DEBLURRING OF IRREGULARLY SAMPLED IMAGES IN SPLINE SPACES

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WHAT IS IRREGULAR SAMPLING ?

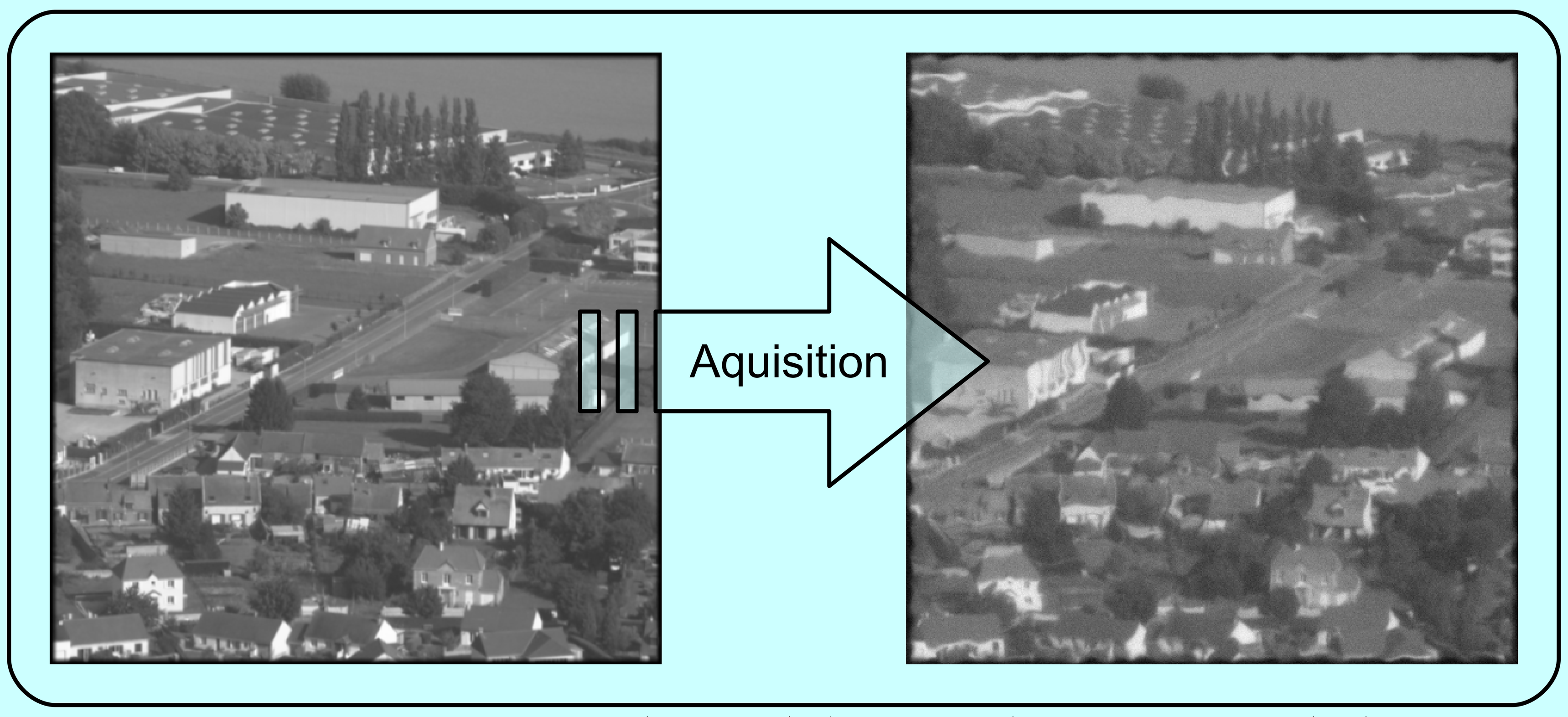

FIGURE 1 : $Z_k = (h*u)(x_k, y_k) + n_k \quad (1)$

$\Lambda = \{(x_k, y_k)\}_{1 \leqslant k \leqslant N_s}$: sampling grid
u : continuous scene
h : impulse response
n : i.i.d. gaussian vector

The irregularity in the sampling grid may come from :

- microvibrations of the push-broom satellite during acquisition
- superresolution by fusion of multiple aliased views of the same scene
- missing pixels

STATE OF THE ART

Recent methods use regularized formulations to solve the irregular sampling linear problem :

$$Argmin_{u \in F} \; E(u) \quad s.t. \quad \| S.u - Z \|^2 \leqslant N_s . \sigma^2$$

where the functional space F may be trigonometric polynomials [1,3,4], or integer shift invariant spaces (splines) [2].

Non-linear regularizers like Total Variation [4] and Frequency Adaptative Regularizers [3] where shown to perform better than linear ones like ACT [1], which are also used in the spline implementation [2].

On the other hand spline-based algorithms are much faster due to their compactly supported base functions. (Sparse operator S) but the spline implementation [2] cannot deal with convolution in addition to sampling.

	Function space	Regularization	Convolution	Fast
[1]	T.P.	linear	yes	+
[3,4]	T.P.	non-linear	yes	-
[2]	Splines	linear	no	++
Ours	Splines	non-linear	yes	+

FIGURE 2 : Regularization methods in irregular sampling

MOTIVATION OF THIS WORK

- Faster restoration from irregular samples via :
  - Compactly supported basis functions instead of trigonometric polynomials
  - Fast Splitting Algorithm
- Extend Arigovindan's work on spline resampling to deconvolution and TV regularization

PROJECTION ON A SHIFT INVARIANT SPACE

$$proj_V(h*u) = \sum_{k,l} a_{k,l} \; proj_V(h*\beta^{(3)})(x-k, y-l)$$
$$proj_V(h*\beta^{(3)}) = Argmin_{v \in V} \left\| (h*\beta^{(3)}) - v \right\|_{L^2} \quad (2)$$

Indeed it is sufficient to compute the projection of only one convolved spline with the Euler-Lagrange equation of (2)

$$\sum_{i,j} \widetilde{H}_{i,j} \langle \beta^{(3)}(x-i, y-j), \beta^{(3)}(x-k, y-l) \rangle = \langle h*\beta^{(3)}(x,y), \beta^{(3)}(x-k, y-l) \rangle \quad , \forall (k,l)$$

where $\widetilde{H}_{i,j}$ are the coefficients of $proj_V(h*\beta^{(3)})$

The associated matrix is circulant, the system is directly inverted with Fourier transforms since

$$FFT(\widetilde{H}) = \frac{FFT(\langle h*\beta^{(3)}(x,y), \beta^{(3)}(x-k, y-l) \rangle)}{FFT(\langle \beta^{(3)}(x,y), \beta^{(3)}(x-k, y-l) \rangle)}$$

(direct computation) $= \beta^{(7)}(k,l)$

By Plancherel's Formula the second scalar product rewrites :

$$\sum_{(q,p) \in \mathbb{Z}^2} \left(\hat{h} . |\widehat{\beta^{(3)}}|^2 \right) \left(q \frac{2\pi}{m}, p \frac{2\pi}{n} \right) e^{-i.(q\frac{2k\pi}{m} + p\frac{2l\pi}{n})}$$

We have good approximation for frequencies in twice the Nyquist range : (q,p) in [-m,m[x[-n,n[(FIG. 5)

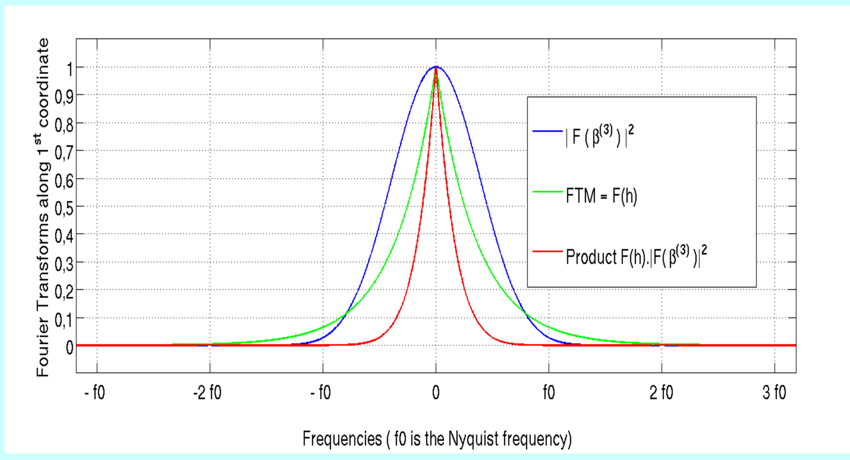

FIGURE 4 : Fourier transforms and summed function

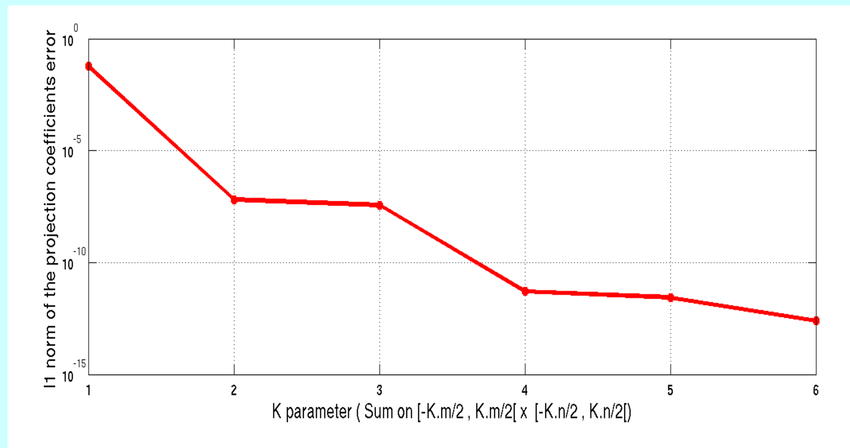

FIGURE 5 : Convergence of the truncated sum

THE SPLINE APPROXIMATION

Spline functions have piecewise polynomial expression and compact support. Here the scene u is assumed to be in a shift invariant, periodic space spanned by tensorial B-splines of order 3, as in [2] .

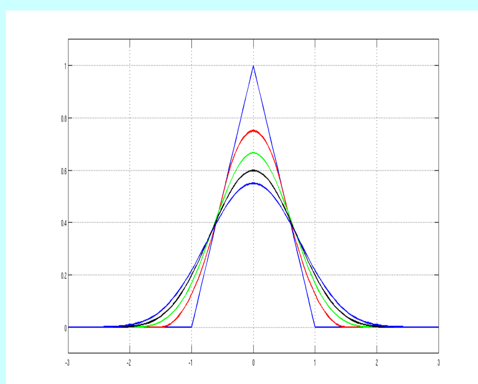

FIGURE 3 : B-Splines order 1 to 5

$$V = span\{\beta^{(3)}(x-k, y-l)\}_{k,l}$$
$$u(x,y) = \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} a_{k,l} \beta^{(3)}(x-k, y-l)$$

- The convolution product in (1) may not belong to a spline space this is why it is projected on V
- Evaluation on the irregular grid is a linear operator (S irr.)

$$(u(x_k, y_k))_{1 \leqslant k < N_s} = S_{irr} . a$$

- Evaluation on the Regular Grid is also necessary (S reg.)

$$U = (u(k,l))_{0 \leqslant k < m, 0 \leqslant l < n} = S_{reg} . a$$

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TV REGULARIZATION

$$\min_u \; TV(U) \quad s.t. \quad \left\| S_{irr}.\widetilde{H}.S_{reg}^{-1}.U - Z \right\|^2 \leqslant N_s.\sigma^2$$

$$TV(U) = \max_{S\in\Gamma} \langle \nabla U, S \rangle \qquad \Gamma = \left\{ S \in (\mathbb{R}^2)^{m\times n} \;/\; \|S_{i,j}\| \leq 1 \quad \forall i,j \right\}$$

$$\begin{cases} (\nabla U)_1(k,l) = \begin{cases} U(k+1,l) - U(k,l) & if\; k \neq m-1 \\ 0 & \text{else} \end{cases} \\ (\nabla U)_2(k,l) = \begin{cases} U(k,l+1) - U(k,l) & if\; l \neq n-1 \\ 0 & \text{else} \end{cases} \end{cases}$$

Lagrangian form : $\exists \lambda > 0 \;\; s.t. \;\; U \;\; minimizes$

$$\underbrace{TV(U)}_{f_1(U)} + \underbrace{\lambda \left\| S_{irr}.\widetilde{H}.S_{reg}^{-1}.U - Z \right\|^2}_{f_2(U)} \qquad (3)$$

This sum of two convexe functions can be minimized by the Forward-Backward splitting algorithm.

MINIMIZATION ALGORITHM

Problem (3) is equivalent to : $0 \in \partial(f_1 + f_2)(U)$

$$<=> \quad -\nabla f_2(U) \in \partial f_1(U)$$

$$<=> \quad U - \tau \nabla f_2(U) \in (Id + \tau\partial f_1)(U) \quad \forall \; \tau > 0$$

$$<=> \quad U = \underbrace{(Id + \tau\partial f_1)^{-1}}_{Backward} \underbrace{(U - \tau\nabla f_2(U))}_{Forward}$$

$(Id + \tau\,\partial f_1)^{-1}$ is the proximal operator of f_1

$$X = (Id + \tau\partial f_1)^{-1}(Y) \;<=>\; X = Argmin_V \; f_1(V) + \frac{1}{2\tau}\|V - Y\|^2 \qquad (4)$$

We use the fixed point algorithm (Forward-Backward)

$$\begin{cases} U^{(k+\frac{1}{2})} = U^{(k)} - \tau \nabla f_2(U^{(k)}) \\ U^{(k+1)} = prox_{TV,\tau}\left(U^{(k+\frac{1}{2})}\right) \end{cases} \qquad (5)$$

and the Chambolle's Algorithm for TV-l2 denoising (4)

Theorem (Combettes et al. [5]) : If $\tau < \frac{2}{C_2}$ where C_2 is the lipschitz constant of ∇f_2 then $U^{(k)}$ defined by (5) converges to a minimizer of (3)

Update of the Lagrange parameter is based on the method noise $S.\widetilde{H}.a - Z$

$$\lambda^{(l+1)} = \lambda^{(l)} \;.\; e^{\left(\frac{\|S.\widetilde{H}.a - Z\|^2}{N_e} - \sigma^2\right)}$$

REFERENCES

[1] H. G. Feichtinger, K. Gröchenig, and T. Strohmer, « Efficient numerical methods in non-uniform sampling theory », *Numerische Mathematik*, vol. 69, no. 4, pp. 423-440, Feb. 1995.

[2] M. Arigovindan, M. Sühling, P. Hunziker, and M. Unser, « Variational image reconstruction form arbitrarily spaced samples: A fast multiresolution spline solution », *IEEE Transaction on Image Processing*, vol 14, no. 4, pp. 450-460, April 2005

[3] G. Facciolo, A. Almansa, J.F. Aujol, and V. Caselles, « Irregular to regular sampling, denoising, and deconvolution », *Multiscale Modeling & Simulation*, vol. 7, no. 4, pp. 1574-1608, 2009

[4] A. Almansa, V. Caselles, G. Haro, and B. Rougé, « Restoration and zoom of irregularly sampled, blurred, and noisy images by accurate total variation minimization with local constraints », *Multiscale Modeling & Simulation*, vol. 5, no. 1, pp. 235-272, 2006

[5] P.L. Combettes and V.R. Wajs, « Signal recovery by proximal forward-backward splitting », *Multiscale Modeling & Simulation*, vol. 4, no. 4, pp. 1168-1200, 2005

EXPERIMENTS AND RESULTS

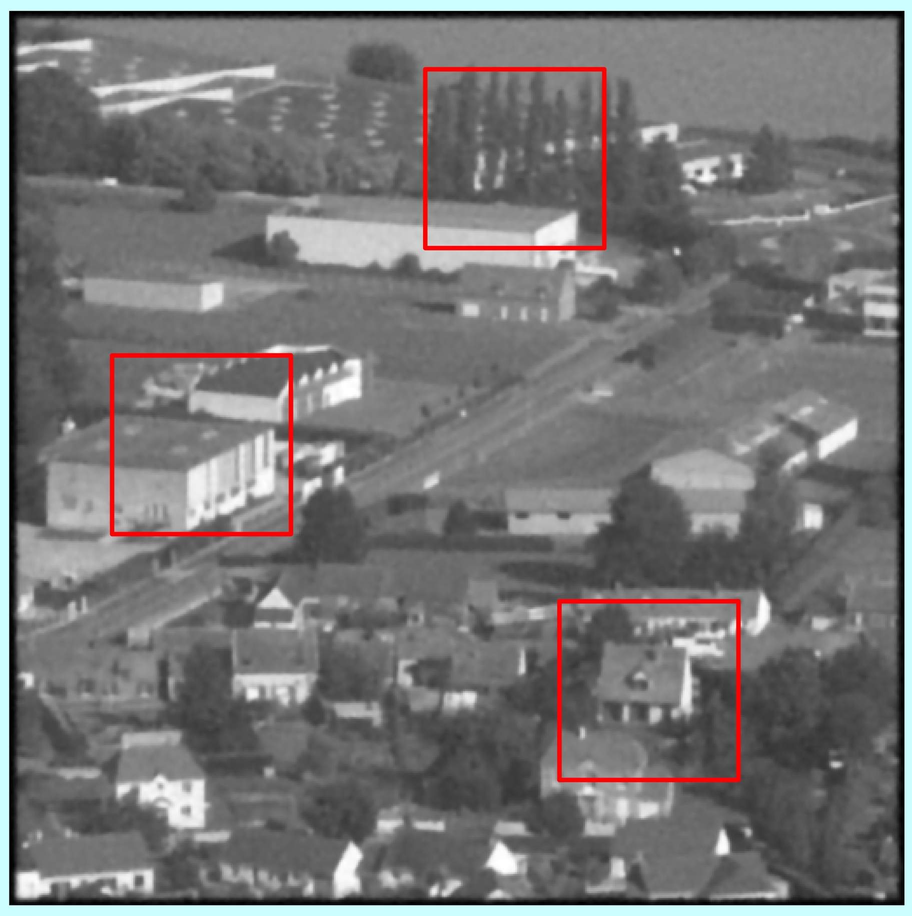

FIGURE 6 : Restored image (size 513x513)$(\sigma = 5)$

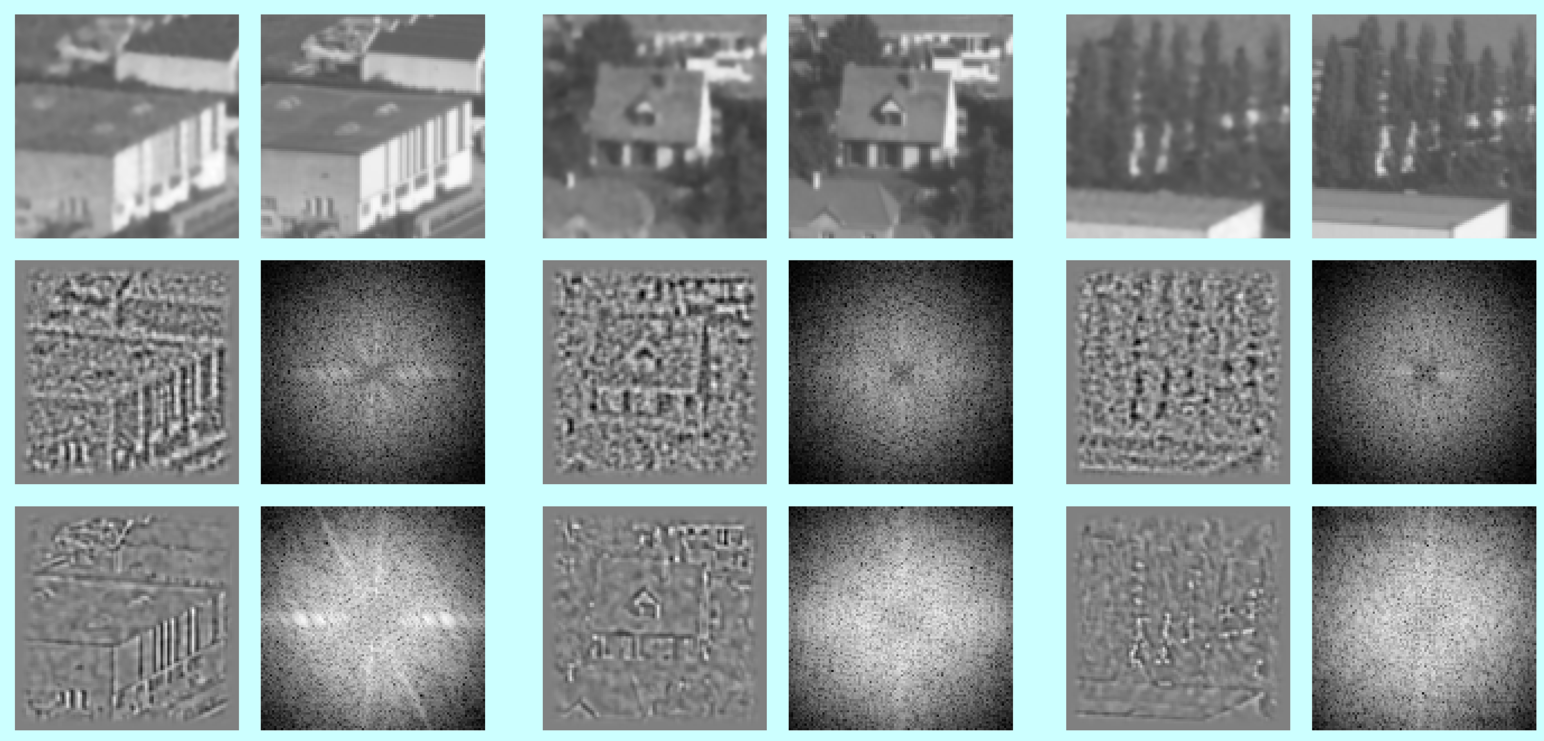

FIGURE 7 :
- Top : restored image (left) and reference (right)
- Middle : method noise and FFT
- Bottom : difference between the reference and the restored image on the regular grid and FFT

Method / Noise level	ACT	FAR	AC-S	Tych-S	TV-S (ours)
$\sigma = 1$	42,36 39s	**43,99** 13mn30s	42,28 2,9s	42,11 35s	**43,88** 41s
$\sigma = 5$	28,27 35s	**35,71** 12mn35s	34,41 1,5s	34,44 10s	**35,62** 39s
$\sigma = 10$	22,24 33s	**32,20** 10mn11s	31,32 1,2s	31,65 34s	**32,11** 53s

- ACT : a truncated conjugate gradient with trigonometric polynomials
- FAR : a non-linear regularizer with trigonometric polynomials and local constraints [3]
- AC-S : same as ACT with splines
- Tych-S : linear regularization (we adapted a version of [2]) with splines and CG
- TV-S : our algorithm, a non-linear TV regularization with splines

FIGURE 8 : PSNR and computation time

CONCLUSION

- Results are very close in terms of PSNR to FAR method [3] which is the best known method at this time and computation time reduced by a factor of approximatively 20 (Fig. 8)
- Compared to linear regularization Tych-S, results are better and computation times are of the same order.
- Method noise (Fig. 7) contains much less structure than the restoration error but could be improved by more sophisticated data fitting constraints (as in [3])

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