Oscillating waves for nonlinear conservation laws

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Habilitation à Diriger des Recherches

Oscillating Waves for Nonlinear Conservation Laws

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Tuesday May 21th, 2013 in front of the jury :

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Acknowledgments

I am greatly indebted to many helpers in all phases of the task of preparing this manuscript: to Thierry Goudon, Bruno Lombard, Yue-Jun Peng and Bernard Rousselet.

Most cordial thanks are due to all the members of the Jury. I am honored by the presence of such mathematicians as Gui-Qiang Chen, Rinaldo Colombo, Isablelle Gallagher and also such acoustician as Vincent Pagneux who represents the GDR US (ultrasonic sound).

It is a pleasure to acknowledge my main collaborators since 2006: Christian Bourdarias and Marguerite Gisclon from Université de Savoie for hyperbolic chromatography studies, Bernard Rousselet from JAD Laboratory for mechanical problems: weak unilateral contact, time reversal mirror and for introducing me to the GDR US, Bruno Lombard from the Marseille LMA and the GDR US for nonlinear cracks problems, Florent Berthelin from JAD Laboratory and Pierre-Emmanuel Jabin from Maryland University for Kinetic equations.

During ten year, before 2006, I was a teacher with not a lot of time to do research works. Nevertheless, I had the chance to work with well established mathematicians as Gui-Qiang Chen, Olivier Guès, Thierry Goudon, Michel Rascle, Giuseppe Toscani. I also thanks other people who trust me to become a mathematician as Jacques Blum, René Lozi, Philippe Maisonobe, Michel Merle, Frédréric Poupaud, Jeffrey Rauch, Pierre Seppecher. I have also to particularly thanks Alain for my first result on geometric optics for hyperbolic PDEs.

I finish my acknowledgments in my native language. Mes plus grands remerciements vont à ma famille: Martine, Guillaume, Vincent, Emily, mes plus grands supporters et ceux qui m'ont le plus supporté depuis si longtemps.

Contents

Chapter 1

Introduction

1.1 Presentation of the manuscript

The manuscript presents my research on hyperbolic Partial Differential Equations (PDE), especially on conservation laws. The corresponding list of publications and preprints is given below on pages 7-8. The papers are available at the URL:

http://math.unice.fr/∼junca/Research/PDEpub.html

My works began with this thought in my mind: "Existence and uniqueness of solutions is not the end but merely the beginning of a theory of differential equations. The really interesting questions concern the behavior of solutions." (P.D. Lax, The formation and decay of shock waves 1974).

To study or highlight some behaviors, I started by working on geometric optics expansions (WKB) for hyperbolic PDEs in chapter 2. For conservation laws, existence of solutions is still a problem (for large data, L^∞ data), so I early learned method of characteristics, Riemann problem, BV spaces, Glimm and Godunov schemes, ...

In this report I emphasize my last works when I became assistant professor.

I use geometric optics method to investigate a conjecture of Lions-Perthame-Tadmor on the maximal smoothing effect for scalar multidimensional conservation laws. This is discussed in chapter 3.

With Christian Bourdarias and Marguerite Gisclon from the LAMA (Laboratoire de Mathématiques de l'Université de Savoie), we have obtained the first mathematical results on a 2×2 system of conservation laws arising in gas chromatography. A series of papers is presented in chapter 4. Of course, I tried to put high oscillations in this system. We have obtained a propagation result in [BGJ4] exhibiting a stratified structure of the velocity, and we have shown that a blow up occurs when there are too high oscillations on the hyperbolic boundary in [BGJ5].

I finish this subject with some works on kinetic equations in chapter 5. In particular, a kinetic formulation of the gas chromatography system, some averaging lemmas for Vlasov equation, and a recent model of a continuous rating system with large interactions are discussed.

Bernard Rousselet (Laboratoire JAD Université de Nice Sophia-Antipolis) introduced me to some periodic solutions related to crak problems and the so called nonlinear normal modes (NNM). Then I became a member of the European GDR: "Wave Propagation in Complex Media for Quantitative and non Destructive Evaluation." In 2008, I started a collaboration with Bruno Lombard, LMA (Laboratoire de Mécanique et d'Acoustique, Marseille). Chapter 6 details mathematical results and challenges we have identified for a linear elasticity model with nonlinear interfaces. It leads to consider original neutral delay differential systems.

Note to the reader: The references of my papers are given in a special bibliography on pages 7-8. The other references which are not my papers are given at the end of chapters 3, 4, 5 and 6.

1.2 Short Curriculum Vitae

- 1993-1994, Military Service in Canjuers and Agrégation de maths (rank 43/2000).
- 1991-1993, 1994-1995, PhD, Optique Géometriques, chocs forts, relaxation.
  Université de Nice Sophia-Antipolis.
  Advisors: Alain Piriou, Michel Rascle
  Jury : Olivier Guès, Fédéric Poupaud, Jeffrey Rauch,
  Monique Sablé-Tougeron, Denis Serre.
- 1996-2006, Teacher (PrAg), IUFM de Nice.
- 2006-. . . , Assistant Professor (MCF), IUFM & Université de Nice Sophia-Antipolis.
- Supervisor for 2 Master Thesis (second year) and for 4 Master Thesis (first year):
  http://math.unice.fr/∼junca/Research/PDEstudents.html

Key words: geometric optics, nonlinear hyperbolic PDEs, conservation laws, gas, chromatography, cracks.

Presented works:

http://math.unice.fr/∼junca/Research/PDEpub.html

[BeJ] Florent Berthelin, Stéphane Junca. *Averaging lemmas with a force term for the transport equation.* Journal de Mathématiques Pures et Appliquées. 93-2, (2010), 113-131.

[BGJ1] Christian Bourdarias, Marguerite Gisclon, Stéphane Junca. *Some mathematical results on a system of transport equations with an algebraic constraint describing fixed-bed adsorption of gases.* Journal of Mathematical Analysis and Applications. 313-2, (2006), 551–571.

[BGJ2] Christian Bourdarias, Marguerite Gisclon, Stéphane Junca. *Existence of weak entropy solutions for gas chromatography system with one or two active species and non convex isotherms.* Communications in Mathematical Sciences. 5-1, (2007), 67-84.

[BGJ3] Christian Bourdarias, Marguerite Gisclon, Stéphane Junca. *Hyperbolic models in gas-solid chromatography.* Boletín de la Sociedad Española de Matemática Aplicada. 43, (2008), 27-55.

[BGJ4] Christian Bourdarias, Marguerite Gisclon, Stéphane Junca. *Strong Stability with respect to weak limit for a Hyperbolic System arising from Gas chromatography.* Methods and Applications of Analysis. 17-3, (2010), 301-330.

[BGJ5] Christian Bourdarias, Marguerite Gisclon, Stéphane Junca. *Blow up at the hyperbolic boundary for a 2×2 system arising from chemical engineering.* Journal of Hyperbolic Differential Equations. 7-2, (2010), 1-20.

[CJR06] Gui-Quiang Chen, Stéphane Junca, Michel Rascle. *Geometric optics for multidimensional conservation law.* Journal of Differential Equations. 222, (2006), 439–475.

[GJ] Thierry Goudon, Stéphane Junca. *Vanishing pressure in gas dynamics equations.* Z. angew. Math. Phys. 51, (2000), 143–148.

[GJT] Thierry Goudon, Stéphane Junca, Giuseppe Toscani. *Fourier-Based distances and Berry-Esseen like inequalities for smooth densities.* Monatshefte fur Mathematik. 135 (2002), 115–136.

[J1] Stéphane Junca. *Réflexions d'ondes oscillantes semilinéaires monodimensionnelles.* Communications in Partial Differential Equations. 23, 3-4, (1998), 727–759.

[J2] Stéphane Junca. *Two-scale convergence and geometric optics for a nonlinear conservation law.* Journal of Asymptotic Analysis. 17, (1998), 221–238.

[J3] Stéphane Junca. *Geometric optics with vanishing viscosity for 1D-semilinear systems.* Revista Mathemática Iberoamericana. 24-2, (2008), 549-566.

[JuLo1] Stéphane Junca, Bruno Lombard. *Dilatation of a one-dimensional nonlinear crack impacted by a periodic elastic wave.* SIAM Journal of Applied Mathematics (SIAP). 70-3, (2009), 735-761.

[JuLo2] Stéphane Junca, Bruno Lombard. *Interaction between periodic elastic waves and two contact nonlinearities.* Mathematical Models and Methods in Applied Sciences. 22-4, (2012), 1-41.

[JR1] Stéphane Junca, Michel Rascle. *Relaxation of the Isothermal Euler-Poisson system to the Drift-Diffusion Equations.* Quaterly of Applied Mathematics. 58-3, (2000), 511–521.

[JR2] Stéphane Junca, Michel Rascle. *Strong relaxation of the isothermal Euler system to the heat equation.* ZAMP (Journal of Applied Mathematics and Physics). 53, (2002), 239–264.

[JRo] Stéphane Junca, Bernard Rousselet. *The method of strained coordinates with weak unilateral springs.* IMA Journal of Applied Mathematics. 76-2, (2011), 251-276.

Preprints

[BGJ6] Christian Bourdarias, Marguerite Gisclon, Stéphane Junca. *Fractional BV spaces and applications to conservation laws.* arXiv:1302.1650, (2013).

[CJ] Pierre Castelli, Stéphane Junca. *Oscillating waves and the maximal smoothing effect for one dimensional nonlinear conservation laws.* arXiv:1302.1345, (2012).

[JJ] Pierre-Emmanuel Jabin, Stéphane Junca. *A continuous rating model.* Preprint, http://math.unice.fr/∼junca/pdffiles/JJ.pdf, (2013).

[J4] Stéphane Junca. *High frequency waves and the maximal smoothing effect for nonlinear scalar conservation laws.* arXiv:1302.2590, (2011).

[JuLo3] Stéphane Junca, Bruno Lombard. *Stability of a critical neutral delay differential equation.* arXiv:1210.5831, (2012).

All the papers, except [JJ], are available at the URL:
http://math.unice.fr/∼junca/Research/PDEpub.html

Chapter 2

Background and results prior to 2006

In this chapter, I recall briefly my works before I became assistant professor. There is no bibliography at the end of this chapter, instead I refer the reader to the bibliography of the corresponding papers.

2.1 PhD Thesis

My thesis was devoted to geometric optics, strong shocks for the isothermal Euler system and an introduction to the relaxation for the one dimensional Isothermal Euler-Poisson system. Publications [J1, J2] are issued from my PhD.

2.1.1 Geometric optics

Advised by Alain Piriou, [J1] is concerned with an oscillating initial boundary value problem of semilinear hyperbolic systems in the spirit of Joly-Métivier-Rauch. In particular a simplified proof of Joly-Métivier-Rauch's result for the Cauchy problem is proposed. Then, I showed the phenomenon of localized oscillations.

The semilinear strictly hyperbolic system is:

$$\left\{\begin{array}{rcll} \dfrac{\partial \mathbf{u}}{\partial t}+\mathbf{A}(t,x)\dfrac{\partial \mathbf{u}}{\partial x} & = & \mathbf{f}(t,x,\mathbf{u}), & t,x>0,\\ \mathbf{u}(0,x) & = & \mathbf{H}\left(x,\dfrac{\varphi_0(x)}{\varepsilon}\right), & x>0,\\ \mathbf{B}\left(t,\mathbf{u}(t,0),\dfrac{\psi_0(t)}{\varepsilon}\right) & = & 0, & t>0. \end{array}\right.$$

I found an optimal construction of all phases issued from the boundary and produced by transmissions, reflexions and resonances.

In [J2], I studied the justification of weakly nonlinear geometric optics for the scalar equation and two scales:

$$\begin{aligned} \partial_t u_\varepsilon+\partial_x f(u_\varepsilon) &= 0,\\ u^\varepsilon(0,x) &= \underline{u}+\varepsilon u_0\left(x,\frac{x}{\varepsilon}\right), \end{aligned}$$

where $\underline{u}$ is a constant, the function u_0 belongs to $BV_{loc}(\mathbb{R}^2, \mathbb{R})$ and is 1-periodic with respect to $\theta := \frac{x}{\varepsilon}$, and $\varepsilon > 0$ is a given small parameter. Thanks to the ε factor, the high frequency oscillations propagate and the natural ansatz is:

$$u^{\varepsilon}(t,x) = \underline{u} + \varepsilon\, U\left(t, x, \frac{x - f'(\underline{u})t}{\varepsilon}\right) + \cdots$$

where $U(t, x, \theta)$ is 1-periodic in θ. The geometric optics expansion is validated for almost all ε. The methods, based on exact profile, reduction of the dimension for Kruzkov solutions, ..., have inspired later works like [CJR06] and [J4].

2.1.2 Strong shocks for isothermal Euler system

Still with Michel Rascle, I studied the stability of strong shocks subjected to small (or not too big) BV perturbation on the one dimensional isothermal Euler system. In Lagrangian variables, the system reads

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{c^2}{v}\right) = 0, \\ \dfrac{\partial v}{\partial t} - \dfrac{\partial u}{\partial x} = 0, \end{cases}$$

where c is the sound speed, u is the velocity and v is the specific volume. The main result is the decay of the Nishida functional (linear Glimm functional) computed outside the strong shocks for the Glimm scheme. This work on the $p-$system has been useful in [JR1, JR2].

2.2 Other works before 2006

2.2.1 Euler system with damping

In [JR1], we consider a classical hydrodynamic model for the transport of electric charges in semi-conductors, namely the Euler-Poisson system, written here in the one-dimensional case :

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}(nu) + \frac{\partial}{\partial x}\left(nu^2 + p(n)\right) = -\frac{nu}{\varepsilon} + nE. \tag{2.2}$$

In this model, n is the density of electrons, u is the velocity, p is the pressure, and E is the electric field. The stiff friction term is intended to describe the collisions between the electrons and the atoms of the crystal. Obviously, equation (2.1) describes the conservation of electrons, and equation (2.2) is the momentum equation, in which the nE term describes the acceleration of the electrons due to the electric field.

This electric field is defined self-consistently by

$$\frac{\partial E}{\partial x} = n - N,$$

where N is the (constant) density of a fixed background charge. The classical work for the isentropic case was done by Marcati and Natalini.

We justify the strong convergence for entropy solutions of the density towards the solution of the drift-diffusion equation when $\varepsilon \to 0$

$$\begin{aligned} \frac{\partial n}{\partial s} + \frac{\partial}{\partial x}\left(nE - \frac{\partial n}{\partial x}\right) &= 0, \\ \frac{\partial E}{\partial x} &= n - N, \end{aligned}$$

together with the initial data

$$\begin{aligned} n(0,x) &= n_0(x), \quad x \in \mathbb{R}, \\ E(s,-\infty) &= E_\infty(s), \quad s \in \mathbb{R}^+, \end{aligned}$$

and the slow time

$$s = \varepsilon\, t. \tag{2.3}$$

The proof relies on estimates from the Glimm scheme and the use of a suitable relative entropy.

When we get rid of the electric field, we obtain the standard Euler system with friction. It is a hydrodynamical model for porous media.

$$\begin{aligned} \frac{\partial \rho^\varepsilon}{\partial t} + \frac{\partial}{\partial x}(\rho^\varepsilon\, u^\varepsilon) &= 0, \\ \frac{\partial}{\partial t}(\rho^\varepsilon\, u^\varepsilon) + \frac{\partial}{\partial x}\left(\rho^\varepsilon (u^\varepsilon)^2 + p(\rho^\varepsilon)\right) &= -\frac{\rho^\varepsilon\, u^\varepsilon}{\varepsilon}. \end{aligned}$$

In this model, ρ^ε is the density, u^ε is the velocity, p is the pressure and ε^{-1} is the friction coefficient. For this system we have obtained global BV estimates and a rate of convergence towards the solution of the porous media equation in the slow time (2.3) :

$$\frac{\partial \rho}{\partial s} - \frac{\partial^2 p(\rho)}{\partial x^2} = 0.$$

The proof also simply applies for the isentropic case $p(\rho) = \rho^\gamma$, $\gamma > 1$, with compactly supported data. Indeed, the proof is reminiscent to compensated compactness techniques, and relies on the tricky introduction of a suitable stream function. This argument has been useful for studying asymptotic problems.

2.2.2 Gas with vanishing pressure

This work was a first attempt to deal with vanishing pressure in the following system of gas dynamics equations

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0 \ \ in\ \mathbb{R}_t^+ \times \mathbb{R}_x, \\ \partial_t(\rho u) + \partial_x(\rho u^2/2 + \varepsilon^2 p(\rho)) = 0 \ \ in\ \mathbb{R}_t^+ \times \mathbb{R}_x. \end{cases}$$

In [GJ] we validated the convergence of smooth solution (ρ, u) on an uniform strip when the sound speed vanishes to the solution of the pressureless gases equations before the shocks. The proof is based on the method of characteristics.

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0 \ \ in\ \mathbb{R}_t^+ \times \mathbb{R}_x, \\ \partial_t(\rho u) + \partial_x(\rho u^2/2) = 0 \ \ in\ \mathbb{R}_t^+ \times \mathbb{R}_x. \end{cases}$$

When this work was achieved, the study of the pressureless gas dynamics was a challenge. For smooth solutions, [GJ] was a first step to justify the approximation for small sound speed. The general problem is still an open problem since solutions become Dirac measure in finite time.

2.2.3 Geometric optics with vanishing viscosity

Numerous physical problems involve parabolic partial differential equations with small viscosity, for instance the Navier-Stokes system with large Reynolds number. It is then of interest to implement the Joly- Métivier-Rauch framework with a small diffusion.

Olivier Guès suggested to study such multiphase expansions for the semilinear case in [J3]. Thus, I investigated the propagation of high frequency oscillations for the following nonlinear parabolic system with highly oscillatory initial data u_0^ε and small positive viscosity ν:

$$\begin{aligned} -\nu\frac{\partial^2 u^\varepsilon}{\partial x^2} + \frac{\partial u^\varepsilon}{\partial t} + A(t,x)\frac{\partial u^\varepsilon}{\partial x} &= F(t,x,u^\varepsilon) \\ u^\varepsilon(0,x) &= u_0^\varepsilon(x). \end{aligned}$$

The matrix $A(t,x)$ is a smooth $N \times N$ real matrix with N distinct real eigenvalues: $\lambda_1(t,x) < \lambda_2(t,x) < \cdots < \lambda_N(t,x)$, in such a way that $\partial_t + A(t,x)\partial_x$ is a *strictly hyperbolic operator.*

I focused my attention on the special case of rapidly oscillating data with a given vector phase $\varphi^0(x)$, frequency $1/\varepsilon$ and a viscosity which is equal to the square of the wavelength:

$$\begin{aligned} u^\varepsilon(0,x) = u_0^\varepsilon(x) &= U_0\left(x, \frac{\varphi^0(x)}{\varepsilon}\right), \\ \nu &= \varepsilon^2. \end{aligned}$$

With such oscillating data, the viscous coefficient has the critical size, so that $\varepsilon^2\partial_x^2 u^\varepsilon$ is expected to be of order one. Interactions of high oscillations with the viscous term is also expected. Indeed, it is the aim of this paper to justify a geometric optics expansion:

$$u^\varepsilon(t,x) = U\left(t, x, \frac{\varphi(t,x)}{\varepsilon}\right) + o(1),$$

where φ is a vector phase as in the hyperbolic case (without viscosity) and the profile U satisfies a new degenerate parabolic modulation system. With such a diagonal viscosity, the WKB expansion is validated in L^2_x with initial data only bounded in $L^2_x \cap L^\infty_x$.

2.2.4 Central limit theorem in Sobolev spaces

In [GJT], we prove a Berry-Esseen like theorem. More precisely let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with values in $\mathbb{R}^N$ where we assume that the common probability density f of the X_n satisfies

$$\int_{\mathbb{R}^N} f\,dx = 1, \qquad \int_{\mathbb{R}^N} xf\,dx = 0, \qquad \int_{\mathbb{R}^N} |x|^2 f\,dx < \infty. \tag{2.4}$$

We denote by f_n the probability density of the normalized partial sum

$$S_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}.$$

Let ω_K be the normal law with zero mean and covariance operator K, whose coefficients are

$$K_{ij} = \int_{\mathbb{R}^N} x_i x_j f(x)\,dx, \qquad \text{for } 1 \le i,j \le N.$$

The classical central limit theorem asserts that f_n converges to the Gaussian density

$$\omega_K(x) = \frac{1}{(2\pi)^{N/2}\sqrt{\det(K)}} \exp\left(-\frac{K^{-1}x\cdot x}{2}\right)$$

as $n \to \infty$. P.L. Lions and G. Toscani proved that, under some additional regularity assumption on f (see condition (2.5) below), the convergence $f_n \to \omega_K$ holds in any Sobolev space

$$H^k(\mathbb{R}^N) = \{\phi \in \mathcal{S}'(\mathbb{R}^N),\ \int_{\mathbb{R}^N} (1+|\xi|^2)^k\,|\widehat{\phi}(\xi)|^2\,d\xi < \infty\}$$

where $\widehat{\phi}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi}\phi(x)\,dx$ stands for the Fourier transform of ϕ. In [GJT], we detail the rate of convergence, i.e. a Berry-Esseen inequality in Sobolev spaces.

Theorem 2.2.1
Let f satisfy (2.4). Moreover, assume there exist $M > 0$ and $k \ge 0$ such that

$$\int_{\mathbb{R}^N} (1+|\xi|^2)^k\,|\widehat{f}(\xi)|^M\,d\xi = C_{M,k} < \infty \tag{2.5}$$

and $\int_{\mathbb{R}^N} |x|^s f(x)\,dx = \rho_s < \infty$ for some $2 < s \le 3$, then $\|f_n - \omega_K\|_{H^k} \le C\frac{1}{n^{(s-2)/2}}$, with the constant C which only depends on K, ρ_s, M and k.

For this purpose, a specific metric on the class $P_s(\mathbb{R}^N)$, $s > 0$, of all probability distributions F on $\mathbb{R}^N$, $N \geq 1$, such that $\displaystyle\int_{\mathbb{R}^N} |x|^s \, dF(x) < \infty$:

$$d_s(F, G) \;=\; \sup_{\xi \in \mathbb{R}^N \setminus \{0\}} \left\{ \frac{|\widehat{f}(\xi) - \widehat{g}(\xi)|}{|\xi|^s} \right\},$$

where f, g are the densities associated to F, G respectively. Such arguments also allow to obtain refined decay estimates for the solution of the heat equation.

Chapter 3

Nonlinear scalar conservation laws

In this chapter, we study highly oscillating waves and smoothing effects satisfied by entropy solutions of nonlinear scalar conservation laws. More precisely, let $\mathbf{F} : \mathbb{R} \to \mathbb{R}^d$ be a smooth flux function, we consider

$$\partial_t u + \mathrm{div}_{\mathbf{x}} \mathbf{F}(u) = 0, \quad u \in \mathbb{R},\, \mathbf{x} \in \mathbb{R}^d. \tag{3.1}$$

The supercritical geometric optics is studied in [CJR06] in an L^∞ framework with multiphase and genuinely nonlinear flux. In [J4], I obtained the most critical high frequency waves propagated by the equations. This family is smooth on a strip $[0, T] \times \mathbb{R}^d$, $T > 0$, with only one phase and has exactly the uniform Sobolev regularity conjectured by Lions, Perthame and Tadmor in [19] for uniformly bounded L^∞ data. To handle all smooth fluxes, a new definition of nonlinear flux is introduced.

In the one dimensional case, we construct special oscillating continuous functions also related to the maximal smoothing effect [CJ]. Indeed, the best smoothing effect in the one dimensional case was already proven in [14] in the class $W^{s,1}$ of Sobolev spaces. The optimality is justified with a piecewise smooth oscillating solution in [9]. We introduce fractional BV spaces in [BGJ6]. Then we obtain the maximal regularity in such a BV^s space with only L^∞ data. Furthermore, the best uniform Sobolev regularity is reached in $W^{s,p}$ with $p = 1/s$. The BV^s optimality is proven in [CJ], see also [9] in Besov spaces. It turns out that proofs are simpler in BV^s framework. The BV^s spaces seem to be promising for other applications to conservations laws.

Section 3.1 presents the weakly nonlinear geometric optics near a constant state in space $\mathbb{R}^d$ with d independent phases and d frequencies for genuinely nonlinear flux.

In section 3.2, geometric optics with the maximal frequency and a well chosen phase highlight the maximal smoothing effect conjectured in [19].

In section 3.3, the fractional BV spaces BV^s are introduced and the first applications to conservation laws are discussed.

3.1 Geometric optics with many frequencies

Weakly nonlinear geometric optics with various frequencies for entropy solutions of multidimensional scalar conservation laws is analyzed in [CJR06]. Weakly means with small amplitude and geometric optics mean with high frequencies. More precisely, we are concerned with nonlinear geometric optics for entropy solutions of multidimensional scalar conservation laws:

$$\partial_t u + \mathrm{div}_{\mathbf{x}} \mathbf{F}(u) = 0, \quad u \in \mathbb{R},\, \mathbf{x} \in \mathbb{R}^d, \tag{3.2}$$

where $\mathbf{F} : \mathbb{R} \to \mathbb{R}^d$ is a smooth flux function. Consider the Cauchy problem (3.2) with oscillatory Cauchy data:

$$u|_{t=0} = u_0^\varepsilon(\mathbf{x}) := \underline{u} + \varepsilon\, U_0 \left(\frac{\phi_1^0(\mathbf{x})}{\varepsilon^{\alpha_1}}, \cdots, \frac{\phi_d^0(\mathbf{x})}{\varepsilon^{\alpha_d}} \right), \tag{3.3}$$

where u_1 is a periodic function of each of its d arguments, the period being denoted by $P = [0,1]^d$, $\underline{u}$ is a constant ground state, the phases $(\phi_1^0, \cdots, \phi_d^0)$ are linear, and $\alpha_i \geq 0$ for $1 \leq i \leq d$.

Assume $\mathbf{a}(\underline{u}) = \mathbf{F}'(\underline{u}) = (0, \cdots, 0)$ without loss of generality. We look for a geometric optics asymptotic expansion:

$$u^\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon v^\varepsilon(t, \mathbf{x}). \tag{3.4}$$

We expect an expansion of v^ε with a profile U independent of ε:

$$u^\varepsilon(t, \mathbf{x}) \;=\; \underline{u} + \varepsilon\, U \left(t, \frac{\phi_1(t, \mathbf{x})}{\varepsilon^{\alpha_1}}, \cdots, \frac{\phi_d(t, \mathbf{x})}{\varepsilon^{\alpha_d}} \right) + o(\varepsilon) \text{ in } L^1_{loc}(]0, +\infty[\times \mathbb{R}^d, \mathbb{R}).$$

We need a general scaling of variables to recover all the numerous cases. Our approach uses an incomplete LU factorization depending on the magnitude of the frequencies. We also preserve the L^1_{loc}-convergence after a triangular scaling of the variables which depends on ε. Using compactness and L^1 stability, we managed all the cases in [CJR06]. The complete proof is quite long and technical. Let us give the genuinely nonlinear assumptions on the flux and a few examples in the 2 dimensional case.

3.1.1 Genuinely nonlinear flux

The phases are supposed linearly independent: $\phi_i := \sum_{j=1}^{d} J_{ij} x_j$, thus $\det J \neq 0$. The independence is not really an assumption. The important fact is the relation between the phases and the flux $\mathbf{F}$. Indeed under a one-to-one *constant* linear change of coordinates $\Phi := \mathbf{J}\mathbf{x}$, equation (3.2) can be rewritten in variable (t, Φ), in the weak form as

$$\partial_t \widetilde{u}^\varepsilon + \mathrm{div}_\Phi (J \mathbf{F}(\widetilde{u}^\varepsilon)) \;=\; 0 \tag{3.5}$$

Thus the flux $\mathbf{F}$ is replaced by $J\,\mathbf{F}$.

We say that the flux is genuinely nonlinear at $u = \underline{u}$ if the following flux matrix is non degenerate:

$$N := \begin{pmatrix} \dfrac{F_1^{(2)}(\underline{u})}{2!} & \cdots & \dfrac{F_1^{(d+1)}(\underline{u})}{(d+1)!} \\ \vdots & \dfrac{F_i^{(j+1)}(\underline{u})}{(j+1)!} & \vdots \\ \dfrac{F_d^{(2)}(\underline{u})}{2!} & \cdots & \dfrac{F_d^{(d+1)}(\underline{u})}{(d+1)!} \end{pmatrix}. \tag{3.6}$$

The possibility that initial high frequency oscillations propagate with (3.2) depends on the powers $\alpha := (\alpha_1, \cdots, \alpha_d)$ of ε involved in (3.3) and on the matrix:

$$M = JN. \tag{3.7}$$

The matrix M is related to the Taylor expansion of the flux in the Φ coordinates : the $j^{\underline{th}}$ column corresponds to the term $\varepsilon^j v_\varepsilon^{j+1}$ in the Taylor expansion of $\mathbf{F}(u^\varepsilon) := \mathbf{F}(\underline{u}+\varepsilon v_\varepsilon)$; while the $i^{\underline{th}}$ row corresponds to the derivatives with respect to the $i^{\underline{th}}$ phase ϕ_i.

The matrix M and the exponents $(\alpha_i)_{i\leq d}$ contain all the information about the profile $U(t, X_1, \cdots, X_d)$. In [CJR06], an (incomplete) LU-type factorization of the matrix M with respect to $(\alpha_i)_{i\leq n}$ gives the profile equation with its initial data which can be a partial or a complete homogenization of U_0.

3.1.2 2D examples

The main result in [CJR06] is Theorem 5.1 (page 467). The analysis is not restricted to a specific space dimension, owing to the identification of a suitable algebraic setting. To give a flavor of this theorem, let us mention few examples for 2D genuinely nonlinear polynomial flux. The conservation law (3.2) is written in phase variables.

$$M := \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad \begin{cases} \partial_t u^\varepsilon + (a\,\partial_\phi + b\,\partial_\psi)[(u^\varepsilon)^2] + (c\,\partial_\phi + d\,\partial_\psi))[(u^\varepsilon)^3] = 0, \\ u^\varepsilon(0,x,y) = 0 + \varepsilon\, U_0\left(\dfrac{\phi(x,y)}{\varepsilon^\alpha}, \dfrac{\psi(x,y)}{\varepsilon^\beta}\right). \end{cases}$$

For each of following examples, we give the geometric optics expansion in $L^1_{loc}(]0,+\infty[\times\mathbb{R}^2)$.

1. $\alpha = 1,\ \beta = 2,\ b = 0$: nonlinear propagation with maximal frequencies and with the orthogonality condition $\nabla_{\mathbf{x}}\psi \perp \mathbf{F}''(\underline{u})$.

$$\partial_t U + a\,\partial_X U^2 + d\,\partial_Y U^3 = 0, \qquad U(0,X,Y) = U_0(X,Y),$$
$$u^\varepsilon(t,x,y) = \underline{u} + \varepsilon\, U\left(t, \frac{\phi(x,y)}{\varepsilon}, \frac{\psi(x,y)}{\varepsilon^2}\right) + o(\varepsilon). \tag{3.8}$$

2. $\alpha, \beta < 1$: linear propagation, small frequencies,

$$\partial_t U = 0, \qquad U(t,X,Y) = U(0,X,Y) = U_0(X,Y),$$
$$u^\varepsilon(t,x,y) = \underline{u} + \varepsilon\, U_0\left(\frac{\phi(x,y)}{\varepsilon^\alpha}, \frac{\psi(x,y)}{\varepsilon^\beta}\right) + o(\varepsilon).$$

3. $\alpha = \beta = 1,\ a \neq 0$: classic geometric optics [11, 15, J2]

$$\partial_t U + a\partial_X U^2 = 0, \qquad U(0,X,Y) = U_0\left(X, \frac{bX - aY}{a}\right),$$
$$u^\varepsilon(t,x,y) = \underline{u} + \varepsilon\, U\left(t, \frac{\phi(x,y)}{\varepsilon}, \frac{b\,\phi(x,y) - a\,\psi(x,y)}{a\,\varepsilon}\right) + o(\varepsilon).$$

4. $\alpha < 1 < \beta,\ b \neq 0$: mix with linear propagation and cancellation,

$$U(t, X, Y) = \overline{U}_0(X) = \int_0^1 U_0(X, Y)\, dY$$
$$u^\varepsilon(t, x, y) = \underline{u} + \varepsilon\, \overline{U}_0\left(\frac{\phi(x, y)}{\varepsilon^\alpha}\right) + o(\varepsilon).$$

5. $\alpha = \beta = 2,\ a \neq 0, b \neq 0,\ \dfrac{a}{b} \notin \mathbb{Q}$: homogenization

$$U(t, X, Y) = \overline{\overline{U}}_0 = \iint_{[0,1]^2} U_0(X, Y)\, dX\, dY$$
$$u^\varepsilon(t, x, y) = \underline{u} + \varepsilon\, \overline{\overline{U}}_0 + o(\varepsilon).$$

6. $\alpha = \beta = 2,\ a \neq 0,\ b \neq 0,\ \dfrac{a}{b} = \dfrac{p}{q} \in \mathbb{Q},\ p, q \in \mathbb{Z} - \{0\}$:
a supercritical geometric optics expansion with an arithmetic condition and without the orthogonality conditions: $0 \neq \nabla_{\mathbf{x}}\phi \cdot \mathbf{F}''(\underline{u}) = a$ and $0 \neq \nabla_{\mathbf{x}}\psi \cdot \mathbf{F}''(\underline{u}) = b$.

$$\partial_t U - \frac{ad - bc}{a}\, \partial_Y U^3 = 0, \quad U(0, X, Y) = U(0, Y) = \int_0^1 U_0(p\theta, q\theta - Y) d\theta,$$
$$u^\varepsilon(t, x, y) = \underline{u} + \varepsilon\, U\left(t, \frac{q\, \phi(x, y) - p\, \psi(x, y)}{p\, \varepsilon^2}\right) + o(\varepsilon). \quad (3.9)$$

3.2 Geometric optics and smoothing effect

In section 3.1, we obtain the supercritical geometric optics expansion for entropy solutions: see e.g. example (3.8). A key ingredient is a specific feature of the phase: the phase gradient has to be orthogonal to the flux derivatives. This observation is improved in [J4]. Firstly, only one phase is enough to the supercritical geometric optics. Secondly, the family of high frequency waves can be smooth on an uniform strip. This last point is surprising since the gradient of the family is unbounded.

Such family of high frequency waves are exactly uniformly bounded in the Sobolev space conjectured by Lions-Perthame-Tadmor in [19]. Thus the best uniform Sobolev exponent conjectured in [19] cannot be exceeded. In the one dimensional case, for power law fluxes, De Lellis and Westdickenberg got the optimality in [9]. In [J4] the optimality is obtained in the multidimensional cases for all smooth fluxes. In the one dimensional case for all smooth fluxes, one can consult [CJ].

3.2.1 Oscillations with unbounded gradient

Consider the problem (3.2) with oscillating initial data:

$$u_\varepsilon(0, \mathbf{x}) = u_0^\varepsilon(\mathbf{x}) := \underline{u} + \varepsilon U_0\left(\frac{\mathbf{v} \cdot \mathbf{x}}{\varepsilon^\gamma}\right), \quad (3.10)$$

where $U_0(\theta)$ is a one periodic function w.r.t. θ, $\gamma > 0$, $\underline{u}$ is a constant ground state, $\underline{u} \in [-M, M]$, $\mathbf{v} \in \mathbb{R}^d$. The case $\gamma = 1$ is the classical geometric optics for scalar conservation laws, ([11, 15, 23]). In this paper we focus on *critical oscillations* when $\gamma > 1$.

One of the two following asymptotic expansions (3.11) or (3.12), is expected for the entropy-solution u_ε of conservation law (3.2) with highly oscillating data (3.10) when ε goes to 0 in $L^1_{loc}(]0, +\infty[\times\mathbb{R}^d, \mathbb{R})$

$$u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon U\left(t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma}\right) + o(\varepsilon) \tag{3.11}$$

$$\text{or} \qquad u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon \overline{U}_0 + o(\varepsilon), \tag{3.12}$$

where the profile $U(t, \theta)$ satisfies a conservation law with initial data $U_0(\theta)$, $\overline{U}_0 = \int_0^1 U_0(\theta) d\theta$ and the phase ϕ satisfies the eikonal equation:

$$\begin{aligned} \partial_t \phi + \mathbf{a}(\underline{u}) \centerdot \nabla_{\mathbf{x}} \phi &= 0, \quad \phi(0, \mathbf{x}) = \mathbf{v} \cdot \mathbf{x}, \\ &\text{i.e.} \quad \phi(t, \mathbf{x}) = \mathbf{v} \cdot (\mathbf{x} - t\, \mathbf{a}(\underline{u})). \end{aligned} \tag{3.13}$$

The behavior described by (3.11) is a lack of compactness for the semi-group $\mathcal{S}_t$ of equation (3.2) since this semi-group propagates data uniformly bounded in $W^{1/\gamma,1}_{loc}$ without improving the Sobolev exponent. Otherwise, if γ is too big ($\gamma\ \alpha_{\sup} > 1$) and the initial oscillating data are not constant, then the high oscillations are canceled out in positive time. The behavior (3.12) means that a nonlinear smoothing effect is associated with the semi-group of equation (3.2).

The propagation of such oscillating data is obtained under the crucial compatibility condition (3.14) below. Otherwise, if the compatibility condition (3.14) is nowhere satisfied, the nonlinear semi-group associated with equation (3.2) cancels these too high oscillations [CJR06, J4]. To decide whether or not assumption (3.14) is satisfied is a key point to characterize nonlinear flux in the next section.

Theorem 3.2.1 [Propagation of smooth high oscillations]
Let γ belong to $]1, +\infty[$ and let q be the integer such that $q - 1 < \gamma \leq q$.
Assume that $\mathbf{F}$ belongs to $C^{q+3}(\mathbb{R}, \mathbb{R}^d)$, $U_0 \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, $\mathbf{v} \neq (0, \cdots, 0)$ and

$$\mathbf{a}^{(k)}(\underline{u}) \centerdot \mathbf{v} = 0, \qquad k = 1, \cdots, q-1. \tag{3.14}$$

Then there exists $T_0 > 0$ such that, for all $\varepsilon \in]0, 1]$, the solutions of the conservation law (3.2) with initial oscillating data (3.10) are smooth on $[0, T_0] \times \mathbb{R}$ and

$$u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon U\left(t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma}\right) + \mathcal{O}(\varepsilon^{1+r}) \ \textit{in}\ C^1([0, T_0] \times \mathbb{R}^d),$$

where $0 < r = \begin{cases} 1 & \textit{if} \quad \gamma = q, \\ q - \gamma & \textit{else}, \end{cases}$ *and the smooth profile U is uniquely determined by the Cauchy problem (3.15), ϕ is given by the eikonal equation (3.13):*

$$\frac{\partial U}{\partial t} + b \frac{\partial U^{q+1}}{\partial \theta} = 0, \qquad U(0, \theta) = U_0(\theta), \tag{3.15}$$

$$\text{with } b = \begin{cases} \frac{1}{(q+1)!}\left(\mathbf{a}^{(q)}(\underline{u}) \cdot \mathbf{v}\right) & if \quad \gamma = q, \\ 0 & else. \end{cases} .$$

We deal with smooth solutions to compute later Sobolev bounds. Indeed, the asymptotic stays valid after shocks formation and for all positive time but in L^1_{loc} instead of L^∞ [CJR06].

3.2.2 Nonlinear flux

The regularizing effect given in [19] is related to the sharp exponent $\alpha = \alpha_{\sup}$ quantifying precisely the nonlinearity of the flux.

Definition 3.2.1 [Nonlinear Flux [19]]
Let M be a positive constant, $\mathbf{F} : \mathbb{R} \to \mathbb{R}^d$ is said to be nonlinear *on $[-M, M]$ if there exist $\alpha > 0$ and $C = C_\alpha > 0$ such that for all $\delta > 0$*

$$\sup_{\tau^2+|\xi|^2=1} |W_\delta(\tau,\xi)| \quad \leq \quad C\,\delta^\alpha, \tag{3.16}$$

where $(\tau,\xi) \in S^d \subset \mathbb{R}^{d+1}$, i.e. $\tau^2+|\xi|^2 = 1$, and $|W_\delta(\tau,\xi)|$ is the one dimensional measure of the singular set:
$W_\delta(\tau,\xi) := \{|v| \leq M, |\tau + \mathbf{a}(v) \cdot \xi| \leq \delta\} \quad \subset \quad [-M, M]$ and $\mathbf{a} = \mathbf{F}'$.

Let us set

$$\alpha_{\sup} \quad = \quad \alpha_{\sup}[\mathbf{F}, M], \qquad \text{the supremum of all } \alpha \text{ satisfying (3.16).} \tag{3.17}$$

There are some examples in [19, 26] where α is computed in dimension two and some insights in [BeJ, 14]. For the first time, we obtain "the sharp α" for all smooth fluxes in Theorem 3.2.2 below.

Another genuinely nonlinear definition for the flux depending on the space dimension d and related to weakly nonlinear geometric optics, is given in [CJR06]. We generalize the definition from [CJR06]. Thanks to Theorem 3.2.2 below, this new definition is equivalent to the classical Definition 3.2.1.

Definition 3.2.2 [Nonlinear flux]
Let the flux $\mathbf{F}$ belong to $C^\infty(\mathbb{R}, \mathbb{R}^d)$ and $I = [c_0, d_0]$, $c_0 < d_0$. The flux is said to be nonlinear on I if, for all $u \in I$, there exists $m \in \mathbb{N}^*$ such that

$$rank\{\mathbf{a}'(u), \cdots, \mathbf{a}^{(m)}(u)\} \quad = \quad d. \tag{3.18}$$

Furthermore, the flux is said to be genuinely nonlinear if $m = d$ is enough in (3.18) for all $u \in I$.

As usual, the non-linearity is a matter of the second derivatives of $\mathbf{F}$: $\mathbf{F}'' = \mathbf{a}'$. Notice that (3.18) implies that $m \geq d$, thus the genuinely nonlinear case is the strongest nonlinear case. The genuinely nonlinear case was first stated in [CJR06] (condition (2.8) and Lemma 2.5 p. 447 therein). The genuinely nonlinear condition $\det(\mathbf{a}', \cdots, \mathbf{a}^{(d)}) \neq 0$ for multidimensional conservations laws was also in [5], see condition (16) p. 84 therein.

The simplest example of velocity $\mathbf{a}$ associated with a genuinely nonlinear flux $\mathbf{F}$ is given in [CJR06, 5, BeJ]:

$$\mathbf{a}(u) \quad = \quad (u, u^2, \cdots, u^d) \qquad \text{with } \mathbf{F}(u) = \left(\frac{u^2}{2}, \cdots, \frac{u^{d+1}}{d+1}\right).$$

To make Definition 3.2.2 more explicit it is convenient to introduce the following characteristic numbers:

$$d_{\mathbf{F}}[u] \quad = \quad \inf\{k \geq 1, rank\{\mathbf{F}''(u), \cdots, \mathbf{F}^{(k+1)}(u)\} = d\} \qquad \geq d, \tag{3.19}$$

$$d_{\mathbf{F}} \quad = \quad \sup_{|u|\leq M} d_{\mathbf{F}}[u] \qquad \in \{d, d+1, \cdots\} \cup \{+\infty\}. \tag{3.20}$$

The integers $d_{\mathbf{F}}[u]$ and $d_{\mathbf{F}}$ characterize the nonlinearity of the flux $\mathbf{F}$ on the interval $I = [-M, M]$.

Indeed, Definition 3.2.2 states that the flux is genuinely nonlinear when $d_{\mathbf{F}}$ reaches its minimal value, $d_{\mathbf{F}} = d$. Conversely, if the flux $\mathbf{F}$ is linear, then $\mathbf{a}$ is a constant vector in $\mathbb{R}^d$ and $d_{\mathbf{F}}$ reaches its maximal value, $d_{\mathbf{F}} = +\infty$. Between $d_{\mathbf{F}} = d$ and $d_{\mathbf{F}} = +\infty$, there is a large variety of nonlinear fluxes.

The following theorem gives the optimal α, see (3.16), for a smooth flux.

Theorem 3.2.2 [Sharp measurement of the flux non-linearity]
Let $\mathbf{F}$ be a smooth flux, $\mathbf{F} \in C^\infty([-M, M], \mathbb{R}^d)$.
The measurement of the flux non-linearity $\alpha_{\sup}$ is given by

$$\alpha_{\sup} \quad = \quad \frac{1}{d_{\mathbf{F}}} \quad \leq \quad \frac{1}{d}.$$

Furthermore, if $\alpha_{\sup} > 0$ there exists $\underline{u} \in [-M, M]$ such that $d_{\mathbf{F}} = d_{\mathbf{F}}[\underline{u}]$.

A similar result for the genuinely nonlinear case: $d_{\mathbf{F}} = d$, can be found in [BeJ].

This theorem is a powerful tool to compute the parameter $\alpha_{\sup}$, for instance:

- if $\mathbf{F}(u) = (\cos(u), \sin(u))$ then $\mathbf{F}$ is genuinely nonlinear and $\alpha_{\sup} = 1/2$.

- if $\mathbf{F}$ is polynomial with degree less or equal to the space dimension d then $\alpha_{\sup} = 0$ and $\mathbf{F}$ does not satisfy Definition 3.2.2.
  It is well known that the "multi-D Burgers" flux $\mathbf{F}(u) = (u^2, \cdots, u^2)$ is not nonlinear when $d \geq 2$. The sequence of oscillations with large amplitude $(u_\varepsilon)_{0<\varepsilon\leq 1}$ given by $u_\varepsilon(t, \mathbf{x}) = \sin\left(\frac{x_1 - x_2}{\varepsilon}\right)$ gives us global smooth solutions while the sequence $(u_\varepsilon)_{0<\varepsilon\leq 1}$ is not bounded in any $W^{s,1}_{loc}$ for all $s > 0$.

- If $\mathbf{F}$ is polynomial such that $deg(\mathbf{F}_i) = 1 + i$, then $\mathbf{F}$ is genuinely nonlinear.

3.2.3 Bounds for the maximal smoothing effect

Having identified $\underline{u}$ in Theorem 3.2.2, we can compute precisely the fractional Sobolev norm of the oscillating sequence arising in Theorem 3.2.1. The gradient of the phase is constrained by (3.14). Then we deduce the bound of the maximal uniform Sobolev smoothing effect.

Theorem 3.2.3 [Bound of the maximal smoothing effect [J4]]
Let $\mathbf{F}$ be a nonlinear flux which belongs to $C^\infty([-M,M],\mathbb{R}^d)$. Let α_{sup} be the sharp measurement of the flux non-linearity. Then there exist a constant $\underline{u} \in [-M,M]$, a time $T_0 > 0$, and a sequence of initial data $(u_0^\varepsilon)_{0<\varepsilon<1}$ such that $\|u_0^\varepsilon - \underline{u}\|_{L^\infty(\mathbb{R}^d)} < \varepsilon$, and the sequence of entropy solutions $(u_\varepsilon)_{0<\varepsilon<1}$ associated with conservation law (3.2) satisfies:

- *for all $s \leq \alpha_{\rm sup}$, the sequence $(u_\varepsilon)_{0<\varepsilon<1}$ is uniformly bounded in $W^{s,1}_{loc}([0,T_0]\times\mathbb{R}^d) \cap C^0([0,T_0], W^{s,1}_{loc}(\mathbb{R}^d))$,*
- *for all $s > \alpha_{\rm sup}$, the sequence $(u_\varepsilon)_{0<\varepsilon<1}$ is* **unbounded** *in $W^{s,1}_{loc}([0,T_0]\times\mathbb{R}^d)$ and in $C^0([0,T_0], W^{s,1}_{loc}(\mathbb{R}^d))$.*

3.3 Fractional BV spaces

We introduce fractional BV spaces in ordrer to investigate fine regularity properties of entropy solutions of conservation laws. The space BV is quite important for conservation laws. BV^s is a promising space in this area. For instance, functions in BV^s are regulated for all $1 > s > 0$. It is not the case in $W^{s,1}$ for all $s < 1$. Furthermore, we already get the best smoothing effect for nonlinear degenerate convex fluxes in BV^s and in $W^{s,p}$ [BGJ6, CJ].

3.3.1 BV^s spaces

Generalizations of the total variation framework naturally yield to BV^s spaces [21]. BV^s spaces are not interpolated spaces ([25]) but BV^s are very close to the Sobolev space $W^{s,1/s}$. Let us define the TV^s variations and the BV^s spaces.

Let I be a non empty interval of $\mathbb{R}$ and $s \in]0,1]$. In the sequel we note $\mathcal{S}(I)$ the set of the subdivisions of I, that is the set of subsets $\sigma = \{x_0, x_1, \cdots, x_n\} \subset I$ with $x_0 < x_1 < \cdots < x_n$.

Definition 3.3.1 *Let $\sigma = \{x_0, x_1, \cdots, x_n\} \in \mathcal{S}(I)$ and let u be a real function on I. The s-total variation of u with respect to σ is*

$$TV^s u\{\sigma\} \;=\; \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})|^{1/s}, \tag{3.21}$$

and the s-total variation of $u(.)$ on I is defined by

$$TV^s u\{I\} = \sup_{\sigma\in\mathcal{S}(I)} TV^s u\{\sigma\}, \tag{3.22}$$

where the supremum is taken over all the subdivisions σ of I. The set $BV^s(I)$ is the set of functions $u : I \to \mathbb{R}$ such that $TV^s u\{I\} < +\infty$. We define the BV^s semi-norm by:

$$|u|_{BV^s(I)} = (TV^s u\{I\})^s . \tag{3.23}$$

The computation of BV^s variations can be quite intricate, and there is no simple criterion as for the standard BV space. For instance, an obvious subdivision or a finer subdivision is not always the right subdivision to estimate the TV^s variation [BGJ6]. Moreover, some criteria valid in BV are not valid in BV^s, $0 < s < 1$. Indeed, the following inequalities can be strict inequalities ([BGJ6]):

$$\begin{aligned} \lim_{n\to+\infty} \sum_{k\in\mathbb{Z}} \left| u\left(\frac{k+1}{n}\right) - u\left(\frac{k}{n}\right) \right|^{1/s} &\leq TV^s u\{\mathbb{R}\}, \\ \sup_{h>0} \frac{1}{h} \int_{\mathbb{R}} |u(x+h) - u(x)|^{1/s} dx &\leq TV^s u\{\mathbb{R}\}. \end{aligned}$$

Previous inequalities are always equalities only for $s = 1$: the BV case. Fortunately, fractional BV spaces are well fitted to study shock waves. BV^s functions are regulated functions for all $0 < s < 1$, i.e. any BV^s function admits right and left limits everywhere. It is a very important trace property shared with $BV^1 = BV$. By contrast, Sobolev spaces $W^{s,p}$ do not share this property ([2]):

- for $p > \dfrac{1}{s}$, $W^{s,p} \subset C^0$,
- for $p = \dfrac{1}{s}$ the Heaviside jump function does not belong to $W^{s,1/s}_{loc}$,
- for $p < \dfrac{1}{s}$ there is no trace.

Nevertheless we can compare BV^s and $W^{s,p}$.

Proposition 3.3.1 (**BV^s and $W^{s,p}$ [BGJ6]**)
Let $I \subset \mathbb{R}$ be a nontrivial bounded interval, then $BV^s(I) \neq W^{s,1/s}(I)$ and for all $\varepsilon > 0$:

$$W^{s,\infty}(I) \subset \quad BV^s(I) \quad \subset W^{s-\varepsilon,1/s}(I).$$

3.3.2 First applications of BV^s spaces

As in BV, the BV^s variation does not increase for any entropy solution and for any flux.

Theorem 3.3.1 *Let $u_0 \in BV^s(\mathbb{R})$, $f \in C^1(\mathbb{R}, \mathbb{R})$ and u be the unique entropy solution on $[0, +\infty[_t \times \mathbb{R}_x$ of*

$$\partial_t u + \partial_x f(u) = 0, \qquad u(0, x) = u_0(x), \tag{3.24}$$

then $\forall t > 0$, $\qquad TV^s u(t,.)(\mathbb{R}) \quad \leq TV^s u_0(.)(\mathbb{R})$.

We recall that the unique entropy solution, i.e. the Kruzkov solution, is the L^∞ function $u(t,x)$ such that for all $\eta'' \geq 0$, $\partial_t \eta(u) + \partial_x q(u) \leq 0$ in the sense of distribution and where $q' = \eta'\ f'$. Furthermore we require that $u_0(x)$ be a strong trace of $u(t,x)$ at $t=0$ in $L^1_{loc}(\mathbb{R})$.

With a general nonlinear degeneracy condition on convex fluxes, we obtain the best smoothing effect in BV^s and then in $W^{s,p}$.

Definition 3.3.2 (Degeneracy of a nonlinear convex flux)
Let f belong to $C^1(I,\mathbb{R})$ where I is an interval of $\mathbb{R}$. We say that the degeneracy of f on I is at least p if the continuous derivative $a(u) = f'(u)$ satisfies:

$$0 < \inf_{I\times I} \frac{|a(u)-a(v)|}{|u-v|^p}. \tag{3.25}$$

We call the lowest such real number p, if it exists, the degeneracy measurement of uniform convexity on I. If there is no p such that (3.25) is satisfied, we set $p = +\infty$.
Let $f \in C^2(I)$. We say that a real number $y \in I$ is a degeneracy point of f in I if $f''(y) = 0$ (i.e. y is a critical point of a).

If $f \in C^2(I)$, we can see easily that $p \geq 1$ [BGJ6, CJ].

Theorem 3.3.2 (BV^s smoothing effect for degenerate convex flux)
Let K be the compact interval $[-M,M]$. Let u_0 belong to $L^\infty(\mathbb{R})$, $f \in C^1(\mathbb{R},\mathbb{R})$ and let u be the entropy solution on $]0,+\infty[_t\times\mathbb{R}_x$ of the scalar conservation law (3.24). Let p be a degeneracy measurement of f on K and $0 < s = \dfrac{1}{p} \leq 1$.
If p is finite and $|u_0| \leq M$ then $\forall\delta > 0$, $u \in \mathrm{Lip}^s([\delta,+\infty[_t, L^{1/s}_{loc}(\mathbb{R}_x,\mathbb{R}))$ and

$$\forall t > 0,\ u(t,.) \in BV^s_{loc}(\mathbb{R}).$$

If u_0 is compactly supported then $\forall\delta > 0$, $u \in \mathrm{Lip}^s([\delta,+\infty[_t, L^{1/s}(\mathbb{R}_x,\mathbb{R}))$ and there exists a constant C such that

$$TV^s u(t,.) \leq C\left(1+\frac{1}{t}\right).$$

Remark 3.3.1 *This theorem gives the regularity conjectured by P.-L. Lions, B. Perthame and E. Tadmor in [19] for a non linear convex flux. This conjecture was stated in Sobolev spaces. The $W^{s,1}$ regularity with only L^∞ initial data was first proved in [14]. We get the best $W^{s,p}$ regularity. Indeed by Proposition 3.3.1, this BV^s regularity gives a $W^{s',1/s}$ smoothing effect for all $s' < s$.*

Remark 3.3.2 *We cannot expect a better regularity. Indeed, C. De Lellis and M. Westdickenberg give in [9] a piecewise smooth entropy solution which does not belong to $W^{s,1/s}$. Recently, in [3, CJ], another examples which are continuous examples are built. Indeed for any $\tau > s$, there exists a smooth solution which belongs to BV^s but not to BV^τ.*

3.4 Prospects

The results of this chapter can be the basis of further investigations.

- Lions, Perthame and Tadmor in [19] page 180, lines 1-5, suggested a proof of the maximal Sobolev smoothing effect for power law flux in the one dimensional case. With my student Pierre Castelli, we are writing a complete proof in the Sobolev $W^{s,1}$ framework for all smooth fluxes.

- With my student Pierre Castelli, we are building a special solution with the maximal regularity for any multidimensional smooth nonlinear flux. It extends the examples in the one dimensional case in [CJ] for all smooth fluxes and in [9] for power law fluxes.

- The critical solutions or family of solutions in [9, J4, CJ] have been constructed near a constant state $\underline{u}$. Jeffrey Rauch asked me what happen near a smooth solution $u_0(t,x)$. It appears, that generically there is no such geometric optics expansions. Indeed, the compatibility conditions (3.14) become overdetermined when $\gamma > 2$ and $\underline{u}$ is replaced by $u_0(t,x)$ since we have to satisfy these conditions for all (t,x) in an open set. As in [15], the constant case $\underline{u}$ is the most degenerate case.

- BV^s smoothing effect is only proved for nonlinear convex (degenerate) fluxes in [BGJ6]. A challenging problem is to study nonconvex fluxes. One conjectures a $BV^{1/2}$ smoothing effect for the cubic flux $f(u) = u^3$ with only L^∞ data. That is to say the same smoothing effect than the convex cubic flux $|u|^3$.

- To the study of the sharp smoothing effect in the nonlinear multidimensional framework, a key example is the 2D genuinely nonlinear example:
$$\partial_t u + \partial_x u^2 + \partial_y u^3 = 0.$$
It was already noticed in [CJR06, 5, J4]. The kinetic methods used in [19, 7, 26, 5] are also valid for weak solutions with bounded entropy production. This class of solutions is a larger set than the set of Kruzkov entropy solutions. For instance, in this larger class, there is no uniqueness for the Cauchy problem. Thus, these results miss some fine properties related to the Kruzkov entropy solutions. Is there another approaches? In [14], the conjectured maximal smoothing effect is reached under an assumption on a generalized Oleinik condition. In all cases, as we said in the previous prospect, we have first to well understand the 1D case:
$$\partial_t u + \partial_y u^3 = 0.$$

- BV^s spaces can be used for systems of hyperbolic P.D.E., for instance for the (PSA) system presented in chapter 4, see comments in 4.6.

- A completely open problem is to understand $BV^s(\mathbb{R}^d)$ in the multidimensional case. Only the classic case $s = 1$ is well known. Indeed, $BV(\mathbb{R})$ was already known in the 19 th century but the multi-dimensional generalization waited for the Italian mathematical school in the 20 th century. How long will we waiting for the multi-dimensional definition of the BV^s spaces?

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Chapter 4

A 2×2 system in gas chromatography

This chapter is devoted to the analysis of a 2×2 hyperbolic system describing an adsorption process used in chemical engineering. This is a joint work with *Christian Bourdarias* and *Marguerite Gisclon*, L.A.M.A. (Laboratoire de Mathématiques de l'Université de Savoie).

Indeed we consider an initial boundary value problem for a 2×2 system of conservation laws modeling heatless adsorption of a gaseous mixture with two species and instantaneous exchange kinetics, close but different to the usual system of chromatography. It is the first mathematical study of this 2×2 system which presents particular features:

- The velocity $u(t, x)$ is not constant, by contrast to the usual chromatography system since the sorption effect is taken into account.
- This problem is naturally set as an initial boundary problem: $t > 0, x > 0$.
- Exchanging the roles of the x, t variables, the system becomes strictly hyperbolic if

$$\inf_{t>0,x>0} u(t,x) \quad > \quad 0.$$

- Zero is a linearly degenerate eigenvalue, so $\{t = 0\}$ is a characteristic boundary.
- The other eigenvalue is generally piecewise nonlinear.
- The concentrations have to satisfy a "maximum principle".

In section 4.1, we briefly set up the model. Next, we investigate the hyperbolicity of the system in section 4.2. The first existence theorem via a Godunov scheme is stated in section 4.3. This theorem needs data which are BV for the concentrations and only L^∞ for the velocity. Some special features of the solutions are discussed in sections 4.4 and 4.5: a stratified structure for the velocity and an example of blow up for the velocity with L^∞ data.

4.1 The model

We derive a system of conservation laws related to a particular isothermal gas-solid chromatography process, called "Pressure Swing Adsorption" (PSA), with two species. Each of the two species simultaneously exists under two phases: a gaseous and movable phase with a common velocity $u(t,x)$ and concentration $c_i(t,x) \geq 0$ and a solid (adsorbed) phase with concentration q_i, $i = 1, 2$. One may consult [15] and [18] for a precise description of the process and [BGJ3] for a survey on various related models.

In gas chromatography, velocity variations occur with changes in gas composition, especially in the case of high concentration solute: it is known as the sorption effect. Here, the sorption effect is taken into account through a constraint on the pressure that is, in this isothermal model, on the total density:

$$c_1 + c_2 = \rho(t), \tag{4.1}$$

where the function ρ is given (this is actually achieved in the experimental or industrial device). This constraint expresses that the total pressure depends only on the time. In the sequel we simply assume that $\rho \equiv 1$.

We assume that mass exchanges between the mobile and the stationary phases are infinitely fast (instantaneous exchange kinetics) thus the two phases are constantly at composition equilibrium: the concentrations in the solid phase are given by some relations $q_i = q_i^*(c_1, c_2)$ where the functions q_i^* are the so-called equilibrium isotherms. A theoretical study of a model with finite exchange kinetics was presented in [1] and a numerical approach was developed in [2].

With these assumptions, the (PSA) system reads:

$$\begin{aligned} \partial_t(c_1 + q_1^*(c_1, c_2)) + \partial_x(u\, c_1) &= 0, & (4.2)\\ \partial_t(c_2 + q_2^*(c_1, c_2)) + \partial_x(u\, c_2) &= 0, & (4.3)\\ c_1 + c_2 &= 1. & (4.4) \end{aligned}$$

Notice that c_1, c_2 must satisfy $0 \leq c_1,\ c_2 \leq 1$.

Adding (4.2) and (4.3) we get, thanks to (4.4):

$$\partial_t(q_1^*(c_1, c_2) + q_2^*(c_1, c_2)) + \partial_x u = 0.$$

Thus the constraint (4.1) leads to an integral dependency of the velocity upon the concentrations. We denote $c = c_1$ then $c_2 = 1 - c$ and the unknowns are the velocity u and the concentration c. We write the (PSA) system under the form:

$$\begin{aligned} \partial_x u + \partial_t h(c) &= 0, & (4.5)\\ \partial_x(u\, c) + \partial_t I(c) &= 0, & (4.6) \end{aligned}$$

with

$$\begin{aligned} h(c) &= q_1^*(c, 1-c) + q_2^*(c, 1-c),\\ I(c) &= c + q_1^*(c, 1-c). \end{aligned}$$

In the sequel we denote

$$q_1(c) = q_1^*(c, 1-c) \quad \text{and} \quad q_2(c) = q_2^*(c, 1-c),$$

thus $h = q_1 + q_2$. Any equilibrium isotherm related to a given species is always non decreasing with respect to the corresponding concentration and non increasing with respect to the others (see [11]) i.e.

$$\frac{\partial q_i^*}{\partial c_i} \geq 0 \text{ and } \frac{\partial q_i^*}{\partial c_j} \leq 0 \text{ for } j \neq i,$$

thus we have immediately:

$$q_1' \geq 0 \geq q_2'. \tag{4.7}$$

(PSA) system (4.5)-(4.6) is completed by initial and boundary data:

$$\begin{cases} c(0,x) &= c_0(x) \in [0,1], \quad x > 0, \\ c(t,0) &= c_b(t) \in [0,1], \quad t > 0, \\ u(t,0) &= u_b(t) \geq \alpha, \quad t > 0, \quad \text{for some constant } \alpha > 0. \end{cases} \tag{4.8}$$

Notice that we assume in (4.8) an influx boundary condition, i.e. $\forall t > 0$, $u_b(t) > 0$ and we choose $]0, +\infty[$ instead of $]0, 1[$ as spatial domain for the sake of simplicity.

4.2 Hyperbolicity

It is well known that it is possible to analyze the system of chromatography, and thus (PSA) system, in terms of hyperbolic system of P.D.E. provided to exchange the time and space variables: see [16] and also [17] for instance. But we have to assume throughout the chapter the positivity of the velocity:

$$u \quad > \quad 0. \tag{4.9}$$

With evolution variable x instead of t, the system is strictly hyperbolic. In particular, the mathematical boundary condition is now related to the set $\{t = 0\}$ and the mathematical initial data to the set $\{x = 0\}$.

In this framework the physical vector state will be $U = (u, m)$ where $m = u\,c$ is the flow rate of the first species. The first component u of this vector must be understood as $u\,\rho$, that is the total flow rate.

4.2.1 Eigenvalues

(PSA) system with conservative variables (u, m) and $u > 0$ takes the form

$$\partial_x U + \partial_t \Phi(U) = 0 \qquad \text{with } \Phi(U) = \left(h\left(\frac{m}{u}\right), I\left(\frac{m}{u}\right) \right). \tag{4.10}$$

It is strictly hyperbolic with variable x as the evolution variable: the two eigenvalues are:

$$0, \qquad \lambda = \frac{H(c)}{u}, \tag{4.11}$$

$$H(c) = 1 + q_1'(c) - ch'(c) = 1 + (1-c)\,q_1'(c) - c\,q_2'(c) \geq 1. \tag{4.12}$$

The eigenvalue 0 is linearly degenerate. $\{t = 0\}$ is a characteristic hyperbolic boundary which is an important feature. In [11], chemists introduced the key function f :

$$f(c) \quad = \quad q_1\,c_2 - q_2\,c_1 \quad = \quad q_1(c) - c\,h(c). \tag{4.13}$$

The right eigenvector $r = (h'(c),\ 1 + q_1'(c))^{\perp}$ associated to λ satisfies $d\lambda \cdot r = \dfrac{H(c)}{u^2}\, f''(c)$, thus λ is genuinely nonlinear in each domain where $f'' \neq 0$.

The simple convex or concave case is first studied in [BGJ1] with sharper results in [BGJ4, BGJ5]. But in applications, f is piecewise convex (concave). So f has many inflexion points, which has a chemical interpretation. Roughly speaking the inflexion points correspond to the formation of layers in the chemical adsorption process ([BGJ3]). The analysis of the piecewise convex case is much more challenging. The first existence result in this case is in [BGJ2].

4.2.2 Riemann invariants

(PSA) system (4.5)-(4.6) admits the two Riemann invariants:

$$c \quad \text{and} \quad W = uG(c), \tag{4.14}$$

$$\text{where } g'(c) = -\frac{h'(c)}{H(c)} \quad \text{and} \quad G = \exp(g). \tag{4.15}$$

We also use the Riemann invariant $w = \ln(W) = \ln(u) + g(c)$. Indeed, for smooth solutions the (PSA) system is reduced to the diagonal system with eigenvalues 0 and λ:

$$\begin{cases} \partial_x W & + & 0 & = & 0, \\ \partial_x c & + & \lambda\,\partial_t c & = & 0, \qquad \lambda = \dfrac{H(c)G(c)}{W} \end{cases} \tag{4.16}$$

4.2.3 Entropies

An entropy is a function $S = S(u,c)$ satisfying a conservation law. Denote $S = S(u,c)$ any smooth entropy and $Q = Q(u,c)$ any associated entropy-flux. Then, for smooth solutions (u,c) of (4.5)-(4.6), $\partial_x S + \partial_t Q = 0$. There is only one family of smooth entropies S for the (PSA) system (4.5)-(4.6) with associated entropy-flux Q defined by:

$$S = u\,\psi(c), \qquad Q'(c) = h'(c)\,\psi(c) + H(c)\,\psi'(c) \tag{4.17}$$

where ψ is any smooth real function.

All entropies $S = u\,\psi(c)$ are linear with respect to the velocity u. This fundamental property has many consequences. For instance, it is an ingredient to prove the convergence of the Godunov scheme with BV concentration but only L^∞ velocity. It gives also the stratified structure of the solutions (see below). Indeed the linearity of the entropies with respect to the velocity is related to the 0 eigenvalue.

As usual the entropy-flux is defined up to a constant. Notice that the entropy-flux $Q(u, c) = Q(c)$ is a function on c only.

For the (PSA) system, it is possible to have $Q \equiv 0$: ψ is a solution of the differential equation (4.17): $0 = h'\psi + H\psi'$. But $\psi = G$ is a non zero solution of this O.D.E. by (4.15). Thus the Riemann invariant W is also an entropy. This is an important feature related to the zero eigenvalue as we can see directly with (4.16).

For 2×2 system we expect two families of entropies. There is only one family for the (PSA) system. It is again a special feature of this system. Indeed W is the entropy related with the second family of entropies [BGJ2]. But it can be interpreted as an entropy for the first family with $S = u\,\psi(c)$ and where $\psi = G$. Thus there is only one family of entropies to consider.

In the theory of conservation laws, convex entropies play a key role to define weak admissible solutions. We have to find all convex entropies. The reader should bear in mind that convexity is a relevant property of the entropy only with respect to the conservative variables (u, m) (see Dafermos [9] section 4.5 page 76 or [BGJ2]). As expected, $S(u, c)$ is degenerate convex (in variables (u, m)) provided $\psi'' \geq 0$. Now, we define entropy solutions. For another definition in the spirit of P. D. Lax see [BGJ1].

Definition 4.2.1 (Entropy solutions)

Let be $T > 0$, $X > 0$, $u \in L^\infty((0, T) \times (0; X); \mathbb{R})$.

If $u > 0$, $0 \leq c(t; x) \leq 1$, for almost all $(t, x) \in (0, T) \times (0, X)$ then (c, u) is called an entropy solution of (PSA) system (4.5)-(4.6) if, for all convex (or degenerate convex) ψ:

$$\partial_x\,(u\,\psi(c)) + \partial_t Q(c) \quad \leq \quad 0 \tag{4.18}$$

in the distribution sense where $Q'(c) = h'\,\psi(c) + H\,\psi'(c)$.

Moreover, if G'' keeps a constant sign on $[0, 1]$, the Riemann invariant $W = u\,G(c)$ has to satisfy:

$$\pm\,\frac{\partial}{\partial x}\,(u\,G(c)) \leq 0, \qquad \text{if } \pm G'' \geq 0 \text{ on } [0, 1]. \tag{4.19}$$

Notice that we recover the (PSA) system with $\psi = \pm 1$ and $\psi = \pm c$.

The monotonicity of W can be an important information. A natural question is when G'' keeps a constant sign. It is not generally the case [BGJ2, BGJ4, BGJ5]. But there are some examples. For instance, for an inert gas ($q_1 = 0$) and an active gas ($q_2 \neq 0$), if the isotherm q_2 is convex or concave, G is concave or convex. The Langmuir isotherm is concave so $\partial_x W \leq 0$. In converse, for ammonia or vapor water W is not decreasing w.r.t. x.

4.2.4 The boundary Riemann problem

For the (PSA) system, it is more convenient to begin by the resolution of the boundary Riemann problem which is characteristic. The boundary Riemann problem (4.5)-(4.6), has the data:

$$c(0,x) = c^0 \in [0,1], \quad x > 0, \qquad \left\{ \begin{array}{lcl} c(t,0) & = & c^+ \in [0,1], \\ u(t,0) & = & u^+ > 0, \end{array} \right. \quad t > 0. \tag{4.20}$$

We are classically looking for a self similar solution, i.e.:

$$c(t,x) = C(z), \quad u(t,x) = U(z) \text{ with } z = \frac{t}{x} > 0 \text{ (see Fig. 4.1).}$$

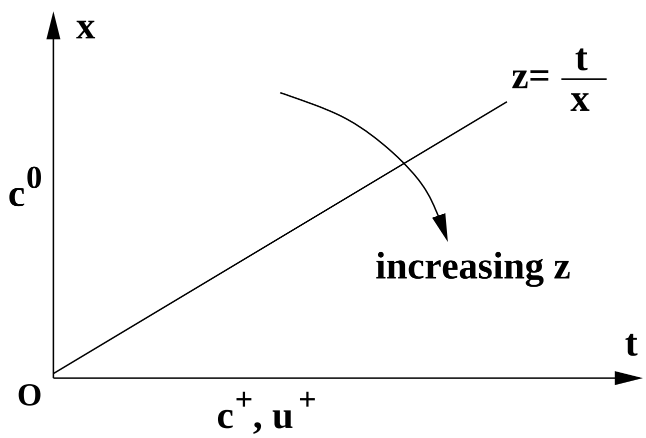

Figure 4.1: *data for the boundary Riemann problem*

In the domain $t > 0$, $x > 0$, the boundary Riemann problem is solved with $\lambda-$waves since λ is the only positive eigenvalue of the system. Let us recall the following three results obtained in [BGJ2], where H and f are given by (4.12, 4.13).

Proposition 4.2.1 ($\lambda-$rarefaction waves)
Any smooth non-constant self-similar solution $(C(z), U(z))$ with data (4.20) in an open domain $\Omega = \{0 \leq \alpha < z < \beta\}$ where $f''(C(z))$ does not vanish, satisfies:

$$\frac{dC}{dz} = \frac{H(C)}{z\, f''(C)}, \quad U(z) = \frac{H(C)}{z}.$$

In particular, $\dfrac{dC}{dz}$ has the same sign as $f''(C)$.
Assume for instance that $0 \leq a < c^0 < c^+ < b \leq 1$ and $f'' > 0$ in $]a,b[$. Then the only smooth self-similar solution with data (4.20) is such that :

$$\left\{ \begin{array}{lclr} C(z) & = & c^0, & 0 < z < z_0, \\ \dfrac{dC}{dz} & = & \dfrac{H(C)}{z\, f''(C)}, & z_0 < z < z_+, \\ C(z) & = & c_+, & z_+ < z, \end{array} \right. \tag{4.21}$$

where $z^+ = \dfrac{H(c^+)}{u^+}$, $z^0 = z^+ e^{-\Phi(c^+)}$ *with* $\Phi(c) = \displaystyle\int_{c^0}^{c} \frac{f''(\xi)}{H(\xi)}\, d\xi$.
Moreover $u^0 = \dfrac{H(c^0)}{z^0}$ *and* U *is given by:*

$$\begin{cases} U(z) &= u_0, & 0 < z < z_0, \\ U(z) &= \dfrac{H(C(z))}{z}, & z_0 < z < z_+, \\ U(z) &= u_+ & z_+ < z. \end{cases} \tag{4.22}$$

The entropy solution with a single shock satisfies the Liu entropy-condition ([12]) which is equivalent to an Oleinik condition for the (PSA) system.

Proposition 4.2.2 ($\lambda-$shock waves)
If (c^0, c^+) *satisfies the following admissibility condition*

$$\forall c \in]c^0, c^+[= \{(1-\tau)c^0 + \tau c^+,\ 0 < \tau < 1\} \quad \frac{f(c^+) - f(c^0)}{c^+ - c^0} \leq \frac{f(c) - f(c^0)}{c - c^0}, \tag{4.23}$$

then the Riemann problem (4.20) is solved by a shock wave defined as:

$$C(z) = \begin{cases} c^0 & \text{if} \quad 0 < z < s, \\ c^+ & \text{if} \quad s < z \end{cases}, \qquad U(z) = \begin{cases} u^0 & \text{if} \quad 0 < z < s, \\ u^+ & \text{if} \quad s < z, \end{cases} \tag{4.24}$$

where u^0 *and the speed* s *of the shock are obtained through*

$$\frac{[f]}{u^0\,[c]} + \frac{1 + h^0}{u^0} = s = \frac{[f]}{u^+\,[c]} + \frac{1 + h^+}{u^+},$$

where $[c] = c^+ - c^0$, $[f] = f^+ - f^0 = f(c^+) - f(c^0)$, $h^+ = h(c^+)$, $h^0 = h(c^0)$.

Proposition 4.2.3 ($\lambda-$contact discontinuity) *Two states* U^0 *and* U^+ *are connected by a* $\lambda-$*contact discontinuity with* $c^0 \neq c^+$ *if and only if* f *is affine between* c^0 *and* c^+.

4.2.5 The Riemann problem

Now, for (PSA) system, we solve the Riemann problem with the following initial data:

$$\begin{cases} c(t,0) &= c^- \in [0,1], \\ u(t,0) &= u^- > 0, \end{cases} t < 0, \quad \begin{cases} c(t,0) &= c^+ \in [0,1], \\ u(t,0) &= u^+ > 0, \end{cases} t > 0. \tag{4.25}$$

Notice the slight difference with the data (4.20) for the boundary Riemann problem. Indeed, a $0-$wave appears on the line $\{t = 0\}$.

Proposition 4.2.4 ($0-$contact discontinuity) *Two distinct states* U^- *and* U^0 *are connected by a* $0-$*contact discontinuity if and only if* $c^- = c^0$ *(with of course* $u^- \neq u^0$*).*

The solution of the Riemann problem for $x > 0$ and a convex function f is

- $(c, u) = (c^-, u^-)$ for $t < 0$,

- a 0−contact discontinuity for $t = 0$,
- a λ−wave for $t > 0$,

see Fig. 4.2 below. In practice, since $c^0 = c^-$, we first solve the boundary Riemann Problem (4.20). Then u^0 is well defined and the 0−contact discontinuity is automatically solved.

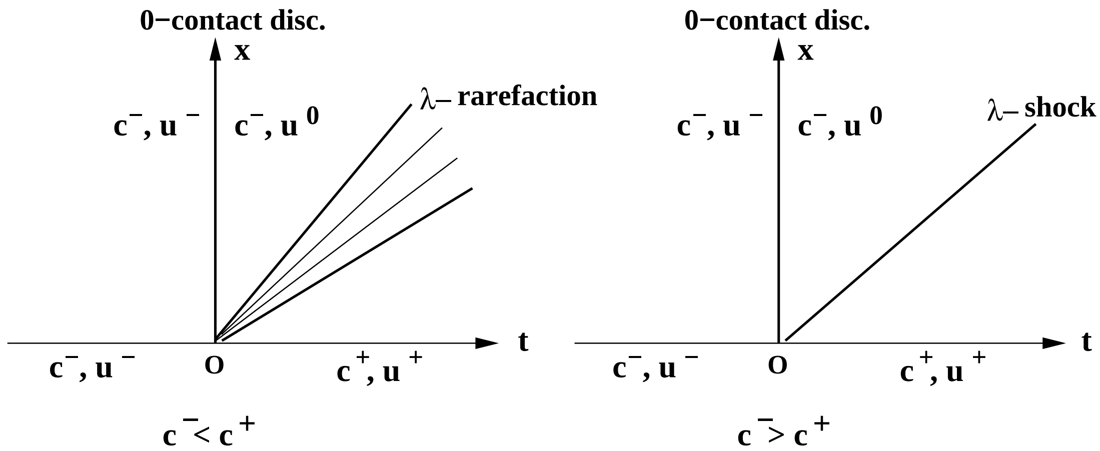

Figure 4.2: solution of the Riemann problem when $f'' > 0$.

Notice that the "maximum principle" is satisfied for the concentration $c(t, x)$ solution of the Riemann problem, since c is monotonic through a λ−wave (and constant through the 0−wave):

$$\min(c^-, c^+) \le c(t, x) \le \max(c^-, c^+). \tag{4.26}$$

Thus, the condition $0 \le c(t, x) \le 1$ is achieved by the solutions ([BGJ2]).

For a piecewise convex function f (4.13), the λ−wave is a composite λ− wave. The admissible composite wave is solved with the function f. There are two cases presented in figure 4.3.

- $c^- < c^+$: we consider the lower convex envelope f_c of the function f (see Fig. 4.3, left). On the subintervals where f is strictly convex (then $f = f_c$) we get a rarefaction wave. Elsewhere we get admissible shock waves (or contact discontinuities if f is affine).
- $c^- > c^+$: we use the upper convex envelope f^c (see Fig. 4.3, right) and get rarefaction waves where f is strictly concave.

The concentration stays monotonic through the composite wave and the maximum principle holds for the concentration ([BGJ2]). Notice that the velocity u is not monotonic through a composite wave. Fortunately, we have the following control of u.

Lemma 4.2.1 ($ln(u)$ estimate throw a λ-wave [BGJ2])
Let U^- and U^+ be two states and U the solution of the associated Riemann Problem. U^+

is connected to an intermediary state U^0 by a λ-composite wave for $z^0 < z < z^+$, and U^0 is connected to U^- by a 0-contact discontinuity ($c^0 = c^-$).

Then, there exists a constant γ depending only on q_1, q_2 and its derivatives such that

$$TV[\ln(u(z)), (z^0, z^+)] \;\; \leq \;\; \gamma|c^+ - c^0| = \gamma|c^+ - c^-| \tag{4.27}$$

Estimate (4.27) is only valid for λ-wave. For 0-wave we have no control of the velocity u.

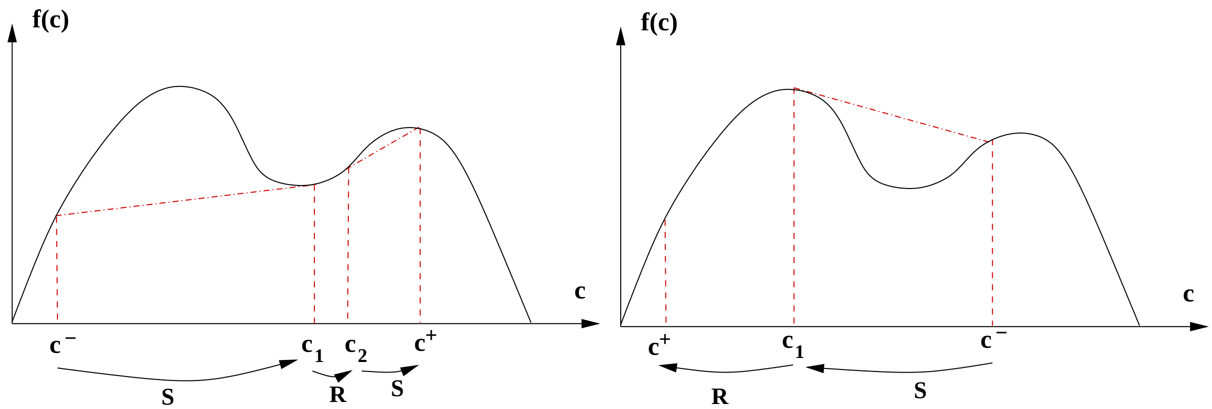

Figure 4.3: Shocks chords are shown as dashed lines. On the left c^- is connected to c^+ via a shock (S), a rarefaction wave (R) and a shock. On the right, c^- is connected to c^+ via a shock and a rarefaction wave.

4.3 Godunov scheme

The convergence of the Godunov scheme is rarely proved for systems. The (PSA) system (4.5)-(4.6) is not hyperbolic with respect to the evolutive variable t. Nevertheless, we adapt the Godunov scheme with the evolutive variable t. The proof of convergence is then simpler since in the time derivative the velocity does not appear. So we can prove the existence of entropy solutions for BV concentration but only L^∞ velocity. Notice that there is no convexity assumptions for the following theorem. Furthermore, we have strong traces to recover initial en boundary data [BGJ2].

Theorem 4.3.1 (Global large weak entropic solution [BGJ2])
Let $X > 0$, $T > 0$. Assume that $\ln(u_b) \in L^\infty(0,T)$, $c_0 \in BV(0,X)$ and $c_b \in BV(0,T)$ satisfy

$$0 \leq c_0 \leq 1 \text{ and } 0 \leq c_b \leq 1.$$

Then the (PSA) system admits a weak entropic solution given by the adapted Godunov scheme. Furthermore, $c \in BV$ and $u \in L^\infty$ satisfy:

$$0 \leq \min\left(\inf_{(0,T)} c_b(t), \inf_{(0,X)} c_0(x)\right) \leq c(t,x) \leq \max\left(\sup_{(0,T)} c_b(t), \sup_{(0,X)} c_0(x)\right) \leq 1, \tag{4.28}$$

$$\inf_{[0,T]\times[0,X]} u(t,x) > 0. \tag{4.29}$$

Let us give a flavor of the proof. The concentration c is constant throughout a $0-$ wave (c is a Riemann invariant). Thus we estimate the concentration throughout the $\lambda-$waves. c is monotonic throughout a $\lambda-$wave, so the total variation of c is less or equal to its initial-boundary total variation. But we need to check the CFL condition. That is to say to bound $\lambda = H(c)/u$. By Lemma 4.2.1 we can estimate $\inf u$. To conclude, there remains to control the projection error in the Godunov scheme and to check the entropy inequalities. These follow from the BV bounds for the concentration and the linearity of entropies w.r.t. the velocity.

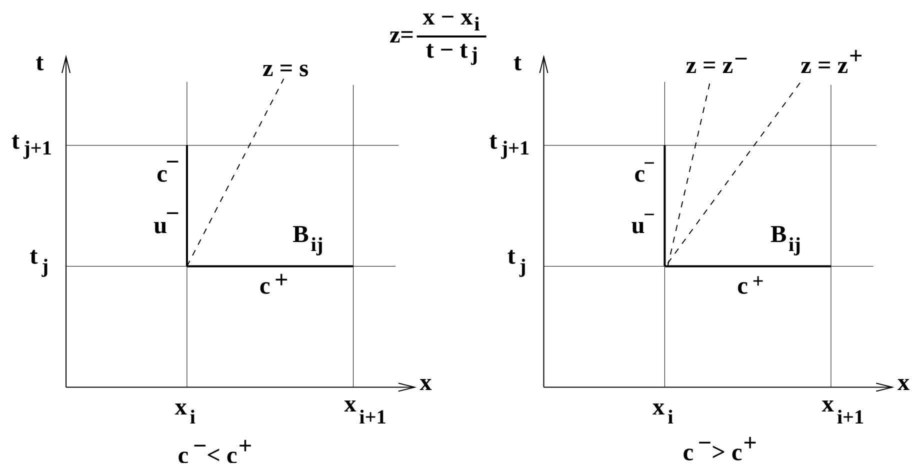

Figure 4.4: Riemann problem in a box B_{ij}

4.4 Regularity and stability

The mains results in [BGJ4] are related to the simple *stratified* structure of the velocity:

$$u(t,x) \quad = \quad u_b(t) \times v(t,x), \quad \text{where} \quad v(.,.) \quad \text{is as regular as} \quad c(.,.). \tag{4.30}$$

Notice that time singularities on $x = 0$ are propagated in the domain. We can also propagate high oscillations with large amplitudes as in [5, 6]. Stratified structure are studied in [7, 8, 13].

For smooth solutions of the (PSA) system, the Riemann invariant $W = u\,G(c)$ satisfies $\partial_x W = 0$, so $v(t,x)$ is simply $G(c_b(t))/G(c(t,x))$. For L^∞ boundary velocity and Lipschitz concentrations (4.30) is still valid ([BGJ4]). There is no restriction on the isotherms q_1, q_2 for such phenomena. The only limitations are the local existence of smooth concentration before shocks.

For entropy solutions, the situation is more delicate. With the Godunov scheme, we are not able to prove that the velocity u belongs to $BV((0,T) \times (0,X))$ if the boundary velocity u_b is in $BV(0,T)$. To get precise estimate on the velocity, we use the Front Tracking Algorithm (FTA), [3, 9]. For this purpose, we restrict ourselves to the case where λ is *genuinely nonlinear*, so with convex assumptions on isotherms. Furthermore we obtain better interaction estimates when the shock and rarefaction curves are monotonic in the coordinates $(c, \ln u)$, (Fig. 4.5). Nevertheless, we conjecture that our result is still valid for general isotherms without convex restrictions. For $\ln u_b \in L^\infty$ and $c_0, c_b \in BV$, we get (4.30) for the velocity. We first precise the notations used in the next theorem. We define the function c_I on $(0,T)$ by

$$c_I(s) = \begin{cases} c_0(s) & \text{if } 0 < s < X \\ c_b(-s) & \text{if } 0 < -s < T \end{cases},$$

and we set $TV c_I = TV c_I[-T, X]$.

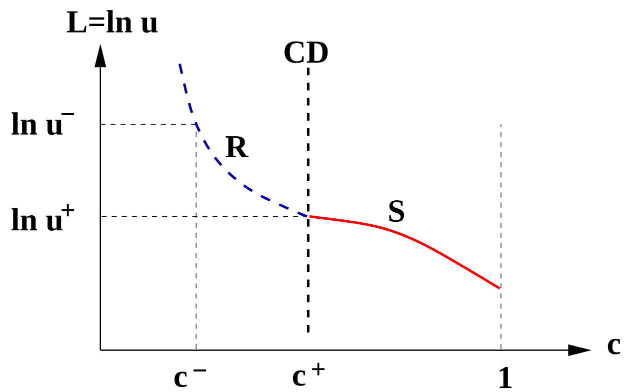

Figure 4.5: structure of shock curves (S), rarefactions (R) and contact discontinuities (CD) in the case $f'' > 0 > h'$.

Theorem 4.4.1 **[BV structure for the velocity]**
We assume the eigenvalue λ to be genuinely nonlinear.
If $\ln u_b \in L^\infty(0,T)$, if c_0, $c_b \in BV$ and if (c,u) is an entropy solution issued from the FTA, then there exist constants γ, $\Gamma > 0$ such that the function v in (4.30) satisfies:

$$\ln v \in \{L^\infty((0,X), BV(0,T)) \cap L^\infty((0,T), BV(0,X))\} \subset BV((0,T)\times(0,X)),$$

$$\begin{aligned} \sup_{0<t<T} TV_x \ln v(t,.)[0,X] &\leq \gamma\, TV c_I, \\ \sup_{0<x<X} TV_t \ln v(.,x)[0,T] &\leq 2\gamma\, TV c_I + \frac{\Gamma}{2}\left(TV c_I\right)^2. \end{aligned}$$

The new result in this theorem is that $\dfrac{u(t,x)}{u_b(t)}$ is BV with respect to time, although u_b is not assumed to be BV, but just in L^∞. The other regularity properties have already been proved in [BGJ1, BGJ2].

4.4.1 Weak limit for velocity with BV concentration

Theorem 4.4.1 yields the following stability result.

Theorem 4.4.2 (Stability with respect to weak limit for the velocity)
Let $(\ln(u_b^\varepsilon))_{0<\varepsilon<1}$ be a bounded sequence in $L^\infty(0,T)$, such that

$$u_b^\varepsilon \rightharpoonup \overline{u}_b \text{ in } L^\infty(0,T) \text{ weak } *.$$

Let $c_0 \in BV((0,X),[0,1])$ *and* $c_b \in BV((0,T),[0,1])$. *Let* $(c^\varepsilon, u^\varepsilon)$ *be an entropy solution of the (PSA) system on* $(0,T)\times(0,X)$ *issued from the FTA with the following initial and boundary values:*

$$\begin{cases} c^\varepsilon(0,x) &=& c_0(x), \quad X > x > 0, \\ c^\varepsilon(t,0) &=& c_b(t), \quad T > t > 0, \\ u^\varepsilon(t,0) &=& u_b^\varepsilon(t)\,, \quad T > t > 0. \end{cases}$$

Then, there exists $(u(t,x), c(t,x))$, *entropy solution of the (PSA) system supplemented by initial and boundary values:*

$$\begin{cases} c(0,x) &=& c_0(x), \quad x > 0, \\ c(t,0) &=& c_b(t), \quad t > 0, \\ u(t,0) &=& \overline{u}_b(t), \quad t > 0, \end{cases}$$

such that, when ε *goes to* 0 *and up to a subsequence:*

$$\begin{array}{lcl} c^\varepsilon(t,x) & \to & c(t,x) \text{ strongly in } L^1([0,T]\times[0,X]), \\ u^\varepsilon(t,x) & \rightharpoonup & u(t,x) \text{ weakly in } L^\infty([0,T]\times[0,X]) \text{ weak } *, \\ u^\varepsilon(t,x) & = & u_b^\varepsilon(t)\times v(t,x) + o(1) \text{ strongly in } L^1([0,T]\times[0,X]), \text{ where } v(t,x) = \dfrac{u(t,x)}{\overline{u}_b(t)}. \end{array}$$

For the convergence of the whole sequence we need the uniqueness of the entropy solution for the (PSA) initial-boundary value problem.

4.4.2 Oscillations for velocity with BV concentration

As an example of weak limit, we consider the case of high oscillations for velocity on the boundary.

Let $u_b(t,\theta) \in L^\infty((0,T), C^0(\mathbb{R}/\mathbb{Z},\mathbb{R}))$, $\overline{u}_b(t) = \int_0^1 u_b(t,\theta)d\theta$ and assume $\inf u_b > 0$. With $u_b^\varepsilon(t) = u_b\left(t, \dfrac{t}{\varepsilon}\right)$ and the same notations as in Theorem 4.4.2, we have:

- first, oscillations do not affect the behavior of the concentration since, up to a subsequence, (c^ε) converges strongly in L^1 towards c and the limiting system depends only on the average $\overline{u}_b$ and not on oscillations, i.e. the (PSA) system with initial boundary velocity $\overline{u}_b$;

- second, up to a subsequence, (u^ε) converges weakly towards $\overline{u}_b(t)\times v(t,x)$ and we have a strong profile for u^ε, $U(t,x,\theta) = u_b(t,\theta)\times v(t,x)$ and

$$\lim_{\varepsilon\to 0}\left\| u^\varepsilon(t,x) - U\left(t,x,\frac{t}{\varepsilon}\right)\right\|_{L^1((0,T)\times(0,X))} = 0.$$

4.5 Blow up

This section deals with an example of a blow up solution [BGJ5]. (PSA) system has the strong property to admit a non decreasing w.r.t. x and positive Riemann invariant $W = u\,G(c) > 0$ which is also an entropy (4.19), under the restrictive assumption $(H1)$. W plays a key-role in the blow up mechanism, so our first assumption is:

$$G'' \quad < \quad 0 \qquad\qquad\qquad (H1).$$

For large data, we need to have monotonic shock and rarefaction curves as in [BGJ4]. Thus we assume:

$$h' \quad \neq \quad 0, \text{ everywhere} \qquad\qquad\qquad (H2).$$

Assumption $(H2)$ means that one gas is more active than the other. Mathematically, it implies the monotonicity of g and G because $g' = -h'/H$. Thanks to $(H2)$, rarefaction curves, which are the level line of W, are monotonic curves in the plane (c, u). We also assume that the eigenvalue $\lambda = H(c)/u$ is genuinely nonlinear, see figure 4.5:

$$f'' \quad \neq \quad 0, \text{ everywhere} \qquad\qquad\qquad (H3).$$

For Temple systems, when rarefaction curves and shocks curves coincide, it is well known that there is no blow up in L^∞, so we assume.

(PSA) system (4.5)-(4.6) is not a Temple system $\qquad (H4).$

These four restrictive hypothesis are satisfied for the case of an inert gas associated with ammonia or water vapor for instance (strictly convex isotherm).

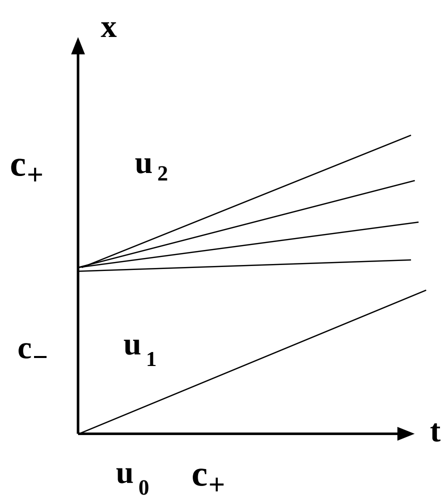

Figure 4.6: the two boundary Riemann problems

Theorem 4.5.1 (Blow up for velocity) *Assume (H1), (H2), (H3) and (H4). Then for any* $T > 0$, $X_\infty > 0$, *there exist* L^∞ *initial and boundary data (4.8) and a corresponding entropy solution on* $[0, T] \times [0, X_\infty[$ *of (PSA) system (4.5)-(4.6) such that*

$$\| \, u \, \|_{L^\infty((0,T)\times(0,X_\infty))} = +\infty.$$

Actually the solution built to prove this theorem has special features.

- The velocity only blows up at the boundary $\{t = 0\}$ when $x \to X_\infty$, with arbitrary small data.

- The system is strictly hyperbolic, but at ($t = 0$, $x = X_\infty$) it becomes degenerate hyperbolic because
$$\lambda = \frac{H(c)}{u}$$
tends to 0 when u tends to $+\infty$. Then the boundary becomes twice characteristic.

- $\forall X \in]0, X_\infty[$, u, $c \in L^\infty(0, T; BV(0, X)) \cap L^\infty(0, X; BV(0, T))$. The concentration c remains bounded while u blows up.

- Let Ω be a neighborhood of the critical point $(t = 0, x = X_\infty)$ such that $\Omega \subset [0, T] \times [0, X_\infty]$. Outside Ω, we can prove that (u, c) has a piecewise smooth structure, so the blow up occurs only at the boundary. Indeed, there is an accumulation of wave-interactions near $(t = 0, x = X_\infty)$.

- To build such solutions, we have necessarily to choose the boundary concentration $c_0(x) = c(0, x)$ not in $BV(0, X_\infty)$. Otherwise there is no blow up, see [BGJ1, BGJ2].

The main ingredient for the construction of the blow up solution consists in the resolution of two consecutive boundary Riemann problems which leads to increase the velocity without increasing the concentrations. For the first boundary Riemann problem, the data are $(c_-,\ c_+,\ u_0)$ chosen in such a way that the solution is a shock wave and we set $u_1 = \mathcal{R}(c_-,\ c_+,\ u_0)$. For the second problem, the data are $(c_+,\ c_-,\ u_1)$, the solution is necessarily a rarefaction wave and we set $u_2 = \mathcal{R}(c_+,\ c_-,\ u_1)$, Fig. 4.6.

Finally we can prove that $u_2 = \mathbf{R}(c_-, c_+)\ u_0$. Notice that the "amplification" coefficient $\mathbf{R}$ depends only on (c_-, c_+).

Lemma 4.5.1 ([BGJ5]) *We assume* $(H1)$, $(H3)$ *and* $(H4)$.
If the function $\mathbf{R}$ *is analytic then* $\mathbf{R}(c_-, c_+) > 1$ *almost everywhere in the domain* $c_- < c_+$.

Now we have to repeat the construction at the boundary and to construct a piecewise smooth entropy solution in $[0, T] \times [0, X_\infty[$ with the Front Tracking Algorithm, (Fig. 4.7). The construction of this example yields the following remarks:

- It is a new physical example of hyperbolic partial differential equations exhibiting a blow up. However, the physical initial data $u_0(x) = u(0, x)$ only blows up. Physically $u_0(x)$ is bounded but for the mathematical model $u_0(x)$ is not a data, it is an unknown. Indeed, u_0 depends essentially on $c_0(x)$ (and also on $c_b(0+), u_b(0+)$).

- It follows from our blow up example that a general mathematical L^∞ theory for (PSA) system assuming only $c_0, c_b, \ln u_b \in L^\infty$ is impossible. Nevertheless, a natural conjecture is: "if $\ln u_0(x)$ remains bounded there is no blow up".

- We also conjecture that our solution built for the blow up can be defined for $x \geq X_\infty$ and $t > 0$ as a piecewise smooth entropy solution outside the boundary, but with infinite velocity on the boundary: $\{0\} \times [X_\infty, +\infty[$.

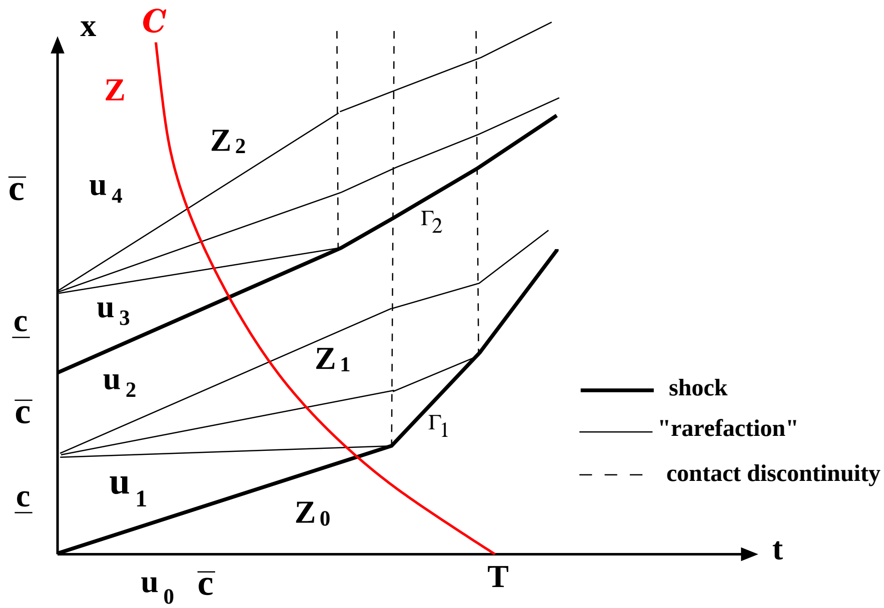

Figure 4.7: first interactions and free domain Z

4.6 Prospects

Many problems are still open for the (PSA) system and some of them could be the object of a PhD thesis.

- The uniqueness for large data is not proven. The Front Tracking Algorithm seems the right tool to handle this problem for general isotherm without convexity assumptions. Moreover, we can improve [BGJ4] by proving that the velocity remains stratified.
- The degenerate eigenvalue can be used by Euler-Lagrange change of variables [28, 14]. Unfortunately, the boundary becomes unknown after this change of variables. But we can expect new features with simpler equations, at least for the Cauchy problem.
- A L^∞ theory seems possible with the following assumption.

  The L^∞ initial-boundary data c_0, c_b, u_b have to satisfy the restricted condition:
  the unknown $u_0(x) = u(0,x)$ satisfies $\ln(u_0) \in L^\infty$.

  Indeed, it is mainly an assumption on c_0 as we can see in the blow up construction [BGJ5]. But we need compactness to obtain existence of solutions for this nonlinear problem. Two directions naturally appear.

  - We restrict our attention with $c_0 \in BV^s$, $s = 1/3$ for convex isotherm or $s = 1/2$ for general isotherm. In this case, one can see that no blow up occurs as in [BGJ5]. Furthermore, a control of BV^s norm of the concentration as in [BGJ6] gives the desired compactness.

  - A more ambitious direction is to prove a smoothing effect for the concentration. At least for a concave or convex isotherm, without blow up of $\ln u$, we can expect a smoothing effect for the concentration in BV as in [16]. Unfortunately we do not have an equivalent of the Lax-Oleinik formula. Nevertheless, possible strategies would be to investigate the generalized characteristics or the Front Tracking Algorithm.
  - We have already studied a compensated compactness approach. Unfortunately, since there is no strictly convex entropy, such approach is limited by the key assumption $\inf u > 0$. With only L^∞ estimates on the concentration, the bound from below on the velocity is an open problem.

- For convex isotherm and L^∞ data, a blow up of velocity can occur. For concave isotherm, the velocity can vanish with a similar construction expounded in [BGJ5]. Indeed, $\ln u$ blows up in this case but not u. Thus the (PSA) system loses its hyperbolicity. Nevertheless, solutions can be defined at the first sight and a theory in this context could be developed.

- Behavior of solutions with non convex isotherms. In chemistry, many isotherms have many inflexion points. An interesting example is an inert gas with a BET gas [11]. In this case the eigenvalue λ is piecewise genuinely nonlinear with an inflexion point. What is the behavior of the solutions for large time? We also conjecture a smoothing effect in $BV^{1/2}$ for L^∞ data.

- Kinetic schemes can be constructed with the kinetic formulation given in [BGJ3], see chapter 5. The main difficulty to study the convergence of such a scheme is to prove that the kinetic scheme is an entropic scheme. It is not so clear. A first approach is to restrict ourselves to convex isotherms.

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Chapter 5

Kinetic Equations

Hyperbolic conservation laws are closely related to kinetic equations. I did some works on this subject collected in this chapter.

Following the chronological order, in section 5.1, the first work is concerned with a kinetic formulation of the (PSA) system expounded in Chapter 4. A second work, related to the Chapter 3, is devoted to the regularity analysis for a Vlasov equation, section 5.2. The third work in section 5.3, deals with a more original topic. With Pierre-Emmanuel Jabin, we have introduced a continuous model with a mean field velocity related to a famous rating system: Elo. This system was first used as the international rating system for professional chess players. Nowadays, it is extended to various domains like on line video games, journals rating, basket teams and so on. For many but selected interactions, we study the asymptotic behavior of the ratings for large time.

5.1 Kinetic formulation of the PSA system

For the (PSA) system expounded in Chapter 4, we have found a kinetic formulation [BGJ3]. The kinetic velocity is defined by for all $\xi \in (0,1)$ by:

$$\xi \in (0,1) \mapsto a(\xi) \quad := \quad H(\xi) - h(\xi). \tag{5.1}$$

where H and h are defined in Chapter 4. Let us introduce the now classic function χ:

$$\chi(c,\xi) := \begin{cases} 1 & \text{if } 0 < \xi < c \\ 0 & \text{else} \end{cases} \tag{5.2}$$

Since $c \in (0,1)$ we define χ only for $(c,\xi) \in (0,1) \times (0,1)$. This function satisfies the fundamental Gibbs property, (also called Brenier's Lemma see [15]): χ minimize the following problem for all $c \in (0,1)$:

$$\inf \left\{ \int_0^1 S'(\xi) f(c,\xi) d\xi, \quad \text{where } 0 \le f \le 1, \text{ and } \int_0^1 f(c,\xi) d\xi = c \right\} \tag{5.3}$$

The kinetic formulation provides a characterization of entropy solutions of the (PSA) system. Unfortunately microscopic variable ξ and macroscopic variables (c,u) are both in the kinetic equation. Thus it is a difficult path to find new theoretical results on the (PSA) system, but a kinetic scheme is conceivable.

Theorem 5.1.1 (Kinetic formulation [BGJ3])
If (u,c) is a weak entropic solution of the (PSA) system, then there exists a nonnegative measure $m(t,x,\xi)$ such that:

$$\partial_x(u\chi(c,.)) + a(\xi)\partial_t\chi(c,.) + \partial_t\left(h(c)\chi(c,.)\right) = \partial_\xi m. \tag{5.4}$$

Conversely, if there exist a positive function u, a nonnegative measure m and a function $f(t,x,\xi)$ such that (u,f,m) satisfies (5.3) with f in place of χ, and if $\ln u \in L^\infty$, $f(t,x,\xi) \in \{0,1\}$ $\forall (t,x,\xi) \in [0,+\infty[\times\mathbb{R}\times[0,1]$, then (u,c) is an entropic solution of the (PSA) system where

$$c(t,x) \;\; := \;\; \int_0^1 f(t,x,\xi)d\xi.$$

5.2 Averaging lemmas

In [BeJ], with Florent Berthelin, we obtain some averaging lemmas for a Vlasov equation:

$$\partial_t f + a(v)\cdot\nabla_x f + F(t,x,v)\cdot\nabla_v f = g, \tag{5.5}$$

We recover the regularity of the free equation ($F \equiv 0$), but with a smoothness assumption on the velocity $a(v)$.

Theorem 5.2.1 (L^2 result)
Let $a \in C^{N+3}(\mathbb{R}^M_v, \mathbb{R}^N_x)$, $F \in C^{N+3}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}^M_v, \mathbb{R}^M_v)$, $f, g \in L^2(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}^M_v)$, satisfying (5.5). Let $A > 0$ and $\psi \in C^{N+2}_c(\mathbb{R}^M_v)$ be such that the support of ψ is included in $[-A,A]^M$. We assume that there exist $0 < \alpha \leq 1$ and $C > 0$ such that for any $(u,\sigma) \in S^N$ and $\varepsilon > 0$,

$$meas\ \left(\{v \in [-A,A]^M\,;\, u-\varepsilon < a(v)\cdot\sigma < u+\varepsilon\}\right) \leq C\varepsilon^\alpha. \tag{5.6}$$

Then the velocity average

$$\rho_\psi(t,x) = \int_{\mathbb{R}^M} f(t,x,v)\psi(v)\,dv$$

lies in $H^{\alpha/2}_{loc}(\mathbb{R}_t \times \mathbb{R}^N_x)$.

We adapt the classical proof of [3] when the test function depends on (t,x,v). For this purpose we need to some local change of variables.

When the force is constant, we can use a stationnary phase argument [16] with an uniform control of the parameters. Then we obtain a global regularity result with a less smooth test function.

Theorem 5.2.2 (L^2 result with F constant)
Let $a \in C^\gamma(\mathbb{R}_v^M, \mathbb{R}_x^N)$ for some positive integer γ, $F(t,x,v) = F \in \mathbb{R}^M$, $F \neq 0$, f, $g \in L^2(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^M)$ satisfy (5.5). We assume that the function $a(\cdot)$ fulfills the following condition
$\forall (v,\sigma) \in \mathbb{R}^M \times S^N, \quad \sigma = (\sigma_0, \sigma_1, \cdots, \sigma_N), \quad \widetilde{\sigma} = (\sigma_1, \cdots, \sigma_N),$

$$|\sigma_0 + a(v).\widetilde{\sigma}| + \sum_{k=1}^{\gamma-1} \left|(F \cdot \nabla_v)^k a(v) \cdot \widetilde{\sigma}\right| > 0. \qquad (\gamma ND) \tag{5.7}$$

Let $\psi \in C_c^1(\mathbb{R}_v^M)$. Then the velocity average

$$\rho_\psi(t,x) = \int_{\mathbb{R}^M} f(t,x,v)\psi(v)\, dv$$

lies in $H^{1/\gamma}(\mathbb{R}_t \times \mathbb{R}_x^N)$.

This result introduces a natural condition between the force F and the transport velocity $a(v)$: condition (5.7). Furthermore, Theorem 5.2.2 is better than theorem 5.2.1 for low velocity dimension, for instance when $M = 1$.

5.3 A rating model

“The Elo rating system is a method for calculating the relative skill levels of players in two-player games such as chess” (Wikipedia). When the players are numerous and interact a lot, we derive a new continuous model. There is a natural question first formulated by Arpad Elo. Does the empirical rating of each player converge towards its real “strength“? We try to answer to this question with a kinetic model. Mathematically we have to study the asymptotic behavior of the density below for large time.

This research started with the training of a student from an engineering school. The student was a strong chess player and we designed and studied a simple discrete model, both in deterministic and stochastic setting (see http://math.unice.fr/ junca/Research/PDE-Master1.html).

5.3.1 Convergence for large time

Let $f(t,r,\rho)$ be the density of players at the time t, with rating r and value ρ for their theoretical strength: the theoretical rating. In [JJ], we derive the new following continuous model for a lot of interacting players:

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial r}(a[f]\, f) \;=\; 0, \tag{5.8}$$
$$f(t=0,r,\rho) \;=\; f^0(r,\rho), \tag{5.9}$$

where the quadratic flux is $a[f]\ f$ and the scalar velocity field $a[f]$ is given by a convolution:

$$a[f] \;=\; a(t,r,\rho) = \iint (w(r-r')\,(b(\rho-\rho') - b(r-r')))\, f(t,r',\rho') d\rho' dr', \tag{5.10}$$

$a = a[f]$ is a linear operator with respect to f.

Notice that model (5.8) does not involve a derivative with respect to the theoretical rating, i.e. $\dfrac{\partial}{\partial \rho}$ does not appear in (5.8).

There and below we adopt the following definition for the variables:

- r is the variable for the rating or empirical rating,
- ρ represents the theoretical rating i.e. the strength of the player,
- $b(.)$ is a "bonus-malus" function. It is the mean centered score predicted by the Elo model, $b(.)$ is an odd increasing function,
- $w(.)$ is an even nonnegative function.

We prove the following theorem:

Theorem 5.3.1 (convergence of f(t,.,.))
There exists a unique non negative measure

$$f_\infty(r,\rho) \;=\; \sum_{i=1}^{n} \delta(r-(\rho+c_i))\; h_i(\rho) \tag{5.11}$$

such that, when $t \to +\infty$, $f(t,r,\rho) \rightharpoonup f_\infty(r,\rho)$ *in* $\mathcal{M}^1(\mathbb{R}_r \times \mathbb{R}_\rho))$ *weak* $*$.

The Theorem states the convergence but it cannot precise the explicit values of $n, c_1, \cdots, c_n$ and $h_1(\rho), \cdots, h_n(\rho)$.

When every players (or teams or particles) interact with all other players, so when $\inf w > 0$, then $n = 1$ in (5.11). Furthermore, the convergence is exponential in this specific case. The proof uses only one energy and a Wasserstein metric. That means that the empirical rating converges towards the theoretical rating (the true rating) up to a constant.

It is well known in the chess community that the round-robin tournament (when each players meet all the other players) is the best way to rank N players. But for large N it is quite difficult in practice to plays the $N^2/2$ games. Thus, players have to meet only a subset of all players. This selection is done by the function w in this model.

The function w has a compact support near zero that means that players only meet players with nearly the same rating. In this case, the convergence of the rating depend strongly on the initial rating and more precisely on the initial distribution. The system becomes slightly unfair. Furthermore, the convergence is slow, not exponential. More precisely, we set the following assumption when the interaction is localized by the support of the pairing function w:

$$w \in C^2(\mathbb{R}, \mathbb{R}^+), \quad w > 0 \text{ on }]-1,1[\quad \text{and } w(x) = 0 \text{ for all } |x| \geq 1. \tag{5.12}$$

The proof of convergence of the density relies on all the entropies and uses the specific mean field structure of the kinetic velocity in equation (5.8). We briefly explain below the formal derivation of the model.

5.3.2 A model derivation

We begin by the discrete model. Next we formally obtain a continuous model. Let two players i and j with rating R_i^n and R_j^n at time $n\Delta t$, $S_{ij}^n \in [-1, 1]$ the result of the interaction (game, confrontation or comparison) $b(D)$ the expected score for two players with D the difference of their theoretical ratings. Notice that the theoretical ratings are usually unknown. Arpad Elo stated the following computation after each game to update the ratings.

$$R_i^{n+1} = R_i^n + K(S_{ij}^n - b(R_i^n - R_j^n)). \qquad (5.13)$$

He expected that the rating of each player converges towards the theoretical rating up to a constant. The model (5.13) is invariant under translation. Thus we only expect to approximate the relative strength $R_i - R_j$.
In this model, the independent random variables S_{ij}^n depend on the theoretical rating difference $\Delta = \rho_i - \rho_j$:

$$\mathbb{E}(S_{ij}^n) = b(\rho_i - \rho_j). \qquad (5.14)$$

To avoid too large fluctuations of the rating, K is usually small relatively to the rating, $\frac{K}{R} < 0.01$. In the sequel we normalize $R \sim 1$ so K is small.

On the web, players can play a large number of game against players with similar ratings. A usual choice is to play with another player if the relative difference is small, and not to play otherwise:

$$W(\Delta r) = \begin{cases} 1 & \text{if} \quad |\Delta r| < 0,05 \\ 0 & \text{else} . \end{cases} \qquad (5.15)$$

Other choices are possible. For instance, $0 \le W(\Delta r) = p < 1$ means that the probability to play with a rating relative difference Δr is p. In the sequel W is often a smooth function.

We propose the following evaluation of the rating.

$$R_i^{n+1} - R_i^n = K \sum_j W(R_i^n - R_j^n)(S_{ij}^n - b(R_i^n - R_j^n)). \qquad (5.16)$$

Usually the rating is updated after each match. But since the change is relatively small, we simplify the model by updating the rating after a global interaction of each player with other players with close ratings.

We also assume that the number $\mathcal{N}_\mathbf{i}$ of confrontations or interactions is large for each player: $\mathcal{N}_\mathbf{i} = \varkappa_i N$ where N is the total number of players and $0 < \varkappa_i < 1$.

For $\mathcal{N}_\mathbf{i}$ large enough, we simplify the system (5.16) by replacing the independent random variables S_{ij}^n by their mean (5.14). We expect that the fluctuations of the score would be compensated by the large number of interactions.

At this stage, stochastic discrete system (5.16) becomes deterministic:

$$R_i^{n+1} - R_i^n = K \sum_j W(R_i^n - R_j^n)(b(\rho_i - \rho_j) - b(R_i^n - R_j^n)). \qquad (5.17)$$

We now look for a continuous deterministic model. First we obtain a large system of differential equations. Rescaling the pairing function

$$w = \frac{K}{\Delta t} W \tag{5.18}$$

we have: $\dfrac{R_i^{n+1} - R_i^n}{\Delta t} = \sum_j w(R_i^n - R_j^n)(b(\rho_i - \rho_j) - b(R_i^n - R_j^n))$. With more interactions, or quick games: $\Delta t << 1$, the discrete system becomes continuous w.r.t. the time and we arrive at:

$$\frac{dR_i(t)}{dt} = \sum_j w(R_i(t) - R_j(t))(b(\rho_i - \rho_j) - b(R_i(t) - R_j(t))), \quad 1 \leq i \leq N. \tag{5.19}$$

Now we introduce the empiric measure .

$$f_N = \frac{1}{N} \sum_i \delta(r - R_i(t)). \tag{5.20}$$

An exact computations leads to the following P.D.E. for the empiric measure, see [3]:

$$\frac{\partial f_N}{\partial t} + \frac{\partial}{\partial r} \left(a[f_N] \times f_N\right) = 0, \tag{5.21}$$

$$a[f](t, r, \rho) = \iint_{\mathbb{R}^2} \left(w(r - r') \left(b(\rho - \rho') - b(r - r')\right)\right) f(t, r', \rho') d\rho' dr', \tag{5.22}$$

$$f_N(t = 0, r, \rho) = f_N^0(r, \rho) = \frac{1}{N} \sum_i \delta(r - R_i(0)). \tag{5.23}$$

This mean field equation is extended for general measure f where the initial atomic measure f_N^0 is replaced by a general measure f^0. We obtain in this way (5.8), (5.9).

5.4 Prospects

- For the (PSA) system, a kinetic scheme can be proposed and at least validated for a convex isotherm. Getting more information on the behavior of the solutions seems challenging with the kinetic formulation (5.4).
- For the Vlasov equation (5.5), the nonlinear condition (5.7) related to the stationary phase method is new. The optimality and its extension to the variable force case is an open problem.
- About the rating model
  - The initial rating is very important to determine the asymptotic rating as we can see in Theorem 5.3.1. This is also already known in the chess players community. The FIDE (world chess federation) uses an estimator of the initial rating which can be interpreted as a mean evaluation with a linearized bonus

fonction b instead of the nonlinear logistic function. I have already found a better nonlinear estimator validated for the discrete deterministic case (5.17) when w is constant. For different pairing function or for the ODE model (5.19) or also for the stochastic model the validation of this estimator has to be done.

– Our model neglects the diffusion, i.e. the variances of random variables. A challenge is to study the asymptotic behavior with some diffusion. As Arpad Elo himself, we could assume the same variance for each random variable. This leads to a simple diffusion in the model.

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Chapter 6

Mechanical waves and cracks

In one dimensional linear elasticity, two models of nonlinear cracks are discussed in this chapter: a piecewise linear model [JRo] and a model with maximal compressibility [JuLo1, JuLo2, JuLo3].

This chapter collects joint works with Bernard Rousselet (Labo. JAD, Université de Nice Sophia Antipolis), Bruno Lombard (L.M.A .: Laboratoire de Mécanique et d'Acoustique, Marseille). This research takes place among the activity of the European GDR: "Wave propagation in complex media for quantitative and non destructive evaluation". In connection to these works, I organized the meeting "Nonlinearities in Acoustics" in Nice (http://math.unice.fr/∼junca/Research/Colloques/NLC-22-23-Mars-Nice-2012.pdf).

In section 6.1, a spring-mass model with piecewise linear stiffness is described. It models small defects in a beam. In [8, 9], it is used for modeling vibrations of solar panels during a launching of a satellite. For the homogeneous system, we perform and justify asymptotic expansions for nonlinear normal mode (NNM).

In sections 6.2, 6.3 and 6.4, a continuous model of interactions between ultrasonic waves and cracks is presented in view of nondestructive testing. Such nonlinear cracks are modeled by nonlinear interfaces. The acoustic wave is a source term. For many cracks, it is a model of heterogeneous media. The features are quite different for one crack in section 6.2 and two cracks in section 6.3. Finally, in section 6.4, we study a critical neutral delay differential equation which concentrates the mathematical problems occurring for two cracks: a weak linear stability, a weak nonlinear stability, and a Diophantine condition to avoid a small divisors problem.

All numerical results have been obtained by my collaborators. The reader can consult [JRo, JuLo1, JuLo2, JuLo3] for more explanations about the numerical methods.

6.1 A discrete piecewise linear model

We study a discrete spring-mass model in [JRo]. The presence of a small piecewise linear rigidity is intended to model a small defect which implies unilateral reactions on the structure. So, the piecewise linear function

$$u_+ \quad = \quad \max(0, u)$$

plays a key role. We are particularly interested in some periodic solutions: "Non linear Normal Modes" (NNM) in the sense of Rosenberg [22]; see [11] for a synthesis on nonlinear normal modes in mechanics.

For a short time, a linearization procedure is enough to compute a relevant approximation of the vibrations. But for large times, nonlinear cumulative effects drastically alter the nature of the solution. We will consider the classical method of strained coordinates, also referred to as the Lindstedt-Poincaré method, to compute asymptotic expansions, see [12, 13, 20] for more details and references. This method is a particular case of multiple scale expansions. The Lindstedt-Poincaré method is a simple and efficient method which gives approximate nonlinear normal modes for systems with several degrees of freedom. This method has been already used in [25] to study NNM of a piecewise linear system with two degrees of freedom. Here, the non linearity is somewhat more general. We consider N dimensional systems. Moreover, we prove rigorously the validity of the expansion.

In subsection 6.1.1, we briefly present the method on the piecewise linear scalar differential equation (6.1). We focus on an equation with one degree of freedom and derive expansions valid for time of order ε^{-1} or, more surprisingly, $\varepsilon^{-1/2}$ for a grazing contact. A key point to validate the method of strained coordinates with piecewise linear contact is to expand $(u + \varepsilon v)_+$ and to get some accurate estimate for the remainder.

In subsection 6.1.2, we extend previous results to systems with N degrees of freedom, first, with the same accuracy for approximate nonlinear normal modes, then, with less accuracy when all modes are excited.

6.1.1 One degree of freedom

We first consider an explicit example to introduce the strained coordinates method that will be used for non explicit and general piecewise linear system in subsection 6.1.2.

Figure 6.1 describes spring-mass system with a one degree of freedom: one spring is linearly elastic and attached to the mass and to a rigid wall, the second, a damaged spring with a small piecewise linear stiffness, is also attached to a rigid wall .

$$\ddot{u} + \omega_0^2 u + \varepsilon u_+ = 0, \qquad \text{with } u_+ = \max(0, u). \tag{6.1}$$

The energy associated to (6.1) reads: $E = (\dot{u}^2 + \omega_0^2 u^2 + \varepsilon (u_+)^2)/2$. The computation of the angular frequency relies on the energy:

$$\omega(\varepsilon) \quad = \quad 2\omega_0 \left(1 + \left(1 + \varepsilon/\omega_0^2\right)^{-1/2}\right)^{-1} = \omega_0 + \frac{\varepsilon}{(4\omega_0)} - \frac{\varepsilon^2}{(8\omega_0^3)} + \mathcal{O}(\varepsilon^3). \tag{6.2}$$

Notice that $\omega(\varepsilon)$ is independent of the energy, due to the homogeneity of (6.1). In general, $\omega(\varepsilon)$ depends on the energy (or amplitude).

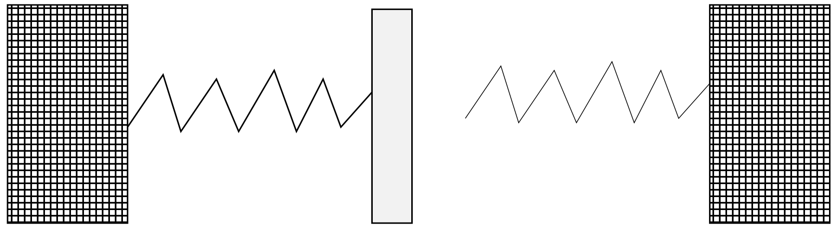

Figure 6.1: Two springs, on the right it has only a unilateral contact.

The method of strained coordinates

The method consists in using a new time variable $s = \omega_\varepsilon\ t$ where ω_ε is an unknown approximation of $\omega(\varepsilon)$, and let $v_\varepsilon(s)$ be $u_\varepsilon(t = s/\omega_\varepsilon)$. In the new time s, the function v is 2π periodic, and equation (6.1) becomes:

$$\omega_\varepsilon^2 v_\varepsilon''(s) + \omega_0^2 v_\varepsilon(s) + \varepsilon(v_\varepsilon(s))_+ \ = \ 0. \tag{6.3}$$

Since solutions of (6.3) are periodic, we restrict ourself to initial data:

$$v_\varepsilon(0) = a_0, \qquad v_\varepsilon'(0) = 0. \tag{6.4}$$

Now, we expand w.r.t. the ε power and use the following **ansatz**

$$\omega_\varepsilon = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2, \qquad v_\varepsilon(s) = v_0(s) + \varepsilon v_1(s) + \varepsilon^2 r_\varepsilon(s). \tag{6.5}$$

Unfortunately, the nonlinear function $v \to v_+$ does not belong to C^2 and we cannot use the Taylor-Lagrange expansion but we can write

$$(u + \varepsilon v)_+ = u_+ + \varepsilon H(u)v + \varepsilon\chi_\varepsilon(u, v),$$

where $H(.)$ is the Heaviside function and the remainder will be estimated in Lemma 6.1.1. Now, replacing ansatz (6.5) in (6.3) we obtain differential equations and initial data for v_0, v_1, r_ε involving the operator $L(v) = -\alpha_0(v'' + v)$:

$$\begin{aligned} L(v_0) &= \quad -\alpha_0(v'' + v) = 0, && v_0(0) = a_0, v_0'(0) = 0, && (6.6)\\ L(v_1) &= \quad (v_0)_+ + \alpha_1 v_0'', && v_1(0) = 0,\ \ v_1'(0) = 0, && (6.7)\\ L(r_\varepsilon) &= \quad H(v_0)v_1 + \alpha_2 v_0'' + \alpha_1 v_1'' + R_\varepsilon(s), && r_\varepsilon(0) = 0,\ \ r_\varepsilon'(0) = 0, && (6.8) \end{aligned}$$

where $\omega_\varepsilon^2 = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \mathcal{O}(\varepsilon^3)$, $\alpha_0 = \omega_0^2$, $\alpha_1 = 2\omega_0\omega_1$, $\alpha_2 = \omega_1^2 + 2\omega_0\omega_2$. Now we compute successively, v_0, α_1, v_1 and then α_2. α_1, α_2 are chosen to avoid resonant or **secular** term in the right-hand-side of equations (6.7), (6.8). We use Fourier expansions to find the function v_1. The piecewise linearity involves an infinite set of frequencies for v_1:

$$v_1(s) \ = \ \frac{-a_0}{\pi\omega_0^2}\left(1 - \cos(s) - 2\sum_{k=1}^{+\infty}\frac{(-1)^k}{(4k^2-1)^2}(\cos(2ks) - \cos(s))\right). \tag{6.9}$$

To remove secular term of order one in (6.8), we have to take α_2 satisfying:

$$0 = \int_0^{2\pi} [H(v_0(s))v_1(s) + \alpha_2 v_0''(s) + \alpha_1 v_1''(s)] \cdot v_0(s)ds \tag{6.10}$$

We display in figure 6.2 the first modes of the Fourier spectra for $v_0(\omega_\varepsilon t) + \varepsilon v_1(\omega_\varepsilon t)$.

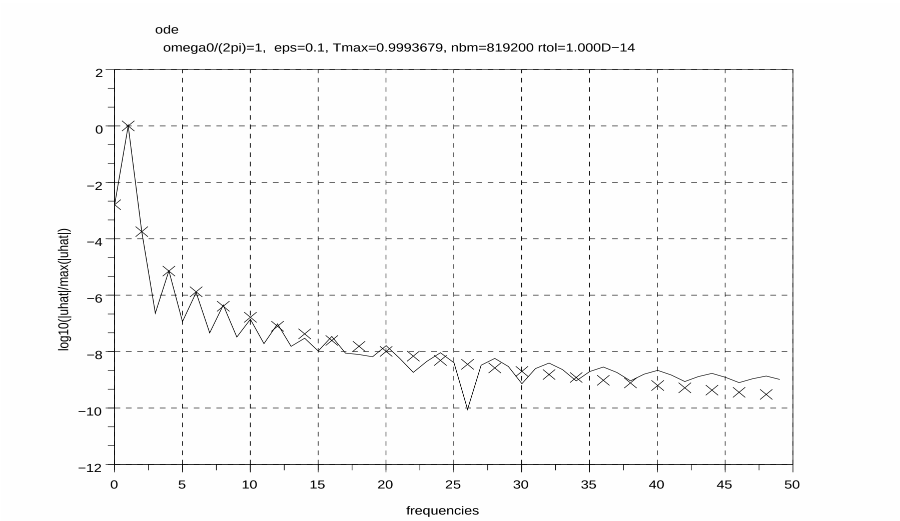

Figure 6.2: $\log_{10}\left(\dfrac{|\widehat{u}_\varepsilon|}{\max|\widehat{u}_\varepsilon|}\right)$

Proposition 6.1.1 *Let u_ε be the solution of (6.1) such that $u_\varepsilon(0) = a_0 > 0$ and $\dot{u}_\varepsilon(0) = 0$. With $\omega_\varepsilon = \omega_0 + \frac{\varepsilon}{4\omega_0} - \frac{\varepsilon^2}{(2\omega_0)^3}$ and $v_1(.)$ given by (6.9), there exists $\gamma > 0$ such that*

$$u_\varepsilon(t) = a_0 \cos(\omega_\varepsilon t) + \varepsilon v_1(\omega_\varepsilon t) + \mathcal{O}(\varepsilon^2) \qquad \text{in } C^2\left(\left[0, \frac{\gamma}{\varepsilon}\right], \mathbb{R}\right).$$

For the general one dof (degree of freedom) case, the method is also validated in [JRo].

$$\ddot{u} + \omega_0^2 u + \varepsilon a(u-b)_+ = 0, \qquad u_\varepsilon(0) = a_0 + \varepsilon a_1,\ \dot{u}_\varepsilon(0) = 0. \tag{6.11}$$

In general v_1 depends on the initial amplitude when $|a_0| > |b|$.

When $|a_0| = |b|$, we have a different asymptotic expansion only valid for shorter time when the unilateral spring slightly interacts with the mass. It is a new feature.

Proposition 6.1.2 (Grazing unilateral contact, shorter time validity) *Let b be a real number, $b \neq 0$, and consider, the solution u_ε of problem (6.11). If $|a_0| = |b|$ and $|a_0 + \varepsilon a_1| > |b|$ then we have*
$u_\varepsilon(t) = (a_0 + \varepsilon a_1)\cos(\omega_0 t) + \mathcal{O}(\varepsilon^2)$, *for* $t \leq T_\varepsilon = \mathcal{O}\left(\dfrac{1}{\sqrt{\varepsilon}}\right)$.

The proof relies on the following technical statements; To validate previous results we need following lemmas [JRo].

Lemma 6.1.1 (Asymptotic expansion for $(u+\varepsilon v)_+$) *Let $T > 0$. Let u, v be two real valued functions defined on $I = [0, T]$, and let H be the Heaviside step function. Then*

$$(u+\varepsilon v)_+ \quad = \quad (u)_+ + \varepsilon H(u) v + \varepsilon \chi_\varepsilon(u, v), \tag{6.12}$$

where $\chi_\varepsilon(u, v)$ is a non negative piecewise linear function which is 1-Lipschitz with respect to v. Let $M > 0$, $J_\varepsilon = \{t \in I, |u(t)| \leq \varepsilon M\}$, $\mu_\varepsilon(T)$ be the measure of the set J_ε.
If $|v(t)| \leq M$ for any $t \in I$, then

$$|\chi_\varepsilon(u, v)| \leq |v| \leq M, \qquad \int_0^T |\chi_\varepsilon(u(t), v(t))| \, dt \leq M \mu_\varepsilon(T). \tag{6.13}$$

The key point in inequality (6.13) is the remainder $\varepsilon\chi_\varepsilon$ is only of order ε in L^∞ but of order $\varepsilon\mu_\varepsilon$ in L^1. μ_ε can be not better than a constant. This is the case for instance for $u \equiv 0$ in (6.12). Generically, μ_ε is of order ε, and for some critical cases of order $\sqrt{\varepsilon}$.

Lemma 6.1.2 (Order of $\mu_\varepsilon(T)$) *Let u be a smooth periodic function, let M be a positive constant and let $\mu_\varepsilon(T)$ be the measure of the set $J_\varepsilon = \{t \in I, |u(t)| \leq \varepsilon M\}$.*
If u has only simple roots on $I = [0, T]$ then $\mu_\varepsilon(T) \leq C\varepsilon\, T$.
If u has double roots then for some positive C, $\mu_\varepsilon(T) \leq C\sqrt{\varepsilon}\, T$.

6.1.2 N degrees of freedom

We now consider the ODE system with the unknown vector valued function $\tilde{U}(t) \in \mathbb{R}^N$:

$$M\ddot{\tilde{U}} + K\tilde{U} + \varepsilon(\tilde{A}\tilde{U} - B)_+ = 0,$$

where, for each component,

$$[(\tilde{A}U - B)_+]_k = \left(\sum_{j=1}^N \tilde{A}_{kj} u_j - b_k \right)_+ .$$

Here, M is a $N \times N$ mass matrix, K is the stiffness matrix they are both symmetric definite positive. Let us introduce the matrix Φ of generalized eigenvectors: $K\Phi = M\Phi\Lambda^2$ with Λ positive diagonal matrix of eigenvalues $\lambda_1, \cdots, \lambda_N$, $\Phi^T M \Phi = Id$, and set $\tilde{U} = \Phi U$, $A = \tilde{A}\Phi$, the system may be written:

$$\ddot{U} + \Lambda^2 U \quad = \quad -\varepsilon \Phi^T (AU - B)_+. \tag{6.14}$$

NNM: Initial condition near an eigenvector

For the system (6.14), we take an initial condition near an eigenmode of the linearized system (possibly at the cost of renumbering the eigenvalues, the eigenmode of interest has index $k = 1$). We thus have:

$$\left\{ \begin{array}{lcl} u_1^\varepsilon(0) & = & a_0 + \varepsilon a_1, \quad \dot{u}_1^\varepsilon(0) = 0, \\ u_k^\varepsilon(0) & = & 0 \ \ + \varepsilon a_k, \quad \dot{u}_k^\varepsilon(0) = 0, \quad for\ k \neq 1. \end{array} \right. \tag{6.15}$$

We shall impose $a_2, \cdots, a_N$ to have a periodic approximation, but a_1 as well as a_0 are free constants. We use the same time $s = \omega_\varepsilon t$ for each component.

Theorem 6.1.1 (NNM)

The Lindstedt-Poincaré expansion is valid on $(0, T_\varepsilon)$, with $T_\varepsilon \to +\infty$ when $\varepsilon \to 0$ under the non resonance assumption $\{\lambda_2, \cdots, \lambda_N\} \cap \lambda_1\mathbb{Z} = \emptyset$:

$$\left\{ \begin{array}{lclclcll} u_1^\varepsilon(t) & = & v_1^0(\omega_\varepsilon t) & + & \varepsilon v_1^1(\omega_\varepsilon t) & + & \mathcal{O}(\varepsilon^2), & \\ u_k^\varepsilon(t) & = & 0 & + & \varepsilon v_k^1(\omega_\varepsilon t) & + & \mathcal{O}(\varepsilon^2), & k \neq 1, \end{array} \right.$$

where $v_1^0(.)$, ω_1, $v_1^1(.)$, $v_k^1(.)$ and a_k for $k \neq 1$, ω_2 are successively given by the method of strained coordinates, see [JRo].

Furthermore, if $(A_{j1}v_1^0 - b_j)$ has only simple roots for all $j = 1, \cdots, N$, then $T_\varepsilon = O(\varepsilon^{-1})$, otherwise $T_\varepsilon = O(\varepsilon^{-1/2})$.

In Theorem 6.1.1, v_k^1 and a_k are obtained by means of Fourier series [JRo].

Quasi periodic approximation

We adapt the method of strained coordinates when all modes are excited:

$$u_k^\varepsilon(0) = a_k, \quad \dot{u}_k^\varepsilon(0) = 0, \qquad k = 1, \cdots, N.$$

We lose one order of accuracy compared to previous results since each mode does not remain periodic and becomes almost-periodic [5].

Let us define N new time variables $s_k = \lambda_k^\varepsilon t$. We use the following ansatz,

$$\begin{array}{rclcrcl} \lambda_k^\varepsilon & = & \lambda_k^0 + \varepsilon\lambda_k^1, & & \lambda_k^0 & = & \lambda_k, \\ u_k^\varepsilon(t) & = & v_k^\varepsilon(\lambda_k^\varepsilon t) & = & v_k^\varepsilon(s_k) & = & v_k^0(s_k) + \varepsilon r_k^\varepsilon(s_k). \end{array}$$

The function v_k^0 are easily obtained by the linearized equation. Indeed, the nonlinear effect is measured for large time by $(\lambda_k^1)_{k=1}^N$.

Theorem 6.1.2 (All modes)
If $\lambda_1, \cdots, \lambda_N$ are $\mathbb{Z}$ independent, then, for any $T_\varepsilon = o(\varepsilon^{-1})$ which means

$$\lim_{\varepsilon \to 0} T_\varepsilon = +\infty, \quad \text{and} \quad \lim_{\varepsilon \to 0} \varepsilon \times T_\varepsilon = 0,$$

we have for all $k = 1, \cdots, N$,

$$\lim_{\varepsilon \to 0} \| u_k^\varepsilon(t) - v_k^0\,(\lambda_k^\varepsilon\; t)\,\|_{W^{2,\infty}(0,T_\varepsilon)} = 0$$

where $\lambda_k^\varepsilon = \lambda_k + \varepsilon\lambda_k^1$, $v_k^0(s) = a_k \cos(s)$, and λ_k^1 is defined by:

$$\lambda_k^1 \;=\; \frac{1}{2\lambda_k a_0} \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left(\sum_{j=1}^N a_{kj} v_j^0 \left(\frac{\lambda_j^0}{\lambda_k^0} s_k \right) - b_k \right)_+ \cos(s) ds.$$

Notice that if only one mode is excited $T_\varepsilon \sim \dfrac{1}{\varepsilon}$ owing to Theorem 6.1.1.

6.2 A nonlinear crack impacted by an acoustic wave

The modeling of interactions between ultrasonic waves and cracks is of great interest in many fields of applied mechanics. When the wavelengths are much larger than the width of the cracks, the latter are usually replaced by zero-thickness interfaces and appropriate jump conditions. Linear models for crack-face interactions do not prevent the nonphysical penetration of crack faces. A realistic hyperbolic model accounting for the finite compressibility of crack faces under normal loading conditions has been presented for applications in engineering [2] and geomechanical contexts [3].

In [16], numerical experiments have brought to light an interesting property of such model. In the case of a sinusoidal incident wave, the mean values of the elastic displacements around a crack were found to be discontinuous. The simulations also indicate that the mean dilatation of the crack increases with the amplitude of the forcing levels. This purely nonlinear phenomenon (the linear models are not able to predict dilatation) is of physical interest: experimenters can measure the dilatation [14] and use the obtained data to deduce the properties of the crack.

Physical modeling

We consider the case of a single crack with rough faces separating two media Ω_0 and Ω_1, which are both linearly elastic and isotropic. Let ρ denote the density and let c denote the elastic speed of the compressional waves. These parameters are piecewise constant and may be discontinuous on the crack: $(\rho_0,\, c_0)$ if $x \in \Omega_0$, $(\rho_1,\, c_1)$ if $x \in \Omega_1$. The media are subject to a constant static stress p. At rest, the distance between the planes of average height is $\xi_0(p) > 0$ (Figure 6.3, left).

Elastic compressional waves are emitted by a singular source of stress at $x = x_s < \alpha$ in Ω_0, where α is a median plane of the actual flaw surface. The wave impacting α gives rise to reflected (in Ω_0) and transmitted (in Ω_1) compressional waves. These perturbations in Ω_0 and Ω_1 are described by the 1-D elastodynamic equations

$$\rho\,\frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}, \qquad \frac{\partial \sigma}{\partial t} = \rho\, c^2\,\frac{\partial v}{\partial x} + S(t)\,\delta(x - x_s), \tag{6.16}$$

where $S(t)$ denotes the causal stress source, $v = \frac{\partial u}{\partial t}$ is the elastic velocity, u is the elastic displacement, and σ is the elastic stress perturbation around p. The dynamic stresses induced by the elastic waves affect the thickness $\xi(t)$ of the crack (Figure 6.3, right). The constraint

$$\xi = \xi_0 + [u] \geq \xi_0 - d > 0 \tag{6.17}$$

must be satisfied, where $[u] = u^+ - u^-$ is the difference between the elastic displacements on the two sides of the crack and $d(p) > 0$ is the *maximum allowable closure* [3]. We also assume that the wavelengths are much larger than ξ, so that the propagation time across the crack is neglected, the latter being replaced by a zero-thickness *interface* at $x = \alpha$: $[u] = [u(\alpha,\, t)] = u(\alpha^+,\, t) - u(\alpha^-,\, t)$.

Two independent jump conditions at α need to be defined to obtain a well-posed problem. The discontinuity of σ is proportional to the mass of the interstitial medium

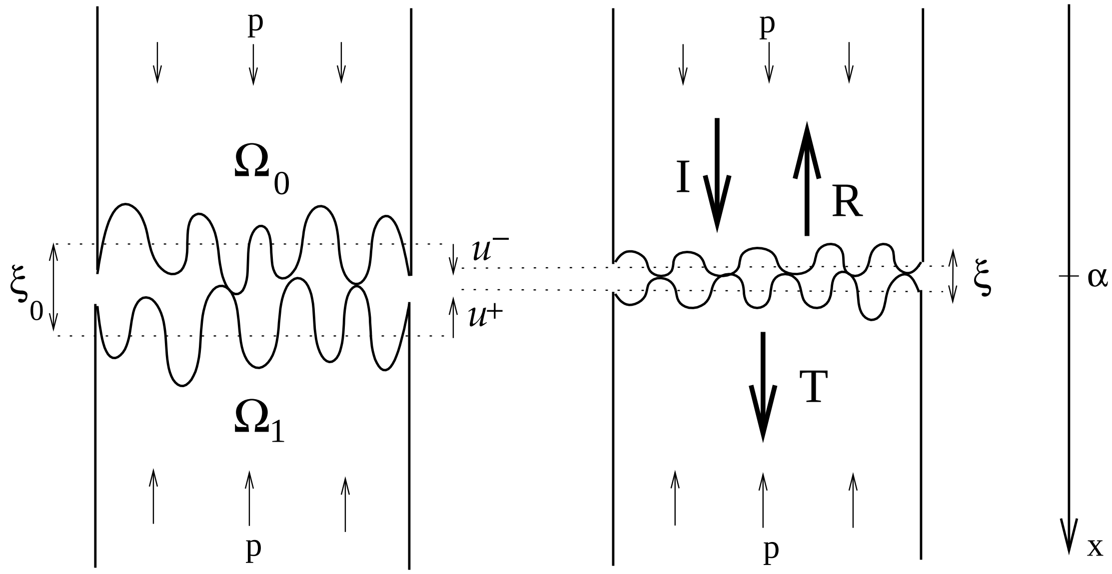

Figure 6.3: Elastic media Ω_0 and Ω_1 with rough contact surfaces, under constant static stress p. Static (left) and dynamic (right) case, with incident (I), reflected (R), and transmitted (T) waves.

between Ω_0 and Ω_1 [21]. Since the crack is dry and contains only air, the density in the crack is much smaller than ρ_0 or ρ_1, while the elastic stress is assumed to be continuous:

$$[\sigma(\alpha,\, t)] = 0 \;\Rightarrow\; \sigma(\alpha^+,\, t) = \sigma(\alpha^-,\, t) = \sigma^*(t). \tag{6.18}$$

Establishing the second jump condition is a more complex task. Experimental and theoretical studies have shown that u is discontinuous and that the discontinuity is proportional to the applied stress. The linear model has often been considered:

$$\sigma^*(t) \;\; = \;\; K\, [u(\alpha,\, t)]\,, \tag{6.19}$$

where K is the *interfacial stiffness*. Welded conditions $[u(\alpha,\, t)] = 0$ are obtained as $K \to +\infty$. However, the linear condition (6.19) violates (6.17) under large compression loading conditions: $\sigma^*(t) < -K\, d \Rightarrow \xi < \xi_0 - d$. The linear condition (6.19) is therefore realistic only with very small perturbations. With larger perturbations, a nonlinear jump condition is required.

To design a relevant condition, it should be noted that compression loading increases the surface area of the contacting faces. Therefore a smaller stress is needed to open a crack than to close. An infinite stress is even required to close the crack completely. In addition, the constraint (6.17) must be satisfied, and the model must comply with (6.19) in the case of small stresses. Lastly, concave stress-closure relations have been observed experimentally [19]. Dimensional analysis shows that the general relation

$$\sigma^*(t) \;\; = \;\; K\, d\, \mathcal{F}\left([u(\alpha,\, t)]/d\right) \tag{6.20}$$

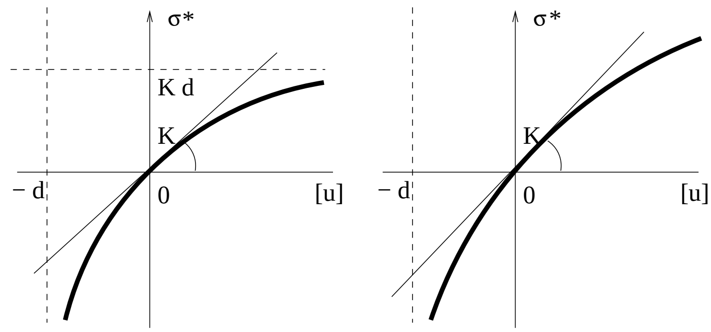

Figure 6.4: Sketch of the nonlinear relation between the stress and the jump of the elastic displacement at α. Left row: model 1 (6.22), right row: model 2 (6.22).

is suitable, where $\mathcal{F}$ is a smooth increasing concave function

$$\begin{aligned} &\mathcal{F} :]-1, +\infty[\rightarrow]-\infty, \mathcal{F}_{\max}[, \quad \lim_{X \to -1} \mathcal{F}(X) = -\infty, \quad 0 < \mathcal{F}_{\max} \leq +\infty, \\ &\mathcal{F}(0) = 0, \quad \mathcal{F}^{'}(0) = 1, \quad \mathcal{F}^{''} < 0 < \mathcal{F}^{'}. \end{aligned} \tag{6.21}$$

More specifically, we can use the following two models (6.20):

$$\text{Model 1: } \mathcal{F}(X) = \frac{X}{1+X}, \qquad \text{Model 2: } \mathcal{F}(X) = \ln(1+X). \tag{6.22}$$

These two models are sketched in Figure 6.4. The straight line with a slope K tangential to the curves at the origin gives the linear jump conditions (6.19).

6.2.1 Numerical experiments

Here, we describe the influence of the nonlinear jump condition (6.20) on the wave scattering. For this purpose, we consider a single nonlinear crack and a sinusoidal source S. The amplitude v_0 of the incident elastic velocity ranges from 10^{-4} m/s to $5\,10^{-3}$ m/s. The maximal amplitude corresponds to a maximal strain $\varepsilon = v_0/c_0 \approx 10^{-6}$, so that the linear elastodynamic equations (6.16) are always valid [1]. The linear first-order hyperbolic system (6.16) and the jump conditions (6.18) and (6.20) are solved numerically on an $(x,\, t)$ grid. For this purpose, a fourth-order finite-difference scheme is combined with an immersed interface method to account for the jump conditions [16]. At each time step, numerical integration of v also gives u.

Figure 6.5 shows snapshots of u after the transients have disappeared and the periodic regime has been established. The mean values of the incident and reflected displacements $(x < \alpha)$ and the transmitted displacement $(x > \alpha)$ are given by horizontal dotted lines. With $v_0 = 10^{-4}$ m/s (a), these mean values are continuous across α. At higher amplitudes, a positive jump from α^- to α^+ is observed. This jump, which amounts to a mean dilatation of the crack, also increases with v_0 (b,c,d).

These properties can be more clearly seen in Figure 6.6, where the time history of $[u]$ is shown, with $v_0 = 2\,10^{-3}$ m/s (a) and $v_0 = 5\,10^{-3}$ m/s (b). It can also be seen from this figure that the maximum value of $[u]$ increases with v_0, whereas the minimum

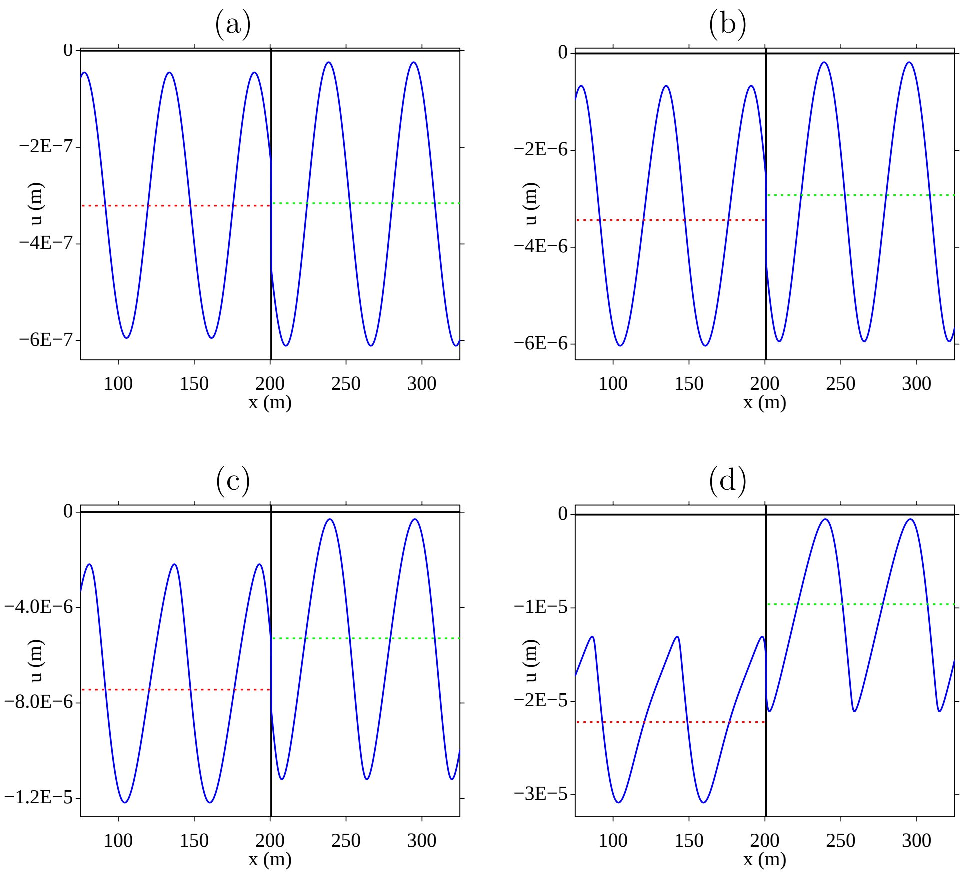

Figure 6.5: Model 1 (6.22): snapshots of the elastic displacement for various amplitudes v_0 of the incident elastic velocity: 10^{-4} m/s (a), 10^{-3} m/s (b), $2\,10^{-3}$ m/s (c), and $5\,10^{-3}$ m/s (d). The vertical solid line denotes the location $\alpha = 200$ m of the crack. The dotted horizontal lines denote the mean value of the elastic displacement on both sides of α.

value of $[u]$ decreases and is bounded by $-d$, as required by (6.17). These numerical findings will be analyzed for general nonlinearity $\mathcal{F}$ that fulfills (6.21) in the following sections.

6.2.2 Mathematical model

Link with an ODE

To explain the numerical findings, we look for an evolution equation satisfied by $[u(\alpha, t)]$. Considering the elastic wave emitted by the source S in (6.16), which impacts a single crack modeled by (6.18) and (6.20), leads to the following proposition.

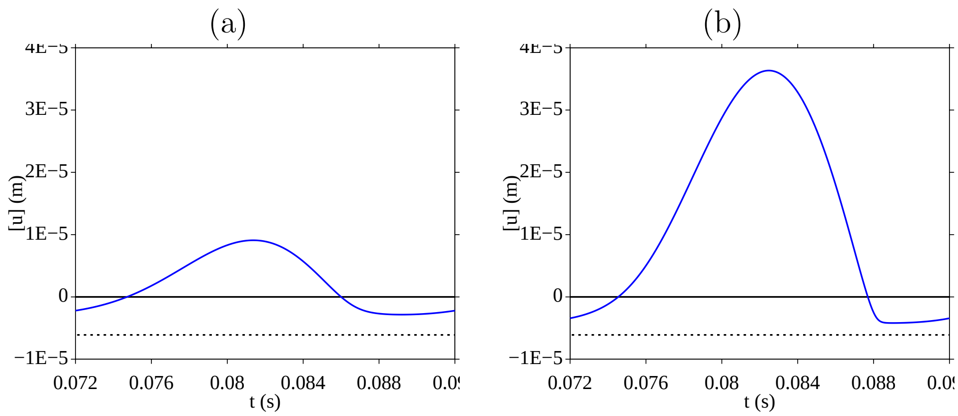

Figure 6.6: Time history of $[u(\alpha, t)]$ at two incident elastic velocities: $v_0 = 2\,10^{-3}$ m/s (a) and $v_0 = 5\,10^{-3}$ m/s (b). The horizontal dotted line denotes $-d$.

Proposition 6.2.1 *Let us define the normalized jump of displacement $y = [u(\alpha, t)]/d$. Let $f = -\mathcal{F}$ such that $\mathcal{F}$ satisfies (6.21), $A\,g(t)$ a $T-$periodic forcing. Then the problem amounts to:*

$$\begin{cases} \dfrac{d\,y}{d\,t} = f(y) + A\,g(t), \qquad \omega > 0, \qquad 0 \leq A < +\infty, \\ f:]-1, +\infty[\rightarrow]f_{\min}, +\infty[, \quad \lim\limits_{y \to -1} f(y) = +\infty, \quad -\infty \leq f_{\min} < 0, \\ f(0) = 0, \quad f^{'}(0) = -1, \quad f^{'}(y) < 0 < f^{''}(y), \\ y(0) = y_0 \in]-1, +\infty[. \end{cases} \tag{6.23}$$

Proposition 6.2.2 *There is a unique T-periodic solution $Y(t)$ of (6.23). This solution is asymptotically stable.*

More precise results are given for sinusoidal forcing in [JuLo1], see also Figure 6.6.

Mean dilatation of the crack

As stated in the introduction, the main aim of this study was to justify that a jump in the mean elastic displacement occurs around the crack, as observed numerically and experimentally [14]. This jump amounts to a mean dilatation of the crack. The next theorem addresses this typically nonlinear phenomenon.

Theorem 6.2.1 *The mean value of the T-periodic solution Y in (6.23) is positive and increases strictly with the forcing amplitude:*

$$\overline{Y} > 0, \qquad \frac{\partial\,\overline{Y}}{\partial\,A} > 0.$$

At small forcing levels, the following local estimate holds:

$$\overline{Y} = \frac{f^{''}(0)}{4}\,\frac{A^2}{1+\omega^2} + \mathcal{O}(A^4). \tag{6.24}$$

The mean dilatation is a physical observable related to a physical constant characterizing the crack: $f''(0)$. We also studied the maximum closure of the crack in [JuLo1].

6.3 Two nonlinear cracks

In this section, we study the interaction between acoustic waves and 2 cracks. We keep the notations of section 6.2. For two cracks the problem becomes more intricate because of reflexions between the cracks, loss of exponential stability, and the occurrence of small divisors.

Physical modeling

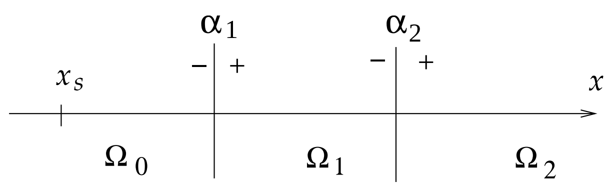

Figure 6.7: Cracks at α_1 and α_2 in elastic media Ω_i; source at $x_s < \alpha_1$.

Let us take two cracks at α_1 and α_2 in the linearly elastic media Ω_0, Ω_1 and Ω_2 (figure 6.7). The density ρ and the elastic speed of the compressional waves c are positive, piecewise constant, and they may be discontinuous across the cracks.

Elastic compressional waves are emitted by a source at $x = x_s < \alpha_1$ in Ω_0, and then they are diffracted by the cracks. Wave propagation is modeled by the 1-D linear elastodynamics (6.16) with boundary conditions (6.18), (6.20) at each crack $x = \alpha_k$. The stress law $\mathcal{F}_k$ depends on the crack $k = 1, 2$.

Scattered waves

In a first series of numerical experiments, we focus on the dynamical behavior of the scattered waves. A source at x_s emits a purely sinusoidal wave with an amplitude of elastic velocity v_0.

Figure 6.8 shows snapshots of u at $t_f = 0.34$ s, obtained by time-domain methods. The mean spatial values $\overline{u}_k$ of the displacements in each of the subdomains are denoted by horizontal dotted lines. We are led to the following observations:

- For large time a periodic solution is found to be reached in each subdomain;
- the scattered displacements are distorted when the forcing level increases;
- null or positive jumps in $\overline{u}_k$ occur across the cracks; null jumps occur at small forcing levels v_0 (a), whereas the jumps increase with the forcing level (b,c,d).

The positive jumps in $\overline{u}_k$ mentioned above are equal to the mean temporal value of jump, and they amount to a mean dilatation of the crack at α_k.

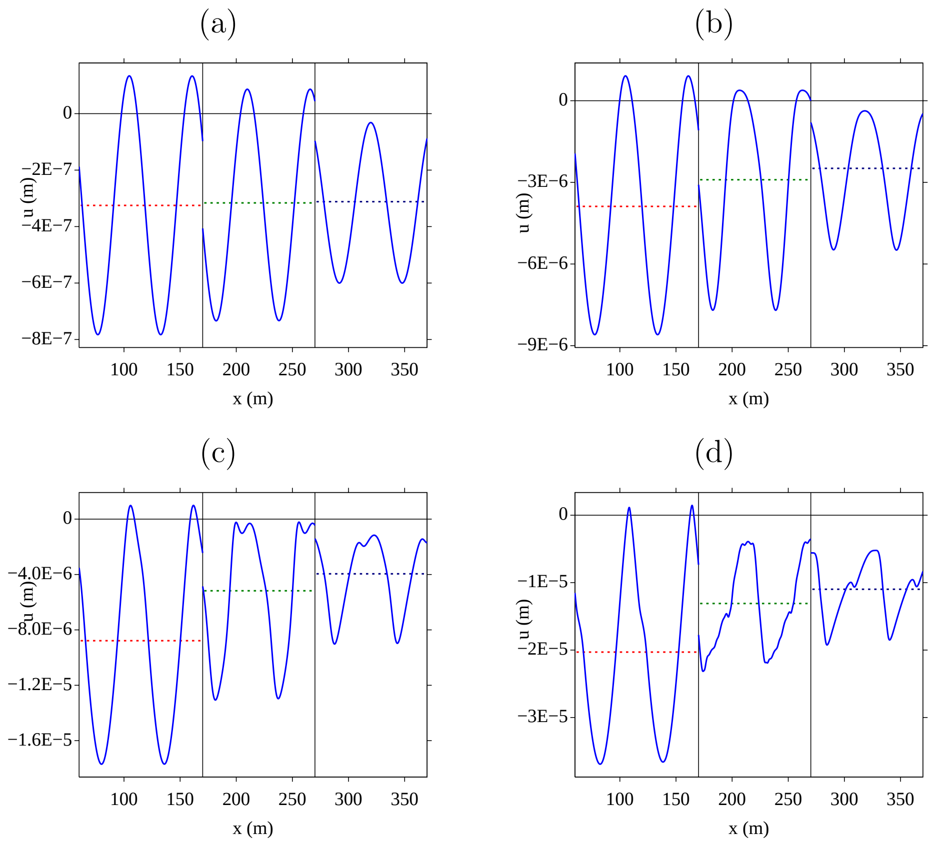

Figure 6.8: Snapshots of the elastic displacement u obtained with model 1 (6.22) and various amplitudes v_0 of the incident elastic velocity: 10^{-4} m/s (a), 10^{-3} m/s (b), $2\,10^{-3}$ m/s (c) and $5\,10^{-3}$ m/s (d). The vertical solid lines denote the locations of the cracks. The red, green and navy dotted horizontal lines denote the mean spatial value $\overline{u}$ of the elastic displacement in each subdomain.

6.3.1 Reduction to a neutral delay differential system

We want to study the solution of the boundary-value problem (6.16), (6.18), (6.20) with two cracks. For this purpose, we focus on the jumps in the elastic displacements across the cracks: $Y_k(t) = [u(\alpha_k, t)]$, where $k = 1, 2$. Based on the method of characteristics, all the scattered fields can be deduced from the known 1-D Green's function and from these $Y_k(t)$. Moreover, it is simpler to study two functions of t than the solution of partial differential equations.

The time histories of Y_k are presented in the left row of figure 6.9. Without surprise, Y_k are null until the wave emitted by the source has reached the first crack at α_1, and subsequently, at α_2. At small forcing levels (a), Y_k are sinusoids centered around a null mean value. As v_0 increases, Y_k are distorted and centered around an increasing positive mean value. The periodic regime also takes longer to be established. Lastly, it is observed that the minimum values of Y_k decrease and are bounded below by $-d_1 = -d_2$, as shown in (e); this property is required by the model.

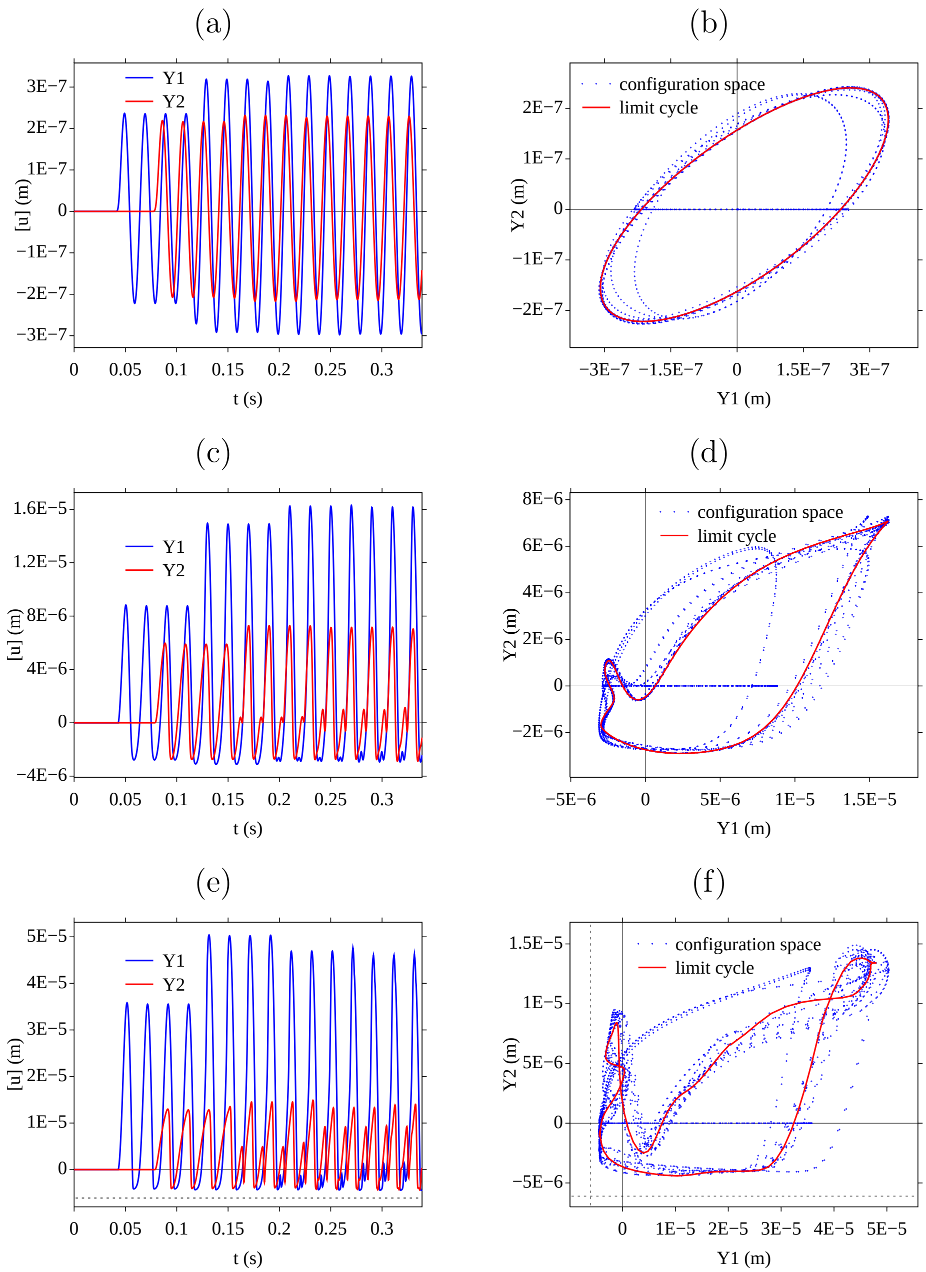

Figure 6.9: Time histories of $Y_k = [u(\alpha_k, t)]$ (left row) and configuration space (Y_1, Y_2) (right row) in the case of model 1 (6.22) and various amplitudes v_0 of the incident elastic velocity: 10^{-4} m/s (a-b), $2\,10^{-3}$ m/s (c-d) and $5\,10^{-3}$ m/s (e-f). In (e-f), the dotted lines denotes $-d_1 = -d_2$.

The configuration space (Y_1, Y_2) is shown in the right row of figure 6.9. The data between $[t_f - T, t_f]$ have been extracted and presented in the form of a continuous red line: at the tested forcing levels, this line is closed, which indicates that a periodic limit cycle has been reached. At small forcing levels (b), a centered elliptic limit cycle is observed, as predicted by the linear theory of oscillators. As v_0 increases, the limit cycle becomes more complex, and even crosses itself (f).

The system is driven by the following quantities.

$$\tau = \frac{\alpha_2 - \alpha_1}{c_1}, \qquad \beta_k = K_k \left(\frac{1}{\rho_k\, c_k} + \frac{1}{\rho_{k-1}\, c_{k-1}} \right), \tag{6.25}$$

$$\gamma_k = (-1)^k K_k \left(\frac{1}{\rho_k\, c_k} - \frac{1}{\rho_{k-1}\, c_{k-1}} \right). \tag{6.26}$$

Here τ is interpreted as the time needed for the wave to travel from a crack to the other. The inequality $0 \leq |\gamma_k| < \beta_k$ comes from their definition and is of crucial importance in following analysis.

In the next proposition, the problem (6.16), (6.18), (6.20) is reduced to a nonlinear *neutral delay differential system*.

Proposition 6.3.1 *The normalized jumps* $y_k(t) = \dfrac{[Y_k(t)]}{d_k}$ *satisfy:*

$$\left\{ \begin{array}{ll} y_1'(t) + y_2'(t-\tau) = \beta_1\, f_1(y_1(t)) + \gamma_2\, f_2(y_2(t-\tau)) + s(t), & t > 0, \\ y_2'(t) + y_1'(t-\tau) = \beta_2\, f_2(y_2(t)) + \gamma_1\, f_1(y_1(t-\tau)) + s(t-\tau), & t > 0, \\ y_k(t) = \phi_k(t) \in C^1\left([-\tau,\, 0],\,]y_{k\,\min},\, +\infty[\right), & -\tau \leq t \leq 0. \end{array} \right. \tag{6.27}$$

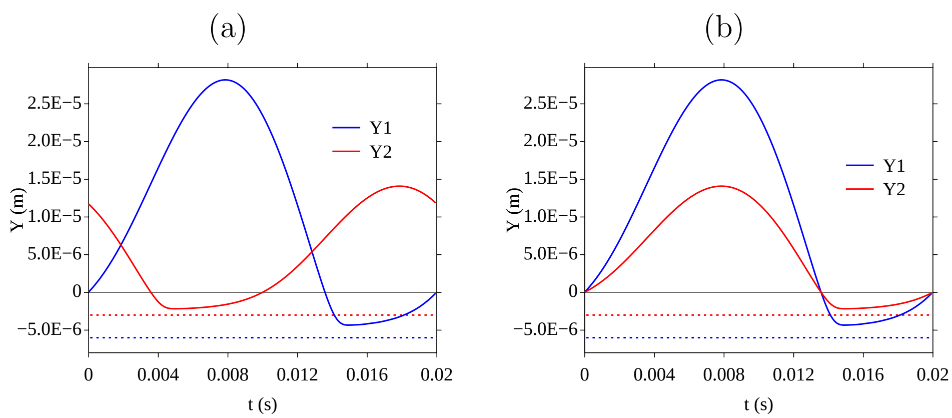

Figure 6.10: Time history of Y_1 and Y_2 during one period, with $\theta = 0.5$ (a) and $\theta = 1$ (b): see theorem 6.3.1. Parameter $v_0 = 5\,10^{-3}$ m/s. The horizontal blue and red dotted lines denote the lower bounds $-d_1$ and $-d_2$, respectively.

Assumptions and notations are defined as follows ($k = 1,\, 2$):

$$\begin{array}{l} \beta_k > 0, \quad 0 \leq |\gamma_k| < \beta_k, \quad y_{k\,\min} < 0, \\ f_k \in C^2\left(]y_{k\,\min},\, +\infty[\rightarrow]f_{k\,\min},\, +\infty[\right), \quad \displaystyle\lim_{y \to y_{k\,\min}} f_k(y) = +\infty, \quad -\infty \leq f_{k\,\min} < 0, \\ f_k(0) = 0, \quad f_k'(0) = -1, \quad q_k = \dfrac{f_k''(0)}{2} > 0, \quad f_k'(y) < 0 < f_k''(y), \\ s \in C^0(\mathbb{R}), \quad s(t+T) = s(t), \quad t > 0, \quad T > 0, \quad \omega = 2\,\pi\,/\,T. \end{array} \tag{6.28}$$

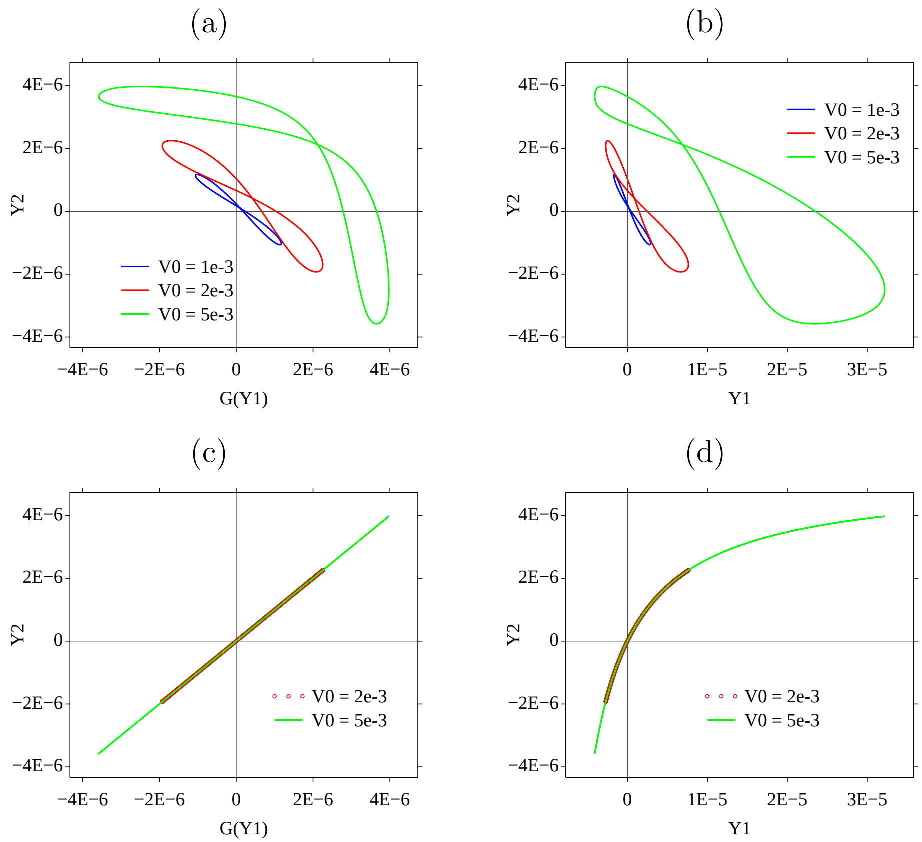

Figure 6.11: Configuration space Γ_G (a-c) and Γ_I (b-d) with $\theta = 0.5$ (a-b) and $\theta = 1$ (c-d): see theorem 6.3.1.

Lastly, the ratio between the travel time and the period of the source is introduced

$$\theta = \frac{\tau}{T} = \frac{\alpha_2 - \alpha_1}{c_1\, T} = \frac{\alpha_2 - \alpha_1}{\lambda_1} > 0, \tag{6.29}$$

where λ_1 is the wavelength in medium Ω_1.

6.3.2 Large periodic solutions for special periods of the source

Special cases occur when twice the travel time of the waves between the cracks coincides with a period of the source. That is to say $2\,\theta$ is an integer, where $\theta = \tau/T$ (6.29). In this case, we have existence and uniqueness of large periodic solution. In this case 2τ is a period of the periodic source s by (6.29). This amounts to say that Ω_1 contains n wavelengths of the source if $\theta = n$. Figures 6.10 and 6.11 illustrate the periodic solutions.

Theorem 6.3.1 *If $2\,\theta \in \mathbb{N}^*$, then there exists a unique periodic solution $\mathbf{y} = (y_1, y_2)^T$ to (6.27)-(6.28) having the same period T as the source. In addition, the increasing diffeomorphism*

$$G(y) = f_2^{-1}\left(\frac{\beta_1 - \gamma_1}{\beta_2 - \gamma_2}\, f_1(y)\right) \equiv f_2^{-1}\left(\frac{K_1}{K_2}\, f_1(y)\right), \qquad G' > 0, \qquad G(0) = 0,$$

ensures that

$$\text{if } \theta = n, \quad n \in \mathbb{N}^*, \text{ then } \qquad y_2(t) = G\left(y_1(t)\right), \tag{6.30}$$

$$\text{if } \theta = n + 1/2, \quad n \in \mathbb{N}^*, \text{ then } \quad y_2(t) = G\left(y_1(t - T/2)\right). \tag{6.31}$$

Lastly, y_1 and y_2 satisfy a scalar non autonomous ODE without any delay as in [JuLo1].

6.3.3 Periodic solution: small forcing

Now, we investigate a larger set of ratios $\theta = \tau\,/\,T$, but restricted to small forcing and a Diophantine condition:

Definition 6.3.1 (Diophantine condition $\mathcal{D}$) *Let*

$$\zeta = \frac{\beta_1 - \gamma_1 + \beta_2 - \gamma_2}{4\,\pi^2}\,\tau > 0. \tag{6.32}$$

A real number $w \in \mathbb{R}^+$ does not satisfy the condition $\mathcal{D}$ if and only if there exists an infinite number of integers $(n,\, k)$ such that

$$2\,w = \frac{n}{k} + \frac{\zeta}{w}\,\frac{1}{k^2} + o\left(\frac{1}{k^2}\right). \tag{6.33}$$

Condition $\mathcal{D}$ in definition 6.3.1 is stated as a negative statement. Condition $\mathcal{D}$ is optimal to avoid a small divisor problem ([JuLo2]). An example of a dense set satisfying $\mathcal{D}$ is $\mathbb{Q}$. Assuming that $\mathcal{D}$ is true and considering a small source s, the following local existence and uniqueness results are obtained. Here H^1_T stands for the Sobolev space of T-periodic functions which are square integrable on a period, as well as their first order derivative.

Theorem 6.3.2 (Existence and uniqueness under a Diophantine condition) *Let $\theta = \tau\,/\,T$ satisfy the Diophantine condition $\mathcal{D}$. Then there exists a neighborhood $V_0 \times V_1 \times V_2$ of the origin in $\left(H^1_T\right)^3$ such that for any s in V_0, there exists a unique periodic solution $\mathbf{y} = (y_1,\, y_2)^T = \mathbf{\Psi}(s)$ to (6.27)-(6.28) on $V_1 \times V_2$. This solution has the same period T as the source, and $\mathbf{\Psi} \in C^1\left(V_0,\, V_1 \times V_2\right)$.*

The proof is quite technical. It relies on a fine analysis of the spectrum of the linearized operator and the delicate estimates of the norm of its inverse.

6.3.4 Mean dilatations of the cracks

The mean dilatation of the cracks $\overline{Y_k}$ has been studied in[JuLo2] with respect to the forcing amplitude and also to $\theta = \tau/T$. The mathematical validation is limited to small forcing by a perturbation analysis. Figures 6.12 and 6.13 illustrate this dependency. Qualitatively, for small amplitude, the dependance of the mean on the amplitude is similar to the one crack case [JuLo1]. For large amplitude, we only know the positivity of the mean.

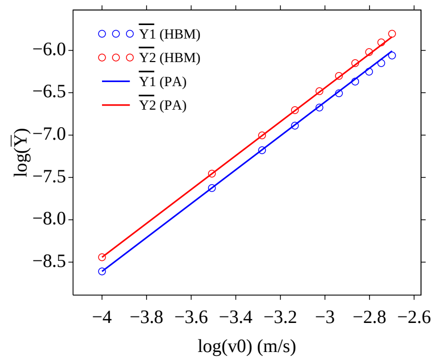

Figure 6.12: Log-log evolution of $\overline{Y_k}$ in terms of the forcing amplitude v_0. Circle: harmonic balance method. Solid line: perturbation analysis [JuLo2].

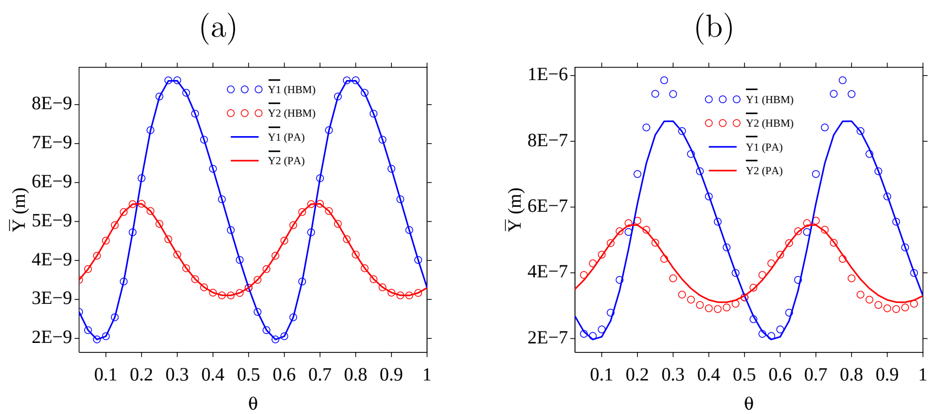

Figure 6.13: Evolution of $\overline{Y_k}$ in terms of the ratio $\theta = \tau\,/\,T$. (a): $v_0 = 10^{-4}$ m/s; (b): $v_0 = 10^{-3}$ m/s. Circle: harmonic balance method. Solid line: perturbation analysis.

6.4 A neutral delay differential equation

In order to understand the features of the model, we consider a scalar nonlinear neutral delay differential equation closely related to the system (6.27). Equation (6.34) is typically issued from hyperbolic partial differential equations with nonlinear boundary conditions [7]. Indeed, equation (6.34) concentrates the main difficulties encountered for two nonlinear cracks in section 6.3.

We look for the solution $y \in H^1_{loc}((-1,\,+\infty),\mathbb{R})$ of

$$\left\{ \begin{array}{l} y^{'}(t) + c\,y^{'}(t-1) + f(y(t)) + g(y(t-1)) = s(t), \quad t > 0, \\ y(t) = y_0(t) \in H^1((-1,\,0),\mathbb{R}), \quad -1 < t < 0, \end{array} \right. \tag{6.34}$$

under the following assumptions:

$$c = \pm 1, \quad f \in C^1(\mathbb{R},\,\mathbb{R}), \;\; f^{'} > 0, \quad f(0) = 0, \quad |g| \leq \gamma\,|f|, \;\; \text{with } 0 \leq \gamma < 1. \tag{6.35}$$

We are also interested by a periodic source term: $s(t+T) = s(t), \quad T = \dfrac{2\,\pi}{\omega} > 0.$

A natural issue when dealing with (6.34) is to prove existence, uniqueness and stability of periodic solutions. Previous works usually address this question by considering a linear version of (6.34) with $|c| < 1$. The critical value $|c| = 1$ raises technical difficulties: the spectrum of the linearized operator is asymptotically close to the left of the imaginary axis, hence exponential stability is lost and small divisors may be encountered.

An analysis of the case $|c| = 1$ dates back to [4, 6]. In the linear case, algebraic rate of convergence is proven. In the nonlinear case with $c = -1$, algebraic convergence is also proven in this reference, but assuming small C^1 data. In both cases, the main tools involve asymptotic expansions of characteristic roots, Laplace transforms and function series. A more recent contribution for linear degenerate retarded differential systems is [26].

Our aim is to push forwards the analysis of (6.34), first by a nonlinear stabilty result. Under the previous assumptions (6.35), we prove the nonlinear stability of the 0 equilibrium state.

Theorem 6.4.1 (Asymptotic stability of the zero solution) *Let y be the solution of (6.34) without source term: $s = 0$. If conditions* (6.35) *are satisfied, then* $\lim\limits_{t\to+\infty} y(t) = 0.$

The stability is proven by an energy method allowing convergence towards zero with an algebraic rate. Indeed, the equilibrium $y = 0$ is not exponentially stable, as we will see in the linear case. This theorem can be extended to the case of a constant source. For a periodic source, the asymptotic stability holds with a restricted stability condition.

Proposition 6.4.1 (Conditional asymptotic stability of a periodic solution) *Let y be the solution of (6.34) with $f \in C^2$ and $g = 0$, and P be a periodic solution. Let $I = conv(y[-1, +\infty[, P(\mathbb{R}))$ be the convex hull of the two sets. If I is bounded and the condition*

$$2\,\sup|P^{'}| < \inf_{w\,\in\,I,\,d\,\in\,P(\mathbb{R})} \frac{(f(w) - f(d))^2}{\left|f(w) - f(d) - (w-d)\,f^{'}(d)\right|} \tag{6.36}$$

is satisfied, then $\lim\limits_{t\to+\infty} (y(t) - P(t)) = 0.$

We also consider the linear NDDE issued from (6.34)

$$y^{'}(t) + c\,y^{'}(t-1) + a\,y(t) + b\,y(t-1) = s(t). \tag{6.37}$$

When $s \equiv 0$, the stability of 0 in (6.37) is analyzed in terms of $a = f^{'}$ and $b = g^{'}$. The cases $|c| < 1$ and $|c| > 1$ are well known in the literature, yielding asymptotic stability and instability, respectively [7]. We focus therefore on the critical case $|c| = 1$ for linear scalar differential-difference equations discussed in [4], p. 191. In this book, it is said that the stability is too much intricate, so the model would never be used by engineers !

This analysis has two interests. Firstly, it highlights the optimality of the energy analysis performed in theorem 6.4.1: as in the nonlinear case, where $|g| < f$ and $f^{'} > 0$, asymptotic stability is proven iff $|b| < a$. Secondly, the lost of exponential stability is explicitly exhibited.

Finally, the nonlinear case with periodic or almost periodic source is studied. A Diophantine condition appears again. More precise results than in [JuLo2] are given in [JuLo3]. In [JuLo3], some arguments seem to indicate that the Diophantine condition is satisfied for almost all period T of the source, but it is still an open problem.

6.5 Prospects

Many prospects are associated with these problems.

6.5.1 N cracks impacted by an acoustic waves

With Bruno Lombard, we wish to study the N cracks problem. We have already derived the $N \times N$ delay differential system for N cracks from the linear elasticity P.D.E. and from the nonlinear boundary conditions at each interface. To simplify the exposition, for an homogeneous medium, N identical cracks, and without source term, the system is:

$$\sum_{j=1}^{N} Y_j'(t - \tau_{ij}) \;=\; 2f(Y_i(t)), \qquad i = 1, \cdots, N.$$

τ_k is the positive travel time between the crack number $k-1$ and the crack number k. τ_{ij} is the travel time between the crack number i and the crack number j. Notice that, if $i < j$ then $\tau_{ij} = \tau_i + \cdots + \tau_j > 0$, $\tau_{ji} = \tau_{ij}$ and also $\tau_{ii} = 0$.

Set $Y = (Y_1, \cdots, Y_N)^\perp$ and $f(Y) = (f(Y_1), \cdots, f(Y_N))^\perp$, the system is shortly written:

$$L\,Y \;=\; 2\,f(Y). \tag{6.38}$$

The operator L as an infinite explicit spectrum on $\iota\,\mathbb{R}$, ($\iota^2 = -1$). Typically, the linearized operator of nonlinear delay differential system (6.38) is $L + Id$. Unfortunately, the spectrum of $L + Id$ is not to be explicitly known. We conjecture that the spectrum is on the left side of the imaginary axis and asymptotically closed to the imaginary axis. To prove this assessment one can use the energy of the initial P.D.E. problem and asymptotic expansions of the eigenvalues for large frequencies.

The nonlinear stability of the state $Y = (0, \cdots, 0)^\perp$ can also be investigated following [JuLo3]. Of course,with a periodic source term, a Diophantine condition occurs for this problem. A Nash-Moser method can be used to prove that a periodic solution is solution of (6.38) with a periodic source term.

Another research concerns the homogenization of (6.38) when $N \to +\infty$. Formally, we obtain the 2×2 hyperbolic system of the nonlinear elasticity.

6.5.2 Unilateral contact

Last year Mathias Legrand (Structural Dynamics and Vibration Laboratory, Mc Gill University) contacted us for an industrial problem in aeronautic with unilateral contact. This work will likely be the subject of further collaborations on nonlinear normal mode with Sokly Heng from Phnom Penh, my former Master student.

6.5.3 Time reversal method

With Bernard Rousselet, I supervised several Master Theses on the time reversal method.

http://math.unice.fr/ junca/Research/PDE.html

This subject has been introduced by Mathias Fink for ultrasonic waves with many concrete applications. It is a way to focalize on the source location the energy emitted by the source. This is possible thanks to the reversibility of the wave equation. With the students, we studied the one dimensional case for the string equation and the Klein-Gordon equation with various boundary conditions. We understood that dissipative boundary conditions are necessary to a sharp focus on the source. A first nonlinear approach can be the one nonlinear crack model with a bump or a Dirac measure as a source. The exponential stability of the 0 state for the model studied in [JuLo1] will be a key ingredient for the validity of the time reversal method.

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