr{M}_{\mathbb{S}}(X)$ such that $L' = H\circ L$.*

Lemma 4.3.5. *If a precongruence*

$$
\begin{tikzcd}
A \arrow[r, shift left, "\sigma_2"] \arrow[r, shift right, "\sigma_1"'] & \mathcal{U}F \arrow[r, "L"] & E
\end{tikzcd}
$$

is a congruence then the morphism $L : F \longrightarrow E$ is a coequalizer rel. to $\mathcal{U}$.

Proof. Obvious: follows from the construction of $(E, \psi_{s,t})$.

■

Lemma 4.3.6 (Linton). *Let $U : D \longrightarrow C$ be a* <u>*faithful*</u> *functor and $\widetilde{f}, \widetilde{g} : A \longrightarrow B$ be a pair of parallel of morphisms in D. Denote by $f = U\widetilde{f}$ and $g = U\widetilde{g}$. Then for $p : A \longrightarrow E$ in D, the following are equivalent:*

- *p is a coequalizer of $(\widetilde{f}, \widetilde{g})$,*
- *p is a coequalizer rel. to U of (f, g).*

Proof. See [63, Lemma 1]. ■

For a given $F \in \mathscr{M}_{\mathbb{S}}(X)$, among the precongruences defined in F we have the ones coming from parallel pair of morphisms in $\mathscr{M}_{\mathbb{S}}(X)$ namely:

$$\mathcal{U}D \underset{\mathcal{U}\widetilde{\sigma}_1}{\overset{\mathcal{U}\widetilde{\sigma}_2}{\rightrightarrows}} \mathcal{U}F$$

for some $D \in \mathscr{M}_{\mathbb{S}}(X)$.

The following lemma tells about these congruences.

Lemma 4.3.7. *Let F be an object of $\mathscr{M}_{\mathbb{S}}(X)$. Consider a precongruence in F*

$$\mathcal{U}D \underset{\mathcal{U}\widetilde{\sigma}_1}{\overset{\mathcal{U}\widetilde{\sigma}_2}{\rightrightarrows}} \mathcal{U}F \xrightarrow{L} E \ .$$

*If there exists a **split** i.e a morphism $p : \mathcal{U}F \longrightarrow \mathcal{U}D$ in $\mathcal{K}_X$ such that $\mathcal{U}\widetilde{\sigma}_1 \circ p = \mathcal{U}\widetilde{\sigma}_2 \circ p = \mathrm{Id}_{\mathcal{U}F}$ then:*

- *the precongruence is a congruence and hence*
- *$L : F \longrightarrow E$ is a coequalizer in $\mathscr{M}_{\mathbb{S}}(X)$ of the pair $(\widetilde{\sigma}_1, \widetilde{\sigma}_2) : D \rightrightarrows F$ (and is obviously preserved by $\mathcal{U}$).*

Proof. We will simply need to show that the equalities of Lemma 4.3.3 holds. We will reduce the proof to the first equalities since the second ones are treated in the same manner by simply permuting $\mathrm{Id}_F \otimes \sigma_i$ to $\sigma_i \otimes \mathrm{Id}_F$.

Setting $\sigma_1 = \mathcal{U}\widetilde{\sigma}_1$ and $\sigma_2 = \mathcal{U}\widetilde{\sigma}_2$, these equalities become:

$$L_{s\otimes t} \circ [\varphi_{s,t} \circ (\mathrm{Id}_{F(s)} \otimes \sigma_1(t))] = L_{s\otimes t} \circ [\varphi_{s,t} \circ (\mathrm{Id}_{F(s)} \otimes \sigma_2(t))]$$

And to simplify the notations we will remove the letters '$s \otimes t, s, t$' but will mention 'φ_F' or 'φ_D' for the laxity maps of F and D respectively. The previous equality will be then written, when there is no confusion, as follows:

$$L \circ [\varphi_F \circ (\mathrm{Id}_F \otimes \sigma_1)] = L \circ [\varphi_F \circ (\mathrm{Id}_F \otimes \sigma_2)].$$

Since $\sigma_1 \circ p = \mathrm{Id}_F = \sigma_2 \circ p$, we have that

$$\begin{aligned} \mathrm{Id}_F \otimes \sigma_1 &= (\sigma_1 \circ p) \otimes (\sigma_1 \circ \mathrm{Id}_D) \\ &= (\sigma_1 \otimes \sigma_1) \circ (p \otimes \mathrm{Id}_D). \end{aligned}$$

Similarly for σ_2: $\mathrm{Id}_F \otimes \sigma_2 = (\sigma_2 \otimes \sigma_2) \circ (p \otimes \mathrm{Id}_D)$.

With these observations we can compute

$$\begin{aligned} L \circ [\varphi_F \circ (\mathrm{Id}_F \otimes \sigma_1)] &= L \circ \varphi_F \circ (\sigma_1 \otimes \sigma_1) \circ (p \otimes \mathrm{Id}_D) \\ &= L \circ \sigma_1 \circ \varphi_D \circ (p \otimes \mathrm{Id}_D) && (a) \\ &= L \circ \sigma_2 \circ \varphi_D \circ (p \otimes \mathrm{Id}_D) && (*) \\ &= L \circ \varphi_F \circ (\sigma_2 \otimes \sigma_2) \circ (p \otimes \mathrm{Id}_D) && (b) \\ &= L \circ [\varphi_F \circ (\mathrm{Id}_F \otimes \sigma_2)] \end{aligned}$$

This gives the desired equalities. We justify the different steps below.

- In (a) and (b) we've used the fact that σ_1 and σ_2 are **morphisms in** $\mathscr{M}_{\mathbb{S}}(X)$, which implies in particular that
$$\varphi_F \circ (\sigma_i \otimes \sigma_i) = \sigma_i \circ \varphi_D .$$
This last equality is equivalent to say that the following diagram commutes:
$$\begin{tikzcd} D(s)\otimes D(t) \arrow[r, "\varphi_D"] \arrow[d, "\sigma_i\otimes\sigma_i"'] & D(s\otimes t) \arrow[d, "\sigma_i"] \\ F(s)\otimes F(t) \arrow[r, dashed, "\varphi_F"] & F(s\otimes t) \end{tikzcd}$$

- In $(*)$ we've used the fact that L is a coequalizer of σ_1 and σ_2 therefore: $L \circ \sigma_1 = L \circ \sigma_2$.

Now from the lemma 4.3.5, we know that L is a coequalizer of $(\mathcal{U}\widetilde{\sigma}_1, \mathcal{U}\widetilde{\sigma}_2)$ rel. to $\mathcal{U}$, since $\mathcal{U}$ is clearly faithful Lemma 4.3.6 applies, hence L is a coequalizer in $\mathscr{M}_{\mathbb{S}}(X)$ of $(\widetilde{\sigma}_1, \widetilde{\sigma}_2)$ (and is obviously preserved by $\mathcal{U}$). ■

Corollary 4.3.8. *$\mathscr{M}_{\mathbb{S}}(X)$ has coequalizers of reflexive pairs and $\mathcal{U}$ preserves them.*

Proof. Given a pair of parallel morphisms $(\widetilde{\sigma}_1, \widetilde{\sigma}_2) : D \rightrightarrows F$ with a split $\widetilde{p}$ in $\mathscr{M}_{\mathbb{S}}(X)$, setting $p = \mathcal{U}\widetilde{p}$, by definition of morphism in $\mathscr{M}_{\mathbb{S}}(X)$, p is a split of $(\mathcal{U}\widetilde{\sigma}_1, \mathcal{U}\widetilde{\sigma}_2)$ and we conclude by the Lemma 4.3.7. ■

4.3.2 Monadicity and Cocompleteness

Theorem 4.3.9. *If $\mathscr{M}$ is cocomplete then $\mathscr{M}_{\mathbb{S}}(X)$ is monadic over $\mathcal{K}_X$.*

Proof. We use Beck's monadicity theorem (see [68, Chap.6, Sec.7, Thm.1]) for $\mathcal{U} : \mathscr{M}_{\mathbb{S}}(X) \longrightarrow \mathcal{K}_X$ since:

- $\mathcal{U}$ has a left adjoint Γ (Lemma 4.2.1),

- $\mathcal{U}$ clearly reflect isomorphisms since by definition a morphism σ is an isomorphism in $\mathscr{M}_{\mathbb{S}}(X)$ if $\mathcal{U}(\sigma)$ is so,

- $\mathscr{M}_{\mathbb{S}}(X)$ has has coequalizers of $\mathcal{U}$-split parallel pair (= reflexive pair) and $\mathcal{U}$ preserves them (Corollary 4.3.8).

■

Theorem 4.3.10. *For a cocomplete symmetric closed monoidal category $\mathscr{M}$, the category $\mathscr{M}_{\mathbb{S}}(X)$ is also cocomplete.*

Proof. By the previous theorem $\mathscr{M}_{\mathbb{S}}(X)$ is equivalent to the category of algebra of the monad $\mathcal{T} = \mathcal{U}\Gamma$ on $\mathcal{K}_X$. Since $\mathcal{K}_X$ is cocomplete and $\mathscr{M}_{\mathbb{S}}(X)$ (hence $\mathcal{T}$-alg) has coequalizer of reflexive pair, a result of Linton [63, Corollary 2] implies that $\mathcal{T}$-alg (hence $\mathscr{M}_{\mathbb{S}}(X)$) is cocomplete. ■

4.4 $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ is locally presentable

Our goal in this section is to prove the following

Theorem 4.4.1. *Let $\mathcal{M}_\bullet$ be a locally presentable $\mathcal{O}$-algebra. For any $\mathcal{O}$-algebra $\mathcal{C}_\bullet$ the category of lax $\mathcal{O}$-morphisms $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ is locally presentable.*

The proof of this theorem is technical and a bit long so we will divide it into small pieces, but for the moment we give hereafter an outline of what we will do.

1. We will construct a left adjoint $\mathbf{F}$, of the forgetful functor $\mathcal{U} : \mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet) \longrightarrow \prod_{i \in C} \mathrm{Hom}(\mathcal{C}_i, \mathcal{M}_i)$.
2. We will show that $\mathcal{U}$ is monadic that is $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ is equivalent to the category of algebra of the induced monad $\mathcal{U}\mathbf{F}$. We will transfer the local-presentability by monadic adjunction following the same idea as Kelly and Lack [50] who proved that $\mathscr{M}$-**Cat** is locally presentable if $\mathscr{M}$ is so. All we need will be to check that the monad is finitary i.e preserve filtered colimits and the result will follow by a classical argument.

Proposition 4.4.2. *Let $\mathcal{O}$ be a multisorted operad and $\mathcal{M}_\bullet$ a cocomplete $\mathcal{O}$-algebra. Then for any $\mathcal{O}$-algebra $\mathcal{C}_\bullet$ the category $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ is cocomplete.*

We will denote by $\mathcal{K}_{\mathcal{C}_\bullet} = \prod_{i \in C} \mathrm{Hom}(\mathcal{C}_i, \mathcal{M}_i)$ and $\mathcal{K}_{\mathcal{C}_i} = \mathrm{Hom}(\mathcal{C}_i, \mathcal{M}_i)$ the ith-factor.
Consider $\mathcal{U} : \mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet) \longrightarrow \mathcal{K}_{\mathcal{C}_\bullet}$ the functor which forgets the laxity maps.
For the proof of the proposition we will establish first the following lemma.

Lemma 4.4.3. *Let $\mathcal{O}$ be a multisorted operad and $\mathcal{M}_\bullet$ a cocomplete $\mathcal{O}$-algebra. Then the functor*

$$\mathcal{U} : \mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet) \longrightarrow \mathcal{K}_{\mathcal{C}_\bullet}$$

has a left adjoint.

As the proof is long we dedicate the next section to it

4.4.1 The functor $\mathcal{U}$ has a left adjoint

In the following we give a 'free algebra construction process' which associates to any family of functors $\mathcal{F}_\bullet = (\mathcal{F}_i)_{i \in C}$, a lax $\mathcal{O}$-morphism $\mathbf{F}\mathcal{F}_\bullet$. One can consider it as the analogue of the 'monadification' of a classical (one-sorted) operad (see for example [56, Part I, Section 3]. For the operad $\mathcal{O}_X$ it will cover the process which associates an $\mathscr{M}$-graph to the corresponding free $\mathscr{M}$-category.

Notation 4.4.1. If $(x, c_1, ..., c_n) \in \mathcal{O}(i_1, ..., i_n; j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n}$, we will write $\otimes_x(c_1, ...c_n) = \rho_{i_\bullet|j}(x, c_1, .., c_n)$.
$\rho^{-1}_{i_\bullet|j}(c)$ = the subcategory of $\mathcal{O}(i_1, ..., i_n; j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n}$ whose objects are $(x, c_1, ..., c_n)$ such that $\otimes_x(c_1, ...c_n) = c$ and morphisms $(f, u_1, ..., u_n)$ such that $\rho_{i_\bullet|j}(f, u_1, ..., u_n) = \mathrm{Id}_c$.
$\mathbf{F}^c_{-} : \mathcal{M}_j \longrightarrow \mathrm{Hom}(\mathcal{C}_j, \mathcal{M}_j)$ = the left adjoint of the evaluation functor at c: $\mathrm{Ev}_c : \mathrm{Hom}(\mathcal{C}_j, \mathcal{M}_j) \longrightarrow \mathcal{M}_j$.

Informally from a family of (abstract) objects $(\mathrm{L}_i)_{i \in C}$, one defines the associated free $\mathcal{O}$-algebra $(\mathbf{F}\mathrm{L}_i)_{i \in C}$ as follows

$$\mathbf{F}\mathrm{L}_j = \coprod_{n \in \mathbb{N}} \Big(\coprod_{(i_1, ..., i_n)} \mathcal{O}(i_1, ..., i_n; j) \boxtimes [\mathrm{L}_{i_1} \otimes \cdots \otimes \mathrm{L}_{i_n}] \Big).$$

Here '⊠' is the action of $\mathcal{O}$ on the category containing the object L_i and $\otimes$ is the internal product of that category. The algebra structure is simply given by the multiplication of $\mathcal{O}$ and shuffles; the reader can find a description of the algebra structure, for the one sorted case, in *loc. cit.*

In our case we want to use the description presented previously, according to which we view a lax $\mathcal{O}$-morphism as an $\mathcal{O}$-algebra of some category $\mathrm{Arr}(\mathbf{Cat})_+$ (see 3.3.1). The action of $\mathcal{O}$ on $\mathrm{Arr}(\mathbf{Cat})_+$ was denoted by '$\odot$'.

So if we start with a family of functors $(\mathcal{F}_i : \mathcal{C}_i \longrightarrow \mathcal{M}_i)_{i \in C}$ we would like to define the associated free algebra by:

$$\text{` } \mathbf{F}\mathcal{F}_j = \coprod_{n \in \mathbb{N}} \Big(\coprod_{(i_1,\dots,i_n)} \mathcal{O}(i_1,\dots,i_n;j) \odot [\mathcal{F}_{i_1} \otimes \cdots \otimes \mathcal{F}_{i_n}] \Big) \text{ '}$$

But as one can see in this coproduct the functors are not defined over the same category, which we want to be $\mathcal{C}_j$, so the previous expression actually doesn't make sense in general. But still it guides us to the correct object which is some left Kan extension of something similar.

For each $(i_1,\dots,i_n) \in C^n$ introduce $\mathbf{Lan}_j(\mathcal{O}, \mathcal{F}_{i_\bullet})$ the left Kan extension of the functor

$$\rho_{i_\bullet|j} \circ [\mathcal{O}(i_1,\dots,i_n;j) \odot \prod \mathcal{F}_i] = \rho_{i_\bullet|j} \circ [\mathrm{Id}_{\mathcal{O}(i_1,\dots,i_n;j)} \times \prod \mathcal{F}_i]$$

along the functor $\theta_{i_\bullet|j}$. This left Kan extension exists since $\mathcal{M}_j$ is cocomplete and we have the following diagram

$$\begin{tikzcd}
\mathcal{O}(i_1,\dots,i_n;j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n} \arrow[r, "\theta_{i_\bullet|j}"] \arrow[d, "\mathrm{Id}_{\mathcal{O}(i_1,\dots,i_n;j)} \times \mathcal{F}_{i_1} \times \cdots \times \mathcal{F}_{i_n}"'] & \mathcal{C}_j \arrow[d, "\mathbf{Lan}_j(\mathcal{O}, \mathcal{F}_{i_\bullet})"] \\
\mathcal{O}(i_1,\dots,i_n;j) \times \mathcal{M}_{i_1} \times \cdots \times \mathcal{M}_{i_n} \arrow[r, "\rho_{i_\bullet|j}"'] \arrow[ur, phantom, "\Nearrow \varepsilon_{i_\bullet|j}"] & \mathcal{M}_j
\end{tikzcd}$$

Here $\varepsilon_{i_\bullet|j}$ is the universal natural transformation arising in the construction of the left Kan extension (see [68, Ch. X]). This diagram represents a morphism in $\mathrm{Arr}(\mathbf{Cat})_+$

$$\Phi(i_1,\dots,i_n;j) : \mathcal{O}(i_1,\dots,i_n;j) \odot [\mathcal{F}_{i_1} \times \cdots \times \mathcal{F}_{i_n}] \longrightarrow \mathbf{Lan}_j(\mathcal{O}, \mathcal{F}_{i_\bullet})$$

which, according to the notations in section (3.3.1), is $(\theta_{i_\bullet|j}; \rho_{i_\bullet|j}; \varepsilon_{i_\bullet|j})$.

Definition of F is $\mathcal{C}_\bullet$ is a free $\mathcal{O}$-algebra

Let $\mathcal{C}_\bullet$ be a free $\mathcal{O}$-algebra. $\mathcal{F}_\bullet \in \mathcal{K}_{\mathcal{C}_\bullet}$. Then we define $\mathbf{F}\mathcal{F}_\bullet$ by

$$\mathbf{F}(\mathcal{F})_j = \coprod_{n \in \mathbb{N}} \coprod_{(i_1,\dots,i_n)} \mathbf{Lan}_j(\mathcal{O}, \mathcal{F}_{i_\bullet}).$$

Note that in this coproduct there is a hidden term for $n = 0$ which is just $\mathcal{F}_j$ itself, since $\mathcal{F}_j$ is the left Kan extension of itself along the identity $\mathrm{Id}_{\mathcal{C}_j}$. The inclusion in the coproduct yields a natural transformation:

$$\eta_j : \mathcal{F}_j \longrightarrow \mathbf{F}(\mathcal{F})_j.$$

We have to specify the morphisms: $\mathcal{O}(i_1,\dots,i_n;j) \odot [\mathbf{F}(\mathcal{F})_{i_1} \times \cdots \times \mathbf{F}(\mathcal{F})_{i_n}] \longrightarrow \mathbf{F}(\mathcal{F})_j$. Before doing this we need to outline some important fact about free $\mathcal{O}$-algebras:

Remark 4.4.1. Since $\mathcal{C}_\bullet$ is a free algebra, $\mathcal{C}_\bullet$ is a defined by a collections of categories $(L_i)_{i\in C}$ and one has

$$\mathcal{C}_j = \coprod_{n\in\mathbb{N}}(\coprod_{(i_1,...,i_n)} \mathcal{O}(i_1,...,i_n;j) \times [\mathrm{L}_{i_1} \times \cdots \times \mathrm{L}_{i_n}]).$$

It follows that each multiplication $\theta_{i_\bullet|j} : \mathcal{O}(i_1,...,i_n;j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n} \longrightarrow \mathcal{C}_j$ is an inclusion to a coproduct, one view it as a 'grafting trees' operation; it's image defines a connected component of $\mathcal{C}_j$. Therefore any $c_j \in \mathrm{Im}(\theta_{i_\bullet|j})$ has a unique presentation $(x, c_1, ...c_n) \in \mathcal{O}(i_1,...,i_n;j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n}$.
Consequently the Kan extension $\mathbf{Lan}_j(\mathcal{O}, \mathcal{F}_{i_\bullet})$ consists to take the image by $(\mathcal{F}_i)$ of the presentation i.e : $\mathbf{Lan}_j(\mathcal{O}, \mathcal{F}_{i_\bullet})c = \rho_{i_\bullet|j}(x, \mathcal{F}_{i_1}c_1, ..., \mathcal{F}_{i_n}c_n)$.

With this description we define the morphism: $\mathcal{O}(i_1,...,i_n;j) \odot [\mathbf{F}(\mathcal{F})_{i_1} \times \cdots \times \mathbf{F}(\mathcal{F})_{i_n}] \longrightarrow \mathbf{F}(\mathcal{F})_j$ as follows.

- First if we expand $\mathcal{O}(i_1,...,i_n;j) \odot [\mathbf{F}(\mathcal{F})_{i_1} \times \cdots \times \mathbf{F}(\mathcal{F})_{i_n}]$ we have:

$$\mathcal{O}(i_\bullet;j) \odot [\mathbf{F}(\mathcal{F})_{i_1} \times \cdots \times \mathbf{F}(\mathcal{F})_{i_n}] = \coprod_{n\in\mathbb{N}} \coprod_{(i_1,...,i_n)} \coprod_{(h_{1,1},...,h_{n,k_n})} \mathrm{Id}_{\mathcal{O}(i_\bullet;j)} \times \prod_i \mathbf{Lan}_i(\mathcal{O}(h_{i,\bullet}|i), \mathcal{F}_{h_{i,l_\bullet}})$$

- Then introduce $\mathbf{Lan}_j[\mathcal{O}(i_\bullet|j), \mathbf{Lan}_i(\mathcal{O}(h_{i,\bullet}|i), \mathcal{F}_{h_{i,l_\bullet}})]$, the left Kan extension of the summand $\mathrm{Id}_{\mathcal{O}(i_\bullet;j)} \times \prod_i \mathbf{Lan}_i(\mathcal{O}(h_{i,\bullet}|i), \mathcal{F}_{h_{i,l_\bullet}})$ along $\theta_{i_\bullet|j}$. This left Kan extension comes equipped with a natural transformation:

$$\delta : \mathrm{Id}_{\mathcal{O}(i_1,...,i_n;j)} \times \prod_i \mathbf{Lan}_i(\mathcal{O}(h_{i,\bullet}|i), \mathcal{F}_{h_{i,l_\bullet}}) \longrightarrow \mathbf{Lan}_j[\mathcal{O}(i_\bullet|j), \mathbf{Lan}_i(\mathcal{O}(h_{i,\bullet}|i), \mathcal{F}_{h_{i,l_\bullet}})].$$

Claim. We have an equality $\mathbf{Lan}_j[\mathcal{O}(i_\bullet|j), \mathbf{Lan}_i(\mathcal{O}(h_{i,\bullet}|i), \mathcal{F}_{h_{i,l_\bullet}})] = \mathbf{Lan}_j[\mathcal{O}(h_{\bullet,l_\bullet}|j), \mathcal{F}_{h_{\bullet,l_\bullet}}]$.

Before telling why the claim holds, one defines the desired map by sending the summand $\mathrm{Id}_{\mathcal{O}(i_\bullet;j)} \times \prod_i \mathbf{Lan}_i(\mathcal{O}(h_{i,\bullet}|i), \mathcal{F}_{h_{i,l_\bullet}})$ of $\mathcal{O}(i_1,...,i_n;j) \odot [\mathbf{F}(\mathcal{F})_{i_1} \times \cdots \times \mathbf{F}(\mathcal{F})_{i_n}]$ to the summand $\mathbf{Lan}_j[\mathcal{O}(h_{\bullet,l_\bullet}|j), \mathcal{F}_{h_{\bullet,l_\bullet}}]$ of $\mathbf{F}(\mathcal{F})_j$ by the composite:

$$\mathrm{Id}_{\mathcal{O}(i_\bullet;j)} \times \prod_i \mathbf{Lan}_i(\mathcal{O}(h_{i,\bullet}|i), \mathcal{F}_{h_{i,l_\bullet}}) \xrightarrow{\delta} \mathbf{Lan}_j[\mathcal{O}(h_{\bullet,l_\bullet}|j), \mathcal{F}_{h_{\bullet,l_\bullet}}] \hookrightarrow \mathbf{F}(\mathcal{F})_j.$$

By the universal property of the coproduct we have a unique map:

$$\Phi_{i_\bullet|j} : \mathcal{O}(i_1,...,i_n;j) \odot [\mathbf{F}(\mathcal{F})_{i_1} \times \cdots \times \mathbf{F}(\mathcal{F})_{i_n}] \longrightarrow \mathbf{F}(\mathcal{F})_j.$$

To see that the claim holds one proceeds as follows. Consider $[x, (x_i, d_{i,1}, ..., d_{i,k_i})_{1\leq i\leq n}]$ an object of $\mathcal{O}(i_\bullet|j) \times [\mathcal{O}(h_{1,\bullet}|i_1) \times \mathcal{C}_{1,\bullet}] \times \cdots \times [\mathcal{O}(h_{n,\bullet}|i_n) \times \mathcal{C}_{n,\bullet}]$. Such presentation defines a unique object $c \in \mathcal{C}_j$, and each $(x_i, d_{i,1}, ..., d_{i,k_i})$ defines a unique object $c_i \in \mathcal{C}_i$.

In the free algebra $\mathcal{C}$, one declares that the following objects are equal to $c \in \mathcal{C}_j$

- $(\gamma(x, x_i); d_{1,1}, ..., d_{n,k_n}) = c$,
- $(x, c_1, ..., c_n) = c$,

Here γ is the substitution in $\mathcal{O}$ and $(x_i; d_{i,1}, ..., d_{i,k_i}) = c_i$. Then one computes in one hand

$$\mathbf{Lan}_j[\mathcal{O}(h_{\bullet,l_\bullet}|j), \mathcal{F}_{h_{\bullet,l_\bullet}}](c) = \otimes_{\gamma(x,x_i)}(\mathcal{F}d_{1,1}, ..., \mathcal{F}d_{n,k_n}).$$

On the other hand one has:

$$\mathbf{Lan}_j[\mathcal{O}(i_{\bullet}|j), \mathbf{Lan}_i(\mathcal{O}(h_{i,\bullet}|i), \mathcal{F}_{h_{i,l_{\bullet}}})](c) = \otimes_x[\otimes_{x_i}(\mathcal{F}(d_{1\bullet}), ..., \otimes_{x_n}(\mathcal{F}d_{n\bullet})].$$

Now as $\mathcal{M}_{\bullet}$ is an $\mathcal{O}$-algebra one has the equality:

$$\otimes_{\gamma(x,x_i)}(\mathcal{F}d_{1,1}, ..., \mathcal{F}d_{n,k_n}) = \otimes_x[\otimes_{x_i}(\mathcal{F}(d_{1\bullet}), ..., \otimes_{x_n}(\mathcal{F}d_{n\bullet})]$$

which means that the two functors are equal as claimed.

By virtue of the previous discussion we have the

Proposition 4.4.4.

1. *The family* $\mathbf{F}(\mathcal{F}_i)_{i\in C}$ *forms a lax* $\mathcal{O}$*-morphism of algebra. Equivalently* $\mathbf{F}(\mathcal{F}_i)_{i\in C}$ *is an* $\mathcal{O}$*-algebra of* $Arr(\mathbf{Cat})_+$.
2. *For any* $\mathcal{G}_{\bullet} = (\mathcal{G}_i)_{i\in C} \in \mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_{\bullet}, \mathcal{M}_{\bullet})$ *we have a functorial isomorphism of sets:*

$$\mathrm{Hom}[\mathbf{F}(\mathcal{F}_i)_{i\in C}, \mathcal{G}_{\bullet}] \cong \prod_i \mathrm{Hom}[\mathcal{F}_i, \mathcal{G}_i]$$

Proof of Proposition 4.4.4. We will simply give a proof of the assertion (1). The statement (2) is tedious but straightforward to check.
The only thing we need to check is the fact that the natural transformations $\Phi_{i_{\bullet}|j}$ fit coherently. We are asked to say if for any $(h_{1,1}, ..., h_{1,l_1}; i_1); \cdots; (h_{n,1}, ..., h_{n,l_n}; i_n)$ the following is commutative:

$$\begin{tikzcd}
\mathcal{O}(i_{\bullet}|j) \times \mathcal{O}(h_{1,\bullet}|i_1) \times \cdots \times \mathcal{O}(h_{n,\bullet}|i_n) \times \prod \mathbf{F}(\mathcal{F}_{h_{\bullet,l_{\bullet}}}) \arrow[r, "\gamma\times \mathrm{Id}"] \arrow[d, swap, "(\mathrm{Id}_{\mathcal{O}(i_{\bullet}|j)} \times \prod \Phi_{h_{i,l_{\bullet}}|i}) \circ \mathrm{shuffle}"] & \mathcal{O}(h_{\bullet,l_{\bullet}}|j) \times \prod \mathbf{F}(\mathcal{F}_{h_{\bullet,l_{\bullet}}}) \arrow[d, "\Phi_{h_{\bullet,l_{\bullet}}|j}"] \\
\mathcal{O}(i_{\bullet}|j) \times \prod \mathbf{F}(\mathcal{F}_i) \arrow[r, swap, "\Phi_{i_{\bullet}|j}"] & \mathbf{F}(\mathcal{F}_j)
\end{tikzcd} \tag{4.4.1.1}$$

But the commutativity of that diagram boils down to check that the following maps are identities:

$$\mathbf{Lan}_j[\mathcal{O}(h_{\bullet,l_{\bullet}}|j), \mathcal{F}_{h_{\bullet,l_{\bullet}}}] \xrightarrow{\text{canonical}} \mathbf{Lan}_j[\mathcal{O}(i_{\bullet}|j), \mathbf{Lan}_i(\mathcal{O}(h_{i,\bullet}|i), \mathcal{F}_{h_{i,l_{\bullet}}})]$$

But this follows from the previous discussion. Consequently the maps $\Phi_{i_{\bullet}|j}$ fit coherently and $(\mathbf{F}\mathcal{F}_i)_{i\in C}$ equipped with $\Phi_{i_{\bullet}|j}$ is a lax $\mathcal{O}$-morphism of algebra.

■

Definition of F for an arbitrary $\mathcal{O}$-algebra $\mathcal{C}$.

Let $\mathcal{C}_{\bullet}$ be an arbitrary $\mathcal{O}$-algebra and $\mathcal{F}_{\bullet} \in \mathcal{K}_{\mathcal{C}_{\bullet}}$. For each $j \in C$ consider,

$$\mathcal{F}_j^1 = \coprod_{n\in\mathbb{N}} \coprod_{(i_1,...,i_n)} \mathbf{Lan}_j(\mathcal{O}, \mathcal{F}_{i_{\bullet}}).$$

This is the same type of expression as before; the inclusion in the coproduct yields a natural transformation:

$$e_j : \mathcal{F}_j \longrightarrow \mathcal{F}_j^1.$$

For each c and each $(x, c_1, ..., c_n) \in \rho^{-1}c$, we have a map:

$$\varepsilon : \otimes_x(\mathcal{F}c_1, ..., \mathcal{F}c_n) \longrightarrow \mathbf{Lan}_j(\mathcal{O}, \mathcal{F}_{i_\bullet})(c) \hookrightarrow \mathcal{F}^1 c.$$

By the adjunction $\mathbf{F}^c \dashv \mathrm{Ev}_c$, the map ε corresponds to a unique map in $\mathcal{K}_j = \mathrm{Hom}(\mathcal{C}_j, \mathcal{M}_j)$:

$$\mathbf{F}^c_{\otimes_x(\mathcal{F}c_1,...,\mathcal{F}c_n)} \longrightarrow \mathcal{F}^1.$$

Let $\mathcal{R}(e; x, c_1, ..., c_n)$ be the object defined by the pushout diagram in $\mathcal{K}_j$:

$$\begin{tikzcd}
\mathbf{F}^c_{\otimes_x(\mathcal{F}c_1,...,\mathcal{F}c_n)} \arrow[r] \arrow[d, "\mathbf{F}^c_{\otimes_x(e_1,...e_n)}"'] & \mathcal{F}^1 \arrow[d, dashed, "p(e;x,c_1,...,c_n)"] \\
\mathbf{F}^c_{\otimes_x(\mathcal{F}^1c_1,...,\mathcal{F}^1c_n)} \arrow[r, dashed] & \mathcal{R}(e; x, c_1, ..., c_n)
\end{tikzcd}$$

An intermediate coherence

Let $[x, (x_i, d_{i,1}, ..., d_{i,k_i})_{1\le i\le n}]$ be an object of $\mathcal{O}(i_\bullet|j) \times [\mathcal{O}(h_{1,\bullet}|i_1) \times \mathcal{C}_{1,\bullet}] \times \cdots \times [\mathcal{O}(h_{n,\bullet}|i_n) \times \mathcal{C}_{n,\bullet}]$ such that:

- $\otimes_{x_i}(d_{i,1}, ..., d_{i,k_i}) = c_i$,
- $\otimes_{\gamma(x,x_i)}(d_{1,1}, ..., d_{n,k_n}) = c$, and
- $\otimes_x(c_1, ..., c_n) = c$; here γ is the substitution in $\mathcal{O}$.

From the map

$$\eta\varepsilon : \otimes_{\gamma(x,x_i)}(\mathcal{F}d_{1,1}, ..., \mathcal{F}d_{n,k_n}) \longrightarrow \mathcal{F}^{1,1}(\otimes_{\gamma(x,x_i)}(d_{1,1}, ..., d_{n,k_n})) = \mathcal{F}^{1,1}c.$$

Using the adjunction $\mathbf{F}^c \dashv \mathrm{Ev}_c$, we define the object $Q(x, x_i; d_{1,1}, ..., d_{n,k_n})$ given by the pushout diagram in $\mathcal{K}_j$:

$$\begin{tikzcd}
\mathbf{F}^c_{\otimes_{\gamma(x,x_i)}(\mathcal{F}d_{1,1},...,\mathcal{F}d_{n,k_n})} \arrow[r] \arrow[d, "\mathbf{F}^c_{\otimes_x[(p\varepsilon)_1,...,(p\varepsilon)_n]}"'] & \mathcal{F}^1 \arrow[d, dotted, "g(x,x_i;d_{1,1},...,d_{n,k_n})"] \\
\mathbf{F}^c_{\otimes_x(\mathcal{R}c_1,...,\mathcal{R}c_n)} \arrow[r, dotted] & Q(x, x_i; d_{1,1}, ..., d_{n,k_n})
\end{tikzcd}$$

Introduce $\mathcal{Z}(x, x_i; d_{1,1}, ..., d_{n,k_n})$ to be the object obtained from the pushout:

$$\begin{tikzcd}
\mathcal{F}^1 \arrow[r, "p"] \arrow[d, "g(x,x_i;d_{1,1},...,d_{n,k_n})"'] & \mathcal{R}(e, \gamma(x, x_i); d_{1,1}, ..., d_{n,k_n}) \arrow[d, dotted, "g'(x,x_i;d_{1,1},...,d_{n,k_n})"] \\
Q(x, x_i; d_{1,1}, ..., d_{n,k_n}) \arrow[r, dotted] & \mathcal{Z}(x, x_i; d_{1,1}, ..., d_{n,k_n})
\end{tikzcd}$$

Denote by $\mathcal{Z}_{h_\bullet,i_\bullet,j}(c) : \rho^{-1}c \longrightarrow \mathcal{F}^1_{/\mathcal{K}_j}$, the functor that takes $[x, (x_i, d_{i,1}, ..., d_{i,k_i})]$ to natural transformation:

$$\mathcal{F}^1 \longrightarrow \mathcal{Z}(x, x_i; d_{1,1}, ..., d_{n,k_n}).$$

Let $\mathcal{F}^{1,c}_{h_\bullet,i_\bullet,j}$ be the colimit of $\mathcal{Z}_{h_\bullet,i_\bullet,j}(c)$ and denote by $\eta_{h_\bullet,i_\bullet,j} : \mathcal{F}^1 \longrightarrow \mathcal{F}^{1,c}_{h_\bullet,i_\bullet,j}$ the canonical map.

Definition 4.4.5. *Define* $\mathcal{F}^{1,1}_{h_\bullet,i_\bullet,j}$ *to be the object obtained by the generalized pushout diagram in* $\mathcal{K}_j$ *as c runs through* $\mathcal{C}_j$*:*

$$\mathcal{F}^{1,1}_{h_\bullet,i_\bullet,j} = colim_{c\in\mathcal{C}_j}\{\mathcal{F}^1 \xrightarrow{\eta_{1,c}} \mathcal{F}^{1,c}_{h_\bullet,i_\bullet,j}\}.$$

We have canonical maps $\eta : \mathcal{F}^1 \longrightarrow \mathcal{F}^{1,1}_{h_\bullet,i_\bullet,j}$ *and* $\delta_+(x,(x_i,d_{i,1},...,d_{i,k_i})) : \mathcal{Z}(x,x_i;d_{1,1},...,d_{n,k_n}) \longrightarrow \mathcal{F}^{1,1}_{h_\bullet,i_\bullet,j}$.
Define $\mathbf{T}(e,\mathcal{F},\mathcal{F}^1)_j$ *to be the object defined also by the generalized pushout:*

$$\mathbf{T}(e,\mathcal{F},\mathcal{F}^1)_j = colim \coprod_{n\in\mathbb{N}} \coprod_{(i_1,...,i_n)} \coprod_{(h_{1,1},...,h_{n,k_n})} \{\eta_{h_\bullet,i_\bullet,j} : \mathcal{F}^1 \longrightarrow \mathcal{F}^{1,1}_{h_\bullet,i_\bullet,j}\}$$

where $e : \mathcal{F} \longrightarrow \mathcal{F}^1$ *is the original morphism from the left Kan extension which gives the first laxity maps* ε.

We will write $\mathcal{F}^2 = \mathbf{T}(e,\mathcal{F},\mathcal{F}^1)$ and $\eta^1 : \mathcal{F}^1 \longrightarrow \mathcal{F}^2$ the canonical map. By construction we end up with new laxity maps $\varepsilon_1 : \otimes_x(\mathcal{F}^1c_1,...,\mathcal{F}^1c_n) \longrightarrow \mathcal{F}^2(c)$ which are not coherent, but we can iterate the process to build an object $\mathcal{F}^3 = \mathbf{T}(\mathcal{F}^1,\mathcal{F}^2)$ which 'bring the coherences of ε_1'. But the new laxity maps are not coherent so we have to repeat the process an infinite number of time.

Proceeding by induction we define for $k \in \lambda$, an object $\mathcal{F}^k$ with maps $\eta^k : \mathcal{F}^k \longrightarrow \mathcal{F}^{k+1}$ by:

1. $\mathcal{F}^0 := \mathcal{F}$ and $\eta^0 = e$.
2. $\mathcal{F}^{k+1} := \mathbf{T}(\eta^{k-1},\mathcal{F}^{k-1},\mathcal{F}^k)$, we have a map $\eta^k : \mathcal{F}^k \longrightarrow \mathcal{F}^{k+1}$ from the construction '$\mathbf{T}$'.

We therefore have a λ-sequence in $\mathcal{K}_j$:

$$\mathcal{F} = \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots \longrightarrow \mathcal{F}^k \longrightarrow \mathcal{F}^{k+1} \longrightarrow \cdots$$

and we can take the colimit $\mathcal{F}^\infty = \mathrm{colim}_{k\in\lambda}\{\mathcal{F}^k \xrightarrow{\eta^k} \mathcal{F}^{k+1}\}$.

Definition 4.4.6. *For a family* $\mathcal{F}_\bullet = (\mathcal{F}_j)_{j\in C}$ *we define* $\mathbf{F}(\mathcal{F}_\bullet)$ *by setting:*

$$\mathbf{F}(\mathcal{F}_\bullet)_j := \mathcal{F}^\infty.$$

We have a canonical map $\eta : \mathcal{F}_j \longrightarrow \mathbf{F}(\mathcal{F}_\bullet)_j$.

We have also sequences of objects $\mathcal{R}^k, Q^k$ and $\mathcal{Z}^k$ which are created step by step and there are also maps induced by universal property from $\mathcal{R}^k \longrightarrow \mathcal{R}^{k+1}$ and similarly for Q^k and $\mathcal{Z}^k$.

Proposition 4.4.7. *For a cocomplete* $\mathcal{O}$*-algebra* $\mathcal{M}_\bullet$ *we have that:*

1. *the family* $\mathbf{F}(\mathcal{F}_\bullet)$ *is a lax morphism from* $\mathcal{C}_\bullet$ *to* $\mathcal{M}_\bullet$ *and*
2. *the functor* $\mathbf{F}$ *is left adjoint of* $\mathcal{U}$.

Sketch of proof. The assertion (2) is straightforward so we leave it to the reader. For the assertion (1) we need to specify the laxity maps and check that they satisfy the coherence conditions.

By construction the following diagram involving $\mathcal{R}^k$ commutes

$$
\begin{tikzcd}
\mathbf{F}^c_{\otimes_x(\mathcal{F}^k c_1,\dots,\mathcal{F} c_n)} \arrow[r] \arrow[d, "\mathbf{F}^c_{\otimes_x(\eta^k_1,\dots\eta^k_n)}"'] & \mathcal{F}^{k+1} \arrow[d, dashed, "p(\eta^k;x,c_1,\dots,c_n)"'] \arrow[dr, dotted, "\eta^{k+1}"] & \\
\mathbf{F}^c_{\otimes_x(\mathcal{F}^{k+1} c_1,\dots,\mathcal{F}^{k+1} c_n)} \arrow[r, dashed] \arrow[d, "\mathbf{F}^c_{\otimes_x(\eta^{k+1}_1,\dots\eta^{k+1}_n)}"'] & \mathcal{R}^{k+1}(\eta^k; x, c_1, ..., c_n) \arrow[r, dotted] \arrow[dr, dotted] & \mathcal{F}^{k+2} \arrow[d, dashed] \\
\mathbf{F}^c_{\otimes_x(\mathcal{F}^{k+2} c_1,\dots,\mathcal{F}^{k+2} c_n)} \arrow[rr, dashed] & & \mathcal{R}^{k+2}(\eta^{k+1}; x, c_1, ..., c_n)
\end{tikzcd}
$$

From this diagram it's easy to see that the sequences $\mathcal{R}^k$ and $\mathcal{F}^k$ 'converge' to the same object, that is they have the same colimit object. A simple analysis shows that also Q^k and $\mathcal{Z}^k$ have as colimit object $\mathcal{F}^\infty$.

Since λ is a regular cardinal for any λ-small cardinal μ the diagonal functor $d : \lambda \longrightarrow \prod_\mu \lambda$ from λ to the product of μ copies of λ is cofinal: a consequence is that diagrams indexed by λ and $\prod_\mu \lambda$ have the same colimits. It follows, in particular, that for any $(i_1, ..., i_n) \in C^n$ the following colimits are the same

$$
\begin{cases}
\operatorname{colim}_{(k_1,\dots,k_n)\in\lambda^n}\{\otimes_x(\mathcal{F}^{k_1}_{i_1} c_1, ..., \mathcal{F}^{k_n}_{i_n} c_n)\} \\
\\
\operatorname{colim}_{k\in\lambda}\{\otimes_x(\mathcal{F}^{k}_{i_1} c_1, ..., \mathcal{F}^{k}_{i_n} c_n)\}
\end{cases}
$$

One of the assumptions on the algebra $\mathcal{M}_\bullet = (\mathcal{M}_i)_{i\in C}$ is the possibility to commute colimits computed in $\mathcal{M}_\bullet$ and the tensor products '$\otimes_x$'.

If we put these together the first colimit is easily computed as:

$$
\operatorname{colim}_{(k_1,\dots,k_n)\in\lambda^n}\{\otimes_x(\mathcal{F}^{k_1}_{i_1} c_1, ..., \mathcal{F}^{k_n}_{i_n} c_n)\} = \otimes_x(\mathbf{F}(\mathcal{F}_{i_1})c_1, ..., \mathbf{F}(\mathcal{F}_{i_n})c_n).
$$

And we deduce that:

$$
\operatorname{colim}_{k\in\lambda}\{\otimes_x(\mathcal{F}^{k}_{i_1} c_1, ..., \mathcal{F}^{k}_{i_n} c_n)\} = \otimes_x(\mathbf{F}(\mathcal{F}_{i_1})c_1, ..., \mathbf{F}(\mathcal{F}_{i_n})c_n).
$$

All these are natural in $(x, c_1, .., c_n)$. One gets the laxity maps by the universal property of the colimit with respect to the compatible cocone which ends at $\mathbf{F}(\mathcal{F}_j)c$

$$
\begin{tikzcd}
\otimes_x(\mathcal{F}^k c_1, ..., \mathcal{F} c_n) \arrow[r] \arrow[d, "\otimes_x(\eta^k_1,\dots\eta^k_n)"'] & \mathcal{F}^{k+1} c \arrow[d, dashed, "p(\eta^k;x,c_1,\dots,c_n)c"'] \arrow[dr, dotted, "\eta^{k+1}"] & & \\
\otimes_x(\mathcal{F}^{k+1} c_1, ..., \mathcal{F}^{k+1} c_n) \arrow[r, dashed] \arrow[drrr, dotted, bend right] & \mathcal{R}^{k+1}(\eta^k; x, c_1, ..., c_n)c \arrow[r, dotted] \arrow[dr, dotted] & \mathcal{F}^{k+2} c \arrow[d, dashed] \arrow[dr, dotted, "\text{canonical}"] & \\
& & \mathcal{R}^{k+2}(\eta^{k+1}; x, c_1, ..., c_n)c \arrow[r, dotted, "\text{canonical}"] & \mathbf{F}(\mathcal{F}_j)c
\end{tikzcd}
$$

So we get a unique map $\varphi^\infty(x, c_1, ..., c_n) : \otimes_x(\mathbf{F}(\mathcal{F}_{i_1})c_1, ..., \mathbf{F}(\mathcal{F}_{i_n})c_n) \longrightarrow \mathbf{F}(\mathcal{F}_j)c$ which makes the obvious diagram commutative. As usual the maps $\varphi^\infty(x, c_1, ..., c_n)$ are natural in $(x, c_1, ..., c_n)$.

The fact that these maps $\varphi^\infty(x, c_1, ..., c_n)$ satisfy the coherence conditions is easy bu tedious to check. One use the diagram involving Q^k and $\mathcal{Z}^k$ and take the colimit everywhere; the universal property of the colimit will force (by uniqueness) the equality between the two maps out of $\otimes_{\gamma(x,x_i)}(\mathbf{F}(\mathcal{F}_{h_{1,1}})d_{1,1}, ..., \mathbf{F}(\mathcal{F}_{h_{n,k_n}})d_{n,k_n})$ and going to $\mathbf{F}(\mathcal{F}_j)c$. For the record these maps are:

- $\varphi^\infty(\gamma(x, x_i), d_{1,1}, ..., d_{n,k_n})$
- $\varphi^\infty(x, c_1, ..., c_n) \circ [\otimes_x(\varphi^\infty(x_1, d_{1,1}, ..., d_{1,k_1}), \cdots, \varphi^\infty(x_n, d_{n,1}, ..., d_{n,k_n}))]$.

■

Remark 4.4.2. As $\mathcal{U}$ has a left adjoint $\mathbf{F}$ we have an induced monad $\mathcal{T} = \mathcal{U}\mathbf{F}$. It's not hard to see that $\mathcal{T}$ automatically preserves the colimits appearing in the definition of $\mathbf{F}$ namely the λ-directed ones. And since directed colimits are the same as filtered ones we deduce that $\mathcal{T}$ preserves filtered colimits as well.

4.4.2 $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ is monadic over $\mathrm{Hom}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$

Let $\sigma_1, \sigma_2 : \mathcal{F}_\bullet \longrightarrow \mathcal{G}_\bullet$ be a pair of parallel morphisms between two lax-morphisms in $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$. Denote by $L : \mathcal{G}_\bullet \longrightarrow \mathcal{E}_\bullet$ the coequalizer of σ_1, σ_2 in $\mathcal{K}_{\mathcal{C}_\bullet} = \prod_{i \in C} \mathrm{Hom}(\mathcal{C}_i, \mathcal{M}_i)$:

$$\mathcal{U}\mathcal{F}_\bullet \underset{\sigma_1}{\overset{\sigma_2}{\rightrightarrows}} \mathcal{U}\mathcal{G}_\bullet \xrightarrow{L} \mathcal{E}_\bullet$$

Note that we've freely identified σ_i and it's image $\mathcal{U}\sigma_i$. The following lemma is the general version of lemma 4.3.3 except that we do not use the language of precongruences.

Lemma 4.4.8. *Consider $\mathcal{F}_\bullet, \mathcal{G}_\bullet, \mathcal{E}_\bullet$ with σ_1, σ_2 and L as before. Assume that for every $(x, c_1, ..., c_n) \in \mathcal{O}(i_1, ..., i_n; j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n}$ with $c = \otimes_x(c_1, ..., c_n)$ and any $l \in \{1, ..., n\}$ the following equality holds:*

$$L_c \circ \varphi_{\mathcal{G}}(x, c_1, ..., c_n) \circ [\otimes_x(\mathrm{Id}_{\mathcal{G}c_1}, ..., \sigma_1(c_l), ..., \mathrm{Id}_{\mathcal{G}c_n})] = L_c \circ \varphi_{\mathcal{G}}(x, c_1, ..., c_n) \circ [\otimes_x(\mathrm{Id}_{\mathcal{G}c_1}, ..., \sigma_2(c_l), ..., \mathrm{Id}_{\mathcal{G}c_n})].$$

Then we have:

1. *$\mathcal{E}_\bullet$ becomes a lax morphism and*
2. *L is the coequalizer of σ_1 , σ_2 in $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$.*

When there is no confusion we will simply write $\varphi_{\mathcal{G}}$ instead of $\varphi_{\mathcal{G}}(x, c_1, ..., c_n)$.

Sketch of proof. The assertion (2) will follow from the proof of (1). To prove (1) we will simply give the laxity maps; the coherence conditions are straightforward.

As mentioned before, the assumptions on $\mathcal{M}_\bullet$ allow to distribute (factor wise) colimits over each tensor product $\otimes_x$. It follows that $\otimes_x(\mathcal{E}c_1, ...\mathcal{E}c_n)$ equipped with the maps $\otimes_x(L_{c_1}, ..., L_{c_n})$ is the colimit of the diagram

$$\epsilon(\sigma_1, \sigma_2; c_1, ..., c_n) = \coprod_{(\tau_1, ..., \tau_n) \in \{1,2\}^n} \{\otimes_x(\sigma_{\tau_1} c_1, ..., \sigma_{\tau_n} c_n) : \otimes_x(\mathcal{F}c_1, ..., \mathcal{F}c_n) \longrightarrow \otimes_x(\mathcal{G}c_1, ..., \mathcal{G}c_n)\}.$$

(For each l, σ_{τ_l} is either σ_1 or σ_2).

For each $(\tau_1,...,\tau_n) \in \{1,2\}^n$ let $\Theta(\tau_1,...,\tau_n) = L_c \circ \varphi_{\mathcal{G}}(x, c_1, ..., c_n) \circ \otimes_x(\sigma_{\tau_1}c_1, ..., \sigma_{\tau_n}c_n)$ be the map illustrated in the diagram below:

$$\begin{tikzcd}
\otimes_x(\mathcal{F}c_1, ..., \mathcal{F}c_n) \arrow[d, "\otimes_x(\sigma_{\tau_1}c_1,...,\sigma_{\tau_n}c_n)"'] & \mathcal{F}c \\
\otimes_x(\mathcal{G}c_1, ..., \mathcal{G}c_n) \arrow[r, "\varphi_{\mathcal{G}}"'] & \mathcal{G}c \arrow[d, "L_c"'] \\
 & \mathcal{E}c
\end{tikzcd}$$

Now we claim that $\Theta(\tau_1,...,\tau_n) = \Theta(\tau'_1,...,\tau'_n)$ for all $(\tau_1,...,\tau_n)$, $(\tau'_1,...,\tau'_n)$ in $\{1,2\}^n$. The claim will holds as soon as we show that for every $l \in \{1,...,n\}$ we have $\Theta(\tau_1,...,\tau_l,...\tau_n) = \Theta(\tau_1,...,\tau'_l,...,\tau_n)$ where τ_l and τ'_l are 'conjugate' that is: if $\tau_l = 1$ then $\tau'_l = 2$ and vice versa. Let's assume that $\tau_l = 1$ (hence $\tau'_l = 2$) that is $\sigma_{\tau_l} = \sigma_1$.

We establish successively the following equalities:

$$\begin{aligned}
\Theta(\tau_1,...,\tau_n) &= L_c \circ \varphi_{\mathcal{G}} \circ \otimes_x(\sigma_{\tau_1}c_1, ..., \sigma_1 c_l, ..., \sigma_{\tau_n}c_n) \\
&= L_c \circ \varphi_{\mathcal{G}} \circ [\otimes_x(\mathrm{Id}_{\mathcal{G}c_1}, ..., \sigma_1 c_l, ..., \mathrm{Id}_{\mathcal{G}c_n}) \circ \otimes_x(\sigma_{\tau_1}c_1, ..., \mathrm{Id}_{\mathcal{F}c_l}, ..., \sigma_{\tau_n}c_n)] \\
&= \{L_c \circ \varphi_{\mathcal{G}} \circ [\otimes_x(\mathrm{Id}_{\mathcal{G}c_1}, ..., \sigma_1 c_l, ..., \mathrm{Id}_{\mathcal{G}c_n})]\} \circ \otimes_x(\sigma_{\tau_1}c_1, ..., \mathrm{Id}_{\mathcal{F}c_l}, ..., \sigma_{\tau_n}c_n) \\
(*) &= \{L_c \circ \varphi_{\mathcal{G}} \circ [\otimes_x(\mathrm{Id}_{\mathcal{G}c_1}, ..., \sigma_2(c_l), ..., \mathrm{Id}_{\mathcal{G}c_n})]\} \circ \otimes_x(\sigma_{\tau_1}c_1, ..., \mathrm{Id}_{\mathcal{F}c_l}, ..., \sigma_{\tau_n}c_n) \\
&= L_c \circ \varphi_{\mathcal{G}} \circ [\otimes_x(\mathrm{Id}_{\mathcal{G}c_1}, ..., \sigma_2 c_l, ..., \mathrm{Id}_{\mathcal{G}c_n}) \circ \otimes_x(\sigma_{\tau_1}c_1, ..., \mathrm{Id}_{\mathcal{F}c_l}, ..., \sigma_{\tau_n}c_n)] \\
&= L_c \circ \varphi_{\mathcal{G}} \circ \otimes_x(\sigma_{\tau_1}c_1, ..., \sigma_2 c_l, ..., \sigma_{\tau_n}c_n) \\
&= \Theta(\tau_1, ..., \tau'_l, ..., \tau_n)
\end{aligned}$$

(In $(*)$ we use the hypothesis to switch σ_1 and σ_2 in the expression contained in '{ }'.) ■

Remark 4.4.3. If $\mathcal{F}_\bullet$ is simply an object of $\mathcal{K}_{\mathcal{C}_\bullet}$ but not necessarily a lax morphism but $\mathcal{G}_\bullet$ is a lax morphism, we will have a precongruence in $\mathcal{G}_\bullet$ and the first assertion of the lemma will hold. The proof will exactly be the same.

The next lemma tells about the existence of coequalizer of a parallel $\mathcal{U}$-split pair. This is the generalization of lemma 4.3.7.

Lemma 4.4.9. *Consider $\mathcal{F}_\bullet, \mathcal{G}_\bullet, \mathcal{E}_\bullet$ with σ_1, σ_2 and L as before. Assume that there is a **$\mathcal{U}$-split** i.e a morphism $p : \mathcal{U}\mathcal{G}_\bullet \longrightarrow \mathcal{U}\mathcal{F}_\bullet$ in $\mathcal{K}_{\mathcal{C}_\bullet}$ such that $\sigma_1 \circ p = \sigma_2 \circ p = \mathrm{Id}_{\mathcal{U}\mathcal{G}_\bullet}$.*
Then we have:

1. *$\mathcal{E}_\bullet$ becomes a lax $\mathcal{O}$-morphism,*

2. *$L : \mathcal{G}_\bullet \longrightarrow \mathcal{E}_\bullet$ is a coequalizer in $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ of the pair (σ_1, σ_2) and $\mathcal{U}$ obviously preserves it (as a coequalizer).*

Proof. All is proved in the same manner as for lemma 4.3.7. We simply have to show that the equalities of lemma 4.4.8 holds.

Since for each $\tau \in \{1,2\})$, $\sigma_\tau \circ p = \mathrm{Id}_{\mathcal{G}}$, for every $(c_1, ..., c_n)$ and every $l \in \{1, ..., n\}$ we have that:

$$\otimes_x(\mathrm{Id}_{\mathcal{G}c_1}, ..., \sigma_\tau c_l, ..., \mathrm{Id}_{\mathcal{G}c_n}) = \otimes_x(\sigma_\tau c_1, ..., \sigma_\tau c_l, ..., \sigma_\tau c_n) \circ \otimes_x(pc_1, ..., \mathrm{Id}_{\mathcal{F}c_l}, ..., pc_n).$$

Moreover as $\sigma_{\mathcal{T}}$ is a morphism of lax-morphisms the following commutes:

$$\begin{tikzcd}
\otimes_x(\mathcal{F}c_1, ..., \mathcal{F}c_n) \arrow[r, "\varphi_{\mathcal{F}}"] \arrow[d, "\otimes_x(\sigma_{\mathcal{T}}c_1,...,\sigma_{\mathcal{T}}c_n)"'] & \mathcal{F}c \arrow[d, "\sigma_{\mathcal{T}}c"] \\
\otimes_x(\mathcal{G}c_1, ..., \mathcal{G}c_n) \arrow[r, "\varphi_{\mathcal{G}}"'] & \mathcal{G}c
\end{tikzcd}$$

which means that we have an equality: $\sigma_{\mathcal{T}}c \circ \varphi_{\mathcal{F}} = \varphi_{\mathcal{G}} \circ \otimes_x(\sigma_{\mathcal{T}}c_1, ..., \sigma_{\mathcal{T}}c_n)$. If we combine all the previous discussion we establish successively the following.

$$\begin{aligned}
L_c \circ \varphi_{\mathcal{G}} \circ [\otimes_x(\mathrm{Id}_{\mathcal{G}c_1}, ..., \sigma_1 c_l, ..., \mathrm{Id}_{\mathcal{G}c_n})] &= L_c \circ \varphi_{\mathcal{G}} \circ [\otimes_x(\sigma_1 c_1, ..., \sigma_1 c_l, ..., \sigma_1 c_n) \circ \otimes_x(pc_1, ..., \mathrm{Id}_{\mathcal{F}c_l}, ..., pc_n)] \\
&= [L_c \circ \underbrace{\varphi_{\mathcal{G}} \circ \otimes_x(\sigma_1 c_1, ..., \sigma_1 c_l, ..., \sigma_1 c_n)}_{=\sigma_1 c \circ \varphi_{\mathcal{F}}}] \circ \otimes_x(pc_1, ..., \mathrm{Id}_{\mathcal{F}c_l}, ..., pc_n) \\
&= [\underbrace{L_c \circ \sigma_1 c}_{=L_c \circ \sigma_2 c} \circ \varphi_{\mathcal{F}}] \circ \otimes_x(pc_1, ..., \mathrm{Id}_{\mathcal{F}c_l}, ..., pc_n) \\
&= [L_c \circ \underbrace{\sigma_2 c \circ \varphi_{\mathcal{F}}}_{=\varphi_{\mathcal{G}} \circ \otimes_x(\sigma_2 c_1, ..., \sigma_2 c_l, ..., \sigma_2 c_n)}] \circ \otimes_x(pc_1, ..., \mathrm{Id}_{\mathcal{F}c_l}, ..., pc_n) \\
&= [L_c \circ \varphi_{\mathcal{G}} \circ \otimes_x(\sigma_2 c_1, ..., \sigma_2 c_l, ..., \sigma_2 c_n)] \circ \otimes_x(pc_1, ..., \mathrm{Id}_{\mathcal{F}c_l}, ..., pc_n) \\
&= L_c \circ \varphi_{\mathcal{G}} \circ [\otimes_x(\sigma_2 c_1, ..., \sigma_2 c_l, ..., \sigma_2 c_n) \circ \otimes_x(pc_1, ..., \mathrm{Id}_{\mathcal{F}c_l}, ..., pc_n)] \\
&= L_c \circ \varphi_{\mathcal{G}} \circ [\otimes_x(\mathrm{Id}_{\mathcal{G}c_1}, ..., \sigma_2 c_l, ..., \mathrm{Id}_{\mathcal{G}c_n})].
\end{aligned}$$

■

Corollary 4.4.10. *Let $\mathcal{M}_\bullet$ be a cocomplete $\mathcal{O}$-algebra. Then we have*

1. *The functor $\mathcal{U} : \mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet) \longrightarrow \mathcal{K}_{\mathcal{C}_\bullet}$ is monadic.*
2. $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ *is cocomplete.*
3. *If moreover $\mathcal{M}_\bullet$ is locally presentable then so is* $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$.

Sketch of proof. The assertion (1) follows from Beck monadicity theorem since:

- $\mathcal{U}$ has a left adjoint $\mathbf{F}$ (Proposition 4.4.2),
- $\mathcal{U}$ clearly reflect isomorphisms,
- $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ has coequalizers of parallel $\mathcal{U}$-split pairs and $\mathcal{U}$ preserves them (Lemma 4.4.9).

It follows that $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ is equivalent to the category $\mathcal{T}$-alg for the monad $\mathcal{T} = \mathcal{U}\mathbf{F}$. The assertion (2) follows from Linton's theorem [63] since $\mathcal{T}$ is defined on $\mathcal{K}_{\mathcal{C}_\bullet}$ which is cocomplete and $\mathcal{T}$-alg has coequalizer of reflexive pair.
From the Remark 4.4.2 we know that $\mathcal{T}$ preserves filtered colimits, and since $\mathcal{K}_{\mathcal{C}_\bullet}$ is locally presentable we know from [1] that $\mathcal{T}$-alg (hence $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$) is automatically locally presentable and the assertion (3) follows. ■

4.5 Some pushouts in $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$

In this section we want to show that for a trivial cofibration $\alpha \in \mathcal{K}_{\mathcal{C}_\bullet}$ then the pushout of $\mathbf{F}\alpha$ is a weak equivalence in $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$, when $\mathcal{M}_\bullet$ is a special Quillen $\mathcal{O}$-algebra (Definition 3.3.9). On $\mathcal{K}_{\mathcal{C}_\bullet}$ we will consider the *injective and projective* model structures; these are product model structures of the ones on each $\mathcal{K}_j = \mathrm{Hom}(\mathcal{C}_j, \mathcal{M}_j)$.

Given a diagram in $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$

$$\begin{tikzcd} \mathbf{F}\mathcal{A} \arrow[r, "\sigma"] \arrow[d, "\mathbf{F}\alpha"] & \mathcal{F} \\ \mathbf{F}\mathcal{B} & \end{tikzcd}$$

with α is a trivial cofibration in $\mathcal{K}_{\mathcal{C}_\bullet}$; if we want to calculate the pushout, then the first thing to do is to consider the pushout in $\mathcal{K}_{\mathcal{C}_\bullet}$ then build the laxity map etc. But the left adjoint $\mathbf{F}$ we've constructed previously, when considered as an endofunctor on $\mathcal{K}_{\mathcal{C}_\bullet}$, may not preserve weak equivalences for arbitrary $\mathcal{O}$-algebra $\mathcal{C}_\bullet$ and $\mathcal{M}_\bullet$. In particular it may not preserve trivial cofibrations. So the pushout of $\mathbf{F}\alpha$ will hardly be a weak equivalence.

The obstruction of $\mathbf{F}$ to be a left Quillen functor can be seen in the following phenomena:

1. first the left Kan extension we've considered to define $\mathcal{F}^1$ may not in general preserves level-wise (trivial) cofibrations:

$$\begin{tikzcd} \mathcal{O}(i_1, ..., i_n; j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n} \arrow[rr, "\theta_{i_\bullet|j}"] \arrow[d, "\mathrm{Id}_{\mathcal{O}(i_1,...,i_n;j)} \times \mathcal{F}_{i_1} \times \cdots \times \mathcal{F}_{i_n}"'] & & \mathcal{C}_j \arrow[d, "\mathbf{Lan}_j(\mathcal{O}, \mathcal{F}_{i_\bullet})"] \\ \mathcal{O}(i_1, ..., i_n; j) \times \mathcal{M}_{i_1} \times \cdots \times \mathcal{M}_{i_n} \arrow[rr, "\rho_{i_\bullet|j}"'] \arrow[urr, phantom, "\Nearrow \varepsilon_{i_\bullet|j}"] & & \mathcal{M}_j \end{tikzcd}$$

2. second the $\mathcal{F}^{1,c}_{h_\bullet,i_\bullet,j}$ appearing in the construction of $\mathbf{F}$ may not be left Quillen functor . In fact $\mathcal{F}^{1,c}_{h_\bullet,i_\bullet,j}$ is a colimit of a functor:

$$\mathcal{Z}_{h_\bullet,i_\bullet,j}(c) : \rho^{-1}c \longrightarrow \mathcal{F}^1/\mathcal{K}_j$$

where the source $\rho^{-1}c$ can *a priori* be any category; so the colimit may not preserve (trivial) cofibrations.

These two facts lead us to some restrictions on our statements, for the moment.
We will reduce our statement to the $\mathcal{O}$-algebra $\mathcal{C}$ such that $\rho^{-1}c$ is a discrete category i.e a set. This way the colimit of $\mathcal{Z}_{h_\bullet,i_\bullet,j}(c)$ is a generalized pushout diagram in $\mathcal{K}_j$; and pushouts interact nicely with (trivial) cofibrations.

So rather than trying to figure out under which conditions $\mathbf{F}$ preserves the level-trivial cofibration as endofunctor on $\mathcal{K}_{\mathcal{C}_\bullet}$, we will work by assuming that it is.

This reduction may appear to be too restrictive, but hopefully the cases we encounter in 'the nature' will be in this situation. Usually this will be the case for all the 'simple objects' we use to built complicated ones eg: the operad $\mathcal{O}_X$, Δ, $\overline{X}$, $\mathbb{S}_{\overline{X}}$, every 1-category $\mathcal{D}$, free $\mathcal{O}$-algebra, etc.

Recall that we introduced previously the

Definition 4.5.1. *Let* $(\mathcal{C}_\bullet, \rho)$ *and* $(\mathcal{M}_\bullet, \theta)$ *be two* $\mathcal{O}$*-algebra.*

1. *Say that* $\mathcal{C}_\bullet$ *is* $\mathcal{O}$***-well-presented****, or* $\mathcal{O}$***-identity-reflecting*** *if: for every* $n+1$*-tuple* $(i_1, ..., i_n; j)$ *the following functor reflects identities*

$$\rho : \mathcal{O}(i_1, ..., i_n; j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n} \longrightarrow \mathcal{C}_j.$$

   This means that the image of $(u, f_1, ..., f_n) \in \mathcal{O}(i_1, ..., i_n; j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n}$ *is an identity morphism in* $\mathcal{C}_j$ *(if and) only if all* $u, f_1, ..., f_n$ *are simultaneously identities.*

2. *Say that* $(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ *is an* $\mathcal{O}$***-homotopy-compatible pair*** *if* $\mathbf{F} : \mathcal{K}_{\mathcal{C}_\bullet} \longrightarrow \mathcal{K}_{\mathcal{C}_\bullet}$ *preserves level-wise trivial cofibrations, where* $\mathcal{K}_{\mathcal{C}_\bullet}$ *is endowed with the injective model structure.*

Remark 4.5.1.

1. A consequence of the definition is that if $\mathcal{C}_\bullet$ is an $\mathcal{O}$-identity-reflecting algebra (henceforth **ir**-$\mathcal{O}$-algebra), then the fiber $\rho^{-1}c = \rho^{-1}\{\mathrm{Id}_c\}$ is a set.

2. Any free $\mathcal{O}$-algebra $\mathcal{C}_\bullet$ is an **ir**-$\mathcal{O}$-algebra; and for any special Quillen $\mathcal{O}$-algebra $\mathcal{M}_\bullet$ having all its objects cofibrant, the pair $(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ is $\mathcal{O}$-homotopy compatible (henceforth $\mathcal{O}$-**hc** pair).

With the previous material we can announce the main result:

Lemma 4.5.2. *Let* $\mathcal{M}_\bullet$ *be a special Quillen* $\mathcal{O}$*-algebra such that all objects of* $\mathcal{M}_\bullet$ *are cofibrant. Let* $\mathcal{C}_\bullet$ *be* **ir**-$\mathcal{O}$*-algebra such that the* $(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ *is an* $\mathcal{O}$-**hc** *pair. Then for any pushout square in* $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$

$$\begin{tikzcd}
\mathbf{F}\mathcal{A} \arrow[r, "\sigma"] \arrow[d, "\mathbf{F}\alpha"'] & \mathcal{F} \arrow[d, "H_\alpha"] \\
\mathbf{F}\mathcal{B} \arrow[r] & \mathcal{G}
\end{tikzcd}$$

$H_\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ *is a level-wise trivial cofibration if* α *is so.*

Proof of the lemma

By the adjunction $\mathbf{F} \dashv \mathcal{U}$ the map $\sigma : \mathbf{F}\mathcal{A} \longrightarrow \mathcal{F}$ in the pushout is induced by a unique map $\mathcal{A} \longrightarrow \mathcal{U}\mathcal{F}$ in $\mathcal{K}_{\mathcal{C}_\bullet}$. Similarly the map $\mathbf{F}\mathcal{B} \longrightarrow \mathcal{G}$ is also induced by a map $\mathcal{B} \longrightarrow \mathcal{U}\mathcal{G}$. We will construct $\mathcal{G}$ out of $\mathcal{F}$ and focus our analysis on the construction of the map $H_\alpha : \mathcal{F} \longrightarrow \mathcal{G}$; the map $\mathcal{B} \longrightarrow \mathcal{U}\mathcal{G}$ will follow automatically. The first thing to do is to consider the pushout square in $\mathcal{K}_{\mathcal{C}_\bullet}$:

$$\begin{tikzcd}
\mathcal{U}\mathbf{F}\mathcal{A} \arrow[r, "\mathcal{U}\sigma"] \arrow[d, "\mathcal{U}\mathbf{F}\alpha"'] & \mathcal{U}\mathcal{F} \arrow[d, "p"] \\
\mathcal{U}\mathbf{F}\mathcal{B} \arrow[r] & \mathcal{E}
\end{tikzcd}$$

Since we assumed that $\mathbf{F}\alpha$ is a level-wise trivial cofibration, the map $p : \mathcal{F} \longrightarrow \mathcal{E}$ is automatically a level-wise trivial cofibration as well.

Intermediate laxity maps Let $(x, c_1, ..., c_n)$ be an object in $\mathcal{O}(i_1, ..., i_n; j) \times \mathcal{C}_{i_1} \times \cdots \times \mathcal{C}_{i_n}$ with $c = \otimes_x(c_1, ..., c_n) \in \mathcal{C}_j$.
Using the adjunction $\mathrm{Ev}_c : \mathrm{Hom}(\mathcal{C}_j, \mathcal{M}_j) \leftrightarrows \mathcal{M}_j : \mathbf{F}^c$ for the laxity map

$$\otimes_x(\mathcal{F}c_1, ..., \mathcal{F}c_n) \longrightarrow \mathcal{F}(\otimes_x(c_1, ..., c_n)) = \mathcal{F}(c)$$

define $\mathcal{R}(p; x, c_1, ..., c_n)$ to be the object we get from the pushout diagram in $\mathcal{K}_j$:

$$L_1 = \begin{tikzcd}
\mathbf{F}^c_{\otimes_x(\mathcal{F}c_1,...,\mathcal{F}c_n)} \arrow[r] \arrow[d, "\mathbf{F}^c_{\otimes_x(p_1,...p_n)}"'] & \mathcal{F} \arrow[d, dotted, "h(x,c_1,...,c_n)"] \\
\mathbf{F}^c_{\otimes_x(\mathcal{E}c_1,...,\mathcal{E}c_n)} \arrow[r, dotted, "q"] & \mathcal{R}(p; x, c_1, ..., c_n)
\end{tikzcd}$$

As each p_k is a trivial cofibration (with cofibrant domain) and since $\mathscr{M}$ is a special Quillen $\mathcal{O}$-algebra, we have that $\otimes_x(p_1, ...p_n)$ is trivial cofibration in $\mathcal{M}_j$. Applying the left Quillen functor $\mathbf{F}^c$, we have that $\mathbf{F}^c_{\otimes_x(p_1,...p_n)}$ is a projective (hence an injective) trivial cofibration. It follows that $h(x, c_1, ..., c_n)$ is also a projective trivial cofibration (as a pushout of such morphism) and therefore a level-wise trivial cofibration.
When the context is clear we will simply write $\mathcal{R}(x, c_\bullet)$ or $\mathcal{R}_{c_\bullet}$, and $p(x, c_\bullet)$, etc.

Intermediate coherences With the 'temporary' laxity maps we need to have a 'temporary coherence' as well. We start with the objects on the fiber $\rho^{-1}c = \otimes^{-1}\{c\}$.

Let $[x, (x_i, d_{i,1}, ..., d_{i,k_i})_{1\leq i\leq n}]$ be an object of $\mathcal{O}(i_\bullet|j) \times [\mathcal{O}(h_{1,\bullet}|i_1) \times \mathcal{C}_{1,\bullet}] \times \cdots \times [\mathcal{O}(h_{n,\bullet}|i_n) \times \mathcal{C}_{n,\bullet}]$ such that:

- $\otimes_{x_i}(d_{i,1}, ..., d_{i,k_i}) = c_i$,
- $\otimes_{\gamma(x,x_i)}(d_{1,1}, ..., d_{n,k_n}) = c$, and
- $\otimes_x(c_1, ..., c_n) = c$.

The coherence condition on the lax morphism $\mathcal{F}$ is equivalent to say that the upper face of the semi-cube below is commutative.

$$\begin{tikzcd}
 & & \mathcal{F}c \arrow[dr, equal, "\mathrm{Id}"] \arrow[dd, dotted] & \\
\otimes_{\gamma(x,x_i)}(\mathcal{F}d_{1,1}, ..., \mathcal{F}d_{n,k_n}) \arrow[urr, "\varphi"] \arrow[dr, "\otimes_x(\varphi_1,...,\varphi_n)"'] \arrow[ddd, "\otimes_{\gamma(x,x_i)}p"'] & & & \mathcal{F}c \\
 & \otimes_x(\mathcal{F}c_1, ..., \mathcal{F}c_n) \arrow[urr, "\varphi"] \arrow[ddd, "\otimes_x(h_{c_i})"] & \mathcal{R}(\gamma(x, x_i), d_{1,1}, ..., d_{n,k_n}) & \\
 & & & \\
\otimes_{\gamma(x,x_i)}(\mathcal{E}d_{1,1}, ..., \mathcal{E}d_{n,k_n}) \arrow[uurr, dotted] \arrow[dr, "①"] & & & \\
 & \otimes_x(\mathcal{R}_{d_{1\bullet}}c_1, ..., \mathcal{R}_{d_{n\bullet}}c_n) & &
\end{tikzcd}$$

Here ① represents the map:

$$\otimes_{\gamma(x,x_i)}(\mathcal{E}d_{1,1}, ..., \mathcal{E}d_{n,k_n}) = \otimes_x[\otimes_{x_1}(\mathcal{E}d_{1\bullet}), ..., \otimes_{x_n}(\mathcal{E}d_{n\bullet})] \xrightarrow{\otimes_x[q_{d_{1\bullet}}, ..., q_{d_{n\bullet}}]} \otimes_x(\mathcal{R}_{d_{1\bullet}}c_1, ..., \mathcal{R}_{d_{n\bullet}}c_n)$$

with $q_{d_{i\bullet}} : \otimes_{x_i}(\mathcal{E}d_{i,1}, ..., \mathcal{E}d_{i,k_i}) \longrightarrow \mathcal{R}(x_i, d_{i,1}, ..., d_{i,k_i})$.

Extend the upper face by the commutative square (L_1) above; then extend the face on the right by taking the pushout of the trivial cofibration $\otimes_x(h_{c_i})$ along the trivial cofibration $\otimes_x(p_{c_i})$. We get a new semi-cube $C(x, x_i, d_i)$ where the face in the back is unchanged.
Since the face in the back is a pushout square and the vertical map in the front is a trivial cofibration, we are in the situation of the Reedy style lemma 3.7.4.

Introduce $O(x, x_i, d_i)$ to be the colimit of the semi-cube $C(x, x_i, d_i)$. By virtue of lemma 3.7.4, the canonical map $\beta : \mathcal{F}c \longrightarrow O(x, x_i, d_i)$ is a trivial cofibration. Applying the left Quillen functor $\mathbf{F}^c$ we get a projective trivial cofibration $\mathbf{F}^c_\beta : \mathbf{F}^c_{\mathcal{F}c} \longrightarrow \mathbf{F}^c_{O(x,x_i,d_i)}$.

The co-unit of the adjunction $\mathbf{F}^c \dashv \mathrm{Ev}_c$ corresponds to a map $e : \mathbf{F}^c_{\mathcal{F}c} \longrightarrow \mathcal{F}$. Define $Q(x, x_i, d_i)$ to be the functor we get by the pushout of $\mathbf{F}^c_\beta$ along e in $\mathcal{K}_j$:

$$\begin{tikzcd}
\mathbf{F}^c_{\mathcal{F}c} \arrow[r, "e"] \arrow[d, hook, "\mathbf{F}^c_\beta"', "\wr"] & \mathcal{F}_j \arrow[d, dotted, "p_{(x,x_i,d_i)}"] \\
\mathbf{F}^c_{O(x,x_i,d_i)} \arrow[r, dotted] & Q(x, x_i, d_i)
\end{tikzcd}$$

Define $\mathcal{F}^{1,c}_j$ to be $\mathrm{colim}_{(x,x_i,d_i)\in\rho^{-1}c}\ \{\mathcal{F}_j \xrightarrow{p_{(x,x_i,d_i)}} Q(x, x_i, d_i)\}$, where:

$$\rho^{-1}c = \coprod_{n\in\mathbb{N}^*} \coprod_{(i_1,\dots,i_n)} \coprod_{(h_{1,1},\dots,h_{n,k_n})} \rho^{-1}_{h_\bullet,i_\bullet,j}\{\mathrm{Id}_c\}.$$

Since we assumed that $\mathcal{C}$ is an **ir**-$\mathcal{O}$-algebra then $\rho^{-1}c$ is a <u>set</u>, therefore the colimit is a generalized pushout diagram in $\mathcal{K}_j$. This is what we called a cone of trivial cofibrations in the model category $\mathcal{K}_{j\text{-inj}}$. By Lemma 4.1.6 we deduce that all the canonical maps going to the colimit are trivial cofibrations in $\mathcal{K}_{j\text{-inj}}$; in particular the map $\iota_c : \mathcal{F}_j \longrightarrow \mathcal{F}^{1,c}_j$ is an injective trivial cofibration.

The construction 'P' Recall that all the previous construction are obtained from the map $p : \mathcal{F} \longrightarrow \mathcal{E}$ which is an object of the under category $\mathcal{F}_{/\mathcal{K}_{\mathcal{C}_\bullet}}$. It's not hard to see that these constructions are functorial in p.

Definition 4.5.3. *For each j, define $\mathbf{P}(j, p, \mathcal{F}, \mathcal{E})$ to be the colimit of the cone of trivial cofibrations in $\mathcal{K}_j$:*

$$\mathbf{P}(j, p, \mathcal{F}, \mathcal{E}) = (\coprod_{c\in Ob(\mathcal{C}_j)} \iota_c : \mathcal{F}_j \longrightarrow \mathcal{F}^{1,c}_j) \cup \{p_j : \mathcal{F} \longrightarrow \mathcal{E}_j\}.$$

Denote by $\eta^1_j : \mathcal{F}_j \longrightarrow \mathbf{P}(j, p, \mathcal{F}, \mathcal{E})$ and $\delta^1_j : \mathcal{E}_j \longrightarrow \mathbf{P}(j, p, \mathcal{F}, \mathcal{E})$ the canonical trivial cofibrations.

By the above remark one clear see that $\mathbf{P}$ is an endofunctor of $\mathcal{F}_{/\mathcal{K}_{\mathcal{C}_\bullet}}$, that takes p to η^1. Moreover for any j the following commutes:

$$\begin{tikzcd}
\mathcal{F}_j \arrow[rr, "\eta^1_j"] \arrow[dr, "p_j"'] & & \mathbf{P}(j, p, \mathcal{F}, \mathcal{E}) \\
& \mathcal{E}_j \arrow[ur, "\delta^1_j"'] &
\end{tikzcd}$$

As $\mathbf{P}$ is an endofunctor, we can repeat the process and apply the previous construction to
$\eta^1 = \{\mathcal{F}_j \xrightarrow{\eta^1_j} \mathbf{P}(j, p, \mathcal{F}, \mathcal{E})\}$ and repeat again and so forth.

Let κ be a regular cardinal. For each j we define a κ-sequence $(\mathcal{F}_j^k)_{k\in\kappa}$ in $\mathcal{K}_j$ as follows.

1. $\mathcal{F}_j = \mathcal{F}_j^0$,
2. $\mathcal{F}_j^1 = \mathcal{E}_j$,
3. $\mathcal{F}_j^k = \mathbf{P}(j, \eta^{k-1}, \mathcal{F}, \mathcal{F}^{k-1})$ for $k \geq 2$,
4. there are canonical maps $\delta^k : \mathcal{F}_j^{k-1} \longrightarrow \mathcal{F}_j^k$ and $\eta^k : \mathcal{F}_j \longrightarrow \mathcal{F}_j^k$ such that $\eta^k = \delta^k \circ \eta^{k-1}$; with $\eta_j^0 = p_j$

We end up with a κ-directed diagram in $\mathcal{K}_j$:

$$\mathcal{F}_j = \mathcal{F}_j^0 \xrightarrow{p_j} \mathcal{F}_j^1 \xrightarrow{\delta_j^1} \cdots \xrightarrow{\delta_j^{k-1}} \mathcal{F}_j^k \xrightarrow{\delta_j^k} \mathcal{F}_j^{k+1} \xrightarrow{\delta_j^{k+1}} \cdots$$

Define $\mathcal{F}_j^\infty$ to be the colimit in $\mathcal{K}_j$ of that κ-sequence and denote by $\eta_j^\infty : \mathcal{F}_j \longrightarrow \mathcal{F}_j^\infty$ the canonical map.

Remark 4.5.2. Since both δ^k and η^k are trivial cofibrations, it follows that η_j^∞ is also a trivial cofibration. Furthermore we have a factorization of η_j^∞ as: $\mathcal{F}_j \overset{p_j}{\hookrightarrow} \mathcal{E}_j \overset{\delta_j^\infty}{\hookrightarrow} \mathcal{F}_j^\infty$. By construction we have also other κ-sequences $(\mathcal{R}^k)_{k\in\kappa}$, $(O^k)_{k\in\kappa}$ and $(Q^k)_{k\in\kappa}$; $\mathcal{R}^k$ bring the laxity maps and Q^k bring the coherences. These objects interact in the semi-cubes $C^k(x, x_i, d_i)$.
For each j and each $c \in \mathcal{C}_j$, all the three sequences $\{\mathcal{R}^k(c)\}_{k\in\kappa}$, $\{O^k(c)\}_{k\in\kappa}$ and $\{Q^k(c)\}_{k\in\kappa}$ have the same colimit object which is $\mathcal{F}_j^\infty(c)$.

We complete the proof with the following

Proposition 4.5.4.

1. *For every laxity map $\otimes_x(\mathcal{F}c_1, ..., \mathcal{F}c_n) \longrightarrow \mathcal{F}(c)$ we have a map $\otimes_x(\mathcal{F}^\infty c_1, ..., \mathcal{F}^\infty c_n) \longrightarrow \mathcal{F}^\infty(c)$ and the following commutes:*

$$\begin{tikzcd} \otimes_x(\mathcal{F}c_1, ..., \mathcal{F}c_n) \arrow[r, "\varphi"] \arrow[d, "\otimes_x(\eta^\infty c_1, ..., \eta^\infty c_n)"'] & \mathcal{F}(c) \arrow[d, "\eta^\infty"] \\ \otimes_x(\mathcal{F}^\infty c_1, ..., \mathcal{F}^\infty c_n) \arrow[r, "\varphi^\infty"] & \mathcal{F}^\infty(c) \end{tikzcd}$$

2. *The maps φ^∞ fit coherently and $(\mathcal{F}_j^\infty)_j$ equipped with φ^∞ is a lax $\mathcal{O}$-morphism i.e an object of* $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$.
3. *The map $\eta^\infty = (\eta_j^\infty) : \mathcal{F} \longrightarrow \mathcal{F}^\infty$ is the pushout in* $\mathrm{Lax}_{\mathcal{O}\text{-alg}}(\mathcal{C}_\bullet, \mathcal{M}_\bullet)$ *of $\mathbf{F}\alpha$ along σ.*
4. *$\mathcal{U}(\eta^\infty)$ is also a level-wise trivial cofibration, so in particular a weak equivalence.*

Sketch of proof. The proof of (1) is exactly the same for the Proposition 4.4.7. One gets the laxity maps by the universal property of the colimit of $\{\otimes_x(\mathcal{F}^k c_1, ..., \mathcal{F}^k c_n)\}_{k\in\kappa}$, with respect to the following compatible cocone which ends at $\mathcal{F}^\infty(c)$ (and starts from $\otimes_x(\mathcal{F}c_1, ..., \mathcal{F}c_n) \longrightarrow \mathcal{F}(c)$):

$$
\begin{tikzcd}
\otimes_x(\mathcal{F}^k c_1, ..., \mathcal{F}^k c_n) \arrow[r] \arrow[d, "\otimes_x(\delta_1^k, ...\delta_n^k)"'] & \mathcal{F}^{k+1}c \arrow[d, dashed, "p(x,c_1,...,c_n)_c"'] \arrow[dr, dotted, "\delta^{k+1}"] & & \\
\otimes_x(\mathcal{F}^{k+1} c_1, ..., \mathcal{F}^{k+1} c_n) \arrow[r, dashed] \arrow[drrr, dotted, bend right] & \mathcal{R}^{k+1}(x, c_1, ..., c_n)c \arrow[r, dotted] \arrow[dr, dotted] & \mathcal{F}^{k+2}c \arrow[d, dashed] \arrow[dr, dotted, "\text{canonical}"] & \\
 & & \mathcal{R}^{k+2}(x, c_1, ..., c_n)c \arrow[r, dotted, "\text{canonical}"] & \mathcal{F}^\infty(c)
\end{tikzcd}
$$

One computed the colimit of $\{\otimes_x(\mathcal{F}^k c_1, ..., \mathcal{F}^k c_n)\}_{k\in\kappa}$ by the same method explained in the proof of Proposition 4.4.7. The map $\varphi^\infty(x, c_1, ..., c_n) : \otimes_x(\mathcal{F}^\infty c_1, ..., \mathcal{F}^\infty c_n) \longrightarrow \mathcal{F}^\infty(c)$ is the unique map which makes everything commutative.

The coherence condition follows by construction; one takes the colimit everywhere in the universal cube defined by the semi-cubes $C^k(x, x_i, d_i)$. The coherence is given by 'the cube at the infinity'. The assertion (3) is easily checked and follows by construction: $\mathcal{F}^\infty$ with the obvious maps satisfies the universal property of the pushout. It's important to notice that this is valid because both $\mathbf{F}\mathcal{A}$ and $\mathbf{F}\mathcal{B}$ are free objects, therefore the map $\mathbf{F}\mathcal{B} \longrightarrow \mathcal{F}^\infty$ is induced by the composite $\mathcal{B} \longrightarrow \mathbf{F}\mathcal{B} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}^\infty$.

The assertion (4) is obvious. ■

4.6 Review of the notion of bicategory

4.6.1 Definitions

Definition 4.6.1. *A small bicategory $\mathcal{C}$ is determined by the following data:*

- *a nonempty set of objects $\underline{\mathcal{C}} = Ob(\mathcal{C})$*
- *a category $\mathcal{C}(A, B)$ of arrows for each pair (A, B) of objects of $\mathcal{C}$*
- *a composition functor $c(A, B, C) : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$ for each triple (A, B, C) of objects of $\mathcal{C}$*
- *an identity arrow $I_A : 1 \longrightarrow \mathcal{C}(A, A)$ for any object A of $\mathcal{C}$*
- *for each quadruple (A, B, C, D) of objects of $\mathcal{C}$ a natural isomorphism $a(A, B, C, D)$, called* ***associativity isomorphism****, between the two composite functors bounding the diagram :*

$$
\begin{tikzcd}
\mathcal{C}(C, D) \times \mathcal{C}(B, C) \times \mathcal{C}(A, B) \arrow[r, "\mathrm{Id} \times c(A,B,C)"] \arrow[d, "c(B,C,D)\times \mathrm{Id}"'] \arrow[dr, bend right] \arrow[dr, bend left] \arrow[dr, phantom, "\Nearrow a(A,B,C,D)"] & \mathcal{C}(C, D) \times \mathcal{C}(A, C) \arrow[d, "c(A,C,D)"] \\
\mathcal{C}(B, D) \times \mathcal{C}(A, B) \arrow[r, "c(A,B,D)"'] & \mathcal{C}(A, D)
\end{tikzcd}
$$

Explicitly :

$$a(A,B,C,D) : c(A,B,D) \circ (c(B,C,D) \times \mathrm{Id}) \longrightarrow c(A,C,D) \circ (\mathrm{Id} \times c(A,B,C))$$

Then if (h,g,f) *is an object of* $\mathcal{C}(C,D) \times \mathcal{C}(B,C) \times \mathcal{C}(A,B)$*, the isomorphism, component of* $a(A,B,C,D)$ *at* (h,g,f) *will be abbreviated into* $a(h,g,f)$ *or even* a *:*

$$a = a(h,g,f) = a(A,B,C,D)(h,g,f) : \ (h \otimes g) \otimes f \xrightarrow{\sim} h \otimes (g \otimes f)$$

– *for each pair* (A,B) *of objects of* $\mathcal{C}$*, two natural isomorphisms* $l(A,B)$ *and* $r(A,B)$ *called* ***left*** *and* ***right*** *identities, between the functors bounding the diagrams:*

$$\begin{tikzcd}
1 \times \mathcal{C}(A,B) \arrow[d, "I_B \times \mathrm{Id}"'] \arrow[dr, "\sim"] & \\
\mathcal{C}(B,B) \times \mathcal{C}(A,B) \arrow[r, "c(A,B,B)"'] \arrow[ur, phantom, "\Nearrow l(A,B)"] & \mathcal{C}(A,B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(A,B) \times 1 \arrow[d, "\mathrm{Id} \times I_A"'] \arrow[dr, "\sim"] & \\
\mathcal{C}(A,B) \times \mathcal{C}(A,A) \arrow[r, "c(A,A,B)"'] \arrow[ur, phantom, "\Nearrow r(A,B)"] & \mathcal{C}(A,B)
\end{tikzcd}$$

If f *is an object of* $\mathcal{C}(A,B)$*, the isomorphism, component of* $l(A,B)$ *at* f

$$l(A,B)(f) : \ I_B \otimes f \xrightarrow{\sim} f$$

is abbreviated into $l(f)$ *or even* l*, and similarly we write*

$$r = r(f) = r(A,B)(f) : \ f \otimes I_A \xrightarrow{\sim} f$$

The natural isomorphisms $a(A,B,C,D)$*,* $l(A,B)$ *and* $r(A,B)$ *are furthermore required to satisfy the following axioms :*

(A. C.): Associativity coherence : If (k,h,g,f) *is an object of* $\mathcal{C}(D,E) \times \mathcal{C}(C,D) \times \mathcal{C}(B,C) \times \mathcal{C}(A,B)$ *the following diagram commutes :*

$$\begin{tikzcd}
((k \otimes h) \otimes g) \otimes f \arrow[rr, "a(k,h,g) \otimes \mathrm{Id}"] \arrow[d, "a(k \otimes h,g,f)"'] & & (k \otimes (h \otimes g)) \otimes f \arrow[d, "a(k,h \otimes g,f)"] \\
(k \otimes h) \otimes (g \otimes f) \arrow[dr, "a(k,h,g \otimes f)"'] & & k \otimes ((h \otimes g) \otimes f) \arrow[dl, "\mathrm{Id} \otimes a(h,g,f)"] \\
& k \otimes (h \otimes (g \otimes f)) &
\end{tikzcd}$$

(I. C.): Identity coherence : If (g,f) *is an object of* $\mathcal{C}(B,C) \times \mathcal{C}(A,B)$*, the following diagram commutes :*

$$\begin{tikzcd}
(g \otimes I_B) \otimes f \arrow[rr, "a(g,I_B,f)"] \arrow[dr, "r(g) \otimes \mathrm{Id}"'] & & g \otimes (I_B \otimes f) \arrow[dl, "\mathrm{Id} \otimes l(f)"] \\
& g \otimes f &
\end{tikzcd}$$

Variant. When all the natural isomorphisms a, l, r are *identities* then $\mathcal{C}$ is said to be a *strict 2-category*

Classically objects of $\mathcal{C}$ are called 0-*cells*, those of each $\mathcal{C}(A, B)$ are called 1-*cells* or 1-*morphisms* and arrows between 1-*morphisms* are called 2-*cells* or 2-*morphisms*.

1. In each $\mathcal{C}(A, B)$:

   – every 1-*cell* f has an identity 2-*cell* :

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & B \end{tikzcd}$$

   – we have a *vertical* composition of 2-*cells*: '$- \star -$'

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\beta\Downarrow"] & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\beta\star\alpha\Downarrow"] & B \end{tikzcd}$$

2. In the *composition functor* we have:

   – a *classical* composition of 1-*cells*: '$- \otimes -$'

$$C \xleftarrow[g]{} B \xleftarrow[f]{} A \rightsquigarrow C \xleftarrow[g \otimes f]{} A$$

   – a *horizontal* composition of 2-*cells*: '$- \otimes -$'

$$\begin{tikzcd} C & B \arrow[l, bend right, "g"'] \arrow[l, bend left, "g'"] \arrow[l, phantom, "\beta\Downarrow"] & A \arrow[l, bend right, "f"'] \arrow[l, bend left, "f'"] \arrow[l, phantom, "\alpha\Downarrow"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} C & A \arrow[l, bend right, "g\otimes f"'] \arrow[l, bend left, "g'\otimes f'"] \arrow[l, phantom, "\beta\otimes\alpha\Downarrow"] \end{tikzcd}$$

$$(\beta \otimes \alpha)(g \otimes f) = \beta(g) \otimes \alpha(f) = g' \otimes f'$$

Example 4.6.2. [Bénabou] Let $(\mathscr{M}, \otimes, I, \alpha, \lambda, \rho)$ be a *monoidal* category. We define a bicategory $\widetilde{\mathscr{M}}$ by:

– $\mathrm{Ob}(\widetilde{\mathscr{M}}) = \{\bigstar\}$

– $\widetilde{\mathscr{M}}(\bigstar, \bigstar) = \mathscr{M}$

– $c(\bigstar, \bigstar, \bigstar) = \otimes$

– $I_{\bigstar} = I$

– $a(\bigstar, \bigstar, \bigstar, \bigstar) = \alpha$

– $l(\bigstar, \bigstar) = \lambda$

– $r(\bigstar, \bigstar) = \rho$

We easily check that the isomorphisms a, l, r satisfy the (A.C.) and (I.C.) axioms since α, λ, ρ satisfy the *associativity* and *identities* axioms of a monoidal category. Conversely every bicategory with one object "is" a monoidal category.

More generally we have:

Proposition 4.6.3. *Let $\mathcal{C}$ be a bicategory and A an object of $\mathcal{C}$, then $\otimes = c(A, A, A)$, $I = I_A$, $\alpha = a(A, A, A, A)$, $\lambda = l(A, A)$, $\rho = r(A, A)$ determine a* monoidal structure *on the category $\mathcal{C}(A, A)$.*

4.6.2 Morphisms of bicategories

Definition 4.6.4. [Lax morphism] *Let $\mathscr{B} = (\underline{\mathscr{B}}, c, , I, a, l, r)$ and $\mathcal{C} = (\underline{\mathcal{C}}, c', I', a', l', r')$ be two small bicategories. A lax morphism $F = (F, \varphi)$ from $\mathscr{B}$ to $\mathcal{C}$ is determined by the following:*

- *A map $F : \underline{\mathscr{B}} \longrightarrow \underline{\mathcal{C}}$, $A \rightsquigarrow FA$*
- *A family functors*
$$F_{AB} = F(A, B) : \mathscr{B}(A, B) \longrightarrow \mathcal{C}(FA, FB),$$
$$f \rightsquigarrow Ff, \alpha \rightsquigarrow F\alpha$$
- *For each object A of $\mathscr{B}$ an arrow of $\mathcal{C}(FA, FA)$ (i.e a* 2-cell *of $\mathcal{C}$) :*
$$\varphi_A : I'_{FA} \longrightarrow F(I_A)$$
- *A family of natural transformations :*
$$\varphi(A, B, C) : c'(FA, FB, FC) \circ (F_{BC} \times F_{AB}) \longrightarrow F_{AC} \circ c(A, B, C)$$
$$\begin{tikzcd}
\mathscr{B}(B,C) \times \mathscr{B}(A,B) \arrow[rr, "c(A,B,C)"] \arrow[d, "F_{BC}\times F_{AB}"'] \arrow[drr, bend right] \arrow[drr, bend left] \arrow[drr, phantom, "\Nearrow \varphi(A,B,C)"] & & \mathscr{B}(A,C) \arrow[d, "F_{AC}"] \\
\mathcal{C}(FB,FC) \times \mathcal{C}(FA,FB) \arrow[rr, "c'(FA,FB,FC)"'] & & \mathcal{C}(FA,FC)
\end{tikzcd}$$
*If (g, f) is an object of $\mathscr{B}(B, C) \times \mathscr{B}(A, B)$, the (g, f)-*component *of $\varphi(A, B, C)$*
$$Fg \otimes Ff \xrightarrow{\varphi(A,B,C)(g,f)} F(g \otimes f)$$
shall be usually abbreviated to φ_{gf} or even φ.

These data are required to satisfy the following coherence axioms:

(M.1): If (h, g, f) is an object of $\mathscr{B}(C, D) \times \mathscr{B}(B, C) \times \mathscr{B}(A, B)$ the following diagram, where the indices A, B, C, D have been omitted, is commutative:
$$\begin{tikzcd}
(Fh \otimes Fg) \otimes Ff \arrow[r, "\varphi_{hg}\otimes \mathrm{Id}"] \arrow[d, "a'(Fh,Fg,Ff)"'] & (F(h \otimes g)) \otimes Ff \arrow[r, "\varphi_{(h\otimes g)f}"] & F((h \otimes g) \otimes f) \arrow[d, "Fa(h,g,f)"] \\
Fh \otimes (Fg \otimes Ff) \arrow[r, "\mathrm{Id}\otimes \varphi_{gf}"] & Fh \otimes (F(g \otimes f)) \arrow[r, "\varphi_{h(g\otimes f)}"] & F(h \otimes (g \otimes f))
\end{tikzcd}$$

(M.2): If f is an object of $\mathscr{B}(A,B)$ the following diagrams commute:

$$\begin{tikzcd}
Ff \otimes I'_{FA} \arrow[r, "\mathrm{Id}\otimes\varphi_A"'] \arrow[d, "r'(Ff)"'] & Ff \otimes FI_A \arrow[r, "\varphi_{fI_A}"'] & F(f\otimes I_A) \arrow[d, "Fr(f)"] \\
Ff \arrow[rr, equal] & & Ff
\end{tikzcd}
\quad
\begin{tikzcd}
I'_{FB} \otimes Ff \arrow[r, "\varphi_B\otimes\mathrm{Id}"'] \arrow[d, "l'(Ff)"'] & FI_B \otimes Ff \arrow[r, "\varphi_{I_B f}"'] & F(I_B\otimes f) \arrow[d, "Fl(f)"] \\
Ff \arrow[rr, equal] & & Ff
\end{tikzcd}$$

Variant.

1. We will say that $F = (F,\varphi)$ is a **colax morphism** if $\varphi(A,B,C)$ and φ_A are in the **opposite sense** i.e
$$Fg \otimes Ff \xleftarrow{\varphi(A,B,C)(g,f)} F(g\otimes f)$$
$$I'_{FA} \xleftarrow{\varphi_A} F(I_A)$$
and all of the horizontal arrows in the diagrams of *(M.1)* and *(M.2)* are in the opposite sense.

2. If $\varphi(A,B,C)$ and φ_A are **natural isomorphisms**, so that $Fg\otimes Ff \xrightarrow{\sim} F(g\otimes f)$ and $I'_{FA} \xrightarrow{\sim} F(I_A)$ then $F = (F,\varphi)$ is called a **homomorphism**.

3. If $\varphi(A,B,C)$ and φ_A are **identities**, so that $Fg\otimes Ff = F(g\otimes f)$ and $I'_{FA} = F(I_A)$ then $F = (F,\varphi)$ is called a **strict homomorphism**.

4.7 The 2-Path-category of a small category

Let $\mathcal{C}$ be a small category. For any pair (A,B) of objects such that $\mathcal{C}(A,B)$ is nonempty, we build **from the composition operation and its properties** a simplicial diagram as follows :

1. If $A \neq B$:

$$\mathcal{C}(A,B) \; \substack{\dashrightarrow \\ \longleftarrow \\ \dashrightarrow} \; \coprod \mathcal{C}(A,A_1)\times\mathcal{C}(A_1,B) \; \substack{\dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow} \; \coprod \mathcal{C}(A,A_1)\times\mathcal{C}(A_1,A_2)\times\mathcal{C}(A_2,B)\cdots$$

2. If $A = B$:

$$\{A\}\cong 1 \xrightarrow{1_A} \mathcal{C}(A,A) \; \substack{\dashrightarrow \\ \longleftarrow \\ \dashrightarrow} \; \coprod \mathcal{C}(A,A_1)\times\mathcal{C}(A_1,A) \; \substack{\dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow} \; \coprod \mathcal{C}(A,A_1)\times\mathcal{C}(A_1,A_2)\times\mathcal{C}(A_2,A)\cdots$$

Here the dotted arrows correspond to add an identity map of an object and the normal arrows correspond to replace composable pair of arrows by their composite.

In each case the diagram "represents" a functor which is a **cosimplicial set**:

- If $A\neq B$: $\mathscr{P}_{AB} : \Delta^+ \longrightarrow \mathrm{Set}$,
- If $A = B$: $\mathscr{P}_{AA} : \Delta \longrightarrow \mathrm{Set}$.

Observations 4.7.1.

1. Here $\mathscr{P}_{AB}(n)$ is the set of n-simplices of the *nerve of* $\mathcal{C}$, with extremal vertices A and B :

$$\mathscr{P}_{AB}(n) = \coprod_{(A=A_0,\ldots,A_n=B)} \mathcal{C}(A_0, A_1) \times \cdots \times \mathcal{C}(A_{n-1}, A_n)$$

   in particular we have : $\mathscr{P}_{AB}(1) = \mathcal{C}(A, B)$.

2. If $A = B$ and for $n = 0$, $\mathscr{P}_{AA}(0)$ has a unique element which is identified with the object A.

3. We will represent an element s of $\mathscr{P}_{AB}(n)$ as a n-tuple

$$s = (A_0 \longrightarrow A_1, \cdots, A_i \longrightarrow A_{i+1}, \cdots, A_{n-1} \longrightarrow A_n)$$

   or as an *oriented graph*

$$s = A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow \cdots \longrightarrow A_{n-1} \longrightarrow A_n.$$

4. For a map $u : n \longrightarrow m$ of Δ, $\mathscr{P}_{AB}(u) : \mathscr{P}_{AB}(n) \longrightarrow \mathscr{P}_{AB}(m)$ is a function which sends a n-simplex to a m-simplex.
   Such function corresponds to :
   - (one or many) insertions of identities if $n < m$
   - (one or many) compositions at some vertices if $n > m$.

Terminology. An element of $\mathscr{P}_{AB}(n)$ will be called a *path or chain of length* n from A to B. When $A = B$ we will call *loops of length* n the elements of $\mathscr{P}_{AA}(n)$. In particular there is a unique path of length 0, which is identified with the object A.

We can rewrite the simplicial diagrams above as :

$$\begin{tikzcd} \mathscr{P}_{AB}(1) \arrow[r, dotted, shift left=2] \arrow[r, dotted, shift right=2] & \mathscr{P}_{AB}(2) \arrow[l] \arrow[r, dotted, shift left=4] \arrow[r, dotted] \arrow[r, dotted, shift right=4] & \mathscr{P}_{AB}(3)\cdots \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{P}_{AA}(0) \arrow[r, dotted, "1_A"] & \mathscr{P}_{AA}(1) \arrow[r, dotted, shift left=2] \arrow[r, dotted, shift right=2] & \mathscr{P}_{AA}(2) \arrow[l] \arrow[r, dotted, shift left=4] \arrow[r, dotted] \arrow[r, dotted, shift right=4] & \mathscr{P}_{AA}(3)\cdots \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

Definition 4.7.1. [Concatenation of paths]
Given s in $\mathscr{P}_{AB}(n)$ and t in $\mathscr{P}_{BC}(m)$

$$s = A \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow \cdots \longrightarrow A_{n-1} \longrightarrow B$$

$$t = B \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_j \longrightarrow B_{j+1} \longrightarrow \cdots \longrightarrow B_{m-1} \longrightarrow C$$

we define the ***concatenation*** *of t and s to be the element of $\mathscr{P}_{AC}(n+m)$:*

$$s * t := \underbrace{A \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_{n-1} \longrightarrow}_{s} B \underbrace{\longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_{m-1} \longrightarrow C}_{t}.$$

It follows from the definition that for any n and for any $s \in \mathscr{P}_{AB}(n)$ we have :

- $s * B = s$,
- $A * s = s$.

The Grothendieck construction In the following we're going to apply the Grothendieck construction to the functors $\mathscr{P}_{AB}$, $\mathscr{P}_{AA}$.

For any pair of objects (A, B) we denote by $\mathscr{P}_{\mathcal{C}}(A, B)$ the *category of elements* or the *Grothendieck integral* of the functor $\mathscr{P}_{AB}$ described as follows.

- The objects of $\mathscr{P}_{\mathcal{C}}(A, B)$ are pairs $[n, s]$, where n is an object of Δ and $s \in \mathscr{P}_{AB}(n)$.
- A morphism $[n, s] \xrightarrow{u} [m, t]$ in $\mathscr{P}_{\mathcal{C}}(A, B)$ is a map $u : n \longrightarrow m$ of Δ such that image of u by $\mathscr{P}_{AB}$ sends s to t :
$$\mathscr{P}_{AB}(u) : \mathscr{P}_{AB}(n) \longrightarrow \mathscr{P}_{AB}(m)$$
and
$$\mathscr{P}_{AB}(u)s = t.$$

We have a forgetful functor $\mathscr{L}_{AB}$ which makes each $\mathscr{P}_{\mathcal{C}}(A, B)$ a category over Δ (or Δ^+) :

$$\mathscr{L}_{AB} : \mathscr{P}_{\mathcal{C}}(A, B) \longrightarrow \Delta$$

with $\mathscr{L}_{AB}([n, s]) = n$ and $\mathscr{L}_{AB}([n, s] \xrightarrow{u} [m, t]) = u$.
The functor $\mathscr{L}_{AB}$ will be called **length** or **degree**.

Remark 4.7.1.

1. The concatenation of paths is a functor. For each triple (A, B, C) of objects of $\mathcal{C}$ we denote by $c(A, B, C)$ that functor:

$$c(A, B, C) : \mathscr{P}_{\mathcal{C}}(B, C) \times \mathscr{P}_{\mathcal{C}}(A, B) \longrightarrow \mathscr{P}_{\mathcal{C}}(A, C)$$
$$\left[\begin{pmatrix} [n', s'] \\ \Big\Downarrow{\scriptstyle u'} \\ [m', t'] \end{pmatrix}, \begin{pmatrix} [n, s] \\ \Big\Downarrow{\scriptstyle u} \\ [m, t] \end{pmatrix} \right] \longmapsto \begin{pmatrix} [n + n', s * s'] \\ \Big\Downarrow{\scriptstyle u+u'} \\ [m + m', t * t'] \end{pmatrix}$$

2. It's easy to check that the concatenation is strictly associative.

Notation 4.7.1. We will use the following notations:
$s' \otimes s := c(A, B, C)(s', s) = s * s'$,

$t' \otimes t := c(A, B, C)(t', t) = t * t'$ and

$u' \otimes u := c(A, B, C)(u', u) = u + u'$

Now we've set up all the tools needed for the definition of the 2-path-category.

Definition 4.7.2. *Let $\mathcal{C}$ be a small category. The* 2-path-category $\mathscr{P}_{\mathcal{C}}$ *of $\mathcal{C}$ is the bicategory given by the following data:*

- *the objects of $\mathscr{P}_{\mathcal{C}}$ are the objects of $\mathcal{C}$*
- *for each pair (A, B) of objects of $\mathscr{P}_{\mathcal{C}}$, the category of arrows of $\mathscr{P}_{\mathcal{C}}$ is the category $\mathscr{P}_{\mathcal{C}}(A, B)$ described above*

- *for each triple* (A, B, C) *the composition functor is given by* the concatenation functor *described in the remark above:*

$$c(A, B, C) : \mathscr{P}_{\mathcal{C}}(B, C) \times \mathscr{P}_{\mathcal{C}}(A, B) \longrightarrow \mathscr{P}_{\mathcal{C}}(A, C)$$

- *for any object* A *of* $\mathcal{C}$ *we have a* strict *identity arrow* $I_A : 1 \longrightarrow \mathscr{P}_C(A, A)$ *which is* $[0, A]$

- *for each quadruple* (A, B, C, D) *of objects of* $\mathcal{C}$ *the associativity natural isomorphism* $a(A, B, C, D)$ *is the identity*

- *the left and right identities natural isomorphisms are the identity for each pair* (A, B) *of objects of* $\mathcal{C}$

These data satisfy clearly the Associativity and Identity Coherence axioms *(A. C.) and (I. C.) so that* $\mathscr{P}_{\mathcal{C}}$ *is even a strict 2-category.*

Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories and F a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$. By definition F commutes with the compositions of $\mathcal{C}$ and D, sends composable arrows of $\mathcal{C}$ to composable arrows of $\mathcal{D}$ and sends identities to identities. We can then easily see that F induces a strict homomorphism $\mathscr{P}_F : \mathscr{P}_{\mathcal{C}} \longrightarrow \mathscr{P}_{\mathcal{D}}$. That is we have a functor:

$$\mathscr{P}_{[-]} : \mathbf{Cat}_{\leq 1} \longrightarrow \mathbf{2\text{-}Cat}$$
$$\mathcal{C} \xrightarrow{F} \mathcal{D} \longmapsto \mathscr{P}_{\mathcal{C}} \xrightarrow{\mathscr{P}_F} \mathscr{P}_{\mathcal{D}}$$

where $\mathbf{Cat}_{\leq 1}$ and $\mathbf{2\text{-}Cat}$ are respectively, the 1-category of small categories and the 1-category of 2-categories (and strict 2-functors).

4.8 Localization and cartesian products

Notation 4.8.1. In this section we will use the following notations.
Cat = the category of small categories.
$\mathrm{Hom}(\mathcal{C}, \mathscr{E})$ = category of functors from $\mathcal{C}$ to $\mathscr{E}$.
$\mathrm{L}_{\mathcal{S}} : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ = a Gabriel-Zisman localization of $\mathcal{C}$ with respect to a class of maps $\mathcal{S}$.
${\mathrm{L}_{\mathcal{S}}}^*(\mathscr{E}) = \mathrm{Hom}(\mathcal{C}[\mathcal{S}^{-1}], \mathscr{E}) \xrightarrow{-\mathrm{o}\mathrm{L}_{\mathcal{S}}} \mathrm{Hom}(\mathcal{C}, \mathscr{E})$.
$\mathrm{Hom}_{\mathcal{S}}(\mathcal{C}, \mathscr{E})$ =the full subcategory of $\mathrm{Hom}(\mathcal{C}, \mathscr{E})$ whose objects are functors which make $\mathcal{S}$ invertible [1] in $\mathcal{D}$.

Note. It is well known that every functor making $\mathcal{S}$ invertible, factorizes in a unique way through $\mathrm{L}_{\mathcal{S}}$, hence ${\mathrm{L}_{\mathcal{S}}}^*$ induces an isomorphism of categories:

$${\mathrm{L}_{\mathcal{S}}}^*(\mathscr{E}) : \mathrm{Hom}(\mathcal{C}[\mathcal{S}^{-1}], \mathscr{E}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{S}}(\mathcal{C}, \mathscr{E}).$$

Our goal here is to prove the following lemma which was established independently and long-time ago before the present paper by Kelly, Lack and Walters in [51, Section 3.1]. We put the proof here for completeness.

Lemma 4.8.1. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two small categories,* $\mathcal{S}$ *and* $\mathcal{T}$ *be respectively two class of morphisms of* $\mathcal{C}$ *and* $\mathcal{D}$*. Choose localizations* $L_{\mathcal{S}} : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ *and* $L_{\mathcal{T}} : \mathcal{D} \longrightarrow \mathcal{D}[\mathcal{T}^{-1}]$*.*

Assume that :

[1] We say that $F : \mathcal{C} \longrightarrow \mathscr{E}$ makes $\mathcal{S}$ invertible if for all $s \in \mathcal{S}$, $F(s)$ is invertible in $\mathscr{E}$.

1. *$\mathcal{S}$ contains all identities of $\mathcal{C}$*

2. *$\mathcal{T}$ contains all identities of $\mathcal{D}$*

Then the canonical functor

$$\mathcal{C} \times \mathcal{D} \xrightarrow{L_{\mathcal{S}} \times L_{\mathcal{T}}} \mathcal{C}[\mathcal{S}^{-1}] \times \mathcal{D}[\mathcal{T}^{-1}]$$

is a localization of $\mathcal{C} \times \mathcal{D}$ with respect to $\mathcal{S} \times \mathcal{T}$.

Observations 4.8.1. From the lemma we have the following consequences.

1. Any object F of $\mathrm{Hom}_{\mathcal{S}\times\mathcal{T}}(\mathcal{C} \times \mathcal{D}, \mathscr{E})$ factorizes uniquely as $F = \overline{F} \circ (\mathrm{L}_{\mathcal{S}} \times \mathrm{L}_{\mathcal{T}})$ where $\overline{F}$ is an object of $\mathrm{Hom}(\mathcal{C}[\mathcal{S}^{-1}] \times \mathcal{D}[\mathcal{T}^{-1}], \mathscr{E})$.

2. We have an isomorphism :

$$(\mathrm{L}_{\mathcal{S}} \times \mathrm{L}_{\mathcal{T}})^*(\mathscr{E}) : \mathrm{Hom}(\mathcal{C}[\mathcal{S}^{-1}] \times \mathcal{D}[\mathcal{T}^{-1}], \mathscr{E}) \xrightarrow{-\circ(\mathrm{L}_{\mathcal{S}} \times \mathrm{L}_{\mathcal{T}})} \mathrm{Hom}_{\mathcal{S}\times\mathcal{T}}(\mathcal{C} \times \mathcal{D}, \mathscr{E})$$

3. For every pair $(\mathscr{E}_1, \mathscr{E}_2)$ of categories and any functors F in $\mathrm{Hom}_{\mathcal{S}}(\mathcal{C}, \mathscr{E}_1)$, G in $\mathrm{Hom}_{\mathcal{T}}(\mathcal{D}, \mathscr{E}_2)$, if we write :
$F = \overline{F} \circ \mathrm{L}_{\mathcal{S}}$,
$G = \overline{G} \circ \mathrm{L}_{\mathcal{T}}$,
then the functor $F \times G$ is in $\mathrm{Hom}_{\mathcal{S}\times\mathcal{T}}(\mathcal{C} \times \mathcal{D}, \mathscr{E}_1 \times \mathscr{E}_2)$ and factorizes (uniquely) as:

$$F \times G = (\overline{F} \times \overline{G}) \circ (\mathrm{L}_{\mathcal{S}} \times \mathrm{L}_{\mathcal{T}}).$$

We therefore have " $\overline{F \times G} = \overline{F} \times \overline{G}$ ".

Proof of Lemma 4.8.1

For the proof of the lemma we will use the following:

- Hom : $\mathrm{Cat}^{op} \times \mathrm{Cat} \longrightarrow \mathrm{Cat}$ is a bifunctor,

- Cat is **symmetric closed** for the cartesian product,

- the universal properties of the Gabriel-Zisman localization.

Cat is symmetric closed

The fact that Cat is symmetric closed means that for every category $\mathscr{B}$ the endofunctor $- \times \mathscr{B} : \mathrm{Cat} \longrightarrow \mathrm{Cat}$ (and also '$\mathscr{B} \times -$') has a right adjoint :

$$\mathrm{Hom}(\mathscr{B}, -) : \mathrm{Cat} \longrightarrow \mathrm{Cat}.$$

The adjunction says that the following functor is an isomorphism:

$$\alpha : \mathrm{Hom}(\mathscr{A} \times \mathscr{B}, \mathscr{E}) \longrightarrow \mathrm{Hom}(\mathscr{A}, \mathrm{Hom}(\mathscr{B}, \mathscr{E}))$$

$$(F : \mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{E}) \longmapsto (\alpha(F) : \mathscr{A} \longrightarrow \mathrm{Hom}(\mathscr{B}, \mathscr{E}))$$

Remark 4.8.1. Given a functor G in $\mathrm{Hom}(\mathscr{A}, \mathrm{Hom}(\mathscr{B}, \mathscr{E}))$, we define $\alpha^{-1}(G) : \mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{E}$ by setting:
$G(A, B) := [GA]B$ on objects
$G(f, g) := [Gf]g$ for every morphism (f, g) of $\mathscr{A} \times \mathscr{B}$.
One can immediately check that this defines indeed a functor from $\mathscr{A} \times \mathscr{B}$ to $\mathscr{E}$. It's obvious that α^{-1} is a functor and for every every $F \in \mathrm{Hom}(\mathscr{A} \times \mathscr{B}, \mathscr{E})$ we have an equality :

$$F = \alpha^{-1}(\alpha(F)).$$

Let $F : \mathcal{C} \times \mathcal{D} \longrightarrow \mathscr{E}$ be an object of $\mathrm{Hom}_{\mathcal{S} \times \mathcal{T}}(\mathcal{C} \times \mathcal{D}, \mathscr{E})$.
We want to show that F factorizes in a unique way as : $F = \overline{F} \circ (\mathrm{L}_{\mathcal{S}} \times \mathrm{L}_{\mathcal{T}})$, with $\overline{F}$ an object of $\mathrm{Hom}(\mathcal{C}[\mathcal{S}^{-1}] \times \mathcal{D}[\mathcal{T}^{-1}], \mathscr{E})$.

Step 1: Factorization of $\alpha(F)$ Consider $\alpha(F) : \mathcal{C} \longrightarrow \mathrm{Hom}(\mathcal{D}, \mathscr{E})$ the functor given by the above adjunction.

Given $s : A \longrightarrow A'$ a morphism of $\mathcal{S}$ and U an object of $\mathcal{D}$, we have $(s, \mathrm{Id}_U) \in \mathcal{S} \times \mathcal{T}$ and by assumption $F(s, \mathrm{Id}_U)$ is invertible in $\mathscr{E}$.

By definition $\alpha(F)s$ is a natural transformation whose component at U is exactly $F(s, \mathrm{Id}_U)$:

$$[\alpha(F)s]_U := F(s, Id_U) : F(A, U) \longrightarrow F(A', U).$$

Then $\alpha(F)s$ is a natural isomorphism which means that $\alpha(F)$ makes $\mathcal{S}$ invertible, hence factorizes uniquely as $\alpha(F) = \overline{\alpha(F)} \circ \mathrm{L}_{\mathcal{S}}$ with $\overline{\alpha(F)} : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathrm{Hom}(\mathcal{D}, \mathscr{E})$.

Now if we apply the inverse 'α^{-1}' to both $\overline{\alpha(F)}$ and $\alpha(F)$ it's easy to see that we have the following equality :

$$F = \overline{F_0} \circ (\mathrm{L}_{\mathcal{S}} \times \mathrm{Id}_D)$$

where $\overline{F_0} = \alpha^{-1}(\overline{\alpha(F)})$ is a functor from $\mathcal{C}[\mathcal{S}^{-1}] \times \mathcal{D}$ to $\mathscr{E}$.

Step 2: Using the symmetry of '$\times$' It suffices to apply the **Step 1** to $\overline{F_0} : \mathcal{C}[\mathcal{S}^{-1}] \times \mathcal{D} \longrightarrow \mathscr{E}$ with $\mathcal{S}_0 \subseteq \mathcal{C}[\mathcal{S}^{-1}]$, $\mathcal{S}_0 := \mathrm{Id}(\mathcal{C}[\mathcal{S}^{-1}])$, and interchanging the role of $\mathcal{T}$ and $\mathcal{S}_0$ using the symmetry of the cartesian product in Cat.
We then have a factorization :

$$\overline{F_0} = \overline{F_1} \circ (\mathrm{Id}_{\mathcal{C}[\mathcal{S}^{-1}]} \times \mathrm{L}_{\mathcal{T}})$$

with $\overline{F_1} : \mathcal{C}[\mathcal{S}^{-1}] \times \mathcal{D}[\mathcal{T}^{-1}] \longrightarrow \mathscr{E}$.

Combining this with the previous equality we have :

$$\begin{aligned} F &= \overline{F_0} \circ (\mathrm{L}_{\mathcal{S}} \times \mathrm{Id}_D) \\ &= [\overline{F_1} \circ (\mathrm{Id}_{\mathcal{C}[\mathcal{S}^{-1}])} \times \mathrm{L}_{\mathcal{T}})] \circ [\mathrm{L}_{\mathcal{S}} \times \mathrm{Id}_D] \\ &= \overline{F_1} \circ [(\mathrm{Id}_{\mathcal{C}[\mathcal{S}^{-1}])} \circ \mathrm{L}_{\mathcal{S}}) \times (\mathrm{L}_{\mathcal{T}} \circ \mathrm{Id}_D)] \\ &= \overline{F_1} \circ (\mathrm{L}_{\mathcal{S}} \times \mathrm{L}_{\mathcal{T}}) \end{aligned}$$

Then $\overline{F} = \overline{F_1}$. ■

4.9 Secondary Localization of a bicategory

In this section we're going to define the Gabriel-Zisman localization of $\mathscr{M}$ with respect to $\mathscr{W}$ when $(\mathscr{M},\mathscr{W})$ is a base of enrichment.

In addition to the previous notations we will write:
$\mathscr{M}_{UV} = \mathscr{M}(U,V)$.
$\mathscr{W}_{UV} = \mathscr{W} \bigcap \mathscr{M}(U,V)$.
$c_{\mathscr{M}} : \mathscr{M}_{VW} \times \mathscr{M}_{UV} \longrightarrow \mathscr{M}_{UW}=$ the composition functor in $\mathscr{M}$; when the context is clear.

Remark 4.9.1. From the assumptions made on $\mathscr{W}$, we clearly see that each $\mathscr{W}_{UV}$ is a subcategory of $\mathscr{M}_{UV}$ having the same objects. Moreover the functor $c_{\mathscr{M}}$ sends $\mathscr{W}_{VW} \times \mathscr{W}_{UV}$ to $\mathscr{W}_{UW}$, so we can view $\mathscr{W}$ as a sub-bicategory $\mathscr{M}$, having the same objects and 1-cells.

Definition 4.9.1. *Let $\mathscr{B}$ be a bicategory and $\Phi : \mathscr{M} \longrightarrow \mathscr{B}$ a homomorphism in the sense of Bénabou [10]. We will say that Φ makes $\mathscr{W}$ invertible if for every pair (U,V) in Ob($\mathscr{M}$), the functor*

$$\Phi_{UV} : \mathscr{M}(U,V) \longrightarrow \mathscr{B}(\Phi U, \Phi V)$$

makes $\mathscr{W}_{UV}$ invertible.

Our purpose is to construct a bicategory $\mathscr{W}^{-1}\mathscr{M}$ with a homomorphism $\mathrm{L}_{\mathscr{W}} : \mathscr{M} \longrightarrow \mathscr{W}^{-1}\mathscr{M}$ which is "universal" among those making $\mathscr{W}$ invertible. The universality here means that for any homomorphism $\Phi : \mathscr{M} \longrightarrow \mathscr{B}$ making $\mathscr{W}$ invertible there is a factorization, unique up-to a transformation[2], $\Phi = \overline{\Phi} \circ \mathrm{L}_{\mathscr{W}}$, where
$\overline{\Phi} : \mathscr{W}^{-1}\mathscr{M} \longrightarrow \mathscr{B}$ is a homomorphism.

Like in the classical case the target bicategory $\mathscr{W}^{-1}\mathscr{M}$ should (essentially) have the same object as $\mathscr{M}$, so that $\mathrm{L}_{\mathscr{W}}$ will be the identity on objects. Moreover if such localization homomorphism $\mathrm{L}_{\mathscr{W}}$ exists, we should have factorizations of its components:

$$\mathrm{L}_{\mathscr{W},UV} : \mathscr{M}_{UV} \xrightarrow{\mathrm{L}_{\mathscr{W}_{UV}}} \mathscr{M}_{UV}[\mathscr{W}_{UV}^{-1}] \xrightarrow{\overline{\mathrm{L}_{\mathscr{W},UV}}} \mathscr{W}^{-1}\mathscr{M}(\mathrm{L}_{\mathscr{W}}U, \mathrm{L}_{\mathscr{W}}V).$$

This suggests to take $\mathscr{M}_{UV}[\mathscr{W}_{UV}^{-1}]$ as category of morphisms in $\mathscr{W}^{-1}\mathscr{M}$ for each (U,V).

Proposition 4.9.2. *Let $(\mathscr{M},\mathscr{W})$ be a base of enrichment. There exists a bicategory $\mathscr{W}^{-1}\mathscr{M}$ together with a homomorphism $L_{\mathscr{W}} : \mathscr{M} \longrightarrow \mathscr{W}^{-1}\mathscr{M}$ such that :*

1. *$L_{\mathscr{W}}$ makes $\mathscr{W}$ invertible,*
2. *any homomorphism $\Phi : \mathscr{M} \longrightarrow \mathscr{B}$ which makes $\mathscr{W}$ invertible factorizes as $\Phi = \overline{\Phi} \circ L_W$ with*
$$\overline{\Phi} : \mathscr{W}^{-1}\mathscr{M} \longrightarrow \mathscr{B}$$
a homomorphism.
3. *$\mathscr{W}^{-1}\mathscr{M}$ is unique up to a biequivalence [3].*

4.9.1 Proof of Proposition 4.9.2

Choose a localization $\mathrm{L}_{\mathscr{W}_{UV}} : \mathscr{M}_{UV} \longrightarrow \mathscr{M}_{UV}[\mathscr{W}_{UV}^{-1}]$ for each pair (U,V) of objects of $\mathscr{M}$.
Set $\mathrm{Ob}(\mathscr{W}^{-1}\mathscr{M}) = \mathrm{Ob}(\mathscr{M})$,
$\mathscr{W}^{-1}\mathscr{M}(U,V) = \mathscr{M}_{UV}[\mathscr{W}_{UV}^{-1}]$.

[2]the transformation is unique up to a **unique** modification
[3]The biequivalence is itself unique up to a unique strong transformation.

Construction of the composition

By applying lemma 4.8.1 for each triple (U, V, W), we have a localization

$$\mathscr{M}_{VW} \times \mathscr{M}_{UV} \xrightarrow{\mathrm{L}_{\mathscr{W}_{VW}} \times \mathrm{L}_{\mathscr{W}_{UV}}} \mathscr{M}_{VW}[\mathscr{W}_{VW}^{-1}] \times \mathscr{M}_{UV}[\mathscr{W}_{UV}^{-1}]$$

of $\mathscr{M}_{VW} \times \mathscr{M}_{UV}$ with respect to $\mathscr{W}_{VW} \times \mathscr{W}_{UV}$.

Since $c_{\mathscr{M}} : \mathscr{M}_{VW} \times \mathscr{M}_{UV} \longrightarrow \mathscr{M}_{UW}$ sends $\mathscr{W}_{VW} \times \mathscr{W}_{UV}$ to $\mathscr{W}_{UW}$, it follows that the composite

$$\mathrm{L}_{\mathscr{W}_{UW}} \circ c_{\mathscr{M}} : \mathscr{M}_{VW} \times \mathscr{M}_{UV} \longrightarrow \mathscr{M}_{UW}[\mathscr{W}_{UW}^{-1}]$$

makes $\mathscr{W}_{VW} \times \mathscr{W}_{UV}$ invertible, hence factorizes as :

$$\begin{tikzcd}
\mathscr{M}_{VW} \times \mathscr{M}_{UV} \arrow[r, "\mathrm{L}_{\mathscr{W}_{UW}} \circ c_{\mathscr{M}}(U,V,W)"] \arrow[d, "\mathrm{L}_{\mathscr{W}_{VW}} \times \mathrm{L}_{\mathscr{W}_{UV}}"'] & \mathscr{M}_{UW}[\mathscr{W}_{UW}^{-1}] \\
\mathscr{M}_{VW}[\mathscr{W}_{VW}^{-1}] \times \mathscr{M}_{UV}[\mathscr{W}_{UV}^{-1}] \arrow[ur, dashed, "c_{\mathscr{W}^{-1}\mathscr{M}}"'] &
\end{tikzcd}$$

which gives the composition functor.

If we follow the notations of the factorization as in lemma 4.8.1 we will write:

$$\mathrm{L}_{\mathscr{W}_{UW}} \circ c_{\mathscr{M}} = \overline{\mathrm{L}_{\mathscr{W}_{UW}} \circ c_{\mathscr{M}}} \circ (\mathrm{L}_{\mathscr{W}_{VW}} \times \mathrm{L}_{\mathscr{W}_{UV}})$$

which means that $c_{\mathscr{W}^{-1}\mathscr{M}} := \overline{\mathrm{L}_{\mathscr{W}_{UW}} \circ c_{\mathscr{M}}}$.

The associativity

We build the following commutative diagram using the universal property of the Gabriel-Zisman localization and lemma 4.8.1.

$$\begin{tikzcd}
 & & \mathscr{M}_{WZ} \times \mathscr{M}_{UW} \arrow[ddr, "c_{\mathscr{M}}(U,W,Z)"] \arrow[dddd, dotted] & \\
\mathscr{M}_{WZ} \times \mathscr{M}_{VW} \times \mathscr{M}_{UV} \arrow[urr, "\mathrm{Id}_{\mathscr{M}_{WZ}} \times c_{\mathscr{M}}(U,V,W)"] \arrow[drrr, bend left, "\eta_1"] \arrow[drrr, bend right, "\eta_2"'] \arrow[drrr, phantom, "\Nearrow a(U,V,W,Z)"] \arrow[ddr, "c_{\mathscr{M}}(V,W,Z) \times \mathrm{Id}_{\mathscr{M}_{UV}}"] \arrow[dddd, dotted, "\mathrm{L}_{\mathscr{W}_{WZ}} \times \mathrm{L}_{\mathscr{W}_{VW}} \times \mathrm{L}_{\mathscr{W}_{UV}}"'] & & & \\
 & & & \mathscr{M}_{UZ} \arrow[dddd, dotted, "\mathrm{L}_{\mathscr{W}_{UZ}}"] \\
 & \mathscr{M}_{VZ} \times \mathscr{M}_{UV} \arrow[urr, "c_{\mathscr{M}}(U,V,Z)"'] \arrow[dddd, dotted] & & \\
 & & \mathscr{M}_{WZ}[\mathscr{W}_{WZ}^{-1}] \times \mathscr{M}_{UW}[\mathscr{W}_{UW}^{-1}] \arrow[ddr, "c_{\mathscr{W}^{-1}\mathscr{M}}(U,W,Z)"] & \\
\mathscr{M}_{WZ}[\mathscr{W}_{WZ}^{-1}] \times \mathscr{M}_{VW}[\mathscr{W}_{VW}^{-1}] \times \mathscr{M}_{UV}[\mathscr{W}_{UV}^{-1}] \arrow[urr, dashed, "\gamma_1"] \arrow[drrr, bend left, "\sigma_1"] \arrow[drrr, bend right, "\sigma_2"'] \arrow[drrr, phantom, "\Nearrow \overline{a(U,V,W,Z)}"] \arrow[ddr, dashed, "\gamma_2"'] & & & \\
 & & & \mathscr{M}_{UZ}[\mathscr{W}_{UZ}^{-1}] \\
 & \mathscr{M}_{VZ}[\mathscr{W}_{VZ}^{-1}] \times \mathscr{M}_{UV}[\mathscr{W}_{UV}^{-1}] \arrow[urr, "c_{\mathscr{W}^{-1}\mathscr{M}}(U,V,Z)"'] & &
\end{tikzcd}$$

We use hereafter the same notations as in lemma 4.8.1. Then for every functor F which factorizes through a localization L, we will denote by $\overline{F}$ the unique functor such that $F = \overline{F} \circ \mathrm{L}$.

- The double dotted vertical maps are localizations given by lemma 4.8.1.
- We've denoted for short $\eta_1 = c_{\mathscr{M}}(U, W, Z) \circ [\mathrm{Id}_{\mathscr{M}_{WZ}} \times c_{\mathscr{M}}(U, V, W)]$.
- Similarly $\eta_2 = [c_{\mathscr{M}}(V, W, Z) \times \mathrm{Id}_{\mathscr{M}_{UV}}] \circ c_{\mathscr{M}}(U, V, Z)$
- γ_1 is by definition $\overline{\mathrm{L}_{\mathscr{W}_{WZ}} \times (\mathrm{L}_{\mathscr{W}_{UW}} \circ [\mathrm{Id}_{\mathscr{M}_{WZ}} \times c_{\mathscr{M}}(U, V, W)]}$ and is given universal property with respect to $\mathrm{L}_{\mathscr{W}_{WZ}} \times \mathrm{L}_{\mathscr{W}_{VW}} \times \mathrm{L}_{\mathscr{W}_{UV}}$.
- γ_2 is $\overline{(\mathrm{L}_{\mathscr{W}_{VZ}} \times \mathrm{L}_{\mathscr{W}_{UV}}) \circ [c_{\mathscr{M}}(V, W, Z) \times \mathrm{Id}_{\mathscr{M}_{UV}}]}$
- $\sigma_1 = \overline{\mathrm{L}_{\mathscr{W}_{UZ}} \circ \eta_1}$
- $\sigma_2 = \overline{\mathrm{L}_{\mathscr{W}_{UZ}} \circ \eta_2}$
- $\overline{a(U, V, W, Z)}$ is the inverse image of $a(U, V, W, Z) \otimes \mathrm{Id}_{\mathrm{L}_{\mathscr{W}_{UZ}}}$, which is an invertible the 2-cell in Cat, by the isomorphism of categories $[\mathrm{L}_{\mathscr{W}_{WZ}} \times \mathrm{L}_{\mathscr{W}_{VW}} \times \mathrm{L}_{\mathscr{W}_{UV}}]^*(\mathscr{M}_{UZ}[\mathscr{W}_{UZ}^{-1}])$.

Recall that $[\mathrm{L}_{\mathscr{W}_{WZ}} \times \mathrm{L}_{\mathscr{W}_{VW}} \times \mathrm{L}_{\mathscr{W}_{UV}}]^*(\mathscr{M}_{UZ}[\mathscr{W}_{UZ}^{-1}])$ is an isomorphism from the hom-category

$$\mathrm{Hom}(\mathscr{M}_{WZ}[\mathscr{W}_{WZ}^{-1}] \times \mathscr{M}_{VW}[\mathscr{W}_{VW}^{-1}] \times \mathscr{M}_{UV}[\mathscr{W}_{UV}^{-1}], \mathscr{M}_{UZ}[\mathscr{W}_{UZ}^{-1}])$$

to the hom-category

$$\mathrm{Hom}_{\mathscr{W}_{WZ} \times \mathscr{W}_{VW} \times \mathscr{W}_{UV}}(\mathscr{M}_{WZ} \times \mathscr{M}_{VW} \times \mathscr{M}_{UV}, \mathscr{M}_{UZ}[\mathscr{W}_{UZ}^{-1}]).$$

It's clear that $\overline{a(U, V, W, Z)}$ is an invertible 2-cell in Cat (a natural isomorphism) from σ_2 to σ_1

We need to show that the following hold

- $\gamma_1 = Id_{\mathscr{M}_{WZ}[\mathscr{W}_{WZ}^{-1}]} \times c_{\mathscr{W}^{-1}\mathscr{M}}(U, V, W)$
- $\gamma_2 = c_{\mathscr{W}^{-1}\mathscr{M}}(V, W, Z) \times Id_{\mathscr{M}_{UV}[\mathscr{W}_{UV}^{-1}]}$
- $\sigma_1 = c_{\mathscr{W}^{-1}\mathscr{M}}(U, W, Z) \circ \gamma_1$
- $\sigma_2 = c_{\mathscr{W}^{-1}\mathscr{M}}(U, V, Z) \circ \gamma_2$

We proof the equality for γ_1 and σ_1, the argument is the same for the remaining cases.

For γ_1 we use the the property '$\overline{F \times G} = \overline{F} \times \overline{G}$' (see Observations 4.8.1). We have

$$\begin{aligned}\gamma_1 &= \overline{\mathrm{L}_{\mathscr{W}_{WZ}} \times (\mathrm{L}_{\mathscr{W}_{UW}} \circ c_{\mathscr{M}}(U, V, W))} \\ &= \overline{\mathrm{L}_{\mathscr{W}_{WZ}}} \times \overline{\mathrm{L}_{\mathscr{W}_{UW}} \circ c_{\mathscr{M}}(U, V, W)}\end{aligned}$$

But since $\mathrm{L}_{\mathscr{W}_{WZ}} = \mathrm{Id}_{\mathscr{M}_{WZ}[\mathscr{W}_{WZ}^{-1}]} \circ \mathrm{L}_{\mathscr{W}_{WZ}}$ then $\overline{\mathrm{L}_{\mathscr{W}_{WZ}}} = \mathrm{Id}_{\mathscr{M}_{WZ}[\mathscr{W}_{WZ}^{-1}]}$.

Combining with the fact that $c_{\mathscr{W}^{-1}\mathscr{M}}(U,V,W) := \overline{\mathrm{L}_{\mathscr{W}_{UW}} \circ c_{\mathscr{M}}(U,V,W)}$, we deduce that

$$\gamma_1 = Id_{\mathscr{M}_{WZ}[\mathscr{W}_{WZ}^{-1}]} \times c_{\mathscr{W}^{-1}\mathscr{M}}(U,V,W)$$

as desired.

For σ_1 we're going to use the commutativity of the vertical faces in the 'cubical' diagram and the fact that Cat is as strict 2-category.

We write

$$\begin{aligned}
[c_{\mathscr{W}^{-1}\mathscr{M}}(U,W,Z) \circ \gamma_1] \circ (\mathrm{L}_{\mathscr{W}_{WZ}} \times \mathrm{L}_{\mathscr{W}_{VW}} \times \mathrm{L}_{\mathscr{W}_{UV}}) &= c_{\mathscr{W}^{-1}\mathscr{M}} \circ [\gamma_1 \circ (\mathrm{L}_{\mathscr{W}_{WZ}} \times \mathrm{L}_{\mathscr{W}_{VW}} \times \mathrm{L}_{\mathscr{W}_{UV}})] \\
&= c_{\mathscr{W}^{-1}\mathscr{M}} \circ [(\mathrm{L}_{\mathscr{W}_{WZ}} \times \mathrm{L}_{\mathscr{W}_{UV}}) \circ (\mathrm{Id}_{\mathscr{M}_{WZ}} \times c_{\mathscr{M}})] \\
&= [c_{\mathscr{W}^{-1}\mathscr{M}} \circ (\mathrm{L}_{\mathscr{W}_{WZ}} \times \mathrm{L}_{\mathscr{W}_{UV}})] \circ [\mathrm{Id}_{\mathscr{M}_{WZ}} \times c_{\mathscr{M}}] \\
&= [\mathrm{L}_{\mathscr{W}_{UZ}} \circ c_{\mathscr{M}}] \circ [\mathrm{Id}_{\mathscr{M}_{WZ}} \times c_{\mathscr{M}}] \\
&= \mathrm{L}_{\mathscr{W}_{UZ}} \circ [c_{\mathscr{M}} \circ (\mathrm{Id}_{\mathscr{M}_{WZ}} \times c_{\mathscr{M}})] \\
&= \mathrm{L}_{\mathscr{W}_{UZ}} \circ \eta_1 \\
&= \sigma_1 \circ (\mathrm{L}_{\mathscr{W}_{WZ}} \times \mathrm{L}_{\mathscr{W}_{VW}} \times \mathrm{L}_{\mathscr{W}_{UV}}).
\end{aligned}$$

The uniqueness of the factorization implies : $\sigma_1 = c_{\mathscr{W}^{-1}\mathscr{M}}(U,W,Z) \circ \gamma_1$.

For the axioms in $\mathscr{W}^{-1}\mathscr{M}$

We give hereafter the argument for the associativity axiom. The argument is the same for the identity axioms.

The idea is to say that these axioms are satisfied in $\mathscr{M}$ and we need to check that they're transferred through the localization and this is true. The reason is that the property '$\overline{F \times G} = \overline{F} \times \overline{G}$' of functors hold also for natural transformations and commute with the composition.

For every objects T, U, V, W of $\mathscr{M}$, the pentagon of associativity from $\mathscr{M}_{WZ} \times \mathscr{M}_{VW} \times \mathscr{M}_{UV} \times \mathscr{M}_{TU}$ to $\mathscr{M}_{TZ}$ gives by composition with $\mathrm{L}_{\mathscr{W}_{TZ}}$ a commutative pentagon from $\mathscr{M}_{WZ} \times \mathscr{M}_{VW} \times \mathscr{M}_{UV} \times \mathscr{M}_{TU}$ to $\mathscr{M}_{TZ}[\mathscr{W}_{TZ}^{-1}]$.

For each vertex other than $\mathscr{M}_{TZ}[\mathscr{W}_{TZ}^{-1}]$, each 'path' from the vertex to $\mathscr{M}_{WZ}[\mathscr{W}_{WZ}^{-1}]$ factorizes through the suitable localization functor.

These factorizations fit together because we have

- a uniqueness of the factorization of the path from $\mathscr{M}_{WZ} \times \mathscr{M}_{VW} \times \mathscr{M}_{UV} \times \mathscr{M}_{TU}$ with respect to the localization $\mathrm{L}_{\mathscr{W}_{WZ}} \times \mathrm{L}_{\mathscr{W}_{VW}} \times \mathrm{L}_{\mathscr{W}_{UV}} \times \mathrm{L}_{\mathscr{W}_{TU}}$.
- for every triple of objects, we have a cubical commutative diagram.

We finally have a pentagon of associativity from $\mathscr{M}_{WZ}[\mathscr{W}_{WZ}^{-1}] \times \mathscr{M}_{VW}[\mathscr{W}_{VW}^{-1}] \times \mathscr{M}_{UV}[\mathscr{W}_{UV}^{-1}] \times \mathscr{M}_{TY}[\mathscr{W}_{TU}^{-1}]$ to $\mathscr{M}_{TZ}[\mathscr{W}_{TZ}^{-1}]$ as desired.

Finally one easily check that these data define a bicategory $\mathscr{W}^{-1}\mathscr{M}$ with a canonical homomorphism $\mathrm{L}_{\mathscr{W}} : \mathscr{M} \longrightarrow \mathscr{W}^{-1}\mathscr{M}$, and that $\mathrm{L}_{\mathscr{W}}$ satisfies the universal property. $\blacksquare$

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Catégories faiblement enrichies sur une catégorie monoïdale symétrique

Résumé Dans cette thèse nous développons une théorie de *catégories faiblement enrichies*. Par 'faiblement' on comprendra ici une catégorie dont la composition de morphismes est associative à homotopie près; à l'inverse d'une catégorie enrichie classique où la composition est strictement associative. Il s'agit donc de notions qui apparaissent dans un contexte homotopique. Nous donnons une notion de *catégorie enrichie de Segal* et une notion de *catégorie enrichie co-Segal*; chacune de ces notions donnant lieu à une structure de catégorie supérieure. L'une des motivations de ce travail était de fournir une théorie de catégories linéaires supérieures, connues pour leur importance dans des différents domaines des mathématiques, notamment dans les géométries algébriques commutative et non-commutative. Les catégories enrichies de Segal généralisent la notion de *monoïde à homotopie près* introduite par Leinster. Les monoïdes de Leinster correspondent précisément aux catégories enrichies de Segal avec un seul objet. Nous montrons comment notre formalisme couvre le cas des catégories de Segal classique, les monoïdes de Leinster et surtout apporte une définition de DG-catégorie de Segal. Les principaux résultats de la thèse sont dans la deuxième partie qui porte sur les catégories enrichies co-Segal. Nous avons introduit ces nouvelles structures lorsqu'on s'est aperçu que les catégories enrichies de Segal ne sont pas faciles à manipuler pour faire une théorie de l'homotopie. En effet il semble devoir imposer une condition supplémentaire qui est trop restrictive dans beaucoup de cas. Ces nouvelles catégories s'obtiennent en 'renversant' la situation du cas Segal, d'où le préfixe 'co' dans 'co-Segal'. Notre résultat principal est l'existence d'une structure de modèles au sens de Quillen sur la catégorie des précatégories co-Segal; avec comme particularité que les objets fibrants sont des catégories co-Segal.

Mots-clés: Catégories enrichies, catégories de Segal, catégories co-Segal, catégories supérieures, catégories de modèles, diagrammes lax, opérades, DG-catégorie co-Segal, enrichissement homotopique.

Weakly enriched categories over a symmetric monoidal category

Abstract In this thesis we develop a theory of *weakly enriched categories*. By 'weakly' we mean an enriched category where the composition is not strictly associative but associative up-to-homotopy. We introduce the notion of *Segal enriched categories* and of *co-Segal categories*. The two notions give rise to higher categorical structures. One of the motivations of this work was to provide an alternative notion of higher linear categories, which are known by the experts to be important in both commutative and noncommutative algebraic geometry. The first part of the thesis is about Segal enriched categories. A Segal enriched category is the generalization of the notion of *up-to-homotopy monoid* introduced by Leinster. The monoids of Leinster correspond precisely to Segal enriched categories having a single object. We show that our formalism cover the definition of classical Segal categories and generalizes Leinster's definition. Furthermore we give a definition of Segal DG-category. The main results of this work are in the second part of the thesis which is about co-Segal categories. The origin of this notion comes from the fact that Segal enriched categories are not easy to manipulate for homotopy theory purposes. In fact when trying to have a model structure on them, it seems important to require an extra hypothesis that can be too restrictive. The co-Segal formalism is obtained by 'reversing' everything in the Segal case, hence the terminology 'co-Segal'. Our main result is the existence of a Quillen model structure on the category of co-Segal precategories; with the property that fibrant objects are co-Segal categories.

Keywords: Enriched categories, Segal categories, co-Segal categories, higher categories, model categories, lax diagrams, operads, co-Segal DG-categories, homotopy enrichment.