Stabilization and approximation of some distributed systems by either dissipative or inde

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« Stabilisation et approximation de certains systèmes distribués par amortisement dissipatif et de signe indéfini »

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Stabilisation et approximation de certains systèmes distribués par amortissement dissipatif et de signe indéfini

Dans cette thèse, nous étudions l'approximation et la stabilisation de certaines équations d'évolution, en utilisant la théorie des semi-groups et l'analyse spectrale. Cette thèse est divisée en deux parties principales. Dans la première partie, comme dans [3, 4], nous considérons l'approximation des équations d'évolution du deuxième ordre modélisant les vibrations de structures élastiques. Il est bien connu que le système approchée par éléments finis ou différences finies n'est pas uniformément exponentiellement ou polynomialement stable par rapport au paramètre de discrétisation, même si le système continu a cette propriété. Dans la première partie, notre objectif est d'amortir les modes parasites à haute fréquence en introduisant des termes de viscosité numérique dans le schéma d'approximation. Avec ces termes de viscosité, nous montrons la décroissance exponentielle ou polynomiale du schéma discret lorsque le problème continu a une telle décroissance et quand le spectre de l'opérateur spatial associé au problème conservatif satisfait la condition du gap généralisée. En utilisant le Théorème de Trotter-Kato, nous montrons la convergence de la solution discrète vers la solution continue. Quelques exemples sont également présentés.

Dans la deuxième partie, comme dans [1, 2], nous étudions la stabilisation des équations des ondes avec amortissement de signe indéterminée. Tout d'abord, nous considérons deux problèmes des ondes dont les termes d'amortissement peuvent changer de signe. En utilisant l'analyse du spectre, on trouve des valeurs critiques des coefficients d'amortissement pour lesquels le problème devient exponentiellement ou polynomialement stable jusqu'à ces valeurs critiques. Ensuite, nous généralisons l'analyse des deux derniers problèmes pour étudier la stabilité des équations des ondes sur un réseau en forme d'étoile en présence de termes d'amortissement de signe indéterminée. Pour ce problème, nous constatons que la condition intuitive sur la positivité de la moyenne ne suffit pas. Que la norme L^∞ des coefficients d'amortissement soit grande ou petite, nous cherchons des conditions suffisantes sur les coefficients d'amortissement pour lesquels le problème devient exponentiellement stable.

Mots-clés : Stabilité, semi-discrétisation, terme de viscosité, gap généralisé, amortissement de signe indéterminée, comportement asymptotique, base de Riesz, réseau en forme d'étoile.

Stabilization and approximation of some distributed systems by either dissipative or indefinite sign damping

In this thesis, we study the approximation and stabilization of some evolution equations, using semigroup theory and some spectral analysis. This Ph.D. thesis is divided into two main parts. In the first part, as in [3, 4], we consider the approximation of second order evolution equations modeling the vibrations of elastic structures. It is well known that the approximated system by finite elements or finite differences is not uniformly exponentially or polynomially stable with respect to the discretization parameter, even if the continuous system has this property. Therefore, our goal is to damp the spurious high frequency modes by introducing numerical viscosity terms in the approximation scheme. With these viscosity terms, we show the exponential or polynomial decay of the discrete scheme when the continuous problem has such a decay and when the spectrum of the spatial operator associated with the undamped problem satisfies the generalized gap condition. By using the Trotter-Kato Theorem, we further show the convergence of the discrete solution to the continuous one. Some illustrative examples are also presented.

In the second part, as in [1, 2], we study the stabilization of wave equations with indefinite sign damping. Here we search for sufficient conditions on the damping coefficients so that the wave equations are either exponentially or polynomially stable. First, we consider two damped wave problems which are either internally or boundary damped and for which the damping terms are allowed to change their sign. Using a careful spectral analysis, we find critical values of the damping coefficients for which the problem becomes exponentially or polynomially stable up to these critical values. Afterwards, we generalize the analysis of the previous two problems to the case of wave equations on a star shaped network in the presence of indefinite sign damping terms. For this problem, we find that the intuitive condition on the positivity of the mean is not sufficient. Whether the L^∞ norm of the damping coefficients is large or small, we search for sufficient conditions on the damping coefficients for which the problem becomes exponentially stable.

Key words : Stability, semi-discretization, viscosity terms, generalized gap condition, indefinite sign damping, asymptotic behavior, Riesz basis, star-shaped network.

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Introduction

Control theory can be described as the process of influencing the behavior of a physical system to achieve a desired goal, primarily through the use of feedback which monitors the effect of a system and modifies its output. It is applied in a diverse range of scientific and engineering disciplines such as the reduction of noise, the vibration of structures like seismic waves and earthquakes, the regulation of biological systems like human cardiovascular system, the design of robotic systems, laser control in quantum mechanical and molecular systems.

In this thesis, we implement the semigroup theory in the spirit of spectral theory to study the approximations and stabilization of some evolution equations. In general, stability results are obtained using different methods like the multipliers method, the frequency domain method, the microlocal analysis, the differential geometry or a combination of them [47,50,77,78]. In this thesis, we use detailed spectral analysis. In fact, this thesis is divided into two parts. In the first part, we consider the approximations of second order evolution equations. Studies and researches have shown that the approximated system by finite element or finite difference is not uniformly exponentially or polynomially stable with respect to the discretization parameter even if the continuous system has this property (see [25, 27, 43, 55, 72, 73, 76, 78]). Therefore, our aim in the first part is to search for a suitable discrete system which approximates the continuous system and most importantly restores the decay rate properties of the continuous one. In the presence of the generalized gap condition, our strategy is based on adding numerical viscosity terms in the approximation schemes to damp out the effect of the high frequencies.

In the second part, we study the stabilization of wave equations with indefinite sign damping. Here, we use a detailed spectral analysis to study the behavior of the spectrum out of which we search for critical values of the damping coefficients so that the wave equations are either exponentially or polynomially stable. First, we consider one dimensional internally and boundary damped wave problems and afterwards we generalize the analysis of indefinite sign damped wave equations to a star shaped network where we find extra conditions to get stability.

0.1 Outline of the thesis

This thesis is divided into four main chapters. In the first chapter, we recall some basic definitions and theorems about the semigroup and spectral analysis theories.

In the second chapter, as in [3] and [4], we consider the approximation of linear equations modeling the vibrations of elastic structures with feedback control. More precisely, let H be a complex Hilbert space with norm and inner product denoted respectively by $\|.\|$ and $(.,.)$. Let $A : \mathcal{D}(A) \to H$ be a densely defined self-adjoint and positive operator with a compact inverse in H. Let $V = \mathcal{D}(A^{\frac{1}{2}})$ be the domain of $A^{\frac{1}{2}}$. Denote by $\mathcal{D}(A^{\frac{1}{2}})'$ the dual space of $\mathcal{D}(A^{\frac{1}{2}})$ obtained by means of the inner product in H.

Furthermore, let U be a complex Hilbert space (which will be identified to its dual space) with norm and inner product denoted respectively by $\|.\|_U$ and $(.,.)_U$

and let $B \in \mathcal{L}(U,\, H)$. We consider the closed loop system

$$\begin{aligned} \ddot{\omega}(t) + A\omega(t) + BB^*\dot{\omega}(t) = 0, \\ \omega(0) = \omega_0,\, \dot{\omega}(0) = \omega_1, \end{aligned} \tag{0.1.1}$$

where $t \in [0,\, \infty)$ represents the time and $\omega\, :\, [0,\, \infty) \to H$ is the state of the system.

We define the energy of system (0.1.1) at time t by

$$E(t) = \frac{1}{2}\left(\|\dot{\omega}(t)\|^2 + \left\| A^{\frac{1}{2}}\omega(t) \right\|^2 \right).$$

Simple formal calculations give

$$E(0) - E(t) = \int_0^t \left(BB^*\dot{\omega}(s),\, \dot{\omega}(s)\right) ds, \quad \forall t \geq 0.$$

This obviously means that the energy is non-increasing. In the second chapter, our goal is to search for a suitable discrete system which first approximates (0.1.1) and second has the same stability properties as (0.1.1). However, in many applications, most of the classical numerical approximation schemes do not possess the same decay rate as that of the continuous problem although the convergence is preserved. At the discrete level, spurious high frequency oscillations are generated and therefore bad behavior of the approximate solution is clearly observed causing a non-uniform decay rate (see [14,19,20,25,27,33,34,43,55,58,72,73,76,78]). For instance, we start the second chapter by considering the vibrations of a flexible string joined at each of its ends. Although the continuous problem is exponentially stable, we show that the finite difference semi-discrete problem is not uniformly exponentially stable ; i.e., there does not exist constants M and $\beta > 0$ independent of the discretization parameter such that

$$E_h(t) \leq Me^{-\beta t}, \quad \text{as} \;\; t \to +\infty,$$

where $E_h(t)$ represents the energy of the semi-discrete system.

Several remedies are proposed to restore the uniform decay rate of the discrete problems like Tychonoff regularization [34, 35, 64, 72], a bi-grid algorithm [32, 58], a mixed finite element method [14, 19, 20, 33, 56], or filtering the high frequencies [43, 49, 76]. As in [64, 72], we introduce artificial numerical viscosity terms in the approximation schemes to rule out the high frequency spurious numerical oscillations and hence restore the uniform decay rate of the discrete scheme. However, contrary to [64] where the standard gap condition is required, we only assume that the spectrum of the operator $A^{1/2}$ satisfies the generalized gap condition. Indeed, if $\{\lambda_k\}_{k\geq 1}$ denotes the set of eigenvalues of $A^{\frac{1}{2}}$ counted with their multiplicities, then we assume that the following generalized gap condition holds :

$$\exists M \in \mathbb{N}^*,\ \exists \gamma_0 > 0,\ \forall k \geq 1,\ \lambda_{k+M} - \lambda_k \geq M\gamma_0.$$

The standard gap condition is satisfied for the particular case when $M = 1$. Therefore, in the second chapter, we treat more general concrete systems.

After recalling the suitable conditions and observability inequalities which lead to the exponential or the polynomial stability of the solution of problem (0.1.1), we search for a suitable discrete system which has the same decay properties under these conditions. For this reason, after finding the suitable discrete system, we use the discrete result of [52] which gives the necessary and sufficient conditions for which an approximate solution is exponentially stable. As for the uniform polynomial stability, we prove a result which gives necessary and sufficient conditions for which a family of semigroups of operators is uniformly polynomially stable. To our knowledge, our work in the second chapter is the first one which addresses the uniform polynomial stability of the discrete schemes.

As for the convergence of the chosen approximate system, we use a general version of the Trotter-Kato Theorem proved in [45] to show that the discrete solution tends to the solution of (0.1.1) as the discretization parameter goes to zero and if the discrete initial data are well chosen. Finally, we end up the second chapter by some illustrative examples which show the limits of the previous work done concerning the approximations and values the attained results of the second chapter.

In the third chapter, we move on to another subject which treats the stabilization of wave equations with indefinite sign damping. As in [1], we analyze the stability of two problems. We consider a one-dimensional wave equation with an indefinite sign damping and a zero order potential term which is internally damped of the form

$$\begin{gathered} u_{tt}(x,t) - u_{xx}(x,t) + 2\chi_{(0,1)}(x)u_t(x,t) + 2\alpha\chi_{(-1,0)}(x)u_t(x,t) = 0,\ x \in (-1,1),\ t > 0, \\ u(1,t) = u(-1,t) = 0,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \end{gathered} \tag{0.1.2}$$

where α is a given constant. Besides, we consider a one-dimensional wave equation with an indefinite sign damping and which is both internally and boundary damped under the form

$$\begin{aligned} &u_{tt}(x,t) - u_{xx}(x,t) + au_t(x,t) = 0, && x \in (0,1),\ t > 0, \\ &u(0,t) = 0,\ u_x(1,t) = -bu_t(1,t), && t > 0, \\ &u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), && \end{aligned} \tag{0.1.3}$$

where $a, b \in \mathbb{R}$.

It is well known that problem (0.1.2) is exponentially stable if the damping term α is non-negative (see [23]). Similarly, if the coefficients a and b are both positive, then, using for instance integral inequalities, one can prove that (0.1.3) is also exponentially stable. In the third chapter, we are interested in the case when the damping terms are allowed to change their sign. Our aim is to analyze to what extent the variation of the sign affects the stability of the problem.

Problem (0.1.2) can be written as a system of the form $U_t = A_\alpha U$ where $U =$

$(u, u_t)^\top$ and the operator $A_\alpha : D(A_\alpha) \to X$ is defined by

$$A_\alpha = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & -2\chi_{(0,1)} - 2\alpha\chi_{(-1,0)} \end{bmatrix}$$

where the energy space $X = H_0^1(-1,1) \times L^2(-1,1)$ is equipped with the usual inner product defined by

$$< (f,g)^\top, (u,v)^\top >= \int_{-1}^{1} (f'\overline{u}' + g\overline{v})dx,$$

and

$$D(A_\alpha) = (H^2(-1,1) \cap H_0^1(-1,1)) \times H_0^1(-1,1).$$

In this case, the energy associated with problem (0.1.2), at time t, is given by

$$E_1(t) = \frac{1}{2}\left(\int_{-1}^{1} (|u_x(x,t)|^2 + |u_t(x,t)|^2)dx\right)$$

with

$$E_1'(t) = -2\left(\int_0^1 |u_t(x,t)|^2 dx + \alpha \int_{-1}^{0} |u_t(x,t)|^2 dx\right), \quad \forall (u_0,u_1) \in D(A_\alpha).$$

Therefore, when $\alpha < 0$, the dissipation of the energy is not trivial. Moreover, the classical techniques which are normally employed to study the stabilization like multipliers method, energy and resolvent methods cannot be well invoked in this case since these methods are based on estimations which involve the absolute value of the damping coefficients. Therefore, the question of the stability of the solution of (0.1.2) in the case of $\alpha < 0$ becomes more involved.

The main motivation behind this question started with a conjecture in [21] by Chen et al. who considered the internally indefinite sign damped wave equation of the form

$$u_{tt} - u_{xx} + 2a(x)u_t = 0,\ x \in (0,1),\ t > 0, \tag{0.1.4}$$

with standard initial conditions and Dirichlet boundary conditions.

It was conjectured that if there exists some $\gamma > 0$ such that for every $n \in \mathbb{N}$ the following condition is satisfied

$$I_n = \int_0^1 a(x)\sin^2(n\pi x)dx \geq \gamma, \tag{0.1.5}$$

then the energy decays exponentially. The hypothesis imposed on the uniform positivity of I_n in (0.1.5) yields the positivity of the average a_0 of a since $I_n \to a_0$, as $n \to +\infty$. However, Freitas in [28] disproves the conjecture of Chen et al. He shows that (0.1.5) is not sufficient to guarantee the exponential stability. He finds out that if $\|a\|_{L^\infty}$ is large then there may exist some positive real eigenvalues (see Theorem 3.6 of [28]). So later on, Freitas and Zuazua in [30] suggest replacing the function $a(x)$ in (0.1.4) by $\epsilon a(x)$ with $\epsilon > 0$ small enough. In this case, the exponential stability is proved under condition (0.1.5) and the additional condition $a \in L^\infty(0,1) \cap BV$ so that its derivative is defined in the weak sense. Furthermore, in [51], the authors find an upper bound of ϵ for which the problem becomes exponentially stable under condition (0.1.5) and the assumption that $a \in L^\infty(0,1)$ without the need for the assumption that $a \in BV$. On the other hand, in [57], Racke and Rivera have removed the factor ϵ and considered the wave equation $u_{tt} - u_{xx} + a(x)u_t = 0$ on $(0, L)$ for some $L > 0$ where $a \in L^\infty(0, L)$ is allowed to change its sign such that its mean value a_0 remains positive. In [57], the exponential stability is proved under one of these conditions : Either $\|a\|_{L^\infty}$ is possibly large with sufficiently small $\|a - a_0\|_{L^2}$ or $\|a\|_{L^\infty}$ is sufficiently small but the pair (a, L) has to satisfy some estimates where it is possible to get a negative moment I_k.

In the third chapter, our work differs from the previous results since we do not want to impose neither a small value of the damping factor a nor a small value of $\|a - a_0\|_{L^2}$. Indeed for system (0.1.2), this mean value is equal to $\sqrt{2}|1 - \alpha|$ which we do not need to be sufficiently small. Moreover, the upper bound of ϵ found in [51] is not easy to check for system (0.1.2).

From the asymptotic behavior of the spectrum of A_α, we find that, according to the value of α, problem (0.1.2) is either unstable or exponentially stable. Using detailed spectral analysis, we find the characteristic equation satisfied by the eigenvalues of A_α and then we show that the root vectors of A_α form a Riesz basis of the energy space. Finally, we find a critical value of α for which the solution of (0.1.2) becomes exponentially stable. Although the critical value which we find for α is not optimal, this value remains coherent with that given by the perturbation theory of semigroups.

In the third chapter, we perform a similar analysis for problem (0.1.3). As usual, by the standard reduction of order method, we can rewrite formally (0.1.3) in the simpler form $U_t = A_a U$, with $U = (u, u_t)^\top$ and the operator $A_a : D(A_a) \to X$ is defined by

$$A_a = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & -a \end{bmatrix} \tag{0.1.6}$$

where the energy space $X = H_l(0,1) \times L^2(0,1)$ is equipped with the usual inner product defined by

$$< (f,g)^\top, (u,v)^\top >= \int_0^1 (f'\overline{u}' + g\overline{v})\, dx,$$

where $H_l(0,1) = \{u \in H^1(0,1);\ u(0) = 0\}$ and therefore, $D(A_a) = \{(u,v)^\top \in H^2(0,1) \cap H_l(0,1) \times H_l(0,1);\ u_x(1) = -bv(1)\}$.

The energy of (0.1.3) is given by

$$E_2(t) = \frac{1}{2}\int_0^1 \left(|u_t|^2 + |u_x|^2\right) dx,$$

and hence formally

$$E_2'(t) = -a\int_0^1 |u_t|^2 dx - b|u_t(1)|^2.$$

From this identity, we remark that A_a is not necessarily dissipative when $ab < 0$. Therefore, we are interested in the case when a and b are of opposite signs. Note that for such a problem, perturbation theory of contractive semigroups cannot be well invoked. Using detailed spectral analysis, we find the conditions that a and b must satisfy so that problem (0.1.3) becomes exponentially or polynomially stable. The particular case $b \in (-1, 0)$ and $a > 0$ retains our attention where we find optimal results for which (0.1.3) is exponentially or polynomially stable.

Finally in the fourth chapter, as in [2], we generalize the analysis of the third chapter to study the exponential stability of the wave equation on a star shaped network with indefinite sign damping which is of the form

$$\begin{cases} u^i_{tt}(x,t) - u^i_{xx}(x,t) + 2\epsilon a_i(x) u^i_t(x,t) &= 0, \qquad x \in (0, L_i),\ t > 0, \\ u^i(L_i, t) &= 0, \\ u^i(0,t) &= u^j(0,t), \quad \forall i \neq j, \\ \sum_{i=1}^{N} u^i_x(0,t) &= 0, \\ u^i(x,0) &= u^i_0(x), \qquad x \in (0, L_i), \\ u^i_x(x,0) &= u^i_1(x), \qquad x \in (0, L_i). \end{cases} \tag{S_1}$$

where $L_i \in \mathbb{R}^+_*$, and $a_i \in W^{1,\infty}(0, L_i)$. We further consider the following hypothesis on the geometry of the domain :

(**H**) There exists $q \in \mathbb{N}^*$ such that for all $i = 2, ..., N$, there exists $p_i \in \mathbb{N}^*$ where $L_i = \dfrac{p_i}{q} L_1$.

In the first part of the fourth chapter, we study the stability of system (S_1) when $\epsilon = 1$. We give necessary and sufficient conditions for which system (S_1) becomes exponentially stable up to a finite dimensional space. The idea is inspired from [65] where the characteristic equation of (S_1) is approximated by another function using the shooting method. This approximation allows us to detect the behavior of the

high frequencies and hence deduce the conditions on the damping coefficients a_i for which the high frequencies are situated to the left of the imaginary axis. Finally, after we prove that the generalized root vectors form a Riesz basis with parentheses, we deduce the exponential stability of (S_1) up to a finite dimensional space generated by the roots vectors of the low frequencies. In the first part, when $N = 2$, we recover the result of Theorem 3.2.1 of this thesis.
In the second part, we consider system (S_1) with ϵ positive but small enough so that we extend the results of Freitas and Zuazua in [30] where $N = 2$. In fact, for $\epsilon > 0$ small enough, unlike [30], we deal with multiple eigenvalues when splitting may occur as ϵ increases. First, we consider $a_i \in \mathbb{R}$ and $L_i = 1$ for all $i = 1, ..., N$ and then we consider $a_i \in L^\infty(0, 1)$. In fact, when $\epsilon > 0$ small enough, the study of the exponential stability of (S_1) enters in the framework of the abstract theory done in [51]. Using the concepts introduced in [46] about the behavior of the spectrum, we shall interpret the hypothesis imposed in [51] to find explicit conditions on the damping coefficients for which (S_1) is exponentially stable. In the presence of a Riesz basis with parenthesis, we search for sufficient conditions for which the eigenvalues are situated strictly to the left of the imaginary axis. We find out that the positivity of the mean of the damping coefficients is not enough to guarantee the exponential stability of (S_1) in the whole energy space. In this second part, we recover the result of Theorem 2.1 of [30] when the damping coefficient is piecewise constant but without the assumption on the positivity of the integrals I_n given in (0.1.5). Finally, we end up the fourth chapter by giving some concrete examples of $\{a_i\}_{i=1}^N$ and N.

0.2 Aims and achieved results

For more coherence, we summarize the main goals and the new results attained in this Ph.D. thesis into the following points :

(i) Search for a suitable approximate system which converges towards problem (0.1.1) and has the same decay properties as (0.1.1) in the presence of the generalized gap condition.

(ii) Analyze the polynomial decay of the discrete schemes when the continuous problem has such a decay and prove a result about uniform polynomial stability for a family of semigroups of operators.

(iii) Use a general version of the Trotter-Kato Theorem proved in [45] to prove the convergence of the discrete solution towards the solution of (0.1.1) as the discretization parameter goes to zero and if the discrete initial data are well chosen.

(iv) Study the stability of wave equations in the presence of indefinite sign damping where the classical methods for studying the stabilization fail to treat such problems.

(v) Consider indefinite sign damping coefficients whose L^∞ norm is not necessarily small.

(vi) Use detailed spectral analysis to find critical values of the damping coefficients for which wave equations with indefinite sign damping become stable.

(vii) Generalize the analysis of the stability of wave equations with indefinite sign damping terms over a star shaped network.

Chapitre 1

Preliminaries

As the analysis done in this Ph.D. thesis is based on the semigroup and spectral analysis theories, we recall, in this chapter, some basic definitions and theorems which will be used in the following chapters. We refer to [8, 18, 24, 36, 37, 42, 62, 63].

1.1 Semigroups

Most of the evolution equations can be reduced to the form

$$\begin{cases} \dot{x}(t) &= \mathcal{A}x(t), \quad t > 0, \\ x(0) &= x_0, \end{cases}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $T(t)$ over a Hilbert space $\mathcal{H}$. Therefore, we start by introducing some basic concepts concerning the semigroups.

Definition 1.1.1. *Let X be a Banach space.*

1) A one parameter family $T(t)$, $t > 0$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if

(i) $T(0) = I$;

(ii) $T(t+s) = T(t)T(s)$ *for every* $s, t \geq 0$.

2) A semigroup of bounded linear operators, $T(t)$*, is uniformly continuous if*

$$\lim_{t\to 0}\|T(t) - I\| = 0.$$

3) A semigroup $T(t)$ *of bounded linear operators on* X *is a strongly continuous semigroup of bounded linear operators or a* C_0 *semigroup if*

$$\lim_{t\to 0} T(t)x = x.$$

4) The linear operator $\mathcal{A}$ *defined by*

$$\mathcal{A}x = \lim_{t\to 0}\frac{T(t)x - x}{t}, \quad \forall x \in D(\mathcal{A}),$$

where

$$D(\mathcal{A}) = \left\{x \in X;\ \lim_{t\to 0}\frac{T(t)x - x}{t} \text{ exists}\right\}$$

is the infinitesimal generator of the semigroup $T(t)$.

Theorem 1.1.2. *Let* $T(t)$ *be a* C_0 *semigroup. Then there exist constants* $\omega \geq 0$ *and* $M \geq 1$ *such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad \forall t > 0.$$

In the above theorem, if $\omega = 0$, then $T(t)$ is called uniformly bounded and if moreover $M = 1$, then $T(t)$ is called a C_0 semigroup of contractions.

For the existence of solutions, we normally use the following Lumer-Phillips Theorem or Hille-Yosida Theorem.

Theorem 1.1.3. *(Lumer-Phillips Theorem) Let* $\mathcal{A}$ *be a linear operator with dense domain* $D(\mathcal{A})$ *in a Hilbert space* X*. If* *(i)* $\mathcal{A}$ *is dissipative; i.e.,* $\Re < \mathcal{A}x, x >_X < 0,\ \forall x \in D(\mathcal{A})$

and if

(ii) there exists a $\lambda_0 > 0$ *such that the range* $\mathcal{R}(\lambda_0 I - \mathcal{A}) = X$,

then $\mathcal{A}$ *generates a* C_0 *semigroup of contractions on* X.

Theorem 1.1.4. *(Hille-Yosida Theorem) Let* $\mathcal{A}$ *be a linear operator on a Banach space* X *and let* $\omega \in \mathbb{R}$, $M \geq 1$ *be constants. Denote by* $\rho(\mathcal{A})$ *the resolvent set of* $\mathcal{A}$. *Then the following properties are equivalent*

(i) $\mathcal{A}$ *generates a* C_0 *semigroup* $T(t)$, $t \geq 0$ *satisfying*

$$\|T(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0;$$

(ii) $\mathcal{A}$ *is closed, densely defined, and for every* $\lambda > \omega$ *one has* $\lambda \in \rho(\mathcal{A})$ *and*

$$\|(\lambda - \omega)^n (\lambda - \mathcal{A})^{-n}\| \leq M, \quad \forall n \in \mathbb{N};$$

(iii) $\mathcal{A}$ *is closed, densely defined, and for every* $\lambda \in \mathbb{C}$ *with* $\Re\lambda > \omega$, *one has* $\lambda \in \rho(\mathcal{A})$ *and*

$$\|(\lambda - \mathcal{A})^{-n}\| \leq \frac{M}{(\Re\lambda - \omega)^n}, \quad \forall n \in \mathbb{N}.$$

If the conditions of the previous two theorems are not clearly satisfied, we may use the following theorem about perturbations by bounded linear operators (see Theorem III.1.1 of [62]).

Theorem 1.1.5. *Let* X *be a Banach space and let* $\mathcal{A}$ *be the infinitesimal generator of a* C_0 *semigroup* $T(t)$ *on* X, *satisfying* $\|T(t)\| \leq Me^{\omega t}$. *If* B *is a bounded linear operator on* X, *then* $A + B$ *is the infinitesimal generator of a* C_0 *semigroup* $S(t)$ *on* X, *satisfying* $\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$.

Now we recall a result in [42, 63] which gives necessary and sufficient conditions for which a semigroup is exponentially stable.

Theorem 1.1.6. *Let $T(t)$ be a C_0 semigroup on a Hilbert space $\mathcal{H}$ and $\mathcal{A}$ be its infinitesimal generator. $T(t)$ is exponentially stable ; i.e., there exists M and α positive constants such that $\|T(t)\| \leq Me^{-\alpha t}$ if and only if*

(i) $i\mathbb{R} \subseteq \rho(\mathcal{A})$, *where $\rho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$*

and

(ii) $\sup\limits_{\omega\in\mathbb{R}} \|(i\omega - \mathcal{A})^{-1}\| < \infty.$

When the exponential stability is attained, we search for the optimal exponential decay rate ; mainly for the spectrum determined growth condition.

Definition 1.1.7. *Let $\mathcal{A}$ be the infinitesimal generator of a C_0 semigroup, $T(t)$, on a Hilbert space $\mathcal{H}$. Consider*

$$\omega(\mathcal{A}) := \inf\{\alpha \in \mathbb{R};\ \|T(t)\| \leq Me^{\alpha t}\} = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|,$$

the growth exponent bound of $T(t)$, and

$$\mu(\mathcal{A}) = \sup\{\Re\lambda;\ \lambda \in \sigma(\mathcal{A})\},$$

the spectral abscissa of the operator $\mathcal{A}$ where $\sigma(\mathcal{A})$ denotes its spectrum. If $\omega(\mathcal{A}) = \mu(\mathcal{A})$, then we say that the spectrum determined growth condition holds.

Remark 1.1.8. *From the Hille-Yosida Theorem, we know that $\mu(\mathcal{A}) \leq \omega(\mathcal{A})$ for any infinitesimal generator of a strongly continuous semigroup. However, in general, $\omega(\mathcal{A}) \leq \mu(\mathcal{A})$ is not always true.*

If the semigroup fails to be exponentially stable, we search for another type of decay rate like the polynomial stability which is characterized by the following Theorem in [18].

Theorem 1.1.9. *Let $(T(t))_{t\geq 0}$ be a bounded C_0 semigroup on a Hilbert space $\mathcal{H}$ with a generator $\mathcal{A}$ such that $i\mathbb{R} \subseteq \rho(\mathcal{A})$. Then for a fixed $\alpha > 0$, the following conditions are equivalent :*

(i)

$$\|(is - \mathcal{A})^{-1}\| = O(|s|^{\alpha}), \quad s \to \infty.$$

(ii)

$$\|T(t)\mathcal{A}^{-\alpha}\| = O(t^{-1}), \quad t \to \infty.$$

(iii)

$$\|T(t)\mathcal{A}^{-1}\| = O(t^{\frac{-1}{\alpha}}), \quad t \to \infty.$$

Note that the notation $A = O(B)$ means that there exists $c > 0$ such that $|A| \leq c|B|$.

1.2 Riesz basis

In the second part, we show that the generalized eigenvectors form a Riesz basis of the energy space. Consequently, we recall basic definitions and theorems needed for Riesz basis generation. We refer to [8, 36, 37].

Definition 1.2.1. *(i) A non-zero element φ in a Hilbert space $\mathcal{H}$ is called a generalized eigenvector of a closed linear operator $\mathcal{A}$, corresponding to an eigenvalue λ of $\mathcal{A}$, if there exists $n \in \mathbb{N}^*$ such that*

$$(\lambda I - \mathcal{A})^n \varphi = 0 \quad \textit{and} \quad (\lambda I - \mathcal{A})^{n-1}\varphi \neq 0.$$

If $n = 1$, then φ is an eigenvector.

(ii) The root subspace of $\mathcal{A}$ corresponding to an eigenvalue λ is defined by

$$\mathcal{N}_\lambda(\mathcal{A}) = \bigcup_{n=1}^{\infty} \ker\left((\lambda I - \mathcal{A})^n\right).$$

(iii) The closed subspace spanned by all the generalized eigenvectors of $\mathcal{A}$ is called the root subspace of $\mathcal{A}$.

Remark 1.2.2. *The family of generalized eigenvectors of $\mathcal{A}$ corresponding to λ forms a basis for the subspace $\mathcal{N}_\lambda(\mathcal{A})$. Denote by m_λ the algebraic multiplicity of λ. In general, these generalized eigenvectors, denoted by $\{\varphi_j,\ 1 \leq j \leq m_\lambda\}$, are constructed by the following procedure :*

$$\begin{cases} \mathcal{A}\varphi_1 &= \lambda\varphi_1 \\ \mathcal{A}\varphi_j &= \lambda\varphi_j + \varphi_{j-1}, \quad j = 2, ..., m_\lambda. \end{cases}$$

Now, we introduce the Riesz basis and then we recall some theorems which help us prove that a family forms a Riesz basis.

Definition 1.2.3. *Let $\Phi = \{\varphi_n\}_{n\in\mathbb{N}}$ be an arbitrary family of vectors in a Hilbert space $\mathcal{H}$.*

(i) The family Φ is said to be a Riesz basis in the closure of its linear span (notation $\Phi \in (LB)$) if Φ is an image by an isomorphic mapping of some orthonormal family. Φ is said to be a Riesz basis if $\Phi \in (LB)$ and Φ is a complete family; i.e., $\overline{Span\{\varphi_n;\ n \in \mathbb{N}\}} = \mathcal{H}$.

(ii) The family Φ is said to be $\omega-$linearly independent if whenever $\sum_{n\in\mathbb{N}} a_n\varphi_n = 0$ for $\sum_{n\in\mathbb{N}} |a_n|^2 < \infty$ then $a_n = 0$ for every $n \in \mathbb{N}$.

(iii) The family Φ is minimal if, for any $n \in \mathbb{N}$, the element φ_n does not belong to the span of all the remaining elements ; i.e., $\varphi_n \notin Span\{\varphi_i;\ i \neq n\}$.

Remark 1.2.4. *(i) If $\Phi \in (LB)$, then Φ is minimal and hence $\omega-$linearly independent.*

(ii) If Φ is minimal, then there exists a family $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ biorthogonal to Φ ; i.e., $< \varphi_j, \psi_i >_{\mathcal{H}} = \delta_{ij}$.

The following proposition and theorems give necessary and sufficient condition so that a family Φ forms a Riesz basis.

Proposition 1.2.5. *(Bari's Theorem, Bari 1951; Gokhberg and krein 1965; Nikolski 1980)*
$\Phi \in (LB)$ if and only if there exists positive constants C_1 and C_2 such that for any sequence $\{\alpha_n\}_{n\in\mathbb{N}}$, we have

$$C_1 \sum_{n\in\mathbb{N}} |\alpha_n|^2 \leq \left\| \sum_{n\in\mathbb{N}} \alpha_n \varphi_n \right\|^2 \leq C_2 \sum_{n\in\mathbb{N}} |\alpha_n|^2.$$

In this case, each element $f \in \overline{Span\{\varphi_n,\ n \in \mathbb{N}\}}$ is written as

$$f = \sum_{n\in\mathbb{N}} < f, \psi_n >_{\mathcal{H}} \varphi_n,$$

where $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ is biorthogonal to $\Phi = \{\varphi_n\}_{n\in\mathbb{N}}$.

In this thesis, we mainly use the following theorems to prove that the generalized eigenvectors form a Riesz basis of the energy space.

Theorem 1.2.6. *(Theorem 2.1 of Chapter VI in [36])*
$\{\phi_n\}$ is a Riesz basis of a Hilbert space $\mathcal{H}$ if and only if $\{\phi_n\}$ is complete in $\mathcal{H}$ and there corresponds to it a complete biorthogonal sequence $\{\psi_n\}$ such that for any $f \in \mathcal{H}$ one has

$$\sum_n |< \phi_n, f >|^2 < \infty, \quad \sum_n |< \psi_n, f >|^2 < \infty. \tag{1.2.1}$$

Theorem 1.2.7. *(Classical Bari's Theorem)*
If $\{\varphi_n\}_{n\in\mathbb{N}}$ is a Riesz basis of a Hilbert space $\mathcal{H}$ and another $\omega-$linearly independent family $\{\psi_n\}_{n\in\mathbb{N}}$ is quadratically close to $\{\varphi_n\}_{n\in\mathbb{N}}$ in the sense that $\sum_{n=1}^{\infty} \|\varphi_n - \psi_n\|^2 < \infty$, then $\{\psi_n\}_{n\in\mathbb{N}}$ also forms a Riesz basis of $\mathcal{H}$.

Normally, there is difficulty in understanding the number of generalized eigenfunctions corresponding to low eigenvalues. This severely limits the application of the above classical Bari's Theorem even if the behavior of the high eigenvalues and their corresponding multiplicities are clearly known. Consequently, in case the behavior of the low eigenvalues is vague, we suggest using Theorem 6.3 of [37] which is a new form of Bari's Theorem (see Theorem 2.3 of Chapter VI in [36]) :

Theorem 1.2.8. *Let $\mathcal{A}$ be a densely defined operator in a Hilbert space $\mathcal{H}$ with a compact resolvent. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a Riesz basis of $\mathcal{H}$. If there are an integer $N \geq 0$ and a sequence of generalized eigenvectors $\{\psi_n\}_{n=N+1}^{\infty}$ of $\mathcal{A}$ such that*

$$\sum_{n=N+1}^{\infty} \| \varphi_n - \psi_n \|^2 < \infty,$$

then the set of generalized eigenvectors of $\mathcal{A}$, $\{\psi_n\}_{n=1}^{\infty}$, forms a Riesz basis of $\mathcal{H}$.

Despite that the proof of Theorem 1.2.8 is found in [37], we give another proof which clarifies the relation between the families $\{\varphi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$. First, we recall Lemma 6.2 of [37].

Lemma 1.2.9. *Let $\{\phi_n\}_{n=1}^{\infty}$ be a Riesz basis of a Hilbert space $\mathcal{H}$. Let $N \geq 0$ and $\{\psi_n\}_{n=N+1}^{\infty}$ be another family such that*

$$\sum_{n=N+1}^{\infty} \| \phi_n - \psi_n \|^2 < \infty.$$

Then there exists $M \geq N$ such that

(i) The set $\{\phi_n\}_{n=1}^{M} \cup \{\psi_n\}_{n=M+1}^{\infty}$ is $\omega-$linearly independent.

(ii) $\{\phi_n\}_{n=1}^{M} \cup \{\psi_n\}_{n=M+1}^{\infty}$ is a Riesz basis of $\mathcal{H}$.

Proof: Once (i) is proved, (ii) follows from Theorem 1.2.7. To prove (i), let $M \geq N$ and $\{c_n\}_n \in l^2(\mathbb{N}^*)$ such that

$$\sum_{n=1}^{M} c_n \phi_n + \sum_{n=M+1}^{\infty} c_n \psi_n = 0.$$

Then

$$\sum_{n=1}^{\infty} c_n \phi_n = - \sum_{n=M+1}^{\infty} c_n (\psi_n - \phi_n). \tag{1.2.2}$$

Since $\{\phi_n\}_{n=1}^{\infty}$ is a Riesz basis, there exists $c_1 > 0$ such that

$$c_1 \sum_{n=1}^{\infty} |c_n|^2 \leq \left\| \sum_{n=1}^{\infty} c_n \phi_n \right\|^2 . \tag{1.2.3}$$

Using Cauchy-Schwarz inequality, we get

$$\left\| \sum_{n=M+1}^{\infty} c_n (\psi_n - \phi_n) \right\|^2 \leq \sum_{n=M+1}^{\infty} |c_n|^2 \sum_{n=M+1}^{\infty} \|\psi_n - \phi_n\|^2 \leq \frac{c_1}{2} \sum_{n=1}^{\infty} |c_n|^2, \tag{1.2.4}$$

provided M is chosen great enough.

Combining (1.2.2), (1.2.3), and (1.2.4) we get $c_n = 0, \ \forall n \in \mathbb{N}^*$. ■

The following theorem clarifies the results of Theorem 1.2.8.

Theorem 1.2.10. *Let $\mathcal{A}$ be a densely defined operator in a Hilbert space $\mathcal{H}$ with a compact resolvent. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a Riesz basis of $\mathcal{H}$. If there are two integers $N_1, N_2 \geq 0$ and a sequence of generalized eigenvectors $\{\psi_n\}_{n=N+1}^{\infty}$ of $\mathcal{A}$ such that*

$$\sum_{n=1}^{\infty} \| \varphi_{n+N_2} - \psi_{n+N_1} \|^2 < \infty, \tag{1.2.5}$$

then the set of generalized eigenvectors (or root vectors) of $\mathcal{A}$, $\{\psi_n\}_{n=1}^{\infty}$ forms a Riesz basis of $\mathcal{H}$.

Proof: The proof is divided into five steps.

First step.

For all $n \geq 1$, we set $\chi_{n+N_2} = \psi_{n+N_1}$. Thus, we have $\chi_m = \psi_{m+N_1-N_2}, \forall m \geq N_2 + 1$. (1.2.5) means that

$$\sum_{n=N_2+1}^{\infty} \| \chi_n - \phi_n \|^2 < \infty. \tag{1.2.6}$$

Consequently, by Lemma 1.2.9, there exists $M \geq N_2$ such that $\{\phi_n\}_{n=1}^{M} \cup \{\psi_n\}_{n=M+N_1-N_2+1}^{\infty}$ is a Riesz basis of $\mathcal{H}$.

We can assume that the eigenvalues corresponding to $\{\psi_n\}$ for $n \leq M+N_1-N_2$ are different from those of $\{\psi_n\}$ for $n \geq M+N_1-N_2+1$.

Second step.

Now, let $V = \overline{\text{span}\{\psi_n\}_{n=M+N_1-N_2+1}^{\infty}}$, then the set of roots vectors $\{\psi_n\}_{n=1}^{M+N_1-N_2}$ is independent and does not belong to V.

We denote by π_V the orthogonal projection on the space V and define

$$H_0 = \text{span}\{\tilde{\psi}_n\}_{n=1}^{M+N_1-N_2},$$

where $\tilde{\psi}_i = \psi_i - \pi_V \psi_i,\ i = 1, ..., M+N_1-N_2$. Clearly, $\{\tilde{\psi}_n\}_{n=1}^{M+N_1-N_2}$ are independent and $\dim H_0 = M + N_1 - N_2$.

Let P be the orthogonal projector on H_0. We have

$$P\tilde{\psi}_i = \tilde{\psi}_i,\ i = 1, ..., M+N_1-N_2 \quad \text{and} \quad P\psi_i = 0,\ \forall i > M+N_1-N_2$$

By the first step, for each $i \in \mathbb{N}^*$, there exists $\{a_n^i\}_{n=1}^{\infty}$ with $a_n^i \in \mathbb{C}$ such that

$$\tilde{\psi}_i = \sum_{n=1}^{M} a_n^i \phi_i + \sum_{n=M+N_1-N_2+1}^{\infty} a_n^i \psi_i.$$

Hence, for $i = 1, ..., M+N_1-N_2$, we get

$$\tilde{\psi}_i = P\tilde{\psi}_i = \sum_{n=1}^{M} a_n^i P\phi_i.$$

This shows that $M \geq M + N_1 - N_2$ or equivalently $N_2 \geq N_1$.

Third step.

If we assume that $N_2 > N_1$, then $H_0 \subsetneq V^{\perp}$. Thus, let $\psi \neq 0 \in V^{\perp} \cap H_0^{\perp}$ and assume for the moment that the set $\{\psi_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}$. We can write

$$\psi = \sum_{n=1}^{\infty} a_n \psi_n.$$

Therefore,

$$0 = P\psi = \sum_{n=1}^{\infty} a_n P\psi_n = \sum_{n=1}^{M+N_1-N_2} a_n P\psi_n.$$

But for $n \leq M + N_1 - N_2$, we have

$$P\psi_n = P(\tilde{\psi}_n + \pi_V \psi_n) = \tilde{\psi}_n.$$

Consequently, we deduce that

$$0 = \sum_{n=1}^{M+N_1-N_2} a_n \tilde{\psi}_n;$$

i.e., $a_n = 0, i = 1, ..., M + N_1 - N_2$. It follows that

$$\psi = \sum_{M+N_1-N_2+1}^{\infty} a_n \psi_n \in V^{\perp} \cap V = \{0\},$$

which is a contradiction. Consequently, $N_1 = N_2$.

Fourth step.

By the first step, $\{\phi_n\}_{n=1}^{M} \cup \{\psi_n\}_{n=M+1}^{\infty}$ forms a Riesz basis of $\mathcal{H}$. Therefore, it remains to prove that $\{\psi_n\}_{n=1}^{\infty}$ is $\omega-$linearly independent so that we can deduce by Theorem 1.2.7 that $\{\psi_n\}_{n=1}^{\infty}$ forms a Riesz basis of $\mathcal{H}$. Indeed, suppose that

$$\sum_{n=1}^{\infty} a_n \psi_n = 0.$$

Hence,

$$0 = \sum_{n=1}^{M} a_n \psi_n + \sum_{M+1}^{\infty} a_n \psi_n.$$

By the first step, we can write

$$\sum_{n=1}^{M} a_n \psi_n = \sum_{n=1}^{M} b_n \phi_n + \sum_{M+1}^{\infty} b_n \psi_n,$$

where for every $n \in \mathbb{N}^*$, $b_n \in \mathbb{C}$. Therefore,

$$0 = \sum_{n=1}^{M} b_n \phi_n + \sum_{n=M+1}^{\infty} (a_n + b_n) \psi_n,$$

which implies that $b_n = 0$ for all $n = 1, ..., M$. Finally, from

$$\sum_{n=1}^{M} a_n \psi_n = \sum_{M+1}^{\infty} b_n \psi_n,$$

we deduce that $\sum_{n=1}^{M} a_n \psi_n = 0$ since $\{\psi_n\}_{n=1}^{M}$ does not belong to V. Hence $a_n = 0, \forall n \in \mathbb{N}^*$.

Last step.

It remains to prove that $\{\psi_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}$. If it is not the case, by the first and second steps, we only know that the family $\{\psi_n\}_{n=1}^{\infty}$ is a Riesz basis for the subspace $H_1 = \overline{Sp(\mathcal{A})}$, the closed subspace spanned by all generalized eigenvectors $\{\psi_n\}_{n=1}^{\infty}$ of $\mathcal{A}$, and that its codimension is finite.

Let $H_2 = H_1^{\perp}$. Thus, $\mathcal{H} = H_1 \oplus H_2$ and $\dim(H_2) < \infty$. Without loss of generality, we may assume that $0 \notin \sigma(\mathcal{A})$; i.e., $\mathcal{A}$ has a compact inverse $\mathcal{A}^{-1} = B$.

Since H_1 is stable by B, then H_2 is stable by B^*. Consequently $B^*_{|H_2}$ admits at least one eigenvalue μ because H_2 is finite dimensional. Thus, there exists $x \neq 0 \in H_2$ such that $B^* x = \mu x$.

We start by proving that necessarily $\mu = 0$. If $\mu \neq 0$ then the complex $\lambda \neq 0$ such that $\mu = \dfrac{1}{\lambda}$ is in $\sigma(\mathcal{A})$. Let H_λ be the root subspace of $\mathcal{A}$ associated to λ and n be the algebraic multiplicity of λ; i.e., the smallest integer such that $H_\lambda = \ker(I - \lambda B)^n$.

Since B is a compact operator, $I - (I - \lambda B)^n$ is also a compact operator and by the Fredholm alternative

$$\ker(I - [I - (I - \lambda B)^n])^{\perp} = R(I - [I - (I - \lambda B)^n])^*;$$

i.e.,

$$H_\lambda^{\perp} = (\ker(I - \lambda B)^n)^{\perp} = R(I - \overline{\lambda} B^*)^n.$$

But $H_\lambda \subseteq H_1$ implies that $x \in H_\lambda^{\perp}$. Consequently, there exists $y \in \mathcal{H},\ y \neq 0$, such that $x = (I - \overline{\lambda} B^*)^n y$.

As $(I - \overline{\lambda}B^*)x = 0$, we get $0 = (I - \overline{\lambda}B^*)^{n+1}y$ with $y \neq 0$. Hence, we get a contradiction since $\frac{1}{\lambda} \in \sigma(B)$ and $\frac{1}{\overline{\lambda}} \in \sigma(B^*)$ have the same algebraic multiplicity.

Therefore, we have $\mu = 0$, but this contradicts the well-known fact that $\mathcal{H} = \overline{R(B)} \oplus \ker(B^*)$ and the assumption $\overline{R(B)} = \overline{D(\mathcal{A})} = \mathcal{H}$. ■

1.3 Riesz basis with parenthesis

Sometimes we fail to prove the existence of a Riesz basis of the energy space or we need some supplementary hypothesis to find a Riesz basis. However, as in the fourth chapter, we can neglect these hypothesis and find a more general basis of the energy space which is called a Riesz basis with parenthesis. According to [75], we recall the definition of a Riesz basis of subspaces and a Riesz basis with parenthesis.

Definition 1.3.1. – *A family of subspaces $\{W_k\}_{k\in\mathbb{N}}$ is called a Riesz basis of subspaces of $\mathcal{H}$ if*

(i) for every $f \in \mathcal{H}$, and every $k \in \mathbb{N}$, there is a unique $f_k \in W_k$ such that $f = \sum_{k\in\mathbb{N}} f_k$, and

(ii) there are positive constants C_1 and C_2 such that

$$C_1 \sum_{k\in\mathbb{N}} \|f_k\|^2 \leq \left\| \sum_{k\in\mathbb{N}} f_k \right\|^2 \leq C_2 \sum_{k\in\mathbb{N}} \|f_k\|^2 .$$

– *A sequence $\{y_i\}_{i\in\mathbb{N}}$ is called a Riesz basis with parenthesis of $\mathcal{H}$ if there is a family $\{W_k\}_{k\in\mathbb{N}}$ of finite-dimensional spaces spanned by some y_i with $W_k \cap W_j = \{0\}$ for $k \neq j$ that forms a Riesz basis of subspaces of $\mathcal{H}$. The spaces W_k are called the parentheses.*

Now, we recall a Theorem which can be proved exactly as Theorem 2 in [71] which gives sufficient conditions for which the generalized eigenfunctions of some operator $\widetilde{A}$ form a Riesz basis with parenthesis. For this aim, we need to group the eigenvalues of $\widetilde{A}$ by packets made of a finite number of eigenvalues and in such a way that the packets remain at a positive distance from each other. Namely for any $r > 0$, we introduce the sets $G_p(r), p \in \mathbb{Z}$ as the connected components of the set $\cup_{\lambda \in \sigma(\widetilde{A})} D_\lambda(r)$,where $D_\lambda(r)$ is the disc with center λ and radius r, as well as the packets of eigenvalues $\Lambda_p(r) = G_p(r) \cap \sigma(\widetilde{A})$. The following Theorem gives sufficient conditions for which the generalized eigenfunctions of a bounded perturbation of a selfadjoint operator form a Riesz basis with parenthesis.

Theorem 1.3.2. *Let T be a selfadjoint operator over a Hilbert space H with discrete spectrum $\{\mu_k\}_{k\in\mathbb{Z}}$ which satisfies the generalized gap condition, i.e, there exists $k_0 > 0$ and $c > 0$ such that*

$$\mu_{k+k_0} - \mu_k > c, \forall k \in \mathbb{Z}.$$

Let B be a bounded operator from H into itself. Then the root vectors of the perturbation $\widetilde{A} = T + B$ form a Riesz basis with parenthesis of H. In this case, only terms corresponding to merging eigenvalues should be put in parenthesis, i.e, there exist $r > 0$ and $N \in \mathbb{N}^*$ such that if we set

$$\Lambda_p = \Lambda_p(r),$$

then

$$\#\Lambda_p \leq N, \forall p \in \mathbb{Z},$$

$$\sigma(T + B) = \cup_{p\in\mathbb{Z}} \Lambda_p,$$

and we can take as parenthesis W_p, $p \in \mathbb{Z}$, the space spanned by the root vectors of $T + B$ corresponding to the eigenvalues in Λ_p where for any $f \in H$, $f_p = \mathbb{P}_p f$ is the the Riesz projection of $T + B$, i.e.,

$$f_p = \mathbb{P}_p f = \frac{1}{2i\pi} \int_{\gamma_p} (\lambda - T - B)^{-1} f \, d\lambda,$$

where γ_p *is a contour surrounding* Λ_p.

Chapitre 2

Uniformly exponentially or polynomially stable approximations for second order evolution equations and some applications

2.1 Introduction and Motivation

Recently, the approximation of second order evolution equations has been extensively studied where misbehavior of the discrete solutions has been remarkably observed (see [25, 27, 55, 72, 73]). Indeed, the discrete schemes, obtained by finite difference, finite element, or finite volume discretization, introduce spurious high oscillations which do not exist at the continuous level and which propagate with group velocity of the order of the mesh size. As a result, even though the numerical scheme converges in the classical sense towards the continuous problem, observability inequalities do not hold uniformly with respect to the discretization parameter and hence, the decay rate of the discrete system turns out to depend on the mesh size. In fact, the uniform decay rate is equivalent to some observability inequalities which estimate the discrete energy by the velocity of the propagation of the oscillations. However, in the presence of high frequency discrete solutions whose velocity is of the order of the mesh size, the observability constants blow up as the mesh size tends to zero (see [55, 73]). Consequently, most of the classical numerical approximation schemes do not possess the same decay rate as that of the continuous problem.

For more coherence, we start with a simple example which studies the approximation properties of a 1-d internally damped wave equation which models the vibrations of a flexible string clamped at each of its ends of the form

$$\begin{cases} y'' - y_{xx} + ay' = 0 & (x,t) \in (0,1) \times (0,\infty), \\ y(0,t) = y(1,t) = 0 & t > 0, \\ y(x,0) = y^0,\ y'(x,0) = y^1, & x \in (0,1), \end{cases} \tag{2.1.1}$$

where $a \geq 0$ such that $a \in L^\infty(0,1)$ and $a(x) \geq a_1 > 0$ for all $x \in I \subsetneq (0,1)$. The symbol ' denotes the partial differentiation with respect to time.

It is well known that for such a choice of a, (2.1.1) is exponentially stable (see [23]). However, referring to [26], we show that for the classical finite difference scheme of (2.1.1), the exponential decay of the discretized energy is non-uniform. For this purpose, given $N \in \mathbb{N}^*$, set $h = \dfrac{1}{N+1}$ and consider the subdivision of $(0,1)$ given by

$$0 = x_0 < x_1 < ... < x_N < x_{N+1} = 1$$

where $x_j = jh$ for $j \in \{0, ..., N+1\}$.

Therefore, the classical finite difference space semi-discretization of problem

(2.1.1) is given by

$$\begin{cases} y_j'' - \dfrac{y_{j+1} - 2y_j + y_{j-1}}{h^2} + a_j y_j' = 0 & t > 0, \ j = 1, ..., N, \\ y_j(t) = 0, & t > 0, \ j = 0, N+1, \\ y_j(0) = y_j^0, \ y_j'(0) = y_j^1 & j = 1, ..., N, \end{cases} \tag{2.1.2}$$

where the function $y_j(t)$ provides an approximation of $y(x_j, t)$ and $a_j = a(jh)$ for all $j = 1, ..., N$. For simplification, we introduce the vector notation where we let $y_h(t) = (y_1(t), ..., y_N(t))^\top$ and $a_h y_h(t) = (a_1 y_1(t), ..., a_N y_N(t))^\top$. Moreover, we define the matrix

$$A_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Then system (2.1.2) reads as

$$\begin{cases} y_h'' + A_h y_h(t) + a_h y_h' = 0 & t > 0, \\ y_h(0) = y_h^0, \ y_h'(0) = y_h^1. \end{cases}$$

The energy of system (2.1.2) is given by

$$E_h(y_h, t) = \frac{h}{2} \sum_{j=0}^{N} \left[\left| y_j'(t) \right|^2 + \left| \frac{y_{j+1} - y_j}{h} \right|^2 \right],$$

which is a natural discretization of the continuous energy and the discrete dissipation law is expressed by

$$E_h'(y_h, t) = -h \sum_{j=0}^{N} a_j |y_j'|^2. \tag{2.1.3}$$

Theorem 2.1.1. *The exponential decay of $E_h(y_h, t)$ to zero is non-uniform with respect to the mesh size h ; i.e., there do not exist positive constants M and ω which* *are independent of h such that for all $h > 0$, and for all initial data $(y_j^0)_j$, and $(y_j^1)_j \in \mathbb{R}^{\mathbb{N}}$, we have*

$$E_h(y_h, t) \leq Me^{-\omega t} E_h(y_h, 0).$$

To prove the above theorem, we need the following two lemmas :

Lemma 2.1.2. *If there exists some positive constants M and ω independent of h such that for all $(y_j^0)_j$, and $(y_j^1)_j \in \mathbb{R}^{\mathbb{N}}$, we have*

$$E_h(y_h, t) \leq Me^{-\omega t} E_h(y_h, 0) \quad \forall t > 0, \tag{2.1.4}$$

then there exists T_0 and C_0, bounded with respect to h, such that for all $(u_j^0)_j$ and $(u_j^1)_j$ in $\mathbb{R}^{\mathbb{N}}$

$$2E_h(u_h, 0) = h\sum_{j=0}^{N} \left[\left|u_j^1\right|^2 + \left| \frac{u_{j+1}^0 - u_j^0}{h} \right|^2 \right] \leq C_0 h \sum_{j=0}^{N} \int_0^{T_0} a_j |u_j'(t)|^2 dt, \tag{2.1.5}$$

where $(u_j)_j$ solves the conservative semi-discrete system given by

$$\begin{cases} u_j'' - \dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = 0, & t > 0,\ j = 1, ..., N, \\ u_j(t) = 0, & t > 0,\ j = 0, N+1, \\ u_j(0) = u_j^0 = y_j^0,\ u_j'(0) = u_j^1 = y_j^1, & j = 1, ..., N. \end{cases} \tag{2.1.6}$$

Proof: The idea of the proof is found in [73]. According to the dissipation law (2.1.3), we have for all $T > 0$,

$$E_h(y_h, 0) - E_h(y_h, T) = h \sum_{j=0}^{N} \int_0^T a_j |y_j'|^2 dt. \tag{2.1.7}$$

If we choose $T \geq \left(\ln(\frac{4M}{3})\right)/\omega$, then (2.1.4) implies that

$$E_h(y_h, T) \leq \frac{3}{4} E_h(y_h, 0) = \frac{3}{4} E_h(u_h, 0).$$

Hence,

$$h\sum_{j=0}^{N}\int_0^T a_j|y_j'|^2dt \geq \frac{1}{4}E_h(u_h,0). \tag{2.1.8}$$

Let $y_h = u_h + v_h$ where v_h solves the complementary semi-discrete system given by

$$\begin{cases} v_j'' - \dfrac{v_{j+1}-2v_j+v_{j-1}}{h^2} + a_j(v_j'+u_j') = 0, & t>0,\ j=1,...,N, \\ v_j(t) = 0, & t>0,\ j=0,N+1, \\ v_j(0) = v_j'(0) = 0, & j=1,...,N. \end{cases} \tag{2.1.9}$$

Since $E_h(v_h,0) = 0$, then

$$E_h(v_h,T) + h\sum_{j=0}^{N}\int_0^T a_j\left|v_j'(t)\right|^2dt = -h\sum_{j=0}^{N}\int_0^T a_j u_j'(t)\overline{v_j'(t)}dt.$$

Hence,

$$h\sum_{j=0}^{N}\int_0^T a_j\left|v_j'(t)\right|^2dt \leq h\sum_{j=0}^{N}\int_0^T a_j\left|u_j'(t)\right|^2dt, \tag{2.1.10}$$

which implies that

$$\begin{aligned} h\sum_{j=0}^{N}\int_0^T a_j\left|y_j'(t)\right|^2dt &\leq 2h\sum_{j=0}^{N}\int_0^T a_j\left|v_j'(t)\right|^2dt + 2h\sum_{j=0}^{N}\int_0^T a_j\left|u_j'(t)\right|^2dt \\ &\leq 4h\sum_{j=0}^{N}\int_0^T a_j\left|u_j'(t)\right|^2dt. \end{aligned}$$

Using the above inequality in (2.1.8) yields

$$4h\sum_{j=0}^{N}\int_0^T a_j\left|u_j'(t)\right|^2dt \geq \frac{1}{4}E_h(u_h,0).$$

■

The following lemma shows that the observability constant C_0 in the discrete observability inequality (2.1.5) blows up as $h \to 0$ and hence the uniform observability fails to hold which implies the lack of a uniform exponential decay rate which completes the proof of Theorem 2.1.1.

Lemma 2.1.3. *For any $T > 0$, we have*

$$\lim_{h \to 0} \inf_{u_h \text{solution of (2.1.6)}} \frac{1}{E_h(u_h, 0)} h \sum_{j=0}^{N} \int_0^T a_j \left| u_j'(t) \right|^2 dt = 0. \qquad (2.1.11)$$

Proof: First, we analyze the spectrum of the matrix A_h; i.e., we consider the eigenvalue problem

$$-\frac{1}{h^2}(w_{j+1} - 2w_j + w_{j-1}) = \lambda w_j, \quad j = 1, ..., N; \quad w_0 = w_{N+1} = 0. \qquad (2.1.12)$$

According to [44], the spectrum is explicitly given by

$$\lambda_h^k = \frac{4}{h^2} \sin^2 \left(\frac{k\pi h}{2} \right), \quad k = 1, ..., N,$$

and the corresponding eigenvectors are given by

$$w_h^k = \left(w_{1,h}^k, ..., w_{N,h}^k \right)^T; \; w_{j,h}^k = \sin(k\pi jh), \quad k, j = 1, ..., N.$$

Obviously, $\lambda_h^k \to \lambda^k = k^2\pi^2$, as $h \to 0$ for each $k \geq 1$ where λ^k is the kth eigenvalue of the continuous wave equation (2.1.1). Moreover, the eigenvectors w_h^k of the discrete system coincide with the restriction to the mesh points of the eigenfunctions $w^k(x) = \sin(k\pi x)$ of the continuous wave equation (2.1.1). Furthermore, we notice that the gap between $\sqrt{\lambda_h^N}$ and $\sqrt{\lambda_h^{N-1}}$ is of the order h while, in the continuous case, the gap between any two consecutive eigenvalues is independent of k (see Figure (a) of 2.1 taken from [26] which shows the square roots of the eigenvalues in the continuous and discrete case via finite difference semi-discretization on the left and the piecewise linear finite element space semi-discretization on the right).

Therefore, according to Theorem 6.9.3 of [74], we choose a discrete solution u_h as a wave package or a superposition of semi-discrete waves corresponding to the last eigenfrequencies of A_h; i.e., we choose $u_h \in Span \left\{ e^{i\sqrt{\lambda_h^k}t} w_h^k : \quad k \sim \frac{\gamma}{h}, \; 0 < \gamma < 1 \right\}$

such that $E_h(u_h, 0) \sim 1$ and $h\sum_{j=0}^{N}\int_0^T a_j \left|u_j'(t)\right|^2 dt \sim h^L,\ L >> 1$ so that the proof of (2.1.11) is complete. We remark that the wave package, u_h, does not penetrate the subinterval I, where the damping coefficient $a(.)$ is effective (see Figure (b) of 2.1 taken from [26] which shows a wave package propagating outside the interval I).

■

In this chapter, as in [4], we consider the approximations of more general abstract second order evolution equations. In other words, let H be a complex Hilbert space with norm and inner product denoted respectively by $\|.\|$ and $(.,.)$. Let $A : \mathcal{D}(A) \to H$ be a densely defined self-adjoint and positive operator with a compact inverse in H. Let $V = \mathcal{D}(A^{\frac{1}{2}})$ be the domain of $A^{\frac{1}{2}}$. Denote by $\mathcal{D}(A^{\frac{1}{2}})'$ the dual space of $\mathcal{D}(A^{\frac{1}{2}})$ obtained by means of the inner product in H.

Furthermore, let U be a complex Hilbert space (which will be identified to its dual space) with norm and inner product denoted respectively by $\|.\|_U$ and $(.,.)_U$ and let $B \in \mathcal{L}(U,\, H)$. We consider the closed loop system

$$\begin{gathered} \ddot{\omega}(t) + A\omega(t) + BB^*\dot{\omega}(t) = 0, \\ \omega(0) = \omega_0,\ \dot{\omega}(0) = \omega_1, \end{gathered} \tag{2.1.13}$$

where $t \in [0,\, \infty)$ represents the time, $\omega : [0,\, \infty) \to H$ is the state of the system. Most of the linear equations modeling the vibrations of elastic structures with feedback control (corresponding to collocated actuators and sensors) can be written in the form (2.1.13), where ω represents the displacement field.

We define the energy of system (2.1.13) at time t by

$$E(t) = \frac{1}{2}\left(\|\dot{\omega}(t)\|^2 + \left\|A^{\frac{1}{2}}\omega(t)\right\|^2\right).$$

Simple formal calculations give

$$E(0) - E(t) = \int_0^t (BB^*\dot{\omega}(s),\, \dot{\omega}(s))\, ds, \quad \forall t \geq 0.$$

This obviously means that the energy is non-increasing.

In many applications, the system (2.1.13) is approximated by finite dimensional systems but usually, as the above simple example shows, if the continuous system is exponentially or polynomially stable, the discrete ones do no more inherit this property due to spurious high frequency modes. Several remedies have been proposed and analyzed to overcome these difficulties. Let us quote the Tychonoff regularization [34, 35, 64, 72], a bi-grid algorithm [32, 58], a mixed finite element method [14, 19, 20, 33, 56], or filtering the high frequencies [43, 49, 76]. These methods provide good numerical results.

As in [64, 72] our goal is to damp the spurious high frequency modes by introducing numerical viscosity terms in the approximation schemes. Though our work in [4] is inspired from [64], it differs from that paper on the following points :

(i) Contrary to [64] where the standard gap condition is required, we only assume that the spectrum of the operator $A^{1/2}$ satisfies the generalized gap condition, allowing to treat more general concrete systems,

(ii) we analyze the polynomial decay of the discrete schemes when the continuous problem has such a decay,

(iii) we prove a result about uniform polynomial stability for a family of semigroups of operators,

(iv) by using a general version of the Trotter-Kato theorem proved in [45], we show that the discrete solution tends to the solution of (2.1.13) as the discretization parameter goes to zero and if the discrete initial data are well chosen.

Consequently, this chapter is divided as follows : After we precise the proper functional setting of the continuous problem (2.1.13) in Section 2.2, we recall some results concerning the stability of (2.1.13) in Section 2.3. In Section 2.4, we introduce the suitable discrete systems and the main results of this chapter. Section 2.5 considers the well-posedness of the discrete systems. Next, in Section 2.6, we show that the generalized gap condition and the observability conditions (2.4.7) and (2.4.8) remain valid for filtered eigenvalues. Section 2.7 first recalls a result about uniform exponential stability for a family of semigroup of operators, and then extends such a result to the case of uniform polynomial stability. Some technical lemmas are proved in Section 2.8. Sections 2.9 and 2.10 are devoted to the proof of Theorem 2.4.1 and 2.4.4 respectively. In Section 2.11, we show that the chosen discrete systems converge towards (2.1.13), as the mesh size goes to zero and if the discrete initial data are well chosen. Finally, we illustrate our results by presenting different examples in Section 2.12. The first application is found in [3].

2.2 The proper functional setting of problem (2.1.13)

Before stating the main results of this chapter, we rewrite problem (2.1.13) in a simplified form. Let $X := V \times H$ be equipped with the inner product

$$((u,v)^\top, (u^*,v^*)^\top)_X = a(u,u^*) + (v,v^*) \quad \forall (u,v)^\top, (u^*,v^*)^\top \in X,$$

where $a(.,.)$ is the sesquilinear form on $V \times V$ defined by

$$a(u,u^*) = (A^{\frac{1}{2}}u,\, A^{\frac{1}{2}}u^*),\, \forall (u,u^*) \in V \times V.$$

Then (2.1.13) is equivalent to

$$\dot{z}(t) = \tilde{A}z(t) \text{ in } X, \quad z(0) = (\omega_0, \omega_1)^\top,$$

where $z(t) = (\omega(t), \dot{\omega}(t))^\top$ and $\tilde{A} : D(\tilde{A}) \to X$ is defined by

$$\tilde{A} = \begin{pmatrix} 0 & I \\ -A & -BB^* \end{pmatrix},$$

with $\mathcal{D}(\tilde{A}) = \mathcal{D}(A) \times V$. It is easy to check that $\tilde{A}$ is a maximal dissipative operator in X. Therefore, according to Lumer-Phillips Theorem, problem (2.1.13) is well-posed. We will denote by $T(t),\ t \geq 0$, the strongly continuous semi-group of contractions generated by $\tilde{A}$.

2.3 Stability of the continuous problem (2.1.13)

Before displaying the suitable approximate system which converges towards (2.1.13) and shares the same stability properties of (2.1.13), we recall some properties concerning the stability of (2.1.13). For this purpose, we start by analyzing the spectrum of the operator $A^{\frac{1}{2}}$.

2.3.1 Spectral Analysis of (2.1.13)

Denote by $\{\lambda_k\}_{k\geq 1}$ the set of eigenvalues of $A^{\frac{1}{2}}$ counted with their multiplicities (i.e. we repeat the eigenvalues according to their multiplicities). We further rewrite the sequence of eigenvalues $\{\lambda_k\}_{k\geq 1}$ as follows :

$$\lambda_{k_1} < \lambda_{k_2} < \ldots < \lambda_{k_i} < \ldots$$

where $k_1 = 1$, k_2 is the lowest index of the second distinct eigenvalue, k_3 is the lowest index of the third distinct eigenvalue, etc. For all $i \in \mathbb{N}^*$, let l_i be the multiplicity of the eigenvalue λ_{k_i}, i.e.

$$\lambda_{k_{i-1}} < \lambda_{k_i} = \lambda_{k_i+1} = \ldots = \lambda_{k_i+l_i-1} < \lambda_{k_i+l_i} = \lambda_{k_{i+1}}.$$

We have $k_1 = 1$, $k_2 = 1 + l_1$, $k_3 = 1 + l_1 + l_2$, etc. Let $\{\varphi_{k_i+j}\}_{0\leq j\leq l_i-1}$ be the orthonormal eigenvectors associated with the eigenvalue λ_{k_i}.

Now, we assume that the following generalized gap condition holds :

$$\exists M \in \mathbb{N}^*,\ \exists \gamma_0 > 0,\ \forall k \geq 1,\ \lambda_{k+M} - \lambda_k \geq M\gamma_0. \tag{2.3.1}$$

Fix a positive real number $\gamma_0' \leq \gamma_0$ and denote by A_k, $k = 1, ..., M$ the set of natural numbers k_m satisfying (see for instance [13])

$$\begin{cases} \lambda_{k_m} - \lambda_{k_{m-1}} \geq \gamma_0' & \\ \lambda_{k_n} - \lambda_{k_{n-1}} < \gamma_0' & \text{for } m+1 \leq n \leq m+k-1, \\ \lambda_{k_{m+k}} - \lambda_{k_{m+k-1}} \geq \gamma_0'. & \end{cases}$$

Then one easily checks that

$$\{k_{m+j} + l \,|k_m \in A_k,\, k \in \{1, ..., M\}\,, j \in \{0, ..., k-1\}\,, l \in \{0, ..., l_{m+j} - 1\}\} = \mathbb{N}^*.$$

Notice that some sets A_k may be empty because, for the generalized gap condition, the choice of M takes into account multiple eigenvalues.

For $k_n \in A_k$, we define $B_{k_n} = (B_{k_n, ij})_{1 \leq i, j \leq k}$ the matrix of size $k \times k$ by

$$B_{k_n, ij} = \begin{cases} \displaystyle\prod_{\substack{q=n \\ q \neq n+i-1}}^{n+j-1} (\lambda_{k_{n+i-1}} - \lambda_{k_q})^{-1} & \text{if } i \leq j,\, (i, j) \neq (1, 1), \\ 1 & \text{if } (i, j) = (1, 1), \\ 0 & \text{else }. \end{cases}$$

More explicitly, we have

$$B_{k_n} = \begin{pmatrix} 1 & \frac{1}{\lambda_{k_n} - \lambda_{k_{n+1}}} & \frac{1}{(\lambda_{k_n} - \lambda_{k_{n+1}})(\lambda_{k_n} - \lambda_{k_{n+2}})} & \cdots & \frac{1}{(\lambda_{k_n} - \lambda_{k_{n+1}}) \cdots (\lambda_{k_n} - \lambda_{k_{n+k-1}})} \\ 0 & \frac{1}{\lambda_{k_{n+1}} - \lambda_{k_n}} & \frac{1}{(\lambda_{k_{n+1}} - \lambda_{k_n})(\lambda_{k_{n+1}} - \lambda_{k_{n+2}})} & \cdots & \frac{1}{(\lambda_{k_{n+1}} - \lambda_{k_n}) \cdots (\lambda_{k_{n+1}} - \lambda_{k_{n+k-1}})} \\ 0 & 0 & \frac{1}{(\lambda_{k_{n+2}} - \lambda_{k_n})(\lambda_{k_{n+2}} - \lambda_{k_{n+1}})} & \cdots & \frac{1}{(\lambda_{k_{n+2}} - \lambda_{k_n}) \cdots (\lambda_{k_{n+2}} - \lambda_{k_{n+k-1}})} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{(\lambda_{k_{n+k-1}} - \lambda_{k_n}) \cdots (\lambda_{k_{n+k-1}} - \lambda_{k_{n+k-2}})} \end{pmatrix}.$$

Lemma 2.3.1. *The inverse matrix of B_{k_n} is given by*

$$B^{-1}_{k_n, ij} = \begin{cases} \displaystyle\prod_{q=n}^{n+i-2} (\lambda_{k_{n+j-1}} - \lambda_{k_q}) & \textit{if } i \leq j,\, i \neq 1, \\ 1 & \textit{if } i = 1, \\ 0 & \textit{else}, \end{cases}$$

that is to say

$$B_{k_n}^{-1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & (\lambda_{k_{n+1}} - \lambda_{k_n}) & \cdots & (\lambda_{k_{n+k-1}} - \lambda_{k_n}) \\ 0 & 0 & \cdots & (\lambda_{k_{n+k-1}} - \lambda_{k_n})(\lambda_{k_{n+k-1}} - \lambda_{k_{n+1}}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\lambda_{k_{n+k-1}} - \lambda_{k_n}) \cdots (\lambda_{k_{n+k-1}} - \lambda_{k_{n+k-2}}) \end{pmatrix},$$

and therefore

$$B_{k_n}^{-1} \to \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}, \text{ when } n \to +\infty.$$

Proof: The form of $B_{k_n}^{-1}$ is obtained by induction on the size k of B_{k_n}. The generalized gap condition (2.3.1) implies that $\lambda_{k_{n+j}} - \lambda_{k_n} \to 0$ as $n \to +\infty$, $\forall 0 \leq j \leq k-1$. This leads to the convergence of $B_{k_n}^{-1}$. ■

Remark 2.3.2. *The structure of the matrix B_{k_n} comes from Ingham's inequality in a Hilbert space H under the generalized gap condition (2.3.1). Indeed, according to Corollary 6.4 of [61], if the sequence $\{\lambda_n\}_{n\geq 1}$ satisfies (2.3.1), then for all sequence $\{\alpha_n\}_{n\in\mathbb{Z}^*}$ in H, the function*

$$f(t) = \sum_{n\in\mathbb{Z}^*} \alpha_n e^{i\lambda_{k_n} t},$$

satisfies the estimate

$$\int_0^T |f(t)|^2 \, dt \sim \sum_{k=1}^{M} \sum_{|k_n|\in A_k} \left\| B_{k_n}^{-1} C_{k_n} \right\|_{H,2}^2,$$

for $T > \dfrac{2\pi}{\gamma_0}$, where $\lambda_{-k_n} = -\lambda_{k_n}$, $C_{k_n} = (\alpha_n, ..., \alpha_{n+k-1})^\top \in H^k$ and $\|.\|_{H,2}$ is the norm in H^k.

2.3.2 Exponential Stability of the energy of (2.1.13)

We recall a sufficient condition for which the energy of (2.1.13) is exponentially stable. Indeed, the approach is based on observability inequalities found in [7] and [61]. For this aim, for $k_n \in A_k$, we define the matrix Φ_{k_n} with coefficients in U and size $k \times L_n$, where $L_n = \sum_{i=1}^{k} l_{n+i-1}$, as follows : for all $i = 1, ..., k$, we set

$$(\Phi_{k_n})_{ij} = \begin{cases} B^* \varphi_{k_{n+i-1}+j-L_{n,\,i-1}-1} & \text{if } L_{n,\,i-1} < j \leq L_{n,\,i}, \\ 0 & \text{else,} \end{cases}$$

where

$$L_{n,\,0} = 0, \quad L_{n,\,i} = \sum_{i'=1}^{i} l_{n+i'-1} \text{ for } i \geq 1. \tag{2.3.2}$$

For a vector $c = (c_l)_{l=1}^m$ in U^m, we set $\|c\|_{U,\,2}$ its norm in U^m defined by

$$\|c\|_{U,\,2}^2 = \sum_{l=1}^{m} \|c_l\|_U^2 \,.$$

Now, we recall Theorem 2.2 of [7] which links the exponential stability of (2.1.13) with some observability property of the associated conservative problem.

Theorem 2.3.3. *Let φ be the solution of the undamped problem*

$$\begin{cases} \ddot{\varphi}(t) + A\varphi(t) = 0, \\ \varphi(0) = \omega_0,\ \dot{\varphi}(0) = \omega_1. \end{cases} \tag{2.3.3}$$

If there exists a time $T > 0$ and a constant $c = c(T) > 0$ such that the observability estimate

$$\|A^{\frac{1}{2}}\omega_0\|_H^2 + \|\omega_1\|_H^2 \leq c \int_0^T \|B^* \dot{\varphi}(t)\|_U^2 dt \tag{2.3.4}$$

holds, then problem (2.1.13) is exponentially stable in the energy space ; i.e., there exist a constant $k > 0$ and $\nu > 0$ such that for all initial data in $X = V \times H$,

$$E(t) \leq kE(0)e^{-\nu t}, \qquad \forall t > 0.$$

We also recall Proposition 6.5 of [61] which gives necessary and sufficient spectral conditions so that the observability estimate (2.3.4) holds.

Proposition 2.3.4. *Assume that the generalized gap condition (2.3.1) holds. There exists a time $T > 0$ and a constant $c = c(T) > 0$ such that the observability estimate (2.3.4) holds if and only if*

$$\exists \alpha_0 > 0,\, \forall k \in \{1, ..., M\},\, \forall k_n \in A_k,\, \forall C \in \mathbb{R}^{L_n},\, \left\| B_{k_n}^{-1} \Phi_{k_n} C \right\|_{U,2} \geq \alpha_0 \left\| C \right\|_2, \tag{2.3.5}$$

where $\|.\|_2$ is the euclidian norm.

Remark 2.3.5. If the standard gap condition

$$\exists \gamma_0 > 0,\, \forall n \geq 1,\, \lambda_{k_{n+1}} - \lambda_{k_n} \geq \gamma_0 \tag{2.3.6}$$

holds, then $A_1 = \mathbb{N}^*$ and $B_1 = 1$. In this case, the assumption (2.3.5) becomes

$$\exists \alpha_0 > 0,\, \forall k_n \geq 1,\, \forall C \in \mathbb{R}^{L_n},\, \left\| \Phi_{k_n} C \right\|_U \geq \alpha_0 \left\| C \right\|_2.$$

Moreover, if the standard gap condition (2.3.6) holds and if the eigenvalues are simple, the assumption (2.3.5) becomes

$$\exists \alpha_0 > 0,\, \forall k \geq 1,\, \| B^* \varphi_k \|_U \geq \alpha_0. \tag{2.3.7}$$

These assumptions are assumed in [64].

In conclusion, if (2.3.5) holds, then problem (2.1.13) is exponentially stable.

2.3.3 Polynomial Stability of the energy of (2.1.13)

Similar to the exponential stability case, we recall a sufficient condition for which the energy of (2.1.13) is polynomially stable. First, we recall Theorem 2.4 of [7] or Theorem 5.3 of [61] which gives the polynomial stability of (2.1.13) based on some observability property of the conservative problem (2.3.3). Then, we recall Proposition 6.8 of [61] which gives necessary and sufficient spectral conditions so that the observability estimate holds.

Theorem 2.3.6. *Let φ be the solution of (2.3.3). If there exists $l \in \mathbb{N}^*$, a time $T > 0$ and a constant $c = c(T) > 0$ such that*

$$\int_0^T \|B^* \dot{\varphi}(t)\|_U^2 dt \geq c \left(\|\omega_0\|^2_{D(A^{\frac{1-l}{2}})} + \|\omega_1\|^2_{D(A^{-\frac{l}{2}})} \right) \tag{2.3.8}$$

holds, then the energy of problem (2.1.13) decays polynomially; i.e., there exists a constant $k = k(l) > 0$ such that for all initial data in $D(\tilde{A})$,

$$E(t) \leq \frac{k}{(1+t)^{\frac{1}{l}}} \|(\omega_0, \omega_1)^\top\|^2_{D(\tilde{A})}, \qquad \forall t > 0.$$

Proposition 2.3.7. *Assume that the generalized gap condition (2.3.1) holds and $(\omega_0, \omega_1)^\top \in X$. There exists $l \in \mathbb{N}^*$, a time $T > 0$ and a constant $c = c(T) > 0$ such that (2.3.8) holds if and only if*

$$\exists \alpha_0 > 0,\, \forall k \in \{1, ..., M\},\, \forall k_n \in A_k,\, \forall C \in \mathbb{R}^{L_n},\, \left\| B_{k_n}^{-1} \Phi_{k_n} C \right\|_{U,2} \geq \frac{\alpha_0}{\lambda_{k_n}^l} \|C\|_2 . \tag{2.3.9}$$

Remark 2.3.8. If the standard gap condition (2.3.6) holds, the assumption (2.3.9) becomes

$$\exists l \in \mathbb{N}^*,\, \exists \alpha_0 > 0,\, \forall k_n \geq 1,\, \forall C \in \mathbb{R}^{L_n},\, \|\Phi_{k_n} C\|_U \geq \frac{\alpha_0}{\lambda_{k_n}^l} \|C\|_2 .$$

Moreover, if the standard gap condition (2.3.6) holds and if the eigenvalues are simple, the assumption (2.3.9) becomes

$$\exists l \in \mathbb{N}^*,\, \exists \alpha_0 > 0,\, \forall k \geq 1,\, \|B^* \varphi_k\|_U \geq \frac{\alpha_0}{\lambda_k^l}. \tag{2.3.10}$$

Remark 2.3.9. Note that the assumption (H) from [7] here holds since A is a positive selfadjoint operator with a compact resolvent and B is bounded. In fact, assumption (H) states that if $\beta > 0$ is fixed and $C_\beta = \{\lambda \in \mathbb{C};\ \Re\lambda = \beta\}$, the function

$$\lambda \in \mathbb{C}_+ = \{\lambda \in \mathbb{C};\ \Re\lambda > 0\} \to H(\lambda) = \lambda B^*(\lambda^2 I + A)^{-1} B \in \mathcal{L}(U)$$

is bounded on C_β. Indeed, if $\lambda = \beta + i\xi$, then $|\lambda_k^2 + \lambda^2|^2 = (\lambda_k^2 - \xi^2 + \beta^2)^2 + 4\beta^2\xi^2$. However,

$$\left\|(\lambda^2 I + A)^{-1}\right\|_{\mathcal{L}(H)} \leq \sup_{k\geq 1} |\lambda_k^2 + \lambda^2|^{-1}.$$

Hence, if $|\xi| > \epsilon$ for some $\epsilon > 0$, then

$$|\lambda| \left\|(\lambda^2 I + A)^{-1}\right\|_{\mathcal{L}(H)} \leq \frac{\sqrt{\beta^2 + \xi^2}}{2\beta|\xi|}$$

which is bounded for every $|\xi| > \epsilon$. On the other hand, if $|\xi| \leq \epsilon$, then $|\lambda_k^2 + \lambda^2| \geq \dfrac{3\beta^2}{4}$, for $\epsilon \leq \dfrac{\beta}{2}$. Therefore,

$$|\lambda| \left\|(\lambda^2 I + A)^{-1}\right\|_{\mathcal{L}(H)} \leq \frac{2\sqrt{5}}{3\beta}$$

which is bounded on C_β.

In conclusion, if (2.3.9) holds, then problem (2.1.13) is polynomially stable with a decay rate of the order $\dfrac{1}{(1+t)^{\frac{1}{l}}}$.

2.4 Approximate system and main results

In this section, we display the suitable discrete system which approximates (2.1.13) and has the same stability properties as (2.1.13). Before stating our main results, let us introduce some notations and assumptions.

We denote by $\|.\|_V$ the norm

$$\|\varphi\|_V = \sqrt{(A^{\frac{1}{2}}\varphi,\, A^{\frac{1}{2}}\varphi)},\, \forall \varphi \in V.$$

Remark that

$$\|\varphi\|_V = \sqrt{(A\varphi,\, \varphi)},\, \forall \varphi \in \mathcal{D}(A).$$

We now assume that $(V_h)_{h>0}$ is a sequence of finite dimensional subspaces of $\mathcal{D}(A^{\frac{1}{2}})$. The inner product in V_h is the restriction of the inner product of H and it is still denoted by $(.,.)$ (since V_h can be seen as a subspace of H). We define the operator $A_h : V_h \to V_h$ by

$$(A_h\varphi_h,\, \psi_h) = (A^{\frac{1}{2}}\varphi_h,\, A^{\frac{1}{2}}\psi_h),\, \forall \varphi_h,\, \psi_h \in V_h. \tag{2.4.1}$$

Let $a(.,.)$ be the sesquilinear form on $V_h \times V_h$ defined by

$$a(\varphi_h,\, \psi_h) = (A^{\frac{1}{2}}\varphi_h,\, A^{\frac{1}{2}}\psi_h),\, \forall (\varphi_h,\, \psi_h) \in V_h \times V_h. \tag{2.4.2}$$

We also define the operators $B_h : U \to V_h$ by

$$B_h u = j_h B u, \quad \forall u \in U, \tag{2.4.3}$$

where j_h is the orthogonal projection of H into V_h with respect to the inner product in H.

The adjoint B_h^* of B_h is then given by the relation

$$B_h^*\varphi_h = B^*\varphi_h, \quad \forall \varphi_h \in V_h.$$

We also suppose that the family of spaces $(V_h)_h$ approximates the space $V = \mathcal{D}(A^{\frac{1}{2}})$. More precisely, if π_h denotes the orthogonal projection of $V = \mathcal{D}(A^{\frac{1}{2}})$ onto V_h, we suppose that there exist $\theta > 0$, $h^* > 0$ and $C_0 > 0$ such that, for all $h \in (0,\, h^*)$, we have :

$$\|\pi_h\varphi - \varphi\|_V \leq C_0 h^\theta \|A\varphi\|,\, \forall \varphi \in \mathcal{D}(A), \tag{2.4.4}$$

$$\|\pi_h \varphi - \varphi\| \leq C_0 h^{2\theta} \|A\varphi\|, \forall \varphi \in \mathcal{D}(A). \tag{2.4.5}$$

Assumptions (2.4.4) and (2.4.5) are, in particular, satisfied in the case of standard finite element approximations of Sobolev spaces.

In this section, we prove two results. The first result gives a necessary and sufficient condition to have the exponential stability of the family of systems

$$\begin{aligned} \ddot{\omega}_h(t) + A_h \omega_h(t) + B_h B_h^* \dot{\omega}_h(t) + h^\theta A_h \dot{\omega}_h(t) = 0 \\ \omega_h(0) = \omega_{0h} \in V_h,\, \dot{\omega}_h(0) = \omega_{1h} \in V_h, \end{aligned} \tag{2.4.6}$$

in the absence of the standard gap condition assumed in [64]. Here and below ω_{0h} (resp. ω_{1h}) is an approximation of ω_0 (resp. ω_1) in V_h. For that purpose, we need to make the following assumption

$$\exists \alpha_0 > 0,\, \forall k \in \{1, ..., M\},\, \forall k_n \in A_k,\, \forall C \in \mathbb{R}^{L_n},\, \left\|B_{k_n}^{-1} \Phi_{k_n} C\right\|_{U,2} \geq \alpha_0 \|C\|_2, \tag{2.4.7}$$

where $\|.\|_2$ is the euclidian norm. The first main result is the following

Theorem 2.4.1. *Suppose that the generalized gap condition (2.3.1) and the assumption (2.4.7) are verified. Assume that the family of subspaces (V_h) satisfies (2.4.4) and (2.4.5). Then the family of systems (2.4.6) is uniformly exponentially stable, in the sense that there exist constants M, α, $h^* > 0$ (independent of h, ω_{0h}, ω_{1h}) such that for all $h \in (0, h^*)$:*

$$\|\dot{\omega}_h(t)\|^2 + a(\omega_h(t), \omega_h(t)) \leq M e^{-\alpha t}(\|\omega_{1h}\|^2 + a(\omega_{0h}, \omega_{0h})),\, \forall t \geq 0.$$

Remark 2.4.2. Note that Theorem 2.4.1 is the discrete counterpart of the exponential decay of the solution of the continuous problem (2.1.13) under the assumptions (2.3.1) and (2.4.7), which follows from Theorem 2.3.3 and Proposition 2.3.4 or Theorem 2.2 of [7] and Proposition 6.5 of [61].

Remark 2.4.3. The uniform exponential stability of the family of systems (2.4.6) has been already proved in Theorem 7.1 of [25] without any assumption on the spectrum of A. The proof of this theorem is based on decoupling of low and high frequencies. More precisely, the author combines a uniform observability estimate for filtered initial data corresponding to low frequencies (see Theorem 1.3 of [25]) together with a result of [27]. Indeed, in [27], after adding the numerical viscosity term, another uniform observability estimate is obtained for the high frequency components. The two established observability inequalities yield the uniform exponential decay of (2.4.6).

If the condition (2.4.7) is not satisfied, we may look at a weaker version. Namely if we assume that

$$\exists l \in \mathbb{N}^*,\, \exists \alpha_0 > 0,\, \forall k \in \{1, ..., M\},\, \forall k_n \in A_k,\, \forall C \in \mathbb{R}^{L_n},\, \left\| B_{k_n}^{-1} \Phi_{k_n} C \right\|_{U,2} \geq \frac{\alpha_0}{\lambda_{k_n}^l} \left\| C \right\|_2, \tag{2.4.8}$$

then we will obtain a polynomial stability for the family of systems

$$\begin{aligned} &\ddot{\omega}_h(t) + (1+h^\theta)^{-2}(I + h^\theta A_h^{\frac{l}{2}})^2 A_h \omega_h(t) \\ &+(I + h^\theta A_h^{\frac{l}{2}})(B_h B_h^* + h^\theta A_h^{1+\frac{l}{2}})(I + h^\theta A_h^{\frac{l}{2}})^{-1} \dot{\omega}_h(t) = 0, \\ &\omega_h(0) = \omega_{0h} \in V_h,\, \dot{\omega}_h(0) = (1+h^\theta)^{-1}(I + h^\theta A_h^{\frac{l}{2}})\omega_{1h} \in V_h. \end{aligned} \tag{2.4.9}$$

The structure of the above discrete system has been inspired from the one introduced in [64] for the exponential stability case where the authors have used system (2.4.6) corresponding to $l = 0$. In both cases, this choice is motivated by the corresponding observability estimates. The numerical viscosity term $(I + h^\theta A_h^{\frac{l}{2}})(B_h B_h^* + h^\theta A_h^{1+\frac{l}{2}})(I + h^\theta A_h^{\frac{l}{2}})^{-1} \dot{\omega}_h(t)$ is added to damp the high frequency modes and as the set of high frequency modes is larger in the polynomial case, the viscosity term is naturally stronger. In the case $l > 0$ the powers of $(I + h^\theta A_h^{\frac{l}{2}})$ have been added to guarantee the uniform boundedness of the resolvent of $\tilde{A}_{l,h}$ (defined below) near zero. The question of the optimality of these viscosity terms remains open.

The second main result of this chapter is the following one.

Theorem 2.4.4. *Suppose that the generalized gap condition (2.3.1) and the assumption (2.4.8) are verified with $l \in \mathbb{N}^*$ even. Assume that the family of subspaces (V_h) satisfies (2.4.4) and (2.4.5). Then the family of systems (2.4.9) is uniformly polynomially stable, in the sense that there exist constants C, $h^* > 0$ (independent of h, ω_{0h}, ω_{1h}) such that for all $h \in (0, h^*)$:*

$$\begin{aligned}
\left\|(I + h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t)\right\|^2 + a(\omega_h(t), \omega_h(t)) &\leq \frac{C}{t^2}\|(\omega_{0h}, \omega_{1h})\|^2_{D(\tilde{A}^{2l}_{l,h})}, \\
\left\|(I + h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t)\right\|^2 + a(\omega_h(t), \omega_h(t)) &\leq \frac{C}{t^{\frac{1}{l}}}\|(\omega_{0h}, \omega_{1h})\|^2_{D(\tilde{A}_{l,h})}, \\
\forall t > 0, \ \forall(\omega_{0h}, \omega_{1h}) &\in V_h \times V_h,
\end{aligned}$$

where for $q \in \mathbb{N}^*$, $\|.\|_{D(\tilde{A}^q_{l,h})}$ is the graph norm of the matrix operator $\tilde{A}^q_{l,h}$ given in (2.5.1) of Section 2.5 below.

For a technical reason, we assume l to be even (see Lemma 2.8.4). If (2.4.8) holds for l odd, then we can still apply the results of Theorem 2.4.4 (see Remark 2.10.1 below).

Remark 2.4.5. As before, Theorem 2.4.4 is the discrete counterpart of the polynomial decay of the solution of the continuous problem (2.1.13) under the assumptions (2.3.1) and (2.4.8), that follows from Theorem 2.3.6 and Proposition 2.3.7 or Theorem 2.4 of [7] and Proposition 6.8 of [61].

2.5 Well-posedness of the discretized problem

From now on, we fix $l \in \mathbb{N}$, l even. We introduce the Hilbert space $X_h = V_h \times V_h$ and the operator $\tilde{A}_{l,h} : X_h \to X_h$ defined by

$$\tilde{A}_{l,h} = \begin{pmatrix} 0 & (1 + h^\theta)^{-1}(I + h^\theta A_h^{\frac{l}{2}}) \\ -(1 + h^\theta)^{-1}(I + h^\theta A_h^{\frac{l}{2}})A_h & -h^\theta A_h^{1+\frac{l}{2}} - B_h B_h^* \end{pmatrix}. \tag{2.5.1}$$

The space X_h is here equipped with the inner product

$$\left(\begin{pmatrix} u_h \\ v_h \end{pmatrix}, \begin{pmatrix} \tilde{u}_h \\ \tilde{v}_h \end{pmatrix} \right)_{X_h} = a(u_h,\, \tilde{u}_h) + (v_h,\, \tilde{v}_h), \quad \forall (u_h,\, v_h),\, (\tilde{u}_h,\, \tilde{v}_h) \in X_h, \tag{2.5.2}$$

with associated norm $\|.\|_{X_h}$. Therefore, the system (2.4.9) is equivalent to the following first order system in X_h :

$$\dot{z}_h(t) = \tilde{A}_{l,h} z_h(t), \quad z_h(0) = z_{0h},$$

where $z_h(t) = \begin{pmatrix} \omega_h(t) \\ (1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1} \dot{\omega}_h(t) \end{pmatrix}$ and $z_{0h} = \begin{pmatrix} \omega_{0h} \\ \omega_{1h} \end{pmatrix}$. Note that we recover the system (2.4.6) in the particular case $l = 0$.

Lemma 2.5.1. *$\tilde{A}_{l,h}$ is maximal dissipative in X_h ; hence, it follows from Lumer-Phillips Theorem that, for every $h > 0$, $\tilde{A}_{l,h}$ generates a C_0 semigroup of contractions in X_h. We will denote this C_0 semigroup by $T_{l,h}(t)$.*

Proof: For the dissipativity of $\tilde{A}_{l,h}$, it is easy to check that $\Re(\tilde{A}_{l,h}U, U) \leq 0$ for every $U \in X_h$. As for the maximality, $\tilde{A}_{l,h}$ is bijective since $0 \in \rho(\tilde{A}_{l,h})$ (see Lemma 2.8.1 below). Therefore, $\tilde{A}_{l,h}$ becomes maximal. Indeed, let $F \in X_h$ and define the operator T on X_h such that $TU = \mu \tilde{A}_{l,h}^{-1} U - \tilde{A}_{l,h}^{-1} F$. For every $U,\ V \in X_h$, we have

$$\|TU - TV\| = \|\mu \tilde{A}_{l,h}^{-1} U - \mu \tilde{A}_{l,h}^{-1} V\| \leq \mu \|\tilde{A}_{l,h}^{-1}\| \|U - V\|.$$

As $\tilde{A}_{l,h}^{-1}$ is bounded since it is linear over a finite dimensional space, we choose $0 < \mu < \dfrac{1}{\|\tilde{A}_{l,h}^{-1}\|}$ so that T becomes a contraction and hence admits a fixed point U. Therefore, there exists $U \in X_h$ and $\mu > 0$ such that $\mu \tilde{A}_{l,h}^{-1} U - \tilde{A}_{l,h}^{-1} F = U$ or $(\mu I - \tilde{A}_{l,h}) U = F$.

■

We shall note here that the discrete energy of system (2.4.9) is given by

$$E_h(t) = \frac{1}{2}\|z_h(t)\|^2_{X_h} = \frac{1}{2}\left(A_h\omega_h, \omega_h\right) + \frac{1}{2}(1+h^\theta)^2\left((I + h^\theta A_h^{\frac{l}{2}})^{-2}\dot{\omega}_h(t), \dot{\omega}_h(t)\right).$$

Therefore, for any $t > 0$, we have

$$E_h'(t) = -(1+h^\theta)^2\left(\left\|B_h^*(I + h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t)\right\|^2 + h^\theta\left\|A_h^{\frac{2+l}{4}}(I + h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t)\right\|^2\right).$$

We notice the additional second term on the right hand side of the above dissipation equality. In fact, this viscosity term helps increase the decay rate of the discrete energy.

2.6 Spectral analysis of the discretized problem

The eigenvalue problem of the discretized problem is the following one : find $\lambda_{k,\,h} \in]0,\, +\infty[$, $\varphi_{k,\,h} \in V_h$, such that

$$a(\varphi_{k,\,h},\, \psi_h) = \lambda^2_{k,\,h}(\varphi_{k,\,h},\, \psi_h),\ \forall \psi_h \in V_h. \tag{2.6.1}$$

Let $N(h)$ be the dimension of V_h. We denote by $\{\lambda^2_{k,\,h}\}_{1\le k\le N(h)}$ the set of eigenvalues of (2.6.1) counted with their multiplicities. Let $\{\varphi_{k,\,h}\}_{1\le k\le N(h)}$ be the orthonormal eigenvectors associated with the eigenvalue $\lambda^2_{k,\,h}$. We define the sesquilinear form $a^l(.,\,.)$ on V_h by

$$a^l(u_h,\, v_h) = \left(A_h^{1+\frac{l}{2}}u_h,\, v_h\right),\ \forall (u_h,\, v_h) \in V_h \times V_h;$$

i.e.,

$$a^l(u_h,\, v_h) = \sum_{k=1}^{N(h)} c_k\overline{d_k}\lambda^{2+l}_{k,\,h},$$

for $u_h = \sum_{k=1}^{N(h)} c_k\varphi_{k,\,h}$ and $v_h = \sum_{k=1}^{N(h)} d_k\varphi_{k,\,h}$. Remark that $a^0(.,.) = a(.,.)$ defined in (2.4.2).

In this Section, we show that the generalized gap condition (2.3.1) and the observability conditions (2.4.7) and (2.4.8) still hold for the approximate problem (uniformly in h), provided that we consider only "low frequencies". More precisely, we have the following first result :

Proposition 2.6.1. *Suppose that the generalized gap condition (2.3.1) and the assumption (2.4.7) are verified. Then, there exist two constants $\epsilon > 0$ and $h^* > 0$, such that, for all $0 < h < h^*$ and for all $k \in \{1, ..., N(h)\}$ satisfying*

$$h^\theta \lambda_k^2 \leq \epsilon, \tag{2.6.2}$$

we have

$$\exists M \in \mathbb{N}^*,\ \exists \gamma > 0,\ \lambda_{k+M,\,h} - \lambda_{k,\,h} \geq M\gamma \tag{2.6.3}$$

and

$$\exists \alpha > 0,\ \forall p \in \{1, ..., M\},\ \forall k_n \in A_{p,h},\ \forall C \in \mathbb{R}^{L_n},\ \left\| B_{k_n}^{-1} \Phi_{k_n,\,h} C \right\|_{U,\,2} \geq \alpha \left\| C \right\|_2 , \tag{2.6.4}$$

where α is independent of h, and where the matrix $\Phi_{k_n,\,h} \in \mathcal{M}_{p,\,L_n}(U)$, with coefficients in U, is defined as follows : for all $i = 1, ..., p$, we set

$$(\Phi_{k_n,\,h})_{ij} = \begin{cases} B_h^* \varphi_{k_{n+i-1}+j-L_{n,\,i-1}-1,\,h} & \text{if } L_{n,\,i-1} < j \leq L_{n,\,i}, \\ 0 & \text{else,} \end{cases}$$

where $L_{n,\,i-1}$ is defined by (2.3.2) and

$$A_{p,h} = \{k_n \in A_p \text{ satisfying (2.6.2) and s.t. } k_{n+p-1} + l_{n+p-1} - 1 \leq N(h)\}\,.$$

For the proof of this proposition, we need a result proved by Babuska and Osborn in [12]. For that purpose, we introduce $\epsilon_h(n,\, j)$ such that

$$\epsilon_h(n,\, j) = \inf_{\varphi \in M_j(\lambda_{k_n})} \inf_{v_h \in V_h} \left\| \varphi - v_h \right\|_V ,$$

where $M_j(\lambda_{k_n}) = \{\varphi \in M(\lambda_{k_n}) : a(\varphi, \varphi_{k_n,h}) = ... = a(\varphi, \varphi_{k_n+j-2,h}) = 0\}$ and $M(\lambda_{k_n}) = \{\varphi : \varphi \text{ is an eigenvector of } A^{\frac{1}{2}} \text{ corresponding to } \lambda_{k_n}, \|\varphi\| = 1\}$. The restrictions $a(\varphi, \varphi_{k_n,h}) = ... = a(\varphi, \varphi_{k_n+j-2,h}) = 0$ are not imposed if $j = 1$. Then, we have the following estimate about the eigenvalue and eigenvector errors for the Galerkin method in terms of the approximability quantities $\epsilon_h(n, j)$.

Theorem 2.6.2. *There are positive constants C and h_0 such that*

$$\lambda_{k_n+j,h} - \lambda_{k_n+j} \leq C\epsilon_h^2(n, j), \quad \forall 0 < h \leq h_0,\ j = 0, ..., l_n - 1,\ k_n + j \leq N(h),\ n \in \mathbb{N}^* \tag{2.6.5}$$

and such that the eigenvectors $\{\varphi_{k_n+j}\}_{0\leq j\leq l_n-1}$ of $A^{\frac{1}{2}}$ can be chosen so that

$$\|\varphi_{k_n+j,h} - \varphi_{k_n+j}\|_V \leq C\epsilon_h(n, j), \quad \forall 0 < h \leq h_0,\ j = 0, ..., l_n-1,\ k_n+j \leq N(h),\ n \in \mathbb{N}^*. \tag{2.6.6}$$

This result is proved by Babuska and Osborn in [12, p. 702] because

$$\lambda_{k_n+j,h}^2 - \lambda_{k_n+j}^2 = (\lambda_{k_n+j,h} - \lambda_{k_n+j})(\lambda_{k_n+j,h} + \lambda_{k_n+j}) \geq 2\lambda_1(\lambda_{k_n+j,h} - \lambda_{k_n+j}).$$

Remark 2.6.3. *Notice that for every $\varphi \in M_j(\lambda_{k_n})$ we have*

$$\begin{aligned} \epsilon_h(n, j) &\leq \inf_{v_h \in V_h} \|\varphi - v_h\|_V \\ &\leq C_0 h^\theta \|A\varphi\| \ \text{by (2.4.4)} \\ &\leq C_0 h^\theta \lambda_{k_n}^2 \|\varphi\| = C_0 h^\theta \lambda_{k_n+j}^2. \end{aligned} \tag{2.6.7}$$

Proof of Proposition 2.6.1. We begin with the proof of the generalized gap condition for the approximate eigenvalues $\lambda_{k,h}$. First, we use the Min-Max principle (see [67]) to obtain

$$\lambda_k \leq \lambda_{k,h},\ \forall k \in \{1, ..., N(h)\}. \tag{2.6.8}$$

Second, we use the estimates (2.6.5) and (2.6.7) and we have

$$\lambda_{k,h} \leq \lambda_k + C(C_0 h^\theta \lambda_k^2)^2 \leq \lambda_k + C(C_0\epsilon)^2 \leq \lambda_k + CC_0^2\epsilon, \tag{2.6.9}$$

for all $k \in \{1, ..., N(h)\}$ verifying (2.6.2) and $\epsilon \leq 1$. Therefore, we may write

$$\lambda_{k+M,h} - \lambda_{k,h} \geq \lambda_{k+M} - \lambda_k - CC_0^2\epsilon \geq M\gamma_0 - CC_0^2\epsilon \geq M\frac{\gamma_0}{2} =: M\gamma$$

for all $k \in \{1, ..., N(h)\}$ satisfying (2.6.2) and for $\epsilon \leq \frac{M\gamma_0}{2CC_0^2}$.

Now, we prove the estimate (2.6.4) which is the approximated version of (2.4.7). Notice that

$$\begin{aligned}
\|\Phi_{k_n,h} - \Phi_{k_n}\|_U &\leq C \max_{i=0,\ldots,p-1} \sum_{j=0}^{l_{n+i}-1} \left\|B^*\varphi_{k_{n+i}+j,h} - B^*\varphi_{k_{n+i}+j}\right\|_U \\
&\leq C \max_{i=0,\ldots,p-1} \sum_{j=0}^{l_{n+i}-1} \|B^*\|_{\mathcal{L}(H,U)} \left\|\varphi_{k_{n+i}+j,h} - \varphi_{k_{n+i}+j}\right\| \\
&\leq C \max_{i=0,\ldots,p-1} \sum_{j=0}^{l_{n+i}-1} \|B^*\|_{\mathcal{L}(H,U)} \left\|\varphi_{k_{n+i}+j,h} - \varphi_{k_{n+i}+j}\right\|_V \\
&\leq C \max_{i=0,\ldots,p-1} \sum_{j=0}^{l_{n+i}-1} \epsilon_h(n+i,\, j) \text{ by } (2.6.6) \\
&\leq C \max_{i=0,\ldots,p-1} \sum_{j=0}^{l_{n+i}-1} h^\theta \lambda_{k_{n+i}+j}^2 \text{ by } (2.6.7).
\end{aligned}$$

Thus, by (2.6.2), we get

$$\|\Phi_{k_n,h} - \Phi_{k_n}\|_U \leq C\epsilon. \tag{2.6.10}$$

Therefore the triangular inequality leads to

$$\begin{aligned}
\left\|B_{k_n}^{-1}\Phi_{k_n,h}C\right\|_{U,2} &= \left\|B_{k_n}^{-1}\Phi_{k_n}C + B_{k_n}^{-1}(\Phi_{k_n,h} - \Phi_{k_n})C\right\|_{U,2} \\
&\geq \left\|B_{k_n}^{-1}\Phi_{k_n}C\right\|_{U,2} - \left\|B_{k_n}^{-1}(\Phi_{k_n,h} - \Phi_{k_n})C\right\|_{U,2} \\
&\geq \alpha_0 \|C\|_2 - \left\|B_{k_n}^{-1}(\Phi_{k_n,h} - \Phi_{k_n})C\right\|_{U,2}
\end{aligned}$$

by (2.4.7). But, as $B_{k_n}^{-1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} + R_{k_n}$, with $R_{k_n} \to 0$, when $k_n \to +\infty$

(see Lemma 2.3.1), we obtain

$$\begin{aligned}\left\|B_{k_n}^{-1}(\Phi_{k_n,h}-\Phi_{k_n})C\right\|_{U,2} &\leq \left\|\begin{pmatrix}1 & 1 & \cdots & 1\\ 0 & 0 & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & & \cdots & 0\end{pmatrix}(\Phi_{k_n,h}-\Phi_{k_n})C\right\|_{U,2}\\ &+ \|R_{k_n}(\Phi_{k_n,h}-\Phi_{k_n})C\|_{U,2}\\ &\leq C\|\Phi_{k_n,h}-\Phi_{k_n}\|_U\|C\|_2+\eta_n\|\Phi_{k_n,h}-\Phi_{k_n}\|_U\|C\|_2\\ &\leq C\epsilon(1+\eta_n)\|C\|_2,\end{aligned} \tag{2.6.11}$$

where $\eta_n = \|R_{k_n}\| \to 0$. Thus

$$\left\|B_{k_n}^{-1}\Phi_{k_n,h}C\right\|_{U,2} \geq (\alpha_0 - C\epsilon(1+\eta_n))\|C\|_2 \geq \frac{\alpha_0}{2}\|C\|_2$$

for $\epsilon \leq \frac{\alpha_0}{2C(1+\max\limits_n(1+\eta_n))}$.■

For the polynomial stability, we have the same kind of result, but more filtering is necessary in order to have the discrete counterpart of the observability condition (2.4.8) (uniformly in h).

Proposition 2.6.4. *Suppose that the generalized gap condition (2.3.1) and the assumption (2.4.8) are verified. Then, there exist two constants $\epsilon > 0$ and $h^* > 0$, such that, for all $0 < h < h^*$ and for all $k \in \{1, ..., N(h)\}$, satisfying*

$$h^\theta \lambda_k^2 \leq \frac{\epsilon}{\lambda_k^l}, \tag{2.6.12}$$

we have (2.6.3) and

$$\exists\alpha > 0,\ \forall p \in \{1,...,M\},\ \forall k_n \in A_{p,h}^{(l)},\ \forall C \in \mathbb{R}^{L_n},\ \left\|B_{k_n}^{-1}\Phi_{k_n,h}C\right\|_{U,2} \geq \frac{\alpha}{\lambda_{k_n}^l}\|C\|_2, \tag{2.6.13}$$

where $A_{p,h}^{(l)} = \{k_n \in A_p$ satisfying (2.6.12) and s.t. $k_{n+p-1} + l_{n+p-1} - 1 \leq N(h)\}$.

Proof: The generalized gap condition for the approximate eigenvalues $\lambda_{k,h}$ is a consequence of Proposition 2.6.1, because $\lambda_k \geq \lambda_1 > 0$.

To prove the estimate (2.6.13) we notice that

$$\|\Phi_{k_n,h} - \Phi_{k_n}\|_U \leq C \max_{i=0,\dots,p-1} \sum_{j=0}^{l_{n+i}-1} h^\theta \lambda^2_{k_{n+i+j}} \leq C\ h^\theta \lambda^2_{k_{n+p-1}}.$$

Moreover by the triangular inequality and (2.4.8), we have

$$\left\|B_{k_n}^{-1}\Phi_{k_n,h}C\right\|_{U,2} \geq \frac{\alpha_0}{\lambda^l_{k_n}}\|C\|_2 - \left\|B_{k_n}^{-1}(\Phi_{k_n,h} - \Phi_{k_n})C\right\|_{U,2}.$$

By (2.6.11) and (2.6.12), we obtain

$$\begin{aligned}\left\|B_{k_n}^{-1}\Phi_{k_n,h}C\right\|_{U,2} &\geq \left(\frac{\alpha_0}{\lambda^l_{k_n}} - \frac{C(1+\eta_n)\epsilon}{\lambda^l_{k_{n+p-1}}}\right)\|C\|_2 \\ &\geq \left(\frac{\alpha_0}{\lambda^l_{k_n}} - \frac{C\epsilon}{\lambda^l_{k_n}+\rho_n}(1+\eta_n)\right)\|C\|_2\,, \text{ with } \rho_n = \lambda^l_{k_{n+p-1}} - \lambda^l_{k_n} \to 0 \\ &\geq \frac{\alpha}{\lambda^l_{k_n}}\|C\|_2\end{aligned}$$

for an appropriate choice of $\epsilon > 0$.

■

Remark 2.6.5. *Proposition 2.6.1 in case $l = 0$ and Proposition 2.6.4 for $l \in \mathbb{N}^*$ show that if $h^\theta \lambda_k^{2+l} \leq \epsilon$, then the discrete version of the observability inequalities is still preserved uniformly in h and hence no problems with the stability of the discrete systems are expected. On the other hand, if $h^\theta \lambda_k^{2+l} \geq \epsilon$, then the viscosity term, $h^\theta A_h^{1+\frac{l}{2}}\dot{\omega}_h(t)$, plays its role in damping the spurious high oscillations. Indeed, if we write*

$$\dot{\omega}_h(t) = \sum_{k=1}^{N(h)} \alpha_{k,h}\varphi_{k,h},$$

then

$$h^\theta A_h^{1+\frac{l}{2}}\dot{\omega}_h(t) = h^\theta \sum_{k=1}^{N(h)} \alpha_{k,h}\lambda^{2+l}_{k,h}\varphi_{k,h}.$$

Hence, if $h^\theta \lambda_k^{2+l} \geq \epsilon$, *then*

$$\left\| h^\theta A_h^{1+\frac{l}{2}} \dot{\omega}_h(t) \right\|^2 \geq \epsilon^2 \|\dot{\omega}_h(t)\|^2.$$

Therefore, in the presence of high frequencies, the viscosity term $h^\theta A_h^{1+\frac{l}{2}} \dot{\omega}_h(t)$ *can be viewed as an additional damping term.*

2.7 Uniform stability results

2.7.1 Exponential stability result

The proof of Theorem 2.4.1 is based on the following result (see Theorem 7.1.3 in [52]) :

Theorem 2.7.1. *Let* $(T_h)_{h>0}$ *be a family of semigroups of contractions on the Hilbert spaces* $(X_h)_{h>0}$ *and let* $(\tilde{A}_h)_{h>0}$ *be the corresponding infinitesimal generators. The family* $(T_h)_{h>0}$ *is uniformly exponentially stable, that is to say there exist constants* $M > 0$, $\alpha > 0$ *(independent of* $h \in (0, h^*)$*) such that*

$$\|T_h(t)\|_{\mathcal{L}(X_h)} \leq M e^{-\alpha t}, \, \forall t \geq 0,$$

if and only if the two following conditions are satisfied :

i) For all $h \in (0, h^*)$, $i\mathbb{R}$ *is contained in the resolvent set* $\rho(\tilde{A}_h)$ *of* $\tilde{A}_h$,

ii) $\sup_{h \in (0,h^*), \omega \in \mathbb{R}} \left\| (i\omega - \tilde{A}_h)^{-1} \right\|_{\mathcal{L}(X_h)} < +\infty.$

2.7.2 Polynomial stability result

The proof of Theorem 2.4.4 is based on the results presented in this section by adapting the results from [18] and from [48] to obtain the (uniform) polynomial stability of the discretized problem (2.4.9). Throughout this section, let $(T_h(t))_{\substack{t \geq 0 \\ h \in (0,h^*)}}$

be a family of uniformly bounded C_0 semigroups on the associated Hilbert spaces $(X_h)_{h\in(0,h^*)}$ (i.e., $\exists M>0,\ \forall h\in(0,h^*),\ \|T_h(t)\|_{\mathcal{L}(X_h)}\leq M$) and let $(\tilde{A}_h)_{h\in(0,h^*)}$ be the corresponding infinitesimal generators.

In the following, for shortness, we denote by $R(\lambda,\tilde{A}_h)$ the resolvent $(\lambda-\tilde{A}_h)^{-1}$; moreover, for any operator mapping X_h into X_h, we skip the index $\mathcal{L}(X_h)$ in its norm, since in the whole section we work in X_h.

Definition 2.7.2. *Assuming that*

$$i\mathbb{R}\subseteq\rho(\tilde{A}_h),\quad \forall h\in(0,h^*), \tag{2.7.1}$$

and that for all $m\geq 1$, there exists $c=c(m)>0$ such that

$$\sup_{\substack{h\in(0,h^*)\\|s|\leq m}}\|R(is,\tilde{A}_h)\|_{\mathcal{L}(X_h)}\leq c, \tag{2.7.2}$$

we define the fractional power $\tilde{A}_h^{-\alpha}$ for $\alpha>0$ and $h\in(0,h^*)$, according to [6] and [24], as

$$\tilde{A}_h^{-\alpha}=\frac{1}{2\pi i}\int_\Gamma\lambda^{-\alpha}(\lambda-\tilde{A}_h)^{-1}d\lambda, \tag{2.7.3}$$

where $\lambda^{-\alpha}=e^{-\alpha\log\lambda}$ and $\mathbb{R}^+$ is taken as the cut branch of the complex log *function and where the curve $\Gamma=\Gamma_1\cup\Gamma_2$ is given by*

$$\Gamma=\{-\epsilon+te^{i\theta},\ t\in[0,+\infty)\}\cup\{-\epsilon-te^{-i\theta},\ t\in(-\infty,0]\} \tag{2.7.4}$$

for some $\epsilon>0$ small enough independent of h and θ is a fixed angle in $\left(0,\frac{\pi}{4}\right)$.

In the sequel, the constant $c>0$ is generic, independent of h, and may change from one line to another.

Remark 2.7.3. *Throughout this section, whenever $\tilde{A}_h^{-\alpha}$ is mentioned, the assumptions (2.7.1) and (2.7.2) are directly taken into consideration since otherwise $\tilde{A}_h^{-\alpha}$ is not well defined.*

In fact, under the assumptions (2.7.1) and (2.7.2), for all $m > 0$ there exists $\epsilon = \epsilon(m) > 0$ such that

$$-\mu + i\beta \in \rho(\tilde{A}_h), \quad \forall h \in (0, h^*),\ \forall 0 \leq \mu \leq \epsilon, \forall |\beta| \leq m.$$

Indeed, for all $m > 0$ such that $|\beta| \leq m$, we have

$$(-\mu + i\beta - \tilde{A}_h)^{-1} = (i\beta - \tilde{A}_h)^{-1}[I_h - \mu(i\beta - \tilde{A}_h)^{-1}]^{-1}$$

and

$$\|\mu(i\beta - \tilde{A}_h)^{-1}\| \leq \mu c.$$

Hence, if $|\beta| \leq m$ and $\mu \leq \epsilon \leq \frac{1}{2c}$, then $(-\mu + i\beta - \tilde{A}_h)$ is invertible and we have

$$\|(-\mu + i\beta - \tilde{A}_h)^{-1}\| \leq 2\|(i\beta - \tilde{A}_h)^{-1}\| \leq 2c,\ \forall h \in (0, h^*). \tag{2.7.5}$$

We choose $m = \Im(-\epsilon + te^{i\theta}) = \epsilon \tan\theta$ when $\Re(-\epsilon + te^{i\theta}) = 0$, i.e. when $t = \frac{\epsilon}{\cos\theta}$. Therefore, by (2.7.5), assumptions (2.7.1) and (2.7.2) imply that there exists $\epsilon > 0$ independent of h such that the curve Γ is included in $\rho(\tilde{A}_h)$ for any $h \in (0, h^*)$, and hence $\tilde{A}_h^{-\alpha}$ is well defined. In fact, if $\xi \in \Gamma$ such that $\Re\xi > 0$, then, by the Hille-Yosida Theorem, $\xi \in \rho(\tilde{A}_h)$, while if $-\epsilon \leq \xi \leq 0$, then, by (2.7.5), $\xi \in \rho(\tilde{A}_h)$.

Proposition 2.7.4. *If the assumptions (2.7.1) and (2.7.2) are satisfied, then $\tilde{A}_h^{-\alpha}$ is bounded independent of $h \in (0, h^*)$.*

Proof: We have

$$\begin{aligned}\tilde{A}_h^{-\alpha} &= \frac{1}{2\pi i}\int_0^{+\infty}(-\epsilon + te^{i\theta})^{-\alpha}(-\epsilon + te^{i\theta} - \tilde{A}_h)^{-1}e^{i\theta}dt \\ &+ \frac{1}{2\pi i}\int_{-\infty}^0(-\epsilon - te^{-i\theta})^{-\alpha}(-\epsilon - te^{-i\theta} - \tilde{A}_h)^{-1}(-e^{-i\theta})dt.\end{aligned} \tag{2.7.6}$$

Since $(T_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ is bounded, then by Hille-Yosida Theorem (see Theorem I.3.1 of [62]) we get

$$\|R(\lambda, \tilde{A}_h)\| \leq \frac{M}{Re\lambda},\ \forall Re\lambda > 0.$$

For $-\epsilon \leq Re\lambda \leq 0$, we have $|\Im\lambda| \leq m$ and therefore, by (2.7.5), we get

$$\|R(\lambda, \tilde{A}_h)\| \leq 2c.$$

Let $t_0 > 0$ be such that $-\epsilon \leq Re(-\epsilon + te^{i\theta}) \leq 0, \forall 0 \leq t \leq t_0 = \dfrac{\epsilon}{\cos\theta}$ and $Re(-\epsilon + te^{i\theta}) \geq 0, \forall t \geq t_0$ and let $t_1 = -\dfrac{\epsilon}{\cos\theta} \leq 0$ be such that $Re(-\epsilon - te^{-i\theta}) \leq 0, \forall t_1 \leq t \leq 0$ and $Re(-\epsilon - te^{-i\theta}) \geq 0,\ \forall t \leq t_1$. Therefore,

$$\begin{aligned}
\tilde{A}_h^{-\alpha} &= \frac{1}{2\pi i}\int_0^{t_0} (-\epsilon + te^{i\theta})^{-\alpha}(-\epsilon + te^{i\theta} - \tilde{A}_h)^{-1} e^{i\theta} dt \\
&+ \frac{1}{2\pi i}\int_{t_0}^{+\infty} (-\epsilon + te^{i\theta})^{-\alpha}(-\epsilon + te^{i\theta} - \tilde{A}_h)^{-1} e^{i\theta} dt \\
&+ \frac{1}{2\pi i}\int_{-\infty}^{t_1} (-\epsilon - te^{-i\theta})^{-\alpha}(-\epsilon - te^{-i\theta} - \tilde{A}_h)^{-1} (-e^{-i\theta}) dt \\
&+ \frac{1}{2\pi i}\int_{t_1}^{0} (-\epsilon - te^{-i\theta})^{-\alpha}(-\epsilon - te^{-i\theta} - \tilde{A}_h)^{-1} (-e^{-i\theta}) dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|\tilde{A}_h^{-\alpha}\right\| &\leq 2c\int_0^{t_0} \mid -\epsilon + te^{i\theta} \mid^{-\alpha} dt + M\int_{t_0}^{+\infty} \frac{1}{\mid -\epsilon + te^{i\theta} \mid^{\alpha} (-\epsilon + t cos\theta)} dt \\
&+ M\int_{-\infty}^{t_1} \frac{1}{\mid \epsilon + te^{-i\theta} \mid^{\alpha} (-\epsilon - t cos\theta)} dt + 2c\int_{t_1}^{0} \mid \epsilon + te^{-i\theta} \mid^{-\alpha} dt,
\end{aligned}$$

which is uniformly bounded with respect to h. ■

The proof of the polynomial stability of $(T_h(t))_{t\geq 0}$ (see Theorem 2.7.9 below) is based on the following three lemmas. The first lemma is the discretized version of

Lemma 3.2 in [48] and the other ones are the discrete versions of similar results of Lemmas 2.1 and 2.3 in [18].

Lemma 2.7.5. *Let* $S = \{\lambda \in \mathbb{C} : \; a \leq Re\lambda \leq b\}$ *be a subset of* $\rho(\tilde{A}_h)$ *for all* $h \in (0, h^*)$ *where* $0 \leq a < b$. *Then if (2.7.1) and (2.7.2) are satisfied and if for some positive constants* α *and* M *we have*

$$\sup_{\substack{h\in(0,h^*)\\ \lambda\in S}} \frac{\|R(\lambda, \tilde{A}_h)\|}{1 + |\lambda|^\alpha} \leq M,$$

then there exists a constant $c > 0$ *independent of* h *such that*

$$\sup_{\substack{h\in(0,h^*)\\ \lambda\in S}} \|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c.$$

Proof: There exists $c > 0$ and $\varphi_0, \; 0 < \varphi_0 < \dfrac{\pi}{2}$, such that

$$|\mu - e^{i\varphi}| \geq c|\mu|, \quad \forall \mu \in \Gamma, \; \forall \varphi_0 < |\varphi| < \pi - \varphi_0 \tag{2.7.7}$$

where the curve Γ is given by (2.7.4). Indeed, if $\mu = -\epsilon + te^{i\theta}$ for some $t > 0$, then

$$|\mu - e^{i\varphi}|^2 = 1 + \epsilon^2 + t^2 - 2tcos(\theta + \varphi) - 2\epsilon tcos\theta + 2\epsilon cos\varphi$$

and

$$|\mu|^2 = \epsilon^2 + t^2 - 2\epsilon tcos\theta.$$

Therefore, whether $t > 0$ is large enough or small , (2.7.7) holds true. Now, Since b is finite, choose N large enough such that whenever $\lambda \in S$ and $|\lambda| > N$ we get both $\varphi_0 < |arg\lambda| < \pi - \varphi_0$ and λ does not belong to the sector bounded by the curve $|\lambda|\Gamma = \{-\epsilon|\lambda| + t|\lambda|e^{i\theta}, \; t \in [0, +\infty)\} \cup \{-\epsilon|\lambda| - t|\lambda|e^{-i\theta}, \; t \in (-\infty, 0]\}$.

For all such choice of $\lambda \in S$, we have according to (2.7.7)

$$|\mu - e^{iarg\lambda}| \geq c|\mu| \quad \forall \mu \in \Gamma. \tag{2.7.8}$$

Consider the following integral for all $\lambda \in S$ with $|\lambda| > N$

$$I_\lambda = \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} d\mu.$$

By the above choice of λ, we have $\lambda \notin \Gamma$ and $\lambda \notin |\lambda|\Gamma$. Consequently, the integral has no singular points between Γ and $|\lambda|\Gamma$. Therefore, by the Cauchy Theorem, we have

$$I_\lambda = \int_{|\lambda|\Gamma} \frac{\mu^{-\alpha}}{\mu - \lambda} d\mu = \frac{1}{|\lambda|^\alpha} \int_\Gamma \frac{\mu^{-\alpha}}{\mu - e^{iarg\lambda}} d\mu.$$

Therefore, by (2.7.8), we get

$$|I_\lambda| \leq \frac{c}{|\lambda|^\alpha}.$$

Now, for $|\lambda| > N$ with $\lambda \in S$, we have by the resolvent identity

$$\begin{aligned} R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha} &= \frac{1}{2\pi i} \int_\Gamma \mu^{-\alpha} R(\lambda, \tilde{A}_h) R(\mu, \tilde{A}_h) d\mu \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} R(\lambda, \tilde{A}_h) d\mu - \frac{1}{2\pi i} \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} R(\mu, \tilde{A}_h) d\mu \\ &= \frac{1}{2\pi i} I_\lambda R(\lambda, \tilde{A}_h) - \frac{1}{2\pi i} \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} R(\mu, \tilde{A}_h) d\mu. \end{aligned}$$

On the other hand, similar to the proof of Proposition 2.7.4,

$$\left| \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} R(\mu, \tilde{A}_h) d\mu \right| \leq c \int_\Gamma \frac{1}{|\mu|^{\alpha+1}} \|R(\mu, \tilde{A}_h)\| d\mu \leq c',$$

where c' is independent of h. Therefore for all $\lambda \in S$, with $|\lambda| > N$, we have

$$\|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq \frac{c}{|\lambda|^\alpha} \|R(\lambda, \tilde{A}_h)\| + c' \leq c\frac{1 + |\lambda|^\alpha}{|\lambda|^\alpha} + c' \leq c''.$$

Now, for $\lambda \in S$ such that $|\lambda| \leq N$, we have

$$\|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq \|R(\lambda, \tilde{A}_h)\| \|\tilde{A}_h^{-\alpha}\| \leq c(1 + |\lambda|^\alpha) \leq c(1 + N^\alpha),$$

which completes the proof of Lemma 2.7.5. ■

Lemma 2.7.6. *If (2.7.1), (2.7.2) and*

$$\sup_{h\in(0,h^*)} \|R(is,\tilde{A}_h)\|_{\mathcal{L}(X_h)} = O(|s|^\alpha),\ |s|\to\infty, \tag{2.7.9}$$

are satisfied, then there exists $c>0$ independent of h such that

$$\sup_{\substack{h\in(0,h^*)\\ Re\lambda>0}} \|R(\lambda,\tilde{A}_h)\tilde{A}_h^{-\alpha}\| \le c. \tag{2.7.10}$$

Proof: For all $h\in(0,h^*)$, $m>0$, and $B>\max\{2m,1\}$, consider $F_h(\lambda) = R(\lambda,\tilde{A}_h)\lambda^{-\alpha}(1-\frac{\lambda^2}{B^2})$ on the domain $D=\left\{\lambda\in\mathbb{C}:\ Re\lambda>0,\ m<|\lambda|\le\dfrac{B}{2}\right\}$. F_h, by the maximum principle, attains its maximum for $|\lambda|=\dfrac{B}{2}$. Therefore,

$$|F_h(\lambda)|\le\frac{c}{Re\lambda}.$$

If there exists $\epsilon>0$ such that $Re\lambda>\epsilon$, then $|F_h(\lambda)|\le c$.

Otherwise, for $0<Re\lambda<\epsilon$, using the resolvent identity

$$R(\lambda,\tilde{A}_h) = R(iIm\lambda,\tilde{A}_h) - Re\lambda R(iIm\lambda,\tilde{A}_h)R(\lambda,\tilde{A}_h) \tag{2.7.11}$$

then, as $|Im\lambda|\ge m-\epsilon$ for all $m>0$, we have

$$\|R(\lambda,\tilde{A}_h)\|\le c|Im\lambda|^\alpha.$$

Therefore,

$$|F_h(\lambda)|\le c|Im\lambda|^\alpha|\lambda|^{-\alpha}\left|1-\frac{\lambda^2}{B^2}\right|\le c.$$

Hence, in all cases, there exists $c>0$ independent of B such that

$$|F_h(\lambda)|\le c.$$

As a result, for all $\lambda\in D$,

$$\|R(\lambda,\tilde{A}_h)\|\le\frac{c|\lambda|^\alpha}{|1-\frac{\lambda^2}{B^2}|}\le c|\lambda|^\alpha\le c(1+|\lambda|^\alpha).$$

If $0 < Re\lambda \leq |\lambda| \leq m$, then by (2.7.11) and assumption (2.7.2), we get

$$\|R(\lambda, \tilde{A}_h)\| \leq c\|R(iIm\lambda, \tilde{A}_h)\| \leq c \leq c(1 + |\lambda|^\alpha).$$

Letting $B \to +\infty$ yields

$$\|R(\lambda, \tilde{A}_h)\| \leq c(1 + |\lambda|^\alpha), \forall Re\lambda > 0.$$

Applying Lemma 2.7.5, we get for $0 \leq Re\lambda \leq m$,

$$\|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c.$$

In addition, if $Re\lambda \geq m$, by the Hille-Yosida theorem and Proposition 2.7.4, there exists some positive constants c_1 and c_2 such that

$$\|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c_1 \frac{\|\tilde{A}_h^{-\alpha}\|}{Re\lambda} \leq c_2.$$

In all cases, we get (2.7.10). ■

The last lemma in this section gives the necessary and sufficient conditions for the boundedness of any family of C_0 semigroups $(S_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$.

Lemma 2.7.7. *Let $(S_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ be a family of C_0 semigroups on the associated Hilbert spaces $(Y_h)_{h\in(0,h^*)}$ and let $(\tilde{E}_h)_{h\in(0,h^*)}$ be the corresponding infinitesimal generators. Then $(S_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ is uniformly bounded if and only if*

(i) $\{\lambda \in \mathbb{C} : \; Re\lambda > 0\} \subseteq \rho(\tilde{E}_h), \forall h \in (0, h^*)$

(ii) There exists $c > 0$ independent of h such that

$$\sup_{\substack{\xi>0\\ h\in(0,h^*)}} \xi \int_{\mathbb{R}} (\|R(\xi + i\eta, \tilde{E}_h)\|^2 + \|R(\xi + i\eta, \tilde{E}_h^*)\|^2) d\eta \leq c.$$

Proof: First, we assume that $(S_h(t))$ is uniformly bounded. Then (i) holds by the Hille-Yosida theorem. As for (ii), we only need to prove that

$$\sup_{\substack{\xi>0 \\ h\in(0,h^*)}} \xi \int_{\mathbb{R}} \|R(\xi+i\eta,\tilde{E}_h)x_h\|^2 d\eta \leq c\|x_h\|^2,\ \forall x_h \in Y_h \tag{2.7.12}$$

because according to the theory of adjoint semigroups, (see [62]), $S^*(t)$ is a C_0 semigroup with the same properties as $S(t)$.

Similar to the proof of Lemma 1 of [42], we have for all $h \in (0,h^*)$, $x_h \in Y_h$

$$\|R(\xi+i\eta,\tilde{E}_h)x_h\|^2 = \int_{\mathbb{R}} e^{-i\eta s} f_h(s)ds,$$

where

$$f_h(s) = \int_{\max\{0,-s\}}^{+\infty} e^{-\xi(s+2u)} < S_h(u+s)x_h, S_h(u)x_h >_{Y_h,Y_h} du.$$

For $s \geq 0$, since $(S_h(t)_{h\in(0,h^*)}$ is uniformly bounded, i.e. $\sup_{h\in(0,h^*)} \|S_h(t)\| \leq M$, we have

$$|f_h(s)| \leq \int_0^{+\infty} M^2\|x_h\|^2 e^{-\xi(s+2u)}du = \frac{M^2\|x_h\|^2}{2\xi}e^{-\xi s} \leq \frac{M^2\|x_h\|^2}{2\xi}.$$

For $s<0$, we have

$$|f_h(s)| \leq \int_{-s}^{+\infty} M^2\|x_h\|^2 e^{-\xi(s+2u)}du = \frac{M^2\|x_h\|^2 e^{\xi s}}{2\xi} \leq \frac{M^2\|x_h\|^2}{2\xi}.$$

Hence, $f_h \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R})$ and

$$\mathfrak{F}(f_h(s)) = \frac{1}{\sqrt{2\pi}}\|R(\xi+i\eta,\tilde{E}_h)x_h\|^2.$$

Using Lemma 21.50 in [40], it follows that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \|R(\xi+i\eta,\tilde{E}_h)x_h\|^2 d\eta = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathfrak{F}(f_h)(\tau)d\tau \leq c\|f_h\|_{L^\infty} \leq \frac{cM^2\|x_h\|^2}{2\xi}.$$

Hence, (2.7.12) is verified.

As for the sufficient condition, since $\{\lambda \in \mathbb{C} : \ Re\lambda > 0\} \subseteq \rho(\tilde{E}_h)$, with $\sigma = \frac{1}{t}$, we get for all $x_h \in Y_h$

$$\begin{aligned} S_h(t)x_h &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}(\lambda - \tilde{E}_h)^{-1}x_h d\lambda, \\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{\lambda t}}{t}(\lambda - \tilde{E}_h)^{-2}x_h d\lambda + \frac{e^{\lambda t}}{t}(\lambda - \tilde{E}_h)^{-1}x_h \,|_{\sigma-i\infty}^{\sigma+i\infty}. \end{aligned}$$

But $\frac{e^{\lambda t}}{t}(\lambda - \tilde{E}_h)^{-1}x_h|_{\sigma-i\infty}^{\sigma+i\infty} = 0$ since according to Lemma 2.1 of [69], under condition (ii), we have $\|R(\lambda, \tilde{E}_h)x_h\| \to 0$ as $|\lambda| \to +\infty$ whenever $Re\lambda > 0$. Therefore,

$$\begin{aligned} \langle S_h(t)x_h, y_h\rangle_{Y_h,Y_h} &= \left\langle \frac{1}{2\pi it}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}(\lambda - \tilde{E}_h)^{-2}x_h d\lambda, y_h \right\rangle_{Y_h,Y_h} \\ &= \frac{1}{2\pi it}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}\left\langle (\lambda - \tilde{E}_h)^{-2}x_h, y_h \right\rangle_{Y_h,Y_h} d\lambda. \end{aligned}$$

Let $\lambda = \frac{1}{t} + i\eta$ with $\eta \in \mathbb{R}$. Then

$$\langle S_h(t)x_h, y_h\rangle_{Y_h,Y_h} = \frac{e}{2\pi t}\int_{\mathbb{R}} e^{i\eta t}\left\langle R^2(\frac{1}{t} + i\eta, \tilde{E}_h)x_h, y_h \right\rangle_{Y_h,Y_h} d\eta.$$

Hölder's inequality yields

$$\begin{aligned} |\langle S_h(t)x_h, y_h\rangle_{Y_h,Y_h}| &= \left| \frac{e}{2\pi t}\int_{\mathbb{R}} e^{i\eta t}\left\langle R(\frac{1}{t} + i\eta, \tilde{E}_h)x_h, R(\frac{1}{t} + i\eta, \tilde{E}_h^*)y_h \right\rangle_{Y_h,Y_h} d\eta \right| \\ &\leq \frac{e}{2\pi t}\left(\int_{\mathbb{R}} \|R(\frac{1}{t} + i\eta, \tilde{E}_h)x_h\|^2 d\eta\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} \|R(\frac{1}{t} + i\eta, \tilde{E}_h^*)y_h\|^2 d\eta\right)^{\frac{1}{2}} \\ &\leq c\|x_h\|\|y_h\|. \end{aligned}$$

Therefore

$$\|S_h(t)\| \leq c, \ \forall h \in (0, h^*).$$

■

Before we give the necessary and sufficient conditions to get the uniform polynomial stability of the discretized problem, we recall Theorem II.5.34 of [24] about the moment inequality.

Theorem 2.7.8. *Let E be the generator of a strongly continuous semigroup and let $\alpha < \beta < \gamma$. Then there exists a constant $L = L(\alpha, \beta, \gamma)$ such that, for every $x \in D(E^\gamma)$, we have*

$$\left\|E^{\beta}x\right\| \leq L \left\|E^{\alpha}x\right\|^{\frac{\gamma-\beta}{\gamma-\alpha}} \left\|E^{\gamma}x\right\|^{\frac{\beta-\alpha}{\gamma-\alpha}}. \tag{2.7.13}$$

Now, we display the main theorem which leads to the uniform polynomial stability of the discretized problem (2.4.9).

Theorem 2.7.9. *Let $(T_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ be a family of uniformly bounded C_0 semigroups on the associated Hilbert spaces $(X_h)_{h\in(0,h^*)}$ and let $(\tilde{A}_h)_{h\in(0,h^*)}$ be the corresponding infinitesimal generators such that (2.7.1) and (2.7.2) are satisfied. Then for a fixed $\alpha > 0$, the following statements are equivalent :*

(i)

$$\sup_{h\in(0,h^*)} \|R(is, \tilde{A}_h)\| = O(|s|^{\alpha}), \quad |s| \to \infty$$

(ii)

$$\sup_{h\in(0,h^*)} \|T_h(t)\tilde{A}_h^{-\alpha}\| = O(t^{-1}), \quad t \to +\infty$$

(iii)

$$\sup_{h\in(0,h^*)} \|T_h(t)\tilde{A}_h^{-1}\| = O(t^{\frac{-1}{\alpha}}), \quad t \to +\infty.$$

Proof: We begin to prove $(ii) \Leftrightarrow (iii)$. We adapt the proof found in Proposition 3.1 of [15] without the discretization parameter h. Given (ii), we have

$$\|T_h(t)\tilde{A}_h^{-\alpha n}\| = \left\|\left[T_h\left(\frac{t}{n}\right)\tilde{A}_h^{-\alpha}\right]^n\right\| \leq c\left(\frac{n}{t}\right)^n \leq c(n)t^{-n}, \forall n \in \mathbb{N}^*,\ h \in (0, h^*),\ t \to +\infty.$$

According to the moment inequality (2.7.13), we remark that there exists a positive constant L independent of h such that, for all $\nu \in (0,1)$, we have

$$\begin{aligned}
\left\|T_h(t)\tilde{A}_h^{-\alpha n\nu}x_h\right\| &= \left\|\tilde{A}_h^{\alpha n(1-\nu)}T_h(t)\tilde{A}_h^{-\alpha n}x_h\right\| \\
&= \left\|\tilde{A}_h^{\beta_1}\tilde{A}_h^{\alpha n(1-\nu)-\beta_1}T_h(t)\tilde{A}_h^{-\alpha n}x_h\right\| \\
&\leq L\left\|\tilde{A}_h^{\alpha_1}y_h\right\|^{\frac{\gamma_1-\beta_1}{\gamma_1-\alpha_1}}\left\|\tilde{A}_h^{\gamma_1}y_h\right\|^{\frac{\beta_1-\alpha_1}{\gamma_1-\alpha_1}} \\
&\leq L\left\|\tilde{A}_h^{\alpha_1+\alpha n(1-\nu)-\beta_1}T_h(t)\tilde{A}_h^{-\alpha n}x_h\right\|^{\nu} \\
&\quad\left\|\tilde{A}_h^{\gamma_1+\alpha n(1-\nu)-\beta_1}T_h(t)\tilde{A}_h^{-\alpha n}x_h\right\|^{1-\nu},
\end{aligned}$$

where $y_h = \tilde{A}_h^{\alpha n(1-\nu)-\beta_1}T_h(t)\tilde{A}_h^{-\alpha n}x_h$. Now, we choose α_1, β_1, and γ_1 such that

$$\begin{cases} \gamma_1 - \beta_1 + \alpha n(1-\nu) = \alpha n \\ \alpha_1 + \alpha n(1-\nu) - \beta_1 = 0; \end{cases}$$

i.e.,

$$\begin{cases} \alpha_1 = \beta_1 - \alpha n(1-\nu) < \beta_1 \\ \gamma_1 = \beta_1 + \alpha n\nu > \beta_1. \end{cases}$$

Therefore,

$$\begin{cases} \gamma_1 - \alpha_1 = \alpha n \\ \gamma_1 - \beta_1 = \alpha n\nu. \end{cases}$$

Finally, we get

$$\begin{aligned}
\left\|T_h(t)\tilde{A}_h^{-\alpha n\nu}x_h\right\| &\leq L\left\|\tilde{A}_h^{\alpha n}T_h(t)\tilde{A}_h^{-\alpha n}x_h\right\|^{1-\nu}\left\|T_h(t)\tilde{A}_h^{-\alpha n}x_h\right\|^{\nu} \\
&\leq LM^{1-\nu}c^{\nu}(n)t^{-n\nu}\|x_h\|, \quad \forall\nu \in (0,1).
\end{aligned}$$

Choose $\nu = \dfrac{1}{\alpha n}$ with $n > \dfrac{1}{\alpha}$ to get

$$\|T_h(t)\tilde{A}_h^{-1}\| \leq ct^{-\frac{1}{\alpha}}.$$

Conversely, assume that (iii) holds. Then

$$\|T_h(t)\tilde{A}_h^{-n}\| = \|[T_h\left(\frac{t}{n}\right)\tilde{A}_h^{-1}]^n\| \leq c\left(\frac{t}{n}\right)^{\frac{-n}{\alpha}} \leq cn^{\frac{n}{\alpha}}t^{-\frac{n}{\alpha}},\ \forall n \in \mathbb{N}^*.$$

Therefore,

$$\begin{aligned}\|T_h(t)\tilde{A}_h^{-n\nu}\| &\leq c\|\tilde{A}_h^n T_h(t)\tilde{A}_h^{-n}\|^{1-\nu}\|T_h(t)\tilde{A}_h^{-n}\|^{\nu}\\ &\leq cM^{1-\nu}c(n)^{\nu}t^{-\frac{n\nu}{\alpha}}, \quad \forall \nu \in (0,1).\end{aligned}$$

Take $\nu = \dfrac{\alpha}{n}$ with $n > \alpha$ to get

$$\sup_{h\in(0,h^*)} \|T_h(t)\tilde{A}_h^{-\alpha}\| = O\left(t^{-1}\right).$$

Now, we prove the implication $(iii) \Rightarrow (i)$ (for the continuous case, see [16]). Given (iii), define

$$m_1(t) = \sup_{\substack{h\in(0,h^*)\\ s\geq t}} \|T_h(s)\tilde{A}_h^{-1}\|.$$

Notice that $m_1(t)$ is non increasing. Let $u_{0h} \in \mathcal{D}(\tilde{A}_h)$, $f_{0h} = (-\tilde{A}_h + i\tau)u_{0h}$, $\tau \in \mathbb{R}$, and let $v_h(t) = e^{it\tau}u_{0h}$. We have

$$\begin{cases} \partial_t v_h - \tilde{A}_h v_h = i\tau e^{it\tau}u_{0h} - \tilde{A}_h(e^{it\tau}u_{0h}) = e^{it\tau}f_{0h}\\ v_h(0) = u_{0h}.\end{cases}$$

By the Duhamel formula,

$$v_h = e^{t\tilde{A}_h}u_{0h} + \int_0^t e^{(t-s)\tilde{A}_h}e^{i\tau s}f_{0h}ds.$$

By the boundedness of the semigroup $(T_h(t))$ and the definition of m_1, we have

$$\begin{aligned}\|u_{0h}\| = \|v_h(t)\| &\leq \|T_h(t)\tilde{A}_h^{-1}\tilde{A}_h u_{0h}\| + c\ t\|f_{0h}\|\\ &\leq m_1(t)\|\tilde{A}_h u_{0h}\| + c\ t\|f_{0h}\|\\ &\leq m_1(t)(\|f_{0h}\| + |\tau|\|u_{0h}\|) + c\ t\|f_{0h}\|.\end{aligned}$$

Apply the above inequality with $t = G(|\tau|)$ where

$$G(\xi) = \begin{cases} m_{1r}^{-1}\left(\dfrac{1}{2(\xi+1)}\right) & if\ \xi > 0 \text{ and } \dfrac{1}{2(\xi+1)} \leq m_1(0), \\ 0 & if\ \xi > 0 \text{ and } \dfrac{1}{2(\xi+1)} > m_1(0), \end{cases}$$

where m_{1r}^{-1} is the right inverse of m_1. Therefore,

$$m_1(t)|\tau| = m_1(G(|\tau|))|\tau| \leq \frac{|\tau|}{2(|\tau|+1)} \leq \frac{1}{2}.$$

Hence,

$$\begin{aligned} \tfrac{1}{2}\|u_{0h}\| &\leq m_1(G(|\tau|))\|f_{0h}\| + c\ G(|\tau|)\|f_{0h}\| \\ &\leq \frac{\|f_{0h}\|}{2(|\tau|+1)} + c\ G(|\tau|)\|f_{0h}\| \\ &\leq (\tfrac{1}{2} + c\ G(|\tau|))\|f_{0h}\|. \end{aligned}$$

Consequently,

$$\|(i\tau - \tilde{A}_h)^{-1}\| \leq 1 + 2c\ G(|\tau|),$$

i.e.,

$$\sup_{h\in(0,h^*)} \|(i\tau - \tilde{A}_h)^{-1}\| \leq 1 + 2c\ G(|\tau|).$$

Since, by (iii),

$$\sup_{h\in(0,h^*)} \|T_h(t)\tilde{A}_h^{-1}\| \leq Mt^{\frac{-1}{\alpha}}, \quad t \to +\infty,$$

then, as m_1 is non-increasing, we get

$$m_1(t) \leq Mt^{\frac{-1}{\alpha}}, \quad t \to +\infty.$$

Besides, as the inverse of $t^{\frac{-1}{\alpha}}$ is $t^{-\alpha}$, then

$$G(\xi) \leq m_{1r}^{-1}\left(\frac{1}{2(\xi+1)}\right) \leq C\left(\frac{1}{2(\xi+1)}\right)^{-\alpha} = C(2(\xi+1))^{\alpha} \leq c\xi^{\alpha}, \quad \xi \to +\infty.$$

Finally, we get

$$\sup_{h\in(0,h^*)} \|(i\tau - \tilde{A}_h)^{-1}\| \leq 1 + 2c|\tau|^{\alpha} \leq c|\tau|^{\alpha}, \quad |\tau| \to +\infty.$$

It remains to prove that $(i) \Rightarrow (ii)$. For this aim, for all $h \in (0, h^*)$, let $\mathbf{X_h} = X_h \times X_h$ and consider the operator $\tilde{\mathbf{A}}_\mathbf{h}$ given by the operator matrix

$$\tilde{\mathbf{A}}_\mathbf{h} = \begin{pmatrix} \tilde{A}_h & \tilde{A}_h^{-\alpha} \\ 0 & \tilde{A}_h \end{pmatrix},$$

where $\mathcal{D}(\tilde{\mathbf{A}}_\mathbf{h}) = \mathcal{D}(\tilde{A}_h) \times \mathcal{D}(\tilde{A}_h)$. For all $h \in (0, h^*)$ and all $\lambda_h \in \rho(\tilde{A}_h)$, we have

$$R(\lambda_h, \tilde{\mathbf{A}}_\mathbf{h}) = \begin{pmatrix} R(\lambda_h, \tilde{A}_h) & R^2(\lambda_h, \tilde{A}_h)\tilde{A}_h^{-\alpha} \\ 0 & R(\lambda_h, \tilde{A}_h) \end{pmatrix}.$$

Indeed,

$$R(\lambda_h, \tilde{\mathbf{A}}_\mathbf{h})(\lambda_h - \tilde{\mathbf{A}}_\mathbf{h}) = (\lambda_h - \tilde{\mathbf{A}}_\mathbf{h})R(\lambda_h, \tilde{\mathbf{A}}_\mathbf{h}) = \begin{pmatrix} I_h & 0 \\ 0 & I_h \end{pmatrix}.$$

Therefore, $\rho(\tilde{\mathbf{A}}_\mathbf{h}) = \rho(\tilde{A}_h)$ and for all $h \in (0, h^*)$, the operator $\tilde{\mathbf{A}}_\mathbf{h}$ is the generator of the C_0 semigroup $(\mathbf{T_h}(t))_{t \geq 0}$ on $\mathbf{X_h}$ defined by

$$\mathbf{T_h}(t) = \begin{pmatrix} T_h(t) & tT_h(t)\tilde{A}_h^{-\alpha} \\ 0 & T_h(t) \end{pmatrix}.$$

In fact,

$$\begin{aligned} \widehat{\mathbf{T_h}(t)} &= \begin{pmatrix} \widehat{T_h(t)} & \widehat{tT_h(t)}\tilde{A}_h^{-\alpha} \\ 0 & \widehat{T_h(t)} \end{pmatrix} \\ &= \begin{pmatrix} R(\lambda_h, \tilde{A}_h) & R^2(\lambda_h, \tilde{A}_h)\tilde{A}_h^{-\alpha} \\ 0 & R(\lambda_h, \tilde{A}_h) \end{pmatrix} \\ &= R(\lambda_h, \tilde{\mathbf{A}}_\mathbf{h}), \end{aligned}$$

where $\widehat{\mathbf{T_h}(t)}$ is the Laplace transform of $\mathbf{T_h}(t)$. Since for all $h \in (0, h^*)$ we have

$$\|R(is, \tilde{A}_h)\| = O(|s|^\alpha), \text{ as } |s| \to +\infty,$$

then by Lemma 2.7.6 we get

$$\sup_{\substack{h\in(0,h^*)\\ Re\lambda>0}} \|R(\lambda,\tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c.$$

Hence, for all $x_h = (x_{1h}, x_{2h}) \in \mathbf{X_h}$, and $Re\lambda_h > 0$, we have

$$\begin{aligned} \|R(\lambda_h,\tilde{\mathbf{A}}_{\mathbf{h}})x_h\|^2 &= \left\| \begin{pmatrix} R(\lambda_h,\tilde{A}_h)x_{1h} + R^2(\lambda_h,\tilde{A}_h)\tilde{A}_h^{-\alpha}x_{2h} \\ R(\lambda_h,\tilde{A}_h)x_{2h} \end{pmatrix} \right\|^2 \\ &\leq c\left(\|R(\lambda_h,\tilde{A}_h)x_{1h}\|^2 + \|R(\lambda_h,\tilde{A}_h)x_{2h}\|^2\right). \end{aligned}$$

Similarly, we have

$$\|R(\lambda_h,\tilde{\mathbf{A}}_{\mathbf{h}}^*)x_h\|^2 \leq c(\|R(\lambda_h,\tilde{A}_h^*)x_{1h}\|^2 + \|R(\lambda_h,\tilde{A}_h^*)x_{2h}\|^2).$$

Indeed, we have

$$\tilde{\mathbf{A}}_{\mathbf{h}}^* = \begin{pmatrix} \tilde{A}_h^* & (\tilde{A}_h^*)^{-\alpha} \\ 0 & \tilde{A}_h^* \end{pmatrix}.$$

In order to get

$$\sup_{\substack{h\in(0,h^*)\\ Re\lambda>0}} \|R(\lambda,\tilde{A}_h^*)(\tilde{A}_h^*)^{-\alpha}\| \leq c,$$

we must have at least

$$\|R(is,\tilde{A}_h^*)\| = O(|s|^\alpha), \text{ as } |s| \to +\infty.$$

Actually, we have

$$R(is,\tilde{A}_h^*) = [(is-\tilde{A}_h^*)]^{-1} = [(is-\tilde{A}_h)^*]^{-1} = R(is,\tilde{A}_h)^*.$$

Therefore, we get

$$\|R(is,\tilde{A}_h^*)\| \leq \|R(is,\tilde{A}_h)\| = O(|s|^\alpha), \text{ as } |s| \to +\infty.$$

Now, by Lemma 2.7.7, since for all $h \in (0, h^*)$, $T_h(t)$ is a uniformly bounded family of C_0 semigroups, we get

$$\sup_{\substack{\xi>0 \\ h\in(0,h^*)}} \xi \int_{\mathbb{R}} (\|R(\xi + i\eta, \tilde{A}_h)x_h\|^2) + (\|R(\xi + i\eta, \tilde{A}_h^*)x_h\|^2) d\eta < \infty, \quad \forall x_h \in X_h.$$

Hence,

$$\sup_{\substack{\xi>0 \\ h\in(0,h^*)}} \xi \int_{\mathbb{R}} (\|R(\xi + i\eta, \tilde{\mathbf{A}}_{\mathbf{h}})x_h\|^2) + (\|R(\xi + i\eta, \tilde{\mathbf{A}}_{\mathbf{h}}^*)x_h\|^2) d\eta < \infty, \quad \forall x_h \in \mathbf{X}_{\mathbf{h}}.$$

Therefore, $(\mathbf{T}_{\mathbf{h}}(t))_{\substack{t\geq 0 \\ h\in(0,h^*)}}$ is uniformly bounded over $(\mathbf{X}_{\mathbf{h}})_{h\in(0,h^*)}$ by Lemma 2.7.7. Since $(T_h(t))_{\substack{t\geq 0 \\ h\in(0,h^*)}}$ is uniformly bounded over $(X_h)_{h\in(0,h^*)}$, the definition of $\mathbf{T}_{\mathbf{h}}(t)$ implies that

$$\sup_{\substack{t\geq 0 \\ h\in(0,h^*)}} \|tT_h(t)\tilde{A}_h^{-\alpha}\| < +\infty.$$

■

2.8 Preliminary lemmas

In this section, we prove that the family $(\tilde{A}_{l,h})_{h\in(0,h^*)}$ defined in (2.5.1) satisfies condition i) in Theorem 2.7.1 and the properties (2.7.1) and (2.7.2) of Subsection 2.7.2. Condition i) in Theorem 2.7.1 or (2.7.1) in Subsection 2.7.2 is satisfied due to the following lemma :

Lemma 2.8.1. *The spectrum of the operator $\tilde{A}_{l,h}$ contains no point on the imaginary axis.*

Proof: Suppose that $\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \in X_h$ and $\omega \in \mathbb{R}$ are such that

$$\tilde{A}_{l,h} \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} = i\omega \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}.$$

Then, by using the definition (2.5.1) of $\tilde{A}_{l,h}$, we have

$$\left\{\begin{array}{c} \psi_h = i\omega(1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\varphi_h \\ -(1+h^\theta)^{-1}(I+h^\theta A_h^{\frac{l}{2}})A_h\varphi_h - i\omega(1+h^\theta)(h^\theta A_h^{1+\frac{l}{2}} + B_hB_h^*)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\varphi_h \\ = -\omega^2(1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\varphi_h. \end{array}\right. \tag{2.8.1}$$

Let $\chi_h = (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\varphi_h$ then the second relation of (2.8.1) becomes

$$(1+h^\theta)^{-2}(I+h^\theta A_h^{\frac{l}{2}})^2 A_h\chi_h + i\omega(h^\theta A_h^{1+\frac{l}{2}} + B_hB_h^*)\chi_h = \omega^2\chi_h. \tag{2.8.2}$$

If $\omega = 0$, then taking the inner product of (2.8.2) with $\chi_h \in V_h$, we get $(I + h^\theta A_h^{\frac{l}{2}})A_h^{\frac{1}{2}}\chi_h = 0$ and hence $\chi_h = 0$ which implies by the definition of χ_h that $\varphi_h = \psi_h = 0$.

It then remains to consider the case $\omega \neq 0$. In that case, we take the imaginary part of the inner product (in H) of (2.8.2) with $\chi_h \in V_h$ to obtain

$$\begin{array}{rcl} 0 & = & \omega h^\theta \left(A_h^{1+\frac{l}{2}}\chi_h,\, \chi_h\right) + \omega\left(B_hB_h^*\chi_h,\, \chi_h\right) \\ & = & \omega h^\theta \left(A_h^{\frac{1}{2}+\frac{l}{4}}\chi_h,\, A_h^{\frac{1}{2}+\frac{l}{4}}\chi_h\right) + \omega\left(B_h^*\chi_h,\, B_h^*\chi_h\right)_U, \end{array}$$

that is to say

$$h^\theta \left\|A_h^{\frac{1}{2}+\frac{l}{4}}\chi_h\right\|^2 + \|B_h^*\chi_h\|_U^2 = 0.$$

This leads to $\chi_h = 0$, and hence $\varphi_h = \psi_h = 0$. ■

Our main goal is to prove condition ii) of Theorem 2.7.1 in the case $l = 0$ and condition i) of Theorem 2.7.9 as well as (2.7.2) in the case $l \geq 2$ and $\alpha = 2l$. In that last case ($l \geq 2$), these two conditions are equivalent to

$$\sup_{h\in(0,h^*),\ s\in\mathbb{R}} (1+|s|^{2l})^{-1}\|R(is,\tilde{A}_{l,h})\|_{\mathcal{L}(X_h)} < \infty. \tag{2.8.3}$$

To prove this above property, we use a contradiction argument. More precisely, we will assume that, for all $n \in \mathbb{N}$, there exist $h_n \in (0,\, h^*)$, $\omega_n \in \mathbb{R}$ and $z_n = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \in$

X_{h_n} such that

$$\|z_n\|^2_{X_{h_n}} = a(\varphi_n,\, \varphi_n) + \|\psi_n\|^2 = 1,\, \forall n \in \mathbb{N}, \tag{2.8.4}$$

and

$$(1 + |\omega_n|^{2l}) \left\| i\omega_n z_n - \tilde{A}_{l,h_n} z_n \right\|_{X_{h_n}} \to 0, \text{ as } n \to \infty, \tag{2.8.5}$$

where $l = 0$ in the setting of Theorem 2.7.1.

Lemma 2.8.2. *Assume that the sequences (h_n), (ω_n), (z_n) satisfy (2.8.4) and (2.8.5). Then, we have*

$$(1 + |\omega_n|^{2l})(h_n^\theta a^l(\psi_n,\, \psi_n) + \left\|B^*_{h_n}\psi_n\right\|^2_U) \to 0, \text{ as } n \to \infty \tag{2.8.6}$$

and

$$\lim_{n\to\infty} a(\varphi_n,\, \varphi_n) = \lim_{n\to\infty} \|\psi_n\|^2 = \frac{1}{2}. \tag{2.8.7}$$

Proof: For (2.8.6), we take the inner product in X_{h_n} of $i\omega_n z_n - \tilde{A}_{l,h_n} z_n$ with z_n and take the real part. We obtain

$$\begin{aligned}
&\Re\left(i\omega_n z_n - \tilde{A}_{l,h_n} z_n,\, z_n\right)_{X_{h_n}} \\
= \;& -\Re\left(\begin{pmatrix} (1+h_n^\theta)^{-1}(I + h_n^\theta A^{\frac{l}{2}}_{h_n})\psi_n \\ -(1+h_n^\theta)^{-1}(I + h_n^\theta A^{\frac{l}{2}}_{h_n})A_{h_n}\varphi_n - h_n^\theta A^{1+\frac{l}{2}}_{h_n}\psi_n - B_{h_n}B^*_{h_n}\psi_n \end{pmatrix}, \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}\right)_{X_{h_n}} \\
= \;& \Re\left(-(1+h_n^\theta)^{-1}\left((I + h_n^\theta A^{\frac{l}{2}}_{h})A_{h_n}\psi_n,\, \varphi_n\right) + (1+h_n^\theta)^{-1}\left((I + h_n^\theta A^{\frac{l}{2}}_{h_n})A_{h_n}\varphi_n,\, \psi_n\right)\right. \\
& \left. +(h_n^\theta A^{1+\frac{l}{2}}_{h_n}\psi_n + B_{h_n}B^*_{h_n}\psi_n,\, \psi_n)\right) \\
= \;& (h_n^\theta A^{1+\frac{l}{2}}_{h_n}\psi_n + B_{h_n}B^*_{h_n}\psi_n,\, \psi_n).
\end{aligned}$$

Then

$$\begin{aligned}
&(1 + |\omega_n|^{2l})\Re\left(i\omega_n z_n - \tilde{A}_{l,h_n} z_n,\, z_n\right)_{X_{h_n}} \\
&= (1 + |\omega_n|^{2l})(h_n^\theta a^l(\psi_n,\, \psi_n) + \left\|B^*_{h_n}\psi_n\right\|^2_U) \to 0 \,\text{by (2.8.5)}.
\end{aligned}$$

In order to prove (2.8.7), we introduce the operator

$$A_{1h_n} = (1+h_n^\theta)^{-1}(I + h_n^\theta A^{\frac{l}{2}}_{h_n}) \begin{pmatrix} 0 & I \\ -A_{h_n} & 0 \end{pmatrix}. \tag{2.8.8}$$

We have

$$\tilde{A}_{l,h_n}\begin{pmatrix}\varphi_n\\ \psi_n\end{pmatrix}=A_{1h_n}\begin{pmatrix}\varphi_n\\ \psi_n\end{pmatrix}-\begin{pmatrix}0\\ h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n+B_{h_n}B_{h_n}^*\psi_n\end{pmatrix},\ \forall\begin{pmatrix}\varphi_n\\ \psi_n\end{pmatrix}\in X_{h_n}.$$

We take the norm $\|.\|_{X_{h_n}}$ of $i\omega_n z_n - A_{1h_n}z_n+\begin{pmatrix}0\\ h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n\end{pmatrix}$ to obtain

$$\begin{aligned}
&(1+|\omega_n|^{2l})\left\|i\omega_n z_n - A_{1h_n}z_n+\begin{pmatrix}0\\ h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n\end{pmatrix}\right\|^2_{X_{h_n}}\\
=\ &(1+|\omega_n|^{2l})\left\|i\omega_n z_n - \tilde{A}_{l,h_n}z_n-\begin{pmatrix}0\\ B_{h_n}B_{h_n}^*\psi_n\end{pmatrix}\right\|^2_{X_{h_n}}\\
\leq\ &2(1+|\omega_n|^{2l})(\left\|i\omega_n z_n-\tilde{A}_{l,h_n}z_n\right\|^2_{X_{h_n}}+\left\|B_{h_n}B_{h_n}^*\psi_n\right\|^2)\\
\leq\ &C(1+|\omega_n|^{2l})(\left\|i\omega_n z_n-\tilde{A}_{l,h_n}z_n\right\|^2_{X_{h_n}}+\left\|B_{h_n}^*\psi_n\right\|^2_U)\to 0,
\end{aligned}$$

by (2.8.5) and (2.8.6). Therefore

$$(1+|\omega_n|^{2l})\left\|i\omega_n z_n - A_{1h_n}z_n+\begin{pmatrix}0\\ h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n\end{pmatrix}\right\|^2_{X_{h_n}}\to 0. \qquad (2.8.9)$$

We can now prove (2.8.7). By Lemma 2.8.3 below, there exists $n_0\in\mathbb{N}$ such that the sequence $(|\omega_n|)_{n\geq n_0}$ is bounded away from zero. Hence, we may write

$$\begin{aligned}
&\Im\left(i\omega_n z_n - A_{1h_n}z_n+\begin{pmatrix}0\\ h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n\end{pmatrix},\frac{1}{\omega_n}\begin{pmatrix}\varphi_n\\ -\psi_n\end{pmatrix}\right)_{X_{h_n}}\\
=\ &\left(\begin{pmatrix}\varphi_n\\ \psi_n\end{pmatrix},\begin{pmatrix}\varphi_n\\ -\psi_n\end{pmatrix}\right)_{X_{h_n}}\\
=\ &a(\varphi_n,\,\varphi_n)-\|\psi_n\|^2_{V_{h_n}}
\end{aligned}$$

and so, by (2.8.9) and (2.8.4), we have

$$\lim_{n\to\infty}(a(\varphi_n,\,\varphi_n)-\|\psi_n\|^2_{V_{h_n}})=0.$$

This relation and (2.8.4) lead to (2.8.7). ■

Lemma 2.8.3. *Assume that (2.8.4) and (2.8.5) hold. Then there exists $n_0 \in \mathbb{N}$ such that the sequence $(|\omega_n|)_{n\geq n_0}$ is uniformly bounded away from zero.*

Proof: By a contradiction argument, we show that the sequence $(\omega_n)_n$ contains no subsequence converging to zero. Namely suppose that such a subsequence exists. For the sake of simplicity, we still denote it by $(\omega_n)_n$. Hence (2.8.9) implies that

$$-A_{1h_n}z_n + \begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n \end{pmatrix} = \begin{pmatrix} -(1+h_n^\theta)^{-1}(I + h_n^\theta A_{h_n}^{\frac{l}{2}})\psi_n \\ (1+h_n^\theta)^{-1}(I + h_n^\theta A_{h_n}^{\frac{l}{2}})A_{h_n}\varphi_n + h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n \end{pmatrix} \to 0 \text{ in } X_{h_n}. \tag{2.8.10}$$

Taking the inner product of first component in (2.8.10) with ψ_n, we get

$$(1+h_n^\theta)^{-1}a\left((I + h_n^\theta A_{h_n}^{\frac{l}{2}})\psi_n,\, \psi_n\right) = (1+h_n^\theta)^{-1}\left(a(\psi_n,\, \psi_n) + h_n^\theta a^l(\psi_n,\, \psi_n)\right) \to 0.$$

As $h_n \leq h^*$, then, by (2.8.6), we get

$$\left\|A_{h_n}^{\frac{1}{2}}\psi_n\right\|^2 = a(\psi_n,\, \psi_n) \to 0. \tag{2.8.11}$$

The convergence of the first component in (2.8.10) implies that

$$\left\|A_{h_n}^{\frac{1}{2}}(I + h_n^\theta A_{h_n}^{\frac{l}{2}})\psi_n\right\|^2 \to 0.$$

Therefore, (2.8.11) yields

$$h_n^\theta A_{h_n}^{\frac{(1+l)}{2}}\psi_n \to 0 \text{ in } H. \tag{2.8.12}$$

The second component in (2.8.10) and the fact that $\alpha\|x\|^2 \leq \|A_h^{\frac{1}{2}}x\|^2 = a(x,x)$ for all $x \in V_h$ imply that

$$(1+h_n^\theta)^{-1}(I + h_n^\theta A_{h_n}^{\frac{l}{2}})A_{h_n}^{\frac{1}{2}}\varphi_n + h_n^\theta A_{h_n}^{\frac{1+l}{2}}\psi_n \to 0 \text{ in } H,$$

which, by (2.8.12), yields

$$(1+h_n^\theta)^{-1}(I+h_n^\theta A_{h_n}^{\frac{l}{2}})A_{h_n}^{\frac{1}{2}}\varphi_n \to 0 \ \text{ in } H.$$

Thus, as $h_n \leq h^*$, we get

$$a(\varphi_n,\, \varphi_n) \to 0.$$

This above relation and (2.8.11) contradict (2.8.4). ■

According to the above lemma, we note that the coefficient $1+|\omega_n|^{2l}$ becomes equivalent to $|\omega_n|^{2l}$. Now, we introduce the operator D_{1h_n} defined by

$$D_{1h_n} = \begin{pmatrix} 0 & I \\ -A_{h_n} & 0 \end{pmatrix}.$$

Note that $A_{1h_n} = (1+h_n^\theta)^{-1}(I+h_n^\theta A_{h_n}^{\frac{l}{2}})D_{1h_n}$. We then use the following spectral basis of the operator D_{1h_n}. Namely, we extend the definitions of $\lambda_{k,\,h_n}$ and of $\varphi_{k,\,h_n}$ for $k \in \{-1, ..., -N(h_n)\}$ by setting $\lambda_{k,\,h_n} = -\lambda_{-k,\,h_n}$ and $\varphi_{k,\,h_n} = \varphi_{-k,\,h_n}$. Then an orthonormal basis of X_{h_n} formed by the eigenvectors of D_{1h_n} is given by

$$\Psi_{k,\,h_n} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{i}{\lambda_{k,\,h_n}}\varphi_{k,\,h_n} \\ \varphi_{k,\,h_n} \end{pmatrix},\ 0<|k|\leq N(h_n), \tag{2.8.13}$$

of associated eigenvalue $i\lambda_{k,\,h_n}$, that is to say

$$D_{1h_n}\Psi_{k,\,h_n} = i\lambda_{k,\,h_n}\Psi_{k,\,h_n}.$$

Consequently, for all $n \in \mathbb{N}$, there exist complex coefficients $(c_k^n)_{0<|k|\leq N(h_n)}$ such that

$$z_n = \sum_{0<|k|\leq N(h_n)} c_k^n \Psi_{k,\,h_n}. \tag{2.8.14}$$

The normalization condition (2.8.4) implies that

$$\sum_{0<|k|\leq N(h_n)} |c_k^n|^2 = 1.$$

Let ϵ be the constant from Proposition 2.6.4 (if $l = 0$, we recover the condition from Proposition 2.6.1). For any $n \in \mathbb{N}$, we define

$$M_l(h_n) = \max\left\{k \in \{1, ..., N(h_n)\} \,\middle|\, h_n^\theta(\lambda_k)^2 \leq \frac{\epsilon}{\lambda_k^l}\right\}, \tag{2.8.15}$$

if $h_n^\theta(\lambda_1)^2 \leq \frac{\epsilon}{\lambda_1^l}$ and $M_l(h_n) = 0$ otherwise.

Lemma 2.8.4. *Suppose that the sequences (h_n), (ω_n), (z_n) satisfy (2.8.4) and (2.8.5). Then, we have*

$$\psi_n = \frac{1}{\sqrt{2}} \sum_{k=1}^{N(h_n)} (c_k^n + c_{-k}^n)\varphi_{k,\,h_n}, \tag{2.8.16}$$

$$\sum_{M_l(h_n)<k\leq N(h_n)} |\omega_n|^{2l} \left|c_k^n + c_{-k}^n\right|^2 \to 0, \tag{2.8.17}$$

and

$$\sum_{0<|k|\leq M_l(h_n)} |\omega_n|^{2l} \left|\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k,\,h_n} + h_n^\theta \lambda_{k,\,h_n}^{1+l})\right|^2 |c_k^n|^2 \to 0. \tag{2.8.18}$$

Proof: Relation (2.8.16) follows directly by taking the second component in (2.8.14) and by using (2.8.13) and the fact that $\varphi_{k,\,h} = \varphi_{-k,\,h}$.

From (2.8.6) and (2.8.16), it follows that

$$|\omega_n|^{2l} h_n^\theta a^l(\psi_n,\, \psi_n) = \frac{1}{2}\sum_{k=1}^{N(h_n)} h_n^\theta \lambda_{k,\,h_n}^{2+l} |\omega_n|^{2l} \left|c_k^n + c_{-k}^n\right|^2 \to 0. \tag{2.8.19}$$

As we have $\lambda_k \leq \lambda_{k,\,h_n}$ for all $k \in \{1, ..., N(h_n)\}$ and by the definition (2.8.15), we obtain (2.8.17).

On the other hand, we use (2.8.14) and the fact that $\Psi_{k,\,h_n}$ is an eigenvector of D_{1h_n} associated with eigenvalue $i\lambda_{k,\,h_n}$ to obtain for all $\tilde{\psi}_{h_n} \in X_{h_n}$

$$\begin{aligned}&\left(i\omega_n z_n - A_{1h_n} z_n,\, \tilde{\psi}_{h_n}\right)_{X_{h_n}} \\ =& \sum_{0<|k|\leq N(h_n)} i\left(\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k,\,h_n} + h_n^\theta\lambda_{k,\,h_n}^{1+l})\right) c_k^n \left(\Psi_{k,\,h_n},\, \tilde{\psi}_{h_n}\right)_{X_{h_n}}.\end{aligned} \tag{2.8.20}$$

By (2.6.9), we have

$$h_n^\theta \lambda_{k,h_n}^2 \le h_n^\theta(\lambda_k + (Ch_n^\theta \lambda_k^2)^2)^2 \le 2h_n^\theta \lambda_k^2 + 2C^4 h_n^\theta (h_n^\theta \lambda_k^2)^4 \le C\frac{\epsilon}{\lambda_k^l} + C\frac{\epsilon^4}{\lambda_k^{4l}} \le C'\frac{\epsilon}{\lambda_k^l} \tag{2.8.21}$$

for $h_n^\theta(\lambda_k)^2 \le \frac{\epsilon}{\lambda_k^l}$. So, by using (2.8.19) and again (2.6.9), there exists a constant C independent of h_n such that

$$\begin{aligned} h_n^{2\theta} \sum_{k=1}^{M_l(h_n)} \lambda_{k,h_n}^{4+2l} |\omega_n|^{2l} \left|c_k^n + c_{-k}^n\right|^2 &\le C \sum_{k=1}^{M_l(h_n)} \epsilon h_n^\theta \lambda_{k,h_n}^{2+l} |\omega_n|^{2l} \left|c_k^n + c_{-k}^n\right|^2 \\ &\le C\epsilon \sum_{k=1}^{M_l(h_n)} h_n^\theta \lambda_{k,h_n}^{2+l} |\omega_n|^{2l} \left|c_k^n + c_{-k}^n\right|^2 \to 0. \end{aligned} \tag{2.8.22}$$

We also have for all $\tilde{\psi}_{h_n} \in X_{h_n}$

$$\left(\begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix}, \tilde{\psi}_{h_n} \right)_{X_{h_n}} = \sum_{0<|k|\le N(h_n)} \frac{h_n^\theta}{2} \lambda_{k,h_n}^{2+l} (c_k^n + c_{-k}^n) \left(\Psi_{k,h_n}, \tilde{\psi}_{h_n} \right)_{X_{h_n}}. \tag{2.8.23}$$

because l is even. Relation (2.8.23) implies that for all $\tilde{\psi}_{h_n} \in X_{h_n}$

$$\begin{aligned} &\left(\begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix} - \sum_{M_l(h_n)<|k|\le N(h_n)} \tfrac{h_n^\theta}{2} \lambda_{k,h_n}^{2+l} (c_k^n + c_{-k}^n) \Psi_{k,h_n}, \tilde{\psi}_{h_n} \right)_{X_{h_n}} \\ &= \sum_{0<|k|\le M_l(h_n)} \tfrac{h_n^\theta}{2} \lambda_{k,h_n}^{2+l} (c_k^n + c_{-k}^n) \Psi_{k,h_n}, \tilde{\psi}_{h_n})_{X_{h_n}}. \end{aligned}$$

However,

$$\begin{aligned} &\left\| |\omega_n|^l \sum_{0<|k|\le M_l(h_n)} \tfrac{h_n^\theta}{2} \lambda_{k,h_n}^{2+l} (c_k^n + c_{-k}^n) \Psi_{k,h_n} \right\|^2 \\ &= |\omega_n|^{2l} \sum_{0<|k|\le M_l(h_n)} \frac{h_n^{2\theta}}{(2\sqrt{2})^2} \lambda_{k,h_n}^{4+2l} |c_k^n + c_{-k}^n|^2 + |\omega_n|^{2l} \sum_{0<|k|\le M_l(h_n)} \frac{h_n^{2\theta}}{(2\sqrt{2})^2} \lambda_{k,h_n}^{4+2l} |c_k^n + c_{-k}^n|^2 \\ &= 2|\omega_n|^{2l} \sum_{0<|k|\le M_l(h_n)} \frac{h_n^{2\theta}}{(2\sqrt{2})^2} \lambda_{k,h_n}^{4+2l} |c_k^n + c_{-k}^n|^2. \end{aligned}$$

Therefore, by (2.8.22), for all $\tilde{\psi}_{h_n} \in X_{h_n}$

$$|\omega_n|^l \left(\left(\begin{array}{c} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{array} \right) - \sum_{M_l(h_n)<|k|\leq N(h_n)} \frac{h_n^\theta}{2} \lambda_{k,\,h_n}^{2+l} (c_k^n + c_{-k}^n) \Psi_{k,\,h_n},\ \tilde{\psi}_{h_n} \right)_{X_{h_n}} \to 0.$$

So, we obtain with (2.8.9), (2.8.20) and the above relation, for all $\tilde{\psi}_{h_n} \in X_{h_n}$, that the inner product in X_{h_n} of $\tilde{\psi}_{h_n}$ with

$$\begin{array}{l} \displaystyle\sum_{0<|k|\leq N(h_n)} i\,|\omega_n|^l \left(\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k,\,h_n} + h_n^\theta \lambda_{k,\,h_n}^{1+l})\right) c_k^n \Psi_{k,\,h_n} \\ \displaystyle + \sum_{M_l(h_n)<|k|\leq N(h_n)} \tfrac{h_n^\theta}{2}\ |\omega_n|^l \lambda_{k,\,h_n}^{2+l} (c_k^n + c_{-k}^n) \Psi_{k,\,h_n} \end{array}$$

tends to zero. As the family $(\Psi_{k,\,h_n})$ is orthogonal, the above relation implies (2.8.18).

■

2.9 Proof of Theorem 2.4.1

We use the results of the previous section with $l = 0$ and set, for shortness, $\tilde{A}_h := \tilde{A}_{0,h}$ and $M(h_n) := M_0(h_n)$.

Proof of Theorem 2.4.1 This proof is based on Theorem 2.7.1. First, for all $h \in (0,\, h^*)$, the family $(e^{t\tilde{A}_h})$ forms a contraction semigroup. The family $(\tilde{A}_h)$ satisfies the condition i) in Theorem 2.7.1 owing to Lemma 2.8.1. To show that the family $(\tilde{A}_h)$ also satisfies the condition ii) in Theorem 2.7.1, we use a contradiction argument. Let $(h_n)_n$, $(\omega_n)_n$ and $z_n = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \in \mathcal{D}(\tilde{A}_{h_n})$ be three sequences satisfying (2.8.4) and (2.8.5). Notice that for $k_m \in A_k$, we have

$$\begin{array}{rcl} \lambda_{k_m,h_n} - \lambda_{k_{m-1}+l_{m-1}-1,h_n} & \geq & \lambda_{k_m} - \lambda_{k_{m-1}+l_{m-1}-1} - c\epsilon \\ & = & \lambda_{k_m} - \lambda_{k_{m-1}} - c\epsilon \geq \gamma_0' - c\epsilon \\ & \geq & \frac{\gamma_0'}{2} =: \gamma' \end{array}$$

for $\epsilon \leq \frac{\gamma_0'}{2c}$ by (2.6.8) and (2.6.9). We now introduce the set

$$\mathcal{F} = \Big\{ n \in \mathbb{N} \,\Big|\, \exists k(n) \in \{1, ..., M\},\, \exists k_{m(n)} \in A_{k(n)},\, \left|k_{m(n)}\right| \leq M(h_n) \text{ and } \left|k_{m(n)+k(n)-1} + l_{m(n)+k(n)-1}\right| \leq N(h_n) \text{ such that } \left|\omega_n - \lambda_{k_{m(n)},\, h_n}\right| < \frac{\gamma'}{2} \Big\}. \tag{2.9.1}$$

We distinguish two cases.

First case : The set $\mathcal{F}$ is infinite. Then, without loss of generality, we can suppose that $\mathcal{F} = \mathbb{N}$ (otherwise we take a subsequence of (ω_n)). Then, by reducing the value of γ' if needed, we can assume that for all $n \in \mathbb{N}$, we have that for all $k_m \in A_{k'}$, $k' = 1, ..., M$ with $m \neq m(n)$,

$$\left|\omega_n - \lambda_{k_{m+j}+l,\, h_n}\right| \geq \frac{\gamma'}{2},\ \forall j = 0, ..., k'-1,\ \forall l = 0, ..., l_{m+j} - 1.$$

By using (2.8.18), we obtain that

$$\sum_{k=1}^{M} \sum_{\substack{k_m \in A_k \\ m \neq m(n)}} \sum_{j=0}^{k-1} \sum_{\substack{l=0 \\ 0 < |k_{m+j} + l_{m+j} - 1| \leq M(h_n)}}^{l_{m+j}-1} \left|c^n_{k_{m+j}+l}\right|^2 \to 0. \tag{2.9.2}$$

Define now

$$\tilde{\psi}_n = \frac{1}{\sqrt{2}} \sum_{j=0}^{k(n)-1} \sum_{l=0}^{l_{m(n)+j}-1} c^n_{k_{m(n)+j}+l}\, \varphi_{k_{m(n)+j}+l,\, h_n}. \tag{2.9.3}$$

We have, by (2.8.16),

$$\psi_n = \frac{1}{\sqrt{2}} \sum_{k=1}^{M} \sum_{k_m \in A_k} \sum_{\substack{j=0 \\ 1 \leq k_{m+j}+l \leq N(h_n)}}^{k-1} \sum_{l=0}^{l_{m+j}-1} (c^n_{k_{m+j}+l} + c^n_{-(k_{m+j}+l)})\varphi_{k_{m+j}+l,\, h_n},$$

and so, by (2.9.2) and (2.8.17), we obtain

$$\left\|\tilde{\psi}_n - \psi_n\right\| \to 0. \tag{2.9.4}$$

Thus, since $(\|B_h^*\|_{\mathcal{L}(V_h,\,U)})_{h\in(0,\,h^*)}$ is bounded, we deduce that

$$\left\|B_{h_n}^*(\tilde{\psi}_n-\psi_n)\right\|_U \to 0. \tag{2.9.5}$$

The above relation and (2.8.6) imply that

$$\left\|B_{h_n}^*\tilde{\psi}_n\right\|_U \to 0. \tag{2.9.6}$$

But

$$\begin{aligned}\left\|B_{h_n}^*\tilde{\psi}_n\right\|_U &= \tfrac{1}{\sqrt{2}}\left\|\sum_{j=0}^{k(n)-1}\sum_{l=0}^{l_{m(n)+j}-1} c^n_{k_{m(n)+j}+l}B_{h_n}^*\varphi_{k_{m(n)+j}+l,\,h_n}\right\|_U\\ &= \tfrac{1}{\sqrt{2}}\left\|\begin{pmatrix} B_{h_n}^*\varphi_{k_{m(n)},\,h_n} & \cdots & B_{h_n}^*\varphi_{k_{m(n)+k(n)-1}+l_{m(n)+k(n)-1}-1,\,h_n}\end{pmatrix}C\right\|_U\\ &= \tfrac{1}{\sqrt{2}}\left\|\begin{pmatrix}1 & \cdots & 1\end{pmatrix}\Phi_{k_{m(n)},\,h_n}C\right\|_U,\end{aligned}$$

where $C = \begin{pmatrix} c_{k_{m(n)}} & \cdots & c_{k_{m(n)}+l_{m(n)}-1} & c_{k_{m(n)+1}} & \cdots & c_{k_{m(n)+k(n)-1}+l_{m(n)+k(n)-1}-1}\end{pmatrix}^T$.
So, we have

$$\left\|B_{h_n}^*\tilde{\psi}_n\right\|_U = \frac{1}{\sqrt{2}}\left\|\begin{pmatrix}1 & 1 & \cdots & 1\\ 0 & 0 & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & & \cdots & 0\end{pmatrix}\Phi_{k_{m(n)},\,h_n}C\right\|_{U,\,2}.$$

We now use Lemma 2.3.1 to have

$$\begin{aligned}\left\|B_{h_n}^*\tilde{\psi}_n\right\|_U &\geq c\left\|B_{k_{m(n)}}^{-1}\Phi_{k_{m(n)},\,h_n}C\right\|_{U,\,2} \text{ for } n \text{ large enough}\\ &\geq c\alpha\left\|C\right\|_2 \text{ by Proposition 2.6.1.}\end{aligned} \tag{2.9.7}$$

Gathering (2.9.3), (2.9.6) and (2.9.7), we obtain that $\tilde{\psi}_n \to 0$ in H. Therefore, by (2.9.4), $\psi_n \to 0$, which contradicts (2.8.7).

Second case : The set $\mathcal{F}$ is finite. Then, we can assume, without loss of generality, that $\mathcal{F}$ is empty (otherwise we take off the finite number of (ω_n)) ; i.e., that for all

$n \in \mathbb{N}$, we have that

$$|\omega_n - \lambda_{k,\,h_n}| \geq \frac{\gamma'}{2} \quad \text{if } 0 < |k| \leq M(h_n).$$

Thus, by (2.8.18) and the above relation, we obtain that

$$\sum_{0<|k|\leq M(h_n)} |c_k^n|^2 \to 0.$$

Therefore, by (2.8.16), (2.8.17) and the above relation, we have $\psi_n \to 0$ in H, which contradicts (2.8.7).

In conclusion, the family $(\tilde{A}_h)$ satisfies the condition ii) in Theorem 2.7.1 and so the family of systems (2.4.6) is uniformly exponentially stable. ■

2.10 Proof of Theorem 2.4.4

Here we use the results of Section 2.8 with $l > 0$ and l even. Without loss of generality, we may assume that $0 < h < h^* = 1$.

Proof of Theorem 2.4.4 and of condition (2.7.2) This proof is based on Theorem 2.7.9. First, for all $h \in (0, h^*)$, $(e^{t\tilde{A}_{l,h}})$ forms a family of contraction semigroups and the family $(\tilde{A}_{l,h})_h$ satisfies (2.7.1). To apply the results of Theorem 2.7.9, the family $(\tilde{A}_{l,h})$ must also satisfy condition i) of Theorem 2.7.9 with $\alpha = 2l$ and condition (2.7.2) or equivalently condition (2.8.3) . We again use a contradiction argument to prove this last condition. Let $(h_n)_n$, $(\omega_n)_n$ and $z_n = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \in X_h$ be three sequences satisfying (2.8.4) and (2.8.5). Notice that for $k_m \in A_k$, we have

$$\begin{aligned}
\lambda_{k_m,h} - \lambda_{k_{m-1}+l_{m-1}-1,h} &\geq \lambda_{k_m} - \lambda_{k_{m-1}+l_{m-1}-1} - \tfrac{c\epsilon}{\lambda^{2l}_{k_{m-1}}} \\
&\geq \lambda_{k_m} - \lambda_{k_{m-1}} - \tfrac{c\epsilon}{\lambda^{2l}_{k_1}} \geq \gamma_0' - \tfrac{c\epsilon}{\lambda^{2l}_{k_1}} \\
&\geq \tfrac{\gamma_0'}{2} =: \gamma'
\end{aligned}$$

for $\epsilon \leq \frac{\gamma_0' \lambda_{k_1}^{2l}}{2c}$ by (2.6.8), (2.6.9) and because $\lambda_{k_m} \geq \lambda_{k_1} > 0$. We introduce the set $\mathcal{F}_2$ like

$$\mathcal{F}_2 = \Big\{ n \in \mathbb{N} \,\Big|\, \exists k(n) \in \{1, ..., M\},\ \exists k_{m(n)} \in A_{k(n)},\ \left|k_{m(n)}\right| \leq M_l(h_n) \text{ and } \left|k_{m(n)+k(n)-1} + l_{m(n)+k(n)-1}\right| \leq N(h_n) \text{ such that } \left|\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k_{m(n)},h_n} + h_n^\theta \lambda_{k_{m(n)},h_n}^{1+l})\right| < \frac{\gamma'}{4} \Big\}. \quad (2.10.1)$$

We distinguish two cases.

First case : The set $\mathcal{F}_2$ is infinite. Then, without loss of generality, we can suppose that $\mathcal{F}_2 = \mathbb{N}$ (otherwise we take a subsequence of $(\omega_n)_n$). Then, by reducing the value of γ' if needed, we can assume that for all $n \in \mathbb{N}$, we have that for all $k_m \in A_{k'}$, $k' = 1, ..., M$ with $m \neq m(n)$, and for all $|k_{m+j} + l| \leq M_l(h_n)$

$$\left|\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k_{m+j}+l,h_n} + h_n^\theta \lambda_{k_{m+j}+l,h_n}^{1+l})\right| \geq \frac{\gamma'}{8},\ \forall j = 0, ..., k'-1,\ \forall l = 0, ..., l_{m+j}-1. \quad (2.10.2)$$

Indeed, similar to (2.8.21), we have

$$\begin{aligned} & \left|\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k_{m+j}+l,h_n} + h_n^\theta \lambda_{k_{m+j}+l,h_n}^{1+l})\right| \\ \geq\ & (1+h_n^\theta)^{-1}\left|\lambda_{k_{m+j}+l,h_n} - \lambda_{k_{m(n)},h_n}\right| - \left|\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k_{m(n)},h_n} + h_n^\theta \lambda_{k_{m(n)},h_n}^{1+l})\right| \\ & -(1+h_n^\theta)^{-1}(h_n^\theta \lambda_{k_{m(n)},h_n}^{1+l} + h_n^\theta \lambda_{k_{m+j}+l,h_n}^{1+l}) \\ \geq\ & \frac{\gamma'}{2} - \frac{\gamma'}{4} - \frac{2C\epsilon}{\lambda_{k_1}}. \end{aligned}$$

So choose again $\epsilon \leq \dfrac{\gamma' \lambda_{k_1}}{16C}$ to get (2.10.2). By using (2.8.18), we obtain that

$$\sum_{k=1}^{M} \sum_{\substack{k_m \in A_k \\ m \neq m(n)}} \sum_{j=0}^{k-1} \sum_{\substack{l=0 \\ 0 < |k_{m+j} + l_{m+j} - 1| \leq M_l(h_n)}}^{l_{m+j}-1} |\omega_n|^{2l} \left|c_{k_{m+j}+l}^n\right|^2 \to 0. \quad (2.10.3)$$

Define now

$$\tilde{\psi}_n = \frac{1}{\sqrt{2}} \sum_{j=0}^{k(n)-1} \sum_{l=0}^{l_{m(n)+j}-1} c^n_{k_{m(n)+j}+l} \varphi_{k_{m(n)+j}+l,\, h_n}. \tag{2.10.4}$$

We have, by (2.8.16),

$$\psi_n = \frac{1}{\sqrt{2}} \sum_{k=1}^{M} \sum_{k_m \in A_k} \sum_{\substack{j=0 \\ 1 \le k_{m+j}+l \le N(h_n)}}^{k-1} \sum_{l=0}^{l_{m+j}-1} (c^n_{k_{m+j}+l} + c^n_{-(k_{m+j}+l)}) \varphi_{k_{m+j}+l,\, h_n},$$

and so, by (2.10.3) and (2.8.17), we obtain

$$|\omega_n|^l \left\| \tilde{\psi}_n - \psi_n \right\| \to 0. \tag{2.10.5}$$

Thus, since $(\|B_h^*\|_{\mathcal{L}(V_h,\, U)})_{h\in(0,\, h^*)}$ is bounded, we deduce that

$$|\omega_n|^l \left\| B^*_{h_n}(\tilde{\psi}_n - \psi_n) \right\|_U \to 0. \tag{2.10.6}$$

The above relation and (2.8.6) imply that

$$|\omega_n|^l \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U \to 0. \tag{2.10.7}$$

But

$$\begin{aligned}
|\omega_n|^l \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U &= \tfrac{|\omega_n|^l}{\sqrt{2}} \left\| \sum_{j=0}^{k(n)-1} \sum_{l=0}^{l_{m(n)+j}-1} c^n_{k_{m(n)+j}+l} B^*_{h_n} \varphi_{k_{m(n)+j}+l,\, h_n} \right\|_U \\
&= \tfrac{|\omega_n|^l}{\sqrt{2}} \left\| \begin{pmatrix} B^*_{h_n}\varphi_{k_{m(n)},\, h_n} & \cdots & B^*_{h_n}\varphi_{k_{m(n)+k(n)-1}+l_{m(n)+k(n)-1}-1,\, h_n} \end{pmatrix} C \right\|_U \\
&= \tfrac{|\omega_n|^l}{\sqrt{2}} \left\| \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \Phi_{k_{m(n)},\, h_n} C \right\|_U,
\end{aligned}$$

where $C = \begin{pmatrix} c_{k_{m(n)}} & \cdots & c_{k_{m(n)}+l_{m(n)}-1} & c_{k_{m(n)+1}} & \cdots & c_{k_{m(n)+k(n)-1}+l_{m(n)+k(n)-1}-1} \end{pmatrix}^T$.
So, we have

$$|\omega_n|^l \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U = \tfrac{|\omega_n|^l}{\sqrt{2}} \left\| \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} \Phi_{k_{m(n)},\, h_n} C \right\|_{U,\,2}.$$

We now use Lemma 2.3.1 to have

$$\begin{aligned} |\omega_n|^l \left\| B^*_{h_n}\tilde{\psi}_n \right\|_U &\geq c\,|\omega_n|^l \left\| B^{-1}_{k_{m(n)}} \Phi_{k_{m(n)},\,h_n} C \right\|_{U,\,2} \text{ for } n \text{ large enough} \\ &\geq c\frac{\alpha|\omega_n|^l}{\lambda^l_{k_{m(n)}}} \|C\|_2 \text{ by Proposition 2.6.4.} \end{aligned}$$

But, ω_n verifies $\left|\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k_{m(n)},\,h_n} + h_n^\theta \lambda^{1+l}_{k_{m(n)},\,h_n})\right| < \frac{\gamma'}{4}$ by definition (2.10.1) of $\mathcal{F}_2$, thus $|\omega_n| \geq (1+h_n^\theta)^{-1}(\lambda_{k_{m(n)},\,h_n} + h_n^\theta \lambda^{1+l}_{k_{m(n)},\,h_n}) - \frac{\gamma'}{4} \geq \frac{1}{2}\lambda_{k_{m(n)},\,h_n} - \frac{\gamma'}{4}$. Therefore, we have

$$\begin{aligned} |\omega_n|^l \left\| B^*_{h_n}\tilde{\psi}_n \right\|_U &\geq \frac{c\alpha}{2^l}\frac{(\lambda_{k_{m(n)},\,h_n} - \frac{\gamma'}{2})^l}{\lambda^l_{k_{m(n)}}} \|C\|_2 \\ &\geq \frac{c\alpha}{2^{2l}}\frac{\lambda^l_{k_{m(n)},\,h_n}}{\lambda^l_{k_{m(n)}}} \|C\|_2 \text{ for } n \text{ large enough} \\ &\geq \frac{c\alpha}{2^{2l}} \|C\|_2 \text{ by (2.6.8).} \end{aligned} \tag{2.10.8}$$

Gathering (2.10.4), (2.10.7) and (2.10.8), we obtain that $\tilde{\psi}_n \to 0$ in H. Therefore, by (2.10.5), $\psi_n \to 0$, which contradicts (2.8.7).

Second case : The set $\mathcal{F}_2$ is finite. We proceed similar to the proof of the second case of Theorem 2.4.1.

In conclusion, the family $(\tilde{A}_{l,h})$ satisfies (2.8.3) ; i.e., the condition (i) in Theorem 2.7.9 with $\alpha = 2l$ when l is even and property (2.7.2) of Subsection 2.7.2.

Remark 2.10.1. *The previous analysis has been held in case $l \in \mathbb{N}^*$ is even. However, in case l is odd, we can still adapt the same analysis to get the same results. Indeed, we consider problem (2.4.9) with powers of $l+1$ instead if l. Besides, whenever $\left\|B^{-1}_{k_n}\Phi_{k_n} C\right\|_{U,\,2} \geq \dfrac{\alpha}{\lambda^l_{k_n}} \|C\|_2$ and $h^\theta \lambda_k^2 \leq \dfrac{\epsilon}{\lambda_k^{l+1}}$, then*

$$\left\|B^{-1}_{k_n}\Phi_{k_n,\,h} C\right\|_{U,\,2} \geq \frac{\alpha}{\lambda^l_{k_n}} \|C\|_2 \,,$$

and

$$\left\|(I + h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t)\right\|^2 + a(\omega_h(t),\, \omega_h(t)) \leq \frac{C}{t^{\frac{1}{l}}} \|(\omega_{0h}, \omega_{1h})\|^2_{D(\tilde{A}_{l,h})}.$$

2.11 Convergence of the discretized problem

Here we want to prove that the solution ω_h of the discrete problem (2.4.9) tends to the solution ω of the continuous problem (2.1.13) in $X := V \times H$ as h goes to zero and if the discrete initial data are well chosen. This is obtained with the help of a general version of the Trotter-Kato Theorem proved in [45] that is appropriated when the approximated semi-groups are defined in proper subspaces of the limit one. The basic idea is that the convergence of the semi-groups is equivalent to the convergence of the resolvent, hence we prove such a convergence result for the resolvents.

First, we recall the Trotter-Kato Theorem proved in [45]. Let Z and X_n be Banach spaces with norms $\|.\|$, $\|.\|_n$, $n = 1, 2, \ldots$, respectively, and X be a closed linear subspace of Z. On X a C_0-semigroup $T(.)$ with infinitesimal generator $\tilde{A}$ is given and on the spaces X_n, the C_0-semigroups $T_n(.)$ are generated by A_n. Suppose that, for every $n \in \mathbb{N}^*$, there exists bounded linear operators $P_n : \ Z \to X_n$ and $E_n : X_n \to Z$ such that the following assumptions hold :

(A1) $\|P_n\| \leq M_1$, $\|E_n\| \leq M_2$, where $M_1, \ M_2$ are independent of n,

(A2) $\|E_n P_n x - x\| \to 0$ as $n \to \infty$ for all $x \in X$,

(A3) $P_n E_n = I_n$, where I_n is the identity operator on X_n.

For all $n \in \mathbb{N}^*$, let $Z_n = \mathit{range}\ E_n$, $\pi_n = E_n P_n$, $\tilde{T}_n(t) = E_n T_n(t) P_n|_{Z_n}$, $\tilde{A}_n = E_n A_n P_n|_{Z_n}$, and $\tilde{I}_n = E_n I_n P_n|_{Z_n}$. The Trotter-Kato Theorem given in Theorem 2.1 of [45] states the following :

Theorem 2.11.1. *(Trotter-Kato). Assume that* $(A_1) - (A_3)$ *are satisfied. Then the following statements are equivalent :*

(a) There exists a $\lambda_0 \in \rho(\tilde{A}) \cap \bigcap_{n=1}^{\infty} \rho(\tilde{A}_n)$ *such that, for all* $x \in X$,

$$\left\|(\lambda_0 \tilde{I}_n - \tilde{A}_n)^{-1} \pi_n x - (\lambda_0 I - \tilde{A})^{-1} x\right\| \to 0, \quad \text{as } n \to \infty.$$

(b) For every $x \in X$ and $t \geq 0$,

$$\|\tilde{T}_n(t)\pi_n x - T(t)x\| \to 0, \quad \text{as } n \to \infty$$

uniformly on bounded $t-$intervals.

Our aim is to prove that condition (a) of Theorem 2.11.1 holds true in order to get the convergence of the solutions. Let us start with some preliminary results.

Lemma 2.11.2. *Let $l \in \mathbb{N}, l \geq 2$. If $f \in V = \mathcal{D}(A^{\frac{1}{2}})$, then*

$$\|(1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f - \pi_h f\|_H \leq Ch^{\frac{\theta}{l}}\|f\|_V, \tag{2.11.1}$$

for some $C > 0$.

Proof: We write

$$\pi_h f = \sum_{k=1}^{N(h)} f_k \varphi_{k,h},$$

with $f_k \in \mathbb{C}$. Hence

$$v_h = (1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f,$$

can be written

$$v_h = \sum_{k=1}^{N(h)} v_k \varphi_{k,h},$$

with $v_k = (1+h^\theta)(1+h^\theta\lambda_{k,h}^l)^{-1} f_k$. Consequently we have

$$\begin{aligned}
\|v_h - \pi_h f\|_H^2 &= \sum_{k=1}^{N(h)} |f_k|^2 \Big((1+h^\theta)(1+h^\theta\lambda_{k,h}^l)^{-1} - 1\Big)^2 \\
&= h^{2\theta}\sum_{k=1}^{N(h)} |f_k|^2 \Big(\frac{1-\lambda_{k,h}^l}{1+h^\theta\lambda_{k,h}^l}\Big)^2 \\
&\leq ch^{2\theta}\sum_{k=1}^{N(h)} |f_k|^2 \frac{\lambda_{k,h}^{2l}}{(1+h^\theta\lambda_{k,h}^l)^2} \\
&\leq ch^{2\theta}\sum_{k=1}^{N(h)} \lambda_{k,h}^2 |f_k|^2 (g(\lambda_{k,h}))^2
\end{aligned}$$

for some $c > 0$ independent of h, where the function $g : [0, \infty) \mapsto \mathbb{R}$ is given by $g(\lambda) = \dfrac{\lambda^{l-1}}{(1 + h^\theta \lambda^l)}$. As the maximum of g is attained at $\lambda_0 > 0$ given by

$$h^\theta \lambda_0^l = l - 1,$$

we get that

$$\|v_h - \pi_h f\|_H^2 \le c c_2^2 h^{\frac{2\theta}{l}} \sum_{k=1}^{N(h)} |f_k|^2 \lambda_{k,h}^2$$

since $\lambda_0 = c_1 h^{-\frac{\theta}{l}}$ and $g(\lambda_0) = c_2 h^{-\frac{\theta(l-1)}{l}}$ with c_1, c_2 two positive constants independent of h. This proves the first estimate since

$$\sum_{k=1}^{N(h)} |f_k|^2 \lambda_{k,h}^2 = \|A_h^{\frac{1}{2}} \pi_h f\|_H^2 = a(\pi_h f, \pi_h f) \le a(f, f) = \|A^{\frac{1}{2}} f\|_H^2.$$

■

Corollary 2.11.3. *Let $l \in \mathbb{N}, l \ge 2$, then for any $f_h \in V_h$ we have*

$$\|(1 + h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1} f_h - f_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} \le C h^{\frac{\theta}{l}} \|f_h\|_H, \tag{2.11.2}$$

for some $C > 0$.

Proof: As in the previous lemma, we have

$$\begin{aligned}
\|(1 + h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1} f_h - f_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})}^2 &= \|A_h^{-\frac{1}{2}} \Big((1 + h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1} f_h - f_h\Big)\|_H^2 \\
&= h^{2\theta} \sum_{k=1}^{N(h)} \lambda_{k,h}^{-2} |f_k|^2 \Big(\frac{1 - \lambda_{k,h}^l}{1 + h^\theta \lambda_{k,h}^l}\Big)^2 \\
&\le c h^{2\theta} \sum_{k=1}^{N(h)} |f_k|^2 (g(\lambda_{k,h}))^2,
\end{aligned}$$

when

$$f_h = \sum_{k=1}^{N(h)} f_k \varphi_{k,h}.$$

We then conclude as before. ■

Lemma 2.11.4. *Let $l \in \mathbb{N}, l \geq 2$ and let $f \in \mathcal{D}(A)$, then*

$$h^{\theta}\|A_h^{1+\frac{l}{2}}(I + h^{\theta}A_h^{\frac{l}{2}})^{-2}\pi_h f\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} \leq Ch^{\frac{\theta}{l}}\|f\|_{\mathcal{D}(A)}, \tag{2.11.3}$$

for some $C > 0$.

Proof: We easily see that

$$\begin{aligned} h^{2\theta}\|A_h^{1+\frac{l}{2}}(I + h^{\theta}A_h^{\frac{l}{2}})^{-2}\pi_h f\|^2_{\mathcal{D}(A_h^{-\frac{1}{2}})} &= h^{2\theta}\|A_h^{-\frac{1}{2}}A_h^{1+\frac{l}{2}}(I + h^{\theta}A_h^{\frac{l}{2}})^{-2}\pi_h f\|^2_H \\ &= h^{2\theta}\sum_{k=1}^{N(h)}|f_k|^2\frac{\lambda_{k,h}^{2l+2}}{(1 + h^{\theta}\lambda_{k,h}^{l})^4} \\ &\leq h^{2\theta}\sum_{k=1}^{N(h)}|f_k|^2\lambda_{k,h}^4(g(\lambda_{k,h}))^2, \end{aligned}$$

and we conclude as before. ■

Lemma 2.11.5. *Let $l \in \mathbb{N}, l \geq 2$ and let $f \in V$, then*

$$\|(1+h^{\theta})(I+h^{\theta}A_h^{\frac{l}{2}})^{-1}B_hB_h^*(1+h^{\theta})(I+h^{\theta}A_h^{\frac{l}{2}})^{-1}\pi_h f - B_hB_h^*\pi_h f\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} \leq Ch^{\frac{\theta}{l}}\|f\|_V, \tag{2.11.4}$$

for some $C > 0$.

Proof: As in Lemma 2.11.2, we set

$$v_h = (1 + h^{\theta})(I + h^{\theta}A_h^{\frac{l}{2}})^{-1}\pi_h f.$$

First, we notice that

$$\|B_hB_h^*(v_h - \pi_h f)\|_H \leq C\|v_h - \pi_h f\|_H,$$

and by Lemma 2.11.2 we get

$$\|B_hB_h^*(v_h - \pi_h f)\|_H \leq Ch^{\frac{\theta}{l}}\|f\|_V.$$

Second, by Corollary 2.11.3, we have

$$\begin{aligned}\|(1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}B_hB_h^*v_h - B_hB_h^*v_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} &\le Ch^{\frac{\theta}{l}}\|B_hB_h^*v_h\|_H \\ &\le Ch^{\frac{\theta}{l}}(\|B_hB_h^*(v_h-\pi_hf)\|_H + \|B_hB_h^*\pi_hf\|_H) \\ &\le Ch^{\frac{\theta}{l}}\|f\|_V,\end{aligned}$$

where we use the fact that $\|\pi_hf\|_H \le c\|\pi_hf\|_V \le c\|f\|_V$. The conclusion follows from the two above estimates. ■

Theorem 2.11.6. *If $z=(f,g)^\top \in \mathcal{D}(A)\times\mathcal{D}(A)$, then*

$$\|(\tilde{A}_{l,h})^{-1}(\pi_hf,\pi_hg)^\top - \tilde{A}^{-1}(f,g)^\top\|_X \to 0 \text{ as } h\to 0.$$

Proof: By the definition of $\tilde{A}_{l,h}$ and $\tilde{A}$, we have

$$(u_h,v_h)^\top = (\tilde{A}_{l,h})^{-1}(\pi_hf,\pi_hg)^\top,$$

and

$$(u,v)^\top = \tilde{A}^{-1}(f,g)^\top,$$

if and only if

$$\begin{cases} v_h &= (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\pi_hf \\ -A_hu_h &= (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}(h^\theta A_h^{1+\frac{l}{2}} + B_hB_h^*)v_h + (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\pi_hg,\end{cases}$$

and

$$\begin{cases} v=f \\ -Au = BB^*v+g.\end{cases}$$

Therefore, we can write

$$-A_hu_h = \pi_hg + B_hB_h^*\pi_hf + r_h,$$

where $r_h \in V_h$ is given by

$$\begin{aligned} r_h &= (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h g - \pi_h g \\ &+ (1+h^\theta)h^\theta(I+h^\theta A_h^{\frac{l}{2}})^{-1}A_h^{1+\frac{l}{2}}v_h \\ &+ (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}B_hB_h^* v_h - B_hB_h^*\pi_h f. \end{aligned}$$

By the previous Lemmas, r_h satisfies

$$\|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} \le Ch^{\frac{\theta}{l}}\|(f,g)^\top\|_{\mathcal{D}(A)\times V}. \tag{2.11.5}$$

Therefore, $u_h \in V_h$ can be seen as the unique solution of

$$a(u_h,w_h) = -(\pi_h g, w_h) - (B_hB_h^*\pi_h f, w_h) - \langle r_h; w_h\rangle \quad \forall w_h \in V_h, \tag{2.11.6}$$

where $\langle ;\rangle$ denotes the dual product in $D(A_h^{-\frac{1}{2}})$. Since $u \in V$ is solution of

$$a(u,w) = -(g,w) - (BB^*f, w) \quad \forall w \in V,$$

we get (recalling that $V_h \subset V$)

$$a(u,w_h) = -(g,w_h) - (BB^*f, w_h) \quad \forall w_h \in V_h.$$

Hence, taking the difference of this identity with (2.11.6), we obtain

$$a(u-u_h,w_h) = (\pi_h g - g, w_h) + (B^*(\pi_h f - f), B^* w_h)_U + \langle r_h; w_h\rangle \quad \forall w_h \in V_h.$$

Consequently, taking $w_h = \pi_h u - u_h$, we get

$$\begin{aligned} a(u-u_h,u-u_h) &= a(u-u_h,u-\pi_h u) + a(u-u_h,\pi_h u - u_h) \\ &= a(u-u_h,u-\pi_h u) + (\pi_h g - g, \pi_h u - u_h) \\ &+ (B^*(\pi_h f - f), B^*(\pi_h u - u_h))_U + \langle r_h; \pi_h u - u_h\rangle. \end{aligned}$$

Hence, by Cauchy-Schwarz's inequality and the boundedness of B^*, we obtain

$$\begin{aligned}\|u-u_h\|_V^2 &= a(u-u_h,u-u_h)\\ &\leq \|u-u_h\|_V\|u-\pi_h u\|_V + C(\|\pi_h g-g\|_H+\|\pi_h f-f\|_H+\|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})})\|\pi_h u-u_h\|_V.\end{aligned}$$

Now, using the triangle inequality, we get

$$\begin{aligned}\|u-u_h\|_V^2 &\leq C\Big((\|u-\pi_h u\|_V+\|\pi_h g-g\|_H+\|\pi_h f-f\|_H+\|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})})\|u-u_h\|_V\\ &+ (\|\pi_h g-g\|_H+\|\pi_h f-f\|_H+\|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})})\|u-\pi_h u\|_V\Big).\end{aligned}$$

Hence, by Young's inequality, we arrive at

$$\begin{aligned}\|u-u_h\|_V^2 &\leq C\Big(\|u-\pi_h u\|_V^2+\|\pi_h g-g\|_H^2+\|\pi_h f-f\|_H^2+\|r_h\|^2_{\mathcal{D}(A_h^{-\frac{1}{2}})}\\ &+ (\|\pi_h g-g\|_H+\|\pi_h f-f\|_H+\|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})})\|u-\pi_h u\|_V\Big).\end{aligned}$$

The estimates (2.4.4), (2.4.5), and (2.11.5) then yield

$$\begin{aligned}\|u-u_h\|_V^2 &\leq C\Big(h^{2\theta}\|u\|^2_{\mathcal{D}(A)}+h^{4\theta}\|f\|^2_{\mathcal{D}(A)}+h^{4\theta}\|g\|^2_{\mathcal{D}(A)}+h^{\frac{2\theta}{l}}\|(f,g)^\top\|^2_{\mathcal{D}(A)\times V} \qquad (2.11.7)\\ &+ (h^{2\theta}\|f\|_{\mathcal{D}(A)}+h^{2\theta}\|g\|_{\mathcal{D}(A)}+h^{\frac{\theta}{l}}\|(f,g)^\top\|_{\mathcal{D}(A)\times V})h^\theta\|u\|_{\mathcal{D}(A)}\Big).\end{aligned}$$

For $v-v_h$, we notice that

$$v-v_h = f-(1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f = f-\pi_h f+\pi_h f-(1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f,$$

and we conclude that it tends to zero in H due to the estimate (2.4.4) and Lemma 2.11.2. ∎

Corollary 2.11.7. *If $z=(f,g)^\top\in V\times H$, recalling that j_h is the projection from H into V_h, we have*

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h f, j_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X \to 0 \text{ as } h\to 0.$$

Proof: First for $z=(f,g)^\top \in \mathcal{D}(A)\times\mathcal{D}(A)$, then

$$\begin{aligned}\|(\tilde{A}_{l,h})^{-1}(\pi_h f, j_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X &\leq \|(\tilde{A}_{l,h})^{-1}(\pi_h f, \pi_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X \\ &+\|(\tilde{A}_{l,h})^{-1}(0, j_h g-\pi_h g)^\top\|_X.\end{aligned}$$

The first term of this right-hand side tends to zero as h goes to zero by the previous Theorem. On the other hand for the second term, as $\tilde{A}_{l,h}$ satisfies (2.7.2) (see Section 2.10), there exists $C>0$ (independent of h) such that for all $h<h^*$

$$\|(\tilde{A}_{l,h})^{-1}(0, j_h g-\pi_h g)^\top\|_X \leq C\|j_h g-\pi_h g\|_H.$$

Hence, by the triangle inequality and the property $\|g-j_h g\|_H \leq \|g-\pi_h g\|_H$ (as j_h in the projection on V_h in H), we get

$$\|(\tilde{A}_{l,h})^{-1}(0, j_h g-\pi_h g)^\top\|_X \leq 2C\|g-\pi_h g\|_H.$$

By the estimate (2.4.5), we then conclude that this second term tends also to zero as h goes to zero.

If $z=(f,g)^\top$ is only in $V\times H$, then for an arbitrary $\varepsilon>0$, we use the density of $\mathcal{D}(A)\times\mathcal{D}(A)$ into $V\times H$ to get $(F,G)^\top\in\mathcal{D}(A)\times\mathcal{D}(A)$ such that

$$\|(f,g)^\top-(F,G)^\top\|_X\leq\varepsilon.$$

Now, by the triangle inequality, we have

$$\begin{aligned}\|(\tilde{A}_{l,h})^{-1}(\pi_h f, j_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X &\leq \|(\tilde{A}_{l,h})^{-1}(\pi_h(f-F), j_h(g-G))^\top\|_X \\ &+ \|\tilde{A}^{-1}(f-F,g-G)^\top\|_X \\ &+ \|(\tilde{A}_{l,h})^{-1}(\pi_h F, j_h G)^\top - \tilde{A}^{-1}(F,G)^\top\|_X.\end{aligned}$$

By the first step, there exists h_ε small enough such that

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h F, j_h G)^\top - \tilde{A}^{-1}(F,G)^\top\|_X\leq\varepsilon, \forall 0<h<h_\varepsilon.$$

For the second term, by the boundedness of $\tilde{A}^{-1}$, we may write

$$\|\tilde{A}^{-1}(f-F,g-G)^\top\|_X \leq C\|(f-F,g-G)^\top\|_X \leq C\varepsilon.$$

Finally for the first term, using the property (2.7.2) and the fact that π_h (resp. j_h) is a projection from V (resp. from H) into V_h, we get for all $h < h^*$

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h(f-F),j_h(g-G))^\top\|_X \leq C\|(\pi_h(f-F),j_h(g-G))^\top\|_X \leq C\|(f-F,g-G)^\top\|_X \leq C\varepsilon.$$

All together we have obtained that

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h f, j_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X \leq (1+2C)\varepsilon, \quad \forall 0 < h < \min\{h_\varepsilon, h^*\}.$$

This proves the result. ■

We are now ready to state the convergence result.

Theorem 2.11.8. *If* $(\omega_0,\omega_1)^\top \in V \times H$, *then*

$$\|T_{l,h}(t)(\pi_h\omega_0, j_h\omega_1)^\top - T(t)(\omega_0,\omega_1)^\top\|_X \to 0 \text{ as } h \to 0. \tag{2.11.8}$$

Proof: We use Theorem 2.1 of [45] or Theorem 2.11.1 with $X = Z = V \times H$, $X_n = V_h \times V_h$, and $P_n : X \to X_n$ defined by

$$P_n(f,g)^\top = (\pi_h f, j_h g)^\top, \forall (f,g)^\top \in X,$$

and $E_n = P_n^*$ that here is the canonical injection of $V_h \times V_h$ into $V \times H$. The assumptions (A1) and (A3) of [45] are trivially satisfied, while the assumption (A2) is a consequence of (2.4.4), (2.4.5) and the density of $\mathcal{D}(A) \times \mathcal{D}(A)$ into $V \times H$.

Since Corollary 2.11.7 shows that condition (a) of Theorem 2.11.1 holds with $\lambda_0 = 0 \in \rho(\tilde{A}) \cap \bigcap_h \rho(\tilde{A}_{l,h})$, we conclude that condition (b) of this Theorem, namely (2.11.8), holds.

■

Remark 2.11.9. *In case $l = 0$, we can still apply the Trotter-Kato Theorem to get the convergence of the discrete problem (2.4.6) towards the continuous one (2.1.13). Indeed, similar to Lemma 2.11.4, we have for $f \in D(A)$, $h^\theta \|A_h \pi_h f\|_H \leq h^\theta \|f\|_{D(A)}$. Moreover, in the proof of Theorem 2.11.6, we get $r_h = h^\theta A_h \pi_h f$, $\|u - u_h\|_V \leq ch^\theta \|(f,g)\|_{D(A)\times D(A)}$, and $\|v - v_h\|_H \leq ch^{2\theta} \|(f,g)\|_{D(A)\times D(A)}$.*

2.12 Examples

2.12.1 Two coupled wave equations

We consider the following system of [3] given by

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha y(x,t) + \beta(x) u_t(x,t) = 0 & \text{in } (0,1)\times\mathbb{R}_+, \\ y_{tt}(x,t) - y_{xx}(x,t) + \alpha u(x,t) + \gamma(x) y_t(x,t) = 0 & \text{in } (0,1)\times\mathbb{R}_+, \\ u(0,t) = u(1,t) = y(0,t) = y(1,t) = 0 & \forall t > 0, \\ u(\cdot,0) = u_0, u_t(\cdot,0) = u_1, y(\cdot,0) = y_0, y_t(\cdot,0) = y_1 & \text{in } (0,1), \end{cases} \tag{2.12.1}$$

when $\alpha \in \mathbb{R}$ such that $\alpha > 0$ is small enough (see below), β and γ are two non-negative bounded functions such that $\beta(x) \geq \beta > 0$ for $x \in I_\beta \subseteq (0,1)$ and $\gamma(x) \geq \gamma > 0$ for $x \in I_\gamma \subseteq (0,1)$ where I_β and I_γ are two open sets such that their measures do not vanish simultaneously. Hence, (2.12.1) is written in the form (2.1.13) with the following choices : Take $H = L^2(0,1)^2$, the operator B as follows :

$$B\omega = \sqrt{\beta(.)}\begin{pmatrix} u \\ 0 \end{pmatrix} + \sqrt{\gamma(.)}\begin{pmatrix} 0 \\ y \end{pmatrix}, \tag{2.12.2}$$

when $\omega = \begin{pmatrix} u \\ y \end{pmatrix}$, which is a bounded operator from H into itself (i.e. $U = H$) and the operator A defined by

$$\mathcal{D}(A) = V \cap H^2(0,1)^2,$$

when $V = H_0^1(0,1)^2$ and

$$A\omega = \begin{pmatrix} -u_{xx} + \alpha y \\ -y_{xx} + \alpha u \end{pmatrix}.$$

If α is small enough, namely if $\alpha < \pi^2$, this operator A is a positive selfadjoint operator in H, since it is the Friedrichs extension of the triple (H, V, a), where the sesquilinear form a is defined by

$$a(\omega, \omega^*) = \int_0^1 (u_x(\overline{u^*})_x + y_x(\overline{y^*})_x + \alpha y\overline{u^*} + \alpha u\overline{y^*})\, dx, \forall \omega = \begin{pmatrix} u \\ y \end{pmatrix}, \omega^* = \begin{pmatrix} u^* \\ y^* \end{pmatrix} \in V.$$

Indeed a is clearly a continuous symmetric sesquilinear form on V and is coercive if $\alpha < \pi^2$ due to Poincaré's inequality

$$\int_0^1 |u_x|^2\, dx \geq \pi^2 \int_0^1 |u|^2\, dx, \quad \forall u \in H_0^1(0,1).$$

Furthermore, A has a compact resolvent since $\mathcal{D}(A)$ is compactly embedded into H. Let us now check that the generalized gap condition (2.3.1) and the assumptions (2.4.7) or (2.4.8) are satisfied for our system (2.12.1). We start by the determination of the spectrum of the operator A. Hence we are looking for $\omega = (u, y)^\top \in V \cap H^2(0,1)^2$ different from 0 and $\lambda^2 > 0$ solution of

$$-u_{xx} + \alpha y = \lambda^2 u \text{ in } (0,1),$$
$$-y_{xx} + \alpha u = \lambda^2 y \text{ in } (0,1).$$

If such a pair exists, we can set

$$s = \frac{u+y}{2}, \quad d = \frac{u-y}{2},$$

and notice that s and d belong to $H_0^1(0,1) \cap H^2(0,1)$ and are solution of

$$-s_{xx} + \alpha s = \lambda^2 s \text{ in } (0,1),$$
$$-d_{xx} - \alpha d = \lambda^2 d \text{ in } (0,1).$$

Hence s (resp. d) is an eigenvector of the Laplace operator $-\frac{d}{dx^2}$ with Dirichlet boundary condition of eigenvalue $\lambda^2-\alpha$ (resp. $\lambda^2+\alpha$). A first choice is then to have for all $k \in \mathbb{N}^*$: $\lambda^2 = k^2\pi^2+\alpha$, $s = \sin(k\pi\cdot)$ and $d = 0$. Coming back to (u, y), we find (since $u = s + d$ and $y = s - d$) a sequence of eigenvalues $\lambda_{+,k}^2 = k^2\pi^2 + \alpha$ of associated eigenvector

$$\omega_{+,k} = (\sin(k\pi\cdot), \sin(k\pi\cdot)).$$

Note that each eigenvalue is simple and that $\omega_{+,k}$ is of norm 1 in H.

A second choice is to take for all $k \in \mathbb{N}^*$: $\lambda^2 = k^2\pi^2 - \alpha$ (which is meaningful since $\alpha < \pi^2$), $s = 0$ and $d = \sin(k\pi\cdot)$. Again coming back to (u, y), we find a sequence of eigenvalues $\lambda_{-,k}^2 = k^2\pi^2 - \alpha$ of associated eigenvector

$$\omega_{-,k} = (\sin(k\pi\cdot), -\sin(k\pi\cdot)).$$

As before each eigenvalue is simple and $\omega_{-,k}$ is of norm 1 in H.

Now we remark that the sequence $\{\omega_{+,k}\}_{k\in\mathbb{N}^*} \cup \{\omega_{-,k}\}_{k\in\mathbb{N}^*}$ is an orthonormal basis of H (because $\omega_{+,k} + \omega_{-,k} = 2(\sin(k\pi\cdot), 0)$ and $\omega_{+,k} - \omega_{-,k} = 2(0, \sin(k\pi\cdot)))$ and therefore we have found all possible eigenvectors of A. We have then shown that the spectrum of A is given by

$$\mathrm{Sp}(A) = \{\lambda_{+,k}^2\}_{k\in\mathbb{N}^*} \cup \{\lambda_{-,k}^2\}_{k\in\mathbb{N}^*},$$

and that each eigenvalue is simple (because the assumption $\alpha < \pi^2$ implies that $k^2\pi^2 + \alpha < (k+1)^2\pi^2 - \alpha$).

We now need to estimate the distance between the consecutive eigenvalues of $A^{1/2}$. We have two different cases to consider :

1. For all $k \in \mathbb{N}^*$, we need to look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k}$. Since

$$\lambda_{+,k} - \lambda_{-,k} = \sqrt{k^2\pi^2+\alpha} - \sqrt{k^2\pi^2-\alpha} = \frac{2\alpha}{\sqrt{k^2\pi^2+\alpha} + \sqrt{k^2\pi^2-\alpha}},$$

we see that this distance goes to zero as k goes to infinity.

2. For all $k \in \mathbb{N}^*$, we look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k+1}$. Here we have

$$\lambda_{-,k+1} - \lambda_{+,k} = \sqrt{(k+1)^2\pi^2 - \alpha} - \sqrt{k^2\pi^2 + \alpha} = \frac{2k\pi^2 + \pi^2 - 2\alpha}{\sqrt{(k+1)^2\pi^2 - \alpha} + \sqrt{k^2\pi^2 + \alpha}},$$

which tends to π as k goes to infinity.

This shows that the generalized gap condition (2.3.1) is satisfied with $M = 2$. Hence, we see that $A_1 = \emptyset$ and $A_2 = \mathbb{N}^*$.

In order to check (2.4.7) or (2.4.8), for all $k \in \mathbb{N}^*$, we set

$$\alpha_k = \lambda_{+,k} - \lambda_{-,k},$$

that behaves like k^{-1} or equivalently like $\lambda_{-,k}^{-1}$. We further need to use the matrix (see Lemma 2.3.1)

$$B_k^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & \alpha_k \end{pmatrix},$$

as well as the matrix Φ_k which here takes the form

$$\Phi_k = \begin{pmatrix} B^*\omega_{-,k} & 0 \\ 0 & B^*\omega_{+,k} \end{pmatrix}.$$

Hence for all $C = (c_1, c_2)^\top \in \mathbb{R}^2$, we have

$$B_k^{-1}\Phi_k C = \begin{pmatrix} c_1 B^*\omega_{-,k} + c_2 B^*\omega_{+,k} \\ \alpha_k c_2 B^*\omega_{+,k} \end{pmatrix},$$

and consequently

$$\begin{aligned} \|B_k^{-1}\Phi_k C\|_{U,2}^2 &= \|c_1 B^*\omega_{-,k} + c_2 B^*\omega_{+,k}\|_2^2 + |\alpha_k|^2|c_2|^2\|B^*\omega_{+,k}\|_2^2 \\ &= |c_1 + c_2|^2 \int_0^1 \beta(x)\sin^2(k\pi x)dx + |c_2 - c_1|^2 \int_0^1 \gamma(x)\sin^2(k\pi x)dx \\ &\quad + |\alpha_k|^2|c_2|^2 \int_0^1 (\beta(x) + \gamma(x))\sin^2(k\pi x)dx. \end{aligned}$$

We have two different cases to consider :

First case : $I_\beta \neq \emptyset$ and $I_\gamma \neq \emptyset$.

In this case, we have

$$\begin{aligned}
&\|B_k^{-1}\Phi_k C\|_{U,2}^2 \\
&\geq \min\{\beta,\gamma\} \min\left\{\int_{I_\beta} \sin^2(k\pi x)dx, \int_{I_\gamma} \sin^2(k\pi x)dx\right\} ((c_1+c_2)^2 + (c_2-c_1)^2) \\
&= 2\min\{\beta,\gamma\} \min\left\{\int_{I_\beta} \sin^2(k\pi x)dx, \int_{I_\gamma} \sin^2(k\pi x)dx\right\} (c_1^2+c_2^2)
\end{aligned}$$

and hence (2.4.7) holds since $\min\left\{\int_{I_\beta} \sin^2(k\pi x)dx, \int_{I_\gamma} \sin^2(k\pi x)dx\right\}$ is uniformly bounded from below. Indeed, as $I_\gamma \neq \emptyset$, there exists $a \in (0,1)$ and $\epsilon > 0$ such that $(a, a+\epsilon) \subset I_\gamma$, and therefore

$$\int_{I_\gamma} \sin^2(k\pi x)dx \geq \frac{\epsilon}{2} + \frac{1}{4k\pi}\left(\sin(2k\pi a) - \sin(2k\pi(a+\epsilon))\right) \geq \frac{\epsilon}{2} - \frac{1}{2k\pi} \geq \frac{\epsilon}{4},$$

for $k \geq \dfrac{2}{\epsilon\pi}$. On the other hand, we clearly have

$$\min_{1\leq k< \frac{2}{\epsilon\pi}} \int_{I_\gamma} \sin^2(k\pi x)dx > 0,$$

which shows that $\int_{I_\gamma} \sin^2(k\pi x)dx$ is uniformly bounded from below.

Second case : $I_\beta = \emptyset$ or $I_\gamma = \emptyset$ (but not empty together). For instance, suppose that $I_\beta = \emptyset$ and $I_\gamma \neq \emptyset$.

As $|\alpha_k| \sim \lambda_{-,k}^{-1}$, we deduce that

$$\|B_k^{-1}\Phi_k C\|_{U,2} \geq \alpha_0 \lambda_{-,k}^{-1} \|C\|_2,$$

for a positive constant α_0, and shows that (2.4.8) holds with $l = 1$.

As stated before, in the first case the system (2.12.1) is exponentially stable, while in the second case (2.12.1) is polynomially stable. We refer to Theorem 2.4 of [7] or to [5, 61] for the proof of these results.

As approximated space V_h, we use the standard one based on $P1$ finite elements. More precisely, for $N \in \mathbb{N}$ and $h = \frac{1}{N+1}$, we define the points $x_j = jh$, $j = 0, 1, ..., N+1$. The space V_h is the linear span of the family of hat functions $(e_i, e_j)_{i,j\in\{1,\dots,N\}}$ such that

$$e_j(x) = \left[1 - \frac{|x - x_j|}{h}\right]^+, \text{ for } j = 1, ..., N.$$

Then, we define the operators A_h and B_h by (2.4.1) and (2.4.3). It is well-known (see [22]) that the operator A and the space V_h satisfy conditions (2.4.4) and (2.4.5) with $\theta = 1$.

Consequently, in the first case ($I_\beta \neq \emptyset$ and $I_\gamma \neq \emptyset$), we can apply Theorem 2.4.1 and thus the family of systems (2.4.6) is uniformly exponentially stable, in the sense that there exist constants M, α, $h^* > 0$ (independent of h, u_{0h}, u_{1h}, y_{0h}, y_{1h}) such that for all $h \in (0,\, h^*)$:

$$\|\dot{\omega}_h(t)\|^2 + a(\omega_h(t),\, \omega_h(t)) \leq Me^{-\alpha t}(\|\omega_{1h}\|^2 + a(\omega_{0h},\, \omega_{0h})),\ \forall t \geq 0,$$

where $\omega_h = (u_h,\, y_h)$, and $\omega_{0h} = (u_{0h}, y_{0h}) \in V_h$ (resp. $\omega_{1h} = (u_{1h}, y_{1h}) \in V_h$) is an approximation of $\omega_0 = (u_0, y_0)$ (resp. $\omega_1 = (u_1, y_1)$).

In the second case ($I_\beta = \emptyset$ and $I_\gamma \neq \emptyset$), we can apply Theorem 2.4.4 and Remark 2.10.1 with $l = 1$ and thus the family of systems (2.4.9) is uniformly polynomially stable, in the sense that, there exist constants C, $h^* > 0$ (independent of h, u_{0h}, u_{1h}, y_{0h}, y_{1h}) such that for all $h \in (0,\, h^*)$:

$$\left\|(I + hA_h)^{-1}\dot{\omega}_h(t)\right\|^2 + a(\omega_h(t),\, \omega_h(t)) \leq \frac{C}{t}\|(\omega_{0h}, \omega_{1h})\|^2_{D(\tilde{A}_{1,h})}\ \forall t > 0, \qquad (2.12.3)$$

where $\tilde{A}_{1,h}$ is given as in (2.5.1) with $l = 1$, $\theta = 1$, and the the graph norm $\|\cdot\|_{D(\tilde{A}_{1,h})}$

is defined by

$$\|(\omega_{0h},\omega_{1h})\|^2_{D(\tilde{A}_{1,h})} = \|(\omega_{0h},\omega_{1h})\|^2_{X_h} + \|\tilde{A}_{1,h}(\omega_{0h},\omega_{1h})\|^2_{X_h}.$$

2.12.2 Two boundary coupled wave equations

We consider the following system

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } (0,1)\times\mathbb{R}_+, \\ y_{tt} - y_{xx} + \beta y_t = 0 & \text{in } (0,1)\times\mathbb{R}_+, \\ u(0,t) = y(0,t) = 0 & \forall t > 0, \\ y_x(1,t) = \alpha u(1,t) & \forall t > 0, \\ u_x(1,t) = \alpha y(1,t) & \forall t > 0, \\ u(\cdot,0) = 0, u_t(\cdot,0) = u_1, y(\cdot,0) = 0, y_t(\cdot,0) = y_1 & \text{in } (0,1), \end{cases} \tag{2.12.4}$$

when $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$ and $\alpha > 0$ small enough (see below). Hence it is written in the form (2.1.13) with the following choices : Take $H = L^2(0,1)^2$, the operator B as follows :

$$B\omega = \sqrt{\beta}\begin{pmatrix} 0 \\ y \end{pmatrix},$$

when $\omega = \begin{pmatrix} u \\ y \end{pmatrix}$, which is a bounded operator from H into itself (i.e. $U = H$) and the operator A defined by

$$\mathcal{D}(A) = \left\{(u,y) \in V \cap H^2(0,1)^2 : \, y_x(1) = \alpha u(1);\, u_x(1) = \alpha y(1)\right\}$$

when $V = \{\omega \in H^1(0,1)^2 : \, \omega(0) = 0\}$ and

$$A\omega = \begin{pmatrix} -u_{xx} \\ -y_{xx} \end{pmatrix}.$$

If α is small enough, namely if $\alpha < 1$, this operator A is a positive selfadjoint operator in H, since it is the Friedrichs extension of the triple (H, V, a), where the sesquilinear form a is defined by

$$a(\omega, \omega^*) \ = \int_0^1 (u_x(\overline{u^*})_x + y_x(\overline{y^*})_x)\, dx - \alpha u(1)\overline{y^*}(1) - \alpha\overline{u^*}(1)y(1),$$

for all $\omega = \begin{pmatrix} u \\ y \end{pmatrix}, \omega^* = \begin{pmatrix} u^* \\ y^* \end{pmatrix} \in V$. Indeed a is clearly a continuous symmetric sesquilinear form on V and is coercive if $\alpha < 1$ due to the trace theorem

$$u(1)^2 \leq \int_0^1 |u_x|^2\, dx, \quad \forall u \in V.$$

In addition to that, the operator A admits a compact resolvent as $\mathcal{D}(A)$ is compactly embedded in H.

Let us now check that the generalized gap condition (2.3.1) and the assumption (2.4.8) are satisfied for our system (2.12.4). We start by the determination of the spectrum of the operator A. Hence we are looking for $\omega = (u, y)^\top \in \mathcal{D}(A)$ different from 0 and $\lambda^2 > 0$ solution of

$$-u_{xx} = \lambda^2 u \text{ in } (0,1),$$
$$-y_{xx} = \lambda^2 y \text{ in } (0,1).$$

Then

$$u(x) = a\sin(\lambda x) \text{ in } (0,1),$$
$$y(x) = b\sin(\lambda x) \text{ in } (0,1).$$

The coupling condition in (2.12.4) gives

$$\begin{cases} a\lambda\cos\lambda = \alpha b\sin\lambda \\ b\lambda\cos\lambda = \alpha a\sin\lambda. \end{cases}$$

Since it is not possible to have $\sin\lambda = 0$ (otherwise $a = b = 0$), we obtain

$$a = \frac{b\lambda\cos\lambda}{\alpha\sin\lambda}, \tag{2.12.5}$$

and then

$$\tan\lambda = \pm\frac{\lambda}{\alpha}, \tag{2.12.6}$$

because $b \neq 0$ (otherwise $u = y = 0$).

We then have two sequences of eigenvalues defined by

$$\lambda_{-,k} = \frac{\pi}{2} + k\pi - \epsilon_{-,k}$$

with $\lim_{k\to+\infty} \epsilon_{-,k} = 0$ and $\epsilon_{-,k} > 0$ for all $k \in \mathbb{N}$, and

$$\lambda_{+,k} = \frac{\pi}{2} + k\pi + \epsilon_{+,k}$$

with $\lim_{k\to+\infty} \epsilon_{+,k} = 0$ and $\epsilon_{+,k} > 0$ for all $k \in \mathbb{N}$. Moreover as $\lambda_{-,k}$ and $\lambda_{+,k}$ satisfies (2.12.6), we can verify that

$$\epsilon_{-,k} = \arctan\left(\frac{\alpha}{\lambda_{-,k}}\right) \text{ and } \epsilon_{+,k} = \arctan\left(\frac{\alpha}{\lambda_{+,k}}\right).$$

By (2.12.5) and (2.12.6), the eigenvector associated with the eigenvalue $\lambda_{+,k}$ is given by

$$\omega_{+,k} = b_{+,k}\sin(\lambda_{+,k}\cdot)(-1,\, 1)^T,$$

and the eigenvector associated with the eigenvalue $\lambda_{-,k}$ is given by

$$\omega_{-,k} = b_{-,k}\sin(\lambda_{-,k}\cdot)(1,\, 1)^T,$$

where $b_{+,k}$, $b_{-,k}$ are chosen to normalize the eigenvectors.

Since we have found all possible eigenvectors of A, we have shown that the spectrum of A is given by

$$\mathrm{Sp}(A) = \{\lambda_{+,k}^2\}_{k\in\mathbb{N}^*} \cup \{\lambda_{-,k}^2\}_{k\in\mathbb{N}^*},$$

and that each eigenvalue is simple.

We again need to estimate the distance between the consecutive eigenvalues of $A^{1/2}$ and as before we consider two different cases :

1. For all $k \in \mathbb{N}^*$, we need to look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k}$. Since

$$\lambda_{+,k} - \lambda_{-,k} = \epsilon_{+,k} + \epsilon_{-,k} = \arctan\left(\frac{\alpha}{\lambda_{+,k}}\right) + \arctan\left(\frac{\alpha}{\lambda_{-,k}}\right),$$

we see that this distance goes to zero as k goes to infinity.

2. For all $k \in \mathbb{N}^*$, we look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k+1}$. Here we have

$$\lambda_{-,k+1} - \lambda_{+,k} = \pi - (\epsilon_{+,k} + \epsilon_{-,k+1}),$$

which tends to π as k goes to infinity.

This shows that the generalized gap condition (2.3.1) is satisfied with $M = 2$.

In order to check (2.4.8), for all $k \in \mathbb{N}^*$, we set

$$\alpha_k = \lambda_{+,k} - \lambda_{-,k},$$

that behaves like k^{-1} or equivalently like $\lambda_{-,k}^{-1}$. As in the previous subsection for all $C = (c_1, c_2)^\top \in \mathbb{R}^2$, we have

$$B_k^{-1}\Phi_k C = \begin{pmatrix} c_1 B^*\omega_{-,k} + c_2 B^*\omega_{+,k} \\ \alpha_k c_2 B^*\omega_{+,k} \end{pmatrix},$$

and consequently

$$\begin{aligned} \|B_k^{-1}\Phi_k C\|_{U,2}^2 &= \|c_1 B^*\omega_{-,k} + c_2 B^*\omega_{+,k}\|_H^2 + |\alpha_k|^2|c_2|^2\|B^*\omega_{+,k}\|_H^2 \\ &= \beta \int_0^1 (b_{-,k}c_1 \sin(\lambda_{-,k}x) + b_{+,k}c_2 \sin(\lambda_{+,k}x))^2\, dx \\ &\quad + \beta|\alpha_k|^2|c_2|^2 b_{+,k}^2 \int_0^1 \sin^2(\lambda_{+,k}x)dx. \end{aligned}$$

By using Young's inequality with $\epsilon > 0$ and the fact that the eigenvectors are normalized (by the choice of $b_{\pm,k}$), we obtain

$$\begin{aligned}\|B_k^{-1}\Phi_k C\|_{U,2}^2 &\geq \beta\left(1-\frac{1}{\epsilon}\right)c_1^2 b_{-,k}^2\int_0^1 \sin^2(\lambda_{-,k}x)dx + \beta\left(1-\epsilon\right)c_2^2 b_{+,k}^2\int_0^1 \sin^2(\lambda_{+,k}x)dx\\ &\quad +\beta|\alpha_k|^2|c_2|^2 b_{+,k}^2\int_0^1 \sin^2(\lambda_{+,k}x)dx\\ &= \frac{\beta}{2}\left(\left(1-\frac{1}{\epsilon}\right)c_1^2+\left(1+\alpha_k^2-\epsilon\right)c_2^2\right).\end{aligned}$$

We then take $\epsilon = 1+\alpha_k^2/2$, which implies

$$1+\alpha_k^2-\epsilon = \frac{\alpha_k^2}{2} \quad \text{and} \quad 1-\frac{1}{\epsilon} > \frac{\alpha_k^2}{4},$$

(since $\alpha_k^2 < 2$). Consequently

$$\|B_k^{-1}\Phi_k C\|_{U,2}^2 \geq \frac{\beta}{8}\alpha_k^2(c_1^2+c_2^2).$$

As $|\alpha_k| \sim \lambda_{-,k}^{-1}$, we deduce that

$$\|B_k^{-1}\Phi_k C\|_{U,2} \geq \alpha_0\lambda_{-,k}^{-1}\|C\|_2,$$

for a positive constant α_0, and shows that (2.4.8) holds with $l=1$.

We construct the space V_h like in the previous subsection, i.e. it is the span of $(e_i, e_j)_{i,j\in\{1,\ldots,N+1\}}$, that still satisfies (2.4.4) and (2.4.5) with $\theta = 1$.

Consequently, we can apply Theorem 2.4.4 and Remark 2.10.1 with $l = 1$ and thus the family of systems (2.4.9) is uniformly polynomially stable, in the sense that the estimate (2.12.3) holds.

2.12.3 A more general wave type system

We consider the following more general system : let $\omega = (\omega_1, \cdots, \omega_N)^T$ be a solution of

$$\begin{cases} \omega_{tt} - \omega_{xx} + M\omega + BB^*\omega_t = 0 & \text{in } (0,1)^N \times \mathbb{R}_+, \\ \omega(0,t) = \omega(1,t) = 0 & \forall t > 0, \\ \omega(\cdot, 0) = \omega^{(0)},\ \omega_t(\cdot, 0) = \omega^{(1)} & \text{in } (0,1)^N, \end{cases} \tag{2.12.7}$$

where $M \in \mathcal{M}_N(\mathbb{R})$ is symmetric and such that $A_0 + M$ is positive definite in $H = L^2(0,1)^N$, when A_0 is the operator of domain $\mathcal{D}(A_0) = H_0^1(0,1)^N \cap H^2(0,1)^N$ and such that $A_0 u = -u_{xx}$, for all $u \in \mathcal{D}(A_0)$; $B \in \mathcal{L}(U, H)$, with U a complex Hilbert space.

Hence, it is written in the form (2.1.13) with the self-adjoint positive operator A defined by $A = A_0 + M$ and $\mathcal{D}(A) = \mathcal{D}(A_0) = V \cap H^2(0,1)^N$, when $V = H_0^1(0,1)^N$. We remark that A admits a compact resolvent since $\mathcal{D}(A)$ is compactly embedded into H.

As M is symmetric, M can be diagonalized by an orthogonal matrix, i.e. there exist a real orthogonal matrix O and a diagonal matrix D such that $O^T M O = D$. We denote by d_i $(i = 1, \cdots, N)$ the coefficients of the diagonal matrix D.

We start by the determination of the spectrum of the operator A. Hence we are looking for $\omega \in V \cap H^2(0,1)^N$ different from 0 and $\lambda^2 > 0$ solution of

$$-\omega_{xx} + M\omega = \lambda^2 \omega.$$

If we denote by $U = O^T \omega$, then $U = (u_1, \cdots, u_N)^T$ satisfies

$$-U_{xx} + DU = \lambda^2 U,$$

which is equivalent to

$$-\frac{d^2}{dx^2} u_i = (\lambda^2 - d_i) u_i, \quad \text{in } (0,1), \quad \forall i = 1, \cdots, N.$$

Hence there exists $c_i \in \mathbb{C}$ such that

$$u_i = \sqrt{2} c_i \sin(k\pi .), \quad \lambda_{i,k}^2 = k^2\pi^2 + d_i, \quad i = 1, \cdots, N.$$

Therefore we have found N families of eigenvectors and eigenvalues :

$$U_{i,k} = \sqrt{2} f_i \sin(k\pi .), \quad \lambda_{i,k}^2 = k^2\pi^2 + d_i, \quad i = 1, \cdots, N,$$

where $(f_i)_{i \in \{1, \cdots, N\}}$ is the canonical basis of $\mathbb{C}^N$. Coming back to the initial eigenvalue problem, we have N families of eigenvectors given by

$$\omega_{i,k} = O U_{i,k}, \quad i = 1, \cdots, N, \tag{2.12.8}$$

and the spectrum of A is given by

$$\mathrm{Sp}(A) = \{\lambda_{1,k}^2\}_{k \in \mathbb{N}^*} \cup \cdots \cup \{\lambda_{N,k}^2\}_{k \in \mathbb{N}^*}.$$

For simplicity, we now assume that all d_i are different and, for instance that

$$d_1 < d_2 < \cdots < d_N.$$

We still have to estimate the distance between the consecutive eigenvalues of $A^{1/2}$:

1. For all $k \in \mathbb{N}^*$, we need to look at the distance between $\lambda_{i,k}$ and $\lambda_{j,k}$ $(i \neq j)$. Since

$$\lambda_{i,k} - \lambda_{j,k} = \sqrt{k^2\pi^2 + d_i} - \sqrt{k^2\pi^2 + d_j} = \frac{d_i - d_j}{\sqrt{k^2\pi^2 + d_i} + \sqrt{k^2\pi^2 + d_j}},$$

we see that this distance goes to zero as k goes to infinity.

2. For all $k \in \mathbb{N}^*$, we look at the distance between $\lambda_{N,k}$ and $\lambda_{1,k+1}$. Here we have

$$\lambda_{1,k+1} - \lambda_{N,k} = \sqrt{(k+1)^2\pi^2 + d_1} - \sqrt{k^2\pi^2 + d_N} = \frac{2k\pi^2 + \pi^2 + d_1 - d_N}{\sqrt{(k+1)^2\pi^2 + d_1} + \sqrt{k^2\pi^2 + d_N}},$$

which tends to π as k goes to infinity.

This shows that the generalized gap condition (2.3.1) is satisfied with $M = N$. With the terminology of Section 1, we see that $A_1 = \cdots = A_{N-1} = \emptyset$ and $A_N = \mathbb{N}^*$. Hence, for $N > 1$, our previous results will allow to obtain stability results for system (2.12.7).

If the eigenvalues are simple (a necessary condition is that all d_i are different), then in order to verify (2.4.7) or (2.4.8), we have to bound from below $\left\| B_k^{-1} \Phi_k C \right\|_{U,2}^2$ with $C = (c_1, \cdots, c_N) \in \mathbb{R}^N$, B_k^{-1} defined in Lemma 2.3.1 and Φ_k given by

$$\Phi_k = \begin{pmatrix} B^*\omega_{1,k} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B^*\omega_{N,k} \end{pmatrix}.$$

Such a lower bound can only be made on some particular examples.

Note that, if $N = 2$, B is defined by (2.12.2) and

$$M = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\alpha > 0$, then we are back to the setting of Subsection 2.12.1. Indeed M is symmetric with $A_0 + M$ positive definite for α small enough, and diagonalized by the orthogonal matrix

$$O = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \ (\text{with } D = \alpha \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}).$$

We then finish this subsection by considering another example. Take $N = 3$ and

$$B \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \sqrt{\beta} \begin{pmatrix} \omega_1 \\ 0 \\ 0 \end{pmatrix} + \sqrt{\gamma} \begin{pmatrix} 0 \\ \omega_2 \\ 0 \end{pmatrix} + \sqrt{\delta} \begin{pmatrix} 0 \\ 0 \\ \omega_3 \end{pmatrix},$$

with non negative real numbers β, γ, δ, which is a bounded operator from H into itself (i.e. $U = H$). We chose the matrix M defined by

$$M = \alpha \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \alpha > 0$$

which is obviously symmetric. As previously we can verify that $A_0 + M$ is positive definite if $\alpha < \pi^2/2$. Moreover M can be diagonalized by the orthogonal matrix

$$O = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix},$$

into

$$D = \begin{pmatrix} -\sqrt{2}\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}\alpha \end{pmatrix}.$$

Then the spectrum of $A = A_0 + M$ is given by

$$\mathrm{Sp}(A) = \{k^2\pi^2 - \sqrt{2}\alpha\}_{k\in\mathbb{N}^*} \cup \{k^2\pi^2\}_{k\in\mathbb{N}^*} \cup \{k^2\pi^2 + \sqrt{2}\alpha\}_{k\in\mathbb{N}^*},$$

and the eigenvalues are simple (because the assumption $\alpha < \pi^2/2$ implies that $k^2\pi^2 + \sqrt{2}\alpha < (k+1)^2\pi^2 - \sqrt{2}\alpha$). Moreover, as we have shown previously, the generalized gap condition (2.3.1) is satisfied with $M = 3$. Thanks to (2.12.8) the normalized eigenvectors are given by

$$\omega_{1,k} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \sin(k\pi\cdot), \qquad \omega_{2,k} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix} \sin(k\pi\cdot),$$

$$\omega_{3,k} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \sin(k\pi\cdot).$$

We set

$$\alpha_k^{(1,2)} = \lambda_{2,k} - \lambda_{1,k}, \quad \alpha_k^{(1,3)} = \lambda_{3,k} - \lambda_{1,k}, \quad \alpha_k^{(2,3)} = \lambda_{3,k} - \lambda_{2,k}.$$

Therefore, for all $C = (c_1, c_2, c_3)^T \in \mathbb{R}^3$, we have

$$\begin{aligned}
&\left\| B_k^{-1} \Phi_k C \right\|_{U,2}^2 \\
= &\left\| \begin{pmatrix} 1 & 1 & 1 \\ 0 & \alpha_k^{(1,2)} & \alpha_k^{(1,3)} \\ 0 & 0 & \alpha_k^{(1,3)}\alpha_k^{(2,3)} \end{pmatrix} \begin{pmatrix} B^*\omega_{1,k} & 0 & 0 \\ 0 & B^*\omega_{2,k} & 0 \\ 0 & 0 & B^*\omega_{3,k} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right\|_{U,2}^2 \\
= &\left\| c_1 B^*\omega_{1,k} + c_2 B^*\omega_{2,k} + c_3 B^*\omega_{3,k} \right\|_H^2 + \left\| c_2 \alpha_k^{(1,2)} B^*\omega_{2,k} + c_3 \alpha_k^{(1,3)} B^*\omega_{3,k} \right\|_H^2 \\
&+ |c_3|^2 \left| \alpha_k^{(1,3)} \alpha_k^{(2,3)} \right|^2 \left\| B^*\omega_{3,k} \right\|_H^2 .
\end{aligned}$$

After some calculations, we obtain

$$\begin{aligned}
\left\| B_k^{-1} \Phi_k C \right\|_{U,2}^2 &= \tfrac{\beta}{4}(c_1 + \sqrt{2}c_2 + c_3)^2 + \tfrac{\gamma}{2}(c_3 - c_1)^2 + \tfrac{\delta}{4}(c_1 - \sqrt{2}c_2 + c_3)^2 \\
&+ \tfrac{\beta}{4}(\sqrt{2}\alpha_k^{(1,2)} c_2 + \alpha_k^{(1,3)} c_3)^2 + \tfrac{\gamma}{2}\left| c_3 \alpha_k^{(1,3)} \right|^2 + \tfrac{\delta}{2}(-\sqrt{2}\alpha_k^{(1,2)} c_2 + \alpha_k^{(1,3)} c_3)^2 \\
&+ \tfrac{|c_3|^2}{2} \left| \alpha_k^{(1,3)} \alpha_k^{(2,3)} \right|^2 (\tfrac{\beta+\delta}{2} + \gamma).
\end{aligned}$$

Hence different decay results can be obtained for system (2.12.7) according to the values of β, γ and δ.

First if $\beta, \gamma, \delta > 0$, then we have

$$\left\| B_k^{-1} \Phi_k C \right\|_{U,2}^2 \geq C(c_1^2 + c_2^2 + c_3^2)$$

for $C > 0$, which shows that (2.4.7) holds and therefore system (2.12.7) is exponentially stable.

Second if $\gamma = 0$ and $\beta, \delta > 0$, we have

$$\begin{aligned}
\left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2 &\geq \tfrac{\min\{\beta,\delta\}}{4}\Bigg(2c_1^2 + 4c_2^2 + 2c_3^2 + 4c_1c_3 + \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 (4c_2^2 + 2c_3^2) \\
&\quad + \min\left\{\alpha_k^{(1,3)}, \alpha_k^{(2,3)}\right\}^4 c_3^2\Bigg) \\
&\geq \tfrac{\min\{\beta,\delta\}}{4}\Bigg(\left(2 - \tfrac{2}{\epsilon}\right) c_1^2 + 4\left(1 + \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2\right) c_2^2 \\
&\quad + \left(2 - 2\epsilon + 2\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2\right) c_3^2\Bigg),
\end{aligned}$$

by Young's inequality with $\epsilon > 0$. We then take $\epsilon = 1 + \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2/2$, which implies

$$2 - \frac{2}{\epsilon} > \frac{\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2}{2}, \quad 2 - 2\epsilon + 2\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 = \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2,$$

if k is large enough. Consequently if k is large enough, we have obtained that

$$\begin{aligned}
\left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2 &\geq \tfrac{\min\{\beta,\delta\}}{4}\Bigg(\frac{\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2}{2} c_1^2 + 4\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 c_2^2 \\
&\quad + \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 c_3^2\Bigg) \\
&\geq \tfrac{\min\{\beta,\delta\}}{8}\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 (c_1^2 + c_2^2 + c_3^2),
\end{aligned}$$

which shows that (2.4.8) holds with $l = 1$, since $\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 \sim \lambda_{1,k}^{-2}$.

We construct the space V_h like in the previous subsection, i.e. it is the span of $(e_i, e_j, e_k)_{i,j,k\in\{1,\ldots,N\}}$, that still satisfies (2.4.4) and (2.4.5) with $\theta = 1$.

Consequently, in the first case ($\beta, \gamma, \delta > 0$), we can apply Theorem 2.4.1 and thus the family of systems (2.4.6) is uniformly exponentially stable. In the second case ($\beta, \delta > 0$ and $\gamma = 0$), we can apply Theorem 2.4.4 and Remark 2.10.1 with $l = 1$ and thus the family of systems (2.4.9) is uniformly polynomially stable, in the sense that (2.12.3) holds.

2.13 Open problem

In this chapter, the stability results of the approximate systems have been studied when the control term B is bounded. The boundedness of the control B is used in (2.9.5) for the proof of Theorem 2.4.1 and in (2.10.6) for the proof of Theorem 2.4.4. An open question here is how we can handle the case when the control B is unbounded. Does the analysis in this chapter after adding suitable viscosity terms remain valid or do we have to search for another method ?

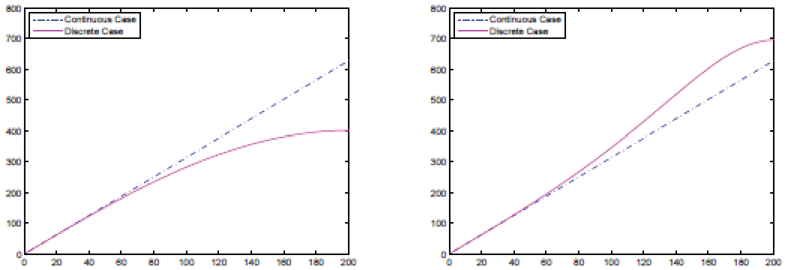

Fig. (a) Left: Square roots of the eigenvalues in the continuous and discrete cases (finite difference semidiscretization). The gaps are clearly independent of k in the continuous case and of order h for large k in the discrete one. Right: Dispersion diagram for the piecewise linear finite element space semidiscretization versus the continuous wave equation.

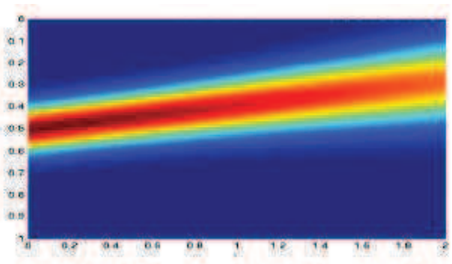

Fig. (b) A discrete wave packet and its propagation. In the horizontal axis we represent the time variable, varying between 0 and 2, and the vertical one the space variable x ranging from 0 to 1.

Figure 2.1 –

Chapitre 3

Spectral analysis and exponential or polynomial stability of some indefinite sign damped problems

3.1 Introduction and Historical background

Systems with indefinite sign damping terms arise in studying the nature of wind loads and their effect on dynamic response. This includes aircraft, buildings, telegraph wires and bridges. For instance, in an air craft, as the speed of the wind increases there may be a point at which structural damping is insufficient to damp out the vibratory motions which are increasing due to aerodynamic energy being added to the structure. The resulting vibrations can cause structural failure. Therefore, in this chapter, our aim is to find critical values of the damping term for which structural failure does not occur. More precisely, as in [1], we consider a one-dimensional wave equation with an indefinite sign damping and a zero order potential term which is either internally damped of the form

$$
\begin{gathered}
u_{tt}(x,t) - u_{xx}(x,t) + 2\chi_{(0,1)}(x)u_t(x,t) + 2\alpha\chi_{(-1,0)}(x)u_t(x,t) = 0,\ x \in (-1,1),\ t > 0,\\
u(1,t) = u(-1,t) = 0,\ t > 0,\\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),
\end{gathered}
\tag{3.1.1}
$$

where α is a given constant or with both internally and boundary damped terms under the form

$$
\begin{aligned}
&u_{tt}(x,t) - u_{xx}(x,t) + au_t(x,t) = 0, && x \in (0,1),\ t > 0,\\
&u(0,t) = 0,\ u_x(1,t) = -bu_t(1,t), && t > 0,\\
&u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),
\end{aligned}
\tag{3.1.2}
$$

where $a, b \in \mathbb{R}$.

It is well known that problem (3.1.1) is exponentially stable if the damping term α is non-negative (see [23]). Similarly, if the coefficients a and b are both positive, then, using for instance integral inequalities, one can prove that (3.1.2) is also exponentially stable.

In this chapter or in [1], we are interested in the case when the damping terms are allowed to change their sign. As mentioned before, such a case occurs, for example, in wind induced oscillations. Problems (3.1.1) and (3.1.2) model the vibrations of flexible structures subject to aerodynamic forces. Our aim is to analyze to what extent the variation of the sign affects the stability of the problem. However, the techniques which are normally employed in the definite case, such as multipliers and resolvent methods cannot be well invoked in case of indefinite sign damping coefficients. Consequently, when the damping coefficients are allowed to change their sign, the question of stability of the solution becomes more interesting.

Such a question was first exposed in a conjecture in [21] by Chen et al. who considered the internally indefinite sign damped wave equation of the form

$$u_{tt} - u_{xx} + 2a(x)u_t = 0,\ x \in (0,1),\ t > 0, \tag{3.1.3}$$

with standard initial conditions and Dirichlet boundary conditions.

It was conjectured that if there exists some $c > 0$ such that for every $n \in \mathbb{N}^*$ the following condition is satisfied

$$I_n = \int_0^1 a(x) \sin^2(n\pi x) dx \geq c, \tag{3.1.4}$$

then, when the function $a \in L^\infty(0,1)$ has an indefinite sign, the energy decays exponentially. The idea of the conjecture is that once the damping term ia allowed to be more positive than negative, then the solution decays as time goes to infinity. The condition on I_n can be interpreted as some sort of positivity condition on the damping term $a(.)$ since $I_n \to a_0$, as $n \to +\infty$, where a_0 is the average of $a(.)$. In fact, problem (3.1.3) can be considered as a perturbation of an undamped problem. Therefore, for a small enough perturbation, the eigenvalues of the associated eigenvalue problem of (3.1.3) are expected to move to the left of the imaginary axis. However, it turns out that this is not enough to ensure stability since the eigenvalues which are to the left for small perturbations may move to the right as perturbation increases. Therefore, Freitas in [28] disproves the conjecture of Chen et al. He shows that (3.1.4) is insufficient to guarantee the exponential stability. Indeed, he finds out that if $\|a\|_{L^\infty}$ is large, then there may exist some positive real eigenvalues (see Theorem 3.6 of [28]). Actually, Freitas in [28] considers the more general wave equation with an additional potential term $b(x)u$ where $b \in L^\infty(0,1)$ and replaces $a(.)$ by $\epsilon a(.)$ where ϵ is a positive parameter ; i.e, Freitas considers the following problem

$$\begin{cases} u_{tt} - \Delta u - b(x)u + \epsilon a(x)u_t = 0, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.1.5}$$

where $\Omega \subseteq \mathbb{R}^n$ is bounded with smooth boundary $\partial\Omega$.

Freitas shows that, under the conditions imposed in the conjecture, the associated eigenvalue problem may admit real positive eigenvalues for $\epsilon > \epsilon_0 > 0$, which shows that the solution of (3.1.5) blows up exponentially as time goes to infinity. The idea of Freitas is based on replacing the eigenvalue problem associated with (3.1.5) by an eigenvalue problem of a selfadjoint operator $L_p = \Delta + pa(x) + b(x)$, where p is a real parameter. This method is presented in [29] and [66]. The asymptotic behavior of the eigenvalues of L_p, as $|p|$ goes to infinity, is studied and characterized. Then a relation is found between the spectrum of L_p and the real eigenvalues associated with problem (3.1.5). Indeed, the eigenvalue problem associated with (3.1.5) is given by

$$\begin{cases} \Delta u - \epsilon a(x)\lambda u + b(x)u = \lambda^2 u, & x \in \Omega,\ \epsilon > 0 \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{3.1.6}$$

Hence, Freitas finds that a real number λ is an eigenvalue associated to the eigenvalue problem (3.1.6) if and only if γ is an eigenvalue of L_p and

$$\begin{cases} \lambda = -\dfrac{p}{\epsilon} \\ \gamma = \dfrac{p^2}{\epsilon^2}. \end{cases}$$

Therefore, the real eigenvalues associated to the eigenvalue problem (3.1.6) correspond to the intersections of the parabola, $\dfrac{p^2}{\epsilon^2}$, with the eigencurves associated with the operator L_p. As ϵ becomes large, the eigencurves intersect the parabola in at least two points and hence a positive eigenvalue λ appears which prevents the solution of (3.1.5) from being stable (see Theorem 3.6 and Corollary 3.7 of [28]).

However, the results of Freitas do not clarify what happens for small values of ϵ. Consequently, later on, Freitas and Zuazua in [30] treat the case when the L^∞ norm of the damping term, $\epsilon a(.)$, with indefinite sign is not large. For $\epsilon > 0$ small enough, they prove that the solution of (3.1.3) is exponentially stable under condition (3.1.4) and the additional condition $a \in BV(0,1)$ so that the derivative of a is defined in the weak sense. Their idea is based on the ansatz suggested by Horn in [41] where the eigenvectors are written in a series form in order to find an approximation of the eigenvalue problem and hence the asymptotic behavior of the large eigenvalues. After using the shooting method and Rouché's Theorem inspired from [23], they prove that there exists $\epsilon_1 > 0$ such that, for every $\epsilon \in (0, \epsilon_1)$, the eigenvalues are asymptotically close to $-\epsilon a_0$, where a_0 is the average of $a(.)$ and therefore the high frequencies admit negative real parts since the hypothesis imposed on the uniform positivity of I_n in (3.1.4) yields the positivity of the average a_0 of $a(.)$. Moreover, the positivity of the integrals I_n ensures that the low frequencies are to the left of the imaginary axis for $\epsilon \in (0, \epsilon_0)$. Finally, by proving that the root vectors form a Riesz basis of the energy space, the exponential stability is established for $\epsilon \in (0, \epsilon_2)$ where $\epsilon_2 = \min\{\epsilon_0, \epsilon_1\}$.

This result is extended in [17] to the case where, in the wave equation, there is an additional zero order potential term $b(x)u(x,t)$ with $b \in L^1(0,1)$. However, the ansatz in Horn does not work any more in this case. Therefore, the authors adapt a shooting method employed in [65] to construct an explicit approximation of the characteristic equation of the underlying system and to find the asymptotic expansion of the eigenvalues and eigenvectors. Under the same assumptions used in [30] on the damping term, $\epsilon a(.)$, and on the integrals, I_n, the authors in [17] establish the exponential stability for $\epsilon > 0$ small enough.

Furthermore, in [51], the authors consider an abstract linear system with perturbation of the form $\frac{d}{dt}y = Ay + \epsilon By$ on a Hilbert space, where A is a skewadjoint operator, B is bounded, and ϵ is a positive parameter. Using an abstract perturbation result and under the hypothesis that the damping operator B is uniformly effective for all normalized linear combinations of eigenvectors corresponding to the eigenvalues located in a neighborhood of any eigenvalue, the authors find an upper bound for ϵ for which the abstract system is exponentially stable. In particular, the authors in [51] find an upper bound of ϵ for which problem (3.1.5) is exponentially stable under condition (3.1.4) and the assumption that $a \in L^\infty(0,1)$ without the need for the assumption that $a \in BV$. On the other hand, for problem (3.1.1), it seems to us that the upper bound of ϵ found in [51] is not easy to check.

Later on, in [57], Racke and Rivera have removed the factor ϵ and considered the wave equation $u_{tt} - u_{xx} + a(x)u_t = 0$ on $(0, L)$ for some $L > 0$ where $a \in L^\infty(0, L)$ is allowed to change its sign such that its mean value a_0 remains positive. In [57], the exponential stability is proved under one of these conditions : Either $\|a\|_{L^\infty}$ is possibly large with sufficiently small $\|a - a_0\|_{L^2}$ or $\|a\|_{L^\infty}$ is sufficiently small but the pair (a, L) has to satisfy some estimates where it is possible to get a negative moment I_n. Note that the second condition in [57] does not contradict the result from [30], because in that case the admissible pairs (a, L), leading to exponential decay, are not independent and, according to Racke and Rivera, the solution is not exponentially decaying if one replaces $a(.)$ by $\epsilon a(.)$. The method in [57] is based on the spectral criteria characterizing exponentially stable semigroups in terms of the spectrum of the generator of the semigroup (see [42]). For instance, for possibly large L^∞ norm of $a(.)$ and small $\|a - a_0\|_{L^2}$, using the fixed point argument, the authors prove that for $\epsilon > 0$,

$$\Gamma_\epsilon^I = \left\{ \epsilon + \alpha + i\beta;\ \alpha > \Re\left(-\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{\pi}{L}\right)^2} \right) \text{ and } \beta \in \mathbb{R} \right\} \subset \rho(A)$$

and that

$$\sup_{\lambda \in \Gamma_\epsilon^I} \left\| (\lambda I - A)^{-1} \right\| < \infty,$$

where A is the generator of the associated semigroup. Furthermore, for the second case, the authors prove that for small $\epsilon_0 > 0$ and any $\epsilon_1 > \epsilon_0$, we have for all

$\epsilon \in [\epsilon_0, \epsilon_1]$

$$\Gamma_\epsilon^{II} = \left\{\epsilon - \frac{a_0}{2} + i\eta;\ \eta \in \mathbb{R}\right\} \subset \rho(A)$$

and

$$\sup_{\lambda\in\Gamma_\epsilon^{II},\ \epsilon\in[\epsilon_0,\epsilon_1]} \left\|(\lambda I - A)^{-1}\right\| < \infty,$$

on the conditions that

$$\|a(.)\|_{L^\infty(0,L)} < \frac{\sinh\left(\frac{\gamma_0}{4}\right)}{\left(\sinh\left(\frac{\gamma_1}{4}\right) + e^{\frac{7}{2}\gamma_1}e^{\frac{7}{2}\gamma_1}\right)},$$

where, for any given $\gamma_0 > 0$ and $\gamma_1 > 0$, we have

$$\gamma_0 \le \int_0^L a(x)dx \le L\|a(.)\|_{L^\infty(0,L)} \le \gamma_1$$

and

$$\frac{\gamma_0}{\|a(.)\|_{L^\infty(0,L)}} < L \le \frac{\gamma_1}{\|a(.)\|_{L^\infty(0,L)}}.$$

Finally, Menz in [54] generalizes the work done in [57] by adding a potential term $b(x)u$. He proves that if the average a_0 is positive, then, for $a(.),\ b(.) \in L^\infty(0,L)$, the exponential stability is proved for small $\|a - a_0\|_{L^2}$ but not necessarily for small $\|a(.)\|_{L^\infty(0,L)}$. Using Gearheart and Huang result, Menz obtains the resolvent estimate for the system where the function $a(.)$ is replaced by its mean value, a_0. Then using a fixed point argument, the exponential stability result is transferred to the original problem with potential term.

3.2 Main results

In this chapter or in [1], our work differs from the previous studies because we do not want to impose neither a small value of the damping factor a nor a small value of $\|a - a_0\|_{L^2}$. Indeed for system (3.1.1), this mean value is equal to $\sqrt{2}|1-\alpha|$ which we do not need to be sufficiently small. Indeed we will show later on that for $\alpha \leq -1$, problem (3.1.1) is never exponentially stable (even up to a finite dimensional space), while for $\alpha > -1$, problem (3.1.1) is exponentially stable up to a finite dimensional space. We even show that there exists a critical value $\alpha_3 \simeq -0.2823$ such that if $\alpha > \alpha_3$, then problem (3.1.1) is exponentially stable. Our method takes advantage of the one-dimensional setting that allows to perform a precise spectral analysis.

Note that these results are coherent with those given by the perturbation theory of contractive semigroups (see [62]). Actually, for system (3.1.1) defined in an appropriate Hilbert setting, we can write the generator of the semigroup as $A_0 + B_- + B_+$ where A_0 is the skew-adjoint operator given by

$$A_0 = \begin{pmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{pmatrix},$$

and $B_\pm$ are bounded operators given by

$$B_+ = \begin{pmatrix} 0 & 0 \\ 0 & -2\chi_{(0,1)} \end{pmatrix}, \quad B_- = \begin{pmatrix} 0 & 0 \\ 0 & -2\alpha\chi_{(-1,0)} \end{pmatrix}.$$

Then applying the perturbation theory of contractive semigroups, in order to get an exponential stability, the condition $\|B_-\| < |\omega(A + B_+)|$ should be imposed. However, according to [23], $\mu(A+B_+) = \omega(A+B_+) \simeq -0.45$ where the approximate value is found numerically. Here, $\mu(\mathcal{A})$ denotes the spectral abscissa of the operator $\mathcal{A}$;

$$\mu(\mathcal{A}) = \sup\{\Re\lambda : \lambda \in \sigma(\mathcal{A})\},$$

$\sigma(\mathcal{A})$ being the spectrum of $\mathcal{A}$, while $\omega(\mathcal{A})$ denotes the growth rate of the evolution equation associated with $\mathcal{A}$ in the Hilbert space X;

$$\omega(\mathcal{A}) = \inf\{\omega : \exists C(\omega) > 0 \text{ s. t. } \|U(t)\|_X^2 \leq C(\omega)\|U(0)\|_X^2 e^{2\omega t}, \forall t > 0,$$
$$\text{for every solution } U \text{ of } U_t(t) = \mathcal{A}U(t), \forall t > 0\}.$$

Therefore $\|B_-\| = 2|\alpha| < |\mu(A + B_+)|$ yields the condition that $\alpha > \alpha_1$, where $\alpha_1 \simeq -0.225$. Our spectral analysis improves this condition and yields a larger range of values of α for which problem (3.1.1) is exponentially stable. However, this result is not optimal since numerical results show that $\alpha > \alpha_2$, where $\alpha_2 \simeq -0.77$, yields the exponential stability of (3.1.1) (see Figure 1).

By a similar approach, we find some exponential or polynomial stability results for the second problem (3.1.2) where a and b are of opposite signs ; the particular case $b \in (-1, 0)$ and $a > 0$ retains our attention. Note that for such a problem, perturbation theory of contractive semigroups cannot be invoked.

This chapter is divided into two main parts. In the first one, we analyze the spectral problem associated with (3.1.1) in order to find a possible range of α for which (3.1.1) is stable. We find and prove the following results of [1] :

Theorem 3.2.1. $\alpha > -1$ *if and only if problem (3.1.1) is exponentially stable up to a finite dimensional space.*

Theorem 3.2.2. *If* $\alpha > \alpha_3$, *where* $\alpha_3 \simeq -0.2823$, *then the solution of problem (3.1.1) is exponentially stable.*

In the second part, we analyze problem (3.1.2) in order to find some conditions that a and b must satisfy to get the stability of (3.1.2). We find out the following results of [1] :

Theorem 3.2.3. *If* $b \notin \{-1, 0, 1\}$, *then* $a > -2\Re \tanh^{-1} \dfrac{1}{b} = -\ln \left| \dfrac{b+1}{b-1} \right|$ *if and only if problem (3.1.2) is exponentially stable up to a finite dimensional space.*

Theorem 3.2.4. *If* $b \in (-1, 0)$, *then* $a > -2 \tanh^{-1} b$ *if and only if problem (3.1.2) is exponentially stable.*

Theorem 3.2.5. *If $b \in (-1,0)$ and $a = -2\tanh^{-1} b$, then there exists $C > 0$ such that for all $U(0) = (u_0, u_1) \in D(A_a)$, we have*

$$E_2(t) \leq C \frac{\|U(0)\|^2_{D(A_a)}}{t}, \forall t > 0,$$

where $E_2(t)$ is the energy of the solution of problem (3.1.2) and A_a is defined in (3.4.2) below.

Note that a statement similar to the one of Theorem 3.2.4 cannot hold in the case $b < -1$ because there exists some $a > -2\Re \tanh^{-1} \frac{1}{b}$ such that some eigenvalues λ of A_a are in $\Re\lambda > 0$ (see Figure 3.6 in the case $b = -2$ and $a = 1.1$).

Before we start our analysis, we introduce some notations used in the remainder of this chapter : On D, the $L^2(D)$-norm will be denoted by $\|\cdot\|_D$. Similarly $(\cdot,\cdot)_D$ means the $L^2(D)$ inner product. Finally, the notation $A \lesssim B$ and $A \simeq B$ means the existence of positive constants C_1 and C_2, which are independent of A and B such that $A \leq C_2 B$ and $C_1 B \leq A \leq C_2 B$, respectively.

3.3 Exponential stability for the indefinite sign internally damped problem (3.1.1)

Since problem (3.1.1) is exponentially stable if the damping term α is non-negative (see [23]), from now on we assume that $\alpha < 0$.

We start by writing problem (3.1.1) as a system of the form $U_t = A_\alpha U$ where $U = (u, u_t)^\top$ and the operator $A_\alpha : D(A_\alpha) \to X$ is defined by

$$A_\alpha = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & -2\chi_{(0,1)} - 2\alpha\chi_{(-1,0)} \end{bmatrix}$$

where the energy space $X = H_0^1(-1,1) \times L^2(-1,1)$ is equipped with the usual inner product defined by

$$< (f,g)^\top, (u,v)^\top >= \int_{-1}^{1} (f'\overline{u}' + g\overline{v})dx,$$

and

$$D(A_\alpha) = (H^2(-1,1) \cap H_0^1(-1,1)) \times H_0^1(-1,1).$$

In this case, the energy associated with problem (3.1.1), at time t, is given by

$$E_1(t) = \frac{1}{2}\left(\int_{-1}^{1} (|u_x(x,t)|^2 + |u_t(x,t)|^2)dx\right)$$

with

$$E_1'(t) = -2\left(\int_0^1 |u_t(x,t)|^2 dx + \alpha \int_{-1}^{0} |u_t(x,t)|^2 dx\right), \quad \forall (u_0, u_1) \in D(A_\alpha).$$

Notice that problem (3.1.1) is well posed since A_α can be written as a sum of the m-dissipative operator A_0 with a bounded operator (see Theorem 1.1.5).

In this section, we start by analyzing the spectrum of the generator A_α. First, we find the asymptotic development of the large eigenvalues. This development shows that the high frequencies approach the line $x = -\frac{\alpha+1}{2}$ (see (3.3.7)). Second, we determine the critical value α_3 of α for which all the eigenvalues of A_α are situated to the left of the imaginary axis for any $\alpha > \alpha_3$. Finally, we show that the generalized eigenvectors of A_α form a Riesz basis of the energy space from which we deduce the exponential stability of problem (3.1.1) for $\alpha > \alpha_3$.

3.3.1 Spectral analysis of problem (3.1.1)

First, we determine the characteristic equation related to problem (3.1.1). If $U = (y,z)^\top \in D(A_\alpha)$ is an eigenvector of A_α associated with the eigenvalue λ, then

$z = \lambda y$ and the eigenvalue problem is given by

$$\begin{cases} y_{xx} - \lambda^2 y - 2\chi_{(0,1)}\lambda y - 2\alpha\chi_{(-1.0)}\lambda y = 0 \text{ in } (-1,1), \\ y(-1) = y(1) = 0. \end{cases} \tag{3.3.1}$$

Clearly $\lambda = 0$ cannot be an eigenvalue of A_α. Similarly the same calculations as below allow to show that $\lambda = -2$ and $\lambda = -2\alpha$ are not eigenvalues of A_α. Now for $\lambda \notin \{0, -2, -2\alpha\}$, setting $y^+ = y_{|(0,1)}$ and $y^- = y_{|(-1,0)}$, we get

$$\begin{cases} y^+_{xx} = (\lambda^2 + 2\lambda)y^+ \text{ in } (0,1), \\ y^+(1) = 0, \end{cases}$$

and consequently,

$$y^+(x) = c^+ \sinh[\sqrt{\lambda^2 + 2\lambda}(x-1)], \tag{3.3.2}$$

for some $c^+ \in \mathbb{C}$. Similarly, we have

$$\begin{cases} y^-_{xx} = (\lambda^2 + 2\alpha\lambda)y^- \text{ in } (-1,0), \\ y^-(-1) = 0, \end{cases}$$

which implies that

$$y^-(x) = c^- \sinh[\sqrt{\lambda^2 + 2\alpha\lambda}(x+1)], \tag{3.3.3}$$

for some $c^- \in \mathbb{C}$. As the differential equation in (3.3.1) yields $y \in H^2(0,1)$ and due to the Sobolev embedding theorem $H^2(0,1) \hookrightarrow C^1[0,1]$, we get

$$\begin{cases} y^+(0) = y^-(0), \\ y^+_x(0) = y^-_x(0), \end{cases}$$

or equivalently

$$\begin{cases} c^+ \sinh(\sqrt{\lambda^2 + 2\lambda}) = -c^- \sinh(\sqrt{\lambda^2 + 2\alpha\lambda}), \\ c^+\sqrt{\lambda^2 + 2\lambda}\cosh(\sqrt{\lambda^2 + 2\lambda}) = c^-\sqrt{\lambda^2 + 2\alpha\lambda}\cosh(\sqrt{\lambda^2 + 2\alpha\lambda}), \end{cases} \tag{3.3.4}$$

i.e.,

$$\mathcal{D}_\alpha(\lambda)(c^+, c^-)^\top = (0,0)^\top, \tag{3.3.5}$$

where

$$\mathcal{D}_\alpha(\lambda) = \begin{pmatrix} \sinh(\sqrt{\lambda^2+2\lambda}) & \sinh(\sqrt{\lambda^2+2\alpha\lambda}) \\ \sqrt{\lambda^2+2\lambda}\cosh(\sqrt{\lambda^2+2\lambda}) & -\sqrt{\lambda^2+2\alpha\lambda}\cosh(\sqrt{\lambda^2+2\alpha\lambda}) \end{pmatrix}.$$

As (3.3.5) admits a non zero solution if and only if $\det \mathcal{D}_\alpha(\lambda) = 0$, the complex number $\lambda \notin \{0, -2, -2\alpha\}$ is an eigenvalue of A_α if and only if it is the root of the characteristic equation

$$\det \mathcal{D}_\alpha(\lambda) = 0.$$

Direct calculations yield

$$\begin{aligned}\det \mathcal{D}_\alpha(\lambda) &= -F_\alpha(\lambda) \\ &= -\sqrt{\lambda^2+2\alpha\lambda}\sinh(\sqrt{\lambda^2+2\lambda})\cosh(\sqrt{\lambda^2+2\alpha\lambda}) \\ &\quad -\sqrt{\lambda^2+2\lambda}\sinh(\sqrt{\lambda^2+2\alpha\lambda})\cosh(\sqrt{\lambda^2+2\lambda}).\end{aligned}$$

Note further that

$$2F_\alpha(\lambda) = g_\alpha(\lambda) = t_2(\lambda)\sinh t_2(\lambda) - t_1(\lambda)\sinh t_1(\lambda),$$

where

$$t_1(\lambda) = \sqrt{\lambda^2+2\lambda} - \sqrt{\lambda^2+2\alpha\lambda},$$

and

$$t_2(\lambda) = \sqrt{\lambda^2+2\lambda} + \sqrt{\lambda^2+2\alpha\lambda}.$$

We have proved the next result.

Lemma 3.3.1. *A_α has a compact inverse and therefore the spectrum of A_α is discrete and its eigenvalues are of finite algebraic multiplicity. Furthermore*

$$\sigma(A_\alpha) = \{\lambda \in \mathbb{C} \setminus \{0, -2, -2\alpha\} \, : \, g_\alpha(\lambda) = 0\}.$$

Remark 3.3.2. *Note that the eigenvalues of A_α depend continuously on α. Indeed, fix α and an eigenvalue λ_0 of A_α. Then as λ_0 is isolated, there exists $\rho > 0$ such that*

$$g_\alpha(z) \neq 0, \forall z \in \mathbb{C} : 0 < |z - \lambda_0| \leq \rho.$$

In particular, as g_α is a continuous function of z, setting $D = \{z \in \mathbb{C} : |z - \lambda_0| = \rho\}$, there exists a positive real number κ such that

$$|g_\alpha(z)| \geq \kappa, \forall z \in D.$$

For a fixed positive real number ϵ_0 we consider the mapping of two variables

$$H : [0, \epsilon_0] \times D \to \mathbb{C} : (\epsilon, z) \to g_\alpha(z) - g_{\alpha+\epsilon}(z).$$

Since it is a uniformly continuous function and since $H(0, z) = 0$ for all z, we deduce the existence of a positive real number δ such that

$$|H(\epsilon, z)| < \kappa, \forall(\epsilon, z) \in [0, \delta] \times D.$$

The two last estimates imply that

$$|g_\alpha(z) - g_{\alpha+\epsilon}(z)| < |g_\alpha(z)|, \forall(\epsilon, z) \in [0, \delta] \times D.$$

Hence, Rouché's theorem allows to conclude that $g_{\alpha+\epsilon}$ has the same number of roots at g_α for all $\epsilon \in [0, \delta]$.

The following Lemma shows the boundedness of the real part of the eigenvalues of the operator A_α and proves that its eigenvalues cannot be real.

Lemma 3.3.3. *Let λ be an eigenvalue of the operator A_α, and $U = y(x, \lambda)(1, \lambda)^\top$ be an associated eigenvector. Then $\lambda \in \mathbb{C} \backslash \mathbb{R}$ with*

$$\Re\lambda = \frac{-\displaystyle\int_0^1 |y(x,\lambda)|^2 dx - \alpha \int_{-1}^0 |y(x,\lambda)|^2 dx}{\displaystyle\int_{-1}^1 |y(x,\lambda)|^2 dx},$$

and

$$(\Im\lambda)^2 = \frac{\displaystyle\int_{-1}^{1} |y_x(x,\lambda)|^2 dx}{\displaystyle\int_{-1}^{1} |y(x,\lambda)|^2 dx} - \left(\frac{\displaystyle\int_{0}^{1} |y(x,\lambda)|^2 dx + \alpha \int_{-1}^{0} |y(x,\lambda)|^2 dx}{\displaystyle\int_{-1}^{1} |y(x,\lambda)|^2 dx} \right)^2.$$

Proof: As A_α is real, in the sense that $\overline{A_\alpha U} = A_\alpha \bar{U}$ for all $U \in D(A_\alpha)$, it follows that $\overline{U} = y(x,\overline{\lambda})(1,\overline{\lambda})$ is an eigenvector of A_α corresponding to the eigenvalue $\overline{\lambda}$. Integrating (3.3.1) against $y(x,\overline{\lambda})$ gives

$$\int_{-1}^{1} |y_x|^2 dx + \lambda^2 \int_{-1}^{1} |y|^2 dx + 2\lambda \left(\int_{0}^{1} |y|^2 dx + \alpha \int_{-1}^{0} |y|^2 dx \right) = 0.$$

Hence,

$$\lambda = \frac{-\displaystyle\int_{0}^{1} |y|^2 dx - \alpha \int_{-1}^{0} |y|^2 dx \pm \left[\left(\int_{0}^{1} |y|^2 dx + \alpha \int_{-1}^{0} |y|^2 dx \right)^2 - \int_{-1}^{1} |y_x|^2 dx \int_{-1}^{1} |y|^2 dx \right]^{\frac{1}{2}}}{\displaystyle\int_{-1}^{1} |y|^2 dx}.$$

If λ is real, then by the Poincaré-Friedrich's inequality

$$\frac{\pi^2}{4} \int_{-1}^{1} |w|^2 dx \leq \int_{-1}^{1} |w_x|^2 dx, \forall w \in H_0^1(-1,1), \tag{3.3.6}$$

we get

$$\begin{aligned} 0 &\leq \left(\int_{0}^{1} |y|^2 dx + \alpha \int_{-1}^{0} |y|^2 dx \right)^2 - \int_{-1}^{1} |y_x|^2 dx \int_{-1}^{1} |y|^2 dx \\ &\leq \left(\int_{-1}^{1} |y|^2 dx \right)^2 - \left(\frac{\pi}{2} \right)^2 \left(\int_{-1}^{1} |y|^2 dx \right)^2 \\ &= \left(1 - \frac{\pi^2}{4} \right) \left(\int_{-1}^{1} |y|^2 dx \right)^2 < 0 \end{aligned}$$

which is impossible. Consequently, $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and the result follows. ∎

Remark 3.3.4. *Note that*

1. $(\Re\lambda)^2 + (\Im\lambda)^2 \geq \dfrac{\pi^2}{4}$ *by Poincaré-Friedrich's inequality (3.3.6).*

2. $|\Re\lambda| \leq \max\{1, |\alpha|\}$.

3. *Denote by* $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ *the spectrum of* A_α. *Since* A_α *is real, without loss of generality we can assume that for all* $k \in \mathbb{Z}^*$, $\lambda_{-k} = \overline{\lambda_k}$.

3.3.1.1 Asymptotic behavior of large eigenvalues

In the sequel, we study the asymptotic behavior of the eigenvalues $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ of A_α as $|\lambda_k| \to +\infty$. According to the previous remark, since $\Re\lambda_k$ is uniformly bounded, then $|\lambda_k| \to +\infty$ is equivalent to $|\Im\lambda_k| \to +\infty$. Moreover, we can restrict our study to the case $k \to +\infty$.

Lemma 3.3.5. *The large eigenvalues of* A_α *are simple. Moreover there exist* $m \in \mathbb{Z}$ *and* $k_0 \in \mathbb{N}$, *such that*

$$\lambda_k = \frac{-\alpha - 1}{2} + i(k+m)\frac{\pi}{2} + O\left(\frac{1}{k}\right), \quad \forall k \geq k_0. \tag{3.3.7}$$

Proof: Let λ be an eigenvalue of A_α or equivalently a root of g_α. For the sake of simplicity, we assume that $\sqrt{\lambda^2} = \lambda$, if this is not the case, the next arguments hold but with t_1 and t_2 replaced by their opposites.

First, we prove that

$$t_1(\lambda) \to 1 - \alpha \text{ and } t_2(\lambda) = 2\lambda + 1 + \alpha + o(1) \text{ as } |\lambda| \to +\infty.$$

Indeed, we write

$$\begin{aligned} t_1(\lambda) &= \frac{(\sqrt{\lambda^2+2\lambda} - \sqrt{\lambda^2+2\alpha\lambda})(\sqrt{\lambda^2+2\lambda} + \sqrt{\lambda^2+2\alpha\lambda})}{\sqrt{\lambda^2+2\lambda} + \sqrt{\lambda^2+2\alpha\lambda}} \\ &= \frac{2\lambda(1-\alpha)}{t_2(\lambda)}. \end{aligned}$$

Therefore, we get [1]

$$t_1(\lambda) \sim \frac{2\lambda(1-\alpha)}{2\lambda} = 1 - \alpha, \quad \text{as } |\Im\lambda| \to +\infty. \tag{3.3.8}$$

1. as usual the notation $h(\lambda) \sim g(\lambda)$ as $|\Im\lambda| \to +\infty$ means that $\lim_{|\Im\lambda|\to+\infty} \frac{h(\lambda)}{g(\lambda)} = 1$

On the other hand,

$$t_2(\lambda) = \lambda\sqrt{1+\frac{2}{\lambda}} + \lambda\sqrt{1+\frac{2\alpha}{\lambda}}.$$

As $|\lambda| \to +\infty$, we write

$$\sqrt{1+\frac{2}{\lambda}} = 1 + \frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right),$$

and

$$\sqrt{1+\frac{2\alpha}{\lambda}} = 1 + \frac{\alpha}{\lambda} + o\left(\frac{1}{\lambda}\right).$$

Therefore,

$$t_2(\lambda) = \lambda\left(2 + \frac{1+\alpha}{\lambda}\right) + o(1) = 2\lambda + 1 + \alpha + o(1) \quad \text{as } |\Im\lambda| \to +\infty. \tag{3.3.9}$$

Since $t_2(\lambda) \neq 0$ for $|\Im\lambda|$ large enough, then from the characteristic equation we can write

$$g_\alpha(\lambda) = t_2(\lambda)\left[\sinh(t_2(\lambda)) - \frac{t_1(\lambda)}{t_2(\lambda)}\sinh(t_1(\lambda))\right] = 0,$$

or equivalently

$$h_\alpha(\lambda) = 0, \tag{3.3.10}$$

where

$$h_\alpha(\lambda) = \sinh(t_2(\lambda)) - \frac{t_1(\lambda)}{t_2(\lambda)}\sinh(t_1(\lambda)). \tag{3.3.11}$$

Now the conclusion follows by using Rouché's Theorem. For this aim, for N large enough, define the curve

$$\Gamma_{\pm n} = \{z : \left|z + \frac{1+\alpha}{2} \mp i\frac{n\pi}{2}\right| = \frac{C_0}{n}\},\ n > N,$$

where C_0 is a positive constant fixed later on in Lemma 3.3.7.

Lemma 3.3.7 below shows, by Rouché's Theorem, that $h_\alpha(z)$ given in (3.3.11) has the same roots as $\sinh(t_2(z))$ in the curve $\Gamma_{\pm n}$, for every $n > N$ where N is large enough. Consequently, we deduce that the large eigenvalues are simple since the roots of $\sinh(t_2(z))$ are simple for $|z|$ large and are situated inside $\Gamma_{\pm n}$ for some n large, which yields (3.3.7). ∎

The next Figures 3.1 and 3.2 illustrate the roots of F_α for $\alpha = -0.75$ and $\alpha = -0.2$ computed using a Newton method, namely a sufficiently large box is decomposed in a relatively fine mesh and each node of the mesh is used as initial value for the Newton method. In these Figures, the asymptotic behavior from the previous lemma is clearly visible.

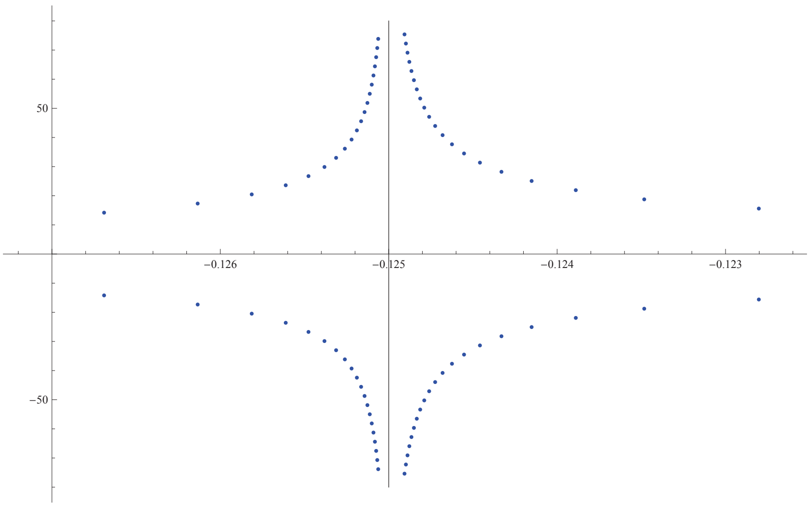

FIGURE 3.1 – Eigenvalues for $\alpha = -0.75$

Remark 3.3.6. *(Increasing the order of the finite expansion)*
If we write $\lambda_k = \dfrac{-\alpha-1}{2} + i(k+m)\dfrac{\pi}{2} + \beta_k + i\varepsilon_k$ *where* $\beta_k, \varepsilon_k \in \mathbb{R}$ *are such that* $\beta_k = O_1\left(\dfrac{1}{k}\right)$ *and* $\varepsilon_k = O_2\left(\dfrac{1}{k}\right)$, *and we substitute this value into* $t_2(\lambda_k)$, $\sinh(t_2(\lambda_k))$,

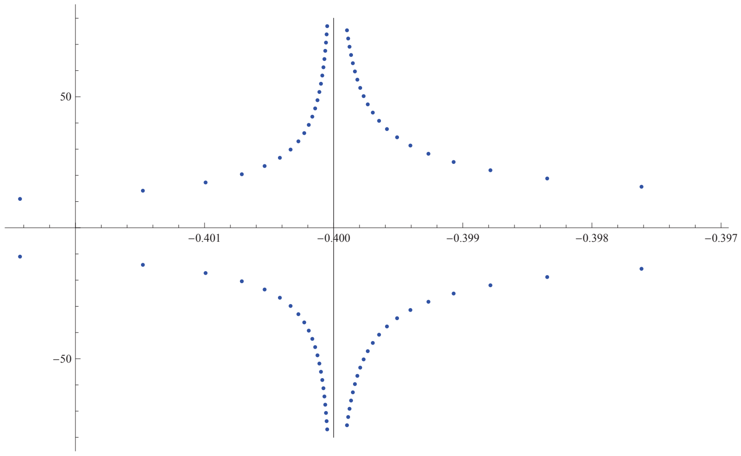

FIGURE 3.2 – Eigenvalues for $\alpha = -0.2$

and $\dfrac{t_1(\lambda_k)}{t_2(\lambda_k)}\sinh(t_1(\lambda_k)$, *then increasing the order of the finite expansion, we get*

$$t_2(\lambda_k) = i(k+m)\pi + 2\beta_k + 2i\varepsilon_k - \frac{1+\alpha^2}{2i(k+m)\pi} + \frac{(1+\alpha^2)(1+\alpha)-2(1+\alpha^3)}{(k+m)^2\pi^2} + o\left(\frac{1}{k^2}\right)$$

$$\sinh(t_2(\lambda_k)) = (-1)^{k+m}\left(2\beta_k + 2i\varepsilon_k - \frac{1+\alpha^2}{i(k+m)\pi} + \frac{(1+\alpha^2)(1+\alpha)-2(1+\alpha^3)}{(k+m)^2\pi^2} + o\left(\frac{1}{k^2}\right)\right)$$

$$\frac{t_1(\lambda_k)}{t_2(\lambda_k)}\sinh(t_1(\lambda_k)) = \frac{(1-\alpha)\sinh(1-\alpha)}{i(k+m)\pi} + \frac{(1-\alpha^2)\sinh(1-\alpha)}{((k+m)\pi)^2} + \frac{(1-\alpha^2)(1-\alpha)\cosh(1-\alpha)}{((k+m)\pi)^2} + o\left(\frac{1}{k^2}\right).$$

From the equality $\sinh(t_2(\lambda_k)) = \dfrac{t_1(\lambda_k)}{t_2(\lambda_k)}\sinh(t_1(\lambda_k))$, *we conclude*

$$\begin{aligned}\varepsilon_k &= -\frac{1+\alpha^2}{2(k+m)\pi} - (-1)^{k+m}\frac{(1-\alpha)\sinh(1-\alpha)}{2(k+m)\pi} + o(\frac{1}{k^2})\\ \beta_k &= -\frac{2(1+\alpha^3)-(1+\alpha)(1+\alpha^2)}{2((k+m)\pi)^2}\\ &\quad + \tfrac{(-1)^{k+m}}{2}\left[\frac{(1-\alpha^2)\sinh(1-\alpha)}{((k+m)\pi)^2} + \frac{(1-\alpha^2)(1-\alpha)\cosh(1-\alpha)}{((k+m)\pi)^2}\right] + o\left(\frac{1}{k^2}\right).\end{aligned}$$

The $(-1)^{k+m}$ *factor appearing in the expression of* β_k *shows that, according to the parity of* $k+m$, *there are eigenvalues to the left and to the right of the axis* $x =$

$-\dfrac{1+\alpha}{2}$. *Hence if* $1+\alpha \leq 0$, *then problem (3.1.1) is never stable, while if* $1+\alpha > 0$, *then we can hope that (3.1.1) is either exponentially stable or unstable but exponentially stable up to a finite dimensional space.*

Lemma 3.3.7. *There exists* $N \in \mathbb{N}$ *large enough such that for every* $n > N$ *and for all* $z \in \Gamma_{\pm n}$, *we have*

$$|h_\alpha(z) - \sinh(t_2(z))| < |\sinh(t_2(z))|.$$

Proof: The proof is divided into two steps. In the first step, for every $n > N$ where N is large enough, we show that if $z \in \Gamma_{\pm n}$, then $|\sinh(t_2(z))| \geq \dfrac{C_0}{|z|}$. In fact, it is enough to consider the case where $z \in \Gamma_n$ since the eigenvalues appear in conjugate pairs. If $z \in \Gamma_n$, then $z = -\dfrac{1+\alpha}{2} + i\dfrac{n\pi}{2} + \rho_n e^{i\theta}$ where $\theta \in [0, 2\pi)$ and $\rho_n = \dfrac{C_0}{n}$. Since n is large enough, then by (3.3.9), we have

$$t_2(z) = 2z + 1 + \alpha + o_1(1) + io_2(1),$$

and

$$\begin{aligned} |\sinh(t_2(z))|^2 &= \sin^2\left(2\rho_n \sin\theta + o_2(1)\right) + \sinh^2\left(2\rho_n \cos\theta + o_1(1)\right) \\ &= \left(2\rho_n \sin\theta + o(1)\right)^2 + \left(2\rho_n \cos\theta + o(1)\right)^2 \\ &= 4\rho_n^2 + o(1). \end{aligned}$$

Hence,

$$\frac{C_0^2}{|z|^2} \leq \frac{C_0^2}{\left(\rho_n \sin\theta + \frac{n\pi}{2}\right)^2} = \frac{4C_0^2}{(n\pi)^2} + o\left(\frac{1}{n^2}\right) \leq \frac{4C_0^2}{n^2} + o(1) = |\sinh(t_2(z))|^2.$$

Now (3.3.8) and (3.3.9) imply that there exists $C_1 > 0$ such that

$$|t_1(z)| < 2|1-\alpha|, \forall z : |\Im z| > C_1,$$

$$|t_2(z)| > |z|, \forall z : |\Im z| > C_1,$$

and therefore there exists $C_0 > 0$ such that

$$|t_1(z)\sinh(t_1(z))| < C_0, \forall z : |\Im z| > C_1.$$

As for $z \in \Gamma_n$, with $n \geq 1$, we have

$$\Im z \geq \frac{n\pi}{2} - \frac{C_0}{n},$$

we need to chose N large enough so that

$$\frac{N\pi}{2} - \frac{C_0}{N} > C_1.$$

With this constraint and by (3.3.11) we see that for $z \in \Gamma_n$, where $n > N$, we have

$$|h_\alpha(z) - \sinh(t_2(z))| \leq \left|\frac{t_1(z)\sinh(t_1(z))}{t_2(z)}\right| < \frac{C_0}{|z|} < |\sinh(t_2(z))|.$$

■

3.3.1.2 Critical value of α

We finish this section by looking for a critical value of α for which we will get an exponential stability of problem (3.1.1). Numerically, as the Figure 3.3 below shows (see also Figures 3.1 and 3.2), for $0 > \alpha > \alpha_2$, with $\alpha_2 \approx -0.77$, the eigenvalues $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ of the operator A_α are all situated to the left of the imaginary axis. However, theoretically we could not hit this range of α. Instead, we prove such a result for $\alpha \in (\alpha_3, 0)$ with $\alpha_3 \approx -0.2823$. In fact, as the value of $\alpha \in (-1, 0)$ decreases, the axis $x = -\dfrac{1+\alpha}{2}$ is shifted to the right and therefore the eigenvalues are shifted near the imaginary axis. Consequently, we try to study the behavior of the eigenvalues on the imaginary axis and then find a critical value of α for which the characteristic equation (3.3.10) has no roots on the imaginary axis.

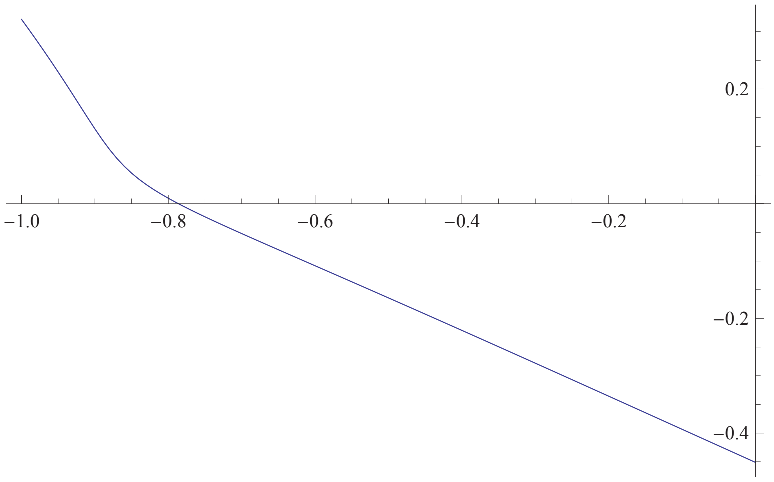

FIGURE 3.3 – Numerical value of $\max\{\Re\lambda : \lambda \in \sigma(A_\alpha)\}$ versus α.

To state properly our result, we introduce the functions

$$\begin{aligned} f_1 \ &: \ (-\infty, 0] \to (0, \infty) \\ &\quad \alpha \to \frac{1}{2}\left[\cosh\left(2\Re(t_2(i\frac{\pi}{2}))\right) - 1\right], \\ f_2 \ &: \ (-\infty, 0] \to (0, \infty) \\ &\quad \alpha \to \frac{1}{2}\frac{\left[(1-\alpha)^2 + \left(\Im(t_1(i\frac{\pi}{2}))\right)^2\right]\left[\cosh\left(2(1-\alpha)\right) - \cos\left(2\Im(t_1(i\frac{\pi}{2}))\right)\right]}{\left|t_2(i\frac{\pi}{2})\right|^2}. \end{aligned}$$

It is easy to check that f_1 (resp. f_2) is increasing (resp. decreasing) and that $f_1(0) > f_2(0)$ (see Figure 3.4), hence there exists a unique negative real number α_3 such that $f_1(\alpha_3) = f_2(\alpha_3)$. Numerically we find that $\alpha_3 \approx -0.2823$.

Theorem 3.3.8. *For all $\alpha > \alpha_3$, any eigenvalue λ of A_α satisfies $\Re\lambda < 0$.*

Proof: Let $\lambda = iy$ with $y \in \mathbb{R}$ be a purely imaginary eigenvalue of A_α. As the complex eigenvalues appear in conjugate pairs, it is enough to consider the case $y > 0$. According to Remark 3.3.4, $y \geq \frac{\pi}{2}$. We start by writing

$$\lambda(2\alpha + \lambda) = -y^2 + 2i\alpha y.$$

Let

$$h(y,\alpha) = \frac{\sqrt{y^2 + y\sqrt{4\alpha^2 + y^2}}}{\sqrt{2}},$$

then, by expanding $\left(\frac{\alpha y}{h(y,\alpha)} + ih(y,\alpha)\right)^2$, we have

$$\sqrt{-y^2 + 2i\alpha y} = \frac{\alpha y}{h(y,\alpha)} + ih(y,\alpha).$$

We note that $h(y,\alpha)$ is non decreasing as a function of y since

$$\frac{\partial}{\partial y} h(y,\alpha) = \frac{1}{\sqrt{2}} \left[\frac{2y + \sqrt{4\alpha^2 + y^2} + \frac{y^2}{\sqrt{4\alpha^2+y^2}}}{2\sqrt{y^2 + y\sqrt{4\alpha^2 + y^2}}} \right],$$

while $\dfrac{h(y,\alpha)}{y}$ is decreasing since

$$\frac{\partial}{\partial y} \frac{h(y,\alpha)}{y} = \frac{-\sqrt{2}\alpha^2}{y\sqrt{4\alpha^2 + y^2}\sqrt{y(y + \sqrt{4\alpha^2 + y^2})}}.$$

Moreover, $t_2(iy)$ is given by

$$t_2(iy) = \frac{\alpha y}{h(y,\alpha)} + \frac{y}{h(y,1)} + i\left(h(y,\alpha) + h(y,1)\right)$$

and $t_1(iy)$ is given by

$$t_1(iy) = -\frac{\alpha y}{h(y,\alpha)} + \frac{y}{h(y,1)} + i\left(-h(y,\alpha) + h(y,1)\right).$$

In the sequel, our aim is to find some bounds for $|\sinh(t_2(iy))|^2$ and $\left|\frac{t_1(iy)}{t_2(iy)} \sinh(t_1(iy))\right|^2$. We start by finding a lower bound for $|\sinh(t_2(iy))|^2$ with $y \geq \frac{\pi}{2}$. We have

$$|\sinh(t_2(iy))|^2 = \frac{1}{2}[\cosh\left(2\Re(t_2(iy))\right) - \cos\left(2\Im(t_2(iy))\right)] \geq \frac{1}{2}[\cosh\left(2\Re(t_2(iy))\right) - 1].$$

But $\Re(t_2(iy))$ is a positive and increasing function of y. In fact,

$$\Re(t_2(iy)) = \frac{y}{h(y,\alpha)} \left[\frac{h(y,\alpha)}{h(y,1)} + \alpha \right].$$

Since $\dfrac{h(y,\alpha)}{h(y,1)}$ is positive then we prove that it is increasing by proving that its square is increasing. Indeed, since $|\alpha| < 1$

$$\frac{\partial}{\partial y}\left(\frac{h(y,\alpha)}{h(y,1)}\right)^2 = \frac{(y+\sqrt{4\alpha^2+y^2})(\sqrt{4+y^2}-\sqrt{4\alpha^2+y^2})}{\sqrt{4+y^2}\sqrt{4\alpha^2+y^2}(y+\sqrt{4+y^2})} > 0.$$

Since

$$\lim_{y\to 0}\frac{h(y,\alpha)}{h(y,1)} = \sqrt{|\alpha|},$$

then

$$\frac{h(y,\alpha)}{h(y,1)} + \alpha > 0, \quad \forall y > 0.$$

Finally, we conclude that $\Re(t_2(iy))$ is positive and an increasing function of y so

$$\Re(t_2(iy)) \geq \Re(t_2(i\frac{\pi}{2})). \tag{3.3.12}$$

Therefore,

$$|\sinh(t_2(iy))|^2 \geq \frac{1}{2}\left[\cosh\left(2\Re(t_2(i\frac{\pi}{2}))\right) - 1\right] = f_1(\alpha). \tag{3.3.13}$$

In the second step, we find an upper bound for $\left|\dfrac{t_1(iy)}{t_2(iy)}\sinh(t_1(iy))\right|^2$. $\Im(t_2(iy))$ is a non decreasing function of y since $h(y,\alpha)$ is non decreasing. So

$$\Im(t_2(iy)) \geq \Im(t_2(i\frac{\pi}{2})). \tag{3.3.14}$$

Both (3.3.12) and (3.3.14) yield

$$|t_2(iy)|^2 \geq \left|t_2(i\frac{\pi}{2})\right|^2. \tag{3.3.15}$$

Next, we find an upper bound for $|t_1(iy)|$. We have

$$\frac{\partial}{\partial y}\Im(t_1(iy)) = \frac{\partial}{\partial y}h(y,1) - \frac{\partial}{\partial y}h(y,\alpha).$$

Knowing that $h(y,\alpha)$ is non decreasing, we compare the difference between the square of $\dfrac{\partial}{\partial y}h(y,1)$ and $\dfrac{\partial}{\partial y}h(y,\alpha)$. We find that

$$\left(\frac{\partial}{\partial y}h(y,\alpha)\right)^2 = \frac{(y+\sqrt{4\alpha^2+y^2})^3}{8(4\alpha^2 y+y^3)} = \varphi(\alpha^2),$$

where

$$\varphi(\beta) = \frac{(y + \sqrt{4\beta + y^2})^3}{8(4\beta y + y^3)}.$$

Deriving with respect to β, we get

$$\varphi'(\beta) = \frac{(y + \sqrt{4\beta + y^2})^2(4\beta + y^2 - 2y\sqrt{4\beta + y^2})}{4y(4\beta + y^2)^{\frac{5}{2}}}.$$

Therefore, $\varphi'(\beta) < 0$ if $y > \frac{2\sqrt{\beta}}{3} = \frac{2|\alpha|}{3}$. But $y \geq \frac{\pi}{2} > \frac{2}{3} > \frac{2|\alpha|}{3}$ which implies that $\varphi'(\beta) < 0$. Hence, for all $y \geq \frac{\pi}{2}$,

$$\left(\frac{\partial}{\partial y}h(y,1)\right)^2 - \left(\frac{\partial}{\partial y}h(y,\alpha)\right)^2 < 0,$$

and so, for all $y \geq \frac{\pi}{2}$, we get

$$\frac{\partial}{\partial y}h(y,1) - \frac{\partial}{\partial y}h(y,\alpha) < 0.$$

Therefore, $\Im(t_1(iy))$ is decreasing and

$$|\Im(t_1(iy))| \leq |\Im(t_1(i\frac{\pi}{2}))|. \tag{3.3.16}$$

On the other hand, we prove that $\Re(t_1(iy))$ is non decreasing and since $\Re(t_1(iy)) \to 1-\alpha$ as $y \to +\infty$, we obtain

$$|\Re(t_1(iy))| < 1 - \alpha. \tag{3.3.17}$$

Consequently, by (3.3.16) and (3.3.17), we get

$$\begin{aligned} |\sinh t_1(iy)|^2 &= \frac{1}{2}\left[\cosh\left(2\Re(t_1(iy))\right) - \cos\left(2\Im(t_1(iy))\right)\right] \\ &\leq \frac{1}{2}\left[\cosh\left(2(1-\alpha)\right) - \cos\left(2\Im(t_1(i\frac{\pi}{2}))\right)\right]. \end{aligned} \tag{3.3.18}$$

Finally, (3.3.15), (3.3.16), (3.3.17), and (3.3.18) yield

$$\begin{aligned} \left|\frac{t_1(iy)}{t_2(iy)}\sinh(t_1(iy))\right|^2 &\leq \frac{1}{2}\frac{\left[(1-\alpha)^2 + \left(\Im(t_1(i\frac{\pi}{2}))\right)^2\right]\left[\cosh\left(2(1-\alpha)\right) - \cos\left(2\Im(t_1(i\frac{\pi}{2}))\right)\right]}{\left|t_2(i\frac{\pi}{2})\right|^2} \\ &= f_2(\alpha). \end{aligned} \tag{3.3.19}$$

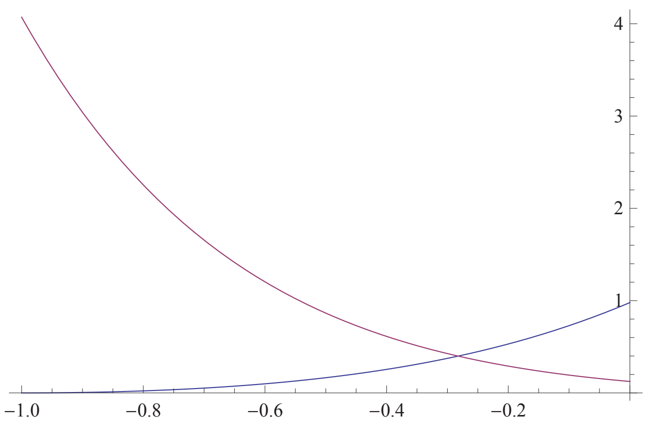

FIGURE 3.4 – Curves of f_1 (blue) and f_2 (red).

In conclusion, using (3.3.13) and (3.3.19) and according to the properties of f_1 and f_2 mentioned before (see Figure 3.4), we find out that if $\alpha > \alpha_3$, then $\left|\dfrac{t_1(iy)}{t_2(iy)}\sinh(t_1(iy))\right|^2 < |\sinh(t_2(iy))|^2$ and therefore the characteristic equation (3.3.10) has no roots on the imaginary axis due to (3.3.11). We deduce that for every $0 > \alpha > \alpha_3$, there does not exist any pure imaginary eigenvalue. By the continuity of the eigenvalues as a function of α, all the eigenvalues are situated to the left of the imaginary axis for every $0 > \alpha > \alpha_3$. Indeed, if we suppose the contrary, namely that there exists $\tilde{\alpha} \in [\alpha_3, 0[$ such that there exists some $\lambda_k(\tilde{\alpha})$ with $\Re\lambda_k(\tilde{\alpha}) > 0$. Since for $\alpha = 0$, all the eigenvalues are to the left of the imaginary axis because of the exponential stability, we deduce by continuity of the eigenvalues as a function of α that there exists $\tilde{\tilde{\alpha}} \in [\alpha_3, 0[$ with $\tilde{\alpha} < \tilde{\tilde{\alpha}} < 0$ such that there exists some pure imaginary eigenvalue associated with $\tilde{\tilde{\alpha}}$ which is a contradiction. ■

3.3.2 Root vectors, Riesz basis, and proof of Theorems 3.2.1 and 3.2.2

3.3.2.1 Root vectors

We start by introducing the root vectors of A_α and A_α^*, the adjoint of A_α. We notice that A_α has a compact resolvent, the geometric multiplicity of each eigenvalue is one, and, from Lemma 3.3.7, the large eigenvalues are simple.

Therefore, as in Section 6 of [23], for any $n \in \mathbb{Z}^*$, we denote the algebraic multiplicity of λ_n by m_n. To λ_n, define the associated Jordan chain of root vectors $\{U_{n,j}\}_{j=0}^{m_n-1}$ by

$$\begin{aligned} U_{n,0} &= y(x,\lambda_n)(1,\lambda_n)^\top \\ A_\alpha U_{n,j} &= \lambda_n U_{n,j} + U_{n,j-1}, \quad j = 0,...,m_n-1 \text{ with } U_{n,-1} = 0. \end{aligned}$$

As usual we choose the generalized eigenvectors such that $< U_{n,j}, U_{n,0} >= 0$, $j = 1,...,m_n-1$. Notice that for $|n|$ large, $m_n = 1$ and the root vectors are limited to the eigenvector $U_{n,0}$. According to Theorem 10.1 of Chapter V of [36], the root vectors of A_α are complete in X since A_α is a bounded perturbation of a skew symmetric operator. Moreover, the root vectors form a basis of the root subspace $\mathcal{L}_n = \{U \in D(A_\alpha);\ (A_\alpha - \lambda_n)^{m_n} U = 0\}$.

We now consider the root vectors of the adjoint of A_α given by

$$A_\alpha^* = \begin{pmatrix} 0 & -I \\ -\dfrac{\partial^2}{\partial x^2} & -2\chi_{(0,1)} - 2\alpha\chi_{(-1,0)} \end{pmatrix}.$$

Since $\sigma(A_\alpha^*) = \overline{\sigma(A_\alpha)} = \sigma(A_\alpha)$, we associate to $\overline{\lambda_n}$ the root vectors of A_α^* as

$$\begin{aligned} W_{n,0} &= y(x,\overline{\lambda_n})\,(1,-\overline{\lambda_n})^\top, \\ A_\alpha^* W_{n,j} &= \overline{\lambda_n} W_{n,j} + W_{n,j-1}, < W_{n,j}, U_{n,m_n-1} >= 0, \quad j = 1,...,m_n-1. \end{aligned}$$

$W_{n,0}$ is an eigenvector of A_α^* and, by completeness, $W_{n,j}$ are uniquely determined since $< W_{n,0}, U_{n,m_n-1} >\neq 0$. Other wise, $U_{n,m_n-1} = 0$ which is impossible.

3.3.2.2 Riesz basis

Here we adapt the results of Section 6 in [23] to prove that the root vectors of the operator A_α form a Riesz basis of the energy space X ; i.e., we prove the following theorem.

Theorem 3.3.9. *The root vectors of the operator A_α form a Riesz basis of the energy space X.*

Proof: of Theorem 3.3.9.

We use the Bari's Theorem given by Theorem 1.2.6 (see Theorem 2.1 of Chapter VI in [36]). First, the completeness in X of the root vectors of A_α follows from Theorem 10.1 of Chapter V of [36]. So it remains to search for a biorthogonal sequence. For that purpose, we can follow the proof of Lemma 6.2 of [23]. From the proof of Lemma 6.2 of [23], we have

$$< U_{n,p}, W_{k,j} >=< U_{n,p}, W_{n,m_n-1-p} > \delta_{n,k}\delta_{m_n-1-p,j},$$

for all $p = 0, ..., m_n-1$, $j = 0, ..., m_k-1$, and for all $n, k \in \mathbb{Z}^*$, with $< U_{n,p}, W_{n,m_n-1-p} >\neq 0$. Indeed, for all $n \neq k$, we have

$$< A_\alpha U_{n,0}, W_{k,0} >= \lambda_n < U_{n,0}, W_{k,0} >= \lambda_k < U_{n,0}, W_{k,0} > .$$

Hence, $< U_{n,0}, W_{k,0} >= 0$. Next,

$$< A_\alpha U_{n,0}, W_{k,1} >= \lambda_n < U_{n,0}, W_{k,1} >= \lambda_k < U_{n,0}, W_{k,1} > + < U_{n,0}, W_{k,0} > .$$

Hence, $(\lambda_n - \lambda_k) < U_{n,0}, W_{k,1} >= 0$ and so $< U_{n,0}, W_{k,1} >= 0$. Proceeding similarly, we prove that

$$< U_{n,0}, W_{k,j} >= 0, \quad \forall j = 0, ..., m_n - 1.$$

Finally, by iteration, we can prove that

$$< U_{n,p}, W_{k,j} >= 0, \quad \forall p, j = 0, ..., m_n - 1, \ \forall n \neq k.$$

Now, if $n = k$, then Fredholm Alternative implies that

$$< U_{n,j}, W_{n,0} >= 0, \quad \forall j = 0, ..., m_n - 2.$$

Hence, by completeness, it follows that

$$< U_{n,m_n-1}, W_{n,0} >\neq 0.$$

Similarly,

$$< U_{n,0}, W_{n,j} >= 0, \quad \forall j = 0, ..., m_n - 2$$

and

$$< U_{n,0}, W_{n,m_n-1} >\neq 0.$$

After comparing $< A_\alpha U_{n,1}, W_{n,m_n-k} >$ with $< U_{n,1}, A^*_\alpha W_{n,m_n-k} >$, we find that

$$< U_{n,1}, W_{n,m_n-k-1} >=< U_{n,0}, W_{n,m_n-k} > .$$

Therefore, $U_{n,1}$ is orthogonal to each $W_{n,j}$ except when $j = m_n - 2$. Finally, by iteration, we find that $U_{n,p}$ is orthogonal to each $W_{n,j}$ except when $j = m_n - p - 1$. In conclusion, we have

$$< U_{n,p}, \frac{W_{n,m_n-1-j}}{< U_{n,j}, W_{n,m_n-1-j} >} >= \delta_{p,j}, \quad \forall p, j = 0, ..., m_n - 1.$$

However, the arguments above are sufficient for the low frequencies but for the high frequencies, in order to replace $\frac{W_{n,0}}{< U_{n,0}, W_{n,0} >}$ by $W_{n,0}$ in (1.2.1), we still need to show that $< U_{n,0}, W_{n,0} >$ does not degenerate as n becomes large. This is our next aim.

According to (3.3.2) and (3.3.3), we choose $U_{n,0}$ such that

$$U_{n,0|\,(0,1)} = y(x, \lambda_n)_{|\,(0,1)}\,(1, \lambda_n)^\top = \frac{b_n^+}{\sqrt{\lambda_n^2 + 2\lambda_n}} \sinh(\sqrt{\lambda_n^2 + 2\lambda_n}(x-1))(1, \lambda_n)^\top$$

and

$$U_{n,0|(-1,0)} = y(x,\lambda_n)_{|(-1,0)}\,(1,\lambda_n)^\top = \frac{b_n^-}{\sqrt{\lambda_n^2+2\alpha\lambda_n}}\sinh(\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1))\,(1,\lambda_n)^\top,$$

where b_n^+ and b_n^- are chosen such that (see (3.3.4))

$$b_n^+ = -b_n^-\frac{\sqrt{\lambda_n^2+2\lambda_n}}{\sqrt{\lambda_n^2+2\alpha\lambda_n}}\frac{\sinh\sqrt{\lambda_n^2+2\alpha\lambda_n}}{\sinh\sqrt{\lambda_n^2+2\lambda_n}}, \tag{3.3.20}$$

with b_n^- fixed such that $< U_{n,0}, U_{n,0} >= 1$.

But we have

$$\begin{aligned}
&< U_{n,0}, W_{n,0} > \\
&= \int_{-1}^{1}\left((y'(x,\lambda_n))^2 - \lambda_n^2(y(x,\lambda_n))^2\right)dx \\
&= (b_n^+)^2\int_0^1\left(\cosh^2(\sqrt{\lambda_n^2+2\lambda_n}(x-1)) - \frac{\lambda_n^2}{\lambda_n^2+2\lambda_n}\sinh^2(\sqrt{\lambda_n^2+2\lambda_n}(x-1))\right)dx \\
&+(b_n^-)^2\int_{-1}^{0}\left(\cosh^2(\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1)) - \frac{\lambda_n^2}{\lambda_n^2+2\alpha\lambda_n}\sinh^2(\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1))\right)dx.
\end{aligned}$$

Furthermore as $\Re\lambda_n$ is uniformly bounded, $\sinh(2\sqrt{\lambda_n^2+2\lambda_n}(x-1))$ and $\sinh(2\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1))$ are bounded (for $-1 \le x \le 1$). Hence,

$$\begin{aligned}
&\int_0^1\left(\cosh^2(\sqrt{\lambda_n^2+2\lambda_n}(x-1)) - \frac{\lambda_n^2}{\lambda_n^2+2\lambda_n}\sinh^2(\sqrt{\lambda_n^2+2\lambda_n}(x-1))\right)dx \\
=\ & 1+o_1(1), \\
&\int_{-1}^0\left(\cosh^2(\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1)) - \frac{\lambda_n^2}{\lambda_n^2+2\alpha\lambda_n}\sinh^2(\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1))\right)dx \\
=\ & 1+o_2(1).
\end{aligned}$$

Therefore,

$$< U_{n,0}, W_{n,0} >= (b_n^+)^2(1+o_1(1)) + (b_n^-)^2(1+o_2(1)). \tag{3.3.21}$$

But owing to (3.3.7), we see that

$$\sqrt{\lambda_n^2+2\lambda_n} = \frac{1-\alpha}{2} + i(n+m)\frac{\pi}{2} + O_1(\frac{1}{n}), \tag{3.3.22}$$

$$\sqrt{\lambda_n^2+2\alpha\lambda_n} = \frac{\alpha-1}{2} + i(n+m)\frac{\pi}{2} + O_2(\frac{1}{n}). \tag{3.3.23}$$

Therefore if $n+m$ is even we deduce that

$$\sinh\sqrt{\lambda_n^2+2\lambda_n} = \sinh(\frac{1-\alpha}{2}+o_3(1))\cos((n+m)\frac{\pi}{2}),$$
$$\sinh\sqrt{\lambda_n^2+2\alpha\lambda_n} = \sinh(\frac{\alpha-1}{2}+o_4(1))\cos((n+m)\frac{\pi}{2}),$$

and therefore

$$\frac{\sinh\sqrt{\lambda_n^2+2\alpha\lambda_n}}{\sinh\sqrt{\lambda_n^2+2\lambda_n}} = -1+o_5(1).$$

Similarly if $n+m$ is odd we show that

$$\frac{\sinh\sqrt{\lambda_n^2+2\alpha\lambda_n}}{\sinh\sqrt{\lambda_n^2+2\lambda_n}} = 1+o_6(1).$$

These asymptotic behaviors in (3.3.20) lead to

$$(b_n^+)^2 = (b_n^-)^2(1+o_7(1)), \tag{3.3.24}$$

and inserting this identity in (3.3.21) we arrive at

$$< U_{n,0}, W_{n,0} >= 2(b_n^-)^2(1+o_8(1)).$$

So we choose $\frac{W_{n,0}}{2(b_n^-)^2(1+o_8(1))}$ instead of $W_{n,0}$, but as mentioned before we have to show that b_n^- does not tend to zero as n tends to infinity. Actually by similar calculations as before, we check that

$$< U_{n,0}, U_{n,0} >= |b_n^+|^2(\delta+o_9(1)) + |b_n^-|^2(\delta+o_{10}(1)),$$

where

$$\delta = \frac{\sinh(1-\alpha)}{1-\alpha},$$

that is positive since $\alpha<0$. Therefore with the help of (3.3.24) we get

$$< U_{n,0}, U_{n,0} >= 2|b_n^-|^2(\delta+o_{11}(1)),$$

and consequently

$$|b_n^-|^2 = \frac{1}{2\delta} + o_{12}(1).$$

In summary, by fixing N large enough such that for $|n| \geq N$, $m_n = 1$, we have proved that the family

$$\left\{ \left\{ \frac{W_{n,m_n-1-j}}{< U_{n,j}, W_{n,m_n-1-j} >} \right\}_{j=0}^{m_n-1} \right\}_{0<|n|<N} \bigcup \left\{ \frac{W_{n,0}}{2(b_n^-)^2(1+o_8(1))} \right\}_{|n|\geq N}$$

is biorthogonal to the set of root vectors of A_α.

It remains to prove (1.2.1). We first prove that for any $(f,g) \in X$ and for all N large, the sum $S = \sum_{n>N} |< U_{n,0}, (f,g) >|^2$ is finite. In fact,

$$\begin{aligned} S &= \sum_{n>N} \Big| \int_0^1 b_n^+ \Big(\cosh(\sqrt{\lambda_n^2 + 2\lambda_n}(x-1)) \overline{f'}(x) \\ &\quad + \frac{\lambda_n}{\sqrt{\lambda_n^2 + 2\lambda_n}} \sinh(\sqrt{\lambda_n^2 + 2\lambda_n}(x-1)) \overline{g}(x) \Big) \, dx \\ &\quad + \int_{-1}^0 b_n^- \Big(\cosh(\sqrt{\lambda_n^2 + 2\alpha\lambda_n}(x+1)) \overline{f'}(x) \\ &\quad + \frac{\lambda_n}{\sqrt{\lambda_n^2 + 2\alpha\lambda_n}} \sinh(\sqrt{\lambda_n^2 + 2\alpha\lambda_n}(x+1)) \overline{g}(x) \Big) \, dx \Big|^2. \end{aligned}$$

Noting that $\dfrac{\lambda_n}{\sqrt{\lambda_n^2 + 2\lambda_n}} = 1 + O_3(\dfrac{1}{n})$ and $\dfrac{\lambda_n}{\sqrt{\lambda_n^2 + 2\alpha\lambda_n}} = 1 + O_4(\dfrac{1}{n})$; hence, in order to prove that the sum S is finite, we will only prove that

$$S_1 = \sum_{n>N} \left| \int_0^1 \cosh(\sqrt{\lambda_n^2 + 2\lambda_n}(x-1)) \overline{f'}(x) \, dx \right|^2 < \infty,$$

since the convergence of the other terms appearing in S follows similarly. According to (3.3.22), we have

$$\sqrt{\lambda_n^2 + 2\lambda_n} = \gamma + i(n+m)\frac{\pi}{2} + \delta_n.$$

where $\gamma = \dfrac{1-\alpha}{2}$ and $\delta_n = O_1(\dfrac{1}{n})$. Therefore, we can write

$$\cosh(\sqrt{\lambda_n^2 + 2\lambda_n}(x-1)) = \frac{1}{2}(e^{(\gamma + i(n+m)\frac{\pi}{2} + \delta_n)(x-1)} + e^{-(\gamma + i(n+m)\frac{\pi}{2} + \delta_n)(x-1)}).$$

So in order to prove that S_1 is finite, we will only prove that

$$\sum_{n>N}\left|\int_0^1 e^{(\gamma+i(n+m)\frac{\pi}{2}+\delta_n)(x-1)}\overline{f'}(x)\right|^2<\infty$$

since the convergence of the other term appearing in S_1 follows similarly. But the convergence of this series is a simple application of Lemma 3.2 of [70] (see also [38, Lemma 4.1]) since the sequence $(\gamma+i(n+m)\frac{\pi}{2}+\delta_n)_{n>N}$ satisfies the conditions of this Lemma and since $f\in H^1(0,1)$. Therefore, $e^{(\gamma+i(n+m)\frac{\pi}{2}+\delta_n)(x-1)}$ is a Bessel sequence. We can also use the mean value Theorem and Fourier series to prove the convergence of this series. Indeed,

$$\begin{aligned}
&\sum_{n>N}\left|\int_0^1 e^{(\gamma+\delta_n+i\frac{(n+m)\pi}{2})(x-1)}\overline{f'}(x)\right|^2\\
&=\sum_{n>N}\left|\int_0^1 (e^{\delta_n(x-1)}-1+1)e^{\gamma(x-1)}e^{i\frac{(n+m)\pi}{2}(x-1)}\overline{f'}(x)\right|^2\\
&\lesssim\sum_{n>N}\int_0^1\left|(e^{\delta_n(x-1)}-1)e^{\gamma(x-1)}\overline{f'}(x)\right|^2+\sum_{n>N}\left|\int_0^1 e^{i\frac{(n+m)\pi}{2}(x-1)}e^{\gamma(x-1)}\overline{f'}(x)\right|^2\\
&\lesssim\sum_{n>N}|\delta_n|^2\int_0^1|(x-1)|^2e^{2\gamma(x-1)}|\overline{f'}(x)|^2dx+\|e^{\gamma(x-1)}f'\|^2_{(0,1)}\\
&\lesssim\|e^{\gamma(x-1)}f'\|^2_{(0,1)}\sum_{n>N}|\delta_n|^2+\|e^{\gamma(x-1)}f'\|^2_{(0,1)}\\
&\lesssim\|e^{\gamma(x-1)}f'\|^2_{(0,1)}\sum_{n>N}\frac{1}{n^2}+\|e^{\gamma(x-1)}f'\|^2_{(0,1)}.
\end{aligned}$$

The convergence of $\sum_{n>N}|< W_{n,0},(f,g)>|^2$ for any $(f,g)\in X$ and for all N large is proved in the same manner since

$$< W_{n,0},(f,g)>=<\bar{U}_{n,0},(f,g)>=\overline{< U_{n,0},(\bar{f},\bar{g})>}.$$

Therefore, the conditions of Theorem 1.2.6 are all fulfilled and hence the root vectors of A_α form a Riesz basis of X. ■

3.3.2.3 Proof of Theorems 3.2.1 and 3.2.2

Since the generalized eigenvectors, $\{U_{n,j}\}_{n,j}$, of A_α form a Riesz basis of X, then given the initial datum

$$U(0) = \sum_{n=\pm 1}^{\pm\infty} \sum_{j=0}^{m_n - 1} \gamma_{n,j} U_{n,j},$$

we can write

$$U(t) = (u, u_t)^\top = \sum_{n=\pm 1}^{\pm\infty} e^{\lambda_n t} \sum_{j=0}^{m_n - 1} \gamma_{n,j} \sum_{k=0}^{j} \frac{t^{j-k}}{(j-k)!} U_{n,k}.$$

Since the low frequencies are of finite multiplicity, then denoting the maximum multiplicity by m, we get for any $\epsilon > 0$

$$E_1(t) \lesssim E_1(0)(1 + t^{2m}) e^{2\mu(A_\alpha)t} \lesssim E_1(0) e^{(2\mu(A_\alpha)+\epsilon)t}. \tag{3.3.25}$$

As $\mu(A_\alpha) < 0$ for $\alpha \in]\alpha_3, 0]$ ($\alpha_3 \approx -0.2823$), we can choose $0 < \epsilon < -\mu(A_\alpha)$ to get the exponential stability of problem (3.1.1) and hence the proof of Theorem 3.2.2 is complete.

The proof of Theorem 3.2.1 is similar since for $\alpha + 1 > 0$, by Remark 3.3.4 and Lemma 3.3.5, at most a finite number of eigenvalues of A_α may be situated on the imaginary axis or to its right; consequently, excluding the finite dimensional space spanned by the corresponding root vectors, we obtain an exponential decay.

3.4 Exponential stability for an indefinite sign internally and boundary damped problem

In this section, we perform a similar analysis for problem (3.1.2) which contains both an internal and a boundary indefinite sign damping term. Recall that (3.1.2) is the problem

$$\begin{aligned} & u_{tt}(x,t) - u_{xx}(x,t) + au_t(x,t) = 0, && x \in (0,1),\ t > 0, \\ & u(0,t) = 0,\ u_x(1,t) = -bu_t(1,t), && t > 0, \\ & u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), && \end{aligned} \tag{3.4.1}$$

where $a, b \in \mathbb{R}$. If a and b are both non negative and one of them is positive, then, using integral inequalities for instance, one can show that (3.4.1) is exponentially stable. Our aim is to find sufficient conditions on a and b so that (3.4.1) is exponentially or polynomially stable whatever the sign of a and b.

The energy of (3.4.1) is given by

$$E_2(t) = \frac{1}{2}\int_0^1 \left(|u_t|^2 + |u_x|^2\right) dx,$$

and hence formally

$$E_2'(t) = -a\int_0^1 |u_t|^2 dx - b|u_t(1)|^2.$$

From this identity, we see that if $a \leq 0$ and $b \leq 0$, then $E_2'(t) \geq 0$. Therefore, the energy increases and no stability can be hoped. Therefore, the only case of interest is the case when a and b are of opposite signs. We, therefore, assume that $ab < 0$. We further assume that $b \notin \{-1, 0, 1\}$. Indeed the case $b = 0$ has no interest since only the case $a > 0$ yields stability results; while the case $b = 1$ or -1 is excluded for technical reasons (see Subsection 3.4.3).

3.4.1 Well-posedness of problem (3.4.1)

As usual, by the standard reduction of order method, we can rewrite formally (3.4.1) in the simpler form $U_t = A_a U$, where $U = (u, u_t)^\top$ and the operator $A_a : D(A_a) \to X$ is defined by

$$A_a = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & -a \end{bmatrix} \tag{3.4.2}$$

where the energy space $X = H_l(0,1) \times L^2(0,1)$ is equipped with the usual inner product defined by

$$< (f,g)^\top, (u,v)^\top >= \int_0^1 (f'\overline{u}' + g\overline{v})\, dx,$$

where $H_l(0,1) = \{u \in H^1(0,1);\ u(0) = 0\}$ and therefore, $D(A_a) = \{(u,v)^\top \in H^2(0,1) \cap H_l(0,1) \times H_l(0,1);\ u_x(1) = -bv(1)\}$.

First, we remark that A_a is not necessarily dissipative so we propose to write $A_a = A_0 - aB$ where

$$A_0 = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \tag{3.4.3}$$

and

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, A_a is a bounded perturbation of a non skew adjoint operator A_0. Nevertheless, we will prove in Subsection 3.4.4 that if $b \neq -1$, then A_0 generates a C_0 semigroup and since $aB \in \mathcal{L}(X)$, A_a will be also a generator of a C_0 semigroup according to Theorem 3.1.1 in [62]. Therefore problem (3.4.1) is a well posed problem (see Theorem 3.4.7).

3.4.2 Eigenvalue Problem

In this part, we find the characteristic equation of the eigenvalues associated with (3.4.1). Let $(y,z)^\top \in D(A_a)$, $(y,z) \neq (0,0)$, such that $A_a(y,z)^\top = \lambda(y,z)^\top$ then $z = \lambda y$ and

$$\begin{aligned} &y_{xx} - \lambda^2 y - a\lambda y = 0 \quad \text{in } (0,1), \\ &y(0) = 0, \quad y_x(1) = -b\lambda y(1). \end{aligned} \tag{3.4.4}$$

First, it is easy to see that $\lambda = 0$ is not an eigenvalue of A_a. Furthermore, if $\lambda = -a$, then $y = cx$ with $c = abc$ which satisfies the boundary condition at 1. Since $y \neq 0$, we have $c \neq 0$, and get $ab = 1$ which is impossible since we have assumed that $ab < 0$.

Now if $\lambda \neq 0$ and $\lambda \neq -a$, then there exists $c_1 \in \mathbb{C}^*$ such that

$$y(x) = c_1 \sinh \sqrt{\lambda^2 + a\lambda} x.$$

Hence, the boundary condition at 1 becomes

$$y_x(1) = c_1\sqrt{\lambda^2 + a\lambda} \cosh \sqrt{\lambda^2 + a\lambda} = -b\lambda c_1 \sinh \sqrt{\lambda^2 + a\lambda}.$$

As $c_1 \neq 0$ then λ is an eigenvalue of A_a if it satisfies the characteristic equation

$$F_a(\lambda) = \sqrt{\lambda^2 + a\lambda} \cosh \sqrt{\lambda^2 + a\lambda} + b\lambda \sinh \sqrt{\lambda^2 + a\lambda} = 0. \tag{3.4.5}$$

Integrating (3.4.4) against $\overline{y(x,\lambda)}$ and performing an integration by parts, we get

$$\lambda^2 \int_0^1 |y|^2 dx + b\lambda |y(1)|^2 + a\lambda \int_0^1 |y|^2 dx + \int_0^1 |y_x|^2 dx = 0.$$

Therefore,

$$\lambda = \frac{-\left[b|y(1)|^2 + a\int_0^1 |y|^2 dx\right] \pm \left[\left(a\int_0^1 |y|^2 dx + b|y(1)|^2\right)^2 - 4\left(\int_0^1 |y_x|^2 dx\right)\left(\int_0^1 |y|^2 dx\right)\right]^{\frac{1}{2}}}{2\int_0^1 |y|^2 dx}.$$

If $\lambda \in \mathbb{C}\backslash\mathbb{R}$, then

$$\Re\lambda = \frac{-\left(b|y(1)|^2 + a\int_0^1 |y|^2 dx\right)}{2\int_0^1 |y|^2 dx} \tag{3.4.6}$$

and since $y \in \{u \in H^1(0,1) : u(0) = 0\}$, we get

$$(\Im\lambda)^2 + (\Re\lambda)^2 = \frac{\int_0^1 |y_x|^2 dx}{\int_0^1 |y|^2 dx} \geq \frac{\pi^2}{4}. \tag{3.4.7}$$

Remark 3.4.1. *If $b < 0$ and $a > 0$, then whenever $\lambda \in \mathbb{C}\backslash\mathbb{R}$, from (3.4.6), we deduce that $\Re\lambda > -\dfrac{a}{2}$. This estimate can be even extended to all eigenvalues of A_a. Indeed, for $(u,v)^\top \in D(A_a)$, by integration by parts, we check that*

$$2\Re < A_a(u,v)^\top, (u,v)^\top >= -a \int_0^1 |v|^2\, dx - b|v(1)|^2.$$

In particular, if λ is an eigenvalue of A_a with associated normalized eigenvector $(u,v)^\top$, we will get

$$2\Re\lambda = -a \int_0^1 |v|^2\, dx - b|v(1)|^2.$$

Since $\displaystyle\int_0^1 |v|^2\, dx \leq < (u,v)^\top, (u,v)^\top >= 1$, and recalling that we here assume that $b < 0$, we then conclude that

$$2\Re\lambda \geq -a \int_0^1 |v|^2\, dx \geq -a.$$

In summary we have proved the next result.

Lemma 3.4.2. *A_a^{-1} is compact. Hence the spectrum of A_a consists of discrete eigenvalues with finite algebraic multiplicity. Furthermore*

$$\sigma(A_a) = \{\lambda \in \mathbb{C} \setminus \{0, -a\} \,:\, \lambda \text{ is a root of } (3.4.5)\}.$$

Concerning the multiplicity of the eigenvalues, we show the following Lemma :

Lemma 3.4.3. *The high frequencies of A_a are simple and there exists at most two double low frequency eigenvalues.*

Proof: We derive (3.4.5) with respect to λ to get

$$\begin{aligned} F_a'(\lambda) &= \frac{\sqrt{\lambda^2 + a\lambda}(2\lambda + a + 2b)\sinh\sqrt{\lambda^2 + a\lambda} + (2\lambda + a)(b\lambda + 1)\cosh\sqrt{\lambda^2 + a\lambda}}{2\sqrt{\lambda^2 + a\lambda}} \\ &= \frac{g(\lambda)}{2\sqrt{\lambda^2 + a\lambda}}. \end{aligned}$$

If $F_a(\lambda) = 0$, then

$$\cosh\sqrt{\lambda^2 + a\lambda} = -\frac{b\lambda \sinh\sqrt{\lambda^2 + a\lambda}}{\sqrt{\lambda^2 + a\lambda}}.$$

Substituting into $g(\lambda)$, we get

$$g(\lambda) = \frac{\lambda \sinh\sqrt{\lambda^2 + a\lambda}}{\sqrt{\lambda^2 + a\lambda}} \left(2(1 - b^2)\lambda^2 + a(3 - b^2)\lambda + a(a + b)\right).$$

Since $\lambda \neq 0$ and $\lambda \neq -a$, then $g(\lambda) = 0$ is equivalent to $2(1 - b^2)\lambda^2 + a(3 - b^2)\lambda + a(a + b) = 0$ which only has two roots. ■

3.4.3 Asymptotic Development of the High Frequencies

In this subsection, using Taylor expansions, we prove that the high frequencies approach the axis $x = -\frac{a}{2} - \Re \tanh^{-1} \frac{1}{b}$. In fact, (3.4.5) implies that for $b \neq 0$

$$-\frac{\sqrt{\lambda^2 + a\lambda}}{b\lambda} = \tanh(\sqrt{\lambda^2 + a\lambda}); \tag{3.4.8}$$

hence,

$$\sqrt{\lambda^2 + a\lambda} = -\tanh^{-1}\left(\frac{\sqrt{\lambda^2 + a\lambda}}{b\lambda}\right) + ik_1\pi, \quad k_1 \in \mathbb{Z}. \tag{3.4.9}$$

For large λ, we write

$$\sqrt{\lambda^2 + a\lambda} = \lambda + \frac{a}{2} - \frac{a^2}{8\lambda} + o\left(\frac{1}{\lambda}\right). \tag{3.4.10}$$

Moreover, for large λ, there exists $k_2 \in \mathbb{Z}$ such that we have

$$\begin{aligned}\tanh^{-1}\left(\frac{\sqrt{\lambda^2+a\lambda}}{b\lambda}\right) &= \tanh^{-1}\left(\frac{1}{b}+\frac{a}{2b\lambda}+o\left(\frac{1}{\lambda}\right)\right)\\ &= \frac{1}{2}\log\left(1+\frac{1}{b}\right)+\frac{1}{2}\log\left(1+\frac{a}{2(b+1)\lambda}+o\left(\frac{1}{\lambda}\right)\right)\\ &-\frac{1}{2}\log\left(1-\frac{1}{b}\right)-\frac{1}{2}\log\left(1-\frac{a}{2(b-1)\lambda}+o\left(\frac{1}{\lambda}\right)\right)+i\pi k_2\\ &= \tanh^{-1}\frac{1}{b}+\frac{a}{4\lambda}\left[\frac{1}{(b+1)}+\frac{1}{(b-1)}\right]+i\pi k_2+o\left(\frac{1}{\lambda}\right)\\ &= \tanh^{-1}\frac{1}{b}+i\pi k_2+\frac{ab}{2\lambda(b^2-1)}+o\left(\frac{1}{\lambda}\right).\end{aligned} \tag{3.4.11}$$

Substituting (3.4.10) and (3.4.11) into (3.4.9), we get that for $\lambda \in \sigma(A_a)$ with $|\lambda|$ large enough, there exists $k \in \mathbb{Z}$ such that

$$\lambda = -\frac{a}{2} - \tanh^{-1}\frac{1}{b} + ik\pi + \frac{a^2}{8\lambda} + \frac{ab}{2\lambda(1-b^2)} + o\left(\frac{1}{\lambda}\right). \tag{3.4.12}$$

In order to get $\Re\lambda < 0$ for λ large, we need that

$$-\frac{a}{2} - \Re\tanh^{-1}\frac{1}{b} < 0. \tag{3.4.13}$$

As before we illustrate in Figures 3.5 and 3.6, the roots of F_a for $a = 1, b = -0.5$ and $a = 1.1, b = -2$ respectively, computed using the same scheme as before. In both cases, $-\frac{a}{2} - \Re\tanh^{-1}\frac{1}{b}$ is negative and the asymptotic behavior (3.4.12) is clearly confirmed. Note that in the first case, all eigenvalues are in the left of the imaginary axis, which is in accordance with Theorem 3.2.4. On the other hand, the second example does not enter in the framework of Theorem 3.2.4 and shows that an exponential stability result cannot hold if $b < -1$ but confirms Theorem 3.2.3.

Note that in Figure 3.6, some eigenvalues with positive real part like

$$\lambda = 0.6910478014752763 \text{ or } \lambda = 0.0123963241846109 01 \pm 2.9711251755632886i,$$

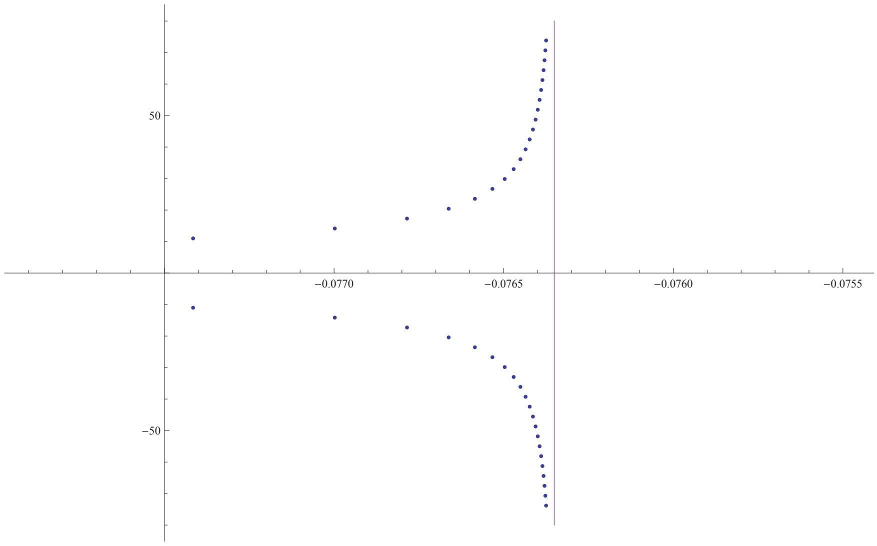

FIGURE 3.5 – Eigenvalues for $a = 1,\ b = -0.5$

are not represented because they are too far from the other eigenvalues. A representation in a larger scale is possible but would avoid to see the main part of the spectrum.

3.4.4 Riesz Basis of X and a note on the well-posedness of problem (3.4.1)

In this part, we prove that the eigenvectors associated with the problem with $a = 0$ form a Riesz basis of X if $b \notin \{-1, 0, 1\}$. In other words, consider the problem

$$\begin{aligned} &\widetilde{u}_{tt}(x,t) - \widetilde{u}_{xx}(x,t) = 0, && x \in (0,1),\ t > 0, \\ &\widetilde{u}(0,t) = 0,\ \widetilde{u}_x(1,t) = -b\widetilde{u}_t(1,t), && t > 0, \\ &\widetilde{u}(x,0) = \widetilde{u}_0(x),\ \widetilde{u}_t(x,0) = \widetilde{u}_1(x). \end{aligned} \tag{3.4.14}$$

(3.4.14) is equivalent to $\widetilde{U}_t = A_0\widetilde{U}$ with $\widetilde{U}_t = (\widetilde{u}, \widetilde{u}_t)^\top$ and A_0 is given by (3.4.3).

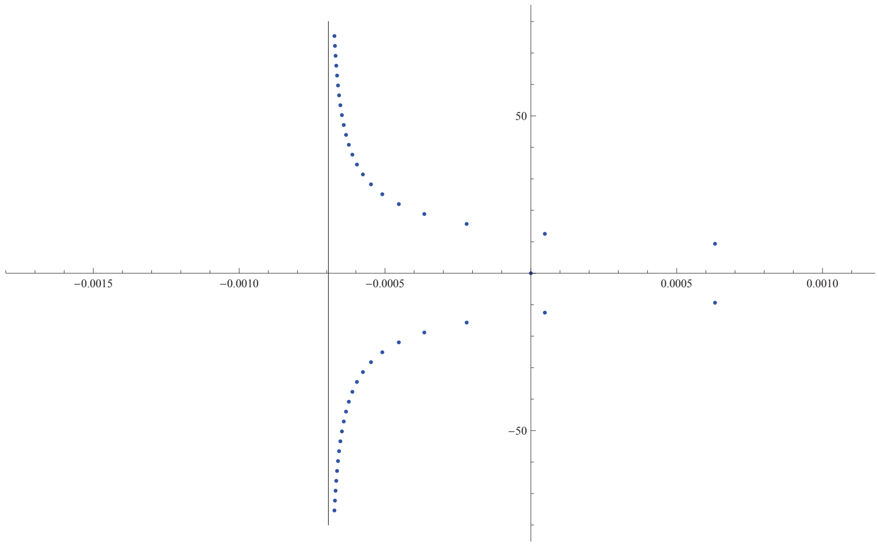

FIGURE 3.6 – Eigenvalues for $a = 1.1,\ b = -2$

By the previous analysis, we know that $\widetilde{\lambda}$ is an eigenvalue of A_0 if and only if

$$\tanh \widetilde{\lambda} = -\frac{1}{b};$$

or equivalently

$$\widetilde{\lambda} = \widetilde{\lambda}_k = -\tanh^{-1}\left(\frac{1}{b}\right) + ik\pi = c(b) + ik\pi, \text{ for some } k \in \mathbb{Z}. \tag{3.4.15}$$

Notice that the root vectors of A_0 are restricted to its eigenvectors since the eigenvalues $\widetilde{\lambda}_k$ are simple. In the sequel, we prove the next Riesz basis property.

Theorem 3.4.4. *The family*

$$\{\widetilde{\phi}_k\}_{k\in\mathbb{Z}} = \{(\widetilde{y}_k, \widetilde{\lambda}_k \widetilde{y}_k)\}_{k\in\mathbb{Z}} = \left\{\left(\frac{1}{\widetilde{\lambda}_k}\sinh(\widetilde{\lambda}_k \cdot), \sinh(\widetilde{\lambda}_k \cdot)\right)\right\}_{k\in\mathbb{Z}}$$

forms a Riesz basis of $X = H_l(0,1) \times L^2(0,1)$.

For this aim, we again use Bari's criterion stated in Theorem 1.2.6.

Lemma 3.4.5. *The sequence $\{\widetilde{\phi}_k\}_k$ is complete in X.*

Proof: It suffices to show that any element of X orthogonal to all the $\widetilde{\phi}_k$ is zero. Hence let $(f,g)^\top \in X$ be such that $< (f,g)^\top, \widetilde{\phi}_k >_X = 0$ for all $k \in \mathbb{Z}$. Then we get

$$\begin{aligned} 0 &= 2\int_0^1 \left(\overline{f_x}\cosh(\widetilde{\lambda}_k x) + \overline{g}\sinh(\widetilde{\lambda}_k x)\right) dx \\ &= \int_0^1 \left((\overline{f_x+g})e^{c(b)x}e^{ik\pi x} + (\overline{f_x-g})e^{-c(b)x}e^{-ik\pi x}\right) dx, \quad \forall k \in \mathbb{Z}. \end{aligned} \tag{3.4.16}$$

In particular, for $k=0$, we have

$$\int_0^1 \left((\overline{f_x+g})e^{c(b)x} + (\overline{f_x-g})e^{-c(b)x}\right) dx = 0. \tag{3.4.17}$$

Moreover, for $k<0$, we write $k=-k'$ with $k' \in \mathbb{N}^*$ to obtain

$$\int_0^1 \left((\overline{f_x+g})e^{c(b)x}e^{-ik'\pi x} + (\overline{f_x-g})e^{-c(b)x}e^{ik'\pi x}\right) dx = 0, \quad \forall k' \in \mathbb{N}^*. \tag{3.4.18}$$

Adding (3.4.16) for $k=k'>0$ with (3.4.18) yields

$$\int_0^1 \overline{h(x)}\left(\frac{e^{ik\pi x}+e^{-ik\pi x}}{2}\right) dx = 0, \quad \forall k \in \mathbb{N}^*,$$

where

$$h(x) = (f_x+g)(x)\overline{e^{c(b)x}} + (f_x-g)(x)\overline{e^{-c(b)x}}.$$

Since $\{\cos(k\pi x)\}_{k\in\mathbb{N}}$ is a basis of $L^2(0,1)$, we get

$$h = 0. \tag{3.4.19}$$

Subtracting (3.4.16) from (3.4.18), we get

$$\int_0^1 \overline{K(x)}\left(\frac{e^{ik\pi x}-e^{-ik\pi x}}{2}\right) dx = 0, \quad \forall k \in \mathbb{N}^*,$$

where

$$K(x) = (f_x+g)(x)\overline{e^{c(b)x}} - (f_x-g)(x)\overline{e^{-c(b)x}}.$$

Since $\{\sin(k\pi x)\}_{k\in\mathbb{N}^*}$ forms a basis of $L^2(0,1)$, we get

$$K = 0. \tag{3.4.20}$$

(3.4.19) and (3.4.20) imply that $f_x = g = 0$ and so $f = 0$ since $f(0) = 0$. ■

In a second step, we search for a sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ biorthogonal to $\{\widetilde{\phi}_k\}_{k\in\mathbb{Z}}$. Here we choose $\{\psi_k\}_{k\in\mathbb{Z}} = \left\{\left(\frac{1}{\overline{\widetilde{\lambda}}_k}\sinh(\overline{\widetilde{\lambda}_k}\cdot), -\sinh(\overline{\widetilde{\lambda}_k}\cdot)\right)\right\}_{k\in\mathbb{Z}}$ where $\overline{\widetilde{\lambda}}_k$ is the conjugate of $\widetilde{\lambda}_k$. The same arguments as before show that this set is complete. Indeed, for $k\in\mathbb{Z}$, ψ_k is an eigenvector of the adjoint of A_0.

Lemma 3.4.6. *The set $\{\psi_k\}_{k\in\mathbb{Z}}$ is biorthogonal to $\{\widetilde{\phi}_k\}_{k\in\mathbb{Z}}$.*

Proof: By definition, we have

$$\begin{aligned}
< \widetilde{\phi}_k, \psi_l >_X &= \int_0^1 \Big(\cosh(\widetilde{\lambda}_k x)\cosh(\widetilde{\lambda}_l x) - \sinh(\widetilde{\lambda}_k x)\sinh(\widetilde{\lambda}_l x)\Big)\,dx \\
&= \int_0^1 \cosh\Big((\widetilde{\lambda}_k - \widetilde{\lambda}_l)x\Big)\,dx \\
&= \int_0^1 \cos\left((k-l)\pi x\right)dx = \delta_{kl}.
\end{aligned}$$

■

Finally, in order to apply Bari's Theorem, it remains to prove (1.2.1). Let $(f,g)^\top \in X$ and consider the following sum

$$\sum_{k\in\mathbb{Z}} \left|< (f,g)^\top, \widetilde{\phi}_k >_X\right|^2 \lesssim \sum_{k\in\mathbb{Z}} \left|(f_x, \cosh(\widetilde{\lambda}_k\cdot))_{(0,1)}\right|^2 + \sum_{k\in\mathbb{Z}} \left|(g, \sinh(\widetilde{\lambda}_k\cdot))_{(0,1)}\right|^2. \tag{3.4.21}$$

By (3.4.15) and Parseval's identity, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left|(f_x,\cosh(\widetilde{\lambda}_k\cdot))_{(0,1)}\right|^2 &\lesssim \sum_{k\in\mathbb{Z}}\left|(f_x\overline{e^{c(b)\cdot}},e^{ik\pi\cdot})_{(0,1)}\right|^2+\sum_{k\in\mathbb{Z}}\left|(f_x\overline{e^{-c(b)\cdot}},e^{-ik\pi\cdot})_{(0,1)}\right|^2\\ &\le \left(\|f_x\overline{e^{c(b)\cdot}}\|^2_{(0,1)}+\|f_x\overline{e^{-c(b)\cdot}}\|^2_{(0,1)}\right)\\ &\lesssim \|f_x\|^2_{(0,1)},\end{aligned} \tag{3.4.22}$$

and

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left|(g,\sinh(\widetilde{\lambda}_k\cdot))_{(0,1)}\right|^2 &\lesssim \sum_{k\in\mathbb{Z}}\left|(g\overline{e^{c(b)\cdot}},e^{ik\pi\cdot})_{(0,1)}\right|^2+\sum_{k\in\mathbb{Z}}\left|(g\overline{e^{-c(b)\cdot}},e^{-ik\pi\cdot})_{(0,1)}\right|^2\\ &\le \left(\|g\overline{e^{c(b)\cdot}}\|^2_{(0,1)}+\|g\overline{e^{-c(b)\cdot}}\|^2_{(0,1)}\right)\\ &\lesssim \|g\|^2_{(0,1)}.\end{aligned} \tag{3.4.23}$$

(3.4.22) and (3.4.23) imply that the right-hand side of (3.4.21) is finite. Similarly, we prove that

$$\sum_{k\in\mathbb{Z}}\left|< (f,g)^\top,\psi_k >_X\right|^2<\infty.$$

In conclusion, by Theorem 1.2.6, the family $\{\tilde{\phi}_k\}_{k\in\mathbb{Z}}$ forms a Riesz basis of X. Theorem 3.4.4 is then proved.

Theorem 3.4.7. *If $b\notin\{-1,0,1\}$, then problem (3.4.1) is well posed.*

Proof: If we consider the problem associated with A_0; i.e.,

$$\begin{cases} U_t(t)=A_0U(t), & t>0,\\ U(0)=U_0, \end{cases} \tag{3.4.24}$$

then writing $U(0)=\sum_{k\in\mathbb{Z}}\alpha_k\widetilde{\phi}_k$ yields

$$U(t)=\sum_{k\in\mathbb{Z}}e^{\widetilde{\lambda}_k t}\alpha_k\widetilde{\phi}_k.$$

Therefore, by the Riesz property of the sequence $\{\widetilde{\phi}_k\}_k$ and (3.4.15), we get

$$\|U(t)\|_X^2\simeq\sum_{k\in\mathbb{Z}}\left|e^{\widetilde{\lambda}_k t}\alpha_k\right|^2\lesssim e^{-2\Re\tanh^{-1}\left(\frac{1}{b}\right)t}\|U(0)\|_X^2. \tag{3.4.25}$$

Hence, if $V=\sum_{k\in\mathbb{Z}}\beta_k\widetilde{\phi}_k$ is given, then we define

$$S(t)V=\sum_{k\in\mathbb{Z}}e^{\widetilde{\lambda}_k t}\beta_k\widetilde{\phi}_k,\quad\forall t>0.$$

According to (3.4.25), we have $S(t)\in\mathcal{L}(X)$ with

$$\|S(t)\|_{\mathcal{L}(X)}\leq Me^{-\Re\tanh^{-1}\left(\frac{1}{b}\right)t},$$

for some positive constant M. Hence we deduce that $(S(t))_{t\geq0}$ is a C_0 semigroup (not necessarily uniformly bounded). As we can write $A_a=A_0+aB$ where A_0 generates the C_0 semigroup $(S(t))_{t\geq0}$ and B is bounded, then, by Theorem 3.1.1 in [62], A_a also generates a C_0 semigroup $(S_a(t))_{t\geq0}$ that satisfies

$$\|S_a(t)\|_{\mathcal{L}(X)}\leq Me^{(-\Re\tanh^{-1}(\frac{1}{b})+|a|\|B\|)t},\quad\forall t>0.$$

By standard semigroup theory, problem (3.4.1) becomes well posed. ■

Remark 3.4.8. *If $b=-1$, our previous considerations show that A_0 has an empty spectrum. Therefore, our method does not allow to prove that it generates a C_0 semigroup and hence, the well posedness of problem (3.4.1) becomes an open question.*

On the other hand, if $b=1$, the operator A_0 generates a C_0 semigroup $(S(t))_{t\geq0}$ that satisfies

$$\|S(t)\|_{\mathcal{L}(X)}\leq Me^{-\omega t},$$

for some positive constants M and ω. Therefore, by the previous arguments, problem (3.4.1) is well posed and is stable if $a < 0$ is small enough by perturbation theory but the question of its stability for $a < 0$ "large" is an open question.

3.4.5 Link between problems (3.1.2) and (3.4.14) and end of the Proof of Theorem 3.2.3

In this part, we prove that the root vectors of A_a form a Riesz basis of X if $b \notin \{-1, 0, 1\}$. For this aim, we will apply Theorem 1.2.8 with the set $\{\widetilde{\phi}_k\}_{k\in\mathbb{Z}}$ the Riesz basis associated with problem (3.4.14) and for an appropriate set of eigenvectors of A_a (corresponding to large eigenvalues). Indeed in view of Lemma 3.4.3 and (3.4.12), we split up the spectrum of A_a into the small and large eigenvalues :

$$\sigma(A_a) = \{\lambda_{k_i}\}_{i=1}^{M} \cup \{\lambda_k\}_{|k|>N}, \tag{3.4.26}$$

where $N \in \mathbb{N}$ is chosen large enough such that, for every $k \in \mathbb{Z}^*$ with $|k| > N$, $|\lambda_k + \frac{a}{2} + \tanh^{-1}\frac{1}{b} - ik\pi| < \frac{\pi}{2}$ and λ_k is simple. Consequently the remaining part of the spectrum $\{\lambda_{k_i}\}_{i=1}^{M}$ is clearly bounded. According to our previous considerations for $|k| > N$, the eigenvector ϕ_k is of the form

$$\phi_k = (y_k, \lambda_k y_k)^\top,$$

with

$$y_k(x) = \frac{1}{\sqrt{\lambda_k^2 + a\lambda_k}} \sinh\left(\sqrt{\lambda_k^2 + a\lambda_k}\, x\right), \forall x \in (0,1).$$

For $|k| > N$, by (3.4.9) and (3.4.11), we have

$$\sqrt{\lambda_k^2 + a\lambda_k} = -\tanh^{-1}\frac{1}{b} + ik\pi + O\left(\frac{1}{\lambda_k}\right) = \tilde{\lambda}_k + O\left(\frac{1}{\lambda_k}\right).$$

Hence, by the mean value theorem, for all $x \in (0,1)$, there exists $\theta_x \in (0,1)$ (depending also on k) such that

$$\begin{aligned}
&\cosh(\sqrt{\lambda_k^2 + a\lambda_k}x) - \cosh(\widetilde{\lambda}_k x) \\
= \quad & \left(\sqrt{\lambda_k^2 + a\lambda_k} - \tilde{\lambda}_k\right) x \sinh\left(\widetilde{\lambda}_k x + \theta_x(\sqrt{\lambda_k^2 + a\lambda_k} - \widetilde{\lambda}_k)x\right).
\end{aligned}$$

Hence, by the previous identity, we find that for all $x \in (0,1)$

$$\left|\cosh(\sqrt{\lambda_k^2 + a\lambda_k}x) - \cosh(\widetilde{\lambda}_k x)\right| \lesssim |\lambda_k|^{-1}| \sinh\left(\widetilde{\lambda}_k x + \theta_x(\sqrt{\lambda_k^2 + a\lambda_k} - \widetilde{\lambda}_k)x\right)|.$$

Moreover, since we assume that $|b| \neq 1$, then $\left|\Re \tanh^{-1} \frac{1}{b}\right|$ is finite and therefore, $\Re\widetilde{\lambda}_k$ remains bounded (independently of k). This implies that

$$\left|\cosh(\sqrt{\lambda_k^2 + a\lambda_k}x) - \cosh(\widetilde{\lambda}_k x)\right| \lesssim |\lambda_k|^{-1}, \forall x \in (0,1).$$

This estimate implies that

$$\|(y_k)_x - (\widetilde{y}_k)_x\|^2_{(0,1)} \lesssim \frac{1}{|\lambda_k|^2}. \tag{3.4.27}$$

Similarly, we can prove that

$$\left\|\lambda_k y_k - \widetilde{\lambda}_k \widetilde{y}_k\right\|^2_{(0,1)} \lesssim \frac{1}{|\lambda_k|^2}. \tag{3.4.28}$$

The estimates (3.4.27) and (3.4.28) yield

$$\sum_{|k|>N} \left\|\phi_k - \widetilde{\phi}_k\right\|^2_X \lesssim \sum_{|k|>N} \frac{1}{|\lambda_k|^2} \lesssim \sum_{|k|>N} \frac{1}{k^2} < \infty.$$

In conclusion, according to Theorem 1.2.8, the root vectors of A_a form a Riesz basis of X.

Similar to (3.3.25), we conclude the proof of Theorem 3.2.3 for all $a \in \mathbb{R}$, $b \notin \{-1,0,1\}$ such that $-\dfrac{a}{2} - \Re \tanh^{-1} \dfrac{1}{b} < 0$ since the high frequencies are situated to the left of the imaginary axis.

3.4.5.1 Further Comments

Theorem 1.2.8 improves Bari's Theorem as it shows that we can neglect any information concerning the low frequencies. However, in the sequel, as we are interested in studying the behavior of the eigenvalues, we give an additional analysis to show that indeed the root vectors corresponding to the low eigenvalues of A_a are in bijection with those of A_0.

For the low frequency modes ($|k| < N$), we fix a sufficiently large rectangle Γ which includes all the low frequencies of $A_{a'}$ for all a' between zero and a and whose edges do not contain any eigenvalue of $A_{a'}$ for any a' between zero and a. This choice of the rectangle is possible by the following arguments. First for the horizontal edges we notice that, by (3.4.12), the horizontal lines

$$y = -\Im \tanh^{-1} \frac{1}{b} \pm \frac{(2k_0+1)\pi}{2}, \text{ for } |k_0| > N$$

are free of eigenvalues of $A_{a'}$ for all $|a'| \leq |a|$ if N is large enough (depending on a). For an upper vertical line, by Theorem 3.4.7, any eigenvalue λ of $A_{a'}$ satisfies

$$\Re\lambda \leq -\Re \tanh^{-1}(\frac{1}{b}) + |a'| \|B\|.$$

Hence, the vertical line

$$x = 1 - \Re \tanh^{-1}(\frac{1}{b}) + |a| \|B\|$$

does not contain any eigenvalues of $A_{a'}$ for all $|a'| \leq |a|$.

Finally, for the lower vertical line, for a fixed a, we denote by

$$m(a) = \min\{\Re\lambda : \lambda \in \sigma(A_a)\},$$

that is clearly finite. We now show that for a fixed a, we have

$$I := \inf_{|a'| \leq |a|} m(a') > -\infty.$$

Indeed, if it would be false, then we would find a sequence $(a_n)_{n\in\mathbb{N}}$ with $|a_n| \leq |a|$ for all n and such that

$$m(a_n) \to -\infty \text{ as } n \to \infty. \tag{3.4.29}$$

Up to a subsequence, still denoted by $(a_n)_{n\in\mathbb{N}}$, $(a_n)_{n\in\mathbb{N}}$ converges to some $a' \in [-|a|, |a|]$. Furthermore, there exists $m \in \mathbb{N}$ such that for all $n \geq m$, there exists an eigenvalue $\lambda_n \in \sigma(A_{a_n})$ such that

$$m(a_n) = \Re\lambda_n.$$

Indeed, if we assume that for all $m \in \mathbb{N}$, there exists $n_m \geq m$ such that for all $\lambda \in \sigma(A_{a_{n_m}})$, $m(a_{n_m}) \neq \Re\lambda$, then necessarily $m(a_{n_m}) = -\frac{a_{n_m}}{2} - \Re\tanh^{-1}\frac{1}{b}$ which is impossible since in this case, as $m \to +\infty$, $m(a_{n_m})$ tends to $-\frac{a'}{2} - \Re\tanh^{-1}\frac{1}{b}$ which is finite and this contradicts (3.4.29). Therefore, there exists $m \in \mathbb{N}$ such that for all $n \geq m$, there exists an eigenvalue $\lambda_n \in \sigma(A_{a_n})$ such that

$$\Re\lambda_n \to -\infty \text{ as } n \to \infty.$$

At this stage we can apply (3.4.12) to λ_n and by taking the real part of this identity we find

$$\Re\lambda_n = -\frac{a_n}{2} - \Re\tanh^{-1}\frac{1}{b} + \left(\frac{a_n^2}{8} + \frac{a_n b}{2(1-b^2)}\right)\Re\frac{1}{\lambda_n} + o\left(\frac{1}{\lambda_n}\right). \tag{3.4.30}$$

Here above $o\left(\frac{1}{\lambda_n}\right)$ depends on a_n but it is easy to see that

$$o\left(\frac{1}{\lambda_n}\right) \to 0 \text{ as } n \to \infty.$$

because the sequence a_n is bounded. Passing to the limit in (3.4.30), we find that the left-hand side tends to $-\infty$ while the right-hand side tends to $-\frac{a'}{2} - \Re\tanh^{-1}\frac{1}{b}$ which is a contradiction.

Accordingly, the line

$$x = -1 + I$$

does not contain any eigenvalues of $A_{a'}$ for all $|a'| \leq |a|$.

Having chosen Γ, we define the operator

$$P(a') = \frac{1}{2\pi i} \oint_{\Gamma} (\xi - A_{a'})^{-1} d\xi. \tag{3.4.31}$$

According to Theorem III.6.17 of [46] or Theorem XII.5 of [68], the above operator is a projection (called eigenprojection) and its range is the set of the generalized eigenspace corresponding to the eigenvalues of $A_{a'}$ inside Γ. Besides, we remark that $A_{a'}$ is closed for any a'.

In the sequel, we prove that the mapping F defined by $F(a') = \dim(P(a')X)$ for all a' between zero and a is continuous.

Lemma 3.4.9. *Fix a_0 between zero and a and let $\xi \in \rho(A_{a_0}) \cap \Gamma$. Then $\xi \in \rho(A_{a'})$ for any a' near a_0 and*

$$(\xi - A_{a'})^{-1} \to (\xi - A_{a_0})^{-1} \quad \textit{as } a' \to a_0.$$

Proof: Without loss of generality, assume that $a_0 = 0$ and let $\xi \in \rho(A_0) \cap \Gamma$. We have

$$\xi - A_{a'} = \xi - A_0 - a'B = (\xi - A_0)[I - a'(\xi - A_0)^{-1}B].$$

Since $a' \to 0$, then we can choose a' such that $\|a'(\xi - A_0)^{-1}B\| < 1$ and hence $I - a'(\xi - A_0)^{-1}B$ is invertible. Consequently, $\xi - A_{a'}$ is invertible which implies that $\xi \in \rho(A_{a'})$ and as $a' \to 0$,

$$\left\|(\xi - A_{a'})^{-1} - (\xi - A_0)^{-1}\right\|_{\mathcal{L}(X)} \leq \left\|(I - a'(\xi - A_0)^{-1}B)^{-1} - I\right\|_{\mathcal{L}(X)} \left\|(\xi - A_0)^{-1}\right\|_{\mathcal{L}(X)} \to 0.$$

■

Now, we recall a result of Kato and Rellich, see the Lemma in page 14 of [68].

Lemma 3.4.10. *If P and Q are two (not necessarily orthogonal) projections in a Hilbert space H and* $\dim(PH) \neq \dim(QH)$*, then* $\|P - Q\| \geq 1$.

Lemma 3.4.9 shows that $P(a')$ is a continuous function of a' while the above Lemma shows that the mapping

$$\{Q \in \mathcal{L}(X);\ Q \text{ is a projection}\} \mapsto \mathbb{N} : Q \to \dim(QX)$$

is continuous. Therefore, the mapping $a' \to P(a') \to \dim(P(a')X)$ is continuous and hence $\dim(P(a')X)$ is constant for every a' between zero and a. Knowing that the eigenvalues of A_0 inside Γ are of finite multiplicity, then we get $\dim P(a)X = \dim P(0)X$. Therefore, we conclude that the number of eigenvalues of A_a is equal to the number of eigenvalues of A_0 inside Γ with the same total number of multiplicity. Consequently, the root vectors corresponding to the eigenvalues of A_a inside Γ are in bijection with those of A_0.

3.5 Proof of Theorem 3.2.4

In this section, we consider the case $b \in (-1, 0)$ and $a \geq -2\tanh^{-1} b$. We prove that all the eigenvalues of A_a are situated to the left of the axis $x = -\dfrac{a}{2} - \Re \tanh^{-1} \dfrac{1}{b} = -\dfrac{a}{2} - \tanh^{-1} b$. Consequently, in the case $a > -2\tanh^{-1} b$, by the arguments of the previous section, we immediately deduce that problem (3.1.2) is exponentially stable in X. In the case $a = -2\tanh^{-1} b$, due to (3.4.12) no exponential decay can be expected but we will show in the last section that a polynomial decay is available.

Lemma 3.5.1. *If $b \in (-1, 0)$ and $a \geq -2\tanh^{-1} b$, then any eigenvalue λ of A_a satisfies*

$$\Re\lambda < -\frac{a}{2} - \tanh^{-1} b.$$

Proof: Recall that from Remark 3.4.1 any eigenvalue λ of A_a satisfies $\Re\lambda > -\dfrac{a}{2}$ (since by our assumptions $b < 0$ and $a > 0$).

First, according to the characteristic equation (3.4.8), we can write

$$\lambda = -\frac{u}{b\tanh u}. \tag{3.5.1}$$

where $u = \sqrt{\lambda^2 + a\lambda}$. Using the identity $\tanh u = \frac{z-1}{z+1}$, with $z = e^{2u}$, (3.5.1) is equivalent to

$$\lambda = -\frac{u(z+1)}{b(z-1)} \text{ with } z = e^{2u}.$$

Substituting this identity into $u^2 = \lambda^2 + a\lambda$ yields $u = g(z)$, where

$$g(z) = ab\left(c_0 - \frac{c_1}{z+b_1} - \frac{c_2}{z+b_2}\right),$$

with $c_0 = \dfrac{1}{1-b^2}$, $c_1 = \dfrac{1}{(1-b)^2}$, $c_2 = \dfrac{1}{(1+b)^2}$, $b_1 = \dfrac{1+b}{1-b}$, and $b_2 = \dfrac{1-b}{1+b}$. Replacing z by e^{2u}, we obtain

$$u = ab\left(c_0 - \frac{c_1}{e^{2u}+b_1} - \frac{c_2}{e^{2u}+b_2}\right). \tag{3.5.2}$$

Remark that for $b \in (-1, 0)$, $0 < b_1 < 1 < b_2$. Note further that the case $e^{2u}+b_1 = 0$ (resp. $e^{2u}+b_2 = 0$) cannot hold ; indeed, we then have $\tanh u = \frac{1}{b}$ (resp. $\tanh u = -\frac{1}{b}$) and therefore, by (3.5.1), $\lambda = -u$ (resp. $\lambda = u$) which yields

$$\lambda^2 = \lambda^2 + a\lambda,$$

and hence $\lambda = 0$. This is impossible since 0 is not an eigenvalue of A_a.

Writing $u = U + iV$, with $U, V \in \mathbb{R}$, we can suppose that $U \geq 0$ and $V \geq 0$ since the complex eigenvalues appear in conjugate pairs. Indeed, $u^2 = \lambda^2 + a\lambda$ implies that $y(2x+a) = 2UV$ where $\lambda = x + iy$. As $x \geq -\dfrac{a}{2}$ so if $y \geq 0$ then U and V have the same sign. Otherwise, we choose $\lambda = x - iy$ to get $U \geq 0$ and $V \geq 0$.

In a first step, we prove that

$$U = \Re u < \frac{\ln b_2}{2} = -\tanh^{-1} b. \tag{3.5.3}$$

For this aim, by setting for $j = 1, 2$,

$$\Sigma_j = \frac{1}{2} - \frac{b_j}{e^{2u} + b_j} = \frac{1}{2}\tanh\left(u - \frac{1}{2}\ln b_j\right), \tag{3.5.4}$$

we notice that (3.5.2) implies that

$$u = \frac{ab}{1-b^2}(\Sigma_1 + \Sigma_2). \tag{3.5.5}$$

Simple calculations show that

$$\Re\Sigma_j = \frac{e^{4U} - b_j^2}{2|e^{2u} + b_j|^2} = \frac{\sinh(2U - \ln b_j)}{2(\cos(2V) + \cosh(2U - \ln b_j))}. \tag{3.5.6}$$

Hence, by the property $0 < b_1 < 1$, we directly see that $\Re\Sigma_1 > 0$.

Now if we suppose that (3.5.3) does not hold, then $U \geq \frac{\ln b_2}{2} \geq 0$ and by (3.5.6), we get $\Re\Sigma_2 \geq 0$. But from (3.5.5) and this property, we deduce that

$$U = \Re u = \frac{ab}{1-b^2}(\Re\Sigma_1 + \Re\Sigma_2) < 0,$$

which is a contradiction. Hence (3.5.3) holds.

In a second step, we check that $U \neq 0$. Indeed if $U = 0$, then $\lambda \in \mathbb{R}$ since by (3.5.1) we find out that

$$\lambda = -\frac{V}{b\tan V}, \tag{3.5.7}$$

with $V \in \mathbb{R} \setminus \{0\}$ (because $\lambda = 0$ and $\lambda = -a$ are not eigenvalues of A_a) such that

$$\sqrt{\lambda^2 + a\lambda} = iV.$$

Hence, we see that

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4V^2}}{2},$$

that is always non positive. This is in contradiction with (3.5.7) because its right-hand side is positive.

In a third step, we show that the eigenvalues of A_a are situated to the left of the axis $-\frac{a}{2} - \Re \tanh^{-1}\frac{1}{b} = -\frac{a}{2} - \tanh^{-1} b$.

Substituting (3.5.2) into (3.5.1), we find, after simple calculations, that

$$\lambda = -\frac{a}{1-b^2} - \frac{ab}{1-b^2}(\Sigma_1 - \Sigma_2). \tag{3.5.8}$$

Hence, summing (3.5.8) with (3.5.5) and subtracting (3.5.8) with (3.5.5), we obtain

$$\Sigma_1 = -\frac{1}{2b} + \frac{b^2-1}{2ab}(\lambda - u), \tag{3.5.9}$$

$$\Sigma_2 = \frac{1}{2b} + \frac{1-b^2}{2ab}(\lambda + u). \tag{3.5.10}$$

Now coming back to (3.5.6), we can write (note that $\Re\Sigma_j \neq 0$, thanks to (3.5.3))

$$\cos(2V) = -\cosh(2U - \ln b_j) + \frac{\sinh(2U - \ln b_j)}{2\Re\Sigma_j}, \text{ for } j = 1, 2.$$

This implies that

$$\Re\Sigma_2\Big(\sinh(2U-\ln b_1)-2\Re\Sigma_1\cosh(2U-\ln b_1)\Big) = \Re\Sigma_1\Big(\sinh(2U-\ln b_2)-2\Re\Sigma_2\cosh(2U-\ln b_2)\Big).$$

Using (3.5.9) and (3.5.10), we get, again after simple calculations, the following relation between $x = \Re\lambda$ and $U = \Re u$:

$$x^2 k_2 \sinh(2U) + x k_1 \sinh(2U) + h_0(U) = 0, \tag{3.5.11}$$

where $k_2 = 4b(b-1)(1+b) > 0$, $k_1 = 2ab(b^2-3) > 0$, and

$$h_0(U) = -2b\left((a^2 - 2U^2 + 2b^2U^2)\sinh(2U) + 2abU\cosh(2U)\right).$$

As by the second step, $U \neq 0$, we can divide (3.5.11) by $\sinh(2U)$ and find

$$k_2 x^2 + k_1 x + k_0(U) = 0, \tag{3.5.12}$$

where

$$k_0(U) = \frac{h_0(U)}{\sinh 2U} = -2b\left(a^2 - 2U^2 + 2b^2U^2 + 2ab\frac{U}{\tanh(2U)}\right).$$

It turns out that $k_0(U) > 0$ for $0 < U \leq -\tanh^{-1} b$ since k_0 is a non increasing function on $(0, \infty)$ and $k_0(-\tanh^{-1} b) > 0$. As the coefficients in (3.5.12) are all positive, $x = \Re\lambda$ has to be negative. In fact, (3.5.12) yields two distinct roots $x_{\pm}(U)$ given by

$$x_{\pm}(U) = \frac{-k_1 \pm \sqrt{k_1^2 - 4k_2 k_0(U)}}{2k_2},$$

such that $x_-(U) \leq x_+(U)$. Again as k_0 is a non increasing function on $(0, \infty)$ and recalling (3.5.3), we get

$$\begin{aligned} x_-(U) \leq x_+(U) < x_+(-\tanh^{-1} b) &= \frac{-k_1 + \sqrt{k_1^2 - 4k_2 k_0(-\tanh^{-1} b)}}{2k_2} \\ &= -\frac{a}{2} - \tanh^{-1} b. \end{aligned} \tag{3.5.13}$$

■

Remark 3.5.2. *Increasing the order of the asymptotic development of the large eigenvalues, we find that for some $N > 0$ large enough and for every $k \in \mathbb{Z}^*$ such that $|k| > N$*

$$\begin{aligned} \lambda_k &= -\frac{a}{2} - \tanh^{-1}\frac{1}{b} + ik\pi + \frac{c}{\lambda} + \frac{\tilde{c}}{\lambda^2} + o(\frac{1}{\lambda^2}) \\ &= -\frac{a}{2} - \tanh^{-1}\frac{1}{b} + ik\pi - \frac{ic}{k\pi} - \frac{c}{k^2\pi^2}\left(\frac{a}{2} + \tanh^{-1}\frac{1}{b}\right) - \frac{\tilde{c}}{k^2\pi^2} + o\left(\frac{1}{k^2}\right), \end{aligned}$$

where $c = \frac{a^2}{8} + \frac{ab}{2(1-b^2)}$ and $\tilde{c} = \frac{a^2}{8}\left(\frac{b(b^2-3)}{(1-b)^2(1+b)^2} - \frac{a}{2}\right)$. We can check that, in case $a > 0$ and $b \in (-1, 0)$, the large eigenvalues approach the axis $x = -\frac{a}{2} - \Re\tanh^{-1}\frac{1}{b} = -\frac{a}{2} - \tanh^{-1} b$ from the left. Indeed, when $a > 0$ and $b \in (-1, 0)$ we can prove that

$$\begin{aligned} &\Re\left(ik\pi - \frac{ic}{k\pi} - \frac{c}{k^2\pi^2}\left(\frac{a}{2} + \tanh^{-1}\frac{1}{b}\right) - \frac{\tilde{c}}{k^2\pi^2}\right) \\ &= -\frac{1}{k^2\pi^2}\left(c\left(\frac{a}{2} + \Re\tanh^{-1}\frac{1}{b}\right) + \tilde{c}\right) < 0. \end{aligned}$$

3.6 Polynomial Stability of problem (3.1.2) and Proof of Theorem 3.2.5

We end up by proving the polynomial stability of problem (3.1.2) in the case $b \in (-1, 0)$ and $a = -2\tanh^{-1} b$. By Lemma 3.5.1, the spectrum of A_a is at the left of the imaginary axis, but approaches this axis. Hence, the decay of the energy depends on the way the large eigenvalues approach this axis. Therefore, we need to precise the asymptotic behavior (3.4.12). Again we use the splitting (3.4.26) of the spectrum of A_a into the small and large eigenvalue.

As before, using Taylor expansion for every $k \in \mathbb{Z}^*$ with $|k| > N$, λ_k is simple and is given by

$$\lambda_k = -\frac{a}{2} - \tanh^{-1}\frac{1}{b} + ik\pi - \frac{ic}{k\pi} - \frac{c}{k^2\pi^2}\left(\frac{a}{2} + \tanh^{-1}\frac{1}{b}\right) - \frac{\tilde{c}}{k^2\pi^2} + o\left(\frac{1}{k^2}\right),$$

where $c = \frac{a^2}{8} + \frac{ab}{2(1-b^2)}$ and $\tilde{c} = \frac{a^2}{8}\left(\frac{b(b^2-3)}{(1-b)^2(1+b)^2} - \frac{a}{2}\right)$. Since $a = -2\tanh^{-1} b$, taking the real part of this expression, we find that

$$\Re\lambda_k = -\frac{\tilde{c}}{k^2\pi^2} + o\left(\frac{1}{k^2}\right). \tag{3.6.1}$$

Note that we can prove that $\tilde{c} > 0$ for every $b \in (-1, 0)$ and $a = -2\tanh^{-1} b$.

If m_{k_i} denotes the multiplicity of λ_{k_i} for every $i = 1, ..., M$, then we denote by $\{\{\varphi_{k_i,j}\}_{j=0}^{m_{k_i}-1}\}_{i=1}^{M} \cup \{\varphi_k\}_{|k|>N}$ the Riesz basis of X formed of the normalized root vectors of A_a (recall that m_{k_i} is one or two). Hence, if we write the initial datum $U(0)$ in this basis

$$U(0) = \sum_{i=1}^{M}\sum_{j=0}^{m_{k_i}-1} \gamma_{k_i,j}\varphi_{k_i,j} + \sum_{|k|>N} \gamma_k\varphi_k, \tag{3.6.2}$$

then the solution $U(t)$ is given by

$$U(t) = \sum_{i=1}^{M} e^{\lambda_{k_i}t} \sum_{j=0}^{m_{k_i}-1} \gamma_{k_i,j} \sum_{n=0}^{j} \frac{t^{j-n}}{(j-n)!}\varphi_{k_i,n} + \sum_{|k|>N} e^{\lambda_k t}\gamma_k\varphi_k.$$

Therefore, for $t > 0$ and $\delta = \dfrac{\tilde{c}}{2\pi^2}$, by (3.6.1) we get

$$\begin{aligned} E(t) = \tfrac{1}{2}\|U(t)\|^2 & \lesssim \sum_{i=1}^{M} e^{2\Re\lambda_{k_i}t} \sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{i=1}^{M} e^{2\Re\lambda_{k_i}t} \sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 t^{2j} \\ & + \sum_{|k|>N} e^{2\Re\lambda_k t}|\gamma_k|^2 \\ & \lesssim \sum_{i=1}^{M} e^{\Re\lambda_{k_i}t} \sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} e^{-\frac{2\delta}{k^2}t}|\gamma_k|^2 \qquad (3.6.3) \\ & \lesssim \frac{1}{t}\left(\sum_{i=1}^{M} \sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} k^2|\gamma_k|^2 \right) \\ & \lesssim \frac{\|U(0)\|^2_{D(A_a)}}{t}, \end{aligned}$$

because

$$e^{-\frac{2\delta}{k^2}t} \lesssim \frac{k^2}{t}, \quad \forall t > 0, k \in \mathbb{N}^*.$$

In the last step above we also use the equivalence

$$\|U(0)\|^2_{D(A_a)} = \|U(0)\|^2_X + \|A_a U(0)\|^2_X \simeq \sum_{i=1}^{M} \sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} k^2|\gamma_k|^2,$$

that follows from the Riesz basis property of $\{\{\varphi_{k_i,j}\}_{j=0}^{m_{k_i}-1}\}_{i=1}^{M} \cup \{\varphi_k\}_{|k|>N}$. Indeed, by (3.6.2), we may write

$$\begin{aligned} A_a U(0) & = \sum_{i=1}^{M} \lambda_{k_i}\gamma_{k_i,0}\varphi_{k_i,0} + \sum_{i=1}^{M} \sum_{j=1}^{m_{k_i}-1} \gamma_{k_i,j}(\lambda_{k_i}\varphi_{k_i,j} + \varphi_{k_i,j-1}) + \sum_{|k|>N} \gamma_k\lambda_k\varphi_k \\ & = \sum_{i=1}^{M} \lambda_{k_i}\gamma_{k_i,0}\varphi_{k_i,0} + \sum_{i=1}^{M} \lambda_{k_i}\gamma_{k_i,m_{k_i}-1}\varphi_{k_i,m_{k_i}-1} + \sum_{i=1}^{M} \gamma_{k_i,1}\varphi_{k_i,0} \\ & + \sum_{i=1}^{M} \sum_{j=1}^{m_{k_i}-2} (\gamma_{k_i,j}\lambda_{k_i} + \gamma_{k_i,j+1})\varphi_{k_i,j} + \sum_{|k|>N} \gamma_k\lambda_k\varphi_k \\ & = \sum_{i=1}^{M} \lambda_{k_i}\gamma_{k_i,m_{k_i}-1}\varphi_{k_i,m_{k_i}-1} + \sum_{i=1}^{M} \sum_{j=0}^{m_{k_i}-2} (\gamma_{k_i,j}\lambda_{k_i} + \gamma_{k_i,j+1})\varphi_{k_i,j} + \sum_{|k|>N} \gamma_k\lambda_k\varphi_k. \end{aligned}$$

As $\{\{\varphi_{k_i,j}\}_{j=0}^{m_{k_i}-1}\}_{i=1}^{M} \cup \{\varphi_k\}_{|k|>N}$ is a Riesz basis of X, we get

$$\begin{aligned}
\|U(0)\|_X^2 &\simeq \sum_{i=1}^{M}\sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} |\gamma_k|^2, \\
\|A_a U(0)\|^2 &\simeq \sum_{i=1}^{M} |\lambda_{k_i}\gamma_{k_i,m_{k_i}-1}|^2 + \sum_{i=1}^{M}\sum_{j=0}^{m_{k_i}-2} |\gamma_{k_i,j}\lambda_{k_i} + \gamma_{k_i,j+1}|^2 + \sum_{|k|>N} |\gamma_k|^2|k|^2.
\end{aligned}$$

These equivalences directly yield

$$\|U(0)\|_{D(A_a)}^2 \gtrsim \sum_{i=1}^{M}\sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} k^2|\gamma_k|^2,$$

while the converse estimate follows from the fact that the set of "small" eigenvalues is bounded.

Remark 3.6.1. *If $b \in (-1,0)$ and $a = -2\tanh^{-1} b$, then, given $U(0) = (u_0,u_1)^\top \in D(A_a^n)$ for some $n \in \mathbb{N}^*$, we get*

$$E(t) \lesssim \frac{\|U(0)\|_{D(A_a^n)}^2}{t^n}. \tag{3.6.4}$$

Consequently, the more regular the initial data is chosen, the faster is the rate of polynomial decay.

Proof: As before we can show that

$$\|U(0)\|_{D(A_a^n)}^2 = \sum_{\ell=0}^{n} \|A_a^\ell U(0)\|_X^2 \simeq \sum_{i=1}^{M}\sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} k^{2n}|\gamma_k|^2.$$

Now, as in (3.6.3), we have

$$E(t) \lesssim \sum_{i=1}^{M} e^{\Re\lambda_{k_i} t} \sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} e^{-\frac{2\delta}{k^2}t}|\gamma_k|^2,$$

and since

$$e^{-\frac{2\delta}{k^2}t} \lesssim \frac{k^{2n}}{t^n}, \quad \forall t > 0, k \in \mathbb{N}^*,$$

we obtain (3.6.4). ∎

3.7 Open questions

The critical value of α found in Theorem 3.3.8, $\alpha_3 \simeq -0.2823$, for which problem (3.1.1) becomes exponentially stable for $\alpha > \alpha_3$ shows that the result given by the perturbation theory of contractive semigroups is not optimal. However, as the numerical result yields a wider range of this critical value, $\alpha > \alpha_2$ where $\alpha_2 \simeq -0.77$, the question of the optimality of α appearing in (3.1.1) remains an open problem.

As for the second problem (3.1.2), necessary and sufficient conditions are found so that (3.1.2) is exponentially or polynomially stable. Optimal results are attained for $b \in (-1, 0)$. If $b \leq -1$, then the question of the stability becomes an open question. Furthermore, the analysis done for problem (3.1.2) can be well adapted to study the stability of the solution of

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + u + au_t(x,t) = 0, & x \in (0,1),\ t > 0, \\ u_x(0,t) = b_0 u_t(0,t),\ u_x(1,t) = -bu_t(1,t), & t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \end{cases}$$

where $bb_0 < 0$ and $a \in \mathbb{R}$.

Chapitre 4

Exponential stability of the wave equation on a star shaped network with indefinite sign damping

4.1 Introduction

As in [2], for $N \geq 2$, we consider the following wave equation on a star shaped network :

$$\begin{cases} u^i_{tt}(x,t) - u^i_{xx}(x,t) + 2a_i(x)u^i_t(x,t) &= 0, \qquad x \in (0, L_i),\ t > 0, i \in \{1, \cdots, N\}, \\ u^i(L_i, t) &= 0, \\ u^i(0,t) &= u^j(0,t), \quad \forall i \neq j, \\ \displaystyle\sum_{i=1}^{N} u^i_x(0,t) &= 0, \\ u^i(x,0) &= u^i_0(x), \quad x \in (0, L_i), \\ u^i_t(x,0) &= u^i_1(x), \quad x \in (0, L_i). \end{cases} \tag{S_1}$$

where $L_i \in \mathbb{R}_*^+$, and $a_i \in W^{1,\infty}(0, L_i)$. This system models the vibrations of a group of strings attached at one extremity. The Kirchoff law $\sum_{i=1}^{N} u_x^i(0,t) = 0$ follows from the principle of stationary action [53, 59].

The main goal of this work is to study the stability of system (S_1) but also to give more precise results when we replace in the system the damping coefficients $a_i(x)$ by $\epsilon a_i(x)$, where the parameter ϵ is positive and small enough. In this case, we will denote this modified system by (S_ϵ) and we only need that $a_i \in L^\infty(0, L_i)$ for all $i \in \{1, \cdots, N\}$.

Using observability inequalities, the stability of the wave equations over a network with positive damping coefficients has been studied in [60]. In the case of one interval, the stability of a wave equation with an indefinite sign damping coefficient has been studied in [1, 28, 30, 51, 54, 57], where it was found that the stability of the wave equation is related to the mean of the damping coefficient. In this chapter, as in [2], using spectral analysis, we find (sufficient) conditions on the damping coefficients to get the exponential stability of (S_1) and (S_ϵ). In fact, we find necessary and sufficient conditions for which (S_1) is exponentially stable up to a finite dimensional space. The idea is inspired from [65] where the characteristic equation of (S_1) is approximated by another one using the shooting method. This approximation allows us to determine the behavior of the high frequencies and hence to deduce the conditions on the damping coefficients $\{a_i\}_{i=1}^N$ for which the high frequencies are situated to the left of the imaginary axis. In a second step, we prove that the generalized root vectors form a Riesz basis with parentheses and then deduce the exponential stability of (S_1) up to a finite dimensional space generated by the roots vectors corresponding to the low frequencies. Note that the shooting method in [23] based on the ansatz of Horn in [41] and used to analyze the high frequencies cannot be easily adapted to our problem as long as the solution in [23] is written in a power series form with unknown coefficients. On the other hand, when $\{a_i\}_{i=1}^N$ is replaced by $\{\epsilon a_i\}_{i=1}^N$ with the parameter ϵ small enough, we search for sufficient conditions for which (S_ϵ) is exponentially stable in the whole energy space. In this case, we note that the positivity of the mean of the damping coefficients in addition to another condition are required (see Theorem 4.1.4 below). In fact, for $\epsilon > 0$ small enough, unlike [23], we deal with multiple eigenvalues. Note that the study of the exponential stability of (S_ϵ) enters in the framework of the abstract theory done in [51]. Using the concepts introduced in [46] about the behavior of the spectrum, we shall interpret the hypothesis imposed in [51] to find explicit conditions on the damping coefficients for which (S_ϵ) is exponentially stable.

Throughout this chapter, we make the following hypothesis on the geometry of the domain :

(**H**) There exists $q \in \mathbb{N}^*$ such that for all $i = 2, \cdots, N$, there exists $p_i \in \mathbb{N}^*$ for which $L_i = \frac{p_i}{q} L_1$.

In applications, the above hypothesis is more realistic. From a mathematical point of view, this above hypothesis is considered since otherwise when some of the lengths take irrational values, then we can find examples for which numerically we see that the spectrum is not structured (for instance there is no asymptotes) and an infinite number of eigenvalues are situated to the right of the imaginary axis (see Figure 4.1). Moreover, hypothesis (H) allows us to find an equivalent and algebraic form of the approximated characteristic equation (see Lemma 4.3.7).

This chapter is divided into three main parts. In the first part, we prove the following theorem :

Theorem 4.1.1. *Under the hypothesis (H), system (S_1) is exponentially stable up*

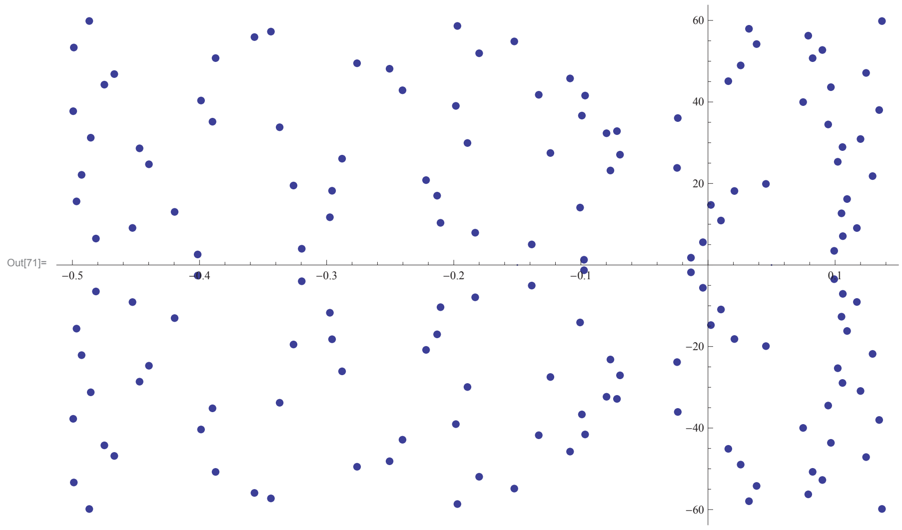

FIGURE 4.1 – $a_1 = a_2 = \dfrac{1}{2},\ a_3 = -\dfrac{1}{3},\ L_i = \sqrt{i},\ i = 1, 2, 3.$

to a finite dimensional space if and only if the roots of the polynomial G defined by

$$G(z) = \sum_{i=1}^{N} \left(e^{\int_0^{L_i} a_i(x)dx} z^{p_i} + e^{-\int_0^{L_i} a_i(x)dx} \right) \prod_{k \neq i, k=1}^{N} \left(e^{\int_0^{L_k} a_k(x)dx} z^{p_k} - e^{-\int_0^{L_k} a_k(x)dx} \right) \tag{4.1.1}$$

are inside the unitary open disk.

If $N = 2$, then according to Theorem 4.1.1, system (S_1) is exponentially stable up to a finite dimensional space if and only if

$$\int_0^{L_1} a_1(x)dx + \int_0^{L_2} a_2(x)dx > 0.$$

Clearly this condition depends only on $\int_0^{L_i} a_i(x)dx$, hence for $N \geq 3$, we may state the following conjecture :

Conjecture 4.1.2. *Although the degree of the polynomial G depends on the lengths L_i, and the coefficients are functions of the parameters a_i and L_i for all $i = 1, \cdots, N$,* *the fact that the roots of* G *are inside the open unitary disk depends only on the values of* $\int_0^{L_i} a_i(x)dx$ *for all* $i = 1, \cdots, N$ *(see the examples of Section 4.7).*

Remark 4.1.3. *If* $N = 2$, $a_1 = 1$, $a_2 = \alpha \in \mathbb{R}$, *and* $L_1 = L_2 = 1$, *then we recover the result of Theorem 1.1 of [1] which states that* (S_1) *is exponentially stable up to a finite dimensional space if and only if* $\alpha > -1$. *Indeed, in this case,* $G(z) = 2e^{1+\alpha}z^2 - 2e^{-1-\alpha}$ *and hence* $G(z) = 0$ *yields* $|z| = e^{-(1+\alpha)}$. *Therefore, by Theorem 4.1.1 above,* (S_1) *is exponentially stable up to a finite dimensional space if and only if* $\alpha > -1$.

In the second part, we consider system (S_ϵ) with $\epsilon > 0$ and prove the following theorem :

Theorem 4.1.4. *Under the hypothesis (H), when* $a_i(x) = a_i \in \mathbb{R}$ *and* $L_i = 1$ *for all* $i = 1, \cdots, N$, *there exists* $\epsilon_0 > 0$ *such that, for all* $\epsilon \in (0, \epsilon_0)$, (S_ϵ) *is exponentially stable if one of the following two conditions holds :*

(i) There exists at most one $j_0 \in \{1, \cdots, N\}$ *such that* $a_{j_0} = 0$ *and* $a_i > 0$ *for all* $i \neq j_0$.

(ii) There exists only one negative damping coefficient a_{i_0} *such that* $a_i > 0$ *for all* $i \neq i_0$, $\sum_{i=1}^{N} a_i > 0$, *and* $\sum_{i=1}^{N} \frac{1}{a_i} < 0$.

Remark 4.1.5. *If* $N = 2$, *then we recover the result of Theorem 2.1 of [30] when the damping coefficient is piecewise constant. However, in this case, Theorem 4.1.4 yields the result of [30] without the assumption on the integrals* I_k *defined in [30].*

Finally, in the third part, we look at some concrete examples of networks and specific values of a_i.

In the whole chapter, we shall use the notation $A \lesssim B$ (resp. $A = O(B)$) for the existence of a positive constant $c > 0$ independent of A and B such that $A \leq cB$ (resp. $|A| \leq c|B|$) and for shortness we will write $\|.\|_\infty$ for $\|.\|_{L^\infty(0,L_i)}$.

4.2 Formulation of the problem

We start by determining the suitable functional setting of system (S_1). If u is a regular solution of (S_1), then the energy of (S_1) is formally given by

$$E(t) = \frac{1}{2}\sum_{i=1}^{N}\int_0^{L_i}(|u_t^i|^2 + |u_x^i|^2)dx,$$

and

$$\frac{d}{dt}E(t) = -\sum_{i=1}^{N}\int_0^{L_i} a_i(x)|u_t^i|^2 dx.$$

Since the signs of the a_i are not specified, the decay of the energy is not guaranteed.

As an energy space, let $\mathcal{H} = V \times H$ where $H = \prod_{i=1}^{N} L^2(0, L_i)$ and

$$V = \left\{ u = (u^1, \cdots, u^N)^\top \in \prod_{i=1}^{N} H^1(0, L_i);\; u^i(0) = u^j(0)\ \forall i \neq j, \right.$$
$$\left. \text{and } u^i(L_i) = 0,\ \forall i = 1, \cdots, N \right\}.$$

The Hilbert space $\mathcal{H}$ is endowed with the inner product

$$< (u,v)^\top, (f,g)^\top >= \sum_{i=1}^{N}\int_0^{L_i}(u_x^i \overline{f_x^i} + v^i \overline{g^i})dx, \forall (u,v)^\top, (f,g)^\top \in \mathcal{H}.$$

Define the operator $A : D(A) \to \mathcal{H}$ by

$$D(A) = \left\{ (u,v)^\top \in V \times V;\; u \in \prod_{i=1}^{N} H^2(0, L_i) \text{ and } \sum_{i=1}^{N} u_x^i(0) = 0 \right\},$$

and for all $(u,v)^\top \in D(A)$

$$A(u,v)^\top = \begin{pmatrix} 0 & A_1^0 \\ A_1^2 & A_{-2a}^0 \end{pmatrix}(u,v)^\top$$

with $A_\alpha^k w = (\alpha_i \partial_x^k w^i)_{i=1}^N$ for $\alpha = (\alpha_i)_{i=1}^N \in \prod_{i=1}^N L^\infty(0, L_i)$ and $w = (w^i)_{i=1}^N \in \prod_{i=1}^N H^k(0, L_i)$, for $k = 0$ or 2.

If u is a sufficiently smooth solution of (S_1), then $U = (u, u_t)^\top \in \mathcal{H}$ satisfies the first order evolution equation

$$\begin{cases} U_t &= AU, \\ U(0) &= (u_0, u_1)^\top. \end{cases} \tag{4.2.1}$$

Using standard semigroup theory, we get the following theorem on the existence, uniqueness, and regularity of the solution of (S_1).

Theorem 4.2.1. *The operator A generates a C_0 semigroup on $\mathcal{H}$ and hence problem (4.2.1) admits a unique solution which implies that (S_1) is well-posed. Moreover, if $U(0) \in \mathcal{H}$, then $U \in C^0([0, +\infty); \mathcal{H})$ and if $U(0) \in D(A)$, then $U \in C^1([0, +\infty); \mathcal{H}) \cap C^0([0, +\infty); D(A))$.*

Proof: The well-posedness of (4.2.1) follows from the fact that the operator A is a bounded perturbation of a skew adjoint operator (see Theorem III.1.1 of [62]), hence it generates a strongly continuous semigroup on $\mathcal{H}$. The regularity results are then a direct consequence of Theorem I.2.4 of [62]. ■

Remark 4.2.2. *Since $D(A)$ is compactly embedded in the energy space $\mathcal{H}$, the spectrum $\sigma(A)$ is discrete and the eigenvalues of A have a finite algebraic multiplicity.*

4.3 High frequencies

In this section, we shall determine the asymptotic behavior of the eigenvalues of the operator A. For this aim, we will adapt the shooting method to our system.

Let λ be an eigenvalue of A and $U = (y, z)$ be an associated eigenfunction. Then,

$z = \lambda y$ and, for all $i = 1, \cdots, N$, we have

$$\left\{ \begin{array}{rcll} y^i_{xx} - 2a_i(x)\lambda y^i - \lambda^2 y^i & = & 0, & x \in (0, L_i), \\ y^i(L_i) & = & 0, & \\ y^i(0) & = & y^j(0), & \forall i \neq j, \\ \displaystyle\sum_{i=1}^{N} y^i_x(0) & = & 0. & \end{array} \right. \tag{4.3.1}$$

It is easy to see that $\lambda = 0$ is not an eigenvalue of A.

Remark 4.3.1. *We have*

$$|\Re\lambda| \leq 2 \max_{i \in \{1, \cdots, N\}} \{\|a_i\|_\infty\}. \tag{4.3.2}$$

Indeed, if we multiply the first identity of (4.3.1) by y^i and then integrate by parts, we get

$$\lambda^2 \sum_{i=1}^{N} \int_0^{L_i} |y^i|^2 dx + 2\lambda \sum_{i=1}^{N} \int_0^{L_i} a_i(x)|y^i|^2 dx + \sum_{i=1}^{N} \int_0^{L_i} |y^i_x|^2 dx = 0.$$

Hence, we have

$$\lambda = \frac{-\displaystyle\sum_{i=1}^{N} \int_0^{L_i} a_i(x)|y^i|^2 dx \pm r(y)^{\frac{1}{2}}}{\displaystyle\sum_{i=1}^{N} \int_0^{L_i} |y^i|^2 dx},$$

with

$$r(y) := \left(\sum_{i=1}^{N} \int_0^{L_i} a_i(x)|y^i|^2 dx \right)^2 - \left(\sum_{i=1}^{N} \int_0^{L_i} |y^i_x|^2 dx \right) \left(\sum_{i=1}^{N} \int_0^{L_i} |y^i|^2 dx \right)$$

and deduce the estimate (4.3.2) by distinguishing the case $r(y) \geq 0$ or not.

Now, we start by searching for the characteristic equation using the shooting method. In order to adapt the shooting method to problem (4.3.1), we first consider the the following separated initial value problems : for all $i = 1, \cdots, N$, let y_1^i and y_2^i be the solution of

$$\begin{cases} y_{1xx}^i - 2a_i(x)\lambda y_1^i - \lambda^2 y_1^i &= 0, \\ y_1^i(0) &= \dfrac{1}{\lambda}, \\ y_{1x}^i(0) &= 0. \end{cases} \tag{4.3.3}$$

$$\begin{cases} y_{2xx}^i - 2a_i(x)\lambda y_2^i - \lambda^2 y_2^i &= 0, \\ y_2^i(0) &= 0, \\ y_{2x}^i(0) &= 1. \end{cases} \tag{4.3.4}$$

The initial conditions are chosen such that the solutions y_1^i and y_2^i are linearly independent. Hence, y^i, the solution of (4.3.1), can be written as $y^i = c_i y_1^i + \alpha_i y_2^i$, where $\alpha_i, c_i \in \mathbb{C}$. By the continuity condition at zero, we get $c_i = c$ for all $i = 1, \cdots, N$, hence

$$y^i(x) = c y_1^i(x) + \alpha_i y_2^i(x). \tag{4.3.5}$$

Moreover, from the transmission condition, $\sum_{i=1}^{N} y_x^i(0) = 0$, we have $\sum_{i=1}^{N} \alpha_i = 0$ and from the boundary condition, $y^i(L_i) = 0$, we get

$$\begin{pmatrix} y_1^1(L_1) & y_2^1(L_1) & 0 & 0 & \cdots & 0 \\ y_1^2(L_2) & 0 & y_2^2(L_2) & 0 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & 0 & \cdots & 0 & \ddots & 0 \\ y_1^N(L_N) & 0 & \cdots & & 0 & y_2^N(L_N) \\ 0 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} c \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_N \end{pmatrix} = 0.$$

Hence a non-zero eigenvector exists if and only if the determinant of the above matrix vanishes, or after some elementary calculations if and only if

$$Y(\lambda) = \sum_{k=1}^{N} y_1^k(L_k) \prod_{l \neq k, l=1}^{N} y_2^l(L_l) = 0. \tag{4.3.6}$$

Recall that G is defined by (4.1.1) and set $d := degree\ G$, the degree of G. Then let $r_j e^{i\varphi_j}, 1 \leq j \leq d$ be the roots of G repeated according to their multiplicity. Without loss of generality we can suppose that the φ_j are non decreasing, namely

$$0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_d \leq 2\pi.$$

Now we can state the following main result :

Theorem 4.3.2. *There exists $k_0 \in \mathbb{N}$ such that for all $j = 1, \cdots, d$ and all $k \in \mathbb{Z}$ such that $|k| > k_0$, the operator A has an eigenvalue $\lambda_{j,k}$ such that*

$$\lambda_{j,k} = \frac{q}{2L_1} \log r_j + \imath \frac{q}{2L_1} \varphi_j + \imath k\pi \frac{q}{L_1} + o_k(1), \tag{4.3.7}$$

where $o_k(1) \to 0$ as $|k| \to \infty$. Moreover the set $\sigma(A) \setminus \cup_{|k|>k_0} \cup_{j=1}^{d} \lambda_{j,k}$ is compact. Therefore, if $r_j < 1$, for all $j = 1, \cdots, d$, then the large eigenvalues of A are situated to the left of the imaginary axis.

Corollary 4.3.3. *There exists $\ell \in \mathbb{N}$ and $\alpha_0 > 0$ such that for all $k \in \mathbb{N}$ with $k > k_0$, we have*

$$\Im(\lambda_{1,k+\ell} - \lambda_{d,k}) \geq \alpha_0,$$
$$\Im(\lambda_{1,-k} - \lambda_{d,-k-\ell}) \geq \alpha_0.$$

This corollary shows that we can group the eigenvalues of A by packets made of a finite number of eigenvalues and in such a way that the packets remain at a positive distance to each other (see section 4.4 below). Namely for any $r > 0$, we can introduce the sets $G_p(r), p \in \mathbb{Z}$ as the connected components of the set $\cup_{\lambda\in\sigma(A)} D_\lambda(r)$ (where $D_\lambda(r)$ is the disc with center λ and radius r), as well as the packets of eigenvalues $\Lambda_p(r) = G_p(r) \cap \sigma(A)$.

Before we prove Theorem 4.3.2, we search for an approximation of the characteristic equation (4.3.6) for all λ large enough. For this aim, the next lemma gives an estimation of y_1^i and y_2^i for all $i = 1, \cdots, N$.

Lemma 4.3.4. *For $i = 1, \cdots, N$ and $\lambda \in \sigma(A)$ large enough, we have*

$$\|y_1^i\|_\infty \lesssim \frac{1}{|\lambda|}, \quad \textit{and} \quad \|y_2^i\|_\infty \lesssim \frac{1}{|\lambda|}.$$

Proof: First, for $i = 1, \cdots, N$ and $\lambda \in \sigma(A)$, we consider the homogenous equation

$$\begin{cases} z_{1xx}^i(x) - \lambda^2 z_1^i(x) &= 0, \quad x \in (0, L_i), \\ z_1^i(0) &= \dfrac{1}{\lambda}, \\ z_{1x}^i(0) &= 0, \end{cases}$$

which yields $z_1^i(x) = \dfrac{1}{\lambda}\cosh(\lambda x)$. Hence, for large enough λ, Remark 4.3.1 yields $\|z_1^i\|_\infty \lesssim \dfrac{1}{|\lambda|}$. Now, by the variation of constants formula, we find that

$$y_1^i(x) = z_1^i(x) + 2\int_0^x \sinh(\lambda(x-s))\, a_i(s) y_1^i(s) ds, \forall x \in (0, L_i).$$

Therefore, by the integral form of Gronwall's Lemma, we get

$$|y_1^i(x)| \leq |z_1^i(x)| + 2\int_0^x \left[|z_1^i(s)| |\sinh(\lambda(x-s))\, a_i(s)| \exp\left(2\int_s^x |\sinh(\lambda(x-r)) a_i(r)| dr\right)\right] ds.$$

The above inequality and Remark 4.3.1 imply that, for λ large enough, $\|y_1^i\|_\infty \lesssim \dfrac{1}{|\lambda|}$.

A similar estimate for y_2^i is obtained by introducing $z_2^i = \dfrac{1}{\lambda}\sinh(\lambda x)$, the solution of

$$\begin{cases} z_{2xx}^i(x) - \lambda^2 z_2^i(x) &= 0, \quad x \in (0, L_i), \\ z_2^i(0) &= 0, \\ z_{2x}^i &= 1, \end{cases}$$

and using that $\|z_2^i\|_\infty \lesssim \dfrac{1}{|\lambda|}$ for λ large enough. ■

Next, we find suitable approximations for y_1^i and y_2^i for $i = 1, \cdots, N$. For this aim we define over $(0, L_i)$, the function

$$\theta^i(x) = \lambda x + \int_0^x a_i(s) ds, \forall x \in (0, L_i),$$

and the functions v_1^i and v_2^i as linear combination of $\sinh\theta^i(x)$ and $\cosh\theta^i(x)$ such that v_1^i satisfies the initial conditions in (4.3.3) and v_2^i satisfies those in (4.3.4). Note that, for $|\lambda| > M$ with $M > \max_i \|a_i(.)\|_\infty$, we have

$$v_1^i(x) = \frac{1}{\lambda}\cosh\theta^i(x), \quad \text{and} \quad v_2^i(x) = \frac{1}{\lambda + a_i(0)}\sinh\theta^i(x), \forall x \in (0, L_i).$$

Note that the functions v_1^i and v_2^i depend on λ.

Lemma 4.3.5. *For all $i = 1, \cdots, N$ and $\lambda \in \sigma(A)$ large enough, we have*

$$\left\|v_1^i - y_1^i\right\|_\infty \lesssim \frac{1}{|\lambda|^2} \quad \textit{and} \quad \left\|v_2^i - y_2^i\right\|_\infty \lesssim \frac{1}{|\lambda|^2}.$$

Proof: For $i = 1, \cdots, N$ and $\varphi^i \in H^2(0, L_i)$, define the function $L^i(\varphi^i) = \varphi^i_{xx} - 2a_i\lambda\varphi^i - \lambda^2\varphi^i$. Then, for all $x \in (0, L_i)$, we have

$$L^i\left(v_1^i(x)\right) = \frac{a_{ix}(x)}{\lambda}\sinh\theta^i(x) + \frac{(a_i(x))^2}{\lambda}\cosh\theta^i(x),$$

and

$$L^i\left(v_2^i(x)\right) = \frac{a_{ix}(x)}{\lambda + a_i(0)}\sinh\theta^i(x) + \frac{(a_i(x))^2}{\lambda + a_i(0)}\cosh\theta^i(x).$$

Therefore, by Remark 4.3.1, we get that for λ large enough

$$\left\|L^i(v_1^i)\right\|_\infty \lesssim \frac{1}{|\lambda|}, \quad \text{and} \quad \left\|L^i(v_2^i)\right\|_\infty \lesssim \frac{1}{|\lambda|}.$$

Since we have

$$\begin{aligned} v^i_{1xx} - 2a_i\lambda v_1^i - \lambda^2 v_1^i &= L^i(v_1^i), \\ v^i_{1x}(0) &= 0, \\ v_1^i(0) &= \frac{1}{\lambda}, \end{aligned}$$

by the variation of constants formula, we get for all $x \in (0, L_i)$

$$v_1^i(x) = y_1^i(x) + \int_0^x y_2^i(x-s)L^i(v_1^i(s))ds.$$

Therefore, by Lemma 4.3.4, we have

$$\left\|v_1^i - y_1^i\right\|_\infty \lesssim \frac{1}{|\lambda|^2}.$$

Similarly, for all $x \in (0, L_i)$, we have

$$v_2^i(x) = y_2^i(x) + \int_0^x y_2^i(x-s)L^i(v_2^i(s))ds,$$

which implies that

$$\left\|v_2^i - y_2^i\right\|_\infty \lesssim \frac{1}{|\lambda|^2},$$

■

Now, we can find an approximation of the characteristic equation (4.3.6) from which we deduce the behavior of the high frequencies. For this aim, we introduce

$$V(\lambda) = \sum_{k=1}^{N} v_1^k(L_k) \prod_{l\neq k, l=1}^{N} v_2^l(L_l)$$

and

$$F(\lambda) = \lambda^{-N} \sum_{k=1}^{N} \cosh \widetilde{\theta}^k(\lambda) \prod_{l\neq k}^{N} \sinh \widetilde{\theta}^l(\lambda), \tag{4.3.8}$$

where, for $z \in \mathbb{C}$, $\widetilde{\theta}^l(z) = zL_l + \int_0^{L_l} a_l(s)ds$, for all $l = 1, \cdots, N$.

Proposition 4.3.6. *For $\lambda \in \sigma(A)$ large enough, we have the following estimate*

$$|Y(\lambda) - F(\lambda)| \lesssim \frac{1}{|\lambda|^{N+1}}. \tag{4.3.9}$$

Proof: Let λ be a large eigenvalue of A. The estimates in Lemmas 4.3.4 and 4.3.5

imply that

$$\begin{aligned}
& |Y(\lambda) - V(\lambda)| \\
= & \left| \sum_{k=1}^{N} \left(y_1^k (L_k) \prod_{l\neq k, l=1}^{N} y_2^l (L_l) - v_1^k (L_k) \prod_{l\neq k, l=1}^{N} v_2^l (L_l) \right) \right| \\
= & \left| \sum_{k=1}^{N} \left(y_1^k (L_k) - v_1^k(L_k) \right) \prod_{l\neq k, l=1}^{N} y_2^l (L_l) + \sum_{k=1}^{N} v_1^k (L_k) \left(\prod_{l\neq k, l=1}^{N} y_2^l (L_l) - \prod_{l\neq k, l=1}^{N} v_2^l (L_l) \right) \right| \\
\lesssim & \frac{1}{|\lambda|^{N+1}}.
\end{aligned} \tag{4.3.10}$$

On the other hand, we readily check that

$$\left| V(\lambda) - \frac{1}{\lambda^N} \sum_{k=1}^{N} \cosh \theta^k(L_k) \prod_{l\neq k}^{N} \sinh \theta^l(L_l) \right| \lesssim \frac{1}{|\lambda|^{N+1}}. \tag{4.3.11}$$

Hence, by (4.3.10) and (4.3.11), we get (4.3.9) since $\theta^k(L_k) = \widetilde{\theta}^k(\lambda)$. ■

Estimation (4.3.9) suggests to apply Rouché's Theorem. Therefore, we are first interested in the roots of F that will be expressed in terms of the roots of the polynomial G given in (4.1.1).

Lemma 4.3.7. *$v \in \mathbb{C}$ is a root of F if and only if $z = e^{\frac{2L_1}{q} v}$ is a root of the polynomial G defined in (4.1.1). Consequently, if $v = x + \imath y$ is a root of F and $r_j e^{i\varphi_j}$ is a root of G for $1 \leq j \leq d$, then $x = \dfrac{q}{2L_1} \log r_j$ and $y = \dfrac{q}{2L_1}\varphi_j + k\pi \dfrac{q}{L_1}$ for some $k \in \mathbb{Z}$.*

Proof: The proof of Lemma 4.3.7 is based on writing F in an exponential form and noting that

$$\begin{aligned}
& 2^N v^N e^{\frac{vL_1 \sum_{i=1}^N p_i}{q}} F(v) \\
= & \sum_{i=1}^{N} \left(e^{\int_0^{L_i} a_i(x)dx} z^{p_i} + e^{-\int_0^{L_i} a_i(x)dx} \right) \prod_{k\neq i, k=1}^{N} \left(e^{\int_0^{L_k} a_k(x)dx} z^{p_k} - e^{-\int_0^{L_k} a_k(x)dx} \right).
\end{aligned}$$

■

Remark 4.3.8. *In the applications, the degree of the polynomial G is high, hence we use the algorithm given by the transformation of Schur (see [31]) that gives a criterion that guarantees that the roots of a given polynomial can be outside the closed unitary disk. Therefore, in applications, we use $G\left(\frac{1}{z}\right)$ instead of $G(z)$.*

Before giving the proof of Theorem 4.3.2, we show that Y has the same number of roots as F in a well chosen domain. Knowing that $\alpha < \Re\lambda < \beta$ where λ is an eigenvalue of A, we consider the rectangle $R_{j,k}$ with vertices $\alpha+\imath\frac{q}{2L_1}\varphi_j + \imath\left(k-\frac{1}{2}\right)\pi\frac{q}{L_1}$, $\alpha+\imath\frac{q}{2L_1}\varphi_j + \imath\left(k+\frac{1}{2}\right)\pi\frac{q}{L_1}$, $\beta+\imath\frac{q}{2L_1}\varphi_j + \imath\left(k-\frac{1}{2}\right)\pi\frac{q}{L_1}$, and $\beta+\imath\frac{q}{2L_1}\varphi_j + \imath\left(k+\frac{1}{2}\right)\pi\frac{q}{L_1}$ where we recall that $r_je^{i\varphi_j}$, $j=1,\cdots,d$, are the roots of G.

Proposition 4.3.9. *There exists $k_0>0$ such that for all $|k|\geq k_0$ and $z\in\partial R_{j,k}$,*

$$|Y(z)-F(z)| < |F(z)|. \tag{4.3.12}$$

Proof: Let $z\in\partial R_{j,k}$ and $|k|\geq k_0$ for some $k_0>0$ large enough. Similar to (4.3.9), we can show that there exists $C>0$ such that

$$|Y(z)-F(z)| \leq \frac{C}{|z|^{N+1}}.$$

Therefore, in order to complete the proof, it is enough to show that for $z\in\partial R_{j,k}$

$$\frac{C}{|z|} < |F_0(z)|,$$

where

$$F_0(z) = \sum_{k=1}^{N}\cosh\widetilde{\theta}^k(z)\prod_{l\neq k}^{N}\sinh\widetilde{\theta}^l(z). \tag{4.3.13}$$

We remark that $|F_0|$ is $i\pi\frac{q}{L_1}$ periodic, hence, $\min_{z\in\partial R_{j,k}}|F_0(z)| = m_j$ is independent of k. Moreover, for $k_0\geq 1$, there exists $\widetilde{C}>0$ such that for $|k|\geq k_0$ and $z\in\partial R_{j,k}$, we have

$$\frac{C}{|z|}\leq\frac{\widetilde{C}}{|k|}.$$

Choosing k_0 large enough, we deduce that

$$\frac{C}{|z|} \leq \frac{\widetilde{C}}{|k|} < m_j$$

and the proof follows. ■

Proof: of Theorem 4.3.2 We shall prove that the large eigenvalues of A are asymptotically close to the roots of F.

First Lemma 4.3.7 yields that all the roots of F are given by

$$z_{j,k} = \frac{q}{2L_1}\log r_j + \imath\frac{q}{2L_1}\varphi_j + \imath k\pi\frac{q}{L_1}.$$

for all $1 \leq j \leq d$, $k \in \mathbb{Z}$.

Let $0 < \rho < \min_j \left\{ \left|\frac{q}{2L_1}\varphi_j + \pi\frac{q}{L_1}\right|, \left|\frac{q}{2L_1}\log r_j\right| \right\}$ so that $B(z_{j,k}, \rho)$ contains only one root of F. From Proposition 4.3.6, in order to prove that $|Y(z) - F(z)| < |F(z)|$ for $z \in \partial B(z_{j,k}, \rho)$, it is enough to show that $\frac{C}{|z|} < |F_0(z)|$ where F_0 was defined by (4.3.13).

Let $h_{j,k}(\rho) = \min_{z\in\partial B(z_{j,k},\rho)} |F_0(z)|$. Since $|F_0|$ is $i\pi\frac{q}{L_1}$ periodic, then $h_{j,k}(\rho)$ is independent of k; i.e., $h_{j,k}(\rho) = h_{j,0}(\rho) = h_j(\rho)$. We denote by $h(\rho) = \min_{1\leq j\leq d} h_j(\rho)$. It is clear that $h(\rho) > 0$ and $h(\rho) \to 0$ as $\rho \to 0$. Therefore, there exists $k_0 > 0$ such that for $|k| > k_0$, $\frac{C}{|z|} \leq \frac{\widetilde{C}}{|k|} < h(\rho)$. Consequently, we define ρ_k by

$$\rho_k = \min_{\rho} \left\{ \frac{\widetilde{C}}{|k|} < h(\rho) \right\}. \tag{4.3.14}$$

We notice that $\rho_k \to 0$ as $|k| \to +\infty$. Therefore, for every $|k| > k_0$ and $z \in \partial B(z_{j,k}, \rho_k)$, we have $\frac{C}{|z|} \leq \frac{\widetilde{C}}{|k|} < h(\rho_k) \leq |F_0(z)|$.

By Rouché's Theorem, we conclude that Y and F have the same roots inside $B(z_{j,k}, \rho_k)$. As Proposition 4.3.9 and again the application of Rouché's theorem imply that Y and F has the same number of roots in $R_{j,k}$, for all $j = 1, \cdots, d$ and $|k| \geq k_0$, we deduce that all eigenvalues of A in $R_{j,k}$ are inside $B(z_{j,k}, \rho_k)$. This completes the proof since $\cup_{|k|\geq k_0} R_{j,k}$ covers the possible set of large eigenvalues of A. ■

Remark 4.3.10. *Using Taylor expansion in ρ, we find that $h(\rho) = O(\rho^{n_0})$ for some $n_0 \in \mathbb{N}^*$. Therefore, according to the definition of ρ_k in (4.3.14), we conclude that $\rho_k = \dfrac{\widetilde{\widetilde{C}}}{|k|^{\frac{1}{n_0}}}$ for some $\widetilde{\widetilde{C}} > 0$. Consequently, there exists some $k_0 \in \mathbb{N}^*$ large enough such that*

$$\lambda_{j,k} = \frac{q}{2L_1} \log r_j + \imath \frac{q}{2L_1} \varphi_j + \imath k\pi \frac{q}{L_1} + O\left(\frac{1}{|k|^{\frac{1}{n_0}}}\right), \forall |k| > k_0.$$

4.4 Riesz basis with parentheses of $\mathcal{H}$ and sine-type functions

In this section, we first prove that the root vectors of A form a Riesz basis with parentheses of $\mathcal{H}$. A direct consequence of Theorem 1.3.2 concerns our operator A :

Proposition 4.4.1. *The family of root vectors of A forms a Riesz basis with parentheses of $\mathcal{H}$, which means that the statements of Theorem 1.3.2 are valid for $\imath A$.*

Proof: It suffices to apply Theorem 1.3.2 with the choice

$$T = i \begin{pmatrix} 0 & A_1^0 \\ A_1^2 & A_0^0 \end{pmatrix} \quad \text{and} \quad B = i \begin{pmatrix} 0 & 0 \\ 0 & A^0_{-2a} \end{pmatrix}.$$

that clearly satisfies the assumptions of Theorem 1.3.2. ■

Another consequence of the previous results is that the packet Λ_p can be splitted up into subpackets, namely there exists $N_p \in \mathbb{N}^*$ with $N_p \leq N$ such that

$$\Lambda_p = \cup_{j=1}^{N_p} \{\lambda_{p,j}\},$$

where each $\lambda_{p,j} \in \sigma(A)$ are different and of multiplicity $m_{p,j}$ (uniformly bounded in p) and therefore

$$\mathbb{P}_p f = \sum_{j=1}^{N_p} \mathbb{P}_{p,j} f,$$

with

$$\mathbb{P}_{p,j} f = \frac{1}{2i\pi} \int_{\gamma_{p,j}} (\lambda - T - B)^{-1} f \, d\lambda,$$

$\gamma_{p,j}$ being a contour surrounding $\lambda_{p,j}$ and small enough so that only the eigenvalue $\lambda_{p,j}$ of A is inside $\gamma_{p,j}$.

In the next section, we also need to show that Y defined in (4.3.6) is a sine-type function in the following sense :

Definition 4.4.2. *Let f be an entire complex valued function. f is said to be of sine-type if*

(a) There exists $l > 0$ such that for all $z \in \mathbb{C}$, $|f(z)| \lesssim e^{l|z|}$.

(b) The zeros of f lie in a strip $\{z \in \mathbb{C};\ |\Re z| \leq c\}$ for some $c > 0$.

(c) There exist constants $c_1, c_2 > 0$ and $x_0 \in \mathbb{R}$ such that for, all $y \in \mathbb{R}$, $c_1 \leq |f(x_0 + iy)| \leq c_2$.

The class of sine-type functions is used to deal with problems of the Riesz basis property of the complex exponentials in $L^2(0,T)$ space, with $T > 0$. When f is a sine-type function, then we can write the explicit expression of f as $f(z) = \lim_{R\to+\infty} \prod_{|\widetilde{\lambda}_k|\leq R} \left(1 - \frac{z}{\widetilde{\lambda}_k}\right)$, where $\{\widetilde{\lambda}_k\}_{k\in\mathbb{Z}}$ is the set of zeros of f (see [8]). If $\widetilde{\lambda}_k = 0$, then we replace the term $\left(1 - \frac{z}{\widetilde{\lambda}_k}\right)$ by z.

In our problem, we remark that the function F defined in the approximated characteristic equation (4.3.8) is a sine-type function. In order to deduce the same property for Y defined in (4.3.6), we recall a Corollary of Section 2 of [11] :

Lemma 4.4.3. *Given* $S(z) = \lim_{R\to+\infty} \prod_{|\widetilde{\lambda}_k|\leq R} \left(1 - \frac{z}{\widetilde{\lambda}_k}\right)$ *a sine-type function, where* $\{\widetilde{\lambda}_k\}_{k\in\mathbb{Z}}$ *is the set of zeros of* $S(z)$*. Then* $S_0(z) = \lim_{R\to+\infty} \prod_{|\widetilde{\lambda}_k|\leq R} \left(1 - \frac{z}{\widetilde{\lambda}_k + \psi_k}\right)$ *is also a sine-type function if* $\{\psi_k\}_{k\in\mathbb{Z}} \in \ell^p$*, for some* $p > 1$*.*

Lemma 4.4.4. Y *defined in (4.3.6) is sine-type, or equivalently the eigenvalues of* A *are the zeros of a sine-type function.*

Proof: According to Theorem 4.3.2 and Remark 4.3.10, the large eigenvalues A are close to the ones of F with a remainder $\{\psi_k\}_{k\in\mathbb{Z}}$ such that

$$\psi_k = O\left(\frac{1}{|k|^{\frac{1}{n_0}}}\right),$$

for $|k| > k_0$ that then belongs to ℓ^{n_0+1}. ■

4.5 Exponential stability of (S_1) and proof of Theorem 4.1.1

Taking advantage of the fact that the root vectors of A form a Riesz basis with parenthesis of $\mathcal{H}$, our aim is now to prove that problem (S_1) is exponentially stable up to a finite dimensional space.

For our proof we recall the following lemma that can be found in Lemma 3.1 of [39].

Lemma 4.5.1. *Let* H *be a separable Hilbert space. Suppose that* $\{e_n(t)\}_{n\in J}$ *forms a Riesz basis for the closed subspace spanned by itself in* $L^2(0,T)$*,* $T > 0$*. Then for any* $\varphi(t) = \sum_{n\in J} e_n(t)\phi_n \in L^2(0,T;H)$*, there exist two positive constants* $C_1(T)$*,* $C_2(T)$

such that

$$C_1(T)\sum_{n\in J}\|\phi_n\|_H^2 \leq \|\varphi\|_{L^2(0,T;H)}^2 \leq C_2(T)\sum_{n\in J}\|\phi_n\|_H^2.$$

To apply the above lemma, we need to search for a Riesz basis in $L^2(0,T)$. Since the eigenvalues are not necessary simple, the family $\{e^{\lambda_k t}\}_{k\in\mathbb{Z}}$ does not form a Riesz basis in $L^2(0,T)$ for any $T>0$. However, as $\sigma(A)$ is a discrete union of separated and finite sets, hence we can use the family of generalized divided differences (see [9,39]).

Definition 4.5.2. *Let $M\in\mathbb{N}^*$ be fixed and let v_k, $k=1,\cdots,M$, be arbitrary complex numbers, not necessarily distinct. Then the generalized divided differences (denoted by GDD) of order $m=0,\cdots,M-1$ are defined by recurrence as follows : the GDD of order zero is defined as $[v_1](t)=e^{v_1 t}$, the GDD of order $m-1, 1\leq m\leq M$ is defined as*

$$[v_1,v_2,\cdots,v_m](t) =: \begin{cases} \dfrac{[v_1,v_2...,v_{m-1}](t)-[v_2,v_3,\cdots,v_m](t)}{v_1-v_m}, & v_1\neq v_m \\ \dfrac{\partial}{\partial v}[v,v_2,\cdots,v_{m-1}](t)\mid_{v=v_1}, & v_1=v_m. \end{cases}$$

An equivalent expression is given by

$$[v_1,v_2,\cdots,v_m](t)=t^{m-1}\int_0^1\int_0^{\tau_1}...\int_0^{\tau_{m-2}} e^{t(v_1+\tau_1(v_2-v_1)+...+\tau_{m-1}(v_m-v_{m-1}))}d\tau_{m-1}...d\tau_2 d\tau_1.$$

Hence, if $\Re v_m\leq\Re v_{m-1}\leq...\leq\Re v_1$, then for all $t\geq 0$

$$|[v_1,v_2...,v_m](t)|\leq t^{m-1}e^{\Re v_1 t}. \tag{4.5.1}$$

Now as some v_j can be repeated, we write $\{v_1,v_2,\cdots,v_M\}=\{w_1,w_2,\cdots,w_n\}$ such that $w_i\neq w_j$ for all $1\leq i,j\leq n$ such that $i\neq j$. Supposing that each w_j is repeated n_j times, i.e, $\sum_{j=1}^{n} n_j=M$, then we can recall Proposition 3.1 of [39] which shows that for any $1\leq k\leq n_l$, $t^{k-1}e^{w_l t}$, $l=1,\cdots,n$ is a linear combination of $[v_1](t),\ [v_1,v_2](t),\cdots,[v_1,v_2,\cdots,v_M](t)$.

Proposition 4.5.3. *Any* $\varphi(t) = \sum_{j=1}^{n} e^{w_j t} \sum_{i=1}^{n_j} a_{ij} t^{i-1}$ *with* $a_{ij} \in \mathcal{H}$ *can be rewritten as*

$$\varphi(t) = \sum_{i=1}^{M} G_i[v_1.v_2, \cdots, v_i](t),$$

with some $G_i \in \mathcal{H}$*, in particular* $G_1 = \sum_{j=1}^{n} a_{1j}$.

If we go back to our problem, for every $p \in \mathbb{Z}$, we construct the family of GDD of the form

$$E_p(t) = \left\{[\lambda_{p,1}](t),\ [\lambda_{p,1}, \lambda_{p,2}], \cdots, [\lambda_{p,1}, \lambda_{p,2}, \cdots, \lambda_{p,M_p}](t)\right\},$$

associated with the set $\lambda_{p,1}, \cdots, \lambda_{p,N_p}$ but the eigenvalues being repeated according to their multiplicity (and consequently $M_p = \sum_{j=1}^{N_p} m_{p,j}$).

Proposition 4.5.4. *There exists* $T > 0$ *such that the family of GDD* $\{E_p(t)\}_{p\in\mathbb{Z}}$ *forms a Riesz basis for the closed subspace spanned by itself in* $L^2(0,T)$*.*

Proof: According to Lemma 4.4.4, the eigenvalues of A are roots of a sine-type function. Hence, the proof becomes a direct consequence of Theorem 3 of [10] where $T > 0$ is chosen large enough (note also that a sine-type function automatically satisfies the Helson-Szego condition due to its equivalent form (condition (A_2) page 2 in [9]) and the condition (c) in our definition 4.4.2). ■

Proof: of Theorem 4.1.1. Given an initial datum $U(0) \in \mathcal{H}$, by Proposition 4.4.1, it can be written as

$$U(0) = \sum_{p\in\mathbb{Z}} \sum_{j=1}^{N_p} \mathbb{P}_{\lambda_{p,j}}(U(0)),$$

where we recall that $\mathbb{P}_{\lambda_{p,j}}$ denotes the Riesz projection of A corresponding to the eigenvalue $\lambda_{p,j}$, then, for any $t > 0$, we have

$$\begin{aligned} e^{tA}U(0) &= \sum_{p\in\mathbb{Z}}\sum_{j=1}^{N_p} e^{\lambda_{p,j}t}\sum_{i=1}^{m_{p,j}} \frac{(A-\lambda_{p,j})^{i-1}}{(i-1)!} t^{i-1}\mathbb{P}_{\lambda_{p,j}}\left(U(0)\right) \\ &= \sum_{p\in\mathbb{Z}}\sum_{j=1}^{N_p} e^{\lambda_{p,j}t}\sum_{i=1}^{m_{p,j}} a_{ij,p}t^{i-1}, \end{aligned} \tag{4.5.2}$$

where $a_{ij,p} = \dfrac{(A-\lambda_{p,j})^{i-1}}{(i-1)!}\mathbb{P}_{\lambda_{p,j}}\left(U(0)\right)$. By Proposition 4.5.3, we get

$$e^{tA}U(0) = \sum_{p\in\mathbb{Z}}\sum_{i=1}^{M_p} G_{p,i}[\lambda_{p,1}, \lambda_{p,2}, \cdots, \lambda_{p,i}](t). \tag{4.5.3}$$

Lemma 4.5.1 and Proposition 4.5.4 yield for some $T > 0$

$$\sum_{p\in\mathbb{Z}}\sum_{i=1}^{M_p} \|G_{p,i}\|_{\mathcal{H}}^2 \lesssim \int_0^T \left\|e^{tA}U(0)\right\|_{\mathcal{H}}^2 \, dt.$$

By the semigroup property, we know that there are $C, \omega > 0$ such that for all $t \geq 0$

$$\|e^{tA}\|_{\mathcal{L}(\mathcal{H})} \leq Ce^{\omega t}.$$

Therefore, the previous estimate becomes

$$\sum_{p\in\mathbb{Z}}\sum_{i=1}^{M_p} \|G_{p,i}\|_{\mathcal{H}}^2 \lesssim \frac{C^2}{2\omega}(e^{2\omega T} - 1)\|U(0)\|_{\mathcal{H}}^2. \tag{4.5.4}$$

Finally, since the root vectors of A form a Riesz basis with parenthesis of $\mathcal{H}$, then by (4.5.1), (4.5.3), and (4.5.4) we get for $t \geq 1$

$$\begin{aligned} \left\|e^{tA}U(0)\right\|_{\mathcal{H}}^2 &\lesssim \sum_{p\in\mathbb{Z}} \left\|\sum_{i=1}^{M_p} G_{p,i}[\lambda_{p,1}, \lambda_{p,2}, \cdots, \lambda_{p,i}](t)\right\|_{\mathcal{H}}^2 \\ &\lesssim \sum_{p\in\mathbb{Z}} t^{2(M_p-1)}e^{2\mu_p t}\sum_{i=1}^{M_p} \|G_{p,i}\|_{\mathcal{H}}^2, \end{aligned} \tag{4.5.5}$$

where $\mu_p = \max_{1\leq j\leq N_p} \Re\lambda_{p,j}$.

Now, by Theorem 4.3.2, we know that if the roots of the polynomial G are in the open unit disk, then there exists $\mu < 0$ and $p_0 \in \mathbb{N}$ such that

$$\mu_p \leq \mu < 0, \forall |p| > p_0.$$

Hence by (4.5.5), we deduce the exponential stability of problem (S_1) up to the finite dimensional space spanned by the roots vectors of A corresponding to the eigenvalues $\lambda_{p,j}$ such that $|p| \leq p_0$. The proof of Theorem 4.1.1 is complete. ■

4.6 Exponential stability of (S_ϵ) for small values of ϵ and proof of Theorem 4.1.4

In this section, we consider constant damping coefficients and equal lengths $L_i = 1$, for all $i = 1, \cdots, N$. Without loss of generality we can assume that the a_i are non decreasing, i.e., $a_1 \leq a_2 \leq \cdots \leq a_N$. In the sequel, we replace the damping coefficients a_i by ϵa_i, where the parameter ϵ is positive. Our goal is to find sufficient conditions for which (S_ϵ) is exponentially stable in the whole energy space for every ϵ small enough.

Based on the results of the previous section, it seems enough to find sufficient conditions on the damping coefficients so that the low eigenvalues have negative real parts for every ϵ small enough. However, we remark that Rouché's Theorem used in the proof of Theorem 4.3.2 yields a constant k_0 dependent of ϵ (mainly of order $\frac{1}{\epsilon}$). Consequently, it seems difficult to separate the large eigenvalues from the low eigenvalues uniformly in ϵ for all ϵ small enough.

As previously mentioned, the exponential stability of (S_ϵ) has been studied in [51] under some abstract hypothesis. Consequently, our aim is to interpret the hypothesis from [51] to find explicit conditions on the damping coefficients. Our strategy is based on the asymptotic behavior of the spectrum of the generator $A = A(\epsilon)$ as a function of ϵ. In the sequel, we use some notations from [46] and we refer the reader to this book for the exact definitions. First, we notice that the generator $A = A(\epsilon)$ is holomorphic of type (A) in the parameter ϵ in the sense of (2.1) of chapter VII.2 in [46]. Indeed, we simply have

$$A(\epsilon) = A(0) + \epsilon B,$$

where $A(0)$ is a skewadjoint operator and B is a bounded selfadjoint operator defined by

$$A(0) = \begin{pmatrix} 0 & A_1^0 \\ A_1^2 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & A_{-2a}^0 \end{pmatrix}.$$

Since $A(0)$ is a skew adjoint operator with a compact resolvent, there is an orthonormal system of eigenvectors of $A(0)$ which is complete in $\mathcal{H}$. The eigenvalues of $A(0)$ are $\lambda_{1,k}(0) = \imath(k\pi + \frac{\pi}{2})$ with multiplicity one, for all $k \in \mathbb{Z}$, and $\lambda_{2,k}(0) = \imath k\pi$ with geometric and algebraic multiplicity $N-1$, for all $k \in \mathbb{Z}^*$. For shortness we write $\{\lambda_k(0)\}_{k\in\mathbb{Z}} = \{\imath k\pi\}_{k\in\mathbb{Z}^*} \cup \left\{\imath(k\pi + \frac{\pi}{2})\right\}_{k\in\mathbb{Z}}$ and we set m_k the multiplity of $\lambda_k(0)$ (hence $m_k = 1$ or $m_k = N-1$).

Now according to Section VII.2.4 in [46], there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, $A(\epsilon)$ has exactly m_k eigenvalues (algebraic multiplicity counted) in $B(\lambda_k(0), \rho)$, with $\rho > 0$ fixed small enough. This set of eigenvalues is called the $\lambda_k(0)$-group eigenvalues of $A(\epsilon)$ generated by the splitting from the common eigenvalue $\lambda_k(0)$ of the unperturbed operator $A(0)$ (see page 74 in [46]). Consequently as ϵ increases, a splitting of the eigenvalues may occur and the eigenvalues of $A(\epsilon)$ can go to the left or to the right of the imaginary axis (or both). Hence, our aim is to find some sufficient conditions for which each $\lambda_k(0)$-group is strictly to the left of the imaginary axis.

For further use, for $\epsilon \in (0, \epsilon_0)$ and all $k \in \mathbb{Z}$, the $\lambda_k(0)$-group eigenvalues of $A(\epsilon)$ will be denoted by $\{\lambda_{k,j}(\epsilon)\}_{j=1}^{m_k}$.

In a first step, consider $\Gamma_{k,\rho}$ a positively-oriented circle around $\lambda_k(0)$ with radius $\rho < \dfrac{\pi}{4}$ such that $\lambda_k(0)$ is isolated. For $\zeta \in \Gamma_{k,\rho}$, we denote by $R(\zeta) = (A(0) - \zeta)^{-1}$. The following lemma gives a uniform estimate of $\|R(\zeta)\|$ for all $\zeta \in \Gamma_{k,\rho}$.

Lemma 4.6.1. *For all $\zeta \in \Gamma_{k,\rho}$, we have*

$$\|R(\zeta)\| = \frac{1}{\rho}, \ \forall \zeta \in \Gamma_{k,\rho}. \tag{4.6.1}$$

Proof: For convenience and for a moment, we rename $\{i\beta_k\}_{k \in \mathbb{Z}^*}$ the set of eigenvalues of $A(0)$ and arrange it in increasing order (i.e $...\beta_{k-1} \leq \beta_k \leq \beta_{k+1}...$). We denote by $\{\phi_k\}_{k \in \mathbb{Z}^*}$ the associated system of eigenvectors which forms an orthonormal basis of $\mathcal{H}$. Let $f = \sum_{k \in \mathbb{Z}^*} f_k \phi_k \in \mathcal{H}$, then by the spectral theorem, for all $\zeta \in \Gamma_{k,\rho}$, we can write

$$R(\zeta) f = \sum_{k \in \mathbb{Z}^*} \frac{f_k}{i\beta_k - \zeta} \phi_k.$$

Since $|i\beta_k - \zeta| = \rho$, for all $k \in \mathbb{Z}^*$, we deduce that

$$\|R(\zeta) f\|^2 = \sum_{k \in \mathbb{Z}^*} \frac{|f_k|^2}{|i\beta_k - \zeta|^2} = \frac{1}{\rho^2} \sum_{k \in \mathbb{Z}^*} |f_k|^2 = \frac{1}{\rho^2} \|f\|^2.$$

This proves (4.6.1) by taking f corresponding to one eigenvector associated with the eigenvalue $\lambda_k(0)$. ■

Now we characterize the asymptotic behaviour of the real parts of the $\lambda_k(0)$-group eigenvalues of $A(\epsilon)$.

Lemma 4.6.2. *There exists* $\epsilon_1 > 0$ *and* $c > 0$ *such that for all* $\epsilon \in (0, \epsilon_1)$*, all* $k \in \mathbb{Z}$ *and all* $j = 1, \cdots, m_k$*,*

$$\Re\lambda_{k,j}(\epsilon) \leq \epsilon \max_{1\leq j\leq m_k} \mu_{k,j} + c\epsilon^2,$$

when $\{\mu_{k,j}\}_{j=1}^{m_k}$ *denotes the set of eigenvalues of* $P_k(0)BP_k(0)$ *and* $P_k(0)$ *denotes the eigenprojection corresponding to* $\lambda_k(0)$*, i.e.,* $P_k(0) = -\frac{1}{2\pi i}\int_{\Gamma_{k,\rho}} R(\xi)d\xi$*.*

Proof: Step 1. Let $\lambda_k(0)$ be an eigenvalue of $A(0)$. Define the space $M_k(\epsilon) = P_k(\epsilon)\mathcal{H}$, where $P_k(\epsilon)$ is the eigenprojection (see (1.16) page 67 of [46]) defined by

$$P_k(\epsilon) = -\frac{1}{2\pi i}\int_{\Gamma_{k,\rho}} (A(\epsilon) - \xi)^{-1}\, d\xi.$$

Notice that $(A(\epsilon) - \xi)^{-1}$ is well defined for $\xi \in \Gamma_{k,\rho}$ when $\epsilon < \dfrac{1}{\|B\|\|R(\xi)\|} = \dfrac{\rho}{\|B\|}$. Indeed, according to (1.13) and (1.14) page 67 of [46], we have by the second Neumann series for the resolvent

$$(A(\epsilon) - \xi)^{-1} = R(\xi)\,(1 + \epsilon BR(\xi))^{-1} = R(\xi)\sum_{n=0}^{\infty}(-\epsilon BR(\xi))^n = R(\xi) + \sum_{n=1}^{\infty}\epsilon^n R_k^{(n)}(\xi), \tag{4.6.2}$$

where

$$R_k^{(n)}(\xi) = R(\xi)(-BR(\xi))^n. \tag{4.6.3}$$

Hence the series on the right-hand side of (4.6.2) converges if $\epsilon < \frac{\rho}{\|B\|}$ (thanks to Lemma 4.6.1, we notice that the upper bound of ϵ is independent of $k \in \mathbb{Z}$) and we get

$$P_k(\epsilon) = P_k(0) + \sum_{n=1}^{\infty}\epsilon^n P_k^{(n)},$$

where $P_k^{(n)} = -\frac{1}{2\pi i}\int_{\Gamma_{k,\rho}} R_k^{(n)}(\xi)d\xi$ and $P_k^{(0)} = P_k(0)$.

As already said before if ϵ is sufficiently small, the eigenvalues of $A(\epsilon)$ lying in $\Gamma_{k,\rho}$ form exactly the $\lambda_k(0)$-group eigenvalues. Therefore, since $\lambda_k(0)$ is semisimple (since it is an eigenvalue of a skewadjoint operator), then according to the identities (5.13) and (5.14) of [46, p. 112], the $\lambda_k(0)$-group eigenvalues of $A(\epsilon)$ are of the form

$$\lambda_{k,j}(\epsilon) = \lambda_k(0) + \epsilon\mu_{k,j}^{(1)}(\epsilon), \quad j = 1, \cdots, m_k, \tag{4.6.4}$$

where $\{\mu_{k,j}^{(1)}(\epsilon)\}_{j=1}^{m_k}$ are the eigenvalues of the operator

$$\widetilde{A}_k^{(1)}(\epsilon) = \epsilon^{-1}(A(\epsilon) - \lambda_k(0))P_k(\epsilon) = -\frac{\epsilon^{-1}}{2\pi i}\int_{\Gamma_{k,\rho}} (\xi - \lambda_k(0))\,(A(\epsilon) - \xi)^{-1}\,d\xi, \tag{4.6.5}$$

in the subspace $M_k(\epsilon) = P_k(\epsilon)\mathcal{H}$. The second equality in (4.6.5) follows from the fact that

$$(A(\epsilon) - \lambda_k(0))\,(A(\epsilon) - \xi)^{-1} = 1 + (\xi - \lambda_k(0))\,(A(\epsilon) - \xi)^{-1}.$$

Step 2. We estimate the difference between $\widetilde{A}_k^{(1)}(\epsilon)$ and $P_k(0)BP_k(0)$. According to the identity (2.16) page 77 of [46], we have

$$(A(\epsilon) - \lambda_k(0))P_k(\epsilon) = (A(0) - \lambda_k(0))P_k(0) + \sum_{n=1}^{\infty}\epsilon^n\widetilde{A}_k^{(n)}, \tag{4.6.6}$$

where

$$\widetilde{A}_k^{(n)} = (-1)^{n+1}\frac{1}{2\pi i}\int_{\Gamma_{k,\rho}} R(\xi)(BR(\xi))^n(\xi - \lambda_k(0))d\xi, \tag{4.6.7}$$

in particular (see (2.19) page 77 of [46])

$$\widetilde{A}_k^{(1)} = P_k(0)BP_k(0). \tag{4.6.8}$$

Since $\lambda_k(0)$ is semisimple, then $A(0)P_k(0) = \lambda_k(0)P_k(0)$. Thus (4.6.6) implies that

$$\widetilde{A}_k^{(1)}(\epsilon) = \sum_{n=0}^{\infty}\epsilon^n\widetilde{A}_k^{(n+1)}.$$

On the other hand, from (4.6.7) and Lemma 4.6.1, we have for all $n \geq 1$

$$\|\widetilde{A}_k^{(n)}\| \leq \frac{\|B\|^n}{\rho^n}.$$

Therefore, for ϵ small enough, there exists $c > 0$ independent of k such that

$$\|\widetilde{A}_k^{(1)}(\epsilon) - \widetilde{A}_k^{(1)}\| \leq \sum_{n=1}^{\infty} \epsilon^n \frac{\|B\|^{n+1}}{\rho^{n+1}} \leq \frac{\|B\|}{\rho} \left(\frac{\epsilon \frac{\|B\|}{\rho}}{1 - \epsilon \frac{\|B\|}{\rho}} \right) \leq c\epsilon. \tag{4.6.9}$$

Step 3. We compare the eigenvalues $\{\mu_{k,j}^{(1)}(\epsilon)\}_{j=1}^{m_k}$ of $\widetilde{A}_k^{(1)}(\epsilon)$ with the eigenvalues of $\widetilde{A}_k^{(1)} = P_k(0)BP_k(0)$. Consider $\mu_{k,j}^{(1)}(\epsilon)$ and $\phi_{k,j}^{(1)}(\epsilon)$ an associated normalized eigenvector, then

$$\widetilde{A}_k^{(1)}(\epsilon)\phi_{k,j}^{(1)}(\epsilon) = \mu_{k,j}^{(1)}(\epsilon)\phi_{k,j}^{(1)}(\epsilon).$$

From (4.6.9), we have

$$\|\widetilde{A}_k^{(1)}(\epsilon)\phi_{k,j}^{(1)}(\epsilon) - \widetilde{A}_k^{(1)}\phi_{k,j}^{(1)}(\epsilon)\| \leq c\epsilon.$$

Thus, by Cauchy-Schwarz's inequality, we have

$$| < \mu_{k,j}^{(1)}(\epsilon)\phi_{k,j}^{(1)}(\epsilon) - \widetilde{A}_k^{(1)}\phi_{k,j}^{(1)}(\epsilon), \phi_{k,j}^{(1)}(\epsilon) > | \leq c\epsilon,$$

or equivalently

$$|\mu_{k,j}^{(1)}(\epsilon) - < \widetilde{A}_k^{(1)}\phi_{k,j}^{(1)}(\epsilon), \phi_{k,j}^{(1)}(\epsilon) > | \leq c\epsilon.$$

Therefore,

$$\Re(\mu_{k,j}^{(1)}(\epsilon)) \leq < \widetilde{A}_k^{(1)}\phi_{k,j}^{(1)}(\epsilon), \phi_{k,j}^{(1)}(\epsilon) > + c\epsilon,$$

or

$$\Re(\mu_{k,j}^{(1)}(\epsilon)) \leq \max_{1 \leq j \leq m_k} \mu_{k,j} + c\epsilon.$$

We conclude by using this estimate and (4.6.4). ■

According to Lemma 4.6.2, to prove that the spectrum of $A(\epsilon)$ is situated to the left of the imaginary axis for $\epsilon > 0$ small enough, we have to prove that, for every $k \in \mathbb{Z}$, the eigenvalues of $\widetilde{A}_k^{(1)} = P_k(0)BP_k(0)$ are strictly to the left of the imaginary axis independently of $k \in \mathbb{Z}$ and $\epsilon > 0$. In fact, the hypothesis imposed in [51] to get the exponential stability of (S_ϵ) can be interpreted as a condition on the negativity of the eigenvalues of $P_k(0)BP_k(0)$. Therefore, our aim in the next two lemmas is to find the eigenvalues of $P_k(0)BP_k(0)$ and to investigate the conditions for which their real parts are negative independently of $k \in \mathbb{Z}$ and $\epsilon > 0$.

Lemma 4.6.3. *If $\mu_{k,0}$ denotes the eigenvalue of $P_k(0)BP_k(0)$, where $P_k(0)$ is the eigenprojection corresponding to $\lambda_k(0) = \imath(k\pi + \frac{\pi}{2})$, with $k \in \mathbb{Z}$, then $\mu_{k,0} = -\frac{1}{N}\sum_{i=1}^{N} a_i$.*

Proof: We recall that $\lambda_k(0) = \imath(k\pi + \frac{\pi}{2})$ is simple for all $k \in \mathbb{Z}$. Some elementary calculations show that the associated normalized eigenvector is of the form

$$\phi_0 = \frac{1}{\sqrt{N}}(u, v)^\top,$$

where, for all $x \in (0,1)$ and $i = 1, \cdots, N$, $u_i(x) = \dfrac{\sinh(\lambda_k(0)(1-x))}{\lambda_k(0)}$ and $v_i(x) = \sinh(\lambda_k(0)(1-x))$. For any $\psi \in \mathcal{H}$, we find that

$$P_k(0)BP_k(0)\psi = -\frac{1}{N}\left(\sum_{i=1}^{N} a_i\right)(\psi, \phi_0)\phi_0,$$

hence ϕ_0 is the eigenvector of $P_k(0)BP_k(0)$ of eigenvalue $-\dfrac{1}{N}\displaystyle\sum_{i=1}^{N} a_i$. ■

Lemma 4.6.4. *If $\{\mu_{k,j}\}_{j=1}^{N-1}$ denotes the set of eigenvalues of $P_k(0)BP_k(0)$, where $P_k(0)$ is the eigenprojection corresponding to $\lambda_k(0) = \imath k\pi$, with $k \in \mathbb{Z}^*$, then $\{\mu_{k,j}\}_{j=1}^{N-1}$ is the set of zeros of the polynomial Q defined by*

$$Q(z) = (z + a_1)(z + a_N)\sum_{i=2}^{N-1}\prod_{\substack{l \neq i \\ l=2}}^{N-1}(z + a_l) + \prod_{l=2}^{N-1}(z + a_l)(2z + a_1 + a_N) \tag{4.6.10}$$

Proof: First, we notice that, for all $k \in \mathbb{Z}^*$, $\lambda_k(0) = \imath k\pi$ is of multiplicity $N-1$ and that the associated eigenvectors are of the form $(u, v)^\top$ where, for $i = 1, \cdots, N$

and $x \in (0,1)$, $u_i(x) = \frac{\alpha_i}{\imath k\pi}\sin(k\pi(1-x))$ and $v_i(x) = \alpha_i \sin(k\pi(1-x))$ with $\alpha = (\alpha_i)_{i=1}^N \in \mathbb{C}^N$ such that $\sum_{i=1}^{N} \alpha_i = 0$. As a basis of the subspace $P_k(0)\mathcal{H}$, we can choose the system of eigenvectors $\{\phi^{(i)}\}_{i=1,\cdots,N-1}$ corresponding to the choice

$$\alpha^{(1)} = (1,-1,0\cdots,0), \alpha^{(2)} = (1,0,-1,0\cdots,0), \cdots, \alpha^{(N-1)} = (1,0,\cdots,0,-1).$$

Therefore, for all $i = 1,\cdots,N-1$, $P_k(0)BP_k(0)\phi^{(i)} = \sum_{k=1}^{N-1} \alpha_{ik}\phi^{(k)}$ where $\alpha_{ik} \in \mathbb{C}$. Moreover, for all $i,j = 1,\cdots,N-1$,

$$\left\langle P_k(0)BP_k(0)\phi^{(i)}, \phi^{(j)}\right\rangle = \left\langle B\phi^{(i)}, \phi^{(j)}\right\rangle = \sum_{k=1}^{N-1} \alpha_{ik}\left\langle \phi^{(k)}, \phi^{(j)}\right\rangle.$$

Hence, $P_k(0)BP_k(0) = \Gamma G^{-1}$, where $\Gamma = (\left\langle B\phi^{(i)}, \phi^{(j)}\right\rangle)_{i,j}$ and G is the Gramian matrix defined by $G = (\left\langle \phi^{(i)}, \phi^{(j)}\right\rangle)_{i,j}$. But some elementary calculations yield

$$\Gamma = \begin{pmatrix} -a_1-a_2 & -a_1 & -a_1 & \cdots & -a_1 \\ -a_1 & -a_1-a_3 & -a_1 & \cdots & -a_1 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & -a_1 \\ -a_1 & -a_1 & \cdots & -a_1 & -a_1-a_N \end{pmatrix}$$

and

$$G = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 2 \end{pmatrix}.$$

Since

$$G^{-1} = \frac{1}{N}\begin{pmatrix} N-1 & -1 & -1 & \cdots & -1 \\ -1 & N-1 & -1 & \cdots & -1 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & -1 \\ -1 & -1 & \cdots & -1 & N-1 \end{pmatrix} = I - \frac{1}{N}\begin{pmatrix} 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix},$$

we get

$$\begin{aligned} & P_k(0)BP_k(0) \\ = & \frac{1}{N}\begin{pmatrix} -a_1-(N-1)a_2 & -a_1+a_2 & -a_1+a_2 & \cdots & -a_1+a_2 \\ -a_1+a_3 & -a_1-(N-1)a_3 & -a_1+a_3 & \cdots & -a_1+a_3 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ -a_1+a_N & -a_1+a_N & \cdots & -a_1+a_N & -a_1-(N-1)a_N \end{pmatrix}. \end{aligned}$$

Therefore, $\{\mu_{k,j}\}_{j=1}^{N-1}$ are the roots of the characteristic polynomial $\det\left(zI - P_k(0)BP_k(0)\right)$ or equivalently

$$Q(z) = \det\begin{pmatrix} z+a_2 & 0 & \cdots & \cdots & -z-a_N \\ 0 & z+a_3 & 0 & \cdots & -z-a_N \\ \vdots & \cdots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & z+a_{N-1} & -z-a_N \\ z+a_1 & z+a_1 & \cdots & z+a_1 & 2z+a_1+a_N \end{pmatrix}.$$

Developing with respect to the last line, we find (4.6.10). ■

Before going on let us notice that the above lemmas show that the eigenvalues $\mu_{k,j}$ of $P_k(0)BP_k(0)$ are independent of k. In the first case we directly find a condition on the damping coefficients to have $\mu_{k,0} < 0$, for the second case we need that the roots of Q are negative. For this aim, we first localize the roots of Q. Before doing so let us introduce the following notation : as the a_i are not necessarily different, we denote by $M \leq N$ the number of different a_i's and denote by $\{b_j\}_{j=1}^M$ the set of the different coefficients in increasing order, which means that

$$\{b_j\}_{j=1}^M = \{a_i\}_{i=1}^N,$$

and

$$b_1 < b_2 < \cdots < b_M.$$

Further for all $j = 1, \cdots M$, denote by k_j the number of repeated values of b_j in the initial set of coefficients a_i, namely

$$k_j = \#\{i \in \{1, \cdots, N\} : b_j = a_i\}.$$

Lemma 4.6.5. *If Q is the polynomial defined by (4.6.10), then its has $N-1$ real roots $\mu_i, i = 1, \cdots, N-1$, in $[-a_N, -a_1]$ such that*

$$-b_{j+1} < \mu_j < -b_j, \forall j = 1, \cdots, M-1,$$

the other roots are $-b_j$ of multiplicity $k_j - 1$, for all $j = 1, \cdots, M$ such that $k_j \geq 2$.

Proof: We first notice that

$$Q(-a_i) = \prod_{\substack{l=1 \\ l \neq i}}^N (a_l - a_i).$$

Hence we see that $-a_i$ is a root of Q if and only if there exists at least one $\ell \neq i$ such that $a_i = a_\ell$. But for a complex number μ such that $\mu \notin \{-a_i\}_{i=1}^N$, we notice that

$$Q(\mu) = \prod_{l=1}^N (\mu + a_l) \left(\sum_{i=1}^N \frac{1}{\mu + a_i} \right). \tag{4.6.11}$$

Therefore $\mu \notin \{-a_i\}_{i=1}^N$ is a root of Q if and only if

$$\widetilde{Q}(\mu) = \sum_{i=1}^{N} \frac{1}{\mu + a_i} = 0.$$

As $\widetilde{Q}$ has vertical asymptotes $\mu = -b_j$, for all $j = 1, \dots, M$ and is a decreasing function on $(-b_{j+1}, -b_j)$, for all $j = 1, \cdots, M-1$ (see Figure 4.2 for the graph of $\widetilde{Q}$ when $N = M = 4,\ a_1 = -2,\ a_2 = 2,\ a_3 = 3,\ a_4 = 4$), we deduce that Q has one and only one real root between two consecutive vertical asymptotes.

Now for $j = 1, \cdots, M$ such that $k_j \geq 2$, we take $\mu \neq -b_j$ but close to it and use the expression (4.6.11) to find that

$$Q(\mu) = \prod_{\ell : a_\ell \neq b_j} (\mu + a_l)\,(\mu + b_j)^{k_j} \left(\frac{k_j}{\mu + b_j} + \sum_{i : a_i \neq b_j} \frac{1}{\mu + a_i} \right) \tag{4.6.12}$$

$$= (\mu + b_j)^{k_j - 1} \prod_{\ell : a_\ell \neq b_j} (\mu + a_l) \left(k_j + (\mu + b_j) \sum_{i : a_i \neq b_j} \frac{1}{\mu + a_i} \right). \tag{4.6.13}$$

Since $\prod_{\ell : a_\ell \neq b_j} (\mu + a_l) \left(k_j + (\mu + b_j) \sum_{i : a_i \neq b_j} \frac{1}{\mu + a_i} \right)$ is holomorphic in a neighborhood of $-b_j$, we deduce that $-b_j$ is a root of Q of multiplicity $k_j - 1$. ■

Corollary 4.6.6. *The polynomial Q defined by (4.6.10) has negative roots if and only if one of the following two conditions is satisfied :*

(i) $a_1 \geq 0$ and $a_i > 0$, for all $i = 2, \cdots, N$,

(ii) $a_1 < 0$, $a_i > 0$, for all $i = 2, \cdots, N$ and

$$\sum_{i=1}^{N} \frac{1}{a_i} < 0.$$

Proof: According to the previous lemma, if $-b_2 \geq 0$, then Q has a positive root, hence b_2 has to be positive. Now if $b_1 = a_1$ is positive, all roots are trivially negative.

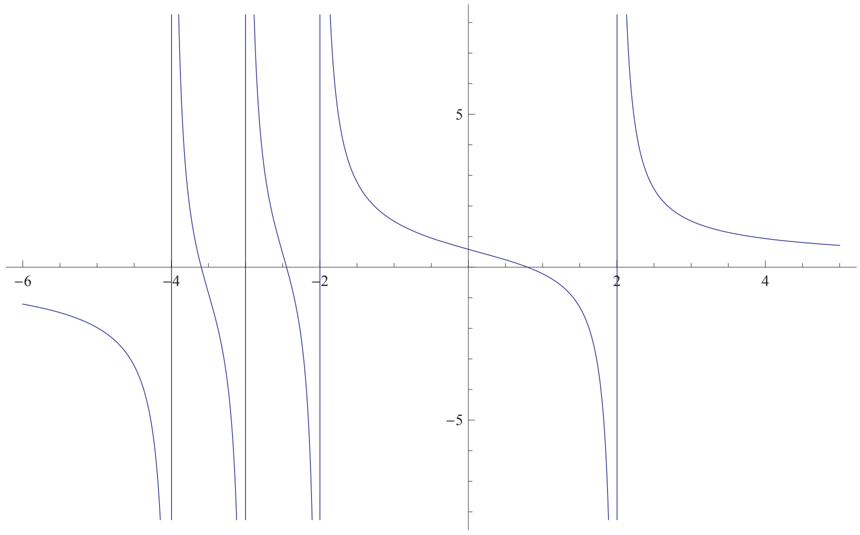

FIGURE 4.2 – $N = 4, a_1 = -2, a_2 = 2, a_3 = 3, a_4 = 4$

On the other hand, if $b_1 \le 0$ with $k_1 > 1$, then Q has a non negative root $-b_1$. Hence k_1 has to be equal to 1. This covers the first item. For the second item, we have $b_1 < 0$ with $k_1 = 1$ and therefore again according to the previous lemma, Q has a root μ (or equivalently $\widetilde{Q}$) between $-a_2 < 0$ and $-a_1 > 0$ that potentially could be positive, but since $\widetilde{Q}$ is decreasing on $(-a_2, -a_1)$ the condition

$$\widetilde{Q}(0) = \sum_{i=1}^{N} \frac{1}{a_i} < 0$$

is a necessary and sufficient condition to get $\mu < 0$. ■

Summing up the results of Lemmas 4.6.2, 4.6.3, and Corollary 4.6.6, we give the proof of Theorem 4.1.4.

Proof: of Theorem 4.1.4 : According to Lemma 4.6.2, if $\max_{k\in\mathbb{Z}} \max_{j=1,\cdots,m_k} \mu_{k,j} = -C < 0$, then there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, and all $k \in \mathbb{Z}$, $\Re\lambda_k(\epsilon) \le -\frac{C}{2}\epsilon$. Using Lemmas 4.6.3 and Corollary 4.6.6, this is satisfied if either one of the two conditions is satisfied :

(i) $a_1 \geq 0$ and $a_i > 0$, for all $i = 2, \cdots, N$,

(ii) $a_1 < 0$, $a_i > 0$, for all $i = 2, \cdots, N$ and

$$\sum_{i=1}^{N} a_i > 0 \text{ as well as } \sum_{i=1}^{N} \frac{1}{a_i} < 0.$$

Since the root vectors of $A(\epsilon)$ form a Riesz basis with parenthesis (see Proposition 4.4.1), we deduce the exponential stability of the solution of (S_ϵ) for all $\epsilon \in (0, \epsilon_0)$ under one of the conditions (i) or (ii) stated above. ■

Remark 4.6.7. *(i) Owing to Lemma 4.6.3 and Corollary 4.6.6, if* $\sum_{i=1}^{N} a_i < 0$ *or if* Q *has a positive root, then* (S_ϵ) *is unstable for all* $\epsilon > 0$ *small enough.*

(ii) If $\sum_{i=1}^{N} a_i = 0$ *or if* Q *has a root equal to zero, then the exponential stability of* (S_ϵ) *for* $\epsilon > 0$ *small enough is an open question. For example, in the case* $N = 3$, $a_1 = a_2 = 1$, *and* $a_3 = -\frac{1}{2}$, *Figure 4.6 shows that there are eigenvalues to the right of the imaginary axis when* $\epsilon = 0.1$.

Remark 4.6.8. *The previous analysis can be adapted to the case when* $a_i \in L^\infty(0,1)$ *and* $L_i = 1$ *for all* $i = 1, \cdots, N$. *As before we can prove that the solution of* (S_ϵ) *is exponentially stable for all* $\epsilon \in (0, \epsilon_0)$ *for* $\epsilon_0 > 0$ *small enough if there exists* $c_0 > 0$ *and* $c_1 > 0$ *such that for all* $k \in \mathbb{Z}$, *one of the following two conditions holds :*

(a) There exists at most one $j_0 \in \{1, \cdots, N\}$ *such that*
$\int_0^1 a_{j_0}(x) \sin^2(k\pi(1-x))dx = 0$, $\int_0^1 a_i(x) \sin^2(k\pi(1-x))dx > c_0$ *for all* $i \neq j_0$ *and* $\sum_{i=1}^{N} \int_0^1 a_i(x) \sin^2((k\pi + \frac{\pi}{2})(1-x))dx > c_0$.

(b) There exists only one $i_0 \in \{1, \cdots, N\}$ *such that*
$\int_0^1 a_{i_0}(x) \sin^2(k\pi(1-x))dx < 0$, $\int_0^1 a_i(x) \sin^2(k\pi(1-x))dx > c_0$ *for all* $i \neq i_0$,

$$\sum_{i=1}^{N}\int_0^1 a_i(x)\sin^2((k\pi+\frac{\pi}{2})(1-x))dx > c_0,\ and$$
$$\sum_{i=1}^{N}\frac{1}{\displaystyle\int_0^1 a_i(x)\sin^2(k\pi(1-x))dx} < -c_1.$$

Indeed, the results of Lemma 4.6.2 still hold. Lemma 4.6.3 also holds but in this case, for all $k \in \mathbb{Z}$, $\mu_{k,0} = -\frac{2}{N}\sum_{i=1}^{N}\int_0^1 a_i(x)\sin^2((k\pi+\frac{\pi}{2})(1-x))dx$. Similarly, in Lemma 4.6.4, we can repeat the same analysis and find that, for all $k \in \mathbb{Z}^*$, $\{\mu_{k,j}\}_{j=1}^{N-1}$ is the set of zeros of

$$\widehat{Q}(z) = (z+I_1)(z+I_N)\sum_{i=2}^{N-1}\prod_{\substack{l\neq i\\ l=2}}^{N-1}(z+I_l) + \prod_{l=2}^{N-1}(z+I_l)(2z+I_1+I_N),$$

where for all $i = 1,\cdots,N$, $I_i = 2\int_0^1 a_i(x)\sin^2(k\pi(1-x))dx$ (which here depends on k). As Lemma 4.6.5 can be used for $\widehat{Q}$, we find the same results but with a_i replaced by I_i for all $i = 1,\cdots,N$. Therefore, thanks to Lemma 4.6.2 and under one of the conditions (a) or (b) stated above, we deduce the existence of $\widehat{C} > 0$ such that for all $k \in \mathbb{Z}$, $\Re\lambda_k(\epsilon) \leq -\epsilon\widehat{C}$ for all $\epsilon \in (0,\epsilon_0)$.

4.7 Examples

In order to illustrate our general results we present some concrete examples where we can give explicit conditions on the damping coefficients to get exponential decay (up to a finite-dimensional space) for both problems (S_1) and (S_ϵ). In the first case, this is reduced to the calculation of the roots of the polynomial G defined by (4.1.1), in the second one since the conditions from Theorem 4.1.4 are easy to check, we concentrate on a limit case (see Remark 4.6.7) and on the characterization of the limit values of ϵ for which the global stability is lost.

4.7.1 Examples for (S_1)

We consider (S_1) with three edges $(N = 3)$ of length $L_i = 1$ and $a_i(.) \in W^{1,\infty}(0,1)$ such that $\int_0^1 a_1(x)dx = \int_0^1 a_2(x)dx = 1$ and $\int_0^1 a_3(x)dx = \alpha \leq 0$. Using Theorem 4.1.1, we will find the critical value of α for which (S_1) is exponentially stable up to a finite dimensional space. Indeed, for this example, the polynomial G is given by

$$G(z) = 3e^{2+\alpha}z^3 - (e^{2-\alpha} + 2e^{\alpha})z^2 - (e^{-2+\alpha} + 2e^{-\alpha})z + 3e^{-2-\alpha}.$$

The roots of G are given by

$$\begin{aligned} z_1 &= e^{-2}, \\ z_2 &= -\frac{e^{-2}}{6} + \frac{e^{-2\alpha}}{6} - \frac{e^{-2-2\alpha}}{6}\sqrt{e^4 + e^{4\alpha} + 34e^{2+2\alpha}}, \\ z_3 & -\frac{e^{-2}}{6} + \frac{e^{-2\alpha}}{6} + \frac{e^{-2-2\alpha}}{6}\sqrt{e^4 + e^{4\alpha} + 34e^{2+2\alpha}}. \end{aligned}$$

Recall that according to Theorem 4.1.1, (S_1) is exponentially stable up to a finite dimensional space if and only if $|z_i| < 1$ for all $i = 1, 2, 3$. Hence we need to analyze the behavior of the z_i with respect to α. Clearly $z_1 < 1$ is independent of α, while the two other ones depend on α. For z_2, we easily check that $z_2 < 0$ is an increasing function of α with $\lim_{\alpha \to -\infty} z_2 = -3e^{-3} > -1$. Hence, $-1 < z_2 < 0$ for all $\alpha \leq 0$. Next, we notice that $|z_2||z_3| = e^{-2-2\alpha}$. So, if $\alpha \leq -1$, then $|z_2||z_3| \geq 1$ which means that $|z_3| \geq 1$. Therefore, to get the exponential stability of (S_1), we must have $\alpha > -1$. In this case, z_3 is a decreasing function of α and for $\alpha_0 = \frac{1}{2}\ln\left(\frac{3+e^2}{1+3e^2}\right)$ we get $z_3 \geq 1$ if $\alpha \leq \alpha_0$ and $0 < z_3 < 1$ if $\alpha > \alpha_0$. In conclusion, (S_1) is exponentially stable up to a finite dimensional space if and only if $\alpha \in (\alpha_0, 0]$.

As a second example, we still take three edges, but consider $L_1 = L_2 = 1$ and $L_3 = 2$ with $\int_0^1 a_1(x)dx = \int_0^1 a_2(x)dx = 1$ and $\int_0^2 a_3(x)dx = \alpha$. With this choice, the polynomial G is given by

$$G(z) = (ez - e^{-1})p(z).$$

where

$$p(z) = 3e^{1+\alpha}z^3 + e^{-1+\alpha}z^2 - e^{1-\alpha}z - 3e^{-1-\alpha}.$$

As the roots of the first factor is $e^{-2} < 1$, we only have to consider the roots of the second factor p. Let $z_i = z_i(\alpha)$ for $i = 1, 2, 3$ be the roots of p and define $\varphi(\alpha) = \max_{i\in\{1,2,3\}} |z_i(\alpha)|$. With the help of a formal computation software (Mathematica), we can find the roots $z_i(\alpha)$ for $i = 1, 2, 3$ as well as $\varphi(\alpha)$.

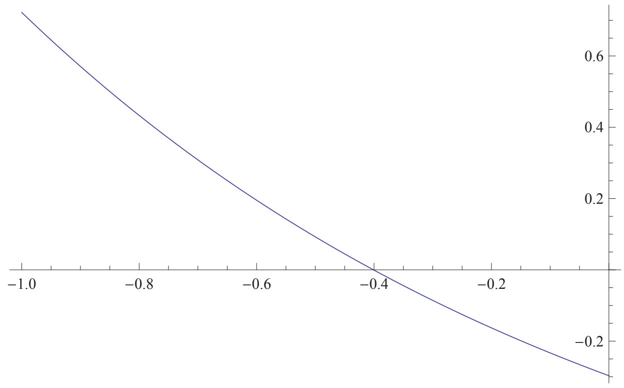

FIGURE 4.3 – Graph of $\varphi(\alpha) - 1$ when $\int_0^1 a_1(x)dx = \int_0^1 a_2(x)dx = 1$, $\int_0^2 a_3(x)dx = \alpha$.

The explicit form of φ allows to check that when $\alpha > \alpha_0 = \frac{1}{2}\ln\left(\frac{3+e^2}{1+3e^2}\right)$ then $\varphi(\alpha) < 1$ (see Figure 4.3). Hence (S_1) is exponentially stable up to a finite dimensional space if and only if $\alpha \in (\alpha_0, 0]$.

The same study can be done when changing L_3, namely by taking $L_3 = \frac{1}{2}$ or $L_3 = 3$ and we surprisingly obtain the same critical value α_0 of α so that (S_1) is exponentially stable up to a finite dimensional space. Moreover, if we choose $L_1 = 1$ and $L_2 = 2$ such that $\int_0^1 a_1(x)dx = \int_0^2 a_2(x)dx = 1$, then for $L_3 = 1$ or $L_3 = 2$, we still obtain the same condition, $\alpha > \alpha_0 = \frac{1}{2}\ln\left(\frac{3+e^2}{1+3e^2}\right)$ to get the exponential stability of (S_1) up to a finite dimensional space. Furthermore, if we change the

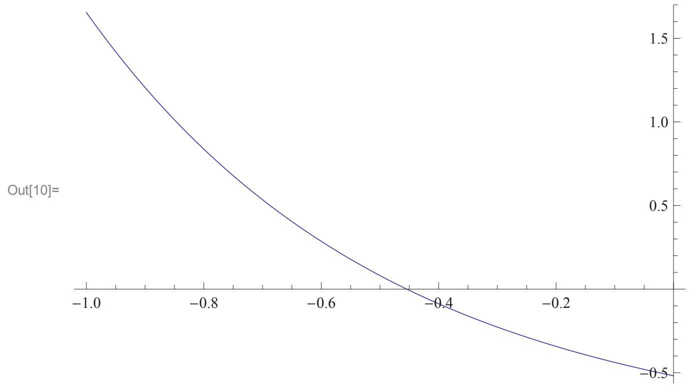

FIGURE 4.4 – Graph of $\varphi(\alpha) - 1$ when $\int_0^1 a_1(x)dx = 1,\ \int_0^1 a_2(x)dx = 2,\ L_3 = 2.$

mean values, by considering $L_1 = L_2 = 1$, but $\int_0^1 a_1(x)dx = 1$ and $\int_0^1 a_2(x)dx = 2$, then whether $L_3 = 1$ or $L_3 = 2$, we still obtain the same critical value α_1 with $0.45 < \alpha_1 < 0.46$ such that (S_1) is exponentially stable up to a finite dimensional space if and only if $\alpha > \alpha_1$ (see Figure 4.4).

In conclusion, we find that the critical value of α depends on $\int_0^{L_1} a_1(x)dx$ and $\int_0^{L_2} a_2(x)dx$ and not on the choice of the lengths. This opens the question whether the abstract condition given in Theorem 4.1.1 can be expressed explicitly in terms of $\int_0^{L_i} a_i(x)dx$ for all $i \in \{1, \cdots, N\}$, see Conjecture 4.1.2.

4.7.2 Examples for problem (S_ϵ)

We start with a limit case in Theorem 4.1.4, namely we take $N = 3$, $a_1 = a_2 = 1$, $a_3 = -\frac{1}{2}$, and $L_1 = L_2 = L_3 = 1$. Hence neither the first condition holds nor the second one since $\sum_{i=1}^{N} a_i > 0$ but $\sum_{i=1}^{N} \frac{1}{a_i} = 0$.

But Lemma 4.6.2 yields that for all $\epsilon > 0$ small enough, the eigenvalues are of the form

$$\begin{aligned}\lambda_{1,k}(\epsilon) &= -\epsilon + \imath k\pi + o(\epsilon),\\ \lambda_{2,k}(\epsilon) &= \imath k\pi + o(\epsilon),\\ \lambda_{3,k}(\epsilon) &= -\frac{\epsilon}{2} + \imath\frac{(2k+1)\pi}{2} + o(\epsilon),\end{aligned}$$

Hence the problem of stability would come from $\lambda_{2,k}(\epsilon)$ but a more precise asymptotic analysis yields $\Re\lambda_{2,k}(\epsilon) = \frac{\epsilon^3}{12} + o(\epsilon^3)$, hence the problem is not exponentially stable for ϵ small. Figure 4.6 shows the existence of a positive asymptote when $\epsilon = 0.1$, since the asymptotes are $x_1 = -0.1,\ x_2 \approx -0.0500833$, and $x_3 \approx 0.000083333$.

Note that for $\epsilon = 1$, then by Theorem 4.1.1 there is a positive asymptote, since the asymptotes are $x_1 = -1,\ x_2 \approx -0.580322$, and $x_3 \approx 0.0803219$ (see Figure 4.5).

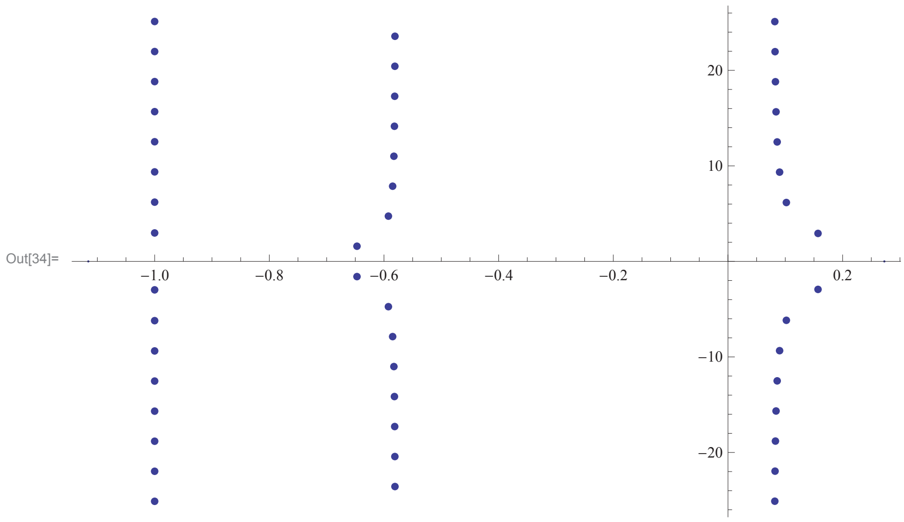

FIGURE 4.5 – $a_1 = a_2 = 1,\ a_3 = -\frac{1}{2},\ \epsilon = 1.$

In general, if we consider $a_1 = a_2 = a$ and $a_3 \neq a$, then according to Theorem 4.1.4, the problem becomes exponentially stable for all ϵ small enough if one of the following two conditions holds :

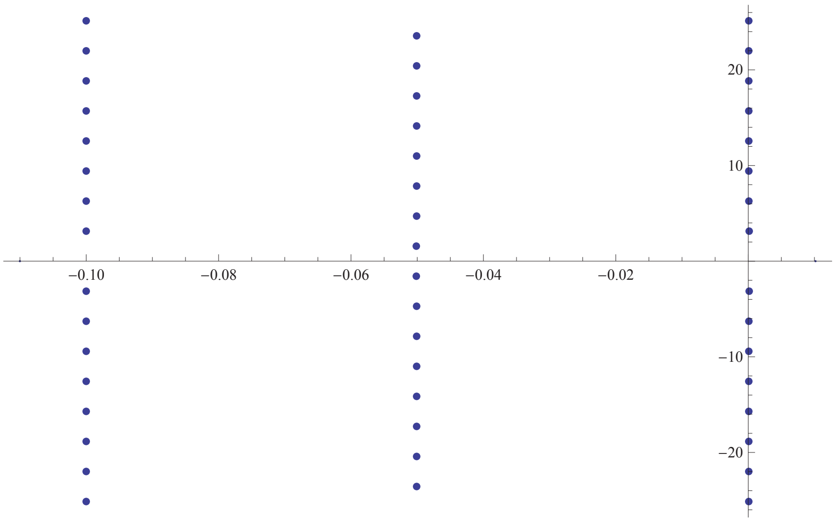

FIGURE 4.6 – $a_1 = a_2 = 1,\ a_3 = -\frac{1}{2},\ \epsilon = 0.1.$

(i) $a > 0$ and $a_3 \geq 0$

(ii) $a > 0$ and $a_3 < 0$ such that $2a + a_3 > 0$ and $a + 2a_3 > 0$.

These results are coherent with the numerical results shown in Figure 4.9 with $a_1 = a_2 = 1,\ a_3 = -\frac{1}{4}$ and $\epsilon = \frac{1}{10}$ where the asymptotes ($x_1 = -0.1,\ x_2 \approx -0.0166184$, and $x_3 \approx -0.0583816$) are to the left of the imaginary axis. If we increase ϵ and take $\epsilon = 1$, then Figure 4.7 still shows the exponential stability in the whole energy space where the asymptotes are $x_1 = -1,\ x_2 \approx -0.630695$, and $x_3 \approx -0.119305$. But for $\epsilon = 1.5$, then Figure 4.8 shows the exponential stability up to a finite dimensional space. Indeed, the asymptotes found in Figure 4.8 are $x_1 = -1.5,\ x_2 \approx -1.02451$, and $x_3 \approx -0.100488$ which show that the large eigenvalues are to the left of the imaginary axis although there are some low eigenvalues with positive real parts. In fact, in the case $a_1 = a_2 = 1,\ a_3 = -\frac{1}{4}$, all the eigenvalues are to the left of the imaginary axis for all $\epsilon < \epsilon_0$, where numerically we have found that $1.30 < \epsilon_0 < 1.31$.

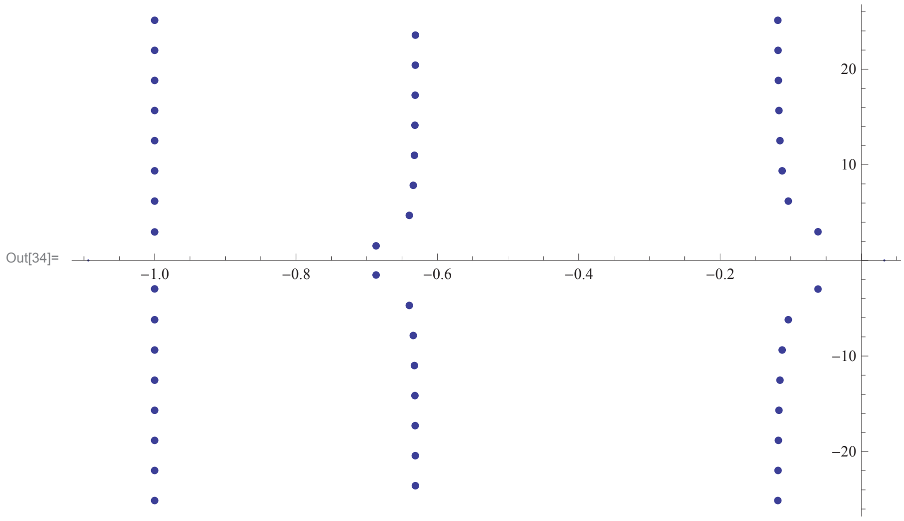

FIGURE 4.7 – $a_1 = a_2 = 1,\ a_3 = -\dfrac{1}{4},\ \epsilon = 1.$

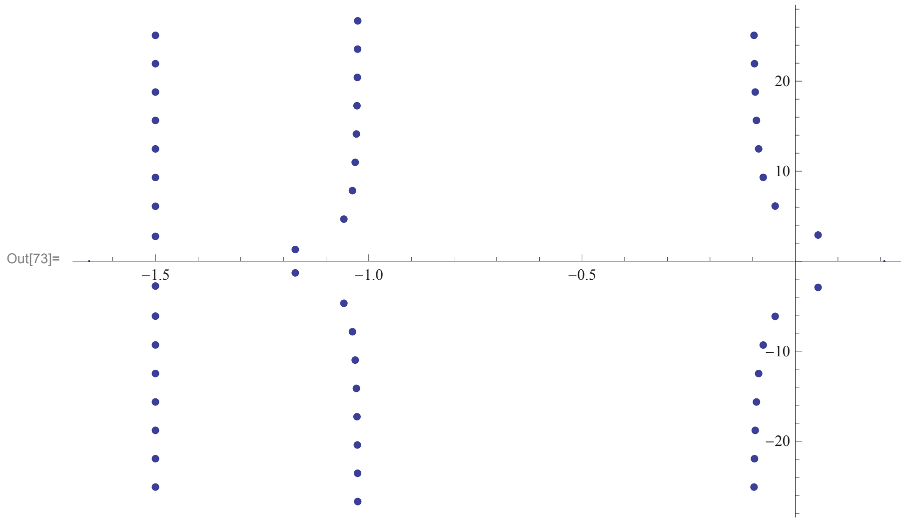

FIGURE 4.8 – $a_1 = a_2 = 1,\ a_3 = -\dfrac{1}{4},\ \epsilon = 1.5.$

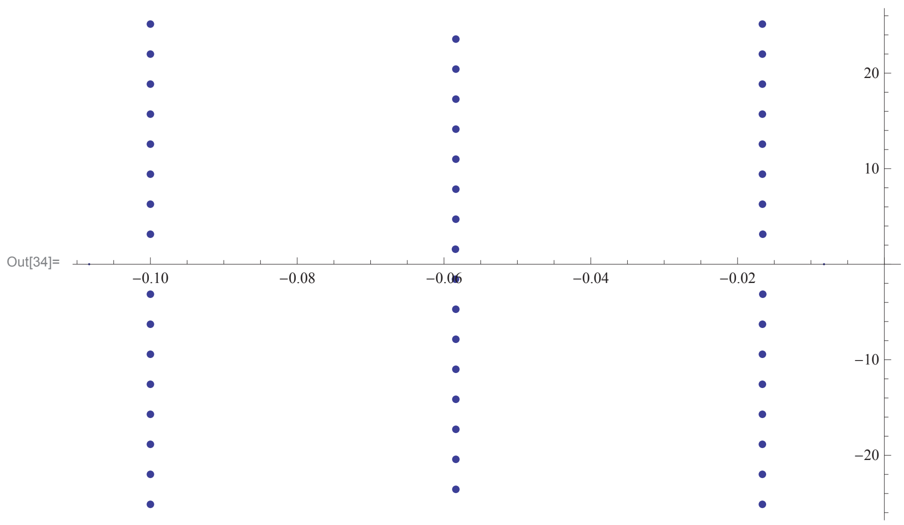

FIGURE 4.9 – $a_1 = a_2 = 1,\ a_3 = -\dfrac{1}{4},\ \epsilon = 1/10.$

Perspective

As a future study, we would like to extend our work about the stability of wave equations with indefinite sign damping over a multi-dimensional space. For instance, let $\Omega = (-1,1) \times (0,1)$ be partitioned into $\Omega_1 = (0,1) \times (0,1)$ and $\Omega_2 = (-1,0) \times (0,1)$. We are interested in studying the stability of the following system

$$\begin{cases} u_{tt} - \Delta u + a_1 u_t &= 0 \quad \text{in } \Omega_1 \times \mathbb{R}, \\ u_{tt} - \Delta u + a_2 u_t &= 0 \quad \text{in } \Omega_2 \times \mathbb{R}, \\ u &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,y,0) = u_0,\ u_t(x,y,0) &= u_1 \quad \text{in } \Omega, \end{cases} \tag{4.7.1}$$

where $a_1 > 0$ and $a_2 < 0$. We write the solution as a Fourier series of the form

$$u(x,y) = 2\sum_{k=1}^{\infty} u_k(x) \sin(k\pi y),\ \forall (x,y) \in \Omega.$$

Then

$$\begin{cases} u_{ktt} - u_{kxx} + k^2\pi^2 u_k + a(x) u_{kt} &= 0 \qquad (x,t) \in (-1,1) \times \mathbb{R}, \\ u_k(-1,t) = u_k(1,t) &= 0 \qquad t \in \mathbb{R}, \\ u_k(x,0) = u_{k0},\ u_{kt}(x,0) &= u_{k1} \quad x \in (-1,1), \end{cases} \tag{4.7.2}$$

where $a(x) = a_1$ if $x \in (0,1)$ and $a(x) = a_2$ if $x \in (-1,0)$. The energy associated with (4.7.2) is given by

$$E_k(t) = \frac{1}{2}\int_{-1}^{1} \left(|u_{kx}|^2 + k^2\pi^2 |u_k|^2 + |u_{kt}|^2\right) dx.$$

Using Parseval's equality, the energy associated with (4.7.1) is given by

$$E(t) = \frac{1}{2}\int_{\Omega}\left(|\nabla u|^2 + |u_t|^2\right)dx = \sum_{k=1}^{\infty} E_k(t).$$

Our aim is to find $c > 0$ independent of k and $\nu_k > 0$ such that

$$E_k(t) \leq ce^{-\nu_k t}E_k(0).$$

Hence,

$$E(t) \leq c\sum_{k=1}^{\infty} e^{-\nu_k t}E_k(0).$$

If $\nu_k \geq \nu > 0$, then $E(t) \leq ce^{-\nu t}E(0)$ and hence system (4.7.1) is exponentially stable. If $\nu_k \cong \frac{1}{k^l}$ for some $l > 0$, then

$$\begin{aligned} E(t) &\lesssim \sum_{k=1}^{\infty} e^{\frac{-t}{k^l}}E_k(0) \\ &\lesssim \sum_{k=1}^{\infty}\frac{k^l}{t}\frac{t}{k^l}e^{\frac{-t}{k^l}}E_k(0) \\ &\lesssim \frac{1}{t}\sum_{k=1}^{\infty}k^l E_k(0) \\ &\lesssim \frac{1}{t}\|u\|_{D(A^{\frac{l}{2}})}, \end{aligned}$$

where the operator A is the generator of the semigroup associated with system (4.7.1).

We are also interested in studying both internally and boundary damped problems of the form

$$\begin{cases} u_{tt} - \Delta u + au_t &= 0 \quad \text{in } \Omega \times \mathbb{R}, \\ u_x &= -bu_t \quad \text{on } \Gamma_N \times \mathbb{R}, \\ u &= 0 \quad \text{on } \Gamma_D \times \mathbb{R}, \\ u(x,y,0) = u_0,\ u_t(x,y,0) &= u_1 \quad \text{in } \Omega, \end{cases} \tag{4.7.3}$$

where $\Gamma_N = \{x = 1\} \times (0, 1)$ and $\Gamma_D = \partial\Omega/\Gamma_N$. We are interested in studying the stability of (4.7.3) when $ab < 0$. If $a = 0$ and $b > 0$ or $a > 0$ and $b = 0$, problem (4.7.3) is polynomially stable and not exponentially stable. Therefore, we expect to find conditions on a and b for which problem (4.7.3) is polynomially stable.

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Stabilisation et approximation de certains systèmes distribués par amortissement dissipatif et de signe indéfini

Dans cette thèse, nous étudions l'approximation et la stabilisation de certaines équations d'évolution, en utilisant la théorie des semi-groups et l'analyse spectrale. Cette thèse est divisée en deux parties principales. Dans la première partie, comme dans [3, 4], nous considérons l'approximation des équations d'évolution du deuxième ordre modélisant les vibrations de structures élastiques. Il est bien connu que le système approchée par éléments finis ou différences finies n'est pas uniformément exponentiellement ou polynomialement stable par rapport au paramètre de discrétisation, même si le système continu a cette propriété. Dans la première partie, notre objectif est d'amortir les modes parasites à haute fréquence en introduisant des termes de viscosité numérique dans le schéma d'approximation. Avec ces termes de viscosité, nous montrons la décroissance exponentielle ou polynomiale du schéma discret lorsque le problème continu a une telle décroissance et quand le spectre de l'opérateur spatial associé au problème conservatif satisfait la condition du gap généralisée. En utilisant le Théorème de Trotter-Kato, nous montrons la convergence de la solution discrète vers la solution continue. Quelques exemples sont également présentés.
Dans la deuxième partie, comme dans [1, 2], nous étudions la stabilisation des équations des ondes avec amortissement de signe indéterminée. Tout d'abord, nous considérons deux problèmes des ondes dont les termes d'amortissement peuvent changer de signe. En utilisant l'analyse du spectre, on trouve des valeurs critiques des coefficients d'amortissement pour lesquels le problème devient exponentiellement ou polynomialement stable jusqu'à ces valeurs critiques. Ensuite, nous généralisons l'analyse des deux derniers problèmes pour étudier la stabilité des équations des ondes sur un réseau en forme d'étoile en présence de termes d'amortissement de signe indéterminée. Pour ce problème, nous constatons que la condition intuitive sur la positivité de la moyenne ne suffit pas. Que la norme L^∞ des coefficients d'amortissement soit grande ou petite, nous cherchons des conditions suffisantes sur les coefficients d'amortissement pour lesquels le problème devient exponentiellement

stable.

Mots-clés : Stabilité, semi-discrétisation, terme de viscosité, gap généralisé, amortissement de signe indéterminée, comportement asymptotique, base de Riesz, réseau en forme d'étoile.

Stabilization and approximation of some distributed systems by either dissipative or indefinite sign damping

In this thesis, we study the approximation and stabilization of some evolution equations, using semigroup theory and some spectral analysis. This Ph.D. thesis is divided into two main parts. In the first part, as in [3, 4], we consider the approximation of second order evolution equations modeling the vibrations of elastic structures. It is well known that the approximated system by finite elements or finite differences is not uniformly exponentially or polynomially stable with respect to the discretization parameter, even if the continuous system has this property. Therefore, our goal is to damp the spurious high frequency modes by introducing numerical viscosity terms in the approximation scheme. With these viscosity terms, we show the exponential or polynomial decay of the discrete scheme when the continuous problem has such a decay and when the spectrum of the spatial operator associated with the undamped problem satisfies the generalized gap condition. By using the Trotter-Kato Theorem, we further show the convergence of the discrete solution to the continuous one. Some illustrative examples are also presented.

In the second part, as in [1, 2], we study the stabilization of wave equations with indefinite sign damping. Here we search for sufficient conditions on the damping coefficients so that the wave equations are either exponentially or polynomially stable. First, we consider two damped wave problems which are either internally or boundary damped and for which the damping terms are allowed to change their sign. Using a careful spectral analysis, we find critical values of the damping coefficients for which the problem becomes exponentially or polynomially stable up to these critical values. Afterwards, we generalize the analysis of the previous two problems to the case of wave equations on a star shaped network in the presence of indefinite sign damping terms. For this problem, we find that the intuitive condition on the positivity of the mean is not sufficient. Whether the L^∞ norm of the damping coefficients is large or small, we search for sufficient conditions on the damping coefficients for which the problem becomes exponentially stable.

Key words : Stability, semi-discretization, viscosity terms, generalized gap condition, indefinite sign damping, asymptotic behavior, Riesz basis, star-shaped network.

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Table des matières

Introduction

Control theory can be described as the process of influencing the behavior of a physical system to achieve a desired goal, primarily through the use of feedback which monitors the effect of a system and modifies its output. It is applied in a diverse range of scientific and engineering disciplines such as the reduction of noise, the vibration of structures like seismic waves and earthquakes, the regulation of biological systems like human cardiovascular system, the design of robotic systems, laser control in quantum mechanical and molecular systems.

In this thesis, we implement the semigroup theory in the spirit of spectral theory to study the approximations and stabilization of some evolution equations. In general, stability results are obtained using different methods like the multipliers method, the frequency domain method, the microlocal analysis, the differential geometry or a combination of them [47,50,77,78]. In this thesis, we use detailed spectral analysis. In fact, this thesis is divided into two parts. In the first part, we consider the approximations of second order evolution equations. Studies and researches have shown that the approximated system by finite element or finite difference is not uniformly exponentially or polynomially stable with respect to the discretization parameter even if the continuous system has this property (see [25, 27, 43, 55, 72, 73, 76, 78]). Therefore, our aim in the first part is to search for a suitable discrete system which approximates the continuous system and most importantly restores the decay rate properties of the continuous one. In the presence of the generalized gap condition, our strategy is based on adding numerical viscosity terms in the approximation schemes to damp out the effect of the high frequencies.

In the second part, we study the stabilization of wave equations with indefinite sign damping. Here, we use a detailed spectral analysis to study the behavior of the spectrum out of which we search for critical values of the damping coefficients so that the wave equations are either exponentially or polynomially stable. First, we consider one dimensional internally and boundary damped wave problems and afterwards we generalize the analysis of indefinite sign damped wave equations to a star shaped network where we find extra conditions to get stability.

0.1 Outline of the thesis

This thesis is divided into four main chapters. In the first chapter, we recall some basic definitions and theorems about the semigroup and spectral analysis theories.

In the second chapter, as in [3] and [4], we consider the approximation of linear equations modeling the vibrations of elastic structures with feedback control. More precisely, let H be a complex Hilbert space with norm and inner product denoted respectively by $\|.\|$ and $(.,.)$. Let $A : \mathcal{D}(A) \to H$ be a densely defined self-adjoint and positive operator with a compact inverse in H. Let $V = \mathcal{D}(A^{\frac{1}{2}})$ be the domain of $A^{\frac{1}{2}}$. Denote by $\mathcal{D}(A^{\frac{1}{2}})'$ the dual space of $\mathcal{D}(A^{\frac{1}{2}})$ obtained by means of the inner product in H.

Furthermore, let U be a complex Hilbert space (which will be identified to its dual space) with norm and inner product denoted respectively by $\|.\|_U$ and $(.,.)_U$

and let $B \in \mathcal{L}(U, H)$. We consider the closed loop system

$$\begin{aligned} \ddot{\omega}(t) + A\omega(t) + BB^*\dot{\omega}(t) = 0, \\ \omega(0) = \omega_0, \, \dot{\omega}(0) = \omega_1, \end{aligned} \tag{0.1.1}$$

where $t \in [0, \infty)$ represents the time and $\omega : [0, \infty) \to H$ is the state of the system.

We define the energy of system (0.1.1) at time t by

$$E(t) = \frac{1}{2}\left(\|\dot{\omega}(t)\|^2 + \left\|A^{\frac{1}{2}}\omega(t)\right\|^2\right).$$

Simple formal calculations give

$$E(0) - E(t) = \int_0^t (BB^*\dot{\omega}(s), \dot{\omega}(s))\, ds, \quad \forall t \geq 0.$$

This obviously means that the energy is non-increasing. In the second chapter, our goal is to search for a suitable discrete system which first approximates (0.1.1) and second has the same stability properties as (0.1.1). However, in many applications, most of the classical numerical approximation schemes do not possess the same decay rate as that of the continuous problem although the convergence is preserved. At the discrete level, spurious high frequency oscillations are generated and therefore bad behavior of the approximate solution is clearly observed causing a non-uniform decay rate (see [14,19,20,25,27,33,34,43,55,58,72,73,76,78]). For instance, we start the second chapter by considering the vibrations of a flexible string joined at each of its ends. Although the continuous problem is exponentially stable, we show that the finite difference semi-discrete problem is not uniformly exponentially stable ; i.e., there does not exist constants M and $\beta > 0$ independent of the discretization parameter such that

$$E_h(t) \leq Me^{-\beta t}, \quad \text{as} \;\; t \to +\infty,$$

where $E_h(t)$ represents the energy of the semi-discrete system.

Several remedies are proposed to restore the uniform decay rate of the discrete problems like Tychonoff regularization [34, 35, 64, 72], a bi-grid algorithm [32, 58], a mixed finite element method [14, 19, 20, 33, 56], or filtering the high frequencies [43, 49, 76]. As in [64, 72], we introduce artificial numerical viscosity terms in the approximation schemes to rule out the high frequency spurious numerical oscillations and hence restore the uniform decay rate of the discrete scheme. However, contrary to [64] where the standard gap condition is required, we only assume that the spectrum of the operator $A^{1/2}$ satisfies the generalized gap condition. Indeed, if $\{\lambda_k\}_{k\geq 1}$ denotes the set of eigenvalues of $A^{\frac{1}{2}}$ counted with their multiplicities, then we assume that the following generalized gap condition holds :

$$\exists M \in \mathbb{N}^*,\ \exists \gamma_0 > 0,\ \forall k \geq 1,\ \lambda_{k+M} - \lambda_k \geq M\gamma_0.$$

The standard gap condition is satisfied for the particular case when $M = 1$. Therefore, in the second chapter, we treat more general concrete systems.

After recalling the suitable conditions and observability inequalities which lead to the exponential or the polynomial stability of the solution of problem (0.1.1), we search for a suitable discrete system which has the same decay properties under these conditions. For this reason, after finding the suitable discrete system, we use the discrete result of [52] which gives the necessary and sufficient conditions for which an approximate solution is exponentially stable. As for the uniform polynomial stability, we prove a result which gives necessary and sufficient conditions for which a family of semigroups of operators is uniformly polynomially stable. To our knowledge, our work in the second chapter is the first one which addresses the uniform polynomial stability of the discrete schemes.

As for the convergence of the chosen approximate system, we use a general version of the Trotter-Kato Theorem proved in [45] to show that the discrete solution tends to the solution of (0.1.1) as the discretization parameter goes to zero and if the discrete initial data are well chosen. Finally, we end up the second chapter by some illustrative examples which show the limits of the previous work done concerning the approximations and values the attained results of the second chapter.

In the third chapter, we move on to another subject which treats the stabilization of wave equations with indefinite sign damping. As in [1], we analyze the stability of two problems. We consider a one-dimensional wave equation with an indefinite sign damping and a zero order potential term which is internally damped of the form

$$\begin{gathered} u_{tt}(x,t) - u_{xx}(x,t) + 2\chi_{(0,1)}(x)u_t(x,t) + 2\alpha\chi_{(-1,0)}(x)u_t(x,t) = 0,\ x \in (-1,1),\ t > 0, \\ u(1,t) = u(-1,t) = 0,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \end{gathered} \tag{0.1.2}$$

where α is a given constant. Besides, we consider a one-dimensional wave equation with an indefinite sign damping and which is both internally and boundary damped under the form

$$\begin{aligned} &u_{tt}(x,t) - u_{xx}(x,t) + au_t(x,t) = 0, \quad && x \in (0,1),\ t > 0, \\ &u(0,t) = 0,\ u_x(1,t) = -bu_t(1,t), && t > 0, \\ &u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \end{aligned} \tag{0.1.3}$$

where $a, b \in \mathbb{R}$.

It is well known that problem (0.1.2) is exponentially stable if the damping term α is non-negative (see [23]). Similarly, if the coefficients a and b are both positive, then, using for instance integral inequalities, one can prove that (0.1.3) is also exponentially stable. In the third chapter, we are interested in the case when the damping terms are allowed to change their sign. Our aim is to analyze to what extent the variation of the sign affects the stability of the problem.

Problem (0.1.2) can be written as a system of the form $U_t = A_\alpha U$ where $U =$

$(u,u_t)^\top$ and the operator $A_\alpha : D(A_\alpha) \to X$ is defined by

$$A_\alpha = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & -2\chi_{(0,1)} - 2\alpha\chi_{(-1,0)} \end{bmatrix}$$

where the energy space $X = H_0^1(-1,1) \times L^2(-1,1)$ is equipped with the usual inner product defined by

$$< (f,g)^\top, (u,v)^\top >= \int_{-1}^{1} (f'\overline{u}' + g\overline{v})dx,$$

and

$$D(A_\alpha) = (H^2(-1,1) \cap H_0^1(-1,1)) \times H_0^1(-1,1).$$

In this case, the energy associated with problem (0.1.2), at time t, is given by

$$E_1(t) = \frac{1}{2}\left(\int_{-1}^{1} (|u_x(x,t)|^2 + |u_t(x,t)|^2)dx\right)$$

with

$$E_1'(t) = -2\left(\int_0^1 |u_t(x,t)|^2 dx + \alpha \int_{-1}^{0} |u_t(x,t)|^2 dx\right), \quad \forall (u_0,u_1) \in D(A_\alpha).$$

Therefore, when $\alpha < 0$, the dissipation of the energy is not trivial. Moreover, the classical techniques which are normally employed to study the stabilization like multipliers method, energy and resolvent methods cannot be well invoked in this case since these methods are based on estimations which involve the absolute value of the damping coefficients. Therefore, the question of the stability of the solution of (0.1.2) in the case of $\alpha < 0$ becomes more involved.

The main motivation behind this question started with a conjecture in [21] by Chen et al. who considered the internally indefinite sign damped wave equation of the form

$$u_{tt} - u_{xx} + 2a(x)u_t = 0, \; x \in (0,1), \; t > 0, \tag{0.1.4}$$

with standard initial conditions and Dirichlet boundary conditions.

It was conjectured that if there exists some $\gamma > 0$ such that for every $n \in \mathbb{N}$ the following condition is satisfied

$$I_n = \int_0^1 a(x)\sin^2(n\pi x)dx \geq \gamma, \tag{0.1.5}$$

then the energy decays exponentially. The hypothesis imposed on the uniform positivity of I_n in (0.1.5) yields the positivity of the average a_0 of a since $I_n \to a_0$, as $n \to +\infty$. However, Freitas in [28] disproves the conjecture of Chen et al. He shows that (0.1.5) is not sufficient to guarantee the exponential stability. He finds out that if $\|a\|_{L^\infty}$ is large then there may exist some positive real eigenvalues (see Theorem 3.6 of [28]). So later on, Freitas and Zuazua in [30] suggest replacing the function $a(x)$ in (0.1.4) by $\epsilon a(x)$ with $\epsilon > 0$ small enough. In this case, the exponential stability is proved under condition (0.1.5) and the additional condition $a \in L^\infty(0,1) \cap BV$ so that its derivative is defined in the weak sense. Furthermore, in [51], the authors find an upper bound of ϵ for which the problem becomes exponentially stable under condition (0.1.5) and the assumption that $a \in L^\infty(0,1)$ without the need for the assumption that $a \in BV$. On the other hand, in [57], Racke and Rivera have removed the factor ϵ and considered the wave equation $u_{tt} - u_{xx} + a(x)u_t = 0$ on $(0, L)$ for some $L > 0$ where $a \in L^\infty(0, L)$ is allowed to change its sign such that its mean value a_0 remains positive. In [57], the exponential stability is proved under one of these conditions : Either $\|a\|_{L^\infty}$ is possibly large with sufficiently small $\|a - a_0\|_{L^2}$ or $\|a\|_{L^\infty}$ is sufficiently small but the pair (a, L) has to satisfy some estimates where it is possible to get a negative moment I_k.

In the third chapter, our work differs from the previous results since we do not want to impose neither a small value of the damping factor a nor a small value of $\|a - a_0\|_{L^2}$. Indeed for system (0.1.2), this mean value is equal to $\sqrt{2}|1-\alpha|$ which we do not need to be sufficiently small. Moreover, the upper bound of ϵ found in [51] is not easy to check for system (0.1.2).

From the asymptotic behavior of the spectrum of A_α, we find that, according to the value of α, problem (0.1.2) is either unstable or exponentially stable. Using detailed spectral analysis, we find the characteristic equation satisfied by the eigenvalues of A_α and then we show that the root vectors of A_α form a Riesz basis of the energy space. Finally, we find a critical value of α for which the solution of (0.1.2) becomes exponentially stable. Although the critical value which we find for α is not optimal, this value remains coherent with that given by the perturbation theory of semigroups.

In the third chapter, we perform a similar analysis for problem (0.1.3). As usual, by the standard reduction of order method, we can rewrite formally (0.1.3) in the simpler form $U_t = A_a U$, with $U = (u, u_t)^\top$ and the operator $A_a : D(A_a) \to X$ is defined by

$$A_a = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & -a \end{bmatrix} \tag{0.1.6}$$

where the energy space $X = H_l(0,1) \times L^2(0,1)$ is equipped with the usual inner product defined by

$$< (f,g)^\top, (u,v)^\top >= \int_0^1 (f'\overline{u}' + g\overline{v})\, dx,$$

where $H_l(0,1) = \{u \in H^1(0,1);\ u(0) = 0\}$ and therefore, $D(A_a) = \{(u,v)^\top \in H^2(0,1) \cap H_l(0,1) \times H_l(0,1);\ u_x(1) = -bv(1)\}$.

The energy of (0.1.3) is given by

$$E_2(t) = \frac{1}{2}\int_0^1 \left(|u_t|^2 + |u_x|^2\right) dx,$$

and hence formally

$$E_2'(t) = -a\int_0^1 |u_t|^2 dx - b|u_t(1)|^2.$$

From this identity, we remark that A_a is not necessarily dissipative when $ab < 0$. Therefore, we are interested in the case when a and b are of opposite signs. Note that for such a problem, perturbation theory of contractive semigroups cannot be well invoked. Using detailed spectral analysis, we find the conditions that a and b must satisfy so that problem (0.1.3) becomes exponentially or polynomially stable. The particular case $b \in (-1, 0)$ and $a > 0$ retains our attention where we find optimal results for which (0.1.3) is exponentially or polynomially stable.

Finally in the fourth chapter, as in [2], we generalize the analysis of the third chapter to study the exponential stability of the wave equation on a star shaped network with indefinite sign damping which is of the form

$$\begin{cases} u^i_{tt}(x,t) - u^i_{xx}(x,t) + 2\epsilon a_i(x) u^i_t(x,t) &= 0, \qquad x \in (0, L_i),\ t > 0, \\ u^i(L_i, t) &= 0, \\ u^i(0,t) &= u^j(0,t), \quad \forall i \neq j, \\ \displaystyle\sum_{i=1}^{N} u^i_x(0,t) &= 0, \\ u^i(x,0) &= u^i_0(x), \qquad x \in (0, L_i), \\ u^i_x(x,0) &= u^i_1(x), \qquad x \in (0, L_i). \end{cases} \tag{S_1}$$

where $L_i \in \mathbb{R}^+_*$, and $a_i \in W^{1,\infty}(0, L_i)$. We further consider the following hypothesis on the geometry of the domain :

(**H**) There exists $q \in \mathbb{N}^*$ such that for all $i = 2, ..., N$, there exists $p_i \in \mathbb{N}^*$ where $L_i = \dfrac{p_i}{q} L_1$.

In the first part of the fourth chapter, we study the stability of system (S_1) when $\epsilon = 1$. We give necessary and sufficient conditions for which system (S_1) becomes exponentially stable up to a finite dimensional space. The idea is inspired from [65] where the characteristic equation of (S_1) is approximated by another function using the shooting method. This approximation allows us to detect the behavior of the

high frequencies and hence deduce the conditions on the damping coefficients a_i for which the high frequencies are situated to the left of the imaginary axis. Finally, after we prove that the generalized root vectors form a Riesz basis with parentheses, we deduce the exponential stability of (S_1) up to a finite dimensional space generated by the roots vectors of the low frequencies. In the first part, when $N = 2$, we recover the result of Theorem 3.2.1 of this thesis.
In the second part, we consider system (S_1) with ϵ positive but small enough so that we extend the results of Freitas and Zuazua in [30] where $N = 2$. In fact, for $\epsilon > 0$ small enough, unlike [30], we deal with multiple eigenvalues when splitting may occur as ϵ increases. First, we consider $a_i \in \mathbb{R}$ and $L_i = 1$ for all $i = 1, ..., N$ and then we consider $a_i \in L^\infty(0, 1)$. In fact, when $\epsilon > 0$ small enough, the study of the exponential stability of (S_1) enters in the framework of the abstract theory done in [51]. Using the concepts introduced in [46] about the behavior of the spectrum, we shall interpret the hypothesis imposed in [51] to find explicit conditions on the damping coefficients for which (S_1) is exponentially stable. In the presence of a Riesz basis with parenthesis, we search for sufficient conditions for which the eigenvalues are situated strictly to the left of the imaginary axis. We find out that the positivity of the mean of the damping coefficients is not enough to guarantee the exponential stability of (S_1) in the whole energy space. In this second part, we recover the result of Theorem 2.1 of [30] when the damping coefficient is piecewise constant but without the assumption on the positivity of the integrals I_n given in (0.1.5). Finally, we end up the fourth chapter by giving some concrete examples of $\{a_i\}_{i=1}^N$ and N.

0.2 Aims and achieved results

For more coherence, we summarize the main goals and the new results attained in this Ph.D. thesis into the following points :

(i) Search for a suitable approximate system which converges towards problem (0.1.1) and has the same decay properties as (0.1.1) in the presence of the generalized gap condition.

(ii) Analyze the polynomial decay of the discrete schemes when the continuous problem has such a decay and prove a result about uniform polynomial stability for a family of semigroups of operators.

(iii) Use a general version of the Trotter-Kato Theorem proved in [45] to prove the convergence of the discrete solution towards the solution of (0.1.1) as the discretization parameter goes to zero and if the discrete initial data are well chosen.

(iv) Study the stability of wave equations in the presence of indefinite sign damping where the classical methods for studying the stabilization fail to treat such problems.

(v) Consider indefinite sign damping coefficients whose L^∞ norm is not necessarily small.

(vi) Use detailed spectral analysis to find critical values of the damping coefficients for which wave equations with indefinite sign damping become stable.

(vii) Generalize the analysis of the stability of wave equations with indefinite sign damping terms over a star shaped network.

Chapitre 1

Preliminaries

As the analysis done in this Ph.D. thesis is based on the semigroup and spectral analysis theories, we recall, in this chapter, some basic definitions and theorems which will be used in the following chapters. We refer to [8, 18, 24, 36, 37, 42, 62, 63].

1.1 Semigroups

Most of the evolution equations can be reduced to the form

$$\begin{cases} \dot{x}(t) &= \mathcal{A}x(t), \quad t > 0, \\ x(0) &= x_0, \end{cases}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $T(t)$ over a Hilbert space $\mathcal{H}$. Therefore, we start by introducing some basic concepts concerning the semigroups.

Definition 1.1.1. *Let X be a Banach space.*

1) A one parameter family $T(t)$, $t > 0$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if

(i) $T(0) = I$;

(ii) $T(t+s) = T(t)T(s)$ *for every* $s, t \geq 0$.

2) A semigroup of bounded linear operators, $T(t)$*, is uniformly continuous if*

$$\lim_{t \to 0} \|T(t) - I\| = 0.$$

3) A semigroup $T(t)$ *of bounded linear operators on* X *is a strongly continuous semigroup of bounded linear operators or a* C_0 *semigroup if*

$$\lim_{t \to 0} T(t)x = x.$$

4) The linear operator $\mathcal{A}$ *defined by*

$$\mathcal{A}x = \lim_{t \to 0} \frac{T(t)x - x}{t}, \quad \forall x \in D(\mathcal{A}),$$

where

$$D(\mathcal{A}) = \left\{ x \in X;\ \lim_{t \to 0} \frac{T(t)x - x}{t} \ \text{exists} \right\}$$

is the infinitesimal generator of the semigroup $T(t)$.

Theorem 1.1.2. *Let* $T(t)$ *be a* C_0 *semigroup. Then there exist constants* $\omega \geq 0$ *and* $M \geq 1$ *such that*

$$\|T(t)\| \leq M e^{\omega t}, \quad \forall t > 0.$$

In the above theorem, if $\omega = 0$, then $T(t)$ is called uniformly bounded and if moreover $M = 1$, then $T(t)$ is called a C_0 semigroup of contractions.

For the existence of solutions, we normally use the following Lumer-Phillips Theorem or Hille-Yosida Theorem.

Theorem 1.1.3. *(Lumer-Phillips Theorem) Let* $\mathcal{A}$ *be a linear operator with dense domain* $D(\mathcal{A})$ *in a Hilbert space* X*. If* *(i)* $\mathcal{A}$ *is dissipative; i.e.,* $\Re < \mathcal{A}x, x >_X < 0, \ \forall x \in D(\mathcal{A})$

and if

(ii) there exists a $\lambda_0 > 0$ *such that the range* $\mathcal{R}(\lambda_0 I - \mathcal{A}) = X$,

then $\mathcal{A}$ *generates a* C_0 *semigroup of contractions on* X.

Theorem 1.1.4. *(Hille-Yosida Theorem) Let* $\mathcal{A}$ *be a linear operator on a Banach space* X *and let* $\omega \in \mathbb{R}$, $M \geq 1$ *be constants. Denote by* $\rho(\mathcal{A})$ *the resolvent set of* $\mathcal{A}$. *Then the following properties are equivalent*

(i) $\mathcal{A}$ *generates a* C_0 *semigroup* $T(t)$, $t \geq 0$ *satisfying*

$$\|T(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0;$$

(ii) $\mathcal{A}$ *is closed, densely defined, and for every* $\lambda > \omega$ *one has* $\lambda \in \rho(\mathcal{A})$ *and*

$$\|(\lambda - \omega)^n (\lambda - \mathcal{A})^{-n}\| \leq M, \quad \forall n \in \mathbb{N};$$

(iii) $\mathcal{A}$ *is closed, densely defined, and for every* $\lambda \in \mathbb{C}$ *with* $\Re\lambda > \omega$, *one has* $\lambda \in \rho(\mathcal{A})$ *and*

$$\|(\lambda - \mathcal{A})^{-n}\| \leq \frac{M}{(\Re\lambda - \omega)^n}, \quad \forall n \in \mathbb{N}.$$

If the conditions of the previous two theorems are not clearly satisfied, we may use the following theorem about perturbations by bounded linear operators (see Theorem III.1.1 of [62]).

Theorem 1.1.5. *Let* X *be a Banach space and let* $\mathcal{A}$ *be the infinitesimal generator of a* C_0 *semigroup* $T(t)$ *on* X, *satisfying* $\|T(t)\| \leq Me^{\omega t}$. *If* B *is a bounded linear operator on* X, *then* $A + B$ *is the infinitesimal generator of a* C_0 *semigroup* $S(t)$ *on* X, *satisfying* $\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$.

Now we recall a result in [42, 63] which gives necessary and sufficient conditions for which a semigroup is exponentially stable.

Theorem 1.1.6. *Let $T(t)$ be a C_0 semigroup on a Hilbert space $\mathcal{H}$ and $\mathcal{A}$ be its infinitesimal generator. $T(t)$ is exponentially stable ; i.e., there exists M and α positive constants such that $\|T(t)\| \leq Me^{-\alpha t}$ if and only if*

(i) $i\mathbb{R} \subseteq \rho(\mathcal{A})$, *where $\rho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$*

and

(ii) $\sup_{\omega \in \mathbb{R}} \|(i\omega - \mathcal{A})^{-1}\| < \infty$.

When the exponential stability is attained, we search for the optimal exponential decay rate ; mainly for the spectrum determined growth condition.

Definition 1.1.7. *Let $\mathcal{A}$ be the infinitesimal generator of a C_0 semigroup, $T(t)$, on a Hilbert space $\mathcal{H}$. Consider*

$$\omega(\mathcal{A}) := \inf\{\alpha \in \mathbb{R};\ \|T(t)\| \leq Me^{\alpha t}\} = \lim_{t \to \infty} \frac{1}{t} \log \|T(t)\|,$$

the growth exponent bound of $T(t)$, and

$$\mu(\mathcal{A}) = \sup\{\Re\lambda;\ \lambda \in \sigma(\mathcal{A})\},$$

the spectral abscissa of the operator $\mathcal{A}$ where $\sigma(\mathcal{A})$ denotes its spectrum. If $\omega(\mathcal{A}) = \mu(\mathcal{A})$, then we say that the spectrum determined growth condition holds.

Remark 1.1.8. *From the Hille-Yosida Theorem, we know that $\mu(\mathcal{A}) \leq \omega(\mathcal{A})$ for any infinitesimal generator of a strongly continuous semigroup. However, in general, $\omega(\mathcal{A}) \leq \mu(\mathcal{A})$ is not always true.*

If the semigroup fails to be exponentially stable, we search for another type of decay rate like the polynomial stability which is characterized by the following Theorem in [18].

Theorem 1.1.9. *Let $(T(t))_{t\geq 0}$ be a bounded C_0 semigroup on a Hilbert space $\mathcal{H}$ with a generator $\mathcal{A}$ such that $i\mathbb{R} \subseteq \rho(\mathcal{A})$. Then for a fixed $\alpha > 0$, the following conditions are equivalent :*

(i)

$$\|(is - \mathcal{A})^{-1}\| = O(|s|^{\alpha}), \quad s \to \infty.$$

(ii)

$$\|T(t)\mathcal{A}^{-\alpha}\| = O(t^{-1}), \quad t \to \infty.$$

(iii)

$$\|T(t)\mathcal{A}^{-1}\| = O(t^{\frac{-1}{\alpha}}), \quad t \to \infty.$$

Note that the notation $A = O(B)$ means that there exists $c > 0$ such that $|A| \leq c|B|$.

1.2 Riesz basis

In the second part, we show that the generalized eigenvectors form a Riesz basis of the energy space. Consequently, we recall basic definitions and theorems needed for Riesz basis generation. We refer to [8, 36, 37].

Definition 1.2.1. *(i) A non-zero element φ in a Hilbert space $\mathcal{H}$ is called a generalized eigenvector of a closed linear operator $\mathcal{A}$, corresponding to an eigenvalue λ of $\mathcal{A}$, if there exists $n \in \mathbb{N}^*$ such that*

$$(\lambda I - \mathcal{A})^n \varphi = 0 \quad \textit{and} \quad (\lambda I - \mathcal{A})^{n-1}\varphi \neq 0.$$

If $n = 1$, then φ is an eigenvector.

(ii) The root subspace of $\mathcal{A}$ corresponding to an eigenvalue λ is defined by

$$\mathcal{N}_{\lambda}(\mathcal{A}) = \bigcup_{n=1}^{\infty} \ker\left((\lambda I - \mathcal{A})^n\right).$$

(iii) *The closed subspace spanned by all the generalized eigenvectors of $\mathcal{A}$ is called the root subspace of $\mathcal{A}$.*

Remark 1.2.2. *The family of generalized eigenvectors of $\mathcal{A}$ corresponding to λ forms a basis for the subspace $\mathcal{N}_\lambda(\mathcal{A})$. Denote by m_λ the algebraic multiplicity of λ. In general, these generalized eigenvectors, denoted by $\{\varphi_j,\ 1 \leq j \leq m_\lambda\}$, are constructed by the following procedure :*

$$\begin{cases} \mathcal{A}\varphi_1 & = \lambda\varphi_1 \\ \mathcal{A}\varphi_j & = \lambda\varphi_j + \varphi_{j-1}, \quad j = 2, ..., m_\lambda. \end{cases}$$

Now, we introduce the Riesz basis and then we recall some theorems which help us prove that a family forms a Riesz basis.

Definition 1.2.3. *Let $\Phi = \{\varphi_n\}_{n\in\mathbb{N}}$ be an arbitrary family of vectors in a Hilbert space $\mathcal{H}$.*

(i) *The family Φ is said to be a Riesz basis in the closure of its linear span (notation $\Phi \in (LB)$) if Φ is an image by an isomorphic mapping of some orthonormal family. Φ is said to be a Riesz basis if $\Phi \in (LB)$ and Φ is a complete family; i.e., $\overline{Span\{\varphi_n;\ n \in \mathbb{N}\}} = \mathcal{H}$.*

(ii) *The family Φ is said to be $\omega-$linearly independent if whenever $\sum_{n\in\mathbb{N}} a_n\varphi_n = 0$ for $\sum_{n\in\mathbb{N}} |a_n|^2 < \infty$ then $a_n = 0$ for every $n \in \mathbb{N}$.*

(iii) *The family Φ is minimal if, for any $n \in \mathbb{N}$, the element φ_n does not belong to the span of all the remaining elements; i.e., $\varphi_n \notin Span\{\varphi_i;\ i \neq n\}$.*

Remark 1.2.4. (i) *If $\Phi \in (LB)$, then Φ is minimal and hence $\omega-$linearly independent.*

(ii) *If Φ is minimal, then there exists a family $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ biorthogonal to Φ; i.e., $< \varphi_j, \psi_i >_{\mathcal{H}} = \delta_{ij}$.*

The following proposition and theorems give necessary and sufficient condition so that a family Φ forms a Riesz basis.

Proposition 1.2.5. *(Bari's Theorem, Bari 1951 ; Gokhberg and krein 1965 ; Nikolski 1980)*
$\Phi \in (LB)$ if and only if there exists positive constants C_1 and C_2 such that for any sequence $\{\alpha_n\}_{n\in\mathbb{N}}$, we have

$$C_1 \sum_{n\in\mathbb{N}} |\alpha_n|^2 \leq \left\| \sum_{n\in\mathbb{N}} \alpha_n \varphi_n \right\|^2 \leq C_2 \sum_{n\in\mathbb{N}} |\alpha_n|^2.$$

In this case, each element $f \in \overline{Span\{\varphi_n,\ n \in \mathbb{N}\}}$ is written as

$$f = \sum_{n\in\mathbb{N}} < f, \psi_n >_{\mathcal{H}} \varphi_n,$$

where $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ is biorthogonal to $\Phi = \{\varphi_n\}_{n\in\mathbb{N}}$.

In this thesis, we mainly use the following theorems to prove that the generalized eigenvectors form a Riesz basis of the energy space.

Theorem 1.2.6. *(Theorem 2.1 of Chapter VI in [36])*
$\{\phi_n\}$ is a Riesz basis of a Hilbert space $\mathcal{H}$ if and only if $\{\phi_n\}$ is complete in $\mathcal{H}$ and there corresponds to it a complete biorthogonal sequence $\{\psi_n\}$ such that for any $f \in \mathcal{H}$ one has

$$\sum_n |< \phi_n, f >|^2 < \infty, \quad \sum_n |< \psi_n, f >|^2 < \infty. \tag{1.2.1}$$

Theorem 1.2.7. *(Classical Bari's Theorem)*
If $\{\varphi_n\}_{n\in\mathbb{N}}$ is a Riesz basis of a Hilbert space $\mathcal{H}$ and another $\omega-$linearly independent family $\{\psi_n\}_{n\in\mathbb{N}}$ is quadratically close to $\{\varphi_n\}_{n\in\mathbb{N}}$ in the sense that $\sum_{n=1}^{\infty} \|\varphi_n - \psi_n\|^2 < \infty$, then $\{\psi_n\}_{n\in\mathbb{N}}$ also forms a Riesz basis of $\mathcal{H}$.

Normally, there is difficulty in understanding the number of generalized eigenfunctions corresponding to low eigenvalues. This severely limits the application of the above classical Bari's Theorem even if the behavior of the high eigenvalues and their corresponding multiplicities are clearly known. Consequently, in case the behavior of the low eigenvalues is vague, we suggest using Theorem 6.3 of [37] which is a new form of Bari's Theorem (see Theorem 2.3 of Chapter VI in [36]) :

Theorem 1.2.8. *Let $\mathcal{A}$ be a densely defined operator in a Hilbert space $\mathcal{H}$ with a compact resolvent. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a Riesz basis of $\mathcal{H}$. If there are an integer $N \geq 0$ and a sequence of generalized eigenvectors $\{\psi_n\}_{n=N+1}^{\infty}$ of $\mathcal{A}$ such that*

$$\sum_{n=N+1}^{\infty} \| \varphi_n - \psi_n \|^2 < \infty,$$

then the set of generalized eigenvectors of $\mathcal{A}$, $\{\psi_n\}_{n=1}^{\infty}$, forms a Riesz basis of $\mathcal{H}$.

Despite that the proof of Theorem 1.2.8 is found in [37], we give another proof which clarifies the relation between the families $\{\varphi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$. First, we recall Lemma 6.2 of [37].

Lemma 1.2.9. *Let $\{\phi_n\}_{n=1}^{\infty}$ be a Riesz basis of a Hilbert space $\mathcal{H}$. Let $N \geq 0$ and $\{\psi_n\}_{n=N+1}^{\infty}$ be another family such that*

$$\sum_{n=N+1}^{\infty} \| \phi_n - \psi_n \|^2 < \infty.$$

Then there exists $M \geq N$ such that

(i) The set $\{\phi_n\}_{n=1}^{M} \cup \{\psi_n\}_{n=M+1}^{\infty}$ is $\omega-$linearly independent.

(ii) $\{\phi_n\}_{n=1}^{M} \cup \{\psi_n\}_{n=M+1}^{\infty}$ is a Riesz basis of $\mathcal{H}$.

Proof: Once (i) is proved, (ii) follows from Theorem 1.2.7. To prove (i), let $M \geq N$ and $\{c_n\}_n \in l^2(\mathbb{N}^*)$ such that

$$\sum_{n=1}^{M} c_n \phi_n + \sum_{n=M+1}^{\infty} c_n \psi_n = 0.$$

Then

$$\sum_{n=1}^{\infty} c_n\phi_n = -\sum_{n=M+1}^{\infty} c_n(\psi_n - \phi_n). \tag{1.2.2}$$

Since $\{\phi_n\}_{n=1}^{\infty}$ is a Riesz basis, there exists $c_1 > 0$ such that

$$c_1 \sum_{n=1}^{\infty} |c_n|^2 \leq \left\| \sum_{n=1}^{\infty} c_n\phi_n \right\|^2. \tag{1.2.3}$$

Using Cauchy-Schwarz inequality, we get

$$\left\| \sum_{n=M+1}^{\infty} c_n(\psi_n - \phi_n) \right\|^2 \leq \sum_{n=M+1}^{\infty} |c_n|^2 \sum_{n=M+1}^{\infty} \|\psi_n - \phi_n\|^2 \leq \frac{c_1}{2} \sum_{n=1}^{\infty} |c_n|^2, \tag{1.2.4}$$

provided M is chosen great enough.

Combining (1.2.2), (1.2.3), and (1.2.4) we get $c_n = 0,\ \forall n \in \mathbb{N}^*$. ■

The following theorem clarifies the results of Theorem 1.2.8.

Theorem 1.2.10. *Let $\mathcal{A}$ be a densely defined operator in a Hilbert space $\mathcal{H}$ with a compact resolvent. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a Riesz basis of $\mathcal{H}$. If there are two integers $N_1, N_2 \geq 0$ and a sequence of generalized eigenvectors $\{\psi_n\}_{n=N+1}^{\infty}$ of $\mathcal{A}$ such that*

$$\sum_{n=1}^{\infty} \| \varphi_{n+N_2} - \psi_{n+N_1} \|^2 < \infty, \tag{1.2.5}$$

then the set of generalized eigenvectors (or root vectors) of $\mathcal{A}$, $\{\psi_n\}_{n=1}^{\infty}$ forms a Riesz basis of $\mathcal{H}$.

Proof: The proof is divided into five steps.

First step.

For all $n \geq 1$, we set $\chi_{n+N_2} = \psi_{n+N_1}$. Thus, we have $\chi_m = \psi_{m+N_1-N_2}, \forall m \geq N_2 + 1$. (1.2.5) means that

$$\sum_{n=N_2+1}^{\infty} \| \chi_n - \phi_n \|^2 < \infty. \tag{1.2.6}$$

Consequently, by Lemma 1.2.9, there exists $M \geq N_2$ such that $\{\phi_n\}_{n=1}^{M} \cup \{\psi_n\}_{n=M+N_1-N_2+1}^{\infty}$ is a Riesz basis of $\mathcal{H}$.

We can assume that the eigenvalues corresponding to $\{\psi_n\}$ for $n \leq M+N_1-N_2$ are different from those of $\{\psi_n\}$ for $n \geq M+N_1-N_2+1$.

Second step.

Now, let $V = \overline{\text{span}\{\psi_n\}_{n=M+N_1-N_2+1}^{\infty}}$, then the set of roots vectors $\{\psi_n\}_{n=1}^{M+N_1-N_2}$ is independent and does not belong to V.

We denote by π_V the orthogonal projection on the space V and define

$$H_0 = \text{span}\{\tilde{\psi}_n\}_{n=1}^{M+N_1-N_2},$$

where $\tilde{\psi}_i = \psi_i - \pi_V \psi_i,\ i = 1, ..., M+N_1-N_2$. Clearly, $\{\tilde{\psi}_n\}_{n=1}^{M+N_1-N_2}$ are independent and $\dim H_0 = M + N_1 - N_2$.

Let P be the orthogonal projector on H_0. We have

$$P\tilde{\psi}_i = \tilde{\psi}_i,\ i = 1, ..., M + N_1 - N_2 \quad \text{and} \quad P\psi_i = 0,\ \forall i > M + N_1 - N_2$$

By the first step, for each $i \in \mathbb{N}^*$, there exists $\{a_n^i\}_{n=1}^{\infty}$ with $a_n^i \in \mathbb{C}$ such that

$$\tilde{\psi}_i = \sum_{n=1}^{M} a_n^i \phi_i + \sum_{n=M+N_1-N_2+1}^{\infty} a_n^i \psi_i.$$

Hence, for $i = 1, ..., M + N_1 - N_2$, we get

$$\tilde{\psi}_i = P\tilde{\psi}_i = \sum_{n=1}^{M} a_n^i P\phi_i.$$

This shows that $M \geq M + N_1 - N_2$ or equivalently $N_2 \geq N_1$.

Third step.

If we assume that $N_2 > N_1$, then $H_0 \subsetneq V^{\perp}$. Thus, let $\psi \neq 0 \in V^{\perp} \cap H_0^{\perp}$ and assume for the moment that the set $\{\psi_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}$. We can write

$$\psi = \sum_{n=1}^{\infty} a_n \psi_n.$$

Therefore,

$$0 = P\psi = \sum_{n=1}^{\infty} a_n P\psi_n = \sum_{n=1}^{M+N_1-N_2} a_n P\psi_n.$$

But for $n \leq M + N_1 - N_2$, we have

$$P\psi_n = P(\tilde{\psi}_n + \pi_V \psi_n) = \tilde{\psi}_n.$$

Consequently, we deduce that

$$0 = \sum_{n=1}^{M+N_1-N_2} a_n \tilde{\psi}_n;$$

i.e., $a_n = 0, i = 1, ..., M + N_1 - N_2$. It follows that

$$\psi = \sum_{M+N_1-N_2+1}^{\infty} a_n \psi_n \in V^{\perp} \cap V = \{0\},$$

which is a contradiction. Consequently, $N_1 = N_2$.

Fourth step.

By the first step, $\{\phi_n\}_{n=1}^{M} \cup \{\psi_n\}_{n=M+1}^{\infty}$ forms a Riesz basis of $\mathcal{H}$. Therefore, it remains to prove that $\{\psi_n\}_{n=1}^{\infty}$ is $\omega-$linearly independent so that we can deduce by Theorem 1.2.7 that $\{\psi_n\}_{n=1}^{\infty}$ forms a Riesz basis of $\mathcal{H}$. Indeed, suppose that

$$\sum_{n=1}^{\infty} a_n \psi_n = 0.$$

Hence,

$$0 = \sum_{n=1}^{M} a_n \psi_n + \sum_{M+1}^{\infty} a_n \psi_n.$$

By the first step, we can write

$$\sum_{n=1}^{M} a_n \psi_n = \sum_{n=1}^{M} b_n \phi_n + \sum_{M+1}^{\infty} b_n \psi_n,$$

where for every $n \in \mathbb{N}^*$, $b_n \in \mathbb{C}$. Therefore,

$$0 = \sum_{n=1}^{M} b_n \phi_n + \sum_{n=M+1}^{\infty} (a_n + b_n) \psi_n,$$

which implies that $b_n = 0$ for all $n = 1, ..., M$. Finally, from

$$\sum_{n=1}^{M} a_n \psi_n = \sum_{M+1}^{\infty} b_n \psi_n,$$

we deduce that $\sum_{n=1}^{M} a_n \psi_n = 0$ since $\{\psi_n\}_{n=1}^{M}$ does not belong to V. Hence $a_n = 0, \forall n \in \mathbb{N}^*$.

Last step.

It remains to prove that $\{\psi_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}$. If it is not the case, by the first and second steps, we only know that the family $\{\psi_n\}_{n=1}^{\infty}$ is a Riesz basis for the subspace $H_1 = \overline{Sp(\mathcal{A})}$, the closed subspace spanned by all generalized eigenvectors $\{\psi_n\}_{n=1}^{\infty}$ of $\mathcal{A}$, and that its codimension is finite.

Let $H_2 = H_1^{\perp}$. Thus, $\mathcal{H} = H_1 \oplus H_2$ and $\dim(H_2) < \infty$. Without loss of generality, we may assume that $0 \notin \sigma(\mathcal{A})$; i.e., $\mathcal{A}$ has a compact inverse $\mathcal{A}^{-1} = B$.

Since H_1 is stable by B, then H_2 is stable by B^*. Consequently $B^*_{|H_2}$ admits at least one eigenvalue μ because H_2 is finite dimensional. Thus, there exists $x \neq 0 \in H_2$ such that $B^* x = \mu x$.

We start by proving that necessarily $\mu = 0$. If $\mu \neq 0$ then the complex $\lambda \neq 0$ such that $\mu = \dfrac{1}{\lambda}$ is in $\sigma(\mathcal{A})$. Let H_λ be the root subspace of $\mathcal{A}$ associated to λ and n be the algebraic multiplicity of λ; i.e., the smallest integer such that $H_\lambda = \ker(I - \lambda B)^n$.

Since B is a compact operator, $I - (I - \lambda B)^n$ is also a compact operator and by the Fredholm alternative

$$\ker(I - [I - (I - \lambda B)^n])^{\perp} = R(I - [I - (I - \lambda B)^n])^*;$$

i.e.,

$$H_\lambda^{\perp} = (\ker(I - \lambda B)^n)^{\perp} = R(I - \overline{\lambda} B^*)^n.$$

But $H_\lambda \subseteq H_1$ implies that $x \in H_\lambda^{\perp}$. Consequently, there exists $y \in \mathcal{H},\ y \neq 0$, such that $x = (I - \overline{\lambda} B^*)^n y$.

As $(I-\overline{\lambda}B^*)x=0$, we get $0=(I-\overline{\lambda}B^*)^{n+1}y$ with $y\neq 0$. Hence, we get a contradiction since $\frac{1}{\lambda}\in\sigma(B)$ and $\frac{1}{\overline{\lambda}}\in\sigma(B^*)$ have the same algebraic multiplicity.

Therefore, we have $\mu=0$, but this contradicts the well-known fact that $\mathcal{H}=\overline{R(B)}\oplus\ker(B^*)$ and the assumption $\overline{R(B)}=\overline{D(\mathcal{A})}=\mathcal{H}$. ■

1.3 Riesz basis with parenthesis

Sometimes we fail to prove the existence of a Riesz basis of the energy space or we need some supplementary hypothesis to find a Riesz basis. However, as in the fourth chapter, we can neglect these hypothesis and find a more general basis of the energy space which is called a Riesz basis with parenthesis. According to [75], we recall the definition of a Riesz basis of subspaces and a Riesz basis with parenthesis.

Definition 1.3.1. – *A family of subspaces* $\{W_k\}_{k\in\mathbb{N}}$ *is called a Riesz basis of subspaces of* $\mathcal{H}$ *if*

(i) for every $f\in\mathcal{H}$*, and every* $k\in\mathbb{N}$*, there is a unique* $f_k\in W_k$ *such that* $f=\sum_{k\in\mathbb{N}}f_k$*, and*

(ii) there are positive constants C_1 *and* C_2 *such that*

$$C_1\sum_{k\in\mathbb{N}}\|f_k\|^2\leq\left\|\sum_{k\in\mathbb{N}}f_k\right\|^2\leq C_2\sum_{k\in\mathbb{N}}\|f_k\|^2.$$

– *A sequence* $\{y_i\}_{i\in\mathbb{N}}$ *is called a Riesz basis with parenthesis of* $\mathcal{H}$ *if there is a family* $\{W_k\}_{k\in\mathbb{N}}$ *of finite-dimensional spaces spanned by some* y_i *with* $W_k\cap W_j=\{0\}$ *for* $k\neq j$ *that forms a Riesz basis of subspaces of* $\mathcal{H}$*. The spaces* W_k *are called the parentheses.*

Now, we recall a Theorem which can be proved exactly as Theorem 2 in [71] which gives sufficient conditions for which the generalized eigenfunctions of some operator $\widetilde{A}$ form a Riesz basis with parenthesis. For this aim, we need to group the eigenvalues of $\widetilde{A}$ by packets made of a finite number of eigenvalues and in such a way that the packets remain at a positive distance from each other. Namely for any $r > 0$, we introduce the sets $G_p(r), p \in \mathbb{Z}$ as the connected components of the set $\cup_{\lambda \in \sigma(\widetilde{A})} D_\lambda(r)$,where $D_\lambda(r)$ is the disc with center λ and radius r, as well as the packets of eigenvalues $\Lambda_p(r) = G_p(r) \cap \sigma(\widetilde{A})$. The following Theorem gives sufficient conditions for which the generalized eigenfunctions of a bounded perturbation of a selfadjoint operator form a Riesz basis with parenthesis.

Theorem 1.3.2. *Let T be a selfadjoint operator over a Hilbert space H with discrete spectrum $\{\mu_k\}_{k\in\mathbb{Z}}$ which satisfies the generalized gap condition, i.e, there exists $k_0 > 0$ and $c > 0$ such that*

$$\mu_{k+k_0} - \mu_k > c, \forall k \in \mathbb{Z}.$$

Let B be a bounded operator from H into itself. Then the root vectors of the perturbation $\widetilde{A} = T + B$ form a Riesz basis with parenthesis of H. In this case, only terms corresponding to merging eigenvalues should be put in parenthesis, i.e, there exist $r > 0$ and $N \in \mathbb{N}^*$ such that if we set

$$\Lambda_p = \Lambda_p(r),$$

then

$$\#\Lambda_p \leq N, \forall p \in \mathbb{Z},$$

$$\sigma(T + B) = \cup_{p\in\mathbb{Z}} \Lambda_p,$$

and we can take as parenthesis W_p, $p \in \mathbb{Z}$, the space spanned by the root vectors of $T + B$ corresponding to the eigenvalues in Λ_p where for any $f \in H$, $f_p = \mathbb{P}_p f$ is the the Riesz projection of $T + B$, i.e.,

$$f_p = \mathbb{P}_p f = \frac{1}{2i\pi} \int_{\gamma_p} (\lambda - T - B)^{-1} f \, d\lambda,$$

where γ_p *is a contour surrounding* Λ_p*.*

Chapitre 2

Uniformly exponentially or polynomially stable approximations for second order evolution equations and some applications

2.1 Introduction and Motivation

Recently, the approximation of second order evolution equations has been extensively studied where misbehavior of the discrete solutions has been remarkably observed (see [25, 27, 55, 72, 73]). Indeed, the discrete schemes, obtained by finite difference, finite element, or finite volume discretization, introduce spurious high oscillations which do not exist at the continuous level and which propagate with group velocity of the order of the mesh size. As a result, even though the numerical scheme converges in the classical sense towards the continuous problem, observability inequalities do not hold uniformly with respect to the discretization parameter and hence, the decay rate of the discrete system turns out to depend on the mesh size. In fact, the uniform decay rate is equivalent to some observability inequalities which estimate the discrete energy by the velocity of the propagation of the oscillations. However, in the presence of high frequency discrete solutions whose velocity is of the order of the mesh size, the observability constants blow up as the mesh size tends to zero (see [55, 73]). Consequently, most of the classical numerical approximation schemes do not possess the same decay rate as that of the continuous problem.

For more coherence, we start with a simple example which studies the approximation properties of a 1-d internally damped wave equation which models the vibrations of a flexible string clamped at each of its ends of the form

$$\begin{cases} y'' - y_{xx} + ay' = 0 & (x,t) \in (0,1) \times (0,\infty), \\ y(0,t) = y(1,t) = 0 & t > 0, \\ y(x,0) = y^0,\ y'(x,0) = y^1, & x \in (0,1), \end{cases} \tag{2.1.1}$$

where $a \geq 0$ such that $a \in L^\infty(0,1)$ and $a(x) \geq a_1 > 0$ for all $x \in I \subsetneq (0,1)$. The symbol ' denotes the partial differentiation with respect to time.

It is well known that for such a choice of a, (2.1.1) is exponentially stable (see [23]). However, referring to [26], we show that for the classical finite difference scheme of (2.1.1), the exponential decay of the discretized energy is non-uniform. For this purpose, given $N \in \mathbb{N}^*$, set $h = \dfrac{1}{N+1}$ and consider the subdivision of $(0,1)$ given by

$$0 = x_0 < x_1 < ... < x_N < x_{N+1} = 1$$

where $x_j = jh$ for $j \in \{0, ..., N+1\}$.

Therefore, the classical finite difference space semi-discretization of problem

(2.1.1) is given by

$$\begin{cases} y_j'' - \dfrac{y_{j+1} - 2y_j + y_{j-1}}{h^2} + a_j y_j' = 0 & t > 0,\ j = 1, ..., N, \\ y_j(t) = 0, & t > 0,\ j = 0, N+1, \\ y_j(0) = y_j^0,\ y_j'(0) = y_j^1 & j = 1, ..., N, \end{cases} \tag{2.1.2}$$

where the function $y_j(t)$ provides an approximation of $y(x_j, t)$ and $a_j = a(jh)$ for all $j = 1, ..., N$. For simplification, we introduce the vector notation where we let $y_h(t) = (y_1(t), ..., y_N(t))^\top$ and $a_h y_h(t) = (a_1 y_1(t), ..., a_N y_N(t))^\top$. Moreover, we define the matrix

$$A_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Then system (2.1.2) reads as

$$\begin{cases} y_h'' + A_h y_h(t) + a_h y_h' = 0 & t > 0, \\ y_h(0) = y_h^0,\ y_h'(0) = y_h^1. \end{cases}$$

The energy of system (2.1.2) is given by

$$E_h(y_h, t) = \frac{h}{2} \sum_{j=0}^{N} \left[\left| y_j'(t) \right|^2 + \left| \frac{y_{j+1} - y_j}{h} \right|^2 \right],$$

which is a natural discretization of the continuous energy and the discrete dissipation law is expressed by

$$E_h'(y_h, t) = -h \sum_{j=0}^{N} a_j |y_j'|^2. \tag{2.1.3}$$

Theorem 2.1.1. *The exponential decay of $E_h(y_h, t)$ to zero is non-uniform with respect to the mesh size h; i.e., there do not exist positive constants M and ω which* *are independent of h such that for all $h > 0$, and for all initial data $(y_j^0)_j$, and $(y_j^1)_j \in \mathbb{R}^{\mathbb{N}}$, we have*

$$E_h(y_h, t) \leq M e^{-\omega t} E_h(y_h, 0).$$

To prove the above theorem, we need the following two lemmas :

Lemma 2.1.2. *If there exists some positive constants M and ω independent of h such that for all $(y_j^0)_j$, and $(y_j^1)_j \in \mathbb{R}^{\mathbb{N}}$, we have*

$$E_h(y_h, t) \leq M e^{-\omega t} E_h(y_h, 0) \quad \forall t > 0, \tag{2.1.4}$$

then there exists T_0 and C_0, bounded with respect to h, such that for all $(u_j^0)_j$ and $(u_j^1)_j$ in $\mathbb{R}^{\mathbb{N}}$

$$2E_h(u_h, 0) = h \sum_{j=0}^{N} \left[\left| u_j^1 \right|^2 + \left| \frac{u_{j+1}^0 - u_j^0}{h} \right|^2 \right] \leq C_0 h \sum_{j=0}^{N} \int_0^{T_0} a_j |u_j'(t)|^2 dt, \tag{2.1.5}$$

where $(u_j)_j$ solves the conservative semi-discrete system given by

$$\begin{cases} u_j'' - \dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = 0, & t > 0, \ j = 1, ..., N, \\ u_j(t) = 0, & t > 0, \ j = 0, N+1, \\ u_j(0) = u_j^0 = y_j^0, \ u_j'(0) = u_j^1 = y_j^1, & j = 1, ..., N. \end{cases} \tag{2.1.6}$$

Proof: The idea of the proof is found in [73]. According to the dissipation law (2.1.3), we have for all $T > 0$,

$$E_h(y_h, 0) - E_h(y_h, T) = h \sum_{j=0}^{N} \int_0^T a_j |y_j'|^2 dt. \tag{2.1.7}$$

If we choose $T \geq \left(\ln(\frac{4M}{3}) \right) / \omega$, then (2.1.4) implies that

$$E_h(y_h, T) \leq \frac{3}{4} E_h(y_h, 0) = \frac{3}{4} E_h(u_h, 0).$$

Hence,

$$h\sum_{j=0}^{N}\int_0^T a_j|y_j'|^2dt \geq \frac{1}{4}E_h(u_h,0). \tag{2.1.8}$$

Let $y_h = u_h + v_h$ where v_h solves the complementary semi-discrete system given by

$$\begin{cases} v_j'' - \dfrac{v_{j+1}-2v_j+v_{j-1}}{h^2} + a_j(v_j'+u_j') = 0, & t>0,\ j=1,...,N,\\ v_j(t)=0, & t>0,\ j=0,N+1,\\ v_j(0)=v_j'(0)=0, & j=1,...,N. \end{cases} \tag{2.1.9}$$

Since $E_h(v_h,0)=0$, then

$$E_h(v_h,T) + h\sum_{j=0}^{N}\int_0^T a_j\left|v_j'(t)\right|^2dt = -h\sum_{j=0}^{N}\int_0^T a_j u_j'(t)\overline{v_j'(t)}dt.$$

Hence,

$$h\sum_{j=0}^{N}\int_0^T a_j\left|v_j'(t)\right|^2dt \leq h\sum_{j=0}^{N}\int_0^T a_j\left|u_j'(t)\right|^2dt, \tag{2.1.10}$$

which implies that

$$\begin{aligned} h\sum_{j=0}^{N}\int_0^T a_j\left|y_j'(t)\right|^2dt &\leq 2h\sum_{j=0}^{N}\int_0^T a_j\left|v_j'(t)\right|^2dt + 2h\sum_{j=0}^{N}\int_0^T a_j\left|u_j'(t)\right|^2dt\\ &\leq 4h\sum_{j=0}^{N}\int_0^T a_j\left|u_j'(t)\right|^2dt. \end{aligned}$$

Using the above inequality in (2.1.8) yields

$$4h\sum_{j=0}^{N}\int_0^T a_j\left|u_j'(t)\right|^2dt \geq \frac{1}{4}E_h(u_h,0).$$

■

The following lemma shows that the observability constant C_0 in the discrete observability inequality (2.1.5) blows up as $h\to 0$ and hence the uniform observability fails to hold which implies the lack of a uniform exponential decay rate which completes the proof of Theorem 2.1.1.

Lemma 2.1.3. *For any $T > 0$, we have*

$$\lim_{h\to 0} \inf_{u_h \text{solution of (2.1.6)}} \frac{1}{E_h(u_h,0)} h\sum_{j=0}^{N}\int_0^T a_j \left|u_j'(t)\right|^2 dt = 0. \tag{2.1.11}$$

Proof: First, we analyze the spectrum of the matrix A_h; i.e., we consider the eigenvalue problem

$$-\frac{1}{h^2}(w_{j+1} - 2w_j + w_{j-1}) = \lambda w_j, \quad j = 1, ..., N; \quad w_0 = w_{N+1} = 0. \tag{2.1.12}$$

According to [44], the spectrum is explicitly given by

$$\lambda_h^k = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right), \quad k = 1, ..., N,$$

and the corresponding eigenvectors are given by

$$w_h^k = \left(w_{1,h}^k, ..., w_{N,h}^k\right)^T; \; w_{j,h}^k = \sin(k\pi j h), \quad k, j = 1, ..., N.$$

Obviously, $\lambda_h^k \to \lambda^k = k^2\pi^2$, as $h \to 0$ for each $k \geq 1$ where λ^k is the kth eigenvalue of the continuous wave equation (2.1.1). Moreover, the eigenvectors w_h^k of the discrete system coincide with the restriction to the mesh points of the eigenfunctions $w^k(x) = \sin(k\pi x)$ of the continuous wave equation (2.1.1). Furthermore, we notice that the gap between $\sqrt{\lambda_h^N}$ and $\sqrt{\lambda_h^{N-1}}$ is of the order h while, in the continuous case, the gap between any two consecutive eigenvalues is independent of k (see Figure (a) of 2.1 taken from [26] which shows the square roots of the eigenvalues in the continuous and discrete case via finite difference semi-discretization on the left and the piecewise linear finite element space semi-discretization on the right).

Therefore, according to Theorem 6.9.3 of [74], we choose a discrete solution u_h as a wave package or a superposition of semi-discrete waves corresponding to the last eigenfrequencies of A_h; i.e., we choose $u_h \in Span\left\{e^{i\sqrt{\lambda_h^k}t} w_h^k : \quad k \sim \frac{\gamma}{h}, \; 0 < \gamma < 1\right\}$

such that $E_h(u_h, 0) \sim 1$ and $h\sum_{j=0}^{N}\int_0^T a_j \left|u_j'(t)\right|^2 dt \sim h^L$, $L >> 1$ so that the proof of (2.1.11) is complete. We remark that the wave package, u_h, does not penetrate the subinterval I, where the damping coefficient $a(.)$ is effective (see Figure (b) of 2.1 taken from [26] which shows a wave package propagating outside the interval I).

■

In this chapter, as in [4], we consider the approximations of more general abstract second order evolution equations. In other words, let H be a complex Hilbert space with norm and inner product denoted respectively by $\|.\|$ and $(.,.)$. Let $A : \mathcal{D}(A) \to H$ be a densely defined self-adjoint and positive operator with a compact inverse in H. Let $V = \mathcal{D}(A^{\frac{1}{2}})$ be the domain of $A^{\frac{1}{2}}$. Denote by $\mathcal{D}(A^{\frac{1}{2}})'$ the dual space of $\mathcal{D}(A^{\frac{1}{2}})$ obtained by means of the inner product in H.

Furthermore, let U be a complex Hilbert space (which will be identified to its dual space) with norm and inner product denoted respectively by $\|.\|_U$ and $(.,.)_U$ and let $B \in \mathcal{L}(U, H)$. We consider the closed loop system

$$\begin{gathered}\ddot{\omega}(t) + A\omega(t) + BB^*\dot{\omega}(t) = 0, \\ \omega(0) = \omega_0, \, \dot{\omega}(0) = \omega_1,\end{gathered} \tag{2.1.13}$$

where $t \in [0, \infty)$ represents the time, $\omega : [0, \infty) \to H$ is the state of the system. Most of the linear equations modeling the vibrations of elastic structures with feedback control (corresponding to collocated actuators and sensors) can be written in the form (2.1.13), where ω represents the displacement field.

We define the energy of system (2.1.13) at time t by

$$E(t) = \frac{1}{2}\left(\|\dot{\omega}(t)\|^2 + \left\|A^{\frac{1}{2}}\omega(t)\right\|^2\right).$$

Simple formal calculations give

$$E(0) - E(t) = \int_0^t (BB^*\dot{\omega}(s), \dot{\omega}(s))\, ds, \quad \forall t \geq 0.$$

This obviously means that the energy is non-increasing.

In many applications, the system (2.1.13) is approximated by finite dimensional systems but usually, as the above simple example shows, if the continuous system is exponentially or polynomially stable, the discrete ones do no more inherit this property due to spurious high frequency modes. Several remedies have been proposed and analyzed to overcome these difficulties. Let us quote the Tychonoff regularization [34, 35, 64, 72], a bi-grid algorithm [32, 58], a mixed finite element method [14, 19, 20, 33, 56], or filtering the high frequencies [43, 49, 76]. These methods provide good numerical results.

As in [64, 72] our goal is to damp the spurious high frequency modes by introducing numerical viscosity terms in the approximation schemes. Though our work in [4] is inspired from [64], it differs from that paper on the following points :

(i) Contrary to [64] where the standard gap condition is required, we only assume that the spectrum of the operator $A^{1/2}$ satisfies the generalized gap condition, allowing to treat more general concrete systems,

(ii) we analyze the polynomial decay of the discrete schemes when the continuous problem has such a decay,

(iii) we prove a result about uniform polynomial stability for a family of semigroups of operators,

(iv) by using a general version of the Trotter-Kato theorem proved in [45], we show that the discrete solution tends to the solution of (2.1.13) as the discretization parameter goes to zero and if the discrete initial data are well chosen.

Consequently, this chapter is divided as follows : After we precise the proper functional setting of the continuous problem (2.1.13) in Section 2.2, we recall some results concerning the stability of (2.1.13) in Section 2.3. In Section 2.4, we introduce the suitable discrete systems and the main results of this chapter. Section 2.5 considers the well-posedness of the discrete systems. Next, in Section 2.6, we show that the generalized gap condition and the observability conditions (2.4.7) and (2.4.8) remain valid for filtered eigenvalues. Section 2.7 first recalls a result about uniform exponential stability for a family of semigroup of operators, and then extends such a result to the case of uniform polynomial stability. Some technical lemmas are proved in Section 2.8. Sections 2.9 and 2.10 are devoted to the proof of Theorem 2.4.1 and 2.4.4 respectively. In Section 2.11, we show that the chosen discrete systems converge towards (2.1.13), as the mesh size goes to zero and if the discrete initial data are well chosen. Finally, we illustrate our results by presenting different examples in Section 2.12. The first application is found in [3].

2.2 The proper functional setting of problem (2.1.13)

Before stating the main results of this chapter, we rewrite problem (2.1.13) in a simplified form. Let $X := V \times H$ be equipped with the inner product

$$((u,v)^\top, (u^*,v^*)^\top)_X = a(u,u^*) + (v,v^*) \quad \forall (u,v)^\top, (u^*,v^*)^\top \in X,$$

where $a(.,.)$ is the sesquilinear form on $V \times V$ defined by

$$a(u,u^*) = (A^{\frac{1}{2}}u,\, A^{\frac{1}{2}}u^*),\, \forall (u,u^*) \in V \times V.$$

Then (2.1.13) is equivalent to

$$\dot{z}(t) = \tilde{A}z(t) \text{ in } X, \quad z(0) = (\omega_0,\omega_1)^\top,$$

where $z(t) = (\omega(t), \dot{\omega}(t))^\top$ and $\tilde{A} : D(\tilde{A}) \to X$ is defined by

$$\tilde{A} = \begin{pmatrix} 0 & I \\ -A & -BB^* \end{pmatrix},$$

with $\mathcal{D}(\tilde{A}) = \mathcal{D}(A) \times V$. It is easy to check that $\tilde{A}$ is a maximal dissipative operator in X. Therefore, according to Lumer-Phillips Theorem, problem (2.1.13) is well-posed. We will denote by $T(t),\ t \geq 0$, the strongly continuous semi-group of contractions generated by $\tilde{A}$.

2.3 Stability of the continuous problem (2.1.13)

Before displaying the suitable approximate system which converges towards (2.1.13) and shares the same stability properties of (2.1.13), we recall some properties concerning the stability of (2.1.13). For this purpose, we start by analyzing the spectrum of the operator $A^{\frac{1}{2}}$.

2.3.1 Spectral Analysis of (2.1.13)

Denote by $\{\lambda_k\}_{k\geq 1}$ the set of eigenvalues of $A^{\frac{1}{2}}$ counted with their multiplicities (i.e. we repeat the eigenvalues according to their multiplicities). We further rewrite the sequence of eigenvalues $\{\lambda_k\}_{k\geq 1}$ as follows :

$$\lambda_{k_1} < \lambda_{k_2} < ... < \lambda_{k_i} < ...$$

where $k_1 = 1$, k_2 is the lowest index of the second distinct eigenvalue, k_3 is the lowest index of the third distinct eigenvalue, etc. For all $i \in \mathbb{N}^*$, let l_i be the multiplicity of the eigenvalue λ_{k_i}, i.e.

$$\lambda_{k_{i-1}} < \lambda_{k_i} = \lambda_{k_i+1} = ... = \lambda_{k_i+l_i-1} < \lambda_{k_i+l_i} = \lambda_{k_{i+1}}.$$

We have $k_1 = 1$, $k_2 = 1 + l_1$, $k_3 = 1 + l_1 + l_2$, etc. Let $\{\varphi_{k_i+j}\}_{0\leq j\leq l_i-1}$ be the orthonormal eigenvectors associated with the eigenvalue λ_{k_i}.

Now, we assume that the following generalized gap condition holds :

$$\exists M \in \mathbb{N}^*,\ \exists \gamma_0 > 0,\ \forall k \geq 1,\ \lambda_{k+M} - \lambda_k \geq M\gamma_0. \tag{2.3.1}$$

Fix a positive real number $\gamma_0' \leq \gamma_0$ and denote by A_k, $k = 1, ..., M$ the set of natural numbers k_m satisfying (see for instance [13])

$$\begin{cases} \lambda_{k_m} - \lambda_{k_{m-1}} \geq \gamma_0' & \\ \lambda_{k_n} - \lambda_{k_{n-1}} < \gamma_0' & \text{for } m+1 \leq n \leq m+k-1, \\ \lambda_{k_{m+k}} - \lambda_{k_{m+k-1}} \geq \gamma_0'. & \end{cases}$$

Then one easily checks that

$$\{k_{m+j} + l \,|k_m \in A_k,\, k \in \{1, ..., M\}\,, j \in \{0, ..., k-1\}\,, l \in \{0, ..., l_{m+j} - 1\}\} = \mathbb{N}^*.$$

Notice that some sets A_k may be empty because, for the generalized gap condition, the choice of M takes into account multiple eigenvalues.

For $k_n \in A_k$, we define $B_{k_n} = (B_{k_n,\, ij})_{1 \leq i,\, j \leq k}$ the matrix of size $k \times k$ by

$$B_{k_n,\, ij} = \begin{cases} \displaystyle\prod_{\substack{q=n \\ q \neq n+i-1}}^{n+j-1} (\lambda_{k_{n+i-1}} - \lambda_{k_q})^{-1} & \text{if } i \leq j,\, (i,\, j) \neq (1,\, 1), \\ 1 & \text{if } (i,\, j) = (1,\, 1), \\ 0 & \text{else }. \end{cases}$$

More explicitly, we have

$$B_{k_n} = \begin{pmatrix} 1 & \frac{1}{\lambda_{k_n} - \lambda_{k_{n+1}}} & \frac{1}{(\lambda_{k_n} - \lambda_{k_{n+1}})(\lambda_{k_n} - \lambda_{k_{n+2}})} & \cdots & \frac{1}{(\lambda_{k_n} - \lambda_{k_{n+1}})\cdots(\lambda_{k_n} - \lambda_{k_{n+k-1}})} \\ 0 & \frac{1}{\lambda_{k_{n+1}} - \lambda_{k_n}} & \frac{1}{(\lambda_{k_{n+1}} - \lambda_{k_n})(\lambda_{k_{n+1}} - \lambda_{k_{n+2}})} & \cdots & \frac{1}{(\lambda_{k_{n+1}} - \lambda_{k_n})\cdots(\lambda_{k_{n+1}} - \lambda_{k_{n+k-1}})} \\ 0 & 0 & \frac{1}{(\lambda_{k_{n+2}} - \lambda_{k_n})(\lambda_{k_{n+2}} - \lambda_{k_{n+1}})} & \cdots & \frac{1}{(\lambda_{k_{n+2}} - \lambda_{k_n})\cdots(\lambda_{k_{n+2}} - \lambda_{k_{n+k-1}})} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{(\lambda_{k_{n+k-1}} - \lambda_{k_n})\cdots(\lambda_{k_{n+k-1}} - \lambda_{k_{n+k-2}})} \end{pmatrix}.$$

Lemma 2.3.1. *The inverse matrix of B_{k_n} is given by*

$$B^{-1}_{k_n,\, ij} = \begin{cases} \displaystyle\prod_{q=n}^{n+i-2} (\lambda_{k_{n+j-1}} - \lambda_{k_q}) & \textit{if } i \leq j,\, i \neq 1, \\ 1 & \textit{if } i = 1, \\ 0 & \textit{else}, \end{cases}$$

that is to say

$$B_{k_n}^{-1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & (\lambda_{k_{n+1}} - \lambda_{k_n}) & \cdots & (\lambda_{k_{n+k-1}} - \lambda_{k_n}) \\ 0 & 0 & \cdots & (\lambda_{k_{n+k-1}} - \lambda_{k_n})(\lambda_{k_{n+k-1}} - \lambda_{k_{n+1}}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\lambda_{k_{n+k-1}} - \lambda_{k_n}) \cdots (\lambda_{k_{n+k-1}} - \lambda_{k_{n+k-2}}) \end{pmatrix},$$

and therefore

$$B_{k_n}^{-1} \to \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}, \text{ when } n \to +\infty.$$

Proof: The form of $B_{k_n}^{-1}$ is obtained by induction on the size k of B_{k_n}. The generalized gap condition (2.3.1) implies that $\lambda_{k_{n+j}} - \lambda_{k_n} \to 0$ as $n \to +\infty$, $\forall 0 \leq j \leq k-1$. This leads to the convergence of $B_{k_n}^{-1}$. ■

Remark 2.3.2. *The structure of the matrix B_{k_n} comes from Ingham's inequality in a Hilbert space H under the generalized gap condition (2.3.1). Indeed, according to Corollary 6.4 of [61], if the sequence $\{\lambda_n\}_{n\geq 1}$ satisfies (2.3.1), then for all sequence $\{\alpha_n\}_{n\in\mathbb{Z}^*}$ in H, the function*

$$f(t) = \sum_{n\in\mathbb{Z}^*} \alpha_n e^{i\lambda_{k_n} t},$$

satisfies the estimate

$$\int_0^T |f(t)|^2 \, dt \sim \sum_{k=1}^{M} \sum_{|k_n|\in A_k} \left\| B_{k_n}^{-1} C_{k_n} \right\|_{H,2}^2,$$

for $T > \dfrac{2\pi}{\gamma_0}$, where $\lambda_{-k_n} = -\lambda_{k_n}$, $C_{k_n} = (\alpha_n, ..., \alpha_{n+k-1})^\top \in H^k$ and $\|.\|_{H,2}$ is the norm in H^k.

2.3.2 Exponential Stability of the energy of (2.1.13)

We recall a sufficient condition for which the energy of (2.1.13) is exponentially stable. Indeed, the approach is based on observability inequalities found in [7] and [61]. For this aim, for $k_n \in A_k$, we define the matrix Φ_{k_n} with coefficients in U and size $k \times L_n$, where $L_n = \sum_{i=1}^{k} l_{n+i-1}$, as follows : for all $i = 1, ..., k$, we set

$$(\Phi_{k_n})_{ij} = \begin{cases} B^* \varphi_{k_{n+i-1}+j-L_{n,\,i-1}-1} & \text{if } L_{n,\,i-1} < j \leq L_{n,\,i}, \\ 0 & \text{else,} \end{cases}$$

where

$$L_{n,\,0} = 0, \quad L_{n,\,i} = \sum_{i'=1}^{i} l_{n+i'-1} \text{ for } i \geq 1. \tag{2.3.2}$$

For a vector $c = (c_l)_{l=1}^{m}$ in U^m, we set $\|c\|_{U,\,2}$ its norm in U^m defined by

$$\|c\|_{U,\,2}^2 = \sum_{l=1}^{m} \|c_l\|_U^2 .$$

Now, we recall Theorem 2.2 of [7] which links the exponential stability of (2.1.13) with some observability property of the associated conservative problem.

Theorem 2.3.3. *Let φ be the solution of the undamped problem*

$$\begin{cases} \ddot{\varphi}(t) + A\varphi(t) = 0, \\ \varphi(0) = \omega_0,\ \dot{\varphi}(0) = \omega_1. \end{cases} \tag{2.3.3}$$

If there exists a time $T > 0$ and a constant $c = c(T) > 0$ such that the observability estimate

$$\|A^{\frac{1}{2}}\omega_0\|_H^2 + \|\omega_1\|_H^2 \leq c \int_0^T \|B^* \dot{\varphi}(t)\|_U^2 dt \tag{2.3.4}$$

holds, then problem (2.1.13) is exponentially stable in the energy space ; i.e., there exist a constant $k > 0$ and $\nu > 0$ such that for all initial data in $X = V \times H$,

$$E(t) \leq kE(0)e^{-\nu t}, \quad \forall t > 0.$$

We also recall Proposition 6.5 of [61] which gives necessary and sufficient spectral conditions so that the observability estimate (2.3.4) holds.

Proposition 2.3.4. *Assume that the generalized gap condition (2.3.1) holds. There exists a time $T > 0$ and a constant $c = c(T) > 0$ such that the observability estimate (2.3.4) holds if and only if*

$$\exists \alpha_0 > 0,\ \forall k \in \{1, ..., M\},\ \forall k_n \in A_k,\ \forall C \in \mathbb{R}^{L_n},\ \left\| B_{k_n}^{-1} \Phi_{k_n} C \right\|_{U,2} \geq \alpha_0 \left\| C \right\|_2, \tag{2.3.5}$$

where $\|.\|_2$ is the euclidian norm.

Remark 2.3.5. If the standard gap condition

$$\exists \gamma_0 > 0,\ \forall n \geq 1,\ \lambda_{k_{n+1}} - \lambda_{k_n} \geq \gamma_0 \tag{2.3.6}$$

holds, then $A_1 = \mathbb{N}^*$ and $B_1 = 1$. In this case, the assumption (2.3.5) becomes

$$\exists \alpha_0 > 0,\ \forall k_n \geq 1,\ \forall C \in \mathbb{R}^{L_n},\ \left\| \Phi_{k_n} C \right\|_U \geq \alpha_0 \left\| C \right\|_2.$$

Moreover, if the standard gap condition (2.3.6) holds and if the eigenvalues are simple, the assumption (2.3.5) becomes

$$\exists \alpha_0 > 0,\ \forall k \geq 1,\ \left\| B^* \varphi_k \right\|_U \geq \alpha_0. \tag{2.3.7}$$

These assumptions are assumed in [64].

In conclusion, if (2.3.5) holds, then problem (2.1.13) is exponentially stable.

2.3.3 Polynomial Stability of the energy of (2.1.13)

Similar to the exponential stability case, we recall a sufficient condition for which the energy of (2.1.13) is polynomially stable. First, we recall Theorem 2.4 of [7] or Theorem 5.3 of [61] which gives the polynomial stability of (2.1.13) based on some observability property of the conservative problem (2.3.3). Then, we recall Proposition 6.8 of [61] which gives necessary and sufficient spectral conditions so that the observability estimate holds.

Theorem 2.3.6. *Let φ be the solution of (2.3.3). If there exists $l \in \mathbb{N}^*$, a time $T > 0$ and a constant $c = c(T) > 0$ such that*

$$\int_0^T \|B^* \dot{\varphi}(t)\|_U^2 dt \geq c \left(\|\omega_0\|^2_{D(A^{\frac{1-l}{2}})} + \|\omega_1\|^2_{D(A^{-\frac{l}{2}})} \right) \tag{2.3.8}$$

holds, then the energy of problem (2.1.13) decays polynomially; i.e., there exists a constant $k = k(l) > 0$ such that for all initial data in $D(\tilde{A})$,

$$E(t) \leq \frac{k}{(1+t)^{\frac{1}{l}}} \|(\omega_0, \omega_1)^\top\|^2_{D(\tilde{A})}, \qquad \forall t > 0.$$

Proposition 2.3.7. *Assume that the generalized gap condition (2.3.1) holds and $(\omega_0, \omega_1)^\top \in X$. There exists $l \in \mathbb{N}^*$, a time $T > 0$ and a constant $c = c(T) > 0$ such that (2.3.8) holds if and only if*

$$\exists \alpha_0 > 0,\, \forall k \in \{1, ..., M\},\, \forall k_n \in A_k,\, \forall C \in \mathbb{R}^{L_n},\, \left\| B_{k_n}^{-1} \Phi_{k_n} C \right\|_{U,2} \geq \frac{\alpha_0}{\lambda_{k_n}^l} \|C\|_2 \,. \tag{2.3.9}$$

Remark 2.3.8. If the standard gap condition (2.3.6) holds, the assumption (2.3.9) becomes

$$\exists l \in \mathbb{N}^*,\, \exists \alpha_0 > 0,\, \forall k_n \geq 1,\, \forall C \in \mathbb{R}^{L_n},\, \|\Phi_{k_n} C\|_U \geq \frac{\alpha_0}{\lambda_{k_n}^l} \|C\|_2 \,.$$

Moreover, if the standard gap condition (2.3.6) holds and if the eigenvalues are simple, the assumption (2.3.9) becomes

$$\exists l \in \mathbb{N}^*,\, \exists \alpha_0 > 0,\, \forall k \geq 1,\, \|B^* \varphi_k\|_U \geq \frac{\alpha_0}{\lambda_k^l}. \tag{2.3.10}$$

Remark 2.3.9. Note that the assumption (H) from [7] here holds since A is a positive selfadjoint operator with a compact resolvent and B is bounded. In fact, assumption (H) states that if $\beta > 0$ is fixed and $C_\beta = \{\lambda \in \mathbb{C};\ \Re\lambda = \beta\}$, the function

$$\lambda \in \mathbb{C}_+ = \{\lambda \in \mathbb{C};\ \Re\lambda > 0\} \to H(\lambda) = \lambda B^*(\lambda^2 I + A)^{-1} B \in \mathcal{L}(U)$$

is bounded on C_β. Indeed, if $\lambda = \beta + i\xi$, then $|\lambda_k^2 + \lambda^2|^2 = (\lambda_k^2 - \xi^2 + \beta^2)^2 + 4\beta^2\xi^2$. However,

$$\left\|(\lambda^2 I + A)^{-1}\right\|_{\mathcal{L}(H)} \leq \sup_{k\geq 1} |\lambda_k^2 + \lambda^2|^{-1}.$$

Hence, if $|\xi| > \epsilon$ for some $\epsilon > 0$, then

$$|\lambda| \left\|(\lambda^2 I + A)^{-1}\right\|_{\mathcal{L}(H)} \leq \frac{\sqrt{\beta^2 + \xi^2}}{2\beta|\xi|}$$

which is bounded for every $|\xi| > \epsilon$. On the other hand, if $|\xi| \leq \epsilon$, then $|\lambda_k^2 + \lambda^2| \geq \dfrac{3\beta^2}{4}$, for $\epsilon \leq \dfrac{\beta}{2}$. Therefore,

$$|\lambda| \left\|(\lambda^2 I + A)^{-1}\right\|_{\mathcal{L}(H)} \leq \frac{2\sqrt{5}}{3\beta}$$

which is bounded on C_β.

In conclusion, if (2.3.9) holds, then problem (2.1.13) is polynomially stable with a decay rate of the order $\dfrac{1}{(1+t)^{\frac{1}{l}}}$.

2.4 Approximate system and main results

In this section, we display the suitable discrete system which approximates (2.1.13) and has the same stability properties as (2.1.13). Before stating our main results, let us introduce some notations and assumptions.

We denote by $\|.\|_V$ the norm

$$\|\varphi\|_V = \sqrt{(A^{\frac{1}{2}}\varphi,\, A^{\frac{1}{2}}\varphi)},\, \forall \varphi \in V.$$

Remark that

$$\|\varphi\|_V = \sqrt{(A\varphi,\, \varphi)},\, \forall \varphi \in \mathcal{D}(A).$$

We now assume that $(V_h)_{h>0}$ is a sequence of finite dimensional subspaces of $\mathcal{D}(A^{\frac{1}{2}})$. The inner product in V_h is the restriction of the inner product of H and it is still denoted by $(.,.)$ (since V_h can be seen as a subspace of H). We define the operator $A_h : V_h \to V_h$ by

$$(A_h\varphi_h,\, \psi_h) = (A^{\frac{1}{2}}\varphi_h,\, A^{\frac{1}{2}}\psi_h),\, \forall \varphi_h,\, \psi_h \in V_h. \tag{2.4.1}$$

Let $a(.,.)$ be the sesquilinear form on $V_h \times V_h$ defined by

$$a(\varphi_h,\, \psi_h) = (A^{\frac{1}{2}}\varphi_h,\, A^{\frac{1}{2}}\psi_h),\, \forall (\varphi_h,\, \psi_h) \in V_h \times V_h. \tag{2.4.2}$$

We also define the operators $B_h : U \to V_h$ by

$$B_h u = j_h B u, \quad \forall u \in U, \tag{2.4.3}$$

where j_h is the orthogonal projection of H into V_h with respect to the inner product in H.

The adjoint B_h^* of B_h is then given by the relation

$$B_h^*\varphi_h = B^*\varphi_h, \quad \forall \varphi_h \in V_h.$$

We also suppose that the family of spaces $(V_h)_h$ approximates the space $V = \mathcal{D}(A^{\frac{1}{2}})$. More precisely, if π_h denotes the orthogonal projection of $V = \mathcal{D}(A^{\frac{1}{2}})$ onto V_h, we suppose that there exist $\theta > 0$, $h^* > 0$ and $C_0 > 0$ such that, for all $h \in (0,\, h^*)$, we have :

$$\|\pi_h\varphi - \varphi\|_V \leq C_0 h^\theta \|A\varphi\|,\, \forall \varphi \in \mathcal{D}(A), \tag{2.4.4}$$

$$\|\pi_h\varphi - \varphi\| \leq C_0 h^{2\theta} \|A\varphi\|, \forall \varphi \in \mathcal{D}(A). \tag{2.4.5}$$

Assumptions (2.4.4) and (2.4.5) are, in particular, satisfied in the case of standard finite element approximations of Sobolev spaces.

In this section, we prove two results. The first result gives a necessary and sufficient condition to have the exponential stability of the family of systems

$$\begin{gathered} \ddot{\omega}_h(t) + A_h\omega_h(t) + B_h B_h^* \dot{\omega}_h(t) + h^\theta A_h \dot{\omega}_h(t) = 0 \\ \omega_h(0) = \omega_{0h} \in V_h,\ \dot{\omega}_h(0) = \omega_{1h} \in V_h, \end{gathered} \tag{2.4.6}$$

in the absence of the standard gap condition assumed in [64]. Here and below ω_{0h} (resp. ω_{1h}) is an approximation of ω_0 (resp. ω_1) in V_h. For that purpose, we need to make the following assumption

$$\exists \alpha_0 > 0,\ \forall k \in \{1, ..., M\},\ \forall k_n \in A_k,\ \forall C \in \mathbb{R}^{L_n},\ \left\|B_{k_n}^{-1}\Phi_{k_n} C\right\|_{U,2} \geq \alpha_0 \|C\|_2, \tag{2.4.7}$$

where $\|.\|_2$ is the euclidian norm. The first main result is the following

Theorem 2.4.1. *Suppose that the generalized gap condition (2.3.1) and the assumption (2.4.7) are verified. Assume that the family of subspaces* (V_h) *satisfies (2.4.4) and (2.4.5). Then the family of systems (2.4.6) is uniformly exponentially stable, in the sense that there exist constants* M, α, $h^* > 0$ *(independent of* h, ω_{0h}, ω_{1h}*) such that for all* $h \in (0, h^*)$ *:*

$$\|\dot{\omega}_h(t)\|^2 + a(\omega_h(t), \omega_h(t)) \leq Me^{-\alpha t}(\|\omega_{1h}\|^2 + a(\omega_{0h}, \omega_{0h})),\ \forall t \geq 0.$$

Remark 2.4.2. Note that Theorem 2.4.1 is the discrete counterpart of the exponential decay of the solution of the continuous problem (2.1.13) under the assumptions (2.3.1) and (2.4.7), which follows from Theorem 2.3.3 and Proposition 2.3.4 or Theorem 2.2 of [7] and Proposition 6.5 of [61].

Remark 2.4.3. The uniform exponential stability of the family of systems (2.4.6) has been already proved in Theorem 7.1 of [25] without any assumption on the spectrum of A. The proof of this theorem is based on decoupling of low and high frequencies. More precisely, the author combines a uniform observability estimate for filtered initial data corresponding to low frequencies (see Theorem 1.3 of [25]) together with a result of [27]. Indeed, in [27], after adding the numerical viscosity term, another uniform observability estimate is obtained for the high frequency components. The two established observability inequalities yield the uniform exponential decay of (2.4.6).

If the condition (2.4.7) is not satisfied, we may look at a weaker version. Namely if we assume that

$$\exists l \in \mathbb{N}^*,\ \exists \alpha_0 > 0,\ \forall k \in \{1, ..., M\},\ \forall k_n \in A_k,\ \forall C \in \mathbb{R}^{L_n},\ \left\| B_{k_n}^{-1} \Phi_{k_n} C \right\|_{U,2} \geq \frac{\alpha_0}{\lambda_{k_n}^l} \left\| C \right\|_2, \tag{2.4.8}$$

then we will obtain a polynomial stability for the family of systems

$$\begin{aligned} &\ddot{\omega}_h(t) + (1+h^\theta)^{-2}(I + h^\theta A_h^{\frac{l}{2}})^2 A_h \omega_h(t) \\ &+(I + h^\theta A_h^{\frac{l}{2}})(B_h B_h^* + h^\theta A_h^{1+\frac{l}{2}})(I + h^\theta A_h^{\frac{l}{2}})^{-1} \dot{\omega}_h(t) = 0, \\ &\omega_h(0) = \omega_{0h} \in V_h,\ \dot{\omega}_h(0) = (1+h^\theta)^{-1}(I + h^\theta A_h^{\frac{l}{2}})\omega_{1h} \in V_h. \end{aligned} \tag{2.4.9}$$

The structure of the above discrete system has been inspired from the one introduced in [64] for the exponential stability case where the authors have used system (2.4.6) corresponding to $l = 0$. In both cases, this choice is motivated by the corresponding observability estimates. The numerical viscosity term $(I + h^\theta A_h^{\frac{l}{2}})(B_h B_h^* + h^\theta A_h^{1+\frac{l}{2}})(I + h^\theta A_h^{\frac{l}{2}})^{-1} \dot{\omega}_h(t)$ is added to damp the high frequency modes and as the set of high frequency modes is larger in the polynomial case, the viscosity term is naturally stronger. In the case $l > 0$ the powers of $(I + h^\theta A_h^{\frac{l}{2}})$ have been added to guarantee the uniform boundedness of the resolvent of $\tilde{A}_{l,h}$ (defined below) near zero. The question of the optimality of these viscosity terms remains open.

The second main result of this chapter is the following one.

Theorem 2.4.4. *Suppose that the generalized gap condition (2.3.1) and the assumption (2.4.8) are verified with $l \in \mathbb{N}^*$ even. Assume that the family of subspaces (V_h) satisfies (2.4.4) and (2.4.5). Then the family of systems (2.4.9) is uniformly polynomially stable, in the sense that there exist constants C, $h^* > 0$ (independent of h, ω_{0h}, ω_{1h}) such that for all $h \in (0, h^*)$:*

$$\begin{aligned}\left\|(I + h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t)\right\|^2 + a(\omega_h(t), \omega_h(t)) &\leq \frac{C}{t^2}\|(\omega_{0h}, \omega_{1h})\|^2_{D(\tilde{A}^{2l}_{l,h})},\\ \left\|(I + h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t)\right\|^2 + a(\omega_h(t), \omega_h(t)) &\leq \frac{C}{t^{\frac{1}{l}}}\|(\omega_{0h}, \omega_{1h})\|^2_{D(\tilde{A}_{l,h})},\\ \forall t > 0, \ \forall(\omega_{0h}, \omega_{1h}) &\in V_h \times V_h,\end{aligned}$$

where for $q \in \mathbb{N}^*$, $\|.\|_{D(\tilde{A}^q_{l,h})}$ is the graph norm of the matrix operator $\tilde{A}^q_{l,h}$ given in (2.5.1) of Section 2.5 below.

For a technical reason, we assume l to be even (see Lemma 2.8.4). If (2.4.8) holds for l odd, then we can still apply the results of Theorem 2.4.4 (see Remark 2.10.1 below).

Remark 2.4.5. As before, Theorem 2.4.4 is the discrete counterpart of the polynomial decay of the solution of the continuous problem (2.1.13) under the assumptions (2.3.1) and (2.4.8), that follows from Theorem 2.3.6 and Proposition 2.3.7 or Theorem 2.4 of [7] and Proposition 6.8 of [61].

2.5 Well-posedness of the discretized problem

From now on, we fix $l \in \mathbb{N}$, l even. We introduce the Hilbert space $X_h = V_h \times V_h$ and the operator $\tilde{A}_{l,h} : X_h \to X_h$ defined by

$$\tilde{A}_{l,h} = \begin{pmatrix} 0 & (1 + h^\theta)^{-1}(I + h^\theta A_h^{\frac{l}{2}}) \\ -(1 + h^\theta)^{-1}(I + h^\theta A_h^{\frac{l}{2}})A_h & -h^\theta A_h^{1+\frac{l}{2}} - B_h B_h^* \end{pmatrix}. \tag{2.5.1}$$

The space X_h is here equipped with the inner product

$$\left(\begin{pmatrix} u_h \\ v_h \end{pmatrix}, \begin{pmatrix} \tilde{u}_h \\ \tilde{v}_h \end{pmatrix} \right)_{X_h} = a(u_h,\, \tilde{u}_h) + (v_h,\, \tilde{v}_h), \quad \forall (u_h,\, v_h),\, (\tilde{u}_h,\, \tilde{v}_h) \in X_h, \quad (2.5.2)$$

with associated norm $\|.\|_{X_h}$. Therefore, the system (2.4.9) is equivalent to the following first order system in X_h :

$$\dot{z}_h(t) = \tilde{A}_{l,h} z_h(t), \quad z_h(0) = z_{0h},$$

where $z_h(t) = \begin{pmatrix} \omega_h(t) \\ (1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t) \end{pmatrix}$ and $z_{0h} = \begin{pmatrix} \omega_{0h} \\ \omega_{1h} \end{pmatrix}$. Note that we recover the system (2.4.6) in the particular case $l = 0$.

Lemma 2.5.1. *$\tilde{A}_{l,h}$ is maximal dissipative in X_h ; hence, it follows from Lumer-Phillips Theorem that, for every $h > 0$, $\tilde{A}_{l,h}$ generates a C_0 semigroup of contractions in X_h. We will denote this C_0 semigroup by $T_{l,h}(t)$.*

Proof: For the dissipativity of $\tilde{A}_{l,h}$, it is easy to check that $\Re(\tilde{A}_{l,h}U, U) \leq 0$ for every $U \in X_h$. As for the maximality, $\tilde{A}_{l,h}$ is bijective since $0 \in \rho(\tilde{A}_{l,h})$ (see Lemma 2.8.1 below). Therefore, $\tilde{A}_{l,h}$ becomes maximal. Indeed, let $F \in X_h$ and define the operator T on X_h such that $TU = \mu \tilde{A}_{l,h}^{-1} U - \tilde{A}_{l,h}^{-1} F$. For every $U,\ V \in X_h$, we have

$$\|TU - TV\| = \|\mu \tilde{A}_{l,h}^{-1} U - \mu \tilde{A}_{l,h}^{-1} V\| \leq \mu \|\tilde{A}_{l,h}^{-1}\| \|U - V\|.$$

As $\tilde{A}_{l,h}^{-1}$ is bounded since it is linear over a finite dimensional space, we choose $0 < \mu < \dfrac{1}{\|\tilde{A}_{l,h}^{-1}\|}$ so that T becomes a contraction and hence admits a fixed point U. Therefore, there exists $U \in X_h$ and $\mu > 0$ such that $\mu \tilde{A}_{l,h}^{-1} U - \tilde{A}_{l,h}^{-1} F = U$ or $(\mu I - \tilde{A}_{l,h}) U = F$.

■

We shall note here that the discrete energy of system (2.4.9) is given by

$$E_h(t) = \frac{1}{2}\|z_h(t)\|^2_{X_h} = \frac{1}{2}(A_h\omega_h, \omega_h) + \frac{1}{2}(1+h^\theta)^2\left((I + h^\theta A_h^{\frac{l}{2}})^{-2}\dot{\omega}_h(t), \dot{\omega}_h(t)\right).$$

Therefore, for any $t > 0$, we have

$$E_h'(t) = -(1+h^\theta)^2\left(\left\|B_h^*(I + h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t)\right\|^2 + h^\theta\left\|A_h^{\frac{2+l}{4}}(I + h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t)\right\|^2\right).$$

We notice the additional second term on the right hand side of the above dissipation equality. In fact, this viscosity term helps increase the decay rate of the discrete energy.

2.6 Spectral analysis of the discretized problem

The eigenvalue problem of the discretized problem is the following one : find $\lambda_{k,h} \in]0, +\infty[$, $\varphi_{k,h} \in V_h$, such that

$$a(\varphi_{k,h}, \psi_h) = \lambda_{k,h}^2(\varphi_{k,h}, \psi_h), \forall\psi_h \in V_h. \tag{2.6.1}$$

Let $N(h)$ be the dimension of V_h. We denote by $\{\lambda_{k,h}^2\}_{1\le k\le N(h)}$ the set of eigenvalues of (2.6.1) counted with their multiplicities. Let $\{\varphi_{k,h}\}_{1\le k\le N(h)}$ be the orthonormal eigenvectors associated with the eigenvalue $\lambda_{k,h}^2$. We define the sesquilinear form $a^l(.,.)$ on V_h by

$$a^l(u_h, v_h) = \left(A_h^{1+\frac{l}{2}}u_h, v_h\right), \forall(u_h, v_h) \in V_h \times V_h;$$

i.e.,

$$a^l(u_h, v_h) = \sum_{k=1}^{N(h)} c_k\overline{d_k}\lambda_{k,h}^{2+l},$$

for $u_h = \sum_{k=1}^{N(h)} c_k\varphi_{k,h}$ and $v_h = \sum_{k=1}^{N(h)} d_k\varphi_{k,h}$. Remark that $a^0(.,.) = a(.,.)$ defined in (2.4.2).

In this Section, we show that the generalized gap condition (2.3.1) and the observability conditions (2.4.7) and (2.4.8) still hold for the approximate problem (uniformly in h), provided that we consider only "low frequencies". More precisely, we have the following first result :

Proposition 2.6.1. *Suppose that the generalized gap condition (2.3.1) and the assumption (2.4.7) are verified. Then, there exist two constants $\epsilon > 0$ and $h^* > 0$, such that, for all $0 < h < h^*$ and for all $k \in \{1, ..., N(h)\}$ satisfying*

$$h^\theta \lambda_k^2 \leq \epsilon, \tag{2.6.2}$$

we have

$$\exists M \in \mathbb{N}^*,\ \exists \gamma > 0,\ \lambda_{k+M,\,h} - \lambda_{k,\,h} \geq M\gamma \tag{2.6.3}$$

and

$$\exists \alpha > 0,\ \forall p \in \{1, ..., M\},\ \forall k_n \in A_{p,h},\ \forall C \in \mathbb{R}^{L_n},\ \left\| B_{k_n}^{-1} \Phi_{k_n,\,h} C \right\|_{U,\,2} \geq \alpha \left\| C \right\|_2, \tag{2.6.4}$$

where α is independent of h, and where the matrix $\Phi_{k_n,\,h} \in \mathcal{M}_{p,\,L_n}(U)$, with coefficients in U, is defined as follows : for all $i = 1, ..., p$, we set

$$(\Phi_{k_n,\,h})_{ij} = \begin{cases} B_h^* \varphi_{k_{n+i-1}+j-L_{n,\,i-1}-1,\,h} & \text{if } L_{n,\,i-1} < j \leq L_{n,\,i}, \\ 0 & \text{else}, \end{cases}$$

where $L_{n,\,i-1}$ is defined by (2.3.2) and

$$A_{p,h} = \{k_n \in A_p \text{ satisfying (2.6.2) and s.t. } k_{n+p-1} + l_{n+p-1} - 1 \leq N(h)\}.$$

For the proof of this proposition, we need a result proved by Babuska and Osborn in [12]. For that purpose, we introduce $\epsilon_h(n,\,j)$ such that

$$\epsilon_h(n,\,j) = \inf_{\varphi \in M_j(\lambda_{k_n})} \inf_{v_h \in V_h} \left\| \varphi - v_h \right\|_V,$$

where $M_j(\lambda_{k_n}) = \{\varphi \in M(\lambda_{k_n}) \,:\, a(\varphi,\, \varphi_{k_n,h}) = ... = a(\varphi,\, \varphi_{k_n+j-2,h}) = 0\}$ and $M(\lambda_{k_n}) = \{\varphi : \varphi \text{ is an eigenvector of } A^{\frac{1}{2}} \text{ corresponding to } \lambda_{k_n},\ \|\varphi\| = 1\}$. The restrictions $a(\varphi,\, \varphi_{k_n,h}) = ... = a(\varphi,\, \varphi_{k_n+j-2,h}) = 0$ are not imposed if $j = 1$. Then, we have the following estimate about the eigenvalue and eigenvector errors for the Galerkin method in terms of the approximability quantities $\epsilon_h(n,\, j)$.

Theorem 2.6.2. *There are positive constants C and h_0 such that*

$$\lambda_{k_n+j,h} - \lambda_{k_n+j} \leq C\epsilon_h^2(n,\, j), \quad \forall 0 < h \leq h_0,\ j = 0, ..., l_n - 1,\ k_n + j \leq N(h),\ n \in \mathbb{N}^* \tag{2.6.5}$$

and such that the eigenvectors $\{\varphi_{k_n+j}\}_{0 \leq j \leq l_n - 1}$ of $A^{\frac{1}{2}}$ can be chosen so that

$$\|\varphi_{k_n+j,h} - \varphi_{k_n+j}\|_V \leq C\epsilon_h(n,\, j), \quad \forall 0 < h \leq h_0,\ j = 0, ..., l_n - 1,\ k_n + j \leq N(h),\ n \in \mathbb{N}^*. \tag{2.6.6}$$

This result is proved by Babuska and Osborn in [12, p. 702] because

$$\lambda_{k_n+j,h}^2 - \lambda_{k_n+j}^2 = (\lambda_{k_n+j,h} - \lambda_{k_n+j})(\lambda_{k_n+j,h} + \lambda_{k_n+j}) \geq 2\lambda_1(\lambda_{k_n+j,h} - \lambda_{k_n+j}).$$

Remark 2.6.3. *Notice that for every $\varphi \in M_j(\lambda_{k_n})$ we have*

$$\begin{aligned} \epsilon_h(n,\, j) &\leq \inf_{v_h \in V_h} \|\varphi - v_h\|_V \\ &\leq C_0 h^\theta \|A\varphi\| \ \text{ by (2.4.4)} \\ &\leq C_0 h^\theta \lambda_{k_n}^2 \|\varphi\| = C_0 h^\theta \lambda_{k_n+j}^2. \end{aligned} \tag{2.6.7}$$

Proof of Proposition 2.6.1. We begin with the proof of the generalized gap condition for the approximate eigenvalues $\lambda_{k,h}$. First, we use the Min-Max principle (see [67]) to obtain

$$\lambda_k \leq \lambda_{k,h},\ \forall k \in \{1, ..., N(h)\}. \tag{2.6.8}$$

Second, we use the estimates (2.6.5) and (2.6.7) and we have

$$\lambda_{k,h} \leq \lambda_k + C(C_0 h^\theta \lambda_k^2)^2 \leq \lambda_k + C(C_0\epsilon)^2 \leq \lambda_k + CC_0^2\epsilon, \tag{2.6.9}$$

for all $k \in \{1, ..., N(h)\}$ verifying (2.6.2) and $\epsilon \leq 1$. Therefore, we may write

$$\lambda_{k+M,h} - \lambda_{k,h} \geq \lambda_{k+M} - \lambda_k - CC_0^2\epsilon \geq M\gamma_0 - CC_0^2\epsilon \geq M\frac{\gamma_0}{2} =: M\gamma$$

for all $k \in \{1, ..., N(h)\}$ satisfying (2.6.2) and for $\epsilon \leq \frac{M\gamma_0}{2CC_0^2}$.

Now, we prove the estimate (2.6.4) which is the approximated version of (2.4.7). Notice that

$$\begin{aligned}
\left\|\Phi_{k_n,h} - \Phi_{k_n}\right\|_U &\leq C \max_{i=0,\dots,p-1} \sum_{j=0}^{l_{n+i}-1} \left\|B^*\varphi_{k_{n+i}+j,h} - B^*\varphi_{k_{n+i}+j}\right\|_U \\
&\leq C \max_{i=0,\dots,p-1} \sum_{j=0}^{l_{n+i}-1} \left\|B^*\right\|_{\mathcal{L}(H,U)} \left\|\varphi_{k_{n+i}+j,h} - \varphi_{k_{n+i}+j}\right\| \\
&\leq C \max_{i=0,\dots,p-1} \sum_{j=0}^{l_{n+i}-1} \left\|B^*\right\|_{\mathcal{L}(H,U)} \left\|\varphi_{k_{n+i}+j,h} - \varphi_{k_{n+i}+j}\right\|_V \\
&\leq C \max_{i=0,\dots,p-1} \sum_{j=0}^{l_{n+i}-1} \epsilon_h(n+i,\, j) \text{ by (2.6.6)} \\
&\leq C \max_{i=0,\dots,p-1} \sum_{j=0}^{l_{n+i}-1} h^\theta \lambda_{k_{n+i}+j}^2 \text{ by (2.6.7).}
\end{aligned}$$

Thus, by (2.6.2), we get

$$\left\|\Phi_{k_n,h} - \Phi_{k_n}\right\|_U \leq C\epsilon. \tag{2.6.10}$$

Therefore the triangular inequality leads to

$$\begin{aligned}
\left\|B_{k_n}^{-1}\Phi_{k_n,h}C\right\|_{U,2} &= \left\|B_{k_n}^{-1}\Phi_{k_n}C + B_{k_n}^{-1}(\Phi_{k_n,h} - \Phi_{k_n})C\right\|_{U,2} \\
&\geq \left\|B_{k_n}^{-1}\Phi_{k_n}C\right\|_{U,2} - \left\|B_{k_n}^{-1}(\Phi_{k_n,h} - \Phi_{k_n})C\right\|_{U,2} \\
&\geq \alpha_0 \left\|C\right\|_2 - \left\|B_{k_n}^{-1}(\Phi_{k_n,h} - \Phi_{k_n})C\right\|_{U,2}
\end{aligned}$$

by (2.4.7). But, as $B_{k_n}^{-1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} + R_{k_n}$, with $R_{k_n} \to 0$, when $k_n \to +\infty$

(see Lemma 2.3.1), we obtain

$$\begin{array}{rcl} \left\|B_{k_n}^{-1}(\Phi_{k_n,h}-\Phi_{k_n})C\right\|_{U,2} & \leq & \left\| \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} (\Phi_{k_n,h}-\Phi_{k_n})C \right\|_{U,2} \\ & + & \left\|R_{k_n}(\Phi_{k_n,h}-\Phi_{k_n})C\right\|_{U,2} \\ & \leq & C\left\|\Phi_{k_n,h}-\Phi_{k_n}\right\|_U \|C\|_2 + \eta_n \left\|\Phi_{k_n,h}-\Phi_{k_n}\right\|_U \|C\|_2 \\ & \leq & C\epsilon(1+\eta_n)\|C\|_2 , \end{array} \tag{2.6.11}$$

where $\eta_n = \|R_{k_n}\| \to 0$. Thus

$$\left\|B_{k_n}^{-1}\Phi_{k_n,h}C\right\|_{U,2} \geq (\alpha_0 - C\epsilon(1+\eta_n))\|C\|_2 \geq \frac{\alpha_0}{2}\|C\|_2$$

for $\epsilon \leq \frac{\alpha_0}{2C(1+\max\limits_n(1+\eta_n))}$. ■

For the polynomial stability, we have the same kind of result, but more filtering is necessary in order to have the discrete counterpart of the observability condition (2.4.8) (uniformly in h).

Proposition 2.6.4. *Suppose that the generalized gap condition (2.3.1) and the assumption (2.4.8) are verified. Then, there exist two constants $\epsilon > 0$ and $h^* > 0$, such that, for all $0 < h < h^*$ and for all $k \in \{1, ..., N(h)\}$, satisfying*

$$h^\theta \lambda_k^2 \leq \frac{\epsilon}{\lambda_k^l}, \tag{2.6.12}$$

we have (2.6.3) and

$$\exists \alpha > 0,\ \forall p \in \{1, ..., M\},\ \forall k_n \in A_{p,h}^{(l)},\ \forall C \in \mathbb{R}^{L_n},\ \left\|B_{k_n}^{-1}\Phi_{k_n,h}C\right\|_{U,2} \geq \frac{\alpha}{\lambda_{k_n}^l}\|C\|_2 , \tag{2.6.13}$$

where $A_{p,h}^{(l)} = \{k_n \in A_p$ satisfying (2.6.12) and s.t. $k_{n+p-1} + l_{n+p-1} - 1 \leq N(h)\}$.

Proof: The generalized gap condition for the approximate eigenvalues $\lambda_{k,h}$ is a consequence of Proposition 2.6.1, because $\lambda_k \geq \lambda_1 > 0$.

To prove the estimate (2.6.13) we notice that

$$\|\Phi_{k_n,h} - \Phi_{k_n}\|_U \leq C \max_{i=0,\ldots,p-1} \sum_{j=0}^{l_{n+i}-1} h^\theta \lambda^2_{k_{n+i+j}} \leq C\ h^\theta \lambda^2_{k_{n+p-1}}.$$

Moreover by the triangular inequality and (2.4.8), we have

$$\left\|B^{-1}_{k_n}\Phi_{k_n,h}C\right\|_{U,2} \geq \tfrac{\alpha_0}{\lambda^l_{k_n}} \|C\|_2 - \left\|B^{-1}_{k_n}(\Phi_{k_n,h} - \Phi_{k_n})C\right\|_{U,2}.$$

By (2.6.11) and (2.6.12), we obtain

$$\begin{aligned}\left\|B^{-1}_{k_n}\Phi_{k_n,h}C\right\|_{U,2} &\geq (\tfrac{\alpha_0}{\lambda^l_{k_n}} - \tfrac{C(1+\eta_n)\epsilon}{\lambda^l_{k_{n+p-1}}}) \|C\|_2 \\ &\geq (\tfrac{\alpha_0}{\lambda^l_{k_n}} - \tfrac{C\epsilon}{\lambda^l_{k_n}+\rho_n}(1+\eta_n)) \|C\|_2, \text{ with } \rho_n = \lambda^l_{k_{n+p-1}} - \lambda^l_{k_n} \to 0 \\ &\geq \tfrac{\alpha}{\lambda^l_{k_n}} \|C\|_2\end{aligned}$$

for an appropriate choice of $\epsilon > 0$.

■

Remark 2.6.5. *Proposition 2.6.1 in case $l = 0$ and Proposition 2.6.4 for $l \in \mathbb{N}^*$ show that if $h^\theta \lambda_k^{2+l} \leq \epsilon$, then the discrete version of the observability inequalities is still preserved uniformly in h and hence no problems with the stability of the discrete systems are expected. On the other hand, if $h^\theta \lambda_k^{2+l} \geq \epsilon$, then the viscosity term, $h^\theta A_h^{1+\frac{l}{2}} \dot{\omega}_h(t)$, plays its role in damping the spurious high oscillations. Indeed, if we write*

$$\dot{\omega}_h(t) = \sum_{k=1}^{N(h)} \alpha_{k,h}\varphi_{k,h},$$

then

$$h^\theta A_h^{1+\frac{l}{2}} \dot{\omega}_h(t) = h^\theta \sum_{k=1}^{N(h)} \alpha_{k,h}\lambda^{2+l}_{k,h}\varphi_{k,h}.$$

Hence, if $h^\theta \lambda_k^{2+l} \geq \epsilon$*, then*

$$\left\| h^\theta A_h^{1+\frac{l}{2}} \dot{\omega}_h(t) \right\|^2 \geq \epsilon^2 \|\dot{\omega}_h(t)\|^2.$$

Therefore, in the presence of high frequencies, the viscosity term $h^\theta A_h^{1+\frac{l}{2}} \dot{\omega}_h(t)$ *can be viewed as an additional damping term.*

2.7 Uniform stability results

2.7.1 Exponential stability result

The proof of Theorem 2.4.1 is based on the following result (see Theorem 7.1.3 in [52]) :

Theorem 2.7.1. *Let* $(T_h)_{h>0}$ *be a family of semigroups of contractions on the Hilbert spaces* $(X_h)_{h>0}$ *and let* $(\tilde{A}_h)_{h>0}$ *be the corresponding infinitesimal generators. The family* $(T_h)_{h>0}$ *is uniformly exponentially stable, that is to say there exist constants* $M > 0$, $\alpha > 0$ *(independent of* $h \in (0, h^*)$*) such that*

$$\|T_h(t)\|_{\mathcal{L}(X_h)} \leq M e^{-\alpha t}, \forall t \geq 0,$$

if and only if the two following conditions are satisfied :

i) For all $h \in (0, h^*)$, $i\mathbb{R}$ *is contained in the resolvent set* $\rho(\tilde{A}_h)$ *of* $\tilde{A}_h$*,*

ii) $\sup_{h\in(0,h^*),\omega\in\mathbb{R}} \left\| (i\omega - \tilde{A}_h)^{-1} \right\|_{\mathcal{L}(X_h)} < +\infty.$

2.7.2 Polynomial stability result

The proof of Theorem 2.4.4 is based on the results presented in this section by adapting the results from [18] and from [48] to obtain the (uniform) polynomial stability of the discretized problem (2.4.9). Throughout this section, let $(T_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$

be a family of uniformly bounded C_0 semigroups on the associated Hilbert spaces $(X_h)_{h\in(0,h^*)}$ (i.e., $\exists M>0,\ \forall h\in(0,h^*),\ \|T_h(t)\|_{\mathcal{L}(X_h)}\le M$) and let $(\tilde{A}_h)_{h\in(0,h^*)}$ be the corresponding infinitesimal generators.

In the following, for shortness, we denote by $R(\lambda,\tilde{A}_h)$ the resolvent $(\lambda-\tilde{A}_h)^{-1}$; moreover, for any operator mapping X_h into X_h, we skip the index $\mathcal{L}(X_h)$ in its norm, since in the whole section we work in X_h.

Definition 2.7.2. *Assuming that*

$$i\mathbb{R}\subseteq\rho(\tilde{A}_h),\quad \forall h\in(0,h^*), \tag{2.7.1}$$

and that for all $m\ge 1$, there exists $c=c(m)>0$ such that

$$\sup_{\substack{h\in(0,h^*)\\ |s|\le m}}\|R(is,\tilde{A}_h)\|_{\mathcal{L}(X_h)}\le c, \tag{2.7.2}$$

we define the fractional power $\tilde{A}_h^{-\alpha}$ for $\alpha>0$ and $h\in(0,h^*)$, according to [6] and [24], as

$$\tilde{A}_h^{-\alpha}=\frac{1}{2\pi i}\int_\Gamma \lambda^{-\alpha}(\lambda-\tilde{A}_h)^{-1}d\lambda, \tag{2.7.3}$$

where $\lambda^{-\alpha}=e^{-\alpha\log\lambda}$ and $\mathbb{R}^+$ is taken as the cut branch of the complex log *function and where the curve $\Gamma=\Gamma_1\cup\Gamma_2$ is given by*

$$\Gamma=\{-\epsilon+te^{i\theta},\ t\in[0,+\infty)\}\cup\{-\epsilon-te^{-i\theta},\ t\in(-\infty,0]\} \tag{2.7.4}$$

for some $\epsilon>0$ small enough independent of h and θ is a fixed angle in $\left(0,\frac{\pi}{4}\right)$.

In the sequel, the constant $c>0$ is generic, independent of h, and may change from one line to another.

Remark 2.7.3. *Throughout this section, whenever $\tilde{A}_h^{-\alpha}$ is mentioned, the assumptions (2.7.1) and (2.7.2) are directly taken into consideration since otherwise $\tilde{A}_h^{-\alpha}$ is not well defined.*

In fact, under the assumptions (2.7.1) and (2.7.2), for all $m > 0$ there exists $\epsilon = \epsilon(m) > 0$ such that

$$-\mu + i\beta \in \rho(\tilde{A}_h), \quad \forall h \in (0, h^*),\ \forall 0 \leq \mu \leq \epsilon, \forall |\beta| \leq m.$$

Indeed, for all $m > 0$ such that $|\beta| \leq m$, we have

$$(-\mu + i\beta - \tilde{A}_h)^{-1} = (i\beta - \tilde{A}_h)^{-1}[I_h - \mu(i\beta - \tilde{A}_h)^{-1}]^{-1}$$

and

$$\|\mu(i\beta - \tilde{A}_h)^{-1}\| \leq \mu c.$$

Hence, if $|\beta| \leq m$ and $\mu \leq \epsilon \leq \frac{1}{2c}$, then $(-\mu + i\beta - \tilde{A}_h)$ is invertible and we have

$$\|(-\mu + i\beta - \tilde{A}_h)^{-1}\| \leq 2\|(i\beta - \tilde{A}_h)^{-1}\| \leq 2c,\ \forall h \in (0, h^*). \tag{2.7.5}$$

We choose $m = \Im(-\epsilon + te^{i\theta}) = \epsilon \tan\theta$ when $\Re(-\epsilon + te^{i\theta}) = 0$, i.e. when $t = \frac{\epsilon}{\cos\theta}$. Therefore, by (2.7.5), assumptions (2.7.1) and (2.7.2) imply that there exists $\epsilon > 0$ independent of h such that the curve Γ is included in $\rho(\tilde{A}_h)$ for any $h \in (0, h^*)$, and hence $\tilde{A}_h^{-\alpha}$ is well defined. In fact, if $\xi \in \Gamma$ such that $\Re\xi > 0$, then, by the Hille-Yosida Theorem, $\xi \in \rho(\tilde{A}_h)$, while if $-\epsilon \leq \xi \leq 0$, then, by (2.7.5), $\xi \in \rho(\tilde{A}_h)$.

Proposition 2.7.4. *If the assumptions (2.7.1) and (2.7.2) are satisfied, then* $\tilde{A}_h^{-\alpha}$ *is bounded independent of* $h \in (0, h^*)$.

Proof: We have

$$\begin{aligned} \tilde{A}_h^{-\alpha} &= \frac{1}{2\pi i}\int_0^{+\infty} (-\epsilon + te^{i\theta})^{-\alpha}(-\epsilon + te^{i\theta} - \tilde{A}_h)^{-1}e^{i\theta}dt \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{0} (-\epsilon - te^{-i\theta})^{-\alpha}(-\epsilon - te^{-i\theta} - \tilde{A}_h)^{-1}(-e^{-i\theta})dt. \end{aligned} \tag{2.7.6}$$

Since $(T_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ is bounded, then by Hille-Yosida Theorem (see Theorem I.3.1 of [62]) we get

$$\|R(\lambda,\tilde{A}_h)\| \leq \frac{M}{Re\lambda},\ \forall Re\lambda > 0.$$

For $-\epsilon \leq Re\lambda \leq 0$, we have $|\Im\lambda| \leq m$ and therefore, by (2.7.5), we get

$$\|R(\lambda,\tilde{A}_h)\| \leq 2c.$$

Let $t_0 > 0$ be such that $-\epsilon \leq Re(-\epsilon + te^{i\theta}) \leq 0, \forall 0 \leq t \leq t_0 = \dfrac{\epsilon}{\cos\theta}$ and $Re(-\epsilon+te^{i\theta}) \geq 0, \forall t \geq t_0$ and let $t_1 = -\dfrac{\epsilon}{\cos\theta} \leq 0$ be such that $Re(-\epsilon - te^{-i\theta}) \leq 0, \forall t_1 \leq t \leq 0$ and $Re(-\epsilon - te^{-i\theta}) \geq 0,\ \forall t \leq t_1$. Therefore,

$$\begin{aligned}
\tilde{A}_h^{-\alpha} &= \frac{1}{2\pi i}\int_0^{t_0}(-\epsilon+te^{i\theta})^{-\alpha}(-\epsilon+te^{i\theta}-\tilde{A}_h)^{-1}e^{i\theta}dt\\
&+ \frac{1}{2\pi i}\int_{t_0}^{+\infty}(-\epsilon+te^{i\theta})^{-\alpha}(-\epsilon+te^{i\theta}-\tilde{A}_h)^{-1}e^{i\theta}dt\\
&+ \frac{1}{2\pi i}\int_{-\infty}^{t_1}(-\epsilon-te^{-i\theta})^{-\alpha}(-\epsilon-te^{-i\theta}-\tilde{A}_h)^{-1}(-e^{-i\theta})dt\\
&+ \frac{1}{2\pi i}\int_{t_1}^{0}(-\epsilon-te^{-i\theta})^{-\alpha}(-\epsilon-te^{-i\theta}-\tilde{A}_h)^{-1}(-e^{-i\theta})dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|\tilde{A}_h^{-\alpha}\right\| &\leq 2c\int_0^{t_0} \mid -\epsilon+te^{i\theta}\mid^{-\alpha} dt + M\int_{t_0}^{+\infty}\frac{1}{\mid -\epsilon+te^{i\theta}\mid^{\alpha}(-\epsilon+tcos\theta)}dt\\
&+M\int_{-\infty}^{t_1}\frac{1}{\mid \epsilon+te^{-i\theta}\mid^{\alpha}(-\epsilon-tcos\theta)}dt + 2c\int_{t_1}^{0}\mid \epsilon+te^{-i\theta}\mid^{-\alpha} dt,
\end{aligned}$$

which is uniformly bounded with respect to h. ■

The proof of the polynomial stability of $(T_h(t))_{t\geq 0}$ (see Theorem 2.7.9 below) is based on the following three lemmas. The first lemma is the discretized version of

Lemma 3.2 in [48] and the other ones are the discrete versions of similar results of Lemmas 2.1 and 2.3 in [18].

Lemma 2.7.5. *Let $S = \{\lambda \in \mathbb{C} : \ a \leq Re\lambda \leq b\}$ be a subset of $\rho(\tilde{A}_h)$ for all $h \in (0, h^*)$ where $0 \leq a < b$. Then if (2.7.1) and (2.7.2) are satisfied and if for some positive constants α and M we have*

$$\sup_{\substack{h\in(0,h^*)\\ \lambda\in S}} \frac{\|R(\lambda, \tilde{A}_h)\|}{1+|\lambda|^{\alpha}} \leq M,$$

then there exists a constant $c > 0$ independent of h such that

$$\sup_{\substack{h\in(0,h^*)\\ \lambda\in S}} \|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c.$$

Proof: There exists $c > 0$ and $\varphi_0,\ 0 < \varphi_0 < \dfrac{\pi}{2}$, such that

$$|\mu - e^{i\varphi}| \geq c|\mu|, \quad \forall \mu \in \Gamma,\ \forall \varphi_0 < |\varphi| < \pi - \varphi_0 \tag{2.7.7}$$

where the curve Γ is given by (2.7.4). Indeed, if $\mu = -\epsilon + te^{i\theta}$ for some $t > 0$, then

$$|\mu - e^{i\varphi}|^2 = 1 + \epsilon^2 + t^2 - 2tcos(\theta + \varphi) - 2\epsilon tcos\theta + 2\epsilon cos\varphi$$

and

$$|\mu|^2 = \epsilon^2 + t^2 - 2\epsilon tcos\theta.$$

Therefore, whether $t > 0$ is large enough or small , (2.7.7) holds true. Now, Since b is finite, choose N large enough such that whenever $\lambda \in S$ and $|\lambda| > N$ we get both $\varphi_0 < |arg\lambda| < \pi - \varphi_0$ and λ does not belong to the sector bounded by the curve $|\lambda|\Gamma = \{-\epsilon|\lambda| + t|\lambda|e^{i\theta},\ t \in [0, +\infty)\} \cup \{-\epsilon|\lambda| - t|\lambda|e^{-i\theta},\ t \in (-\infty, 0]\}$.

For all such choice of $\lambda \in S$, we have according to (2.7.7)

$$|\mu - e^{iarg\lambda}| \geq c|\mu| \ \ \forall \mu \in \Gamma. \tag{2.7.8}$$

Consider the following integral for all $\lambda \in S$ with $|\lambda| > N$

$$I_\lambda = \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} d\mu.$$

By the above choice of λ, we have $\lambda \notin \Gamma$ and $\lambda \notin |\lambda|\Gamma$. Consequently, the integral has no singular points between Γ and $|\lambda|\Gamma$. Therefore, by the Cauchy Theorem, we have

$$I_\lambda = \int_{|\lambda|\Gamma} \frac{\mu^{-\alpha}}{\mu - \lambda} d\mu = \frac{1}{|\lambda|^\alpha} \int_\Gamma \frac{\mu^{-\alpha}}{\mu - e^{iarg\lambda}} d\mu.$$

Therefore, by (2.7.8), we get

$$|I_\lambda| \leq \frac{c}{|\lambda|^\alpha}.$$

Now, for $|\lambda| > N$ with $\lambda \in S$, we have by the resolvent identity

$$\begin{aligned} R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha} &= \frac{1}{2\pi i} \int_\Gamma \mu^{-\alpha} R(\lambda, \tilde{A}_h) R(\mu, \tilde{A}_h) d\mu \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} R(\lambda, \tilde{A}_h) d\mu - \frac{1}{2\pi i} \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} R(\mu, \tilde{A}_h) d\mu \\ &= \frac{1}{2\pi i} I_\lambda R(\lambda, \tilde{A}_h) - \frac{1}{2\pi i} \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} R(\mu, \tilde{A}_h) d\mu. \end{aligned}$$

On the other hand, similar to the proof of Proposition 2.7.4,

$$\left| \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} R(\mu, \tilde{A}_h) d\mu \right| \leq c \int_\Gamma \frac{1}{|\mu|^{\alpha+1}} \|R(\mu, \tilde{A}_h)\| d\mu \leq c',$$

where c' is independent of h. Therefore for all $\lambda \in S$, with $|\lambda| > N$, we have

$$\|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq \frac{c}{|\lambda|^\alpha} \|R(\lambda, \tilde{A}_h)\| + c' \leq c \frac{1 + |\lambda|^\alpha}{|\lambda|^\alpha} + c' \leq c''.$$

Now, for $\lambda \in S$ such that $|\lambda| \leq N$, we have

$$\|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq \|R(\lambda, \tilde{A}_h)\| \|\tilde{A}_h^{-\alpha}\| \leq c(1 + |\lambda|^\alpha) \leq c(1 + N^\alpha),$$

which completes the proof of Lemma 2.7.5. ■

Lemma 2.7.6. *If (2.7.1), (2.7.2) and*

$$\sup_{h\in(0,h^*)} \|R(is,\tilde{A}_h)\|_{\mathcal{L}(X_h)} = O(|s|^\alpha),\ |s| \to \infty, \tag{2.7.9}$$

are satisfied, then there exists $c > 0$ independent of h such that

$$\sup_{\substack{h\in(0,h^*)\\ Re\lambda>0}} \|R(\lambda,\tilde{A}_h)\tilde{A}_h^{-\alpha}\| \le c. \tag{2.7.10}$$

Proof: For all $h \in (0,h^*)$, $m > 0$, and $B > \max\{2m,1\}$, consider $F_h(\lambda) = R(\lambda,\tilde{A}_h)\lambda^{-\alpha}(1-\frac{\lambda^2}{B^2})$ on the domain $D = \left\{\lambda \in \mathbb{C}:\ Re\lambda > 0,\ m < |\lambda| \le \dfrac{B}{2}\right\}$. F_h, by the maximum principle, attains its maximum for $|\lambda| = \dfrac{B}{2}$. Therefore,

$$|F_h(\lambda)| \le \frac{c}{Re\lambda}.$$

If there exists $\epsilon > 0$ such that $Re\lambda > \epsilon$, then $|F_h(\lambda)| \le c$.

Otherwise, for $0 < Re\lambda < \epsilon$, using the resolvent identity

$$R(\lambda,\tilde{A}_h) = R(iIm\lambda,\tilde{A}_h) - Re\lambda R(iIm\lambda,\tilde{A}_h)R(\lambda,\tilde{A}_h) \tag{2.7.11}$$

then, as $|Im\lambda| \ge m-\epsilon$ for all $m > 0$, we have

$$\|R(\lambda,\tilde{A}_h)\| \le c|Im\lambda|^\alpha.$$

Therefore,

$$|F_h(\lambda)| \le c|Im\lambda|^\alpha|\lambda|^{-\alpha}\left|1-\frac{\lambda^2}{B^2}\right| \le c.$$

Hence, in all cases, there exists $c > 0$ independent of B such that

$$|F_h(\lambda)| \le c.$$

As a result, for all $\lambda \in D$,

$$\|R(\lambda,\tilde{A}_h)\| \le \frac{c|\lambda|^\alpha}{|1-\frac{\lambda^2}{B^2}|} \le c|\lambda|^\alpha \le c(1+|\lambda|^\alpha).$$

If $0 < Re\lambda \leq |\lambda| \leq m$, then by (2.7.11) and assumption (2.7.2), we get

$$\|R(\lambda, \tilde{A}_h)\| \leq c\|R(iIm\lambda, \tilde{A}_h)\| \leq c \leq c(1 + |\lambda|^\alpha).$$

Letting $B \to +\infty$ yields

$$\|R(\lambda, \tilde{A}_h)\| \leq c(1 + |\lambda|^\alpha), \forall Re\lambda > 0.$$

Applying Lemma 2.7.5, we get for $0 \leq Re\lambda \leq m$,

$$\|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c.$$

In addition, if $Re\lambda \geq m$, by the Hille-Yosida theorem and Proposition 2.7.4, there exists some positive constants c_1 and c_2 such that

$$\|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c_1 \frac{\|\tilde{A}_h^{-\alpha}\|}{Re\lambda} \leq c_2.$$

In all cases, we get (2.7.10). ■

The last lemma in this section gives the necessary and sufficient conditions for the boundedness of any family of C_0 semigroups $(S_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$.

Lemma 2.7.7. *Let* $(S_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ *be a family of* C_0 *semigroups on the associated Hilbert spaces* $(Y_h)_{h\in(0,h^*)}$ *and let* $(\tilde{E}_h)_{h\in(0,h^*)}$ *be the corresponding infinitesimal generators. Then* $(S_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ *is uniformly bounded if and only if*

(i) $\{\lambda \in \mathbb{C} : \ Re\lambda > 0\} \subseteq \rho(\tilde{E}_h), \forall h \in (0, h^*)$

(ii) There exists $c > 0$ *independent of* h *such that*

$$\sup_{\substack{\xi>0\\ h\in(0,h^*)}} \xi \int_{\mathbb{R}} (\|R(\xi + i\eta, \tilde{E}_h)\|^2 + \|R(\xi + i\eta, \tilde{E}_h^*)\|^2) d\eta \leq c.$$

Proof: First, we assume that $(S_h(t))$ is uniformly bounded. Then (i) holds by the Hille-Yosida theorem. As for (ii), we only need to prove that

$$\sup_{\substack{\xi>0 \\ h\in(0,h^*)}} \xi \int_{\mathbb{R}} \|R(\xi+i\eta,\tilde{E}_h)x_h\|^2 d\eta \le c\|x_h\|^2,\ \forall x_h \in Y_h \tag{2.7.12}$$

because according to the theory of adjoint semigroups, (see [62]), $S^*(t)$ is a C_0 semigroup with the same properties as $S(t)$.
Similar to the proof of Lemma 1 of [42], we have for all $h \in (0,h^*)$, $x_h \in Y_h$

$$\|R(\xi+i\eta,\tilde{E}_h)x_h\|^2 = \int_{\mathbb{R}} e^{-i\eta s} f_h(s)ds,$$

where

$$f_h(s) = \int_{\max\{0,-s\}}^{+\infty} e^{-\xi(s+2u)} < S_h(u+s)x_h, S_h(u)x_h >_{Y_h,Y_h} du.$$

For $s \ge 0$, since $(S_h(t)_{h\in(0,h^*)}$ is uniformly bounded, i.e. $\sup\limits_{h\in(0,h^*)} \|S_h(t)\| \le M$, we have

$$|f_h(s)| \le \int_0^{+\infty} M^2\|x_h\|^2 e^{-\xi(s+2u)}du = \frac{M^2\|x_h\|^2}{2\xi}e^{-\xi s} \le \frac{M^2\|x_h\|^2}{2\xi}.$$

For $s<0$, we have

$$|f_h(s)| \le \int_{-s}^{+\infty} M^2\|x_h\|^2 e^{-\xi(s+2u)}du = \frac{M^2\|x_h\|^2 e^{\xi s}}{2\xi} \le \frac{M^2\|x_h\|^2}{2\xi}.$$

Hence, $f_h \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R})$ and

$$\mathfrak{F}(f_h(s)) = \frac{1}{\sqrt{2\pi}}\|R(\xi+i\eta,\tilde{E}_h)x_h\|^2.$$

Using Lemma 21.50 in [40], it follows that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \|R(\xi+i\eta,\tilde{E}_h)x_h\|^2 d\eta = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathfrak{F}(f_h)(\tau)d\tau \le c\|f_h\|_{L^\infty} \le \frac{cM^2\|x_h\|^2}{2\xi}.$$

Hence, (2.7.12) is verified.

As for the sufficient condition, since $\{\lambda \in \mathbb{C}: \ Re\lambda > 0\} \subseteq \rho(\tilde{E}_h)$, with $\sigma = \frac{1}{t}$, we get for all $x_h \in Y_h$

$$\begin{aligned} S_h(t)x_h &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}(\lambda - \tilde{E}_h)^{-1}x_h d\lambda, \\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{\lambda t}}{t}(\lambda - \tilde{E}_h)^{-2}x_h d\lambda + \frac{e^{\lambda t}}{t}(\lambda - \tilde{E}_h)^{-1}x_h \,|_{\sigma-i\infty}^{\sigma+i\infty}. \end{aligned}$$

But $\frac{e^{\lambda t}}{t}(\lambda - \tilde{E}_h)^{-1}x_h|_{\sigma-i\infty}^{\sigma+i\infty} = 0$ since according to Lemma 2.1 of [69], under condition (ii), we have $\|R(\lambda, \tilde{E}_h)x_h\| \to 0$ as $|\lambda| \to +\infty$ whenever $Re\lambda > 0$. Therefore,

$$\begin{aligned} \langle S_h(t)x_h, y_h\rangle_{Y_h,Y_h} &= \left\langle \frac{1}{2\pi i t}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}(\lambda - \tilde{E}_h)^{-2}x_h d\lambda, y_h \right\rangle_{Y_h,Y_h} \\ &= \frac{1}{2\pi i t}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}\left\langle (\lambda - \tilde{E}_h)^{-2}x_h, y_h \right\rangle_{Y_h,Y_h} d\lambda. \end{aligned}$$

Let $\lambda = \frac{1}{t} + i\eta$ with $\eta \in \mathbb{R}$. Then

$$\langle S_h(t)x_h, y_h\rangle_{Y_h,Y_h} = \frac{e}{2\pi t}\int_{\mathbb{R}} e^{i\eta t}\left\langle R^2(\frac{1}{t} + i\eta, \tilde{E}_h)x_h, y_h \right\rangle_{Y_h,Y_h} d\eta.$$

Hölder's inequality yields

$$\begin{aligned} |\langle S_h(t)x_h, y_h\rangle_{Y_h,Y_h}| &= \left| \frac{e}{2\pi t}\int_{\mathbb{R}} e^{i\eta t}\left\langle R(\frac{1}{t} + i\eta, \tilde{E}_h)x_h, R(\frac{1}{t} + i\eta, \tilde{E}_h^*)y_h \right\rangle_{Y_h,Y_h} d\eta \right| \\ &\leq \frac{e}{2\pi t}\left(\int_{\mathbb{R}} \|R(\frac{1}{t} + i\eta, \tilde{E}_h)x_h\|^2 d\eta\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \|R(\frac{1}{t} + i\eta, \tilde{E}_h^*)y_h\|^2 d\eta\right)^{\frac{1}{2}} \\ &\leq c\|x_h\|\|y_h\|. \end{aligned}$$

Therefore

$$\|S_h(t)\| \leq c, \ \forall h \in (0, h^*).$$

■

Before we give the necessary and sufficient conditions to get the uniform polynomial stability of the discretized problem, we recall Theorem II.5.34 of [24] about the moment inequality.

Theorem 2.7.8. *Let E be the generator of a strongly continuous semigroup and let $\alpha < \beta < \gamma$. Then there exists a constant $L = L(\alpha, \beta, \gamma)$ such that, for every $x \in D(E^\gamma)$, we have*

$$\left\|E^{\beta} x\right\| \leq L \left\|E^{\alpha} x\right\|^{\frac{\gamma-\beta}{\gamma-\alpha}} \left\|E^{\gamma} x\right\|^{\frac{\beta-\alpha}{\gamma-\alpha}}. \tag{2.7.13}$$

Now, we display the main theorem which leads to the uniform polynomial stability of the discretized problem (2.4.9).

Theorem 2.7.9. *Let $(T_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ be a family of uniformly bounded C_0 semigroups on the associated Hilbert spaces $(X_h)_{h\in(0,h^*)}$ and let $(\tilde{A}_h)_{h\in(0,h^*)}$ be the corresponding infinitesimal generators such that (2.7.1) and (2.7.2) are satisfied. Then for a fixed $\alpha > 0$, the following statements are equivalent :*

(i)

$$\sup_{h\in(0,h^*)} \|R(is, \tilde{A}_h)\| = O(|s|^{\alpha}), \quad |s| \to \infty$$

(ii)

$$\sup_{h\in(0,h^*)} \|T_h(t)\tilde{A}_h^{-\alpha}\| = O(t^{-1}), \quad t \to +\infty$$

(iii)

$$\sup_{h\in(0,h^*)} \|T_h(t)\tilde{A}_h^{-1}\| = O(t^{\frac{-1}{\alpha}}), \quad t \to +\infty.$$

Proof: We begin to prove $(ii) \Leftrightarrow (iii)$. We adapt the proof found in Proposition 3.1 of [15] without the discretization parameter h. Given (ii), we have

$$\|T_h(t)\tilde{A}_h^{-\alpha n}\| = \left\|\left[T_h\left(\frac{t}{n}\right)\tilde{A}_h^{-\alpha}\right]^n\right\| \leq c\left(\frac{n}{t}\right)^n \leq c(n)t^{-n}, \forall n \in \mathbb{N}^*,\ h \in (0, h^*),\ t \to +\infty.$$

According to the moment inequality (2.7.13), we remark that there exists a positive constant L independent of h such that, for all $\nu \in (0,1)$, we have

$$\begin{aligned}
\left\|T_h(t)\tilde{A}_h^{-\alpha n\nu}x_h\right\| &= \left\|\tilde{A}_h^{\alpha n(1-\nu)}T_h(t)\tilde{A}_h^{-\alpha n}x_h\right\| \\
&= \left\|\tilde{A}_h^{\beta_1}\tilde{A}_h^{\alpha n(1-\nu)-\beta_1}T_h(t)\tilde{A}_h^{-\alpha n}x_h\right\| \\
&\leq L\left\|\tilde{A}_h^{\alpha_1}y_h\right\|^{\frac{\gamma_1-\beta_1}{\gamma_1-\alpha_1}}\left\|\tilde{A}_h^{\gamma_1}y_h\right\|^{\frac{\beta_1-\alpha_1}{\gamma_1-\alpha_1}} \\
&\leq L\left\|\tilde{A}_h^{\alpha_1+\alpha n(1-\nu)-\beta_1}T_h(t)\tilde{A}_h^{-\alpha n}x_h\right\|^{\nu} \\
&\quad \left\|\tilde{A}_h^{\gamma_1+\alpha n(1-\nu)-\beta_1}T_h(t)\tilde{A}_h^{-\alpha n}x_h\right\|^{1-\nu},
\end{aligned}$$

where $y_h = \tilde{A}_h^{\alpha n(1-\nu)-\beta_1}T_h(t)\tilde{A}_h^{-\alpha n}x_h$. Now, we choose α_1, β_1, and γ_1 such that

$$\begin{cases} \gamma_1 - \beta_1 + \alpha n(1-\nu) = \alpha n \\ \alpha_1 + \alpha n(1-\nu) - \beta_1 = 0; \end{cases}$$

i.e.,

$$\begin{cases} \alpha_1 = \beta_1 - \alpha n(1-\nu) < \beta_1 \\ \gamma_1 = \beta_1 + \alpha n\nu > \beta_1. \end{cases}$$

Therefore,

$$\begin{cases} \gamma_1 - \alpha_1 = \alpha n \\ \gamma_1 - \beta_1 = \alpha n\nu. \end{cases}$$

Finally, we get

$$\begin{aligned}
\left\|T_h(t)\tilde{A}_h^{-\alpha n\nu}x_h\right\| &\leq L\left\|\tilde{A}_h^{\alpha n}T_h(t)\tilde{A}_h^{-\alpha n}x_h\right\|^{1-\nu}\left\|T_h(t)\tilde{A}_h^{-\alpha n}x_h\right\|^{\nu} \\
&\leq LM^{1-\nu}c^{\nu}(n)t^{-n\nu}\|x_h\|, \quad \forall\nu \in (0,1).
\end{aligned}$$

Choose $\nu = \dfrac{1}{\alpha n}$ with $n > \dfrac{1}{\alpha}$ to get

$$\|T_h(t)\tilde{A}_h^{-1}\| \leq ct^{-\frac{1}{\alpha}}.$$

Conversely, assume that (iii) holds. Then

$$\|T_h(t)\tilde{A}_h^{-n}\| = \|[T_h\left(\frac{t}{n}\right)\tilde{A}_h^{-1}]^n\| \leq c\left(\frac{t}{n}\right)^{\frac{-n}{\alpha}} \leq cn^{\frac{n}{\alpha}}t^{-\frac{n}{\alpha}},\ \forall n \in \mathbb{N}^*.$$

Therefore,

$$\begin{aligned}\|T_h(t)\tilde{A}_h^{-n\nu}\| &\leq c\|\tilde{A}_h^{n}T_h(t)\tilde{A}_h^{-n}\|^{1-\nu}\|T_h(t)\tilde{A}_h^{-n}\|^{\nu}\\ &\leq cM^{1-\nu}c(n)^{\nu}t^{-\frac{n\nu}{\alpha}}, \quad \forall\nu \in (0,1).\end{aligned}$$

Take $\nu = \dfrac{\alpha}{n}$ with $n > \alpha$ to get

$$\sup_{h\in(0,h^*)} \|T_h(t)\tilde{A}_h^{-\alpha}\| = O\left(t^{-1}\right).$$

Now, we prove the implication $(iii) \Rightarrow (i)$ (for the continuous case, see [16]). Given (iii), define

$$m_1(t) = \sup_{\substack{h\in(0,h^*)\\ s\geq t}} \|T_h(s)\tilde{A}_h^{-1}\|.$$

Notice that $m_1(t)$ is non increasing. Let $u_{0h} \in \mathcal{D}(\tilde{A}_h),\ f_{0h} = (-\tilde{A}_h + i\tau)u_{0h},\ \tau \in \mathbb{R}$, and let $v_h(t) = e^{it\tau}u_{0h}$. We have

$$\begin{cases}\partial_t v_h - \tilde{A}_h v_h &= i\tau e^{it\tau}u_{0h} - \tilde{A}_h(e^{it\tau}u_{0h}) = e^{it\tau}f_{0h}\\ v_h(0) &= u_{0h}.\end{cases}$$

By the Duhamel formula,

$$v_h = e^{t\tilde{A}_h}u_{0h} + \int_0^t e^{(t-s)\tilde{A}_h}e^{i\tau s}f_{0h}ds.$$

By the boundedness of the semigroup $(T_h(t))$ and the definition of m_1, we have

$$\begin{aligned}\|u_{0h}\| = \|v_h(t)\| &\leq \|T_h(t)\tilde{A}_h^{-1}\tilde{A}_h u_{0h}\| + c\ t\|f_{0h}\|\\ &\leq m_1(t)\|\tilde{A}_h u_{0h}\| + c\ t\|f_{0h}\|\\ &\leq m_1(t)(\|f_{0h}\| + |\tau|\|u_{0h}\|) + c\ t\|f_{0h}\|.\end{aligned}$$

Apply the above inequality with $t = G(|\tau|)$ where

$$G(\xi) = \begin{cases} m_{1r}^{-1}\left(\dfrac{1}{2(\xi+1)}\right) & if\ \xi > 0 \text{ and } \dfrac{1}{2(\xi+1)} \leq m_1(0), \\ 0 & if\ \xi > 0 \text{ and } \dfrac{1}{2(\xi+1)} > m_1(0), \end{cases}$$

where m_{1r}^{-1} is the right inverse of m_1. Therefore,

$$m_1(t)|\tau| = m_1(G(|\tau|))|\tau| \leq \frac{|\tau|}{2(|\tau|+1)} \leq \frac{1}{2}.$$

Hence,

$$\begin{aligned} \tfrac{1}{2}\|u_{0h}\| &\leq m_1(G(|\tau|))\|f_{0h}\| + c\ G(|\tau|)\|f_{0h}\| \\ &\leq \frac{\|f_{0h}\|}{2(|\tau|+1)} + c\ G(|\tau|)\|f_{0h}\| \\ &\leq (\tfrac{1}{2} + c\ G(|\tau|))\|f_{0h}\|. \end{aligned}$$

Consequently,

$$\|(i\tau - \tilde{A}_h)^{-1}\| \leq 1 + 2c\ G(|\tau|),$$

i.e.,

$$\sup_{h\in(0,h^*)} \|(i\tau - \tilde{A}_h)^{-1}\| \leq 1 + 2c\ G(|\tau|).$$

Since, by (iii),

$$\sup_{h\in(0,h^*)} \|T_h(t)\tilde{A}_h^{-1}\| \leq M t^{\frac{-1}{\alpha}}, \quad t \to +\infty,$$

then, as m_1 is non-increasing, we get

$$m_1(t) \leq M t^{\frac{-1}{\alpha}}, \quad t \to +\infty.$$

Besides, as the inverse of $t^{\frac{-1}{\alpha}}$ is $t^{-\alpha}$, then

$$G(\xi) \leq m_{1r}^{-1}\left(\frac{1}{2(\xi+1)}\right) \leq C\left(\frac{1}{2(\xi+1)}\right)^{-\alpha} = C(2(\xi+1))^{\alpha} \leq c\xi^{\alpha}, \quad \xi \to +\infty.$$

Finally, we get

$$\sup_{h\in(0,h^*)} \|(i\tau - \tilde{A}_h)^{-1}\| \leq 1 + 2c|\tau|^{\alpha} \leq c|\tau|^{\alpha}, \quad |\tau| \to +\infty.$$

It remains to prove that $(i) \Rightarrow (ii)$. For this aim, for all $h \in (0, h^*)$, let $\mathbf{X_h} = X_h \times X_h$ and consider the operator $\tilde{\mathbf{A}}_\mathbf{h}$ given by the operator matrix

$$\tilde{\mathbf{A}}_\mathbf{h} = \begin{pmatrix} \tilde{A}_h & \tilde{A}_h^{-\alpha} \\ 0 & \tilde{A}_h \end{pmatrix},$$

where $\mathcal{D}(\tilde{\mathbf{A}}_\mathbf{h}) = \mathcal{D}(\tilde{A}_h) \times \mathcal{D}(\tilde{A}_h)$. For all $h \in (0, h^*)$ and all $\lambda_h \in \rho(\tilde{A}_h)$, we have

$$R(\lambda_h, \tilde{\mathbf{A}}_\mathbf{h}) = \begin{pmatrix} R(\lambda_h, \tilde{A}_h) & R^2(\lambda_h, \tilde{A}_h)\tilde{A}_h^{-\alpha} \\ 0 & R(\lambda_h, \tilde{A}_h) \end{pmatrix}.$$

Indeed,

$$R(\lambda_h, \tilde{\mathbf{A}}_\mathbf{h})(\lambda_h - \tilde{\mathbf{A}}_\mathbf{h}) = (\lambda_h - \tilde{\mathbf{A}}_\mathbf{h})R(\lambda_h, \tilde{\mathbf{A}}_\mathbf{h}) = \begin{pmatrix} I_h & 0 \\ 0 & I_h \end{pmatrix}.$$

Therefore, $\rho(\tilde{\mathbf{A}}_\mathbf{h}) = \rho(\tilde{A}_h)$ and for all $h \in (0, h^*)$, the operator $\tilde{\mathbf{A}}_\mathbf{h}$ is the generator of the C_0 semigroup $(\mathbf{T_h}(t))_{t \geq 0}$ on $\mathbf{X_h}$ defined by

$$\mathbf{T_h}(t) = \begin{pmatrix} T_h(t) & tT_h(t)\tilde{A}_h^{-\alpha} \\ 0 & T_h(t) \end{pmatrix}.$$

In fact,

$$\begin{aligned} \widehat{\mathbf{T_h}(t)} &= \begin{pmatrix} \widehat{T_h(t)} & \widehat{tT_h(t)}\tilde{A}_h^{-\alpha} \\ 0 & \widehat{T_h(t)} \end{pmatrix} \\ &= \begin{pmatrix} R(\lambda_h, \tilde{A}_h) & R^2(\lambda_h, \tilde{A}_h)\tilde{A}_h^{-\alpha} \\ 0 & R(\lambda_h, \tilde{A}_h) \end{pmatrix} \\ &= R(\lambda_h, \tilde{\mathbf{A}}_\mathbf{h}), \end{aligned}$$

where $\widehat{\mathbf{T_h}(t)}$ is the Laplace transform of $\mathbf{T_h}(t)$. Since for all $h \in (0, h^*)$ we have

$$\|R(is, \tilde{A}_h)\| = O(|s|^\alpha), \text{ as } |s| \to +\infty,$$

then by Lemma 2.7.6 we get

$$\sup_{\substack{h\in(0,h^*)\\ Re\lambda>0}} \|R(\lambda,\tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c.$$

Hence, for all $x_h = (x_{1h}, x_{2h}) \in \mathbf{X_h}$, and $Re\lambda_h > 0$, we have

$$\begin{aligned}\|R(\lambda_h,\tilde{\mathbf{A}}_\mathbf{h})x_h\|^2 &= \left\|\begin{pmatrix} R(\lambda_h,\tilde{A}_h)x_{1h} + R^2(\lambda_h,\tilde{A}_h)\tilde{A}_h^{-\alpha}x_{2h} \\ R(\lambda_h,\tilde{A}_h)x_{2h}\end{pmatrix}\right\|^2 \\ &\leq c\left(\|R(\lambda_h,\tilde{A}_h)x_{1h}\|^2 + \|R(\lambda_h,\tilde{A}_h)x_{2h}\|^2\right).\end{aligned}$$

Similarly, we have

$$\|R(\lambda_h,\tilde{\mathbf{A}}_\mathbf{h}^*)x_h\|^2 \leq c(\|R(\lambda_h,\tilde{A}_h^*)x_{1h}\|^2 + \|R(\lambda_h,\tilde{A}_h^*)x_{2h}\|^2).$$

Indeed, we have

$$\tilde{\mathbf{A}}_\mathbf{h}^* = \begin{pmatrix} \tilde{A}_h^* & (\tilde{A}_h^*)^{-\alpha} \\ 0 & \tilde{A}_h^*\end{pmatrix}.$$

In order to get

$$\sup_{\substack{h\in(0,h^*)\\ Re\lambda>0}} \|R(\lambda,\tilde{A}_h^*)(\tilde{A}_h^*)^{-\alpha}\| \leq c,$$

we must have at least

$$\|R(is,\tilde{A}_h^*)\| = O(|s|^\alpha), \text{ as } |s| \to +\infty.$$

Actually, we have

$$R(is,\tilde{A}_h^*) = [(is - \tilde{A}_h^*)]^{-1} = [(is - \tilde{A}_h)^*]^{-1} = R(is,\tilde{A}_h)^*.$$

Therefore, we get

$$\|R(is,\tilde{A}_h^*)\| \leq \|R(is,\tilde{A}_h)\| = O(|s|^\alpha), \text{ as } |s| \to +\infty.$$

Now, by Lemma 2.7.7, since for all $h \in (0, h^*)$, $T_h(t)$ is a uniformly bounded family of C_0 semigroups, we get

$$\sup_{\substack{\xi>0 \\ h\in(0,h^*)}} \xi \int_{\mathbb{R}} (\|R(\xi+i\eta, \tilde{A}_h)x_h\|^2) + (\|R(\xi+i\eta, \tilde{A}_h^*)x_h\|^2) d\eta < \infty, \quad \forall x_h \in X_h.$$

Hence,

$$\sup_{\substack{\xi>0 \\ h\in(0,h^*)}} \xi \int_{\mathbb{R}} (\|R(\xi+i\eta, \tilde{\mathbf{A}}_{\mathbf{h}})x_h\|^2) + (\|R(\xi+i\eta, \tilde{\mathbf{A}}_{\mathbf{h}}^*)x_h\|^2) d\eta < \infty, \quad \forall x_h \in \mathbf{X}_{\mathbf{h}}.$$

Therefore, $(\mathbf{T}_{\mathbf{h}}(t))_{\substack{t\geq 0 \\ h\in(0,h^*)}}$ is uniformly bounded over $(\mathbf{X}_{\mathbf{h}})_{h\in(0,h^*)}$ by Lemma 2.7.7. Since $(T_h(t))_{\substack{t\geq 0 \\ h\in(0,h^*)}}$ is uniformly bounded over $(X_h)_{h\in(0,h^*)}$, the definition of $\mathbf{T}_{\mathbf{h}}(t)$ implies that

$$\sup_{\substack{t\geq 0 \\ h\in(0,h^*)}} \|tT_h(t)\tilde{A}_h^{-\alpha}\| < +\infty.$$

■

2.8 Preliminary lemmas

In this section, we prove that the family $(\tilde{A}_{l,h})_{h\in(0,h^*)}$ defined in (2.5.1) satisfies condition i) in Theorem 2.7.1 and the properties (2.7.1) and (2.7.2) of Subsection 2.7.2. Condition i) in Theorem 2.7.1 or (2.7.1) in Subsection 2.7.2 is satisfied due to the following lemma :

Lemma 2.8.1. *The spectrum of the operator $\tilde{A}_{l,h}$ contains no point on the imaginary axis.*

Proof: Suppose that $\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \in X_h$ and $\omega \in \mathbb{R}$ are such that

$$\tilde{A}_{l,h} \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} = i\omega \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}.$$

Then, by using the definition (2.5.1) of $\tilde{A}_{l,h}$, we have

$$\left\{\begin{array}{c}\psi_h = i\omega(1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\varphi_h \\ -(1+h^\theta)^{-1}(I+h^\theta A_h^{\frac{l}{2}})A_h\varphi_h - i\omega(1+h^\theta)(h^\theta A_h^{1+\frac{l}{2}} + B_hB_h^*)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\varphi_h \\ = -\omega^2(1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\varphi_h.\end{array}\right. \tag{2.8.1}$$

Let $\chi_h = (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\varphi_h$ then the second relation of (2.8.1) becomes

$$(1+h^\theta)^{-2}(I+h^\theta A_h^{\frac{l}{2}})^2 A_h\chi_h + i\omega(h^\theta A_h^{1+\frac{l}{2}} + B_hB_h^*)\chi_h = \omega^2\chi_h. \tag{2.8.2}$$

If $\omega = 0$, then taking the inner product of (2.8.2) with $\chi_h \in V_h$, we get $(I + h^\theta A_h^{\frac{l}{2}})A_h^{\frac{1}{2}}\chi_h = 0$ and hence $\chi_h = 0$ which implies by the definition of χ_h that $\varphi_h = \psi_h = 0$.

It then remains to consider the case $\omega \neq 0$. In that case, we take the imaginary part of the inner product (in H) of (2.8.2) with $\chi_h \in V_h$ to obtain

$$\begin{aligned} 0 &= \omega h^\theta \left(A_h^{1+\frac{l}{2}}\chi_h,\, \chi_h\right) + \omega\left(B_hB_h^*\chi_h,\, \chi_h\right) \\ &= \omega h^\theta \left(A_h^{\frac{1}{2}+\frac{l}{4}}\chi_h,\, A_h^{\frac{1}{2}+\frac{l}{4}}\chi_h\right) + \omega\left(B_h^*\chi_h,\, B_h^*\chi_h\right)_U, \end{aligned}$$

that is to say

$$h^\theta \left\|A_h^{\frac{1}{2}+\frac{l}{4}}\chi_h\right\|^2 + \|B_h^*\chi_h\|_U^2 = 0.$$

This leads to $\chi_h = 0$, and hence $\varphi_h = \psi_h = 0$. ■

Our main goal is to prove condition ii) of Theorem 2.7.1 in the case $l = 0$ and condition i) of Theorem 2.7.9 as well as (2.7.2) in the case $l \geq 2$ and $\alpha = 2l$. In that last case ($l \geq 2$), these two conditions are equivalent to

$$\sup_{h\in(0,h^*),\ s\in\mathbb{R}} (1+|s|^{2l})^{-1}\|R(is, \tilde{A}_{l,h})\|_{\mathcal{L}(X_h)} < \infty. \tag{2.8.3}$$

To prove this above property, we use a contradiction argument. More precisely, we will assume that, for all $n \in \mathbb{N}$, there exist $h_n \in (0, h^*)$, $\omega_n \in \mathbb{R}$ and $z_n = \begin{pmatrix}\varphi_n \\ \psi_n\end{pmatrix} \in$

X_{h_n} such that

$$\|z_n\|^2_{X_{h_n}} = a(\varphi_n,\, \varphi_n) + \|\psi_n\|^2 = 1,\, \forall n \in \mathbb{N}, \tag{2.8.4}$$

and

$$(1 + |\omega_n|^{2l}) \left\| i\omega_n z_n - \tilde{A}_{l,h_n} z_n \right\|_{X_{h_n}} \to 0, \text{ as } n \to \infty, \tag{2.8.5}$$

where $l = 0$ in the setting of Theorem 2.7.1.

Lemma 2.8.2. *Assume that the sequences* (h_n), (ω_n), (z_n) *satisfy (2.8.4) and (2.8.5). Then, we have*

$$(1 + |\omega_n|^{2l})(h_n^\theta a^l(\psi_n,\, \psi_n) + \left\| B^*_{h_n}\psi_n \right\|^2_U) \to 0, \text{ as } n \to \infty \tag{2.8.6}$$

and

$$\lim_{n\to\infty} a(\varphi_n,\, \varphi_n) = \lim_{n\to\infty} \|\psi_n\|^2 = \frac{1}{2}. \tag{2.8.7}$$

Proof: For (2.8.6), we take the inner product in X_{h_n} of $i\omega_n z_n - \tilde{A}_{l,h_n} z_n$ with z_n and take the real part. We obtain

$$\begin{aligned}
&\Re\left(i\omega_n z_n - \tilde{A}_{l,h_n} z_n,\, z_n\right)_{X_{h_n}} \\
= \;& -\Re\left(\begin{pmatrix} (1+h_n^\theta)^{-1}(I + h_n^\theta A_{h_n}^{\frac{l}{2}})\psi_n \\ -(1+h_n^\theta)^{-1}(I + h_n^\theta A_{h_n}^{\frac{l}{2}})A_{h_n}\varphi_n - h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n - B_{h_n}B^*_{h_n}\psi_n \end{pmatrix}, \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}\right)_{X_{h_n}} \\
= \;& \Re\Big(-(1+h_n^\theta)^{-1}\left((I + h_n^\theta A_h^{\frac{l}{2}})A_{h_n}\psi_n,\, \varphi_n\right) + (1+h_n^\theta)^{-1}\left((I + h_n^\theta A_{h_n}^{\frac{l}{2}})A_{h_n}\varphi_n,\, \psi_n\right) \\
& +(h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n + B_{h_n}B^*_{h_n}\psi_n,\, \psi_n)\Big) \\
= \;& (h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n + B_{h_n}B^*_{h_n}\psi_n,\, \psi_n).
\end{aligned}$$

Then

$$\begin{aligned}
&(1 + |\omega_n|^{2l})\Re\left(i\omega_n z_n - \tilde{A}_{l,h_n} z_n,\, z_n\right)_{X_{h_n}} \\
&= (1 + |\omega_n|^{2l})(h_n^\theta a^l(\psi_n,\, \psi_n) + \left\| B^*_{h_n}\psi_n \right\|^2_U) \to 0 \text{ by } (2.8.5).
\end{aligned}$$

In order to prove (2.8.7), we introduce the operator

$$A_{1h_n} = (1 + h_n^\theta)^{-1}(I + h_n^\theta A_{h_n}^{\frac{l}{2}}) \begin{pmatrix} 0 & I \\ -A_{h_n} & 0 \end{pmatrix}. \tag{2.8.8}$$

We have

$$\tilde{A}_{l,h_n}\begin{pmatrix}\varphi_n\\ \psi_n\end{pmatrix} = A_{1h_n}\begin{pmatrix}\varphi_n\\ \psi_n\end{pmatrix} - \begin{pmatrix}0\\ h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n + B_{h_n}B_{h_n}^*\psi_n\end{pmatrix},\ \forall \begin{pmatrix}\varphi_n\\ \psi_n\end{pmatrix} \in X_{h_n}.$$

We take the norm $\|.\|_{X_{h_n}}$ of $i\omega_n z_n - A_{1h_n}z_n + \begin{pmatrix}0\\ h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n\end{pmatrix}$ to obtain

$$\begin{aligned}
&(1+|\omega_n|^{2l})\left\|i\omega_n z_n - A_{1h_n}z_n + \begin{pmatrix}0\\ h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n\end{pmatrix}\right\|_{X_{h_n}}^2\\
= \ & (1+|\omega_n|^{2l})\left\|i\omega_n z_n - \tilde{A}_{l,h_n}z_n - \begin{pmatrix}0\\ B_{h_n}B_{h_n}^*\psi_n\end{pmatrix}\right\|_{X_{h_n}}^2\\
\leq \ & 2(1+|\omega_n|^{2l})(\left\|i\omega_n z_n - \tilde{A}_{l,h_n}z_n\right\|_{X_{h_n}}^2 + \left\|B_{h_n}B_{h_n}^*\psi_n\right\|^2)\\
\leq \ & C(1+|\omega_n|^{2l})(\left\|i\omega_n z_n - \tilde{A}_{l,h_n}z_n\right\|_{X_{h_n}}^2 + \left\|B_{h_n}^*\psi_n\right\|_U^2) \to 0,
\end{aligned}$$

by (2.8.5) and (2.8.6). Therefore

$$(1+|\omega_n|^{2l})\left\|i\omega_n z_n - A_{1h_n}z_n + \begin{pmatrix}0\\ h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n\end{pmatrix}\right\|_{X_{h_n}}^2 \to 0. \tag{2.8.9}$$

We can now prove (2.8.7). By Lemma 2.8.3 below, there exists $n_0 \in \mathbb{N}$ such that the sequence $(|\omega_n|)_{n\geq n_0}$ is bounded away from zero. Hence, we may write

$$\begin{aligned}
&\Im\left(i\omega_n z_n - A_{1h_n}z_n + \begin{pmatrix}0\\ h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n\end{pmatrix}, \tfrac{1}{\omega_n}\begin{pmatrix}\varphi_n\\ -\psi_n\end{pmatrix}\right)_{X_{h_n}}\\
= \ & \left(\begin{pmatrix}\varphi_n\\ \psi_n\end{pmatrix}, \begin{pmatrix}\varphi_n\\ -\psi_n\end{pmatrix}\right)_{X_{h_n}}\\
= \ & a(\varphi_n, \varphi_n) - \|\psi_n\|_{V_{h_n}}^2
\end{aligned}$$

and so, by (2.8.9) and (2.8.4), we have

$$\lim_{n\to\infty}(a(\varphi_n, \varphi_n) - \|\psi_n\|_{V_{h_n}}^2) = 0.$$

This relation and (2.8.4) lead to (2.8.7). ■

Lemma 2.8.3. *Assume that (2.8.4) and (2.8.5) hold. Then there exists $n_0 \in \mathbb{N}$ such that the sequence $(|\omega_n|)_{n\geq n_0}$ is uniformly bounded away from zero.*

Proof: By a contradiction argument, we show that the sequence $(\omega_n)_n$ contains no subsequence converging to zero. Namely suppose that such a subsequence exists. For the sake of simplicity, we still denote it by $(\omega_n)_n$. Hence (2.8.9) implies that

$$-A_{1h_n} z_n + \begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix} = \begin{pmatrix} -(1+h_n^\theta)^{-1}(I + h_n^\theta A_{h_n}^{\frac{l}{2}})\psi_n \\ (1+h_n^\theta)^{-1}(I + h_n^\theta A_{h_n}^{\frac{l}{2}}) A_{h_n}\varphi_n + h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n \end{pmatrix} \to 0 \text{ in } X_{h_n}. \tag{2.8.10}$$

Taking the inner product of first component in (2.8.10) with ψ_n, we get

$$(1+h_n^\theta)^{-1} a\left((I + h_n^\theta A_{h_n}^{\frac{l}{2}})\psi_n,\, \psi_n\right) = (1+h_n^\theta)^{-1}\left(a(\psi_n,\, \psi_n) + h_n^\theta a^l(\psi_n,\, \psi_n)\right) \to 0.$$

As $h_n \leq h^*$, then, by (2.8.6), we get

$$\left\| A_{h_n}^{\frac{1}{2}} \psi_n \right\|^2 = a(\psi_n,\, \psi_n) \to 0. \tag{2.8.11}$$

The convergence of the first component in (2.8.10) implies that

$$\left\| A_{h_n}^{\frac{1}{2}} (I + h_n^\theta A_{h_n}^{\frac{l}{2}})\psi_n \right\|^2 \to 0.$$

Therefore, (2.8.11) yields

$$h_n^\theta A_{h_n}^{\frac{(1+l)}{2}} \psi_n \to 0 \ \text{ in } H. \tag{2.8.12}$$

The second component in (2.8.10) and the fact that $\alpha\|x\|^2 \leq \|A_h^{\frac{1}{2}} x\|^2 = a(x,x)$ for all $x \in V_h$ imply that

$$(1+h_n^\theta)^{-1}(I + h_n^\theta A_{h_n}^{\frac{l}{2}}) A_{h_n}^{\frac{1}{2}} \varphi_n + h_n^\theta A_{h_n}^{\frac{1+l}{2}} \psi_n \to 0 \ \text{ in } H,$$

which, by (2.8.12), yields

$$(1+h_n^\theta)^{-1}(I+h_n^\theta A_{h_n}^{\frac{l}{2}})A_{h_n}^{\frac{1}{2}}\varphi_n \to 0 \text{ in } H.$$

Thus, as $h_n \leq h^*$, we get

$$a(\varphi_n,\, \varphi_n) \to 0.$$

This above relation and (2.8.11) contradict (2.8.4). ■

According to the above lemma, we note that the coefficient $1+|\omega_n|^{2l}$ becomes equivalent to $|\omega_n|^{2l}$. Now, we introduce the operator D_{1h_n} defined by

$$D_{1h_n} = \begin{pmatrix} 0 & I \\ -A_{h_n} & 0 \end{pmatrix}.$$

Note that $A_{1h_n} = (1+h_n^\theta)^{-1}(I+h_n^\theta A_{h_n}^{\frac{l}{2}})D_{1h_n}$. We then use the following spectral basis of the operator D_{1h_n}. Namely, we extend the definitions of $\lambda_{k,\,h_n}$ and of $\varphi_{k,\,h_n}$ for $k \in \{-1, ..., -N(h_n)\}$ by setting $\lambda_{k,\,h_n} = -\lambda_{-k,\,h_n}$ and $\varphi_{k,\,h_n} = \varphi_{-k,\,h_n}$. Then an orthonormal basis of X_{h_n} formed by the eigenvectors of D_{1h_n} is given by

$$\Psi_{k,\,h_n} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{i}{\lambda_{k,\,h_n}}\varphi_{k,\,h_n} \\ \varphi_{k,\,h_n} \end{pmatrix},\ 0<|k|\leq N(h_n), \tag{2.8.13}$$

of associated eigenvalue $i\lambda_{k,\,h_n}$, that is to say

$$D_{1h_n}\Psi_{k,\,h_n} = i\lambda_{k,\,h_n}\Psi_{k,\,h_n}.$$

Consequently, for all $n \in \mathbb{N}$, there exist complex coefficients $(c_k^n)_{0<|k|\leq N(h_n)}$ such that

$$z_n = \sum_{0<|k|\leq N(h_n)} c_k^n \Psi_{k,\,h_n}. \tag{2.8.14}$$

The normalization condition (2.8.4) implies that

$$\sum_{0<|k|\leq N(h_n)} |c_k^n|^2 = 1.$$

Let ϵ be the constant from Proposition 2.6.4 (if $l = 0$, we recover the condition from Proposition 2.6.1). For any $n \in \mathbb{N}$, we define

$$M_l(h_n) = \max\left\{k \in \{1, \ldots, N(h_n)\} \,\middle|\, h_n^\theta(\lambda_k)^2 \leq \frac{\epsilon}{\lambda_k^l}\right\}, \tag{2.8.15}$$

if $h_n^\theta(\lambda_1)^2 \leq \frac{\epsilon}{\lambda_1^l}$ and $M_l(h_n) = 0$ otherwise.

Lemma 2.8.4. *Suppose that the sequences (h_n), (ω_n), (z_n) satisfy (2.8.4) and (2.8.5). Then, we have*

$$\psi_n = \frac{1}{\sqrt{2}} \sum_{k=1}^{N(h_n)} (c_k^n + c_{-k}^n)\varphi_{k,\, h_n}, \tag{2.8.16}$$

$$\sum_{M_l(h_n)<k\leq N(h_n)} |\omega_n|^{2l} \left|c_k^n + c_{-k}^n\right|^2 \to 0, \tag{2.8.17}$$

and

$$\sum_{0<|k|\leq M_l(h_n)} |\omega_n|^{2l} \left|\omega_n - (1 + h_n^\theta)^{-1}(\lambda_{k,\, h_n} + h_n^\theta \lambda_{k,\, h_n}^{1+l})\right|^2 |c_k^n|^2 \to 0. \tag{2.8.18}$$

Proof: Relation (2.8.16) follows directly by taking the second component in (2.8.14) and by using (2.8.13) and the fact that $\varphi_{k,\, h} = \varphi_{-k,\, h}$.

From (2.8.6) and (2.8.16), it follows that

$$|\omega_n|^{2l} h_n^\theta a^l(\psi_n, \psi_n) = \frac{1}{2} \sum_{k=1}^{N(h_n)} h_n^\theta \lambda_{k,\, h_n}^{2+l} |\omega_n|^{2l} \left|c_k^n + c_{-k}^n\right|^2 \to 0. \tag{2.8.19}$$

As we have $\lambda_k \leq \lambda_{k,\, h_n}$ for all $k \in \{1, \ldots, N(h_n)\}$ and by the definition (2.8.15), we obtain (2.8.17).

On the other hand, we use (2.8.14) and the fact that $\Psi_{k,\, h_n}$ is an eigenvector of D_{1h_n} associated with eigenvalue $i\lambda_{k,\, h_n}$ to obtain for all $\tilde{\psi}_{h_n} \in X_{h_n}$

$$\begin{aligned} &\left(i\omega_n z_n - A_{1h_n} z_n,\, \tilde{\psi}_{h_n}\right)_{X_{h_n}} \\ = &\sum_{0<|k|\leq N(h_n)} i\left(\omega_n - (1 + h_n^\theta)^{-1}(\lambda_{k,\, h_n} + h_n^\theta \lambda_{k,\, h_n}^{1+l})\right) c_k^n \left(\Psi_{k,\, h_n},\, \tilde{\psi}_{h_n}\right)_{X_{h_n}}. \end{aligned} \tag{2.8.20}$$

By (2.6.9), we have

$$h_n^\theta \lambda_{k,h_n}^2 \leq h_n^\theta(\lambda_k + (Ch_n^\theta \lambda_k^2)^2)^2 \leq 2h_n^\theta \lambda_k^2 + 2C^4 h_n^\theta (h_n^\theta \lambda_k^2)^4 \leq C\frac{\epsilon}{\lambda_k^l} + C\frac{\epsilon^4}{\lambda_k^{4l}} \leq C'\frac{\epsilon}{\lambda_k^l} \tag{2.8.21}$$

for $h_n^\theta(\lambda_k)^2 \leq \frac{\epsilon}{\lambda_k^l}$. So, by using (2.8.19) and again (2.6.9), there exists a constant C independent of h_n such that

$$\begin{aligned} h_n^{2\theta} \sum_{k=1}^{M_l(h_n)} \lambda_{k,h_n}^{4+2l} |\omega_n|^{2l} \left|c_k^n + c_{-k}^n\right|^2 &\leq C \sum_{k=1}^{M_l(h_n)} \epsilon h_n^\theta \lambda_{k,h_n}^{2+l} |\omega_n|^{2l} \left|c_k^n + c_{-k}^n\right|^2 \\ &\leq C\epsilon \sum_{k=1}^{M_l(h_n)} h_n^\theta \lambda_{k,h_n}^{2+l} |\omega_n|^{2l} \left|c_k^n + c_{-k}^n\right|^2 \to 0. \end{aligned} \tag{2.8.22}$$

We also have for all $\tilde{\psi}_{h_n} \in X_{h_n}$

$$\left(\begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix}, \tilde{\psi}_{h_n}\right)_{X_{h_n}} = \sum_{0<|k|\leq N(h_n)} \frac{h_n^\theta}{2} \lambda_{k,h_n}^{2+l} (c_k^n + c_{-k}^n) \left(\Psi_{k,h_n}, \tilde{\psi}_{h_n}\right)_{X_{h_n}}. \tag{2.8.23}$$

because l is even. Relation (2.8.23) implies that for all $\tilde{\psi}_{h_n} \in X_{h_n}$

$$\begin{aligned} &\left(\begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix} - \sum_{M_l(h_n)<|k|\leq N(h_n)} \tfrac{h_n^\theta}{2} \lambda_{k,h_n}^{2+l} (c_k^n + c_{-k}^n) \Psi_{k,h_n}, \tilde{\psi}_{h_n}\right)_{X_{h_n}} \\ &= \sum_{0<|k|\leq M_l(h_n)} \tfrac{h_n^\theta}{2} \lambda_{k,h_n}^{2+l} (c_k^n + c_{-k}^n) \Psi_{k,h_n}, \tilde{\psi}_{h_n})_{X_{h_n}}. \end{aligned}$$

However,

$$\begin{aligned} &\left\| |\omega_n|^l \sum_{0<|k|\leq M_l(h_n)} \tfrac{h_n^\theta}{2} \lambda_{k,h_n}^{2+l} (c_k^n + c_{-k}^n) \Psi_{k,h_n} \right\|^2 \\ &= |\omega_n|^{2l} \sum_{0<|k|\leq M_l(h_n)} \frac{h_n^{2\theta}}{(2\sqrt{2})^2} \lambda_{k,h_n}^{4+2l} |c_k^n + c_{-k}^n|^2 + |\omega_n|^{2l} \sum_{0<|k|\leq M_l(h_n)} \frac{h_n^{2\theta}}{(2\sqrt{2})^2} \lambda_{k,h_n}^{4+2l} |c_k^n + c_{-k}^n|^2 \\ &= 2\,|\omega_n|^{2l} \sum_{0<|k|\leq M_l(h_n)} \frac{h_n^{2\theta}}{(2\sqrt{2})^2} \lambda_{k,h_n}^{4+2l} |c_k^n + c_{-k}^n|^2. \end{aligned}$$

Therefore, by (2.8.22), for all $\tilde{\psi}_{h_n} \in X_{h_n}$

$$|\omega_n|^l \left(\begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix} - \sum_{M_l(h_n)<|k|\le N(h_n)} \frac{h_n^\theta}{2} \lambda_{k,\,h_n}^{2+l} (c_k^n + c_{-k}^n) \Psi_{k,\,h_n},\ \tilde{\psi}_{h_n} \right)_{X_{h_n}} \to 0.$$

So, we obtain with (2.8.9), (2.8.20) and the above relation, for all $\tilde{\psi}_{h_n} \in X_{h_n}$, that the inner product in X_{h_n} of $\tilde{\psi}_{h_n}$ with

$$\sum_{0<|k|\le N(h_n)} i\,|\omega_n|^l \left(\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k,\,h_n} + h_n^\theta \lambda_{k,\,h_n}^{1+l})\right) c_k^n \Psi_{k,\,h_n}$$
$$+ \sum_{M_l(h_n)<|k|\le N(h_n)} \frac{h_n^\theta}{2} |\omega_n|^l \lambda_{k,\,h_n}^{2+l} (c_k^n + c_{-k}^n) \Psi_{k,\,h_n}$$

tends to zero. As the family $(\Psi_{k,\,h_n})$ is orthogonal, the above relation implies (2.8.18). ■

2.9 Proof of Theorem 2.4.1

We use the results of the previous section with $l = 0$ and set, for shortness, $\tilde{A}_h := \tilde{A}_{0,h}$ and $M(h_n) := M_0(h_n)$.

Proof of Theorem 2.4.1 This proof is based on Theorem 2.7.1. First, for all $h \in (0,\, h^*)$, the family $(e^{t\tilde{A}_h})$ forms a contraction semigroup. The family $(\tilde{A}_h)$ satisfies the condition i) in Theorem 2.7.1 owing to Lemma 2.8.1. To show that the family $(\tilde{A}_h)$ also satisfies the condition ii) in Theorem 2.7.1, we use a contradiction argument. Let $(h_n)_n$, $(\omega_n)_n$ and $z_n = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \in \mathcal{D}(\tilde{A}_{h_n})$ be three sequences satisfying (2.8.4) and (2.8.5). Notice that for $k_m \in A_k$, we have

$$\begin{aligned} \lambda_{k_m,h_n} - \lambda_{k_{m-1}+l_{m-1}-1,h_n} &\ge \lambda_{k_m} - \lambda_{k_{m-1}+l_{m-1}-1} - c\epsilon \\ &= \lambda_{k_m} - \lambda_{k_{m-1}} - c\epsilon \ge \gamma_0' - c\epsilon \\ &\ge \tfrac{\gamma_0'}{2} =: \gamma' \end{aligned}$$

for $\epsilon \leq \frac{\gamma_0'}{2c}$ by (2.6.8) and (2.6.9). We now introduce the set

$$\mathcal{F} = \Big\{ n \in \mathbb{N} \,\Big|\, \exists k(n) \in \{1, ..., M\},\ \exists k_{m(n)} \in A_{k(n)},\ \left|k_{m(n)}\right| \leq M(h_n) \text{ and } \left|k_{m(n)+k(n)-1} + l_{m(n)+k(n)-1}\right| \leq N(h_n) \text{ such that } \left|\omega_n - \lambda_{k_{m(n)},h_n}\right| < \frac{\gamma'}{2} \Big\}. \quad (2.9.1)$$

We distinguish two cases.

First case : The set $\mathcal{F}$ is infinite. Then, without loss of generality, we can suppose that $\mathcal{F} = \mathbb{N}$ (otherwise we take a subsequence of (ω_n)). Then, by reducing the value of γ' if needed, we can assume that for all $n \in \mathbb{N}$, we have that for all $k_m \in A_{k'}$, $k' = 1, ..., M$ with $m \neq m(n)$,

$$\left|\omega_n - \lambda_{k_{m+j}+l,h_n}\right| \geq \frac{\gamma'}{2},\ \forall j = 0, ..., k'-1,\ \forall l = 0, ..., l_{m+j}-1.$$

By using (2.8.18), we obtain that

$$\sum_{k=1}^{M} \sum_{\substack{k_m \in A_k \\ m \neq m(n)}} \sum_{j=0}^{k-1} \sum_{\substack{l=0 \\ 0 < |k_{m+j}+l_{m+j}-1| \leq M(h_n)}}^{l_{m+j}-1} \left|c^n_{k_{m+j}+l}\right|^2 \to 0. \quad (2.9.2)$$

Define now

$$\tilde{\psi}_n = \frac{1}{\sqrt{2}} \sum_{j=0}^{k(n)-1} \sum_{l=0}^{l_{m(n)+j}-1} c^n_{k_{m(n)+j}+l} \varphi_{k_{m(n)+j}+l,h_n}. \quad (2.9.3)$$

We have, by (2.8.16),

$$\psi_n = \frac{1}{\sqrt{2}} \sum_{k=1}^{M} \sum_{k_m \in A_k} \sum_{\substack{j=0 \\ 1 \leq k_{m+j}+l \leq N(h_n)}}^{k-1} \sum_{l=0}^{l_{m+j}-1} (c^n_{k_{m+j}+l} + c^n_{-(k_{m+j}+l)}) \varphi_{k_{m+j}+l,h_n},$$

and so, by (2.9.2) and (2.8.17), we obtain

$$\left\|\tilde{\psi}_n - \psi_n\right\| \to 0. \quad (2.9.4)$$

Thus, since $(\|B_h^*\|_{\mathcal{L}(V_h, U)})_{h\in(0,h^*)}$ is bounded, we deduce that

$$\left\|B_{h_n}^*(\tilde{\psi}_n - \psi_n)\right\|_U \to 0. \tag{2.9.5}$$

The above relation and (2.8.6) imply that

$$\left\|B_{h_n}^*\tilde{\psi}_n\right\|_U \to 0. \tag{2.9.6}$$

But

$$\begin{aligned}
\left\|B_{h_n}^*\tilde{\psi}_n\right\|_U &= \tfrac{1}{\sqrt{2}}\left\|\sum_{j=0}^{k(n)-1}\sum_{l=0}^{l_{m(n)+j}-1} c^n_{k_{m(n)+j}+l} B_{h_n}^*\varphi_{k_{m(n)+j}+l,\,h_n}\right\|_U \\
&= \tfrac{1}{\sqrt{2}}\left\|\begin{pmatrix} B_{h_n}^*\varphi_{k_{m(n)},\,h_n} & \cdots & B_{h_n}^*\varphi_{k_{m(n)+k(n)-1}+l_{m(n)+k(n)-1}-1,\,h_n}\end{pmatrix} C\right\|_U \\
&= \tfrac{1}{\sqrt{2}}\left\|\begin{pmatrix} 1 & \cdots & 1\end{pmatrix}\Phi_{k_{m(n)},\,h_n} C\right\|_U,
\end{aligned}$$

where $C = \begin{pmatrix} c_{k_{m(n)}} & \cdots & c_{k_{m(n)}+l_{m(n)}-1} & c_{k_{m(n)+1}} & \cdots & c_{k_{m(n)+k(n)-1}+l_{m(n)+k(n)-1}-1}\end{pmatrix}^T$. So, we have

$$\left\|B_{h_n}^*\tilde{\psi}_n\right\|_U = \tfrac{1}{\sqrt{2}}\left\|\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}\Phi_{k_{m(n)},\,h_n} C\right\|_{U,2}.$$

We now use Lemma 2.3.1 to have

$$\begin{aligned}
\left\|B_{h_n}^*\tilde{\psi}_n\right\|_U &\geq c\left\|B_{k_{m(n)}}^{-1}\Phi_{k_{m(n)},\,h_n} C\right\|_{U,2} \text{ for } n \text{ large enough} \\
&\geq c\alpha\,\|C\|_2 \text{ by Proposition 2.6.1.}
\end{aligned} \tag{2.9.7}$$

Gathering (2.9.3), (2.9.6) and (2.9.7), we obtain that $\tilde{\psi}_n \to 0$ in H. Therefore, by (2.9.4), $\psi_n \to 0$, which contradicts (2.8.7).

<u>Second case</u> : The set $\mathcal{F}$ is finite. Then, we can assume, without loss of generality, that $\mathcal{F}$ is empty (otherwise we take off the finite number of (ω_n)) ; i.e., that for all

$n \in \mathbb{N}$, we have that

$$|\omega_n - \lambda_{k,\,h_n}| \geq \frac{\gamma'}{2} \quad \text{if } 0 < |k| \leq M(h_n).$$

Thus, by (2.8.18) and the above relation, we obtain that

$$\sum_{0<|k|\leq M(h_n)} |c_k^n|^2 \to 0.$$

Therefore, by (2.8.16), (2.8.17) and the above relation, we have $\psi_n \to 0$ in H, which contradicts (2.8.7).

In conclusion, the family $(\tilde{A}_h)$ satisfies the condition ii) in Theorem 2.7.1 and so the family of systems (2.4.6) is uniformly exponentially stable. ■

2.10 Proof of Theorem 2.4.4

Here we use the results of Section 2.8 with $l > 0$ and l even. Without loss of generality, we may assume that $0 < h < h^* = 1$.

Proof of Theorem 2.4.4 and of condition (2.7.2) This proof is based on Theorem 2.7.9. First, for all $h \in (0, h^*)$, $(e^{t\tilde{A}_{l,h}})$ forms a family of contraction semigroups and the family $(\tilde{A}_{l,h})_h$ satisfies (2.7.1). To apply the results of Theorem 2.7.9, the family $(\tilde{A}_{l,h})$ must also satisfy condition i) of Theorem 2.7.9 with $\alpha = 2l$ and condition (2.7.2) or equivalently condition (2.8.3) . We again use a contradiction argument to prove this last condition. Let $(h_n)_n$, $(\omega_n)_n$ and $z_n = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \in X_h$ be three sequences satisfying (2.8.4) and (2.8.5). Notice that for $k_m \in A_k$, we have

$$\begin{aligned} \lambda_{k_m,h} - \lambda_{k_{m-1}+l_{m-1}-1,h} &\geq \lambda_{k_m} - \lambda_{k_{m-1}+l_{m-1}-1} - \tfrac{c\epsilon}{\lambda_{k_{m-1}}^{2l}} \\ &\geq \lambda_{k_m} - \lambda_{k_{m-1}} - \tfrac{c\epsilon}{\lambda_{k_1}^{2l}} \geq \gamma_0' - \tfrac{c\epsilon}{\lambda_{k_1}^{2l}} \\ &\geq \tfrac{\gamma_0'}{2} =: \gamma' \end{aligned}$$

for $\epsilon \leq \frac{\gamma_0' \lambda_{k_1}^{2l}}{2c}$ by (2.6.8), (2.6.9) and because $\lambda_{k_m} \geq \lambda_{k_1} > 0$. We introduce the set $\mathcal{F}_2$ like

$$\mathcal{F}_2 = \Big\{ n \in \mathbb{N} \,\Big|\, \exists k(n) \in \{1, ..., M\},\ \exists k_{m(n)} \in A_{k(n)},\ \left|k_{m(n)}\right| \leq M_l(h_n) \text{ and } \left|k_{m(n)+k(n)-1} + l_{m(n)+k(n)-1}\right| \leq N(h_n) \text{ such that } \left|\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k_{m(n)},h_n} + h_n^\theta \lambda_{k_{m(n)},h_n}^{1+l})\right| < \frac{\gamma'}{4} \Big\}. \quad (2.10.1)$$

We distinguish two cases.

First case : The set $\mathcal{F}_2$ is infinite. Then, without loss of generality, we can suppose that $\mathcal{F}_2 = \mathbb{N}$ (otherwise we take a subsequence of $(\omega_n)_n$). Then, by reducing the value of γ' if needed, we can assume that for all $n \in \mathbb{N}$, we have that for all $k_m \in A_{k'}$, $k' = 1, ..., M$ with $m \neq m(n)$, and for all $|k_{m+j} + l| \leq M_l(h_n)$

$$\left|\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k_{m+j}+l,h_n} + h_n^\theta \lambda_{k_{m+j}+l,h_n}^{1+l})\right| \geq \frac{\gamma'}{8},\ \forall j = 0, ..., k'-1,\ \forall l = 0, ..., l_{m+j}-1. \quad (2.10.2)$$

Indeed, similar to (2.8.21), we have

$$\begin{aligned} & \left|\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k_{m+j}+l,h_n} + h_n^\theta \lambda_{k_{m+j}+l,h_n}^{1+l})\right| \\ \geq\ & (1+h_n^\theta)^{-1}\left|\lambda_{k_{m+j}+l,h_n} - \lambda_{k_{m(n)},h_n}\right| - \left|\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k_{m(n)},h_n} + h_n^\theta \lambda_{k_{m(n)},h_n}^{1+l})\right| \\ & -(1+h_n^\theta)^{-1}(h_n^\theta \lambda_{k_{m(n)},h_n}^{1+l} + h_n^\theta \lambda_{k_{m+j}+l,h_n}^{1+l}) \\ \geq\ & \frac{\gamma'}{2} - \frac{\gamma'}{4} - \frac{2C\epsilon}{\lambda_{k_1}}. \end{aligned}$$

So choose again $\epsilon \leq \dfrac{\gamma' \lambda_{k_1}}{16C}$ to get (2.10.2). By using (2.8.18), we obtain that

$$\sum_{k=1}^{M} \sum_{\substack{k_m \in A_k \\ m \neq m(n)}} \sum_{j=0}^{k-1} \sum_{\substack{l=0 \\ 0 < |k_{m+j} + l_{m+j} - 1| \leq M_l(h_n)}}^{l_{m+j}-1} |\omega_n|^{2l} \left|c_{k_{m+j}+l}^n\right|^2 \to 0. \quad (2.10.3)$$

Define now

$$\tilde{\psi}_n = \frac{1}{\sqrt{2}} \sum_{j=0}^{k(n)-1} \sum_{l=0}^{l_{m(n)+j}-1} c^n_{k_{m(n)+j}+l} \varphi_{k_{m(n)+j}+l,\, h_n}. \tag{2.10.4}$$

We have, by (2.8.16),

$$\psi_n = \frac{1}{\sqrt{2}} \sum_{k=1}^{M} \sum_{k_m \in A_k} \sum_{\substack{j=0 \\ 1 \le k_{m+j}+l \le N(h_n)}}^{k-1} \sum_{l=0}^{l_{m+j}-1} (c^n_{k_{m+j}+l} + c^n_{-(k_{m+j}+l)}) \varphi_{k_{m+j}+l,\, h_n},$$

and so, by (2.10.3) and (2.8.17), we obtain

$$|\omega_n|^l \left\| \tilde{\psi}_n - \psi_n \right\| \to 0. \tag{2.10.5}$$

Thus, since $(\|B_h^*\|_{\mathcal{L}(V_h,\, U)})_{h \in (0,\, h^*)}$ is bounded, we deduce that

$$|\omega_n|^l \left\| B^*_{h_n} (\tilde{\psi}_n - \psi_n) \right\|_U \to 0. \tag{2.10.6}$$

The above relation and (2.8.6) imply that

$$|\omega_n|^l \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U \to 0. \tag{2.10.7}$$

But

$$\begin{aligned} |\omega_n|^l \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U &= \frac{|\omega_n|^l}{\sqrt{2}} \left\| \sum_{j=0}^{k(n)-1} \sum_{l=0}^{l_{m(n)+j}-1} c^n_{k_{m(n)+j}+l} B^*_{h_n} \varphi_{k_{m(n)+j}+l,\, h_n} \right\|_U \\ &= \frac{|\omega_n|^l}{\sqrt{2}} \left\| \begin{pmatrix} B^*_{h_n} \varphi_{k_{m(n)},\, h_n} & \cdots & B^*_{h_n} \varphi_{k_{m(n)+k(n)-1} + l_{m(n)+k(n)-1} - 1,\, h_n} \end{pmatrix} C \right\|_U \\ &= \frac{|\omega_n|^l}{\sqrt{2}} \left\| \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \Phi_{k_{m(n)},\, h_n} C \right\|_U , \end{aligned}$$

where $C = \begin{pmatrix} c_{k_{m(n)}} & \cdots & c_{k_{m(n)}+l_{m(n)}-1} & c_{k_{m(n)+1}} & \cdots & c_{k_{m(n)+k(n)-1}+l_{m(n)+k(n)-1}-1} \end{pmatrix}^T$.
So, we have

$$|\omega_n|^l \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U = \frac{|\omega_n|^l}{\sqrt{2}} \left\| \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} \Phi_{k_{m(n)},\, h_n} C \right\|_{U,\,2} .$$

We now use Lemma 2.3.1 to have

$$\begin{aligned} |\omega_n|^l \left\| B_{h_n}^* \tilde{\psi}_n \right\|_U &\geq c\,|\omega_n|^l \left\| B_{k_{m(n)}}^{-1} \Phi_{k_{m(n)},\,h_n} C \right\|_{U,\,2} \text{ for } n \text{ large enough} \\ &\geq c\frac{\alpha|\omega_n|^l}{\lambda_{k_{m(n)}}^l} \|C\|_2 \text{ by Proposition 2.6.4.} \end{aligned}$$

But, ω_n verifies $\left|\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k_{m(n)},\,h_n} + h_n^\theta \lambda_{k_{m(n)},\,h_n}^{1+l})\right| < \frac{\gamma'}{4}$ by definition (2.10.1) of $\mathcal{F}_2$, thus $|\omega_n| \geq (1+h_n^\theta)^{-1}(\lambda_{k_{m(n)},\,h_n} + h_n^\theta \lambda_{k_{m(n)},\,h_n}^{1+l}) - \frac{\gamma'}{4} \geq \frac{1}{2}\lambda_{k_{m(n)},\,h_n} - \frac{\gamma'}{4}$. Therefore, we have

$$\begin{aligned} |\omega_n|^l \left\| B_{h_n}^* \tilde{\psi}_n \right\|_U &\geq \frac{c\alpha}{2^l} \frac{(\lambda_{k_{m(n)},\,h_n} - \frac{\gamma'}{2})^l}{\lambda_{k_{m(n)}}^l} \|C\|_2 \\ &\geq \frac{c\alpha}{2^{2l}} \frac{\lambda_{k_{m(n)},\,h_n}^l}{\lambda_{k_{m(n)}}^l} \|C\|_2 \text{ for } n \text{ large enough} \\ &\geq \frac{c\alpha}{2^{2l}} \|C\|_2 \text{ by (2.6.8).} \end{aligned} \tag{2.10.8}$$

Gathering (2.10.4), (2.10.7) and (2.10.8), we obtain that $\tilde{\psi}_n \to 0$ in H. Therefore, by (2.10.5), $\psi_n \to 0$, which contradicts (2.8.7).

Second case : The set $\mathcal{F}_2$ is finite. We proceed similar to the proof of the second case of Theorem 2.4.1.

In conclusion, the family $(\tilde{A}_{l,h})$ satisfies (2.8.3) ; i.e., the condition (i) in Theorem 2.7.9 with $\alpha = 2l$ when l is even and property (2.7.2) of Subsection 2.7.2.

Remark 2.10.1. *The previous analysis has been held in case $l \in \mathbb{N}^*$ is even. However, in case l is odd, we can still adapt the same analysis to get the same results. Indeed, we consider problem (2.4.9) with powers of $l+1$ instead if l. Besides, whenever $\left\|B_{k_n}^{-1}\Phi_{k_n} C\right\|_{U,\,2} \geq \dfrac{\alpha}{\lambda_{k_n}^l} \|C\|_2$ and $h^\theta \lambda_k^2 \leq \dfrac{\epsilon}{\lambda_k^{l+1}}$, then*

$$\left\|B_{k_n}^{-1}\Phi_{k_n,\,h} C\right\|_{U,\,2} \geq \frac{\alpha}{\lambda_{k_n}^l} \|C\|_2\,,$$

and

$$\left\|(I + h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t)\right\|^2 + a(\omega_h(t),\,\omega_h(t)) \leq \frac{C}{t^{\frac{1}{l}}} \|(\omega_{0h}, \omega_{1h})\|_{D(\tilde{A}_{l,h})}^2.$$

2.11 Convergence of the discretized problem

Here we want to prove that the solution ω_h of the discrete problem (2.4.9) tends to the solution ω of the continuous problem (2.1.13) in $X := V \times H$ as h goes to zero and if the discrete initial data are well chosen. This is obtained with the help of a general version of the Trotter-Kato Theorem proved in [45] that is appropriated when the approximated semi-groups are defined in proper subspaces of the limit one. The basic idea is that the convergence of the semi-groups is equivalent to the convergence of the resolvent, hence we prove such a convergence result for the resolvents.

First, we recall the Trotter-Kato Theorem proved in [45]. Let Z and X_n be Banach spaces with norms $\|.\|$, $\|.\|_n$, $n = 1, 2, ...$, respectively, and X be a closed linear subspace of Z. On X a C_0-semigroup $T(.)$ with infinitesimal generator $\tilde{A}$ is given and on the spaces X_n, the C_0-semigroups $T_n(.)$ are generated by A_n. Suppose that, for every $n \in \mathbb{N}^*$, there exists bounded linear operators $P_n : \ Z \to X_n$ and $E_n : X_n \to Z$ such that the following assumptions hold :

(A1) $\|P_n\| \leq M_1$, $\|E_n\| \leq M_2$, where $M_1,\ M_2$ are independent of n,

(A2) $\|E_n P_n x - x\| \to 0$ as $n \to \infty$ for all $x \in X$,

(A3) $P_n E_n = I_n$, where I_n is the identity operator on X_n.

For all $n \in \mathbb{N}^*$, let $Z_n = range\ E_n$, $\pi_n = E_n P_n$, $\tilde{T}_n(t) = E_n T_n(t) P_n|_{Z_n}$, $\tilde{A}_n = E_n A_n P_n|_{Z_n}$, and $\tilde{I}_n = E_n I_n P_n|_{Z_n}$. The Trotter-Kato Theorem given in Theorem 2.1 of [45] states the following :

Theorem 2.11.1. *(Trotter-Kato). Assume that $(A_1) - (A_3)$ are satisfied. Then the following statements are equivalent :*

(a) There exists a $\lambda_0 \in \rho(\tilde{A}) \cap \bigcap_{n=1}^{\infty} \rho(\tilde{A}_n)$ such that, for all $x \in X$,

$$\left\|(\lambda_0 \tilde{I}_n - \tilde{A}_n)^{-1} \pi_n x - (\lambda_0 I - \tilde{A})^{-1} x\right\| \to 0, \quad \textit{as } n \to \infty.$$

(b) For every $x \in X$ and $t \geq 0$,

$$\|\tilde{T}_n(t)\pi_n x - T(t)x\| \to 0, \quad \textit{as } n \to \infty$$

uniformly on bounded $t-$intervals.

Our aim is to prove that condition (a) of Theorem 2.11.1 holds true in order to get the convergence of the solutions. Let us start with some preliminary results.

Lemma 2.11.2. *Let $l \in \mathbb{N}, l \geq 2$. If $f \in V = \mathcal{D}(A^{\frac{1}{2}})$, then*

$$\|(1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f - \pi_h f\|_H \leq C h^{\frac{\theta}{l}} \|f\|_V, \tag{2.11.1}$$

for some $C > 0$.

Proof: We write

$$\pi_h f = \sum_{k=1}^{N(h)} f_k \varphi_{k,h},$$

with $f_k \in \mathbb{C}$. Hence

$$v_h = (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f,$$

can be written

$$v_h = \sum_{k=1}^{N(h)} v_k \varphi_{k,h},$$

with $v_k = (1+h^\theta)(1+h^\theta \lambda_{k,h}^l)^{-1} f_k$. Consequently we have

$$\begin{aligned} \|v_h - \pi_h f\|_H^2 &= \sum_{k=1}^{N(h)} |f_k|^2 \Big((1+h^\theta)(1+h^\theta\lambda_{k,h}^l)^{-1} - 1\Big)^2 \\ &= h^{2\theta} \sum_{k=1}^{N(h)} |f_k|^2 \Big(\frac{1-\lambda_{k,h}^l}{1+h^\theta \lambda_{k,h}^l}\Big)^2 \\ &\leq c h^{2\theta} \sum_{k=1}^{N(h)} |f_k|^2 \frac{\lambda_{k,h}^{2l}}{(1+h^\theta\lambda_{k,h}^l)^2} \\ &\leq c h^{2\theta} \sum_{k=1}^{N(h)} \lambda_{k,h}^2 |f_k|^2 (g(\lambda_{k,h}))^2 \end{aligned}$$

for some $c > 0$ independent of h, where the function $g : [0, \infty) \mapsto \mathbb{R}$ is given by $g(\lambda) = \dfrac{\lambda^{l-1}}{(1 + h^\theta \lambda^l)}$. As the maximum of g is attained at $\lambda_0 > 0$ given by

$$h^\theta \lambda_0^l = l - 1,$$

we get that

$$\|v_h - \pi_h f\|_H^2 \leq c c_2^2 h^{\frac{2\theta}{l}} \sum_{k=1}^{N(h)} |f_k|^2 \lambda_{k,h}^2$$

since $\lambda_0 = c_1 h^{-\frac{\theta}{l}}$ and $g(\lambda_0) = c_2 h^{-\frac{\theta(l-1)}{l}}$ with c_1, c_2 two positive constants independent of h. This proves the first estimate since

$$\sum_{k=1}^{N(h)} |f_k|^2 \lambda_{k,h}^2 = \|A_h^{\frac{1}{2}} \pi_h f\|_H^2 = a(\pi_h f, \pi_h f) \leq a(f, f) = \|A^{\frac{1}{2}} f\|_H^2.$$

■

Corollary 2.11.3. *Let $l \in \mathbb{N}, l \geq 2$, then for any $f_h \in V_h$ we have*

$$\|(1 + h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1} f_h - f_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} \leq C h^{\frac{\theta}{l}} \|f_h\|_H, \tag{2.11.2}$$

for some $C > 0$.

Proof: As in the previous lemma, we have

$$\begin{aligned}
\|(1 + h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1} f_h - f_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})}^2 &= \|A_h^{-\frac{1}{2}} \Big((1 + h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1} f_h - f_h\Big)\|_H^2 \\
&= h^{2\theta} \sum_{k=1}^{N(h)} \lambda_{k,h}^{-2} |f_k|^2 \Big(\frac{1 - \lambda_{k,h}^l}{1 + h^\theta \lambda_{k,h}^l}\Big)^2 \\
&\leq c h^{2\theta} \sum_{k=1}^{N(h)} |f_k|^2 (g(\lambda_{k,h}))^2,
\end{aligned}$$

when

$$f_h = \sum_{k=1}^{N(h)} f_k \varphi_{k,h}.$$

We then conclude as before. ■

Lemma 2.11.4. *Let $l \in \mathbb{N}, l \geq 2$ and let $f \in \mathcal{D}(A)$, then*

$$h^{\theta} \|A_h^{1+\frac{l}{2}}(I + h^{\theta} A_h^{\frac{l}{2}})^{-2}\pi_h f\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} \leq C h^{\frac{\theta}{l}} \|f\|_{\mathcal{D}(A)}, \tag{2.11.3}$$

for some $C > 0$.

Proof: We easily see that

$$\begin{aligned} h^{2\theta} \|A_h^{1+\frac{l}{2}}(I + h^{\theta} A_h^{\frac{l}{2}})^{-2}\pi_h f\|^2_{\mathcal{D}(A_h^{-\frac{1}{2}})} &= h^{2\theta} \|A_h^{-\frac{1}{2}} A_h^{1+\frac{l}{2}}(I + h^{\theta} A_h^{\frac{l}{2}})^{-2}\pi_h f\|^2_H \\ &= h^{2\theta} \sum_{k=1}^{N(h)} |f_k|^2 \frac{\lambda_{k,h}^{2l+2}}{(1 + h^{\theta}\lambda_{k,h}^{l})^4} \\ &\leq h^{2\theta} \sum_{k=1}^{N(h)} |f_k|^2 \lambda_{k,h}^4 (g(\lambda_{k,h}))^2, \end{aligned}$$

and we conclude as before. ■

Lemma 2.11.5. *Let $l \in \mathbb{N}, l \geq 2$ and let $f \in V$, then*

$$\|(1+h^{\theta})(I+h^{\theta}A_h^{\frac{l}{2}})^{-1}B_h B_h^*(1+h^{\theta})(I+h^{\theta}A_h^{\frac{l}{2}})^{-1}\pi_h f - B_h B_h^* \pi_h f\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} \leq C h^{\frac{\theta}{l}} \|f\|_V, \tag{2.11.4}$$

for some $C > 0$.

Proof: As in Lemma 2.11.2, we set

$$v_h = (1 + h^{\theta})(I + h^{\theta} A_h^{\frac{l}{2}})^{-1}\pi_h f.$$

First, we notice that

$$\|B_h B_h^*(v_h - \pi_h f)\|_H \leq C\|v_h - \pi_h f\|_H,$$

and by Lemma 2.11.2 we get

$$\|B_h B_h^*(v_h - \pi_h f)\|_H \leq C h^{\frac{\theta}{l}} \|f\|_V.$$

Second, by Corollary 2.11.3, we have

$$\begin{aligned}\|(1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}B_hB_h^*v_h - B_hB_h^*v_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} &\leq Ch^{\frac{\theta}{l}}\|B_hB_h^*v_h\|_H \\ &\leq Ch^{\frac{\theta}{l}}(\|B_hB_h^*(v_h-\pi_hf)\|_H + \|B_hB_h^*\pi_hf\|_H) \\ &\leq Ch^{\frac{\theta}{l}}\|f\|_V,\end{aligned}$$

where we use the fact that $\|\pi_hf\|_H \leq c\|\pi_hf\|_V \leq c\|f\|_V$. The conclusion follows from the two above estimates. ■

Theorem 2.11.6. *If $z=(f,g)^\top \in \mathcal{D}(A)\times\mathcal{D}(A)$, then*

$$\|(\tilde{A}_{l,h})^{-1}(\pi_hf,\pi_hg)^\top - \tilde{A}^{-1}(f,g)^\top\|_X \to 0 \text{ as } h\to 0.$$

Proof: By the definition of $\tilde{A}_{l,h}$ and $\tilde{A}$, we have

$$(u_h,v_h)^\top = (\tilde{A}_{l,h})^{-1}(\pi_hf,\pi_hg)^\top,$$

and

$$(u,v)^\top = \tilde{A}^{-1}(f,g)^\top,$$

if and only if

$$\begin{cases} v_h &= (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\pi_hf \\ -A_hu_h &= (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}(h^\theta A_h^{1+\frac{l}{2}} + B_hB_h^*)v_h + (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\pi_hg,\end{cases}$$

and

$$\begin{cases} v=f \\ -Au = BB^*v+g.\end{cases}$$

Therefore, we can write

$$-A_hu_h = \pi_hg + B_hB_h^*\pi_hf + r_h,$$

where $r_h \in V_h$ is given by

$$\begin{array}{rcl} r_h &=& (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h g - \pi_h g \\ &+& (1+h^\theta)h^\theta(I+h^\theta A_h^{\frac{l}{2}})^{-1}A_h^{1+\frac{l}{2}}v_h \\ &+& (1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}B_hB_h^*v_h - B_hB_h^*\pi_h f. \end{array}$$

By the previous Lemmas, r_h satisfies

$$\|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} \leq Ch^{\frac{\theta}{l}}\|(f,g)^\top\|_{\mathcal{D}(A)\times V}. \tag{2.11.5}$$

Therefore, $u_h \in V_h$ can be seen as the unique solution of

$$a(u_h, w_h) = -(\pi_h g, w_h) - (B_hB_h^*\pi_h f, w_h) - \langle r_h; w_h\rangle \quad \forall w_h \in V_h, \tag{2.11.6}$$

where $\langle ; \rangle$ denotes the dual product in $D(A_h^{-\frac{1}{2}})$. Since $u \in V$ is solution of

$$a(u, w) = -(g, w) - (BB^*f, w) \quad \forall w \in V,$$

we get (recalling that $V_h \subset V$)

$$a(u, w_h) = -(g, w_h) - (BB^*f, w_h) \quad \forall w_h \in V_h.$$

Hence, taking the difference of this identity with (2.11.6), we obtain

$$a(u-u_h, w_h) = (\pi_h g - g, w_h) + (B^*(\pi_h f - f), B^*w_h)_U + \langle r_h; w_h\rangle \quad \forall w_h \in V_h.$$

Consequently, taking $w_h = \pi_h u - u_h$, we get

$$\begin{array}{rcl} a(u-u_h, u-u_h) &=& a(u-u_h, u-\pi_h u) + a(u-u_h, \pi_h u - u_h) \\ &=& a(u-u_h, u-\pi_h u) + (\pi_h g - g, \pi_h u - u_h) \\ &+& (B^*(\pi_h f - f), B^*(\pi_h u - u_h))_U + \langle r_h; \pi_h u - u_h\rangle. \end{array}$$

Hence, by Cauchy-Schwarz's inequality and the boundedness of B^*, we obtain

$$\begin{aligned}\|u-u_h\|_V^2 &= a(u-u_h,u-u_h)\\ &\leq \|u-u_h\|_V\|u-\pi_h u\|_V + C(\|\pi_h g-g\|_H+\|\pi_h f-f\|_H+\|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})})\|\pi_h u-u_h\|_V.\end{aligned}$$

Now, using the triangle inequality, we get

$$\begin{aligned}\|u-u_h\|_V^2 &\leq C\Big((\|u-\pi_h u\|_V+\|\pi_h g-g\|_H+\|\pi_h f-f\|_H+\|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})})\|u-u_h\|_V\\ &+ (\|\pi_h g-g\|_H+\|\pi_h f-f\|_H+\|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})})\|u-\pi_h u\|_V\Big).\end{aligned}$$

Hence, by Young's inequality, we arrive at

$$\begin{aligned}\|u-u_h\|_V^2 &\leq C\Big(\|u-\pi_h u\|_V^2+\|\pi_h g-g\|_H^2+\|\pi_h f-f\|_H^2+\|r_h\|^2_{\mathcal{D}(A_h^{-\frac{1}{2}})}\\ &+ (\|\pi_h g-g\|_H+\|\pi_h f-f\|_H+\|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})})\|u-\pi_h u\|_V\Big).\end{aligned}$$

The estimates (2.4.4), (2.4.5), and (2.11.5) then yield

$$\begin{aligned}\|u-u_h\|_V^2 &\leq C\Big(h^{2\theta}\|u\|^2_{\mathcal{D}(A)}+h^{4\theta}\|f\|^2_{\mathcal{D}(A)}+h^{4\theta}\|g\|^2_{\mathcal{D}(A)}+h^{\frac{2\theta}{l}}\|(f,g)^\top\|^2_{\mathcal{D}(A)\times V}\\ &+ (h^{2\theta}\|f\|_{\mathcal{D}(A)}+h^{2\theta}\|g\|_{\mathcal{D}(A)}+h^{\frac{\theta}{l}}\|(f,g)^\top\|_{\mathcal{D}(A)\times V})h^\theta\|u\|_{\mathcal{D}(A)}\Big).\end{aligned} \tag{2.11.7}$$

For $v-v_h$, we notice that

$$v-v_h = f-(1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f = f-\pi_h f+\pi_h f-(1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f,$$

and we conclude that it tends to zero in H due to the estimate (2.4.4) and Lemma 2.11.2. ■

Corollary 2.11.7. *If $z=(f,g)^\top\in V\times H$, recalling that j_h is the projection from H into V_h, we have*

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h f, j_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X \to 0 \text{ as } h\to 0.$$

Proof: First for $z = (f, g)^\top \in \mathcal{D}(A) \times \mathcal{D}(A)$, then

$$\begin{aligned} \|(\tilde{A}_{l,h})^{-1}(\pi_h f, j_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X &\leq \|(\tilde{A}_{l,h})^{-1}(\pi_h f, \pi_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X \\ &+\|(\tilde{A}_{l,h})^{-1}(0, j_h g - \pi_h g)^\top\|_X. \end{aligned}$$

The first term of this right-hand side tends to zero as h goes to zero by the previous Theorem. On the other hand for the second term, as $\tilde{A}_{l,h}$ satisfies (2.7.2) (see Section 2.10), there exists $C > 0$ (independent of h) such that for all $h < h^*$

$$\|(\tilde{A}_{l,h})^{-1}(0, j_h g - \pi_h g)^\top\|_X \leq C\|j_h g - \pi_h g\|_H.$$

Hence, by the triangle inequality and the property $\|g - j_h g\|_H \leq \|g - \pi_h g\|_H$ (as j_h in the projection on V_h in H), we get

$$\|(\tilde{A}_{l,h})^{-1}(0, j_h g - \pi_h g)^\top\|_X \leq 2C\|g - \pi_h g\|_H.$$

By the estimate (2.4.5), we then conclude that this second term tends also to zero as h goes to zero.

If $z = (f, g)^\top$ is only in $V \times H$, then for an arbitrary $\varepsilon > 0$, we use the density of $\mathcal{D}(A) \times \mathcal{D}(A)$ into $V \times H$ to get $(F, G)^\top \in \mathcal{D}(A) \times \mathcal{D}(A)$ such that

$$\|(f, g)^\top - (F, G)^\top\|_X \leq \varepsilon.$$

Now, by the triangle inequality, we have

$$\begin{aligned} \|(\tilde{A}_{l,h})^{-1}(\pi_h f, j_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X &\leq \|(\tilde{A}_{l,h})^{-1}(\pi_h(f-F), j_h(g-G))^\top\|_X \\ &+ \|\tilde{A}^{-1}(f-F, g-G)^\top\|_X \\ &+ \|(\tilde{A}_{l,h})^{-1}(\pi_h F, j_h G)^\top - \tilde{A}^{-1}(F,G)^\top\|_X. \end{aligned}$$

By the first step, there exists h_ε small enough such that

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h F, j_h G)^\top - \tilde{A}^{-1}(F,G)^\top\|_X \leq \varepsilon, \forall 0 < h < h_\varepsilon.$$

For the second term, by the boundedness of $\tilde{A}^{-1}$, we may write

$$\|\tilde{A}^{-1}(f-F,g-G)^\top\|_X \leq C\|(f-F,g-G)^\top\|_X \leq C\varepsilon.$$

Finally for the first term, using the property (2.7.2) and the fact that π_h (resp. j_h) is a projection from V (resp. from H) into V_h, we get for all $h < h^*$

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h(f-F),j_h(g-G))^\top\|_X \leq C\|(\pi_h(f-F),j_h(g-G))^\top\|_X \leq C\|(f-F,g-G)^\top\|_X \leq C\varepsilon.$$

All together we have obtained that

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h f, j_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X \leq (1+2C)\varepsilon, \quad \forall 0 < h < \min\{h_\varepsilon, h^*\}.$$

This proves the result. ∎

We are now ready to state the convergence result.

Theorem 2.11.8. *If* $(\omega_0,\omega_1)^\top \in V \times H$, *then*

$$\|T_{l,h}(t)(\pi_h\omega_0, j_h\omega_1)^\top - T(t)(\omega_0,\omega_1)^\top\|_X \to 0 \text{ as } h \to 0. \tag{2.11.8}$$

Proof: We use Theorem 2.1 of [45] or Theorem 2.11.1 with $X = Z = V \times H$, $X_n = V_h \times V_h$, and $P_n : X \to X_n$ defined by

$$P_n(f,g)^\top = (\pi_h f, j_h g)^\top, \forall (f,g)^\top \in X,$$

and $E_n = P_n^*$ that here is the canonical injection of $V_h \times V_h$ into $V \times H$. The assumptions (A1) and (A3) of [45] are trivially satisfied, while the assumption (A2) is a consequence of (2.4.4), (2.4.5) and the density of $\mathcal{D}(A) \times \mathcal{D}(A)$ into $V \times H$.

Since Corollary 2.11.7 shows that condition (a) of Theorem 2.11.1 holds with $\lambda_0 = 0 \in \rho(\tilde{A}) \cap \bigcap_h \rho(\tilde{A}_{l,h})$, we conclude that condition (b) of this Theorem, namely (2.11.8), holds.

∎

Remark 2.11.9. *In case $l = 0$, we can still apply the Trotter-Kato Theorem to get the convergence of the discrete problem (2.4.6) towards the continuous one (2.1.13). Indeed, similar to Lemma 2.11.4, we have for $f \in D(A)$, $h^\theta \|A_h \pi_h f\|_H \leq h^\theta \|f\|_{D(A)}$. Moreover, in the proof of Theorem 2.11.6, we get $r_h = h^\theta A_h \pi_h f$, $\|u - u_h\|_V \leq ch^\theta \|(f,g)\|_{D(A)\times D(A)}$, and $\|v - v_h\|_H \leq ch^{2\theta} \|(f,g)\|_{D(A)\times D(A)}$.*

2.12 Examples

2.12.1 Two coupled wave equations

We consider the following system of [3] given by

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha y(x,t) + \beta(x) u_t(x,t) = 0 & \text{in } (0,1)\times\mathbb{R}_+, \\ y_{tt}(x,t) - y_{xx}(x,t) + \alpha u(x,t) + \gamma(x) y_t(x,t) = 0 & \text{in } (0,1)\times\mathbb{R}_+, \\ u(0,t) = u(1,t) = y(0,t) = y(1,t) = 0 & \forall t > 0, \\ u(\cdot,0) = u_0, u_t(\cdot,0) = u_1, y(\cdot,0) = y_0, y_t(\cdot,0) = y_1 & \text{in } (0,1), \end{cases} \tag{2.12.1}$$

when $\alpha \in \mathbb{R}$ such that $\alpha > 0$ is small enough (see below), β and γ are two non-negative bounded functions such that $\beta(x) \geq \beta > 0$ for $x \in I_\beta \subseteq (0,1)$ and $\gamma(x) \geq \gamma > 0$ for $x \in I_\gamma \subseteq (0,1)$ where I_β and I_γ are two open sets such that their measures do not vanish simultaneously. Hence, (2.12.1) is written in the form (2.1.13) with the following choices : Take $H = L^2(0,1)^2$, the operator B as follows :

$$B\omega = \sqrt{\beta(.)}\begin{pmatrix} u \\ 0 \end{pmatrix} + \sqrt{\gamma(.)}\begin{pmatrix} 0 \\ y \end{pmatrix}, \tag{2.12.2}$$

when $\omega = \begin{pmatrix} u \\ y \end{pmatrix}$, which is a bounded operator from H into itself (i.e. $U = H$) and the operator A defined by

$$\mathcal{D}(A) = V \cap H^2(0,1)^2,$$

when $V = H_0^1(0,1)^2$ and

$$A\omega = \begin{pmatrix} -u_{xx} + \alpha y \\ -y_{xx} + \alpha u \end{pmatrix}.$$

If α is small enough, namely if $\alpha < \pi^2$, this operator A is a positive selfadjoint operator in H, since it is the Friedrichs extension of the triple (H, V, a), where the sesquilinear form a is defined by

$$a(\omega, \omega^*) = \int_0^1 (u_x(\overline{u^*})_x + y_x(\overline{y^*})_x + \alpha y\overline{u^*} + \alpha u\overline{y^*})\, dx, \forall \omega = \begin{pmatrix} u \\ y \end{pmatrix}, \omega^* = \begin{pmatrix} u^* \\ y^* \end{pmatrix} \in V.$$

Indeed a is clearly a continuous symmetric sesquilinear form on V and is coercive if $\alpha < \pi^2$ due to Poincaré's inequality

$$\int_0^1 |u_x|^2\, dx \geq \pi^2 \int_0^1 |u|^2\, dx, \quad \forall u \in H_0^1(0,1).$$

Furthermore, A has a compact resolvent since $\mathcal{D}(A)$ is compactly embedded into H. Let us now check that the generalized gap condition (2.3.1) and the assumptions (2.4.7) or (2.4.8) are satisfied for our system (2.12.1). We start by the determination of the spectrum of the operator A. Hence we are looking for $\omega = (u, y)^\top \in V \cap H^2(0,1)^2$ different from 0 and $\lambda^2 > 0$ solution of

$$-u_{xx} + \alpha y = \lambda^2 u \text{ in } (0,1),$$
$$-y_{xx} + \alpha u = \lambda^2 y \text{ in } (0,1).$$

If such a pair exists, we can set

$$s = \frac{u+y}{2}, \quad d = \frac{u-y}{2},$$

and notice that s and d belong to $H_0^1(0,1) \cap H^2(0,1)$ and are solution of

$$-s_{xx} + \alpha s = \lambda^2 s \text{ in } (0,1),$$
$$-d_{xx} - \alpha d = \lambda^2 d \text{ in } (0,1).$$

Hence s (resp. d) is an eigenvector of the Laplace operator $-\frac{d}{dx^2}$ with Dirichlet boundary condition of eigenvalue $\lambda^2-\alpha$ (resp. $\lambda^2+\alpha$). A first choice is then to have for all $k\in\mathbb{N}^*$: $\lambda^2=k^2\pi^2+\alpha$, $s=\sin(k\pi\cdot)$ and $d=0$. Coming back to (u,y), we find (since $u=s+d$ and $y=s-d$) a sequence of eigenvalues $\lambda^2_{+,k}=k^2\pi^2+\alpha$ of associated eigenvector

$$\omega_{+,k}=(\sin(k\pi\cdot),\sin(k\pi\cdot)).$$

Note that each eigenvalue is simple and that $\omega_{+,k}$ is of norm 1 in H.

A second choice is to take for all $k\in\mathbb{N}^*$: $\lambda^2=k^2\pi^2-\alpha$ (which is meaningful since $\alpha<\pi^2$), $s=0$ and $d=\sin(k\pi\cdot)$. Again coming back to (u,y), we find a sequence of eigenvalues $\lambda^2_{-,k}=k^2\pi^2-\alpha$ of associated eigenvector

$$\omega_{-,k}=(\sin(k\pi\cdot),-\sin(k\pi\cdot)).$$

As before each eigenvalue is simple and $\omega_{-,k}$ is of norm 1 in H.

Now we remark that the sequence $\{\omega_{+,k}\}_{k\in\mathbb{N}^*}\cup\{\omega_{-,k}\}_{k\in\mathbb{N}^*}$ is an orthonormal basis of H (because $\omega_{+,k}+\omega_{-,k}=2(\sin(k\pi\cdot),0)$ and $\omega_{+,k}-\omega_{-,k}=2(0,\sin(k\pi\cdot)))$ and therefore we have found all possible eigenvectors of A. We have then shown that the spectrum of A is given by

$$\mathrm{Sp}(A)=\{\lambda^2_{+,k}\}_{k\in\mathbb{N}^*}\cup\{\lambda^2_{-,k}\}_{k\in\mathbb{N}^*},$$

and that each eigenvalue is simple (because the assumption $\alpha<\pi^2$ implies that $k^2\pi^2+\alpha<(k+1)^2\pi^2-\alpha$).

We now need to estimate the distance between the consecutive eigenvalues of $A^{1/2}$. We have two different cases to consider :

1. For all $k\in\mathbb{N}^*$, we need to look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k}$. Since

$$\lambda_{+,k}-\lambda_{-,k}=\sqrt{k^2\pi^2+\alpha}-\sqrt{k^2\pi^2-\alpha}=\frac{2\alpha}{\sqrt{k^2\pi^2+\alpha}+\sqrt{k^2\pi^2-\alpha}},$$

we see that this distance goes to zero as k goes to infinity.

2. For all $k \in \mathbb{N}^*$, we look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k+1}$. Here we have

$$\lambda_{-,k+1} - \lambda_{+,k} = \sqrt{(k+1)^2\pi^2 - \alpha} - \sqrt{k^2\pi^2 + \alpha} = \frac{2k\pi^2 + \pi^2 - 2\alpha}{\sqrt{(k+1)^2\pi^2 - \alpha} + \sqrt{k^2\pi^2 + \alpha}},$$

which tends to π as k goes to infinity.

This shows that the generalized gap condition (2.3.1) is satisfied with $M = 2$. Hence, we see that $A_1 = \emptyset$ and $A_2 = \mathbb{N}^*$.

In order to check (2.4.7) or (2.4.8), for all $k \in \mathbb{N}^*$, we set

$$\alpha_k = \lambda_{+,k} - \lambda_{-,k},$$

that behaves like k^{-1} or equivalently like $\lambda_{-,k}^{-1}$. We further need to use the matrix (see Lemma 2.3.1)

$$B_k^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & \alpha_k \end{pmatrix},$$

as well as the matrix Φ_k which here takes the form

$$\Phi_k = \begin{pmatrix} B^*\omega_{-,k} & 0 \\ 0 & B^*\omega_{+,k} \end{pmatrix}.$$

Hence for all $C = (c_1, c_2)^\top \in \mathbb{R}^2$, we have

$$B_k^{-1}\Phi_k C = \begin{pmatrix} c_1 B^*\omega_{-,k} + c_2 B^*\omega_{+,k} \\ \alpha_k c_2 B^*\omega_{+,k} \end{pmatrix},$$

and consequently

$$\begin{aligned} \|B_k^{-1}\Phi_k C\|_{U,2}^2 &= \|c_1 B^*\omega_{-,k} + c_2 B^*\omega_{+,k}\|_2^2 + |\alpha_k|^2|c_2|^2\|B^*\omega_{+,k}\|_2^2 \\ &= |c_1 + c_2|^2 \int_0^1 \beta(x)\sin^2(k\pi x)dx + |c_2 - c_1|^2 \int_0^1 \gamma(x)\sin^2(k\pi x)dx \\ &\quad + |\alpha_k|^2|c_2|^2 \int_0^1 (\beta(x) + \gamma(x))\sin^2(k\pi x)dx. \end{aligned}$$

We have two different cases to consider :

First case : $I_\beta \neq \emptyset$ and $I_\gamma \neq \emptyset$.

In this case, we have

$$\begin{aligned}
&\|B_k^{-1}\Phi_k C\|_{U,2}^2 \\
&\geq \min\{\beta,\gamma\} \min\left\{\int_{I_\beta} \sin^2(k\pi x)dx, \int_{I_\gamma} \sin^2(k\pi x)dx\right\}((c_1+c_2)^2+(c_2-c_1)^2) \\
&= 2\min\{\beta,\gamma\} \min\left\{\int_{I_\beta} \sin^2(k\pi x)dx, \int_{I_\gamma} \sin^2(k\pi x)dx\right\}(c_1^2+c_2^2)
\end{aligned}$$

and hence (2.4.7) holds since $\min\left\{\int_{I_\beta} \sin^2(k\pi x)dx, \int_{I_\gamma} \sin^2(k\pi x)dx\right\}$ is uniformly bounded from below. Indeed, as $I_\gamma \neq \emptyset$, there exists $a \in (0,1)$ and $\epsilon > 0$ such that $(a, a+\epsilon) \subset I_\gamma$, and therefore

$$\int_{I_\gamma} \sin^2(k\pi x)dx \geq \frac{\epsilon}{2} + \frac{1}{4k\pi}\left(\sin(2k\pi a) - \sin(2k\pi(a+\epsilon))\right) \geq \frac{\epsilon}{2} - \frac{1}{2k\pi} \geq \frac{\epsilon}{4},$$

for $k \geq \dfrac{2}{\epsilon\pi}$. On the other hand, we clearly have

$$\min_{1\leq k<\frac{2}{\epsilon\pi}} \int_{I_\gamma} \sin^2(k\pi x)dx > 0,$$

which shows that $\int_{I_\gamma} \sin^2(k\pi x)dx$ is uniformly bounded from below.

Second case : $I_\beta = \emptyset$ or $I_\gamma = \emptyset$ (but not empty together). For instance, suppose that $I_\beta = \emptyset$ and $I_\gamma \neq \emptyset$.

As $|\alpha_k| \sim \lambda_{-,k}^{-1}$, we deduce that

$$\|B_k^{-1}\Phi_k C\|_{U,2} \geq \alpha_0 \lambda_{-,k}^{-1}\|C\|_2,$$

for a positive constant α_0, and shows that (2.4.8) holds with $l = 1$.

As stated before, in the first case the system (2.12.1) is exponentially stable, while in the second case (2.12.1) is polynomially stable. We refer to Theorem 2.4 of [7] or to [5, 61] for the proof of these results.

As approximated space V_h, we use the standard one based on $P1$ finite elements. More precisely, for $N \in \mathbb{N}$ and $h = \frac{1}{N+1}$, we define the points $x_j = jh$, $j = 0, 1, ..., N+1$. The space V_h is the linear span of the family of hat functions $(e_i, e_j)_{i,j\in\{1,...,N\}}$ such that

$$e_j(x) = \left[1 - \frac{|x - x_j|}{h}\right]^+, \text{ for } j = 1, ..., N.$$

Then, we define the operators A_h and B_h by (2.4.1) and (2.4.3). It is well-known (see [22]) that the operator A and the space V_h satisfy conditions (2.4.4) and (2.4.5) with $\theta = 1$.

Consequently, in the first case ($I_\beta \neq \emptyset$ and $I_\gamma \neq \emptyset$), we can apply Theorem 2.4.1 and thus the family of systems (2.4.6) is uniformly exponentially stable, in the sense that there exist constants M, α, $h^* > 0$ (independent of h, u_{0h}, u_{1h}, y_{0h}, y_{1h}) such that for all $h \in (0, h^*)$:

$$\|\dot{\omega}_h(t)\|^2 + a(\omega_h(t), \omega_h(t)) \leq Me^{-\alpha t}(\|\omega_{1h}\|^2 + a(\omega_{0h}, \omega_{0h})), \forall t \geq 0,$$

where $\omega_h = (u_h, y_h)$, and $\omega_{0h} = (u_{0h}, y_{0h}) \in V_h$ (resp. $\omega_{1h} = (u_{1h}, y_{1h}) \in V_h$) is an approximation of $\omega_0 = (u_0, y_0)$ (resp. $\omega_1 = (u_1, y_1)$).

In the second case ($I_\beta = \emptyset$ and $I_\gamma \neq \emptyset$), we can apply Theorem 2.4.4 and Remark 2.10.1 with $l = 1$ and thus the family of systems (2.4.9) is uniformly polynomially stable, in the sense that, there exist constants C, $h^* > 0$ (independent of h, u_{0h}, u_{1h}, y_{0h}, y_{1h}) such that for all $h \in (0, h^*)$:

$$\left\|(I + hA_h)^{-1}\dot{\omega}_h(t)\right\|^2 + a(\omega_h(t), \omega_h(t)) \leq \frac{C}{t}\|(\omega_{0h}, \omega_{1h})\|^2_{D(\tilde{A}_{1,h})} \forall t > 0, \quad (2.12.3)$$

where $\tilde{A}_{1,h}$ is given as in (2.5.1) with $l = 1$, $\theta = 1$, and the the graph norm $\|\cdot\|_{D(\tilde{A}_{1,h})}$

is defined by

$$\|(\omega_{0h},\omega_{1h})\|^2_{D(\tilde{A}_{1,h})} = \|(\omega_{0h},\omega_{1h})\|^2_{X_h} + \|\tilde{A}_{1,h}(\omega_{0h},\omega_{1h})\|^2_{X_h}.$$

2.12.2 Two boundary coupled wave equations

We consider the following system

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } (0,1)\times\mathbb{R}_+, \\ y_{tt} - y_{xx} + \beta y_t = 0 & \text{in } (0,1)\times\mathbb{R}_+, \\ u(0,t) = y(0,t) = 0 & \forall t > 0, \\ y_x(1,t) = \alpha u(1,t) & \forall t > 0, \\ u_x(1,t) = \alpha y(1,t) & \forall t > 0, \\ u(\cdot,0) = 0, u_t(\cdot,0) = u_1, y(\cdot,0) = 0, y_t(\cdot,0) = y_1 & \text{in } (0,1), \end{cases} \tag{2.12.4}$$

when $\alpha,\beta \in \mathbb{R}$ with $\beta > 0$ and $\alpha > 0$ small enough (see below). Hence it is written in the form (2.1.13) with the following choices : Take $H = L^2(0,1)^2$, the operator B as follows :

$$B\omega = \sqrt{\beta}\begin{pmatrix} 0 \\ y \end{pmatrix},$$

when $\omega = \begin{pmatrix} u \\ y \end{pmatrix}$, which is a bounded operator from H into itself (i.e. $U = H$) and the operator A defined by

$$\mathcal{D}(A) = \left\{(u,y) \in V \cap H^2(0,1)^2 : \, y_x(1) = \alpha u(1); \, u_x(1) = \alpha y(1)\right\}$$

when $V = \{\omega \in H^1(0,1)^2 : \, \omega(0) = 0\}$ and

$$A\omega = \begin{pmatrix} -u_{xx} \\ -y_{xx} \end{pmatrix}.$$

If α is small enough, namely if $\alpha < 1$, this operator A is a positive selfadjoint operator in H, since it is the Friedrichs extension of the triple (H, V, a), where the sesquilinear form a is defined by

$$a(\omega, \omega^*) \;=\; \int_0^1 (u_x(\overline{u^*})_x + y_x(\overline{y^*})_x)\, dx - \alpha u(1)\overline{y^*}(1) - \alpha\overline{u^*}(1)y(1),$$

for all $\omega = \begin{pmatrix} u \\ y \end{pmatrix}, \omega^* = \begin{pmatrix} u^* \\ y^* \end{pmatrix} \in V$. Indeed a is clearly a continuous symmetric sesquilinear form on V and is coercive if $\alpha < 1$ due to the trace theorem

$$u(1)^2 \leq \int_0^1 |u_x|^2\, dx, \quad \forall u \in V.$$

In addition to that, the operator A admits a compact resolvent as $\mathcal{D}(A)$ is compactly embedded in H.

Let us now check that the generalized gap condition (2.3.1) and the assumption (2.4.8) are satisfied for our system (2.12.4). We start by the determination of the spectrum of the operator A. Hence we are looking for $\omega = (u, y)^\top \in \mathcal{D}(A)$ different from 0 and $\lambda^2 > 0$ solution of

$$-u_{xx} = \lambda^2 u \text{ in } (0,1),$$
$$-y_{xx} = \lambda^2 y \text{ in } (0,1).$$

Then

$$u(x) = a\sin(\lambda x) \text{ in } (0,1),$$
$$y(x) = b\sin(\lambda x) \text{ in } (0,1).$$

The coupling condition in (2.12.4) gives

$$\begin{cases} a\lambda\cos\lambda = \alpha b\sin\lambda \\ b\lambda\cos\lambda = \alpha a\sin\lambda. \end{cases}$$

Since it is not possible to have $\sin\lambda = 0$ (otherwise $a = b = 0$), we obtain

$$a = \frac{b\lambda\cos\lambda}{\alpha\sin\lambda}, \tag{2.12.5}$$

and then

$$\tan\lambda = \pm\frac{\lambda}{\alpha}, \tag{2.12.6}$$

because $b \neq 0$ (otherwise $u = y = 0$).

We then have two sequences of eigenvalues defined by

$$\lambda_{-,k} = \frac{\pi}{2} + k\pi - \epsilon_{-,k}$$

with $\lim_{k\to+\infty}\epsilon_{-,k} = 0$ and $\epsilon_{-,k} > 0$ for all $k \in \mathbb{N}$, and

$$\lambda_{+,k} = \frac{\pi}{2} + k\pi + \epsilon_{+,k}$$

with $\lim_{k\to+\infty}\epsilon_{+,k} = 0$ and $\epsilon_{+,k} > 0$ for all $k \in \mathbb{N}$. Moreover as $\lambda_{-,k}$ and $\lambda_{+,k}$ satisfies (2.12.6), we can verify that

$$\epsilon_{-,k} = \arctan\left(\frac{\alpha}{\lambda_{-,k}}\right) \text{ and } \epsilon_{+,k} = \arctan\left(\frac{\alpha}{\lambda_{+,k}}\right).$$

By (2.12.5) and (2.12.6), the eigenvector associated with the eigenvalue $\lambda_{+,k}$ is given by

$$\omega_{+,k} = b_{+,k}\sin(\lambda_{+,k}\cdot)(-1,\,1)^T,$$

and the eigenvector associated with the eigenvalue $\lambda_{-,k}$ is given by

$$\omega_{-,k} = b_{-,k}\sin(\lambda_{-,k}\cdot)(1,\,1)^T,$$

where $b_{+,k}$, $b_{-,k}$ are chosen to normalize the eigenvectors.

Since we have found all possible eigenvectors of A, we have shown that the spectrum of A is given by

$$\mathrm{Sp}(A) = \{\lambda_{+,k}^2\}_{k\in\mathbb{N}^*} \cup \{\lambda_{-,k}^2\}_{k\in\mathbb{N}^*},$$

and that each eigenvalue is simple.

We again need to estimate the distance between the consecutive eigenvalues of $A^{1/2}$ and as before we consider two different cases :

1. For all $k \in \mathbb{N}^*$, we need to look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k}$. Since

$$\lambda_{+,k} - \lambda_{-,k} = \epsilon_{+,k} + \epsilon_{-,k} = \arctan\left(\frac{\alpha}{\lambda_{+,k}}\right) + \arctan\left(\frac{\alpha}{\lambda_{-,k}}\right),$$

we see that this distance goes to zero as k goes to infinity.

2. For all $k \in \mathbb{N}^*$, we look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k+1}$. Here we have

$$\lambda_{-,k+1} - \lambda_{+,k} = \pi - (\epsilon_{+,k} + \epsilon_{-,k+1}),$$

which tends to π as k goes to infinity.

This shows that the generalized gap condition (2.3.1) is satisfied with $M = 2$.

In order to check (2.4.8), for all $k \in \mathbb{N}^*$, we set

$$\alpha_k = \lambda_{+,k} - \lambda_{-,k},$$

that behaves like k^{-1} or equivalently like $\lambda_{-,k}^{-1}$. As in the previous subsection for all $C = (c_1, c_2)^\top \in \mathbb{R}^2$, we have

$$B_k^{-1}\Phi_k C = \begin{pmatrix} c_1 B^*\omega_{-,k} + c_2 B^*\omega_{+,k} \\ \alpha_k c_2 B^*\omega_{+,k} \end{pmatrix},$$

and consequently

$$\begin{aligned} \|B_k^{-1}\Phi_k C\|_{U,2}^2 &= \|c_1 B^*\omega_{-,k} + c_2 B^*\omega_{+,k}\|_H^2 + |\alpha_k|^2|c_2|^2\|B^*\omega_{+,k}\|_H^2 \\ &= \beta\int_0^1 \left(b_{-,k}c_1\sin(\lambda_{-,k}x) + b_{+,k}c_2\sin(\lambda_{+,k}x)\right)^2 dx \\ &\quad +\beta|\alpha_k|^2|c_2|^2 b_{+,k}^2\int_0^1 \sin^2(\lambda_{+,k}x)dx. \end{aligned}$$

By using Young's inequality with $\epsilon > 0$ and the fact that the eigenvectors are normalized (by the choice of $b_{\pm,k}$), we obtain

$$\begin{aligned}\|B_k^{-1}\Phi_k C\|_{U,2}^2 &\geq \beta\left(1-\frac{1}{\epsilon}\right)c_1^2 b_{-,k}^2\int_0^1 \sin^2(\lambda_{-,k}x)dx + \beta\left(1-\epsilon\right)c_2^2 b_{+,k}^2\int_0^1 \sin^2(\lambda_{+,k}x)dx \\ &\quad +\beta|\alpha_k|^2|c_2|^2 b_{+,k}^2\int_0^1 \sin^2(\lambda_{+,k}x)dx \\ &= \frac{\beta}{2}\left(\left(1-\frac{1}{\epsilon}\right)c_1^2 + \left(1+\alpha_k^2-\epsilon\right)c_2^2\right).\end{aligned}$$

We then take $\epsilon = 1+\alpha_k^2/2$, which implies

$$1+\alpha_k^2-\epsilon = \frac{\alpha_k^2}{2} \quad \text{and} \quad 1-\frac{1}{\epsilon} > \frac{\alpha_k^2}{4},$$

(since $\alpha_k^2 < 2$). Consequently

$$\|B_k^{-1}\Phi_k C\|_{U,2}^2 \geq \frac{\beta}{8}\alpha_k^2(c_1^2+c_2^2).$$

As $|\alpha_k| \sim \lambda_{-,k}^{-1}$, we deduce that

$$\|B_k^{-1}\Phi_k C\|_{U,2} \geq \alpha_0\lambda_{-,k}^{-1}\|C\|_2,$$

for a positive constant α_0, and shows that (2.4.8) holds with $l = 1$.

We construct the space V_h like in the previous subsection, i.e. it is the span of $(e_i, e_j)_{i,j\in\{1,\ldots,N+1\}}$, that still satisfies (2.4.4) and (2.4.5) with $\theta = 1$.

Consequently, we can apply Theorem 2.4.4 and Remark 2.10.1 with $l = 1$ and thus the family of systems (2.4.9) is uniformly polynomially stable, in the sense that the estimate (2.12.3) holds.

2.12.3 A more general wave type system

We consider the following more general system : let $\omega = (\omega_1, \cdots, \omega_N)^T$ be a solution of

$$\begin{cases} \omega_{tt} - \omega_{xx} + M\omega + BB^*\omega_t = 0 & \text{in } (0,1)^N \times \mathbb{R}_+, \\ \omega(0,t) = \omega(1,t) = 0 & \forall t > 0, \\ \omega(\cdot,0) = \omega^{(0)},\ \omega_t(\cdot,0) = \omega^{(1)} & \text{in } (0,1)^N, \end{cases} \tag{2.12.7}$$

where $M \in \mathcal{M}_N(\mathbb{R})$ is symmetric and such that $A_0 + M$ is positive definite in $H = L^2(0,1)^N$, when A_0 is the operator of domain $\mathcal{D}(A_0) = H_0^1(0,1)^N \cap H^2(0,1)^N$ and such that $A_0 u = -u_{xx}$, for all $u \in \mathcal{D}(A_0)$; $B \in \mathcal{L}(U, H)$, with U a complex Hilbert space.

Hence, it is written in the form (2.1.13) with the self-adjoint positive operator A defined by $A = A_0 + M$ and $\mathcal{D}(A) = \mathcal{D}(A_0) = V \cap H^2(0,1)^N$, when $V = H_0^1(0,1)^N$. We remark that A admits a compact resolvent since $\mathcal{D}(A)$ is compactly embedded into H.

As M is symmetric, M can be diagonalized by an orthogonal matrix, i.e. there exist a real orthogonal matrix O and a diagonal matrix D such that $O^T M O = D$. We denote by d_i $(i = 1, \cdots, N)$ the coefficients of the diagonal matrix D.

We start by the determination of the spectrum of the operator A. Hence we are looking for $\omega \in V \cap H^2(0,1)^N$ different from 0 and $\lambda^2 > 0$ solution of

$$-\omega_{xx} + M\omega = \lambda^2 \omega.$$

If we denote by $U = O^T \omega$, then $U = (u_1, \cdots, u_N)^T$ satisfies

$$-U_{xx} + DU = \lambda^2 U,$$

which is equivalent to

$$-\frac{d^2}{dx^2} u_i = (\lambda^2 - d_i) u_i, \quad \text{in } (0,1), \quad \forall i = 1, \cdots, N.$$

Hence there exists $c_i \in \mathbb{C}$ such that

$$u_i = \sqrt{2} c_i \sin(k\pi .), \quad \lambda_{i,k}^2 = k^2\pi^2 + d_i, \quad i = 1, \cdots, N.$$

Therefore we have found N families of eigenvectors and eigenvalues :

$$U_{i,k} = \sqrt{2} f_i \sin(k\pi .), \quad \lambda_{i,k}^2 = k^2\pi^2 + d_i, \quad i = 1, \cdots, N,$$

where $(f_i)_{i\in\{1,\cdots,N\}}$ is the canonical basis of $\mathbb{C}^N$. Coming back to the initial eigenvalue problem, we have N families of eigenvectors given by

$$\omega_{i,k} = OU_{i,k}, \quad i = 1, \cdots, N, \tag{2.12.8}$$

and the spectrum of A is given by

$$\mathrm{Sp}(A) = \{\lambda_{1,k}^2\}_{k\in\mathbb{N}^*} \cup \cdots \cup \{\lambda_{N,k}^2\}_{k\in\mathbb{N}^*}.$$

For simplicity, we now assume that all d_i are different and, for instance that

$$d_1 < d_2 < \cdots < d_N.$$

We still have to estimate the distance between the consecutive eigenvalues of $A^{1/2}$:

1. For all $k \in \mathbb{N}^*$, we need to look at the distance between $\lambda_{i,k}$ and $\lambda_{j,k}$ $(i \neq j)$. Since

$$\lambda_{i,k} - \lambda_{j,k} = \sqrt{k^2\pi^2 + d_i} - \sqrt{k^2\pi^2 + d_j} = \frac{d_i - d_j}{\sqrt{k^2\pi^2 + d_i} + \sqrt{k^2\pi^2 + d_j}},$$

we see that this distance goes to zero as k goes to infinity.

2. For all $k \in \mathbb{N}^*$, we look at the distance between $\lambda_{N,k}$ and $\lambda_{1,k+1}$. Here we have

$$\lambda_{1,k+1} - \lambda_{N,k} = \sqrt{(k+1)^2\pi^2 + d_1} - \sqrt{k^2\pi^2 + d_N} = \frac{2k\pi^2 + \pi^2 + d_1 - d_N}{\sqrt{(k+1)^2\pi^2 + d_1} + \sqrt{k^2\pi^2 + d_N}},$$

which tends to π as k goes to infinity.

This shows that the generalized gap condition (2.3.1) is satisfied with $M = N$. With the terminology of Section 1, we see that $A_1 = \cdots = A_{N-1} = \emptyset$ and $A_N = \mathbb{N}^*$. Hence, for $N > 1$, our previous results will allow to obtain stability results for system (2.12.7).

If the eigenvalues are simple (a necessary condition is that all d_i are different), then in order to verify (2.4.7) or (2.4.8), we have to bound from below $\left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2$ with $C = (c_1, \cdots, c_N) \in \mathbb{R}^N$, B_k^{-1} defined in Lemma 2.3.1 and Φ_k given by

$$\Phi_k = \begin{pmatrix} B^*\omega_{1,k} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B^*\omega_{N,k} \end{pmatrix}.$$

Such a lower bound can only be made on some particular examples.

Note that, if $N = 2$, B is defined by (2.12.2) and

$$M = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\alpha > 0$, then we are back to the setting of Subsection 2.12.1. Indeed M is symmetric with $A_0 + M$ positive definite for α small enough, and diagonalized by the orthogonal matrix

$$O = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \text{ (with } D = \alpha \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\text{)}.$$

We then finish this subsection by considering another example. Take $N = 3$ and

$$B \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \sqrt{\beta} \begin{pmatrix} \omega_1 \\ 0 \\ 0 \end{pmatrix} + \sqrt{\gamma} \begin{pmatrix} 0 \\ \omega_2 \\ 0 \end{pmatrix} + \sqrt{\delta} \begin{pmatrix} 0 \\ 0 \\ \omega_3 \end{pmatrix},$$

with non negative real numbers β, γ, δ, which is a bounded operator from H into itself (i.e. $U = H$). We chose the matrix M defined by

$$M = \alpha \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \alpha > 0$$

which is obviously symmetric. As previously we can verify that $A_0 + M$ is positive definite if $\alpha < \pi^2/2$. Moreover M can be diagonalized by the orthogonal matrix

$$O = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix},$$

into

$$D = \begin{pmatrix} -\sqrt{2}\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}\alpha \end{pmatrix}.$$

Then the spectrum of $A = A_0 + M$ is given by

$$\mathrm{Sp}(A) = \{k^2\pi^2 - \sqrt{2}\alpha\}_{k\in\mathbb{N}^*} \cup \{k^2\pi^2\}_{k\in\mathbb{N}^*} \cup \{k^2\pi^2 + \sqrt{2}\alpha\}_{k\in\mathbb{N}^*},$$

and the eigenvalues are simple (because the assumption $\alpha < \pi^2/2$ implies that $k^2\pi^2 + \sqrt{2}\alpha < (k+1)^2\pi^2 - \sqrt{2}\alpha$). Moreover, as we have shown previously, the generalized gap condition (2.3.1) is satisfied with $M = 3$. Thanks to (2.12.8) the normalized eigenvectors are given by

$$\omega_{1,k} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \sin(k\pi\cdot), \qquad \omega_{2,k} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix} \sin(k\pi\cdot),$$

$$\omega_{3,k} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \sin(k\pi\cdot).$$

We set

$$\alpha_k^{(1,2)} = \lambda_{2,k} - \lambda_{1,k}, \quad \alpha_k^{(1,3)} = \lambda_{3,k} - \lambda_{1,k}, \quad \alpha_k^{(2,3)} = \lambda_{3,k} - \lambda_{2,k}.$$

Therefore, for all $C = (c_1, c_2, c_3)^T \in \mathbb{R}^3$, we have

$$\begin{aligned}
&\left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2 \\
= &\left\| \begin{pmatrix} 1 & 1 & 1 \\ 0 & \alpha_k^{(1,2)} & \alpha_k^{(1,3)} \\ 0 & 0 & \alpha_k^{(1,3)}\alpha_k^{(2,3)} \end{pmatrix} \begin{pmatrix} B^*\omega_{1,k} & 0 & 0 \\ 0 & B^*\omega_{2,k} & 0 \\ 0 & 0 & B^*\omega_{3,k} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right\|_{U,2}^2 \\
= &\left\|c_1 B^*\omega_{1,k} + c_2 B^*\omega_{2,k} + c_3 B^*\omega_{3,k}\right\|_H^2 + \left\|c_2\alpha_k^{(1,2)} B^*\omega_{2,k} + c_3\alpha_k^{(1,3)} B^*\omega_{3,k}\right\|_H^2 \\
&+ |c_3|^2 \left|\alpha_k^{(1,3)}\alpha_k^{(2,3)}\right|^2 \left\|B^*\omega_{3,k}\right\|_H^2 .
\end{aligned}$$

After some calculations, we obtain

$$\begin{aligned}
\left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2 &= \tfrac{\beta}{4}(c_1 + \sqrt{2}c_2 + c_3)^2 + \tfrac{\gamma}{2}(c_3 - c_1)^2 + \tfrac{\delta}{4}(c_1 - \sqrt{2}c_2 + c_3)^2 \\
&+ \tfrac{\beta}{4}(\sqrt{2}\alpha_k^{(1,2)} c_2 + \alpha_k^{(1,3)} c_3)^2 + \tfrac{\gamma}{2}\left|c_3\alpha_k^{(1,3)}\right|^2 + \tfrac{\delta}{2}(-\sqrt{2}\alpha_k^{(1,2)} c_2 + \alpha_k^{(1,3)} c_3)^2 \\
&+ \tfrac{|c_3|^2}{2}\left|\alpha_k^{(1,3)}\alpha_k^{(2,3)}\right|^2 (\tfrac{\beta+\delta}{2} + \gamma).
\end{aligned}$$

Hence different decay results can be obtained for system (2.12.7) according to the values of β, γ and δ.

First if $\beta, \gamma, \delta > 0$, then we have

$$\left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2 \geq C(c_1^2 + c_2^2 + c_3^2)$$

for $C > 0$, which shows that (2.4.7) holds and therefore system (2.12.7) is exponentially stable.

Second if $\gamma = 0$ and $\beta, \delta > 0$, we have

$$\begin{aligned}
\left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2 &\geq \tfrac{\min\{\beta,\delta\}}{4}\Bigg(2c_1^2 + 4c_2^2 + 2c_3^2 + 4c_1c_3 + \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 (4c_2^2 + 2c_3^2) \\
&\quad + \min\left\{\alpha_k^{(1,3)}, \alpha_k^{(2,3)}\right\}^4 c_3^2\Bigg) \\
&\geq \tfrac{\min\{\beta,\delta\}}{4}\Bigg(\left(2 - \tfrac{2}{\epsilon}\right) c_1^2 + 4\left(1 + \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2\right) c_2^2 \\
&\quad + \left(2 - 2\epsilon + 2\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2\right) c_3^2\Bigg),
\end{aligned}$$

by Young's inequality with $\epsilon > 0$. We then take $\epsilon = 1 + \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2/2$, which implies

$$2 - \frac{2}{\epsilon} > \frac{\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2}{2}, \quad 2 - 2\epsilon + 2\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 = \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2,$$

if k is large enough. Consequently if k is large enough, we have obtained that

$$\begin{aligned}
\left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2 &\geq \tfrac{\min\{\beta,\delta\}}{4}\Bigg(\tfrac{\min\left\{\alpha_k^{(1,2)},\alpha_k^{(1,3)}\right\}^2}{2} c_1^2 + 4\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 c_2^2 \\
&\quad + \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 c_3^2\Bigg) \\
&\geq \tfrac{\min\{\beta,\delta\}}{8}\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 (c_1^2 + c_2^2 + c_3^2),
\end{aligned}$$

which shows that (2.4.8) holds with $l = 1$, since $\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 \sim \lambda_{1,k}^{-2}$.

We construct the space V_h like in the previous subsection, i.e. it is the span of $(e_i, e_j, e_k)_{i,j,k\in\{1,\ldots,N\}}$, that still satisfies (2.4.4) and (2.4.5) with $\theta = 1$.

Consequently, in the first case ($\beta, \gamma, \delta > 0$), we can apply Theorem 2.4.1 and thus the family of systems (2.4.6) is uniformly exponentially stable. In the second case ($\beta, \delta > 0$ and $\gamma = 0$), we can apply Theorem 2.4.4 and Remark 2.10.1 with $l = 1$ and thus the family of systems (2.4.9) is uniformly polynomially stable, in the sense that (2.12.3) holds.

2.13 Open problem

In this chapter, the stability results of the approximate systems have been studied when the control term B is bounded. The boundedness of the control B is used in (2.9.5) for the proof of Theorem 2.4.1 and in (2.10.6) for the proof of Theorem 2.4.4. An open question here is how we can handle the case when the control B is unbounded. Does the analysis in this chapter after adding suitable viscosity terms remain valid or do we have to search for another method ?

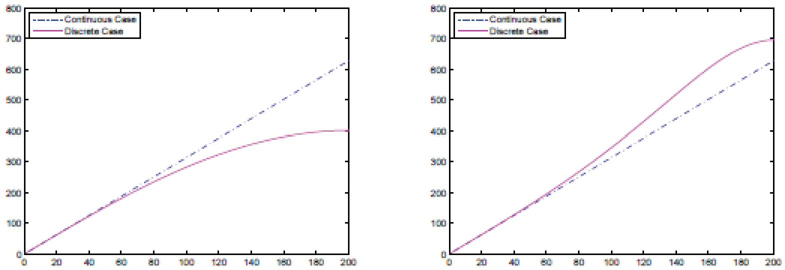

Fig. (a) Left: Square roots of the eigenvalues in the continuous and discrete cases (finite difference semidiscretization). The gaps are clearly independent of k in the continuous case and of order h for large k in the discrete one. Right: Dispersion diagram for the piecewise linear finite element space semidiscretization versus the continuous wave equation.

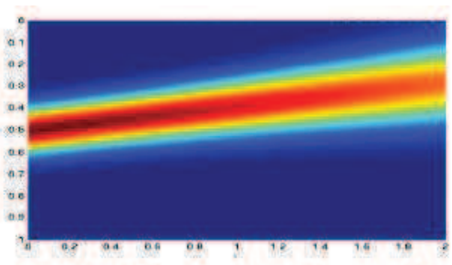

Fig. (b) A discrete wave packet and its propagation. In the horizontal axis we represent the time variable, varying between 0 and 2, and the vertical one the space variable x ranging from 0 to 1.

FIGURE 2.1 –

Chapitre 3

Spectral analysis and exponential or polynomial stability of some indefinite sign damped problems

3.1 Introduction and Historical background

Systems with indefinite sign damping terms arise in studying the nature of wind loads and their effect on dynamic response. This includes aircraft, buildings, telegraph wires and bridges. For instance, in an air craft, as the speed of the wind increases there may be a point at which structural damping is insufficient to damp out the vibratory motions which are increasing due to aerodynamic energy being added to the structure. The resulting vibrations can cause structural failure. Therefore, in this chapter, our aim is to find critical values of the damping term for which structural failure does not occur. More precisely, as in [1], we consider a one-dimensional wave equation with an indefinite sign damping and a zero order potential term which is either internally damped of the form

$$\begin{aligned} u_{tt}(x,t) - u_{xx}(x,t) + 2\chi_{(0,1)}(x)u_t(x,t) + 2\alpha\chi_{(-1,0)}(x)u_t(x,t) = 0,\ x \in (-1,1),\ t > 0, \\ u(1,t) = u(-1,t) = 0,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \end{aligned} \tag{3.1.1}$$

where α is a given constant or with both internally and boundary damped terms under the form

$$\begin{aligned} & u_{tt}(x,t) - u_{xx}(x,t) + au_t(x,t) = 0, && x \in (0,1),\ t > 0, \\ & u(0,t) = 0,\ u_x(1,t) = -bu_t(1,t), && t > 0, \\ & u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \end{aligned} \tag{3.1.2}$$

where $a, b \in \mathbb{R}$.

It is well known that problem (3.1.1) is exponentially stable if the damping term α is non-negative (see [23]). Similarly, if the coefficients a and b are both positive, then, using for instance integral inequalities, one can prove that (3.1.2) is also exponentially stable.

In this chapter or in [1], we are interested in the case when the damping terms are allowed to change their sign. As mentioned before, such a case occurs, for example, in wind induced oscillations. Problems (3.1.1) and (3.1.2) model the vibrations of flexible structures subject to aerodynamic forces. Our aim is to analyze to what extent the variation of the sign affects the stability of the problem. However, the techniques which are normally employed in the definite case, such as multipliers and resolvent methods cannot be well invoked in case of indefinite sign damping coefficients. Consequently, when the damping coefficients are allowed to change their sign, the question of stability of the solution becomes more interesting.

Such a question was first exposed in a conjecture in [21] by Chen et al. who considered the internally indefinite sign damped wave equation of the form

$$u_{tt} - u_{xx} + 2a(x)u_t = 0,\ x \in (0,1),\ t > 0, \tag{3.1.3}$$

with standard initial conditions and Dirichlet boundary conditions.

It was conjectured that if there exists some $c > 0$ such that for every $n \in \mathbb{N}^*$ the following condition is satisfied

$$I_n = \int_0^1 a(x) \sin^2(n\pi x) dx \geq c, \tag{3.1.4}$$

then, when the function $a \in L^\infty(0,1)$ has an indefinite sign, the energy decays exponentially. The idea of the conjecture is that once the damping term ia allowed to be more positive than negative, then the solution decays as time goes to infinity. The condition on I_n can be interpreted as some sort of positivity condition on the damping term $a(.)$ since $I_n \to a_0$, as $n \to +\infty$, where a_0 is the average of $a(.)$. In fact, problem (3.1.3) can be considered as a perturbation of an undamped problem. Therefore, for a small enough perturbation, the eigenvalues of the associated eigenvalue problem of (3.1.3) are expected to move to the left of the imaginary axis. However, it turns out that this is not enough to ensure stability since the eigenvalues which are to the left for small perturbations may move to the right as perturbation increases. Therefore, Freitas in [28] disproves the conjecture of Chen et al. He shows that (3.1.4) is insufficient to guarantee the exponential stability. Indeed, he finds out that if $\|a\|_{L^\infty}$ is large, then there may exist some positive real eigenvalues (see Theorem 3.6 of [28]). Actually, Freitas in [28] considers the more general wave equation with an additional potential term $b(x)u$ where $b \in L^\infty(0,1)$ and replaces $a(.)$ by $\epsilon a(.)$ where ϵ is a positive parameter ; i.e, Freitas considers the following problem

$$\begin{cases} u_{tt} - \Delta u - b(x)u + \epsilon a(x)u_t = 0, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.1.5}$$

where $\Omega \subseteq \mathbb{R}^n$ is bounded with smooth boundary $\partial\Omega$.

Freitas shows that, under the conditions imposed in the conjecture, the associated eigenvalue problem may admit real positive eigenvalues for $\epsilon > \epsilon_0 > 0$, which shows that the solution of (3.1.5) blows up exponentially as time goes to infinity. The idea of Freitas is based on replacing the eigenvalue problem associated with (3.1.5) by an eigenvalue problem of a selfadjoint operator $L_p = \Delta + pa(x) + b(x)$, where p is a real parameter. This method is presented in [29] and [66]. The asymptotic behavior of the eigenvalues of L_p, as $|p|$ goes to infinity, is studied and characterized. Then a relation is found between the spectrum of L_p and the real eigenvalues associated with problem (3.1.5). Indeed, the eigenvalue problem associated with (3.1.5) is given by

$$\begin{cases} \Delta u - \epsilon a(x)\lambda u + b(x)u = \lambda^2 u, & x \in \Omega,\ \epsilon > 0 \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{3.1.6}$$

Hence, Freitas finds that a real number λ is an eigenvalue associated to the eigenvalue problem (3.1.6) if and only if γ is an eigenvalue of L_p and

$$\begin{cases} \lambda = -\dfrac{p}{\epsilon} \\ \gamma = \dfrac{p^2}{\epsilon^2}. \end{cases}$$

Therefore, the real eigenvalues associated to the eigenvalue problem (3.1.6) correspond to the intersections of the parabola, $\dfrac{p^2}{\epsilon^2}$, with the eigencurves associated with the operator L_p. As ϵ becomes large, the eigencurves intersect the parabola in at least two points and hence a positive eigenvalue λ appears which prevents the solution of (3.1.5) from being stable (see Theorem 3.6 and Corollary 3.7 of [28]).

However, the results of Freitas do not clarify what happens for small values of ϵ. Consequently, later on, Freitas and Zuazua in [30] treat the case when the L^∞ norm of the damping term, $\epsilon a(.)$, with indefinite sign is not large. For $\epsilon > 0$ small enough, they prove that the solution of (3.1.3) is exponentially stable under condition (3.1.4) and the additional condition $a \in BV(0,1)$ so that the derivative of a is defined in the weak sense. Their idea is based on the ansatz suggested by Horn in [41] where the eigenvectors are written in a series form in order to find an approximation of the eigenvalue problem and hence the asymptotic behavior of the large eigenvalues. After using the shooting method and Rouché's Theorem inspired from [23], they prove that there exists $\epsilon_1 > 0$ such that, for every $\epsilon \in (0, \epsilon_1)$, the eigenvalues are asymptotically close to $-\epsilon a_0$, where a_0 is the average of $a(.)$ and therefore the high frequencies admit negative real parts since the hypothesis imposed on the uniform positivity of I_n in (3.1.4) yields the positivity of the average a_0 of $a(.)$. Moreover, the positivity of the integrals I_n ensures that the low frequencies are to the left of the imaginary axis for $\epsilon \in (0, \epsilon_0)$. Finally, by proving that the root vectors form a Riesz basis of the energy space, the exponential stability is established for $\epsilon \in (0, \epsilon_2)$ where $\epsilon_2 = \min\{\epsilon_0, \epsilon_1\}$.

This result is extended in [17] to the case where, in the wave equation, there is an additional zero order potential term $b(x)u(x,t)$ with $b \in L^1(0,1)$. However, the ansatz in Horn does not work any more in this case. Therefore, the authors adapt a shooting method employed in [65] to construct an explicit approximation of the characteristic equation of the underlying system and to find the asymptotic expansion of the eigenvalues and eigenvectors. Under the same assumptions used in [30] on the damping term, $\epsilon a(.)$, and on the integrals, I_n, the authors in [17] establish the exponential stability for $\epsilon > 0$ small enough.

Furthermore, in [51], the authors consider an abstract linear system with perturbation of the form $\frac{d}{dt}y = Ay + \epsilon By$ on a Hilbert space, where A is a skewadjoint operator, B is bounded, and ϵ is a positive parameter. Using an abstract perturbation result and under the hypothesis that the damping operator B is uniformly effective for all normalized linear combinations of eigenvectors corresponding to the eigenvalues located in a neighborhood of any eigenvalue, the authors find an upper bound for ϵ for which the abstract system is exponentially stable. In particular, the authors in [51] find an upper bound of ϵ for which problem (3.1.5) is exponentially stable under condition (3.1.4) and the assumption that $a \in L^\infty(0,1)$ without the need for the assumption that $a \in BV$. On the other hand, for problem (3.1.1), it seems to us that the upper bound of ϵ found in [51] is not easy to check.

Later on, in [57], Racke and Rivera have removed the factor ϵ and considered the wave equation $u_{tt} - u_{xx} + a(x)u_t = 0$ on $(0, L)$ for some $L > 0$ where $a \in L^\infty(0, L)$ is allowed to change its sign such that its mean value a_0 remains positive. In [57], the exponential stability is proved under one of these conditions : Either $\|a\|_{L^\infty}$ is possibly large with sufficiently small $\|a - a_0\|_{L^2}$ or $\|a\|_{L^\infty}$ is sufficiently small but the pair (a, L) has to satisfy some estimates where it is possible to get a negative moment I_n. Note that the second condition in [57] does not contradict the result from [30], because in that case the admissible pairs (a, L), leading to exponential decay, are not independent and, according to Racke and Rivera, the solution is not exponentially decaying if one replaces $a(.)$ by $\epsilon a(.)$. The method in [57] is based on the spectral criteria characterizing exponentially stable semigroups in terms of the spectrum of the generator of the semigroup (see [42]). For instance, for possibly large L^∞ norm of $a(.)$ and small $\|a - a_0\|_{L^2}$, using the fixed point argument, the authors prove that for $\epsilon > 0$,

$$\Gamma_\epsilon^I = \left\{ \epsilon + \alpha + i\beta;\ \alpha > \Re\left(-\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{\pi}{L}\right)^2} \right) \text{ and } \beta \in \mathbb{R} \right\} \subset \rho(A)$$

and that

$$\sup_{\lambda \in \Gamma_\epsilon^I} \left\| (\lambda I - A)^{-1} \right\| < \infty,$$

where A is the generator of the associated semigroup. Furthermore, for the second case, the authors prove that for small $\epsilon_0 > 0$ and any $\epsilon_1 > \epsilon_0$, we have for all

$\epsilon \in [\epsilon_0, \epsilon_1]$

$$\Gamma_\epsilon^{II} = \left\{\epsilon - \frac{a_0}{2} + i\eta;\ \eta \in \mathbb{R}\right\} \subset \rho(A)$$

and

$$\sup_{\lambda \in \Gamma_\epsilon^{II},\ \epsilon \in [\epsilon_0, \epsilon_1]} \left\|(\lambda I - A)^{-1}\right\| < \infty,$$

on the conditions that

$$\|a(.)\|_{L^\infty(0,L)} < \frac{\sinh\left(\frac{\gamma_0}{4}\right)}{\left(\sinh\left(\frac{\gamma_1}{4}\right) + e^{\frac{7}{2}\gamma_1} e^{\frac{7}{2}\gamma_1}\right)},$$

where, for any given $\gamma_0 > 0$ and $\gamma_1 > 0$, we have

$$\gamma_0 \leq \int_0^L a(x)dx \leq L\|a(.)\|_{L^\infty(0,L)} \leq \gamma_1$$

and

$$\frac{\gamma_0}{\|a(.)\|_{L^\infty(0,L)}} < L \leq \frac{\gamma_1}{\|a(.)\|_{L^\infty(0,L)}}.$$

Finally, Menz in [54] generalizes the work done in [57] by adding a potential term $b(x)u$. He proves that if the average a_0 is positive, then, for $a(.),\ b(.) \in L^\infty(0, L)$, the exponential stability is proved for small $\|a - a_0\|_{L^2}$ but not necessarily for small $\|a(.)\|_{L^\infty(0,L)}$. Using Gearheart and Huang result, Menz obtains the resolvent estimate for the system where the function $a(.)$ is replaced by its mean value, a_0. Then using a fixed point argument, the exponential stability result is transferred to the original problem with potential term.

3.2 Main results

In this chapter or in [1], our work differs from the previous studies because we do not want to impose neither a small value of the damping factor a nor a small value of $\|a - a_0\|_{L^2}$. Indeed for system (3.1.1), this mean value is equal to $\sqrt{2}|1-\alpha|$ which we do not need to be sufficiently small. Indeed we will show later on that for $\alpha \leq -1$, problem (3.1.1) is never exponentially stable (even up to a finite dimensional space), while for $\alpha > -1$, problem (3.1.1) is exponentially stable up to a finite dimensional space. We even show that there exists a critical value $\alpha_3 \simeq -0.2823$ such that if $\alpha > \alpha_3$, then problem (3.1.1) is exponentially stable. Our method takes advantage of the one-dimensional setting that allows to perform a precise spectral analysis.

Note that these results are coherent with those given by the perturbation theory of contractive semigroups (see [62]). Actually, for system (3.1.1) defined in an appropriate Hilbert setting, we can write the generator of the semigroup as $A_0 + B_- + B_+$ where A_0 is the skew-adjoint operator given by

$$A_0 = \begin{pmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{pmatrix},$$

and $B_\pm$ are bounded operators given by

$$B_+ = \begin{pmatrix} 0 & 0 \\ 0 & -2\chi_{(0,1)} \end{pmatrix}, \quad B_- = \begin{pmatrix} 0 & 0 \\ 0 & -2\alpha\chi_{(-1,0)} \end{pmatrix}.$$

Then applying the perturbation theory of contractive semigroups, in order to get an exponential stability, the condition $\|B_-\| < |\omega(A + B_+)|$ should be imposed. However, according to [23], $\mu(A+B_+) = \omega(A+B_+) \simeq -0.45$ where the approximate value is found numerically. Here, $\mu(\mathcal{A})$ denotes the spectral abscissa of the operator $\mathcal{A}$;

$$\mu(\mathcal{A}) = \sup\{\Re\lambda : \lambda \in \sigma(\mathcal{A})\},$$

$\sigma(\mathcal{A})$ being the spectrum of $\mathcal{A}$, while $\omega(\mathcal{A})$ denotes the growth rate of the evolution equation associated with $\mathcal{A}$ in the Hilbert space X;

$$\omega(\mathcal{A}) = \inf\{\omega : \exists C(\omega) > 0 \text{ s. t. } \|U(t)\|_X^2 \leq C(\omega)\|U(0)\|_X^2 e^{2\omega t}, \forall t > 0,$$
$$\text{for every solution } U \text{ of } U_t(t) = \mathcal{A}U(t), \forall t > 0\}.$$

Therefore $\|B_-\| = 2|\alpha| < |\mu(A + B_+)|$ yields the condition that $\alpha > \alpha_1$, where $\alpha_1 \simeq -0.225$. Our spectral analysis improves this condition and yields a larger range of values of α for which problem (3.1.1) is exponentially stable. However, this result is not optimal since numerical results show that $\alpha > \alpha_2$, where $\alpha_2 \simeq -0.77$, yields the exponential stability of (3.1.1) (see Figure 1).

By a similar approach, we find some exponential or polynomial stability results for the second problem (3.1.2) where a and b are of opposite signs ; the particular case $b \in (-1, 0)$ and $a > 0$ retains our attention. Note that for such a problem, perturbation theory of contractive semigroups cannot be invoked.

This chapter is divided into two main parts. In the first one, we analyze the spectral problem associated with (3.1.1) in order to find a possible range of α for which (3.1.1) is stable. We find and prove the following results of [1] :

Theorem 3.2.1. $\alpha > -1$ *if and only if problem (3.1.1) is exponentially stable up to a finite dimensional space.*

Theorem 3.2.2. *If* $\alpha > \alpha_3$*, where* $\alpha_3 \simeq -0.2823$*, then the solution of problem (3.1.1) is exponentially stable.*

In the second part, we analyze problem (3.1.2) in order to find some conditions that a and b must satisfy to get the stability of (3.1.2). We find out the following results of [1] :

Theorem 3.2.3. *If* $b \notin \{-1, 0, 1\}$*, then* $a > -2\Re \tanh^{-1} \dfrac{1}{b} = -\ln \left| \dfrac{b+1}{b-1} \right|$ *if and only if problem (3.1.2) is exponentially stable up to a finite dimensional space.*

Theorem 3.2.4. *If* $b \in (-1, 0)$*, then* $a > -2 \tanh^{-1} b$ *if and only if problem (3.1.2) is exponentially stable.*

Theorem 3.2.5. *If $b \in (-1, 0)$ and $a = -2\tanh^{-1} b$, then there exists $C > 0$ such that for all $U(0) = (u_0, u_1) \in D(A_a)$, we have*

$$E_2(t) \le C \frac{\|U(0)\|^2_{D(A_a)}}{t}, \forall t > 0,$$

where $E_2(t)$ is the energy of the solution of problem (3.1.2) and A_a is defined in (3.4.2) below.

Note that a statement similar to the one of Theorem 3.2.4 cannot hold in the case $b < -1$ because there exists some $a > -2\Re \tanh^{-1} \frac{1}{b}$ such that some eigenvalues λ of A_a are in $\Re\lambda > 0$ (see Figure 3.6 in the case $b = -2$ and $a = 1.1$).

Before we start our analysis, we introduce some notations used in the remainder of this chapter : On D, the $L^2(D)$-norm will be denoted by $\| \cdot \|_D$. Similarly $(\cdot, \cdot)_D$ means the $L^2(D)$ inner product. Finally, the notation $A \lesssim B$ and $A \simeq B$ means the existence of positive constants C_1 and C_2, which are independent of A and B such that $A \le C_2 B$ and $C_1 B \le A \le C_2 B$, respectively.

3.3 Exponential stability for the indefinite sign internally damped problem (3.1.1)

Since problem (3.1.1) is exponentially stable if the damping term α is non-negative (see [23]), from now on we assume that $\alpha < 0$.

We start by writing problem (3.1.1) as a system of the form $U_t = A_\alpha U$ where $U = (u, u_t)^\top$ and the operator $A_\alpha : D(A_\alpha) \to X$ is defined by

$$A_\alpha = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & -2\chi_{(0,1)} - 2\alpha\chi_{(-1,0)} \end{bmatrix}$$

where the energy space $X = H_0^1(-1,1) \times L^2(-1,1)$ is equipped with the usual inner product defined by

$$< (f,g)^\top, (u,v)^\top >= \int_{-1}^{1} (f'\overline{u}' + g\overline{v})dx,$$

and

$$D(A_\alpha) = (H^2(-1,1) \cap H_0^1(-1,1)) \times H_0^1(-1,1).$$

In this case, the energy associated with problem (3.1.1), at time t, is given by

$$E_1(t) = \frac{1}{2}\left(\int_{-1}^{1} (|u_x(x,t)|^2 + |u_t(x,t)|^2)dx\right)$$

with

$$E_1'(t) = -2\left(\int_0^1 |u_t(x,t)|^2 dx + \alpha \int_{-1}^{0} |u_t(x,t)|^2 dx\right), \quad \forall (u_0,u_1) \in D(A_\alpha).$$

Notice that problem (3.1.1) is well posed since A_α can be written as a sum of the m-dissipative operator A_0 with a bounded operator (see Theorem 1.1.5).

In this section, we start by analyzing the spectrum of the generator A_α. First, we find the asymptotic development of the large eigenvalues. This development shows that the high frequencies approach the line $x = -\frac{\alpha+1}{2}$ (see (3.3.7)). Second, we determine the critical value α_3 of α for which all the eigenvalues of A_α are situated to the left of the imaginary axis for any $\alpha > \alpha_3$. Finally, we show that the generalized eigenvectors of A_α form a Riesz basis of the energy space from which we deduce the exponential stability of problem (3.1.1) for $\alpha > \alpha_3$.

3.3.1 Spectral analysis of problem (3.1.1)

First, we determine the characteristic equation related to problem (3.1.1). If $U = (y,z)^\top \in D(A_\alpha)$ is an eigenvector of A_α associated with the eigenvalue λ, then

$z = \lambda y$ and the eigenvalue problem is given by

$$\begin{cases} y_{xx} - \lambda^2 y - 2\chi_{(0,1)}\lambda y - 2\alpha\chi_{(-1.0)}\lambda y = 0 \text{ in } (-1,1), \\ y(-1) = y(1) = 0. \end{cases} \tag{3.3.1}$$

Clearly $\lambda = 0$ cannot be an eigenvalue of A_α. Similarly the same calculations as below allow to show that $\lambda = -2$ and $\lambda = -2\alpha$ are not eigenvalues of A_α. Now for $\lambda \notin \{0, -2, -2\alpha\}$, setting $y^+ = y_{|(0,1)}$ and $y^- = y_{|(-1,0)}$, we get

$$\begin{cases} y^+_{xx} = (\lambda^2 + 2\lambda) y^+ \text{ in } (0,1), \\ y^+(1) = 0, \end{cases}$$

and consequently,

$$y^+(x) = c^+ \sinh[\sqrt{\lambda^2 + 2\lambda}(x-1)], \tag{3.3.2}$$

for some $c^+ \in \mathbb{C}$. Similarly, we have

$$\begin{cases} y^-_{xx} = (\lambda^2 + 2\alpha\lambda) y^- \text{ in } (-1,0), \\ y^-(-1) = 0, \end{cases}$$

which implies that

$$y^-(x) = c^- \sinh[\sqrt{\lambda^2 + 2\alpha\lambda}(x+1)], \tag{3.3.3}$$

for some $c^- \in \mathbb{C}$. As the differential equation in (3.3.1) yields $y \in H^2(0,1)$ and due to the Sobolev embedding theorem $H^2(0,1) \hookrightarrow C^1[0,1]$, we get

$$\begin{cases} y^+(0) = y^-(0), \\ y^+_x(0) = y^-_x(0), \end{cases}$$

or equivalently

$$\begin{cases} c^+ \sinh(\sqrt{\lambda^2 + 2\lambda}) = -c^- \sinh(\sqrt{\lambda^2 + 2\alpha\lambda}), \\ c^+ \sqrt{\lambda^2 + 2\lambda} \cosh(\sqrt{\lambda^2 + 2\lambda}) = c^- \sqrt{\lambda^2 + 2\alpha\lambda} \cosh(\sqrt{\lambda^2 + 2\alpha\lambda}), \end{cases} \tag{3.3.4}$$

i.e.,

$$\mathcal{D}_\alpha(\lambda)(c^+, c^-)^\top = (0,0)^\top, \tag{3.3.5}$$

where

$$\mathcal{D}_\alpha(\lambda) = \begin{pmatrix} \sinh(\sqrt{\lambda^2+2\lambda}) & \sinh(\sqrt{\lambda^2+2\alpha\lambda}) \\ \sqrt{\lambda^2+2\lambda}\cosh(\sqrt{\lambda^2+2\lambda}) & -\sqrt{\lambda^2+2\alpha\lambda}\cosh(\sqrt{\lambda^2+2\alpha\lambda}) \end{pmatrix}.$$

As (3.3.5) admits a non zero solution if and only if $\det \mathcal{D}_\alpha(\lambda) = 0$, the complex number $\lambda \notin \{0, -2, -2\alpha\}$ is an eigenvalue of A_α if and only if it is the root of the characteristic equation

$$\det \mathcal{D}_\alpha(\lambda) = 0.$$

Direct calculations yield

$$\begin{aligned} \det \mathcal{D}_\alpha(\lambda) &= -F_\alpha(\lambda) \\ &= -\sqrt{\lambda^2+2\alpha\lambda}\sinh(\sqrt{\lambda^2+2\lambda})\cosh(\sqrt{\lambda^2+2\alpha\lambda}) \\ &\quad -\sqrt{\lambda^2+2\lambda}\sinh(\sqrt{\lambda^2+2\alpha\lambda})\cosh(\sqrt{\lambda^2+2\lambda}). \end{aligned}$$

Note further that

$$2F_\alpha(\lambda) = g_\alpha(\lambda) = t_2(\lambda)\sinh t_2(\lambda) - t_1(\lambda)\sinh t_1(\lambda),$$

where

$$t_1(\lambda) = \sqrt{\lambda^2+2\lambda} - \sqrt{\lambda^2+2\alpha\lambda},$$

and

$$t_2(\lambda) = \sqrt{\lambda^2+2\lambda} + \sqrt{\lambda^2+2\alpha\lambda}.$$

We have proved the next result.

Lemma 3.3.1. *A_α has a compact inverse and therefore the spectrum of A_α is discrete and its eigenvalues are of finite algebraic multiplicity. Furthermore*

$$\sigma(A_\alpha) = \{\lambda \in \mathbb{C} \setminus \{0, -2, -2\alpha\} \,:\, g_\alpha(\lambda) = 0\}.$$

Remark 3.3.2. *Note that the eigenvalues of A_α depend continuously on α. Indeed, fix α and an eigenvalue λ_0 of A_α. Then as λ_0 is isolated, there exists $\rho > 0$ such that*

$$g_\alpha(z) \neq 0, \forall z \in \mathbb{C} : 0 < |z - \lambda_0| \leq \rho.$$

In particular, as g_α is a continuous function of z, setting $D = \{z \in \mathbb{C} : |z - \lambda_0| = \rho\}$, there exists a positive real number κ such that

$$|g_\alpha(z)| \geq \kappa, \forall z \in D.$$

For a fixed positive real number ϵ_0 we consider the mapping of two variables

$$H : [0, \epsilon_0] \times D \to \mathbb{C} : (\epsilon, z) \to g_\alpha(z) - g_{\alpha+\epsilon}(z).$$

Since it is a uniformly continuous function and since $H(0, z) = 0$ for all z, we deduce the existence of a positive real number δ such that

$$|H(\epsilon, z)| < \kappa, \forall (\epsilon, z) \in [0, \delta] \times D.$$

The two last estimates imply that

$$|g_\alpha(z) - g_{\alpha+\epsilon}(z)| < |g_\alpha(z)|, \forall (\epsilon, z) \in [0, \delta] \times D.$$

Hence, Rouché's theorem allows to conclude that $g_{\alpha+\epsilon}$ has the same number of roots at g_α for all $\epsilon \in [0, \delta]$.

The following Lemma shows the boundedness of the real part of the eigenvalues of the operator A_α and proves that its eigenvalues cannot be real.

Lemma 3.3.3. *Let λ be an eigenvalue of the operator A_α, and $U = y(x, \lambda)(1, \lambda)^\top$ be an associated eigenvector. Then $\lambda \in \mathbb{C} \backslash \mathbb{R}$ with*

$$\Re\lambda = \frac{-\displaystyle\int_0^1 |y(x, \lambda)|^2 dx - \alpha \int_{-1}^0 |y(x, \lambda)|^2 dx}{\displaystyle\int_{-1}^1 |y(x, \lambda)|^2 dx},$$

and

$$(\Im\lambda)^2 = \frac{\displaystyle\int_{-1}^{1}|y_x(x,\lambda)|^2dx}{\displaystyle\int_{-1}^{1}|y(x,\lambda)|^2dx} - \left(\frac{\displaystyle\int_{0}^{1}|y(x,\lambda)|^2dx + \alpha\int_{-1}^{0}|y(x,\lambda)|^2dx}{\displaystyle\int_{-1}^{1}|y(x,\lambda)|^2dx}\right)^2.$$

Proof: As A_α is real, in the sense that $\overline{A_\alpha U} = A_\alpha \bar{U}$ for all $U \in D(A_\alpha)$, it follows that $\overline{U} = y(x,\overline{\lambda})(1,\overline{\lambda})$ is an eigenvector of A_α corresponding to the eigenvalue $\overline{\lambda}$. Integrating (3.3.1) against $y(x,\overline{\lambda})$ gives

$$\int_{-1}^{1}|y_x|^2dx + \lambda^2\int_{-1}^{1}|y|^2dx + 2\lambda\left(\int_{0}^{1}|y|^2dx + \alpha\int_{-1}^{0}|y|^2dx\right) = 0.$$

Hence,

$$\lambda = \frac{-\displaystyle\int_{0}^{1}|y|^2dx - \alpha\int_{-1}^{0}|y|^2dx \pm \left[\left(\int_{0}^{1}|y|^2dx + \alpha\int_{-1}^{0}|y|^2dx\right)^2 - \int_{-1}^{1}|y_x|^2dx\int_{-1}^{1}|y|^2dx\right]^{\frac{1}{2}}}{\displaystyle\int_{-1}^{1}|y|^2dx}.$$

If λ is real, then by the Poincaré-Friedrich's inequality

$$\frac{\pi^2}{4}\int_{-1}^{1}|w|^2dx \le \int_{-1}^{1}|w_x|^2dx, \forall w \in H_0^1(-1,1), \tag{3.3.6}$$

we get

$$\begin{aligned} 0 &\le \left(\int_{0}^{1}|y|^2dx + \alpha\int_{-1}^{0}|y|^2dx\right)^2 - \int_{-1}^{1}|y_x|^2dx\int_{-1}^{1}|y|^2dx \\ &\le \left(\int_{-1}^{1}|y|^2dx\right)^2 - \left(\frac{\pi}{2}\right)^2\left(\int_{-1}^{1}|y|^2dx\right)^2 \\ &= \left(1-\frac{\pi^2}{4}\right)\left(\int_{-1}^{1}|y|^2dx\right)^2 < 0 \end{aligned}$$

which is impossible. Consequently, $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and the result follows. ∎

Remark 3.3.4. *Note that*

1. $(\Re\lambda)^2 + (\Im\lambda)^2 \ge \dfrac{\pi^2}{4}$ *by Poincaré-Friedrich's inequality (3.3.6).*

2. $|\Re\lambda| \leq \max\{1, |\alpha|\}$.

3. *Denote by* $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ *the spectrum of* A_α. *Since* A_α *is real, without loss of generality we can assume that for all* $k \in \mathbb{Z}^*$, $\lambda_{-k} = \overline{\lambda_k}$.

3.3.1.1 Asymptotic behavior of large eigenvalues

In the sequel, we study the asymptotic behavior of the eigenvalues $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ of A_α as $|\lambda_k| \to +\infty$. According to the previous remark, since $\Re\lambda_k$ is uniformly bounded, then $|\lambda_k| \to +\infty$ is equivalent to $|\Im\lambda_k| \to +\infty$. Moreover, we can restrict our study to the case $k \to +\infty$.

Lemma 3.3.5. *The large eigenvalues of* A_α *are simple. Moreover there exist* $m \in \mathbb{Z}$ *and* $k_0 \in \mathbb{N}$, *such that*

$$\lambda_k = \frac{-\alpha - 1}{2} + i(k+m)\frac{\pi}{2} + O\left(\frac{1}{k}\right), \quad \forall k \geq k_0. \tag{3.3.7}$$

Proof: Let λ be an eigenvalue of A_α or equivalently a root of g_α. For the sake of simplicity, we assume that $\sqrt{\lambda^2} = \lambda$, if this is not the case, the next arguments hold but with t_1 and t_2 replaced by their opposites.

First, we prove that

$$t_1(\lambda) \to 1 - \alpha \text{ and } t_2(\lambda) = 2\lambda + 1 + \alpha + o(1) \text{ as } |\lambda| \to +\infty.$$

Indeed, we write

$$\begin{aligned} t_1(\lambda) &= \frac{(\sqrt{\lambda^2 + 2\lambda} - \sqrt{\lambda^2 + 2\alpha\lambda})(\sqrt{\lambda^2 + 2\lambda} + \sqrt{\lambda^2 + 2\alpha\lambda})}{\sqrt{\lambda^2 + 2\lambda} + \sqrt{\lambda^2 + 2\alpha\lambda}} \\ &= \frac{2\lambda(1-\alpha)}{t_2(\lambda)}. \end{aligned}$$

Therefore, we get [1]

$$t_1(\lambda) \sim \frac{2\lambda(1-\alpha)}{2\lambda} = 1 - \alpha, \quad \text{ as } |\Im\lambda| \to +\infty. \tag{3.3.8}$$

1. as usual the notation $h(\lambda) \sim g(\lambda)$ as $|\Im\lambda| \to +\infty$ means that $\lim_{|\Im\lambda|\to+\infty} \frac{h(\lambda)}{g(\lambda)} = 1$

On the other hand,

$$t_2(\lambda) = \lambda\sqrt{1+\frac{2}{\lambda}} + \lambda\sqrt{1+\frac{2\alpha}{\lambda}}.$$

As $|\lambda| \to +\infty$, we write

$$\sqrt{1+\frac{2}{\lambda}} = 1 + \frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right),$$

and

$$\sqrt{1+\frac{2\alpha}{\lambda}} = 1 + \frac{\alpha}{\lambda} + o\left(\frac{1}{\lambda}\right).$$

Therefore,

$$t_2(\lambda) = \lambda\left(2 + \frac{1+\alpha}{\lambda}\right) + o(1) = 2\lambda + 1 + \alpha + o(1) \quad \text{as } |\Im\lambda| \to +\infty. \tag{3.3.9}$$

Since $t_2(\lambda) \neq 0$ for $|\Im\lambda|$ large enough, then from the characteristic equation we can write

$$g_\alpha(\lambda) = t_2(\lambda)\left[\sinh(t_2(\lambda)) - \frac{t_1(\lambda)}{t_2(\lambda)}\sinh(t_1(\lambda))\right] = 0,$$

or equivalently

$$h_\alpha(\lambda) = 0, \tag{3.3.10}$$

where

$$h_\alpha(\lambda) = \sinh(t_2(\lambda)) - \frac{t_1(\lambda)}{t_2(\lambda)}\sinh(t_1(\lambda)). \tag{3.3.11}$$

Now the conclusion follows by using Rouché's Theorem. For this aim, for N large enough, define the curve

$$\Gamma_{\pm n} = \{z : \left|z + \frac{1+\alpha}{2} \mp i\frac{n\pi}{2}\right| = \frac{C_0}{n}\},\ n > N,$$

where C_0 is a positive constant fixed later on in Lemma 3.3.7.

Lemma 3.3.7 below shows, by Rouché's Theorem, that $h_\alpha(z)$ given in (3.3.11) has the same roots as $\sinh(t_2(z))$ in the curve $\Gamma_{\pm n}$, for every $n > N$ where N is large enough. Consequently, we deduce that the large eigenvalues are simple since the roots of $\sinh(t_2(z))$ are simple for $|z|$ large and are situated inside $\Gamma_{\pm n}$ for some n large, which yields (3.3.7). ■

The next Figures 3.1 and 3.2 illustrate the roots of F_α for $\alpha = -0.75$ and $\alpha = -0.2$ computed using a Newton method, namely a sufficiently large box is decomposed in a relatively fine mesh and each node of the mesh is used as initial value for the Newton method. In these Figures, the asymptotic behavior from the previous lemma is clearly visible.

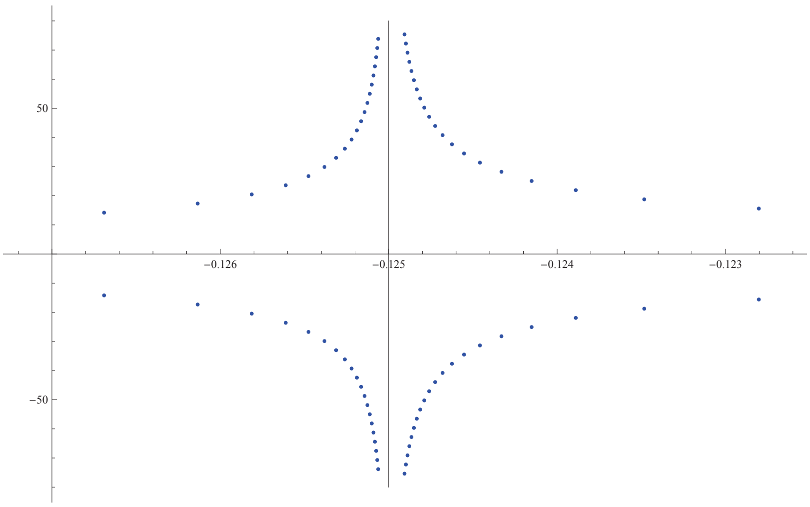

FIGURE 3.1 – Eigenvalues for $\alpha = -0.75$

Remark 3.3.6. *(Increasing the order of the finite expansion)*
If we write $\lambda_k = \dfrac{-\alpha-1}{2} + i(k+m)\dfrac{\pi}{2} + \beta_k + i\varepsilon_k$ *where* $\beta_k, \varepsilon_k \in \mathbb{R}$ *are such that* $\beta_k = O_1\left(\dfrac{1}{k}\right)$ *and* $\varepsilon_k = O_2\left(\dfrac{1}{k}\right)$, *and we substitute this value into* $t_2(\lambda_k)$, $\sinh(t_2(\lambda_k))$,

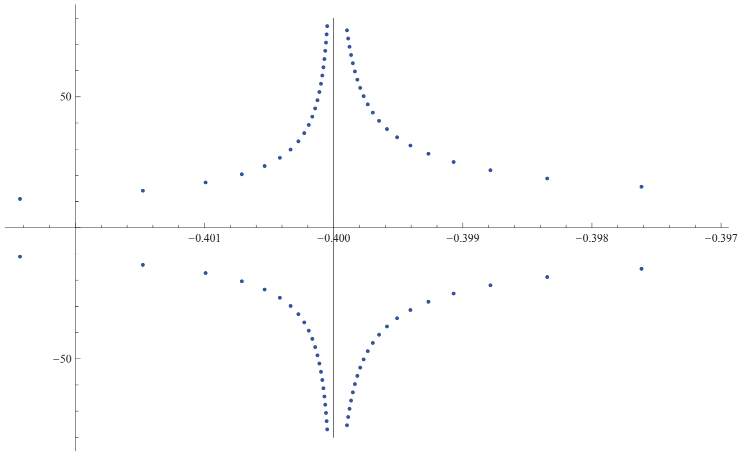

FIGURE 3.2 – Eigenvalues for $\alpha = -0.2$

and $\dfrac{t_1(\lambda_k)}{t_2(\lambda_k)}\sinh(t_1(\lambda_k)$, *then increasing the order of the finite expansion, we get*

$$
\begin{aligned}
t_2(\lambda_k) &= i(k+m)\pi + 2\beta_k + 2i\varepsilon_k - \frac{1+\alpha^2}{2i(k+m)\pi} + \frac{(1+\alpha^2)(1+\alpha)-2(1+\alpha^3)}{(k+m)^2\pi^2} + o\left(\frac{1}{k^2}\right)\\
\sinh(t_2(\lambda_k)) &= (-1)^{k+m}\left(2\beta_k + 2i\varepsilon_k - \frac{1+\alpha^2}{i(k+m)\pi} + \frac{(1+\alpha^2)(1+\alpha)-2(1+\alpha^3)}{(k+m)^2\pi^2} + o\left(\frac{1}{k^2}\right)\right)\\
\frac{t_1(\lambda_k)}{t_2(\lambda_k)}\sinh(t_1(\lambda_k)) &= \frac{(1-\alpha)\sinh(1-\alpha)}{i(k+m)\pi} + \frac{(1-\alpha^2)\sinh(1-\alpha)}{((k+m)\pi)^2}\\
&\quad + \frac{(1-\alpha^2)(1-\alpha)\cosh(1-\alpha)}{((k+m)\pi)^2} + o\left(\frac{1}{k^2}\right).
\end{aligned}
$$

From the equality $\sinh(t_2(\lambda_k)) = \dfrac{t_1(\lambda_k)}{t_2(\lambda_k)}\sinh(t_1(\lambda_k))$, *we conclude*

$$
\begin{aligned}
\varepsilon_k &= -\frac{1+\alpha^2}{2(k+m)\pi} - (-1)^{k+m}\frac{(1-\alpha)\sinh(1-\alpha)}{2(k+m)\pi} + o(\frac{1}{k^2})\\
\beta_k &= -\frac{2(1+\alpha^3)-(1+\alpha)(1+\alpha^2)}{2((k+m)\pi)^2}\\
&\quad + \tfrac{(-1)^{k+m}}{2}\left[\frac{(1-\alpha^2)\sinh(1-\alpha)}{((k+m)\pi)^2} + \frac{(1-\alpha^2)(1-\alpha)\cosh(1-\alpha)}{((k+m)\pi)^2}\right] + o\left(\frac{1}{k^2}\right).
\end{aligned}
$$

The $(-1)^{k+m}$ *factor appearing in the expression of* β_k *shows that, according to the parity of* $k+m$, *there are eigenvalues to the left and to the right of the axis* $x =$

$-\dfrac{1+\alpha}{2}$. *Hence if* $1+\alpha \leq 0$, *then problem (3.1.1) is never stable, while if* $1+\alpha > 0$, *then we can hope that (3.1.1) is either exponentially stable or unstable but exponentially stable up to a finite dimensional space.*

Lemma 3.3.7. *There exists* $N \in \mathbb{N}$ *large enough such that for every* $n > N$ *and for all* $z \in \Gamma_{\pm n}$, *we have*

$$|h_\alpha(z) - \sinh(t_2(z))| < |\sinh(t_2(z))|.$$

Proof: The proof is divided into two steps. In the first step, for every $n > N$ where N is large enough, we show that if $z \in \Gamma_{\pm n}$, then $|\sinh(t_2(z))| \geq \dfrac{C_0}{|z|}$. In fact, it is enough to consider the case where $z \in \Gamma_n$ since the eigenvalues appear in conjugate pairs. If $z \in \Gamma_n$, then $z = -\dfrac{1+\alpha}{2} + i\dfrac{n\pi}{2} + \rho_n e^{i\theta}$ where $\theta \in [0, 2\pi)$ and $\rho_n = \dfrac{C_0}{n}$. Since n is large enough, then by (3.3.9), we have

$$t_2(z) = 2z + 1 + \alpha + o_1(1) + i o_2(1),$$

and

$$\begin{aligned} |\sinh(t_2(z))|^2 &= \sin^2\left(2\rho_n \sin\theta + o_2(1)\right) + \sinh^2\left(2\rho_n \cos\theta + o_1(1)\right) \\ &= \left(2\rho_n \sin\theta + o(1)\right)^2 + \left(2\rho_n \cos\theta + o(1)\right)^2 \\ &= 4\rho_n^2 + o(1). \end{aligned}$$

Hence,

$$\frac{C_0^2}{|z|^2} \leq \frac{C_0^2}{\left(\rho_n \sin\theta + \frac{n\pi}{2}\right)^2} = \frac{4C_0^2}{(n\pi)^2} + o\left(\frac{1}{n^2}\right) \leq \frac{4C_0^2}{n^2} + o(1) = |\sinh(t_2(z))|^2.$$

Now (3.3.8) and (3.3.9) imply that there exists $C_1 > 0$ such that

$$|t_1(z)| < 2|1-\alpha|, \forall z : |\Im z| > C_1,$$

$$|t_2(z)| > |z|, \forall z : |\Im z| > C_1,$$

and therefore there exists $C_0 > 0$ such that

$$|t_1(z)\sinh(t_1(z))| < C_0, \forall z : |\Im z| > C_1.$$

As for $z \in \Gamma_n$, with $n \geq 1$, we have

$$\Im z \geq \frac{n\pi}{2} - \frac{C_0}{n},$$

we need to chose N large enough so that

$$\frac{N\pi}{2} - \frac{C_0}{N} > C_1.$$

With this constraint and by (3.3.11) we see that for $z \in \Gamma_n$, where $n > N$, we have

$$|h_\alpha(z) - \sinh(t_2(z))| \leq \left|\frac{t_1(z)\sinh(t_1(z))}{t_2(z)}\right| < \frac{C_0}{|z|} < |\sinh(t_2(z))|\,.$$

■

3.3.1.2 Critical value of α

We finish this section by looking for a critical value of α for which we will get an exponential stability of problem (3.1.1). Numerically, as the Figure 3.3 below shows (see also Figures 3.1 and 3.2), for $0 > \alpha > \alpha_2$, with $\alpha_2 \approx -0.77$, the eigenvalues $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ of the operator A_α are all situated to the left of the imaginary axis. However, theoretically we could not hit this range of α. Instead, we prove such a result for $\alpha \in (\alpha_3, 0)$ with $\alpha_3 \approx -0.2823$. In fact, as the value of $\alpha \in (-1, 0)$ decreases, the axis $x = -\dfrac{1+\alpha}{2}$ is shifted to the right and therefore the eigenvalues are shifted near the imaginary axis. Consequently, we try to study the behavior of the eigenvalues on the imaginary axis and then find a critical value of α for which the characteristic equation (3.3.10) has no roots on the imaginary axis.

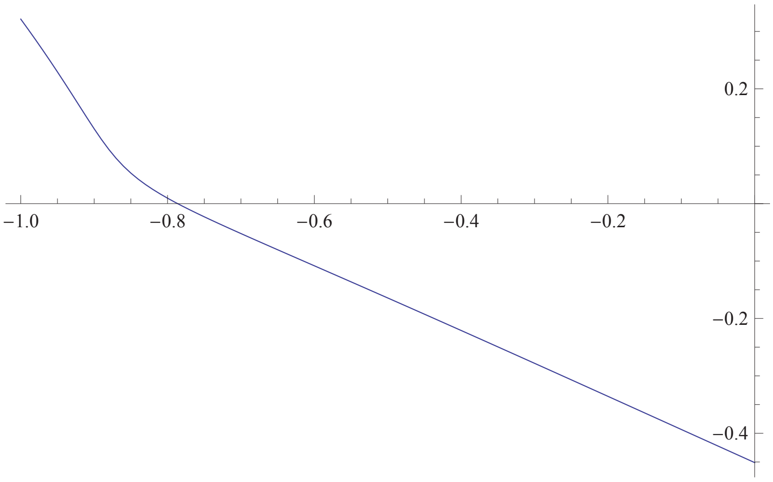

FIGURE 3.3 – Numerical value of $\max\{\Re\lambda : \lambda \in \sigma(A_\alpha)\}$ versus α.

To state properly our result, we introduce the functions

$$\begin{aligned} f_1 \; &: \; (-\infty, 0] \to (0, \infty) \\ &\alpha \to \frac{1}{2}\left[\cosh\left(2\Re(t_2(i\frac{\pi}{2}))\right) - 1\right], \\ f_2 \; &: \; (-\infty, 0] \to (0, \infty) \\ &\alpha \to \frac{1}{2}\frac{\left[(1-\alpha)^2 + \left(\Im(t_1(i\frac{\pi}{2}))\right)^2\right]\left[\cosh\left(2(1-\alpha)\right) - \cos\left(2\Im(t_1(i\frac{\pi}{2}))\right)\right]}{\left|t_2(i\frac{\pi}{2})\right|^2}. \end{aligned}$$

It is easy to check that f_1 (resp. f_2) is increasing (resp. decreasing) and that $f_1(0) > f_2(0)$ (see Figure 3.4), hence there exists a unique negative real number α_3 such that $f_1(\alpha_3) = f_2(\alpha_3)$. Numerically we find that $\alpha_3 \approx -0.2823$.

Theorem 3.3.8. *For all $\alpha > \alpha_3$, any eigenvalue λ of A_α satisfies $\Re\lambda < 0$.*

Proof: Let $\lambda = iy$ with $y \in \mathbb{R}$ be a purely imaginary eigenvalue of A_α. As the complex eigenvalues appear in conjugate pairs, it is enough to consider the case $y > 0$. According to Remark 3.3.4, $y \geq \frac{\pi}{2}$. We start by writing

$$\lambda(2\alpha + \lambda) = -y^2 + 2i\alpha y.$$

Let

$$h(y,\alpha)=\frac{\sqrt{y^2+y\sqrt{4\alpha^2+y^2}}}{\sqrt{2}},$$

then, by expanding $\left(\dfrac{\alpha y}{h(y,\alpha)}+ih(y,\alpha)\right)^2$, we have

$$\sqrt{-y^2+2i\alpha y}=\frac{\alpha y}{h(y,\alpha)}+ih(y,\alpha).$$

We note that $h(y,\alpha)$ is non decreasing as a function of y since

$$\frac{\partial}{\partial y}h(y,\alpha)=\frac{1}{\sqrt{2}}\left[\frac{2y+\sqrt{4\alpha^2+y^2}+\frac{y^2}{\sqrt{4\alpha^2+y^2}}}{2\sqrt{y^2+y\sqrt{4\alpha^2+y^2}}}\right],$$

while $\dfrac{h(y,\alpha)}{y}$ is decreasing since

$$\frac{\partial}{\partial y}\frac{h(y,\alpha)}{y}=\frac{-\sqrt{2}\alpha^2}{y\sqrt{4\alpha^2+y^2}\sqrt{y(y+\sqrt{4\alpha^2+y^2})}}.$$

Moreover, $t_2(iy)$ is given by

$$t_2(iy)=\frac{\alpha y}{h(y,\alpha)}+\frac{y}{h(y,1)}+i\left(h(y,\alpha)+h(y,1)\right)$$

and $t_1(iy)$ is given by

$$t_1(iy)=-\frac{\alpha y}{h(y,\alpha)}+\frac{y}{h(y,1)}+i\left(-h(y,\alpha)+h(y,1)\right).$$

In the sequel, our aim is to find some bounds for $|\sinh(t_2(iy))|^2$ and $\left|\dfrac{t_1(iy)}{t_2(iy)}\sinh(t_1(iy))\right|^2$. We start by finding a lower bound for $|\sinh(t_2(iy))|^2$ with $y\geq\dfrac{\pi}{2}$. We have

$$|\sinh(t_2(iy))|^2=\frac{1}{2}[\cosh\left(2\Re(t_2(iy))\right)-\cos\left(2\Im(t_2(iy))\right)]\geq\frac{1}{2}[\cosh\left(2\Re(t_2(iy))\right)-1].$$

But $\Re(t_2(iy))$ is a positive and increasing function of y. In fact,

$$\Re(t_2(iy))=\frac{y}{h(y,\alpha)}\left[\frac{h(y,\alpha)}{h(y,1)}+\alpha\right].$$

Since $\dfrac{h(y,\alpha)}{h(y,1)}$ is positive then we prove that it is increasing by proving that its square is increasing. Indeed, since $|\alpha| < 1$

$$\frac{\partial}{\partial y}\left(\frac{h(y,\alpha)}{h(y,1)}\right)^2 = \frac{(y+\sqrt{4\alpha^2+y^2})(\sqrt{4+y^2}-\sqrt{4\alpha^2+y^2})}{\sqrt{4+y^2}\sqrt{4\alpha^2+y^2}(y+\sqrt{4+y^2})} > 0.$$

Since

$$\lim_{y\to 0}\frac{h(y,\alpha)}{h(y,1)} = \sqrt{|\alpha|},$$

then

$$\frac{h(y,\alpha)}{h(y,1)} + \alpha > 0, \quad \forall y > 0.$$

Finally, we conclude that $\Re(t_2(iy))$ is positive and an increasing function of y so

$$\Re(t_2(iy)) \geq \Re(t_2(i\frac{\pi}{2})). \tag{3.3.12}$$

Therefore,

$$|\sinh(t_2(iy))|^2 \geq \frac{1}{2}\left[\cosh\left(2\Re(t_2(i\frac{\pi}{2}))\right) - 1\right] = f_1(\alpha). \tag{3.3.13}$$

In the second step, we find an upper bound for $\left|\dfrac{t_1(iy)}{t_2(iy)}\sinh(t_1(iy))\right|^2$. $\Im(t_2(iy))$ is a non decreasing function of y since $h(y,\alpha)$ is non decreasing. So

$$\Im(t_2(iy)) \geq \Im(t_2(i\frac{\pi}{2})). \tag{3.3.14}$$

Both (3.3.12) and (3.3.14) yield

$$|t_2(iy)|^2 \geq \left|t_2(i\frac{\pi}{2})\right|^2. \tag{3.3.15}$$

Next, we find an upper bound for $|t_1(iy)|$. We have

$$\frac{\partial}{\partial y}\Im(t_1(iy)) = \frac{\partial}{\partial y}h(y,1) - \frac{\partial}{\partial y}h(y,\alpha).$$

Knowing that $h(y,\alpha)$ is non decreasing, we compare the difference between the square of $\dfrac{\partial}{\partial y}h(y,1)$ and $\dfrac{\partial}{\partial y}h(y,\alpha)$. We find that

$$\left(\frac{\partial}{\partial y}h(y,\alpha)\right)^2 = \frac{(y+\sqrt{4\alpha^2+y^2})^3}{8(4\alpha^2 y+y^3)} = \varphi(\alpha^2),$$

where

$$\varphi(\beta) = \frac{(y + \sqrt{4\beta + y^2})^3}{8(4\beta y + y^3)}.$$

Deriving with respect to β, we get

$$\varphi'(\beta) = \frac{(y + \sqrt{4\beta + y^2})^2(4\beta + y^2 - 2y\sqrt{4\beta + y^2})}{4y(4\beta + y^2)^{\frac{5}{2}}}.$$

Therefore, $\varphi'(\beta) < 0$ if $y > \frac{2\sqrt{\beta}}{3} = \frac{2|\alpha|}{3}$. But $y \geq \frac{\pi}{2} > \frac{2}{3} > \frac{2|\alpha|}{3}$ which implies that $\varphi'(\beta) < 0$. Hence, for all $y \geq \frac{\pi}{2}$,

$$\left(\frac{\partial}{\partial y} h(y,1)\right)^2 - \left(\frac{\partial}{\partial y} h(y,\alpha)\right)^2 < 0,$$

and so, for all $y \geq \frac{\pi}{2}$, we get

$$\frac{\partial}{\partial y} h(y,1) - \frac{\partial}{\partial y} h(y,\alpha) < 0.$$

Therefore, $\Im(t_1(iy))$ is decreasing and

$$|\Im(t_1(iy))| \leq |\Im(t_1(i\frac{\pi}{2}))|. \tag{3.3.16}$$

On the other hand, we prove that $\Re(t_1(iy))$ is non decreasing and since $\Re(t_1(iy)) \to 1 - \alpha$ as $y \to +\infty$, we obtain

$$|\Re(t_1(iy))| < 1 - \alpha. \tag{3.3.17}$$

Consequently, by (3.3.16) and (3.3.17), we get

$$\begin{aligned} |\sinh t_1(iy)|^2 &= \frac{1}{2}\left[\cosh\left(2\Re(t_1(iy))\right) - \cos\left(2\Im(t_1(iy))\right)\right] \\ &\leq \frac{1}{2}\left[\cosh\left(2(1-\alpha)\right) - \cos\left(2\Im(t_1(i\frac{\pi}{2}))\right)\right]. \end{aligned} \tag{3.3.18}$$

Finally, (3.3.15), (3.3.16), (3.3.17), and (3.3.18) yield

$$\begin{aligned} \left|\frac{t_1(iy)}{t_2(iy)}\sinh(t_1(iy))\right|^2 &\leq \frac{1}{2}\frac{\left[(1-\alpha)^2 + \left(\Im(t_1(i\frac{\pi}{2}))\right)^2\right]\left[\cosh\left(2(1-\alpha)\right) - \cos\left(2\Im(t_1(i\frac{\pi}{2}))\right)\right]}{\left|t_2(i\frac{\pi}{2})\right|^2} \\ &= f_2(\alpha). \end{aligned} \tag{3.3.19}$$

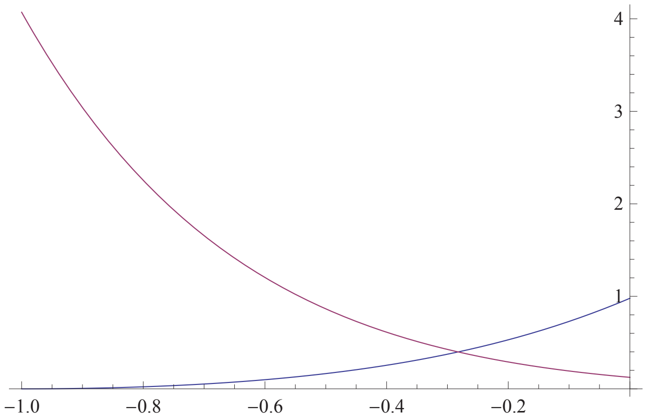

FIGURE 3.4 – Curves of f_1 (blue) and f_2 (red).

In conclusion, using (3.3.13) and (3.3.19) and according to the properties of f_1 and f_2 mentioned before (see Figure 3.4), we find out that if $\alpha > \alpha_3$, then $\left|\frac{t_1(iy)}{t_2(iy)}\sinh(t_1(iy))\right|^2 < |\sinh(t_2(iy))|^2$ and therefore the characteristic equation (3.3.10) has no roots on the imaginary axis due to (3.3.11). We deduce that for every $0 > \alpha > \alpha_3$, there does not exist any pure imaginary eigenvalue. By the continuity of the eigenvalues as a function of α, all the eigenvalues are situated to the left of the imaginary axis for every $0 > \alpha > \alpha_3$. Indeed, if we suppose the contrary, namely that there exists $\tilde{\alpha} \in [\alpha_3, 0[$ such that there exists some $\lambda_k(\tilde{\alpha})$ with $\Re\lambda_k(\tilde{\alpha}) > 0$. Since for $\alpha = 0$, all the eigenvalues are to the left of the imaginary axis because of the exponential stability, we deduce by continuity of the eigenvalues as a function of α that there exists $\tilde{\tilde{\alpha}} \in [\alpha_3, 0[$ with $\tilde{\alpha} < \tilde{\tilde{\alpha}} < 0$ such that there exists some pure imaginary eigenvalue associated with $\tilde{\tilde{\alpha}}$ which is a contradiction. ■

3.3.2 Root vectors, Riesz basis, and proof of Theorems 3.2.1 and 3.2.2

3.3.2.1 Root vectors

We start by introducing the root vectors of A_α and A_α^*, the adjoint of A_α. We notice that A_α has a compact resolvent, the geometric multiplicity of each eigenvalue is one, and, from Lemma 3.3.7, the large eigenvalues are simple.

Therefore, as in Section 6 of [23], for any $n \in \mathbb{Z}^*$, we denote the algebraic multiplicity of λ_n by m_n. To λ_n, define the associated Jordan chain of root vectors $\{U_{n,j}\}_{j=0}^{m_n-1}$ by

$$\begin{aligned} U_{n,0} &= y(x,\lambda_n)(1,\lambda_n)^\top \\ A_\alpha U_{n,j} &= \lambda_n U_{n,j} + U_{n,j-1}, \quad j = 0, ..., m_n - 1 \text{ with } U_{n,-1} = 0. \end{aligned}$$

As usual we choose the generalized eigenvectors such that $< U_{n,j}, U_{n,0} >= 0$, $j = 1, ..., m_n - 1$. Notice that for $|n|$ large, $m_n = 1$ and the root vectors are limited to the eigenvector $U_{n,0}$. According to Theorem 10.1 of Chapter V of [36], the root vectors of A_α are complete in X since A_α is a bounded perturbation of a skew symmetric operator. Moreover, the root vectors form a basis of the root subspace $\mathcal{L}_n = \{U \in D(A_\alpha);\ (A_\alpha - \lambda_n)^{m_n} U = 0\}$.

We now consider the root vectors of the adjoint of A_α given by

$$A_\alpha^* = \begin{pmatrix} 0 & -I \\ -\dfrac{\partial^2}{\partial x^2} & -2\chi_{(0,1)} - 2\alpha\chi_{(-1,0)} \end{pmatrix}.$$

Since $\sigma(A_\alpha^*) = \overline{\sigma(A_\alpha)} = \sigma(A_\alpha)$, we associate to $\overline{\lambda_n}$ the root vectors of A_α^* as

$$\begin{aligned} W_{n,0} &= y(x,\overline{\lambda_n})\,(1,-\overline{\lambda_n})^\top, \\ A_\alpha^* W_{n,j} &= \overline{\lambda_n} W_{n,j} + W_{n,j-1}, < W_{n,j}, U_{n,m_n-1} >= 0, \quad j = 1, ..., m_n - 1. \end{aligned}$$

$W_{n,0}$ is an eigenvector of A_α^* and, by completeness, $W_{n,j}$ are uniquely determined since $< W_{n,0}, U_{n,m_n-1} >\neq 0$. Other wise, $U_{n,m_n-1} = 0$ which is impossible.

3.3.2.2 Riesz basis

Here we adapt the results of Section 6 in [23] to prove that the root vectors of the operator A_α form a Riesz basis of the energy space X; i.e., we prove the following theorem.

Theorem 3.3.9. *The root vectors of the operator A_α form a Riesz basis of the energy space X.*

Proof: of Theorem 3.3.9.

We use the Bari's Theorem given by Theorem 1.2.6 (see Theorem 2.1 of Chapter VI in [36]). First, the completeness in X of the root vectors of A_α follows from Theorem 10.1 of Chapter V of [36]. So it remains to search for a biorthogonal sequence. For that purpose, we can follow the proof of Lemma 6.2 of [23]. From the proof of Lemma 6.2 of [23], we have

$$< U_{n,p}, W_{k,j} >=< U_{n,p}, W_{n,m_n-1-p} > \delta_{n,k}\delta_{m_n-1-p,j},$$

for all $p = 0, ..., m_n-1$, $j = 0, ..., m_k-1$, and for all $n, k \in \mathbb{Z}^*$, with $< U_{n,p}, W_{n,m_n-1-p} >\neq 0$. Indeed, for all $n \neq k$, we have

$$< A_\alpha U_{n,0}, W_{k,0} >= \lambda_n < U_{n,0}, W_{k,0} >= \lambda_k < U_{n,0}, W_{k,0} > .$$

Hence, $< U_{n,0}, W_{k,0} >= 0$. Next,

$$< A_\alpha U_{n,0}, W_{k,1} >= \lambda_n < U_{n,0}, W_{k,1} >= \lambda_k < U_{n,0}, W_{k,1} > + < U_{n,0}, W_{k,0} > .$$

Hence, $(\lambda_n - \lambda_k) < U_{n,0}, W_{k,1} >= 0$ and so $< U_{n,0}, W_{k,1} >= 0$. Proceeding similarly, we prove that

$$< U_{n,0}, W_{k,j} >= 0, \quad \forall j = 0, ..., m_n - 1.$$

Finally, by iteration, we can prove that

$$< U_{n,p}, W_{k,j} >= 0, \quad \forall p, j = 0, ..., m_n - 1, \ \forall n \neq k.$$

Now, if $n = k$, then Fredholm Alternative implies that

$$< U_{n,j}, W_{n,0} >= 0, \quad \forall j = 0, ..., m_n - 2.$$

Hence, by completeness, it follows that

$$< U_{n,m_n-1}, W_{n,0} > \neq 0.$$

Similarly,

$$< U_{n,0}, W_{n,j} >= 0, \quad \forall j = 0, ..., m_n - 2$$

and

$$< U_{n,0}, W_{n,m_n-1} > \neq 0.$$

After comparing $< A_\alpha U_{n,1}, W_{n,m_n-k} >$ with $< U_{n,1}, A_\alpha^* W_{n,m_n-k} >$, we find that

$$< U_{n,1}, W_{n,m_n-k-1} >=< U_{n,0}, W_{n,m_n-k} > .$$

Therefore, $U_{n,1}$ is orthogonal to each $W_{n,j}$ except when $j = m_n - 2$. Finally, by iteration, we find that $U_{n,p}$ is orthogonal to each $W_{n,j}$ except when $j = m_n - p - 1$. In conclusion, we have

$$< U_{n,p}, \frac{W_{n,m_n-1-j}}{< U_{n,j}, W_{n,m_n-1-j} >} >= \delta_{p,j}, \quad \forall p, j = 0, ..., m_n - 1.$$

However, the arguments above are sufficient for the low frequencies but for the high frequencies, in order to replace $\dfrac{W_{n,0}}{< U_{n,0}, W_{n,0} >}$ by $W_{n,0}$ in (1.2.1), we still need to show that $< U_{n,0}, W_{n,0} >$ does not degenerate as n becomes large. This is our next aim.

According to (3.3.2) and (3.3.3), we choose $U_{n,0}$ such that

$$U_{n,0|\,(0,1)} = y(x, \lambda_n)_{|\,(0,1)} \,(1, \lambda_n)^\top = \frac{b_n^+}{\sqrt{\lambda_n^2 + 2\lambda_n}} \sinh(\sqrt{\lambda_n^2 + 2\lambda_n}(x-1))(1, \lambda_n)^\top$$

and

$$U_{n,0|(-1,0)} = y(x,\lambda_n)_{|(-1,0)}\,(1,\lambda_n)^\top = \frac{b_n^-}{\sqrt{\lambda_n^2+2\alpha\lambda_n}}\sinh(\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1))\,(1,\lambda_n)^\top,$$

where b_n^+ and b_n^- are chosen such that (see (3.3.4))

$$b_n^+ = -b_n^-\frac{\sqrt{\lambda_n^2+2\lambda_n}}{\sqrt{\lambda_n^2+2\alpha\lambda_n}}\frac{\sinh\sqrt{\lambda_n^2+2\alpha\lambda_n}}{\sinh\sqrt{\lambda_n^2+2\lambda_n}}, \tag{3.3.20}$$

with b_n^- fixed such that $< U_{n,0}, U_{n,0} >= 1$.

But we have

$$\begin{aligned}
&< U_{n,0}, W_{n,0} > \\
&= \int_{-1}^{1}\left((y'(x,\lambda_n))^2 - \lambda_n^2(y(x,\lambda_n))^2\right)dx \\
&= (b_n^+)^2\int_0^1\left(\cosh^2(\sqrt{\lambda_n^2+2\lambda_n}(x-1)) - \frac{\lambda_n^2}{\lambda_n^2+2\lambda_n}\sinh^2(\sqrt{\lambda_n^2+2\lambda_n}(x-1))\right)dx \\
&+(b_n^-)^2\int_{-1}^0\left(\cosh^2(\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1)) - \frac{\lambda_n^2}{\lambda_n^2+2\alpha\lambda_n}\sinh^2(\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1))\right)dx.
\end{aligned}$$

Furthermore as $\Re\lambda_n$ is uniformly bounded, $\sinh(2\sqrt{\lambda_n^2+2\lambda_n}(x-1))$ and $\sinh(2\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1))$ are bounded (for $-1 \le x \le 1$). Hence,

$$\begin{aligned}
&\int_0^1\left(\cosh^2(\sqrt{\lambda_n^2+2\lambda_n}(x-1)) - \frac{\lambda_n^2}{\lambda_n^2+2\lambda_n}\sinh^2(\sqrt{\lambda_n^2+2\lambda_n}(x-1))\right)dx \\
=\;& 1+o_1(1), \\
&\int_{-1}^0\left(\cosh^2(\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1)) - \frac{\lambda_n^2}{\lambda_n^2+2\alpha\lambda_n}\sinh^2(\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1))\right)dx \\
=\;& 1+o_2(1).
\end{aligned}$$

Therefore,

$$< U_{n,0}, W_{n,0} >= (b_n^+)^2(1+o_1(1)) + (b_n^-)^2(1+o_2(1)). \tag{3.3.21}$$

But owing to (3.3.7), we see that

$$\sqrt{\lambda_n^2+2\lambda_n} = \frac{1-\alpha}{2} + i(n+m)\frac{\pi}{2} + O_1(\frac{1}{n}), \tag{3.3.22}$$

$$\sqrt{\lambda_n^2+2\alpha\lambda_n} = \frac{\alpha-1}{2} + i(n+m)\frac{\pi}{2} + O_2(\frac{1}{n}). \tag{3.3.23}$$

Therefore if $n+m$ is even we deduce that

$$\sinh\sqrt{\lambda_n^2+2\lambda_n}=\sinh(\frac{1-\alpha}{2}+o_3(1))\cos((n+m)\frac{\pi}{2}),$$
$$\sinh\sqrt{\lambda_n^2+2\alpha\lambda_n}=\sinh(\frac{\alpha-1}{2}+o_4(1))\cos((n+m)\frac{\pi}{2}),$$

and therefore

$$\frac{\sinh\sqrt{\lambda_n^2+2\alpha\lambda_n}}{\sinh\sqrt{\lambda_n^2+2\lambda_n}}=-1+o_5(1).$$

Similarly if $n+m$ is odd we show that

$$\frac{\sinh\sqrt{\lambda_n^2+2\alpha\lambda_n}}{\sinh\sqrt{\lambda_n^2+2\lambda_n}}=1+o_6(1).$$

These asymptotic behaviors in (3.3.20) lead to

$$(b_n^+)^2=(b_n^-)^2(1+o_7(1)), \tag{3.3.24}$$

and inserting this identity in (3.3.21) we arrive at

$$< U_{n,0}, W_{n,0} >= 2(b_n^-)^2(1+o_8(1)).$$

So we choose $\frac{W_{n,0}}{2(b_n^-)^2(1+o_8(1))}$ instead of $W_{n,0}$, but as mentioned before we have to show that b_n^- does not tend to zero as n tends to infinity. Actually by similar calculations as before, we check that

$$< U_{n,0}, U_{n,0} >= |b_n^+|^2(\delta+o_9(1))+|b_n^-|^2(\delta+o_{10}(1)),$$

where

$$\delta=\frac{\sinh(1-\alpha)}{1-\alpha},$$

that is positive since $\alpha<0$. Therefore with the help of (3.3.24) we get

$$< U_{n,0}, U_{n,0} >= 2|b_n^-|^2(\delta+o_{11}(1)),$$

and consequently

$$|b_n^-|^2 = \frac{1}{2\delta} + o_{12}(1).$$

In summary, by fixing N large enough such that for $|n| \geq N$, $m_n = 1$, we have proved that the family

$$\left\{ \left\{ \frac{W_{n,m_n-1-j}}{< U_{n,j}, W_{n,m_n-1-j} >} \right\}_{j=0}^{m_n-1} \right\}_{0<|n|<N} \bigcup \left\{ \frac{W_{n,0}}{2(b_n^-)^2(1+o_8(1))} \right\}_{|n|\geq N}$$

is biorthogonal to the set of root vectors of A_α.

It remains to prove (1.2.1). We first prove that for any $(f,g) \in X$ and for all N large, the sum $S = \sum_{n>N} |< U_{n,0}, (f,g) >|^2$ is finite. In fact,

$$\begin{aligned} S &= \sum_{n>N} \Big| \int_0^1 b_n^+ \Big(\cosh(\sqrt{\lambda_n^2 + 2\lambda_n}(x-1))\overline{f'}(x) \\ &\quad + \frac{\lambda_n}{\sqrt{\lambda_n^2+2\lambda_n}} \sinh(\sqrt{\lambda_n^2+2\lambda_n}(x-1))\overline{g}(x) \Big) \, dx \\ &\quad + \int_{-1}^0 b_n^- \Big(\cosh(\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1))\overline{f'}(x) \\ &\quad + \frac{\lambda_n}{\sqrt{\lambda_n^2+2\alpha\lambda_n}} \sinh(\sqrt{\lambda_n^2+2\alpha\lambda_n}(x+1))\overline{g}(x) \Big) \, dx \Big|^2. \end{aligned}$$

Noting that $\dfrac{\lambda_n}{\sqrt{\lambda_n^2+2\lambda_n}} = 1 + O_3(\dfrac{1}{n})$ and $\dfrac{\lambda_n}{\sqrt{\lambda_n^2+2\alpha\lambda_n}} = 1 + O_4(\dfrac{1}{n})$; hence, in order to prove that the sum S is finite, we will only prove that

$$S_1 = \sum_{n>N} \left| \int_0^1 \cosh(\sqrt{\lambda_n^2+2\lambda_n}(x-1))\overline{f'}(x)\, dx \right|^2 < \infty,$$

since the convergence of the other terms appearing in S follows similarly. According to (3.3.22), we have

$$\sqrt{\lambda_n^2+2\lambda_n} = \gamma + i(n+m)\frac{\pi}{2} + \delta_n.$$

where $\gamma = \dfrac{1-\alpha}{2}$ and $\delta_n = O_1(\dfrac{1}{n})$. Therefore, we can write

$$\cosh(\sqrt{\lambda_n^2+2\lambda_n}(x-1)) = \frac{1}{2}(e^{(\gamma+i(n+m)\frac{\pi}{2}+\delta_n)(x-1)} + e^{-(\gamma+i(n+m)\frac{\pi}{2}+\delta_n)(x-1)}).$$

So in order to prove that S_1 is finite, we will only prove that

$$\sum_{n>N}\left|\int_0^1 e^{(\gamma+i(n+m)\frac{\pi}{2}+\delta_n)(x-1)}\overline{f'}(x)\right|^2<\infty$$

since the convergence of the other term appearing in S_1 follows similarly. But the convergence of this series is a simple application of Lemma 3.2 of [70] (see also [38, Lemma 4.1]) since the sequence $(\gamma+i(n+m)\frac{\pi}{2}+\delta_n)_{n>N}$ satisfies the conditions of this Lemma and since $f\in H^1(0,1)$. Therefore, $e^{(\gamma+i(n+m)\frac{\pi}{2}+\delta_n)(x-1)}$ is a Bessel sequence. We can also use the mean value Theorem and Fourier series to prove the convergence of this series. Indeed,

$$\begin{aligned}
&\sum_{n>N}\left|\int_0^1 e^{(\gamma+\delta_n+i\frac{(n+m)\pi}{2})(x-1)}\overline{f'}(x)\right|^2\\
&=\sum_{n>N}\left|\int_0^1 (e^{\delta_n(x-1)}-1+1)e^{\gamma(x-1)}e^{i\frac{(n+m)\pi}{2}(x-1)}\overline{f'}(x)\right|^2\\
&\lesssim\sum_{n>N}\int_0^1\left|(e^{\delta_n(x-1)}-1)e^{\gamma(x-1)}\overline{f'}(x)\right|^2+\sum_{n>N}\left|\int_0^1 e^{i\frac{(n+m)\pi}{2}(x-1)}e^{\gamma(x-1)}\overline{f'}(x)\right|^2\\
&\lesssim\sum_{n>N}|\delta_n|^2\int_0^1|(x-1)|^2e^{2\gamma(x-1)}|\overline{f'}(x)|^2dx+\|e^{\gamma(x-1)}f'\|^2_{(0,1)}\\
&\lesssim\|e^{\gamma(x-1)}f'\|^2_{(0,1)}\sum_{n>N}|\delta_n|^2+\|e^{\gamma(x-1)}f'\|^2_{(0,1)}\\
&\lesssim\|e^{\gamma(x-1)}f'\|^2_{(0,1)}\sum_{n>N}\frac{1}{n^2}+\|e^{\gamma(x-1)}f'\|^2_{(0,1)}.
\end{aligned}$$

The convergence of $\sum_{n>N}|< W_{n,0},(f,g)>|^2$ for any $(f,g)\in X$ and for all N large is proved in the same manner since

$$< W_{n,0},(f,g)>=<\bar{U}_{n,0},(f,g)>=\overline{< U_{n,0},(\bar{f},\bar{g})>}.$$

Therefore, the conditions of Theorem 1.2.6 are all fulfilled and hence the root vectors of A_α form a Riesz basis of X. ■

3.3.2.3 Proof of Theorems 3.2.1 and 3.2.2

Since the generalized eigenvectors, $\{U_{n,j}\}_{n,j}$, of A_α form a Riesz basis of X, then given the initial datum

$$U(0) = \sum_{n=\pm 1}^{\pm\infty} \sum_{j=0}^{m_n-1} \gamma_{n,j} U_{n,j},$$

we can write

$$U(t) = (u, u_t)^\top = \sum_{n=\pm 1}^{\pm\infty} e^{\lambda_n t} \sum_{j=0}^{m_n-1} \gamma_{n,j} \sum_{k=0}^{j} \frac{t^{j-k}}{(j-k)!} U_{n,k}.$$

Since the low frequencies are of finite multiplicity, then denoting the maximum multiplicity by m, we get for any $\epsilon > 0$

$$E_1(t) \lesssim E_1(0)(1+t^{2m})e^{2\mu(A_\alpha)t} \lesssim E_1(0)e^{(2\mu(A_\alpha)+\epsilon)t}. \tag{3.3.25}$$

As $\mu(A_\alpha) < 0$ for $\alpha \in]\alpha_3, 0]$ ($\alpha_3 \approx -0.2823$), we can choose $0 < \epsilon < -\mu(A_\alpha)$ to get the exponential stability of problem (3.1.1) and hence the proof of Theorem 3.2.2 is complete.

The proof of Theorem 3.2.1 is similar since for $\alpha + 1 > 0$, by Remark 3.3.4 and Lemma 3.3.5, at most a finite number of eigenvalues of A_α may be situated on the imaginary axis or to its right ; consequently, excluding the finite dimensional space spanned by the corresponding root vectors, we obtain an exponential decay.

3.4 Exponential stability for an indefinite sign internally and boundary damped problem

In this section, we perform a similar analysis for problem (3.1.2) which contains both an internal and a boundary indefinite sign damping term. Recall that (3.1.2) is the problem

$$\begin{aligned} &u_{tt}(x,t) - u_{xx}(x,t) + au_t(x,t) = 0, && x \in (0,1),\ t > 0, \\ &u(0,t) = 0,\ u_x(1,t) = -bu_t(1,t), && t > 0, \\ &u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), && \end{aligned} \tag{3.4.1}$$

where $a, b \in \mathbb{R}$. If a and b are both non negative and one of them is positive, then, using integral inequalities for instance, one can show that (3.4.1) is exponentially stable. Our aim is to find sufficient conditions on a and b so that (3.4.1) is exponentially or polynomially stable whatever the sign of a and b.

The energy of (3.4.1) is given by

$$E_2(t) = \frac{1}{2} \int_0^1 \left(|u_t|^2 + |u_x|^2 \right) dx,$$

and hence formally

$$E_2'(t) = -a \int_0^1 |u_t|^2 dx - b|u_t(1)|^2.$$

From this identity, we see that if $a \leq 0$ and $b \leq 0$, then $E_2'(t) \geq 0$. Therefore, the energy increases and no stability can be hoped. Therefore, the only case of interest is the case when a and b are of opposite signs. We, therefore, assume that $ab < 0$. We further assume that $b \notin \{-1, 0, 1\}$. Indeed the case $b = 0$ has no interest since only the case $a > 0$ yields stability results; while the case $b = 1$ or -1 is excluded for technical reasons (see Subsection 3.4.3).

3.4.1 Well-posedness of problem (3.4.1)

As usual, by the standard reduction of order method, we can rewrite formally (3.4.1) in the simpler form $U_t = A_a U$, where $U = (u, u_t)^\top$ and the operator $A_a : D(A_a) \to X$ is defined by

$$A_a = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & -a \end{bmatrix} \tag{3.4.2}$$

where the energy space $X = H_l(0,1) \times L^2(0,1)$ is equipped with the usual inner product defined by

$$< (f,g)^\top, (u,v)^\top >= \int_0^1 (f'\overline{u}' + g\overline{v})\, dx,$$

where $H_l(0,1) = \{u \in H^1(0,1);\ u(0) = 0\}$ and therefore, $D(A_a) = \{(u,v)^\top \in H^2(0,1) \cap H_l(0,1) \times H_l(0,1);\ u_x(1) = -bv(1)\}$.

First, we remark that A_a is not necessarily dissipative so we propose to write $A_a = A_0 - aB$ where

$$A_0 = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \tag{3.4.3}$$

and

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, A_a is a bounded perturbation of a non skew adjoint operator A_0. Nevertheless, we will prove in Subsection 3.4.4 that if $b \neq -1$, then A_0 generates a C_0 semigroup and since $aB \in \mathcal{L}(X)$, A_a will be also a generator of a C_0 semigroup according to Theorem 3.1.1 in [62]. Therefore problem (3.4.1) is a well posed problem (see Theorem 3.4.7).

3.4.2 Eigenvalue Problem

In this part, we find the characteristic equation of the eigenvalues associated with (3.4.1). Let $(y,z)^\top \in D(A_a)$, $(y,z) \neq (0,0)$, such that $A_a(y,z)^\top = \lambda(y,z)^\top$ then $z = \lambda y$ and

$$\begin{aligned} &y_{xx} - \lambda^2 y - a\lambda y = 0 \quad \text{in } (0,1), \\ &y(0) = 0, \quad y_x(1) = -b\lambda y(1). \end{aligned} \tag{3.4.4}$$

First, it is easy to see that $\lambda = 0$ is not an eigenvalue of A_a. Furthermore, if $\lambda = -a$, then $y = cx$ with $c = abc$ which satisfies the boundary condition at 1. Since $y \neq 0$, we have $c \neq 0$, and get $ab = 1$ which is impossible since we have assumed that $ab < 0$.

Now if $\lambda \neq 0$ and $\lambda \neq -a$, then there exists $c_1 \in \mathbb{C}^*$ such that

$$y(x) = c_1 \sinh \sqrt{\lambda^2 + a\lambda} x.$$

Hence, the boundary condition at 1 becomes

$$y_x(1) = c_1\sqrt{\lambda^2 + a\lambda} \cosh \sqrt{\lambda^2 + a\lambda} = -b\lambda c_1 \sinh \sqrt{\lambda^2 + a\lambda}.$$

As $c_1 \neq 0$ then λ is an eigenvalue of A_a if it satisfies the characteristic equation

$$F_a(\lambda) = \sqrt{\lambda^2 + a\lambda} \cosh \sqrt{\lambda^2 + a\lambda} + b\lambda \sinh \sqrt{\lambda^2 + a\lambda} = 0. \tag{3.4.5}$$

Integrating (3.4.4) against $\overline{y(x,\lambda)}$ and performing an integration by parts, we get

$$\lambda^2 \int_0^1 |y|^2 dx + b\lambda |y(1)|^2 + a\lambda \int_0^1 |y|^2 dx + \int_0^1 |y_x|^2 dx = 0.$$

Therefore,

$$\lambda = \frac{-\left[b|y(1)|^2 + a\int_0^1 |y|^2 dx\right] \pm \left[\left(a\int_0^1 |y|^2 dx + b|y(1)|^2\right)^2 - 4\left(\int_0^1 |y_x|^2 dx\right)\left(\int_0^1 |y|^2 dx\right)\right]^{\frac{1}{2}}}{2\int_0^1 |y|^2 dx}.$$

If $\lambda \in \mathbb{C}\backslash\mathbb{R}$, then

$$\Re\lambda = \frac{-\left(b|y(1)|^2 + a\int_0^1 |y|^2 dx\right)}{2\int_0^1 |y|^2 dx} \tag{3.4.6}$$

and since $y \in \{u \in H^1(0,1) : u(0) = 0\}$, we get

$$(\Im\lambda)^2 + (\Re\lambda)^2 = \frac{\int_0^1 |y_x|^2 dx}{\int_0^1 |y|^2 dx} \geq \frac{\pi^2}{4}. \tag{3.4.7}$$

Remark 3.4.1. *If $b < 0$ and $a > 0$, then whenever $\lambda \in \mathbb{C}\backslash\mathbb{R}$, from (3.4.6), we deduce that $\Re\lambda > -\frac{a}{2}$. This estimate can be even extended to all eigenvalues of A_a. Indeed, for $(u,v)^\top \in D(A_a)$, by integration by parts, we check that*

$$2\Re < A_a(u,v)^\top, (u,v)^\top >= -a \int_0^1 |v|^2\, dx - b|v(1)|^2.$$

In particular, if λ is an eigenvalue of A_a with associated normalized eigenvector $(u,v)^\top$, we will get

$$2\Re\lambda = -a \int_0^1 |v|^2\, dx - b|v(1)|^2.$$

Since $\int_0^1 |v|^2\, dx \leq< (u,v)^\top, (u,v)^\top >= 1$, and recalling that we here assume that $b < 0$, we then conclude that

$$2\Re\lambda \geq -a \int_0^1 |v|^2\, dx \geq -a.$$

In summary we have proved the next result.

Lemma 3.4.2. *A_a^{-1} is compact. Hence the spectrum of A_a consists of discrete eigenvalues with finite algebraic multiplicity. Furthermore*

$$\sigma(A_a) = \{\lambda \in \mathbb{C} \setminus \{0, -a\} \,:\, \lambda \text{ is a root of } (3.4.5)\}.$$

Concerning the multiplicity of the eigenvalues, we show the following Lemma :

Lemma 3.4.3. *The high frequencies of A_a are simple and there exists at most two double low frequency eigenvalues.*

Proof: We derive (3.4.5) with respect to λ to get

$$\begin{aligned} F_a'(\lambda) &= \frac{\sqrt{\lambda^2 + a\lambda}(2\lambda + a + 2b)\sinh\sqrt{\lambda^2 + a\lambda} + (2\lambda + a)(b\lambda + 1)\cosh\sqrt{\lambda^2 + a\lambda}}{2\sqrt{\lambda^2 + a\lambda}} \\ &= \frac{g(\lambda)}{2\sqrt{\lambda^2 + a\lambda}}. \end{aligned}$$

If $F_a(\lambda) = 0$, then

$$\cosh\sqrt{\lambda^2 + a\lambda} = -\frac{b\lambda\sinh\sqrt{\lambda^2 + a\lambda}}{\sqrt{\lambda^2 + a\lambda}}.$$

Substituting into $g(\lambda)$, we get

$$g(\lambda) = \frac{\lambda\sinh\sqrt{\lambda^2 + a\lambda}}{\sqrt{\lambda^2 + a\lambda}}\left(2(1-b^2)\lambda^2 + a(3-b^2)\lambda + a(a+b)\right).$$

Since $\lambda \neq 0$ and $\lambda \neq -a$, then $g(\lambda) = 0$ is equivalent to $2(1-b^2)\lambda^2 + a(3-b^2)\lambda + a(a+b) = 0$ which only has two roots. ■

3.4.3 Asymptotic Development of the High Frequencies

In this subsection, using Taylor expansions, we prove that the high frequencies approach the axis $x = -\frac{a}{2} - \Re\tanh^{-1}\frac{1}{b}$. In fact, (3.4.5) implies that for $b \neq 0$

$$-\frac{\sqrt{\lambda^2 + a\lambda}}{b\lambda} = \tanh(\sqrt{\lambda^2 + a\lambda}); \tag{3.4.8}$$

hence,

$$\sqrt{\lambda^2 + a\lambda} = -\tanh^{-1}\left(\frac{\sqrt{\lambda^2 + a\lambda}}{b\lambda}\right) + ik_1\pi, \quad k_1 \in \mathbb{Z}. \tag{3.4.9}$$

For large λ, we write

$$\sqrt{\lambda^2 + a\lambda} = \lambda + \frac{a}{2} - \frac{a^2}{8\lambda} + o\left(\frac{1}{\lambda}\right). \tag{3.4.10}$$

Moreover, for large λ, there exists $k_2 \in \mathbb{Z}$ such that we have

$$\begin{aligned}
\tanh^{-1}\left(\frac{\sqrt{\lambda^2+a\lambda}}{b\lambda}\right) &= \tanh^{-1}\left(\frac{1}{b}+\frac{a}{2b\lambda}+o\left(\frac{1}{\lambda}\right)\right)\\
&= \frac{1}{2}\log\left(1+\frac{1}{b}\right)+\frac{1}{2}\log\left(1+\frac{a}{2(b+1)\lambda}+o\left(\frac{1}{\lambda}\right)\right)\\
&-\frac{1}{2}\log\left(1-\frac{1}{b}\right)-\frac{1}{2}\log\left(1-\frac{a}{2(b-1)\lambda}+o\left(\frac{1}{\lambda}\right)\right)+i\pi k_2\\
&= \tanh^{-1}\frac{1}{b}+\frac{a}{4\lambda}\left[\frac{1}{(b+1)}+\frac{1}{(b-1)}\right]+i\pi k_2+o\left(\frac{1}{\lambda}\right)\\
&= \tanh^{-1}\frac{1}{b}+i\pi k_2+\frac{ab}{2\lambda(b^2-1)}+o\left(\frac{1}{\lambda}\right).
\end{aligned} \tag{3.4.11}$$

Substituting (3.4.10) and (3.4.11) into (3.4.9), we get that for $\lambda \in \sigma(A_a)$ with $|\lambda|$ large enough, there exists $k \in \mathbb{Z}$ such that

$$\lambda = -\frac{a}{2}-\tanh^{-1}\frac{1}{b}+ik\pi+\frac{a^2}{8\lambda}+\frac{ab}{2\lambda(1-b^2)}+o\left(\frac{1}{\lambda}\right). \tag{3.4.12}$$

In order to get $\Re\lambda < 0$ for λ large, we need that

$$-\frac{a}{2}-\Re\tanh^{-1}\frac{1}{b}<0. \tag{3.4.13}$$

As before we illustrate in Figures 3.5 and 3.6, the roots of F_a for $a=1, b=-0.5$ and $a=1.1, b=-2$ respectively, computed using the same scheme as before. In both cases, $-\frac{a}{2}-\Re\tanh^{-1}\frac{1}{b}$ is negative and the asymptotic behavior (3.4.12) is clearly confirmed. Note that in the first case, all eigenvalues are in the left of the imaginary axis, which is in accordance with Theorem 3.2.4. On the other hand, the second example does not enter in the framework of Theorem 3.2.4 and shows that an exponential stability result cannot hold if $b<-1$ but confirms Theorem 3.2.3.

Note that in Figure 3.6, some eigenvalues with positive real part like

$$\lambda = 0.6910478014752763 \text{ or } \lambda = 0.0123963241846109 01 \pm 2.9711251755632886i,$$

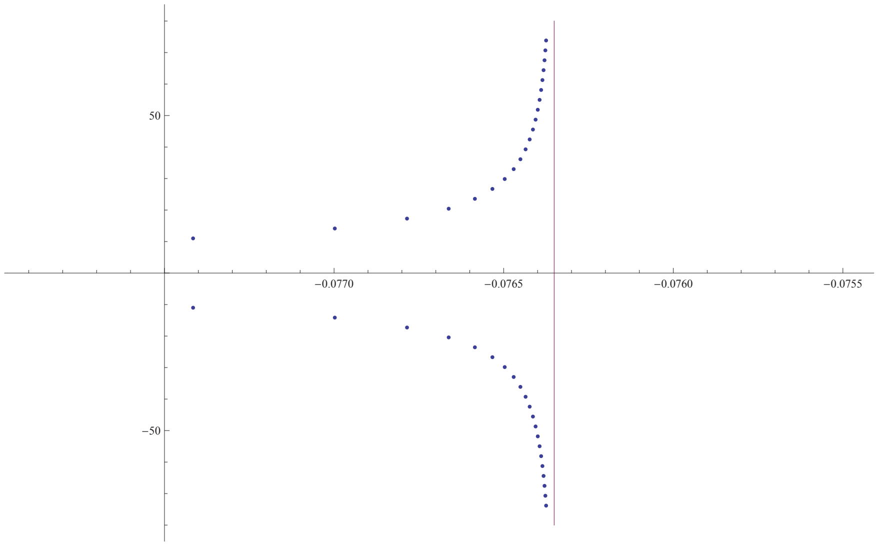

FIGURE 3.5 – Eigenvalues for $a = 1,\ b = -0.5$

are not represented because they are too far from the other eigenvalues. A representation in a larger scale is possible but would avoid to see the main part of the spectrum.

3.4.4 Riesz Basis of X and a note on the well-posedness of problem (3.4.1)

In this part, we prove that the eigenvectors associated with the problem with $a = 0$ form a Riesz basis of X if $b \notin \{-1, 0, 1\}$. In other words, consider the problem

$$\begin{aligned} &\widetilde{u}_{tt}(x,t) - \widetilde{u}_{xx}(x,t) = 0, && x \in (0,1),\ t > 0, \\ &\widetilde{u}(0,t) = 0,\ \widetilde{u}_x(1,t) = -b\widetilde{u}_t(1,t), && t > 0, \\ &\widetilde{u}(x,0) = \widetilde{u}_0(x),\ \widetilde{u}_t(x,0) = \widetilde{u}_1(x). && \end{aligned} \tag{3.4.14}$$

(3.4.14) is equivalent to $\widetilde{U}_t = A_0\widetilde{U}$ with $\widetilde{U}_t = (\widetilde{u}, \widetilde{u}_t)^\top$ and A_0 is given by (3.4.3).

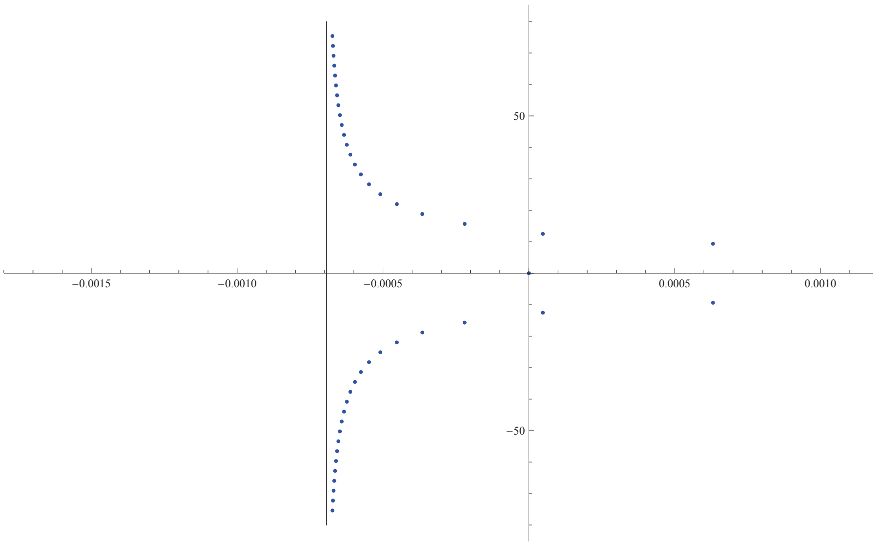

FIGURE 3.6 – Eigenvalues for $a = 1.1,\ b = -2$

By the previous analysis, we know that $\widetilde{\lambda}$ is an eigenvalue of A_0 if and only if

$$\tanh \widetilde{\lambda} = -\frac{1}{b};$$

or equivalently

$$\widetilde{\lambda} = \widetilde{\lambda}_k = -\tanh^{-1}\left(\frac{1}{b}\right) + ik\pi = c(b) + ik\pi, \text{ for some } k \in \mathbb{Z}. \tag{3.4.15}$$

Notice that the root vectors of A_0 are restricted to its eigenvectors since the eigenvalues $\widetilde{\lambda}_k$ are simple. In the sequel, we prove the next Riesz basis property.

Theorem 3.4.4. *The family*

$$\{\widetilde{\phi}_k\}_{k\in\mathbb{Z}} = \{(\widetilde{y}_k, \widetilde{\lambda}_k \widetilde{y}_k)\}_{k\in\mathbb{Z}} = \left\{\left(\frac{1}{\widetilde{\lambda}_k}\sinh(\widetilde{\lambda}_k \cdot), \sinh(\widetilde{\lambda}_k \cdot)\right)\right\}_{k\in\mathbb{Z}}$$

forms a Riesz basis of $X = H_l(0,1) \times L^2(0,1)$.

For this aim, we again use Bari's criterion stated in Theorem 1.2.6.

Lemma 3.4.5. *The sequence $\{\widetilde{\phi}_k\}_k$ is complete in X.*

Proof: It suffices to show that any element of X orthogonal to all the $\widetilde{\phi}_k$ is zero. Hence let $(f,g)^\top \in X$ be such that $< (f,g)^\top, \widetilde{\phi}_k >_X = 0$ for all $k \in \mathbb{Z}$. Then we get

$$\begin{aligned} 0 &= 2\int_0^1 \left(\overline{f_x}\cosh(\widetilde{\lambda}_k x) + \overline{g}\sinh(\widetilde{\lambda}_k x)\right) dx \\ &= \int_0^1 \left((\overline{f_x+g})e^{c(b)x}e^{ik\pi x} + (\overline{f_x-g})e^{-c(b)x}e^{-ik\pi x}\right) dx, \quad \forall k \in \mathbb{Z}. \end{aligned} \tag{3.4.16}$$

In particular, for $k=0$, we have

$$\int_0^1 \left((\overline{f_x+g})e^{c(b)x} + (\overline{f_x-g})e^{-c(b)x}\right) dx = 0. \tag{3.4.17}$$

Moreover, for $k<0$, we write $k=-k'$ with $k' \in \mathbb{N}^*$ to obtain

$$\int_0^1 \left((\overline{f_x+g})e^{c(b)x}e^{-ik'\pi x} + (\overline{f_x-g})e^{-c(b)x}e^{ik'\pi x}\right) dx = 0, \quad \forall k' \in \mathbb{N}^*. \tag{3.4.18}$$

Adding (3.4.16) for $k=k'>0$ with (3.4.18) yields

$$\int_0^1 \overline{h(x)}\left(\frac{e^{ik\pi x}+e^{-ik\pi x}}{2}\right) dx = 0, \quad \forall k \in \mathbb{N}^*,$$

where

$$h(x) = (f_x+g)(x)\overline{e^{c(b)x}} + (f_x-g)(x)\overline{e^{-c(b)x}}.$$

Since $\{\cos(k\pi x)\}_{k\in\mathbb{N}}$ is a basis of $L^2(0,1)$, we get

$$h = 0. \tag{3.4.19}$$

Subtracting (3.4.16) from (3.4.18), we get

$$\int_0^1 \overline{K(x)}\left(\frac{e^{ik\pi x}-e^{-ik\pi x}}{2}\right) dx = 0, \quad \forall k \in \mathbb{N}^*,$$

where

$$K(x) = (f_x+g)(x)\overline{e^{c(b)x}} - (f_x-g)(x)\overline{e^{-c(b)x}}.$$

Since $\{\sin(k\pi x)\}_{k\in\mathbb{N}^*}$ forms a basis of $L^2(0,1)$, we get

$$K = 0. \tag{3.4.20}$$

(3.4.19) and (3.4.20) imply that $f_x = g = 0$ and so $f = 0$ since $f(0) = 0$. ■

In a second step, we search for a sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ biorthogonal to $\{\widetilde{\phi}_k\}_{k\in\mathbb{Z}}$. Here we choose $\{\psi_k\}_{k\in\mathbb{Z}} = \left\{\left(\frac{1}{\overline{\widetilde{\lambda}}_k}\sinh(\overline{\widetilde{\lambda}_k}\cdot), -\sinh(\overline{\widetilde{\lambda}_k}\cdot)\right)\right\}_{k\in\mathbb{Z}}$ where $\overline{\widetilde{\lambda}}_k$ is the conjugate of $\widetilde{\lambda}_k$. The same arguments as before show that this set is complete. Indeed, for $k\in\mathbb{Z}$, ψ_k is an eigenvector of the adjoint of A_0.

Lemma 3.4.6. *The set $\{\psi_k\}_{k\in\mathbb{Z}}$ is biorthogonal to $\{\widetilde{\phi}_k\}_{k\in\mathbb{Z}}$.*

Proof: By definition, we have

$$\begin{aligned} <\widetilde{\phi}_k, \psi_l>_X &= \int_0^1 \Big(\cosh(\widetilde{\lambda}_k x)\cosh(\widetilde{\lambda}_l x) - \sinh(\widetilde{\lambda}_k x)\sinh(\widetilde{\lambda}_l x)\Big)\, dx \\ &= \int_0^1 \cosh\Big((\widetilde{\lambda}_k - \widetilde{\lambda}_l)x\Big)\, dx \\ &= \int_0^1 \cos\left((k-l)\pi x\right) dx = \delta_{kl}. \end{aligned}$$

■

Finally, in order to apply Bari's Theorem, it remains to prove (1.2.1). Let $(f,g)^\top \in X$ and consider the following sum

$$\sum_{k\in\mathbb{Z}} \left| <(f,g)^\top, \widetilde{\phi}_k>_X \right|^2 \lesssim \sum_{k\in\mathbb{Z}} \left|(f_x, \cosh(\widetilde{\lambda}_k\cdot))_{(0,1)}\right|^2 + \sum_{k\in\mathbb{Z}} \left|(g, \sinh(\widetilde{\lambda}_k\cdot))_{(0,1)}\right|^2. \tag{3.4.21}$$

By (3.4.15) and Parseval's identity, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left|(f_x,\cosh(\widetilde{\lambda}_k\cdot))_{(0,1)}\right|^2 &\lesssim \sum_{k\in\mathbb{Z}}\left|(f_x\overline{e^{c(b)\cdot}},e^{ik\pi\cdot})_{(0,1)}\right|^2+\sum_{k\in\mathbb{Z}}\left|(f_x\overline{e^{-c(b)\cdot}},e^{-ik\pi\cdot})_{(0,1)}\right|^2\\ &\le\left(\|f_x\overline{e^{c(b)\cdot}}\|^2_{(0,1)}+\|f_x\overline{e^{-c(b)\cdot}}\|^2_{(0,1)}\right)\\ &\lesssim \|f_x\|^2_{(0,1)},\end{aligned} \tag{3.4.22}$$

and

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left|(g,\sinh(\widetilde{\lambda}_k\cdot))_{(0,1)}\right|^2 &\lesssim \sum_{k\in\mathbb{Z}}\left|(g\overline{e^{c(b)\cdot}},e^{ik\pi\cdot})_{(0,1)}\right|^2+\sum_{k\in\mathbb{Z}}\left|(g\overline{e^{-c(b)\cdot}},e^{-ik\pi\cdot})_{(0,1)}\right|^2\\ &\le\left(\|g\overline{e^{c(b)\cdot}}\|^2_{(0,1)}+\|g\overline{e^{-c(b)\cdot}}\|^2_{(0,1)}\right)\\ &\lesssim \|g\|^2_{(0,1)}.\end{aligned} \tag{3.4.23}$$

(3.4.22) and (3.4.23) imply that the right-hand side of (3.4.21) is finite. Similarly, we prove that

$$\sum_{k\in\mathbb{Z}}\left|<(f,g)^\top,\psi_k>_X\right|^2<\infty.$$

In conclusion, by Theorem 1.2.6, the family $\{\tilde{\phi}_k\}_{k\in\mathbb{Z}}$ forms a Riesz basis of X. Theorem 3.4.4 is then proved.

Theorem 3.4.7. *If $b\notin\{-1,0,1\}$, then problem (3.4.1) is well posed.*

Proof: If we consider the problem associated with A_0; i.e.,

$$\begin{cases} U_t(t)=A_0U(t), & t>0,\\ U(0)=U_0, \end{cases} \tag{3.4.24}$$

then writing $U(0) = \sum_{k\in\mathbb{Z}} \alpha_k \widetilde{\phi}_k$ yields

$$U(t) = \sum_{k\in\mathbb{Z}} e^{\widetilde{\lambda}_k t} \alpha_k \widetilde{\phi}_k.$$

Therefore, by the Riesz property of the sequence $\{\widetilde{\phi}_k\}_k$ and (3.4.15), we get

$$\|U(t)\|_X^2 \simeq \sum_{k\in\mathbb{Z}} \left| e^{\widetilde{\lambda}_k t} \alpha_k \right|^2 \lesssim e^{-2\Re \tanh^{-1}\left(\frac{1}{b}\right)t} \|U(0)\|_X^2. \tag{3.4.25}$$

Hence, if $V = \sum_{k\in\mathbb{Z}} \beta_k \widetilde{\phi}_k$ is given, then we define

$$S(t)V = \sum_{k\in\mathbb{Z}} e^{\widetilde{\lambda}_k t} \beta_k \widetilde{\phi}_k, \quad \forall t > 0.$$

According to (3.4.25), we have $S(t) \in \mathcal{L}(X)$ with

$$\|S(t)\|_{\mathcal{L}(X)} \leq M e^{-\Re \tanh^{-1}\left(\frac{1}{b}\right)t},$$

for some positive constant M. Hence we deduce that $(S(t))_{t\geq 0}$ is a C_0 semigroup (not necessarily uniformly bounded). As we can write $A_a = A_0 + aB$ where A_0 generates the C_0 semigroup $(S(t))_{t\geq 0}$ and B is bounded, then, by Theorem 3.1.1 in [62], A_a also generates a C_0 semigroup $(S_a(t))_{t\geq 0}$ that satisfies

$$\|S_a(t)\|_{\mathcal{L}(X)} \leq M e^{(-\Re \tanh^{-1}(\frac{1}{b}) + |a|\|B\|)t}, \quad \forall t > 0.$$

By standard semigroup theory, problem (3.4.1) becomes well posed. ∎

Remark 3.4.8. *If $b = -1$, our previous considerations show that A_0 has an empty spectrum. Therefore, our method does not allow to prove that it generates a C_0 semigroup and hence, the well posedness of problem (3.4.1) becomes an open question.*

On the other hand, if $b = 1$, the operator A_0 generates a C_0 semigroup $(S(t))_{t\geq 0}$ that satisfies

$$\|S(t)\|_{\mathcal{L}(X)} \leq M e^{-\omega t},$$

for some positive constants M and ω. Therefore, by the previous arguments, problem (3.4.1) is well posed and is stable if $a < 0$ is small enough by perturbation theory but the question of its stability for $a < 0$ "large" is an open question.

3.4.5 Link between problems (3.1.2) and (3.4.14) and end of the Proof of Theorem 3.2.3

In this part, we prove that the root vectors of A_a form a Riesz basis of X if $b \notin \{-1, 0, 1\}$. For this aim, we will apply Theorem 1.2.8 with the set $\{\widetilde{\phi}_k\}_{k\in\mathbb{Z}}$ the Riesz basis associated with problem (3.4.14) and for an appropriate set of eigenvectors of A_a (corresponding to large eigenvalues). Indeed in view of Lemma 3.4.3 and (3.4.12), we split up the spectrum of A_a into the small and large eigenvalues :

$$\sigma(A_a) = \{\lambda_{k_i}\}_{i=1}^{M} \cup \{\lambda_k\}_{|k|>N}, \tag{3.4.26}$$

where $N \in \mathbb{N}$ is chosen large enough such that, for every $k \in \mathbb{Z}^*$ with $|k| > N$, $|\lambda_k + \frac{a}{2} + \tanh^{-1}\frac{1}{b} - ik\pi| < \frac{\pi}{2}$ and λ_k is simple. Consequently the remaining part of the spectrum $\{\lambda_{k_i}\}_{i=1}^{M}$ is clearly bounded. According to our previous considerations for $|k| > N$, the eigenvector ϕ_k is of the form

$$\phi_k = (y_k, \lambda_k y_k)^\top ,$$

with

$$y_k(x) = \frac{1}{\sqrt{\lambda_k^2 + a\lambda_k}} \sinh\left(\sqrt{\lambda_k^2 + a\lambda_k x}\right), \forall x \in (0, 1).$$

For $|k| > N$, by (3.4.9) and (3.4.11), we have

$$\sqrt{\lambda_k^2 + a\lambda_k} = -\tanh^{-1}\frac{1}{b} + ik\pi + O\left(\frac{1}{\lambda_k}\right) = \tilde{\lambda}_k + O\left(\frac{1}{\lambda_k}\right).$$

Hence, by the mean value theorem, for all $x \in (0,1)$, there exists $\theta_x \in (0,1)$ (depending also on k) such that

$$\begin{aligned}&\cosh(\sqrt{\lambda_k^2 + a\lambda_k}x) - \cosh(\widetilde{\lambda}_k x)\\ = &\left(\sqrt{\lambda_k^2 + a\lambda_k} - \tilde{\lambda}_k\right) x \sinh\left(\widetilde{\lambda}_k x + \theta_x(\sqrt{\lambda_k^2 + a\lambda_k} - \widetilde{\lambda}_k)x\right).\end{aligned}$$

Hence, by the previous identity, we find that for all $x \in (0,1)$

$$\left|\cosh(\sqrt{\lambda_k^2 + a\lambda_k}x) - \cosh(\widetilde{\lambda}_k x)\right| \lesssim |\lambda_k|^{-1}|\sinh\left(\widetilde{\lambda}_k x + \theta_x(\sqrt{\lambda_k^2 + a\lambda_k} - \widetilde{\lambda}_k)x\right)|.$$

Moreover, since we assume that $|b| \neq 1$, then $\left|\Re \tanh^{-1} \frac{1}{b}\right|$ is finite and therefore, $\Re\widetilde{\lambda}_k$ remains bounded (independently of k). This implies that

$$\left|\cosh(\sqrt{\lambda_k^2 + a\lambda_k}x) - \cosh(\widetilde{\lambda}_k x)\right| \lesssim |\lambda_k|^{-1}, \forall x \in (0,1).$$

This estimate implies that

$$\|(y_k)_x - (\widetilde{y}_k)_x\|^2_{(0,1)} \lesssim \frac{1}{|\lambda_k|^2}. \tag{3.4.27}$$

Similarly, we can prove that

$$\left\|\lambda_k y_k - \widetilde{\lambda}_k \widetilde{y}_k\right\|^2_{(0,1)} \lesssim \frac{1}{|\lambda_k|^2}. \tag{3.4.28}$$

The estimates (3.4.27) and (3.4.28) yield

$$\sum_{|k|>N} \left\|\phi_k - \widetilde{\phi}_k\right\|^2_X \lesssim \sum_{|k|>N} \frac{1}{|\lambda_k|^2} \lesssim \sum_{|k|>N} \frac{1}{k^2} < \infty.$$

In conclusion, according to Theorem 1.2.8, the root vectors of A_a form a Riesz basis of X.

Similar to (3.3.25), we conclude the proof of Theorem 3.2.3 for all $a \in \mathbb{R}$, $b \notin \{-1,0,1\}$ such that $-\frac{a}{2} - \Re \tanh^{-1} \frac{1}{b} < 0$ since the high frequencies are situated to the left of the imaginary axis.

3.4.5.1 Further Comments

Theorem 1.2.8 improves Bari's Theorem as it shows that we can neglect any information concerning the low frequencies. However, in the sequel, as we are interested in studying the behavior of the eigenvalues, we give an additional analysis to show that indeed the root vectors corresponding to the low eigenvalues of A_a are in bijection with those of A_0.

For the low frequency modes ($|k| < N$), we fix a sufficiently large rectangle Γ which includes all the low frequencies of $A_{a'}$ for all a' between zero and a and whose edges do not contain any eigenvalue of $A_{a'}$ for any a' between zero and a. This choice of the rectangle is possible by the following arguments. First for the horizontal edges we notice that, by (3.4.12), the horizontal lines

$$y = -\Im \tanh^{-1}\frac{1}{b} \pm \frac{(2k_0+1)\pi}{2}, \text{ for } |k_0| > N$$

are free of eigenvalues of $A_{a'}$ for all $|a'| \leq |a|$ if N is large enough (depending on a). For an upper vertical line, by Theorem 3.4.7, any eigenvalue λ of $A_{a'}$ satisfies

$$\Re\lambda \leq -\Re \tanh^{-1}(\frac{1}{b}) + |a'|\|B\|.$$

Hence, the vertical line

$$x = 1 - \Re \tanh^{-1}(\frac{1}{b}) + |a|\|B\|$$

does not contain any eigenvalues of $A_{a'}$ for all $|a'| \leq |a|$.

Finally, for the lower vertical line, for a fixed a, we denote by

$$m(a) = \min\{\Re\lambda : \lambda \in \sigma(A_a)\},$$

that is clearly finite. We now show that for a fixed a, we have

$$I := \inf_{|a'|\leq|a|} m(a') > -\infty.$$

Indeed, if it would be false, then we would find a sequence $(a_n)_{n\in\mathbb{N}}$ with $|a_n| \leq |a|$ for all n and such that

$$m(a_n) \to -\infty \text{ as } n \to \infty. \tag{3.4.29}$$

Up to a subsequence, still denoted by $(a_n)_{n\in\mathbb{N}}$, $(a_n)_{n\in\mathbb{N}}$ converges to some $a' \in [-|a|, |a|]$. Furthermore, there exists $m \in \mathbb{N}$ such that for all $n \geq m$, there exists an eigenvalue $\lambda_n \in \sigma(A_{a_n})$ such that

$$m(a_n) = \Re\lambda_n.$$

Indeed, if we assume that for all $m \in \mathbb{N}$, there exists $n_m \geq m$ such that for all $\lambda \in \sigma(A_{a_{n_m}})$, $m(a_{n_m}) \neq \Re\lambda$, then necessarily $m(a_{n_m}) = -\frac{a_{n_m}}{2} - \Re\tanh^{-1}\frac{1}{b}$ which is impossible since in this case, as $m \to +\infty$, $m(a_{n_m})$ tends to $-\frac{a'}{2} - \Re\tanh^{-1}\frac{1}{b}$ which is finite and this contradicts (3.4.29). Therefore, there exists $m \in \mathbb{N}$ such that for all $n \geq m$, there exists an eigenvalue $\lambda_n \in \sigma(A_{a_n})$ such that

$$\Re\lambda_n \to -\infty \text{ as } n \to \infty.$$

At this stage we can apply (3.4.12) to λ_n and by taking the real part of this identity we find

$$\Re\lambda_n = -\frac{a_n}{2} - \Re\tanh^{-1}\frac{1}{b} + \left(\frac{a_n^2}{8} + \frac{a_n b}{2(1-b^2)}\right)\Re\frac{1}{\lambda_n} + o\left(\frac{1}{\lambda_n}\right). \tag{3.4.30}$$

Here above $o\left(\frac{1}{\lambda_n}\right)$ depends on a_n but it is easy to see that

$$o\left(\frac{1}{\lambda_n}\right) \to 0 \text{ as } n \to \infty.$$

because the sequence a_n is bounded. Passing to the limit in (3.4.30), we find that the left-hand side tends to $-\infty$ while the right-hand side tends to $-\frac{a'}{2} - \Re\tanh^{-1}\frac{1}{b}$ which is a contradiction.

Accordingly, the line

$$x = -1 + I$$

does not contain any eigenvalues of $A_{a'}$ for all $|a'| \leq |a|$.

Having chosen Γ, we define the operator

$$P(a') = \frac{1}{2\pi i} \oint_{\Gamma} (\xi - A_{a'})^{-1} d\xi. \tag{3.4.31}$$

According to Theorem III.6.17 of [46] or Theorem XII.5 of [68], the above operator is a projection (called eigenprojection) and its range is the set of the generalized eigenspace corresponding to the eigenvalues of $A_{a'}$ inside Γ. Besides, we remark that $A_{a'}$ is closed for any a'.

In the sequel, we prove that the mapping F defined by $F(a') = \dim(P(a')X)$ for all a' between zero and a is continuous.

Lemma 3.4.9. *Fix a_0 between zero and a and let $\xi \in \rho(A_{a_0}) \cap \Gamma$. Then $\xi \in \rho(A_{a'})$ for any a' near a_0 and*

$$(\xi - A_{a'})^{-1} \to (\xi - A_{a_0})^{-1} \quad \text{as } a' \to a_0.$$

Proof: Without loss of generality, assume that $a_0 = 0$ and let $\xi \in \rho(A_0) \cap \Gamma$. We have

$$\xi - A_{a'} = \xi - A_0 - a'B = (\xi - A_0)[I - a'(\xi - A_0)^{-1}B].$$

Since $a' \to 0$, then we can choose a' such that $\|a'(\xi - A_0)^{-1}B\| < 1$ and hence $I - a'(\xi - A_0)^{-1}B$ is invertible. Consequently, $\xi - A_{a'}$ is invertible which implies that $\xi \in \rho(A_{a'})$ and as $a' \to 0$,

$$\left\|(\xi - A_{a'})^{-1} - (\xi - A_0)^{-1}\right\|_{\mathcal{L}(X)} \leq \left\|(I - a'(\xi - A_0)^{-1}B)^{-1} - I\right\|_{\mathcal{L}(X)} \left\|(\xi - A_0)^{-1}\right\|_{\mathcal{L}(X)} \to 0.$$

■

Now, we recall a result of Kato and Rellich, see the Lemma in page 14 of [68].

Lemma 3.4.10. *If P and Q are two (not necessarily orthogonal) projections in a Hilbert space H and* $\dim(PH) \neq \dim(QH)$*, then* $\|P - Q\| \geq 1$.

Lemma 3.4.9 shows that $P(a')$ is a continuous function of a' while the above Lemma shows that the mapping

$$\{Q \in \mathcal{L}(X);\ Q \text{ is a projection}\} \mapsto \mathbb{N} : Q \to \dim(QX)$$

is continuous. Therefore, the mapping $a' \to P(a') \to \dim(P(a')X)$ is continuous and hence $\dim(P(a')X)$ is constant for every a' between zero and a. Knowing that the eigenvalues of A_0 inside Γ are of finite multiplicity, then we get $\dim P(a)X = \dim P(0)X$. Therefore, we conclude that the number of eigenvalues of A_a is equal to the number of eigenvalues of A_0 inside Γ with the same total number of multiplicity. Consequently, the root vectors corresponding to the eigenvalues of A_a inside Γ are in bijection with those of A_0.

3.5 Proof of Theorem 3.2.4

In this section, we consider the case $b \in (-1, 0)$ and $a \geq -2\tanh^{-1} b$. We prove that all the eigenvalues of A_a are situated to the left of the axis $x = -\frac{a}{2} - \Re\tanh^{-1}\frac{1}{b} = -\frac{a}{2} - \tanh^{-1} b$. Consequently, in the case $a > -2\tanh^{-1} b$, by the arguments of the previous section, we immediately deduce that problem (3.1.2) is exponentially stable in X. In the case $a = -2\tanh^{-1} b$, due to (3.4.12) no exponential decay can be expected but we will show in the last section that a polynomial decay is available.

Lemma 3.5.1. *If $b \in (-1, 0)$ and $a \geq -2\tanh^{-1} b$, then any eigenvalue λ of A_a satisfies*

$$\Re\lambda < -\frac{a}{2} - \tanh^{-1} b.$$

Proof: Recall that from Remark 3.4.1 any eigenvalue λ of A_a satisfies $\Re\lambda > -\dfrac{a}{2}$ (since by our assumptions $b < 0$ and $a > 0$).

First, according to the characteristic equation (3.4.8), we can write

$$\lambda = -\frac{u}{b\tanh u}. \tag{3.5.1}$$

where $u = \sqrt{\lambda^2 + a\lambda}$. Using the identity $\tanh u = \frac{z-1}{z+1}$, with $z = e^{2u}$, (3.5.1) is equivalent to

$$\lambda = -\frac{u(z+1)}{b(z-1)} \text{ with } z = e^{2u}.$$

Substituting this identity into $u^2 = \lambda^2 + a\lambda$ yields $u = g(z)$, where

$$g(z) = ab\left(c_0 - \frac{c_1}{z+b_1} - \frac{c_2}{z+b_2}\right),$$

with $c_0 = \dfrac{1}{1-b^2}$, $c_1 = \dfrac{1}{(1-b)^2}$, $c_2 = \dfrac{1}{(1+b)^2}$, $b_1 = \dfrac{1+b}{1-b}$, and $b_2 = \dfrac{1-b}{1+b}$. Replacing z by e^{2u}, we obtain

$$u = ab\left(c_0 - \frac{c_1}{e^{2u}+b_1} - \frac{c_2}{e^{2u}+b_2}\right). \tag{3.5.2}$$

Remark that for $b \in (-1,0)$, $0 < b_1 < 1 < b_2$. Note further that the case $e^{2u}+b_1 = 0$ (resp. $e^{2u}+b_2 = 0$) cannot hold ; indeed, we then have $\tanh u = \frac{1}{b}$ (resp. $\tanh u = -\frac{1}{b}$) and therefore, by (3.5.1), $\lambda = -u$ (resp. $\lambda = u$) which yields

$$\lambda^2 = \lambda^2 + a\lambda,$$

and hence $\lambda = 0$. This is impossible since 0 is not an eigenvalue of A_a.

Writing $u = U + iV$, with $U, V \in \mathbb{R}$, we can suppose that $U \geq 0$ and $V \geq 0$ since the complex eigenvalues appear in conjugate pairs. Indeed, $u^2 = \lambda^2 + a\lambda$ implies that $y(2x+a) = 2UV$ where $\lambda = x + iy$. As $x \geq -\dfrac{a}{2}$ so if $y \geq 0$ then U and V have the same sign. Otherwise, we choose $\lambda = x - iy$ to get $U \geq 0$ and $V \geq 0$.

In a first step, we prove that

$$U = \Re u < \frac{\ln b_2}{2} = -\tanh^{-1} b. \tag{3.5.3}$$

For this aim, by setting for $j = 1, 2$,

$$\Sigma_j = \frac{1}{2} - \frac{b_j}{e^{2u} + b_j} = \frac{1}{2} \tanh\left(u - \frac{1}{2} \ln b_j\right), \tag{3.5.4}$$

we notice that (3.5.2) implies that

$$u = \frac{ab}{1 - b^2}(\Sigma_1 + \Sigma_2). \tag{3.5.5}$$

Simple calculations show that

$$\Re\Sigma_j = \frac{e^{4U} - b_j^2}{2|e^{2u} + b_j|^2} = \frac{\sinh(2U - \ln b_j)}{2(\cos(2V) + \cosh(2U - \ln b_j))}. \tag{3.5.6}$$

Hence, by the property $0 < b_1 < 1$, we directly see that $\Re\Sigma_1 > 0$.

Now if we suppose that (3.5.3) does not hold, then $U \geq \frac{\ln b_2}{2} \geq 0$ and by (3.5.6), we get $\Re\Sigma_2 \geq 0$. But from (3.5.5) and this property, we deduce that

$$U = \Re u = \frac{ab}{1 - b^2}(\Re\Sigma_1 + \Re\Sigma_2) < 0,$$

which is a contradiction. Hence (3.5.3) holds.

In a second step, we check that $U \neq 0$. Indeed if $U = 0$, then $\lambda \in \mathbb{R}$ since by (3.5.1) we find out that

$$\lambda = -\frac{V}{b \tan V}, \tag{3.5.7}$$

with $V \in \mathbb{R} \setminus \{0\}$ (because $\lambda = 0$ and $\lambda = -a$ are not eigenvalues of A_a) such that

$$\sqrt{\lambda^2 + a\lambda} = iV.$$

Hence, we see that

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4V^2}}{2},$$

that is always non positive. This is in contradiction with (3.5.7) because its right-hand side is positive.

In a third step, we show that the eigenvalues of A_a are situated to the left of the axis $-\frac{a}{2} - \Re \tanh^{-1} \frac{1}{b} = -\frac{a}{2} - \tanh^{-1} b$.

Substituting (3.5.2) into (3.5.1), we find, after simple calculations, that

$$\lambda = -\frac{a}{1-b^2} - \frac{ab}{1-b^2}(\Sigma_1 - \Sigma_2). \tag{3.5.8}$$

Hence, summing (3.5.8) with (3.5.5) and subtracting (3.5.8) with (3.5.5), we obtain

$$\Sigma_1 = -\frac{1}{2b} + \frac{b^2-1}{2ab}(\lambda - u), \tag{3.5.9}$$

$$\Sigma_2 = \frac{1}{2b} + \frac{1-b^2}{2ab}(\lambda + u). \tag{3.5.10}$$

Now coming back to (3.5.6), we can write (note that $\Re\Sigma_j \neq 0$, thanks to (3.5.3))

$$\cos(2V) = -\cosh(2U - \ln b_j) + \frac{\sinh(2U - \ln b_j)}{2\Re\Sigma_j}, \text{ for } j = 1, 2.$$

This implies that

$$\Re\Sigma_2\Big(\sinh(2U - \ln b_1) - 2\Re\Sigma_1 \cosh(2U - \ln b_1)\Big) = \Re\Sigma_1\Big(\sinh(2U - \ln b_2) - 2\Re\Sigma_2 \cosh(2U - \ln b_2)\Big).$$

Using (3.5.9) and (3.5.10), we get, again after simple calculations, the following relation between $x = \Re\lambda$ and $U = \Re u$:

$$x^2 k_2 \sinh(2U) + x k_1 \sinh(2U) + h_0(U) = 0, \tag{3.5.11}$$

where $k_2 = 4b(b-1)(1+b) > 0$, $k_1 = 2ab(b^2-3) > 0$, and

$$h_0(U) = -2b\left((a^2 - 2U^2 + 2b^2U^2)\sinh(2U) + 2abU\cosh(2U)\right).$$

As by the second step, $U \neq 0$, we can divide (3.5.11) by $\sinh(2U)$ and find

$$k_2 x^2 + k_1 x + k_0(U) = 0, \tag{3.5.12}$$

where

$$k_0(U) = \frac{h_0(U)}{\sinh 2U} = -2b\left(a^2 - 2U^2 + 2b^2U^2 + 2ab\frac{U}{\tanh(2U)}\right).$$

It turns out that $k_0(U) > 0$ for $0 < U \leq -\tanh^{-1} b$ since k_0 is a non increasing function on $(0,\infty)$ and $k_0(-\tanh^{-1} b) > 0$. As the coefficients in (3.5.12) are all positive, $x = \Re\lambda$ has to be negative. In fact, (3.5.12) yields two distinct roots $x_\pm(U)$ given by

$$x_\pm(U) = \frac{-k_1 \pm \sqrt{k_1^2 - 4k_2k_0(U)}}{2k_2},$$

such that $x_-(U) \leq x_+(U)$. Again as k_0 is a non increasing function on $(0,\infty)$ and recalling (3.5.3), we get

$$\begin{aligned} x_-(U) \leq x_+(U) < x_+(-\tanh^{-1} b) &= \frac{-k_1 + \sqrt{k_1^2 - 4k_2k_0(-\tanh^{-1} b)}}{2k_2} \\ &= -\frac{a}{2} - \tanh^{-1} b. \end{aligned} \tag{3.5.13}$$

■

Remark 3.5.2. *Increasing the order of the asymptotic development of the large eigenvalues, we find that for some $N > 0$ large enough and for every $k \in \mathbb{Z}^*$ such that $|k| > N$*

$$\begin{aligned} \lambda_k &= -\frac{a}{2} - \tanh^{-1}\frac{1}{b} + ik\pi + \frac{c}{\lambda} + \frac{\tilde{c}}{\lambda^2} + o(\frac{1}{\lambda^2}) \\ &= -\frac{a}{2} - \tanh^{-1}\frac{1}{b} + ik\pi - \frac{ic}{k\pi} - \frac{c}{k^2\pi^2}\left(\frac{a}{2} + \tanh^{-1}\frac{1}{b}\right) - \frac{\tilde{c}}{k^2\pi^2} + o\left(\frac{1}{k^2}\right), \end{aligned}$$

where $c = \frac{a^2}{8} + \frac{ab}{2(1-b^2)}$ and $\tilde{c} = \frac{a^2}{8}\left(\frac{b(b^2-3)}{(1-b)^2(1+b)^2} - \frac{a}{2}\right)$. We can check that, in case $a > 0$ and $b \in (-1, 0)$, the large eigenvalues approach the axis $x = -\frac{a}{2} - \Re\tanh^{-1}\frac{1}{b} = -\frac{a}{2} - \tanh^{-1} b$ from the left. Indeed, when $a > 0$ and $b \in (-1, 0)$ we can prove that

$$\begin{aligned} &\Re\left(ik\pi - \frac{ic}{k\pi} - \frac{c}{k^2\pi^2}\left(\frac{a}{2} + \tanh^{-1}\frac{1}{b}\right) - \frac{\tilde{c}}{k^2\pi^2}\right) \\ &= -\frac{1}{k^2\pi^2}\left(c\left(\frac{a}{2} + \Re\tanh^{-1}\frac{1}{b}\right) + \tilde{c}\right) < 0. \end{aligned}$$

3.6 Polynomial Stability of problem (3.1.2) and Proof of Theorem 3.2.5

We end up by proving the polynomial stability of problem (3.1.2) in the case $b \in (-1, 0)$ and $a = -2\tanh^{-1} b$. By Lemma 3.5.1, the spectrum of A_a is at the left of the imaginary axis, but approaches this axis. Hence, the decay of the energy depends on the way the large eigenvalues approach this axis. Therefore, we need to precise the asymptotic behavior (3.4.12). Again we use the splitting (3.4.26) of the spectrum of A_a into the small and large eigenvalue.

As before, using Taylor expansion for every $k \in \mathbb{Z}^*$ with $|k| > N$, λ_k is simple and is given by

$$\lambda_k = -\frac{a}{2} - \tanh^{-1}\frac{1}{b} + ik\pi - \frac{ic}{k\pi} - \frac{c}{k^2\pi^2}\left(\frac{a}{2} + \tanh^{-1}\frac{1}{b}\right) - \frac{\tilde{c}}{k^2\pi^2} + o\left(\frac{1}{k^2}\right),$$

where $c = \frac{a^2}{8} + \frac{ab}{2(1-b^2)}$ and $\tilde{c} = \frac{a^2}{8}\left(\frac{b(b^2-3)}{(1-b)^2(1+b)^2} - \frac{a}{2}\right)$. Since $a = -2\tanh^{-1} b$, taking the real part of this expression, we find that

$$\Re\lambda_k = -\frac{\tilde{c}}{k^2\pi^2} + o\left(\frac{1}{k^2}\right). \tag{3.6.1}$$

Note that we can prove that $\tilde{c} > 0$ for every $b \in (-1, 0)$ and $a = -2\tanh^{-1} b$.

If m_{k_i} denotes the multiplicity of λ_{k_i} for every $i = 1, ..., M$, then we denote by $\{\{\varphi_{k_i,j}\}_{j=0}^{m_{k_i}-1}\}_{i=1}^{M} \cup \{\varphi_k\}_{|k|>N}$ the Riesz basis of X formed of the normalized root vectors of A_a (recall that m_{k_i} is one or two). Hence, if we write the initial datum $U(0)$ in this basis

$$U(0) = \sum_{i=1}^{M}\sum_{j=0}^{m_{k_i}-1}\gamma_{k_i,j}\varphi_{k_i,j} + \sum_{|k|>N}\gamma_k\varphi_k, \tag{3.6.2}$$

then the solution $U(t)$ is given by

$$U(t) = \sum_{i=1}^{M} e^{\lambda_{k_i}t}\sum_{j=0}^{m_{k_i}-1}\gamma_{k_i,j}\sum_{n=0}^{j}\frac{t^{j-n}}{(j-n)!}\varphi_{k_i,n} + \sum_{|k|>N} e^{\lambda_k t}\gamma_k\varphi_k.$$

Therefore, for $t > 0$ and $\delta = \dfrac{\tilde{c}}{2\pi^2}$, by (3.6.1) we get

$$\begin{aligned}
E(t) = \tfrac{1}{2}\|U(t)\|^2 \ &\lesssim \sum_{i=1}^{M} e^{2\Re\lambda_{k_i} t} \sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{i=1}^{M} e^{2\Re\lambda_{k_i} t} \sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 t^{2j} \\
&+ \sum_{|k|>N} e^{2\Re\lambda_k t} |\gamma_k|^2 \\
&\lesssim \sum_{i=1}^{M} e^{\Re\lambda_{k_i} t} \sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} e^{-\frac{2\delta}{k^2} t} |\gamma_k|^2 \qquad (3.6.3) \\
&\lesssim \frac{1}{t} \left(\sum_{i=1}^{M} \sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} k^2 |\gamma_k|^2 \right) \\
&\lesssim \frac{\|U(0)\|^2_{D(A_a)}}{t},
\end{aligned}$$

because

$$e^{-\frac{2\delta}{k^2} t} \lesssim \frac{k^2}{t}, \quad \forall t > 0, k \in \mathbb{N}^*.$$

In the last step above we also use the equivalence

$$\|U(0)\|^2_{D(A_a)} = \|U(0)\|^2_X + \|A_a U(0)\|^2_X \simeq \sum_{i=1}^{M} \sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} k^2 |\gamma_k|^2,$$

that follows from the Riesz basis property of $\{\{\varphi_{k_i,j}\}_{j=0}^{m_{k_i}-1}\}_{i=1}^{M} \cup \{\varphi_k\}_{|k|>N}$. Indeed, by (3.6.2), we may write

$$\begin{aligned}
A_a U(0) \ &= \sum_{i=1}^{M} \lambda_{k_i} \gamma_{k_i,0} \varphi_{k_i,0} + \sum_{i=1}^{M} \sum_{j=1}^{m_{k_i}-1} \gamma_{k_i,j} (\lambda_{k_i} \varphi_{k_i,j} + \varphi_{k_i,j-1}) + \sum_{|k|>N} \gamma_k \lambda_k \varphi_k \\
&= \sum_{i=1}^{M} \lambda_{k_i} \gamma_{k_i,0} \varphi_{k_i,0} + \sum_{i=1}^{M} \lambda_{k_i} \gamma_{k_i,m_{k_i}-1} \varphi_{k_i,m_{k_i}-1} + \sum_{i=1}^{M} \gamma_{k_i,1} \varphi_{k_i,0} \\
&+ \sum_{i=1}^{M} \sum_{j=1}^{m_{k_i}-2} (\gamma_{k_i,j} \lambda_{k_i} + \gamma_{k_i,j+1}) \varphi_{k_i,j} + \sum_{|k|>N} \gamma_k \lambda_k \varphi_k \\
&= \sum_{i=1}^{M} \lambda_{k_i} \gamma_{k_i,m_{k_i}-1} \varphi_{k_i,m_{k_i}-1} + \sum_{i=1}^{M} \sum_{j=0}^{m_{k_i}-2} (\gamma_{k_i,j} \lambda_{k_i} + \gamma_{k_i,j+1}) \varphi_{k_i,j} + \sum_{|k|>N} \gamma_k \lambda_k \varphi_k.
\end{aligned}$$

As $\{\{\varphi_{k_i,j}\}_{j=0}^{m_{k_i}-1}\}_{i=1}^{M} \cup \{\varphi_k\}_{|k|>N}$ is a Riesz basis of X, we get

$$\begin{aligned}
\|U(0)\|_X^2 &\simeq \sum_{i=1}^{M}\sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} |\gamma_k|^2, \\
\|A_a U(0)\|^2 &\simeq \sum_{i=1}^{M} |\lambda_{k_i}\gamma_{k_i,m_{k_i}-1}|^2 + \sum_{i=1}^{M}\sum_{j=0}^{m_{k_i}-2} |\gamma_{k_i,j}\lambda_{k_i} + \gamma_{k_i,j+1}|^2 + \sum_{|k|>N} |\gamma_k|^2|k|^2.
\end{aligned}$$

These equivalences directly yield

$$\|U(0)\|_{D(A_a)}^2 \gtrsim \sum_{i=1}^{M}\sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} k^2|\gamma_k|^2,$$

while the converse estimate follows from the fact that the set of "small" eigenvalues is bounded.

Remark 3.6.1. *If $b \in (-1,0)$ and $a = -2\tanh^{-1} b$, then, given $U(0) = (u_0, u_1)^\top \in D(A_a^n)$ for some $n \in \mathbb{N}^*$, we get*

$$E(t) \lesssim \frac{\|U(0)\|_{D(A_a^n)}^2}{t^n}. \tag{3.6.4}$$

Consequently, the more regular the initial data is chosen, the faster is the rate of polynomial decay.

Proof: As before we can show that

$$\|U(0)\|_{D(A_a^n)}^2 = \sum_{\ell=0}^{n} \|A_a^\ell U(0)\|_X^2 \simeq \sum_{i=1}^{M}\sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} k^{2n}|\gamma_k|^2.$$

Now, as in (3.6.3), we have

$$E(t) \lesssim \sum_{i=1}^{M} e^{\Re\lambda_{k_i} t} \sum_{j=0}^{m_{k_i}-1} |\gamma_{k_i,j}|^2 + \sum_{|k|>N} e^{-\frac{2\delta}{k^2}t} |\gamma_k|^2,$$

and since

$$e^{-\frac{2\delta}{k^2}t} \lesssim \frac{k^{2n}}{t^n}, \quad \forall t > 0, k \in \mathbb{N}^*,$$

we obtain (3.6.4). ■

3.7 Open questions

The critical value of α found in Theorem 3.3.8, $\alpha_3 \simeq -0.2823$, for which problem (3.1.1) becomes exponentially stable for $\alpha > \alpha_3$ shows that the result given by the perturbation theory of contractive semigroups is not optimal. However, as the numerical result yields a wider range of this critical value, $\alpha > \alpha_2$ where $\alpha_2 \simeq -0.77$, the question of the optimality of α appearing in (3.1.1) remains an open problem.

As for the second problem (3.1.2), necessary and sufficient conditions are found so that (3.1.2) is exponentially or polynomially stable. Optimal results are attained for $b \in (-1, 0)$. If $b \leq -1$, then the question of the stability becomes an open question. Furthermore, the analysis done for problem (3.1.2) can be well adapted to study the stability of the solution of

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + u + a u_t(x,t) = 0, & x \in (0,1),\ t > 0, \\ u_x(0,t) = b_0 u_t(0,t),\ u_x(1,t) = -b u_t(1,t), & t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \end{cases}$$

where $bb_0 < 0$ and $a \in \mathbb{R}$.

Chapitre 4

Exponential stability of the wave equation on a star shaped network with indefinite sign damping

4.1 Introduction

As in [2], for $N \geq 2$, we consider the following wave equation on a star shaped network :

$$\left\{\begin{array}{rll} u^i_{tt}(x,t) - u^i_{xx}(x,t) + 2a_i(x)u^i_t(x,t) & = 0, & x \in (0, L_i),\ t > 0, i \in \{1, \cdots, N\}, \\ u^i(L_i, t) & = 0, & \\ u^i(0,t) & = u^j(0,t), & \forall i \neq j, \\ \displaystyle\sum_{i=1}^{N} u^i_x(0,t) & = 0, & \\ u^i(x,0) & = u^i_0(x), & x \in (0, L_i), \\ u^i_t(x,0) & = u^i_1(x), & x \in (0, L_i). \end{array}\right. \tag{S_1}$$

where $L_i \in \mathbb{R}_*^+$, and $a_i \in W^{1,\infty}(0, L_i)$. This system models the vibrations of a group of strings attached at one extremity. The Kirchoff law $\sum_{i=1}^{N} u_x^i(0,t) = 0$ follows from the principle of stationary action [53, 59].

The main goal of this work is to study the stability of system (S_1) but also to give more precise results when we replace in the system the damping coefficients $a_i(x)$ by $\epsilon a_i(x)$, where the parameter ϵ is positive and small enough. In this case, we will denote this modified system by (S_ϵ) and we only need that $a_i \in L^\infty(0, L_i)$ for all $i \in \{1, \cdots, N\}$.

Using observability inequalities, the stability of the wave equations over a network with positive damping coefficients has been studied in [60]. In the case of one interval, the stability of a wave equation with an indefinite sign damping coefficient has been studied in [1, 28, 30, 51, 54, 57], where it was found that the stability of the wave equation is related to the mean of the damping coefficient. In this chapter, as in [2], using spectral analysis, we find (sufficient) conditions on the damping coefficients to get the exponential stability of (S_1) and (S_ϵ). In fact, we find necessary and sufficient conditions for which (S_1) is exponentially stable up to a finite dimensional space. The idea is inspired from [65] where the characteristic equation of (S_1) is approximated by another one using the shooting method. This approximation allows us to determine the behavior of the high frequencies and hence to deduce the conditions on the damping coefficients $\{a_i\}_{i=1}^N$ for which the high frequencies are situated to the left of the imaginary axis. In a second step, we prove that the generalized root vectors form a Riesz basis with parentheses and then deduce the exponential stability of (S_1) up to a finite dimensional space generated by the roots vectors corresponding to the low frequencies. Note that the shooting method in [23] based on the ansatz of Horn in [41] and used to analyze the high frequencies cannot be easily adapted to our problem as long as the solution in [23] is written in a power series form with unknown coefficients. On the other hand, when $\{a_i\}_{i=1}^N$ is replaced by $\{\epsilon a_i\}_{i=1}^N$ with the parameter ϵ small enough, we search for sufficient conditions for which (S_ϵ) is exponentially stable in the whole energy space. In this case, we note that the positivity of the mean of the damping coefficients in addition to another condition are required (see Theorem 4.1.4 below). In fact, for $\epsilon > 0$ small enough, unlike [23], we deal with multiple eigenvalues. Note that the study of the exponential stability of (S_ϵ) enters in the framework of the abstract theory done in [51]. Using the concepts introduced in [46] about the behavior of the spectrum, we shall interpret the hypothesis imposed in [51] to find explicit conditions on the damping coefficients for which (S_ϵ) is exponentially stable.

Throughout this chapter, we make the following hypothesis on the geometry of the domain :

(**H**) There exists $q \in \mathbb{N}^*$ such that for all $i = 2, \cdots, N$, there exists $p_i \in \mathbb{N}^*$ for which $L_i = \frac{p_i}{q} L_1$.

In applications, the above hypothesis is more realistic. From a mathematical point of view, this above hypothesis is considered since otherwise when some of the lengths take irrational values, then we can find examples for which numerically we see that the spectrum is not structured (for instance there is no asymptotes) and an infinite number of eigenvalues are situated to the right of the imaginary axis (see Figure 4.1). Moreover, hypothesis (H) allows us to find an equivalent and algebraic form of the approximated characteristic equation (see Lemma 4.3.7).

This chapter is divided into three main parts. In the first part, we prove the following theorem :

Theorem 4.1.1. *Under the hypothesis (H), system (S_1) is exponentially stable up*

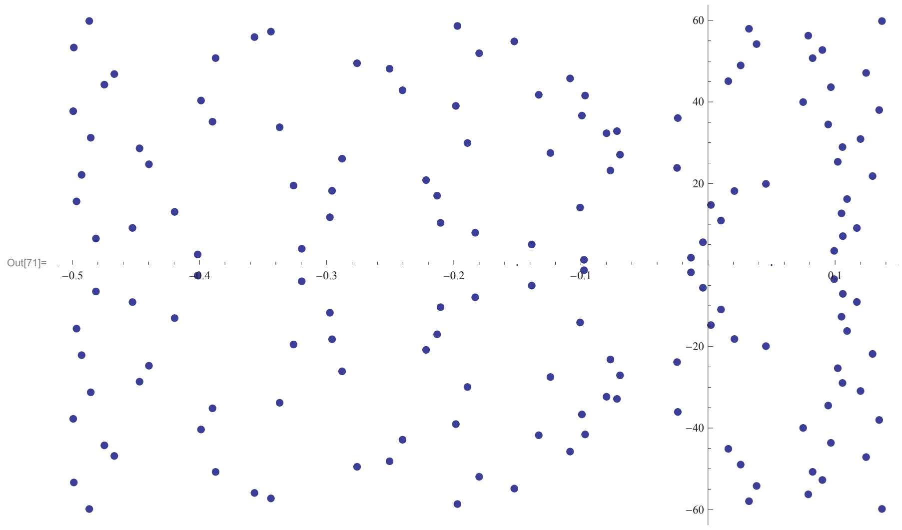

FIGURE 4.1 – $a_1 = a_2 = \dfrac{1}{2},\ a_3 = -\dfrac{1}{3},\ L_i = \sqrt{i},\ i = 1, 2, 3.$

to a finite dimensional space if and only if the roots of the polynomial G defined by

$$G(z) = \sum_{i=1}^{N} \left(e^{\int_0^{L_i} a_i(x)dx} z^{p_i} + e^{-\int_0^{L_i} a_i(x)dx} \right) \prod_{k\neq i, k=1}^{N} \left(e^{\int_0^{L_k} a_k(x)dx} z^{p_k} - e^{-\int_0^{L_k} a_k(x)dx} \right) \tag{4.1.1}$$

are inside the unitary open disk.

If $N = 2$, then according to Theorem 4.1.1, system (S_1) is exponentially stable up to a finite dimensional space if and only if

$$\int_0^{L_1} a_1(x)dx + \int_0^{L_2} a_2(x)dx > 0.$$

Clearly this condition depends only on $\int_0^{L_i} a_i(x)dx$, hence for $N \geq 3$, we may state the following conjecture :

Conjecture 4.1.2. *Although the degree of the polynomial G depends on the lengths L_i, and the coefficients are functions of the parameters a_i and L_i for all $i = 1, \cdots, N$,* *the fact that the roots of G are inside the open unitary disk depends only on the values of* $\int_0^{L_i} a_i(x)dx$ *for all* $i = 1, \cdots, N$ *(see the examples of Section 4.7).*

Remark 4.1.3. *If* $N = 2$, $a_1 = 1$, $a_2 = \alpha \in \mathbb{R}$, *and* $L_1 = L_2 = 1$, *then we recover the result of Theorem 1.1 of [1] which states that* (S_1) *is exponentially stable up to a finite dimensional space if and only if* $\alpha > -1$. *Indeed, in this case,* $G(z) = 2e^{1+\alpha}z^2 - 2e^{-1-\alpha}$ *and hence* $G(z) = 0$ *yields* $|z| = e^{-(1+\alpha)}$. *Therefore, by Theorem 4.1.1 above,* (S_1) *is exponentially stable up to a finite dimensional space if and only if* $\alpha > -1$.

In the second part, we consider system (S_ϵ) with $\epsilon > 0$ and prove the following theorem :

Theorem 4.1.4. *Under the hypothesis (H), when* $a_i(x) = a_i \in \mathbb{R}$ *and* $L_i = 1$ *for all* $i = 1, \cdots, N$, *there exists* $\epsilon_0 > 0$ *such that, for all* $\epsilon \in (0, \epsilon_0)$, (S_ϵ) *is exponentially stable if one of the following two conditions holds :*

(i) There exists at most one $j_0 \in \{1, \cdots, N\}$ *such that* $a_{j_0} = 0$ *and* $a_i > 0$ *for all* $i \neq j_0$.

(ii) There exists only one negative damping coefficient a_{i_0} *such that* $a_i > 0$ *for all* $i \neq i_0$, $\sum_{i=1}^{N} a_i > 0$, *and* $\sum_{i=1}^{N} \frac{1}{a_i} < 0$.

Remark 4.1.5. *If* $N = 2$, *then we recover the result of Theorem 2.1 of [30] when the damping coefficient is piecewise constant. However, in this case, Theorem 4.1.4 yields the result of [30] without the assumption on the integrals* I_k *defined in [30].*

Finally, in the third part, we look at some concrete examples of networks and specific values of a_i.

In the whole chapter, we shall use the notation $A \lesssim B$ (resp. $A = O(B)$) for the existence of a positive constant $c > 0$ independent of A and B such that $A \leq cB$ (resp. $|A| \leq c|B|$) and for shortness we will write $\|.\|_\infty$ for $\|.\|_{L^\infty(0,L_i)}$.

4.2 Formulation of the problem

We start by determining the suitable functional setting of system (S_1). If u is a regular solution of (S_1), then the energy of (S_1) is formally given by

$$E(t) = \frac{1}{2}\sum_{i=1}^{N}\int_0^{L_i}(|u_t^i|^2 + |u_x^i|^2)dx,$$

and

$$\frac{d}{dt}E(t) = -\sum_{i=1}^{N}\int_0^{L_i} a_i(x)|u_t^i|^2 dx.$$

Since the signs of the a_i are not specified, the decay of the energy is not guaranteed.

As an energy space, let $\mathcal{H} = V \times H$ where $H = \prod_{i=1}^{N} L^2(0, L_i)$ and

$$V = \left\{ u = (u^1, \cdots, u^N)^\top \in \prod_{i=1}^{N} H^1(0, L_i);\ u^i(0) = u^j(0)\ \forall i \neq j, \right. \\ \left. \text{and } u^i(L_i) = 0,\ \forall i = 1, \cdots, N \right\}.$$

The Hilbert space $\mathcal{H}$ is endowed with the inner product

$$< (u, v)^\top, (f, g)^\top >= \sum_{i=1}^{N}\int_0^{L_i}(u_x^i\overline{f_x^i} + v^i\overline{g^i})dx, \forall (u, v)^\top, (f, g)^\top \in \mathcal{H}.$$

Define the operator $A : D(A) \to \mathcal{H}$ by

$$D(A) = \left\{ (u, v)^\top \in V \times V;\ u \in \prod_{i=1}^{N} H^2(0, L_i) \text{ and } \sum_{i=1}^{N} u_x^i(0) = 0 \right\},$$

and for all $(u, v)^\top \in D(A)$

$$A(u, v)^\top = \begin{pmatrix} 0 & A_1^0 \\ A_1^2 & A_{-2a}^0 \end{pmatrix} (u, v)^\top$$

with $A_\alpha^k w = (\alpha_i \partial_x^k w^i)_{i=1}^N$ for $\alpha = (\alpha_i)_{i=1}^N \in \prod_{i=1}^N L^\infty(0, L_i)$ and $w = (w^i)_{i=1}^N \in \prod_{i=1}^N H^k(0, L_i)$, for $k = 0$ or 2.

If u is a sufficiently smooth solution of (S_1), then $U = (u, u_t)^\top \in \mathcal{H}$ satisfies the first order evolution equation

$$\begin{cases} U_t &= AU, \\ U(0) &= (u_0, u_1)^\top. \end{cases} \tag{4.2.1}$$

Using standard semigroup theory, we get the following theorem on the existence, uniqueness, and regularity of the solution of (S_1).

Theorem 4.2.1. *The operator A generates a C_0 semigroup on $\mathcal{H}$ and hence problem (4.2.1) admits a unique solution which implies that (S_1) is well-posed. Moreover, if $U(0) \in \mathcal{H}$, then $U \in C^0([0, +\infty); \mathcal{H})$ and if $U(0) \in D(A)$, then $U \in C^1([0, +\infty); \mathcal{H}) \cap C^0([0, +\infty); D(A))$.*

Proof: The well-posedness of (4.2.1) follows from the fact that the operator A is a bounded perturbation of a skew adjoint operator (see Theorem III.1.1 of [62]), hence it generates a strongly continuous semigroup on $\mathcal{H}$. The regularity results are then a direct consequence of Theorem I.2.4 of [62]. ■

Remark 4.2.2. *Since $D(A)$ is compactly embedded in the energy space $\mathcal{H}$, the spectrum $\sigma(A)$ is discrete and the eigenvalues of A have a finite algebraic multiplicity.*

4.3 High frequencies

In this section, we shall determine the asymptotic behavior of the eigenvalues of the operator A. For this aim, we will adapt the shooting method to our system.

Let λ be an eigenvalue of A and $U = (y, z)$ be an associated eigenfunction. Then,

$z = \lambda y$ and, for all $i = 1, \cdots, N$, we have

$$\begin{cases} y^i_{xx} - 2a_i(x)\lambda y^i - \lambda^2 y^i &= 0, \qquad x \in (0, L_i), \\ y^i(L_i) &= 0, \\ y^i(0) &= y^j(0), \quad \forall i \neq j, \\ \sum_{i=1}^{N} y^i_x(0) &= 0. \end{cases} \tag{4.3.1}$$

It is easy to see that $\lambda = 0$ is not an eigenvalue of A.

Remark 4.3.1. *We have*

$$|\Re\lambda| \leq 2 \max_{i\in\{1,\cdots,N\}} \{\|a_i\|_\infty\}. \tag{4.3.2}$$

Indeed, if we multiply the first identity of (4.3.1) by y^i and then integrate by parts, we get

$$\lambda^2 \sum_{i=1}^{N} \int_0^{L_i} |y^i|^2 dx + 2\lambda \sum_{i=1}^{N} \int_0^{L_i} a_i(x)|y^i|^2 dx + \sum_{i=1}^{N} \int_0^{L_i} |y^i_x|^2 dx = 0.$$

Hence, we have

$$\lambda = \frac{-\sum_{i=1}^{N} \int_0^{L_i} a_i(x)|y^i|^2 dx \pm r(y)^{\frac{1}{2}}}{\sum_{i=1}^{N} \int_0^{L_i} |y^i|^2 dx},$$

with

$$r(y) := \left(\sum_{i=1}^{N} \int_0^{L_i} a_i(x)|y^i|^2 dx\right)^2 - \left(\sum_{i=1}^{N} \int_0^{L_i} |y^i_x|^2 dx\right)\left(\sum_{i=1}^{N} \int_0^{L_i} |y^i|^2 dx\right)$$

and deduce the estimate (4.3.2) by distinguishing the case $r(y) \geq 0$ or not.

Now, we start by searching for the characteristic equation using the shooting method. In order to adapt the shooting method to problem (4.3.1), we first consider the the following separated initial value problems : for all $i = 1, \cdots, N$, let y_1^i and y_2^i be the solution of

$$\begin{cases} y_{1xx}^i - 2a_i(x)\lambda y_1^i - \lambda^2 y_1^i &= 0, \\ y_1^i(0) &= \dfrac{1}{\lambda}, \\ y_{1x}^i(0) &= 0. \end{cases} \tag{4.3.3}$$

$$\begin{cases} y_{2xx}^i - 2a_i(x)\lambda y_2^i - \lambda^2 y_2^i &= 0, \\ y_2^i(0) &= 0, \\ y_{2x}^i(0) &= 1. \end{cases} \tag{4.3.4}$$

The initial conditions are chosen such that the solutions y_1^i and y_2^i are linearly independent. Hence, y^i, the solution of (4.3.1), can be written as $y^i = c_i y_1^i + \alpha_i y_2^i$, where $\alpha_i, c_i \in \mathbb{C}$. By the continuity condition at zero, we get $c_i = c$ for all $i = 1, \cdots, N$, hence

$$y^i(x) = c y_1^i(x) + \alpha_i y_2^i(x). \tag{4.3.5}$$

Moreover, from the transmission condition, $\sum_{i=1}^{N} y_x^i(0) = 0$, we have $\sum_{i=1}^{N} \alpha_i = 0$ and from the boundary condition, $y^i(L_i) = 0$, we get

$$\begin{pmatrix} y_1^1(L_1) & y_2^1(L_1) & 0 & 0 & \cdots & 0 \\ y_1^2(L_2) & 0 & y_2^2(L_2) & 0 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & 0 & \cdots & 0 & \ddots & 0 \\ y_1^N(L_N) & 0 & \cdots & & 0 & y_2^N(L_N) \\ 0 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} c \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_N \end{pmatrix} = 0.$$

Hence a non-zero eigenvector exists if and only if the determinant of the above matrix vanishes, or after some elementary calculations if and only if

$$Y(\lambda) = \sum_{k=1}^{N} y_1^k(L_k) \prod_{l \neq k, l=1}^{N} y_2^l(L_l) = 0. \tag{4.3.6}$$

Recall that G is defined by (4.1.1) and set $d := degree\ G$, the degree of G. Then let $r_j e^{i\varphi_j}, 1 \leq j \leq d$ be the roots of G repeated according to their multiplicity. Without loss of generality we can suppose that the φ_j are non decreasing, namely

$$0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_d \leq 2\pi.$$

Now we can state the following main result :

Theorem 4.3.2. *There exists $k_0 \in \mathbb{N}$ such that for all $j = 1, \cdots, d$ and all $k \in \mathbb{Z}$ such that $|k| > k_0$, the operator A has an eigenvalue $\lambda_{j,k}$ such that*

$$\lambda_{j,k} = \frac{q}{2L_1} \log r_j + \imath \frac{q}{2L_1} \varphi_j + \imath k\pi \frac{q}{L_1} + o_k(1), \tag{4.3.7}$$

where $o_k(1) \to 0$ as $|k| \to \infty$. Moreover the set $\sigma(A) \setminus \cup_{|k|>k_0} \cup_{j=1}^{d} \lambda_{j,k}$ is compact. Therefore, if $r_j < 1$, for all $j = 1, \cdots, d$, then the large eigenvalues of A are situated to the left of the imaginary axis.

Corollary 4.3.3. *There exists $\ell \in \mathbb{N}$ and $\alpha_0 > 0$ such that for all $k \in \mathbb{N}$ with $k > k_0$, we have*

$$\Im(\lambda_{1,k+\ell} - \lambda_{d,k}) \geq \alpha_0,$$

$$\Im(\lambda_{1,-k} - \lambda_{d,-k-\ell}) \geq \alpha_0.$$

This corollary shows that we can group the eigenvalues of A by packets made of a finite number of eigenvalues and in such a way that the packets remain at a positive distance to each other (see section 4.4 below). Namely for any $r > 0$, we can introduce the sets $G_p(r), p \in \mathbb{Z}$ as the connected components of the set $\cup_{\lambda\in\sigma(A)} D_\lambda(r)$ (where $D_\lambda(r)$ is the disc with center λ and radius r), as well as the packets of eigenvalues $\Lambda_p(r) = G_p(r) \cap \sigma(A)$.

Before we prove Theorem 4.3.2, we search for an approximation of the characteristic equation (4.3.6) for all λ large enough. For this aim, the next lemma gives an estimation of y_1^i and y_2^i for all $i = 1, \cdots, N$.

Lemma 4.3.4. *For $i = 1, \cdots, N$ and $\lambda \in \sigma(A)$ large enough, we have*

$$\|y_1^i\|_\infty \lesssim \frac{1}{|\lambda|}, \quad \text{and} \quad \|y_2^i\|_\infty \lesssim \frac{1}{|\lambda|}.$$

Proof: First, for $i = 1, \cdots, N$ and $\lambda \in \sigma(A)$, we consider the homogenous equation

$$\begin{cases} z_{1xx}^i(x) - \lambda^2 z_1^i(x) &= 0, \quad x \in (0, L_i), \\ z_1^i(0) &= \dfrac{1}{\lambda}, \\ z_{1x}^i(0) &= 0, \end{cases}$$

which yields $z_1^i(x) = \dfrac{1}{\lambda} \cosh(\lambda x)$. Hence, for large enough λ, Remark 4.3.1 yields $\|z_1^i\|_\infty \lesssim \dfrac{1}{|\lambda|}$. Now, by the variation of constants formula, we find that

$$y_1^i(x) = z_1^i(x) + 2\int_0^x \sinh(\lambda(x-s))\, a_i(s) y_1^i(s) ds, \forall x \in (0, L_i).$$

Therefore, by the integral form of Gronwall's Lemma, we get

$$|y_1^i(x)| \le |z_1^i(x)| + 2\int_0^x \left[|z_1^i(s)| |\sinh(\lambda(x-s))\, a_i(s)| \exp\left(2\int_s^x |\sinh(\lambda(x-r)) a_i(r)| dr \right) \right] ds.$$

The above inequality and Remark 4.3.1 imply that, for λ large enough, $\|y_1^i\|_\infty \lesssim \dfrac{1}{|\lambda|}$.

A similar estimate for y_2^i is obtained by introducing $z_2^i = \dfrac{1}{\lambda} \sinh(\lambda x)$, the solution of

$$\begin{cases} z_{2xx}^i(x) - \lambda^2 z_2^i(x) &= 0, \quad x \in (0, L_i), \\ z_2^i(0) &= 0, \\ z_{2x}^i &= 1, \end{cases}$$

and using that $\|z_2^i\|_\infty \lesssim \dfrac{1}{|\lambda|}$ for λ large enough. ■

Next, we find suitable approximations for y_1^i and y_2^i for $i = 1, \cdots, N$. For this aim we define over $(0, L_i)$, the function

$$\theta^i(x) = \lambda x + \int_0^x a_i(s) ds, \forall x \in (0, L_i),$$

and the functions v_1^i and v_2^i as linear combination of $\sinh\theta^i(x)$ and $\cosh\theta^i(x)$ such that v_1^i satisfies the initial conditions in (4.3.3) and v_2^i satisfies those in (4.3.4). Note that, for $|\lambda| > M$ with $M > \max_i \|a_i(.)\|_\infty$, we have

$$v_1^i(x) = \frac{1}{\lambda}\cosh\theta^i(x), \quad \text{and} \quad v_2^i(x) = \frac{1}{\lambda + a_i(0)}\sinh\theta^i(x), \forall x \in (0, L_i).$$

Note that the functions v_1^i and v_2^i depend on λ.

Lemma 4.3.5. *For all $i = 1, \cdots, N$ and $\lambda \in \sigma(A)$ large enough, we have*

$$\left\|v_1^i - y_1^i\right\|_\infty \lesssim \frac{1}{|\lambda|^2} \quad \text{and} \quad \left\|v_2^i - y_2^i\right\|_\infty \lesssim \frac{1}{|\lambda|^2}.$$

Proof: For $i = 1, \cdots, N$ and $\varphi^i \in H^2(0, L_i)$, define the function $L^i(\varphi^i) = \varphi^i_{xx} - 2a_i\lambda\varphi^i - \lambda^2\varphi^i$. Then, for all $x \in (0, L_i)$, we have

$$L^i\left(v_1^i(x)\right) = \frac{a_{ix}(x)}{\lambda}\sinh\theta^i(x) + \frac{(a_i(x))^2}{\lambda}\cosh\theta^i(x),$$

and

$$L^i\left(v_2^i(x)\right) = \frac{a_{ix}(x)}{\lambda + a_i(0)}\sinh\theta^i(x) + \frac{(a_i(x))^2}{\lambda + a_i(0)}\cosh\theta^i(x).$$

Therefore, by Remark 4.3.1, we get that for λ large enough

$$\left\|L^i(v_1^i)\right\|_\infty \lesssim \frac{1}{|\lambda|}, \quad \text{and} \quad \left\|L^i(v_2^i)\right\|_\infty \lesssim \frac{1}{|\lambda|}.$$

Since we have

$$\begin{aligned} v^i_{1xx} - 2a_i\lambda v_1^i - \lambda^2 v_1^i &= L^i(v_1^i), \\ v^i_{1x}(0) &= 0, \\ v_1^i(0) &= \frac{1}{\lambda}, \end{aligned}$$

by the variation of constants formula, we get for all $x \in (0, L_i)$

$$v_1^i(x) = y_1^i(x) + \int_0^x y_2^i(x-s)L^i(v_1^i(s))ds.$$

Therefore, by Lemma 4.3.4, we have

$$\left\|v_1^i - y_1^i\right\|_\infty \lesssim \frac{1}{|\lambda|^2}.$$

Similarly, for all $x \in (0, L_i)$, we have

$$v_2^i(x) = y_2^i(x) + \int_0^x y_2^i(x-s)L^i(v_2^i(s))ds,$$

which implies that

$$\left\|v_2^i - y_2^i\right\|_\infty \lesssim \frac{1}{|\lambda|^2},$$

■

Now, we can find an approximation of the characteristic equation (4.3.6) from which we deduce the behavior of the high frequencies. For this aim, we introduce

$$V(\lambda) = \sum_{k=1}^{N} v_1^k(L_k) \prod_{l\neq k, l=1}^{N} v_2^l(L_l)$$

and

$$F(\lambda) = \lambda^{-N} \sum_{k=1}^{N} \cosh \widetilde{\theta}^k(\lambda) \prod_{l\neq k}^{N} \sinh \widetilde{\theta}^l(\lambda), \tag{4.3.8}$$

where, for $z \in \mathbb{C}$, $\widetilde{\theta}^l(z) = zL_l + \int_0^{L_l} a_l(s)ds$, for all $l = 1, \cdots, N$.

Proposition 4.3.6. *For $\lambda \in \sigma(A)$ large enough, we have the following estimate*

$$|Y(\lambda) - F(\lambda)| \lesssim \frac{1}{|\lambda|^{N+1}}. \tag{4.3.9}$$

Proof: Let λ be a large eigenvalue of A. The estimates in Lemmas 4.3.4 and 4.3.5

imply that

$$
\begin{aligned}
& |Y(\lambda) - V(\lambda)| \\
= & \left| \sum_{k=1}^{N} \left(y_1^k(L_k) \prod_{l\neq k, l=1}^{N} y_2^l(L_l) - v_1^k(L_k) \prod_{l\neq k, l=1}^{N} v_2^l(L_l) \right) \right| \\
= & \left| \sum_{k=1}^{N} \left(y_1^k(L_k) - v_1^k(L_k) \right) \prod_{l\neq k, l=1}^{N} y_2^l(L_l) + \sum_{k=1}^{N} v_1^k(L_k) \left(\prod_{l\neq k, l=1}^{N} y_2^l(L_l) - \prod_{l\neq k, l=1}^{N} v_2^l(L_l) \right) \right| \\
\lesssim & \frac{1}{|\lambda|^{N+1}}.
\end{aligned}
\tag{4.3.10}
$$

On the other hand, we readily check that

$$
\left| V(\lambda) - \frac{1}{\lambda^N} \sum_{k=1}^{N} \cosh \theta^k(L_k) \prod_{l\neq k}^{N} \sinh \theta^l(L_l) \right| \lesssim \frac{1}{|\lambda|^{N+1}}. \tag{4.3.11}
$$

Hence, by (4.3.10) and (4.3.11), we get (4.3.9) since $\theta^k(L_k) = \widetilde{\theta}^k(\lambda)$. ■

Estimation (4.3.9) suggests to apply Rouché's Theorem. Therefore, we are first interested in the roots of F that will be expressed in terms of the roots of the polynomial G given in (4.1.1).

Lemma 4.3.7. *$v \in \mathbb{C}$ is a root of F if and only if $z = e^{\frac{2L_1}{q}v}$ is a root of the polynomial G defined in (4.1.1). Consequently, if $v = x + \imath y$ is a root of F and $r_j e^{i\varphi_j}$ is a root of G for $1 \leq j \leq d$, then $x = \frac{q}{2L_1} \log r_j$ and $y = \frac{q}{2L_1}\varphi_j + k\pi\frac{q}{L_1}$ for some $k \in \mathbb{Z}$.*

Proof: The proof of Lemma 4.3.7 is based on writing F in an exponential form and noting that

$$
\begin{aligned}
& 2^N v^N e^{\frac{vL_1 \sum_{i=1}^N p_i}{q}} F(v) \\
= & \sum_{i=1}^{N} \left(e^{\int_0^{L_i} a_i(x)dx} z^{p_i} + e^{-\int_0^{L_i} a_i(x)dx} \right) \prod_{k\neq i, k=1}^{N} \left(e^{\int_0^{L_k} a_k(x)dx} z^{p_k} - e^{-\int_0^{L_k} a_k(x)dx} \right).
\end{aligned}
$$

■

Remark 4.3.8. *In the applications, the degree of the polynomial G is high, hence we use the algorithm given by the transformation of Schur (see [31]) that gives a criterion that guarantees that the roots of a given polynomial can be outside the closed unitary disk. Therefore, in applications, we use $G\left(\frac{1}{z}\right)$ instead of $G(z)$.*

Before giving the proof of Theorem 4.3.2, we show that Y has the same number of roots as F in a well chosen domain. Knowing that $\alpha < \Re\lambda < \beta$ where λ is an eigenvalue of A, we consider the rectangle $R_{j,k}$ with vertices $\alpha + \imath\frac{q}{2L_1}\varphi_j + \imath\left(k-\frac{1}{2}\right)\pi\frac{q}{L_1}$, $\alpha + \imath\frac{q}{2L_1}\varphi_j + \imath\left(k+\frac{1}{2}\right)\pi\frac{q}{L_1}$, $\beta + \imath\frac{q}{2L_1}\varphi_j + \imath\left(k-\frac{1}{2}\right)\pi\frac{q}{L_1}$, and $\beta + \imath\frac{q}{2L_1}\varphi_j + \imath\left(k+\frac{1}{2}\right)\pi\frac{q}{L_1}$ where we recall that $r_j e^{i\varphi_j}$, $j = 1, \cdots, d$, are the roots of G.

Proposition 4.3.9. *There exists $k_0 > 0$ such that for all $|k| \geq k_0$ and $z \in \partial R_{j,k}$,*

$$|Y(z) - F(z)| < |F(z)|. \tag{4.3.12}$$

Proof: Let $z \in \partial R_{j,k}$ and $|k| \geq k_0$ for some $k_0 > 0$ large enough. Similar to (4.3.9), we can show that there exists $C > 0$ such that

$$|Y(z) - F(z)| \leq \frac{C}{|z|^{N+1}}.$$

Therefore, in order to complete the proof, it is enough to show that for $z \in \partial R_{j,k}$

$$\frac{C}{|z|} < |F_0(z)|,$$

where

$$F_0(z) = \sum_{k=1}^{N} \cosh \widetilde{\theta}^k(z) \prod_{l \neq k}^{N} \sinh \widetilde{\theta}^l(z). \tag{4.3.13}$$

We remark that $|F_0|$ is $i\pi\frac{q}{L_1}$ periodic, hence, $\min\limits_{z \in \partial R_{j,k}} |F_0(z)| = m_j$ is independent of k. Moreover, for $k_0 \geq 1$, there exists $\widetilde{C} > 0$ such that for $|k| \geq k_0$ and $z \in \partial R_{j,k}$, we have

$$\frac{C}{|z|} \leq \frac{\widetilde{C}}{|k|}.$$

Choosing k_0 large enough, we deduce that

$$\frac{C}{|z|} \leq \frac{\widetilde{C}}{|k|} < m_j$$

and the proof follows. ■

Proof: of Theorem 4.3.2 We shall prove that the large eigenvalues of A are asymptotically close to the roots of F.

First Lemma 4.3.7 yields that all the roots of F are given by

$$z_{j,k} = \frac{q}{2L_1}\log r_j + \imath\frac{q}{2L_1}\varphi_j + \imath k\pi\frac{q}{L_1}.$$

for all $1 \leq j \leq d$, $k \in \mathbb{Z}$.

Let $0 < \rho < \min_j \left\{ \left|\frac{q}{2L_1}\varphi_j + \pi\frac{q}{L_1}\right|, \left|\frac{q}{2L_1}\log r_j\right| \right\}$ so that $B(z_{j,k}, \rho)$ contains only one root of F. From Proposition 4.3.6, in order to prove that $|Y(z) - F(z)| < |F(z)|$ for $z \in \partial B(z_{j,k}, \rho)$, it is enough to show that $\frac{C}{|z|} < |F_0(z)|$ where F_0 was defined by (4.3.13).

Let $h_{j,k}(\rho) = \min_{z\in\partial B(z_{j,k},\rho)} |F_0(z)|$. Since $|F_0|$ is $i\pi\frac{q}{L_1}$ periodic, then $h_{j,k}(\rho)$ is independent of k; i.e., $h_{j,k}(\rho) = h_{j,0}(\rho) = h_j(\rho)$. We denote by $h(\rho) = \min_{1\leq j\leq d} h_j(\rho)$. It is clear that $h(\rho) > 0$ and $h(\rho) \to 0$ as $\rho \to 0$. Therefore, there exists $k_0 > 0$ such that for $|k| > k_0$, $\frac{C}{|z|} \leq \frac{\widetilde{C}}{|k|} < h(\rho)$. Consequently, we define ρ_k by

$$\rho_k = \min_{\rho} \left\{ \frac{\widetilde{C}}{|k|} < h(\rho) \right\}. \tag{4.3.14}$$

We notice that $\rho_k \to 0$ as $|k| \to +\infty$. Therefore, for every $|k| > k_0$ and $z \in \partial B(z_{j,k}, \rho_k)$, we have $\frac{C}{|z|} \leq \frac{\widetilde{C}}{|k|} < h(\rho_k) \leq |F_0(z)|$.

By Rouché's Theorem, we conclude that Y and F have the same roots inside $B(z_{j,k}, \rho_k)$. As Proposition 4.3.9 and again the application of Rouché's theorem imply that Y and F has the same number of roots in $R_{j,k}$, for all $j = 1, \cdots, d$ and $|k| \geq k_0$, we deduce that all eigenvalues of A in $R_{j,k}$ are inside $B(z_{j,k}, \rho_k)$. This completes the proof since $\cup_{|k|\geq k_0} R_{j,k}$ covers the possible set of large eigenvalues of A. ■

Remark 4.3.10. *Using Taylor expansion in ρ, we find that $h(\rho) = O(\rho^{n_0})$ for some $n_0 \in \mathbb{N}^*$. Therefore, according to the definition of ρ_k in (4.3.14), we conclude that $\rho_k = \dfrac{\widetilde{\widetilde{C}}}{|k|^{\frac{1}{n_0}}}$ for some $\widetilde{\widetilde{C}} > 0$. Consequently, there exists some $k_0 \in \mathbb{N}^*$ large enough such that*

$$\lambda_{j,k} = \frac{q}{2L_1}\log r_j + \imath\frac{q}{2L_1}\varphi_j + \imath k\pi\frac{q}{L_1} + O\left(\frac{1}{|k|^{\frac{1}{n_0}}}\right), \forall |k| > k_0.$$

4.4 Riesz basis with parentheses of $\mathcal{H}$ and sine-type functions

In this section, we first prove that the root vectors of A form a Riesz basis with parentheses of $\mathcal{H}$. A direct consequence of Theorem 1.3.2 concerns our operator A :

Proposition 4.4.1. *The family of root vectors of A forms a Riesz basis with parentheses of $\mathcal{H}$, which means that the statements of Theorem 1.3.2 are valid for $\imath A$.*

Proof: It suffices to apply Theorem 1.3.2 with the choice

$$T = i\begin{pmatrix} 0 & A_1^0 \\ A_1^2 & A_0^0 \end{pmatrix} \quad \text{and} \quad B = i\begin{pmatrix} 0 & 0 \\ 0 & A^0_{-2a} \end{pmatrix}.$$

that clearly satisfies the assumptions of Theorem 1.3.2. ■

Another consequence of the previous results is that the packet Λ_p can be splitted up into subpackets, namely there exists $N_p \in \mathbb{N}^*$ with $N_p \leq N$ such that

$$\Lambda_p = \cup_{j=1}^{N_p}\{\lambda_{p,j}\},$$

where each $\lambda_{p,j} \in \sigma(A)$ are different and of multiplicity $m_{p,j}$ (uniformly bounded in p) and therefore

$$\mathbb{P}_p f = \sum_{j=1}^{N_p} \mathbb{P}_{p,j} f,$$

with

$$\mathbb{P}_{p,j} f = \frac{1}{2i\pi} \int_{\gamma_{p,j}} (\lambda - T - B)^{-1} f \, d\lambda,$$

$\gamma_{p,j}$ being a contour surrounding $\lambda_{p,j}$ and small enough so that only the eigenvalue $\lambda_{p,j}$ of A is inside $\gamma_{p,j}$.

In the next section, we also need to show that Y defined in (4.3.6) is a sine-type function in the following sense :

Definition 4.4.2. *Let f be an entire complex valued function. f is said to be of sine-type if*

(a) There exists $l > 0$ such that for all $z \in \mathbb{C}$, $|f(z)| \lesssim e^{l|z|}$.

(b) The zeros of f lie in a strip $\{z \in \mathbb{C};\ |\Re z| \leq c\}$ for some $c > 0$.

(c) There exist constants $c_1, c_2 > 0$ and $x_0 \in \mathbb{R}$ such that for, all $y \in \mathbb{R}$, $c_1 \leq |f(x_0 + iy)| \leq c_2$.

The class of sine-type functions is used to deal with problems of the Riesz basis property of the complex exponentials in $L^2(0, T)$ space, with $T > 0$. When f is a sine-type function, then we can write the explicit expression of f as $f(z) = \lim_{R \to +\infty} \prod_{|\widetilde{\lambda}_k| \leq R} \left(1 - \frac{z}{\widetilde{\lambda}_k}\right)$, where $\{\widetilde{\lambda}_k\}_{k \in \mathbb{Z}}$ is the set of zeros of f (see [8]). If $\widetilde{\lambda}_k = 0$, then we replace the term $\left(1 - \frac{z}{\widetilde{\lambda}_k}\right)$ by z.

In our problem, we remark that the function F defined in the approximated characteristic equation (4.3.8) is a sine-type function. In order to deduce the same property for Y defined in (4.3.6), we recall a Corollary of Section 2 of [11] :

Lemma 4.4.3. *Given* $S(z) = \lim_{R\to+\infty} \prod_{|\widetilde{\lambda}_k|\leq R} \left(1 - \frac{z}{\widetilde{\lambda}_k}\right)$ *a sine-type function, where* $\{\widetilde{\lambda}_k\}_{k\in\mathbb{Z}}$ *is the set of zeros of* $S(z)$. *Then* $S_0(z) = \lim_{R\to+\infty} \prod_{|\widetilde{\lambda}_k|\leq R} \left(1 - \frac{z}{\widetilde{\lambda}_k + \psi_k}\right)$ *is also a sine-type function if* $\{\psi_k\}_{k\in\mathbb{Z}} \in \ell^p$, *for some* $p > 1$.

Lemma 4.4.4. Y *defined in (4.3.6) is sine-type, or equivalently the eigenvalues of* A *are the zeros of a sine-type function.*

Proof: According to Theorem 4.3.2 and Remark 4.3.10, the large eigenvalues A are close to the ones of F with a remainder $\{\psi_k\}_{k\in\mathbb{Z}}$ such that

$$\psi_k = O\left(\frac{1}{|k|^{\frac{1}{n_0}}}\right),$$

for $|k| > k_0$ that then belongs to ℓ^{n_0+1}. ■

4.5 Exponential stability of (S_1) and proof of Theorem 4.1.1

Taking advantage of the fact that the root vectors of A form a Riesz basis with parenthesis of $\mathcal{H}$, our aim is now to prove that problem (S_1) is exponentially stable up to a finite dimensional space.

For our proof we recall the following lemma that can be found in Lemma 3.1 of [39].

Lemma 4.5.1. *Let* H *be a separable Hilbert space. Suppose that* $\{e_n(t)\}_{n\in J}$ *forms a Riesz basis for the closed subspace spanned by itself in* $L^2(0,T)$, $T > 0$. *Then for any* $\varphi(t) = \sum_{n\in J} e_n(t)\phi_n \in L^2(0,T;H)$, *there exist two positive constants* $C_1(T)$, $C_2(T)$

such that

$$C_1(T)\sum_{n\in J}\|\phi_n\|_H^2 \le \|\varphi\|^2_{L^2(0,T;H)} \le C_2(T)\sum_{n\in J}\|\phi_n\|_H^2.$$

To apply the above lemma, we need to search for a Riesz basis in $L^2(0,T)$. Since the eigenvalues are not necessary simple, the family $\{e^{\lambda_k t}\}_{k\in\mathbb{Z}}$ does not form a Riesz basis in $L^2(0,T)$ for any $T>0$. However, as $\sigma(A)$ is a discrete union of separated and finite sets, hence we can use the family of generalized divided differences (see [9,39]).

Definition 4.5.2. *Let $M\in\mathbb{N}^*$ be fixed and let v_k, $k=1,\cdots,M$, be arbitrary complex numbers, not necessarily distinct. Then the generalized divided differences (denoted by GDD) of order $m=0,\cdots,M-1$ are defined by recurrence as follows : the GDD of order zero is defined as $[v_1](t)=e^{v_1 t}$, the GDD of order $m-1, 1\le m\le M$ is defined as*

$$[v_1,v_2,\cdots,v_m](t) =: \begin{cases} \dfrac{[v_1,v_2...,v_{m-1}](t)-[v_2,v_3,\cdots,v_m](t)}{v_1-v_m}, & v_1\neq v_m \\ \dfrac{\partial}{\partial v}[v,v_2,\cdots,v_{m-1}](t)\mid_{v=v_1}, & v_1=v_m. \end{cases}$$

An equivalent expression is given by

$$[v_1,v_2,\cdots,v_m](t)=t^{m-1}\int_0^1\int_0^{\tau_1}...\int_0^{\tau_{m-2}} e^{t(v_1+\tau_1(v_2-v_1)+...+\tau_{m-1}(v_m-v_{m-1}))}d\tau_{m-1}...d\tau_2 d\tau_1.$$

Hence, if $\Re v_m\le\Re v_{m-1}\le...\le\Re v_1$, then for all $t\ge 0$

$$|[v_1,v_2...,v_m](t)|\le t^{m-1}e^{\Re v_1 t}. \tag{4.5.1}$$

Now as some v_j can be repeated, we write $\{v_1,v_2,\cdots,v_M\}=\{w_1,w_2,\cdots,w_n\}$ such that $w_i\neq w_j$ for all $1\le i,j\le n$ such that $i\neq j$. Supposing that each w_j is repeated n_j times, i.e, $\sum_{j=1}^{n} n_j=M$, then we can recall Proposition 3.1 of [39] which shows that for any $1\le k\le n_l$, $t^{k-1}e^{w_l t}$, $l=1,\cdots,n$ is a linear combination of $[v_1](t),\ [v_1,v_2](t),\cdots,[v_1,v_2,\cdots,v_M](t)$.

Proposition 4.5.3. *Any* $\varphi(t) = \sum_{j=1}^{n} e^{w_j t} \sum_{i=1}^{n_j} a_{ij} t^{i-1}$ *with* $a_{ij} \in \mathcal{H}$ *can be rewritten as*

$$\varphi(t) = \sum_{i=1}^{M} G_i[v_1.v_2, \cdots, v_i](t),$$

with some $G_i \in \mathcal{H}$*, in particular* $G_1 = \sum_{j=1}^{n} a_{1j}$.

If we go back to our problem, for every $p \in \mathbb{Z}$, we construct the family of GDD of the form

$$E_p(t) = \left\{[\lambda_{p,1}](t),\ [\lambda_{p,1}, \lambda_{p,2}], \cdots, [\lambda_{p,1}, \lambda_{p,2}, \cdots, \lambda_{p,M_p}](t)\right\},$$

associated with the set $\lambda_{p,1}, \cdots, \lambda_{p,N_p}$ but the eigenvalues being repeated according to their multiplicity (and consequently $M_p = \sum_{j=1}^{N_p} m_{p,j}$).

Proposition 4.5.4. *There exists* $T > 0$ *such that the family of GDD* $\{E_p(t)\}_{p\in\mathbb{Z}}$ *forms a Riesz basis for the closed subspace spanned by itself in* $L^2(0, T)$*.*

Proof: According to Lemma 4.4.4, the eigenvalues of A are roots of a sine-type function. Hence, the proof becomes a direct consequence of Theorem 3 of [10] where $T > 0$ is chosen large enough (note also that a sine-type function automatically satisfies the Helson-Szego condition due to its equivalent form (condition (A_2) page 2 in [9]) and the condition (c) in our definition 4.4.2). ■

Proof: of Theorem 4.1.1. Given an initial datum $U(0) \in \mathcal{H}$, by Proposition 4.4.1, it can be written as

$$U(0) = \sum_{p\in\mathbb{Z}} \sum_{j=1}^{N_p} \mathbb{P}_{\lambda_{p,j}}(U(0)),$$

where we recall that $\mathbb{P}_{\lambda_{p,j}}$ denotes the Riesz projection of A corresponding to the eigenvalue $\lambda_{p,j}$, then, for any $t > 0$, we have

$$\begin{aligned} e^{tA}U(0) &= \sum_{p\in\mathbb{Z}}\sum_{j=1}^{N_p} e^{\lambda_{p,j}t}\sum_{i=1}^{m_{p,j}} \frac{(A-\lambda_{p,j})^{i-1}}{(i-1)!} t^{i-1}\mathbb{P}_{\lambda_{p,j}}(U(0)) \\ &= \sum_{p\in\mathbb{Z}}\sum_{j=1}^{N_p} e^{\lambda_{p,j}t}\sum_{i=1}^{m_{p,j}} a_{ij,p}t^{i-1}, \end{aligned} \tag{4.5.2}$$

where $a_{ij,p} = \dfrac{(A-\lambda_{p,j})^{i-1}}{(i-1)!}\mathbb{P}_{\lambda_{p,j}}(U(0))$. By Proposition 4.5.3, we get

$$e^{tA}U(0) = \sum_{p\in\mathbb{Z}}\sum_{i=1}^{M_p} G_{p,i}[\lambda_{p,1}, \lambda_{p,2}, \cdots, \lambda_{p,i}](t). \tag{4.5.3}$$

Lemma 4.5.1 and Proposition 4.5.4 yield for some $T > 0$

$$\sum_{p\in\mathbb{Z}}\sum_{i=1}^{M_p} \|G_{p,i}\|_{\mathcal{H}}^2 \lesssim \int_0^T \left\|e^{tA}U(0)\right\|_{\mathcal{H}}^2 dt.$$

By the semigroup property, we know that there are $C, \omega > 0$ such that for all $t \geq 0$

$$\|e^{tA}\|_{\mathcal{L}(\mathcal{H})} \leq Ce^{\omega t}.$$

Therefore, the previous estimate becomes

$$\sum_{p\in\mathbb{Z}}\sum_{i=1}^{M_p} \|G_{p,i}\|_{\mathcal{H}}^2 \lesssim \frac{C^2}{2\omega}(e^{2\omega T} - 1)\|U(0)\|_{\mathcal{H}}^2. \tag{4.5.4}$$

Finally, since the root vectors of A form a Riesz basis with parenthesis of $\mathcal{H}$, then by (4.5.1), (4.5.3), and (4.5.4) we get for $t \geq 1$

$$\begin{aligned} \left\|e^{tA}U(0)\right\|_{\mathcal{H}}^2 &\lesssim \sum_{p\in\mathbb{Z}} \left\| \sum_{i=1}^{M_p} G_{p,i}[\lambda_{p,1}, \lambda_{p,2}, \cdots, \lambda_{p,i}](t) \right\|_{\mathcal{H}}^2 \\ &\lesssim \sum_{p\in\mathbb{Z}} t^{2(M_p-1)} e^{2\mu_p t} \sum_{i=1}^{M_p} \|G_{p,i}\|_{\mathcal{H}}^2, \end{aligned} \tag{4.5.5}$$

where $\mu_p = \max_{1\leq j\leq N_p} \Re\lambda_{p,j}$.

Now, by Theorem 4.3.2, we know that if the roots of the polynomial G are in the open unit disk, then there exists $\mu < 0$ and $p_0 \in \mathbb{N}$ such that

$$\mu_p \leq \mu < 0, \forall |p| > p_0.$$

Hence by (4.5.5), we deduce the exponential stability of problem (S_1) up to the finite dimensional space spanned by the roots vectors of A corresponding to the eigenvalues $\lambda_{p,j}$ such that $|p| \leq p_0$. The proof of Theorem 4.1.1 is complete. ■

4.6 Exponential stability of (S_ϵ) for small values of ϵ and proof of Theorem 4.1.4

In this section, we consider constant damping coefficients and equal lengths $L_i = 1$, for all $i = 1, \cdots, N$. Without loss of generality we can assume that the a_i are non decreasing, i.e., $a_1 \leq a_2 \leq \cdots \leq a_N$. In the sequel, we replace the damping coefficients a_i by ϵa_i, where the parameter ϵ is positive. Our goal is to find sufficient conditions for which (S_ϵ) is exponentially stable in the whole energy space for every ϵ small enough.

Based on the results of the previous section, it seems enough to find sufficient conditions on the damping coefficients so that the low eigenvalues have negative real parts for every ϵ small enough. However, we remark that Rouché's Theorem used in the proof of Theorem 4.3.2 yields a constant k_0 dependent of ϵ (mainly of order $\frac{1}{\epsilon}$). Consequently, it seems difficult to separate the large eigenvalues from the low eigenvalues uniformly in ϵ for all ϵ small enough.

As previously mentioned, the exponential stability of (S_ϵ) has been studied in [51] under some abstract hypothesis. Consequently, our aim is to interpret the hypothesis from [51] to find explicit conditions on the damping coefficients. Our strategy is based on the asymptotic behavior of the spectrum of the generator $A = A(\epsilon)$ as a function of ϵ. In the sequel, we use some notations from [46] and we refer the reader to this book for the exact definitions. First, we notice that the generator $A = A(\epsilon)$ is holomorphic of type (A) in the parameter ϵ in the sense of (2.1) of chapter VII.2 in [46]. Indeed, we simply have

$$A(\epsilon) = A(0) + \epsilon B,$$

where $A(0)$ is a skewadjoint operator and B is a bounded selfadjoint operator defined by

$$A(0) = \begin{pmatrix} 0 & A_1^0 \\ A_1^2 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & A_{-2a}^0 \end{pmatrix}.$$

Since $A(0)$ is a skew adjoint operator with a compact resolvent, there is an orthonormal system of eigenvectors of $A(0)$ which is complete in $\mathcal{H}$. The eigenvalues of $A(0)$ are $\lambda_{1,k}(0) = \imath(k\pi + \frac{\pi}{2})$ with multiplicity one, for all $k \in \mathbb{Z}$, and $\lambda_{2,k}(0) = \imath k\pi$ with geometric and algebraic multiplicity $N-1$, for all $k \in \mathbb{Z}^*$. For shortness we write $\{\lambda_k(0)\}_{k\in\mathbb{Z}} = \{\imath k\pi\}_{k\in\mathbb{Z}^*} \cup \left\{\imath(k\pi + \frac{\pi}{2})\right\}_{k\in\mathbb{Z}}$ and we set m_k the multiplity of $\lambda_k(0)$ (hence $m_k = 1$ or $m_k = N-1$).

Now according to Section VII.2.4 in [46], there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, $A(\epsilon)$ has exactly m_k eigenvalues (algebraic multiplicity counted) in $B(\lambda_k(0), \rho)$, with $\rho > 0$ fixed small enough. This set of eigenvalues is called the $\lambda_k(0)$-group eigenvalues of $A(\epsilon)$ generated by the splitting from the common eigenvalue $\lambda_k(0)$ of the unperturbed operator $A(0)$ (see page 74 in [46]). Consequently as ϵ increases, a splitting of the eigenvalues may occur and the eigenvalues of $A(\epsilon)$ can go to the left or to the right of the imaginary axis (or both). Hence, our aim is to find some sufficient conditions for which each $\lambda_k(0)$-group is strictly to the left of the imaginary axis.

For further use, for $\epsilon \in (0, \epsilon_0)$ and all $k \in \mathbb{Z}$, the $\lambda_k(0)$-group eigenvalues of $A(\epsilon)$ will be denoted by $\{\lambda_{k,j}(\epsilon)\}_{j=1}^{m_k}$.

In a first step, consider $\Gamma_{k,\rho}$ a positively-oriented circle around $\lambda_k(0)$ with radius $\rho < \dfrac{\pi}{4}$ such that $\lambda_k(0)$ is isolated. For $\zeta \in \Gamma_{k,\rho}$, we denote by $R(\zeta) = (A(0) - \zeta)^{-1}$. The following lemma gives a uniform estimate of $\|R(\zeta)\|$ for all $\zeta \in \Gamma_{k,\rho}$.

Lemma 4.6.1. *For all $\zeta \in \Gamma_{k,\rho}$, we have*

$$\|R(\zeta)\| = \frac{1}{\rho}, \ \forall \zeta \in \Gamma_{k,\rho}. \tag{4.6.1}$$

Proof: For convenience and for a moment, we rename $\{i\beta_k\}_{k\in\mathbb{Z}^*}$ the set of eigenvalues of $A(0)$ and arrange it in increasing order (i.e $...\beta_{k-1} \leq \beta_k \leq \beta_{k+1}...$). We denote by $\{\phi_k\}_{k\in\mathbb{Z}^*}$ the associated system of eigenvectors which forms an orthonormal basis of $\mathcal{H}$. Let $f = \sum_{k\in\mathbb{Z}^*} f_k\phi_k \in \mathcal{H}$, then by the spectral theorem, for all $\zeta \in \Gamma_{k,\rho}$, we can write

$$R(\zeta)f = \sum_{k\in\mathbb{Z}^*} \frac{f_k}{i\beta_k - \zeta}\phi_k.$$

Since $|i\beta_k - \zeta| = \rho$, for all $k \in \mathbb{Z}^*$, we deduce that

$$\|R(\zeta)f\|^2 = \sum_{k\in\mathbb{Z}^*} \frac{|f_k|^2}{|i\beta_k - \zeta|^2} = \frac{1}{\rho^2}\sum_{k\in\mathbb{Z}^*} |f_k|^2 = \frac{1}{\rho^2}\|f\|^2.$$

This proves (4.6.1) by taking f corresponding to one eigenvector associated with the eigenvalue $\lambda_k(0)$. ∎

Now we characterize the asymptotic behaviour of the real parts of the $\lambda_k(0)$-group eigenvalues of $A(\epsilon)$.

Lemma 4.6.2. *There exists* $\epsilon_1 > 0$ *and* $c > 0$ *such that for all* $\epsilon \in (0, \epsilon_1)$*, all* $k \in \mathbb{Z}$ *and all* $j = 1, \cdots, m_k$*,*

$$\Re\lambda_{k,j}(\epsilon) \leq \epsilon \max_{1\leq j\leq m_k} \mu_{k,j} + c\epsilon^2,$$

when $\{\mu_{k,j}\}_{j=1}^{m_k}$ *denotes the set of eigenvalues of* $P_k(0)BP_k(0)$ *and* $P_k(0)$ *denotes the eigenprojection corresponding to* $\lambda_k(0)$*, i.e.,* $P_k(0) = -\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_{k,\rho}} R(\xi)d\xi$*.*

Proof: Step 1. Let $\lambda_k(0)$ be an eigenvalue of $A(0)$. Define the space $M_k(\epsilon) = P_k(\epsilon)\mathcal{H}$, where $P_k(\epsilon)$ is the eigenprojection (see (1.16) page 67 of [46]) defined by

$$P_k(\epsilon) = -\frac{1}{2\pi i}\int_{\Gamma_{k,\rho}} (A(\epsilon) - \xi)^{-1} d\xi.$$

Notice that $(A(\epsilon) - \xi)^{-1}$ is well defined for $\xi \in \Gamma_{k,\rho}$ when $\epsilon < \dfrac{1}{\|B\|\|R(\xi)\|} = \dfrac{\rho}{\|B\|}$. Indeed, according to (1.13) and (1.14) page 67 of [46], we have by the second Neumann series for the resolvent

$$(A(\epsilon) - \xi)^{-1} = R(\xi)\left(1 + \epsilon BR(\xi)\right)^{-1} = R(\xi)\sum_{n=0}^{\infty}(-\epsilon BR(\xi))^n = R(\xi) + \sum_{n=1}^{\infty}\epsilon^n R_k^{(n)}(\xi), \tag{4.6.2}$$

where

$$R_k^{(n)}(\xi) = R(\xi)(-BR(\xi))^n. \tag{4.6.3}$$

Hence the series on the right-hand side of (4.6.2) converges if $\epsilon < \frac{\rho}{\|B\|}$ (thanks to Lemma 4.6.1, we notice that the upper bound of ϵ is independent of $k \in \mathbb{Z}$) and we get

$$P_k(\epsilon) = P_k(0) + \sum_{n=1}^{\infty}\epsilon^n P_k^{(n)},$$

where $P_k^{(n)} = -\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_{k,\rho}} R_k^{(n)}(\xi)d\xi$ and $P_k^{(0)} = P_k(0)$.

As already said before if ϵ is sufficiently small, the eigenvalues of $A(\epsilon)$ lying in $\Gamma_{k,\rho}$ form exactly the $\lambda_k(0)$-group eigenvalues. Therefore, since $\lambda_k(0)$ is semisimple (since it is an eigenvalue of a skewadjoint operator), then according to the identities (5.13) and (5.14) of [46, p. 112], the $\lambda_k(0)$-group eigenvalues of $A(\epsilon)$ are of the form

$$\lambda_{k,j}(\epsilon) = \lambda_k(0) + \epsilon\mu_{k,j}^{(1)}(\epsilon), \quad j = 1, \cdots, m_k, \tag{4.6.4}$$

where $\{\mu_{k,j}^{(1)}(\epsilon)\}_{j=1}^{m_k}$ are the eigenvalues of the operator

$$\widetilde{A}_k^{(1)}(\epsilon) = \epsilon^{-1}(A(\epsilon) - \lambda_k(0))P_k(\epsilon) = -\frac{\epsilon^{-1}}{2\pi i}\int_{\Gamma_{k,\rho}} (\xi - \lambda_k(0))\,(A(\epsilon) - \xi)^{-1}\, d\xi, \tag{4.6.5}$$

in the subspace $M_k(\epsilon) = P_k(\epsilon)\mathcal{H}$. The second equality in (4.6.5) follows from the fact that

$$(A(\epsilon) - \lambda_k(0))\,(A(\epsilon) - \xi)^{-1} = 1 + (\xi - \lambda_k(0))\,(A(\epsilon) - \xi)^{-1}.$$

Step 2. We estimate the difference between $\widetilde{A}_k^{(1)}(\epsilon)$ and $P_k(0)BP_k(0)$. According to the identity (2.16) page 77 of [46], we have

$$(A(\epsilon) - \lambda_k(0))P_k(\epsilon) = (A(0) - \lambda_k(0))P_k(0) + \sum_{n=1}^{\infty}\epsilon^n\widetilde{A}_k^{(n)}, \tag{4.6.6}$$

where

$$\widetilde{A}_k^{(n)} = (-1)^{n+1}\frac{1}{2\pi i}\int_{\Gamma_{k,\rho}} R(\xi)(BR(\xi))^n(\xi - \lambda_k(0))d\xi, \tag{4.6.7}$$

in particular (see (2.19) page 77 of [46])

$$\widetilde{A}_k^{(1)} = P_k(0)BP_k(0). \tag{4.6.8}$$

Since $\lambda_k(0)$ is semisimple, then $A(0)P_k(0) = \lambda_k(0)P_k(0)$. Thus (4.6.6) implies that

$$\widetilde{A}_k^{(1)}(\epsilon) = \sum_{n=0}^{\infty}\epsilon^n\widetilde{A}_k^{(n+1)}.$$

On the other hand, from (4.6.7) and Lemma 4.6.1, we have for all $n \geq 1$

$$\|\widetilde{A}_k^{(n)}\| \leq \frac{\|B\|^n}{\rho^n}.$$

Therefore, for ϵ small enough, there exists $c > 0$ independent of k such that

$$\|\widetilde{A}_k^{(1)}(\epsilon) - \widetilde{A}_k^{(1)}\| \leq \sum_{n=1}^{\infty} \epsilon^n \frac{\|B\|^{n+1}}{\rho^{n+1}} \leq \frac{\|B\|}{\rho} \left(\frac{\epsilon \frac{\|B\|}{\rho}}{1 - \epsilon \frac{\|B\|}{\rho}} \right) \leq c\epsilon. \tag{4.6.9}$$

Step 3. We compare the eigenvalues $\{\mu_{k,j}^{(1)}(\epsilon)\}_{j=1}^{m_k}$ of $\widetilde{A}_k^{(1)}(\epsilon)$ with the eigenvalues of $\widetilde{A}_k^{(1)} = P_k(0)BP_k(0)$. Consider $\mu_{k,j}^{(1)}(\epsilon)$ and $\phi_{k,j}^{(1)}(\epsilon)$ an associated normalized eigenvector, then

$$\widetilde{A}_k^{(1)}(\epsilon)\phi_{k,j}^{(1)}(\epsilon) = \mu_{k,j}^{(1)}(\epsilon)\phi_{k,j}^{(1)}(\epsilon).$$

From (4.6.9), we have

$$\|\widetilde{A}_k^{(1)}(\epsilon)\phi_{k,j}^{(1)}(\epsilon) - \widetilde{A}_k^{(1)}\phi_{k,j}^{(1)}(\epsilon)\| \leq c\epsilon.$$

Thus, by Cauchy-Schwarz's inequality, we have

$$| < \mu_{k,j}^{(1)}(\epsilon)\phi_{k,j}^{(1)}(\epsilon) - \widetilde{A}_k^{(1)}\phi_{k,j}^{(1)}(\epsilon), \phi_{k,j}^{(1)}(\epsilon) > | \leq c\epsilon,$$

or equivalently

$$|\mu_{k,j}^{(1)}(\epsilon) - < \widetilde{A}_k^{(1)}\phi_{k,j}^{(1)}(\epsilon), \phi_{k,j}^{(1)}(\epsilon) > | \leq c\epsilon.$$

Therefore,

$$\Re(\mu_{k,j}^{(1)}(\epsilon)) \leq < \widetilde{A}_k^{(1)}\phi_{k,j}^{(1)}(\epsilon), \phi_{k,j}^{(1)}(\epsilon) > + c\epsilon,$$

or

$$\Re(\mu_{k,j}^{(1)}(\epsilon)) \leq \max_{1\leq j\leq m_k} \mu_{k,j} + c\epsilon.$$

We conclude by using this estimate and (4.6.4). ∎

According to Lemma 4.6.2, to prove that the spectrum of $A(\epsilon)$ is situated to the left of the imaginary axis for $\epsilon > 0$ small enough, we have to prove that, for every $k \in \mathbb{Z}$, the eigenvalues of $\widetilde{A}_k^{(1)} = P_k(0)BP_k(0)$ are strictly to the left of the imaginary axis independently of $k \in \mathbb{Z}$ and $\epsilon > 0$. In fact, the hypothesis imposed in [51] to get the exponential stability of (S_ϵ) can be interpreted as a condition on the negativity of the eigenvalues of $P_k(0)BP_k(0)$. Therefore, our aim in the next two lemmas is to find the eigenvalues of $P_k(0)BP_k(0)$ and to investigate the conditions for which their real parts are negative independently of $k \in \mathbb{Z}$ and $\epsilon > 0$.

Lemma 4.6.3. *If $\mu_{k,0}$ denotes the eigenvalue of $P_k(0)BP_k(0)$, where $P_k(0)$ is the eigenprojection corresponding to $\lambda_k(0) = \imath(k\pi + \frac{\pi}{2})$, with $k \in \mathbb{Z}$, then $\mu_{k,0} = -\frac{1}{N}\sum_{i=1}^{N} a_i$.*

Proof: We recall that $\lambda_k(0) = \imath(k\pi + \frac{\pi}{2})$ is simple for all $k \in \mathbb{Z}$. Some elementary calculations show that the associated normalized eigenvector is of the form

$$\phi_0 = \frac{1}{\sqrt{N}}(u, v)^\top,$$

where, for all $x \in (0, 1)$ and $i = 1, \cdots, N$, $u_i(x) = \dfrac{\sinh(\lambda_k(0)(1-x))}{\lambda_k(0)}$ and $v_i(x) = \sinh(\lambda_k(0)(1-x))$. For any $\psi \in \mathcal{H}$, we find that

$$P_k(0)BP_k(0)\psi = -\frac{1}{N}\left(\sum_{i=1}^{N} a_i\right)(\psi, \phi_0)\phi_0,$$

hence ϕ_0 is the eigenvector of $P_k(0)BP_k(0)$ of eigenvalue $-\dfrac{1}{N}\displaystyle\sum_{i=1}^{N} a_i$. ■

Lemma 4.6.4. *If $\{\mu_{k,j}\}_{j=1}^{N-1}$ denotes the set of eigenvalues of $P_k(0)BP_k(0)$, where $P_k(0)$ is the eigenprojection corresponding to $\lambda_k(0) = \imath k\pi$, with $k \in \mathbb{Z}^*$, then $\{\mu_{k,j}\}_{j=1}^{N-1}$ is the set of zeros of the polynomial Q defined by*

$$Q(z) = (z + a_1)(z + a_N)\sum_{i=2}^{N-1}\prod_{\substack{l \neq i \\ l=2}}^{N-1}(z + a_l) + \prod_{l=2}^{N-1}(z + a_l)(2z + a_1 + a_N) \qquad (4.6.10)$$

Proof: First, we notice that, for all $k \in \mathbb{Z}^*$, $\lambda_k(0) = \imath k\pi$ is of multiplicity $N - 1$ and that the associated eigenvectors are of the form $(u, v)^\top$ where, for $i = 1, \cdots, N$

and $x \in (0,1)$, $u_i(x) = \frac{\alpha_i}{\imath k\pi}\sin(k\pi(1-x))$ and $v_i(x) = \alpha_i \sin(k\pi(1-x))$ with $\alpha = (\alpha_i)_{i=1}^N \in \mathbb{C}^N$ such that $\sum_{i=1}^{N}\alpha_i = 0$. As a basis of the subspace $P_k(0)\mathcal{H}$, we can choose the system of eigenvectors $\{\phi^{(i)}\}_{i=1,\cdots,N-1}$ corresponding to the choice

$$\alpha^{(1)} = (1,-1,0\cdots,0), \alpha^{(2)} = (1,0,-1,0\cdots,0), \cdots, \alpha^{(N-1)} = (1,0,\cdots,0,-1).$$

Therefore, for all $i = 1,\cdots,N-1$, $P_k(0)BP_k(0)\phi^{(i)} = \sum_{k=1}^{N-1}\alpha_{ik}\phi^{(k)}$ where $\alpha_{ik} \in \mathbb{C}$. Moreover, for all $i,j = 1,\cdots,N-1$,

$$\left\langle P_k(0)BP_k(0)\phi^{(i)},\phi^{(j)}\right\rangle = \left\langle B\phi^{(i)},\phi^{(j)}\right\rangle = \sum_{k=1}^{N-1}\alpha_{ik}\left\langle \phi^{(k)},\phi^{(j)}\right\rangle.$$

Hence, $P_k(0)BP_k(0) = \Gamma G^{-1}$, where $\Gamma = (\left\langle B\phi^{(i)},\phi^{(j)}\right\rangle)_{i,j}$ and G is the Gramian matrix defined by $G = (\left\langle \phi^{(i)},\phi^{(j)}\right\rangle)_{i,j}$. But some elementary calculations yield

$$\Gamma = \begin{pmatrix} -a_1-a_2 & -a_1 & -a_1 & \cdots & -a_1 \\ -a_1 & -a_1-a_3 & -a_1 & \cdots & -a_1 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & -a_1 \\ -a_1 & -a_1 & \cdots & -a_1 & -a_1-a_N \end{pmatrix}$$

and

$$G = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 2 \end{pmatrix}.$$

Since

$$G^{-1} = \frac{1}{N}\begin{pmatrix} N-1 & -1 & -1 & \cdots & -1 \\ -1 & N-1 & -1 & \cdots & -1 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & -1 \\ -1 & -1 & \cdots & -1 & N-1 \end{pmatrix} = I - \frac{1}{N}\begin{pmatrix} 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix},$$

we get

$$\begin{aligned} & P_k(0)BP_k(0) \\ = & \frac{1}{N}\begin{pmatrix} -a_1-(N-1)a_2 & -a_1+a_2 & -a_1+a_2 & \cdots & -a_1+a_2 \\ -a_1+a_3 & -a_1-(N-1)a_3 & -a_1+a_3 & \cdots & -a_1+a_3 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ -a_1+a_N & -a_1+a_N & \cdots & -a_1+a_N & -a_1-(N-1)a_N \end{pmatrix}. \end{aligned}$$

Therefore, $\{\mu_{k,j}\}_{j=1}^{N-1}$ are the roots of the characteristic polynomial $\det\left(zI - P_k(0)BP_k(0)\right)$ or equivalently

$$Q(z) = \det\begin{pmatrix} z+a_2 & 0 & \cdots & \cdots & -z-a_N \\ 0 & z+a_3 & 0 & \cdots & -z-a_N \\ \vdots & \cdots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & z+a_{N-1} & -z-a_N \\ z+a_1 & z+a_1 & \cdots & z+a_1 & 2z+a_1+a_N \end{pmatrix}.$$

Developing with respect to the last line, we find (4.6.10). ∎

Before going on let us notice that the above lemmas show that the eigenvalues $\mu_{k,j}$ of $P_k(0)BP_k(0)$ are independent of k. In the first case we directly find a condition on the damping coefficients to have $\mu_{k,0} < 0$, for the second case we need that the roots of Q are negative. For this aim, we first localize the roots of Q. Before doing so let us introduce the following notation : as the a_i are not necessarily different, we denote by $M \leq N$ the number of different a_i's and denote by $\{b_j\}_{j=1}^M$ the set of the different coefficients in increasing order, which means that

$$\{b_j\}_{j=1}^M = \{a_i\}_{i=1}^N,$$

and

$$b_1 < b_2 < \cdots < b_M.$$

Further for all $j = 1, \cdots M$, denote by k_j the number of repeated values of b_j in the initial set of coefficients a_i, namely

$$k_j = \#\{i \in \{1, \cdots, N\} : b_j = a_i\}.$$

Lemma 4.6.5. *If Q is the polynomial defined by (4.6.10), then its has $N-1$ real roots $\mu_i, i = 1, \cdots, N-1$, in $[-a_N, -a_1]$ such that*

$$-b_{j+1} < \mu_j < -b_j, \forall j = 1, \cdots, M-1,$$

the other roots are $-b_j$ of multiplicity $k_j - 1$, for all $j = 1, \cdots, M$ such that $k_j \geq 2$.

Proof: We first notice that

$$Q(-a_i) = \prod_{\substack{l=1 \\ l \neq i}}^N (a_l - a_i).$$

Hence we see that $-a_i$ is a root of Q if and only if there exists at least one $\ell \neq i$ such that $a_i = a_\ell$. But for a complex number μ such that $\mu \notin \{-a_i\}_{i=1}^N$, we notice that

$$Q(\mu) = \prod_{l=1}^N (\mu + a_l) \left(\sum_{i=1}^N \frac{1}{\mu + a_i} \right). \tag{4.6.11}$$

Therefore $\mu \notin \{-a_i\}_{i=1}^N$ is a root of Q if and only if

$$\widetilde{Q}(\mu) = \sum_{i=1}^{N} \frac{1}{\mu + a_i} = 0.$$

As $\widetilde{Q}$ has vertical asymptotes $\mu = -b_j$, for all $j = 1, \dots, M$ and is a decreasing function on $(-b_{j+1}, -b_j)$, for all $j = 1, \cdots, M-1$ (see Figure 4.2 for the graph of $\widetilde{Q}$ when $N = M = 4,\ a_1 = -2,\ a_2 = 2,\ a_3 = 3,\ a_4 = 4$), we deduce that Q has one and only one real root between two consecutive vertical asymptotes.

Now for $j = 1, \cdots, M$ such that $k_j \geq 2$, we take $\mu \neq -b_j$ but close to it and use the expression (4.6.11) to find that

$$\begin{aligned} Q(\mu) &= \prod_{\ell : a_\ell \neq b_j} (\mu + a_l)\, (\mu + b_j)^{k_j} \left(\frac{k_j}{\mu + b_j} + \sum_{i : a_i \neq b_j} \frac{1}{\mu + a_i} \right) \qquad (4.6.12) \\ &= (\mu + b_j)^{k_j - 1} \prod_{\ell : a_\ell \neq b_j} (\mu + a_l) \left(k_j + (\mu + b_j) \sum_{i : a_i \neq b_j} \frac{1}{\mu + a_i} \right). \qquad (4.6.13) \end{aligned}$$

Since $\prod_{\ell : a_\ell \neq b_j} (\mu + a_l) \left(k_j + (\mu + b_j) \sum_{i : a_i \neq b_j} \frac{1}{\mu + a_i} \right)$ is holomorphic in a neighborhood of $-b_j$, we deduce that $-b_j$ is a root of Q of multiplicity $k_j - 1$. ■

Corollary 4.6.6. *The polynomial Q defined by (4.6.10) has negative roots if and only if one of the following two conditions is satisfied :*

(i) $a_1 \geq 0$ and $a_i > 0$, for all $i = 2, \cdots, N$,

(ii) $a_1 < 0$, $a_i > 0$, for all $i = 2, \cdots, N$ and

$$\sum_{i=1}^{N} \frac{1}{a_i} < 0.$$

Proof: According to the previous lemma, if $-b_2 \geq 0$, then Q has a positive root, hence b_2 has to be positive. Now if $b_1 = a_1$ is positive, all roots are trivially negative.

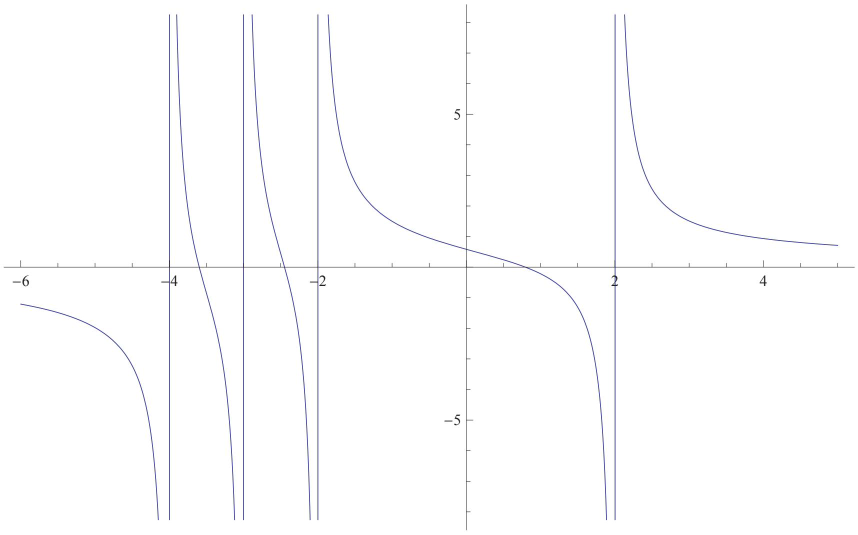

FIGURE 4.2 – $N = 4, a_1 = -2, a_2 = 2, a_3 = 3, a_4 = 4$

On the other hand, if $b_1 \leq 0$ with $k_1 > 1$, then Q has a non negative root $-b_1$. Hence k_1 has to be equal to 1. This covers the first item. For the second item, we have $b_1 < 0$ with $k_1 = 1$ and therefore again according to the previous lemma, Q has a root μ (or equivalently $\widetilde{Q}$) between $-a_2 < 0$ and $-a_1 > 0$ that potentially could be positive, but since $\widetilde{Q}$ is decreasing on $(-a_2, -a_1)$ the condition

$$\widetilde{Q}(0) = \sum_{i=1}^{N} \frac{1}{a_i} < 0$$

is a necessary and sufficient condition to get $\mu < 0$. ■

Summing up the results of Lemmas 4.6.2, 4.6.3, and Corollary 4.6.6, we give the proof of Theorem 4.1.4.

Proof: of Theorem 4.1.4 : According to Lemma 4.6.2, if $\max_{k\in\mathbb{Z}} \max_{j=1,\cdots,m_k} \mu_{k,j} = -C < 0$, then there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, and all $k \in \mathbb{Z}$, $\Re\lambda_k(\epsilon) \leq -\frac{C}{2}\epsilon$. Using Lemmas 4.6.3 and Corollary 4.6.6, this is satisfied if either one of the two conditions is satisfied :

(i) $a_1 \geq 0$ and $a_i > 0$, for all $i = 2, \cdots, N$,

(ii) $a_1 < 0$, $a_i > 0$, for all $i = 2, \cdots, N$ and

$$\sum_{i=1}^{N} a_i > 0 \text{ as well as } \sum_{i=1}^{N} \frac{1}{a_i} < 0.$$

Since the root vectors of $A(\epsilon)$ form a Riesz basis with parenthesis (see Proposition 4.4.1), we deduce the exponential stability of the solution of (S_ϵ) for all $\epsilon \in (0, \epsilon_0)$ under one of the conditions (i) or (ii) stated above. ■

Remark 4.6.7. *(i) Owing to Lemma 4.6.3 and Corollary 4.6.6, if* $\sum_{i=1}^{N} a_i < 0$ *or if* Q *has a positive root, then* (S_ϵ) *is unstable for all* $\epsilon > 0$ *small enough.*

(ii) If $\sum_{i=1}^{N} a_i = 0$ *or if* Q *has a root equal to zero, then the exponential stability of* (S_ϵ) *for* $\epsilon > 0$ *small enough is an open question. For example, in the case* $N = 3$, $a_1 = a_2 = 1$, *and* $a_3 = -\frac{1}{2}$, *Figure 4.6 shows that there are eigenvalues to the right of the imaginary axis when* $\epsilon = 0.1$.

Remark 4.6.8. *The previous analysis can be adapted to the case when* $a_i \in L^\infty(0,1)$ *and* $L_i = 1$ *for all* $i = 1, \cdots, N$. *As before we can prove that the solution of* (S_ϵ) *is exponentially stable for all* $\epsilon \in (0, \epsilon_0)$ *for* $\epsilon_0 > 0$ *small enough if there exists* $c_0 > 0$ *and* $c_1 > 0$ *such that for all* $k \in \mathbb{Z}$, *one of the following two conditions holds :*

(a) There exists at most one $j_0 \in \{1, \cdots, N\}$ *such that*
$\int_0^1 a_{j_0}(x) \sin^2(k\pi(1-x))dx = 0$, $\int_0^1 a_i(x) \sin^2(k\pi(1-x))dx > c_0$ *for all* $i \neq j_0$ *and* $\sum_{i=1}^{N} \int_0^1 a_i(x) \sin^2((k\pi + \frac{\pi}{2})(1-x))dx > c_0$.

(b) There exists only one $i_0 \in \{1, \cdots, N\}$ *such that*
$\int_0^1 a_{i_0}(x) \sin^2(k\pi(1-x))dx < 0$, $\int_0^1 a_i(x) \sin^2(k\pi(1-x))dx > c_0$ *for all* $i \neq i_0$,

$$\sum_{i=1}^{N}\int_0^1 a_i(x)\sin^2((k\pi+\frac{\pi}{2})(1-x))dx > c_0, \ and$$
$$\sum_{i=1}^{N}\frac{1}{\displaystyle\int_0^1 a_i(x)\sin^2(k\pi(1-x))dx} < -c_1.$$

Indeed, the results of Lemma 4.6.2 still hold. Lemma 4.6.3 also holds but in this case, for all $k\in\mathbb{Z}$, $\mu_{k,0} = -\frac{2}{N}\sum_{i=1}^{N}\int_0^1 a_i(x)\sin^2((k\pi+\frac{\pi}{2})(1-x))dx$. Similarly, in Lemma 4.6.4, we can repeat the same analysis and find that, for all $k\in\mathbb{Z}^*$, $\{\mu_{k,j}\}_{j=1}^{N-1}$ is the set of zeros of

$$\widehat{Q}(z) = (z+I_1)(z+I_N)\sum_{i=2}^{N-1}\prod_{\substack{l\neq i\\ l=2}}^{N-1}(z+I_l) + \prod_{l=2}^{N-1}(z+I_l)(2z+I_1+I_N),$$

where for all $i=1,\cdots,N$, $I_i = 2\int_0^1 a_i(x)\sin^2(k\pi(1-x))dx$ (which here depends on k). As Lemma 4.6.5 can be used for $\widehat{Q}$, we find the same results but with a_i replaced by I_i for all $i=1,\cdots,N$. Therefore, thanks to Lemma 4.6.2 and under one of the conditions (a) or (b) stated above, we deduce the existence of $\widehat{C}>0$ such that for all $k\in\mathbb{Z}$, $\Re\lambda_k(\epsilon)\leq -\epsilon\widehat{C}$ for all $\epsilon\in(0,\epsilon_0)$.

4.7 Examples

In order to illustrate our general results we present some concrete examples where we can give explicit conditions on the damping coefficients to get exponential decay (up to a finite-dimensional space) for both problems (S_1) and (S_ϵ). In the first case, this is reduced to the calculation of the roots of the polynomial G defined by (4.1.1), in the second one since the conditions from Theorem 4.1.4 are easy to check, we concentrate on a limit case (see Remark 4.6.7) and on the characterization of the limit values of ϵ for which the global stability is lost.

4.7.1 Examples for (S_1)

We consider (S_1) with three edges $(N = 3)$ of length $L_i = 1$ and $a_i(.) \in W^{1,\infty}(0,1)$ such that $\int_0^1 a_1(x)dx = \int_0^1 a_2(x)dx = 1$ and $\int_0^1 a_3(x)dx = \alpha \leq 0$. Using Theorem 4.1.1, we will find the critical value of α for which (S_1) is exponentially stable up to a finite dimensional space. Indeed, for this example, the polynomial G is given by

$$G(z) = 3e^{2+\alpha}z^3 - (e^{2-\alpha} + 2e^{\alpha})z^2 - (e^{-2+\alpha} + 2e^{-\alpha})z + 3e^{-2-\alpha}.$$

The roots of G are given by

$$\begin{aligned} z_1 &= e^{-2}, \\ z_2 &= -\frac{e^{-2}}{6} + \frac{e^{-2\alpha}}{6} - \frac{e^{-2-2\alpha}}{6}\sqrt{e^4 + e^{4\alpha} + 34e^{2+2\alpha}}, \\ z_3 & \; -\frac{e^{-2}}{6} + \frac{e^{-2\alpha}}{6} + \frac{e^{-2-2\alpha}}{6}\sqrt{e^4 + e^{4\alpha} + 34e^{2+2\alpha}}. \end{aligned}$$

Recall that according to Theorem 4.1.1, (S_1) is exponentially stable up to a finite dimensional space if and only if $|z_i| < 1$ for all $i = 1, 2, 3$. Hence we need to analyze the behavior of the z_i with respect to α. Clearly $z_1 < 1$ is independent of α, while the two other ones depend on α. For z_2, we easily check that $z_2 < 0$ is an increasing function of α with $\lim_{\alpha \to -\infty} z_2 = -3e^{-3} > -1$. Hence, $-1 < z_2 < 0$ for all $\alpha \leq 0$. Next, we notice that $|z_2||z_3| = e^{-2-2\alpha}$. So, if $\alpha \leq -1$, then $|z_2||z_3| \geq 1$ which means that $|z_3| \geq 1$. Therefore, to get the exponential stability of (S_1), we must have $\alpha > -1$. In this case, z_3 is a decreasing function of α and for $\alpha_0 = \frac{1}{2}\ln\left(\frac{3+e^2}{1+3e^2}\right)$ we get $z_3 \geq 1$ if $\alpha \leq \alpha_0$ and $0 < z_3 < 1$ if $\alpha > \alpha_0$. In conclusion, (S_1) is exponentially stable up to a finite dimensional space if and only if $\alpha \in (\alpha_0, 0]$.

As a second example, we still take three edges, but consider $L_1 = L_2 = 1$ and $L_3 = 2$ with $\int_0^1 a_1(x)dx = \int_0^1 a_2(x)dx = 1$ and $\int_0^2 a_3(x)dx = \alpha$. With this choice, the polynomial G is given by

$$G(z) = (ez - e^{-1})p(z).$$

where

$$p(z) = 3e^{1+\alpha}z^3 + e^{-1+\alpha}z^2 - e^{1-\alpha}z - 3e^{-1-\alpha}.$$

As the roots of the first factor is $e^{-2} < 1$, we only have to consider the roots of the second factor p. Let $z_i = z_i(\alpha)$ for $i = 1, 2, 3$ be the roots of p and define $\varphi(\alpha) = \max_{i\in\{1,2,3\}} |z_i(\alpha)|$. With the help of a formal computation software (Mathematica), we can find the roots $z_i(\alpha)$ for $i = 1, 2, 3$ as well as $\varphi(\alpha)$.

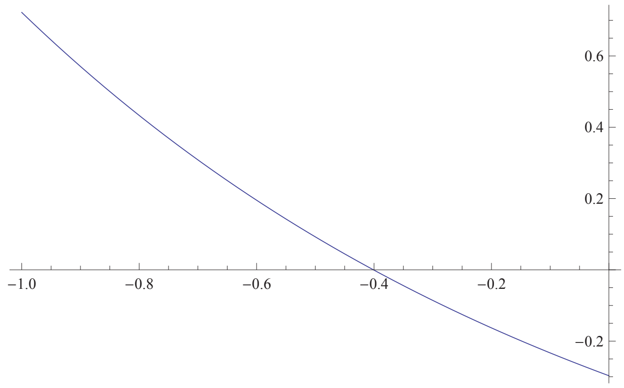

FIGURE 4.3 – Graph of $\varphi(\alpha) - 1$ when $\int_0^1 a_1(x)dx = \int_0^1 a_2(x)dx = 1$, $\int_0^2 a_3(x)dx = \alpha$.

The explicit form of φ allows to check that when $\alpha > \alpha_0 = \frac{1}{2}\ln\left(\frac{3+e^2}{1+3e^2}\right)$ then $\varphi(\alpha) < 1$ (see Figure 4.3). Hence (S_1) is exponentially stable up to a finite dimensional space if and only if $\alpha \in (\alpha_0, 0]$.

The same study can be done when changing L_3, namely by taking $L_3 = \frac{1}{2}$ or $L_3 = 3$ and we surprisingly obtain the same critical value α_0 of α so that (S_1) is exponentially stable up to a finite dimensional space. Moreover, if we choose $L_1 = 1$ and $L_2 = 2$ such that $\int_0^1 a_1(x)dx = \int_0^2 a_2(x)dx = 1$, then for $L_3 = 1$ or $L_3 = 2$, we still obtain the same condition, $\alpha > \alpha_0 = \frac{1}{2}\ln\left(\frac{3+e^2}{1+3e^2}\right)$ to get the exponential stability of (S_1) up to a finite dimensional space. Furthermore, if we change the

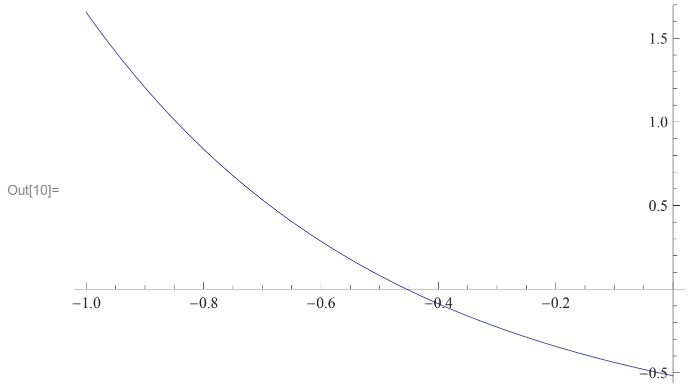

FIGURE 4.4 – Graph of $\varphi(\alpha)-1$ when $\int_0^1 a_1(x)dx = 1,\ \int_0^1 a_2(x)dx = 2,\ L_3 = 2.$

mean values, by considering $L_1 = L_2 = 1$, but $\int_0^1 a_1(x)dx = 1$ and $\int_0^1 a_2(x)dx = 2$, then whether $L_3 = 1$ or $L_3 = 2$, we still obtain the same critical value α_1 with $0.45 < \alpha_1 < 0.46$ such that (S_1) is exponentially stable up to a finite dimensional space if and only if $\alpha > \alpha_1$ (see Figure 4.4).

In conclusion, we find that the critical value of α depends on $\int_0^{L_1} a_1(x)dx$ and $\int_0^{L_2} a_2(x)dx$ and not on the choice of the lengths. This opens the question whether the abstract condition given in Theorem 4.1.1 can be expressed explicitly in terms of $\int_0^{L_i} a_i(x)dx$ for all $i \in \{1, \cdots, N\}$, see Conjecture 4.1.2.

4.7.2 Examples for problem (S_ϵ)

We start with a limit case in Theorem 4.1.4, namely we take $N = 3$, $a_1 = a_2 = 1$, $a_3 = -\frac{1}{2}$, and $L_1 = L_2 = L_3 = 1$. Hence neither the first condition holds nor the second one since $\sum_{i=1}^{N} a_i > 0$ but $\sum_{i=1}^{N} \frac{1}{a_i} = 0$.

But Lemma 4.6.2 yields that for all $\epsilon > 0$ small enough, the eigenvalues are of the form

$$\begin{aligned}\lambda_{1,k}(\epsilon) &= -\epsilon + \imath k\pi + o(\epsilon),\\ \lambda_{2,k}(\epsilon) &= \imath k\pi + o(\epsilon),\\ \lambda_{3,k}(\epsilon) &= -\frac{\epsilon}{2} + \imath\frac{(2k+1)\pi}{2} + o(\epsilon),\end{aligned}$$

Hence the problem of stability would come from $\lambda_{2,k}(\epsilon)$ but a more precise asymptotic analysis yields $\Re\lambda_{2,k}(\epsilon) = \frac{\epsilon^3}{12} + o(\epsilon^3)$, hence the problem is not exponentially stable for ϵ small. Figure 4.6 shows the existence of a positive asymptote when $\epsilon = 0.1$, since the asymptotes are $x_1 = -0.1$, $x_2 \approx -0.0500833$, and $x_3 \approx 0.000083333$.

Note that for $\epsilon = 1$, then by Theorem 4.1.1 there is a positive asymptote, since the asymptotes are $x_1 = -1$, $x_2 \approx -0.580322$, and $x_3 \approx 0.0803219$ (see Figure 4.5).

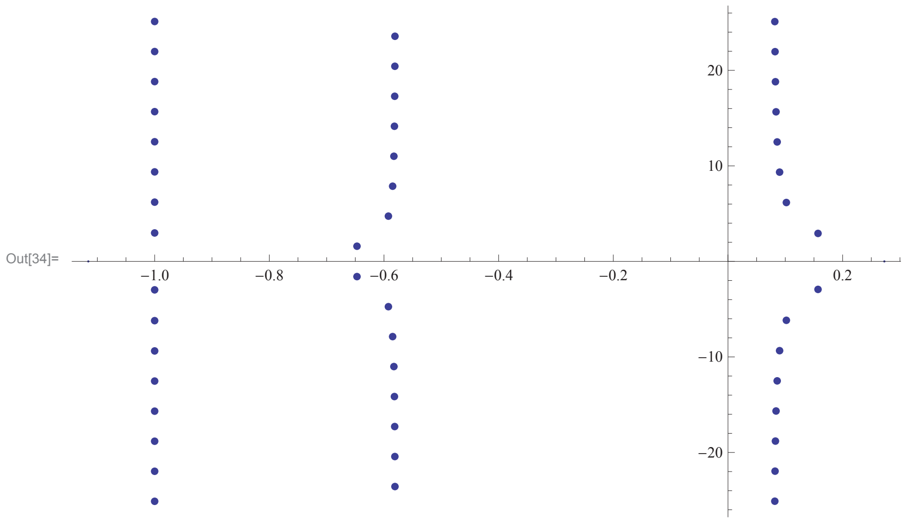

FIGURE 4.5 – $a_1 = a_2 = 1,\ a_3 = -\frac{1}{2},\ \epsilon = 1.$

In general, if we consider $a_1 = a_2 = a$ and $a_3 \neq a$, then according to Theorem 4.1.4, the problem becomes exponentially stable for all ϵ small enough if one of the following two conditions holds :

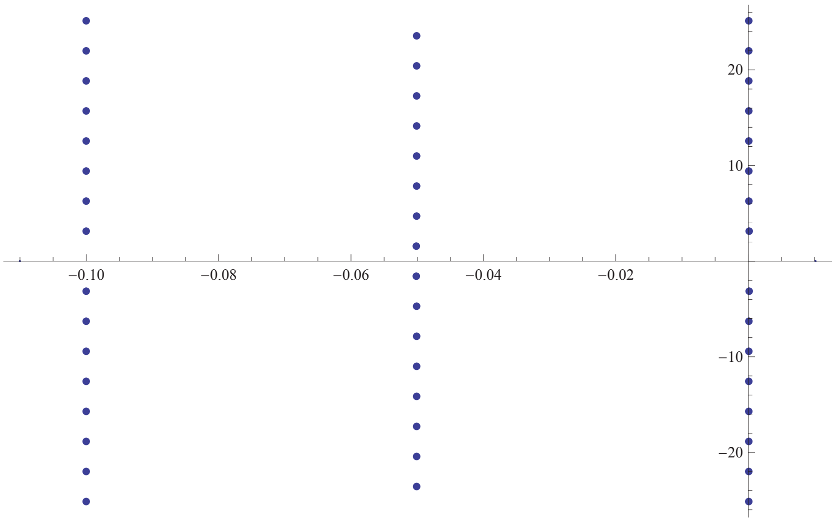

FIGURE 4.6 – $a_1 = a_2 = 1,\ a_3 = -\frac{1}{2},\ \epsilon = 0.1.$

(i) $a > 0$ and $a_3 \geq 0$

(ii) $a > 0$ and $a_3 < 0$ such that $2a + a_3 > 0$ and $a + 2a_3 > 0$.

These results are coherent with the numerical results shown in Figure 4.9 with $a_1 = a_2 = 1,\ a_3 = -\frac{1}{4}$ and $\epsilon = \frac{1}{10}$ where the asymptotes ($x_1 = -0.1,\ x_2 \approx -0.0166184$, and $x_3 \approx -0.0583816$) are to the left of the imaginary axis. If we increase ϵ and take $\epsilon = 1$, then Figure 4.7 still shows the exponential stability in the whole energy space where the asymptotes are $x_1 = -1,\ x_2 \approx -0.630695$, and $x_3 \approx -0.119305$. But for $\epsilon = 1.5$, then Figure 4.8 shows the exponential stability up to a finite dimensional space. Indeed, the asymptotes found in Figure 4.8 are $x_1 = -1.5,\ x_2 \approx -1.02451$, and $x_3 \approx -0.100488$ which show that the large eigenvalues are to the left of the imaginary axis although there are some low eigenvalues with positive real parts. In fact, in the case $a_1 = a_2 = 1,\ a_3 = -\frac{1}{4}$, all the eigenvalues are to the left of the imaginary axis for all $\epsilon < \epsilon_0$, where numerically we have found that $1.30 < \epsilon_0 < 1.31$.

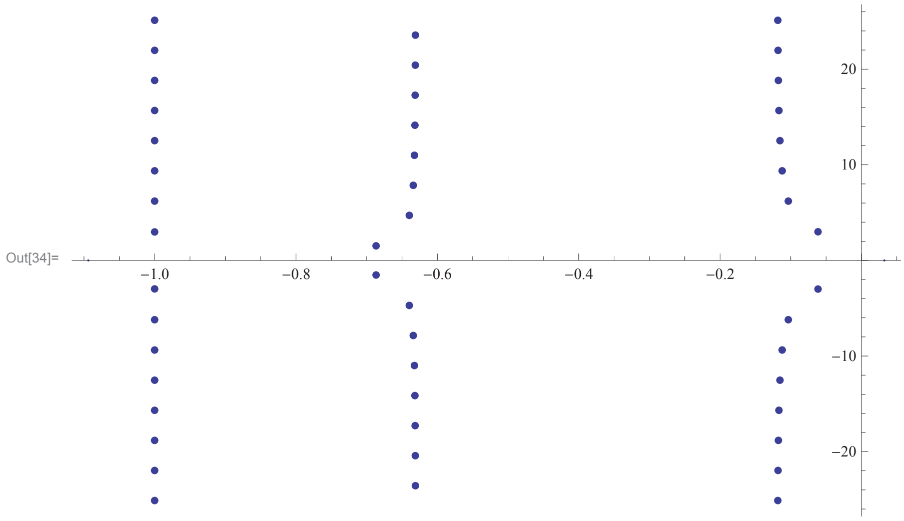

FIGURE 4.7 – $a_1 = a_2 = 1,\ a_3 = -\dfrac{1}{4},\ \epsilon = 1.$

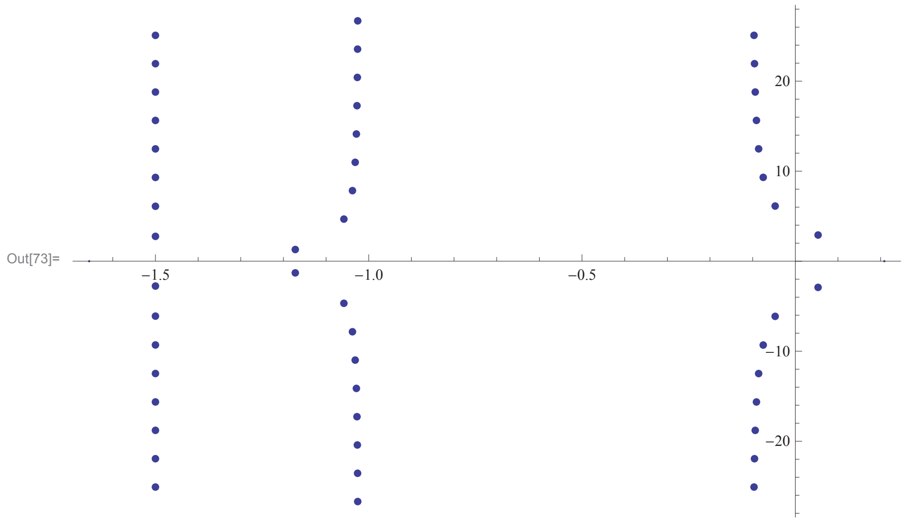

FIGURE 4.8 – $a_1 = a_2 = 1,\ a_3 = -\dfrac{1}{4},\ \epsilon = 1.5.$

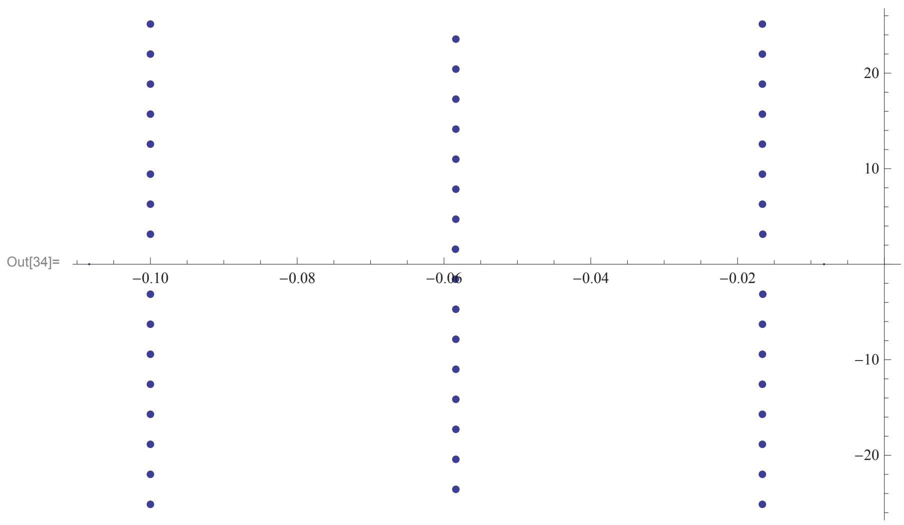

FIGURE 4.9 – $a_1 = a_2 = 1,\ a_3 = -\dfrac{1}{4},\ \epsilon = 1/10.$

Perspective

As a future study, we would like to extend our work about the stability of wave equations with indefinite sign damping over a multi-dimensional space. For instance, let $\Omega = (-1,1)\times(0,1)$ be partitioned into $\Omega_1 = (0,1)\times(0,1)$ and $\Omega_2 = (-1,0)\times(0,1)$. We are interested in studying the stability of the following system

$$\begin{cases} u_{tt} - \Delta u + a_1 u_t &= 0 \quad \text{in } \Omega_1 \times \mathbb{R}, \\ u_{tt} - \Delta u + a_2 u_t &= 0 \quad \text{in } \Omega_2 \times \mathbb{R}, \\ u &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,y,0) = u_0,\ u_t(x,y,0) &= u_1 \quad \text{in } \Omega, \end{cases} \tag{4.7.1}$$

where $a_1 > 0$ and $a_2 < 0$. We write the solution as a Fourier series of the form

$$u(x,y) = 2\sum_{k=1}^{\infty} u_k(x)\sin(k\pi y),\ \forall (x,y)\in\Omega.$$

Then

$$\begin{cases} u_{ktt} - u_{kxx} + k^2\pi^2 u_k + a(x)u_{kt} &= 0 \qquad (x,t)\in(-1,1)\times\mathbb{R}, \\ u_k(-1,t) = u_k(1,t) &= 0 \qquad t\in\mathbb{R}, \\ u_k(x,0) = u_{k0},\ u_{kt}(x,0) &= u_{k1} \quad x\in(-1,1), \end{cases} \tag{4.7.2}$$

where $a(x) = a_1$ if $x\in(0,1)$ and $a(x) = a_2$ if $x\in(-1,0)$. The energy associated with (4.7.2) is given by

$$E_k(t) = \frac{1}{2}\int_{-1}^{1}\left(|u_{kx}|^2 + k^2\pi^2|u_k|^2 + |u_{kt}|^2\right)dx.$$

Using Parseval's equality, the energy associated with (4.7.1) is given by

$$E(t) = \frac{1}{2}\int_{\Omega} \left(|\nabla u|^2 + |u_t|^2\right) dx = \sum_{k=1}^{\infty} E_k(t).$$

Our aim is to find $c > 0$ independent of k and $\nu_k > 0$ such that

$$E_k(t) \leq ce^{-\nu_k t} E_k(0).$$

Hence,

$$E(t) \leq c\sum_{k=1}^{\infty} e^{-\nu_k t} E_k(0).$$

If $\nu_k \geq \nu > 0$, then $E(t) \leq ce^{-\nu t}E(0)$ and hence system (4.7.1) is exponentially stable. If $\nu_k \cong \dfrac{1}{k^l}$ for some $l > 0$, then

$$\begin{aligned} E(t) &\lesssim \sum_{k=1}^{\infty} e^{\frac{-t}{k^l}} E_k(0) \\ &\lesssim \sum_{k=1}^{\infty} \frac{k^l}{t}\frac{t}{k^l} e^{\frac{-t}{k^l}} E_k(0) \\ &\lesssim \frac{1}{t}\sum_{k=1}^{\infty} k^l E_k(0) \\ &\lesssim \frac{1}{t}\|u\|_{D(A^{\frac{l}{2}})}, \end{aligned}$$

where the operator A is the generator of the semigroup associated with system (4.7.1).

We are also interested in studying both internally and boundary damped problems of the form

$$\begin{cases} u_{tt} - \Delta u + au_t &= 0 \quad \text{in } \Omega \times \mathbb{R}, \\ u_x &= -bu_t \quad \text{on } \Gamma_N \times \mathbb{R}, \\ u &= 0 \quad \text{on } \Gamma_D \times \mathbb{R}, \\ u(x, y, 0) = u_0,\ u_t(x, y, 0) &= u_1 \quad \text{in } \Omega, \end{cases} \tag{4.7.3}$$

where $\Gamma_N = \{x = 1\} \times (0, 1)$ and $\Gamma_D = \partial\Omega/\Gamma_N$. We are interested in studying the stability of (4.7.3) when $ab < 0$. If $a = 0$ and $b > 0$ or $a > 0$ and $b = 0$, problem (4.7.3) is polynomially stable and not exponentially stable. Therefore, we expect to find conditions on a and b for which problem (4.7.3) is polynomially stable.

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