Somme connesse generalizzate per problemi della geometria

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SOMME CONNESSE GENERALIZZATE PER PROBLEMI DELLA GEOMETRIA

SOMMES CONNEXES GÉNÉRALISÉES POUR DES PROBLÈMES ISSUS DE LA GÉOMÉTRIE

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Introduzione

1 Somme connesse generalizzate

In questi ultimi due decenni le tecniche di somma connessa, basate essenzialmente su strumenti di natura analitica, hanno permesso di fare importanti progressi nella comprensione di svariati problemi non lineari derivati dalla geometria (studio di metriche a curvatura scalare costante in geometria Riemanniana [16], [21], [24], metriche autoduali [31], metriche con gruppi di olonomia speciali [17], [20], metriche estremali in geometria Kähleriana [2], [3], equazioni di Yang-Mills [11], studio di ipersuperfici minime [34] e di superfici a curvatura media costante [22],[23], metriche di Einstein [1],...). Queste tecniche si sono rivelate essere uno strumento potente per dimostrare l'esistenza di soluzioni di problemi altamente non lineari.

La somma connessa (ossia l'aggiunta di un manico) è un'operazione topologica che consiste nel prendere due varietà M_1 e M_2, rimuovere da ciascuna di esse una piccola palla geodetica e identificare i bordi (i.e., due sfere) che si sono formati al fine di ottenere una nuova varietà $M_1 \sharp M_2$ che, in generale, sarà topologicamente diversa dalle due varietà iniziali. Più in generale si può considerare la sommma connessa di due varietà M_1 ed M_2 lungo una sottovarietà K (somma connessa generalizzata). In questo caso si rimuove un piccolo intorno tubolare di K nelle due varietà iniziali e si identificano i bordi ottenuti per costruire $M_1 \sharp_K M_2$. Osserviamo che per effettuare una tale costruzione bisogna richiedere che i fibrati normali di K in M_1 ed M_2 siano diffeomorfi.

Le cose si complicano quando le due varietà iniziali sono munite di una particolare struttura (come nel caso di varietà con metriche a curvatura scalare costante, o varietà che sono superfici minime,...) e si vuole preservare questa struttura, o quando sulle varietà iniziali esistono soluzioni di certe equazioni non lineari e si vogliono risolvere le stesse equazioni sulla somma connessa delle due varietà M_1 e M_2 (come ad esempio le equazioni di Yang-Mills).

Se da un lato le tecniche che permettono di effettuare le somme connesse in punti isolati sono state ben comprese e frequentemente utilizzate, dall'altro non si ha ancora un'effettiva padronanza delle tecniche che permettono di effettuare la somma connessa lungo sottovarietà. Il principale obiettivo di questo lavoro è quello di colmare (parzialmente) questa lacuna, sviluppando questo tipo di tecnologie nel quadro delle metriche a curvatura scalare costante e nel quadro delle equazioni di vincolo di Einstein, in relatività generale.

2 Il problema di Yamabe

Il problema di Yamabe in dimensione $m \geq 3$ consiste nel cercare, partendo da una metrica Riemanniana g su una varietà compatta M, un fattore conforme $u > 0$ tale che la metrica $\tilde{g} = u^{\frac{4}{m-2}} g$ sia a curvatura scalare costante. Dal punto di vista analitico, questo problema corrisponde a trovare una soluzione positiva dell'equazione

$$-\frac{4(m-1)}{m-2}\,\Delta_g u \,+\, R_g u = R_{\tilde{g}}\, u^{\frac{m+2}{m-2}} \tag{1}$$

dove $R_{\tilde{g}}$ indica la curvatura scalare della metrica $\tilde{g} := u^{\frac{4}{m-2}} g$ e R_g la curvatura scalare della metrica iniziale g (il nostro Laplaciano è definito negativo).

Questo problema è stato risolto grazie ai contributi di H. Yamabe [33], N. Trudinger [32] (nel caso delle metriche a curvatura scalare negativa), T. Aubin [4] (nel caso delle metriche non localmente conformemente piatte a curvatura scalare positiva e in dimensione $m \geq 6$) e R. Schoen [29] (nei casi restanti, cioè per metriche g a curvatura scalare positiva e in dimensione $m = 3, 4$ e 5, o localmente conformemente piatte). Come conseguenza sappiamo che su una varietà compatta esiste una metrica a curvatura scalare costante in ogni classe conforme. Inoltre tale metrica è unica nel caso della curvatura scalare negativa.

Teorema 1 (Aubin, Schoen, Trudinger, Yamabe). *Sia (M, g) una varietà Riemanniana compatta di dimensione $m \geq 3$, allora esiste sempre su M una metrica $\tilde{g}$ a curvatura scalare costante conforme a g.*

La dimostrazione di questo Teorema purtroppo è ben lontana dall'essere costruttiva, pertanto da essa non si può ricavare alcuna informazione sulla struttura delle metriche a curvatura scalare costante effettivamente ottenute. Allo scopo di migliorare la comprensione di tali metriche, D. Joyce s'è interessato alla somma connessa puntuale di varietà a curvatura scalare costante. In questo modo è riuscito a fornire una descrizione abbastanza precisa di alcune soluzioni dell'equazione di Yamabe. L'idea è quella di partire da due soluzioni note del problema di Yamabe per produrre poi nuovi esempi di metriche a curvatura scalare costante sulla somma connessa delle due varietà, perturbando le metriche iniziali. Nella prossima sezione, decriveremo più dettagliatamente i risultati di D. Joyce.

3 Il risultato di D. Joyce

D. Joyce in [16] costruisce delle famiglie di metriche a curvatura scalare costante sulla somma connessa puntuale di varietà Riemanniane compatte munite di metriche a curvatura scalare costante. Nella prima parte di questa tesi ci proponiamo di generalizzare questo risultato al caso delle somme connesse lungo sottovarietà.

Ci è sembrato opportuno fornire qui una sintetica descrizione del metodo utilizzato da D. Joyce, dal momento che, nelle sue linee guida, tale metodo è comune alla maggior parte dei risultati di somma connessa. Ci accontentiamo di descrivere i risultati di D. Joyce nel caso in cui le due metriche sulle varietà M_1 e M_2 abbiano la stessa curvatura scalare costante, visto che questa è la situazione più vicina ai risultati contenuti in questa tesi. Precisiamo che Joyce tratta anche la somma connessa puntuale di metriche iniziali più generali, ma queste costruzioni non sembrano estendersi in modo naturale al caso delle somme connesse generalizzate. In ogni caso, per maggiori dettagli, rinviamo direttamente il lettore all'articolo di Joyce sopra citato.

Il punto di partenza è il dato di due varietà Riemanniane (M_1, g_1) e (M_2, g_2) di dimensione $m \geq 3$ aventi la stessa curvatura scalare costante. Si rimuove una piccola palla di raggio ε da ciascuna varietà e si identificano i bordi che si sono formati con i bordi di un "collo" $[-T, T] \times S^{m-1}$. Questo "collo" è munito di una versione riscalata (il fattore di riscalamento dipenderà da ε) della metrica di Schwarzischild

$$g_{Sch} \;=\; \left[\cosh\left(\tfrac{m-2}{2}t\right)\right]^{\frac{4}{m-2}} \cdot \left(dt^2 + g_{S^{m-1}} \right) \tag{2}$$

la qual cosa lo rende a curvatura scalare nulla. Utilizzando delle funzioni cut-off si costruisce una famiglia (parametrizzata da $\varepsilon \in (0,1)$) di metriche che non sono a curvatura scalare costante, ma che rappresentano delle soluzioni approssimate del problema. Queste nuove metriche $(g_\varepsilon)_{\varepsilon\in(0,1)}$ sono identiche alle metriche di partenza su tutta la nuova varietà $M_1 \sharp M_2$ salvo un piccolo anello situato fra i bordi di identificazione. Il passo successivo consiste nel perturbare, per ε abbastanza piccolo, le soluzioni approssimate, in modo da ottenere delle metriche a curvatura scalare costante.

Una volta costruita la famiglia delle funzioni approssimate, il problema diventa quello di cercare un fattore conforme u_ε vicino ad 1, tale che la metrica $\tilde{g}_\varepsilon := u_\varepsilon^{\frac{4}{m-2}}\, g_\varepsilon$ abbia curvatura scalare costante. Il fatto che il fattore conforme sia vicino ad 1 permette di controllare la struttura delle metriche a curvatura scalare costante ottenute. Evidenziamo il fatto che tale controllo sul fattore conforme è essenziale in questo tipo di studio. Infatti, se ci si affranca da questo vincolo, è sufficiente applicare il Teorema 1 per ottenere direttamente l'esistenza di una metrica a curvatura scalare costante su $M_1 \sharp M_2$. Così facendo, però si perde il controllo sul fattore conforme u_ε e di conseguenza sulla struttura della metrica finale.

Possiamo ora enunciare e commentare i risultati di Joyce, iniziando dalla somma connessa di due varietà entrambe con curvatura scalare costante uguale a $R < 0$:

Teorema 2 (Joyce). *Siano (M_1, g_1) e (M_2, g_2) due varietà Riemanniane compatte di dimensione $m \geq 3$, munite di metriche la cui curvatura scalare è costante ed uguale a $R < 0$. Denotiamo con g_ε la metrica (soluzione approssimata) definita su $M := M_1 \sharp_\varepsilon M_2$, la somma connessa di M_1 e M_2 ottenuta rimuovendo piccole palle di raggio ε da ogni varietà ed identificando i due bordi.*

Sotto queste ipotesi, per ogni ε sufficientemente piccolo, è possibile dotare M di una metrica $\tilde{g}_\varepsilon$ a curvatura scalare costante R. Tali metriche sono conformi alle metriche iniziali lontano dai bordi di identificazione. Inoltre questo fattore conforme u_ε è vicino ad 1, nel senso che

$$\|1 - u_\varepsilon\|_{W^{1,2}(M, g_\varepsilon)} \quad \leq \quad C\,\varepsilon^2$$

dove $C > 0$ è una costante positiva e g_ε è la metrica soluzione approssimata costruita esplicitamente su M.

Come già detto, la dimostrazione di questo risultato si basa su un argomento di perturbazione, che permette di passare da una soluzione approssimata g_ε ad una soluzione esatta $\tilde{g}_\varepsilon$ utilizzando un cambio conforme. Per fare ciò, si risolve l'equazione di Yamabe (1) con $R_{\tilde{g}_\varepsilon} \equiv R < 0$, cercando una soluzione vicina alla funzione costante 1. Detto altrimenti, si cerca la soluzione u_ε sotto la forma $u_\varepsilon = 1 + v$, dove la funzione v è piccola (in un senso da precisare). Siamo perciò ricondotti a risolvere il problema non lineare

$$\begin{aligned} \left(\Delta_{g_\varepsilon} + \tfrac{R}{m-1}\right) v \;&=\; c_m\,(R - R_{g_\varepsilon}) + c_m\,(R - R_{g_\varepsilon})\,v \\ &+\; c_m\,R\left((1+v)^{\frac{m+2}{m-2}} - 1 - \tfrac{m+2}{m-2}v\right) \end{aligned} \tag{3}$$

dove $c_m = -\frac{m-2}{4(m-1)}$.

Poniamo il termine di destra uguale a $F_\varepsilon(v)$. Osserviamo che $F_\varepsilon(0)$ misura di quanto la metrica g_ε fallisce dall'essere a curvatura scalare costante uguale ad R. L'operatore lineare L_{g_ε} che appare nel membro a sinistra è l'operatore di Yamabe linearizzato attorno alla funzione costante 1.

A questo punto si costruiscono degli spazi di funzioni dove poter stimare il termine di errore in funzione di ε e la norma dell'inverso dell'operatore L_{g_ε} sempre in funzione di ε. Essenzialmente si deve garantire che per ε sufficientemente piccolo la taglia dell'errore sia molto più piccola della taglia della norma dell'inverso di L_{g_ε}. Fatto questo si può risolvere il problema (3) per mezzo di un teorema di punto fisso per contrazioni.

$$v \quad = \quad L_{g_\varepsilon}^{-1} \circ F_\varepsilon(v) \tag{4}$$

Osserviamo che, poiché $-\Delta_{g_\varepsilon}$ è un operatore positivo, il potenziale $\frac{R}{m-1}$ non è mai nel suo spettro quando $R < 0$. Di conseguenza, l'inversione di L_{g_ε} non presenta alcuna difficoltà in questo caso. La questione è diversa nel caso in cui le metriche iniziali sono a curvatura scalare positiva. Qui occorre introdurre un'ipotesi di non degenerazione sugli operatori di Yamabe linearizzati per le metriche g_1 e g_2, come si vede nel seguente enunciato:

Teorema 3 (Joyce). *Riprendendo le notazioni e le ipotesi del Teorema 2 con $R > 0$, supponiamo anche che $\frac{R}{m-1}$ non sia nello spettro di $-\Delta_{g_i}$, per $i = 1, 2$. Allora, per ogni ε sufficientemente piccolo, è possibile dotare M di una metrica $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{m-2}}\, g_\varepsilon$ a curvatura scalare $\equiv R$. Inoltre u_ε è tale che*

$$\|1 - u_\varepsilon\|_{W^{1,2}(M,g_\varepsilon)} \quad \leq \quad C\,\varepsilon^2$$

dove $C > 0$ è una costante positiva e g_ε è la metrica soluzione approssimata costruita esplicitamente su M.

Sotto queste ipotesi si dimostra che se L_{g_1} e L_{g_2} sono invertibili, allora anche L_{g_ε} è invertibile, per ogni ε sufficientemente piccolo.

Nel caso in cui le metriche iniziali sono a curvatura scalare nulla, bisogna tener conto del fatto che gli operatori $L_{g_i} = \Delta_{g_i}$, $i = 1, 2$ hanno un nucleo non banale costituito dalle funzioni costanti. In particolare, la questione dell'inversione dell'operatore linearizzato attorno alla soluzione approssimata per cui si vuole ottenere una buona stima *a priori* è in questo caso più delicata. In un primo momento si osserva che l'operatore di Yamabe linearizzato è essenzialmente uguale a Δ_{g_ε}, nel cui nucleo ci sono evidentemente le costanti.

L'idea è pertanto quella di lavorare ortogonalmente alle costanti introducendo un parametro. Più precisamente, in questo caso non si mira a costruire delle metriche a curvatura scalare nulla, ma delle metriche $\tilde{g}_\varepsilon$ a curvatura scalare $R = R(\varepsilon)$ costante e vicina a zero.

Un'ulteriore difficoltà nasce dal fatto che l'operatore Δ_{g_ε} sviluppa un autovalore λ_ε vicino a 0. Si tratta di un autovalore associato ad un'autofunzione β_ε che è essenzialmente uguale ad una costante su M_1 e ad un'altra costante (di segno opposto) su M_2. Al fine di ottenere delle buone stime sull'immagine dell'errore mediante l'inverso dell'operatore linearizzato, è importante poter lavorare sull'ortogonale di β_ε. Per fare ciò è sufficiente supporre che i due volumi delle metriche iniziali siano uguali. Si ottiene allora il:

Teorema 4 (Joyce). *Siano (M_1, g_1) et (M_2, g_2) due varietà Riemanniane compatte di dimensione $m \geq 3$ tali che $R_{g_1} = 0 = R_{g_2}$ e $\mathrm{vol}_{g_1}(M_1) = \mathrm{vol}_{g_2}(M_2)$ e sia $M = M_1 \sharp_\varepsilon M_2$ la somma connessa di M_1 e M_2 ottenuta rimuovendo una piccola palla di raggio ε da ogni varietà, munita della famiglia di metriche soluzioni approssimate $(g_\varepsilon)_{\varepsilon \in (0,1)}$. Allora, per ogni ε sufficientemente piccolo, è possibile munire M di una metrica $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{m-2}} g_\varepsilon$ a curvatura scalare constante $R = R(\varepsilon)$ e conforme alle metriche iniziali lontano dai bordi di identificazione. Inoltre questo fattore conforme u_ε è tale che*

$$\|1 - u_\varepsilon\|_{W^{1,2}(M, g_\varepsilon)} \;\; \leq \;\; C \, \varepsilon^\alpha$$

dove $C > 0$ e $\alpha = \alpha(m) > 0$ sono delle costanti positive e g_ε è la metrica soluzione approssimata costruita esplicitamente su M. Infine si trova che la curvatura scalare finale $R(\varepsilon)$ è un $\mathcal{O}\left(\varepsilon^{m-2}\right)$.

4 Somme connesse generalizzate e curvatura scalare positiva

Utilizzando la nozione di somma connessa generalizzata, M. Gromov e H. B. Lawson da una parte [12] e R. Schoen e S. T. Yau dall'altra [30] hanno analizzato, all'inizio degli anni '80, la struttura delle varietà Riemanniane che ammettono una metrica a curvatura scalare positiva. La costruzione presentata da M. Gromov e H. B. Lawson tratta solo il caso di somma connessa lungo sfere, mentre R. Schoen e S. T. Yau costruiscono una metrica a curvatura scalare positiva sulla somma connessa lungo una qualunque sottovarietà di due varietà a curvatura scalare positiva. In particolare dimostrano il seguente:

Teorema 5 (Schoen, Yau). *Siano M_1 e M_2 due varietà compatte di dimensione m, munite di metriche a curvatura scalare positiva, e siano K_1 e K_2 due sottovarietà compatte (rispettivamente di M_1 e di M_2) di dimensione k e codimensione $n := m-k \geq 3$. Supponiamo anche che esista un diffeomorfismo fra il fibrato normale di K_1 in M_1 e quello di K_2 in M_2, che preservi le fibre. Allora, la somma connessa generalizzata di M_1 e M_2 lungo K_1 e K_2 ammette una metrica a curvatura scalare positiva.*

Per ottenere questo risultato è necessario supporre che la codimensione $n := m-k$ sia ≥ 3. Il ruolo di tale ipotesi si evidenzia in un risultato intermedio della dimostrazione, in cui, con un cambio conforme, si passa dalle metriche iniziali (a curvatura scalare positiva) a due metriche scalarmente piatte su $M_i \setminus K_i$, $i = 1, 2$ (proiezione stereografica). Il resto della dimostrazione consiste nel modificare molto attentamente le due proiezioni stereografiche in modo da ottenere una metrica a curvatura scalare positiva sulla nuova varietà.

5 Somme connesse generalizzate e curvatura scalare costante

In questa sezione presentiamo la prima parte dei risultati di questa tesi. Il nostro obiettivo, come annunciato, è quello di generalizzare al caso di somme connesse lungo sottovarietà i risultati ottenuti da D. Joyce nel caso di somme connesse puntuali per metriche a curvatura scalare costante. Il nostro studio è diviso in due lavori. Nel primo [25] studiamo il caso in cui la curvatura scalare è non nulla, mentre nel secondo [26] affrontiamo il caso delle metriche scalarmente piatte. Se da una parte la costruzione geometrica è essenzialmente identica, dall'altra tuttavia l'analisi è piuttosto differente nei due casi, come già abbiamo avuto modo di osservare presentando i lavori di D. Joyce.

Osserviamo che la somma connessa di due varietà Riemanniane (M_1, g_1) e (M_2, g_2) lungo una comune sottovarietà K è in generale un'operazione meno flessibile della somma connessa in punti. Infatti, come si può constatare dall'enunciato del Teorema 5, sono neccessarie diverse ipotesi topologico-differenziali sui dati iniziali per poter effettuare una tale costruzione. Ad esempio è necessario che i fibrati normali di K in M_1 e in M_2 siano diffeomorfi. In più la codimensione di K in M_1 e in M_2 deve essere $n := m - k \geq 3$.

Per meglio comprendere la natura di quest'ultima ipotesi è utile pensare la somma connessa generalizzata come una somma connessa puntuale effettuata su ogni fibra del fibrato normale di K in M_1 e in M_2. Ispirandoci ai risultati di D. Joyce, costruiamo una famiglia di metriche $(g_\varepsilon)_{\varepsilon\in(0,1)}$, che fibra per fibra assomigliano alla metrica di Schwarzschild n-dimensionale (2) e che rappresentano delle soluzioni approssimate del nostro problema. Tale costruzione ci fornisce una buona piattaforma di partenza per tutte le costruzioni che affronteremo in questa tesi. Osserviamo che tale costruzione ci induce a lavorare in codimensione alta, $n \geq 3$.

Per la nostra costruzione avremo bisogno anche di un'altra ipotesi geometrica fondamentale. Più precisamente supporremo che la sottovarietà K sia immersa isometricamente in M_1 e in M_2. Per motivare questa assunzione (che non appariva nel risultato di Schoen-Yau e Gromov-Lawson), dobbiamo fare una piccola digressione. La strategia generale che intendiamo adottare è quella utilizzata da D. Joyce e descritta nella Sezione 3. Come già detto, dovremo risolvere l'equazione (3) al fine di trovare una metrica a curvatura scalare costante su $M_1\sharp_K M_2$ che sia vicina alle metriche iniziali al tendere a zero del parametro ε. Abbiamo anche già osservato che nell'equazione (3) il primo addendo del termine di destra misura il fallimento delle metriche g_ε dall'essere una soluzione del nostro problema. Per poter procedere alla costruzione è fondamentale che tale errore sia sufficientemente piccolo. L'ipotesi dell'immersione isometrica di K nelle due varietà iniziali è ciò che in pratica ci permette di avere un buon controllo sull'errore $R - R_{g_\varepsilon}$.

Osserviamo che, nel caso di somma connessa puntuale, la costruzione delle metriche g_ε è più semplice e l'errore $R - R_{g_\varepsilon}$ resta uniformemente limitato al tendere a zero del parametro ε. Ciò traduce analiticamente il fatto che in questo caso le metriche g_ε sono già molto vicine a delle metriche a curvatura scalare costante. Nel nostro caso, invece, la costruzione geometrica produce un errore di maggiore entità, ma localizzato al centro del "collo". Per misurare questo errore e controllare l'inverso dell'operatore linearizzato abbiamo scelto di lavorare in spazi funzionali pesati.

Per poter risolvere il problema di punto fisso (4) dobbiamo invertire l'operatore di Yamabe linearizzato L_{g_ε} e fornire delle buone stime *a priori* per soluzioni dell'equazione

$$L_{g_\varepsilon}\, v = f\,. \tag{5}$$

Se le stime sono tali che l'immagine dei termini d'errore $F_\varepsilon(v)$ (fra cui il principale è $R - R_{g_\varepsilon}$) mediante l'inverso dell'operatore L_{g_ε} è di piccola taglia, allora saremo in grado di risolvere il problema (4) utilizzando un argomento di punto fisso.

Ciò che rende diversi i casi $R \neq 0$ e $R \equiv 0$ è proprio l'inversione del linearizzato. Nel primo caso è sufficiente supporre che gli operatori di Yamabe linearizzati relativi alle metriche iniziali $L_{g_i} := \Delta_{g_i} + \frac{R}{m-1}$, $i = 1, 2$, siano iniettivi (evidenziamo ancora una volta che tale ipotesi è sempre verificata quando $R < 0$). Sotto questa ipotesi siamo in grado di concludere che l'operatore L_{g_ε} è invertibile e siamo pure in grado di ottenere delle stime precise sull'inverso di questo operatore al tendere a zero di ε. L'enunciato del nostro Teorema è il seguente:

Teorema 6. *Siano (M_1, g_1) e (M_2, g_2) due varietà Riemanniane compatte di dimensione $m \geq 3$, munite di metriche la cui curvatura scalare è costante e uguale a $R \neq 0$. Sia (K, g_K) una varietà Riemanniana compatta di dimensione k immersa isometricamente in (M_1, g_1) e in (M_2, g_2). Supponiamo inoltre che $n := m - k \geq 3$ e che i due operatori $\Delta_{g_i} + \frac{R}{m-1}$, $i = 1, 2$, siano iniettivi. Indichiamo con g_ε la metrica (soluzione approssimata) definita su $M_\varepsilon := M_1 \sharp_{K,\varepsilon} M_2$, la somma connessa generalizzata di M_1 e M_2 ottenuta rimuovendo dei piccoli intorni tubolari di K di raggio ε in ciascuna varietà e identificando i due bordi rimasti.*

Per ogni ε abbastanza piccolo, è possibile dotare M_ε di una metrica $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{m-2}} g_\varepsilon$ a curvatura scalare costante R. Tali metriche sono conformi alle metriche iniziali fuori da un piccolo intorno tubolare di K. Inoltre il fattore conforme u_ε è vicino a 1 nel senso che

$$\|1 - u_\varepsilon\|_{L^\infty(M_\varepsilon)} \leq C\, \varepsilon^{\frac{n-2}{2} - \delta}$$

dove $C > 0$ è una costante positiva che non dipende da ε e il peso δ può essere scelto nell'intervallo $\max\{0, \frac{n-4}{2}\} < \delta < \frac{n-2}{2}$.

Nel caso in cui $R \equiv 0$, come già osservato nella Sezione 3, il linearizzato dell'operatore di Yamabe è il Laplaciano. In particolare l'ipotesi di non degenerazione che permetteva di concludere nel caso precedente non è mai verificata. Per poter sperare di invertire il Laplaciano bisogna lavorare nello spazio delle funzioni ortogonali alle costanti. Il fatto di lavorare ortogonalmente alle costanti si traduce nel fatto che non potremo più controllare il valore esatto della curvatura scalare costante che otterremo sulla somma connessa delle due varietà. Infatti le funzioni curvatura scalare che otterremo saranno ortogonali allo spazio delle funzioni ortogonali alle costanti.

Come per la somma connessa in punti, troveremo il Laplaciano delle metriche g_ε sviluppa su M_ε un'autofunzione β_ε che assomoglia molto da vicino ad una costante (ad esempio) positiva su M_1 e ad una costante negativa su M_2 (le due costanti saranno scelte in modo che $\int_{M_\varepsilon} \beta_\varepsilon \; \mathrm{d}vol_{g_\varepsilon} = 0$). Quando $\varepsilon \to 0$, l'autovalore λ_ε tende verso 0 e l'autofunzione associata β_ε tende verso una funzione che è costante su ciascuna delle due componenti connesse dellle varietà limite M_0. L'esistenza di questa autofunzione associata ad un piccolo autovalore ha un'importante conseguenza: diventa infatti necessario lavorare nello spazio delle funzioni ortogonali sia alle costanti che a β_ε. In particolare le metriche g_ε devono essere costruite in modo che l'errore $R - R_{g_\varepsilon}$ sia ortogonale a β_ε.

Per le ragioni che abbiamo riportato, adotteremo ora una strategia leggermente diversa da quella utilizzata per ottenere il risultato precedente. Al posto dell'equazione (3), risolveremo in un primo tempo l'equazione non lineare

$$L_{g_\varepsilon} v \;=\; F_\varepsilon(v) \,-\, \lambda_{F_\varepsilon(v)}\, \beta_\varepsilon \tag{6}$$

dove $\lambda_{F_\varepsilon(v)}$ rappresenta la proiezione dei termini d'errore $F_\varepsilon(v)$ lungo β_ε. Puntualizziamo che in questo contesto $L_{g_\varepsilon} = \Delta_{g_\varepsilon}$ e $F_\varepsilon(v)$ è data da

$$\begin{aligned} F_\varepsilon(v) \;&=\; c_m\,(R - R_{g_\varepsilon}) \,+\, c_m\,(R - R_{g_\varepsilon})\,v \,-\, \tfrac{R}{m-1}\,v \\ &+\; c_m\, R\,\Big((1+v)^{\frac{m+2}{m-2}} - 1 - \tfrac{m+2}{m-2} v\Big) \end{aligned}$$

dove $c_m = -\frac{m-2}{4(m-1)}$. Qui $R = R(\varepsilon)$ diventa un parametro che rappresenta geometricamente la curvatura scalare finale e sarà scelto in modo che l'errore sia ortogonale alle funzioni costanti $\int_{M_\varepsilon} F_\varepsilon(v) \; \mathrm{d}vol_{g_\varepsilon} = 0$. Di fatto troveremo che $R = \mathcal{O}\left(\varepsilon^{n-2}\right)$, in analogia con quanto avveniva per la somma connessa puntuale di due varietà scalarmente piatte.

Stavolta, per risolvere l'equazione (6) utilizzeremo un punto fisso per contrazioni. Ciò ci permette di affermare, in un primo momento, che la soluzione ottenuta dipende in modo continuo dai dati del problema. In un secondo momento, quindi, effettueremo dei piccoli cambi conformi (la cui taglia tende a zero con ε) localizzati lontano dal "collo" che ci permetteranno di annullare il valore della proiezione $\lambda_{F_\varepsilon(v)}$. Come nel caso della somma connessa in punti, è necessario che i volumi delle due varietà iniziali siano uguali.

Teorema 7. *Siano* (M_1, g_1) *e* (M_2, g_2) *due varietà Riemanniane compatte di dimensione* $m \geq 3$*, munite di metriche la cui curvatura scalare è costante e uguale a zero. Sia* (K, g_K) *una varietà Riemanniana compatta di dimensione* k *immersa isometricamente in* (M_1, g_1) *e in* (M_2, g_2)*. Supponiamo inoltre che* $n := m - k \geq 3$ *e che i due volumi* $\mathrm{vol}_{g_1}(M_1)$ *e* $\mathrm{vol}_{g_2}(M_2)$ *siano uguali. Indichiamo con* g_ε *la metrica (soluzione approssimata) definita su* $M_\varepsilon := M_1 \sharp_{K,\varepsilon} M_2$*, la somma connessa generalizzata di* M_1 *e* M_2 *ottenuta rimuovendo dei piccoli intorni tubolari di* K *di raggio* ε *in ciascuna varietà e identificando i due bordi rimasti.*

Per ogni ε *abbastanza piccolo, è possibile dotare* M_ε *di una metrica* $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{m-2}} g_\varepsilon$ *a curvatura scalare costante in generale non nulla* $R = \mathcal{O}(\varepsilon^{n-2})$*. Tali metriche sono conformi alle metriche iniziali fuori da un piccolo intorno tubolare di* K*. Inoltre il fattore conforme* u_ε *è vicino a* 1 *nel senso che*

$$\|1 - u_\varepsilon\|_{L^\infty(M_\varepsilon)} \quad \leq \quad C\, \varepsilon^\gamma$$

dove $C > 0$ *è una costante positiva che non dipende da* ε *e il peso* γ *può essere scelto nell'intervallo* $0 < \gamma < \frac{1}{4}$*.*

Nel caso in cui le metriche iniziali siano scalarmente piatte, ma non Ricci piatte, possiamo migliorare il risultato precedente e ottenere una metrica a curvatura scalare nulla sulla somma connessa (generalizzata). L'idea è quella di garantire al contempo le condizioni $\int_{M_\varepsilon} F_\varepsilon(v)\ d vol_{g_\varepsilon} = 0$ e $\lambda_{F_\varepsilon(v)} = 0$ utilizzando delle piccole perturbazioni non conformi (di taglia $\mathcal{O}(\varepsilon^{n-2})$) delle metriche g_1 e g_2 localizzate fuori di un intorno tubolare di K. Più precisamente, se le due metriche iniziali sono non Ricci piatte, abbiamo gli sviluppi:

$$\begin{aligned} R_{\tilde{g}_\varepsilon} &= R_{g_\varepsilon} + R_{g_1 + r h_1} + R_{g_2 + s h_2} \\ &= R_{g_\varepsilon} + r\, K_1 + \mathcal{O}\left(r^2\right) + s\, K_2 + \mathcal{O}\left(s^2\right) \end{aligned}$$

dove

$$K_i \;=\; \Delta_{g_i}(\mathrm{tr}_{g_i} h_i) + \delta_{g_i}(\delta_{g_i} h_i) + g_i\,(\mathrm{Ric}_{g_i}, h_i) \qquad\qquad i = 1, 2$$

L'idea è dunque quella di utilizzare r e s come parametri in modo da controllare le quantità $\int_{M_\varepsilon} F_\varepsilon(v)\ d vol_{g_\varepsilon}$ e $\lambda_{F_\varepsilon(v)}$ e annullarle. Otteniamo allora il seguente enunciato:

Teorema 8. *Siano* (M_1, g_1) *e* (M_2, g_2) *due varietà Riemanniane compatte di dimensione* $m \geq 3$*, munite di metriche la cui curvatura scalare è costante e* *uguale a zero. Sia (K, g_K) una varietà Riemanniana compatta di dimensione k immersa isometricamente in (M_1, g_1) e in (M_2, g_2). Supponiamo inoltre che $n := m - k \geq 3$ e che le due metriche iniziali g_1 e g_2 siano non Ricci piatte. Indichiamo con $M_\varepsilon := M_1 \sharp_{K,\varepsilon} M_2$ la somma connessa generalizzata di M_1 e M_2 ottenuta rimuovendo dei piccoli intorni tubolari di K di raggio ε in ciascuna varietà e identificando i due bordi rimasti.*

Sotto queste ipotesi, per ogni ε abbastanza piccolo, è possibile dotare M_ε di una metrica $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{m-2}} g_\varepsilon$ la cui curvatura scalare è costante e uguale a zero. Inoltre tali metriche tendono verso le metriche iniziali su ogni compatto di M_i, $i = 1, 2$, quando $\varepsilon \to 0$.

Osserviamo che l'esistenza di una metrica scalarmente piatta ma non Ricci piatta implica l'appartenenza della varietà alla classe (1_+) nella classificazione di J. L. Kazdan e F. Warner [18], [19]. Per una varietà M nella classe (1_+), ogni funzione $f \in \mathcal{C}^\infty(M)$ può essere realizzata come curvatura scalare di un'opportuna metrica Riemanniana definita su M. Si può dunque dire che le varietà che stanno in (1_+) sono le più malleabili per quanto riguarda la curvatura scalare. Solo per tali varietà, infatti, si riesce a costruire una metrica a curvatura scalare nulla sulla somma connessa (generalizzata). Come vedremo in seguito, il Teorema 8 avrà un'applicazione interessante in relatività generale.

6 Equazioni di vincolo di Einstein

La strategia perturbativa messa in atto da D. Joyce nell'ambito della somma connessa puntuale per metriche a curvatura scalare costante è stata anche applicata allo studio di problemi di relatività generale [14], [15], [13]. Il problema studiato in questi lavori è quello delle "equazioni di vincolo di Einstein". Per meglio comprendere le motivazioni fisiche che stanno all'origine di questi studi, presentiamo qui brevemente il problema di Cauchy in relatività generale e rinviamo il lettore a [6] per maggiori informazioni a riguardo.

Sia (Z, γ) una varietà Lorentziana di dimensione $(m + 1)$, l'equazione di Einstein per lo spazio-tempo vuoto (vacuum space-time) è allora data da

$$\mathrm{Ric}_\gamma = 0 \tag{7}$$

Un' ipersuperficie di Cauchy in (Z, γ) è una sottovarietà M di dimensione m di tipo spazio (space-like), i.e., il cui campo di vettori normali ha norma negativa, è tale che ogni curva inestensibile di tipo tempo (time-like), i.e., il cui vettore tangente ha norma negativa, ha una e una sola intersezione con M. Osserviamo che la proprietà di ammettere un'ipersuperficie di Cauchy non è comune a tutte le varietà Lorentziane. Quelle che possiedono una tale ipersuperficie sono dette "globalmente iperboliche".

Supponiamo che (Z, γ) sia globalmente iperbolica e supponiamo anche che l'equazione (7) sia verificata. Sia M un'ipersuperficie di Cauchy in Z. Indichiamo con g la metrica (Riemanniana) indotta da γ su M e Π la seconda forma fondamentale di M in (Z, γ). Combinando l'equazione (7) con le equazioni di Gauss-Codazzi, si trova che g e Π devono verificare il seguente sistema che chiameremo "equazioni di vincolo di Einstein"

$$\mathrm{div}_g \, \Pi - \mathrm{d}\,(\mathrm{tr}_g \, \Pi) \;=\; 0 \tag{8}$$
$$R_g - |\Pi|_g^2 + (\mathrm{tr}_g \, \Pi)^2 \;=\; 0 \tag{9}$$

Il problema di Cauchy in relatività generale si pone nel seguente modo. Il dato iniziale è la tripla (M, g, Π), dove (M, g) è una varietà Riemanniana di dimensione m e Π è un 2-tensore symmetrico definito su M. Si cerca allora una varietà Lorentziana di dimensione $(m+1)$ che soddisfi l'equazione (7), ed un'immersione $\iota : M \hookrightarrow Z$ tale che $\iota(M)$ è un'ipersuperficie di Cauchy in (Z, γ) e γ induce g e Π su M. La coppia (Z, γ) è detta sviluppo di Cauchy di (M, g, Π) e per definizione (Z, γ) è globalmente iperbolica. Se ogni altro sviluppo di Cauchy di (M, g, Π) può essere immerso isometricamente in (Z, γ), si dice allora che (Z, γ) è lo sviluppo massimale di (M, g, Π).

Come già detto, le equazioni di vincolo di Einstein (8) e (9) sono condizioni necessarie per l'esistenza di uno sviluppo di Cauchy di (M, g, Π). Il seguente Teorema, dovuto a Y. Choquet-Bruhat, ci assicura che tali condizioni sono anche sufficienti

Teorema 9 (Choquet-Bruhat). *Supponiamo che (M, g, Π), ove (M, g) è una varietà Riemannian di dimensione m e Π è un 2-tensore simmetrico su M, verifichi le equazioni (8) e (9), allora esiste una varietà Lorentziana (Z, γ) di dimensione $(m+1)$ che è lo sviluppo di Cauchy massimale di (M, g, Π).*

Questo risultato ci fornisce un importante strumento per costruire dei modelli di spazio-tempo. Ci si può quindi concentrare direttamente sul sistema dei vincoli di Einstein (che è ellittico), anziché affrontare lo studio dell'equazione $\mathrm{Ric}_\gamma = 0$ (che è un sistema iperbolico). Nel caso particolare in cui si cerchino soluzioni delle equazioni di vincolo di Einstein a curvatura media costante (i.e., per cui $\mathrm{tr}_g \Pi \equiv \tau$), si ha a disposizione una strategia molto potente, nota come "metodo conforme".

Al fine di costruire delle soluzioni delle equazioni di vincolo di Einstein, proponiamo, nella seconda parte di questa tesi [27], una generalizzazione del risultato ottenuto da J. Isenberg, R. Mazzeo et D. Pollack [14], nel caso in cui i dati iniziali siano a curvatura media costante (CMC). Questo studio si basa fondamentalmente sul metodo conforme, di cui forniamo qui di seguito una breve descrizione.

Se si cercano delle soluzioni di (8)-(9) a curvatura media costante uguale a τ, si può scomporre la seconda forma fondamentale Π scrivendo

$$\Pi \;=\; \mu + (\tau/m)\, g$$

ove μ è un 2-tensore simmetrico a traccia nulla $\mathrm{tr}_g \mu = 0$. In seguito si considerano i due cambi conformi

$$\begin{aligned} g &= u^{\frac{4}{m-2}}\, \bar{g} \\ \mu &= u^{-2}\, \bar{\mu}\,, \end{aligned}$$

Facendo questo, il sistema delle equazioni di vincolo di Einstein diventa

$$\mathrm{div}_{\bar{g}}\, \bar{\mu} \;=\; 0 \tag{10}$$

$$\Delta_{\bar{g}} u + c_m R_{\bar{g}} u - c_m |\bar{\mu}|^2_{\bar{g}} u^{-\frac{3m-2}{m-2}} + c_m \frac{m-1}{m} \tau^2 u^{\frac{m+2}{m-2}} \;=\; 0 \tag{11}$$

dove $c_m = -\frac{m-2}{4(m-1)}$ e, come sempre, il nostro Laplaciano ha spettro negativo. Osserviamo che il sistema è debolmente disaccoppiato, dal momento che si può risolvere in un primo tempo l'equazione (10). Quindi, inserendo il valore di $|\bar{\mu}|_{\bar{g}}$ nell'equazione (11), si risolve nell'incognita u questa seconda equazione. Osserviamo anche che il dato $(M, \bar{g}, \bar{\mu}, u, \tau)$ permette di ricostruire (M, g, Π).

A questo punto abbiamo introdotto tutte le nozioni che ci permettono di comprendere il risultato di J. Isenberg, R. Mazzeo e D. Pollack [14]. In questo articolo i tre autori costruiscono nuove soluzioni per le equazioni di vincolo di Einstein facendo la somma connessa di due soluzioni note (o aggiungendo un manico ad una soluzione nota). Il risultato basilare è per dati iniziali definiti su uno spazio compatto e a curvatura media costante. In seguito sono state ottenute varie generalizzazioni di questo risultato. Ad esempio lo si è esteso a dati iniziali definiti su spazi asintoticamente euclidei e asintoticamente iperbolici [14], inoltre è stata rilassata l'ipotesi di curvatura media costante [15], infine, raffinando la costruzione, sono state prodotte nuove soluzioni che coincidono esattamente con quelle di partenza, fuori dalla zona di incollamento [9]. Nei paragrafi che seguono ci limiteremo a descrivere il risultato base di [14], dal momento che questa situazione è la più vicina ai risultati di questa tesi.

Si parte da due soluzioni (M_1, g_1, Π_1) e (M_2, g_2, Π_2) delle equazioni di vincolo di Einstein, in cui le varietà sono compatte ed hanno la stessa curvatura media costante uguale a τ. Questo corrisponde a considerare due soluzioni $(M_1, \bar{g}_1, \bar{\mu}_1, u_1, \tau)$ et $(M_2, \bar{g}_2, \bar{\mu}_2, u_2, \tau)$ delle equazioni (10) e (11).

Sia ora p_1 un punto di M_1 e p_2 un punto di M_2. Si effettua la somma connessa $M_\varepsilon = M_1 \sharp_\varepsilon M_2$ delle due varietà rimuovendo due piccole palle di raggio ε centrate in p_1 e p_2 e identificando i bordi. Per mezzo di funzioni cut-off si costruisce una famiglia di metriche soluzioni approssimate $(g_\varepsilon)_{\varepsilon \in (0,1)}$ modellate sulla metrica cilindrica

$$g_{Cyl} = \mathrm{d}t^2 + g_{S^{m-1}}$$

all'interno del collo e coincidenti con le metriche iniziali al di fuori. Sempre utilizzando delle funzioni cut-off si costruisce anche una famiglia di 2-tensori simmetrici $\mu = \mu(\varepsilon)$ tali che $\mathrm{tr}_{g_\varepsilon}(\mu) = 0$ per ogni $\varepsilon \in (0, 1)$.

In generale $\mathrm{div}_{g_\varepsilon} \mu \neq 0$ ma si può sempre trovare una (piccola) correzione σ_ε tale che $\mathrm{div}_{g_\varepsilon}(\mu + \sigma_\varepsilon) = 0$. Al fine di ottenere delle buone stime sulla correzione fatta (i.e., delle stime che ci assicurino che la norma della correzione tenda a zero con ε) è utile cercare σ_ε nella forma $\sigma_\varepsilon = D_{g_\varepsilon} X$, dove X è un campo di vettori e D_{g_ε} è l'operatore di Killing conforme per la metrica g_ε, definito da

$$D_{g_\varepsilon} X = \frac{1}{2} \mathcal{L}_X g_\varepsilon - \frac{1}{m} (\mathrm{div}_{g_\varepsilon} X) \cdot g_\varepsilon \qquad (12)$$

Qui $\mathcal{L}_X g_\varepsilon$ indica la derivata di Lie della metrica g_ε rispetto al campo di vettori X.

La ricerca di un termine correttivo si riconduce pertanto alla ricerca di un campo di vettori X che verifichi l'equazione

$$\mathfrak{L}_{g_\varepsilon} X = \mathrm{div}_{g_\varepsilon} \mu \qquad (13)$$

dove $\mathfrak{L}_{g_\varepsilon}$ è il Laplaciano vettoriale, definito da $\mathfrak{L}_{g_\varepsilon} := -\mathrm{div}_{g_\varepsilon} \circ D_{g_\varepsilon}$. Essendo questo operatore lineare ed ellittico ed essendo il termine di destra la divergenza di un tensore simmetrico, la nostra equazione ammette sempre soluzione su una varietà compatta. Inoltre, sotto opportune ipotesi di non degenerazione delle metriche iniziali $\bar{g}_1$ e $\bar{g}_2$, si è in grado di produrre le stime desiderate su X (e di conseguenza su σ_ε).

L'ipotesi di non degenerazione di cui hanno bisogno J. Isenberg, R. Mazzeo et D. Pollack è la seguente: se X è un campo di vettori che sta nel nucleo dell'operatore $D_{\bar{g}_i}$ (i.e., un campo di Killing conforme) e se $X(p_i) = 0$, allora $X \equiv 0$, per $i = 1, 2$. Questa ipotesi è abbastanza naturale, dal momento che su una varietà compatta il nucleo del Laplaciano vettoriale è contenuto nel nulceo dell'operatore di Killing conforme, essendo $\mathrm{div}_g = -D_g^*$. Sotto questa ipotesi è possibile costruire e stimare una soluzione di (10) (con $\bar{g} = g_\varepsilon$) la cui traccia è nulla

$$\mu_\varepsilon := \mu(\varepsilon) + \sigma_\varepsilon \tag{14}$$

Una volta ottenuta μ_ε, si può affrontare lo studio dell'equazione di Lichnerowicz (11) con $\bar{g} = g_\varepsilon$ e $\bar{\mu} = \mu_\varepsilon$. Al fine di risolvere questa equazione garantendo al contempo che la soluzione sia molto vicina alle soluzioni iniziali u_1 e u_2, si utilizza un argomento di perturbazione simile a quello già descritto per l'equazione di Yamabe, visto che le due equazioni hanno essenzialmente la stessa struttura. Il solo punto delicato è l'inversione dell'operatore di Lichnerowicz linearizzato. Per fare ciò è sufficiente supporre l'iniettività dei due operatori di Lichnerowicz linearizzati per le metriche iniziali, vale a dire

$$\Delta_{g_i} - |\mu_i|^2_{g_i} - \frac{1}{m}\tau^2 \qquad\qquad i = 1, 2 \tag{15}$$

Tale condizione è implicata, modulo il principio del massimo, dalla condizione $\Pi_1 \neq 0$ e $\Pi_2 \neq 0$.

Si ottiene così il seguente risultato:

Teorema 10 (Isenberg, Mazzeo, Pollack). *Siano (M_1, g_1, Π_1) e (M_2, g_2, Π_2) due soluzioni delle equazioni di vincolo di Einstein. Supponiamo che per $i = 1, 2$ le M_i siano compatte, di uguale dimensione $m \geq 3$ e uguale curvatura media costante pari a τ. Sia $M_\varepsilon := M_1 \sharp_\varepsilon M_2$ la somma connessa di due varietà ottenuta rimuovendo una piccola palla di raggio ε attorno ai punti $p_i \in M_i$, $i = 1, 2$, e identificando i bordi. Supponiamo che ogni campo di Killing conforme X che si annulla in p_i è in realtà nullo su M_i e supponiamo anche che $\Pi_i \neq 0$, per $i = 1, 2$.*

Allora, per ogni ε sufficientemente piccolo, si può costruire su M_ε una metrica $\tilde{g}_\varepsilon$ e un 2-tensore simmetrico $\tilde{\Pi}_\varepsilon$, tali che $(M_\varepsilon, \tilde{g}_\varepsilon, \tilde{\Pi}_\varepsilon)$ sia una soluzione delle equazioni di vincolo di Einstein la cui curvatura media è costante e uguale a τ. Inoltre, quando ε tende verso zero, $\tilde{g}_\varepsilon$ e $\tilde{\Pi}_\varepsilon$ tendono verso g_i e Π_i su $M_i \setminus B_R(p_i)$, dove $B_R(p_i)$ è una palla di raggio R fissato (e piccolo) centrata in p_i, $i = 1, 2$.

Osserviamo che, nel caso in cui $\Pi_1 \equiv 0 \equiv \Pi_2$, le equazioni di vincolo di Einstein diventano semplicemente $R_{g_i} = 0$, $i = 1, 2$ e per ragioni fisiche si è indotti a parlare di "dati iniziali simmetrici rispetto al tempo" (time-symmetric initial data). Se supponiamo allora di avere due dati iniziali simmetrici rispetto al tempo e con curvatura di Ricci non nulla, il Teorema 8 produce immediatamente delle nuove soluzioni per le equazioni di vincolo di Einstein sulla somma connessa generalizzata delle due varietà iniziali.

Infine facciamo notare che i risultati di [14] hanno dato luogo a molteplici applicazioni d'interesse fisico. Effettuando la somma connessa di due dati iniziali, si possono costruire molti esempi di soluzioni per le equazioni di vincolo di Einstein. Questo ha permesso da una parte di comprendere meglio la struttura topologica dello spazio-tempo [15], dall'altra di approfondire alcuni aspetti legati alla presenza e al comportamento dei buchi neri [10].

7 Somme connesse generalizzate e equazioni di vincolo di Einstein

In questa sezione presentiamo la costruzione di nuove soluzioni per le equazioni di vincolo di Einstein sulla somma connessa generalizzata di due dati iniziali (M_1, g_1, Π_1) e (M_2, g_2, Π_2). Il nostro risultato e la strategia della dimostrazione sono nello spirito di [14]. Tuttavia la nostra costruzione differisce da quellla di J. Isenberg, R. Mazzeo e D. Pollack su parecchi punti.

L'aspetto più evidente è la differente costruzione delle metriche soluzioni approssimate $(g_\varepsilon)_{\varepsilon \in (0,1)}$. Se si cerca di trasporre fedelmente la costruzione fatta in [14], si è indotti a fabbricare lungo il collo delle metriche la cui componente normale (alla sottovarietà K) è modellata sulla metrica cilindrica (modulo un fattore di scala dipendente da ε). Sfortunatamente questa costruzione non permette di trattare agevolmente il termine di errore che appare nell'equazione di Lichnerowicz

$$
\begin{aligned}
F_\varepsilon(v) \;\; := \;\; & c_m \,(R_{g_1} - R_{g_\varepsilon})\,\chi_1 \, - \, c_m \,(|\mu_1|^2_{g_1} - |\mu_\varepsilon|^2_{g_\varepsilon})\,\chi_1 \\
& + c_m \,(R_{g_2} - R_{g_\varepsilon})\,\chi_2 \, - \, c_m \,(|\mu_2|^2_{g_2} - |\mu_\varepsilon|^2_{g_\varepsilon})\,\chi_2 \\
& + c_m \,(R_{g_1} - R_{g_\varepsilon})\,\chi_1\, v \, + \, b_m \,(|\mu_1|^2_{g_1} - |\mu_\varepsilon|^2_{g_\varepsilon})\,\chi_1\, v \\
& + c_m \,(R_{g_2} - R_{g_\varepsilon})\,\chi_2\, v \, + \, b_m \,(|\mu_2|^2_{g_2} - |\mu_\varepsilon|^2_{g_\varepsilon})\,\chi_2\, v \\
& + c_m \,|\mu_\varepsilon|^2_{g_\varepsilon}\,\mathcal{O}\left(v^2\right) \, - \, c_m \,\frac{m-1}{m}\,\tau^2\,\mathcal{O}\left(v^2\right)
\end{aligned}
\qquad (16)
$$

dove $c_m = -(m-2)/[4(m-1)]$, $b_m = c_m \cdot (3m-2)/(m-2)$ e $\{\chi_1, \chi_2\}$ è una partizione dell'unità. In particolare sono i termini $R_{g_i} - R_{g_\varepsilon}$ che pongono i maggiori problemi, per $i = 1, 2$.

D'altra parte, questo termine di curvatura scalare risulta ben stimato se si utilizzano come metriche soluzioni approssimate le metriche g_ε che abbiamo già utilizzato per risolvere l'equazione di Yamabe [25], vale a dire metriche la cui componenete normale (alla sotovarietà K) è modellata sulla metrica di Schwarzschild.

L'utilizzo di questo tipo di metriche ci obbliga a riottenere una stima per il termine correttivo σ_ε (quello che produce la soluzione dell'equazione $\mathrm{div}_{g_\varepsilon}\,\mu_\varepsilon = 0$, con $\mu_\varepsilon = \mu(\varepsilon) + \sigma_\varepsilon$). A questo fine adottiamo in [27] una tecnica piuttosto differente da quella adottata in [14]. In particolare, poiché cerchiamo σ_ε della forma $D_{g_\varepsilon} X$, siamo indotti a studiare il problema ellittico

$$
\mathfrak{L}_{g_\varepsilon}\, X \;\; = \;\; \mathrm{div}_{g_\varepsilon}\, \mu \qquad (17)
$$

Anziché stimare il primo autovalore del Laplaciano vettoriale $\mathfrak{L}_{g_\varepsilon}$ (come si fa in [14]), produciamo direttamente una stima *a priori* dellla soluzione X in termini di μ. Per questa ragione lavoriamo ancora una volta in spazi pesati e osserviamo che la stima che troviamo per X (e dunque anche per σ_ε e per il termine μ_ε che appare nel termine d'errore $F_\varepsilon(v)$ (16)) è di fatto uniforme rispetto al parametro ε. Questo risultato ci permette poi di risolvere abbastanza agevolmente l'equazione nonlineare di Lichnerowicz, utilizzando un teorema di punto fisso.

Per ottenere la stima *a priori* per X, utilizziamo in [27] un argomento per assurdo. Questo ci porta a considerare tre problemi limite omogenei (con differenti condizioni di decadimento sullle soluzioni). Questi problemi, di cui si vuole provare la non esistenza di soluzioni, sono

$$\mathfrak{L}_{g_i} X = 0 \qquad \text{su } M_i \setminus K,\ i = 1,2 \tag{18}$$

$$\mathfrak{L}_{\mathbb{R}^k \times \mathbb{S}^n} X = 0 \qquad \text{su } \mathbb{R}^k \times \mathbb{S}^n \tag{19}$$

$$\mathfrak{L}_{\mathbb{R}^k \times \mathbb{R}^n} X = 0 \qquad \text{su } \mathbb{R}^k \times \mathbb{R}^n \tag{20}$$

dove $\mathbb{S}^n$ indica lo spazio di Schwarzschild di dimensione $n = m - k$ e gli ultimi due Laplaciani vettoriali sono relativi alle due metriche prodotto $g_{\mathbb{R}^k} + g_{\mathbb{S}^n}$ e $g_{\mathbb{R}^k} + g_{\mathbb{R}^n}$ rispettivamente.

Per quanto riguarda gli ultimi due problemi, siamo in grado di dedurre direttamente una contraddizione combinando risultati di b-calculus [28] con l'analisi di Fourier, mentre nel primo caso abbiamo bisogno di un'ipotesi di non degenerazione molto simile a quellla utilizzata in [14]. L'ipotesi di cui abbiamo bisogno è che l'operatore di Killing conforme D_{g_i} sia iniettivo su M_i, per $i = 1, 2$.

Il nostro risultato è il seguente

Teorema 11. *Siano (M_1, g_1, Π_1) e (M_2, g_2, Π_2) due soluzioni delle equazioni di vincolo di Einstein aventi la stessa curvatura media costante τ e la stessa dimensione $m \geq 3$. Supponiamo anche che le due varietà M_i siano compatte. Sia (K, g_K) una sottovarietà compatta di dimensione k immersa isometricamente in (M_1, g_1) e (M_2, g_2) e sia $n := m - k \geq 3$. Supponiamo inoltre che i fibrati normali di K in M_1 e M_2 siano diffeomorfi e denotiamo con $M_\varepsilon := M_1 \sharp_{K,\varepsilon} M_2$ la somma connessa generalizzata di M_1 e M_2 ottenuta rimuovendo dei piccoli intorni tubolari di K di raggio ε da ogni varietà e identificando i due bordi. Supponiamo inoltre che ogni campo di vettori di Killing conforme X sia banale su M_i e che $\Pi_i \neq 0$, per $i = 1, 2$.*

Allora, per ogni ε sufficientemente piccolo, si può costruire su M_ε una metrica $\tilde{g}_\varepsilon$ e un 2-tensore simmetrico $\tilde{\Pi}_\varepsilon$, tali che $(M_\varepsilon, \tilde{g}_\varepsilon, \tilde{\Pi}_\varepsilon)$ sia una soluzione delle equazioni di vincolo di Einstein con curvatura media costante uguale a τ.
Inoltre quando ε tende verso 0, $\tilde{g}_\varepsilon$ e $\tilde{\Pi}_\varepsilon$ tendono verso g_i e Π_i all'infuori di un intorno tubolare di K fissato.

Osserviamo che come in [14], questo Teorema stabilisce un risultato di base che può essere esteso in diverse direzioni. Per esempio si può rilassare l'ipotesi di compattezza su K, richiedendo che il raggio di iniettività di K sia limitato inferiormente. Un'altra possibile estensione di questo risultato è al caso di dati iniziali asintoticamente euclidei e asintoticamente iperbolici. Ancora, si può cercare di localizzare la costruzione al fine di lasciare inalterate le soluzioni iniziali al di fuori di un intorno tubolare di K fissato, così come è già stato fatto per il risultato di [14].

In conclusione osserviamo che un'interessante direzione di ricerca consiste nell'approfondire lo studio della geometria della regione di incollamento. In particolare si hanno delle buone ragioni per credere che al centro del collo si trovino degli "orizzonti apparenti". In alcuni casi ciò implica che lo sviluppo di Cauchy di un tale dato iniziale contiene un buco nero. In quest'ottica la nostra costruzione potrebbe allora essere utilizzata per produrre, in dimensione ≥ 4, degli esempi di buchi neri pluridimensionali dalla topologia piuttosto varia, cosa che potrebbe interessare gli specialisti di teoria delle stringhe.

References

[1] M. Anderson, *Dehn filling and Einstein metrics in higher dimensions*, arXiv:math.DG/0303260 (2003).

[2] C. Arezzo, F. Pacard, *Blowing up Kähler manifolds with constant scalar curvature II*, accettato da Ann. of Math.

[3] C. Arezzo, F. Pacard, *Blowing up and desingularizing Kähler manifolds of constant scalar curvature*, Acta Math. 196, no.2, (2006), 179-228.

[4] T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. 55 (1976), 269–296.

[5] T. Aubin, *Some nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics, Springer, (1998).

[6] R. Bartnik, J. Isenberg, *The constraint equations*, The Einstein equations and the large scale behavior of gravitational fields, eds. P. T. Chrusciel and H. Friedrich, Birkhäuser, Basel, (2004), 1–39.

[7] Y. Choquet-Bruhat, *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires*, Acta Math. 88 (1952), 141–225.

[8] P. T. Chruściel, E. Delay, *On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications*, Mém. Soc. Math. de France 94 (2003), 1–103.

[9] P. T. Chruściel, J. Isenberg, D. Pollack, *Initial data engineering*, Comm. Math. Phys. 257 (2005), 29–42.

[10] P. T. Chruściel, R. Mazzeo, *On many-black-hole vacuum spacetimes*, Class. Quantum Grav. 20 (2003), 729–754

[11] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Diff. Geom. 18 (1983), 279–315.

[12] M. Gromov, H. B. Lawson, *The classification of simply connected manifolds of positive scalar curvature.* Ann. of Math. (2) 111 (1980), no. 3, 423–434.

[13] J. Isenberg, D. Maxwell, D. Pollack, *A gluing construction for non vacuum solutions of the Einstein constraint equations*, Adv. Theor. Phys. 9 (2005), 129–172.

[14] J. Isenberg, R. Mazzeo, D. Pollack, *Gluing and wormholes for the Einstein constraint equations*, Comm. Math. Phys. 231 (2002), 529–568.

[15] J. Isenberg, R. Mazzeo, D. Pollack, *On the topology of vacuum spacetime*, Ann. H. Poincaré 4 (2003), 369–383.

[16] D. Joyce, *Constant scalar curvature metrics on connected sums.* Int. J. Math. Math. Sci. (2003), no. 7, 405–450.

[17] D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford University Press, (2000).

[18] J. L. Kazdan, F. Warner, *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature*, Ann. of Math. 101 (1975), 317–331.

[19] J. L. Kazdan, F. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, J. Diff. Geom. 10 (1975), 113–134.

[20] A. G. Kovalev, *Twisted connected sums and special Riemannian holonomy.* , J. Reine Angew. Math. 565 (2003), 125–160.

[21] R. Mazzeo, F. Pacard, *Constant scalar curvature metrics with isolated singularities*, Duke Math. Journal 99 No. 3 (1999) 353–418.

[22] R. Mazzeo, F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, Comm. Anal. Geom. 9 No. 1 (2001) 169–237.

[23] R. Mazzeo, F. Pacard, D. Pollack, *Connected sums of constant mean curvature surfaces in Euclidean 3 space*, J. Reine Angew. Math. 536 (2001), 115–165.

[24] R. Mazzeo, D. Pollack, K. Uhlenbeck, *Connected sums constructions for constant scalar curvature metrics*, Topological Method in Nonlinear Analysis 6 (1995), 207–233.

[25] L. Mazzieri, *Generalized connected sum construction for constant scalar curvature metrics* accettato da in Communication in Partial Differential Equations.

[26] L. Mazzieri, *Generalized connected sum construction for scalar flat metrics*, arXiv:math.DG/0611778 (2006).

[27] L. Mazzieri, *Generalized gluing for Einstein constraint equations*, Preprint (2007).

[28] R. B. Melrose, *The Atiyah-Patodi-Singer Index Theorem*, Res. Notes Math. 4, A. K. Peters, Wellesly, Mass. (1993).

[29] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Diff. Geom. 20 (1984), 479–495.

[30] R. Schoen, S. T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. 28 (1979), no. 1-3, 159–183.

[31] C. H. Taubes, *The existence of anti-self dual conformal structures*, J. Diff. Geom., 36b(1) (1992), 163–253.

[32] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa 22 (1968), 265–274.

[33] H. Yamabe, *On a deformation of Riemannian structure on compact manifolds*, Osaka Math. J. 12 (1960), 21–37.

[34] S. D. Yang, *A connected sum construction for complete minimal surfaces with finite total curvature,* Commun. Anal. Geom. 9, No.1 (2001), 115–167.

Introduction

1 Sommes connexes généralisées

Ces deux dernières décennies, les techniques de somme connexe essentiellement basées sur des outils d'analyse ont permis de faire des progrès importants dans la compréhension de nombreux problèmes non linéaires issus de la géométrie (étude des métriques à courbure scalaire constante en géométrie Riemannienne [16], [21], [24], métriques auto-duales [31], métriques ayant des groupes d'holonomie spéciaux [17], [20], métriques extrémales en géométrie Kählérienne [2], [3], équations de Yang-Mills [11], étude des surfaces minimales [34] et des surfaces à courbure moyenne constante [22], [23], métriques d'Einstein [1], ...). Ces techniques se sont avérées être un outil puissant pour démontrer l'existence de solutions à des problèmes hautement non linéaires.

La somme connexe (ou bien l'ajout d'une 1-anse) est une opération topologique qui consiste à prendre deux variétés M_1 et M_2, exciser une petite boule géodésique de chacune des deux variétés et identifier les deux bords (deux sphères) obtenus après excision afin d'obtenir une variété $M_1 \sharp M_2$ qui, en général, sera topologiquement différente des deux variétés initiales. Plus généralement, on peut aussi considérer la somme connexe de deux variétés M_1 et M_2 le long d'une sous-variété K (ou "somme connexe généralisée"). Dans ce cas, on excise un petit voisinage tubulaire d'un plongement de K dans les deux variétés initiales et on identifie les bords obtenus pour construire $M_1 \sharp_K M_2$. Remarquons que dans une telle construction, il est nécessaire que les fibrés normaux des plongements de K dans les deux variétés soient difféomorphes.

Les choses se compliquent quand les variétés initiales sont munies d'une structure particulière (comme par exemple le cas où les variétés sont munies de métriques à courbure scalaire constante, le cas où les variétés sont des surfaces minimales, ...) et que l'on souhaite préserver cette structure, ou quand on a des solutions d'une certaine équation non linéaire sur les variétés initiales et que l'on souhaite résoudre cette équation sur la somme connexe des deux variétés M_1 et M_2 (comme par exemple les équations de Yang-Mills).

Si les techniques permettant d'effectuer des sommes connexes en des points isolés sont bien comprises et fréquemment utiliées, les techniques permettant d'effectuer des sommes connexes le long de sous variétés ne sont pas encore bien maîtrisées. Le principal objectif de ce travail est de combler (partiellement) cette lacune en développant de telles techniques applicables dans le cadre de l'étude des métriques à courbure scalaire constante et aussi dans le cadre de l'étude des équations de compatibilité d'Einstein en relativité générale.

2 Le problème de Yamabe

Le problème de Yamabe en dimension $m \geq 3$ consiste, partant d'une métrique g sur une variété compacte M, à chercher un facteur conforme $u > 0$ tel que la nouvelle métrique $\tilde{g} = u^{\frac{4}{m-2}}\, g$ soit à courbure scalaire constante. D'un point de vu analytique, ce problème revient à trouver une solution positive de l'équation

$$-\frac{4(m-1)}{m-2}\,\Delta_g\, u \,+\, R_g\, u = R_{\tilde{g}}\, u^{\frac{m+2}{m-2}} \tag{1}$$

où $R_{\tilde{g}}$ désigne la courbure scalaire de la métrique $\tilde{g} := u^{\frac{4}{m-2}}\, g$ et R_g la courbure scalaire de la métrique g initiale (notre Laplacien est défini négatif).

Ce problème a été résolu grâce aux efforts conjugués de H. Yamabe [33], N. Trudinger [32] (dans le cas des métriques à courbure scalaire négative), T. Aubin [4] (dans le cas des métriques non localement conformement plates à courbure scalaire positive et en dimension $m \geq 6$) et enfin R. Schoen [29] (dans le cas restants, c'est à dire pour des métriques g à courbure scalaire positive et en dimensions $m = 3, 4$ et 5, où bien g localement conformement plates). On sait maintenant que, sur une variété compacte, il existe une métrique à courbure scalaire constante dans chaque classe conforme. De plus, cette métrique est unique dans le cas de la courbure scalaire négative.

Théorème 1 (Aubin, Schoen, Trudinger, Yamabe). *Soit (M, g) une variété Riemannienne compacte de dimension $m \geq 3$, alors il existe toujours sur M une metrique $\tilde{g}$ à courbure scalaire constante qui est conforme à g.*

La démonstration de ce résultat n'est pas du tout constructive et, par conséquent, elle ne donne aucune information sur la structure des métriques à courbure scalaire constante obtenues. Afin de mieux comprendre la structure de ces métriques D. Joyce s'est intéressé à la somme connexe en des points de variété possédant des métriques à courbure scalaire constante. Ce faisant, il a donné une description assez précise de certaines solutions de l'équation de Yamabe. L'idée principale est, partant de deux solutions du problème de Yamabe, de produire ensuite de nouveaux exemples de métriques à courbure scalaire constante sur la somme connexe des deux variétés en construisant ces métriques comme perturbations des métriques initiales. Nous décrivons plus précisement les résultats de D. Joyce dans la section qui suit.

3 Les résultats de D. Joyce

D. Joyce construit dans [16] des familles de métriques à courbure scalaire constante sur la somme connexe en des points points de variétés Riemanniennes compactes munies de métriques à courbure scalaire constante. Dans la première partie de cette thèse nous nous proposons de généraliser ce résultat dans le cadre des sommes connexes le long de sous-variétés.

Il nous a semblé opportun de présenter brièvement ici les points principaux de la méthode utilisée par D. Joyce, étant donné que cette méthode est commune à la plupart des résultats de somme connexe. Nous nous contentons de décrire les résultats de D. Joyce dans le cas où les deux métriques sur les variétés M_1 et M_2 sont toutes les deux à courbure scalaire constante, étant donné que cette situation est la plus proche des résultats de cette thèse. Précisions que D. Joyce envisage aussi la somme connexe en des points pour des métriques initiales plus générales mais étant donné que ces résultats ne semblent pas s'étendre aisément au cas de sommes connexes généralisées, nous préférons renvoyer directement le lecteur à l'article de D. Joyce pour de plus amples précisions.

Le point de départ est la donnée de deux variétés Riemanniennes (M_1, g_1) et (M_2, g_2) de dimension $m \geq 3$ ayant la même courbure scalaire constante. On enlève une petite boule de rayon ε de chacune de ces variétés et on identifie les deux bords avec les bords d'un "cou" $[-T, T] \times S^{m-1}$. Ce "cou" est muni d'une version réduite (l'échelle dépendra de ε) de la métrique de Schwarzschild

$$g_{Sch} \;=\; \left(\cosh\left(\tfrac{m-2}{2}\,t\right)\right)^{\frac{4}{m-2}}\;(\,dt^2 \;+\; g_{S^{m-1}}\,)\,, \tag{2}$$

ce qui lui donne la propriété d'être à courbure scalaire nulle. En utilisant des fonctions troncature on construit ensuite une famille de métriques (paramétrée par $\varepsilon \in (0,1)$) qui ne sont pas à courbure scalaire constante, mais qui représentent des solutions approchées du problème. Ces nouvelles métriques $(g_\varepsilon)_{\varepsilon\in(0,1)}$ sont identiques aux métriques de départ sur la variété $M_1 \sharp M_2$ en dehors d'un petit anneau situé entre les bords d'identification. L'étape suivante consiste à perturber, pour tout ε assez petit, les solutions approchées de façon à obtenir des métriques à courbure scalaire constante.

Une fois la famille de solutions approchées construite, le problème revient à rechercher un facteur conforme u_ε proche de 1 tel que la métrique $\tilde{g}_\varepsilon := u_\varepsilon^{\frac{4}{m-2}}\, g_\varepsilon$ ait une courbure scalaire constante. Le fait que le facteur conforme soit proche de 1 permet de contrôler la structure des métriques à courbure scalaire constante obtenues. Soulignons que ce contrôle sur le facteur conforme u_ε est essentiel dans ce type d'étude. En effet, si l'on s'affranchit de cette contrainte, il suffit d'appliquer le Théorème 1 qui nous assure directement l'existence d'une métrique à courbure scalaire constante sur $M_1 \sharp M_2$, sans autre précision sur la structure du facteur conforme u_ε.

Nous pouvons maintentant énoncer et commenter les résultats de D. Joyce, en commençant par la somme connexe de deux variétés à courbure scalaire constante toutes deux égales à $R < 0$:

Théorème 2 (Joyce). *Soient (M_1, g_1) et (M_2, g_2) deux variétés Riemanniennes compactes de dimension $m \geq 3$, munies de métriques dont la courbure scalaire est constante égale à $R < 0$. On note g_ε la métrique (solution apporchée) définie sur $M := M_1 \sharp_\varepsilon M_2$, la somme connexe de M_1 et M_2 obtenue en excisant de petites boules de rayon ε de chaque variété et en identifiant les deux bords.*

Sous ces hypothèses, pour tout ε assez petit, il est possible de munir M d'une métrique $\tilde{g}_\varepsilon$ à courbure scalaire constante R. Ces métriques sont conforme aux métriques initiales loin des bords d'identification. En outre ce facteur conforme u_ε est proche de 1 au sens où

$$\|1 - u_\varepsilon\|_{W^{1,2}(M,g_\varepsilon)} \quad \leq \quad C\,\varepsilon^2$$

où $C > 0$ est une constante positive et g_ε est la métrique solution approchée construite explicitement sur M.

Comme nous l'avons déjà mentionné, la démonstration de ce résultat repose sur un argument de perturbation qui permet de passer d'une solution approchée g_ε à une solution exacte $\tilde{g}_\varepsilon$ en utilisant un changement conforme. Pour ce faire, on résout l'équation de Yamabe (1) avec $R_{\tilde{g}_\varepsilon} \equiv R < 0$ en recherchant une solution proche de la fonction constante égale à 1. Autrement dit, on recherche une solution u_ε sous la forme $u_\varepsilon = 1 + v$ où la fonction v est petite (en un sens à préciser). Nous sommes alors amenés à résoudre le problème non linéaire

$$\begin{aligned} \left(\Delta_{g_\varepsilon} + \tfrac{R}{m-1}\right) v \;&=\; c_m\,(R - R_{g_\varepsilon}) + c_m\,(R - R_{g_\varepsilon})\,v \\ &+\; c_m\,R\left((1+v)^{\frac{m+2}{m-2}} - 1 - \tfrac{m+2}{m-2}v\right) \end{aligned} \qquad (3)$$

où $c_m = -\frac{m-2}{4(m-1)}$.

On note $F_\varepsilon(v)$ le terme de droite. Remarquons que $F_\varepsilon(0)$ représente l'érreur commise en considérant que la métrque g_ε est à courbure scalaire égale à R. L'opérateur L_{g_ε} qui apparaît dans le terme de gauche est l'opérateur de Yamabe linéarisé autour de la fonction constante égale à 1.

À présent, il nous faut construire des espaces fonctionnels dans lesquels on estime le terme d'erreur en fonction de ε et dans lesquels on peut estimer la norme de l'inverse de l'opérateur L_{g_ε} en fonction de ε. Essentiellement, il faut s'assurer que, pour ε assez petit, la taille de l'erreur est beaucoup plus petite que la norme de l'inverse de l'opérateur L_{g_ε}. Une fois ce travail effectué, on peut résoudre le problème (3) en utilisant un théorème de point fixe pour les applications contractantes

$$v \;=\; L_{g_\varepsilon}^{-1} \circ F_\varepsilon(v) \tag{4}$$

Remarquons que, puisque $-\Delta_{g_\varepsilon}$ est un opérateur positif, le potentiel $\frac{R}{m-1}$ n'est jamais dans son spectre dans le cas où $R < 0$. En particulier, l'inversion de L_{g_ε} ne présente aucune difficulté dans ce premier cas. La question est différente dans le cas où les métriques initiales sont à courbure scalaire positive et il faut alors introduire une hypothèse de non dégénérescence, comme on peut voir dans le résultat suivant :

Théorème 3 (Joyce). *On reprend les notations et les hypothèses du Théorème 2 dans le cas où $R > 0$. On suppose en outre que $\frac{R}{m-1}$ n'est pas dans le spectre de $-\Delta_{g_i}$, pour $i = 1, 2$. Alors pour tout ε assez petit, il est possible de munir M d'une métrique $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{m-2}}\, g_\varepsilon$ à courbure scalaire $\equiv R$. De plus u_ε est tel que*

$$\|1 - u_\varepsilon\|_{W^{1,2}(M,g_\varepsilon)} \leq C\,\varepsilon^2 \,,$$

où $C > 0$ est une constante positive et g_ε est la métrique solution approchée construite explicitement sur M.

Sous ces hypothèses, on démontre que si les opérateurs L_{g_1} et L_{g_2} sont inversibles, alors L_{g_ε} est aussi inversible pour tout ε est assez petit.

Dans le cas où les métrique initiales sont à courbure scalaire nulle, il faut tenir compte du fait que les opérateurs $L_{g_i} = \Delta_{g_i}$, $i = 1, 2$ ont un noyau non trivial constitué par les fonctions constantes. En particulier la question de l'inversion de l'opérateur linéarisé autour de la solution approchée pour lequel on veut obtenir une bonne estimation *a priori* devient dans ce cas beaucoup plus délicate. Dans un premier temps on constate que l'opérateur de Yamabe linéarisé est essentiellement égal à Δ_{g_ε}, qui a visiblement les constantes dans son noyau. L'idée est de travailler dans l'espace des fonctions orthogonales aux fonctions constantes en introduisant un paramètre. Plus précisément, dans ce cas, on ne se propose plus de construire des métriques a courbure scalaire nulle mais des métriques $\tilde{g}_\varepsilon$ à courbure scalaire $R = R(\varepsilon)$ constante (proche de 0), dont la valeur n'est pas fixée et peut dépendre de ε.

Une difficulté supplémentaire (moins évidente) apparaît car il se trouve que l'opérateur $-\Delta_{g_\varepsilon}$ développe une valeur propre λ_ε proche de de 0. Il s'agit d'une valeur propre associée à une fonction propre β_ε qui est essentiellement égale à une constante positive sur M_1 et une autre constante négative sur M_2. Il est important de pouvoir travailler sur l'orthogonal de β_ε afin d'obtenir de bonnes estimations sur l'image de l'erreur par l'inverse de l'opérateur linéarisé. Pour ce faire, il suffit en fait de supposer que les volumes des métriques initiales sont égaux. On obtient alors le :

Théorème 4 (Joyce). *Soient (M_1, g_1) et (M_2, g_2) deux variétés Riemanniennes compactes de dimension $m \geq 3$ telles que $R_{g_1} = 0 = R_{g_2}$ et $\mathrm{vol}_{g_1}(M_1) = \mathrm{vol}_{g_2}(M_2)$ et soit $M = M_1 \sharp_\varepsilon M_2$ la somme connexe de M_1 et M_2 obtenue en enlevant une petite boulle de rayon ε de chaque variété, munie de la suite de métriques approchées g_ε. Alors, pour tout ε assez petit, il est possible de munir M d'une métrique $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{m-2}} g_\varepsilon$ à courbure scalaire constante $R = R(\varepsilon)$ et conforme aux métriques initiales loin des bords d'identification. En outre ce facteur conforme u_ε est tel que*

$$\|1 - u_\varepsilon\|_{W^{1,2}(M,g_\varepsilon)} \leq C\, \varepsilon^\alpha$$

où $C > 0$ et $\alpha = \alpha(m) > 0$ sont des constantes positives et g_ε est la métrique solution approchée construite explicitement sur M. Enfin on trouve que la courbure scalaire finale $R(\varepsilon)$ est un $\mathcal{O}(\varepsilon^{m-2})$.

4 Sommes connexes généralisées et courbure scalaire positive

C'est en utilisant la notion de somme connexe généralisée qu'au début des années 80 M. Gromov et H. B. Lawson d'une part [12] et R. Schoen et S. T. Yau [30] d'autre part ont analysée la structure des variétés Riemannienne qui possèdent une métrique à courbure scalaire positive. La construction présentée par M. Gromov et H. B. Lawson traite exclusivement le cas de somme connexe le long de sphères tandis que R. Schoen et S. T. Yau construisent une métrique à courbure scalaire positive sur la somme connexe le long d'une sous variété quelconque de deux variétés à courbure scalaire positive. En particulier ils démontrent le résultat suivant :

Théorème 5 (Schoen, Yau). *Soient M_1 et M_2 deux variétés compactes de dimension m, munies de métriques à courbure scalaire positive, et soient K_1 et K_2 deux sous variétés compactes (respectivement de M_1 et M_2) de dimension k et de codimension $n := m - k \geq 3$. Supposons aussi qu'il existe un difféomorphisme entre le fibré normal de K_1 dans M_1 et celui de K_2 dans M_2, qui préserve les fibres. Alors la somme connexe généralisée de M_1 et M_2 le long de K_1 et K_2 admet une métrique à courbure scalaire positive.*

Dans ce résultat il est essentiel de faire l'hypothèse que la codimension $n = m - k \geq 3$. Cette restriction est visible dans un des résultats intermédiaires utilsés dans la démonstration, résultat qui consiste à effectuer une transformation conforme des deux métriques initiales à courbure scalaire positive, afin de se ramener au cas de métriques à courbure scalaire nulle sur $M_i \setminus K_i$, $i = 1, 2$ (projection stéréographique). Le reste de la démonstration consiste à modifier soigneusement ces projections stéréographiques de façon à obtenir une métrique à courbure scalaire positive sur la nouvelle variété.

5 Sommes connexes généralisées et courbure scalaire constante

Dans cette section nous présentons la première partie des résultats contenus dans cette thèse. Notre objectif étant de généraliser, au cas des sommes connexes le long de sous-variétés, les résultats obtenus par D. Joyce pour les sommes connexes en des points pour la construction de métriques à courbure scalaire constante. Nous avons divisé cette étude en deux travaux. Dans un premier travail nous étudions le cas de la courbure scalaire non nulle [25] et dans un deuxième travail le cas de métriques à courbure scalaire nulle [26]. Si la partie géométrique de la construction est essentiellement identique dans les deux cas, l'analyse quant à elle est assez différente, comme nous avons déjà observé en présentant les résultats de D. Joyce.

Remarquons que la somme connexe de deux variétés Riemanniennes (M_1, g_1) et (M_2, g_2) le long d'une même sous variété K est une opération moins flexible que la somme connexe en des points. En effet, comme on peut le constater dans l'énoncé du Théoréme 5, on a besoin de plusieurs hypothèses topologiques sur les données initiales afin de pouvoir effectuer une telle opération. Par exemple il faut que les fibrés normaux des plongements de K dans M_1 et dans M_2 soient difféomorphes. De plus, il faut que la codimension de K dans M_1 et M_2 soit $n := m - k \geq 3$.

Pour mieux comprendre cette hypothèse il est utile de penser à la somme connexe généralisée comme étant une somme connexe ponctuelle effectué sur chaque fibre du fibré normal de K dans M_1 et dans M_2. En nous inspirant des résultats de D. Joyce, nous construisons $(g_\varepsilon)_{\varepsilon \in (0,1)}$ une famille de métriques qui ressemblent sur chaque fibre normale à la métrique de Schwarzschild n-dimensionnelle (2) et qui sont solutions approchées de notre problème. Cette construction va nous donner le bon point de départ pour toutes les constructions que nous allons entreprendre. Remarquons que nous devons maintenant travailler en grande dimension étant donné que $n \geq 3$.

Nous avons besoin, pour notre construction d'une autre hypothèse géométrique fondamentale qui est la sous-variété K doit être plongée isométriquement dans M_1 et M_2. Pour bien comprendre la raison de cette nouvelle hypothèse (qui n'apparait pas dans les résultats de Schoen-Yau et Gromov-Lawson), il nous faut faire une petite digression. La stratégie générales que nous comptons mettre en œuvre est celle qui a été utilisée par D. Joyce et qui est décrite dans la Section 3. Comme nous l'avons déjà expliqué, il nous faut résoudre l'équation (3), afin de trouver une métrique a courbure scalaire constante sur la nouvelle variété $M = M_1 \sharp_K M_2$ qui soit proche des métriques initiales, quand le paramètre ε tends vers zéro. Nous avons déjà mentionné que, dans l'équation (3), le premier membre du terme de droite représente l'erreur commise en considérant que la métrique g_ε est une solution de notre problème. Il est essentiel que cette erreure soit suffisament petite afin de procéder à la construction. L'hypothèse de plongement isométrique de K dans les deux variétés initiales est exactement l'hypothèse qu'il nous faut pour avoir un bon contrôle de l'erreur $R - R_{g_\varepsilon}$.

Remarquons que, dans le cas de la somme connexe en des points, la construction des métriques g_ε est plus simple et l'erreur $R - R_{g_\varepsilon}$ est uniformement bornée quand ε tend vers 0. Ce qui traduit le fait que les métriques g_ε sont dans ce cas très proches de métriques à courbure scalaire constante. Dans notre cas, la construction géométrique produit une erreur sur la courbure scalaire beaucoup plus importante mais localisée au milieu du "cou". Pour mesurer cette érreur et contrôler l'inverse de l'opérateur linéarisé, nous avons été amenés à travailler dans des espaces fonctionnels avec poids.

Afin de résoudre le problème de point fixe (4) nous devons inverser le linéarisé de l'opérateur de Yamabe L_{g_ε} et fournir des bonnes estimations *a priori* pour les solutions de

$$L_{g_\varepsilon} v = f \,. \tag{5}$$

Si les estimations sont telles que l'image des termes d'erreur $F_\varepsilon(v)$ (dont le principal terme est $R - R_{g_\varepsilon}$) par l'inverse de l'opérateur L_{g_ε} est de taille petite, alors on sera en mesure de résoudre (4) en utilisant un argument de point fixe.

Ce qui rend différents les cas $R \neq 0$ et $R \equiv 0$, c'est justement l'inversion du linéarisé. Dans le premier cas, il est suffisant de supposer que les opérateurs de Yamabe linéarisés pour les métriques initiales, $L_{g_i} := \Delta_{g_i} + \frac{R}{m-1}$, $i = 1, 2$, sont injectifs (soulignons une fois de plus que cette hypothèse est toujours vérifiée quand $R < 0$). Sous cette hypothèse, nous pouvons conclure que l'opérateur L_{g_ε} est inversible et d'autre part, nous pourrons obtenir des estimations précises sur l'inverse de cet opérateur quand ε tend vers 0. L'enoncé de notre Théorème est le suivant :

Théorème 6. *Soient (M_1, g_1) et (M_2, g_2) deux variétés Riemanniennes compactes de dimension $m \geq 3$, munies de métriques dont la courbure scalaire est constante égale à $R \neq 0$. Soit (K, g_K) une variété Riemannienne compacte de dimension k plongée isométriquement dans (M_1, g_1) et dans (M_2, g_2). Supposons de plus que $n := m - k \geq 3$ et que les deux opérateurs $\Delta_{g_i} + \frac{R}{m-1}$, $i = 1, 2$, sont injectifs. On note g_ε la métrique (solution approchée) définie sur $M_\varepsilon := M_1 \sharp_{K,\varepsilon} M_2$, la somme connexe généralisée de M_1 et M_2 obtenue en excisant de petites voisinages tubulaires de K de rayon ε de chaque variété et en identifiant les deux bords.*

Pour tout ε assez petit, il est possible de munir M_ε d'une métrique $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{m-2}} g_\varepsilon$ à courbure scalaire constante R. Ces métriques sont conforme aux métriques initiales en dehors d'un voisinage tubulaire de K. En outre, le facteur conforme u_ε est proche de 1 au sens où

$$\|1 - u_\varepsilon\|_{L^\infty(M_\varepsilon)} \;\leq\; C\, \varepsilon^{\frac{n-2}{2} - \delta}$$

où $C > 0$ est une constante positive qui ne depend pas de ε et le poids δ peut être choisi dans l'interval $\max\{0, (n-4)/2\} < \delta < (n-2)/2$.

Dans le cas où $R \equiv 0$, comme nous l'avons déjà remarqué dans la Section 3, le linéarisé de l'opérateur de Yamabe est le Laplacien. En particulier, l'hypothèse de non dégénérescence, qui permettait de conclure dans le cas précédent, n'est jamais vérifiée. Pour avoir des chances d'inverser le Laplacien il faut travailler dans l'espace des fonctions orthogonales aux fonctions constantes. Le fait de travailler de manière orthogonale aux fonctions constante va se traduire dans les résultats par le fait que nous n'allons plus pouvoir contrôler la valeur de la courbure scalaire que nous allons obtenir sur la somme connexe des deux variétés. En fait nous allons obtenir des fonctions courbures scalaires qui sont nulles dans l'espace des fonctions orthogonales aux fonctions constantes.

Comme dans la somme connexe en des points, il se trouve que le Laplacian des métriques g_ε développe sur M_ε une fonction propre β_ε qui est très proche d'une fonction égale à une constante positive sur M_1 et à une constante négative sur M_2 (les deux constantes étant choisies de façon à ce que $\int_{M_\varepsilon} \beta_\varepsilon \, dvol_{g_\varepsilon} = 0$). Quand $\varepsilon \to 0$, la valeur propre λ_ε tend vers 0 et la fonction propre associée β_ε converge vers une fonction constante sur les deux composantes connexes de la variété limite M_0. L'existence de cette fonction propre associée à une petite valeur propre a une conséquence importante : on essaie maintenant de travailler dans l'orthogonal des fonctions constantes mais aussi dans l'orthogonal de la fonction propre β_ε. En particulier, les métriques g_ε doivent être construites de façon à ce que l'erreur commise $R - R_{g_\varepsilon}$ soit orthogonale à β_ε.

Pour les raisons évoquées ci-dessus, on adopte maintenant une stratégie légèrement différente de celle utlisée dans le résultat précédent. Au lieu de l'équation (3), on résout dans un premier temps l'équation nonlinéaire

$$L_{g_\varepsilon} v \;=\; F_\varepsilon(v) - \lambda_{F_\varepsilon(v)} \beta_\varepsilon \tag{6}$$

où $\lambda_{F_\varepsilon(v)}$ représente la valeur de la projection des termes d'erreur $F_\varepsilon(v)$ sur β_ε. Remarquons que dans ce contexte $L_{g_\varepsilon} = \Delta_{g_\varepsilon}$ et $F_\varepsilon(v)$ est donné par

$$\begin{aligned} F_\varepsilon(v) \;&=\; c_m\,(R - R_{g_\varepsilon}) + c_m\,(R - R_{g_\varepsilon})\, v - \tfrac{R}{m-1} v \\ &+\; c_m\, R \left((1+v)^{\frac{m+2}{m-2}} - 1 - \tfrac{m+2}{m-2} v \right) \end{aligned}$$

où $c_m = -\frac{m-2}{4(m-1)}$. Ici, $R = R(\varepsilon)$ est maintenant un paramètre, qui géométriquement représente la courbure scalaire finale, et qui est choisi de façon à ce que l'erreur soit orthogonale aux fonctions constantes, $\int_{M_\varepsilon} F_\varepsilon(v)\ dvol_{g_\varepsilon} = 0$. On trouve en fait que $R = \mathcal{O}\left(\varepsilon^{n-2}\right)$, comme dans le cas de la somme connexe en des points pour des variétés à courbure scalaire nulle.

Cette fois ci, on utilise pour résoudre (6), un Théorème de point fixe pour les applications contractantes. Ceci nous permet dans un premier temps d'affirmer que la solution obtenue dépend de manière continue des données du problème et dans un deuxième temps de faire des petites modifications conformes (dont la taille tends ver zéro quand $\varepsilon \to 0$) loin du "cou" pour faire en sorte que le paramètre $\lambda_{F_\varepsilon(v)}$ soit égal à 0. Comme dans le cas de la somme connexe en des points, il est nécessaire de supposer que les deux volumes des variétés initiales sont égaux.

Théorème 7. *Soient (M_1, g_1) et (M_2, g_2) deux variétés Riemanniennes compactes de dimension $m \geq 3$, munies de métriques dont la courbure scalaire est constante égale à zéro. Soit (K, g_K) une variété Riemannienne compacte de dimension k plongée isométriquement dans (M_1, g_1) et dans (M_2, g_2). Supposons que $n := m-k \geq 3$ et que les deux volumes $\mathrm{vol}_{g_1}(M_1)$ et $\mathrm{vol}_{g_2}(M_2)$ soient égaux. On note g_ε la métrique (solution apporchée) définie sur $M_\varepsilon := M_1 \sharp_{K,\varepsilon} M_2$, la somme connexe généralisée de M_1 et M_2 obtenue en excisant de petites voisinages tubulaires de K de rayon ε de chaque variété et en identifiant les deux bords.*

Sous ces hypothèses, pour tout ε assez petit, il est possible de munir M_ε d'une métrique $\tilde{g}_\varepsilon = u_\varepsilon^{4/(m-2)}\, g_\varepsilon$ à courbure scalaire constante (en générale non nulle) $R = \mathcal{O}\left(\varepsilon^{n-2}\right)$. Ces métriques sont conforme aux métriques initiales en dehors d'un voisinage tubulaire des K. En outre, ce facteur conforme u_ε est proche de 1 au sens où

$$\|1 - u_\varepsilon\|_{L^\infty(M_\varepsilon)} \quad \leq \quad C\, \varepsilon^\gamma$$

où $C > 0$ est une constante positive qui ne depend pas de ε et le poids γ peut être choisi dans l'interval $0 < \gamma < 1/4$.

Dans le cas où les deux métriques initiales sont à courbure scalaire nulle mais ne sont pas à courbure de Ricci nulle, nous pouvons améliorer le résultat précédent et obtenir une métrique à courbure scalaire nulle sur la somme connexe (généralisée). L'idée est de pouvoir garantir au même temps le deux conditions $\int_{M_\varepsilon} F_\varepsilon(v)\ \mathrm{d}vol_{g_\varepsilon} = 0$ et $\lambda_{F_\varepsilon(v)} = 0$ en utilisant de petites modifications non conformes des métriques g_1 et g_2 (modifications dont la taille est de l'ordre de ε^{n-2}) en dehors d'un voisinage tubulaire de K. Plus précisément, si les deux métriques initiales ne sont pas à courbure de Ricci nulle, nous avons les développements :

$$\begin{aligned} R_{\tilde{g}_\varepsilon} &= R_{g_\varepsilon} + R_{g_1+rh_1} + R_{g_2+sh_2} \\ &= R_{g_\varepsilon} + r\,K_1 + \mathcal{O}\left(r^2\right) + s\,K_2 + \mathcal{O}\left(s^2\right) \end{aligned}$$

où

$$K_i = \Delta_{g_i}(\mathrm{tr}_{g_i} h_i) + \delta_{g_i}(\delta_{g_i} h_i) + g_i\,(\mathrm{Ric}_{g_i}, h_i) \qquad i = 1, 2$$

L'idée est alors d'utiliser r et s comme des paramètres nous permettant de contrôler les quantités $\int_{M_\varepsilon} F_\varepsilon(v)\ \mathrm{d}vol_{g_\varepsilon}$ et $\lambda_{F_\varepsilon(v)}$ et de les annuler. Nous obtenons alors l'énoncé suivant :

Théorème 8. *Soient (M_1, g_1) et (M_2, g_2) deux variétés Riemanniennes compactes de dimension $m \geq 3$, munies de métriques dont la courbure scalaire est constante égale à zéro. Soit (K, g_K) une variété Riemannienne compacte de dimension k plongée isométriquement dans (M_1, g_1) et dans (M_2, g_2). Supposons que $n := m - k \geq 3$ et que les deux métriques initiales g_1 et g_2 ne soient pas à courbure de Ricci nulle. On note $M_\varepsilon := M_1 \sharp_{K,\varepsilon} M_2$ la somme connexe généralisée de M_1 et M_2 obtenue en excisant de petites voisinages tubulaires de K de rayon ε de chaque variété et en identifiant les deux bords.*

Sous ces hypothèses, pour tout ε assez petit, il est possible de munir M_ε d'une métrique $\tilde{g}_\varepsilon$ à courbure scalaire nulle. En outre ces métriques tendent vers le métriques initiales sur tout compact de $M_i \setminus K$, $i = 1, 2$, quand $\varepsilon \to 0$.

Remarquons que l'existence d'une métrique dont la courbure scalaire est nulle, mais qui 'est pas à courbure de Ricci nulle force la variété à appartenir à la classe (1_+) dans la classification de J. L. Kazdan et F. Warner [18], [19]. Pour une variété M dans la classe (1_+), chaque fonction $f \in \mathcal{C}^\infty(M)$ est la courbure scalaire d'une métrique Riemannienne définie sur M. On peut donc dire que les variétés dans (1_+) sont, en ce qui concerne la courbure scalaire, les plus malléables. C'est seulement pour de telles variétés que nous arrivons à construire une métrique à courbure scalaire nulle sur la somme connexe (généralisée). Comme nous le verrons par la suite, le Théorème 8 aura une application intéressante en relativité générale.

6 Équations de compatibilité d'Einstein

La procédure de perturbation pour des sommes connexes en des points, utilisée par D. Joyce dans le cadre des métriques à courbure scalaire constante, a aussi été appliquée à l'étude de problèmes issus de la relativité générale [14], [15], [13]. Le problème étudié dans ce contexte est celui des "équations de compatibilité d'Einstein". Pour mieux comprendre les motivations physiques à l'origine de ces études, nous présentons brièvement le problème de Cauchy en relativité générale et nous renvoyons le lecteur à [6] pour plus d'informations sur ce sujet.

Soit (Z, γ) une variété Lorentzienne de dimension $(m+1)$, l'équation d'Einstein pour l'espace-temps vide (vacuum space-time) est donnée par

$$\mathrm{Ric}_\gamma = 0 \tag{7}$$

Une hyper-surface de Cauchy dans (Z, γ) est une sous-variété M de dimension m de type espace (space-like), i.e., dont le champ de vecteurs normal a une norme négative, telle que toute courbe inextensible de type temps (time-like), i.e., dont le vecteur tangent a norme négative, a une et une seule intersection avec M. Remarquons que la propriété d'avoir une hyper-surface de Cauchy n'est pas commune à toutes les variétés Lorentziennes. Celles qui admettent une telle hyper-surface sont dites "globalement hyperboliques".

Supposons que (Z, γ) est globalement hyperbolique et supposons de plus que l'équation (7) est vérifiée. Soit M une hyper-surface de Cauchy dans Z. On note g la métrique (Riemannienne) induite par γ sur M et Π la deuxième forme fondamentale de M dans (Z, γ). En combinant l'équation (7) avec les équations de Gauss-Codazzi, on trouve que g et Π doivent vérifier le suivant système, que nous désignerons par "équations de compatibilité d'Einstein"

$$\mathrm{div}_g \, \Pi - \mathrm{d}\,(\mathrm{tr}_g \, \Pi) = 0 \tag{8}$$

$$R_g - |\Pi|_g^2 + (\mathrm{tr}_g \, \Pi)^2 = 0 \tag{9}$$

Le problème de Cauchy en relativité générale se pose de la manière suivante. La donnée initiale est un triplet (M, g, Π), où (M, g) est une variété Riemannienne de dimension m et Π est un 2-tenseur symétrique défini sur M. On cherche alors une variété Lorentzienne de dimension $(m+1)$ satisfaisant l'équation (7), et un plongement $\iota : M \hookrightarrow Z$ tel que $\iota(M)$ est une hyper-surface de Cauchy dans (Z, γ) et γ induit g et Π sur M. Le couple

(Z,γ) est appelé développement de Cauchy de (M, g, Π) et, par définition, (Z,γ) est globalement hyperbolique. Si tout autre développement de Cauchy de (M, g, Π) peut être plongé isométriquement dans (Z,γ), on dit alors que (Z,γ) est le développement maximale de (M, g, Π).

Comme nous avons déjà mentionné, les équations de compatibilité d'Einstein (8) et (9) sont des conditions nécessaires pour l'existence d'un développement de Cauchy de (M, g, Π). Le Théorème suivant, dû à Y. Choquet-Bruhat, nous assure que ces conditions sont aussi des conditions suffisantes

Théorème 9 (**Choquet-Bruhat**). *Supposons que* (M, g, Π)*, où* (M, g) *est une variété Riemannienne de dimension* m *et* Π *est un 2-tenseur symétrique sur* M*, vérifie les équations (8) et (9), alors il existe une variété Lorentzienne* (Z,γ) *de dimension* $(m+1)$ *qui est le développement de Cauchy maximal de* (M, g, Π).

Ce résultat nous donne un important outil pour construire des modèles d'espace-temps. On peut dès lors se concentrer directement sur le système de compatibilité d'Einstein (qui est elliptique) au lieu d'envisager l'étude de l'équation $\mathrm{Ric}_\gamma = 0$ (qui est un système hyperbolique). Dans le cas particulier où l'on cherche des solutions des équations de compatibilité d'Einstein à courbure moyenne constante (i.e., pour lesquelles $\mathrm{tr}_g \, \Pi \equiv \tau$), on a à notre disposition une méthode très puissante pour étudier le système (8)-(9), méthode connue sous le nom de "méthode conforme".

Afin de construire des solutions des équations de compatibilité d'Einstein, nous proposons dans la deuxième partie de cette thèse, une généralisation du résultat obtenu par J. Isenberg, R. Mazzeo et D. Pollack [14], dans le cas où les données initiales sont à courbure moyenne constante (CMC). Cet étude repose largement sur la méthode conforme dont nous donnons ici une brève déscription.

Si l'on cherche des solutions de (8)-(9) à courbure moyenne constante égale à τ, on peut décomposer la deuxième forme fondamentale Π en écrivant

$$\Pi \;=\; \mu + (\tau/m)\, g$$

où μ est un 2-tenseur symétrique à trace nulle $\mathrm{tr}_g \, \mu = 0$. Ensuite on considère les deux changements conformes

$$\begin{aligned} g &= u^{\frac{4}{m-2}} \, \bar{g} \\ \mu &= u^{-2} \, \bar{\mu} \end{aligned}$$

En faisant ça, le système des équations de compatibilité devient

$$\begin{aligned} \mathrm{div}_{\bar g}\,\bar\mu &= 0 \qquad (10)\\ \Delta_{\bar g}\,u + c_m\,R_{\bar g}\,u - c_m\,|\,\bar\mu\,|^2_{\bar g}\,u^{-\frac{3m-2}{m-2}} + c_m\,\frac{m-1}{m}\,\tau^2\,u^{\frac{m+2}{m-2}} &= 0 \qquad (11)\end{aligned}$$

où $c_m = -\frac{m-2}{4\,(m-1)}$ et, comme toujours, notre Laplacien est à spectre négatif. Remarquons que le systême est faiblement couplé car on peut résoudre dans un premier temps l'équation (10), ce qui nous donne la valeur de $|\bar\mu|_{\bar g}$ dans l'équation (11). Enfin, on résoud la deuxième équation en u. Remarquons que la donnée de $(M, \bar g, \bar\mu, u, \tau)$ permet de reconstruire (M, g, Π).

Maintenant nous avons introduites toutes les notions permettant de comprendre le résultat de J. Isenberg, R. Mazzeo et D. Pollack [14]. Dans cet article, ils construisent de nouvelles solutions pour les équations de compatibilité d'Einstein en faisant la somme connexe de deux solutions connues (où bien en ajoutant une anse à une solution connue). Le résultat de base concerne le cas des données initiales définies sur un espace compact et à courbure moyenne constante. Des généralisations de ce résultat ont ensuite été obtenues pour par exemple prendre en compte le cas de données initiales définies sur des espaces asymptotiquement euclidiens, ou asymptotiquement hyperboliques [14], on peut aussi relaxer l'hypothèse de courbure moyenne constante [15], ou bien raffiner la construction afin de ne pas modifier les deux solutions connues au dehors du cou d'identification [9]. Nous nous contenterons de décrire du résultat fondamental de [14], étant donné que cette situation est la plus proche des résultats de cette thèse.

On se donne deux solutions (M_1, g_1, Π_1) et (M_2, g_2, Π_2) des équations de compatibilité d'Einstein, où les variétés sont compactes et l'on suppose que les même courbures moyenne sont constantes toutes deux égales à τ. Ceci revient (modulo les changements conformes décrits ci-dessus) à considérer deux solutions $(M_1, \bar g_1, \bar\mu_1, u_1, \tau)$ et $(M_2, \bar g_2, \bar\mu_2, u_2, \tau)$ des équations (10) et (11).

Soit p_1 un point de M_1 et p_2 un point de M_2. On fait la somme connexe $M_\varepsilon = M_1 \sharp_\varepsilon M_2$ des deux variétés M_1 et M_2 en excisant deux petites boules de rayon ε autour de p_1 et p_2 et en identifiant les bords. En utilisant des fonctions troncature, on construit une famille de métriques solutions approchées $(g_\varepsilon)_{\varepsilon\in(0,1)}$ modelées sur la métrique cylindrique

$$g_{Cyl} = \mathrm{d}t^2 + g_{S^{m-1}}$$

à l'intérieur du cou et égales aux métriques initiales au dehors. En utilisant toujours des fonctions troncature on fabrique aussi une famille de 2-tenseurs symétriques $\mu = \mu(\varepsilon)$ tels que $\mathrm{tr}_{g_\varepsilon}(\mu) = 0$ pour tout $\varepsilon \in (0, 1)$.

En général $\mathrm{div}_{g_\varepsilon} \mu \neq 0$ mais on peut toujours trouver une (petite) correction σ_ε telle que $\mathrm{div}_{g_\varepsilon}(\mu + \sigma_\varepsilon) = 0$. Afin d'obtenir de bonnes estimations sur la correction faite (i.e., des estimations qui nous assurent que la norme de la correction tend vers 0 quand ε tends vers 0) il est utile de chercher σ_ε sous la forme $\sigma_\varepsilon = D_{g_\varepsilon} X$, oú X est un champ de vecteurs et D_{g_ε} est l'opérateur de Killing conforme pour la métrique g_ε, défini par

$$D_{g_\varepsilon} X = \frac{1}{2}\,\mathcal{L}_X\, g_\varepsilon \,-\, \frac{1}{m}\,(\mathrm{div}_{g_\varepsilon} X)\,\cdot\, g_\varepsilon \tag{12}$$

Ici $\mathcal{L}_X\, g_\varepsilon$ indique la dérivée de Lie de la métrique g_ε par rapport au champ de vecteurs X.

La recherche d'un terme correctif revient donc à trouver un champ de vecteurs X vérifiant l'équation

$$\mathfrak{L}_{g_\varepsilon} X = \mathrm{div}_{g_\varepsilon} \mu \tag{13}$$

où $\mathfrak{L}_{g_\varepsilon}$ est le Laplacien vectoriel, defini par $\mathfrak{L}_{g_\varepsilon} := -\,\mathrm{div}_{g_\varepsilon} \circ D_{g_\varepsilon}$. Étant donné que cet opérateur est linéaire et elliptique et étant donné que le terme de droite est la divergence d'un tenseur symétrique, on peut toujours résoudre cette équation sur une variété compacte. En outre, sous une certaine hypothèse de non dégénérescence des deux métriques initiales $\bar{g}_1$ et $\bar{g}_2$, on arrive à trouver les estimations désirées pour X (et par conséquent pour σ_ε).

L'hypothèse de non dégénérescence dont ils ont besoin, J. Isenberg, R. Mazzeo et D. Pollack, est la suivante : si X est un champ de vecteurs dans le noyau de l'opératuer $D_{\bar{g}_i}$ (i.e., un champ de Killing conforme) et si $X(p_i) = 0$, alors $X \equiv 0$, pour $i = 1, 2$. Cette hypothèse est assez naturelle car, sur une variété compacte, le noyau du Laplacien vectoriel est contenu dans le noyau de l'opérateur de Killing conforme, étant donné que $\mathrm{div}_g = -D_g^*$. Sous cette hypothèse, on peut construire et donner une estimation d'une solution de (10) (avec $\bar{g} = g_\varepsilon$) dont la trace est nulle

$$\mu_\varepsilon := \mu(\varepsilon) \,+\, \sigma_\varepsilon \tag{14}$$

Une fois que l'on a obtenu μ_ε, on peut envisager l'étude de l'équation de Lichnerowicz (11) avec $\bar{g} = g_\varepsilon$ et $\bar{\mu} = \mu_\varepsilon$. Afin de résoudre cette équation et être assuré de trouver une solution très proche des solutions initiales u_1 et u_2, on utilise une argument de perturbation tel que celui que l'on a déjà décrit pour l'équation de Yamabe, étant donné que les deux équations ont essentiellement la même structure. Le seul point délicat est l'inversion de l'opérateur de Lichnerowicz linéarisé. Pour ce faire, il suffit de supposer l'injectivité des deux opérateurs de Lichnerowicz linéarisés pour les deux métriques initiales, c'est-à-dire

$$\Delta_{g_i} - |\mu_i|^2_{g_i} - \frac{1}{m}\tau^2 \qquad\qquad i = 1, 2 \qquad (15)$$

Cette condition est impliquée, modulo le principe du maximum, par la condition $\Pi_1 \neq 0$ et $\Pi_2 \neq 0$.

On a alors le résultat suivant :

Théorème 10 (Isenberg, Mazzeo, Pollack). *Soient* (M_1, g_1, Π_1) *et* (M_2, g_2, Π_2) *deux solutions des équations de compatibilité d'Einstein. On suppose que* M_i *sont compactes, de même dimension* $m \geq 3$ *et que les courbures moyennes sont constantes toutes les deux égales à* τ*. Soit* $M_\varepsilon := M_1 \sharp_\varepsilon M_2$ *la somme connexe de ces deux variétés obtenue en excisant une petite boule de rayon* ε *autour des points* $p_i \in M_i$*,* $i = 1, 2$*, et en identifiant les bords. Supposons que tout champ de vecteurs de Killing conforme* X*, qui s'annulle en* p_i*, est trivial sur* M_i *et que* $\Pi_i \neq 0$*, pour* $i = 1, 2$*.*

Alors, pour tout ε *assez petit, on peut construire sur* M_ε *une métrique* $\tilde{g}_\varepsilon$ *et un 2-tenseur symétrique* $\tilde{\Pi}_\varepsilon$*, tels que* $(M_\varepsilon, \tilde{g}_\varepsilon, \tilde{\Pi}_\varepsilon)$ *est une solution des équations de compatibilité d'Einstein dont la courbure moyenne constante est égale à* τ*. De plus, quand* ε *tend vers zéro,* $\tilde{g}_\varepsilon$ *et* $\tilde{\Pi}_\varepsilon$ *tendent vers* g_i *et* Π_i *sur* $M_i \setminus B_R(p_i)$*, où* $B_R(p_i)$ *est une boule de rayon* R *fixé (petit) centrée en* p_i*,* $i = 1, 2$*.*

Remarquons que dans le cas où $\Pi_1 \equiv 0 \equiv \Pi_2$ les équations de compatibilité d'Einstein deviennent simplement $R_{g_i} = 0$, $i = 1, 2$ et, pour des raisons physiques, on parle alors de "données symétriques par rapport au temps" (time-symmetric initial data). Si l'on suppose les deux données initiales sont symétriques par rapport au temps et ont une courbure de Ricci non nulle, le Théorème 8 produit immédiatement de nouvelles solutions des équations de compatibilité d'Einstein sur la somme connexe généralisée des deux variétés initiales.

Enfin soulignons que les résulats de [14] ont donné lieu à plusieurs applications d'intérêt physique. En faisant la somme connexe de deux données initiales, on peut maintenant construire une multitude d'exemples de solutions des équations de compatibilité d'Einstein, ce qui a permis d'une part de mieux comprendre la structure topologique de l'éspace-temps [15] et d'autre part de mieux appréhendre certains aspects liés au comportement des trous noirs [10].

7 Sommes connexes généralisées et équations de compatibilité d'Einstein

Dans cette section nous présentons la construction de nouvelles solutions pour les équations de compatibilité d'Einstein sur la somme connexe généralisée de deux donné initiales (M_1, g_1, Π_1) et (M_2, g_2, Π_2). Notre résultat et la stratégie de la preuve suivent l'esprit du résultat de [14]. Néanmoins, notre construction diffère en plusieurs points de celle de J. Isenberg, R. Mazzeo et D. Pollack.

L'aspect le plus évident est la différence de construction des métriques approchées $(g_\varepsilon)_{\varepsilon\in(0,1)}$. Si on cherche à transposer la construction de [14] on est conduit à construire sur le cou des métriques dont la composante normale (à la sous-variété K) est modelée sur la métrique cylindrique (modulo un facteur d'échelle qui dépend de ε). Malheureusement cette construction ne permet pas de traiter aisément le terme d'erreur qui apparaît dans l'équation de Lichnerowicz

$$\begin{aligned} F_\varepsilon(v) \;:=\; & c_m\,(R_{g_1}-R_{g_\varepsilon})\,\chi_1 \;-\; c_m\,(|\mu_1|^2_{g_1}-|\mu_\varepsilon|^2_{g_\varepsilon})\,\chi_1 \\ & + c_m\,(R_{g_2}-R_{g_\varepsilon})\,\chi_2 \;-\; c_m\,(|\mu_2|^2_{g_2}-|\mu_\varepsilon|^2_{g_\varepsilon})\,\chi_2 \\ & + c_m\,(R_{g_1}-R_{g_\varepsilon})\,\chi_1\,v \;+\; b_m\,(|\mu_1|^2_{g_1}-|\mu_\varepsilon|^2_{g_\varepsilon})\,\chi_1\,v \\ & + c_m\,(R_{g_2}-R_{g_\varepsilon})\,\chi_2\,v \;+\; b_m\,(|\mu_2|^2_{g_2}-|\mu_\varepsilon|^2_{g_\varepsilon})\,\chi_2\,v \\ & + c_m\,|\mu_\varepsilon|^2_{g_\varepsilon}\,\mathcal{O}\left(v^2\right) \;-\; c_m\,\frac{m-1}{m}\,\tau^2\,\mathcal{O}\left(v^2\right) \end{aligned} \tag{16}$$

où $c_m = -\,(m-2)/[4(m-1)]$, $b_m = c_m\cdot(3m-2)/(m-2)$ et $\{\chi_1,\chi_2\}$ est une partition de l'unité. En particulier ce sont les termes $R_{g_i}-R_{g_\varepsilon}$ qui posent des problèmes, pour $i=1,2$.

En revanche, ce terme de courbure scalaire est bien estimé si l'on utilise comme métriques approchées les métriques g_ε que nous avons déjà utilisées pour résoudre l'équation de Yamabe [25], c'est-à-dire des métriques dont la composante normale (à la sous-variété K) est modelée sur la métrique de Schwarzschild.

L'utilisation de ces métriques approchées nous oblige à obtenir une estimation pour le terme correctif σ_ε (celui qui produit la solution de l'équation $\mathrm{div}_{g_\varepsilon} \mu_\varepsilon = 0$, avec $\mu_\varepsilon = \mu(\varepsilon) + \sigma_\varepsilon$). Pour ce faire, nous adoptons dans [27] une stratégie très différente de celle utilisée dans [14]. En particulier, comme nous cherchons σ_ε sous la forma $D_{g_\varepsilon} X$, nous devons étudier le problème elliptique

$$\mathfrak{L}_{g_\varepsilon} X \;=\; \mathrm{div}_{g_\varepsilon} \mu \tag{17}$$

Au lieu d'estimer la première valeur propre de $\mathfrak{L}_{g_\varepsilon}$ (comme cela est fait dans [14]) nous produisons directement une estimation *a priori* de la solution X en termes de μ. Pour cela, nous travaillons une fois de plus dans des espaces à poids, nous constatons que l'estimation que nous trouvons pour X (et donc aussi pour σ_ε et pour le terme μ_ε qui apparaît dans le terme d'erreur $F_\varepsilon(v)$ (16)) est en fait uniforme par rapport au paramètre ε. Ce résultat nous permet ensuite de résoudre assez aisément l'équation nonlinéaire de Lichnerowicz en utilisant un théorème de point fixe.

Pour obtenir l'estimation *a priori* pour X, nous utilisons dans [27] un argument par contradiction. Ceci nous conduit à considérer trois problèmes limites homogènes (avec des différentes conditions de décroissance pour les solutions). Ces problèmes limites, pour lesquels nous voulons démontrer qu'ils n'ont pas de solution, sont

$$\mathfrak{L}_{g_i} X \;=\; 0 \qquad \text{sur } M_i \setminus K,\ i = 1, 2 \tag{18}$$

$$\mathfrak{L}_{\mathbb{R}^k \times \mathbb{S}^n} X \;=\; 0 \qquad \text{sur } \mathbb{R}^k \times \mathbb{S}^n \tag{19}$$

$$\mathfrak{L}_{\mathbb{R}^k \times \mathbb{R}^n} X \;=\; 0 \qquad \text{sur } \mathbb{R}^k \times \mathbb{R}^n \tag{20}$$

où $\mathbb{S}^n$ indique l'espace de Schwarzschild de dimension $n = m - k$ et les deux dernier Laplaciens vectoriels correspondent aux deux métriques produit $g_{\mathbb{R}^k} + g_{\mathbb{S}^n}$ et $g_{\mathbb{R}^k} + g_{\mathbb{R}^n}$, respectivement.

En ce qui concerne les deux derniers problèmes, nous déduisons une contradiction directement en utilisant des résultats issus du b-calculus [28] et de l'analyse de Fourier, tandis que dans le premier cas nous avons besoin d'une hypothèse de non dégénérescence très proche de celle utilisée dans [14]. L'hypothè-se dont nous avons besoin est que l'opérateur de Killing conforme D_{g_i} soit injectif sur M_i, pour $i = 1, 2$.

Notre résultat est le suivant

Théorème 11. *Soient* (M_1, g_1, Π_1) *et* (M_2, g_2, Π_2) *deux solutions des équations de compatibilité d'Einstein avec la même courbure moyenne constante* τ *et la même dimension* $m \geq 3$. *On suppose que les variétés* M_i *sont compactes. Soit* (K, g_k) *une sous-variété compacte de dimension* k *plongée isométriquement dans* (M_1, g_1) *et* (M_2, g_2) *et soit* $n := m - k \geq 3$. *En outre supposons que les fibrés normaux de* K *dans* M_1 *et dans* M_2 *soient difféomorphes. On note* $M_\varepsilon := M_1 \sharp_{K,\varepsilon} M_2$ *la somme connexe généralisée de* M_1 *et* M_2 *obtenue en excisant de petites voisinages tubulaires de* K *de rayon* ε *de chaque variété et en identifiant les deux bords. Supposons en outre que tout champ de vecteurs de Killing conforme* X *soit trivial sur* M_i *et que* $\Pi \neq 0$, *pour* $i = 1, 2$.

Alors, pour tout ε *assez petit, on peut construire sur* M_ε *une métrique* $\tilde{g}_\varepsilon$ *et un 2-tenseur symétrique* $\tilde{\Pi}_\varepsilon$, *tels que* $(M_\varepsilon, \tilde{g}_\varepsilon, \tilde{\Pi}_\varepsilon)$ *soit une solution des équations de compatibilité d'Einstein avec courbure moyenne constante égale à* τ. *De plus, quand* ε *tend vers* 0, $\tilde{g}_\varepsilon$ *et* $\tilde{\Pi}_\varepsilon$ *tendent vers* g_i *et* Π_i *loin d'un voisinage tubulaire de* K *fixé.*

Remarquons que comme dans [14], ce Théorème établi un résultat de base qui peut être étendu dans plusieurs directions. Par example, on peut relaxer l'hypothèse de compacité sur K en demandant que le rayon d'injectivité soit borné inférieurement.On peut étendre assez aisément ce résultat au cas de données initiales asymptotiquement euclidiennes et asymptotiquement hyperboliques.On peut aussi chercher à localiser la construction afin de ne pas modifier les solutions initiales hors d'un voisinage tubulaire de K fixé, comme cela a déjà été fait pour le résultat de [14].

Enfin remarquons qu'une direction de recherche intéressante consiste à approfondir l'étude de la géométrie du cou. En particulier nous avons des bonnes raisons de croire qu'au milieu du cou il est possible de repérer des “horizons apparents”. Ceci impliquerait, dans certains cas, que le développement de Cauchy d'une telle donnée initiale présente des trous noirs. Notre construction pourrait alors être utilisée pour produire, en dimension ≥ 4, des modèles de trous noirs pluri-dimensionnels, avec des topologies assez variées, ce qui pourrait intéresser les spécialistes de la théorie des cordes.

References

[1] M. Anderson, *Dehn filling and Einstein metrics in higher dimensions*, arXiv:math.DG/0303260 (2003).

[2] C. Arezzo, F. Pacard, *Blowing up Kähler manifolds with constant scalar curvature II*, à paraître dans Ann. of Maths.

[3] C. Arezzo, F. Pacard, *Blowing up and desingularizing Kähler manifolds of constant scalar curvature*, Acta Math. 196, no 2, (2006) 179-228.

[4] T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. 55 (1976), 269–296.

[5] T. Aubin, *Some nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics, Springer, (1998).

[6] R. Bartnik, J. Isenberg, *The constraint equations*, The Einstein equations and the large scale behavior of gravitational fields, eds. P. T. Chrusciel and H. Friedrich, Birkhäuser, Basel, (2004), 1–39.

[7] Y. Choquet-Bruhat, *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires*, Acta Math. 88 (1952), 141–225.

[8] P. T. Chruściel, E. Delay, *On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications*, Mém. Soc. Math. de France 94 (2003), 1–103.

[9] P. T. Chruściel, J. Isenberg, D. Pollack, *Initial data engineering*, Comm. Math. Phys. 257 (2005), 29–42.

[10] P. T. Chruściel, R. Mazzeo, *On many-black-hole vacuum spacetimes*, Class. Quantum Grav. 20 (2003), 729–754

[11] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Diff. Geom. 18 (1983), 279–315.

[12] M. Gromov, H. B. Lawson, *The classification of simply connected manifolds of positive scalar curvature.* Ann. of Math. (2) 111 (1980), no. 3, 423–434.

[13] J. Isenberg, D. Maxwell, D. Pollack, *A gluing construction for non vacuum solutions of the Einstein constraint equations*, Adv. Theor. Phys. 9 (2005), 129–172.

[14] J. Isenberg, R. Mazzeo, D. Pollack, *Gluing and wormholes for the Einstein constraint equations*, Comm. Math. Phys. 231 (2002), 529–568.

[15] J. Isenberg, R. Mazzeo, D. Pollack, *On the topology of vacuum spacetime*, Ann. H. Poincaré 4 (2003), 369–383.

[16] D. Joyce, *Constant scalar curvature metrics on connected sums.* Int. J. Math. Math. Sci. (2003), no. 7, 405–450.

[17] D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford University Press, (2000).

[18] J. L. Kazdan, F. Warner, *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature*, Ann. of Math. 101 (1975), 317–331.

[19] J. L. Kazdan, F. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, J. Diff. Geom. 10 (1975), 113–134.

[20] A. G. Kovalev, *Twisted connected sums and special Riemannian holonomy.* , J. Reine Angew. Math. 565 (2003), 125–160.

[21] R. Mazzeo, F. Pacard, *Constant scalar curvature metrics with isolated singularities*, Duke Math. Journal 99 No. 3 (1999) 353–418.

[22] R. Mazzeo, F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, Comm. Anal. Geom. 9 No. 1 (2001) 169–237.

[23] R. Mazzeo, F. Pacard, D. Pollack, *Connected sums of constant mean curvature surfaces in Euclidean 3 space*, J. Reine Angew. Math. 536 (2001), 115–165.

[24] R. Mazzeo, D. Pollack, K. Uhlenbeck, *Connected sums constructions for constant scalar curvature metrics*, Topological Method in Nonlinear Analysis 6 (1995), 207–233.

[25] L. Mazzieri, *Generalized connected sum construction for constant scalar curvature metrics* à paraître dans Communication in Partial Differential Equations.

[26] L. Mazzieri, *Generalized connected sum construction for scalar flat metrics*, arXiv:math.DG/0611778 (2006).

[27] L. Mazzieri, *Generalized gluing for Einstein constraint equations*, Preprint (2007).

[28] R. B. Melrose, *The Atiyah-Patodi-Singer Index Theorem*, Res. Notes Math. 4, A. K. Peters, Wellesly, Mass. (1993).

[29] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Diff. Geom. 20 (1984), 479–495.

[30] R. Schoen, S. T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. 28 (1979), no. 1-3, 159–183.

[31] C. H. Taubes, *The existence of anti-self dual conformal structures*, J. Diff. Geom., 36b(1) (1992), 163–253.

[32] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa 22 (1968), 265–274.

[33] H. Yamabe, *On a deformation of Riemannian structure on compact manifolds*, Osaka Math. J. 12 (1960), 21–37.

[34] S. D. Yang, *A connected sum construction for complete minimal surfaces with finite total curvature,* Commun. Anal. Geom. 9, No.1 (2001), 115–167.

Articolo 1 / Article 1

Generalized connected sum construction for nonzero constant scalar curvature metrics

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Abstract

In this paper we construct constant scalar curvature metrics on the generalized connected sum $M = M_1 \sharp_K M_2$ of two compact Riemannian manifolds (M_1, g_1) and (M_2, g_2) along a common Riemannian submanifold (K, g_K), in the case where the codimension of K is ≥ 3 and the manifolds M_1 and M_2 carry the same nonzero constant scalar curvature S. This yields a generalization of the D. Joyce's results for point-wise connected sums.

Key Words: scalar curvature, connected sum, nonlinear elliptic PDE's on manifolds, conformal geometry

AMS subject classification: 53C21, 58J60, 53A30, 57R65

1 Introduction and statement of the result

Connected sum of solutions of nonlinear problems has revealed to be a very powerful tool in understanding solutions of many geometric problems (minimal and constant mean curvature surfaces [6], [7], constant scalar curvature metrics [4], [8], [5], and recently even Einstein metrics [1]). However, generalized connected sums along a submanifold have not been addressed so much, probably because these constructions are less flexible.

In this paper we consider the problem of constructing solutions to the Yamabe equation (i.e. conformal constant scalar curvature metrics) on the generalized connected sum $M = M_1 \sharp_K M_2$ of two compact m-dimensional Riemannian manifolds (M_1, g_1) and (M_2, g_2) along a common (isometrically embedded) submanifold (K, g_K) of codimension ≥ 3. We are able to perform this generalized connected sum under the assumptions that the two initial Riemannian metrics have the same constant scalar curvature S and the linearized Yamabe operators about the metrics g_i (i.e. the operators $\Delta_{g_i} + S/(m-1)$) have trivial kernels, for $i = 1, 2$.

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To put this result in perspective, let us recall the classical result of Schoen and Yau [10] and Gromov and Lawson [9] which ensures that if the manifolds M_1 and M_2 carry positive scalar curvature metrics, then so does the generalized connected sum $M = M_1 \sharp_K M_2$ along a submanifold K of codimension ≥ 3 and, thanks to the resolution of the Yamabe problem by T. Aubin and R. Schoen, M can be endowed with a constant positive scalar curvature metric. This result however does not give the precise structure of the constant scalar curvature metric one obtains on the generalized connected sum M. In particular, one would like to know how does the constant scalar curvature metric on the connected sum looks like in terms of the constant scalar curvature metric on the summands. Our result does not cover all cases covered by the above mentioned result but, as it is typical for most of the gluing results, we have a very precise description of the metric on the connected sum in terms of the metric on the summands. Indeed, away from the region where the generalized connected sum takes place, we obtain metrics on M which are conformal to the metrics g_i with some conformal factor as close to the constant function 1 as we want.

In the case of connected sum at points a result analogous to ours had been obtained by D. Joyce [4]. Our strategy is roughly speaking the same: we first write down a one dimensional family of approximate solutions metrics $(g_\varepsilon)_{\varepsilon\in(0,1)}$ (where the parameter ε represent the size of the tubular neighborhood we excise from each manifold in order to perform the generalized connected sum), then we study the linearized scalar curvature operator about the metric g_ε and, for all sufficiently small ε, we find suitable conformal factors u_ε such that the metrics $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{m-2}} g_\varepsilon$ have constant scalar curvature S, using a simple fixed point argument. Let us now describe our result more precisely.

Let (M_1, g_1) and (M_2, g_2) be two m-dimensional compact Riemannian manifolds with constant scalar curvature S, and suppose that there exists a k-dimensional Riemannian manifold (K, g_K) which is isometrically embedded in each (M_i, g_i), for $i = 1, 2$, $m \geq 3$, $n := m - k \geq 3$. We also assume that the normal bundles of K in (M_i, g_i) can be diffeomorphically identified. Finally, we assume that on both manifolds, the operator

$$L_{g_i} \quad := \quad \Delta_{g_i} + \frac{S}{m-1} \,, \qquad\qquad i = 1, 2 \tag{1}$$

is injective. Notice that when $S < 0$ this is always the case.

Let $M = M_1 \sharp_K M_2$ be the generalized connected sum of (M_1, g_1) and (M_2, g_2) along K which is obtained by removing an ε-tubular neighborhood of K from each M_i and identifying the two boundaries.

Our main result reads:

Theorem 1. *Under the above assumptions, there exists a real number ε_0 such that for every $\varepsilon \in (0, \varepsilon_0)$ it is possible to endow the generalized connected sum $M = M_1 \sharp_{K,\varepsilon} M_2$ of M_1 and M_2 along K with a constant scalar curvature metric $\tilde{g}_\varepsilon$ whose scalar curvature $S_{\tilde{g}_\varepsilon}$ is constant and equal to S. In addition, the following holds:*

(i) - The metric $\tilde{g}_\varepsilon$ is conformal to the metrics g_i away from a fixed (small) tubular neighborhood of K in M_i, $i = 1, 2$, for a conformal factor u_ε which can be chosen so

that

$$\|u_\varepsilon - 1\|_{L^\infty(M)} \quad \leq \quad c \cdot \varepsilon^{\frac{n-2}{2}-\delta}\,, \tag{2}$$

where $n := m - k$*,* $\max\{\,0, (n-4)/2\,\} < \delta < (n-2)/2$ *and the positive constant* c *does not depend on* ε*.*

(ii) - As ε *tends to* 0*, the metrics* $\tilde{g}_\varepsilon$ *converge to* g_i *on the compact sets of* $M_i \setminus K$*,* $i = 1, 2$*.*

A typical case where our result applies is when both $(M_1, g_1) = (M_2, g_2)$ and K is any submanifold of codimension ≥ 3, provided the operator L_{g_i} has no nontrivial kernel, for $i = 1, 2$.

There are some main technical differences between our construction and D. Joyce's construction in the connected sum case [4]. Our construction seems to be less flexible in the sense that more hypothesis are needed on the summands to obtain the result. In particular (so far) the construction only holds when (K, g_K) is isometrically embedded in both (M_i, g_i) and if this is not the case it seems harder to construct a reasonable approximate solution g_ε to our problem. However, when $k = 1$, the hypothesis on the submanifold K are not so restrictive and we are allowed to glue along any couple of curves, provided they have the same length. The second difference comes from the analysis of the operator L_{g_ε}, the linearized scalar curvature operator about the metric g_ε. As in the connected sum case, the derivation of the estimates of the solution of $L_{g_\varepsilon}\, u = f$ follows from application of the maximum principle. However, in the generalized connected sum case, the estimates for the partial derivatives of the solution u are not as nicely behaved as in the connected sum case. Hopefully, the scalar curvature equation is a semilinear elliptic equation, hence the nonlinear part of this equation only involves the function u and not its partial derivatives.

It is possible to extend our result to the case where $S = 0$ relaxing the fact that the scalar curvature one obtains on the generalized connected sum is equal to 0. Indeed, in this case, the scalar curvature obtained on M might not be in general equal to 0 but will be a constant close to 0.

2 Building the metrics

Let (K, g_K) be a k-dimensional Riemannian manifold isometrically embedded in both the n-dimensional Riemannian manifolds (M_1, g_1) and (M_2, g_2),

$$\iota_i : K \hookrightarrow M_i$$

We assume that the isometric map $\iota_1^{-1} \circ \iota_2 : \iota_1(K) \to \iota_2(K)$ extends to a diffeomorphism between the normal bundles of $\iota_i(K)$ in (M_i, g_i), $i = 1, 2$. We further assume that the metrics g_1 and g_2 have the same constant scalar curvature S. In this section our aim is to perform a generalized connected sum of (M_1, g_1) and (M_2, g_2) along (K, g_K) and to construct on the new manifold $M = M_1 \sharp_K M_2$ a family of metrics $(g_\varepsilon)_{\varepsilon \in (0,1)}$, whose scalar curvature is close to S.

For a fixed $\varepsilon \in (0,1)$, we describe the generalized connected sum construction and the definition of the metric g_ε in local coordinates, the fact that this construction yields a globally defined metric will follow at once.

Let U^k be an open set of $\mathbb{R}^k_z$, and let B^{m-k} be the $(m-k)$-dimensional open ball $(m-k \geq 3)$ of $\mathbb{R}^{m-k}_x$. For $i = 1,2$, the map $F_i : U^k \times B^{m-k} \to W_i \subset M_i$, given by

$$F_i(z,x) \quad := \quad \exp^{M_i}_{(z,0)}(x) \ ,$$

defines local Fermi coordinates near the coordinate patches $F_i(\cdot,0)\,(U) \subset \iota_i(K) \subset M_i$. In these coordinates, the metric g_i can be decomposed as

$$g_i(z,x) \quad = \quad g^{(i)}_{j_1 j_2} dz^{j_1} \otimes dz^{j_2} + g^{(i)}_{\alpha\beta} dx^\alpha \otimes dx^\beta + g^{(i)}_{j\alpha} dz^j \otimes dx^\alpha$$

and it is well known that in this coordinate system

$$g^{(i)}_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}\left(|x|^2\right) \qquad \text{and} \qquad g^{(i)}_{j\alpha} = \mathcal{O}\left(|x|\right) \ .$$

In order to perform the identification between W_1 and W_2 and in order to glue the metrics together and define g_ε, we partially change the coordinate system, by setting

$$\begin{aligned} x &= \varepsilon\, \mathrm{e}^{-t} \cdot \theta && \text{on } F_1^{-1}(W_1) \\ x &= \varepsilon\, \mathrm{e}^{t} \cdot \theta && \text{on } F_2^{-1}(W_2) \ , \end{aligned}$$

for $\varepsilon \in (0,1)$, $\log \varepsilon < t < -\log \varepsilon$, $\theta \in S^{m-k-1}$.

Using these changes of coordinates the expressions of the two metrics g_1 and g_2 on $U^k \times A^1_{\varepsilon^2}$, where $A^1_{\varepsilon^2}$ is the annulus $\{\, \varepsilon^2 < |x| < 1 \,\}$, become respectively

$$\begin{aligned} g_1(z,t,\theta) \quad &= \quad g^{(1)}_{ij} dz^i \otimes dz^j \\ &+ \quad u^{(1)}_\varepsilon{}^{\frac{4}{n-2}} \left[\left(dt \otimes dt + g^{(1)}_{\lambda\mu} d\theta^\lambda \otimes d\theta^\mu \right) + g^{(1)}_{t\theta} dt \ltimes d\theta \right] \\ &+ \quad g^{(1)}_{it} dz^i \otimes dt + g^{(1)}_{i\lambda} dz^i \otimes d\theta^\lambda \end{aligned}$$

and

$$\begin{aligned} g_2(z,t,\theta) \quad &= \quad g^{(2)}_{ij} dz^i \otimes dz^j \\ &+ \quad u^{(2)}_\varepsilon{}^{\frac{4}{n-2}} \left[\left(dt \otimes dt + g^{(2)}_{\lambda\mu} d\theta^\lambda \otimes d\theta^\mu \right) + g^{(2)}_{t\theta} dt \ltimes d\theta \right] \\ &+ \quad g^{(2)}_{it} dz^i \otimes dt + g^{(2)}_{i\lambda} dz^i \otimes d\theta^\lambda \ , \end{aligned}$$

where by the compact notation $dt \ltimes d\theta$ we indicate the general component of the normal metric tensor (i.e., it involves $dt \otimes dt$, $d\theta^\lambda \otimes d\theta^\mu$ and $dt \otimes d\theta^\lambda$ components), whereas the coefficients $g_{t\theta}$ multiplied by $u^{(i)}_\varepsilon{}^{\frac{4}{n-2}}$, $i = 1,2$ represent the correction to the Euclidean metric $u^{(i)}_\varepsilon{}^{\frac{4}{n-2}} \left[\left(dt \otimes dt + g^{(i)}_{\lambda\mu} d\theta^\lambda \otimes d\theta^\mu \right) \right]$, $i = 1,2$ in our coordinate system.

Observe that for $j = 1,2$ we have

$$\begin{aligned} g^{(j)}_{\lambda\mu} &= \mathcal{O}\left(1\right) & \qquad g^{(j)}_{t\theta} &= \mathcal{O}\left(|x|^2\right) \\ g^{(j)}_{it} &= \mathcal{O}\left(|x|^2\right) & \qquad g^{(j)}_{i\lambda} &= \mathcal{O}\left(|x|^2\right) \end{aligned}$$

and

$$u_\varepsilon^{(1)}(t) = \varepsilon^{\frac{n-2}{2}} \mathrm{e}^{-\frac{n-2}{2}t} \qquad \text{and} \qquad u_\varepsilon^{(2)}(t) = \varepsilon^{\frac{n-2}{2}} \mathrm{e}^{\frac{n-2}{2}t} \ .$$

We choose a cut-off function $\chi : (\log\varepsilon, -\log\varepsilon) \to [0,1]$ to be a non increasing smooth function which is identically equal to 1 in $(\log\varepsilon, -1]$ and 0 in $[1, -\log\varepsilon)$ and we choose another cut-off function $\eta : (\log\varepsilon, -\log\varepsilon) \to [0,1]$ to be a non increasing smooth function which is identically equal to 1 in $(\log\varepsilon, -\log\varepsilon - 1]$ and which satisfies $\lim_{t\to -\log\varepsilon} \eta = 0$. Using these two cut-off functions, we can define a new normal conformal factor u_ε by

$$u_\varepsilon(t) := \eta(t)\, u_\varepsilon^{(1)}(t) + \eta(-t)\, u_\varepsilon^{(2)}(t) \tag{3}$$

and the metric g_ε by

$$\begin{aligned} g_\varepsilon(z,t,\theta) \ &:= \ \left(\chi g_{ij}^{(1)} + (1-\chi) g_{ij}^{(2)}\right) dz^i \otimes dz^j \\ &+ \ u_\varepsilon^{\frac{4}{n-2}} \Big[dt \otimes dt + \left(\chi g_{\lambda\mu}^{(1)} + (1-\chi) g_{\lambda\mu}^{(2)}\right) d\theta^\lambda \otimes d\theta^\mu \\ &\qquad\qquad + \left(\chi g_{t\theta}^{(1)} + (1-\chi) g_{t\theta}^{(2)}\right) dt \ltimes d\theta \Big] \\ &+ \ \left(\chi g_{it}^{(1)} + (1-\chi) g_{it}^{(2)}\right) dz^i \otimes dt \\ &+ \ \left(\chi g_{i\lambda}^{(1)} + (1-\chi) g_{i\lambda}^{(2)}\right) dz^i \otimes d\theta^\lambda \ . \end{aligned} \tag{4}$$

Closer inspection of this expression shows that the only objects that are not *a priori* globally defined on the identification of the tubular neighborhoods of $\iota_1(K)$ in M_1 and $\iota_2(K)$ in M_2 are the functions χ and u_ε (since η is used in the construction). However, observe that both cut-off functions can easily be expressed as functions of the Riemannian distance to K in the respective manifolds. Hence they are globally defined and the metric g_ε - whose definition can be obviously completed by setting $g_\varepsilon \equiv g_1$ and $g_\varepsilon \equiv g_2$ out of the polyneck - is a Riemannian metric which is globally defined on the manifold M.

3 Estimate of the scalar curvature

Now we want to estimate the difference $S_{g_\varepsilon} - S$ on the polyneck (which, in the above coordinates, corresponds to $\log\varepsilon + 1 \le t \le -\log\varepsilon - 1$). To begin with, we restrict our attention to the case where $\log\varepsilon + 1 \le t \le -1$. Here the normal conformal factor can be written down as $u_\varepsilon = u_\varepsilon^{(1)} \left(1 + u_\varepsilon^{(2)}/u_\varepsilon^{(1)}\right)$ so, if we define $h = u_\varepsilon^{(2)}/u_\varepsilon^{(1)}$ the metric g_ε looks like

$$g_\varepsilon(z,t,\theta) = g_{ij}^{(1)} dz^i \otimes dz^j + (1+h)^{\frac{4}{n-2}}\, g_{\alpha\beta}^{(1)} dx^\alpha \otimes dx^\beta + g_{i\alpha}^{(1)} dz^i \otimes dx^\alpha \ ,$$

where in fact $h = \mathrm{e}^{(n-2)t} = \varepsilon^{(n-2)} |x|^{2-n}$.

In order to simplify the notations, let us drop the upper $^{(1)}$ indices and simply write

$$g(z,x,h) \ = \ g_{ij} dz^i \otimes dz^j + (1+h)^{\frac{4}{n-2}}\, g_{\alpha\beta} dx^\alpha \otimes dx^\beta + g_{i\alpha} dz^i \otimes dx^\alpha \ .$$

Recall that the following expansions hold

$$\begin{aligned} g_{ij} &= g^K_{ij}(z) + \mathcal{O}(|x|) \\ g_{\alpha\beta} &= \delta_{\alpha\beta} + \mathcal{O}\left(|x|^2\right) \\ g_{i\alpha} &= \mathcal{O}(|x|) \end{aligned}$$

In the following computation we will use the notations

$$\begin{aligned} g_h(z,x) &:= g(z,x,h) \\ g_0(z,x) &:= g(z,x,0) \\ \tilde{g}_h(z) &:= g(z,0,h) \\ \tilde{g}_0(z) &:= g(z,0,0) \end{aligned}$$

and their respective scalar curvature will be denoted by

$$\begin{aligned} S_h &:= S_{g_h} \\ S_0 &:= S_{g_0} \\ \tilde{S}_h &:= S_{\tilde{g}_h} \\ \tilde{S}_0 &:= S_{\tilde{g}_0} \ . \end{aligned}$$

The idea is to estimate the difference between the scalar curvatures of the metrics g_h and g_0 by first estimating the differences with the scalar curvature of the Riemannian product metrics $\tilde{g}_h$ and $\tilde{g}_0$. In fact, we can easily obtain

$$\tilde{S}_h = \tilde{S}_0 + c_n^{-1}\,(1+h)^{-\frac{n+2}{n-2}}\,\Delta^{(x)}_{eucl} h \ .$$

where $c_n = -(n-2)/4(n-1)$.

Next we consider the term $S_h - \tilde{S}_h$. To keep notations short, we agree that $A^{(j)}_l = A^{(j)}_l(z,x,h)$, $j,l \in \mathbb{N}$ is a function, a row vector or a matrix whose coefficients satisfy

$$\begin{aligned} \left|A^{(j)}_l(z,x,h)\right| &\leq C\,|x|^l \\ \left|A^{(j)}_l(z,x,h) - A^{(j)}_l(z,x,0)\right| &\leq C\,|x|^l\,|h| \end{aligned}$$

for some positive constant $C = C(j)$.

We start with the expansions of the coefficients of the metrics g_h (and hence also g_0 which corresponds to g_h when $h=0$) and their inverses in terms of $|x|$

$$\begin{aligned} g^{(h)}_{ij} &= \tilde{g}^{(h)}_{ij}(z) + \mathcal{O}(|x|) \\ g^{(h)}_{\alpha\beta} &= \tilde{g}^{(h)}_{\alpha\beta} + \mathcal{O}\left(|x|^2\right) \\ g^{(h)}_{i\alpha} &= \mathcal{O}(|x|) \end{aligned}$$

and

$$\begin{aligned} g^{ij}_{(h)} &= \tilde{g}^{ij}_{(h)}(z) + A^{(1)}_1 \\ g^{\alpha\beta}_{(h)} &= \tilde{g}^{\alpha\beta}_{(h)} + A^{(2)}_1 \\ g^{i\alpha}_{(h)} &= A^{(3)}_1 \ . \end{aligned}$$

To estimate the Christoffel symbols of the metric g_h, observe that

$$\begin{aligned} g_{(h)}^{\cdot\cdot} \frac{\partial g^{(h)}_{\cdots}}{\partial \ldots} &= \left(\tilde{g}_{(h)}^{\cdot\cdot} + A_1^{(4)}\right)\left(\frac{\partial \tilde{g}^{(h)}_{\cdots}}{\partial \ldots} + A^{(5)} + A_1^{(6)}\,[\nabla h]\right) \\ &= \tilde{g}_{(h)}^{\cdot\cdot} \frac{\partial \tilde{g}^{(h)}_{\cdots}}{\partial \ldots} + A_0^{(7)} + A_1^{(8)}\,[\nabla h] \ . \end{aligned}$$

As a consequence we have that

$$\Gamma(h, \nabla h) \quad = \quad \tilde{\Gamma}(h, \nabla h) + A_0^{(9)} + A_1^{(10)}\,[\nabla h] \ .$$

Moreover, it is straightforward to check that

$$\tilde{\Gamma}(h, \nabla h) \quad = \quad A_0^{(11)} + A_0^{(12)}\,[\nabla h] \ .$$

Proceeding with the computation we get

$$\begin{aligned} \frac{\partial \Gamma}{\partial \ldots}(h, \nabla h) &= \frac{\partial \tilde{\Gamma}}{\partial \ldots}(h, \nabla h) + A_0^{(13)}\,[\nabla h] + A_1^{(14)}\,[\nabla h, \nabla h] + A_1^{(15)}\,\left[\nabla^2 h\right] \\ \frac{\partial \tilde{\Gamma}}{\partial \ldots}(h, \nabla h) &= A_0^{(16)}\,[\nabla h] + A_0^{(17)}\,[\nabla h, \nabla h] + A_0^{(18)}\,\left[\nabla^2 h\right] \ , \end{aligned}$$

while for the product of Christoffel symbols, we get

$$\Gamma\,\Gamma(h, \nabla h) \quad = \quad \tilde{\Gamma}\,\tilde{\Gamma}(h, \nabla h) + A_0^{(19)} + A_0^{(20)}[\nabla h] + A_1^{(21)}\,[\nabla h, \nabla h]$$

and hence we get for the coefficients of the curvature tensors

$$\begin{aligned} R(h, \nabla h, \nabla^2 h) &= \tilde{R}(h, \nabla h, \nabla^2 h) + A_0^{(22)} + A_0^{(23)}\,[\nabla h] \\ &+ A_1^{(24)}\,[\nabla h, \nabla h] + A_1^{(25)}\,\left[\nabla^2 h\right] \\ \tilde{R}(h, \nabla h, \nabla^2 h) &= A_0^{(26)} + A_0^{(27)}\,[\nabla h] + A_0^{(28)}\,[\nabla h, \nabla h] + A_0^{(29)}\,\left[\nabla^2 h\right] \ . \end{aligned}$$

Finally, observing that

$$g_h^{\cdot\cdot}\, g_h^{\cdot\cdot} \quad = \quad \tilde{g}_h^{\cdot\cdot}\, \tilde{g}_h^{\cdot\cdot} + A_1^{(30)}$$

and contracting twice the Riemann tensor, we get the expression for the scalar curvature

$$S_h \quad = \quad \tilde{S}_h + A_0^{(31)} + A_0^{(32)}\,[\nabla h] + A_1^{(33)}\,[\nabla h, \nabla h] + A_1^{(34)}\,\left[\nabla^2 h\right] \ .$$

Choosing $h \equiv 0$ in the previous computation we get immediately

$$S_0 \quad = \quad \tilde{S}_0 + A_0^{(35)}(z, x, 0) \ .$$

Hence we have obtained

$$\begin{aligned} S_h \quad = \quad S_0 \quad &+ \quad c_n^{-1}(1+h)^{-\frac{n+2}{n-2}} \Delta^{(x)}_{eucl} h + A_0^{(36)}(z, x, h) - A_0^{(36)}(z, x, 0) \\ &+ \quad A_0^{(37)}\,[\nabla h] + A_1^{(38)}\,[\nabla h, \nabla h] + A_1^{(39)}\,\left[\nabla^2 h\right] \ . \end{aligned}$$

Since $h = \varepsilon^{n-2}|x|^{2-n}$ is $\Delta^{(x)}_{eucl}$-harmonic we conclude that

$$
\begin{aligned}
S_h - S_0 &= A_0^{(40)}\mathcal{O}\left(|h|\right) + A_0^{(41)}[\nabla h] + A_1^{(42)}[\nabla h, \nabla h] + A_1^{(43)}[\nabla^2 h] \\
&= \mathcal{O}\left(\varepsilon^{n-2}|x|^{1-n}\right) \;=\; \mathcal{O}\left(\varepsilon^{-1}\mathrm{e}^{(n-1)t}\right)
\end{aligned}
$$

We remark that, when $t = \log\varepsilon + 1$, we get the estimate $S_{g_\varepsilon} - S_{g_1} = \mathcal{O}(\varepsilon^{n-2})$. On the other hand it is straightforward to check that the same estimate holds for $\log\varepsilon < t < \log\varepsilon + 1$, since the cut-off η is bounded with bounded derivatives.

Let us now treat the case where $-1 \le t \le 0$. The action of the cut-off functions is effective here, hence *a priori* we have to handle the full expression of g_ε. In any case, it is easy to see that one can always write for $-1 \le t \le 0$

$$
\begin{aligned}
g_\varepsilon(z,t,\theta) &= \left(g^{(1)}_{ij} + \mathcal{O}\left(|x|\right)\right) dz^i \otimes dz^j \\
&+ (1+h)^{\frac{4}{n-2}}\left(g^{(1)}_{\alpha\beta} + \mathcal{O}\left(|x|\right)\right) dx^\alpha \otimes dx^\beta \\
&+ \left(g^{(1)}_{i\alpha} + \mathcal{O}\left(|x|\right)\right) dz^i \otimes dx^\alpha \; .
\end{aligned}
$$

Hence, if we take $g(z,x,h) = g_\varepsilon$ and $g(z,x,0) = g_1 + \mathcal{O}\left(|x|\right)$ in the previous computation we get immediately $S_{g_\varepsilon} - S_{g_1 + \mathcal{O}(|x|)} = \mathcal{O}\left(\varepsilon^{n-2}|x|^{1-n}\right)$.

Now we observe that in general if we have two metrics g and $\hat{g}$ such that $\hat{g} = g + \mathcal{O}\left(|x|\right)$, then $\hat{\Gamma} = \Gamma + \mathcal{O}\left(1\right)$ and $\hat{R} = R + \mathcal{O}\left(|x|^{-1}\right)$, thence the scalar curvatures of g and $\hat{g}$ are related by $\hat{S} = S + \mathcal{O}\left(|x|^{-1}\right)$.

To conclude, we have that

$$
S_{g_\varepsilon} - S_{g_1} = \mathcal{O}\left(|x|^{-1}\right) = \mathcal{O}\left(\varepsilon^{-1}\mathrm{e}^{t}\right) \;,
$$

for $-1 \le t \le 0$. In particular, when $t = 0$ we get $S_{g_\varepsilon} - S_{g_1} = \mathcal{O}\left(\varepsilon^{-1}\right)$. Similar estimates hold for $S_{g_\varepsilon} - S_{g_2}$ when $0 \le t \le -\log\varepsilon$. To summarize the computation above, we state the following

Proposition 2 (Estimate of the scalar curvature). *There exists a constant $C > 0$ independent of $\varepsilon \in (0,1)$ such that*

$$
|S_{g_\varepsilon} - S| \;\le\; C \cdot \varepsilon^{-1}\left(\mathrm{ch}\, t\right)^{1-n} \;, \tag{5}
$$

for $|t| \le |\log\varepsilon|$.

4 Analysis of a linear operator

In order to carry out the proof of Theorem 1, we want to solve, using a perturbation argument, the Yamabe equation

$$
\Delta_{g_\varepsilon} u + c_m\, S_{g_\varepsilon}\, u \;=\; c_m\, S\, u^{\frac{m+2}{m-2}} \;, \tag{6}
$$

where $c_m = -(m-2)/4(m-1)$ (notice that our Laplacian is conventionally the negative definite one). If we are able to find such a solution u, then, by performing the conformal change $\tilde{g}_\varepsilon = u^{\frac{4}{m-2}} g_\varepsilon$ we get a metric $\tilde{g}_\varepsilon$, whose scalar curvature is constant and equal to S.

We write $u = 1 + v$ where v is a small function ($|v| \leq 1/2$) so that the equation becomes

$$\Delta_{g_\varepsilon} v - \frac{4\,c_m}{m-2}\, S\, v \;=\; c_m\,(S - S_{g_\varepsilon}) + c_m\,(S - S_{g_\varepsilon})\, v + c_m\, S \left[(1+v)^{\frac{m+2}{m-2}} - 1 - \frac{m+2}{m-2}\, v \right] . \tag{7}$$

The reason for doing this change of variable is that we are looking for a conformal factor which is very close to 1, thence the more v is close to zero, the more the final metric is close to the starting ones and we have a precise notion of its structure.

We define the linearized scalar curvature operator by

$$L_{g_\varepsilon} \;:=\; \Delta_{g_\varepsilon} - \frac{4\,c_m}{m-2}\, S \;=\; \Delta_{g_\varepsilon} + \frac{S}{m-1} \tag{8}$$

Our aim in this section is to study the operator L_{g_ε} and provide an *a priori* estimate for the solutions of the linear problem

$$L_{g_\varepsilon} v \;=\; f \tag{9}$$

This is the starting point and the key-tool for the nonlinear perturbation argument, which will produce a solution to equation (7).

Unfortunately a global *a priori* estimate is not immediately available for solution to the equation (9). We will be able to obtain such an estimate using an argument by contradiction, once a local *a priori* estimate is obtained for the solutions of the linearized problem on the polyneck.

4.1 Local expression for Δ_{g_ε} on the polyneck and barrier functions

The first step is to write down the local expression for the g_ε-Laplacian, which is the principal part of our operator, on the polyneck. Clearly, we can restrict ourselves to the set $\{\log\varepsilon + 1 \leq t \leq 0\}$, where $|x| = \varepsilon \mathrm{e}^{-t}$. We have at hand the expansions

$$\begin{aligned}
g^\varepsilon_{ij} &= g^K_{ij}(z) + \mathcal{O}\left(|x|\right) \\
g^\varepsilon_{it} &= \mathcal{O}\left(|x|^2\right) \\
g^\varepsilon_{i\lambda} &= \mathcal{O}\left(|x|^2\right) \\
g^\varepsilon_{tt} &= u_\varepsilon^{\frac{4}{n-2}} \left(1 + \mathcal{O}\left(|x|^2\right)\right) \\
g^\varepsilon_{t\lambda} &= u_\varepsilon^{\frac{4}{n-2}}\, \mathcal{O}\left(|x|^2\right) \\
g^\varepsilon_{\lambda\mu} &= u_\varepsilon^{\frac{4}{n-2}} \left(g_{\lambda\mu}(\theta) + \mathcal{O}\left(|x|^2\right)\right) ,
\end{aligned}$$

where $g_{\lambda\mu}(\theta)$ is the common value of $g^{(1)}_{\lambda\mu}(\theta)$ and $g^{(2)}_{\lambda\mu}(\theta)$. Hence

$$\sqrt{g_\varepsilon} \;=\; \sqrt{\det(g^K_{ij}(z))} \cdot \sqrt{\det(g_{\lambda\mu}(\theta))} \cdot u_\varepsilon^{\frac{2n}{n-2}}(t) \cdot \left[\, 1 + \mathcal{O}\left(|x|\right) \right] .$$

Therefore, for the coefficients of the inverse matrix we have the expansions

$$
\begin{aligned}
g_\varepsilon^{ij} &= g_K^{ij}(z) + \mathcal{O}\left(|x|\right) \\
g_\varepsilon^{it} &= \mathcal{O}\left(|x|^2\right) \\
g_\varepsilon^{i\lambda} &= \mathcal{O}\left(|x|^2\right) \\
g_\varepsilon^{tt} &= u_\varepsilon^{-\frac{4}{n-2}}\left[1 + \mathcal{O}\left(|x|\right)\right] \\
g_\varepsilon^{t\lambda} &= \mathcal{O}\left(|x|^2\right) \\
g_\varepsilon^{\lambda\mu} &= u_\varepsilon^{-\frac{4}{n-2}} g^{\lambda\mu}\left[1 + \mathcal{O}\left(|x|\right)\right] .
\end{aligned}
$$

A straightforward computation yields the expression we were looking for

$$
\begin{aligned}
\Delta_{g_\varepsilon} = u_\varepsilon^{-\frac{4}{n-2}} \Bigg[& \partial_t^2 + (n-2)\,\mathrm{th}\left(\frac{n-2}{2}\,t\right) \cdot \partial_t \\
& + \Delta_{S^{n-1}}^{(\theta)} + u_\varepsilon^{\frac{4}{n-2}} \cdot \Delta_K^{(z)} + \mathcal{O}\left(|x|\right) \cdot \Phi(\nabla, \nabla^2) \Bigg] ,
\end{aligned}
$$

where $\Phi(\nabla, \nabla^2)$ is a nonlinear differential operator involving first order and second order partial derivatives with respect to t, θ^λ and z^j and whose coefficients are bounded uniformly on the polyneck, as $\varepsilon \in (0, 1)$.

To obtain the local *a priori* estimates, the key tools are the maximum principle for the g_ε-Laplacian and the construction of barrier functions. In order to find the latter, let us remark that

$$
\begin{aligned}
\partial_t^2 + \left(\frac{n-2}{2}\right)^2 \cdot \left[\mathrm{ch}\left(\frac{n-2}{2}t\right) \cdot u\right] = \; & \mathrm{ch}\left(\frac{n-2}{2}t\right) \partial_t^2 \cdot u \\
& + (n-2)\,\mathrm{sh}\left(\frac{n-2}{2}\,t\right) \cdot u .
\end{aligned}
$$

Hence we can conjugate the g_ε-Laplacian by a multiple of the function $\mathrm{ch}(t(n-2)/2)$ (and of course, in particular, by u_ε) to obtain the following identity

$$
\Delta_{g_\varepsilon} = u_\varepsilon^{-\frac{n+2}{n-2}} \mathcal{L}_\varepsilon \left(u_\varepsilon \,\cdot\right) , \tag{10}
$$

where

$$
\begin{aligned}
\mathcal{L}_\varepsilon = \; & \partial_t^2 - \left(\frac{n-2}{2}\right)^2 + \Delta_{S^{n-1}}^{(\theta)} \\
& + u_\varepsilon^{\frac{4}{n-2}} \cdot \Delta_K^{(z)} + \mathcal{O}\left(|x|\right) \cdot \tilde{\Phi}(\nabla, \nabla^2) ,
\end{aligned} \tag{11}
$$

where the linear second order differential operator $\tilde{\Phi}(\nabla, \nabla^2)$ enjoys similar properties as the operator Φ above. For $(2-n)/2 \leq \delta \leq 0$ we have that

$$
\mathcal{L}_\varepsilon \left(\mathrm{ch}t\right)^\delta = \left[\delta^2 - \left(\frac{n-2}{2}\right)^2 + \mathcal{O}\left(|x|\right)\right] \cdot (\mathrm{ch}t)^\delta + \left(\delta - \delta^2\right) \cdot (\mathrm{ch}t)^{\delta-2} .
$$

The choice of the parameter $\delta \in ((2-n)/2, 0)$ obviously implies that

$$
\delta - \delta^2 \leq 0 \qquad \text{and} \qquad \delta^2 - \left(\frac{n-2}{2}\right)^2 \leq 0 .
$$

In order to estimate the term $\mathcal{O}(|x|)$ let us take $\alpha > 0$ and let $\varepsilon_\alpha \in (0,1)$ be chosen so that $\log \varepsilon_\alpha + \alpha < 0$ or equivalently $\varepsilon_\alpha \mathrm{e}^{\alpha} < 1$, then it is easy to see that $|x| \leq \mathrm{e}^{-\alpha}$ for every $\varepsilon \in (0, \varepsilon_\alpha)$ and every $t \in [\log \varepsilon + \alpha, 0]$. Finally, if we choose $\alpha > 0$ such that

$$\mathrm{e}^{-\alpha} \quad \leq \quad -\frac{1}{2}\left[\delta^2 - \left(\frac{n-2}{2}\right)^2\right],$$

we obtain that, for every $\varepsilon \in (0, \varepsilon_\alpha)$ and for $t \in [\log \varepsilon + \alpha, 0]$

$$\mathcal{L}_\varepsilon \cdot (\mathrm{ch} t)^\delta \quad \leq \quad \frac{1}{2}\left[\delta^2 - \left(\frac{n-2}{2}\right)^2\right] \cdot (\mathrm{ch} t)^\delta .$$

When $0 \leq \delta \leq (n-2)/2$ we use the function $\mathrm{ch}\,\delta t$ and we get

$$\begin{aligned} \mathcal{L}_\varepsilon \cdot (\mathrm{ch}\,\delta t) \quad &= \quad \left[\delta^2 - \left(\frac{n-2}{2}\right)^2 + \mathcal{O}(|x|)\right] \cdot \mathrm{ch}\,\delta t \\ &\leq \quad \frac{1}{2}\left[\delta^2 - \left(\frac{n-2}{2}\right)^2\right] \cdot \mathrm{ch}\,\delta t \end{aligned}$$

with similar restrictions on ε and t.

We define the function φ_δ by

$$\begin{aligned} \varphi_\delta \quad &= \quad u_\varepsilon^{-1} \cdot (\mathrm{ch} t)^\delta \qquad \text{if} \quad -\frac{n-2}{2} \leq \delta \leq 0 \\ \varphi_\delta \quad &= \quad u_\varepsilon^{-1} \cdot \mathrm{ch} \delta t \qquad \text{if} \quad 0 \leq \delta \leq \frac{n-2}{2} \end{aligned}$$

and taking into account the conjugation described above (10), we can state the following

Lemma 3. *Given $\delta \in (-\frac{n-2}{2}, \frac{n-2}{2})$ there exist a real number $\alpha = \alpha(n,\delta) > 0$ and a constant $C = C(n,\delta) \geq 0$ such that for every $\varepsilon \in (0, \varepsilon_\alpha)$ we have*

$$\Delta_{g_\varepsilon} \varphi_\delta \quad \leq \quad - C \cdot u_\varepsilon^{-\frac{4}{n-2}} \cdot \varphi_\delta \tag{12}$$

in the set $T_\alpha^\varepsilon = \{\log \varepsilon + \alpha \leq t \leq -\log \varepsilon - \alpha\}$.

In particular the functions φ_δ can be used as barrier functions in the set T_α^ε.

4.2 Local *a priori* estimate using the maximum principle

We first provide a local *a priori* estimate for the g_ε-Laplacian, then we will observe that a similar estimate holds for the operator L_{g_ε}. Let us assume that v, f are bounded functions satisfying $\Delta_{g_\varepsilon} v = f$ in T_α^ε. The inequality found in Lemma 3 multiplied by a nonnegative real constant $a \geq 0$ yields

$$\Delta_{g_\varepsilon}(a\,\varphi_\delta - v) \quad \leq \quad - a\, C \cdot u_\varepsilon^{-\frac{4}{n-2}} \cdot \varphi_\delta \; - \; f$$

If we choose

$$a := C' \left[\sup_{T_\alpha^\varepsilon} \left| u_\varepsilon^{\frac{4}{n-2}} \cdot \varphi_\delta^{-1} f \right| + \sup_{\partial T_\alpha^\varepsilon} \left| \varphi_\delta^{-1} v \right| \right] ,$$

where $C' = \max\{1, C^{-1}\}$ and $\partial T_\alpha^\varepsilon = \{t = \pm \log \varepsilon \pm \alpha\}$, we immediately get

$$\begin{aligned} \Delta_{g_\varepsilon} (a\varphi_\delta - v) &\leq 0 \quad \text{in} \quad T_\alpha^\varepsilon \\ a\varphi_\delta - v &\geq 0 \quad \text{on} \quad \partial T_\alpha^\varepsilon . \end{aligned}$$

Hence, by the maximum principle $a\,\varphi_\delta - v \geq 0$ on T_α^ε. In particular, we obtain

$$\sup_{T_\alpha^\varepsilon} \left| \varphi_\delta^{-1} v \right| \leq C' \left[\sup_{T_\alpha^\varepsilon} \left| u_\varepsilon^{\frac{4}{n-2}} \cdot \varphi_\delta^{-1} f \right| + \sup_{\partial T_\alpha^\varepsilon} \left| \varphi_\delta^{-1} v \right| \right] . \tag{13}$$

In order to simplify the expression above, which is the estimate we were looking for, it is sufficient to replace u_ε by its expression and to observe that for every $\lambda \in \mathbb{R}$ there exist two constants $K_1(\lambda), K_2(\lambda) \geq 0$ such that

$$K_1(\lambda)\,(\mathrm{ch}t)^\lambda \leq \mathrm{ch}\lambda t \leq K_2(\lambda)\,(\mathrm{ch}t)^\lambda$$

for $t \in \mathbb{R}$. Performing simple manipulations, the estimate (13) can be written as

$$\sup_{T_\alpha^\varepsilon} \left| \psi_\varepsilon^{\frac{n-2}{2}-\delta} v \right| \leq C_{n,\delta} \cdot \left[\sup_{T_\alpha^\varepsilon} \left| \psi_\varepsilon^{\frac{n+2}{2}-\delta} f \right| + \sup_{\partial T_\alpha^\varepsilon} \left| \psi_\varepsilon^{\frac{n-2}{2}-\delta} v \right| \right] , \tag{14}$$

where $\psi_\varepsilon = \varepsilon\,\mathrm{ch}t$ and $C_{n,\delta}$ is a positive constant depending on n and δ.

Let us assume now that $v, f \in \mathcal{C}^0(T_\alpha^\varepsilon)$ are functions verifying $L_{g_\varepsilon} v = f$, then it easily follows from (14) that

$$\begin{aligned} \sup_{T_\alpha^\varepsilon} \left| \psi_\varepsilon^{\frac{n-2}{2}-\delta} v \right| \leq C_{n,\delta} \cdot \Bigg[& \sup_{T_\alpha^\varepsilon} \left| \psi_\varepsilon^{\frac{n+2}{2}-\delta} f \right| + \sup_{T_\alpha^\varepsilon} \left| \psi_\varepsilon^{\frac{n+2}{2}-\delta} S v \right| \\ & + \sup_{\partial T_\alpha^\varepsilon} \left| \psi_\varepsilon^{\frac{n-2}{2}-\delta} v \right| \Bigg] . \end{aligned}$$

If we write the term $\left| \psi_\varepsilon^{\frac{n+2}{2}-\delta} S\,v \right|$ as $\left| \psi_\varepsilon^2 S \right| \left| \psi_\varepsilon^{\frac{n-2}{2}-\delta} v \right|$, the only term to control is the factor $\left| \psi_\varepsilon^2 S \right|$, but it can be easily seen that, for a suitable constant $C'' > 0$

$$\left| \psi_\varepsilon^2 S \right| \leq C'' \cdot (\, \varepsilon^2 + \mathrm{e}^{-2\alpha} \,)$$

for all $\varepsilon \in (0, \varepsilon_\alpha)$. Hence, for large enough $\alpha > 0$, we get

$$C_{n,\delta} \cdot \sup_{T_\alpha^\varepsilon} \left| \psi_\varepsilon^{\frac{n+2}{2}-\delta} S\,v \right| \leq \frac{1}{2} \sup_{T_\alpha^\varepsilon} \left| \psi_\varepsilon^{\frac{n-2}{2}-\delta} v \right| .$$

Introducing this information back in the above estimate, we get

Proposition 4 (**Local *a priori* estimate**). *Given $\delta \in (-\frac{n-2}{2}, \frac{n-2}{2})$, there exist a real number $\alpha = \alpha(n,\delta) > 0$ and a constant $C_{n,\delta} \geq 0$ such that for all $\varepsilon \in (0, \varepsilon_\alpha)$ and all $v, f \in \mathcal{C}^0(T_\alpha^\varepsilon)$ satisfying $L_{g_\varepsilon} v = f$, the following estimate holds*

$$\sup_{T_\alpha^\varepsilon} \left| \psi_\varepsilon^{\frac{n-2}{2}-\delta} v \right| \leq C_{n,\delta} \cdot \left[\sup_{T_\alpha^\varepsilon} \left| \psi_\varepsilon^{\frac{n+2}{2}-\delta} f \right| + \sup_{\partial T_\alpha^\varepsilon} \left| \psi_\varepsilon^{\frac{n-2}{2}-\delta} v \right| \right] , \tag{15}$$

where $\psi_\varepsilon = \varepsilon\,\mathrm{ch}t$.

4.3 Global *a priori* estimate

Thanks to the previous local result, we will be able to prove a global *a priori* estimate. To introduce the result, we define a smooth distance function ψ_ε by

$$\psi_\varepsilon := \begin{cases} \varepsilon \mathrm{ch} t & \text{in } T_1^\varepsilon \\ 1 & \text{in } M \setminus T_0^\varepsilon \end{cases},$$

where $T_\rho^\varepsilon := \{\log\varepsilon + \rho \leq t \leq -\log\varepsilon - \rho\}$, for $\rho \geq 0$ and ψ_ε interpolate smoothly between these definitions in $T_0^\varepsilon \setminus T_1^\varepsilon$.

Proposition 5 (Global *a priori* estimate). *Let $M_\varepsilon := M_1 \sharp_{K,\varepsilon} M_2$ (briefly M) be the generalized connected sum obtained by removing an ε-tubular neighborhood $V_i(\varepsilon)$ of $\iota_i(K)$ from each M_i, $i = 1, 2$ and identifying the two boundaries. Suppose that both L_{g_1} and L_{g_2} have trivial kernel on M_1 and on M_2 respectively, then for every $\delta \in \left(-\frac{n-2}{2}, \frac{n-2}{2}\right)$ there exist a real number $\alpha = \alpha(n, \delta) > 0$ and a constant $C_{n,\delta} \geq 0$ such that for every $\varepsilon \in (0, \varepsilon_\alpha)$ and every functions $v, f \in \mathcal{C}^0(M)$ satisfying $L_{g_\varepsilon} v = f$, the following estimate holds*

$$\sup_M \left| \psi_\varepsilon^{\frac{n-2}{2}-\delta} v \right| \leq C_{n,\delta} \cdot \left[\sup_M \left| \psi_\varepsilon^{\frac{n+2}{2}-\delta} f \right| \right]. \tag{16}$$

The proof is by contradiction. Let us assume that the statement is false. Then for every $j \in \mathbb{N}$ we can find a triple $(\varepsilon_j, v_j, f_j)$ such that

1. $\varepsilon_j < \mathrm{e}^{-j}$
2. $L_{g_{\varepsilon_j}} v_j = f_j$
3. $\sup_M \left| \psi_{\varepsilon_j}^{\frac{n-2}{2}-\delta} v_j \right| = 1$
4. $\lim_{j\to\infty} \sup_M \left| \psi_{\varepsilon_j}^{\frac{n+2}{2}} f_j \right| = 0$.

For every $j \in \mathbb{N}$ we consider a point p_j such that $\left| \psi_{\varepsilon_j}^{\frac{n-2}{2}-\delta}(p_j) \cdot v_j(p_j) \right| = 1$, then (up to a subsequence) we have to distinguish two cases:

Case 1. $p_j \in M \setminus T_\alpha^{\varepsilon_j}$ for every $j \in \mathbb{N}$.

Case 2. $p_j \in T_\alpha^{\varepsilon_j}$ for every $j \in \mathbb{N}$.

Without loss of generality we can assume (up to a subsequence) that $p_j \in M_1 \setminus V_1(\varepsilon_j)$, for all $j \in \mathbb{N}$, where the notation $V_1(\rho)$ indicates a tubular neighborhood of K of size $\rho > 0$ in M_1. Hence, in the first case, all the p_j's lie in the compact set $Q_1(\mathrm{e}^{-\alpha}) := M_1 \setminus V_1(\mathrm{e}^{-\alpha})$, then (up to a subsequence) they must converge to a point $p_\infty \in Q_1(\mathrm{e}^{-\alpha})$. We prove now that, for every compact set $Q(\sigma) := Q_1(\sigma) \cup Q_2(\sigma) = (M_1 \setminus V_1(\sigma)) \cup (M_2 \setminus V_2(\sigma))$, $\sigma > 0$, the sequence of functions $(v_j)_{j\in\mathbb{N}}$ converges (up to a subsequence) to a function v_∞ in $L^\infty(Q(\sigma))$. In particular this implies that $|v_\infty(p_\infty)| > 0$.

In order to prove the uniform convergence of the v_j's on the compact $Q(\sigma)$, we start by observing that

$$|v_j| \leq \left(\inf_{Q(\sigma)} \left| \psi_j^{\frac{n-2}{2}-\delta} \right| \right)^{-1} \quad \leq \quad 2\,/\,\sigma$$

and hence $\|v_j\|_{L^\infty(Q^\sigma)} \leq 2/\sigma$. The next step is to provide the ∇v_j's with an $L^\infty(Q(\sigma))$-uniform bound. For this reason let us quote the following L^p-regularity result [3] for solutions of linear elliptic equations

Theorem 6 (**L^p-regularity for linear elliptic equations**). *Let be $L = a^{ij}\partial_{ij} + b^i\partial_i + c$, where the a, b, c's are functions defined on an open domain $\Omega \subset \mathbb{R}^m$, let be $1 < p < \infty$ and let be $u \in W^{2,p}_{loc}(\Omega) \cap L^p(\Omega)$. Moreover suppose that:*

1. *$a^{ij} \in \mathcal{C}^0(\Omega)$; $b^j, c \in L^\infty(\Omega)$; $f \in L^p(\Omega)$*
2. *There exist $\lambda, \Lambda > 0$ such that $|a^{ij}|, |b^j|, |c| \leq \Lambda$ and $a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ for every $\xi \in \mathbb{R}^n$*
3. *$Lu = f$*

then, for every $\Omega' \subset\subset \Omega$, the following estimate holds:

$$\|u\|_{W^{2,p}(\Omega')} \quad \leq \quad C \cdot \left[\ \|u\|_{L^p(\Omega)} \ + \ \|f\|_{L^p(\Omega)}\ \right]\ , \tag{17}$$

for a suitable constant C.

This result can be restated in our context by saying:

Corollary 7. *Let be $\sigma > 0$ and suppose that the linear elliptic differential operator $L_g = \Delta_g + c$ is defined on a geodesic ball $B_{\sigma/2}$ of the Riemannian manifold (M, g), where c is a continuous bounded function on $B_{\sigma/2}$. Moreover let be $1 < p < \infty$ and let be $u \in W^{2,p}_{loc}(B_{\sigma/2}) \cap L^p(B_{\sigma/2})$, $f \in L^p(B_{\sigma/2})$ such that $L_g u = f$, then for every $0 < r < \sigma/2$ the following estimate holds*

$$\|u\|_{W^{2,p}(B_r)} \quad \leq \quad C \cdot \left[\ \|u\|_{L^p(B_{\sigma/2})} \ + \ \|f\|_{L^p(B_{\sigma/2})}\ \right]\ , \tag{18}$$

for a suitable constant C (depending on σ).

In our case it is convenient to cover the compact set $Q(\sigma)$ by finitely many geodesic balls of radius $r = \sigma/4$. We deduce that there exist positive constants C_0, C_1 and C_2 such that

$$\begin{aligned} \|v_j\|_{W^{2,p}(B_{\sigma/4})} &\leq C_0 \cdot \left[\ \|v_j\|_{L^p(B_{\sigma/2})} \ + \ \|f_j\|_{L^p(B_{\sigma/2})}\ \right] \\ &\leq C_1 \cdot \left[\ \|v_j\|_{L^\infty(B_{\sigma/2})} \ + \ \|f_j\|_{L^\infty(B_{\sigma/2})}\ \right] \\ &\leq C_2\,/\,\sigma\ . \end{aligned}$$

Thanks to Sobolev Embedding Theorem, the space $W^{2,p}$ is continuously embedded in L^∞ for $p > m/2$, in particular there exists a positive constant C_3 such that $\|\nabla v_j\|_{L^\infty(B_{\sigma/4})} \leq C_3/\sigma$ for every $j \in \mathbb{N}$. Hence, thanks to the Ascoli's Theorem, we conclude that (up to a subsequence) the sequence $(v_j)_{j\in\mathbb{N}}$ uniformly converges to a function v_∞ on every ball $B_{\sigma/4}$ of the finite covering. Using a classical diagonal argument we obtain the uniform convergence on each $Q(\sigma)$.

To summarize, in the **Case 1**, we have found for every fixed $\sigma > 0$ a subsequence of the v_j's that converges to a function v_∞ with respect to the norm $L^\infty(Q(\sigma))$, thence $v_\infty \in \mathcal{C}^0(Q(\sigma))$ and, for $\sigma = \mathrm{e}^{-\alpha}$, we get $|v_\infty(p_\infty)| > 0$, as announced.

Let us consider now the **Case 2**. Since each p_j is in $T_\alpha^{\varepsilon_j}$, we can apply the local *a priori* estimate (15) obtained in the previous section to deduce that

$$\begin{aligned} C_{n,\delta}^{-1} &\leq \sup_{T_\alpha^{\varepsilon_j}} \left| \psi_{\varepsilon_j}^{\frac{n+2}{2}-\delta} f_j \right| + \sup_{\partial T_\alpha^{\varepsilon_j}} \left| \psi_{\varepsilon_j}^{\frac{n-2}{2}-\delta} v_j \right| \\ &\leq \sup_M \left| \psi_{\varepsilon_j}^{\frac{n+2}{2}-\delta} f_j \right| + \max_{\partial V_1(\mathrm{e}^{-\alpha})} \left| \psi_{\varepsilon_j}^{\frac{n-2}{2}-\delta} v_j \right| + \max_{\partial V_2(\mathrm{e}^{-\alpha})} \left| \psi_{\varepsilon_j}^{\frac{n-2}{2}-\delta} v_j \right| . \end{aligned}$$

This shows that we can choose a sequence of points $(q_j)_{j\in\mathbb{N}} \subset \partial V_1(\mathrm{e}^{-\alpha}) \cup \partial V_2(\mathrm{e}^{-\alpha})$ such that

$$\lim_{j\to\infty} \left| \psi_{\varepsilon_j}^{\frac{n-2}{2}-\delta}(q_j) \cdot v_j(q_j) \right| = C_{n,\delta}^{-1} .$$

In particular we have that $\lim_{j\to\infty} |v_j(q_j)| = C_4 > 0$, for a suitable positive constant C_4 depending on n, δ and α. Using then the L^∞-convergence to v_∞ on the compact set $Q(\mathrm{e}^{-\alpha})$, it is easy to conclude that also $|v_\infty(q_\infty)| > 0$, where $q_\infty \in \partial V_1(\mathrm{e}^{-\alpha}) \cup \partial V_2(\mathrm{e}^{-\alpha})$ is the limit (up to a subsequence) of the sequence $(q_j)_{j\in\mathbb{N}}$.

Hence, in both the cases, we have found a point $P \in M \setminus T_\alpha^\varepsilon$ such that $v_\infty(P) \neq 0$. Without loss of generality we can suppose that $P \in M_1 \setminus \iota_1(K)$: if we prove that $L_{g_1} v_\infty = 0$ on M_1, then v_∞ must be identically zero, because of the hypothesis (1) on the kernel of L_{g_1} and we have a contradiction.

The final task is then to prove that v_∞ lies in the kernel of L_{g_1}. This will be achieved in two steps. The first one amounts to say that $L_{g_1} v_\infty$ is zero as a distribution on $M_1 \setminus \iota_1(K)$, the second one amounts to estimate the growth of v_∞ near $\iota_1(K)$ and then to conclude by means of the following classical result

Theorem 8 (**Removable singularities**). *Suppose that*

$$\begin{cases} L_{g_1} u = 0 & \textit{in } \mathfrak{D}'(M_1 \setminus \iota_1(K)) \\ |u| \leq C \cdot \mathrm{d}_{g_1}(\,\cdot\,, \iota_1(K))^{-\gamma} & \textit{in } V_1(\rho) , \end{cases}$$

for $0 < \gamma < n-2$*, a suitable real number* $\rho > 0$ *and a constant* $C \geq 0$*, then* $u \in \mathcal{C}^\infty(M_1)$ *and satisfies* $L_{g_1} u = 0$ *on* M_1*.*

The idea of the proof of Theorem 8 is quite natural. First, using standard elliptic estimates, one can get the bound $|\nabla u| \leq C \cdot \mathrm{d}_{g_1}(\,\cdot\,, \iota_1(K))^{-\gamma-1}$ on $V_1(\rho/2)$. This allows one to prove that $L_{g_1} u = 0$ on M_1 in the sense of distributions, via integration by parts. Finally, using elliptic regularity (see [2], for example), one can obtain $u \in \mathcal{C}^\infty(M_1)$.

To complete the proof of Proposition 5, we choose $\varphi \in \mathfrak{D}(M_1 \setminus \iota_1(K))$ and $\sigma > 0$ such that $\operatorname{supp}\varphi \subset Q_1(\sigma)$. We claim that

$$\int_{M_1} v_\infty L_{g_1}\varphi \,\mathrm{d}vol_{g_1} \quad = \quad 0 \;.$$

This identity is obtained by taking the limit, as ε_j tends to 0 in the expression

$$\int_M v_j \, L_{g_{\varepsilon_j}}\varphi \,\mathrm{d}vol_{g_{\varepsilon_j}} = \int_M f_j \,\varphi \,\mathrm{d}vol_{g_{\varepsilon_j}} \;.$$

Clearly, the right hand side of this expression tends to zero as ε_j tends to 0. As far as the right hand side is concerned the metrics g_{ε_j}'s converge in the $\mathcal{C}^2$ topology to the metric g_1 on the compact set $Q_1(\sigma)$. This implies that the $L_{g_{\varepsilon_j}}\varphi$'s uniformly converge to $L_{g_1}\varphi$ on this set, hence the left hand side converges to the required expression as ε_j tends to 0.

To conclude we have to control the growth of v_∞ near $\iota_1(K)$. The simple remark that, on $V_1(\rho)$, $\rho \in (0,1)$

$$\frac{1}{2}\,|x| \;\leq\; \psi_{\varepsilon_j} \;\leq\; 2\,|x|$$

for every $j \in \mathbb{N}$ implies that, for a suitable constant $C_5 \,>\, 0$,

$$|x|^{\frac{n-2}{2}-\delta}\,|v_j| \;\leq\; C_5 \;.$$

As a consequence we have that

$$|v_\infty| \;\leq\; C_5 \cdot |x|^{\delta-\frac{n-2}{2}} \;=\; C \cdot |x|^{-\gamma} \;,$$

where $\gamma \,:=\, (n-2)/2 \,-\, \delta$. Since $(2-n)\,/\,2 \,<\, \delta \,<\, (n-2)/2$, then $0 \,<\, \gamma \,<\, n-2$, as needed.

5 Nonlinear analysis: a fixed point argument

We are now ready to solve equation (7). It is clear that our goal is achieved if we are able to exhibit a function $v_\varepsilon \in L^\infty(M)$ which verifies

$$v_\varepsilon = L_{g_\varepsilon}^{-1} \circ F_\varepsilon\,(v_\varepsilon)\;, \tag{19}$$

where

$$F_\varepsilon(v) \;:=\; c_m\,(S - S_{g_\varepsilon}) + c_m\,(S - S_{g_\varepsilon})\,v + \,c_m\,S\,\left[\;(1+v)^{\frac{m+2}{m-2}} \;-\; 1 \;-\; \frac{m+2}{m-2}\,v\;\right]\;.$$

In other words we are looking for a fixed point of the operator $L_{g_\varepsilon}^{-1} \circ F_\varepsilon$ (observe that, as a consequence of the Proposition 5, the operator L_{g_ε} is injective for sufficiently small ε; since it is also self-adjoint, then it is invertible).

We claim that, for a suitable choice of δ and for sufficiently small ε there exists a real number $r_\varepsilon > 0$ such that

$$\|v\|_{L^\infty(M)} \;\leq\; r_\varepsilon \quad\Longrightarrow\quad \|L_{g_\varepsilon}^{-1} \circ F_\varepsilon(v)\|_{L^\infty(M)} \;\leq\; r_\varepsilon \tag{20}$$

Indeed, using the scalar curvature estimates of Proposition 2 it is easy to see that for suitable $C_0 > 0$,

$$\sup_M \left| \psi_\varepsilon^{\frac{n+2}{2}-\delta} F_\varepsilon(v) \right| \;\; \leq \;\; C_0 \cdot \Big[\, \varepsilon^{n-2} \, + \, \varepsilon^{\frac{n}{2}-\delta} \, + \, \|v\|^2_{L^\infty(M)} \, \Big]$$

Moreover, there exist a positive constant C_1 such that

$$\psi_\varepsilon^{\delta-\frac{n-2}{2}} \left| \varepsilon^{n-2} + \varepsilon^{\frac{n}{2}-\delta} + \|v\|^2_{L^\infty(M)} \right| \;\; \leq \;\; C_1 \cdot \Big[\, \varepsilon^{\frac{n-2}{2}+\delta} \, + \, \varepsilon \, + \, \varepsilon^{\delta-\frac{n-2}{2}} \cdot \|v\|^2_{L^\infty(M)} \, \Big]$$

Therefore, using the global *a priori* estimate (16) obtained in Proposition 5 and the hypothesis of the claim (20) we get

$$\|L^{-1}_{g_\varepsilon} \circ F_\varepsilon(v)\|_{L^\infty(M)} \;\; \leq \;\; C_2 \cdot \Big[\, \varepsilon^{\frac{n-2}{2}+\delta} \, + \, \varepsilon \, + \, \varepsilon^{\delta-\frac{n-2}{2}} \cdot r_\varepsilon^2 \, \Big] \; ,$$

where the positive constant C_2 is the product of C_0, C_1 and the constant of the estimate (16). To conclude the proof of (20) it is sufficient to choose $r_\varepsilon > 0$ such that

$$\varepsilon^{\delta-\frac{n-2}{2}} \cdot r_\varepsilon^2 \; \leq \; r_\varepsilon/(2\,C_2) \qquad \text{and} \qquad \varepsilon^{\frac{n-2}{2}+\delta} \, + \, \varepsilon \; \leq \; r_\varepsilon/(2\,C_2) \; .$$

The first condition is satisfied if we choose $r_\varepsilon := \varepsilon^{\frac{n-2}{2}-\delta}/(2C_2)$. With this choice, the second inequality becomes

$$\varepsilon^{2\delta} + \varepsilon^{\delta-\left(\frac{n-2}{2}-1\right)} \leq 1/(2\,C_2)^2 \; .$$

Now it is clear that if $\max\{0, (n-2)/2 - 1\} < \delta < (n-2)/2$, then it is possible to find $\varepsilon_0 \in (0, \varepsilon_\alpha)$ such that the last inequality is verified for all $\varepsilon \in (0, \varepsilon_0)$. For these ε's and the corresponding $r_\varepsilon := \varepsilon^{\frac{n-2}{2}-\delta}/(2\,C_2)$ the claim follows. Thence, if $\|v\|_{L^\infty(M)}$, also

$$\|L^{-1}_{g_\varepsilon} \circ F_\varepsilon(v)\|_{L^\infty(M)} \leq r_\varepsilon \; ,$$

when $\varepsilon \, < \, \varepsilon_0$.

It is easy to check that the mapping

$$v \in L^\infty(M) \longmapsto L^{-1}_{g_\varepsilon} \circ F_\varepsilon(v) \in L^\infty(M)$$

is continuous and compact. This later property follows from the fact that the equation we want to solve is a semilinear equation and hence, if $v \in L^\infty(M)$, then $L^{-1}_{g_\varepsilon} \circ F_\varepsilon(v) \in W^{2,p}(M)$ for all $p > 1$, and the embedding $W^{2,p}(M) \hookrightarrow L^\infty(M)$ is compact, provided $p > m/2$. The well known Schauder's Theorem guarantees the existence of a fixed point $v_\varepsilon \in L^\infty(M)$, namely

$$v_\varepsilon \; = \; L^{-1}_{g_\varepsilon} \circ F_\varepsilon\,(v_\varepsilon) \; ,$$

which satisfies $\|v_\varepsilon\|_{L^\infty(M)} \leq r_\varepsilon$.

A priori the function v_ε is only bounded, but a simple boot-strap argument (based on Corollary 7) shows that $v_\varepsilon \in \mathcal{C}^\infty(M)$.

Finally, observe that as $\varepsilon \to 0$, then $r_\varepsilon \to 0$ and consequently so does $\|v_\varepsilon\|_{L^\infty(M)}$. This shows that the conformal factor $u_\varepsilon \; = \; 1 + v_\varepsilon$ is as close to 1 as we want and more precisely the definition of r_ε implies the estimate (2). This completes the proof of Theorem 1.

The reason why we will treat the scalar flat case independently (i.e., the case where the initial manifolds have the same constant scalar curvature $S = 0$) is that the analysis needed to handle this case is rather different. In fact the assumption that the linearized Yamabe operators $\Delta_{g_i} + S/(m-1)$ of the initial metrics g_i, $i = 1, 2$ are injective, which is crucial to carry out the proof of Theorem 1, does not make sense in the scalar flat case. Here the linearized operator reduces to the Laplacian of the initial metric, for which this assumption is not fulfilled. This will force us to work orthogonally with respect to the kernel of the Laplacian (namely the space of constant functions). In addition, it turns out that, on the generalized connected sum, the first non zero eigenvalue of the Laplacian is very small and actually tends to zero as $\varepsilon \to 0$. This fact will make the search of suitable *a priori* estimates for the linearized operator harder. Because of this, in performing the nonlinear analysis, one also has to take some care in estimating the projection of the error terms over the eigenfunction associated to the small eigenvalue.

References

[1] M. Anderson, *Dehn filling and Einstein metrics in higher dimensions*, arxiv:math.DG/0303260, (2003).

[2] T. Aubin, *Some nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics, Springer, (1998).

[3] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equation of Second Order*, Springer, (1983).

[4] D. Joyce, *Constant scalar curvature metrics on connected sums.* Int. J. Math. Math. Sci. 7, (2003), 405–450.

[5] R. Mazzeo, F. Pacard, *Constant scalar curvature metrics with isolated singularities*, Duke Math. Journal 99 No. 3, (1999), 353–418.

[6] R. Mazzeo, F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, Comm. Anal. Geom. 9 No. 1, (2001), 169–237.

[7] R. Mazzeo, F. Pacard, D. Pollack, *Connected sums of constant mean curvature surfaces in Euclidean 3 space*, J. Reine Angew. Math. 536, (2001), 115–165.

[8] R. Mazzeo, D. Pollack, K. Uhlenbeck, *Connected sums constructions for constant scalar curvature metrics*, Topological Method in Nonlinear Analysis 6, (1995), 207–233.

[9] M. Gromov, H. B. Lawson, *The classification of simply connected manifolds of positive scalar curvature.*, Ann. of Math. (2) 111, No. 3, (1980), 423–434.

[10] Schoen, R. and Yau, S. T. *On the structure of manifolds with positive scalar curvature.*, Manuscripta Math. 28, No. 1-3, (1979), 159–183

Articolo 2 / Article 2

Generalized connected sum construction for scalar flat metrics

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Abstract

In this paper we construct constant scalar curvature metrics on the generalized connected sum (also known as fiber sum) $M = M_1 \sharp_K M_2$ of two compact Riemannian scalar flat manifolds (M_1, g_1) and (M_2, g_2) along a common Riemannian submanifold (K, g_K) whose codimension is ≥ 3. Here we present two constructions : the first one produces a family of "small" (in general non zero) constant scalar curvature metrics on the generalized connected sum of M_1 and M_2. It yields an extension of Joyce's result [6] for point-wise connected sums in the spirit of the work presented in [16] for nonzero constant scalar curvature metrics.

When the initial manifolds are not Ricci flat, and in particular they belong to the (1_+) class in the Kazdan-Warner classification, we refine the first construction in order to produce a family of scalar flat metrics on M. As a consequence we get new solutions to the Einstein Constraint equations on the generalized connected sum of two compact time symmetric initial data sets, extending the Isenberg-Mazzeo-Pollack gluing construction [10].

Key Words: scalar curvature, connected sum, nonlinear elliptic PDE's on manifolds, conformal geometry, Einstein constraint equations

AMS subject classification: 53C21, 58J60, 53A30, 57R65, 83C05

1 Introduction and statement of the results

This last two decades gluing techniques for solutions of nonlinear problems has been successfully applied in several situations. They has been used to understand solutions to problems arising from the geometry (minimal and constant mean curvature surfaces [13], [14], constant scalar curvature metrics [6], [15], [12], and recently even Einstein metrics [1]) and from physics (Einstein constraint equations [10] and [9]). However

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most of the results are concerned with the connected sum at points (point-wise connected sum), whereas the case of connected sum along a submanifold (generalized connected sum or fiber sum) has received less attention. This kind of construction is clearly less flexible than the first one, nevertheless it has revealed to be a very powerful tool in studying for example the structure of the manifolds with positive scalar curvature (see [17] and [18]).

In this paper we will show how the generalized connected sum construction for nonzero constant scalar curvature metrics introduced in [16] can be extended to the case where the initial manifolds carry scalar flat metrics. In other words we produce a family of new solutions to the Yamabe equation by gluing together two compact scalar flat Riemennian manifolds (M_1, g_1) and (M_2, g_2), along a common submanifold (K, g_K) of codimension ≥ 3. The reason to requiring high codimension lies in the geometry of the construction. In fact in order to produce the generalized connected sum $M = M_1 \sharp_K M_2$ we assume that the normal bundles of K in M_1 and M_2 are diffeomorphic, then we remove from both the initial manifolds a tubular neighborhood of K of size $\varepsilon > 0$ and we perform a fiber-wise identification between the left over boundaries (fiber sum). This way we obtain a sort of polyneck which glues together the differential structures of M_1 and M_2, but we still need to identify the Riemannian structure. If m is the dimension of the starting manifolds, k is the dimension of K and $n := m - k$ its codimension in M_i, $i = 1, 2$, it turns out (see [6] and [16]) that a good choice is to model the metric of the polyneck fiber-wise around an n-dimensional Schwarzschild metric, whose existence requires that n has to be ≥ 3.

The reason why we treat the scalar flat case independently from the non zero constant scalar curvature case (i.e. when the initial manifolds have the same constant scalar curvature $S \neq 0$) is that the analysis needed to handle this case is rather different. In fact the assumption that the linearized Yamabe operators $\Delta_{g_i} + S/(m-1)$ of the initial metrics g_i, $i = 1, 2$ are injective, is crucial to carry out the proof in the non zero case. In the scalar flat situation, the linearized operator reduces to the Laplacian for which this assumption is not fulfilled and this will force us to work orthogonally with respect to the kernel of the Laplacian (namely the space of constant functions). In addition, it turns out that, on the generalized connected sum, the first non zero eigenvalue of the Laplacian is very small and actually tends to zero as $\varepsilon \to 0$. This fact will make the search of suitable *a priori* estimates for the linearized operator harder. Because of this, when we will perform the nonlinear analysis, we will have to take some care in estimating the projection of the error terms over the eigenfunction associated to the small eigenvalue. We will show that, if the construction is done with care, such a projection can be chosen to be zero.

We present in this paper two kinds of construction. The first one is is more general than the second one but it has a major drawback. In fact following this method we are not allowed to choose a scalar flat metric on the generalized connected sum, although the error can be chosen as small as we want (notice that similar phenomena happen in the point-wise connected sum case [6]). The second construction is an improvement of the first one and enables us to obtain scalar flat curvature metric on the final manifold, but it requires the hypothesis that the summands are non Ricci flat. In particular, in order to obtain a scalar flat generalized connected sums, it is necessary that both the manifolds M_1 and M_2 belong to the (1_+) class in the Kazdan-Warner classification [7], [8]. An important corollary of the second construction is that it provides a gluing construction for time symmetric initial data sets in the context of the Einstein Constraint equations. In this sense our result partially completes the work of Isenberg-Mazzeo-Pollack [10], which treats the point-wise connected sum of non time symmetric Cauchy data.

In sections 2-5 we present the first method. As in the non scalar flat case, we write down a family of approximate solution metrics $(g_\varepsilon)_{\varepsilon\in(0,1)}$ (where the parameter $\varepsilon > 0$ represents the size of the tubular neighborhood we excise from each manifold in order to perform the generalized connected sum) and then we find out a conformal factor u_ε such that for sufficiently small $\varepsilon > 0$ the metrics $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{m-2}} g_\varepsilon$, $\varepsilon \in (0,1)$, are (small) constant scalar curvature metrics. As mentioned before, by this method it is not possible to ensure that the scalar curvature $S = S_{\tilde{g}_\varepsilon}$ of the metrics we obtain is exactly zero. However we will show that the scalar curvature of the metrics we obtain are as close to zero as we want, namely $S = \mathcal{O}\left(\varepsilon^{n-2}\right)$. Let us now describe this result more precisely.

Let (M_1, g_1) and (M_2, g_2) be two m-dimensional compact Riemannian manifolds with scalar flat metric, and suppose that there exists a k-dimensional Riemannian manifold (K, g_K) which is isometrically embedded in both (M_1, g_1) and M_2, g_2, $m \geq 3$, $n := m - k \geq 3$. We also assume that the normal bundles of K in (M_i, g_i) can be diffeomorphically identified (this is necessary to perform the fiber sum, see [18]). Another natural assumption is that the starting manifolds have the same volume and in particular we assume that $\mathrm{vol}_{g_1}(M_1) = 1 = \mathrm{vol}_{g_2}(M_2)$. Notice that this condition turns out to be necessary also in the case of point-wise connected sum of two scalar flat metrics [6].

Let $M_\varepsilon = M_1 \sharp_{K,\varepsilon} M_2$ (briefly M) be the generalized connected sum of (M_1, g_1) and (M_2, g_2) along K which is obtained by removing an ε-tubular neighborhood of K from each M_i and identifying the two left over boundaries. Our main result reads :

Theorem 1. *Under the above assumptions, there exists a real number $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ it is possible to endow M with a constant scalar curvature metric $\bar{g}_\varepsilon$, whose scalar curvature $S_{\bar{g}_\varepsilon}$ is a $\mathcal{O}\left(\varepsilon^{n-2}\right)$. In addition the metric $\bar{g}_\varepsilon$ is conformal to the metrics g_i away from a fixed (small) tubular neighborhood of K in M_i, $i = 1, 2$ for a conformal factor u_ε which can be chosen so that*

$$\|u_\varepsilon - 1\|_{L^\infty(M)} \quad \leq \quad C \cdot \varepsilon^\gamma \tag{1}$$

where $C > 0$ and $\gamma \in (0, 1/4)$. In particular the new metrics tend to the old ones on the compact sets of $M_i \setminus K$, $i = 1, 2$ in the $\mathcal{C}^2$ topology, as $\varepsilon \to 0$.

Section 6 is devoted to the description of the second construction, which works in the non Ricci flat case. In this situation we will be able to improve the construction of the approximate solution metrics in order to obtain a scalar flat metric on the generalized connected sum. In fact, if the starting manifolds are non Ricci flat, we are allowed to create two correction terms by means of slight non conformal modifications of the initial metrics away from the gluing locus, what enable us to ensure that the error terms are orthogonal to the space generated by the constant functions and - roughly speaking - to the first non constant eigenfunction (the one whose eigenvalue tends to zero in the limit for $\varepsilon \to 0$). This is enough to carry out the analysis and to construct on the generalized connected sum a solution of the Yamabe equation with prescribed zero scalar curvature. As already mentioned, the non Ricci flat condition on a scalar flat manifold implies that such a manifold belongs to the (1_+) class in the Kazdan-Warner classification. This seems to be quite natural since this class of manifolds are in some sense more flexible as far as the prescription of the scalar curvature is concerned. In other words, if a manifold M is in class (1_+), any smooth function on it can be viewed as the scalar curvature of some Riemannian metric [7], [8].

The statement of our second result is the following:

Theorem 2. *Let M be the generalized connected sum of two Riemannian scalar flat, non Ricci flat manifolds (M_1, g_1) and (M_2, g_2) of dimension $m \geq 3$ along a common isometrically embedded submanifold (K, g_K) of codimension at least 3 (recall that in order to perform the generalized connected sum it is necessary that the normal bundles of K in M_1 and M_2 are diffeomorphic). Under these assumptions, there exists a real number $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, it is possible to endow M with a scalar flat metrics $\bar{g}_\varepsilon$. Moreover the new metrics tend to the old ones on the compact sets of $M_i \setminus K$, $i = 1, 2$ in the $\mathcal{C}^2$ topology, as $\varepsilon \to 0$.*

In section 7 we will show how Theorem 2 applies to a problem of physical interest. It is well known [4] a solution to the hyperbolic Einstein system in general relativity can be found by evolving a suitable initial data set (or Cauchy data set). More precisely a space-like m-dimensional hypersurface M in a $(m+1)$-dimensional Lorentzian manifold (Z, γ) do evolve to a solution of $\mathrm{Ric}_\gamma = 0$ if and only if the following Einstein constraint equations are satisfied

$$\mathrm{div}_g \, \Pi - d\,(\mathrm{tr}_g \, \Pi) = 0 \tag{2}$$

$$S_g - |\Pi|_g^2 + (\mathrm{tr}_g \, \Pi)^2 = 0 \tag{3}$$

where g and Π represent the induced Riemannian metric and the second fundamental form of M respectively, whereas S_g is the scalar curvature of (M, g, Π). In the case where $\Pi \equiv 0$ the Cauchy data set is said to be time symmetric and the system above reduces to the vanishing of the scalar curvature. Therefore Theorem 2 automatically provides a generalized gluing construction for non Ricci flat initial data sets, in the spirit of [10].

2 Geometric construction

The geometric construction we use here is essentially the same we used in [16], but in order to fix the notation it is useful to transfer it, paying attention to the appropriate adjustments which are needed in our construction.

Let (K, g_K) be a k-dimensional Riemannian manifold isometrically embedded in both the n-dimensional Riemannian manifolds (M_1, g_1) and (M_2, g_2), through the maps

$$\iota_i \quad : \quad K \hookrightarrow M_i \qquad\qquad , \; i = 1, 2.$$

We assume that the isometric map $\iota_1^{-1} \circ \iota_2 : \iota_1(K) \to \iota_2(K)$ extends to a diffeomorphism between the normal bundles of $\iota_i(K)$ in (M_i, g_i), $i = 1, 2$. We further assume that both the metrics g_1 and g_2 have zero constant scalar curvature. In this section our aim is to perform a generalized connected sum (or fiber sum) of the differentiable structures of M_1 and M_2 along the submanifold K. At the same time we construct on the new manifold $M = M_1 \sharp_K M_2$ a family of metrics $(g_\varepsilon)_{\varepsilon \in (0,1)}$, whose scalar curvature is close to zero in a suitable sense.

For a fixed $\varepsilon \in (0, 1)$, we describe the generalized connected sum construction and the definition of the metric g_ε in a local coordinate system, the fact that this construction yields a globally defined metric will follow at once.

Let U^k be an open set of $\mathbb{R}^k$, B^{m-k} the $(m-k)$-dimensional open ball $(m - k \geq 3)$. For $i = 1, 2$, $F_i : U^k \times B^{m-k} \to W_i \subset M_i$ given by

$$F_i(z, x) \quad := \quad \exp_z^{M_i}(x)$$

defines local Fermi coordinates near the coordinate patches $F_i(\cdot, 0)\,(U) \subset \iota_i(K) \subset M_i$. In these coordinates, the metric g_i can be decomposed as

$$g_i(z, x) \quad = \quad g^{(i)}_{j\,l}\, dz^j \otimes dz^l \,+\, g^{(i)}_{\alpha\beta}\, dx^\alpha \otimes dx^\beta \,+\, g^{(i)}_{j\alpha}\, dz^j \otimes dx^\alpha$$

and it is well known that, in this coordinate system,

$$g^{(i)}_{\alpha\beta} \,=\, \delta_{\alpha\beta} \,+\, \mathcal{O}\left(|x|^2\right) \qquad \text{and} \qquad g^{(i)}_{j\alpha} \,=\, \mathcal{O}\left(|x|\right)$$

In order to perform the identification between W_1 and W_2 and in order to glue the metrics together and define g_ε, we partially change the coordinate system, by setting

$$\begin{aligned} x &\;=\; \varepsilon\, \mathrm{e}^{-t} \cdot \theta && \text{on } F_1^{-1}(W_1) \\ x &\;=\; \varepsilon\, \mathrm{e}^{t} \cdot \theta && \text{on } F_2^{-1}(W_2) \end{aligned}$$

for $\varepsilon \in (0, 1)$, $\log \varepsilon < t < -\log \varepsilon$, $\theta \in S^{m-k-1}$.

Using these changes of coordinates the expressions of the two metrics g_1 and g_2 on $U^k \times A(\varepsilon^2, 1)$, where $A(\varepsilon^2, 1)$ is the annulus $\{\varepsilon^2 < |x| < 1\}$ become respectively

$$\begin{aligned} g_1(z, t, \theta) \;&=\; g^{(1)}_{ij} dz^i \otimes dz^j \\ &+\; u_\varepsilon^{(1)\,\frac{4}{n-2}} \left[\left(dt \otimes dt + g^{(1)}_{\lambda\mu} d\theta^\lambda \otimes d\theta^\mu \right) + g^{(1)}_{t\theta}\, dt \ltimes d\theta \right] \\ &+\; g^{(1)}_{it} dz^i \otimes dt + g^{(1)}_{i\lambda} dz^i \otimes d\theta^\lambda \end{aligned}$$

and

$$\begin{aligned} g_2(z, t, \theta) \;&=\; g^{(2)}_{ij} dz^i \otimes dz^j \\ &+\; u_\varepsilon^{(2)\,\frac{4}{n-2}} \left[\left(dt \otimes dt + g^{(2)}_{\lambda\mu} d\theta^\lambda \otimes d\theta^\mu \right) + g^{(2)}_{t\theta}\, dt \ltimes d\theta \right] \\ &+\; g^{(2)}_{it} dz^i \otimes dt + g^{(2)}_{i\lambda} dz^i \otimes d\theta^\lambda \end{aligned}$$

where by the compact notation $dt \ltimes d\theta$ we indicate the general component of the normal metric tensor (i.e., it involves $dt \otimes dt$, $d\theta^\lambda \otimes d\theta^\mu$ and $dt \otimes d\theta^\lambda$ components), whereas the coefficients $g_{t\theta}$ multiplied by $u_\varepsilon^{(i)\,\frac{4}{n-2}}$, $i = 1, 2$ represent the correction to the Euclidean metric $u_\varepsilon^{(i)\,\frac{4}{n-2}} \left[\left(dt \otimes dt + g^{(i)}_{\lambda\mu} d\theta^\lambda \otimes d\theta^\mu \right) \right]$, $i = 1, 2$ in our coordinate system.

Observe that for $j = 1, 2$ we have

$$\begin{aligned} g^{(j)}_{\lambda\mu} &= \mathcal{O}(1) & \qquad g^{(j)}_{t\theta} &= \mathcal{O}\left(|x|^2\right) \\ g^{(j)}_{it} &= \mathcal{O}\left(|x|^2\right) & \qquad g^{(j)}_{i\lambda} &= \mathcal{O}\left(|x|^2\right) \end{aligned}$$

and

$$u_\varepsilon^{(1)}(t) = \varepsilon^{\frac{n-2}{2}} \mathrm{e}^{-\frac{n-2}{2}t} \qquad \text{and} \qquad u_\varepsilon^{(2)}(t) = \varepsilon^{\frac{n-2}{2}} \mathrm{e}^{\frac{n-2}{2}t}$$

We choose a cut-off function $\zeta : (\log\varepsilon, -\log\varepsilon) \to [0, 1]$ to be a non increasing smooth function which is identically equal to 1 in $(\log\varepsilon, -1]$ and 0 in $[1, -\log\varepsilon)$ and we choose another cut-off function $\eta : (\log\varepsilon, -\log\varepsilon) \to [0, 1]$ to be a non increasing smooth function which is identically equal to 1 in $(\log\varepsilon, -\log\varepsilon - 1]$ and which satisfies $\lim_{t \to -\log\varepsilon} \eta = 0$. Using these two cut-off functions, we can define a new normal conformal factor u_ε by

$$u_\varepsilon(t) := \eta(t)\, u_\varepsilon^{(1)}(t) + \eta(-t)\, u_\varepsilon^{(2)}(t) \tag{4}$$

and the metric g_ε by

$$\begin{aligned} g_\varepsilon(z, t, \theta) \;:=\; & \left(\zeta g^{(1)}_{ij} + (1-\zeta) g^{(2)}_{ij} \right) dz^i \otimes dz^j \\ + \; & u_\varepsilon^{\frac{4}{n-2}} \Big[dt \otimes dt + \left(\zeta g^{(1)}_{\lambda\mu} + (1-\zeta) g^{(2)}_{\lambda\mu} \right) d\theta^\lambda \otimes d\theta^\mu \\ & \qquad\qquad + \left(\zeta g^{(1)}_{t\theta} + (1-\zeta) g^{(2)}_{t\theta} \right) dt \ltimes d\theta \Big] \\ + \; & \left(\zeta g^{(1)}_{it} + (1-\zeta) g^{(2)}_{it} \right) dz^i \otimes dt \\ + \; & \left(\zeta g^{(1)}_{i\lambda} + (1-\zeta) g^{(2)}_{i\lambda} \right) dz^i \otimes d\theta^\lambda \end{aligned} \tag{5}$$

Closer inspection of this expression shows that the metric g_ε (whose definition can be obviously completed by setting $g_\varepsilon \equiv g_1$ and $g_\varepsilon \equiv g_2$ out of the poly-neck) is a Riemannian metric which is globally defined on the manifold M.

In the following we also need to consider some slight conformal (Section 5) and non conformal (Section 6) perturbations of the approximate solution metrics g_ε's away from the gluing locus. However, since such adjustments do not modify at all the linear analysis, we prefer to introduce them later, for seek of simplicity.

Following [16] it is easy to obtain the estimate for the scalar curvature of the approximate solution metric.

Proposition 3 (**Estimate of the scalar curvature**). *There exists a constant $C > 0$ independent of $\varepsilon \in (0,1)$ such that*

$$|S_{g_\varepsilon}| \quad \leq \quad C \cdot \varepsilon^{-1} \, (\mathrm{ch}\, t)^{1-n} \tag{6}$$

for $|t| \leq |\log \varepsilon|$.

Another useful tool we can recover from [16] is the expression for the g_ε-laplacian on the polyneck

$$\Delta_{g_\varepsilon} \quad = \quad u_\varepsilon^{-\frac{4}{n-2}} \left[\partial_t^2 + (n-2)\, \mathrm{th} \left(\frac{n-2}{2} t \right) \partial_t + \Delta^{(\theta)}_{S^{n-1}} + u_\varepsilon^{\frac{4}{n-2}} \Delta^{(z)}_K + \mathcal{O}\left(|x|\right) \Phi(\nabla, \nabla^2) \right]$$

where $\Phi(\nabla, \nabla^2)$ is a nonlinear differential operator involving first order and second order partial derivatives with respect to t, θ^λ and z^j and whose coefficients are bounded uniformly on the polyneck, as $\varepsilon \in (0,1)$.

3 Analysis of a linear operator

Our aim is now to solve the Yamabe equation

$$\Delta_{g_\varepsilon} u \, + \, c_m \, S_{g_\varepsilon} \, u \quad = \quad c_m \, S \, u^{\frac{m+2}{m-2}} \tag{7}$$

where $c_m = -(m-2)/4(m-1)$ and $S = S(\varepsilon)$ is a suitable constant. If u is a solution to this equation, then the metric $\bar{g}_\varepsilon \, = \, u^{4/(m-2)} \, g_\varepsilon$ has constant scalar curvature equal to S. Therefore, when solving the equation (7) we want to guarantee that this tends to zero as ε goes to zero, as stated in Theorem 1.

Since we want to preserve the structure of the two initial metrics far away from the gluing locus, we are looking for a conformal factor u which is as close to 1 as we want. For these reasons it is natural to consider the change $u = 1 + v$ and consequently the equation

$$\begin{aligned} \Delta_{g_\varepsilon} v \quad &= \quad c_m \, (S - S_{g_\varepsilon}) \, + \, c_m \, (S - S_{g_\varepsilon}) \, v \, + \, c_m \, \frac{4}{m-2} \, S \, v \, + \, c_m \, S \, f(v) \\ &=: \quad F_\varepsilon(v) \end{aligned} \tag{8}$$

where $f(v) \, = \, (1+v)^{\frac{m+2}{m-2}} \, - \, 1 \, - \, \frac{m+2}{m-2} \, v$.

As already mentioned, the natural linearized operator to consider in a scalar flat context turns out to be the Lapalcian Δ_{g_ε}. Since we want to invert it, we are forced to work orthogonally to the space of constant functions. Another problem is that on the generalized connected sum the first non zero eigenvalue of Δ_{g_ε} tends to zero as ε tends to zero. Roughly speaking, such an eigenvalue is produced by a function which takes approximately the value 1 on M_1 and -1 on M_2 (since $\mathrm{vol}_{g_1}(M_1) \, = \, 1 \, = \, \mathrm{vol}_{g_2}(M_2)$) and interpolates smoothly between these two values on the polyneck. As ε tends to zero, the generalized connected sum degenerates into the two initial manifolds and the eigenfunction described above converges to a function which is the constant 1 on M_1 and the constant -1 on M_2. The corresponding eigenvalue is forced to tend to 0 in the limit. Notice that this reasoning can be made precise by adapting the argument presented in the appendix of [6]. Because of this fact it is not possible to provide (in natural function spaces) an *a priori* estimate which is uniformly bounded with respect to the parameter ε for solutions of the linearized equation $\Delta_{g_\varepsilon} u = f$. Therefore we will adopt the following strategy: we first produce an approximate non constant eigenfunction β_ε as explained above, then we obtain ε-uniform *a priori* estimate for solutions to the projected linearized equation

$$\Delta_{g_\varepsilon} u \quad = \quad f - \lambda \beta_\varepsilon \tag{9}$$

where f is a function such that $\int_M f \,\mathrm{dvol}_{g_\varepsilon} = 0$. Here we are looking for a suitable constant λ (which roughly speaking is the projection of f along β_ε) and a solution u which, up to a constant, can be chosen such that $\int_M u \,\mathrm{dvol}_{g_\varepsilon} = 0$.

Combining the *a priori* estimate for equation (9) and the estimate of the scalar curvature S_{g_ε} obtained in proposition 3 we will then be able to solve the (nonlinear) fixed point problem

$$\Delta_{g_\varepsilon} v \quad = \quad F_\varepsilon(v) \, - \, \lambda_{F_\varepsilon(v)} \, \beta_\varepsilon \tag{10}$$

The final step is then to discuss the conditions which ensure the vanishing of the rough projection of the error term $\lambda_{F_\varepsilon(v)}$, providing a solution to equation (8).

As suggested by the title, this section is devoted to the solution of the linear problem (9). To begin with, let us fix the functional setting by recalling the following result from [16]:

Proposition 4 (**Local *a priori* estimate**). *Given $\gamma \in (0, n-2)$, there exist real numbers $\alpha_1 = \alpha_1(n,\gamma) > 0$, $\alpha_2 = \alpha_2(n,\gamma) > 0$ and a constant $C_{n,\gamma} \geq 0$ such that for all $\varepsilon \in (0, \mathrm{e}^{-\max\{\alpha_1,\alpha_2\}})$ and all $v, f \in \mathcal{C}^0(T^\varepsilon(\alpha_1,\alpha_2))$ satisfying $\Delta_{g_\varepsilon} v = f$, the following estimate holds*

$$\sup_{T^\varepsilon(\alpha_1,\alpha_2)} |\psi_\varepsilon^\gamma v| \quad \leq \quad C_{n,\gamma} \left[\sup_{T^\varepsilon(\alpha_1,\alpha_2)} \left|\psi_\varepsilon^{\gamma+2} f\right| \, + \, \sup_{\partial T^\varepsilon(\alpha_1,\alpha_2)} |\psi_\varepsilon^\gamma v| \right] \tag{11}$$

where $T^\varepsilon(\rho,\sigma) := \{\log\varepsilon + \rho \leq t \leq -\log\varepsilon - \sigma\}$, for $\rho, \sigma > 0$ and ψ_ε is the distance function defined as

$$\psi_\varepsilon := \begin{cases} \varepsilon \mathrm{ch} t & \text{in } T^\varepsilon(1,1) \\ 1 & \text{in } M \setminus T^\varepsilon(0,0) \end{cases}$$

which interpolates smoothly between these two definitions.
(Observe that the statement is true for any couple of sufficiently large real numbers (α_1, α_2)).

Having at hand this result and working orthogonally to the kernel of Δ_{g_ε}, it becomes natural to consider the weighted Banach spaces of continuous functions defined by:

$$\mathcal{C}^0_\delta(M) \; := \; \left\{ v \in \mathcal{C}^0(M) \; : \; \|v\|_{\mathcal{C}^0_\delta(M)} \; \text{ and } \; \int_M v \; \mathrm{dvol}_{g_\varepsilon} = 0 \right\}$$

where $\|v\|_{\mathcal{C}^0_\delta(M)} := sup_M |\psi_\varepsilon^\delta\, v|$, and $\delta \in \mathbb{R}$ is the weight. In our context we consider functions $f \in \mathcal{C}^0_{\gamma+2}(M)$ and we look for solutions $u \in \mathcal{C}^0_\gamma(M)$, $\gamma \in (0, n-2)$.

Let us describe now more precisely the function β_ε. For the reasons explained above it is useful to think of it as an approximation of the degenerate eigenfunction of Δ_{g_ε}, whose associated eigenvalue tends to 0 as $\varepsilon \to 0$. We simply define β_ε as:

$$\beta_\varepsilon \quad := \quad \chi_1 \, - \, \chi_2 \tag{12}$$

where χ_1 and χ_2 are functions defined by

$$\chi_1 \;:=\; \begin{cases} 1 & \text{on} \quad M_1 \setminus T^\varepsilon(0,0) \\ 1 & \text{on} \;\; \{\log\varepsilon \,<\, t \,<\, \log\varepsilon + \alpha_1\} \\ 0 & \text{on} \;\; \{\log\varepsilon + \alpha_1 + 1 \,<\, t \,<\, 0\} \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_2 \;:=\; \begin{cases} 1 & \text{on} \quad M_2 \setminus T^\varepsilon(0,0) \\ 1 & \text{on} \;\; \{-\log\varepsilon + \alpha_2 \,<\, t \,<\, -\log\varepsilon\} \\ 0 & \text{on} \;\; \{0 \,<\, t \,<\, -\log\varepsilon - \alpha_2 - 1\} \\ 0 & \text{otherwise} \end{cases}$$

which interpolate smoothly between these definitions.

Since by hypothesis $\mathrm{vol}(M_1) = 1 = \mathrm{vol}(M_2)$, it is always possible to choose two real numbers α_1 and α_2 such that

$$\int_M \chi_1 - \chi_2 \;\, \mathrm{dvol}_{g_\varepsilon} \quad = \quad 0$$

This implies that the approximate eigenfunction β_ε is orthogonal to the constants.

As a first step towards the solution of the problem (9) we will prove the following:

Lemma 5. *Given a function $f \in \mathcal{C}^0_{\gamma+2}(M)$, it is possible to find a real number $\lambda = \lambda(f)$, an approximate solution $u \in \mathcal{C}^0_\gamma(M)$ and an error term $R \in \mathcal{C}^0_{\gamma+2}(M)$ that verify*

$$\Delta_{g_\varepsilon} u \quad = \quad f - \lambda\beta_\varepsilon + R \tag{13}$$

Moreover u, f and R enjoy the following estimates:

$$\|u\|_{\mathcal{C}^0_\gamma(M)} \;\;\leq\;\; A \,\cdot\, \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \tag{14}$$

$$|\lambda| \;\;\leq\;\; B \,\cdot\, \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \tag{15}$$

$$\|R\|_{\mathcal{C}^0_{\gamma+2}(M)} \;\;\leq\;\; C \,\cdot\, \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \,\cdot\, \varepsilon^{\beta\cdot\gamma} \tag{16}$$

where the positive constants A, B and C depend on K, n, γ, α_1 and α_2, the weight γ lies in $(0, n-2)$ and the real parameter β can be chosen in $(0,1)$.

The proof of Lemma 5 consists in building an approximate solution u to equation (9) and in estimating the remaining terms, collected in the error R. In order to do so, let us consider a non negative smooth function χ_P such that the triple $\{\chi_1, \chi_P, \chi_2\}$ is a partition of the unity. It is useful to split f into

$$f = f\chi_1 + f\chi_P + f\chi_2 = f_1 + f_P + f_2$$

As a first step we want to build a good approximate solution on the polyneck. It is well known that the problem

$$\begin{cases} \Delta_{g_\varepsilon} v = f_P & \text{on } T^\varepsilon(\alpha_1, \alpha_2) \\ \\ v = 0 & \text{on } \partial T^\varepsilon(\alpha_1, \alpha_2) \end{cases}$$

admits a solution and we call it $\tilde{u}_P$. Moreover, if f_P is continuous, so is $\tilde{u}_P$ and thanks to Proposition 4, if we choose large enough α_1 and α_2, we get immediately the estimate

$$\|\tilde{u}_P\|_{\mathcal{C}^0_\gamma(T^\varepsilon(\alpha_1,\alpha_2))} \quad \leq \quad A_P \cdot \|f_P\|_{\mathcal{C}^0_{\gamma+2}(T^\varepsilon(\alpha_1,\alpha_2))} \tag{17}$$

In fact the boundary conditions allow us to drop out the term $\|\tilde{u}_P\|_{\mathcal{C}^0_\gamma(\partial T^\varepsilon(\alpha_1,\alpha_2))}$ in the estimate (11). Also notice that the positive constant A_P only depends on n, γ, α_1 and α_2, so that the bound is uniform with respect to the parameter ε.

Let us smooth the polyneck solution just obtained, by defining $u_P := \chi_P \tilde{u}_P$. As a consequence we have

$$\begin{aligned} \Delta_{g_\varepsilon} u_P &= \Delta_{g_\varepsilon} \tilde{u}_P - \Delta_{g_\varepsilon} (1 - \chi_P)\tilde{u}_P \\ &= f_P - \Delta_{g_\varepsilon}(\chi_1 \tilde{u}_P) - \Delta_{g_\varepsilon}(\chi_2 \tilde{u}_P) \\ &= f_P - q_1 - q_2 \end{aligned}$$

where $q_i := \Delta_{g_\varepsilon}(\chi_i \tilde{u}_P)$, $i = 1, 2$.

As a second step we want now to construct approximate solutions on the pieces of M coming from M_1 and M_2. To this purpose, let us consider, for $i = 1, 2$, the functions $\tilde{f}_i := f_i + q_i$ and $\tilde{f} := \tilde{f}_1 + \tilde{f}_2$. Since $\int_M f \, \mathrm{dvol}_{g_\varepsilon} = 0$, it is easy to check that $\int_M \tilde{f} \, \mathrm{dvol}_{g_\varepsilon} = 0$. We also set $h_i := \tilde{f}_i + (-1)^i \lambda \chi_i$ for $i = 1, 2$ and $h := h_1 + h_2 = \tilde{f} - \lambda \beta_\varepsilon$. Obviously we have $\int_M h \, \mathrm{dvol}_{g_\varepsilon} = 0$ and $\int_M h_1 \, \mathrm{dvol}_{g_\varepsilon} = -\int_M h_2 \, \mathrm{dvol}_{g_\varepsilon}$.

Moreover

$$\int_M h_1 \, \mathrm{dvol}_{g_\varepsilon} - \int_M h_2 \, \mathrm{dvol}_{g_\varepsilon} = \int_M \tilde{f}_1 - \tilde{f}_2 \, \mathrm{dvol}_{g_\varepsilon} - \lambda \int_M \chi_1 + \chi_2 \, \mathrm{dvol}_{g_\varepsilon}$$

Hence, by setting

$$\lambda \quad := \quad \frac{\int_M \tilde{f}_1 - \tilde{f}_2 \, \mathrm{dvol}_{g_\varepsilon}}{\int_M \chi_1 + \chi_2 \, \mathrm{dvol}_{g_\varepsilon}} \tag{18}$$

it follows at once that $\int_M h_i \, \mathrm{dvol}_{g_\varepsilon} = 0$, $i = 1, 2$. Notice that the definition (18) allows us to think of λ as the rough projection of f along β_ε.

By slight modifications of very classical results (see [2]) we are allowed to consider solutions $\tilde{u}_i$, $i = 1, 2$ to the problems

$$\Delta_{g_i} \tilde{u}_i \quad = \quad h_i - b_i \, \delta_K \tag{19}$$

where δ_K is the Dirac distribution supported on the submanifold K and the constants b_i, $i = 1, 2$ are forced to be

$$b_i \quad = \quad \frac{\int_{M_i} h_i \,\mathrm{dvol}_{g_i}}{\mathrm{vol}_{g_K}(K)} \tag{20}$$

It is rather simple to describe how for example the function $\tilde{u}_1$ approximately look like, in fact we can write (notice that the following remarks still hold for $i = 2$)

$$\Delta_{g_1} \tilde{u}_1 = h_1 + \frac{1}{V_1} \int_M h_1 \,\mathrm{dvol}_{g_1} - \frac{1}{V_1} \int_M h_1 \,\mathrm{dvol}_{g_1} - b_1 \delta_K$$

where V_1 is the short notation for $\mathrm{vol}_{g_1}(M_1)$. To proceed it is useful to consider the splitting $\tilde{u}_1 = \bar{u}_1 + \hat{u}_1$ where

$$\begin{cases} \Delta_{g_1} \bar{u}_1 = h_1 - \frac{1}{V_1} \int_M h_1 \,\mathrm{dvol}_{g_1} \\[2ex] \Delta_{g_1} \hat{u}_1 = \frac{1}{V_1} \int_M h_1 \,\mathrm{dvol}_{g_1} - b_1 \delta_K \end{cases}$$

We can think of $\bar{u}_1$ as the finite part and of $\hat{u}_1$ as the pure Green function part of $\tilde{u}_1$. In particular $\hat{u}_1$ has the following shape in a neighborhood of K:

$$\hat{u}_1 \quad = \quad \Omega_{n,K} \cdot \int_M h_1 \,\mathrm{dvol}_{g_1} \cdot \left[\, |x|^{2-n} \; + \; \mathcal{O}\left(|x|^{3-n} \right) \, \right] \tag{21}$$

where $\Omega_{n,K} \; := \; [\mathrm{vol}_{g_K}(K) \cdot (n-2) \cdot \omega_{n-1}]^{-1}$.

In order to estimate $\hat{u}_1$ it is useful to remember that, on the region $T^\varepsilon(0,0) \setminus T^\varepsilon(\alpha_1, 0)$, the definition of the metric g_ε implies that:

$$\sqrt{g_\varepsilon} \quad = \quad \sqrt{g_1} \, + \, \mathcal{O}\left(\mathrm{e}^{(n-2)\,t} \right)$$

Hence, thanks to the fact that $\int_M h_1 \,\mathrm{dvol}_{g_\varepsilon} = 0$, we can write

$$\begin{aligned} \int_{M_1} h_1 \,\mathrm{dvol}_{g_1} \quad &= \quad \int_{M_1} h_1 \,\mathrm{dvol}_{g_1} \; - \; \int_{M_1} h_1 \,\mathrm{dvol}_{g_\varepsilon} \\ &= \quad \int_{T^\varepsilon(0,0) \setminus T^\varepsilon(\alpha_1, 0)} h_1 \, (\sqrt{g_1} - \sqrt{g_\varepsilon} \,) \, dz^1 \dots dz^k \, dt \, d\theta^1 \dots d\theta^{n-1} \end{aligned}$$

In the following, to keep short the notations, we indicate by $\mathrm{dvol}_{g_1 \,-\, g_\varepsilon}$ the volume element $(\sqrt{g_1} - \sqrt{g_\varepsilon} \,) \, dz^1, \dots, dz^k \, dt \, d\theta^1 \dots d\theta^{n-1}$

Let us recall that we have by definition $h_1 \; = \; f_1 + q_1 - \lambda \, \chi_1$. Concerning the piece coming from f_1 it is straightforward to check that there exists a positive constant $\hat{A}'_{K,n,\gamma,\alpha_1}$ such that

$$\int_{T^\varepsilon(0,0) \setminus T^\varepsilon(\alpha_1+1, 0)} f_1 \;\; \mathrm{dvol}_{g_1 \,-\, g_\varepsilon} \quad \leq \quad \hat{A}'_{K,n,\gamma,\alpha_1} \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \cdot \varepsilon^{n-2}$$

To analyze the contribution of $q_1 \; := \; \Delta_{g_\varepsilon}(\chi_1 \, \tilde{u}_P)$ it is convenient to write explicitly

$$q_1 \quad = \quad (\Delta_{g_\varepsilon} \chi_1) \, \tilde{u}_P \; + \; 2 \, g_\varepsilon \, (d\chi_1, d\tilde{u}_P) \; + \; \chi_1 \, (\Delta_{g_\varepsilon} \tilde{u}_P)$$

Thanks to Proposition 4 and using the fact that $\Delta_{g_\varepsilon} \tilde{u}_P = f_P$, it is easy to see that there exists a positive constant $\hat{A}''_{K,n,\gamma,\alpha_1}$ such that

$$\int_{T^\varepsilon(\alpha_1,0)\,\setminus\, T^\varepsilon(\alpha_1+1,0)} (\Delta_{g_\varepsilon}\chi_1)\,\tilde{u}_P \quad + \quad \chi_1\,(\Delta_{g_\varepsilon}\tilde{u}_P) \quad \mathrm{dvol}_{g_1\,-\,g_\varepsilon}$$
$$\leq \quad \hat{A}''_{K,n,\gamma,\alpha_1} \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \cdot \varepsilon^{n-2}$$

To treat the remaining term it is convenient to integrate by parts, then using the fact that $\partial_t\,\chi_1$ vanish on $\partial\,[\,T^\varepsilon(\alpha_1,0)\,\setminus\,T^\varepsilon(\alpha_1+1,0)\,]$ and Proposition 4 again, we deduce that

$$\int_{T^\varepsilon(\alpha_1,0)\,\setminus\, T^\varepsilon(\alpha_1+1,0)} 2\,g_\varepsilon\,(d\chi_1, d\tilde{u}_P) \;\; \mathrm{dvol}_{g_1\,-\,g_\varepsilon} \quad \leq \quad \hat{A}'''_{K,n,\gamma,\alpha_1} \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \cdot \varepsilon^{n-2}$$

for some positive constant $\hat{A}'''_{K,n,\gamma,\alpha_1}$. Notice that a similar estimate also follows from the fact that, up to a careful choice of the cut off χ_1, the term $g_\varepsilon\,(d\chi_1, d\tilde{u}_P)$ enjoys the following inequality

$$|\,g_\varepsilon\,(d\chi_1, d\tilde{u}_P)\,| \quad \leq \quad C_{m,\gamma,\alpha_1} \cdot \|f_P\|_{\mathcal{C}^0_{\gamma+2}(T^\varepsilon(\alpha_1,\alpha_2))} \tag{22}$$

In fact, adapting to the Riemannian setting the very classical gradient estimate for bounded solutions of the Poisson equation [5] and recalling that $\tilde{u}_P$ is a bounded solution of $\Delta_{g_\varepsilon}\,\tilde{u}_P = f_P$ on the domain $D_{\alpha_1} := T^\varepsilon(\alpha_1,0) \setminus T^\varepsilon(\alpha_1+1,0)$, we get the bound

$$(y,\partial D_{\alpha_1}) \cdot |d\tilde{u}_P|\,(y) \quad \leq \quad C_{n,\alpha_1} \cdot \Big[\,\|\tilde{u}_P\|_{\mathcal{C}^0(D_{\alpha_1})} \,+\, \|f_P\|_{\mathcal{C}^0(D_{\alpha_1})}\,\Big]$$

where y is a point in D_{α_1} and $(y,\partial D_{\alpha_1})$ represents the distance from y to the boundary of D_{α_1}. Having this at hand we immediately get

$$|\,g_\varepsilon\,(d\chi_1, d\tilde{u}_P)\,|\,(y) \quad \leq \quad C_{m,\gamma,\alpha_1} \cdot \Big[\,\|\tilde{u}_P\|_{\mathcal{C}^0_\gamma(D_{\alpha_1})} \,+\, \|f_P\|_{\mathcal{C}^0_{\gamma+2}(D_{\alpha_1})}\,\Big] \cdot \frac{|d\chi_1|\,(y)}{(y,\partial D_{\alpha_1})}$$

If χ_1 is sufficiently smooth, then the last factor in the right hand side is bounded in D_{α_1} and Proposition 4 yields the estimate (22).

The definition of the rough projection λ (18) obviously implies (modulo the same computation on M_2) the estimate (15) in the statement of Lemma 5 and, as a consequence, an analogue of the estimates above for the term $-\lambda\,\chi_1$ which appears in the expression of $\int_{M_1} h_1 \;\mathrm{dvol}_{g_1}$.

Hence, recalling the expression (21) of $\hat{u}_1$, we conclude that there exists a positive constant $\hat{A}^1_{K,n,\gamma,\alpha_1}$ such that

$$|\hat{u}_1| \quad \leq \quad \hat{A}^1_{K,n,\gamma,\alpha_1} \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \cdot \mathrm{e}^{(n-2)\,t} \tag{23}$$

Also notice that formula (21) implies at once analogous estimates for the derivatives of $\hat{u}_1$ with respect to the variables t, θ^λ, z^i, for $\lambda = 1,\ldots,n-1$ and $i = 1,\ldots,k$.

Let us look now to the finite part of $\tilde{u}_1$, namely $\bar{u}_1$. If we define $\bar{h}_1$ as

$$\bar{h}_1 \quad := \quad h_1 \,-\, \frac{1}{V_1}\int_{M_1} h_1 \;\mathrm{dvol}_{g_1}$$

then $\Delta_{g_1} \bar{u}_1 = \bar{h}_1$ and the classical Green representation formula (see for example [2]) for $\bar{u}_1$ automatically yields the estimate

$$\|\bar{u}_1\|_{\mathcal{C}^1(M_1)} \leq \|\bar{h}_1\|_{\mathcal{C}^0(M_1)} \tag{24}$$

Applying the remark (22) and Proposition 4, we deduce that there exists a positive constant $\bar{A}'_{K,n,\gamma,\alpha_1}$ such that

$$|q_1| \leq \bar{A}'_{K,n,\gamma,\alpha_1} \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)}$$

Hence, also the $\mathcal{C}^0$ norm of $\bar{h}_1$ is bounded by

$$\|\bar{h}_1\|_{\mathcal{C}^0(M_1)} \leq \bar{A}''_{K,n,\gamma,\alpha_1} \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)}$$

for some positive constant $\bar{A}''_{K,n,\gamma,\alpha_1}$. This implies that there exists $\bar{A}^1_{K,n,\gamma,\alpha_1} > 0$ such that

$$|\bar{u}_1| \leq \bar{A}^1_{K,n,\gamma,\alpha_1} \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \tag{25}$$

and the same is true for the derivatives of $\bar{u}_1$ with respect to the variables t, θ^λ, z^i, for $\lambda = 1, \dots, n-1$ and $i = 1, \dots, k$.

To summarize, we obtain from (25) and (23) that there exists a positive constant $A^1_{K,n,\gamma,\alpha_1}$ such that the function $\tilde{u}_1 = \bar{u}_1 + \hat{u}_1$ is bounded by

$$|\tilde{u}_1| \leq A^1_{K,n,\gamma,\alpha_1} \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \tag{26}$$

and the same is true for its derivatives with respect to the variables t, θ^λ, z^i, for $\lambda = 1, \dots, n-1$ and $i = 1, \dots, k$.

Following the same strategy it is straightforward to obtain a similar result for a function $\tilde{u}_2$, which is the analogue of $\tilde{u}_1$ on M_2. Now, using $\tilde{u}_1, \tilde{u}_2$ and u_P (which is nothing but the polyneck solution $\tilde{u}_P$ smoothed down), we are ready to produce the approximate solution u of Lemma 5. To do that, let us introduce, for $\beta \in (0,1)$ the smooth cut off functions ϕ_1 and ϕ_2 as follows

$$\phi_1 := \begin{cases} 1 & \text{on } \; M_1 \setminus T^\varepsilon(0,0) \\ 1 & \text{on } \; \{\log\varepsilon < t < (1-\beta)\log\varepsilon\} \\ 0 & \text{on } \; \{(1-\beta)\log\varepsilon + 1 < t < 0\} \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2 := \begin{cases} 1 & \text{on } \; M_2 \setminus T^\varepsilon(0,0) \\ 1 & \text{on } \; \{-(1-\beta)\log\varepsilon < t < -\log\varepsilon\} \\ 0 & \text{on } \; \{0 < t < -(1-\beta)\log\varepsilon - 1\} \\ 0 & \text{otherwise} \end{cases}$$

To complete the description of the cut-offs we assume that ϕ_1 interpolates smoothly between $(1-\beta)\log\varepsilon$ and $(1-\beta)\log\varepsilon + 1$ and ϕ_2 does the same between $-(1-\beta)\log\varepsilon - 1$ and $-(1-\beta)\log\varepsilon$. Notice that for small enough ε, we have that supp $(d\phi_i) \subset T^\varepsilon(\alpha_1, \alpha_2)$, for $i = 1, 2$ (in the following we will always assume that).

Let us define now the approximate solution as

$$u \;:=\; \phi_1 \tilde{u}_1 \,+\, u_P \,+\, \phi_2 \tilde{u}_2 \tag{27}$$

Notice that the estimate (17) and the estimate (26), with its analogue for $\tilde{u}_2$, imply at once the estimate (14) in the statement of Lemma 5, namely there exists a positive constant $A_{K,n,\gamma,\alpha_1,\alpha_2}$ such that

$$\|u\|_{\mathcal{C}^0_\gamma(M)} \;\leq\; A_{K,n,\gamma,\alpha_1,\alpha_2} \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \tag{28}$$

To define the error term R of Lemma 5, we compute

$$\begin{aligned}
\Delta_{g_\varepsilon} u \;&=\; \Delta_{g_\varepsilon} u_P \,+\, \Delta_{g_\varepsilon}(\phi_1 \tilde{u}_1) \,+\, \Delta_{g_\varepsilon}(\phi_2 \tilde{u}_2) \\
&=\; f_P \,-\, q_1 \,-\, q_2 \\
&\quad\; + \phi_1\,(h_1 - b_1 \delta_K) \,+\, (\Delta_{g_\varepsilon}\phi_1)\,\tilde{u}_1 \,+\, g_\varepsilon\,(d\phi_1, d\tilde{u}_1) \\
&\quad\; + \phi_2\,(h_2 - b_2 \delta_K) \,+\, (\Delta_{g_\varepsilon}\phi_2)\,\tilde{u}_2 \,+\, g_\varepsilon\,(d\phi_2, d\tilde{u}_2) \\
&=\; f \,-\, \lambda \beta_\varepsilon \\
&\quad\; + (\Delta_{g_\varepsilon}\phi_1)\,\tilde{u}_1 \,+\, g_\varepsilon\,(d\phi_1, d\tilde{u}_1) \\
&\quad\; + (\Delta_{g_\varepsilon}\phi_2)\,\tilde{u}_2 \,+\, g_\varepsilon\,(d\phi_2, d\tilde{u}_2)
\end{aligned}$$

At this point it is quite natural to define $E_i := (\Delta_{g_\varepsilon}\phi_i)\tilde{u}_i + g_\varepsilon(\nabla\phi_i, \nabla\tilde{u}_i)$, $i = 1, 2$ and $R := E_1 + E_2$, so that u, λ and R satisfy the equation (13)

$$\Delta_{g_\varepsilon} u \;=\; f \,-\, \lambda\beta_\varepsilon \,+\, R \tag{29}$$

The last task in order to complete the proof of the Lemma 5 is to provide R with the estimate (16). Without loss of generality, let us look for example at the error term E_1.

First notice that since supp $(\Delta_{g_\varepsilon}\phi_1)$ and supp $(d\phi_1)$ are both included in $[\,(1-\beta)\,\log\varepsilon\,, (1-\beta)\,\log\varepsilon\,+\,1\,]$, the term E_1 is supported here as well. Considering this fact and the estimates obtained for $\tilde{u}_1$ and its derivatives, it is straightforward to deduce that, for a suitable positive constant $C^1_{K,n,\gamma,\alpha_1}$,

$$\|E_1\|_{\mathcal{C}^0_{\gamma+2}(M)} \;\leq\; C^1_{K,n,\gamma,\alpha_1} \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \cdot \varepsilon^{\beta\,\gamma} \tag{30}$$

This estimate and its counterpart for E_2 imply the estimate (16) for R and this completes the proof of Lemma 5.

Now we want to look at equation (9). The idea is to solve it by means of an induction process. We start by setting $f^{(0)} := f$ and thanks to Lemma 5 we obtain a triple $(\lambda^{(0)}, u^{(0)}, R^{(0)})$ satisfying

$$\Delta_{g_\varepsilon} u^{(0)} \;=\; f^{(0)} - \lambda^{(0)}\beta_\varepsilon + R^{(0)}$$

and the estimates (14), (15) and (16). Then, setting $f^{(1)} := -R^{(0)}$, we find another triple $(\lambda^{(1)}, u^{(1)}, R^{(1)})$ with the same properties as the first one and so on. In general, for every $j \in \mathbb{N}$, we have $f^{(j)} := -R^{(j-1)}$ and a triple $(\lambda^{(j)}, u^{(j)}, R^{(j)})$ satisfying the equation

$$\Delta_{g_\varepsilon} u^{(j)} \;=\; f^{(j)} - \lambda^{(j)} \beta_\varepsilon + R^{(j)} \tag{31}$$

and the estimates (14), (15) and (16) of Lemma 5.

Taking the sum of the equations (31) we have that, for every $N \in \mathbb{N}$

$$\begin{aligned}
\Delta_{g_\varepsilon} \sum_{j=0}^{N} u^{(j)} &= \sum_{j=0}^{N} f^{(j)} \,-\, \beta_\varepsilon \sum_{j=0}^{N} \lambda^{(j)} \,+\, \sum_{j=0}^{N} R^{(j)} \\
&= f \,-\, \beta_\varepsilon \sum_{j=0}^{N} \lambda^{(j)} \,+\, R^{(N)}
\end{aligned}$$

We can rephrase this by saying

$$\Delta_{g_\varepsilon} v^{(N)} \;=\; f \,-\, \mu^{(N)} \beta_\varepsilon \,+\, R^{(N)}$$

where

$$v^{(N)} \,:=\, \sum_{j=0}^{N} u^{(j)} \qquad \text{and} \qquad \mu^{(N)} \,:=\, \sum_{j=0}^{N} \lambda^{(j)}$$

From the estimates of Lemma 5 it easily follows that

$$\begin{aligned}
\|f^{(j)}\|_{\mathcal{C}^0_{\gamma+2}(M)} \;=\; \|R^{(j-1)}\|_{\mathcal{C}^0_{\gamma+2}(M)} &\leq C \cdot \|f^{(j-1)}\|_{\mathcal{C}^0_{\gamma+2}(M)} \cdot \varepsilon^{\beta\,\gamma} \\
&\leq (C\,\varepsilon^{\beta\,\gamma})^j \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)}
\end{aligned}$$

$$\begin{aligned}
\|u^{(j)}\|_{\mathcal{C}^0_{\gamma}(M)} &\leq A \cdot \|f^{(j)}\|_{\mathcal{C}^0_{\gamma+2}(M)} \\
&\leq A \cdot \big(C\,\varepsilon^{\beta\,\gamma}\big)^j \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)}
\end{aligned}$$

$$\begin{aligned}
|\lambda^{(j)}| &\leq B \cdot \|f^{(j)}\|_{\mathcal{C}^0_{\gamma+2}(M)} \\
&\leq B \cdot \big(C\,\varepsilon^{\beta\,\gamma}\big)^j \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)}
\end{aligned}$$

It is clear that, for sufficiently small $\varepsilon > 0$, there exist a real number $\lambda \in \mathbb{R}$ and function $u \in \mathcal{C}^0_\gamma$ such that

$$R^{(N)} \;\overset{\|\cdot\|_{\mathcal{C}^0_{\gamma+2}}}{\longrightarrow}\; 0$$

$$v^{(N)} \;\overset{\|\cdot\|_{\mathcal{C}^0_{\gamma}}}{\longrightarrow}\; u$$

$$\mu^{(N)} \;\longrightarrow\; \lambda$$

Moreover there exist positive constants A' and B' depending on K, n, γ, α_1 and α_2 such that

$$\|u\|_{\mathcal{C}^0_\gamma(M)} \;\leq\; B' \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \qquad \text{and} \qquad |\lambda| \;\leq\; C' \|f\|_{\mathcal{C}^0_{\gamma+2}(M)}$$

Hence

$$\Delta_{g_\varepsilon} v^{(N)} \overset{\|\cdot\|_{\mathcal{C}^0_{\gamma+2}}}{\longrightarrow} f - \lambda\beta_\varepsilon \qquad \text{and} \qquad v^{(N)} \overset{\|\cdot\|_{\mathcal{C}^0_{\gamma}}}{\longrightarrow} u$$

On the other hand we have that, for every $N \in \mathbb{N}$ and for every $\phi \in \mathcal{C}^\infty(M)$

$$\int_M v^{(N)}\, \Delta_{g_\varepsilon}\phi \;\, \mathrm{dvol}_{g_\varepsilon} \;=\; \int_M (f - \mu^{(N)}\beta_\varepsilon + R^{(N)})\,\phi \;\, \mathrm{dvol}_{g_\varepsilon}$$

Hence, by taking the limit for $N \to +\infty$ we find, for every $\phi \in \mathcal{C}^\infty(M)$, the expression

$$\int_M u\, \Delta_{g_\varepsilon}\phi \;\, \mathrm{dvol}_{g_\varepsilon} \;=\; \int_M (f - \lambda\beta_\varepsilon)\,\phi \;\, \mathrm{dvol}_{g_\varepsilon}$$

In other words the following identity

$$\Delta_{g_\varepsilon} u \;=\; f - \lambda\beta_\varepsilon$$

holds in the sense of the distributions.

Thanks to the elliptic regularity (see for example [2] and [5]), if we suppose $f \in \mathcal{C}^{0,\alpha}(M)$, then $u \in \mathcal{C}^{2,\alpha}$ and the expression above is a point-wise identity.

To conclude this section we summarize our results in the following

Proposition 6. *Given a function $f \in \mathcal{C}^0_{\gamma+2}(M)$, it is possible to find a real number λ and a function $u \in \mathcal{C}^0_\gamma(M)$ such that the equation*

$$\Delta_{g_\varepsilon} u \;=\; f - \lambda\beta_\varepsilon \tag{32}$$

is satisfied in the sense of the distributions and the following estimates hold

$$\|u\|_{\mathcal{C}^0_\gamma(M)} \;\leq\; A' \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \tag{33}$$

$$|\lambda| \;\leq\; B' \cdot \|f\|_{\mathcal{C}^0_{\gamma+2}(M)} \tag{34}$$

for suitable positive constants A' and B' depending on K, n, γ, α_1 and α_2.
Moreover, if $f \in \mathcal{C}^{0,\alpha}(M)$, then $u \in \mathcal{C}^{2,\alpha}(M)$ and the identity above holds point-wise.

4 Nonlinear analysis: a fixed point argument

The aim of this section is to solve the fixed point problem (10), namely

$$\Delta_{g_\varepsilon} v \;=\; F_\varepsilon(v) - \lambda_{F_\varepsilon(v)}\beta_\varepsilon$$

We will be able to do this using a fixed point theorem for contracting mappings, provided the $\mathcal{C}^0_\gamma(M)$-norm of v is small enough.

Before starting, let us remark that in the expression of $F_\varepsilon(v)$ (8) it is always possible to choose $S = S(\varepsilon, v)$ in such a way that $\int_M F_\varepsilon \,\mathrm{dvol}_{g_\varepsilon} = 0$, in fact it is sufficient to set

$$S \;:=\; \frac{\int_M S_{g_\varepsilon}\,(1+v)\;\, \mathrm{dvol}_{g_\varepsilon}}{\int_M 1 + \frac{m+2}{m-2} + f(v)\;\, \mathrm{dvol}_{g_\varepsilon}} \tag{35}$$

Using the estimate of the scalar curvature (6) of Proposition 3 it is not hard to check that $S = \mathcal{O}\left(\varepsilon^{n-2}\right)$.

Observe that in Section 5 and in Section 6, we will need to introduce some slight modifications of the approximate solution metrics g_ε away from the gluing locus, in order to kill the rough projection $\lambda_{F_\varepsilon(v)}$. This will induce some small changes in the expression of S_{g_ε} and, consequently, the rough estimate for S will be strictly worse than the one we obtained in the statement of Theorem 1. More precisely, we will now find that the estimate for $S = \mathcal{O}\left(\varepsilon^{(n-2)/2}\right)$ instead of $S = \mathcal{O}\left(\varepsilon^{n-2}\right)$. However the fixed point argument we are going to describe will still hold in this modified context,provided some adjustments are made. More precisely, once a solution to equation (10) is obtained in the new setting, it will be possible to improve the approximate solution in order to get $S = \mathcal{O}\left(\varepsilon^{n-2}\right)$ and hence the fixed point argument will follow from the one we discuss now. For these reasons and for seek of simplicity, we prefer to prove the existence of a fixed point for the problem above in the current situation.

To begin with, let us introduce the maps

$$\begin{array}{rrcl} H_\varepsilon \ : \ \mathcal{C}^0_\gamma(M) & \longrightarrow & \mathcal{C}^0_{\gamma+2}(M) \\ v & \longmapsto & F_\varepsilon(v) - \lambda(\varepsilon, v)\beta_\varepsilon \end{array}$$

$$\begin{array}{rrcl} \Delta^{-1}_{g_\varepsilon} \ : \ \mathcal{C}^0_{\gamma+2}(M) & \longrightarrow & \mathcal{C}^0_\gamma(M) \\ w & \longmapsto & \Delta^{-1}_{g_\varepsilon} w \end{array}$$

$$\begin{array}{rrcl} P_\varepsilon \ : \ \mathcal{C}^0_\gamma(M) & \longrightarrow & \mathcal{C}^0_\gamma(M) \\ v & \longmapsto & \Delta^{-1}_{g_\varepsilon} \circ H_\varepsilon(v) \end{array}$$

We can now state the following:

Lemma 7. *For $\gamma \in (0, 1/2)$ and for sufficiently small $\varepsilon > 0$ there exists a radius $r_\varepsilon := \varepsilon^{2\gamma}$ such that $P_\varepsilon\left(B_\gamma(r_\varepsilon)\right) \subset B_\gamma(r_\varepsilon)$, where $B_\gamma(r_\varepsilon) := \left\{u \in \mathcal{C}^0_\gamma(M) \ : \ \|u\|_{\mathcal{C}^0_\gamma(M)} \le r_\varepsilon\right\}$. In other words:*

$$\|v\|_{\mathcal{C}^0_\gamma(M)} \le r_\varepsilon \quad \Longrightarrow \quad \|P_\varepsilon(v)\|_{\mathcal{C}^0_\gamma(M)} \le r_\varepsilon \tag{36}$$

In order to prove the statement, we start by observing that the choice of S and the estimate of S_{g_ε} obtained in Proposition 3 imply at once that $F_\varepsilon(v) \in \mathcal{C}^0_{\gamma+2}$, for $v \in B_\gamma(r_\varepsilon)$. Then, using the estimate (33) of Proposition 6 we immediately get

$$\|P_\varepsilon(v)\|_{\mathcal{C}^0_\gamma(M)} \quad \le \quad D \cdot \|F_\varepsilon(v)\|_{\mathcal{C}^0_{\gamma+2}(M)}$$

for a suitable positive constant D.

Now, we have to estimate the right hand side of the above expression. Recalling the definition of r_ε and the estimate (6), we compute

$$\begin{aligned}
|F_\varepsilon(v)\psi_\varepsilon^{\gamma+2}| &\leq c_1\,|S|\,\psi_\varepsilon^{\gamma+2} + c_2\,|S_{g_\varepsilon}|\,\psi_\varepsilon^{\gamma+2} + c_3\,|S|\,\psi_\varepsilon^{\gamma+2}\,|v| + c_4\,|S_{g_\varepsilon}|\,\psi_\varepsilon^{\gamma+2}\,|v| \\
&\qquad + c_5\,|S|\,|f(v)|\,\psi_\varepsilon^{\gamma+2} \\
&\leq c_6\varepsilon^{n-2} + c_7\left[\varepsilon^{n-2} + \varepsilon^{1+\gamma}\right] + c_8\,\varepsilon^{n-2+\gamma} + c_9\left[\varepsilon^{n-2} + \varepsilon^{1+\gamma}\right]\varepsilon^{\gamma} \\
&\qquad + c_{10}\,\varepsilon^{n-2+2\gamma} \\
&\leq c_{11}\,\varepsilon^{n-2} + c_{12}\,\varepsilon^{1+\gamma} \\
&\leq \left[c_{13}\,\varepsilon^{n-2-2\gamma} + \varepsilon^{1-\gamma}\right] r_\varepsilon
\end{aligned}$$

for suitable positive constants c_j's.

Therefore, the Lemma is proved, provided $\gamma \in (0, 1/2)$. Let us insist on the fact that the range of the admissible γ's will be smaller and the rough estimate of S will not be as good when, in the following sections, we will consider slight modifications of the approximate solutions (namely we will obtain the condition $\gamma \in (0, 1/4)$, as in the statement of Theorem 1). Nevertheless, the statement of Lemma 7 remains true also in the setting of Section 5 and Section 6.

At this point, our aim is to prove that the sequence

$$v^j \quad := \quad P_\varepsilon^j(0) \qquad\qquad j \in \mathbb{N} \tag{37}$$

converges to a function $v_\varepsilon \in B_\gamma(r_\varepsilon)$ with respect to the norm $\|\cdot\|_{\mathcal{C}^0_\gamma(M)}$.

Since we want to use a contraction mapping argument, we need to provide an estimate for $\|P_\varepsilon(u) - P_\varepsilon(v)\|_{\mathcal{C}^0_\gamma(M)}$ in terms of $\|u - v\|_{\mathcal{C}^0_\gamma(M)}$, where $u, v \in B_\gamma(r_\varepsilon)$. In fact, since $0 \in B_\gamma(r_\varepsilon)$, all the terms belong to $B_\gamma(r_\varepsilon)$, because of Lemma 7.

First notice that

$$\begin{aligned}
\Delta_{g_\varepsilon}\,\left(P_\varepsilon(u) - P_\varepsilon(v)\right) &= H_\varepsilon(u) - H_\varepsilon(v) \\
&:= F_\varepsilon(u) - F_\varepsilon(v) - \left(\lambda_{F_\varepsilon(u)} - \lambda_{F_\varepsilon(v)}\right)\beta_\varepsilon
\end{aligned}$$

As it is not hard to check that $f \mapsto \lambda_f$, where λ_f is the rough projection defined in Proposition 6, is a linear map, hence

$$\lambda_{F_\varepsilon(u)} - \lambda_{F_\varepsilon(v)} \quad = \quad \lambda_{F_\varepsilon(u)-F_\varepsilon(v)} \tag{38}$$

As a consequence of the estimate (33) we obtain

$$\|P_\varepsilon(u) - P_\varepsilon(v)\|_{\mathcal{C}^0_\gamma(M)} \quad \leq \quad C_0 \cdot \|F_\varepsilon(u) - F_\varepsilon(v)\|_{\mathcal{C}^0_{\gamma+2}(M)}$$

for some suitable positive constant C_0.

Since the function f which appears in the definition of $F_\varepsilon(v)$ satisfies the following inequality

$$|f(u) - f(v)| \quad \leq \quad \left[C_1\,(|u| + |v|) + C_2\left(|u|^{\frac{4}{m-2}} - |v|^{\frac{4}{m-2}}\right)\right]|u - v|$$

for suitable positive constants C_1 and C_2, we can proceed with the estimate of the term $F_\varepsilon(u) - F_\varepsilon(v)$. The condition $\gamma \in (0, 1/2)$ is a sufficient condition to ensure that

$$\begin{aligned}
\psi_\varepsilon^{\gamma+2} \, |F_\varepsilon(u) - F_\varepsilon(v)| \;\; &\leq \;\; \psi_\varepsilon^{\gamma+2} \left[c_m |S| \, |u-v| + c_m |S_{g_\varepsilon}| \, |u-v| + \frac{|S|}{m-1} \, |u-v| \right] \\
&\quad + \psi_\varepsilon^{\gamma+2} \, c_m \, |S| \left[C_1 \, (|u| + |v|) \right. \\
&\qquad\qquad \left. + \, C_2 \, \left(|u|^{\frac{4}{m-2}} - |v|^{\frac{4}{m-2}} \right) \right] |u-v| \\
&\leq \;\; C_3 \, \varepsilon \cdot \|u-v\|_{\mathcal{C}^0_\gamma(M)} \\
&\quad + C_4 \, \varepsilon^{n-2} \cdot \left[\|u\|_{\mathcal{C}^0_\gamma(M)} + \|v\|_{\mathcal{C}^0_\gamma(M)} \right] \cdot \|u-v\|_{\mathcal{C}^0_\gamma(M)} \\
&\quad + C_5 \, \varepsilon^{n-2} \cdot \left[\|u\|^{\frac{4}{m-2}}_{\mathcal{C}^0_\gamma(M)} + \|v\|^{\frac{4}{m-2}}_{\mathcal{C}^0_\gamma(M)} \right] \cdot \|u-v\|_{\mathcal{C}^0_\gamma(M)}
\end{aligned}$$

for suitable positive C_j's. (Again, the condition on γ becomes slightly different for the metrics we will consider in the next sections, namely $\gamma \in (0, 1/4)$).

Hence, for $u, v \in B_\gamma(r_\varepsilon)$ and $\varepsilon > 0$ small enough , we obtain the inequality

$$\|P_\varepsilon(u) - P_\varepsilon(v)\|_{\mathcal{C}^0_\gamma(M)} \;\; \leq \;\; C_6 \, \varepsilon \cdot \|u-v\|_{\mathcal{C}^0_\gamma(M)} \tag{39}$$

Therefore, if we choose two integers $p \leq q$, we have that

$$\begin{aligned}
\|v^q - v^p\|_{\mathcal{C}^0_\gamma(M)} \;\; &\leq \;\; \sum_{j=1}^{p-q} \|v^{p+j} - v^{p+j-1}\|_{\mathcal{C}^0_\gamma(M)} \\
&\leq \;\; (C_6 \, \varepsilon)^p \cdot \sum_{j=0}^{+\infty} (C_6 \, \varepsilon)^j \cdot \|v^1 - v^0\|_{\mathcal{C}^0_\gamma(M)}
\end{aligned}$$

Therefore, the sequence (v^j) is a Cauchy sequence and it must converge to a function $v_\varepsilon \in B_\gamma(r_\varepsilon)$ which is the fixed point of P_ε in $B_\gamma(r_\varepsilon)$, namely

$$P_\varepsilon(v_\varepsilon) \;\; = \;\; v_\varepsilon \tag{40}$$

Recalling the definition of P_ε, it is straightforward to see that in other words v_ε is a solution to problem (10)

$$\Delta_{g_\varepsilon} v_\varepsilon \;\; = \;\; F_\varepsilon(v_\varepsilon) - \lambda_{F_\varepsilon(v_\varepsilon)} \, \beta_\varepsilon \tag{41}$$

Notice that by means of a classical boot strap argument one can prove that v_ε is actually a smooth function.

To conclude this section, let us remark that since v_ε has been found by means of a contraction mapping argument, it also depends continuously on the data of our problem.

5 Vanishing of the rough projection $\lambda_{F_\varepsilon(v_\varepsilon)}$

In this section we want to discuss the vanishing of $\lambda_{F_\varepsilon(v_\varepsilon)}$. This is the last step needed to complete the proof of Theorem 1. In the previous section, our main purpose was to produce a solution to equation (8). Recall that if $F_\varepsilon(v)$ represents the error term in this equation, then we can think of $\lambda_{F_\varepsilon(v)}$ as the rough projection of the error term along the corresponding eigenfunction β_ε of the linearized operator Δ_{g_ε}. For the time being we have produced a solution v_ε to the equation (10), if we are able to ensure the vanishing of $\lambda_{F_\varepsilon(v_\varepsilon)}$, then we will be done. To do so, we consider some slight conformal modifications of the initial metrics g_1 and g_2 supported away from the gluing locus.

More precisely let $\bar{w}_1$ and $\bar{w}_2$ be two smooth functions supported on $M_1 \setminus W_1$ and $M_2 \setminus W_2$ respectively (with the notation introduced in Section 2). Then, we set

$$w_1 \;:=\; a\,\varepsilon^{(n-2)/2}\cdot\bar{w}_1 \qquad (42)$$

$$w_2 \;:=\; b\,\varepsilon^{(n-2)/2}\cdot\bar{w}_2 \qquad (43)$$

where a and b are real parameters. Having defined w_1 and w_2, we introduce the modified metrics

$$\tilde{g}_1 \;:=\; (1+w_1)^{\frac{4}{m-2}}\,g_1 \qquad (44)$$

$$\tilde{g}_2 \;:=\; (1+w_2)^{\frac{4}{m-2}}\,g_2 \qquad (45)$$

Using $\tilde{g}_1$ and $\tilde{g}_2$ instead of g_1 and g_2 in the geometric construction presented in Section 2 we obtain a family of modified approximate solution metrics $(\tilde{g}_\varepsilon)_{\varepsilon\in(0,1)}$ and it is easy to check that the linear analysis remains unchanged for these new metrics. Namely Proposition 6 still holds with $\tilde{g}_\varepsilon$ instead of g_ε (because of the support of w_i, $i=1,2$). Also observe that, because of the smallness of the modification introduced, it is always possible to chose α_1 and α_2 in the definition of χ_1 and χ_2 such that

$$\int_M \chi_1 - \chi_2 \;\mathrm{dvol}_{\tilde{g}_\varepsilon} \;=\; 0$$

It turns out that the scalar curvature of $\tilde{g}_i$ is supported on supp $(\bar{w}_i)$, and that it is given by

$$S_{\tilde{g}_i} \;=\; (1+w_i)^{-\frac{m+2}{m-2}}\cdot\Delta_{g_i}w_i \qquad (46)$$

for $i=1,2$, then $S_{\tilde{g}_i}=\mathcal{O}\left(\varepsilon^{(n-2)/2}\right)$. From this it is easy to deduce an analogue of Proposition 3 for the scalar curvature $S_{\tilde{g}_\varepsilon}$. In other words $S_{\tilde{g}_\varepsilon}$ enjoys an estimate similar to the one of S_{g_ε} (namely the estimate (6)) on the polyneck and an estimate which turns out to be a $\mathcal{O}\left(\varepsilon^{(n-2)/2}\right)$ on the supports of the w_i's. As a consequence, we also deduce a rough estimate for S, namely $S=\mathcal{O}\left(\varepsilon^{(n-2)/2}\right)$. In the following, once a new fixed point is found in the modified setting, we will also improve this estimate, in order to get $S=\mathcal{O}\left(\varepsilon^{n-2}\right)$, as required in the statement of Theorem 1.

As mentioned in the previous section, up to choosing $\gamma\in(0,1/4)$, we can reproduce with slight modifications the fixed point argument of Section 4, in order to obtain a solution to the equation

$$\Delta_{\tilde{g}_\varepsilon} v_\varepsilon \quad = \quad \tilde{F}_\varepsilon(v_\varepsilon) - \lambda_{\tilde{F}_\varepsilon(v_\varepsilon)}\, \beta_\varepsilon \tag{47}$$

where the explicit expression for $\tilde{F}_\varepsilon(v_\varepsilon)$ is

$$\tilde{F}_\varepsilon(v_\varepsilon) \quad = \quad c_m\, S\left[1 + \frac{m+2}{m-2}\, v_\varepsilon \;+\; f(v_\varepsilon)\right] \;-\; c_m\, S_{\tilde{g}_\varepsilon}\;[\,1 \;+\; v_\varepsilon\,] \tag{48}$$

and S is given by

$$S \quad = \quad \frac{\int_M S_{\tilde{g}_\varepsilon}\,(1 \;+\; v_\varepsilon)\;\mathrm{dvol}_{\tilde{g}_\varepsilon}}{\int_M 1 \;+\; \frac{m+2}{m-2}\, v_\varepsilon \;+\; f(v_\varepsilon)\;\mathrm{dvol}_{\tilde{g}_\varepsilon}} \tag{49}$$

$$= \quad \frac{\int_M (\,S_{\tilde{g}_1} \;+\; S_{\tilde{g}_\varepsilon} \;+\; S_{\tilde{g}_2}\,)\,(1 \;+\; v_\varepsilon)\;\mathrm{dvol}_{\tilde{g}_\varepsilon}}{\mathrm{vol}_{\tilde{g}_\varepsilon}\,(M)} \;+\; \mathcal{O}\left(\varepsilon^{(n-2)/2\,+\,\gamma}\right) \tag{50}$$

Also notice that the fact that v_ε lies in $B_\gamma(r_\varepsilon)$ implies at once the estimate (1) which is required in the statement of Theorem 1.

The task is now to show that $\lambda_{\tilde{F}_\varepsilon(v_\varepsilon)}$ can be chosen to be zero. Since all the quantities which appear in the expression (47) depend smoothly on the real parameters a and b introduced in the definitions of the w_i's, our goal is achieved if we prove that we can control the sign of the rough projection $\lambda_{\tilde{F}_\varepsilon(v_\varepsilon)}$ by means of a and b.

Following the proof of Proposition 6, we can think of $\lambda_{\tilde{F}_\varepsilon(v_\varepsilon)}$ as a sum of the series

$$\lambda_{\tilde{F}_\varepsilon(v_\varepsilon)} \quad = \quad \sum_{j=0}^{\infty} \lambda^{(j)}_{\tilde{F}_\varepsilon(v_\varepsilon)} \tag{51}$$

where the real numbers $\lambda^{(j)}_{\tilde{F}_\varepsilon(v_\varepsilon)}$ are built on as in the mentioned proposition and consequently enjoy the estimate

$$|\lambda^{(j)}_{\tilde{F}_\varepsilon(v_\varepsilon)}| \quad \leq \quad B \cdot \left(C\,\varepsilon^{\beta\,\gamma}\right)^j \cdot \|\tilde{F}_\varepsilon(v_\varepsilon)\|_{\mathcal{C}^0_{\gamma+2}(M)} \tag{52}$$

where B and C are positive constants depending on K, n, γ, α_1 and α_2 and the real parameter β lies in $(0,1)$. As indicated by this estimate, the leading term in the expression for $\lambda_{\tilde{F}_\varepsilon(v_\varepsilon)}$ is given by the first summand. More precisely, using the explicit definition of the $\lambda^{(j)}_{\tilde{F}_\varepsilon(v_\varepsilon)}$'s, it is not hard to show that, for a suitable choice of the parameter β, all these terms are $o(\varepsilon^{n-2})$, for $j > 1$. We deduce that the sign of the rough projection is determined by the one of $\lambda^{(0)}_{\tilde{F}_\varepsilon(v_\varepsilon)}$, which is explicitly given by

$$\begin{aligned}\lambda^{(0)}_{\tilde{F}_\varepsilon(v_\varepsilon)} \quad &= \quad \frac{1}{\int_M \chi_1 + \chi_2\;\mathrm{dvol}_{\tilde{g}_\varepsilon}} \int_M \tilde{F}_\varepsilon(v_\varepsilon)\;(\,\chi_1 \;-\; \chi_2\,)\;\mathrm{dvol}_{\tilde{g}_\varepsilon} \\ &+ \quad \frac{1}{\int_M \chi_1 + \chi_2\;\mathrm{dvol}_{\tilde{g}_\varepsilon}} \int_M \Delta_{g_\varepsilon}(\chi_1 \tilde{u}^\varepsilon_P) - \Delta_{g_\varepsilon}(\chi_2 \tilde{u}^\varepsilon_P)\;\mathrm{dvol}_{\tilde{g}_\varepsilon}\end{aligned} \tag{53}$$

where $\tilde{u}^\varepsilon_P$ is the solution of the problem

$$\begin{cases} \Delta_{g_\varepsilon} \tilde{u}^\varepsilon_P \;=\; \chi_P \cdot \tilde{F}_\varepsilon(v_\varepsilon) & \text{on } T^\varepsilon(\alpha_1, \alpha_2) \\ \tilde{u}^\varepsilon_P \;=\; 0 & \text{on } \partial T^\varepsilon(\alpha_1, \alpha_2) \end{cases}$$

Let us focus now on the term $\int_M \tilde{F}_\varepsilon(v_\varepsilon)\,(1+v_\varepsilon)\ \mathrm{dvol}_{g_\varepsilon}$ which appears in (53). Recalling the expression for $\tilde{F}_\varepsilon(v_\varepsilon)$, it becomes

$$
\begin{aligned}
c_m\, S \int_M \frac{m+2}{m-2}\, v_\varepsilon\,(\,\chi_1\ -\ \chi_2\,) + \ f(v_\varepsilon)\,(\,\chi_1\ -\ \chi_2\,)\ \mathrm{dvol}_{\tilde{g}_\varepsilon} \\
-\, c_m \int_M (\,S_{\tilde{g}_1}\ +\ S_{g_\varepsilon}\ + S_{\tilde{g}_2}\,)\,(1+v_\varepsilon)\,(\,\chi_1\ -\ \chi_2\,)\ \mathrm{dvol}_{\tilde{g}_\varepsilon}
\end{aligned}
$$

It is not hard to show that

$$
\begin{aligned}
\int_M S_{g_\varepsilon}\,(1+v_\varepsilon)\,\chi_1\ \mathrm{dvol}_{\tilde{g}_\varepsilon} &= -\,4n\,\mathrm{vol}_{g_K}(K)\,\omega_{n-1}\cdot\varepsilon^{n-2}\ +\ \mathcal{O}\left(\mathrm{e}^{-\alpha_1}\,\varepsilon^{n-2}\right) \\
\int_M S_{g_\varepsilon}\,(1+v_\varepsilon)\,\chi_2\ \mathrm{dvol}_{\tilde{g}_\varepsilon} &= -\,4n\,\mathrm{vol}_{g_K}(K)\,\omega_{n-1}\cdot\varepsilon^{n-2}\ +\ \mathcal{O}\left(\mathrm{e}^{-\alpha_2}\,\varepsilon^{n-2}\right)
\end{aligned}
$$

Moreover it is also straightforward to see that

$$
\begin{aligned}
\int_M S_{\tilde{g}_1}\,\chi_1\ \mathrm{dvol}_{\tilde{g}_\varepsilon} &= -\int_{M_1} |dw_1|^2_{g_1}\ \mathrm{dvol}_{g_1}\ +\ \mathcal{O}\left(\varepsilon^{\frac{3(n-2)}{2}}\right) \\
\int_M S_{\tilde{g}_2}\,\chi_2\ \mathrm{dvol}_{\tilde{g}_\varepsilon} &= -\int_{M_2} |dw_2|^2_{g_2}\ \mathrm{dvol}_{g_2}\ +\ \mathcal{O}\left(\varepsilon^{\frac{3(n-2)}{2}}\right)
\end{aligned}
$$

The estimate of the term $\int_M(\,S_{\tilde{g}_1}\,\chi_1 + S_{\tilde{g}_2}\,\chi_2\,)\,v_\varepsilon\ \mathrm{dvol}_{\tilde{g}_\varepsilon}$ is more delicate. Let us look for example at the term $\int_M S_{\tilde{g}_1}\,v_\varepsilon\ \mathrm{dvol}_{\tilde{g}_\varepsilon}$. A direct computation shows that

$$
\int_M S_{\tilde{g}_1}\,v_\varepsilon\ \mathrm{dvol}_{\tilde{g}_\varepsilon} = \int_{M_1} w_1\ (\Delta_{g_1} v_\varepsilon)\ \mathrm{dvol}_{g_1}\ +\ \mathcal{O}\left(\varepsilon^{n-2+\gamma}\right) \tag{54}
$$

Using simple Taylor expansions we obtain (on the support of w_1)

$$
\begin{aligned}
\Delta_{g_1} v_\varepsilon &= \left[1+\frac{4}{m-2}\,w_1+\mathcal{O}\left(\varepsilon^{n-2}\right)\right]\cdot\Delta_{\tilde{g}_\varepsilon} v_\varepsilon \\
&\quad -2\,g_1\,(dw_1,dv_\varepsilon)\ +\ 2\,w_1\,g_1\,(dw_1,dv_\varepsilon)+\mathcal{O}\left(\varepsilon^{\frac{3(n-2)}{2}}\right)
\end{aligned}
$$

Integrating against w_1 gives

$$
\int_M S_{\tilde{g}_1}\,v_\varepsilon\ \mathrm{dvol}_{\tilde{g}_\varepsilon} = \int_{M_1} w_1\,\tilde{F}_\varepsilon(v_\varepsilon)\ \mathrm{dvol}_{g_1}\ -\ \lambda_{\tilde{F}_\varepsilon(v_\varepsilon)}\cdot\int_{M_1} w_1\ \mathrm{dvol}_{g_1}\ +\ \mathcal{O}\left(\varepsilon^{n-2+\gamma}\right)
$$

Performing a rough estimate one can easily see that $\tilde{F}_\varepsilon(v_\varepsilon)\ =\ \mathcal{O}\left(\varepsilon^{(n-2)/2}\right)$ on the support of w_1 and also $\lambda_{\tilde{F}_\varepsilon(v_\varepsilon)}\ =\ \mathcal{O}\left(\varepsilon^{(n-2)/2}\right)$. Combining these facts with the computation above, we improve the rough estimate for S and we immediately obtain $S=\mathcal{O}\left(\varepsilon^{n-2}\right)$.

Again, using the explicit formula for $\tilde{F}_\varepsilon$, one can refine (54) obtaining

$$
\int_M S_{\tilde{g}_1}\,v_\varepsilon\ \mathrm{dvol}_{\tilde{g}_\varepsilon} = c_m\int_{M_1} |dw_1|^2_{g_1}\ \mathrm{dvol}_{g_1}\ +\ \lambda_{\tilde{F}_\varepsilon(v_\varepsilon)}\cdot\mathcal{O}\left(\varepsilon^{\frac{n-2}{2}}\right)\ +\ \mathcal{O}\left(\varepsilon^{\frac{3(n-2)}{2}}\right)
$$

Collecting all the information obtained we deduce that

$$\begin{aligned}\int_M \tilde{F}_\varepsilon(v_\varepsilon)\cdot(\chi_1 - \chi_2)\ \mathrm{dvol}_{\tilde{g}_\varepsilon} &= (c_m^2 - c_m)\cdot\int_M |dw_2|^2_{g_2}\ \mathrm{dvol}_{g_2} \\ &\quad - (c_m^2 - c_m)\cdot\int_M |dw_1|^2_{g_1}\ \mathrm{dvol}_{g_1} \\ &\quad + \lambda_{\tilde{F}_\varepsilon(v_\varepsilon)}\cdot\mathcal{O}\left(\varepsilon^{(n-2)/2)}\right) \\ &\quad + \mathcal{O}\left(\mathrm{e}^{-\alpha_1}\,\varepsilon^{n-2}\right) + \mathcal{O}\left(\mathrm{e}^{-\alpha_2}\,\varepsilon^{n-2}\right)\end{aligned} \tag{55}$$

If we write $\Delta_{g_\varepsilon}(\chi_i\tilde{u}^\varepsilon_P) = \chi_i\,(\Delta_{g_\varepsilon}\tilde{u}^\varepsilon_P) + 2\,[\,g_\varepsilon\,(d\chi_i, d\tilde{u}^\varepsilon_P) + \tilde{u}^\varepsilon_P\,(\Delta_{g_\varepsilon}\chi_i)\,] - \tilde{u}^\varepsilon_P\,(\Delta_{g_\varepsilon}\chi_i)$, then Green's formula implies, for $i = 1, 2$

$$\int_M \Delta_{g_\varepsilon}(\chi_i\tilde{u}^\varepsilon_P)\ \mathrm{dvol}_{\tilde{g}_\varepsilon} = \int_M \chi_1\ \chi_P\,\tilde{F}_\varepsilon(v_\varepsilon)\ \mathrm{dvol}_{g_\varepsilon} - \int_M \tilde{u}^\varepsilon_P\ (\Delta_{g_\varepsilon}\chi_i)\ \mathrm{dvol}_{g_\varepsilon}$$

Following the proof of Proposition 4 contained in [16] and taking into account the shape of $\tilde{F}_\varepsilon(v_\varepsilon)$, one gets the following bound for $\tilde{u}^\varepsilon_P$

$$|\tilde{u}^\varepsilon_P| \leq C\cdot\varepsilon^{n-2}\cdot\psi_\varepsilon^{\gamma-(n-2)}$$

where the positive constant C, does not depend on ε.

Using this fact it is straightforward to deduce that

$$\begin{aligned}\int_M \Delta_{g_\varepsilon}(\chi_1\tilde{u}^\varepsilon_P)\ \mathrm{dvol}_{\tilde{g}_\varepsilon} &= \mathcal{O}\left(\mathrm{e}^{-\alpha_1}\,\varepsilon^{n-2}\right) \\ \int_M \Delta_{g_\varepsilon}(\chi_i\tilde{u}^\varepsilon_P)\ \mathrm{dvol}_{\tilde{g}_\varepsilon} &= \mathcal{O}\left(\mathrm{e}^{-\alpha_2}\,\varepsilon^{n-2}\right)\end{aligned}$$

These estimates and the expression (55) imply that the sign of $\lambda_{\tilde{F}_\varepsilon(v_\varepsilon)}$ is determined, for small ε and sufficiently large α_1 and α_2, by the term

$$(c_m^2 - c_m)\cdot\int_M |dw_2|^2_{g_2}\ \mathrm{dvol}_{g_2} - (c_m^2 - c_m)\cdot\int_M |dw_1|^2_{g_1}\ \mathrm{dvol}_{g_1} \tag{56}$$

Hence it is clear that if we move the real parameters a and b in the definition of the w_i's, the sign of the rough projection changes. Since the solution depends continuously on these parameters, we conclude that for a suitable choice of a and b the rough projection $\lambda_{\tilde{F}_\varepsilon(v_\varepsilon)}$ of the error term $\tilde{F}_\varepsilon(v_\varepsilon)$ along β_ε is zero and Theorem 1 is proved.

6 Getting $S = 0$: the non Ricci flat case

In this section we will prove Theorem 2. As claimed in the statement, when both the initial metrics are scalar flat but non Ricci flat it is possible to construct a zero scalar curvature metric on the generalized connected sum. The idea consists in doing slight non conformal modifications of the approximate solution metrics g_ε's away from the gluing locus. By means of these modifications it is possible to obtain at once the orthogonality to the constant functions of the new error term $\tilde{F}_\varepsilon(v_\varepsilon)$ and the vanishing of its rough projection along the approximate degenerate eigenfunction β_ε of the linearized operator $\Delta_{\tilde{g}_\varepsilon}$. Remember that in the proof of Theorem 1, we used the nonzero constant scalar curvature $S = S(\varepsilon, v_\varepsilon)$ to insure the first condition and slight conformal modifications of the g_ε's to get the second one.

Let us describe the construction. Instead of the metrics g_ε's let us consider the new approximate solution metric

$$\tilde{g}_\varepsilon(r, s) \quad = \quad g_\varepsilon \, + \, r\, h_1 \, + \, s\, h_2 \tag{57}$$

where h_1 and h_2 are positive definite symmetric tensors supported respectively on the manifolds M_1 and M_2 away from the polyneck, and r and s are real parameters. Also remark that the h_i's are non conformal to g_i's, $i = 1, 2$.

The equation we need to solve as a first step is the following

$$\Delta_{\tilde{g}_\varepsilon} v \quad = \quad \tilde{F}_\varepsilon(v, r, s) \, - \, \lambda_{\tilde{F}_\varepsilon(v,r,s)} \cdot \beta_\varepsilon \tag{58}$$

where the new error term is given by $\tilde{F}_\varepsilon(v, r, s) \, := \, -c_m \, S_{\tilde{g}_\varepsilon} \, (1 + v)$. Notice that this definition automatically imposes that the scalar curvature we are going to achieve is constant and equal to zero. Again we assume that $\int_M \beta_\varepsilon \; \mathrm{dvol}_{\tilde{g}_\varepsilon} \, = \, 0$.

Once a solution $v_\varepsilon(r, s)$ is obtained, we will discuss the vanishing of the rough projection $\lambda_{\tilde{F}_\varepsilon(v_\varepsilon,r,s)}$. This will complete the proof of Theorem 2.

As in the previous case, we will be able to find a solution to the nonlinear problem (58) using a fixed point argument for contraction mapping which will produce a fixed point for the equation (58) as a solution of an iteration scheme.

Concerning the linear analysis, notice that the construction above allows us to use the results of Proposition 6, once the orthogonality of the error term $\tilde{F}_\varepsilon(v, r, s)$ to the constant functions is provided.

Let us therefore focus on the condition

$$\int_M \tilde{F}_\varepsilon(v, r, s) \; \mathrm{dvol}_{\tilde{g}_\varepsilon} \quad = \quad 0 \tag{59}$$

Since $\int_M S_{g_\varepsilon} \; \mathrm{dvol}_{\tilde{g}_\varepsilon} \, = \, \mathcal{O}\left(\varepsilon^{n-2}\right)$, we are allowed to choose the correction parameters of the same size, namely $r, s = \mathcal{O}\left(\varepsilon^{n-2}\right)$.

Before starting the calculation let us make some remarks concerning the scalar curvature of the metric $\tilde{g}_\varepsilon$, in order to get more information about $\tilde{F}_\varepsilon$. Since the supports of S_{g_ε}, h_1 and h_2 are disjoint and $S_{g_1} = 0 = S_{g_2}$, we can write, following [3]

$$\begin{aligned} S_{\tilde{g}_\varepsilon} \quad &= \quad S_{g_\varepsilon} \, + \, S_{g_1 + r h_1} \, + \, S_{g_2 + s h_2} \\ &= \quad S_{g_\varepsilon} \, + \, r\, K_1 \, + \, \mathcal{O}\left(r^2\right) \, + \, s\, K_2 \, + \, \mathcal{O}\left(s^2\right) \end{aligned}$$

where

$$K_i \quad = \quad \Delta_{g_i}(\mathrm{tr}_{g_i} h_i) \, + \, \delta_{g_i}(\delta_{g_i} h_i) \, + \, g_i\,(\mathrm{Ric}_{g_i}, h_i) \qquad\qquad i = 1, 2 \tag{60}$$

In the notation above δ_{g_i} indicates the divergence of a symmetric tensor with respect to the metric g_i, and Ric_{g_i} is the Ricci tensor of the metric g_i, for $i = 1, 2$.

Integrating over M gives

$$\begin{aligned} \int_M K_i \ \mathrm{dvol}_{\tilde{g}_\varepsilon} &= \int_{M_i} K_i \ \mathrm{dvol}_{\tilde{g}_i} \\ &= \int_{M_i} g_i \left(\mathrm{Ric}_{g_i}, h_i\right) \ \mathrm{dvol}_{g_i} \ + \mathcal{O}\left(\varepsilon^{n-2}\right) \end{aligned}$$

where $\tilde{g}_1 = g_1 + r\, h_1$ and $\tilde{g}_2 = g_2 + s\, h_2$. Notice that, in the Ricci flat case, the integral above is zero and there is no chance to correct the term $\int_M S_{g_\varepsilon}(1+v) \ \mathrm{dvol}_{g_\varepsilon}$ in order to get the right orthogonality condition.

For $v \in B_\gamma(r_\varepsilon)$ (where $r_\varepsilon = \varepsilon^{2\gamma}$, as in Section 4), let us define the map $G_{\varepsilon,v}(r,s)$ as follows:

$$\begin{aligned} G_{\varepsilon,v}(r,s) &:= \int_M S_{\tilde{g}_\varepsilon}(1+v) \ \mathrm{dvol}_{\tilde{g}_\varepsilon} \\ &= \int_M S_{g_\varepsilon} \ \mathrm{dvol}_{g_\varepsilon} \ + r \int_{M_1} K_1 \ \mathrm{dvol}_{g_1} \ + s \int_{M_2} K_2 \ \mathrm{dvol}_{g_2} \\ &\quad + L_v\left(r,s\right) \ + \ Q_v\left(r,s\right) \end{aligned}$$

where

$$\begin{aligned} L_v(r,s) &:= \int_M S_{g_\varepsilon} v \ \mathrm{dvol}_{g_\varepsilon} \ + r \int_{M_1} K_1\, v \ \mathrm{dvol}_{g_1} + s \int_{M_2} K_2\, v \ \mathrm{dvol}_{g_2} \\ Q_v(r,s) &:= \int_{M_1} S_{g_1 + r h_1}\left(1+v\right) \ \mathrm{dvol}_{g_1} - r \int_{M_1} K_1\left(1+v\right) \ \mathrm{dvol}_{g_1} \\ &\quad + \int_{M_2} S_{g_2 + s h_2}\left(1+v\right) \ \mathrm{dvol}_{g_2} \ - \ s \int_{M_2} K_2\left(1+v\right) \ \mathrm{dvol}_{g_2} \\ &\quad + \mathcal{O}\left(\varepsilon^{2(n-2)}\right) \end{aligned}$$

at this point our purpose is to describe the set where $G_{\varepsilon,v}(r,s)$ is zero.

To proceed, let us consider the map

$$H_\varepsilon(r,s) \quad := \quad G_{\varepsilon,v}(r,s) \ - \ L_v(r,s) \ - \ Q_v(r,s) \tag{61}$$

In order to simplify the following computation, we can assume that the symmetric tensors h_1 and h_2 are chosen so that $\int_{M_1} K_1 \, \mathrm{dvol}_{g_1} \ = 1 = \int_{M_2} K_2 \, \mathrm{dvol}_{g_2}$. We can further assume (up to consider $-g_\varepsilon$ instead of g_ε) that $\int_M S_{g_\varepsilon} \, \mathrm{dvol}_{g_\varepsilon} < 0$ and since $\int_M S_{g_\varepsilon} \, \mathrm{dvol}_{g_\varepsilon} = \mathcal{O}\left(\varepsilon^{n-2}\right)$ we can set, up to normalize, $\int_M S_{g_\varepsilon} \, \mathrm{dvol}_{g_\varepsilon} = -\varepsilon^{n-2}$. With these assumption, the expression for H_ε becomes then the following

$$H_\varepsilon(r,s) \quad = \quad -\varepsilon^{n-2} + r + s$$

The set where H_ε vanishes is given by $\{(r,s) \in \mathbb{R}^2 \quad | \quad r + s = \varepsilon^{n-2}\}$. We will show that the set where $G_{\varepsilon,v}$ vanishes is uniformly close to the set $\{H_\varepsilon = 0\}$ with respect to $v \in B_\gamma(r_\varepsilon)$.

Since $r, s = \mathcal{O}\left(\varepsilon^{n-2}\right)$ it is easy to see that there exist two positive constants C_L and C_Q such that, for all $v \in B_\gamma(r_\varepsilon)$

$$\begin{aligned} L_v(r,s) &\leq C_L \cdot \varepsilon^{n-2+\gamma} \\ Q_v(r,s) &\leq C_Q \cdot \varepsilon^{2(n-2)+\gamma} \end{aligned}$$

In particular, for an arbitrarily small fixed constant $c > 0$ and sufficiently small $\varepsilon > 0$ we have

$$\begin{aligned} |L_v(r,s)| &\leq (c/2)\, \varepsilon^{n-2} \\ |Q_v(r,s)| &\leq (c/2)\, \varepsilon^{n-2} \end{aligned}$$

At this point, it is immediate to check that for all $v \in B_\gamma(r_\varepsilon)$

$$\begin{aligned} \{ G_{\varepsilon,v}(r,s) \,=\, 0 \} &= \left\{ (r,s) \in \mathbb{R}^2 \;\; | \;\; r+s \,=\, \varepsilon^{n-2} - L_v(r,s) - Q_v(r,s) \right\} \\ &\subseteq \left\{ (r,s) \in \mathbb{R}^2 \;\; | \;\; (1-c)\,\varepsilon^{n-2} \,\leq\, r+s \,\leq\, (1+c)\,\varepsilon^{n-2} \right\} \;=:\; Z_\varepsilon \end{aligned}$$

If we set $r_0 := \varepsilon^{n-2}/2$ for every $v \in B_\gamma(r_\varepsilon)$, there must exist a real number $s_0(v)$ such that $(r_0, s_0(v)) \in Z_\varepsilon$ and $G_{\varepsilon,v}(r_0, s_0(v)) = 0$.

Obviously the functions $G_{\varepsilon,v}$ depend smoothly on variables r and s and it is not hard to show that their partial derivatives at the origin are uniformly bounded with respect to $v \in B_\gamma(r_\varepsilon)$. We can also provide the partial derivatives $\partial_r \cdot G_{\varepsilon,v}\,(0,0)$ with a uniform lower bound. In fact, for sufficiently small ε, we can compute

$$\begin{aligned} |\, \partial_r \cdot G_{\varepsilon,j}\,(0,0)\,| &= \left| \int_{M_1} K_1 \,\cdot\, (1+v_j)\; \mathrm{dvol}_{g_1} \right| \\ &\geq \left| \int_{M_1} K_1 \;\mathrm{dvol}_{g_1} \right| - \int_{M_1} |K_1||v_j| \;\mathrm{dvol}_{g_1} \\ &\geq \left| \int_{M_1} K_1 \;\mathrm{dvol}_{g_1} \right| - \|v_j\|_{\mathcal{C}^0(M)} \int_{M_1} |K_1| \;\mathrm{dvol}_{g_1} \\ &\geq \frac{1}{2} \left| \int_{M_1} K_1 \;\mathrm{dvol}_{g_1} \right| \end{aligned}$$

Observe that the bound does not depend on v and that the same is true for $\partial_s \cdot G_{\varepsilon,v}$.

Arguing by contradiction and using these estimates it is possible to deduce that there exists a positive constant $C > 0$ and a positive real number $R > 0$ such that both the first partial derivatives $|\,\partial_r \cdot G_{\varepsilon,v}\,|$ and $|\,\partial_s \cdot G_{\varepsilon,v}\,|$ are greater than C in $B_R\,(0,0)$, for every $v \in B_\gamma(r_\varepsilon)$.

Provided ε is sufficiently small, we see that the set $Z_\varepsilon \,\cap\, \{r, s \geq 0\}$ lies in the ball of radius R centered at the origin, hence it is possible to apply the implicit function Theorem to the functions $G_{\varepsilon,v}$ around the points $(r_0, s_0(v))$. As a consequence, we obtain, for every $v \in B_\gamma(r_\varepsilon)$, an open neighborhood $U(v)$ of r_0, an open neighborhood $V(v)$ of $s_0(v)$ and a smooth function $f_v : U(v) \longrightarrow V(v)$ such that $G_{\varepsilon,v}(r, f_v(r)) \,=\, 0$ for every $r \in U(v)$.

Since it is possible to extend each implicit function f_v to the interval $(0, (1-c)\,\varepsilon^{n-2})$, we can suppose that there exists an open neighborhood U of r_0 and an open neighborhood V of every $s_0(v)$ such that it is possible to choose $U(v) = U$ and $V(v) = V$ for every $v \in B_\gamma(r_\varepsilon)$.

Let us fucus now on the family of functions $\{f_v\}_{v \in B_\gamma(r_\varepsilon)}$. Since each f_v is a uniformly continuous function, we can extend all of them to the compact set $\overline{U}$. This way we have obtained a family of functions $f_v : \overline{U} \longrightarrow \overline{V}$ defined on the same compact set $\overline{U}$ and all bounded by the same constant, namely $(1+c)\,\varepsilon^{n-2}$.

Our aim is now to show that the f_v's admit the same Lipschitz's constant. First remember that the f_v's satisfy

$$f_v(r) \quad = \quad \int_M S_{g_\varepsilon} \ \mathrm{dvol}_{g_\varepsilon} \, - \, r \, + \, L_v(r, f_v(r)) \, + \, Q_v(r, f_v(r)) \tag{62}$$

As a consequence, for $r, r' \in \overline{U}$ and suitable positive constants C_1 and C_2, we have

$$\begin{aligned} |f_j(r) - f_j(r')| \quad &\leq \quad |r - r'| + |\, L_v(r, f_v(r)) \, - \, L_v(r', f_v(r'))\,| \\ &\qquad\qquad + \, |Q_v(r, f_v(r)) \, - \, Q_v(r', f_v(r'))| \\ &\leq \quad |r - r'| + \int_{M_1} |K_1| \ \mathrm{dvol}_{g_1} \cdot \|v\|_{\mathcal{C}^0(M)} \cdot |r - r'| \\ &\qquad + \int_{M_2} |K_2| \ \mathrm{dvol}_{g_2} \cdot \|v\|_{\mathcal{C}^0(M)} \cdot |\, f_v(r) \, - \, f_v(r')\,| \\ &\qquad\qquad + C_1\,\varepsilon^{n-2} \cdot |r - r'| \, + \, C_2\,\varepsilon^{n-2} \cdot |f_j(r) - f_j(r')| \end{aligned}$$

It follows that for small enough ε

$$\begin{aligned} |f_j(r) - f_j(r')| \quad &\leq \quad \frac{1 \, + \, \|v\|_{\mathcal{C}^0(M)} \int_{M_1} |K_1| \ \mathrm{dvol}_{g_1} \, + \, C_1\,\varepsilon^{n-2}}{1 \, - \, \|v\|_{\mathcal{C}^0(M)} \int_{M_2} |K_2| \ \mathrm{dvol}_{g_2} \, + \, C_2\,\varepsilon^{n-2}} \cdot |\, r - r'\,| \\ &\leq \quad 4 \cdot |\, r - r'\,| \end{aligned}$$

Thanks to the Ascoli-Arzelà Theorem, any sequence of functions contained in the family $\{f_v\}_{v \in B_\gamma(r_\varepsilon)}$ converges (up to a subsequence) to a continuous function f with respect to the norm $\|\cdot\|_{\mathcal{C}^0(\overline{U})}$. Moreover f has the same bound and the same Lipschitz's constant as the f_v's. In the following we will use the continuity of f to kill the rough projection of the error term in the equation (58).

We summarize the results obtained so far in this section : forall function $v \in B_\gamma(r_\varepsilon)$ we have found a smooth function f_v defined on a neighborhood $\overline{U}$ of $r_0 = \varepsilon^{n-2}/2$ such that the condition

$$\int_M \tilde{F}_\varepsilon(v, r, f_v(r)) \ \mathrm{dvol}_{\tilde{g}_\varepsilon} \quad = \quad 0 \tag{63}$$

is verified for all $r \in \overline{U}$.

Having the orthogonality condition, we can define the operator $\tilde{P}_\varepsilon$ as in Section 4, namely

$$\tilde{P}_\varepsilon \quad := \quad \Delta_{\tilde{g}_\varepsilon}^{-1} \circ \tilde{F}_\varepsilon \tag{64}$$

For such an operator it is easy to obtain (modulo obvious modifications) the analogue of Lemma 7, with the same definition of the radius r_ε.

It is also immediate to prove that for sufficiently small $\varepsilon > 0$, $\tilde{P}_\varepsilon$ is a contraction mapping and more precisely

$$\|\tilde{P}_\varepsilon(u) - \tilde{P}_\varepsilon(v)\|_{\mathcal{C}^0_\gamma(M)} \quad \leq \quad D\,\varepsilon \cdot \|u - v\|_{\mathcal{C}^0_\gamma(M)} \tag{65}$$

for a suitable constant $D > 0$. In particular the sequence defined by $v_j := \tilde{P}^j_\varepsilon(0)$ converges with respect to the norm $\|\cdot\|_{\mathcal{C}^0_\gamma(M)}$ to a function $v_\varepsilon \in B_\gamma(r_\varepsilon)$. Also notice that at the same time (up to consider a subsequence of the v_j's) the sequence of functions f_{v_j} converges as well to a continuous function f. Hence, for every $r \in \overline{U}$, v_ε verifies the identity

$$\Delta_{\tilde{g}_\varepsilon} v_\varepsilon \quad = \quad \tilde{F}_\varepsilon(v_\varepsilon, r, f(r)) \, - \, \lambda_{\tilde{F}_\varepsilon(v_\varepsilon, r, f(r))}\, \beta_\varepsilon \tag{66}$$

We are now ready to discuss the sign of the term $\lambda_{\tilde{F}_\varepsilon(v_\varepsilon, r, f(r))}$ which appears in this formula.

As in the general case, it turns out that the sign of the rough projection is determined by the sign of

$$\begin{aligned}
\lambda^{(0)}_{\tilde{F}_\varepsilon(v_\varepsilon, r, f(r))} \quad &= \quad \frac{1}{\int_M \chi_1 + \chi_2 \;\mathrm{dvol}_{\tilde{g}_\varepsilon}} \int_M \tilde{F}_\varepsilon(v_\varepsilon)\,\left(\,\chi_1 \,-\, \chi_2\,\right)\;\mathrm{dvol}_{\tilde{g}_\varepsilon} \\
&+ \quad \frac{1}{\int_M \chi_1 + \chi_2 \;\mathrm{dvol}_{\tilde{g}_\varepsilon}} \int_M \Delta_{g_\varepsilon}(\chi_1 \tilde{u}^\varepsilon_P) - \Delta_{g_\varepsilon}(\chi_2 \tilde{u}^\varepsilon_P)\;\mathrm{dvol}_{\tilde{g}_\varepsilon}
\end{aligned} \tag{67}$$

where $\tilde{F}_\varepsilon(v_\varepsilon) = -\,c_m \cdot (S_{g_1 + r\,h_1} \,+\, S_{g_\varepsilon} \,+\, S_{g_2 + f(r)\,h_2}) \cdot (1 + v_\varepsilon)$ and $\tilde{u}^\varepsilon_P$ is the same as in Section 5.

As in the previous section we have

$$\begin{aligned}
\int_M \Delta_{g_\varepsilon}\,(\chi_1 \tilde{u}^\varepsilon_P)\;\mathrm{dvol}_{\tilde{g}_\varepsilon} \quad &= \quad \mathcal{O}\left(\mathrm{e}^{-\alpha_1}\,\varepsilon^{n-2}\right) \\
\int_M \Delta_{g_\varepsilon}\,(\chi_i \tilde{u}^\varepsilon_P)\;\mathrm{dvol}_{\tilde{g}_\varepsilon} \quad &= \quad \mathcal{O}\left(\mathrm{e}^{-\alpha_2}\,\varepsilon^{n-2}\right)
\end{aligned}$$

Concerning the other summand, we compute

$$\begin{aligned}
-\,\frac{1}{c_m} \int_M \tilde{F}_\varepsilon(v_\varepsilon)\,\left(\,\chi_1 \,-\, \chi_2\,\right)\;\mathrm{dvol}_{\tilde{g}_\varepsilon} \quad &= \quad r \int_{M_1} K_1\;\mathrm{dvol}_{g_1} \,-\, f(r) \int_{M_2} K_2\;\mathrm{dvol}_{g_2} \\
&\qquad + \int_M S_{g_\varepsilon}\,\left(\,\chi_1 \,-\, \chi_2\,\right)\;\mathrm{dvol}_{\tilde{g}_\varepsilon} \,+\, \mathcal{O}\left(\varepsilon^{n-2+\gamma}\right) \\
&= \quad r \,-\, f(r) \,+\, \mathcal{O}\left(\mathrm{e}^{-\alpha_1}\,\varepsilon^{n-2}\right) \,+\, \mathcal{O}\left(\mathrm{e}^{-\alpha_2}\,\varepsilon^{n-2}\right) \\
&\qquad + \mathcal{O}\left(\varepsilon^{n-2+\gamma}\right)
\end{aligned}$$

Hence it is clear that for small ε and large enough α_1 and α_2, the sign of the rough projection is determined by the term $r \,-\, f(r)$.

Since it is always possible to choose r either in a region where $f(r) \,>\, r$ or in a region where $f(r) \,<\, r$ and f is a continuous function, we conclude that there exist $\bar{r} \in \overline{U}$ such that

$$\lambda_{\tilde{F}_\varepsilon(v_\varepsilon, \bar{r}, f(\bar{r}))} \quad = \quad 0 \tag{68}$$

This completes the proof of Theorem 2.

To conclude this section, we would like to make some comment about the non Ricci flat hypothesis. Following [7] and [8], the compact Riemannian manifolds without boundary can be divided in the following three classes:

(1_+) Manifolds admitting a Riemannian metric whose scalar curvature is non-negative and not identically zero.

(1_0) Manifolds admitting a Riemannian metric with non-negative scalar curvature, but not in class (1_+).

(1_-) Manifolds not in classes (1_+) or (1_0).

This classification is justified by the following classical result

Theorem 8 (**Trichotomy Theorem, [7], [8]**). *Let M be a compact connected Riemannian manifold without boundary, of dimension ≥ 3, then we have*

1. *If M belongs to class (1_+), every smooth function is realized as the scalar curvature of some Riemannian metric on M.*
2. *If M belongs to class (1_0), then a function $S \in \mathcal{C}^\infty(M)$ is the scalar curvature of some Riemannian metric on M if and only if either $S(p) < 0$ for some point $p \in M$, or else $S \equiv 0$. Moreover, if the scalar curvature of some metric g vanishes identically, then g is Ricci flat.*
3. *If M belongs to class (1_-), then $S \in \mathcal{C}^\infty(M)$ is the scalar curvature of some metric if and only if $S(p) < 0$ for some point $p \in M$.*

In our situation, the initial manifolds M_1 and M_2 carry a scalar flat metric, then *a priori* they might belong to class (1_+) or to class (1_0). Because of the Trichotomy Theorem the non Ricci flat hypothesis implies that they must belong to class (1_+). As it is stated in the Theorem above, manifolds in class (1_+) are the ones for which the prescribed scalar curvature problem has no obstructions. This yields a philosophical justification of the non Ricci flat hypothesis. In other word, if we want to construct a scalar flat metric on the generalized connected sum, we need to handle manifolds which are very flexible concerning the scalar curvature.

7 Generalized gluing for time symmetric initial data

In this section we will discuss an interesting physical application of Theorem 2. To fix the setting, let (Z, γ) be an $(m+1)$-dimensional Lorentzian manifold. The hyperbolic Einstein system for the vacuum spacetime is given by

$$\mathrm{Ric}_\gamma \quad = \quad 0 \qquad (69)$$

In the early 50's Y. Choquet-Bruhat showed in a famous paper [4] that a solution to this system can be obtained from a suitable initial data set. Such an initial data set consists of an m-dimensional space-like hypersurface $M \subset Z$ and two symmetric tensors g and Π (which correspond to the induced metric and to the second fundamental form of M, respectively) verifying the following system, also known as the Einstein constraint equations

$$\begin{aligned} \mathrm{div}_g \Pi - d(\mathrm{tr}_g \Pi) &= 0 \qquad (70)\\ S_g - |\Pi|_g^2 + (\mathrm{tr}_g \Pi)^2 &= 0 \qquad (71) \end{aligned}$$

where S_g indicates the (intrinsic) scalar curvature of the Riemannian manifold (M, g).

A natural idea is to produce new solutions to (69) by gluing together two suitable initial data (or Cauchy data) sets. This has been done in the case of the connected sum at points of two constant mean curvature (briefly CMC) solutions to the constraints, with second fundamental form Π non identically zero (see [10]).

For physical reason, when the second fundamental form Π is identically equal to zero, the Cauchy data set (M, g, Π) is said to be time symmetric (roughly, a time symmetric slice). In this case, it is immediate to check that the system (70)-(71) simply reduces to

$$S_g = 0 \qquad (72)$$

Hence the (generalized) gluing of two time symmetric initial data set (M_1, g_1, Π_1) and (M_2, g_2, Π_2) simply reduces as well to the construction of a scalar flat metric on the (generalized) connected sum of two scalar flat Riemannian manifolds. In fact the second fundamental form on the generalized connected sum can be defined to be identically zero and this trivially yields a gluing when both Π_1 and Π_2 are identically zero.

If we consider two time symmetric initial data which are non Ricci flat, then Theorem 2 provide us with a generalized gluing construction for such Cauchy data sets. Hence we can state the following

Corollary 9. *Let (M_1, g_1, Π_1) and (M_2, g_2, Π_2) be two m-dimensional non Ricci flat solutions to the system (70)-(71) with $\Pi_1 \equiv 0 \equiv \Pi_2$ and let (K, g_K) be a common isometrically embedded Riemannian submanifold whose dimension k is such that $n := m - k \geq 3$. Moreover suppose that the normal bundles of K in M_1 and M_2 are diffeomorphic, then there exists a real number ε_0 such that, for every $\varepsilon \in (0, \varepsilon_0)$, the generalized connected sum $M_\varepsilon = M_1 \sharp_{K,\varepsilon} M_2$ of M_1 and M_2 along K, obtained by excising a tubular neighborhood of K of size ε from both the initial manifolds and identifying the left over boundaries, can be endowed with a new metric $\bar{g}_\varepsilon$ and a new second fundamental forms $\bar{\Pi}_\varepsilon \equiv 0$ such that $(M_\varepsilon, \bar{g}_\varepsilon, \bar{\Pi}_\varepsilon)$ is still a solution of the Einstein constraints (70)-(71). Moreover the new metrics tend to the old ones on the compact sets of $M_i \setminus K$, $i = 1, 2$ in the $\mathcal{C}^2$ topology, as $\varepsilon \to 0$.*

References

[1] M. Anderson, *Dehn filling and Einstein metrics in higher dimensions*, arxiv:math.DG/0303260 (2003).

[2] T. Aubin, *Some nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics, Springer, (1998).

[3] A. L. Besse, *Einstein Manifolds*, Spinger, (1987).

[4] Y. Choquet-Bruhat, *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires*, Acta Math. 88 (1952), 141–225.

[5] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equation of Second Order*, Springer, (1983).

[6] D. Joyce *Constant scalar curvature metrics on connected sums.* Int. J. Math. Math. Sci. (2003), no. 7, 405–450.

[7] J. L. Kazdan, F. Warner, *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature*, Ann. of Math. 101 (1975), 317–331.

[8] J. L. Kazdan, F. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, J. Diff. Geom. 10 (1975), 113–134.

[9] J. Isenberg, D. Maxwell, D. Pollack, *A gluing construction for non vacuum solutions of the Einstein constraint equations*, Adv. Theor. Phys. 9 (2005), 129–172.

[10] J. Isenberg, R. Mazzeo, D. Pollack, *Gluing and wormholes for the Einstein constraint equations*, Comm. Math. Phys. 231 (2002), 529–568.

[11] J. M. Lee, T. H. Parker, *The Yamabe Problem*, Bull. A.M.S. 17 (1987), 37–91.

[12] R. Mazzeo, F. Pacard, *Constant scalar curvature metrics with isolated singularities*, Duke Math. Journal 99 No. 3 (1999), 353–418.

[13] R. Mazzeo, F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, Comm. Anal. Geom. 9 No. 1 (2001), 169–237.

[14] R. Mazzeo, F. Pacard, D. Pollack, *Connected sums of constant mean curvature surfaces in Euclidean 3 space*, J. Reine Angew. Math. 536 (2001), 115–165.

[15] R. Mazzeo, D. Pollack, K. Uhlenbeck, *Connected sums constructions for constant scalar curvature metrics*, Topological Method in Nonlinear Analysis 6 (1995), 207–233.

[16] L. Mazzieri *Generalized connected sum construction for constant scalar curvature metrics* to appear in Communication in Partial Differential Equations.

[17] M. Gromov, H. B. Lawson *The classification of simply connected manifolds of positive scalar curvature.* Ann. of Math. (2) 111, No. 3 (1980), 423–434.

[18] R. Schoen, S. T. Yau *On the structure of manifolds with positive scalar curvature.* Manuscripta Math. 28 No. 1-3 (1979), 159–183.

Articolo 3 / Article 3

Generalized gluing for the Einstein constraint equations

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Abstract

In this paper we construct a family of new (topologically distinct) solutions to the Einstein constraint equations by performing the generalized connected sum (or fiber sum) of two known compact m-dimensional constant mean curvature solutions (M_1, g_1, Π_1) and (M_2, g_2, Π_2) along a common isometrically embedded k-dimensional sub-manifold (K, g_K). Away from the gluing locus the metric and the second fundamental form of the new solutions can be chosen as close as desired to the ones of the original solutions. The proof is essentially based on the conformal method and the geometric construction produces a polyneck between M_1 and M_2 whose metric is modeled fiber-wise (i. e. along the slices of the normal fiber bundle of K) around a Schwarzschild metric; for these reasons the codimension $n := m - k$ of K in M_1 and M_2 is required to be ≥ 3. In this sense our result is a generalization of the Isenberg-Mazzeo-Pollack gluing, which works for connected sum at points and in dimension 3. The solutions we obtain for the Einstein constraint equations can be used to produce new short time vacuum solutions of the Einstein system on a Lorentzian $(m + 1)$-dimensional manifold, as guaranteed by a well known result of Choquet-Bruhat.

Key Words: Einstein constraint equations, connected sum, conformal method, non-linear elliptic PDE's on manifolds

AMS subject classification: 53C21, 58J60, 83C05, 53A30, 57R65

1 Introduction and statement of the result

1.1 CMC solutions and conformal method

It is well known [4] that short time vacuum solutions for the Einstein hyperbolic system on a Lorentzian $(m + 1)$-dimensional manifold (Z, γ) may be obtained from

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solutions of the Einstein constraint equations on a m-dimensional space-like Riemannian submanifold (M, g) of Z. In fact the solutions to the constraints form a suitable set of vacuum initial data for the hyperbolic Cauchy problem (for further details see [3]). More precisely, when we are talking about a solution of the constraints we refer to a triple (M, g, Π), where M is a smooth manifold and g and Π are symmetric $(2,0)$ tensors (respectively the induced Riemannian metric and the second fundamental form), verifying the relationships

$$\begin{aligned} \mathrm{div}_g\,\Pi - d\,(\mathrm{tr}_g\,\Pi) &= 0 \qquad (1)\\ R_g - |\Pi|_g^2 + (\mathrm{tr}_g\,\Pi)^2 &= 0\,, \qquad (2)\end{aligned}$$

where div_g and tr_g are respectively the divergence operator and the trace operator computed with respect to the metric g and R_g is the scalar curvature of the metric g.

In the case we are looking for constant mean curvature (CMC) solutions of the constraints (i.e. when $\tau := \mathrm{tr}_g\,\Pi$ is a constant) the system above becomes equivalent to an uncoupled system by means of the so called conformal method. Following [4], [10] and [11], one can split the second fundamental form Π into trace free and pure trace parts

$$\Pi = \mu + \frac{\tau}{m}\,g\,, \qquad (3)$$

where μ is a symmetric 2-tensor such that $\mathrm{tr}_g\mu = 0$.
Then it is convenient to consider the double conformal change

$$\begin{aligned} g &= u^{\frac{4}{m-2}}\,\bar g \qquad (4)\\ \mu &= u^{-2}\,\bar\mu\,, \qquad (5)\end{aligned}$$

where the conformal factor u is a positive smooth function on M.

It is now straightforward to see that g and Π verify the Einstein constraint equations (1) and (2) if and only if the following holds for $\bar g, \bar\mu$ and u

$$\begin{aligned} \mathrm{tr}_{\bar g}\,\bar\mu &= 0 \qquad (6)\\ \mathrm{div}_{\bar g}\,\bar\mu &= 0 \qquad (7)\\ \mathrm{Lic}_{\bar g}(u) &= 0\,, \qquad (8)\end{aligned}$$

where Lic is the semi-linear elliptic operator given by

$$\mathrm{Lic}_{\bar g}(u) = \Delta_{\bar g}\,u + c_m\,R_{\bar g}\,u - c_m\,|\,\bar\mu\,|_{\bar g}^2\,u^{-\frac{3m-2}{m-2}} + c_m\,\frac{m-1}{m}\,\tau^2\,u^{\frac{m+2}{m-2}} \qquad (9)$$

with $c_m = -(m-2)/[4\,(m-1)]$ (also notice that our Laplacian is negative definite).

Therefore, if we start with a metric $\bar g$ and a real number τ, in order to produce a τ-CMC solution for the Einstein constraints it is sufficient to provide a symmetric $\bar g$-transverse (7) $\bar g$-traceless (6) tensor (briefly TT-tensor) and the right conformal factor, it is to say a solution of the Lichnerowicz equation (8).

In this context and because of their physical meaning [3], we will refer in the following to the equation (1) (or equivalently to the equation (7)) as the momentum constraint and to the equation (2) (or equivalently to the equation (8)) as the Hamiltonian or energy constraint.

1.2 Strategy of the gluing and statement of the main result

In the spirit of [11] suppose now that we start with two Cauchy data sets, namely two solutions $(M_i,\, g_i,\, \mu_i,\, u_i,\, \tau)$, $i = 1,2$ to equations (6), (7) and (8) (notice that this corresponds, modulo the conformal changes $\tilde{g}_i = u_i^{4/(m-2)}\, g_i$ and $\tilde{\mu}_i = u_i^{-2}\, \mu_i$, to considering two sets of τ-CMC solutions $(M_i,\, \tilde{g}_i,\, \tilde{\Pi}_i = \tilde{\mu}_i + (\tau/m)\tilde{g}_i)$, $i = 1,2$ to equations (1), and (2)), and suppose that we construct the generalized connected sum of the compact m-dimensional manifolds M_1 and M_2 along a common isometrically embedded k-dimensional Riemannian submanifold (K, g_K). This construction consists in excising a small ε-tubular neighborhood (i.e. a tubular neighborhood of size $\varepsilon \in (0,1)$) of K in both the starting manifolds and in identifying the differentiable structures along the leftover boundaries as explained in [17] and summarized in section 2. The purpose is then to endow - in correspondence to each value of ε - the new manifold $M_\varepsilon = M_1 \sharp_{K,\varepsilon} M_2$ with a Riemannian structure g_ε and a symmetric TT-tensor μ_ε such that a solution u_ε to equation (8) can be found, with the same τ as the starting Cauchy data sets.

As is typical of the gluing results, the new solution has to preserve the information about the starting solutions insofar as is possible. In our case the metric g_ε will coincide by construction with the metrics g_1 and g_2 away from the gluing locus. Moreover, as the geometric parameter ε tends to zero, the metric g_ε tends to the metric g_i on the compact sets of $M_i \setminus K$, with respect to the $\mathcal{C}^2$ topology, for $i = 1,2$. The TT-tensor μ_ε too tends to μ_i away from the gluing locus, as ε tends to zero (in the following discussion we suppose the injectivity radius of K in (M_1, g_1) and (M_2, g_2) to be greater than one so that the gluing locus can be chosen to be the size one tubular neighborhood of K in both the (M_i, g_i)'s in order to simplify the notations; however it is clear that the gluing locus can be chosen to be arbitrarily small, by fixing its radius at the beginning of the construction). In addition, we can make the conformal factor u_ε as close to the constant one as we want, by choosing ε to be small. In this sense, we are allowed to look at the metric g_ε and at the TT-tensor μ_ε as an approximate solution of the system (1)-(2), which can be made exact by means of a small conformal perturbation $u_\varepsilon \simeq 1$

$$\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{m-2}}\, g_\varepsilon\,, \tag{10}$$

$$\tilde{\Pi}_\varepsilon = u_\varepsilon^{-2}\, \mu_\varepsilon + \frac{\tau}{m}\, u_\varepsilon^{\frac{4}{m-2}}\, g_\varepsilon\,. \tag{11}$$

As already explained, the real advantage in considering CMC solutions is that one has an uncoupled system (6)-(7)-(8) to solve instead of the system (1)-(5). In particular, once an approximate solution metric g_ε is available, the natural way to proceed is to solve first the equations (6) and (7), and then to put the solution μ_ε in the equation (8) and solve this one for u_ε. Since the latter equation is nonlinear and we wish to solve it by means of a perturbation argument which allows us to obtain a new solution which is as close as we want to the starting ones when ε tends to zero (notice that this corresponds to $u_\varepsilon \to 1$ as $\varepsilon \to 0$), we are led to linearize the Lichnerowicz operator around the constant one and to consider the leftover terms as error terms. Among these error terms, the squared norm of μ_ε plays a significant role.

As a consequence of this special role of μ_ε, it is crucial to get an ε-uniform bound for solutions of the equation (7). In this form the momentum constraint is a linear system of partial differential equations and there is a standard two step procedure to produce trace free solutions of it [20]. In our case we will proceed as follows. Starting with μ_1 and μ_2 and using suitable cut-off functions, we produce a g_ε-trace free symmetric 2-tensor μ, which in general is not a solution to (7) (notice that μ actually depends on ε as long as it has to be trace free with respect to the metric g_ε). The second step consists then in finding a correction term σ_ε which repairs the momentum constraint (i.e., $\mathrm{div}_{g_\varepsilon}\,(\mu + \sigma_\varepsilon) = 0$). Since the system is largely underdetermined, we may force the solution to have a special shape. In particular we look for a solution of the form $\sigma_\varepsilon = D_{g_\varepsilon} X$, where X is a vector field on M and D_{g_ε} is the so called conformal Killing operator for the metric g_ε. The conformal Killing operator acts in general as a map from vector fields to symmetric trace free 2-tensors and for an arbitrary metric g is defined as

$$D_g X \;=\; \frac{1}{2}\,\mathcal{L}_X\, g - \frac{1}{m}\,(\mathrm{div}_g X)\cdot g\,, \tag{12}$$

where $\mathcal{L}$ is the Lie derivative. This operator enjoys a nice algebraic property: it is the negative of the formal adjoint of the divergence applied to symmetric trace free 2-tensors. More precisely

$$D_g \;=\; -\,(\sharp\,\mathrm{div}_g)^*\,. \tag{13}$$

As a consequence, when we perform the repair of the momentum constraint, we are induced to consider the elliptic self-adjoint operator $L_g := -\sharp\,\mathrm{div}_g \circ D_g = D_g^* \circ D_g$, known as the vector Laplacian, and to solve the equation

$$L_g X \;=\; \sharp\,\mathrm{div}_g\, \mu \tag{14}$$

with respect to each g_ε metric, therefore providing the solutions with an *a priori* ε-uniform bound.

The vector fields in the kernel of D_g are called conformal Killing vector fields. In fact their flow leaves the metric invariant up to conformal changes; in other words they preserve the conformal class of the metric. For technical reasons, in order to deduce the ε-uniform estimate, we have to require a non-degeneracy assumption about the conformal Killing vector fields of the starting manifolds. The hypothesis we need is the following:

Non-degeneracy condition. *There are no nontrivial conformal Killing vector fields on either* (M_1, g_1) *or* (M_2, g_2).

Notice that because of the different geometric construction this assumption is slightly different from the non-degeneracy condition required in [11]. In fact the IMP gluing works under the assumption that there are no nontrivial conformal Killing vector fields on (M_1, g_1) and (M_2, g_2) wich vanish at the excised points.

In analogy with [11] we also need the assumption that both $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ are non identically zero. This guarantees, via the maximum principle, the injectivity of the linearized Lichnerowicz operators of the metrics g_i around the constant one, namely

$$\Delta_{g_i} - |\mu_i|^2_{g_i} - \tau^2/m, \qquad i = 1, 2. \tag{15}$$

Notice that in the case where $\tilde{\Pi}_1 \equiv 0 \equiv \tilde{\Pi}_2$ the Einstein constraint equations reduces to $R_{g_1} = 0$ and $R_{g_2} = 0$. In this situation the starting Cauchy data sets are said to be time symmetric and our problem reduces to constructing a scalar flat metric on the generalized connected sum of two scalar flat Riemannian manifolds. This can be done if both (M_1, g_1) and (M_2, g_2) are non Ricci flat, as shown in [18].

Following the strategy summarized above, we can prove the following result

Theorem 1. *Let $(M_1, \tilde{g}_1, \tilde{\Pi}_1)$ and $(M_2, \tilde{g}_2, \tilde{\Pi}_2)$ be two compact m-dimensional CMC solutions to the Einstein constraint equations (1)-(2) having the same constant mean curvature τ and verifying the non-degeneracy condition. Moreover suppose that both $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ are non identically zero and let $(K, \tilde{g}_K)$ be a common isometrically embedded k-dimensional sub-manifold with codimension $n := m - k \geq 3$ such that the normal bundles of K in M_1 and in M_2 are diffeomorphic. Then there exists a real value $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ it is possible to endow the ε-generalized connected sum $M_\varepsilon = M_1 \sharp_{K,\varepsilon} M_2$ of M_1 and M_2 along K with a metric $\tilde{g}_\varepsilon$ and a second fundamental form $\tilde{\Pi}_\varepsilon$ such that the triple $(M_\varepsilon, \tilde{g}_\varepsilon, \tilde{\Pi}_\varepsilon)$ is still a τ-CMC solution to the Einstein constraint equations.*
Moreover the new metric $\tilde{g}_\varepsilon$ tends to the starting metric $\tilde{g}_i$ in the $\mathcal{C}^2$ topology on the compact sets of $M_i \setminus K$, for $i = 1, 2$, as the geometric parameter ε tends to zero. The symmetric TT-tensor $\tilde{\mu}_\varepsilon$ does the same away from a fixed tubular neighborhood of K (gluing locus) whose radius can be chosen to be arbitrarily small.

2 The geometric construction

The aim of this section is to give a precise description of the generalized connected sum and to present a way to construct a family of approximate solution metrics $(g_\varepsilon)_{\varepsilon \in (0,1)}$; these are metrics which, when ε varies in a sufficiently small range, can be perturbed to the final metric $\tilde{g}_\varepsilon$ by means of a small (i.e. close to one) conformal factor u_ε, this one being a solution to the Lichnerowicz equation with respect to the metric g_ε, the constant mean curvature τ and a suitable TT-tensor μ_ε. At the end of this section we also present the construction of a symmetric g_ε-trace free tensor $\mu = \mu(\varepsilon)$ by means of a warped cut-off method (then, repairing this μ by means of a suitable symmetric tensor σ_ε, we will find the TT-tensor $\mu_\varepsilon := \mu + \sigma_\varepsilon$ mentioned above).

The construction we present here is the same as[17]. Nevertheless, in order to make the exposition self contained and to fix the notation, we recall it here. The reason why this construction yields a good ansatz relies on the fact that the hamiltonian constraint is very similar to the Yamabe equation treated in [17], and since we want to produce analogous results (i.e. a conformal factor very close to one), we choose to solve our equation using the analytical tools and the geometric construction which have been successful with the Yamabe problem of [17].

Let (K, g_K) be a k-dimensional Riemannian manifold isometrically embedded in both the m-dimensional Riemannian manifolds (M_1, g_1) and (M_2, g_2), we label the embedding maps as follows

$$\iota_i : K \hookrightarrow M_i \,.$$

We assume that the isometric map $\iota_1^{-1} \circ \iota_2 : \iota_1(K) \to \iota_2(K)$ extends to a diffeomorphism between the normal bundles of $\iota_i(K)$ in (M_i, g_i), $i = 1, 2$. To simplify the notations and the computations, here and in the following the injectivity radius of K in both the manifolds is supposed to be greater than one, so that we are allowed to manipulate the differential and the metric structure on a fixed tubular neighborhood of K in M_1 and M_2 of size one (gluing locus). This construction can obviously be modified in order make the gluing locus as small as desired, and in particular smaller than the injectivity radii. For a fixed $\varepsilon \in (0, 1)$, we describe the construction of the generalized connected sum (or fiber sum) of M_1 and M_2 along K and the definition of the metric g_ε in local coordinates. The fact that this construction yields a globally defined metric will follow at once.

Let U^k be an open set of $\mathbb{R}^k_z$, B^n the n-dimensional open ball of radius one in $\mathbb{R}^n_x$, where $n := m - k \geq 3$ is the codimension of K. For $i = 1, 2$, $F_i : U^k \times B^n \to W_i \subset M_i$ given by

$$F_i(z, x) \quad := \quad \exp^{M_i}_{(z,0)}(x) \,, \tag{16}$$

defines local Fermi coordinates near the coordinate patches $F_i(\cdot, 0)\,(U) \subset \iota_i(K) \subset M_i$. In these coordinates, the metric g_i can be decomposed as

$$g_{(i)}(z, x) \quad = \quad g^{(i)}_{jl} dz^j \otimes dz^l + g^{(i)}_{\alpha\beta} dx^\alpha \otimes dx^\beta + g^{(i)}_{j\alpha} dz^j \otimes dx^\alpha \,, \tag{17}$$

and it is well known that in this coordinate system

$$g^{(i)}_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}\left(|x|^2\right) \qquad \text{and} \qquad g^{(i)}_{j\alpha} = \mathcal{O}\left(|x|\right) \,.$$

In order to perform the identification between W_1 and W_2 and in order to glue the metrics together and define g_ε, we partially change the coordinate system, by setting $x = \varepsilon\, \mathrm{e}^{-t}\, \theta$ on $F_1^{-1}(W_1)$ and $x = \varepsilon\, \mathrm{e}^{t} \theta$ on $F_2^{-1}(W_2)$, for $\varepsilon \in (0, 1)$, $\log \varepsilon < t < -\log \varepsilon$, $\theta \in S^{m-k-1}$. Actually, we will modify the starting metrics only on the hollow domain $U^k \times A^n\left(\varepsilon^2, 1\right)$ (where $A^n\left(r, s\right)$ is the n-dimensional annulus $\{r < |x| < s\}$).

Using the changes of coordinates just described the expressions of the two metrics g_1 and g_2 on the hollow domain become respectively

$$\begin{aligned} g_1(z, t, \theta) \quad &= \quad g^{(1)}_{ij} dz^i \otimes dz^j \\ &+ \quad u_\varepsilon^{(1)\,\frac{4}{n-2}} \Big[\Big(dt \otimes dt + g^{(1)}_{\lambda\mu} d\theta^\lambda \otimes d\theta^\mu \Big) + g^{(1)}_{t\theta} dt \ltimes d\theta \Big] \\ &+ \quad g^{(1)}_{it} dz^i \otimes dt + g^{(1)}_{i\lambda} dz^i \otimes d\theta^\lambda \end{aligned} \tag{18}$$

and

$$
\begin{aligned}
g_2(z,t,\theta) &= g^{(2)}_{ij} dz^i \otimes dz^j \\
&+ u^{(2)}_\varepsilon {}^{\frac{4}{n-2}} \left[\left(dt \otimes dt + g^{(2)}_{\lambda\mu} d\theta^\lambda \otimes d\theta^\mu \right) + g^{(2)}_{t\theta} dt \ltimes d\theta \right] \\
&+ g^{(2)}_{it} dz^i \otimes dt + g^{(2)}_{i\lambda} dz^i \otimes d\theta^\lambda ,
\end{aligned}
\tag{19}
$$

where by the compact notation $dt \ltimes d\theta$ we indicate the general component of the normal metric tensor (i.e., it involves $dt \otimes dt$, $d\theta^\lambda \otimes d\theta^\mu$ and $dt \otimes d\theta^\lambda$ components), whereas the coefficients $g_{t\theta}$ multiplied by $u^{(i)}_\varepsilon {}^{\frac{4}{n-2}}$, $i = 1, 2$ represent the correction to the Euclidean metric $u^{(i)}_\varepsilon {}^{\frac{4}{n-2}} \left[\left(dt \otimes dt + g^{(i)}_{\lambda\mu} d\theta^\lambda \otimes d\theta^\mu \right) \right]$, $i = 1, 2$ in our coordinate system.

Remark that for $j = 1, 2$ we have

$$
\begin{aligned}
g^{(j)}_{\lambda\mu} &= \mathcal{O}(1) & \qquad g^{(j)}_{t\theta} &= \mathcal{O}\left(|x|^2\right) \\
g^{(j)}_{it} &= \mathcal{O}\left(|x|^2\right) & \qquad g^{(j)}_{i\lambda} &= \mathcal{O}\left(|x|^2\right)
\end{aligned}
$$

and

$$
u^{(1)}_\varepsilon(t) = \varepsilon^{\frac{n-2}{2}} \mathrm{e}^{-\frac{n-2}{2} t} \qquad \text{and} \qquad u^{(2)}_\varepsilon(t) = \varepsilon^{\frac{n-2}{2}} \mathrm{e}^{\frac{n-2}{2} t} .
$$

We choose a cut-off function $\chi : (\log \varepsilon, -\log \varepsilon) \to [0,1]$ to be a non increasing smooth function which is identically equal to 1 in $(\log \varepsilon, -1]$ and 0 in $[1, -\log \varepsilon)$ and we choose another cut-off function $\eta : (\log \varepsilon, -\log \varepsilon) \to [0,1]$ to be a non increasing smooth function which is identically equal to 1 in $(\log \varepsilon, -\log \varepsilon - 1]$ and which satisfies $\lim_{t \to -\log \varepsilon} \eta = 0$. Using these two cut-off functions, we can define a new normal conformal factor u_ε by

$$
u_\varepsilon(t) \;:=\; \eta(t)\, u^{(1)}_\varepsilon(t) \;+\; \eta(-t)\, u^{(2)}_\varepsilon(t) , \tag{20}
$$

and the metric g_ε by

$$
\begin{aligned}
g_\varepsilon(z,t,\theta) &:= \left(\chi g^{(1)}_{ij} + (1-\chi) g^{(2)}_{ij} \right) dz^i \otimes dz^j \\
&+ u_\varepsilon^{\frac{4}{n-2}} \Big[dt \otimes dt + \left(\chi g^{(1)}_{\lambda\mu} + (1-\chi) g^{(2)}_{\lambda\mu} \right) d\theta^\lambda \otimes d\theta^\mu \\
&\qquad\qquad + \left(\chi g^{(1)}_{t\theta} + (1-\chi) g^{(2)}_{t\theta} \right) dt \ltimes d\theta \Big] \\
&+ \left(\chi g^{(1)}_{it} + (1-\chi) g^{(2)}_{it} \right) dz^i \otimes dt \\
&+ \left(\chi g^{(1)}_{i\lambda} + (1-\chi) g^{(2)}_{i\lambda} \right) dz^i \otimes d\theta^\lambda .
\end{aligned}
\tag{21}
$$

Closer inspection of this expression shows that the only objects that are not *a priori* globally defined on the identification of the tubular neighborhoods (poly-neck) of $\iota_1(K)$ in M_1 and $\iota_2(K)$ in M_2 are the functions χ and u_ε (since the cut-off η is involved in the definition). However, observe that both cut-off functions can easily be expressed as functions of the Riemannian distance to K in the respective manifolds.

Hence they are globally defined and the metric g_ε - whose definition can be obviously completed by setting $g_\varepsilon \equiv g_1$ and $g_\varepsilon \equiv g_2$ out of the polyneck - is a Riemannian metric which is globally defined on the manifold M_ε.

We conclude this section with the definition of the proto-TT-tensor $\mu = \mu(\varepsilon)$, which is the symmetric g_ε-trace free tensor which will be corrected to a symmetric g_ε-TT-tensor by adding an ε-uniformly bounded term σ_ε. In order to do that we describe a warped cut off procedure on the side of the polyneck coming from M_1. The same manipulation on the other side provides us with the complete definition of μ.

Let a g_1-trace-free symmetric tensor μ_1 be given on M_1. In local coordinates this reads

$$g_{ij}^{(1)}\,\mu_1^{ij} \,+\, 2\,g_{i\alpha}^{(1)}\,\mu_1^{i\alpha} \,+\, g_{\alpha\beta}^{(1)}\mu_1^{\alpha\beta} \quad = \quad 0\,. \tag{22}$$

We are looking for a symmetric tensor μ which is trace free with respect to the metric g_ε. To do that we set

$$\begin{aligned} \mu^{ij} &= a(t)\cdot\mu_1^{ij} \\ \mu^{i\alpha} &= a(t)\cdot\mu_1^{i\alpha} \\ \mu^{\alpha\beta} &= b(t)\cdot\mu_1^{\alpha\beta}\,, \end{aligned} \tag{23}$$

where a and b are smooth radial cut-off functions which are equal to one on M_1 and which vanish for $t > 1 + \log\varepsilon$. The definitions of a and b are made more precise below; however we remark that the warped cut-off still guarantees the symmetry of μ. Taking into account (22) and the definition of the metric g_ε on the region where μ is not identically zero, the condition $\mathrm{tr}_{g_\varepsilon}\mu = 0$ is equivalent to

$$\begin{aligned} 0 &= a\,g_{ij}^{(1)}\,\mu_1^{ij} \,+\, 2\,a\,g_{i\alpha}^{(1)}\,\mu_1^{i\alpha} \,+\, b\,\phi^2\,g_{\alpha\beta}^{(1)}\mu_1^{\alpha\beta} \\ &= \big[b\,\phi^2 \,-\, a\big]\cdot g_{\alpha\beta}^{(1)}\mu_1^{\alpha\beta}\,, \end{aligned}$$

where the normal conformal factor ϕ^2 is by definition

$$\phi^2 \quad := \quad \Big[1 \,+\, \eta(-t)\cdot(u_\varepsilon^{(2)}/u_\varepsilon^{(1)})\Big]^{\frac{4}{n-2}}\,.$$

It is now straightforward to verify that one can always choose two smooth cut-off functions satisfying the conditions above and such that $a = \phi^2\,b$. Notice that since ϕ depends on ε, a and b do as well, but they admit an ε-uniform bound, as do their derivatives. Finally, let us observe that, for every $k \geq 0$, $\big|\nabla^k\,\mathrm{div}_{g_\varepsilon}\mu\big|_{g_\varepsilon}(t) \to \big|\nabla^k\,\mathrm{div}_{g_1}\mu_1\big|_{g_1}(\log\varepsilon) = 0$, as t tends to $\log\varepsilon$. Moreover $|\mathrm{div}_{g_\varepsilon}\mu|_{g_\varepsilon}$ and $|\nabla\,\mathrm{div}_{g_\varepsilon}\mu|_{g_\varepsilon}$ are ε-uniformly bounded in the interior of the polyneck.

3 The momentum constraint

3.1 The vector Laplacian L_{g_ε}

As explained in the introduction, the next task is now to repair the momentum constraint. We start from μ defined in the previous section, which is symmetric and trace free, but in general does not satisfy the equation $\operatorname{div}_{g_\varepsilon} \mu = 0$. We want to replace it by a symmetric divergence free tensor μ_ε whose ε-trace is still zero. The way to do that is to find a symmetric trace-free correction term σ_ε whose norm $|\sigma_\varepsilon|^2_{g_\varepsilon}$ admits a bound which is uniform with respect to ε. This way we are allowed to choose $\mu_\varepsilon := \mu + \sigma_\varepsilon$ to be a TT-tensor and to put the term $|\mu_\varepsilon|^2_{g_\varepsilon}$ in the nonlinear equation $\operatorname{Lic}_{g_\varepsilon} u = 0$. Then the uniform bound enables us to get an appropriate estimate for the error term of the latter equation and then solve it by means of a perturbation argument.

As discussed above, a good way to proceed is to seek a correction term of the form $\sigma_\varepsilon = D_{g_\varepsilon} X$, where D_{g_ε} is the conformal deformation operator with respect to the metric g_ε. This automatically guarantees that σ_ε is symmetric and trace free, as it is easy to see from the local expression of this operator

$$(D_{g_\varepsilon})_{jk} \quad = \quad \frac{1}{2}\,[\,(\nabla_j X)_k + (\nabla_k X)_j\,] - \frac{1}{m}\,(\nabla_l X)^l \cdot g^{(\varepsilon)}_{jk}\,, \tag{24}$$

where ∇ is the Levi-Civita connection of the metric g_ε, and the indices has been lowered by means of the metric g_ε, where needed.

The problem we are led to consider is then the vector equation

$$L_{g_\varepsilon} X \quad = \quad \sharp \operatorname{div}_{g_\varepsilon} \mu\,, \tag{25}$$

where the operator involved - the so called vector Laplacian - is defined as $L_{g_\varepsilon} := (D_{g_\varepsilon})^* \cdot D_{g_\varepsilon} = -\sharp \operatorname{div}_{g_\varepsilon} \cdot D_{g_\varepsilon}$. As it is easy to verify, L_{g_ε} is a linear elliptic second order partial differential operator with smooth coefficients and it is formally self-adjoint. We can think of the vector Laplacian as acting between the spaces of sections with Hölder regularity

$$L_{g_\varepsilon} \;:\; \mathcal{C}^{2,\alpha}(M_\varepsilon, TM_\varepsilon) \;\longrightarrow\; \mathcal{C}^{0,\alpha}(M_\varepsilon, TM_\varepsilon)\,. \tag{26}$$

In Section 3.2, in order to produce an ε-uniform *a priori* bound for solutions to (25), we introduce a more sophisticated functional setting (i.e., weighted Hölder spaces of sections of fiber bundles); but now the following definitions are sufficient. For a general tensor field T we define the $\mathcal{C}^k$ norm of T with respect to a Riemmannian metric g as

$${}^{g}\|T\|_{\mathcal{C}^k} \quad := \quad \sum_{j=0}^{k} \sup_M \left|\nabla^j T\right|_g\,, \tag{27}$$

where ∇ is the Levi-Civita connection of g; we define the Hölder seminorm of the k-th derivative with exponent $\alpha \in (0,1)$ as

$${}^{g}\left[\nabla^k T\right]_\alpha \quad := \quad \sup_{p\neq q} \frac{\left|\nabla^k T(p) - \nabla^k T(q)\right|_g}{d_g(p\,,\,q)^\alpha}\,, \tag{28}$$

where the distance $d_g(p\,,\,q)$ is supposed to be smaller than the injectivity radius and with abuse of notation the term $\nabla^k T(q)$ is interpreted as its parallel transport from q to p along the unique geodesic joining p and q (in order to give sense to the subtraction which appears in the numerator above). The definition of the $\mathcal{C}^{k,\alpha}$-Hölder norm follows obviously.

The general theory of elliptic operators between vector bundles and in particular the Fredholm alternative guarantees the existence of a solution to the equation $L_{g_\varepsilon} X = W$, provided that the right hand term W is orthogonal to $\operatorname{Ker} L^*_{g_\varepsilon}$. In our case we have to check the vanishing of the L^2-product

$$-\langle \sharp \operatorname{div}_{g_\varepsilon} \mu \, , \, Y \rangle \;\; := \;\; - \int_{M_\varepsilon} g_\varepsilon(\sharp \operatorname{div}_{g_\varepsilon} \mu \, , \, Y) \; dvol_{g_\varepsilon}$$
$$= \;\; \int_{M_\varepsilon} g_\varepsilon(\mu \, , \, D_{g_\varepsilon} Y) \; dvol_{g_\varepsilon} \, ,$$

where Y is an element of $\operatorname{Ker} L^*_{g_\varepsilon} = \operatorname{Ker} L_{g_\varepsilon}$, by self-adjointness. Since for each fixed $\varepsilon \in (0,1)$ the generalized connected sum M_ε is a compact manifold, the integration by part yields

$$0 \;\; = \;\; \langle L_{g_\varepsilon} Y \, , \, Y \rangle \;\; = \;\; {}^{\varepsilon}\| D_{g_\varepsilon} Y \|^2_{L^2} \, ,$$

and the orthogonality is then proved. Hence, for each $\varepsilon \in (0,1)$ we can get a vector field X_ε satisfying the equation (25).

3.2 *A priori* uniform bound for solutions of $L_{g_\varepsilon} X = \sharp \operatorname{div}_{g_\varepsilon} \mu$

This section is devoted to providing the existence of solutions X_ε of the equations $L_{g_\varepsilon} X = \sharp \operatorname{div}_{g_\varepsilon} \mu$ with an *a priori* bound which is uniform in $\varepsilon \in (0,1)$. As noted earlier, a more sophisticated functional setting is needed. In particular the weighted Hölder spaces turn out to be the crucial tools needed to get the estimate we want. Using the definition of t from Section 2, we define the distance function ρ_ε to be $\rho_\varepsilon := \varepsilon \cosh t$ for $(\log \varepsilon) + 1 < t < -(\log \varepsilon) - 1$ (i.e., in the middle of the polyneck), to be $\rho_\varepsilon \equiv 1$ out of the radius one tubular neighborhoods of K in M_1 and in M_2 and to be a monotone radial smooth interpolation in between these regions. Having introduced a radial distance function, we can define, for $k \in \mathbb{N}$ and $\alpha \in (0,1)$, the weighted $\mathcal{C}^k$-norms and the weighted Hölder α-seminorms for a general tensor field T on M_ε with respect to the metric g_ε. The definition of the weighted $\mathcal{C}^{k,\alpha}$-Hölder norm follows at once. For a general weight $\gamma \in \mathbb{R}$ let us set

$${}^{\varepsilon}\|T\|_{\mathcal{C}^k_\gamma} \;\; := \;\; \sum_{j=0}^{k} \sup_{M_\varepsilon} \left\{ \rho_\varepsilon^{-\gamma+j} \cdot \left| \nabla^j T \right|_{g_\varepsilon} \right\} \tag{29}$$

$${}^{\varepsilon}\left[\nabla^k T \right]_{\alpha,\gamma} \;\; := \;\; \sup_{p \neq q} \left\{ \left| \rho_\varepsilon(p) \wedge \rho_\varepsilon(q) \right|^{-\gamma+k} \cdot \frac{\left| \nabla^k T(p) - \nabla^k T(q) \right|_{g_\varepsilon}}{d_{g_\varepsilon}(p \, , \, q)^\alpha} \right\} , \tag{30}$$

where ∇ indicates the Levi-Civita connection of g_ε, $|\rho_\varepsilon(p) \wedge \rho_\varepsilon(q)|$ is the minimum between $\rho_\varepsilon(p)$ and $\rho_\varepsilon(q)$, and the conventions used in (28) are still valid for (30). In the following we indicate by $\rho_\varepsilon^\gamma \cdot \mathcal{C}^{k,\alpha}(M_\varepsilon, TM_\varepsilon)$ the space of tensor fields X such that the norm ${}^{\varepsilon}\|X\|_{\mathcal{C}^{k,\alpha}_\gamma} := {}^{\varepsilon}\|X\|_{\mathcal{C}^k_\gamma} + {}^{\varepsilon}\left[\nabla^k X \right]_{\alpha,\gamma}$ is well defined and finite. In this context it is convenient to think of L_{g_ε} as acting between the spaces

$$L_{g_\varepsilon} \; : \; \rho_\varepsilon^\delta \cdot \mathcal{C}^{2,\alpha}(M_\varepsilon, TM_\varepsilon) \; \longrightarrow \; \rho_\varepsilon^{\delta-2} \cdot \mathcal{C}^{0,\alpha}(M_\varepsilon, TM_\varepsilon) \, , \tag{31}$$

for a suitable weight $\delta \in \mathbb{R}$. Notice that, for fixed ε, the functional setting of (31) is strictly equivalent to the one of (26). In particular, the existence result of the previous section still holds. The reason for introducing weighted spaces is that uniform estimates are not available in the old context, since the geometry of our construction becomes singular when the parameter ε tends to zero. Having introduced these new analytical devices, we can now state the following:

Proposition 2. *Let $X \in \rho_\varepsilon^\delta \cdot \mathcal{C}^{2,\alpha}(M_\varepsilon, TM_\varepsilon)$ and $W \in \rho_\varepsilon^{\delta-2} \cdot \mathcal{C}^{0,\alpha}(M_\varepsilon, TM_\varepsilon)$ be vector fields satisfying the equation $L_{g_\varepsilon} X = W$. Moreover suppose that W is of the form $W = \sharp \operatorname{div}_{g_\varepsilon} \mu$, for some symmetric 2-tensor μ. Then, if the weight δ is chosen to be in $(\,(2-n)/2\,,\,0\,)$, there exists a constant $C > 0$ independent of $\varepsilon \in (0,1)$ such that*

$$ {}^{\varepsilon}\|X\|_{\mathcal{C}^1_\delta} \quad \leq \quad C \cdot {}^{\varepsilon}\|W\|_{\mathcal{C}^{0,\alpha}_{\delta-2}} \,. \tag{32} $$

(Remember that n is the codimension of K in M_i, $i = 1, 2$ and it is supposed to be greater than 3).

The proof is by contradiction. If such a constant C does not exist, we can find out for every $j \in \mathbb{N}$ a triple $(\varepsilon_j, X_j, W_j)$ such that

1. $\quad \varepsilon_j \rightarrow 0 \quad$ as $j \rightarrow +\infty$

2. $\quad L_{\varepsilon_j} X_j = W_j \quad$ for every $j \in \mathbb{N}$

3. $\quad {}^{\varepsilon_j}\|X_j\|_{\mathcal{C}^1_\delta} = 1 \quad$ for every $j \in \mathbb{N}$

4. $\quad {}^{\varepsilon_j}\|W_j\|_{\mathcal{C}^{0,\alpha}_{\delta-2}} \rightarrow 0 \quad$ as $j \rightarrow +\infty$

For technical reasons we prefer to replace condition **3**. with the following

3.bis $\quad \sup_{M_j} \left\{ \rho_j^{-\delta} \cdot |X_j|_{\varepsilon_j} + \rho_j^{-\delta+1} \cdot |\nabla X_j|_{\varepsilon_j} \right\} = 1 \,, \qquad$ for every $j \in \mathbb{N}$.

Notice that this can be done because condition **3.bis** is nothing but an equivalent way to define the norm ${}^{\varepsilon_j}\|X_j\|_{\mathcal{C}^1_\delta}$. Since each M_j is compact we can look now at the points p_j's where this maximum is achieved, so that

$$ \left\{ \rho_j^{-\delta} \cdot |X_j|_{\varepsilon_j} + \rho_j^{-\delta+1} \cdot |\nabla X|_{\varepsilon_j} \right\} (p_j) \quad = \quad 1 \tag{33} $$

To carry out the proof we are going to take the limit of the expression **2**. It is then clear that the issue reduces to investigating whether the homogeneous limit problem admits nontrivial solutions with prescribed decay, where the non triviality strictly depends on the behavior at the limit of the p_j's and the decay is prescribed by the weight. This leads us to distinguish three different limit situations, depending on how the geometric structure degenerates near the p_j's, as $j \rightarrow +\infty$.

Case 1. The p_j's converge, up to a subsequence, to a point p_∞ which lies in $M_1 \setminus K$ (or analogously in $M_2 \setminus K$), as $j \rightarrow +\infty$. In this case, as we discuss below, we are induced to look for nontrivial solutions of the homogeneous problem

$$\begin{cases} L_{g_1} X \, = \, 0 & \text{on } M_1 \setminus K \\ |X|_{g_1} \, \leq \, A \, \cdot \, r^{\delta} \\ |\nabla X|_{g_1} \, \leq \, B \, \cdot \, r^{\delta - 1} \end{cases} \tag{34}$$

where $r \, := \, d_{g_1}(\,\cdot\,, K)$, and $A, B > 0$ are positive constants.

Case 2. The p_j's converge, up to a subsequence, to a point p_∞ which lies in K with the same speed as the radius of the excised tubular neighborhood: $d_{g_1}(p_j, K) = \mathcal{O}(\varepsilon_j)$, as $j \to +\infty$ (notice that, up to a subsequence, we can suppose without loss of generality that all the p_j's lie in the side coming from M_1). In this case, by means of a blow-up method, we are induced to look for nontrivial solutions of the homogeneous problem

$$\begin{cases} L_{\mathbb{R}^k \times \mathbb{S}^n} X \, = \, 0 & \text{on } \mathbb{R}^k_z \times \mathbb{S}^n_{t,\theta} \\ |X|_{\mathbb{R}^k \times \mathbb{S}^n} \, \leq \, C \, \cdot \, (\cosh t)^{\delta} \end{cases} \tag{35}$$

where $C > 0$ is a positive constant and $\mathbb{S}^n$ denotes the n-dimensional Schwarzschild space. Moreover, by the expressions $L_{\mathbb{R}^k \times \mathbb{S}^n}$ and $|\cdot|_{\mathbb{R}^k \times \mathbb{S}^n}$, we indicate respectively the vector Laplacian and the norm of the product metric $g_{\mathbb{R}^k} + g_{\mathbb{S}^n}$.

Case 3. The p_j's converge, up to a subsequence, to a point p_∞ which lies in K with a lower speed than the radius of the excised tubular neighborhood: $d_{g_1}(p_j, K) \,/\, \varepsilon_j \to +\infty$, as $j \to +\infty$ (notice that, up to a subsequence, we can suppose without loss of generality that all the p_j's lie in the side coming from M_1). In this case, refining the blow-up method of the previous case, we are induced to look for nontrivial solutions of the homogeneous problem

$$\begin{cases} L_{\mathbb{R}^k \times \mathbb{R}^n} X \, = \, 0 & \text{on } \mathbb{R}^k_z \times (\mathbb{R}^n_x \setminus \{0\}) \\ |X|_{\mathbb{R}^k \times \mathbb{R}^n} \, \leq \, D \, \cdot \, |x|^{\delta} \end{cases} \tag{36}$$

where $D > 0$ is a positive constant.

The choice of the weight δ in the right indicial interval $(\,(2-n)/2\,,\,0\,)$ leads us to a contradiction in all of the three cases. In other words the homogeneous problems with prescribed decay (34),(35) and (36) only admit trivial solutions. In the following we analyze one by one the three cases presented above.

3.2.1 Case 1: the equation $L_{g_1} X \, = \, 0$ and the non-degeneracy condition

In this first case we assume that (up to a subsequence) the p_j's tend to a point $p_\infty \, \in \, M_1 \setminus K$. As it is easy to check from the expression (21), we have that the metrics g_{ε_j}'s converge to the metric g_1 with respect to to the $\mathcal{C}^2$-topology on the compact sets of $M_1 \setminus K$. Hence on a fixed compact set $Q \, \subset \, M_1 \setminus K$ the weighted

g_{ε_j} norms are all equivalent to the standard g_1 norm. More precisely, there exist two positive constants $A(Q), B(Q) > 0$ such that

$$A(Q) \cdot {}^{\varepsilon_j}\|X\|_{\mathcal{C}^1_\delta(Q)} \;\leq\; {}^{g_1}\|X\|_{\mathcal{C}^1(Q)} \;\leq B(Q) \cdot {}^{\varepsilon_j}\|X\|_{\mathcal{C}^1_\delta(Q)} \tag{37}$$

In particular, if $p_\infty \in Q$, we have that for every sufficiently large $j \in \mathbb{N}$

$$A(Q) \;\leq\; {}^{g_1}\|X_j\|_{\mathcal{C}^1(Q)} \;\leq B(Q)\,. \tag{38}$$

Our task is now to show that the vector fields X_j converge to a nontrivial vector field X with respect to the $\mathcal{C}^1(Q)$ topology. Thanks to the Ascoli-Arzelá Theorem and to the estimate (38), our goal is achieved if we can produce a j-uniform bound for ${}^{g_1}\|X_j\|_{\mathcal{C}^{1,\alpha}(Q)}$. In order to do that we invoke the following result from [6], which is a Schauder interior estimate for second order linear elliptic systems in divergence form.

Proposition 3. *Let $X \in W^{1,2}_{loc}(\Omega)$ be a solution to*

$$\nabla_\sigma \left(A^{\sigma\tau}_{ij} \, \nabla_\tau \, X^j \right) \;=\; -\nabla_\sigma \, F^\sigma_i$$

with $A^{\sigma\tau}_{ij} \in \mathcal{C}^{0,\alpha}(\Omega)$, for $1 \leq i, j \leq m$ and $1 \leq \sigma, \tau \leq 2$, satisfying the Legendre-Hadamard condition

$$A^{\sigma\tau}_{ij} \, \xi_\sigma \xi_\tau \eta^i \eta^j \;\;\geq\;\; \lambda \, |\xi|^2 |\eta|^2 \qquad , \forall \, \xi \in \mathbb{R}^2 \,, \; \forall \, \eta \in \mathbb{R}^n.$$

If $F^i_\sigma \in \mathcal{C}^{0,\alpha}(\Omega)$, then we have $\nabla X \in \mathcal{C}^{0,\alpha}(\Omega)$, $\alpha \in (0,1)$. Moreover, for every compact set $K \subset \Omega$

$$\|\nabla X\|_{\mathcal{C}^{0,\alpha}(K)} \;\;\leq\;\; C \cdot \left\{ \, \|\nabla X\|_{L^2(\Omega)} \, + \, \|F\|_{\mathcal{C}^{0,\alpha}(\Omega)} \, \right\} , \tag{39}$$

with C depending on K, the ellipticity constant λ and the Hölder norm of the coefficients $A^{\sigma\tau}_{ij}$.

Since Q is compact we can cover it with finitely many small balls, and then we can cover each of these balls with another ball with a little larger radius. Using Proposition 3, we can easily get the desired j-uniform $\mathcal{C}^{1,\alpha}$ bound on each small ball. In fact, since the metrics g_{ε_j}'s converge to g_1, the coefficients of the operators $L_{g_{\varepsilon_j}}$ too will converge to the coefficients of L_{g_1}. In particular the ellipticity constant can be chosen to be the same and consequently also the constant in the estimate (39) can be chosen to be the same. Up to taking the maximum of the finitely many constants obtained as a new constant, we have that there exists a real number $C > 0$ depending only on the compact set Q and on the ellipticity constant of the operator L_{g_1} on Q, such that

$${}^{g_1}\|X_j\|_{\mathcal{C}^{1,\alpha}(Q)} \;\;\leq\;\; C \cdot \left\{ \, {}^{g_1}\|X_j\|_{\mathcal{C}^0(Q')} \, + \, {}^{g_1}\|W_j\|_{\mathcal{C}^{0,\alpha}(Q')} \, \right\} \tag{40}$$

where Q' is another compact set of $M_1 \setminus K$ including Q and the larger small balls. Hence, taking into account the hypothesis of the argument by contradiction and the inequality (38), there must exists a constant $C' > 0$ depending on C and Q such that

$${}^{g_1}\|X_j\|_{\mathcal{C}^{1,\alpha}(Q)} \;\;\leq\;\; C' \,. \tag{41}$$

Hence we can conclude that there exists a vector field X defined on $M_1 \setminus K$ such that, up to a subsequence, $X_j \to X$ with respect to the $\mathcal{C}^1$-topology on the compact sets of $M_1 \setminus K$. Therefore X must satisfy the homogeneous problem (34). Moreover, the inequality (38) guarantees that X is non identically zero.

We are now ready to perform the integration by parts which concludes the discussion of this case. Let us denote by V_R a tubular neighborhood of K in M_1 of size $R > 0$ and by U_R the set $M_1 \setminus V_R$. Then the integration by parts yields

$$\langle L_{g_1} X \, , \, X \rangle_{L^2(U_R)} \quad = \quad \int_{U_R} |D_{g_1} X|^2 \, dvol_{g_1} + \int_{\partial U_R} (D_{g_1} X) \, \cdot \, (X, \nu_{g_1}) \, dvol_{\iota_R^* g_1} \, ,$$

where $\iota_R : \partial U_R \hookrightarrow M_1$ and ν_{g_1} is the inward g_1-normal vector field. The prescriptions on the decay of $|X|_{g_1}$ and $|\nabla X|_{g_1}$ imply the following estimate for the integrand of the boundary term

$$\left| (D_{g_1} X) \, \cdot \, (X, \nu_{g_1}) \, \sqrt{\iota_R^* g_1} \right| \quad \leq \quad C \, \cdot \, R^{2\delta + n - 2} \, .$$

Since $\delta > -(n-2)/2$ the boundary value term goes to zero as $R \to 0$, and consequently $\|D_{g_1} X\|_{L^2} = 0$. Hence X is a nontrivial conformal Killing vector field on M_1, which is excluded by the non-degeneracy condition.

3.2.2 Case 2: the equation $L_{\mathbb{R}^k \times \mathbb{S}^n} X \, = \, 0$ and the blow-up method

In this case we suppose that up to a subsequence the p_j's tend to a point $p_\infty \in K$ lying on the side of M_1 with the higher velocity allowed (i.e. $d_{g_1}(p_j, p_\infty) = \mathcal{O}(\varepsilon_j)$). Since as $\varepsilon \to 0$ the geometry of our construction becomes singular we are induced to perform a blow-up around the point p_∞ in order to investigate the analytic behavior of the objects we are considering. Let us fix then a local system of Fermi coordinates centered at p_∞, so that $z(p_\infty) = 0 = x(p_\infty)$ and let us focus on the region $B^k(1) \times A^n(\varepsilon, 1)$, where $B^k(1)$ and $A^n(\varepsilon, 1)$ are respectively the k-dimensional unit ball and the n-dimensional annulus centered at p_∞. We introduce now a family of diffeomorphisms $(\phi_\varepsilon)_{\varepsilon \in (0,1)}$, defined as

$$\begin{array}{rcl} \phi_\varepsilon \, : \, B^k(1/\varepsilon) \times A^n(1, 1/\varepsilon) & \longrightarrow & B^k(1) \times A^n(\varepsilon, 1) \\ (\, z \, , \, x \,) & \longmapsto & (\, \varepsilon z \, , \, \varepsilon x \,) \, . \end{array} \tag{42}$$

Using the ϕ_ε's, we define on the new domains the blow-up metrics $\mathring{g}_\varepsilon$ by setting

$$\mathring{g}_\varepsilon \quad := \quad \frac{1}{\varepsilon^2} \, \cdot \, [\phi_\varepsilon^* \, \cdot \, g_\varepsilon] \, . \tag{43}$$

In the limit for $\varepsilon \to 0$ the domain of definition of the ϕ_ε's becomes $\mathbb{R}^k \times (\mathbb{R}^n \setminus B^n(1))$ and the blow-up metrics $\mathring{g}_\varepsilon$ tend to $g_{\mathbb{R}^k} + g_{\mathbb{S}^n}$ on the compact sets of $\mathbb{R}^k \times (\mathbb{R}^n \setminus B^n(1))$

in the $\mathcal{C}^2$ topology. In fact the local expression for $\overset{\circ}{g}_\varepsilon$ reads

$$\begin{aligned}
\overset{\circ}{g}_\varepsilon\ (z,x) &= \frac{1}{\varepsilon^2}\cdot\left\{\, g^K_{ij}(\varepsilon z)\,+\,\mathcal{O}\left(\varepsilon|x|\right)\right\}\,\cdot\, d(\varepsilon z^i)\otimes d(\varepsilon z^j)\\
&+\ \frac{1}{\varepsilon^2}\cdot\mathcal{O}\left(\varepsilon|x|\right)\,\cdot\, d(\varepsilon z^i)\otimes d(\varepsilon x^\alpha)\\
&+\ \frac{1}{\varepsilon^2}\cdot\left(|x|^{\frac{n-2}{2}}\,+\,|x|^{-\frac{n-2}{2}}\right)^{\frac{4}{n-2}}\cdot\left\{\,|x|^{-2}\delta_{\alpha\beta}\,+\,\mathcal{O}\left(\varepsilon^2\right)\right\}\,\cdot\, d(\varepsilon x^\alpha)\otimes d(\varepsilon x^\beta)\\
&= \left\{\,\delta_{ij}\,+\,\mathcal{O}\left(\varepsilon|x|\right)\,+\,\mathcal{O}\left(\varepsilon^2|z|^2\right)\right\}\,\cdot\, dz^i\otimes dz^j\\
&+\ \mathcal{O}\left(\varepsilon|x|\right)\,\cdot\, dz^i\otimes dx^\alpha\\
&+\ \left\{\, g^{\mathbb{S}^n}_{\alpha\beta}(x)\,+\,\mathcal{O}\left(\varepsilon^2\right)\right\}\,\cdot\, dx^\alpha\otimes dx^\beta\,.
\end{aligned}$$

(Notice that the blow-up construction described here obviously applies to both the sides of the polyneck in order to get the whole Schwarzschild space as limit manifold and to the operator $L_{\mathbb{R}^k\times\mathbb{S}^n}$ as limit operator. The description of the blow-up procedure in terms of (z,x) coordinates makes clearer the analogies and the differences between this blow-up and the one we use in treating the third case. Nevertheless it is possible to give the description of the same procedure in terms of (z,t,θ) coordinates. In this case it is sufficient to remember that for the metric g_ε we have at hand the expression

$$\begin{aligned}
g_\varepsilon(z,t,\theta) &:= \{\delta_{ij}\,+\,\mathcal{O}\left(\varepsilon\cosh t\right)\,\mathcal{O}\left(|z|^2\right)\}\cdot dz^i\otimes dz^j\\
&+\ \left[\varepsilon^{\frac{n-2}{2}}\,\cosh\left(\frac{n-2}{2}t\right)\right]^{\frac{4}{n-2}}\left\{dt^2\,+\,d\theta^2\,+\,\mathcal{O}\left(\varepsilon^2\cosh^2 t\right)dt\ltimes d\theta\right\}\\
&+\ \mathcal{O}\left(\varepsilon^2\cosh^2 t\right)\cdot dz^i\otimes dt\\
&+\ \mathcal{O}\left(\varepsilon^2\cosh^2 t\right)\cdot dz^i\otimes d\theta^\lambda\,,
\end{aligned}$$

where $d\theta^2$ is the round metric on S^{n-1}. Then using the blow-up diffeomorphisms

$$\begin{aligned}
\psi_\varepsilon\,:\,B^k(1/\varepsilon)\times(\log\varepsilon,-\log\varepsilon)\times S^{n-1} &\longrightarrow B^k(1)\times(\log\varepsilon,-\log\varepsilon)\times S^{n-1}\\
(\,z\,,\,t\,,\,\theta\,) &\longmapsto (\,\varepsilon z\,,\,t\,,\,\theta\,)
\end{aligned}$$

and defining the blow-up metrics as

$$\overset{\circ}{g}_\varepsilon\ :=\ \frac{1}{\varepsilon^2}\cdot[\psi^*_\varepsilon\cdot g_\varepsilon]\,,$$

one can easily obtain the same results). Hence the coefficients of the operators $L_{\overset{\circ}{g}_\varepsilon}$ tend to the coefficients of the operator $L_{\mathbb{R}^k\times\mathbb{S}^n}$. Moreover, if we consider for every $j\in\mathbb{N}$ the vector fields

$$\overset{\circ}{X}_j\ :=\ \varepsilon_j^{-\delta+1}\,\phi^*_{\varepsilon_j}\,\cdot\,X_j \tag{44}$$

$$\overset{\circ}{W}_j\ :=\ \varepsilon_j^{-\delta+1}\,\phi^*_{\varepsilon_j}\,\cdot\,W_j\,, \tag{45}$$

we have that the triples $(\varepsilon_j,\overset{\circ}{X}_j,\overset{\circ}{W}_j)$ verify the properties **1. - 4.** with respect to the blow-up metrics $\overset{\circ}{g}_{\varepsilon_j}$ and the distance function(s)

$$\overset{\circ}{\rho}_{\varepsilon_j}\ (t)\ :=\ \cosh t\,. \tag{46}$$

Using the same argument as in the previous section it is easy to show that the vector fields $\overset{\circ}{X}_j$ converge on the compact sets to a vector field X with respect to the $\mathcal{C}^1$-topology of the metric $g_{\mathbb{R}^K} + g_{\mathbb{S}^n}$ and the sequence of problems

$$\overset{\circ}{L}_j \, \overset{\circ}{X}_j \;=\; \overset{\circ}{W}_j \tag{47}$$

converges to the homogeneous problem (35).

Moreover, since $d_{g_1}(p_j, K) = \mathcal{O}(\varepsilon_j)$, we have that the sequence of points $q_j := \phi_{\varepsilon_j}^{-1}(p_j)$ lies in a compact region of $\mathbb{R}^k \times (\mathbb{R}^n \setminus B^n(1))$ and converges, up to a subsequence, to a point q_∞, so that the solution of the limit problem X must be non identically zero.

Our task is now to show that the homogeneous problem for the operator $L_{\mathbb{R}^k \times \mathbb{S}^n}$ with prescribed decay does not admit nontrivial solutions. In order to do that we write down the explicit expression of our equation, using the fact that we are dealing with the product metric $g_{\mathbb{R}^k} + g_{\mathbb{S}^n}$. If we set $X = U + V$ with $U \in \mathbb{R}^k$ and $V \in \mathbb{S}^n$, the vector Laplacian decomposes as follows

$$\begin{aligned}
\left[L_{\mathbb{R}^k \times \mathbb{S}^n} X\right]^{\mathbb{R}^k} \;&=\; L_{\mathbb{R}^k} U - \left(\frac{m-k}{m \cdot k}\right) \cdot \mathrm{grad}_{\mathbb{R}^k} \circ \mathrm{div}_{\mathbb{R}^k} U \qquad (48)\\
&\quad - \frac{1}{2} \cdot \Delta_{\mathbb{S}^n} U - \left(\frac{m-2}{m \cdot 2}\right) \cdot \mathrm{grad}_{\mathbb{R}^k} \circ \mathrm{div}_{\mathbb{S}^n} V \\
\left[L_{\mathbb{R}^k \times \mathbb{S}^n} X\right]^{\mathbb{S}^n} \;&=\; L_{\mathbb{S}^n} V - \left(\frac{m-n}{m \cdot n}\right) \cdot \mathrm{grad}_{\mathbb{S}^n} \circ \mathrm{div}_{\mathbb{S}^n} V \qquad (49)\\
&\quad - \frac{1}{2} \cdot \Delta_{\mathbb{R}^k} V - \left(\frac{m-2}{m \cdot 2}\right) \cdot \mathrm{grad}_{\mathbb{S}^n} \circ \mathrm{div}_{\mathbb{R}^k} U \,,
\end{aligned}$$

where in general we indicate by $\Delta_g W$ the (negative definite) Laplace-Beltrami operator of a Riemannian metric g applied to the components of the vector field W.

The idea we mean to use to carry out the analysis of this operator is to perform a Fourier transform along the $\mathbb{R}^k$ components. This allows us to decouple the system above into two simpler vector equations. In order to fix a suitable functional setting, we consider for $\beta \in \mathbb{R}$ the weighted Lebesgue spaces

$$L^2_\beta\left(\mathbb{S}^n, \mathbb{R}^k \otimes T\,\mathbb{S}^n\right) \;:=\; (\cosh t)^\beta \cdot L^2\left(\mathbb{S}^n, \mathbb{R}^k \otimes T\,\mathbb{S}^n\right)$$

(since in the following, we want to use a partial Fourier transform, it turns out to be useful to consider the complexification $\mathbb{C}^k \otimes T^{\mathbb{C}}\mathbb{S}^n$ of the tangent bundle $\mathbb{R}^k \otimes T\,\mathbb{S}^n$). We also define the Schwartz space $\mathcal{S}\big(\mathbb{R}^k, L^2_\beta\left(\mathbb{S}^n, \mathbb{C}^k \otimes T^{\mathbb{C}}\mathbb{S}^n\right)\big)$, briefly $\mathcal{S}(\beta)$, as the $L^2_\beta\left(\mathbb{S}^n, \mathbb{C}^k \otimes T^{\mathbb{C}}\mathbb{S}^n\right)$-valued smooth functions Y such that for every couple of multi-indices $j, l \in \mathbb{N}^k$

$$\sup_{z \in \mathbb{R}^k} \left| z^j \, \|\nabla^l Y\|_{L^2_\beta}(z) \right| \;<\; +\infty\,.$$

For $Y \in \mathcal{S}(\beta)$ it is natural to define the partial Fourier transform with respect to the z variable as

$$\widehat{Y}(\zeta, x) \;:=\; \widehat{Y}^j(\zeta, x) \cdot \partial_{\zeta^j} + \widehat{Y}^\alpha(\zeta, x) \cdot \partial_{x^\alpha}\,, \tag{50}$$

where

$$\widehat{Y}^j(\zeta,x) \quad := \quad (2\pi)^{-k/2}\int_{\mathbb{R}^k} Y^j(z,x)\cdot \mathrm{e}^{-i\,\langle\zeta,z\rangle}\;dz \tag{51}$$

$$\widehat{Y}^\alpha(\zeta,x) \quad := \quad (2\pi)^{-k/2}\int_{\mathbb{R}^k} Y^\alpha(z,x)\cdot \mathrm{e}^{-i\,\langle\zeta,z\rangle}\;dz\,. \tag{52}$$

If we denote by $\mathcal{S}'\big(\mathbb{R}^k, L^2_\beta\left(\mathbb{S}^n, \mathbb{C}^k\otimes T^{\mathbb{C}}\mathbb{S}^n\right)\big)$, briefly $\mathcal{S}'(\beta)$, the $L^2_\beta\left(\mathbb{S}^n, \mathbb{C}^k\otimes T^{\mathbb{C}}\mathbb{S}^n\right)$-valued temperate distributions, which is nothing but the space of the continuous linear functionals on $\mathcal{S}(\beta)$, then the partial Fourier transform defined above on the Schwartz space $\mathcal{S}(\beta)$ extends by transposition to the temperate distributions $\mathcal{S}'(\beta)$.

In our case we observe that the decay prescription $|X|_{\mathbb{R}^k\times\mathbb{S}^n} \,\leq\, C\cdot(\cosh t)^\delta$ implies that $X \,\in\, L^\infty\big(\mathbb{R}^k_z, L^2_{-\gamma+n/2}\left(\mathbb{S}^n, \mathbb{R}^k\otimes T\,\mathbb{S}^n\right)\big)$ for every $\gamma < -\delta$. Hence in general $\widehat{X} \,\in\, \mathcal{S}'\big(\mathbb{R}^k_\zeta, L^2_{-\gamma+n/2}\left(\mathbb{S}^n, \mathbb{C}^k\otimes T^{\mathbb{C}}\mathbb{S}^n\right)\big)$, briefly $\mathcal{S}'(-\gamma+n/2)$, so that our first aim is to give a precise definition of $\widehat{L}_{\mathbb{R}^k\times\mathbb{S}^n}$ as an operator acting on the space of the $L^2_{-\gamma+n/2}\left(\mathbb{S}^n\,,\,\mathbb{C}^k\otimes T^{\mathbb{C}}\mathbb{S}^n\right)$-valued temperate distributions. First of all we describe the action of $\widehat{L}_{\mathbb{R}^k\times\mathbb{S}^n}$ as a continuous linear operator between the Schwartz spaces $\mathcal{S}(\beta)$ and $\mathcal{S}(-\beta)$, i.e.,

$$\widehat{L}_{\mathbb{R}^k\times\mathbb{S}^n}\,:\,\mathcal{S}\big(\mathbb{R}^k_\zeta, L^2_\beta\left(\mathbb{S}^n, \mathbb{C}^k\otimes T^{\mathbb{C}}\mathbb{S}^n\right)\big)\;\longrightarrow\;\mathcal{S}\big(\mathbb{R}^k_\zeta, L^2_{-\beta}\left(\mathbb{S}^n, \mathbb{C}^k\otimes T^{\mathbb{C}}\mathbb{S}^n\right)\big)\,. \tag{53}$$

Given a vector field $W\in\mathcal{S}(\beta)$ we define $\widehat{L}_{\mathbb{R}^k\times\mathbb{S}^n}\,W$ as the unique vector field $Y\in\mathcal{S}(-\beta)$ such that

$$\big(W(\zeta)\,,\,\widehat{L}_{\mathbb{R}^k\times\mathbb{S}^n}\,Z(\zeta)\big)_{L^2} \quad = \quad \big(Y(\zeta)\,,\,Z(\zeta)\big)_{L^2}\,, \tag{54}$$

for every $\zeta\in\mathbb{R}^k$ and every test vector field $Z\in\mathcal{S}\big(\,\mathbb{R}^k_\zeta\,,\,\mathcal{C}^\infty_0\left(\,\mathbb{S}^n\,,\,\mathbb{C}^k\otimes T^{\mathbb{C}}\mathbb{S}^n\,\right)\,\big)$. The transposed operator $\widehat{L}'_{\mathbb{R}^k\times\mathbb{S}^n}\,:\,\mathcal{S}'(-\beta)\,\rightarrow\,\mathcal{S}'(\beta)$ is defined by

$$\langle\,\widehat{L}'_{\mathbb{R}^k\times\mathbb{S}^n}\,\Phi\,,\,Y\,\rangle_{\mathcal{S}'(\beta)\times\mathcal{S}(-\beta)} \quad := \quad \langle\,\Phi\,,\,\widehat{L}_{\mathbb{R}^k\times\mathbb{S}^n}\,Y\,\rangle_{\mathcal{S}'(-\beta)\times\mathcal{S}'(\beta)}\,, \tag{55}$$

for every $\Phi\,\in\,\mathcal{S}'\big(\mathbb{R}^k_\zeta, L^2_{-\beta}\left(\mathbb{S}^n, \mathbb{C}^k\otimes T^{\mathbb{C}}\mathbb{S}^n\right)\big)$ and $Y\,\in\,\mathcal{S}\big(\mathbb{R}^k_\zeta, L^2_\beta\left(\mathbb{S}^n, \mathbb{C}^k\otimes T^{\mathbb{C}}\mathbb{S}^n\right)\big)$. Since $\widehat{L}'_{\mathbb{R}^k\times\mathbb{S}^n}\big|_{\mathcal{S}(-\beta)}\equiv\widehat{L}_{\mathbb{R}^k\times\mathbb{S}^n}$, it defines a continuous linear map from $\mathcal{S}(-\beta)$ to $\mathcal{S}(\beta)$, hence it can be used to extend $\widehat{L}_{\mathbb{R}^k\times\mathbb{S}^n}$ to the tempered distributions by transposition. In fact the formula

$$\langle\,\widehat{L}_{\mathbb{R}^k\times\mathbb{S}^n}\,\Phi\,,\,W\,\rangle_{\mathcal{S}'(-\beta)\times\mathcal{S}(\beta)} \quad := \quad \langle\,\Phi\,,\,\widehat{L}'_{\mathbb{R}^k\times\mathbb{S}^n}\,W\,\rangle_{\mathcal{S}'(\beta)\times\mathcal{S}'(-\beta)} \tag{56}$$

for every $\Phi\,\in\,\mathcal{S}'\big(\mathbb{R}^k_\zeta, L^2_\beta\left(\mathbb{S}^n, \mathbb{C}^k\otimes T^{\mathbb{C}}\mathbb{S}^n\right)\big)$ and $W\,\in\,\mathcal{S}\big(\mathbb{R}^k_\zeta, L^2_{-\beta}\left(\mathbb{S}^n, \mathbb{C}^k\otimes T^{\mathbb{C}}\mathbb{S}^n\right)\big)$, yields the definition of $\widehat{L}_{\mathbb{R}^k\times\mathbb{S}^n}\,:\,\mathcal{S}'(\beta)\,\rightarrow\,\mathcal{S}'(-\beta)$ we are looking for. In particular we are interested in the case $\beta=-\gamma+n/2$ with $\gamma<-\delta$.

Since $L_{\mathbb{R}^k\times\mathbb{S}^n}\,X \,=\, 0$ we have that

$$\begin{aligned} 0 \quad &= \quad \langle\,\widehat{L}_{\mathbb{R}^k\times\mathbb{S}^n}\,\widehat{X}\,,\,W\,\rangle_{\mathcal{S}'(\gamma-\frac{n}{2})\times\mathcal{S}(-\gamma+\frac{n}{2})} \\ &= \quad \langle\,\widehat{X}\,,\,\widehat{L}_{\mathbb{R}^k\times\mathbb{S}^n}\,W\,\rangle_{\mathcal{S}'(-\gamma+\frac{n}{2})\times\mathcal{S}(\gamma-\frac{n}{2})}\,, \end{aligned}$$

for every $W \in \mathcal{S}(-\gamma + n/2)$. If we prove that $\widehat{L}_{\mathbb{R}^k \times \mathbb{S}^n} : \mathcal{S}(-\gamma + n/2) \to \mathcal{S}(\gamma - n/2)$ is surjective, then for every $Y \in \mathcal{S}(\gamma - n/2)$

$$\langle \widehat{X} \, , Y \rangle_{\mathcal{S}'(-\gamma+\frac{n}{2}) \times \mathcal{S}(\gamma-\frac{n}{2})} \;=\; 0\,, \tag{57}$$

which implies that $\widehat{X} = 0$, and thence $X = 0$, which is a contradiction. Therefore the problem reduces to proving that for every $Y \in \mathcal{S}(\gamma - n/2)$ there exists a $W \in \mathcal{S}(-\gamma + n/2)$ such that

$$\widehat{L}_{\mathbb{R}^k \times \mathbb{S}^n} W \;=\; Y\,. \tag{58}$$

The advantage of having performed a partial Fourier transform is that now we can solve the equation above, for every fixed $\zeta \in \mathbb{R}^k$. Writing $W(\zeta) = U(\zeta) + V(\zeta)$ with $V(\zeta) \in L^2_{-\gamma+n/2}(\mathbb{S}^n, T^{\mathbb{C}}\mathbb{S}^n)$ and $U(\zeta) \in L^2_{-\gamma+n/2}(\mathbb{S}^n, \mathbb{C}^k)$, and $Y(\zeta) = A(\zeta) + B(\zeta)$ with $B(\zeta) \in L^2_{\gamma-n/2}(\mathbb{S}^n, T^{\mathbb{C}}\mathbb{S}^n)$ and $A(\zeta) \in L^2_{\gamma-n/2}(\mathbb{S}^n, \mathbb{C}^k)$, we are induced to solve for U and V the system

$$\begin{aligned} A \;&=\; -\frac{1}{2}\,\Delta_{\mathbb{S}^n} U \;+\; \left(\frac{m-2}{2\,m}\right) \cdot \langle \zeta, U \rangle \cdot \zeta \\ &+\; \frac{1}{2} \cdot |\zeta|^2\, U \;-\; i\,\left(\frac{m-2}{2\,m}\right) \cdot \mathrm{div}_{\mathbb{S}^n} V \cdot \zeta \end{aligned} \tag{59}$$

$$\begin{aligned} B \;&=\; L_{\mathbb{S}^n} V \;-\; \left(\frac{m-n}{m \cdot n}\right) \cdot \mathrm{grad}_{\mathbb{S}^n} \circ \mathrm{div}_{\mathbb{S}^n} V \\ &+\; \frac{1}{2}\,|\zeta|^2 \cdot V \;-\; i\,\left(\frac{m-2}{2\,m}\right) \cdot \mathrm{grad}_{\mathbb{S}^n} \langle \zeta, U \rangle \end{aligned} \tag{60}$$

If we look for solutions which in addition satisfy the condition

$$\langle \zeta, U \rangle \;=\; 0 \tag{61}$$

$$\mathrm{div}_{\mathbb{S}^n} V \;=\; 0\,, \tag{62}$$

then the system decouples and we have

$$-\Delta_{\mathbb{S}^n} U \,+\, |\zeta|^2\, U \;=\; A \tag{63}$$

$$L_{\mathbb{S}^n} V \,+\, \frac{1}{2} \cdot |\zeta|^2\, V \;=\; B\,. \tag{64}$$

Since both the operators

$$P_\zeta \;:=\; -\Delta_{\mathbb{S}^n} \,+\, |\zeta|^2 \;:\; L^2_{-\gamma+n/2}(\mathbb{S}^n, \mathbb{C}^k) \;\longrightarrow\; L^2_{\gamma-n/2}(\mathbb{S}^n, \mathbb{C}^k) \tag{65}$$

$$Q_\zeta \;:=\; L_{\mathbb{S}^n} \,+\, \frac{1}{2}\,|\zeta|^2 \;:\; L^2_{-\gamma+n/2}(\mathbb{S}^n, T^{\mathbb{C}}\mathbb{S}^n) \;\longrightarrow\; L^2_{\gamma-n/2}(\mathbb{S}^n, T^{\mathbb{C}}\mathbb{S}^n) \tag{66}$$

are self-adjoint (with respect to the L^2-product), they are surjective if and only if they are injective. Therefore what we have to prove is that

$$P_\zeta\, U \,=\, 0 \quad,\, U \in L^2_{-\gamma+n/2}(\mathbb{S}^n, \mathbb{C}^k) \quad\Longrightarrow\quad U \,=\, 0 \tag{67}$$

$$Q_\zeta\, V \,=\, 0 \quad,\, V \in L^2_{-\gamma+n/2}(\mathbb{S}^n, T^{\mathbb{C}}\mathbb{S}^n) \quad\Longrightarrow\quad V \,=\, 0 \tag{68}$$

The first equation splits into k identical scalar problems

$$\begin{cases} -\Delta_{\mathbb{S}^n} u + |\zeta|^2 u = 0 & \text{on } \mathbb{R}_t \times S^{n-1}_\theta \\ \\ u \in L^2_{-\gamma+\frac{n}{2}}(\mathbb{S}^n, \mathbb{C}), \end{cases} \tag{69}$$

where $\Delta_{\mathbb{S}^n}$ is the (negative definite) scalar Laplacian of the Schwarzschild metric $g_{\mathbb{S}^n}$. If we choose γ sufficiently close to $-\delta$ (remember that here $\gamma < -\delta$), the condition $u \in L^2_{-\gamma+n/2}$ implies decay at infinity, then the injectivity of P_ζ for every $\zeta \in \mathbb{R}^k$ easily follows from the classical maximum principle. Let us focus now on the problem

$$\begin{cases} L_{\mathbb{S}^n} V + \frac{1}{2}|\zeta|^2 V = 0 & \text{on } \mathbb{R}_t \times S^{n-1}_\theta \\ \\ V \in L^2_{-\gamma+\frac{n}{2}}(\mathbb{S}^n, T^{\mathbb{C}}\mathbb{S}^n). \end{cases} \tag{70}$$

If the integration by parts is allowed (i.e., if the boundary term goes to zero), then we get immediately

$$\|D_{\mathbb{S}^n} V\|^2_{L^2} \;=\; -\frac{|\zeta|^2}{2} \cdot \|V\|^2_{L^2}. \tag{71}$$

Hence if $\zeta \neq 0$ then $V = 0$, whereas if $\zeta = 0$ we deduce $D_{\mathbb{S}^n} V = 0$. Therefore V has to be a conformal Killing vector field. However it is well known that there are no nontrivial conformal Killing vector field which decay at infinity on $\mathbb{S}^n$ (notice that if one chooses γ sufficiently close to $-\delta$, then the condition $V \in L^2_{-\gamma+n/2}$ implies decay at infinity). Thence $V = 0$ and the Q_ζ's are injective for every $\zeta \in \mathbb{R}^k$.

What remains to prove is that the boundary term in the integration by parts actually goes to zero, namely that

$$\lim_{R\to+\infty} \int_{\partial S^{n-1}_R} (D_{\mathbb{S}^n} V) \cdot (V, \nu_{\mathbb{S}^n})\, dvol_{\iota_R^* \mathbb{S}^n} \;=\; 0, \tag{72}$$

where $\iota_R : S^{n-1}_R \hookrightarrow \mathbb{S}^n$ is the $(n-1)$-dimensional sphere of radius R. Notice that the condition $\delta > (2-n)/2$ which allowed us to do the integration by parts in the previous case is precisely what prevents us from reaching the desired conclusion immediately. In fact, since now $R \to +\infty$, we would like to have $\delta < (2-n)/2$. Nevertheless, since the Schwarzschild space is asymptotically Euclidian, the indicial roots of both $L_{\mathbb{S}^n}$ and $L_{\mathbb{R}^n}$ are the same. In particular, a direct computation shows that the set of the admissible rate of growth (or rate of decay) at infinity for solutions to the homogeneous equation is a translated of the indicial roots set, and is given by $I(n) := \{ j \in \mathbb{Z} : j \leq (2-n) \} \cup \{ j \in \mathbb{Z} : j \geq 0 \}$. Hence if V is a solution to $L_{\mathbb{S}^n} V = 0$ (or to $Q_\zeta V = 0$, since the principal part is the same) which lies in $L^2_{-\gamma+n/2}$ with γ close enough to $-\delta$, then the first term allowed in its polyhomogeneous expansion decays as $(\cosh t)^{2-n}$, which allows us to do the desired integration by parts.

3.2.3 Case 3: the equation $L_{\mathbb{R}^k \times \mathbb{R}^n} X = 0$ and the refined blow-up

The analysis of the third case is very similar to the analysis of the second one, with the only substantial difference of the blow-up construction. Roughly speaking the slower velocity of the p_j's in tending to p_∞ forces us to refine the blow-up procedure used in the previous section. In particular, if we want to control the behavior of the p_j's (which we need to do in order to carry out the argument by contradiction, since it guarantees the non-triviality of the solution), we need to choose their rate of approach to p_∞ as rate of the blow-up of the other objects involved. If we set $\omega_j := d_{g_1}(p_j, p_\infty)$, for $j \in \mathbb{N}$, we have that in this case $\varepsilon_j/\omega_j \to 0$ as $j \to 0$. Now, with the notation introduced in the previous section, it is quite natural to define a new family of diffeomorphisms $(\phi_{\omega_j})_{j\in\mathbb{N}}$ via

$$\begin{aligned} \phi_{\omega_j} \,:\, B^k(1/\omega_j) \times A^n(\varepsilon_j/\omega_j \,,\, 1/\omega_j) \quad &\longrightarrow \quad B^k(1) \times A^n(\varepsilon_j \,,\, 1) \\ (\,z\,,\,x\,) \quad &\longmapsto \quad (\,\omega_j\, z\,,\,\omega_j\, x\,)\,, \end{aligned} \tag{73}$$

and consequently define a family of new metrics $\overset{\circ}{g}_{\omega_j}$ as follows:

$$\overset{\circ}{g}_{\omega_j} \;:=\; \frac{1}{\omega_j^2} \cdot \Big[\, \phi^*_{\omega_j} \,\cdot\, g_{\varepsilon_j} \,\Big]\,. \tag{74}$$

In the limit $j \to +\infty$, the domain of definition of the ϕ_{ω_j}'s becomes $\mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\})$ and the blow-up metrics $\overset{\circ}{g}_{\omega_j}$ tend to $g_{\mathbb{R}^k} \times g_{\mathbb{R}^n}$ on the compact sets of $\mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\})$ in the $\mathcal{C}^2$ topology. In fact the local expression for the $\overset{\circ}{g}_{\omega_j}$ reads

$$\begin{aligned} \overset{\circ}{g}_{\omega_j}\,(z,x) \;&=\; \Big\{\, \delta_{il} \,+\, \mathcal{O}\left(\omega_j|x|\right) \,+\, \mathcal{O}\left(\omega_j^2|z|^2\right) \Big\} \,\cdot\, dz^i \otimes dz^l \\ &+\; \mathcal{O}\left(\omega_j|x|\right) \,\cdot\, dz^i \otimes dx^\alpha \\ &+\; \left[\, |x|^{\frac{n-2}{2}} \,+\, \left(\frac{\varepsilon_j}{\omega_j}\right)^{n-2} |x|^{-\,\frac{n-2}{2}} \,\right]^{\frac{4}{n-2}} \,\cdot\, \Big\{\, |x|^{-2}\delta_{\alpha\beta} \,+\, \mathcal{O}\left(\omega_j^2\right) \Big\} \,\cdot\, dx^\alpha \otimes dx^\beta\,. \end{aligned}$$

Proceeding as in the previous case we are led to the problem (36). In particular, we notice that the p_j's lie in a compact region of $\mathbb{R}^k \times S^{n-1}$. Hence, up to a subsequence, they converge to a point p_∞. The rest of the analysis, *mutatis mutandis*, is still the same as for the second case excepted the discussion of the problem

$$\left\{ \begin{array}{ll} L_{\mathbb{R}^n}\, V \,+\, \frac{1}{2}\,|\zeta|^2\, V = 0 & \text{on } \mathbb{R}^+_r \times S^{n-1}_\theta \\ & \\ V \in L^2_{-\gamma+\frac{n}{2}}(\mathbb{R}^n \setminus \{0\}, \mathbb{C}^n) & \end{array} \right. \tag{75}$$

which is the analogue of the problem (70) (here $r = |x|$). If we choose γ close enough to $-\delta$ (i.e., $(2-n)/2 < \delta < -\gamma < 0$), the indicial roots analysis shows that a nontrivial solution to the problem above has to decay faster than r^{2-n} as $r \to +\infty$ and it cannot grow faster than r^0 as $r \to 0$. Hence on the one hand the terms allowed in the polyhomogeneous expansion of V include power of r which are less or equal to $2-n$, whereas on the other hand the power of r greater or equal than 0, which is a contradiction. Therefore V has to be zero.

This completes the proof of Proposition 2. As a consequence we immediately get an ε-uniform bound for the correction term σ_ε, and then for the TT-tensor $\mu_\varepsilon = \mu + \sigma_\varepsilon$, which solves the equations

$$\mathrm{tr}_{g_\varepsilon}\, \mu_\varepsilon \;=\; 0\,, \tag{76}$$

$$\mathrm{div}_{g_\varepsilon}\, \mu_\varepsilon \;=\; 0\,. \tag{77}$$

In the next section, we put μ_ε in the Lichnerowicz equation (for the metric g_ε and the constant mean curvature τ), and we use the ε-uniform bound of Proposition 2 to carry out a perturbation argument. By this method we will produce a solution u_ε to the Lichnerowicz such that the more ε is close to zero, the more u_ε is close to one. As explained in the Introduction, this means that the new solution of the constraint approaches the starting ones, as ε tends to zero.

4 The energy constraint

The aim of this section is to produce a solution to the ε-parameterized equation

$$\Delta_{g_\varepsilon} u + c_m R_{g_\varepsilon} u - c_m \, |\, \mu_\varepsilon \,|^2_{g_\varepsilon} u^{-\frac{3m-2}{m-2}} + c_m \frac{m-1}{m} \tau^2 u^{\frac{m+2}{m-2}} \quad = \quad 0 \,, \tag{78}$$

where μ_ε is the TT-tensor obtained in the Section 3. As claimed above, we also provide the solution with suitable estimates so that we have a control of the new Cauchy data set in terms of the old ones, as ε tends to zero.

Our goal is achieved by means of a perturbation argument analogous to the one developed in [17]. Since the equation we are interested in is nonlinear, the first step consists in linearizing the Lichnerowicz operator around the constant one, this is reasonable since we want the solution to be as close as possible to the starting ones, as ε approaches to zero. What we obtain is the linear operator

$$\mathcal{L}_{g_\varepsilon} \quad = \quad \Delta_{g_\varepsilon} - \chi_1 \cdot \left(|\mu_1|^2_{g_1} + \tau^2/m \right) - \chi_2 \cdot \left(|\mu_2|^2_{g_2} + \tau^2/m \right) \,, \tag{79}$$

where χ_1 and χ_2 are the smooth cut-off functions defined in Section 2, and the error term

$$\begin{aligned} F_\varepsilon(v) \quad := \quad & c_m \,(R_{g_1} - R_{g_\varepsilon})\, \chi_1 - c_m \,(|\mu_1|^2_{g_1} - |\mu_\varepsilon|^2_{g_\varepsilon})\, \chi_1 \\ & + c_m \,(R_{g_2} - R_{g_\varepsilon})\, \chi_2 - c_m \,(|\mu_2|^2_{g_2} - |\mu_\varepsilon|^2_{g_\varepsilon})\, \chi_2 \\ & + c_m \,(R_{g_1} - R_{g_\varepsilon})\, \chi_1 \, v + b_m \,(|\mu_1|^2_{g_1} - |\mu_\varepsilon|^2_{g_\varepsilon})\, \chi_1 \, v \\ & + c_m \,(R_{g_2} - R_{g_\varepsilon})\, \chi_2 \, v + b_m \,(|\mu_2|^2_{g_2} - |\mu_\varepsilon|^2_{g_\varepsilon})\, \chi_2 \, v \\ & + c_m \, |\mu_\varepsilon|^2_{g_\varepsilon} \, h(v) - c_m \frac{m-1}{m} \tau^2 f(v) \,, \end{aligned} \tag{80}$$

where $c_m = -\,(m-2)/[4(m-1)]$, $b_m = c_m \cdot (3m-2)/(m-2)$ and

$$\begin{aligned} h(v) \quad &= \quad \left[(1+v)^{-\frac{3m-2}{m-2}} - 1 + \left(\frac{3m-2}{m-2} \right) v \right] \\ f(v) \quad &= \quad \left[(1+v)^{\frac{m+2}{m-2}} - 1 - \left(\frac{m+2}{m-2} \right) v \right] . \end{aligned}$$

Hence both h and f are $\mathcal{O}\left(|v|^2\right)$.

The second step amounts to produce ε-uniform *a priori* estimate for solutions of the linear equation

$$\mathcal{L}_{g_\varepsilon} v \quad = \quad w \,. \tag{81}$$

Notice that since g_ε tends to the metric g_i in the $\mathcal{C}^2$-topology on the compact sets of $M_i \setminus K$, for $i = 1, 2$, then the coefficient of $\mathcal{L}_{g_\varepsilon}$ tend to the coefficients of the operators $\mathcal{L}_{g_i}$ defined in (15) on the compact sets of $M_i \setminus K$, $i = 1, 2$. Once this is done we seek a suitable estimate for the error term $F_\varepsilon(v)$. Having the *a priori* estimate and the estimate of the error term we can solve the equation

$$\mathcal{L}_{g_\varepsilon} v \quad = \quad F_\varepsilon(v) \tag{82}$$

by means of a fixed point argument.

As noted above, since the equation we want to solve is very similar to the Yamabe equation and since the linearized Lichnerowicz operators for the starting initial data set are injective (15), it is sufficient to adapt the argument used in [17] to our case. Let us focus, for instance, on the part of the error term $F_\varepsilon(v)$ which is supported on M_1 (the same is true for the other part of the error term). In order to apply successfully the Schauder fixed point theorem as in [17], it suffices that the "zero order" term in $F_\varepsilon(v)$ satisfies the estimate

$$(R_{g_1} - R_{g_\varepsilon}) \; - \; \big(|\mu_1|^2_{g_1} - |\mu_\varepsilon|^2_{g_\varepsilon}\big) \quad \leq \quad C \cdot \varepsilon^{n-2} \, \rho_\varepsilon^{1-n} \, , \tag{83}$$

for some positive constant $C > 0$. Concerning the piece $R_{g_1} - R_{g_\varepsilon}$ this is exactly the estimate of the scalar curvature obtained in [17]. Since $\mu_\varepsilon = \mu + \sigma_\varepsilon$, the other piece is dominated by $|\sigma_\varepsilon|^2_{g_\varepsilon} + |\mu - \mu_1|^2_{g_\varepsilon}$. Since $|\mu - \mu_1|^2_{g_\varepsilon}$ is zero outside the boundary of the polyneck and since it is clearly bounded in the middle, we can concentrate on the squared norm of $\sigma_\varepsilon = D_{g_\varepsilon} X$, where $L_{g_\varepsilon} X = \sharp \, \mathrm{div}_{g_\varepsilon} \mu$.

It follows from proposition (2) that there exists a constant $C_0 > 0$ independent of ε such that

$$|D_{g_\varepsilon} X|_{g_\varepsilon} \quad \leq \quad C_0 \cdot {}^{\varepsilon}\|\mathrm{div}_{g_\varepsilon} \mu\|_{\mathcal{C}^1_{\delta-2}} \cdot \rho_\varepsilon^{\delta-1} \, . \tag{84}$$

Since our aim is to get the bound $|D_{g_\varepsilon} X|^2 \;\leq\; C \cdot \varepsilon^{n-2} \, \rho_\varepsilon^{1-n}$, it is sufficient to prove that

$${}^{\varepsilon}\|\mathrm{div}_{g_\varepsilon} \mu\|_{\mathcal{C}^1_{\delta-2}} \quad \leq \quad C_1 \cdot \varepsilon^{\frac{n-2}{2}} \, \rho_\varepsilon^{\frac{3-n}{2} - \delta} \, , \tag{85}$$

for some positive constant $C_1 > 0$. If $n = 3$ (thence $-1/2 < \delta < 0$) this reduces to proving that

$$\begin{cases} |\mathrm{div}_{g_\varepsilon} \mu|_{g_\varepsilon} \;\leq\; C_2 \cdot \varepsilon^{\frac{n-2}{2} - \delta} \, \rho_\varepsilon^{\delta-2} \\[2ex] |\nabla \, \mathrm{div}_{g_\varepsilon} \mu_\varepsilon|_{g_\varepsilon} \;\leq\; C_3 \cdot \varepsilon^{\frac{n-2}{2} - \delta} \, \rho_\varepsilon^{\delta-3} \end{cases} \tag{86}$$

for some positive constants $C_2, C_3 > 0$. Since by construction both the $|\mathrm{div}_{g_\varepsilon} \mu|_{g_\varepsilon}$ and $|\nabla \, \mathrm{div}_{g_\varepsilon} \mu_\varepsilon|_{g_\varepsilon}$ terms are zero outside the boundary of the gluing region and bounded in the middle of the polyneck, it is always possible to find such constants for every $\delta \in (-1/2, 0)$. If $n \geq 4$ and if we require that the weight $\delta \in ((2-n)/2, 0)$ be greater than or equal to $(3-n)/2$, the condition reduces to finding two positive constants

$C_4, C_5 > 0$ such that

$$\begin{cases} |\mathrm{div}_{g_\varepsilon}\mu|_{g_\varepsilon} \;\leq\; C_4 \cdot \varepsilon^{\frac{n-2}{2}}\, \rho_\varepsilon^{\delta-2} \\ \\ |\nabla\,\mathrm{div}_{g_\varepsilon}\mu_\varepsilon|_{g_\varepsilon} \;\leq\; C_5 \cdot \varepsilon^{\frac{n-2}{2}}\, \rho_\varepsilon^{\delta-3} \end{cases} \tag{87}$$

With the choice $(3-n)/2 \leq \delta < \min\{0, (6-n)/2\}$, it is always possible to find such constants, since by construction $|\mathrm{div}_{g_\varepsilon}\mu|_{g_\varepsilon}$ and $|\nabla\,\mathrm{div}_{g_\varepsilon}\mu_\varepsilon|_{g_\varepsilon}$ satisfy the properties mentioned above.

What remains to prove is the *a priori* estimate for solutions to the linearized problem $\mathcal{L}_{g_\varepsilon} v = w$. Following [17], we want to prove that for every $\gamma \in (2-n, 0)$ there exists a positive constant $C_{n,\gamma} > 0$ and a real number $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \in (0, \varepsilon_0)$, if $v, w \in \mathcal{C}^0(M_\varepsilon)$ are functions satisfying $\mathcal{L}_{g_\varepsilon} v = w$, then

$${}^{\varepsilon}\|v\|_{\mathcal{C}^0_\gamma(M_\varepsilon)} \;\;\leq\;\; C \cdot {}^{\varepsilon}\|w\|_{\mathcal{C}^0_{\gamma-2}(M_\varepsilon)} \,. \tag{88}$$

As shown in [17], this result can be achieved as a consequence of an analogous local *a priori* estimate. More precisely it is sufficient to prove that for every $\gamma \in (2-n, 0)$ there exist a real number $\alpha = \alpha(n,\gamma) > 0$ and a positive constant $C_{n,\gamma}$ such that for all $\varepsilon \in (0, \mathrm{e}^{-\alpha})$, if $v, w \in \mathcal{C}^0(M_\varepsilon)$ are functions verifying $\mathcal{L}_{g_\varepsilon} v = w$, then

$${}^{\varepsilon}\|v\|_{\mathcal{C}^0_\gamma(T^\varepsilon_\alpha)} \;\;\leq\;\; C_{n,\gamma} \cdot {}^{\varepsilon}\|w\|_{\mathcal{C}^0_{\gamma-2}(T^\varepsilon_\alpha)} + {}^{\varepsilon}\|v\|_{\mathcal{C}^0_\gamma(\partial T^\varepsilon_\alpha)} \,, \tag{89}$$

where T^ε_α is the portion of the poly-neck where $\alpha + \log\varepsilon \leq t \leq -\alpha - \log\varepsilon$. Since in [17] an analog of the estimate (89) is obtained for solutions to the equation $\Delta_{g_\varepsilon} v = w$, it is sufficient to adapt it to our case. Since for α large enough and ε sufficiently small we have that

$$\sup_{T^\varepsilon_\alpha} \left|\rho_\varepsilon^{-\gamma+2}\, |\mu_1|^2_{g_1}\, v\right| \;\;\leq\;\; \frac{1}{4}\, \sup_{T^\varepsilon_\alpha} \left|\rho_\varepsilon^{-\gamma}\, v\right| \tag{90}$$

$$\sup_{T^\varepsilon_\alpha} \left|\rho_\varepsilon^{-\gamma+2}\, \left(\tau^2/m\right)\, v\right| \;\;\leq\;\; \frac{1}{4}\, \sup_{T^\varepsilon_\alpha} \left|\rho_\varepsilon^{-\gamma}\, v\right| \,, \tag{91}$$

this implies (89). Having obtained all the estimates required, it is now possible to apply the Schauder fixed point theorem as in [17], provided that $\max\{-1, (2-n)/2\} < \gamma < 0$. Hence we find a solution v_ε to the equation (82). Moreover we have that $\|v_\varepsilon\|_{L^\infty(M)} = \mathcal{O}\left(\varepsilon^{-\gamma}\right)$ and this completes the proof of Theorem 1.

5 Conclusions and further directions

The result provided in Theorem 1 allows one to build a new zoo of solutions to the vacuum Einstein equation, hence it provides a good instrument to investigate the structure of the space-time. Notice that in the case of the classical $(3+1)$-dimensional space-time, our result reduces to the IMP gluing, because of the hypothesis on the codimension of K. However people who study string theory, might find our result of some interest on the physical point of view.

Following [11] our result can be extended to the case of Asymptotically Euclidean (AE) and Asymptotically Hyperbolic (AH) initial data set without difficulty. In fact to adapt the proof, it is sufficient to slightly modify the functional setting, in order to guarantee the existence of an inverse for the vector Laplacian and for the linearized Lichnerowicz operator.

Another possible improvement of our result consists in localizing the construction in order to produce a new initial data set which is exactly like the starting ones out of the polyneck, as it has already been done for the IMP gluing [5].

Finally (ongoing work) the structure of the polyneck should be further investigated. In particular we expect that in certain cases it is possible to find an apparent horizon in the middle of the polyneck, hence the space-time development of such initial data sets is forced to contain multidimensional black holes with possibly non trivial topology.

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References

[1] T. Aubin, *Some nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics, Springer, (1998).

[2] R. Beig, P. T. Chruściel, R. Schoen, *KIDs are non-generic* Ann. Henry Poincaré 6 (2005), 155–194.

[3] R. Bartnik and J. Isenberg, *The constraint equations*, The Einstein equationsand the large scale behavior of gravitational fields, eds. P. T. Chrusciel and H. Friedrich, Birkhäuser, Basel, (2004), 1–39.

[4] Y. Choquet-Bruhat, *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires*, Acta Math. 88 (1952), 141–225.

[5] P. T. Chruściel, J. Isenberg, D. Pollack, *Initial data engineering*, Comm. Math. Phys. 257 (2005), 29–42.

[6] M. Giaquinta, L. Martinazzi, *An introduction to the regularity thoery for elliptic systems, harmonic maps and minimal graphs*, Edizioni della Normale (2005).

[7] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equation of Second Order*, Springer, (1983).

[8] M. Gromov, H. B. Lawson *The classification of simply connected manifolds of positive scalar curvature.* Ann. of Math. (2) 111 (1980), no. 3, 423–434.

[9] D. Joyce *Constant scalar curvature metrics on connected sums.* Int. J. Math. Math. Sci. (2003), no. 7, 405–450.

[10] J. Isenberg, D. Maxwell, D. Pollack, *A gluing construction for non vacuum solutions of the Einstein constraint equations*, Adv. Theor. Phys. 9 (2005), 129–172.

[11] J. Isenberg, R. Mazzeo, D. Pollack, *Gluing and wormholes for the Einstein constraint equations*, Comm. Math. Phys. 231 (2002), 529–568.

[12] J. M. Lee, T. H. Parker, *The Yamabe Problem*, Bull. A.M.S. 17 (1987), 37–91.

[13] R. Mazzeo, F. Pacard, *Constant scalar curvature metrics with isolated singularities*, Duke Math. Journal 99 No. 3 (1999) 353–418.

[14] R. Mazzeo, F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, Comm. Anal. Geom. 9 No. 1 (2001) 169–237.

[15] R. Mazzeo, F. Pacard, D. Pollack, *Connected sums of constant mean curvature surfaces in Euclidean 3 space*, J. Reine Angew. Math. 536 (2001), 115–165.

[16] R. Mazzeo, D. Pollack, K. Uhlenbeck, *Connected sums constructions for constant scalar curvature metrics*, Topological Method in Nonlinear Analysis 6 (1995), 207–233.

[17] L. Mazzieri *Generalized connected sum construction for nonzero constant scalar curvature metrics* to appear in Communication in Partial Differential Equations.

[18] L. Mazzieri *Generalized connected sum construction for scalar flat metrics*, arXiv:math.DG/0611778 (2006).

[19] R. Schoen, S. T. Yau *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. 28 (1979), no. 1-3, 159–183.

[20] M. Taylor *Partial differential equations III: nonlinear equations*, Appl. Math. Sci. 117, New York: Springer-Verlag, (1996).